Quantitative Risk Management and Pricing for Equity-Based Insurance Guarantees

by

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Declaration

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Abstract

Equity-based insurance guarantees also known as unit-linked annuities are annuities with embedded exotic, long-term and path-dependent options which can be categorised into variable and equity indexed annuities, whereby investors participate in the security markets through insurance companies that guarantee them a minimum of their invested premiums. The difference between the financial options and options embedded in equity-based policies is that financial ones are financed by the option buyers' premiums, whereas options of the equity-based policies are financed by also continuous fees that follow the premium paid first by the policyholders during the life of the contracts. Other important dissimilarities are that equity-based policies do not give the owner the right to sell the contract, and carry not just security market related risk, but also insurance related risks such as the selection rate, behavioural, mortality, others and the systematic longevity. Thus equity-based annuities are much complicated insurance products to precisely value and hedge. For insurance companies to successfully fulfil their promise of eventually returning at least initially invested amount to the policyholders, they have to be able to measure and manage risks within the equity-based policies. So in this thesis, we do fair pricing of the variable and equity indexed annuities, then discuss management of financial market and insurance risks management.

Uittreksel

Aandeel-gebaseerde versekering waarborg ook bekend as eenheid-gekoppelde annuiteite is eksotiese, langtermyn-en pad-afhanklike opsies wat in veranderlike en gelykheid geindekseer annuiteite, waardeur beleggers neem in die sekuriteit markte deur middel van versekering maatskappye wat waarborg hulle 'n minimum van geklassifiseer kan word hulle belê premies. Die verskil tussen die finansiële opsies en opsies is ingesluit in aandele-gebaseerde beleid is dat die finansiële mense is gefinansier deur die opsie kopers se premies, terwyl opsies van die aandele-gebaseerde beleid word deur ook deurlopende fooie wat volg op die premie wat betaal word eers deur die polishouers gefinansier gedurende die lewe van die kontrakte. Ander belangrike verskille is dat aandele-gebaseerde beleid gee nie die eienaar die reg om die kontrak te verkoop, en dra nie net markverwante risiko sekuriteit, maar ook versekering risiko's, soos die seleksie koers, gedrags, sterftes, ander en die sistematiese langslewendheid. So aandeel-gebaseerde annuiteite baie ingewikkeld versekering produkte om presies waarde en heining. Vir versekeringsmaatskappye suksesvol te vervul hul belofte van uiteindelik ten minste aanvanklik belê bedrag terug te keer na die polishouers, hulle moet in staat wees om te meet en te bestuur risiko's binne die aandeel-gebaseerde beleid. So in hierdie tesis, ons doen billike pryse van die veranderlike en gelykheid geïndekseer annuiteite, bespreek dan die bestuur van finansiele markte en versekering risiko's bestuur.

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Finally, I thank my family for the support I get that always give me courage to study further.

Dedications

This thesis is dedicated to all my family.

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Chapter 1

Introduction

When approaching retirement, people are confronted by a range of financial risks and uncertainties in their lives ahead. A primary concern for them is gaining insubstantial income and the associated problem of outliving their capital. So retirement savings tend to be what people rely on in their older ages. As they enjoy their retirement savings, the question is do these investments provide effectual protections against unfavourable conditions of the market or retirement income risk? As another means of dealing with such a challenge equity-based annuities, also called unit-linked annuities, form part of safer investment and retirement advance arrangements.

Increased life expectancy as well as reduction of the state retirement pensions in several countries led to the rapid growth of equity-based annuities. They have advantageous tax treatment of the proceeds, and at the same time allows participation in the financial/security markets, see Mackenzie (2010). These are specifically retirement designed, long-term financial deals made between investors and investment/insurance companies whereby the companies concur to provide a lump sum payment to some, or payments periodically, starting either immediately or in future time with guaranteed benefits.

Some of the key reasons making equity-based annuities attractive are that: investors finally have a solution to a reduced mortality rate, replacement to pensions, participation in the security market, and alleviation of many investment risks. Even through the recent recession of 2008 there has been a strong demand to these insurance products. These types of investments were introduced in the United States in the 1970s, then in the early 1990s insurance companies started including some guarantees in policies of that nature, see Holz *et al.* (2012). However, the idea of a retirement income came from the Romans with the jurist turned annuity dealer "Gnaeus Domitius Annius Ulpianis" who also wrote the first mortality table.

The equity-based types of contracts are divided into two categories, namely, *variable annuities* (VA) and *equity indexed annuities* (EIA). Insurance companies take a certain percentage of the policyholders' premiums and invest it on their behalf in a portfolio which gives returns that match with a stock index returns, see Marshall (2011). These portfolio returns that the policyholder benefits from are attainable if the market is booming, and when the market condition is unfavourable the policyholder's investment account does not see a growth but is still protected as an equity-based annuity account.

In VA, the return on the portfolio that the insurance company gets after participating on behalf of the policyholder is entirely transferred into the policyholder's account, whereas in EIA we can have a combination of VA and fixed annuities but the return is capped and floored within a certain interval for the policyholder's account. These insurance policies come when other means of retirement income becoming increasingly unsustainable.

Because they provide a primal form of lucrative and protected investment, VA and EIA are very popular in the United States and their use now has spread to Japan and many European countries. In 2012, *LIMRA Secure Retirement Institute* published a report revealing that \$159 billion was invested in variable annuities in US by year end of 2011, see LIMRA (2012). The *Milliman Incorporated* is an independent international actuarial and consulting firm that published in August 2011 a survey report on variable annuity development in Japan. This was led by a principal and senior consultant *Ino Rikiya*, and revealed that about \$216.5 billion were invested in variable annuities by March 2011.

In 2010-2011, a financial institution of the European Union called the *European Insurance and Occupational Pensions Authority* (EIOPA) conducted a survey to find out the size of the variable annuities market in Europe and the findings estimated about €188 billion invested in variable annuities, see EIOPA (2011). A commission created by government congress in the US called *Securities and Exchange Commission* that regulates security markets, stated that about \$25 billion was invested in EIA in 2007, and in 2008 the Commission estimated about \$123 billion invested. In South Africa, the available latest data from the *Financial Services Board* (FSB) that was analysed by the department of national treasury revealed in 2012 that the annuity market had increased from R8 billion invested in 2003 to R31 billion in 2011, see Treasury (2012). We are yet to see the latest reports estimating total annuity fund invested for 2013-2014 in these countries.

The insurance institutions operating in the world's three largest annuity markets that offer equity-based policies include among others as displayed in Table 1.1.

US	Japan	Europe
Jackson National	Hartford Life	Aegon
Prudential Financial	Nippon Life	Lincoln
Met Life	Sumitomo Life	Met Life
Nationwide Financial	AIG Fuji Life	Allianz
Transamerica Corporation	Nipponkoa Insurance	Swiss Life
Riversource Annuities	Sompo Japan	Bayern LB
AIG	Tokio Marine	AXA Protection
TIAA-CREF	Dai-ichi Life	Generali Group
Ameriprise Financial	Manulife Financial	Deutsche Bank
Hartford	Mitsui Mutual Life	ING Group

Table 1.1: Insurance Companies in Top Three Markets

Equity-based products mitigate the risk that many retirees will outlast their retirement savings. At inception of the deal, an investor either chooses to make a periodic payment or a lump sum premium to the investment account for later retirement withdrawals. These VA and EIA products have become people's hope to getting an assured pay-check and something that provides a substitute for the pension. Since they have guarantees, they do not fall under securities in the markets, but insurance policies for some regulatory reasons.

The equity-based products are planned out with an intent to protect the income against the effect of any unfavourable market condition. Within these equity-based insurance policies tax is included, but it is only taken after withdrawal is made and not as the annuity occurs. What is so advantageous about this is that the amount that would have been deducted for tax annually is compounding and accrues within the annuity. This also suggests that it is lucrative to defer withdrawals rather than making immediate exhaustion.

The equity-based products also include a refund feature to a contract. They are successful because they meet most of the requirements and needs of the investors and make easier the decisions about trading them. When entering an annuity contract, there are some options on how long the annuity should last, such as the annuitant's lifetime, beneficiary's lifetime, or a pre-specified duration.

When valuing the equity-based products, it is important that people first understand the contract benefits in an insurance policy. In the case where the insurer pays a benefit when a policyholder is still alive, the policyholder benefits from a stream of payments or a lump sum where the value G_T is received at each maturity date $T = t$ until the expiration date $T = T_*$. Thus G_T is a maturity or a living benefit. However in the case where the policyholder dies first, the benefit as a lump sum is paid to the beneficiary, which is the value

G_t received at t^{th} time if the policyholder dies between $t-1$ and t, where $t \in \mathbb{Z}_+$.

In Table 1.2 we show the advantages and disadvantages of the equity-based annuities to both the insurance company and policyholder.

Insured	Insurer
Advantages	**Advantages**
Numerous investment choices Participation in the security market Potential for unrestricted asset growth Tax-deferred amount accumulation Fully protected from investment risks Rest assured to receive income benefits	Clients pay for guarantees Surpass insurers without VA&EIA
Disadvantages	**Disadvantages**
Advisory fees Administrative fees Over limit withdrawal penalty charge Dissuasive surrender charges Death forfeits lifelong benefit balance Early withdrawal charges imposed	Many risks to deal with Obligation to pay claims

Table 1.2: Advantages and disadvantages of equity-based annuities

Surrender and Cliquet Option Features

For insurance companies to meet the needs of customers, equity-based annuity products have had to evolve over time. They are not similar to each other in the way guaranteed amount is decided. Some have features such as *cliquet/ratchet* option, where the guarantee base/balance is reset to equal the level of the accrued account value during the life of the contract if the annuitant wants to increase the annual withdrawal percentage.

These features depend on the value of the account at the maturity or death period. The annuitant can make a choice of stepping up payments after a certain period agreed by both parties but with some charges. If for example, the value of the account \$45,000 exceeds the guarantee base \$34,000, then the guarantee base can be set again to equal the value of the account \$45,000. From this information, clearly, it is not wise if not impossible to reset when the base exceeds the account value. For further reading, see Liu (2010).

Another feature which is of greater concern to the insurer is the *surrender* option. This is because the insurer is forced to consider due to unpredictable behaviours of policyholders and circumstances they find themselves in. In this case, due to several personal reasons and unexpected situations, the policyholder may want to surrender the policy. By surrender we refer to an act of deciding to terminate the contract. On account of the costs the policy provider/insurer experienced when providing the contract, it will cost the policyholder some charges to surrender. The amount of the surrender cash value that the policyholder receives depends on the period the policy is to be surrendered.

The earlier the policyholder surrenders the policy the higher the charges, simply because it has never satisfied the insurer's expectations yet. Also, it can be optimal to surrender the contract if the account value W_t accrued to exceed the guarantee base H_t. In many companies the surrender value is acquired in the policy if the premiums were paid regularly for at least 3 years. When surrendering the contract, all benefits that are associated also terminate. So it is important that the policyholder considers terminating by means of surrender when the policy does not live up to its promises.

Among the factors that trigger the surrenders by policyholders, we have the change in GDP, inflation and unemployment. These can cause the interest rates in the market to drive policyholders to end up surrendering their contracts. Usually, when there is an increase in interest rates many people surrender their contracts and as a result the insurance company's surplus worsens.

When interest rates decrease, the number of people who surrender their contracts also falls because surrendering at that period is a lose situation for the policyholder since the account value is low. To model the surrender rates, Kim (2005) suggested that a logit function or what is called a logistic regression model is a suitable model. In this thesis we represent the function by

$$\begin{aligned}\ln\left(\frac{\pi}{1-\pi}\right) &= b_0 + b_1\cdot \mathrm{GDP} + b_2\cdot \mathrm{Inflation} + b_3\cdot \mathrm{Unemployment}\\ &\quad + b_4\cdot \mathrm{Difference} \qquad (1.0.1)\end{aligned}$$

where the surrender rate is denoted by π, the coefficients that should be estimated are b_1, b_2, b_3 and b_4, and the factors that represent explanatory variables are as written. The difference of rates represent the interest rates from the security market minus the annuity crediting rate from the insurance company credited to the policyholder's account. These explanatory variables we mentioned help to explain any change in the value of the response variable which in our case is the logit function of the surrender rates.

The analysis is as displayed in Section A.1, where the data used for GDP growth, unemployment and inflation are taken from the *US Bureau of Labor Statistics Department (2014)*. The surrender rates data is collected from LIMRA *Secure Retirement Institute*. The difference of rates is the 30 years Treasury yield rates minus the annuity crediting rates assumed to be 7,5%. The Treasury yield data is from *Yahoo*.

Background Review on Policy Products

In this thesis, the focus of our study is based on pricing and risk management framework of variable annuities and equity indexed annuities, where we will be giving fair prices to these insurance products.

Starting with the Guaranteed Minimum Death Benefit (GMDB), academic researchers and market practitioners such as Mudavanhu and Zhuo (2002) made a contribution to pricing the death benefit by analysing the benefit with a lapse option as a strategy to increase the account value and expose the insurer to the fee loss. Piscopo (2009) also made analysis of variable annuities and embedded option that included the GMDB. In their paper, Marshall *et al.* (2010) decompose a payoff of the Guaranteed Minimum Income Benefit (GMIB) to analyse its value. Two years later, they again examined the static hedge effectiveness, see Marshall *et al.* (2012).

Kélani and Quittard-Pinon (2014) developed a unified framework of pricing, hedging and assessing the risk existing in variable annuity guarantees including the Guaranteed Minimum Accumulation Benefit (GMAB) in a Lévy market. The recent complex variable annuity product is the Guaranteed Minimum Withdrawal Benefit (GMWB). A breakthrough in the pricing of this benefit was made by Milevsky and Salisbury (2006). In their paper they considered a geometric Brownian motion for investment fund process, and suggested withdrawals are continuous. Their work was partitioned into two strategies of policyholder's behaviour.

Bauer *et al.* (2008) generalize a finite mesh discretization technique to model and define a fair price to the GMWB product. Dai *et al.* (2008) used optimal withdrawal strategy by maximizing the expectation of the discounted value of the cash inflows, and further explored by also applying the penalty charge method suggested to solve singular stochastic models.

Discrete pricing to the product was also part of the framework in Dai *et al.* (2008). In their work, some important features such as surrenders and resets were included. The most recent popular variable annuity product is the GMWB for life abbreviated as GLWB, short for Guaranteed Lifelong Withdrawal Benefit. The product is an extension of GMWB. Holz *et al.* (2012) priced the product by including different features and taking into account how retiree's behaviour has an impact to the policy.

In other papers studying the variable annuities such as Ngai and Sherris (2011), the strategies in managing mortality risk embedded in variable annuities was investigated. Many other articles published thereafter to improve the framework and broaden the strategies in pricing variable annuity products.

For equity indexed annuities that will be discussed in this thesis, more information is found in papers such as Hardy (2003), Nielsen and Sandmanne (2002), Bacinello (2003) and many others.

The structure of this thesis is as follows: in Chapter 1, we already gave a necessary introduction that also include a review to the study of equity-based annuities, their advantages, disadvantages and features. In Chapter 2, we begin the study of variable annuities by first stating the assumptions for the valuation of all equity-based annuities, then pricing and hedging the first introduced VA benefit called the GMDB. In Chapter 3, we do valuation and hedging of the GMIB and GMAB. In Chapter 4, we introduce the most popular variable annuity called the GMWB product, giving a detailed explanation of its features and examples.

With the use of data from Yahoo (2010), we also examine sustainability of withdrawals for a certain chosen exhaustion rate, and eventually formulate pricing model to valuing the GMWB product. We further include its extended form GLWB product in pricing. Lastly, In Chapter 5, we base our study on valuing equity indexed annuities with cliquet and surrender feature.

Chapter 2

Valuation of Variable Annuities

Before getting to the pricing of variable annuities, let us consider the following assumptions underlying the process of pricing the variable annuities.

Assume $(\Omega, \mathbb{F}, \mathbb{Q})$ is a probability space where Ω is a sample space, the filtration $(\mathbb{F}_t)_{t\in[0,T_*]}$, and $\mathbb{Q}$ is the risk neutral probability measure on $(\Omega, \mathbb{F})$. The risk neutral probability is known in finance as the probability of future outcomes adjusted for risk, which helps in computing the expectation of asset values. Thus we will make valuations of payment streams under risk neutral measure as the expectation of the discounted values. This assumption also implies that security markets where financial agents are trading is frictionless with no arbitrage opportunities, see Bauer *et al.* (2008).

We assume also under the risk neutral measure $\mathbb{Q}$ that the reference equity index value S evolves according to a geometric Brownian motion

$$dS_t = (r - \delta)S_t dt + \sigma S_t dB_t, \quad S_0 = 1, \tag{2.0.1}$$

which has the solution

$$S_t = \exp[(r - \delta - \frac{1}{2}\sigma^2)t + \sigma B_t]. \tag{2.0.2}$$

Here B denotes the standard Brownian motion, the fee rate by δ, the risk-free return rate by r, and lastly σ the volatility of the equity returns. It is in line with the literature of investment account modeling as shown in Windcliff *et al.* (2001) and Gerber and Pafumi (2000) to assume that the geometric Brownian motion describes the index dynamics.

Variable annuities can be understood to be a combination of separate/investment accounts with guarantees where the policyholder can choose the asset category he would link to, for example, the NASDAQ, S&P 500, Bond Index or other assets combination. The *investment account* consists of sub-accounts where all annuitants make premium payments during the accumulation phase.

Simulations in the valuation of variable annuities are often the only option since they are complex, exotic, long-term, path dependent and some have no closed form solutions as in standard vanilla options. The other reason the embedded options in VA differ from standard vanilla options is that the charges are deducted periodically. Modeling of these charges and fees assumes they are taken as dividends. These insurance benefits are offered mostly by life insurance companies. The valuation of these benefits requires the application of derivatives techniques as they are derivative oriented. To some, annuitants are allowed to access their accounts every time, but surrendering the contract and making withdrawals exceeding yearly guaranteed amount may have harsh penalty charges. As for the *guarantees*, calculations of the guarantees to be withdrawn are made with reference to their guarantee base. Guarantees in variable annuities are provided even if the account value has gone low. Unlike equity indexed annuities EIA, variable annuities VA have no cap/ceiling on the investment growth.

VA policies have many choices making them even more complex and attractive to investors. There are two main types of VA guaranteed minimum benefits: a death benefit and four living benefits as listed below.

(i). **Guaranteed Minimum Death Benefit (GMDB)** - Here an assured lump sum is being given to the beneficiary when the policyholder dies. The GMDB and other variable annuity benefits that include a death cover have a stochastic maturity due at the end of the spontaneous exercise period, and also are increasingly put options. The payoff is given by

$$B^D = \max(W_0 e^{gT_*}, W_{T_*}), \tag{2.0.3}$$

where W_0 is the initial account value at time $t = 0$, g is the guaranteed rate of growth, and W_{T_*} is the account value at a random maturity time T_* when the policyholder dies.

(ii). **Guaranteed Minimum Income Benefit (GMIB)** - This type of investment is suitable for people who plan to annuitize their contracts. Money saved in the account is annuitized to a stream of guaranteed income for life at a certain point in the future as the maturity after the deferral period. By *annuitization* we mean to start a stream of payments from the money that has been invested and accumulated. If the policyholder dies before the conversion period, the beneficiary receives payment by the GMDB whereas after the conversion, the beneficiary is no longer part of the deal. Once annuitization has been triggered it is irreversible, and that means the policyholder has no access to the account value except from receiving a stream of fixed guaranteed income benefits G annually.

(iii). **Guaranteed Minimum Accumulation Benefit (GMAB)** - For this contract, a lump sum guaranteed as the minimum of all deposits is given to the policyholder at a contract maturity date, regardless of the performance of the fund. This accumulation benefit GMAB is also known as the maturity benefit GMMB with a cliquet feature in it, and is a similar contract to the GMDB except the assumption that the policyholder is alive at the maturity date. See Kélani and Quittard-Pinon (2014) and Quittard-Pinon and Randrianarivony (2009) on how the GMMB/GMAB is linked to the GMDB.

(iv). **Guaranteed Minimum Withdrawal Benefit (GMWB)** - This contract is different from GMIB in that it can allow immediate withdrawals, while assuming the retiree is still alive at expiration date. Depending on how much he invested, the policyholder is guaranteed to receive a certain amount each year usually less than 8% of the nest egg invested. Mathematically, we can express it as follows: let $G = wH_0 = wW_0$, for w the percentage rate of withdrawal, be the guaranteed annual amount to be withdrawn as long as the guarantee base H is not exhausted at each yearly withdrawal maturity date T before the expiration date T_*. Then the final withdrawal at time T_* is given by

$$B^W = \max(G, W_{T_*}), \tag{2.0.4}$$

which is the greater of the yearly minimum withdrawal and the remaining account value at expiration date.

(v). **Guaranteed Life Withdrawal Benefit (GLWB)** - The most recent guarantee introduced in 2004 is a hybrid of GMIB and GMWB. The difference is that it is only immediate, and the policyholder can only withdraw a fixed annual amount G for the remaining lifespan without the limit on the total amount that can be withdrawn. This product is usually given to people who wish to start withdrawals from the age of 65.

In the following Section 2.1, we start the pricing of variable annuities with the death benefit.

2.1 Valuation of the GMDB

The Guaranteed Minimum Death Benefit (GMDB) product is a withdrawal-deferred annuity contract whereby the policyholder makes a lump sum payment once at contract inception or through periodic payments to the insurance company as investment premiums. Should the policyholder die, the amount as a lump sum which is the minimum of invested premium, is given to the beneficiary. Main GMDB literature includes that of Mudavanhu and Zhuo

(2002), Milevsky (2006), Hardy (2003) and Piscopo (2009).

In this thesis, in a situation where we have only one maturity as in a death benefit GMDB, $T = T_*$ because the maturity date will also be an expiration date. We can also think of T_* conforming to T_{a*} in actuarial literature, as the remaining future time of life random variable which takes any time t for policyholder aged a, with $F_a(t)$ and $f_a(t)$ as its cumulative distribution function and probability density function respectively. Then the probability that a person aged a dies before reaching the time t is given by

$$\begin{aligned} F_a(t) &= P(T_* \leq t) \\ &= {}_tq_a \\ &= 1 - {}_tp_a \\ &= 1 - \frac{n-a-t}{n-a}. \end{aligned} \tag{2.1.1}$$

Here ${}_tp_a$ is the probability that the policyholder aged a will still be living at age $a+t$, for $t = 0, 1, 2, \cdots, n-a$, and n is the terminal age above which aliveness is impossible. Denote again by q_{a+t} the probability that the policyholder of age $a+t$ dies during the course of the following year, then

$$\begin{aligned} q_{a+t} &= 1 - p_{a+t} \\ &= \frac{1}{n-a-t}. \end{aligned} \tag{2.1.2}$$

Therefore, ${}_tp_a q_{a+t} = ({}_{t|1}q_a)$ is the probability that the policyholder of age a, will die between time $a+t$ and $a+t+1$. This survival model is for illustration only and should not be used for any applications. See Dickson *et al.* (2013) and Hardy (2003) for more on probabilities of survival and death.

In the case of GMDB, we do not have the annual withdrawal guarantee we usually denote by G. Here the personal annuity account value W_t obeys the SDE

$$dW_t = (r-\delta)W_t dt + \sigma W_t dB_t,$$

where other parameters are as mentioned in equation (2.0.1), see Chu and Kwok (2004) for account and equity values dynamics.

In Hardy (2003), the payoff to the GMDB product at a maturity T_* is understood to be

$$B^D_{T_*} = \max(W_0 e^{gT_*}, W_{T_*}), \tag{2.1.3}$$

where g is the guaranteed rate of growth. This payoff can have a valuation that resembles a put option as

$$B^D_{T_*} = \max(W_0 e^{gT_*} - W_{T_*}, 0) + W_{T_*}. \tag{2.1.4}$$

This suggests the GMDB at a maturity T_* is the sum of the account value and an European put option that has a strike price $W_0 e^{gT_*}$ and the asset price W_{T_*}.

Let the underlying stock process be S_t, so $W_{T_*} = S_{T_*} e^{-\delta T_*}$, and since S_t is an index we can set S_0 to be whatever we want here. Set $W_0 = S_0$. Since we have a stochastic maturity T_* and the investment account value W_{T_*} that are independent of each other, the time zero value of the GMDB product can be determined by

$$
\begin{aligned}
B_0^D &= \mathbb{E}_t[\mathbb{E}^{\mathbb{Q}}[e^{-rT_*} B_{T_*}^D |_{T_*=t}]] && (2.1.5)\\
&= \mathbb{E}_t[\mathbb{E}^{\mathbb{Q}}[e^{-rT_*}(\max(W_0 e^{gT_*} - W_{T_*}, 0) + W_{T_*})|_{T_*=t}]] && (2.1.6)\\
&= \mathbb{E}_t[\mathbb{E}^{\mathbb{Q}}[e^{-rT_*}(\max(S_0 e^{gT_*} - S_{T_*} e^{-\delta T_*}, 0) + S_0 e^{-\delta T_*})]], && (2.1.7)
\end{aligned}
$$

where the expectation inside is taken on W_{T_*} conditional on the fixed value of $T_* = t$, whereas the expectation outside is taken on all possible values of T_*, see Carr and Wu (2004). Let the inside expectation with the embedded put option in equation (2.1.7) be denoted as follows

$$
\mathbf{EEP} = \mathbb{E}^{\mathbb{Q}}[e^{-rT_*}(\max(S_0 e^{gT_*} - S_{T_*} e^{-\delta T_*}, 0) + S_0 e^{-\delta T_*})]. \qquad (2.1.8)
$$

By using the Black and Scholes (1973) model to find the embedded put option price, equation (2.1.8) becomes

$$
\begin{aligned}
\mathbf{EEP} &= W_0 e^{gT_*} e^{-rT_*} \Phi(-\mathbf{d_B}) - W_0 e^{-\delta T_*} \Phi(-\mathbf{d_A}) + W_0 e^{-\delta T_*}\\
&= W_0 (e^{-\delta T_*} \Phi(\mathbf{d_A}) + e^{(g-r)T_*} \Phi(-\mathbf{d_B})) && (2.1.9)
\end{aligned}
$$

where

$$
\mathbf{d_A} = \frac{(r - \delta - g)T_* + \frac{1}{2}\sigma^2 T_*}{\sigma\sqrt{T_*}}, \quad \text{and} \quad \mathbf{d_B} = \mathbf{d_A} - \sigma\sqrt{T_*}. \qquad (2.1.10)
$$

Substituting equation (2.1.9) into equation (2.1.7), we have as expressed in general form

$$
B_0^D = \int_0^{n-a} f_a(t) W_0 \Big(\mathbf{BS}(r, g, \delta, \sigma, t) + e^{-\delta t}\Big) dt, \qquad (2.1.11)
$$

where $\mathbf{BS}(r, g, \delta, \sigma, t)$ represents the Black and Scholes (1973) put option price. In discrete form, we can express it as

$$
B_0^D = \sum_{t=1}^{n-a} ({}_t p_a q_{a+t}) W_0 \Big(\mathbf{BS}(r, g, \delta, \sigma, t) + e^{-\delta t}\Big). \qquad (2.1.12)
$$

Setting the parameters $r = 0.05$, $\sigma = 0.2$, $\delta = 0.02$, $g = 0.09$, $W_0 = 100$ and $n = 100$, yields diagrams in Figure 2.1 showing the GMDB values' dependence on the age of the policyholder at contract inception.

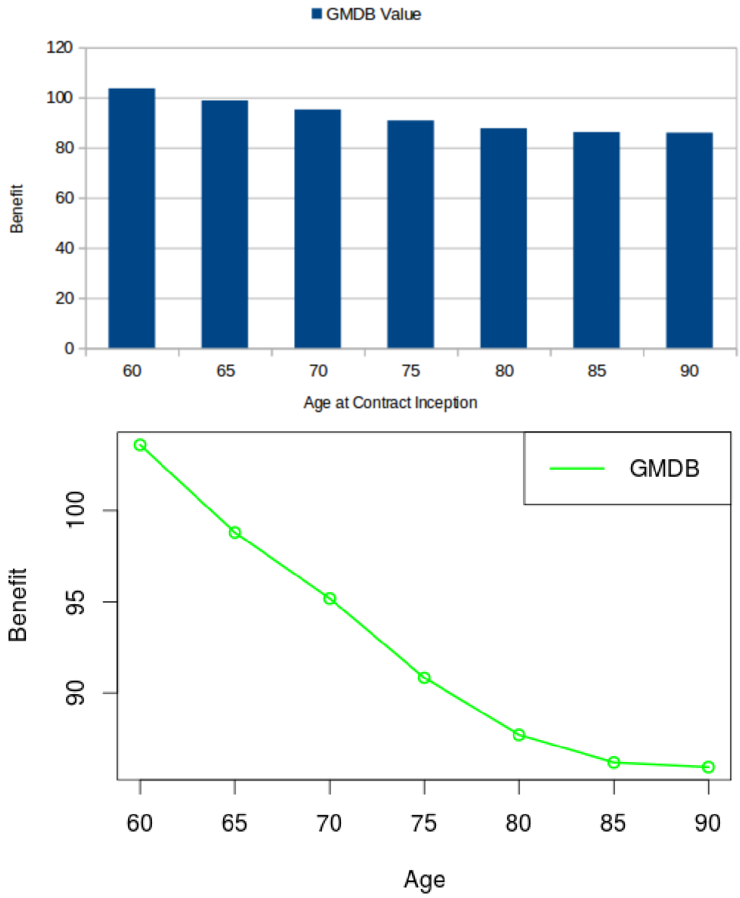

Figure 2.1: GMDB Present Values

Chapter 3

Valuation of GMIB and GMAB

The Guaranteed Minimum Income Benefit (GMIB) and the Guaranteed Minimum Accumulation Benefit (GMAB) are some of the benefits introduced after the Guaranteed Minimum Death Benefit (GMDB). In this Chapter, we price the two survival benefits starting with the GMIB in Section 3.1, then GMAB in Section 3.2.

3.1 The Pricing GMIB Model

The GMIB is a withdrawal deferred insurance policy whereby the policyholder pays a premium payment to the insurer in the form of a lump sum single premium or periodic payments. This will then be invested in the security market to accumulate over time before it can be converted to a stream of annual income at retirement time.

Before the conversion of the accumulated amount to an annual income stream by means of annuitization, there is also a choice to withdraw all the accrued amount if the calculation reveals a much higher account value as compared to the guarantee base. This type of benefit is similar to what was known before in the UK as the Guaranteed Annuity Option (GAO), except that the conversion rate applies to the account value or the guarantee base, depending on which one is maximum, yielding a fixed annual income stream, see Hardy (2003) and Kling *et al.* (2014).

As the income benefit guarantees a lifelong fixed annual payment after annuitization has been made, it implies that the investor is also protected against the longevity risk. During the weak security market conditions, the benefit still serves as a protection during the accumulation phase of the contract. Thus it is a rather challenging contract to hedge with such characteristics.

In this Section, we value the GMIB using the arbitrage free methodology and other model assumptions made in Chapter 2.

Suppose that the policyholder aged a is paid at a retirement date T, either by a lump sum which is equal to the investment value W_T linked to the reference equity fund S_T, or chose to annuitize for the remaining lifetime and get a stream of guaranteed annual income $\mathbf{c_r}H_T$. Here $\mathbf{c_r}$ is the conversion rate at which the policyholder converts the guarantee base H_T into annuity if the investment fund is equal or lower compared to the base.

As shown in Kling *et al.* (2014), here the account value W_T can be expressed as $A_0\frac{S_T}{S_0}$, where $A_0 = W_0 - \delta$ as the amount left after the fee has been deducted by the insurer from the premium. Then the income benefit (GMIB) payoff is expressed by

$$\begin{aligned} B_T^I &= \max(\mathbf{c_r}H_T\nu_{a+T} - W_T, 0) \\ &= S_T \max\left(\mathbf{c_r}\frac{H_T}{S_T}\nu_{a+T} - K, 0\right), \end{aligned} \tag{3.1.1}$$

where the strike price $K = \frac{A_0}{S_0}$, and ν_{a+T} representing the annuity factor which is given by

$$\nu_{a+T} = \sum_{t=0}^{n-(a+T)} {}_tp_{a+T}\mathbf{P}(T, T+t). \tag{3.1.2}$$

Here the denotation $\mathbf{P}(T, T+t)$ represents the time T zero coupon bond with maturity $T+t$, and ${}_tp_{a+T}$ is the probability that the person aged $a+T$ still lives at year t.

It is possible in a GMIB contract for the policyholder to annuitize if the investment fund is less than or equal to the guarantee base. So we can assume $H_T = S_T$, which implies that equation (3.1.1) can be written as

$$B_T^I = S_T \max\left(\sum_{t=0}^{n-(a+T)} \mathbf{P}(T, T+t)\mathbf{c_r}{}_tp_{a+T} - K, 0\right). \tag{3.1.3}$$

Applying the decomposition suggested by Jamshidian (1989), we may rewrite the GMIB contract payoff that is generated by the zero coupon bond portfolio with K_t strike prices, and ${}_tp_{a+T}$ survival probabilities as weights. Hence we can find from $t = 0, \cdots, n-(a+T)$ the interest rate critical value r^* such that

$$K = \sum_{t=0}^{n-(a+T)} \mathbf{P}^*(T, T+t)\mathbf{c_r}{}_tp_{a+T}.$$

From this we define the bond price corresponding with interest rate critical value by K_t which is the new strike price that is artificially introduced as

$$K_t = \mathbf{P}^*(T, T+t).$$

We know that the bond price is a monotonic function of the interest rate. Equivalently $\mathbf{P}(T, T+t)$ are decreasing functions of interest rate r. This implies that if $r* > r$, then $\sum_{t=0}^{n-(a+T)} \mathbf{P}(T, T+t)\mathbf{c}_{\mathbf{r}t}p_{a+T} > K$ and also $\mathbf{P}(T, T+t) > K_t$. Thus

$$\max\left(\sum_{t=0}^{n-(a+T)} \mathbf{P}(T, T+t)\mathbf{c}_{\mathbf{r}t}p_{a+T} - K, 0\right) = \sum_{t=0}^{n-(a+T)} \mathbf{c}_{\mathbf{r}t}p_{a+T} \max(\mathbf{P}(T, T+t) - K_t, 0),$$

and equation (3.1.3) becomes

$$B_T^I = S_T \sum_{t=0}^{n-(a+T)} \mathbf{c}_{\mathbf{r}t}p_{a+T} \max(\mathbf{P}(T, T+t) - K_t, 0). \tag{3.1.4}$$

The present value under the martingale framework valuation for the GMIB contract of the policyholder aged a at time 0 with maturity T is given by

$$\begin{aligned} B_0^I &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds\right) {}_Tp_a B_T^I\right] \\ &= {}_Tp_a \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds} S_T \sum_{t=0}^{n-(a+T)} \mathbf{c}_{\mathbf{r}t}p_{a+T} \max(\mathbf{P}(T, T+t) - K_t, 0)\right]. \end{aligned} \tag{3.1.5}$$

In order to find analytical solutions to equation (3.1.5), it is suitable to measure the payments in stock units instead of values of the money market. So we are to establish a numeraire as the equity price S_T and switch from a risk neutral measure $\mathbb{Q}$ fixed in the market to equity price measure $\mathbb{Q}^s$ corresponding to S_T. To comply with Geman *et al.* (1995) about the change of numeraire, for martingale probability measure $\mathbb{Q}^s$ equivalent to $\mathbb{Q}$, we have the density process defined as

$$\psi_T = \frac{d\mathbb{Q}^s}{d\mathbb{Q}} |_{\mathbb{F}_T} \tag{3.1.6}$$

$$= \exp\left(-\int_0^T r_s ds\right) \frac{S_T}{S_0}. \tag{3.1.7}$$

This reduces equation (3.1.5) under the new measure $\mathbb{Q}^s$ to

$$B_0^I = {}_Tp_a S_0 e^{-\delta T} \sum_{t=0}^{n-(a+T)} \mathbf{c}_{\mathbf{r}t}p_{a+T} \mathbb{E}^{\mathbb{Q}^s}\left[\max(\mathbf{P}(T, T+t) - K_t, 0)\right], \tag{3.1.8}$$

where the expectation in equation (3.1.8) is taken under the equity price measure $\mathbb{Q}^s$.

To obtain the expectation of a call option in equation (3.1.8), Vasicek (1977) made the assumption that the term structure of interest rates through the short rate r_t evolves as Ornstein-Uhlenbeck process where the bond options are explicitly calculated. The process is expressed as

$$dr_t = \varphi(\Theta - r_t)dt + \sigma dB_t \tag{3.1.9}$$

where φ, Θ, and σ are positive real constants. The standard solution for the SDE (3.1.9) is given by

$$r_t = r_s e^{-\varphi(t-s)} + \Theta(1 - e^{-\varphi(t-s)}) + \sigma \int_s^t e^{-\varphi(t-u)} dB_u. \tag{3.1.10}$$

We have

$$(r_t \mid \mathbb{F}_s) \sim N\Big(\mathbb{E}\{r_t \mid \mathbb{F}_s\} = \mu_r, \mathrm{Var}\{r_t \mid \mathbb{F}_s\} = \sigma_r^2\Big),$$

where

$$\begin{aligned} \mu_r &= r_s e^{-\varphi(t-s)} + \Theta(1 - e^{-\varphi(t-s)}) \\ \sigma_r^2 &= \frac{\sigma^2}{2\varphi}(1 - e^{-2\varphi(t-s)}). \end{aligned}$$

A zero coupon bond $\mathbf{P}(T, T+t)$ in equation (3.1.8) with expiration $T+t$ at time T is given by

$$\mathbf{P}(T, T+t) = e^{A(T,T+t) - B(T,T+t)r_T}, \tag{3.1.11}$$

where

$$\begin{aligned} B(T, T+t) &= \frac{1}{\varphi}\Big[1 - e^{-\varphi((T+t)-T)}\Big] = \frac{1}{\varphi}\Big[1 - e^{-\varphi t}\Big] \\ A(T, T+t) &= \frac{2\varphi^2(\Theta B(T, T+t) - \Theta t) + (t - B(T, T+t))\sigma^2}{2\varphi^2} - \frac{B^2(T, T+t)\sigma^2}{4\varphi}, \end{aligned}$$

see Björk (2004). For a normally distributed r_T, the bond price $\mathbf{P}(T, T+t)$ is distributed log-normally with mean $M = A(T, T+t) - B(T, T+t)\mu_r$ and variance $V = B^2(T, T+t)\sigma_r^2$. Then the expectation of a call option payoff via the Black and Scholes (1973) model is given by

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^s}\left[\max(\mathbf{P}(T, T+t) - K_t, 0)\right] &= \mathbf{FT}\Phi\left(\frac{M - \ln(K_t) + V^2}{V}\right) - K_t\Phi\left(\frac{M - \ln(K_t)}{V}\right) \\ &= \mathbf{FT}\Phi\left(\mathbf{d_A}\right) - K_t\Phi\left(\mathbf{d_B}\right) \end{aligned}$$

where $\mathbf{FT} = e^{M + \frac{1}{2}V^2}$. Thus from equation (3.1.8), the GMIB present value with a closed form solution to the Black and Scholes (1973) bond option price is given by

$$B_0^I = {}_Tp_a S_0 e^{-\delta T} \sum_{t=0}^{n-(a+T)} \mathbf{c_r} {}_tp_{a+T} \Big[\mathbf{FT}\Phi\left(\mathbf{d_A}\right) - K_t\Phi\left(\mathbf{d_B}\right)\Big]. \tag{3.1.12}$$

Consider the following parameter values: $M = 6.3$, $V = 0.4$, $K_t = 90$, $S_0 = 1$, $\mathbf{c_r} = 0.07$, $\delta = 0.001$, $n = 100$, $T = 65$, and the policyholder's age at contract inception from $a = 23$ until $a = 30$. As depicted in Figure 3.1 we show the GMIB present values for policyholders of different ages.

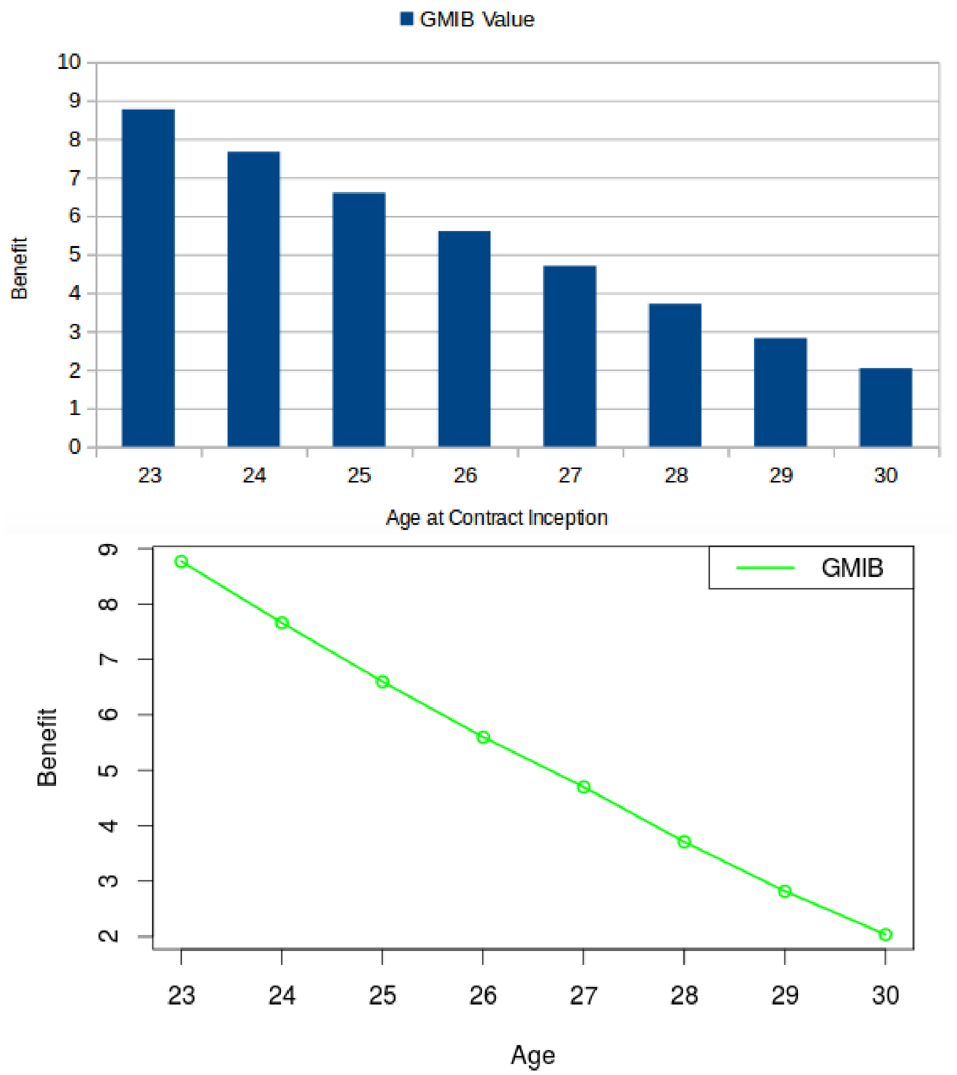

Figure 3.1: GMIB Present values

3.1.1 Hedging GMIB Through Replicating Portfolio

To hedge the GMIB embedded option using the Greeks can be complicated since they are dependent on the interest rate model. If in reality the interest rates and stock prices fluctuations are not reasonably approximated by the model, the hedging via Greeks will not work as explained in *Section 3.1* of Marshall (2011) . So the insurer has to, on behalf of the policyholder, make an investment in a portfolio that is replicating where the fee $\delta = \delta^*$ is fair if and only if the payoff of the variable annuity with a GMIB option embedded is equal to the total premium expressed as

$$B_0(\delta^*) = W_0. \tag{3.1.13}$$

A detailed explanation on how financial industries use replicating portfolios for risk management can be found in the research report Milliman (2009).

3.2 The Pricing of GMAB

In this Section, we price the GMAB variable annuity, taking into account a contract with cliquet/reset feature. The GMAB is a survival benefit which needs a fairly long deferred withdrawal period which happens once. If the policyholder dies prior to the expiration date, the contract takes the form of GMDB contract as a lump sum will be given to the beneficiary just as in a death benefit after a waiting period. It is usually called a maturity benefit GMMB with a cliquet option. In our valuation, we assume that the policyholder will still be alive at expiration date and that the contract has a cliquet feature at any chosen anniversary dates on the interval $t \in (t_0, t_n]$.

We apply here as well the assumptions of arbitrage free pricing made in Chapter 2. Let $W^b_{t_i}$ and $W^a_{t_i}$ be the account value before and after a cliquet option is exercised at an agreed anniversary date t_i, then

$$W^b_{t_i} = W^b_{t_{i-1}} e^{m(t_i - t_{i-1})} \frac{S_{t_i}}{S_{t_{i-1}}}, \tag{3.2.1}$$

where m is the rate at which insurer charges the policyholder as also can be seen in Kélani and Quittard-Pinon (2014). The GMAB at time t_i is given by

$$B^A_{t_i} = \max(B^A_{t_{i-1}}, W^b_{t_i}). \tag{3.2.2}$$

The after reset account value is given by

$$W^a_{t_i} = W^b_{t_i} + \max(B^A_{t_{i-1}} - W^b_{t_i}, 0). \tag{3.2.3}$$

Under the risk neutral measure, the time zero value for the expiration T_* of the GMAB is given by

$$B^A_{\text{pv}} = \mathbb{E}[e^{-\int_0^t r_s ds} \times \max(B^A_{t_{i-1}} - W^b_{t_i}, 0)]. \tag{3.2.4}$$

Suppose that the interest rate r is constant. Then for the chosen two anniversary dates $t = t_1$ and $t = t_2$, we have the present value

$$B^A_{\text{pv}} = \mathbb{E}[e^{-rt_1} \max(B^A_{t_{i-1}} - W^b_{t_1}, 0)] + \mathbb{E}[e^{-rt_2} \max(B^A_{t_{i-1}} - W^b_{t_2}, 0)]. \tag{3.2.5}$$

Let us exclude the *mortality* and *lapse* forces, and use the assumption that the holder is still alive at the expiration date. Also, $S_{t_0} = W_{t_0} = B^A_{t_0}$. Then by applying the Black and Scholes (1973) model, we arrive at a closed form solution for the present value of the anniversaries

$$\begin{aligned} B^A_{\text{pv}} = {} & e^{-rt_1} B^A_{t_0} \Phi(-\mathbf{d_B}) - W^b_{t_0} e^{-m(t_1 - t_0)} \Phi(-\mathbf{d_A}) \\ & + e^{-rt_2} B^A_{t_1} \Phi(-\mathbf{d_B}) - W^b_{t_0} e^{-m(t_2 - t_1)} \Phi(-\mathbf{d_A}) \end{aligned} \tag{3.2.6}$$

where

$$\mathbf{d_A} = \frac{\ln(\frac{W_{t_0}}{B^A_{t_{i-1}}}) + (r + \frac{1}{2}\sigma^2)(t_i - t_{i-1})}{\sigma\sqrt{t_i - t_{i-1}}}$$
$$\mathbf{d_B} = \mathbf{d_A} - \sigma\sqrt{t_i - t_{i-1}}. \tag{3.2.7}$$

For example, suppose the policyholder at contract inception invested a single premium amount of $W_{t_0} = 100$, and the other parameters are $r = 0.04$, $m = 0.2$, and $\sigma = 0.4$. The two reset times are $t_1 = 3$ and $t_2 = 7$, then equation (3.2.6) yields the present value

$$B^A_{\text{pv}} = 63.5$$

Chapter 4

Valuation of GMWB

In this Chapter, we begin by explaining the withdrawal benefit and sustainable withdrawal rates in Section 4.1. We then formulate the model to price the minimum withdrawal and the lifelong withdrawal benefits starting with the static pricing in Subsection 4.2.1 under the policyholder's perspective and the dynamic pricing in Subsection 4.2.2, then in Section 4.3 we employ static pricing under the insurer's perspective. In Section 4.4, we use the tree methodology by incorporating Cox *et al.* (1979) binomial, the bino-trinomial, and the stair tree structures by Dai and Lyuu (2010) to find the continuation value of the withdrawal benefit. In Section 4.5 we make valuation of GMWB for life or what is called lifelong benefit GLWB.

4.1 The Benefit Itself

One of the variable annuities people consider purchasing more often for their retirement is the GMWB. Irrespective of whether the account value has decreased or not, the GMWB product gives the annuitant the possibility of withdrawals that are guaranteed during the life of the deal. In this type of a contract, the annuitant withdraws a certain amount both parties agree upon at contract initiation or can increase withdrawal accepting some penalty. In an instance, when the annuitant dies, the GMWB product will give the beneficiary any amount left in the account if the contract is still alive. With the GMWB variable annuity, investors are also enabled to invest in the markets with stocks and bonds but under insurance regulations since they involve guarantees. To impress investors, sometimes the insurer offers that the annuitant can get bonuses if they do not withdraw within a certain period from the contract initiation. That is, for having a deferred annuity contract with withdrawals at $T \leq t \leq T_*$, where T_* is the expiration date and the maturity T denotes the time when the policyholder started annual withdrawal after a deferred period $t \in [0, T]$.

4.1.1 Sustainable Withdrawal Rates

Before getting deeper into the GMWB study, let us try to understand the calculus of sustainability of withdrawal rates by explicitly explaining and eventually presenting things pictorially.

When a retiree invests in a portfolio and wants to make withdrawals, it is obvious that he may meet the possibility of having his nest egg (total premium invested) exhausted before the expiration date. It should be taken into consideration as an important element to determine withdrawal rates that are sustainable.

One of the questions that may be raised is what withdrawal percentage can be chosen on the retirement savings in order to sustain withdrawal throughout the contract period? This is what investors can ask themselves, since for them to have a greater income in retirement periods, they must choose making higher withdrawal rates from the account. But it has to be noted that the standard of living for such withdrawal rates cannot be sustainable for a longer period. On the other hand, the lower rates of withdrawal would diminish the retirement income but help to reduce the risk of depleting funds in a short period of time.

In a popular paper of financial valuation of GMWB written by Milevsky and Salisbury (2006), it is recommended that for half a percentage of asset allocation, the withdrawal rate should not exceed 7% annually as this will shorten the withdrawal period by leading to surrendering of the contract. We will investigate how to wisely choose the exhaustion rate using the calculus of sustainable withdrawal rate, where the main idea is to supply investors with an tool to examine different withdrawal rate sustainability for a particular nest egg.

As we proceed below, we make some illustrations on how to estimate the probability of withdrawal success rate and the ruin rate. For example, let us take a retiree who is 45 years old, with 30 years as the median remaining lifetime. The present value of lifetime withdrawals is not normally distributed but distributed closer to a gamma distribution, see Milevsky (2007). Hence, the withdrawal ruin rate probability for a continuous random variable Y which takes any withdrawal rate value y with parameters k and ϕ has a probability density function given by

$$f(y; k, \phi) = \frac{1}{\Gamma(k)\phi^k} y^{k-1} e^{-\frac{y}{\phi}} \quad k, \phi \geq 0, \tag{4.1.1}$$

with mean $k\phi$ and variance $k\phi^2$.

The parameters k and ϕ are given by

$$k = \frac{2\mu + 4\eta}{\sigma^2 + \eta} - 1, \quad \phi = \frac{\sigma^2 + \eta}{2} \tag{4.1.2}$$

where η is the mortality rate, μ is the expected return rate, and σ represent the volatility of the investment returns. The withdrawal success rate is given by

$$\text{P(Success Rate)} = 1 - \text{P(Ruin Rate)}. \tag{4.1.3}$$

As we continue with our example, let us assume that the nest egg that should finance the stream of annuity withdrawals is \$300,000 invested in the NASDAQ Index. Taking the Yahoo (2010) monthly data from (January 1981-December 2010), we can use these values of the parameters k, ϕ, μ and σ. The mortality rate η is given by

$$\eta = \frac{\ln 2}{\text{median remaining lifetime}},$$

which is $\eta = \frac{\ln 2}{30} = 0.023$ in our case.

As displayed in Table B.1 of Section B.1, we show the calculated withdrawal success rate probabilities for every exhaustion rate mentioned in percentages. In Table B.2, we report the values of the investment portfolio accumulated for every exhaustion rate in every 5 years overlapping end-of-period. Those values are calculated by using the formula

$$W_t = W_{t-1}(1 + R_t) - G, \tag{4.1.4}$$

where W_t is the current remaining value of the investment account at the end of the period and W_{t-1} is the previous account value at the beginning of the period. The return rate R_t of the investment portfolio for period t is calculated by taking the average of all monthly returns within the specified periods.

Looking at the results in Table B.1, we can see the combination of nine rates of exhaustion. For the first five years, based on the withdrawal/exhaustion rate of 2%, the retiree has a withdrawal success rate probability of 87%. It shows the probability percentage of success dropping and eventually declines for the retiree who choose the higher rate of exhaustion over 30 years. Although a little more returns can increment the probability of a success rate as displayed on the results, it is also clear that it can never be optimal to increase the rate of withdrawal, for it shortens the retirement income receiving periods and eventually depletes the nest egg before the expiration date. However in equity-based annuities such as GMWB of the variable annuity class, this can never be the case for rational policyholders since they are guaranteed to get the annual fixed withdrawals until the expiration date. However if they are not behaving rationally and withdraw above the limit, then this might result to depletion and eventual surrender. The policyholder will have to pay a penalty charge every time he decides to withdraw any amount above the annual fixed guaranteed amount agreed upon at contract inception. All the results are also represented in Figure 4.1 showing success/ruin rate probability percentages for each exhaustion rate if chosen over a period of 30 years.

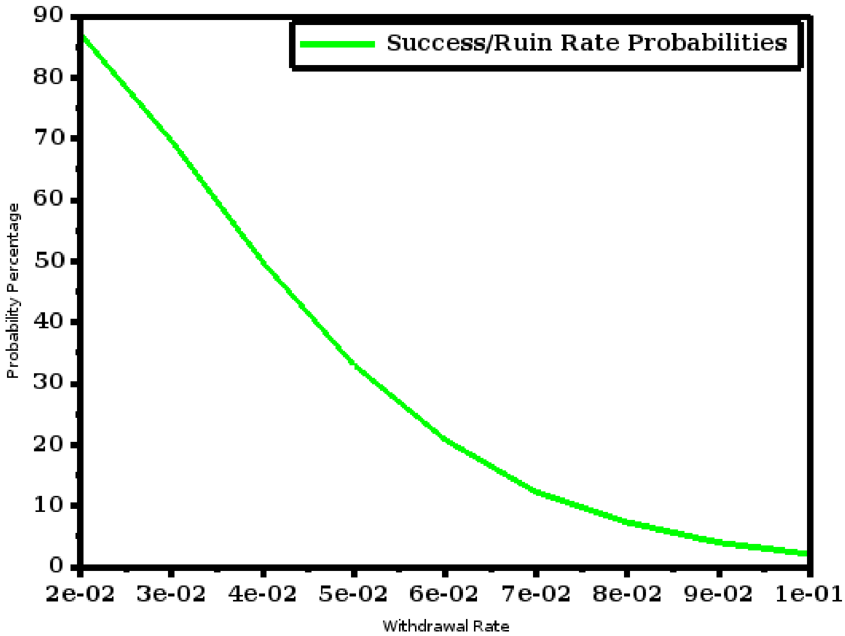

Figure 4.1: Success/Ruin Rate Probability Percentages.

Assuming non-adjusted withdrawals, let us take a constant exhaustion rate of 4% for our example with a nest egg of \$300,000. Table 4.1 below reports the results for constant annual withdrawals where W^b, W^a, and H respectively represent the account value before withdrawal, account value after withdrawal, and the guarantee base.

Years	$\mu(\%)$	W^b	Withdrawn	W^a	H
5	0.95	302849.96	12000	290849.96	288000
10	0.69	302069.96	24000	278069.96	264000
15	1.07	303209.96	36000	267209.96	228000
20	1.3	303899.96	48000	255899.96	180000
25	1.03	303089.96	60000	243089.96	120000
30	0.93	302789.96	72000	230789.96	48000

Table 4.1: Constant Exhaustion Rate.

As reported in Table 4.1, our example of 30 years contract deal, shows that for 4% constant exhaustion rate, a retiree's withdrawal rate is sustainable. Therefore, in such a case, he can choose to take all remaining amount or extend the contract and step-up/reset since after the 30^{th} year there is still some amount left.

In Figure 4.2 we represent the guarantee base values and the account value under constant exhaustion rate of 4% for our nest egg of \$300,000.

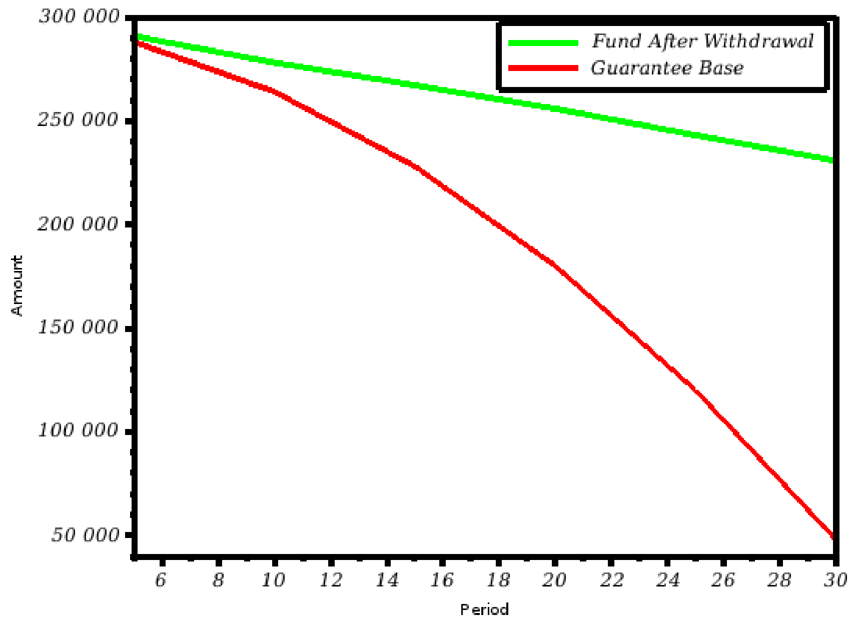

Figure 4.2: Base and Account Values under Constant Exhaustion Rate of 4%.

4.2 Model under Policyholder's Perspective

Before the formulation of the model in Subsection 4.2.1, we first recall all the assumptions made for the pricing of variable annuities in Chapter 2.

4.2.1 Fundamental GMWB Static Pricing

Let W_t denote the GMWB value of the personal annuity account linked to the portfolio at time t, and assume that the initial value of the account is denoted by W_0, where the expiration date is $T_* = \frac{W_0}{G} = \frac{1}{w}$ with G representing the guaranteed fixed annual withdrawal amount. If a withdrawal exceeding G is made for some reason, there will be a penalty charge. See Wenger (2012) for more details.

The personal account value W_t of the variable annuity with the effects namely the change in guarantee withdrawal base denoted by $dH_t = -Gdt$ and the contract fee rate δ, satisfies the SDE

$$dW_t = [(r-\delta)W_t - G]dt + \sigma W_t dB_t, \quad 0 \leq t \leq T_*, \quad W_t > 0. \tag{4.2.1}$$

If W_t ever hits zero, it stays there. To find the solution of the SDE in equation (4.2.1), we follow techniques by the Itô formula. Define the integrating factor by

$$F_t = e^{(-(r-\delta)+\frac{1}{2}\sigma^2)t-\sigma B_t}. \tag{4.2.2}$$

For the Itô processes F and W, the product rule yields

$$\begin{aligned} d(F_tW_t) &= F_t dW_t + W_t dF_t + d\langle F, W\rangle_t \\ &= -GF_t dt. \end{aligned} \tag{4.2.3}$$

Therefore, the solution to equation (4.2.1) at the expiration time $t = T_*$ is given by

$$\begin{aligned} F_{T_*}W_{T_*} &= F_0W_0 + \int_0^{T_*} -GF_s ds \\ W_{T_*} &= W_0F_{T_*}^{-1} - GF_{T_*}^{-1}\int_0^{T_*} F_s ds \\ &= W_0 e^{(r-\delta-\frac{1}{2}\sigma^2)T_*+\sigma B_{T_*}} - Ge^{(r-\delta-\frac{1}{2}\sigma^2)T_*+\sigma B_{T_*}}\int_0^{T_*} e^{-(r-\delta-\frac{1}{2}\sigma^2)s-\sigma B_s} ds \\ &= e^{(r-\delta-\frac{1}{2}\sigma^2)T_*+\sigma B_{T_*}}\left(W_0 - \frac{W_0}{T_*}\int_0^{T_*} e^{-(r-\delta-\frac{1}{2}\sigma^2)s-\sigma B_s} ds\right). \end{aligned} \tag{4.2.4}$$

Let $X_{T_*} = e^{-(r-\delta-\frac{1}{2}\sigma^2)T_*-\sigma B_{T_*}}$ which can be perceived as the monetary units that can be bought with a dollar in the annuity account, identical to Euros that can be purchased with a dollar in the foreign exchange market. Then equation (4.2.4) can be written resembling an Asian-Quanto put option as

$$\begin{aligned} W_{T_*} &= \frac{1}{X_{T_*}}\left(W_0 - \frac{W_0}{T_*}\int_0^{T_*} X_s ds\right) \\ &= \frac{W_0}{X_{T_*}}\left(1 - \frac{1}{T_*}\int_0^{T_*} X_s ds\right). \end{aligned} \tag{4.2.5}$$

An Asian option is one which, unlike European and American ones, has the payoff determined by the average of the underlying asset prices taken over a pre-specified period of time. On the other hand, a Quanto option is one which is expressed in terms of a foreign monetary unit/currency but at an exercise date is converted with a fixed exchange rate to the investor's home currency, see Datey *et al.* (2003) to read more on Asian and Quanto options. Hence an Asian-Quanto option is an option of Asian type which is expressed in foreign monetary unit and can be converted at a fixed rate of exchange to investor's home currency. Moreover, a Quanto option explains that if, for example, an European investor invests in an American stock with NASDAQ index, he is exposed to the rise and fall happening in the NASDAQ and Euro/Dollar exchange rate.

Let the value of the account's average be given by

$$\bar{X} = \frac{1}{T_*}\int_0^{T_*} X_s ds. \tag{4.2.6}$$

We know that if W_t reaches zero it stays, and that model equation (4.2.1) holds for $W_t > 0$, $\forall\, t \geq 0$. Then under such constraint we can write equation (4.2.5) as

$$\text{AQPP} = \mathbf{FX} \cdot \max(0, 1 - \bar{X}). \tag{4.2.7}$$

Here AQPP denotes the Asian-Quanto put option payoff, where $\mathbf{FX} = \frac{W_0}{X_{T_*}}$ is the foreign exchange rate, and W_0 represents the foreign monetary unit whereas X_{T_*} is the domestic monetary unit.

For a GMWB policy, the retiree gets the annual withdrawal guarantee and the remaining account value at the expiration date. So the maturity value of the yearly withdrawals or what is also called the term certain annuity is given by

$$\int_0^{T_*} G e^{rs} ds = \frac{W_0}{T_* r}(e^{rT_*} - 1). \tag{4.2.8}$$

Consequently, the present value of the total cash inflow to a GMWB policy under the policyholder's perspective is given by the arbitrage free formula

$$\begin{aligned}
\mathbf{CF}_{\mathrm{P}} &= e^{-\int_0^{T_*} r_s ds}\left\{\mathbb{E}[W_{T_*}] + \int_0^{T_*} G e^{rs} ds\right\} \\
&= e^{-rT_*}\mathbb{E}[W_{T_*}] + \frac{W_0}{T_* r}(1 - e^{-rT_*}) \\
&= e^{-rT_*}\mathbb{E}[\mathbf{FX} \cdot \max(0, 1 - \bar{X})] + \frac{W_0}{T_* r}(1 - e^{-rT_*}),
\end{aligned} \tag{4.2.9}$$

where $\mathbb{E}[.]$ is the expectation taken under risk neutral measure. Equation (4.2.9) implies that the cash inflow package of the GMWB under the static valuation is the sum of an Asian put option and a term certain annuity.

As an example, let us take \$120 to be the value of the account at contract inception denoted by W_0, the fixed annual withdrawal amount denoted by G be \$8, with the interest rate r of 8%. Then making calculations, the term certain annuity part in equation (4.2.9) is \$69.88. That means the option can be purchased with \$50.12. Therefore, the GMWB consists of 42% option constituent element and 58% of a term certain annuity part.

For $\bar{X} < 1$ in equation (4.2.7), we have that

$$\begin{aligned}
\mathbf{CF}_{\mathrm{P}} &= e^{-rT_*}\mathbb{E}[\mathbf{FX}\max(0, 1-\bar{X})] + \frac{W_0}{T_* r}(1-e^{-rT_*}) \\
&= e^{-rT_*}\mathbb{E}\left[\frac{W_0}{X_{T_*}}\left(1-\frac{1}{T_*}\int_0^{T_*} X_s ds\right)\right] + \frac{W_0}{T_* r}(1-e^{-rT_*}) \\
&= e^{-rT_*}W_0\left[e^{(r-\delta)T_*} - \left(-\frac{w}{r-\delta} + \frac{we^{-(r-\delta)T_*}}{r-\delta}\right)\right] + \frac{wW_0}{r}(1-e^{-rT_*}) \\
&= W_0\left[e^{-\delta T_*} + \left(\frac{we^{-(2r-\delta)T_*}}{r-\delta} - \frac{we^{-rT_*}}{r-\delta}\right)\right] + \frac{wW_0}{r}(1-e^{-rT_*}), \qquad (4.2.10)
\end{aligned}$$

whereas for $\bar{X} \geq 1$, the option becomes zero in equation (4.2.7).

Lastly, to fairly value the GMWB product, we equate the total cash inflow $\mathbf{CF}_{\mathrm{P}}$ to the policyholder's investment premium W_0 as

$$W_0 = e^{-rT_*}\mathbb{E}[\mathbf{FX}\cdot\max(0, 1-\bar{X})] + \frac{W_0}{T_* r}(1-e^{-rT_*}), \qquad (4.2.11)$$

whereby we can find the fair fee as the solution of the expression

$$e^{-rT_*}\mathbb{E}\left[\frac{\mathbf{FX}\cdot\max(0, 1-\bar{X})}{W_0}\right] + \frac{1}{T_* r}(1-e^{-rT_*}) - 1 = 0. \qquad (4.2.12)$$

In Table 4.2 we display the possible fee δ for a single premium of $W_0 = 100$ for interest rate $r = 0.07$ and different withdrawal rates G.

G	T	δ
5.5	18.2	0.0451
6.0	16.7	0.053
6.5	13.3	0.061
7.0	14.3	0.069
7.5	13.3	0.079
8.0	12.5	0.088
8.5	11.8	0.096

Table 4.2: Fee δ applied on an investment with single premium and varying withdrawals.

4.2.2 GMWB Dynamic Pricing

Under the dynamic pricing, we assume that the policyholder makes withdrawals above, below or sometimes equal to the level G. As also can be seen in Dai *et al.* (2008), the dynamics of the account value W obeys

$$\begin{aligned} dW_t &= (r-\delta)W_t dt + dH_t + \sigma W_t dB_t, \quad 0 \le t \le T_*, \quad W_t \ge 0 \\ H_t &= H_0 - \int_0^t \lambda_s ds, \quad 0 \le \lambda_s \le \gamma, \end{aligned} \tag{4.2.13}$$

where H_t is the guarantee account balance at time t, denotation λ_s is the rate at which withdrawal is made, and γ is the upper bound. The rest is defined as in equation (4.2.1).

The penalty charge q is deducted for any withdrawal made exceeding the annual fixed withdrawal guarantee value G. On the process of λ exceeding G (i.e on $\lambda > G$), the retiree is certain to get $G + (1-q)(\lambda - G)$.

For a cash flow rate $g(\lambda)$ that the retiree receives from a continuous process of withdrawal, we have that

$$\begin{aligned} g(\lambda) &= \begin{cases} \lambda & \text{for} \quad 0 \le \lambda \le G \\ G + (1-q)(\lambda - G) & \text{for} \quad \lambda > G \end{cases} \\ &= \lambda - q \max(\lambda - G, 0). \end{aligned} \tag{4.2.14}$$

The income is received by the retiree throughout the life of the deal, and the account value remaining at an expiration date if it exceeds 0 (i.e if $W_{T_*} \ge 0$). We assume the retiree is wise and would like to maximise the present value of the cash inflow by choosing withdrawals that are optimal based on the restriction $0 \le \lambda \le \gamma$. Then, the GMWB has arbitrage free value at time t given by

$$\vartheta(W,H,t) = \max_\lambda \mathbb{E}_t[e^{-r(T_*-t)} \max(0, W_{T_*}) + \int_t^{T_*} e^{-r(s-t)} g(\lambda_s) ds], \tag{4.2.15}$$

where $\mathbb{E}_t$ is the conditional expectation of the expression inside taken under risk neutral measure based on the information at time t.

The pricing formula for the GMWB is found when $W = 0$, if there is no annuity account participation in the security market any more. Denote by $\vartheta_0(H,t)$ the GMWB value when $W = 0$, which is what we are looking for.

To solve equation (4.2.15), we employ the standard procedures that are used when we derive the Hamilton-Jacobi-Bellman equation for problems in stochastic control theory, see Dai *et al.* (2008). Here ϑ evolves according to

$$\frac{\partial \vartheta}{\partial t} + \mathcal{L}\vartheta + \max_\lambda f(\lambda) = 0, \tag{4.2.16}$$

with

$$\mathcal{L}\vartheta = \frac{1}{2}\sigma^2 W^2 \frac{\partial^2 \vartheta}{\partial W^2} + (r-\delta)W\frac{\partial \vartheta}{\partial W} - r\vartheta, \tag{4.2.17}$$

and

$$\begin{aligned} f(\lambda) &= g(\lambda) - \lambda\frac{\partial \vartheta}{\partial W} - \lambda\frac{\partial \vartheta}{\partial H} \\ &= \begin{cases} \lambda(1-A) & \text{for} \quad \lambda \in [0, G) \\ qG + \lambda(1-q-A) & \text{for} \quad \lambda \geq G, \end{cases} \end{aligned} \tag{4.2.18}$$

where $A = \frac{\partial \vartheta}{\partial W} + \frac{\partial \vartheta}{\partial H}$.

We can obtain the maximum of $f(\lambda)$ on $\lambda = 0$ or G or γ, since it is piecewise linear. Then, the maximum of $f(\lambda)$ is given by

$$\max_{\lambda} f(\lambda) = \begin{cases} qG + (1-q-A)\gamma & \text{for} \quad 1-A \geq q \\ (1-A)G & \text{for} \quad 1-A \in (0,q) \\ 0 & \text{for} \quad 1-A \leq 0. \end{cases} \tag{4.2.19}$$

Substituting equation (4.2.19) into equation (4.2.16), we get that

$$\frac{\partial \vartheta}{\partial t} + \mathcal{L}\vartheta + \gamma \max(1-q-A, 0) + \min[\max(1-A,0), q]G = 0. \tag{4.2.20}$$

When the upper bound approaches infinity (i.e $\gamma \to \infty$), we get as solved in Section B.2 that

$$\min\left[-\frac{\partial \vartheta}{\partial t} - \mathcal{L}\vartheta - \max(1-A,0)G, -(1-q) + A\right] = 0. \tag{4.2.21}$$

At time T_*, the annuitant receives the maximum of the remaining account value W and the remaining guarantee base of the charges (i.e $\vartheta(W, H, T_*) = \max(W, (1-q)H))$. When the guarantee base $H = 0$, we have that $\vartheta(W, 0, t) = We^{-r(T_*-t)}$ as the annuity value whereby $e^{-r(T_*-t)}$ is a discounting factor at rate r. So the function $\vartheta_0(H, t)$ can be obtained by finding a solution to equation (4.2.22) below, which is a reduced form of equation (4.2.21)

$$\min\left[-\frac{\partial \vartheta_0}{\partial t} + r\vartheta_0 - \max(1 - \frac{\partial \vartheta_0}{\partial H}, 0)G, -(1-q) + \frac{\partial \vartheta_0}{\partial H}\right] = 0. \tag{4.2.22}$$

We now have the solution found in Section B.2, the stopping point solution for H optimal withdrawals when $W = 0$ is given by

$$\vartheta_0(H,t) = (1-q)\max(H - G\jmath, 0) + \frac{G}{r}[1 - e^{-r\min(\frac{H}{G},\jmath)}] \tag{4.2.23}$$

For $H \leq G\jmath$, we have

$$\vartheta_0(H,t) = \frac{G}{r}[1 - e^{-rH}], \tag{4.2.24}$$

where

$$\jmath = \min\left[(T_* - t), -\frac{1}{r}\ln(1-q)\right]. \tag{4.2.25}$$

As an interpretation, equation (4.2.23) implies that for a large value of guarantee base H, the strategy the policyholder considers optimal is to withdraw a portion at time t of the guarantee base plus the value G of the remaining account value. But if the guarantee base at time t is significantly small, the option is to withdraw G of the remaining amount.

Setting the parameters $W_0 = 100$, $r = 0.08$, $T_* = 15$, $t = 3$ and $q = 0.1$ in equation (B.2.15), the desirable withdrawal of the portion of H given $G = 8$ is

$$H \leq -\frac{G}{r}\ln(1-q) \simeq 10.54.$$

Table 4.3 shows all values for different penalty charges q

q	H	$\vartheta_0(H,t)$
0.1	8.6	56.9
0.2	22.3	83.2
0.3	35.7	94.2

Table 4.3: Optimal withdrawal values for portion of H when $W = 0$.

4.3 Model under Insurer's Perspective

Consider now the insurer's perspective of an account value W_t that is completely depleted before the expiration date T_* by not a fixed but deterministic withdrawals represented by $0 \leq G_t \leq W_t$, and the short rate process r_t is governed as in equation (3.1.9). Denote by T_0 the time when the account value becomes depleted (i.e when $W_t = 0$), in which case the insurer will no longer deduct the fee δ from the account but keep on paying the claims made by the policyholder. Then the account value is governed by

$$\begin{aligned} dW_t &= [(r_t - \delta)W_t - G_t]dt + \sigma W_t dB_t, \quad t \in [0, T_0) \\ W_t &= 0, \quad t \geq T_0. \end{aligned} \tag{4.3.1}$$

Recall equation (4.2.9). Then from the insurer's perspective, the dynamics of the short rate r_t and X_t are needed for the valuation of a AQPP and can be simplified by switching from $\mathbb{Q}$ to a new measure $\mathbb{Q}^s$. Firstly, let us denote the money market account by

$$A_{T_*} = e^{\int_0^{T_*} r_s ds}.$$

To shift from measure $\mathbb{Q}$ to $\mathbb{Q}^s$, we apply the derivative as in equation (3.1.7), which here we denote by the density process ψ_{T_*} as follows

$$\begin{aligned}\psi_{T_*} &= \frac{d\mathbb{Q}^s}{d\mathbb{Q}} |_{\mathbb{F}_T} \\ &= \exp\left(-\int_0^{T_*} r_s ds\right)\frac{S_{T_*}}{S_0} \\ &= \frac{S_{T_*}/S_0}{A_{T_*}/A_0},\end{aligned}$$

so that the option embedded part of equation (4.2.9), can be expressed as

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{T_*} r_s ds}\frac{W_0}{X_{T_*}}\max(0, 1-\bar{X})\right] = e^{-\delta T_*}\mathbb{E}^{\mathbb{Q}^s}\Big[W_0 \max(0, 1-\bar{X})\Big].$$

Now for a deterministic G_t, the whole of equation (4.2.9) including the term certain annuity part can be written as

$$\mathbf{CF}_\mathrm{I} = e^{-\delta T_*}\mathbb{E}^{\mathbb{Q}^s}\Big[W_0 \max(0, 1-\bar{X})\Big] - \frac{1}{r}(G_{T_*}e^{-rT_*} - G_0). \tag{4.3.2}$$

If we discount equation (4.3.1) and integrate from 0 to $T_0 \wedge T_*$ (i.e from zero to the maximum time between T_0 and T_*), the insurer's liability can be obtained as we show

$$\begin{aligned}\int_0^{T_0\wedge T_*} \delta e^{-\int_0^t r_s ds} W_t dt &= \int_0^{T_0\wedge T_*} r_t e^{-\int_0^t r_s ds} W_t dt - \int_0^{T_0\wedge T_*} e^{-\int_0^t r_s ds} G_t dt \\ &\quad + \int_0^{T_0\wedge T_*} \sigma e^{-\int_0^t r_s ds} W_t dB_t \\ &= -\int_0^{T_0\wedge T_*} d\left(e^{-\int_0^t r_s ds} W_t\right) - \int_0^{T_0\wedge T_*} e^{-\int_0^t r_s ds} G_t dt \\ &\quad + \int_0^{T_0\wedge T_*} \sigma e^{-\int_0^t r_s ds} W_t dB_t \\ &= -W_{T_0\wedge T_*} e^{-\int_0^{T_0\wedge T_*} r_s ds} + W_0 - \int_0^{T_0\wedge T_*} e^{-\int_0^t r_s ds} G_t dt \\ &\quad + \int_0^{T_0\wedge T_*} \sigma e^{-\int_0^t r_s ds} W_t dB_t \\ &= -W_{T_0\wedge T_*} e^{-\int_0^{T_0\wedge T_*} r_s ds} + W_0 \\ &\quad - \left[\int_0^{T_*} e^{-\int_0^t r_s ds} G_t dt - \int_{T_0\wedge T_*}^{T_*} e^{-\int_0^t r_s ds} G_t dt\right] \\ &\quad + \int_0^{T_0\wedge T_*} \sigma e^{-\int_0^t r_s ds} W_t dB_t,\end{aligned} \tag{4.3.3}$$

which implies

$$\int_{T_0\wedge T_*}^{T_*} e^{-\int_0^t r_s ds} G_t dt - \int_0^{T_0\wedge T_*} \delta e^{-\int_0^t r_s ds} W_t dt = W_{T_0\wedge T_*} e^{-\int_0^{T_0\wedge T_*} r_s ds} - W_0 - \int_0^{T_0\wedge T_*} \sigma e^{-\int_0^t r_s ds} W_t dB_t + \int_0^{T_*} e^{-\int_0^t r_s ds} G_t dt. \quad (4.3.4)$$

By noting that $W_{T_0\wedge T_*} = W_{T_*}$, we can express equation (4.3.4) as

$$\int_{T_0\wedge T_*}^{T_*} e^{-\int_0^t r_s ds} G_t dt - \int_0^{T_0\wedge T_*} \delta e^{-\int_0^t r_s ds} W_t dt = W_{T_*} e^{-\int_0^{T_*} r_s ds} - W_0 - \int_0^{T_0\wedge T_*} \sigma e^{-\int_0^t r_s ds} W_t dB_t + \int_0^{T_*} e^{-\int_0^t r_s ds} G_t dt. \quad (4.3.5)$$

The present value of the GMWB associated liability to the insurer is the amount stream that the insurer should pay from the depletion period of the policyholder's account value until the expiration T_* minus the fees proportion the insurer deducts before the policyholder's account value W_t is entirely exhausted. The present value of the liability denoted by $\mathbf{L}$ is obtained by taking the expectation of (4.3.5) as follows

$$\begin{aligned}\mathbf{L} &= \mathbb{E}\left[\int_{T_0\wedge T_*}^{T_*} G_t e^{-\int_0^t r_s ds} dt - \int_0^{T_0\wedge T_*} \delta W_t e^{-\int_0^t r_s ds} dt\right] \\ &= \mathbb{E}\left[W_{T_*} e^{-\int_0^{T_*} r_s ds} - W_0 + \int_0^{T_*} e^{-\int_0^t r_s ds} G_t dt \right. \\ &\quad \left. - \int_0^{T_0\wedge T_*} \sigma e^{-\int_0^t r_s ds} W_t dB_t\right]. \end{aligned} \quad (4.3.6)$$

Equations (4.2.5) and (4.2.7) allow us to express equation (4.3.6) by

$$\begin{aligned}\mathbf{L} &= \mathbb{E}\left[\frac{W_0}{X_{T_*}} \max(0, 1-\bar{X}) e^{-\int_0^{T_*} r_s ds} - W_0 - \frac{1}{r}\int_0^{T_*} d(G_t e^{-\int_0^t r_s ds}) \right. \\ &\quad \left. - \int_0^{T_0\wedge T_*} \sigma e^{-\int_0^t r_s ds} W_t dB_t\right] && (4.3.7) \\ &= e^{-\delta T_*}\mathbb{E}[W_0 \max(0, 1-\bar{X})] - W_0 - \frac{1}{r}(G_{T_*} e^{-rT_*} - G_0) && (4.3.8) \\ &= \mathbf{CF}_{\mathrm{I}} - W_0. && (4.3.9)\end{aligned}$$

This achievement in equation (4.3.9) satisfies the intuition in finance that the present value of the GMWB cash inflow is the present value of the insurer's liability plus the policyholder's initially invested premium.

Suppose the parameters are contract period of $T_* = 17$, premium $W_0 = 60$, time zero withdrawal $G_0 = 0$, interest rate $r = 0.08$, $G_{T_*} = \frac{W_0}{T_*}$, and that $\bar{X} < 1$. Then the GMWB present values with different fee applied is as displayed in Table 4.4

δ	B_0^W
0.02	7725.5
0.03	7002.9
0.04	6362.6
0.05	5794.3

Table 4.4: GMWB present values for varying fees and constant interest rate.

4.4 Tree Methodology for Pricing the GMWB

One of the interesting techniques used in finance to price and model is the tree methodology. This involves the application of Cox *et al.* (1979) binomial, bino-trinomial, and the stair tree structures used in Dai and Lyuu (2010). In this Section, we will incorporate these three types of tree structures in order to avoid significant errors when pricing the GMWB with trees.

The tree modelling in this study is needed to find the continuation values of the benefit. From the initial investment account value W_0, the account value varies because of the return on an investment and diminishes as the retiree keeps on withdrawing the amount G. Using the tree structure, we can model the account value on the interval $[0, T_*]$ at each time step $\frac{T_*}{n}$ denoted by Δt, with n the number of steps. Let us consider an example as portrayed on the structure below in Figure 4.3.

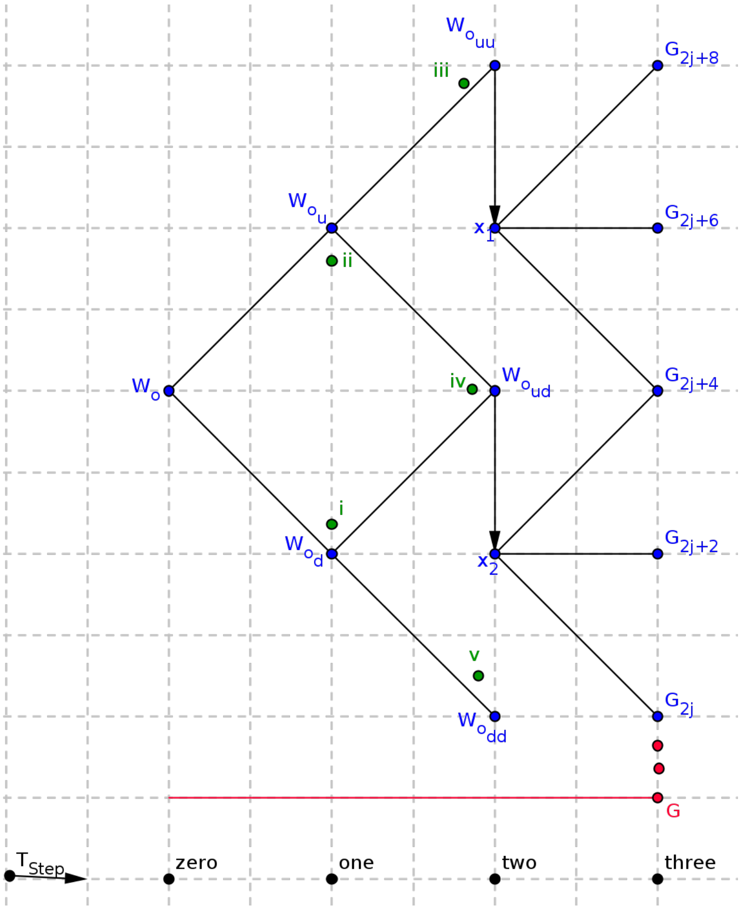

Figure 4.3: Main idea of incorporated Tree Structures

Figure 4.3 displays an idea of incorporated three tree structures. The account value at the initial state, denoted by W_0, moves up with probability p or down with probability $1-p$, depending on the investment return after withdrawal G. The log-prices of W_0 at a given node on the tree, with $2\sigma\sqrt{\frac{T_*}{n}}$ distance from each other at the same time step are denoted by Roman figures, and are calculated as follows

$$W_{0_d}^{\ln} = -\sigma\sqrt{\frac{T_*}{n}},\ W_{0_u}^{\ln} = \sigma\sqrt{\frac{T_*}{n}},\ W_{0_{uu}}^{\ln} = 2\sigma\sqrt{\frac{T_*}{n}},$$
$$W_{0_{ud}}^{\ln} = 0, \text{ and } W_{0_{dd}}^{\ln} = -2\sigma\sqrt{\frac{T_*}{n}}.$$

The jumps made from a tree structure to the other can be represented by involving the stair tree structure which also reflect withdrawals that are discrete, via the formula

$$W_t^a = W_t^b - G. \tag{4.4.1}$$

The notations W_t^a and W_t^b is the representation of account value after and before withdrawal G respectively.

The idea of a stair tree structure employed is displayed in Figure 4.3 by a downward jump from $W_{0_{uu}}$ to node x_1, with which a trinomial tree branches from and connects to step 3 nodes. The same is applied with $W_{0_{ud}}$ to node x_2. The final step has the value G set in a manner that the critical position is eventually hit, which is the stopping withdrawal point at an expiration stage. This is displayed and denoted by dots down until the red line showing the level G. The log-prices at the expiration stage for each node are as follows:

$$G^{\ln}_{j+8} = (j+8)\sigma\sqrt{\tfrac{T_*}{n}}, \quad G^{\ln}_{j+6} = (j+6)\sigma\sqrt{\tfrac{T_*}{n}}, \quad G^{\ln}_{j+4} = (j+4)\sigma\sqrt{\tfrac{T_*}{n}}$$
$$G^{\ln}_{j+2} = (j+2)\sigma\sqrt{\tfrac{T_*}{n}}, \quad G^{\ln}_{j} = j\sigma\sqrt{\tfrac{T_*}{n}}, \quad \text{and} \quad G^{\ln} = \ln\tfrac{G}{G} = 0$$

for some even integer j.

We are now going to model the account value for the withdrawal period on the interval $[0, T_*]$ with a more detailed different diagram example from the first. Consider the tree in Figure 4.4 below

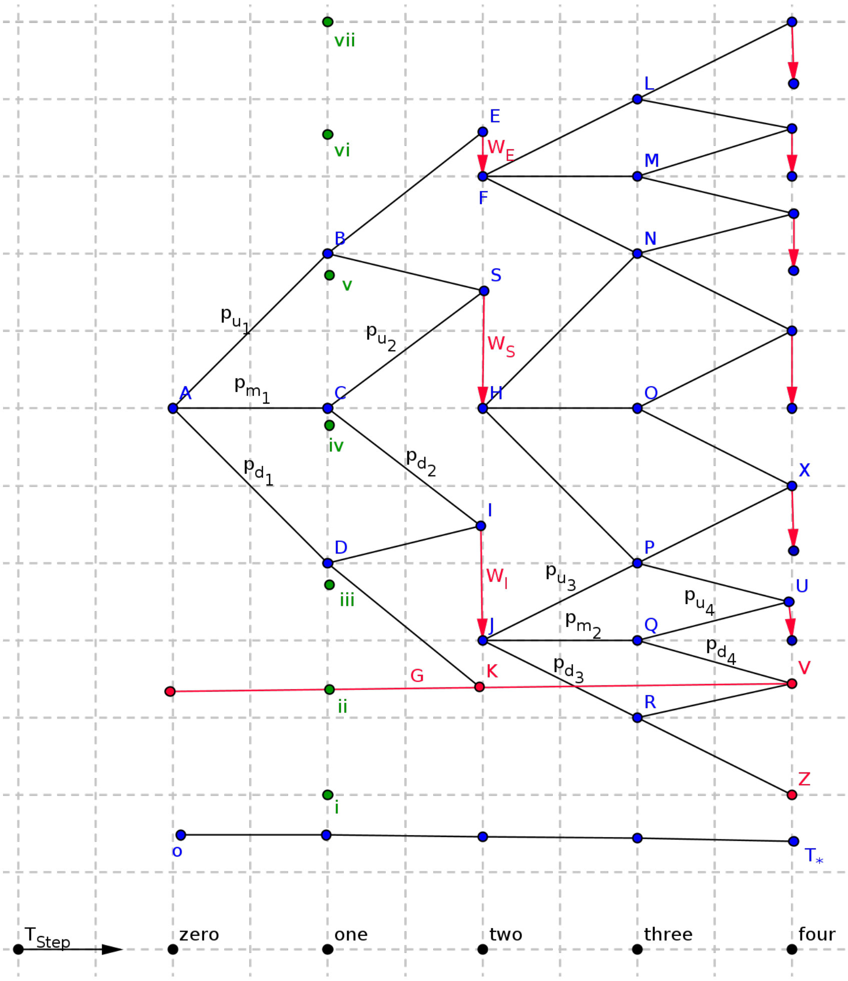

Figure 4.4: Withdrawal Period Tree Structure on $[0, T_*]$

Figure 4.4 can be explained as thus: firstly, before the withdrawal, we have the initial account value W_0^b denoted on the tree by node A. Then A branches a trinomial with upward, middle, and downward probabilities p_u, p_m, and p_d respectively. With the sizes of the up, middle, and down jumps

$$u = e^{\sigma\sqrt{2\Delta t}},\ m = 1, \quad \text{and } d = e^{-\sigma\sqrt{2\Delta t}},$$

we have, as proposed by Boyle (1986), the upward probability of

$$p_u = \left(\frac{e^{\frac{(r-\delta)\Delta t}{2}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}}{e^{\sigma\sqrt{\frac{\Delta t}{2}}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}} \right)^2, \tag{4.4.2}$$

and the downward probability

$$p_d = \left(\frac{e^{\sigma\sqrt{\frac{\Delta t}{2}}} - e^{\frac{(r-\delta)\Delta t}{2}}}{e^{\sigma\sqrt{\frac{\Delta t}{2}}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}} \right)^2, \tag{4.4.3}$$

and the middle probability given by

$$p_m = 1 - (p_u + p_d). \tag{4.4.4}$$

Note p_u, p_m, and p_d are within the interval (0,1) and $\Delta t < \frac{2\sigma^2}{(r-\delta)^2}$.

The implementation of the jumps on a stair tree structure to model discrete withdrawals, appears represented with the red downward jump arrows. For example, take node I with the account value before withdrawal $W^b(I)$, then after withdrawing G, it jumps downwards to node J which has the discrete withdrawal formula

$$J = W^a(I) = W^b(I) - G. \tag{4.4.5}$$

Looking at the values of the account at critical position nodes K,V, and Z, we notice that they lie below the level of the guaranteed withdrawal G. Therefore, until the expiration date T_* is reached, the retiree still receives the guarantee G even if the account value declines as in K. Now, on our tree in Figure 4.4, if that happens in critical positions before expiration date T_* as in K, the account value there becomes 0 after payment G and no branching out from such 0 amount account value to connect with next time step nodes. That is because it is still on the way to reaching the expiration date T_* and we have to consider branching only from nodes with account values exceeding withdrawal guarantee G. The node V is the first at the expiration date T_* that indicates when $G \leq 0$, which is the convergence destination.

However if there is an amount left in the account, even after the contract is over, the retiree has the right to receive the remainder as part of the deal in GMWB variable annuity. Algebraically, this means that the GMWB at the expiration date is given by

$$\max(W^b_{T_*} - G, 0) + G = \max(W^a_{T_*}, 0) + G, \tag{4.4.6}$$

which in our case in Figure 4.4, we have as

$$\max(V, 0) + G. \tag{4.4.7}$$

The formula (4.4.6) of the GMWB is not the same as the ones we used before the expiration date T_*. If the retiree dies before the expiration date is reached, the continuation value of the GMWB is obtained as follows. Let us pick a node Q for example, then if the retiree dies at that time step, the continuation value of the GMWB is given by

$$GMWB_{t_Q} = e^{-r\Delta t}(p_u GMWB_{T_{*_U}} + p_d GMWB_{T_{*_V}}). \tag{4.4.8}$$

For a doubling period to expiration date, for example, if this happens at node J, the continuation value is given by

$$\begin{aligned} GMWB_{t_J} = e^{-3r\Delta t}(p_u^2 GMWB_{T_{*X}} + p_u p_d GMWB_{T_{*U}} + p_u p_m GMWB_{T_{*U}} \\ + p_m p_d GWMB_{T_{*V}} + p_u p_d GWMB_{T_{*V}} + p_u^2 GWMB_{T_{*Z}}), \quad (4.4.9) \end{aligned}$$

where r denotes the return rate. See equations (B.3.4) and (B.3.5) in Section B.3 for clarity to above equations (4.4.8) and (4.4.9), and also Cox *et al.* (1979).

Because of the nature of the Cox *et al.* (1979) tree structure at steps 1 and 3, we have the log-prices in Figure 4.4 represented in green dotted Roman figures, where

$$\text{i}= \ln(\tfrac{G}{W_0^a}) - \sigma\sqrt{\tfrac{T_*}{n}}, \quad \text{ii}= \ln(\tfrac{G}{W_0^a}), \quad \text{iii}= \ln(\tfrac{G}{W_0^a}) + \sigma\sqrt{\tfrac{T_*}{n}}, \ \text{iv}= \ln(\tfrac{G}{W_0^a}) + 3\sigma\sqrt{\tfrac{T_*}{n}},$$
$$\text{v}= \ln(\tfrac{G}{W_0^a}) + 5\sigma\sqrt{\tfrac{T_*}{n}}, \quad \text{vi}= \ln(\tfrac{G}{W_0^a}) + 7\sigma\sqrt{\tfrac{T_*}{n}}$$

and so forth with the distance of $2\sigma\sqrt{\frac{T_*}{n}}$ between each other, see Yang and Dai (2013). These are the same log-prices at time step 3 on same level.

As a numerical illustration to equation (4.4.9), let us use equations (4.4.2), (4.4.3) and (4.4.4), and assume $p_d = 0.6$, $p_m = 0.1$, $p_u = 0.3$ and other parameters $r = 0.07$, $T = 15$, $n = 2$ such that $\Delta t = 7.2$. Again, suppose that the GMWB values at point X, U, V and Z in Figure 4.4 are respectively given by 0, 22, 40 and 62. Then the point J continuation value of the GMWB is expressed as

$$\begin{aligned} GMWB_{t_J} &= e^{-(3\times 0.07\times 7.2)}\Big[(0.3^2\times 0) + (0.3\times 0.6\times 22) + (0.3\times 0.1\times 22) \\ &\quad + (0.1\times 0.6\times 40) + (0.3\times 0.6\times 40) + (0.3^2\times 60)\Big] \\ &= 4.33. \end{aligned}$$

4.5 GMWB for Life Valuation

The GMWB is made complex by also extending it to the GMWB for life or what is known as the GLWB, short for Guaranteed Lifelong Withdrawal Benefit. This takes the form of GMIB after annuitization. The GLWB product provides a lifespan income for the retirees who are aged 65 and above in many insurance companies. If the nest egg is exhausted prior to the death of the policyholder, the withdrawals that will be made after are liabilities of the insurer.

As in the GMWB, the GLWB can be decomposed into static and dynamic pricing, even though we show an example for static pricing only in this Section. The difference will be that in GLWB, it is quite necessary to involve the survival and death probabilities in pricing of the model. In GLWB, there is no limit on how much total to withdraw because even if the guarantee base H becomes zero, the policyholder is entitled to withdraw the yearly guaranteed amount agreed upon at contract inception. This means that the yearly withdrawal guarantee is specified but the total is never limited.

4.5.1 Pricing the GLWB

Recall that the equity S evolves according to equation (2.0.1) and all related assumptions about the state of the market and pricing in Chapter 2.

The policyholder gets the yearly guaranteed amount if still alive and the beneficiary receives the remaining account value should the policyholder die before the account value is depleted. The GLWB discounted value at time $t = 0$ is given by the discounted values sum of the benefits the policyholder gets when alive and those the beneficiary receives when dead. Let us denote the value at time zero of the GLWB by Υ_0, then

$$\Upsilon_0 = B_0^\ell + B_0^D, \tag{4.5.1}$$

where B_0^ℓ and B_0^D are, respectively, the living benefit and the remaining benefit received by the beneficiary at death that are discounted at time zero.

Recall equation (2.1.1) with all the probabilities associated to it, and that $T_* = T_{a*}$ is the remaining future time of life random variable for the policyholder aged a at contract inception. Now we can express the time zero discounted living benefit or present value of yearly guaranteed withdrawal $G = wW_0$ that are weighted with probability of survival as

$$B_0^\ell = \sum_{t=1}^{n-a} wW_0 e^{-rt} ({}_t p_a), \tag{4.5.2}$$

where w is as mentioned in Chapter 2. For a stochastic expiration date T_* and the account value W_{T_*} that are independent, the death benefit value discounted at time $t = 0$ for maturity $T = t$ is expressed as in Mudavanhu and Zhuo (2002) by

$$B_0^D(T = t) = \mathbb{E}_0[\mathbb{E}^{\mathbb{Q}}[e^{-rt} B_T^D|_{T=t}]], \tag{4.5.3}$$

where the expectation taken for T and W_T under risk neutral measure.

But fixing the time to be at expiration T_*, we can use the Itô formula to calculate the death benefit and get the solution of equation (4.2.1) under the withdrawal model for life as an Asian Quanto Put

$$B_{T_*}^D = e^{(r-\delta-\frac{1}{2}\sigma^2)T_*+\sigma B_{T_*}} \max\left[\left(W_0 - \frac{W_0}{T_*}\int_0^{T_*} e^{-(r-\delta-\frac{1}{2}\sigma^2)s-\sigma B_s} ds\right), 0\right].$$

The arbitrage free present value at time $t = 0$ for maturity $T = T_*$ of the death benefit is given by

$$B_0^D(T = T_*) = e^{-rT_*}\mathbb{E}^{\mathbb{Q}}[B_{T_*}^D], \tag{4.5.4}$$

where

$$\mathbb{E}[B_{T_*}^D] = \mathbb{E}^{\mathbb{Q}}\left[e^{(r-\delta-\frac{1}{2}\sigma^2)T_*+\sigma B_{T_*}} \max\left[\left(W_0 - \frac{W_0}{T_*}\int_0^{T_*} e^{-(r-\delta-\frac{1}{2}\sigma^2)s-\sigma B_s} ds\right), 0\right]\right].$$

Consider equation (4.5.3), then we have for maturity $T = t$ that

$$B_0^D = \int_0^{n-a} f_a(t)\mathbb{E}^{\mathbb{Q}}(B_t^D)dt, \tag{4.5.5}$$

where $f_a(t)$ is the probability density function for the remaining future time of life random variable T_*. It can be expressed in discrete form as

$$B_0^D = \sum_{t=0}^{n-a} {}_tp_a\theta_{a+t}\mathbb{E}^{\mathbb{Q}}(B_t^D). \tag{4.5.6}$$

With equations (4.5.2) and (4.5.6), we have the GLWB present value of all the cash inflow from $t = 0$ to $t = (T_* = n - a)$ as

$$B_0^L = \sum_{t=0}^{n-a}[wW_0e^{-rt}{}_tp_a + {}_tp_a\theta_{a+t}\mathbb{E}^{\mathbb{Q}}(B_t^D)]. \tag{4.5.7}$$

Setting the parameters for equation (4.5.7) as follows: $r = 5\%$, $w = 7\%$, $W_0 = 1000$, $\delta = 1\%$, $n = 100$, and $a = 65$ until $a = 95$, the sum was calculated and the results in Figure 4.5 show the present values for GLWB.

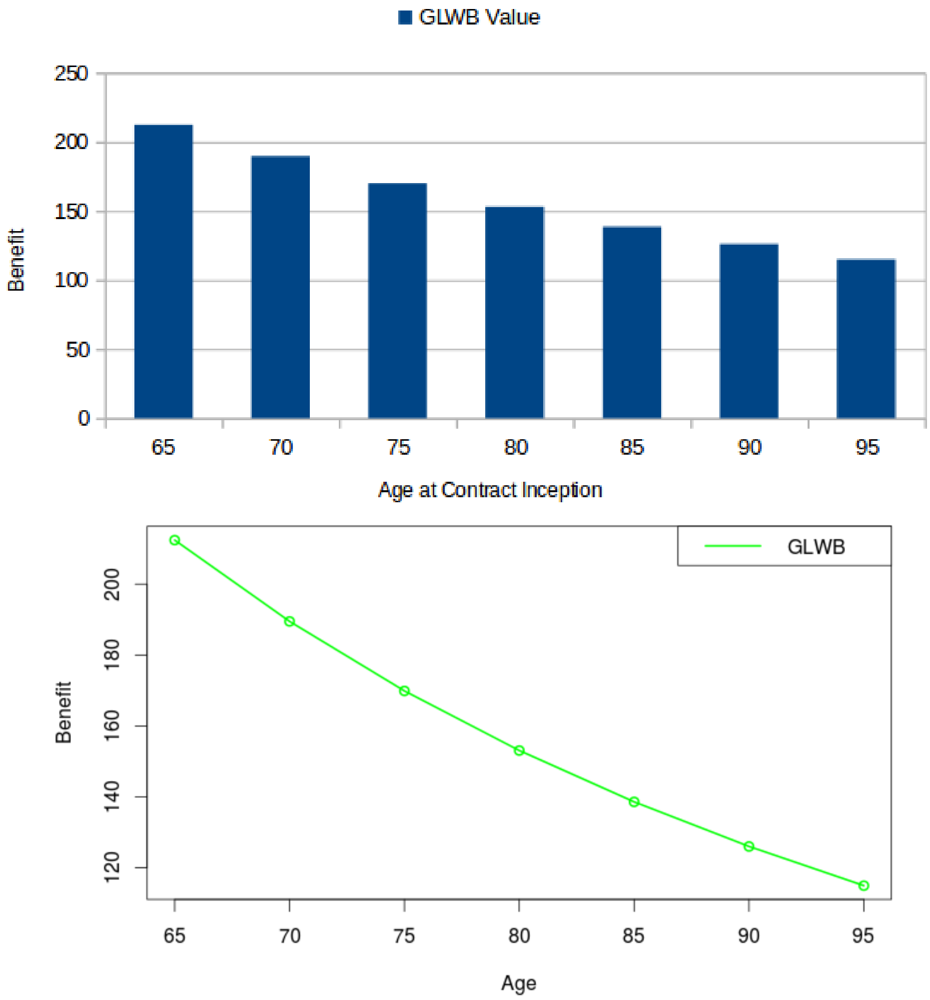

Figure 4.5: GLWB Present Values

Chapter 5

Valuation of Equity-Indexed Annuities

In this Chapter, the main aim is to value an equity-indexed policy which is mostly known as purchased with a cliquet feature involved, but we will also include surrender feature. There are some differences between the variable annuities and equity-indexed annuities. EIA is a life income annuity whereby it is guaranteed that the return on investment never falls to be below a certain level regarded as the minimum, which is not the case when it comes to VA income products, see Bernard and Boyle (2010). They have variable and fixed annuities characteristics, but their returns vary more than in fixed annuity and not more than the way they do in variable annuities. The EIA contract is relatively short and usually (5-10) years, whereas in variable annuities the contract period can be much longer. The embedded options of the equity-based annuities in general, as compared to traded options make it impossible to determine what the market volatility would be for equity-based options. If the portfolio fund performs well, additional interests are credited to the policyholder's account as a bonus. The profit gained by the insurance company is shared to the holder of the policy. But since the market has uncertainties, the chances that the policyholder will get additional interests are slim as the fund performance varies. Nevertheless, this does not have any severe impact to the annual benefit since the minimum guarantee helps to protect the income against any condition or performance of the fund, as this is part of the deal at contract inception. The policyholder gets the benefit at each withdrawal date, at a death (received by the beneficiary), and when he decide to surrender the contract. The guaranteed minimum we are talking about makes the equity-indexed policy to qualify and be recognised as an insurance policy. The premiums of the equity-indexed policy are not invested in a separate account as in variable annuities, but those premiums contribute to the formation of the general fund of the insurer. The EIA was first marketed by the Keyport Life in 1995, see MacKay (2011).

5.1 Equity Indexed Annuities with Cliquet Option

In this Section, we value the equity indexed policy of a living benefit with a cliquet option. The equity indexed insurance policies not only offer the protection to the investment income, but also give a chance to participate in the market index such as NASDAQ, S&P 500, DOW JONES, NYSE and others, see Hardy (2004). If the index linked goes up, the policyholder get additional interests depending on the rate at which annuity participates.

The cliquet option also allows the policyholder to reset the guarantee base to equal the account value linked to the index, and all that depend on the growth of the index at each time. So in our study we are going to separate the cliquet contracts in two different types, namely the simple cliquet contract and the compound cliquet contract.

The index linked S evolves as in equation (2.0.1) with a drift $r-\delta$. Suppose for cliquet option that the annual return rates linked to the index (e.g NASDAQ and S&P 500) are given by

$$R_t = \frac{S_t}{S_{t-1}} \tag{5.1.1}$$

which implies

$$\begin{aligned} R_t &= e^{[(r-\delta-\frac{1}{2}\sigma^2)t+\sigma B_t]} \times e^{-[(r-\delta-\frac{1}{2}\sigma^2)(t-1)+\sigma B_{t-1}]} \\ &= e^{[(r-\delta-\frac{1}{2}\sigma^2)(t-(t-1))+\sigma(B_t-B_{t-1})]} \\ &= e^{[(r-\delta-\frac{1}{2}\sigma^2)+\sigma(B_t-B_{t-1})]}. \end{aligned} \tag{5.1.2}$$

The return rates are independently and identically distributed with their logarithms normally distributed as follows

$$\ln(R_t) \sim N(r-\delta-\frac{1}{2}\sigma^2, \sigma^2) \tag{5.1.3}$$

with mean $r-\delta-\frac{1}{2}\sigma^2$ and variance σ^2.

Following Hsieh and Chiu (2007), let us first denote by R^* the cliquet option contract return and define it at time t as

$$R_t^* = 1 + \min\left(c, \max(f, \varsigma(R_t - 1))\right) \tag{5.1.4}$$

where the rates of ceiling and floor are denoted by c and f respectively, and the rate at which the annuity participates in the index by ς. The floor and the ceiling rates are, respectively, the minimum and the maximum rates of the interest earned by the annuity. If, for example, the index linked gains a growth rate of 11% and the annuity participated 80%, then the rate of interest for the annuity is $0.11 \times 0.8 = 8.8\%$. So if the annuity has the ceiling rate of 7%, only 7% will be credited to its account and 1.8% is taken by the insurance company and that is how both parties benefit.

By considering equation (5.1.4), we define the following payoffs of the cliquet option contracts. Denote by S_p, the simple cliquet contract payoff, the compound cliquet contract payoff by C_p, and the single premium by W_0. Then

$$\begin{aligned} S_p &= W_0 \left(1 + \sum_{t=1}^{T_*} \min(c, \max(f, \varsigma(R_t - 1))) \right) \\ &= W_0 \left(1 + \sum_{t=1}^{T_*} (R_t^* - 1) \right) \\ &= W_0 \left((1 - T_*) + \sum_{t=1}^{T_*} R_t^* \right) \end{aligned} \tag{5.1.5}$$

and

$$\begin{aligned} C_p &= W_0 \prod_{t=1}^{T_*} (1 + \min(c, \max(f, \varsigma(R_t - 1))) \\ &= W_0 \prod_{t=1}^{T_*} R_t^* \end{aligned} \tag{5.1.6}$$

where the expiration date $T_* \in \mathbb{Z}_+$.

Now, we are to discount the cliquet option contracts payoffs under risk neutral measure to find the T_* years values of the contracts. We have by using the martingale framework, the value of the simple cliquet option contract as

$$\begin{aligned} SV &= \mathbb{E}[e^{-rT_*} S_p] \\ &= \mathbb{E}\left[e^{-rT_*} W_0 \left(1 + \sum_{t=1}^{T_*} \min(c, \max(f, \varsigma(R_t - 1))) \right) \right] \end{aligned} \tag{5.1.7}$$

and the value of the compound cliquet option contract given by

$$\begin{aligned} CV &= \mathbb{E}[e^{-rT_*} C_p] \\ &= \mathbb{E}\left[e^{-rT_*} W_0 \prod_{t=1}^{T_*} [1 + \min(c, \max(f, \varsigma(R_t - 1)))] \right]. \end{aligned} \tag{5.1.8}$$

Set $f_\varsigma = 1 + \frac{f}{\varsigma}$ and $c_\varsigma = 1 + \frac{c}{\varsigma}$. Then equation (5.1.4) can be written as

$$\begin{aligned} R_t^* &= 1 - \varsigma + \varsigma \min\left(1 + \frac{c}{\varsigma}, \max(1 + \frac{f}{\varsigma}, R_t)\right) \\ &= 1 - \varsigma + \varsigma \min(c_\varsigma, \max(f_\varsigma, R_t)). \end{aligned} \tag{5.1.9}$$

With equations (5.1.5) and (5.1.7), we have that

$$\begin{aligned} SV &= \mathbb{E}[e^{-rT_*} S_p] \\ &= \mathbb{E}\left[e^{-rT_*} W_0 \left((1 - T_*) + \sum_{t=1}^{T_*} R_t^*\right)\right] \\ &= \mathbb{E}\left[e^{-rT_*} W_0 \left((1 - T_*) + \sum_{t=1}^{T_*} (1 - \varsigma + \varsigma \min(c_\varsigma, \max(f_\varsigma, R_t)))\right)\right] \\ &= \mathbb{E}\left[e^{-rT_*} W_0 \left((1 - T_*) + T_* - \varsigma T_* + \varsigma \sum_{t=1}^{T_*} \min(c_\varsigma, \max(f_\varsigma, R_t))\right)\right] \end{aligned} \tag{5.1.10}$$

and equations (5.1.6) and (5.1.8) give

$$\begin{aligned} CV &= \mathbb{E}[e^{-rT_*} C_p] \\ &= \mathbb{E}[e^{-rT_*} W_0 \prod_{t=1}^{T_*} R_t^*] \\ &= \mathbb{E}[e^{-rT_*} W_0 \prod_{t=1}^{T_*} (1 - \varsigma + \varsigma \min(c_\varsigma, \max(f_\varsigma, R_t)))] \end{aligned} \tag{5.1.11}$$

To arrive at the analytical solutions of the simple and compound cliquet option contracts values, we have to first find $\mathbb{E}[\min(c_\varsigma, \max(f_\varsigma, R_t))]$. Let $Z_t = \min(c_\varsigma, \max(f_\varsigma, R_t))$, then clearly it can be observed that Z_t's are random variables that are independent and log-normally distributed with values censored on the interval $[f_\varsigma, c_\varsigma]$.

Assuming the expected values of random variables Z_t's are equal constants, we can represent a constant by $\mathbb{E}Z_I$ and express equations (5.1.10) and (5.1.11) as

$$SV = e^{-rT_*} W_0 \Big(1 - \varsigma T_* + \varsigma T_* \mathbb{E}[Z_I]\Big)$$

and

$$CV = e^{-rT_*} W_0 \Big(1 - \varsigma + \varsigma \mathbb{E}[Z_I]\Big)^{T_*},$$

where

$$Z_I = \begin{cases} f_\varsigma & \text{for} \quad R_I \leq f_\varsigma, \\ R_I & \text{for} \quad R_I \in [f_\varsigma, c_\varsigma], \\ c_\varsigma & \text{for} \quad R_I \geq c_\varsigma. \end{cases} \tag{5.1.12}$$

The logarithmic annual returns in equation (5.1.3) linked to the index are normally distributed with the probability density function $h_R(z)$. Then we can write the expression

$$\begin{aligned} \mathbb{E}[Z_I] &= f_\varsigma P(R_I \leq f_\varsigma) + \mathbb{E}[R_I | f_\varsigma \leq R_I \leq c_\varsigma] \times P(f_\varsigma \leq R_I \leq c_\varsigma) \\ &\quad + c_\varsigma P(R_I \geq c_\varsigma) \\ &= f_\varsigma \int_0^{f_\varsigma} h_R(z)dz + \int_{f_\varsigma}^{c_\varsigma} z h_R(z)dz + c_\varsigma \int_{c_\varsigma}^{\infty} h_R(z)dz \\ &= f_\varsigma \Phi(\mathbf{d_A}) + e^r[\Phi(\mathbf{d_B}) - \Phi(\mathbf{d_A})] + c_\varsigma(1 - \Phi(\mathbf{d_B})) \end{aligned} \tag{5.1.13}$$

where the denotation $\Phi(.)$ represents the cumulative probability function for the standard normal distribution. The denotations $\mathbf{d_A}$ and $\mathbf{d_B}$ found in Subsection C.1 are given by

$$\mathbf{d_A} = \frac{\ln(f_\varsigma) - r + \delta + \frac{1}{2}\sigma^2}{\sigma} \tag{5.1.14}$$

and

$$\mathbf{d_B} = \frac{\ln(c_\varsigma) - r + \delta + \frac{1}{2}\sigma^2}{\sigma}. \tag{5.1.15}$$

Then equations (5.1.10) and (5.1.11) become the simple and compound cliquet contracts values for T_* years as expressed below

$$\begin{aligned} SV = e^{-rT_*} W_0 \Big(1 - \varsigma T_* + \varsigma T_* \Big[& f_\varsigma \Phi(\mathbf{d_A}) \\ & + e^r[\Phi(\mathbf{d_B}) - \Phi(\mathbf{d_A})] + c_\varsigma(1 - \Phi(\mathbf{d_B})) \Big] \Big) \end{aligned} \tag{5.1.16}$$

and

$$\begin{aligned} CV = e^{-rT_*} W_0 \Big(1 - \varsigma + \varsigma \Big[& f_\varsigma \Phi(\mathbf{d_A}) \\ & + e^r[\Phi(\mathbf{d_B}) - \Phi(\mathbf{d_A})] + c_\varsigma(1 - \Phi(\mathbf{d_B})) \Big] \Big)^{T_*}. \end{aligned} \tag{5.1.17}$$

The analytical solutions to equations (5.1.16) and (5.1.17) are as shown in Subsection C.1.1.

5.2 Equity-indexed annuities with surrender option

We now look at the equity-indexed policy with a surrender option. To value the policy, let us consider and recall equation (2.0.1) and every model assumption. Also, suppose that the index linked annual return rates are as in equation (5.1.1).

To calculate the contract policy value, let us assume that it was initiated at time $t = 0$ and matures at time T_*. Denote the benefit at time $t = 0$ by U_0, rate of participation ς, and since EIA can also be a fixed annuity we may assume the guaranteed growth rate g where the benefit is without floor and ceiling. The benefit whereby the additional interest is credited to the policyholder, we may call it a bonus and denote it by $\mathbf{b}_t$ expressed as in Calidonio-Aguilar and Xu (2011) and Bacinello (2001) as follows

$$\mathbf{b}_t = \max(0, \varsigma R_t - g). \tag{5.2.1}$$

The benefit U at time t is given as the previous benefit multiplied by the minimum guarantee rate plus additional interest as thus:

$$U_t = U_{t-1}(1 + g + \mathbf{b}_t), \quad \text{for} \quad t \in [0, T_*]. \tag{5.2.2}$$

Equation (5.2.2) can actually be expressed in terms of a fundamental benefit with the relations from equations (5.2.1) and (5.2.2) as we show below.

$$\begin{aligned}
U_1 &= U_0(1 + g + \mathbf{b}_1) \\
U_2 &= U_1(1 + g + \mathbf{b}_2) \\
\vdots &= \quad \vdots \\
U_{t-1} &= U_{t-2}(1 + g + \mathbf{b}_{t-1}) \\
U_t &= U_{t-1}(1 + g + \mathbf{b}_t)
\end{aligned}$$

then,

$$\begin{aligned}
U_1 \times U_2 \times \cdots U_t = U_0(1 + g + \mathbf{b}_1) \times \cdots U_{t-2}(1 + g + \mathbf{b}_{t-1}) \\
\times U_{t-1}(1 + g + \mathbf{b}_t).
\end{aligned} \tag{5.2.3}$$

Therefore, considering from equation (5.2.1) the case when $R_t \geq g$, we have the benefit as

$$\begin{aligned} U_t &= U_0 \prod_{i=1}^{t} (1 + g + \mathbf{b}_i) \\ &= U_0 \prod_{i=1}^{t} (1 + g + \varsigma R_i - g) \\ &= U_0 \prod_{i=1}^{t} (1 + g) \left(1 + \frac{\varsigma R_i - g}{1+g}\right) \\ &= U_0 \prod_{i=1}^{t} (1 + g)(1 + \xi_i) \end{aligned} \tag{5.2.4}$$

where

$$\xi_i = \max\left(0, \frac{\varsigma R_i - g}{1+g}\right). \tag{5.2.5}$$

Now, denote by $I(U_t)$ the present value of the benefit U_t under the martingale framework for estimating contingent claims. Then, we can express $I(U_t)$ as

$$I(U_t) = \mathbb{E}[e^{-rt} U_t], \quad \text{for} \quad t \in [0, T_*]. \tag{5.2.6}$$

and the expectation $\mathbb{E}[.]$ is taken under the risk neutral measure Q. Substituting equation (5.2.4) into equation (5.2.6), we have that

$$\begin{aligned} I(U_t) &= \mathbb{E}[e^{-rt} U_0 \prod_{i=1}^{t} (1+g)(1+\xi_i)] \\ &= \mathbb{E}\left[e^{-rt} U_0 \prod_{i=1}^{t} (1+g)\left(1 + \max\left(0, \frac{\varsigma R_i - g}{1+g}\right)\right)\right] \\ &= U_0 \prod_{i=1}^{t} \mathbb{E}\left[e^{-r}(1+g) + e^{-r}\left(\max\left(0, \varsigma R_i - g\right)\right)\right] \\ &= U_0 \prod_{i=1}^{t} \left(e^{-r}(1+g)\right. \\ &\quad \left. + \varsigma \mathbb{E}\left[e^{-r} \max\left(0, (1+R_i) - (1 + \frac{g}{\varsigma})\right)\right]\right). \end{aligned} \tag{5.2.7}$$

The expected value under the risk neutral measure taken in equation (5.2.7) can be regarded as European call option at time $t = 0$, which has an asset price equal to 2 and exercise price of $1 + \frac{g}{\varsigma}$. Let that expected value or a European call option price be denoted by C_i, then

$$I(U_t) = U_0 \prod_{i=1}^{t} \left[e^{-r}(1+g) + \varsigma C_i\right], \quad \text{for} \quad t \in [0, T_*]. \tag{5.2.8}$$

By using Black and Scholes (1973), we define the value of C_t as follows

$$C_t = \Phi(\mathbf{d_A}) - (1 + \frac{g}{\varsigma})e^{-r}\Phi(\mathbf{d_B}), \tag{5.2.9}$$

where

$$\mathbf{d_A} = \frac{\ln\left(\frac{1}{1+\frac{g}{\varsigma}}\right) + (r + \delta + \frac{1}{2}\sigma^2)}{\sigma}, \quad \mathbf{d_B} = \mathbf{d_A} - \sigma \tag{5.2.10}$$

with $\Phi(.)$ representing the cumulative probability function for standard normal distribution.

As we continue, the value of the policy is the present value sum of all the benefits that are given to the policyholder that are weighted with probabilities of life, surrender and death. So we have to first find the present value which involves taking into account the surrender option. This surrender value is given by a certain percentage from the benefit accrued at the time the policyholder chooses to surrender, and the longer the contract exist the more increased it becomes. We can denote by j_t that percentage at time t, then the surrender value can be expressed as

$$\bar{V} = j_t U_t, \quad \text{for} \quad t \in [0, T_*]. \tag{5.2.11}$$

Under the risk neutral measure Q, the surrender value at time $t = 0$ denoted by $I(\bar{V})$ is expressed as below

$$\begin{aligned} I(\bar{V}) &= \mathbb{E}[e^{-rt}\bar{V}] \\ &= \mathbb{E}[e^{-rt}(j_t U_t)] \\ &= j_t I(U_t). \end{aligned} \tag{5.2.12}$$

More about surrendering of a contract is well explained in Chapter 1 where we have also shown how modeling of surrender rates is done. Insurance companies usually use historical data to estimate the probability that investors might choose to surrender their contracts. For example, when modelling surrender rates we used historical data for logistic regression model. So to get a fair value for the policy with a surrender feature, we assume the historical data is used to estimate the likelihood of surrendering by the policyholder. Therefore we will have the fair value that includes the probabilities of surrender, life and death which are described as follows:

- The probability that surrendering by policyholder aged a is made at year t is denoted by ${}_t\Omega_a$.
- The probability that the policyholder aged a might die at year t is denoted by ${}_tq_a$.

- The probability that the holder aged a is still living at $(T_* - 1)^{th}$ year is denoted by ${}_{(T_*-1)}p_a$.
- The probability that surrendering the contract has never made at $(T_* - 1)^{th}$ year is denoted by ${}_{(T_*-1)}\chi_a$.

Then the value of the benefit with a surrender feature for the EIA can be found by summing all the present values $I(U_t)$ and $I(\bar{V})$ weighted with all the probabilities as we do

$$\begin{aligned} P\bar{V} = &\sum_{t=t}^{T_*-1} ({}_t\Omega_a) I(U_t) + \sum_{t=t}^{T_*-1} ({}_tp_a)({}_t\Omega_a) j_t I(U_t) \\ &+ ({}_{(T_*-1)}p_a)({}_{(T_*-1)}\chi_a) I(U_{T_*}). \end{aligned} \tag{5.2.13}$$

The costs amount resulted when the policyholder dies and surrender the contract prior to the maturity are respectively represented by the first and second terms that are weighted with probabilities of paying those amounts (which are the probabilities of death or surrender). The final term represents the costs amount payable when the policyholder who is alive at maturity never surrendered the policy.

Suppose that age $a = 50$ with $T_* = 8$ as the maturity date, and percentage $j_t = 0.8$ for $t \in [5, 8]$ since surrenders usually made not at earlier periods. The probabilities of surrender considered are as follows: ${}_t\Omega_{50} = 0.2$ for $t \in [5, 8]$ and ${}_{(T_*-1)}\chi_{50} = {}_7\chi_{50} = 1- {}_7\Omega_{50} = 0.8$. The probabilities of life and death considered as follows: ${}_tp_{50} = \frac{n-a-t}{n-a} = \frac{100-50-t}{50}$ where n is the terminal age of the investor. Now by considering the surrender periods 5-8 and assuming $I(U) = 0.7$ within the periods mentioned, equation (5.2.13) yields the results as follows:

$$P\bar{V} = 0.56 + 0.38896 + 0.48160 = 1.43$$

Chapter 6

VA/EIA Investment Embedded Risks

In this Chapter we mention important risks that are mostly experienced by the policyholders and the insurance companies. It is important that the insurers not only measure the risks, but know how to also manage them. Hedging of insurance benefits involves the use of both quantitative finance and actuarial science because of the existing inseparable insurance and financial security risks within the guaranteed benefits. Among the risks known the following are of greater concern:

(i) **Mortality risk**- It is the risk that insurance companies experience too many claims due to high policyholders' death rate in a short period of time

• This risk can be managed by issuing contracts to many retirees, as it gives an insurer a pledge in estimating. For example, if one person out of 300 policyholders dies, then the insurer can collect from 300 policyholders the premiums including that of the deceased to pay claim made by the beneficiary.

(ii) **Behavioural risk** - This is the risk affecting the insurer because of the policyholder not making decisions aligned with the insurer's assumptions. It includes eventually lapsing the contract as a result of policyholder's failure to pay premiums as agreed, and others such as surrendering the contract. To manage the risks associated with behaviour, the following should be among what can be done:

• Issuing the policy to a larger number of people can mitigate the risk, since the aggregate of the whole population's behaviour is of greater importance to the policy issuer.
• It is a usual thing that most people below age 60 defer income, but older ones make immediate exhaustion. So with that kind of behaviour, the insurer knows which class of individuals suitable to get the type of contract he issues.

• Making restrictions on the behaviour of the policyholder. For example, the maximum withdrawal to be made each year. Also requiring the policyholder a certain waiting period before starting income withdrawal or a certain age to start income withdrawal.

(iii) **Selection rate risk** - This is the risk that insurers bear for giving policyholders an option on when to start making withdrawals and flexibility on how much they would like to withdraw. This can entail greater loss to the insurance company since early and high withdrawals can cause a severe loss to the company.

• Managing such risk can be done through improved contract terms for policyholders who choose to defer withdrawals, and a cliquet option can be helpful in reduction of an income selection.

(iv) **Market risk** - This is the risk of losses in position resulted from the characteristic behaviour of the market prices. It includes for equity-based annuities, the change in volatility, the rate of interest and equity returns. There are many ways of dealing with that kind of risk, for example:

• Hedging strategies such as the Greeks to measure sensitivity of embedded options to the interest, volatility and time.
• Diversification of investments to reduce the volatility of asset values.

(v) **Longevity risk** - This is one of them and considered the primary and systematic risk since it is difficult to efficiently deal with. To the policyholders, longevity risk is viewed as a risk that they might live too long and eventually outlive their savings. So with the equity-based investments it is understood that longevity risk as a challenge to the policyholder it can be managed. But longevity risk as a problem to the insurance company is still a primary concern because it has to pay every claim policyholders make as long as they live according to the contract obligation agreed upon at contract initiation. Thus it poses a greater challenge to the management of such risk.

Chapter 7

Conclusion

In this thesis, the main objective was to carry out the risk management, valuation and analysis of the equity-based insurance policies which gained popularity during the 1990s in United States, Europe, Japan and other developed countries. These kind of policies are categorised into Variable Annuity (VA) and Equity Indexed Annuity (EIA). The four complex VA benefits differ from EIA in that the return on an investment is not restricted to floors and caps.

As can be viewed from valuation results of the priced benefits, we conclude that indeed the pricing formulae are fair as there was no negative present values. This means that despite the fees deducted, it is possible for the policyholder to receive what he initially invested, either by lump sum or periodic withdrawals.

Based on risk management, since it is such a greater challenge for insurers to efficiently manage the undiversifiable longevity risk, in conclusion statistical models are better candidates to the estimation of certain factors that will help the company make roughly good decisions when issuing a contract. Factors such as *risk premium* which we can compute and estimate how much risk acceptable to both the policyholder and the insurer when getting into a contract. This risk premium is a compensation for taking additional risk and is key to every model in finance associated with risk and return. It is intuitive that for highly risky investment there should be high expected returns compared to less risky investment. Thus we have the expected return as the sum of risk free return and risk premium compensated for risk taken by investor. The importance of risk premium is that it reflects basic judgements to be made on how large the risk projected in the equity market and what associated price we should attach to it. This can be included in the future work using models that better estimate risk premium.

Another factor that is important for the management of longevity risk is the *mortality*. Important thing to note about mortalities in annuities is that each year not provisional added for life expectancy may increase the policy provider's liabilities. The line of management to be used falls under the definition of product design where there should be some age restrictions and requirements for a particular contract to be issued to the buyers. Such designs help the insurance companies minimise the time length for which they will pay the income claims made by the policyholders. The idea here would be to make projections about the mortality rates of any country we choose in order to help the insurer decide when and to which age class should the contract be issued. An important assumption would be that a large number of population in a country we are interested in usually make their savings for retirement in the equity-based insurance policies. This also can be done in the future study of the equity based annuities using good models to forecast the mortality rates.

Appendices

Appendix A

Modelling Surrender Rates

A.1 Analysis of Surrender Rates

The plots in Figures A.1, A.2, A.3 and A.4, describe the nature of the data used for both variable annuities and equity indexed annuities, as well as independent variables that can be seen in equation (1.0.1).

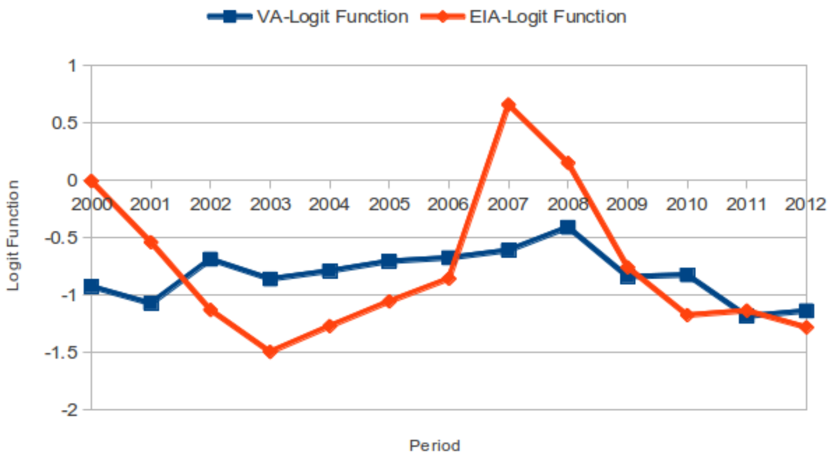

Figure A.1: Logit Surrender Rates Response Variables for VA and EIA

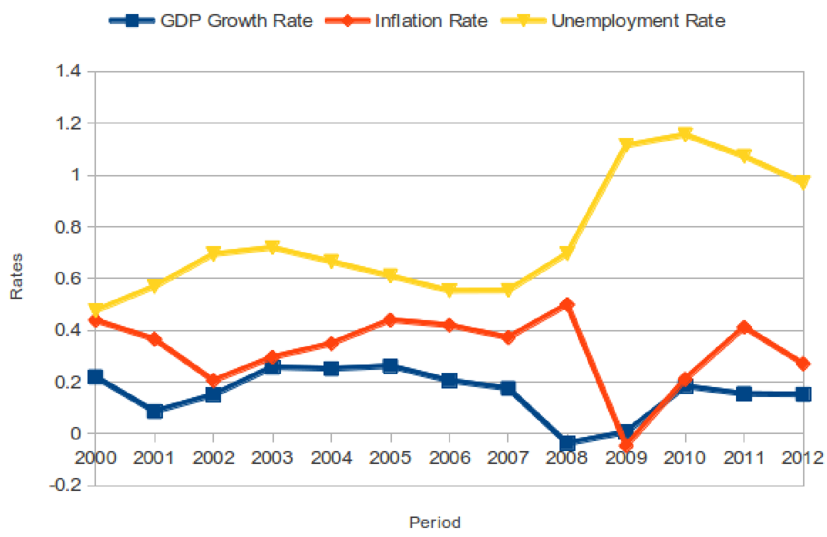

Figure A.2: GDP, Inflation and Unemployment data description

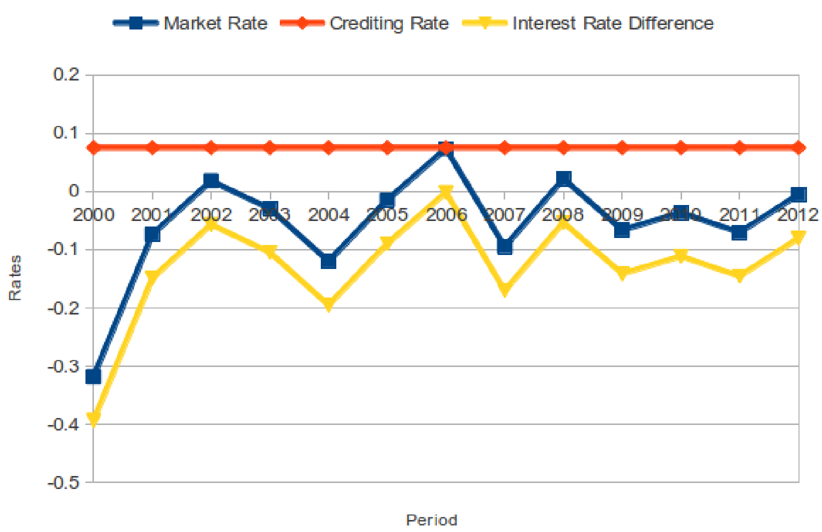

Figure A.3: Interest Rates data description

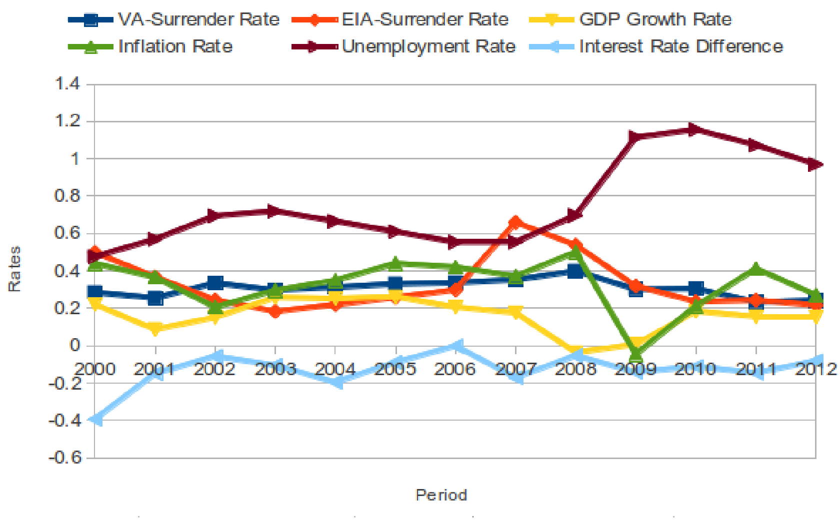

Figure A.4: Description of non-logit Surrender Rates vs. All Explanatory Variables

The analysis output for the logistic regression of both VA and EIA surrender rates data is as displayed in Figures A.5 and A.6

```
Residuals:
     Min       1Q   Median       3Q      Max
-0.37359 -0.03238  0.01910  0.09689  0.22593

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -0.1497     0.4373  -0.342    0.741
GDP          -0.5923     0.7109  -0.833    0.429
INF          -0.1502     0.5513  -0.273    0.792
UNE          -0.5557     0.3441  -1.615    0.145
DIFF          0.8494     0.6730   1.262    0.242

Residual standard error: 0.213 on 8 degrees of freedom
Multiple R-squared: 0.3669, Adjusted R-squared: 0.05037
F-statistic: 1.159 on 4 and 8 DF,  p-value: 0.3964
```

Figure A.5: VA Logit Function Analysis Output

```
Residuals:
     Min       1Q   Median       3Q      Max 
-0.42756 -0.20268 -0.07073  0.03304  1.10120 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)  
(Intercept)   0.3162     0.9733   0.325   0.7536  
GDP          -4.5854     1.5821  -2.898   0.0199 *
INF           0.8419     1.2269   0.686   0.5120  
UNE          -1.2467     0.7658  -1.628   0.1422  
DIFF         -2.4944     1.4978  -1.665   0.1344  
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.4741 on 8 degrees of freedom
Multiple R-squared: 0.645, Adjusted R-squared: 0.4675 
F-statistic: 3.634 on 4 and 8 DF,  p-value: 0.05685
```

Figure A.6: EIA Logit Function Analysis Output

The Tables A.1 and A.2 also describe the data for all factors used in the model.

	VA-Logit	EIA-Logit	GDP	INF	UNEMP	DIFF
Average	-0.825	-0.761	0.160	0.326	0.757	-0.130
S.Dev	0.210	0.624	0.089	0.138	0.226	0.091

Table A.1: Variables Averages and Standard deviations

	VA-Logit	EIA-Logit	GDP	INF	UNEMP	DIFF
VA-Logit	1					
EIA-Logit	0.444	1				
GDP	-0.173	-0.391	1			
INF	0.175	0.333	0.257	1		
UNEMP	-0.383	-0.450	-0.301	-0.621	1	
DIFF	0.302	-0.342	-0.212	-0.094	0.216	1

Table A.2: Correlation for Logit function of VA and EIA with explanatory variables

Appendix B

Withdrawal rates, Dynamic Solution and Continuation Value for GMWB

B.1 Sustainable Withdrawal Probabilities

Tables B.1 and B.2 below correspond to what is explained for equation (4.1.4) in Subsection 4.1.1.

(Years) Periods	μ(%)	σ	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
1981-1985	0.95	0.050	87.1	69.6	49.8	33.0	20.8	12.2	7.3	4.0	2.1
1981-1990	0.69	0.056	85.2	65.5	45.5	29.3	17.8	10.3	6.1	3.2	1.9
1981-1995	1.07	0.052	89.0	71.5	52.0	34.9	22.1	13.2	7.9	4.4	2.4
1981-2000	1.3	0.063	91.1	75.2	56.2	38.8	25.1	15.4	9.1	5.3	2.9
1981-2005	1.03	0.066	89.0	71.5	52.0	34.9	22.1	13.2	7.9	4.4	2.4
1981-2010	0.93	0.065	87.1	69.6	49.8	33.0	20.8	12.2	7.3	4.0	2.1

Table B.1: Withdrawal Success Rate Probabilities (%).

(Year) Periods	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
1981-1985	302849.98	302849.97	302849.96	302849.95	302849.94	302849.93	302849.92	302849.91	302849.9
1981-1990	302069.98	302069.97	302069.96	302069.95	302069.94	302069.93	302069.92	302069.91	302069.9
1981-1999	303209.98	303209.97	303209.96	303209.95	303209.94	303209.93	303209.92	303209.91	303209.9
1981-2000	303899.98	303899.97	303899.96	303899.95	303899.94	303899.93	303899.92	303899.91	303899.9
1981-2005	303089.98	303089.97	303089.96	303089.95	303089.94	303089.93	303089.92	303089.91	303089.9
1981-2010	302789.98	302789.97	302789.96	302789.95	302789.94	302789.93	302789.92	302789.91	302789.9

Table B.2: Accumulated Period-End Investment Account Values in Dollars ($).

B.2 Solving for Solution to Dynamic GMWB

From equation (4.2.21), below is how we came about equation (4.2.22) when $\gamma \to \infty$.
First, let us consider a case where $1 - A \leq q$, then the term $\gamma \max(1 - q - A, 0)$ will all disappear. So we will be left with

$$\frac{\partial \vartheta}{\partial t} + \mathcal{L}\vartheta + \min[\max(1 - A, 0), q]G = 0 \tag{B.2.1}$$

$$\Rightarrow \frac{\partial \vartheta}{\partial t} + \mathcal{L}\vartheta + \max(1 - A, 0)G = 0 \tag{B.2.2}$$

following $-1 + A + q \geq 0$.

Knowing that $\min(a, b) + c = \min(a + c, b + c)$, thus equation (4.2.21) yields equation (4.2.22) written as

$$\min\left[-\frac{\partial \vartheta}{\partial t} - \mathcal{L}\vartheta - \max(1 - A, 0)G, -(1 - q) + A\right] = 0. \tag{B.2.3}$$

Recall that when $W = 0$ we have $\vartheta_0(H, T_*) = H(1 - q)$ and $\vartheta_0(0, t) = 0$. Now to get $\vartheta_0(H, t)$, let us consider these scenarios below:

First Scenario: Consider when

$$1 \geq \frac{\partial \vartheta_0}{\partial H} > (1 - q) \tag{B.2.4}$$

then, equation (4.2.22) or (B.2.3) becomes

$$-\frac{\partial \vartheta_0}{\partial t} + r\vartheta_0 - G + G\frac{\partial \vartheta_0}{\partial H} = 0. \tag{B.2.5}$$

Define

$$\gamma_0(H, t) = \vartheta_0(H, t)e^{r(T_* - t)} - G\int_t^{T_*} e^{r(T_* - s)} ds, \tag{B.2.6}$$

then $\gamma_0(H, t)$ satisfies

$$\frac{\partial \gamma_0(H, t)}{\partial t} - G\frac{\partial \gamma_0(H, t)}{\partial H} = 0 \tag{B.2.7}$$

with conditions

$$\begin{aligned} &\text{(i).} \quad \gamma_0(H, T_*) = H(1 - q), \\ &\text{(ii).} \quad \gamma_0(0, t) = -G\int_t^{T_*} e^{r(T_* - s)} ds \end{aligned} \tag{B.2.8}$$

The function $\gamma_0(H,t)$ has a solution of the kind

$$\gamma_0(H,t) = F(\upsilon) \tag{B.2.9}$$

where $\upsilon = t + \frac{H}{G}$ and the function F will be determined from the conditions above. Equation (B.2.7) has characteristics given by $T_* \le \upsilon$ and $T_* > \upsilon$.

We know that $\vartheta_0(H,T_*) = H(1-q)$ for time $t = T_*$, so $\frac{\partial \vartheta_0(H,T_*)}{\partial H} = 1-q$ which opposes the first scenario. This means that the condition (i) in equation (B.2.8) is also rejected since we will have $\gamma_0(H,T_*) = F(T_* + \frac{H}{G}) = H(1-q)$. Thus it does not hold for characteristic $T_* \le t + \frac{H}{G}$

So, we will be focusing on the situation when $t = t$, that is only on condition (ii).

$$\gamma_0(0,t) = -G\int_t^{T_*} e^{r(T_*-s)}ds$$

which implies that when $H = 0$, equation (B.2.9) becomes

$$F(t+\frac{0}{G}) = -G\int_t^{T_*} e^{r(T_*-s)}ds \tag{B.2.10}$$

and consequently, for characteristic $T_* > t + \frac{H}{G}$ we have from equation (B.2.9) that

$$\gamma_0(H,t) = F(t+\frac{H}{G}) \tag{B.2.11}$$

$$= -G\int_{t+\frac{H}{G}}^{T_*} e^{r(T_*-s)}ds. \tag{B.2.12}$$

Thus from equation (B.2.6), we have that

$$\begin{aligned}
\vartheta_0(H,t) &= e^{-r(T_*-t)}G\left[-\int_{\upsilon}^{T_*} e^{r(T_*-s)}ds + \int_t^{T_*} e^{r(T_*-s)}ds\right] \\
&= G\left[\int_{T_*}^{\upsilon} e^{r(t-s)}ds + \int_t^{T_*} e^{r(t-s)}ds\right] \\
&= G\int_t^{\upsilon} e^{r(s-t)}ds, \qquad \text{(B.2.13)}
\end{aligned}$$

and thus $\frac{\partial \vartheta_0(H,t)}{\partial H} = e^{-r\frac{H}{G}} < 1$ satisfying one part of first scenario $1 \ge \frac{\partial \vartheta_0(H,t)}{\partial H}$. Which means that to satisfy the whole of our first scenario we need that

$$e^{-r\frac{H}{G}} > (1-q), \tag{B.2.14}$$

and that is when $H < -\frac{G}{r}\ln(1-q)$. Now combining the characteristic region $H < G(T_*-t)$ and $H < -\frac{G}{r}\ln(1-q)$, we have for (H,t) the optimal boundary for withdrawal starting at

$$H \le \min[G(T_*-t), -\frac{G}{r}\ln(1-q)]. \tag{B.2.15}$$

Set $\jmath = \min[(T_* - t), -\frac{1}{r}\ln(1-q)]$.

Second Scenario: We consider when

$$\frac{\partial \vartheta_0}{\partial H} = 1 - q. \tag{B.2.16}$$

Then, we have $\vartheta_0(H,t) = H(1-q) + C_t$ where C_t is an arbitrary function of t. So, taking $H < \jmath G$ we have

$$\begin{aligned} \jmath G(1-q) + C_t &= \vartheta_0(\jmath G, t) \\ &= G\int_t^{t+\jmath} e^{-r(s-t)} ds \\ &= \frac{G}{r}\left[1 - e^{-r\jmath}\right] \end{aligned} \tag{B.2.17}$$

which implies that

$$\vartheta_0(H,t) = H(1-q) + G\int_t^{t+\jmath} e^{-r(s-t)} ds - \jmath G(1-q). \tag{B.2.18}$$

Combining solutions (B.2.13) and (B.2.18) to both scenarios, we have that

$$\vartheta_0(H,t) = (1-q)\max(H - \jmath G, 0) + \frac{G}{r}\left[1 - e^{-r\min(\jmath, \frac{H}{G})}\right]. \tag{B.2.19}$$

B.3 GMWB Continuation Value

Recall the formulae in equations (4.4.8) and (4.4.9) for the continuation values of the GMWB. Let us denote the GMWB continuation value by Λ at any node before the maturity date. Then, we have the following formulae

$$\Lambda_u = e^{-\mu\Delta t}(p_u\Lambda_{uu} + p_d\Lambda_{um}), \tag{B.3.1}$$

$$\Lambda_m = e^{-\mu\Delta t}(p_u\Lambda_{um} + p_d\Lambda_{md}), \tag{B.3.2}$$

$$\Lambda_d = e^{-\mu\Delta t}(p_u\Lambda_{md} + p_d\Lambda_{dd}). \tag{B.3.3}$$

Then for a doubling period to the maturity date T_*, we have substitute equations (B.3.1), (B.3.2), and (B.3.3) in the following GMWB continuation value formula in equation (B.3.4) of a single period successor node:

$$\Lambda = e^{-\mu\Delta t}(p_u\Lambda_u + p_m\Lambda_m + p_d\Lambda_d), \tag{B.3.4}$$

so that the doubling period as in Figure 4.4 (from J to the maturity date) has the continuation value as

$$\begin{aligned} \Lambda &= e^{-\mu\Delta t}[p_u e^{-\mu\Delta t}(p_u\Lambda_{uu} + p_d\Lambda_{um}) + p_m e^{-\mu\Delta t}(p_u\Lambda_{um} + p_d\Lambda_{md}) \\ &\quad + p_d e^{-\mu\Delta t}(p_u\Lambda_{md} + p_d\Lambda_{dd})] \\ &= e^{-3\mu\Delta t}[p_u^2\Lambda_{uu} + p_u p_d\Lambda_{um} \\ &\quad + p_m p_u\Lambda_{um} + p_m p_d\Lambda_{md} + p_d p_u\Lambda_{md} + p_d^2\Lambda_{dd}]. \end{aligned} \tag{B.3.5}$$

Appendix C

Valuation Analysis for EIA

C.1 Analysis of Equity-Indexed Policies

In equations (5.1.13), (5.1.16) and (5.1.17), $\mathbf{d_A}$ and $\mathbf{d_B}$ are derived as follows: Firstly,

Definition C.1 *For a log-normal distributed variable Y with parameters denoted by μ and σ which are, respectively, the mean and standard deviation of the variable's natural logarithm (i.e the variable's logarithm is normally distributed), it implies $Y = e^{\mu+\sigma Z}$, where Z is a standard normal variable with mean 0 and variance 1. Then expected value is given by $E[Y] = e^{\mu+\frac{1}{2}\sigma^2}$, See Serfling (2002).*

Knowing the definition, we then solve the probability terms to get equation (5.1.13) as follows

$$\begin{aligned} P(R_I \le f_\varsigma) &= P(e^{\mu+\sigma N(0,1)} \le f_\varsigma) \\ &= P(\mu + \sigma N(0,1) \le \ln f_\varsigma) \\ &= P\left(N(0,1) \le \frac{\ln f_\varsigma - (r - \delta - \frac{1}{2}\sigma^2)}{\sigma}\right) \\ &= P(N(0,1) \le \mathbf{d_A}) \\ &= \Phi(\mathbf{d_A}), \end{aligned} \tag{C.1.1}$$

then

$$\begin{aligned} P(R_I \ge c_\varsigma) &= P(e^{\mu+\sigma N(0,1)} \ge c_\varsigma) \\ &= P(\mu + \sigma N(0,1) \ge \ln c_\varsigma) \\ &= P\left(N(0,1) \ge \frac{\ln c_\varsigma - (r - \delta - \frac{1}{2}\sigma^2)}{\sigma}\right) \\ &= P(N(0,1) \ge d_2) \\ &= P(N(0,1) \le (-\mathbf{d_B})) \\ &= 1 - \Phi(\mathbf{d_B}), \end{aligned} \tag{C.1.2}$$

and

$$\begin{aligned} P(f_\varsigma \le R_I \le c_\varsigma) &= P(R_I \le c_\varsigma) - P(R_I \le f_\varsigma) \\ &= P(e^{\mu+\sigma N(0,1)} \le c_\varsigma) - P(e^{\mu+\sigma N(0,1)} \le f_\varsigma) \\ &= \Phi(\mathbf{d_B}) - \Phi(\mathbf{d_A}). \end{aligned} \tag{C.1.3}$$

C.1.1 Analytical EIA Values

Analytical values to the EIA's simple and compound cliquet options in equations (5.1.16) and (5.1.17) are as in Tables C.1 and C.2, where we have taken into account the following assumptions: (i). Duration of the contract is 14 years; (ii). Initial investment W_0 is \$100; (iii). Risk free interest rate of 4%; (iv). Fee rate of 0.1% (v). Floor rate of 0%; and (vi). Volatility of 30%.

Now, with the rates of participation ς from 0.7 to 1.0, and the ceiling rates c from 12.5% to 15.5%, we have

$\varsigma \diagdown c$	12.5%	13.5%	14.5%	15.5%
0.7	90.55	91.42	92.67	93.47
0.8	92.75	94.08	95.51	96.47
0.9	94.61	96.36	97.68	99.11
1.0	96.12	98.03	99.77	108.68

Table C.1: Simple EIA Cliquet Values

$\varsigma \diagdown c$	12.5%	13.5%	14.5%	15.5%
0.7	101.34	102.83	105.02	106.42
0.8	105.15	107.52	110.12	111.91
0.9	108.48	111.72	114.18	116.94
1.0	111.26	114.86	118.23	136.99

Table C.2: Compound EIA Cliquet Values

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