THE RADON-NIKODÝM AND THE KREIN-MILMAN PROPERTIES IN BANACH SPACES

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The Radon-Nikodým and the Krein-Milman Properties in Banach spaces

by

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Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own work and that I have not previously in its entirety or partly submitted it at any university for a degree.

Abstract

A Banach space X over the field of real numbers $\mathbb{R}$ has the Radon-Nikodým property (RNP) if for each finite positive measure space (Ω, Σ, μ) and each X-valued, μ-continuous measure ν on Σ with bounded variation $|\nu|$, there exists a Bochner integrable function $f : \Omega \to X$ such that $\nu(E) = \int_E f \, \mathrm{d}\mu$ for $E \in \Sigma$.

The RNP has become a geometrical property when the following result was introduced: A Banach space X has the RNP if and only if each non-empty bounded subset of X is dentable.

Futhermore, a Banach space X has the Krein-Milman property (KMP) if each closed bounded convex subset of X is the closed convex hull of its extreme points.

Lindenstrauss proved that if each nonempty closed bounded convex subset of a Banach space X contains an extreme point, then X has the Krein-Milman property. In particular, a Banach space with the RNP has the KMP. The converse remains an open question. In this thesis we examine conditions under which the KMP implies the RNP.

Opsomming

Gestel X is 'n Banach-ruimte oor die liggaam $\mathbb{R}$. Dan het X die Radon-Nikodým-eienskap as vir elke eindige positive maatruimte (Ω, Σ, μ) en vir elke μ-kontinue aftelbaar additiewe maat $\nu : \Sigma \to X$ met begrensde variasie $|\nu|$, daar 'n Bochner-integreerbare funksie $f : \Omega \to X$ bestaan sodanig dat $\nu(E) = \int_E f d\mu$ vir elke $E \in \Sigma$.

Die RN-eienskap het van die maatteoretiese na die meetkundige verskuif toe Rieffel aangetoon het dat 'n Banach-ruimte X die Radon-Nikodým eienskap het as en slegs as elke nie-leë begrensde deelversameling van X induikbaar is.

Verder, 'n Banach-ruimte X het die Krein-Milman-eienskap as elke nie-leë geslote begrensde konvekse deelversameling van X gelyk is aan die geslote konvekse omhulsel van sy ekstreempunte. Lindenstrauss het bewys dat as elke nie-leë geslote begrensde konvekse deelversameling van 'n Banach-ruimte X 'n ekstreempunt bevat, dan het X die Krein-Milman-eienskap. In die besonder geld dat 'n Banach-ruimte met die Radon-Nikodým eienskap ook die Krein-Milman-eienskap het.

Omdat die omgekeerde in die algemeen nie geld nie, word in hierdie tesis ondersoek ingestel na voorwaardes waaronder Krein-Milman wel vir Radon-Nikodým impliseer.

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Thabang J. M. Nthebe

Contents

Historical Background

There appear to be three aspects to the theory of differentiation of vector-valued measures, namely, **analytic**, **operator theoretic** and **geometric**. While these aspects are mutually interrelated, we discuss the analytic and geometric aspects only, and separately in this thesis.

The Radon-Nikodým theorem (also known as the Lebesgue-Nikodým theorem), was proved first by H. Lebesgue in 1904, in [29], where he gave a necessary and sufficient condition for a function defined on [0,1] to be expressible as an indefinite integral. In the following year, G. Vitali in [52] characterised such functions as the now familiar absolutely continuous functions.

These results were extended by J. Radon in 1913 in [39], for a Borel measure in Euclidean space. They were further extended by O. M. Nikodým in the general form in [34].

N. Dunford and B. J. Pettis in 1940, extended the Radon-Nikodým theorem for vector measures m with values in a separable dual Banach space, absolutely continuous with respect to a positive measure μ, such that $\|m(E)\| < a\mu(E)$ for $E \in R$ (a ring) and for some $a > 0$.

An extension of the Radon-Nikodým theorem for finitely additive measures was given by S. Bochner in [3], and by S. Bochner and R. S. Phillips in [4].

Not too surprisingly, the start of the theory of a vector-valued Radon-Nikodým theorem coincides with the introduction of the first vector-valued integration theory by S. Bochner. Bochner notes that if every X-valued function of bounded variation defined on [0,1] is differentiable almost everywhere then each X-valued absolutely continuous function on [0,1] can be recovered from its derivative via the Bochner integral, where X is a Banach space. It was left open, however, whether any infinite dimensional Banach space had the aforementioned property (called by some the Gel'fand-Fréchet property, abbreviated GFP).

J. Clarkson showed then that every uniformly convex Banach space is a Gel'fand-Fréchet (GF) space. He also observed that l_1 is a GF space, but c_0 and $L_1[0,1]$ are not.

After the results of N. Dunford and B. J. Pettis [16], and those of Phillips [37] in 1940, on the representability of linear operators on $L^1(\mu)$ as Bochner integrals, however, few

direct substantive advances were noted until the late 1960's. N. Dunford and M. Morse extended Clarkson's observation on l_1 to the class of Banach spaces with boundedly complete Schauder basis, (the Schauder basis (x_n) is boundedly complete if for any sequence $(a_n)_{n\geq 1}$ of scalars such that $\sup_{n\to\infty} \|\sum_{n=1}^{\infty} a_n x_n\| < \infty$, then $\sum_{n=1}^{\infty} a_n x_n$ converges). They also showed that Banach spaces with boundedly complete bases are GF spaces.
I. M. Gel'fand showed that $L_1[0,1]$ was not isomorphic to any dual space. Furthermore, Bochner and Taylor has shown that the GF spaces were the same as those Banach spaces X with the property that, given a countably additive X-valued map F defined on a σ-algebra, possessing finite variation $|F|$, then there exists a Bochner $|F|$-integrable function f such that $F(A) = \int_A f d|F|$ for each A in the domain of F (this property is called the Radon-Nikodým Property). It is then from this point that we will build our discussion on the analytical aspects.

On the geometric aspects, a great break-through in the theory of Radon-Nikodým was due to M. A. Rieffel who recovered a classical differentiation theorem of Phillips by introducing the geometric notion of dentability.

In 1967, M. A Rieffel tied the Radon-Nikodým theorem (RN-theorem) in Banach spaces to the geometry via the notion of dentability, in [40]. The establishment of a close interrelationship between the Radon-Nikodým theorem and the Radon-Nikodým Property, and the geometry of Banach spaces, emerged from Rieffel's efforts. This is then the aspect of the RN-theorem that has seen the most spectacular advances in the recent years.
An immediate consequence of Rieffel's Dentability theory is as follows:
If every bounded subset of a Banach space is dentable, then the Banach space possesses the RNP.
After this result appeared there was a period of absorption of the notion of dentability. Then in 1973, Hugh Maynard introduced the notion of σ-dentability and characterised Banach spaces with the RNP as spaces where bounded sets are σ-dentable.
During 1973, W. J. Davis and R. R. Phelps, and independently E. Huff, proved the converse of Rieffel's results and it states the following:

If a Banach space possesses the RNP, then every bounded non-empty subset of the Banach space is dentable. The combined effect, is thus:

A Banach space possesses the RNP if and only if every bounded non-empty subset is dentable.

The concurrent geometrical advances of J. Lindenstrauss and others on the existence of extreme points for not necessarily compact (closed bounded) convex subsets of certain Banach spaces, induced considerable research activity. A coherent picture of Banach spaces having the RNP developed from operator, martingale and geometrical perspectives.

By the mid 1970's, the broad strokes were in place and a period of re-evaluation and refinement began. One of the upshots was the localisation of many of the early theorems on Banach spaces to the corresponding ones for closed bounded convex sets. What emerged was an elegant and comprehensive theory of sets with the RNP. By the late 1970's, emphasis shifted from the study of sets with the RNP, which until then had flourished, to an analysis of their place in the larger functonal analysis picture, especially their role in the structure of Banach spaces.

This is where our discussion will commence, by using most of these results without proving them. This historical background was extracted from [14], which is an excellent source of information on this topic.

Introduction

In this thesis we discuss the equivalence between the Radon-Nikodým Property (RNP) and the Krein-Milman Property (KMP) in Banach spaces and in their dual spaces. We will investigate when and in what spaces these properties are and are not equivalent. Our main aim is to investigate the conditions under which these two properties are equivalent in a Banach space.

The first chapter deals predominantly with the concepts and definitions necessary for a reader to get acquainted with the subject, and it lays a firm foundation for the rest of this thesis. As mentioned before, the Radon-Nikodým theorem is important and it is the base of our discussion for the RNP. The Radon-Nikodým Property and all the known properties that are equivalent to the RNP, and those that are implied by the RNP, in both a Banach space X and its topological dual Banach space X^*, are being discussed in the first chapter. The concept of **dentability,** introduced by M. A. Rieffel, will be discussed in an attempt to find properties equivalent to the RNP in a Banach space. On the other hand the **separability** property will be the backbone of those properties, in the dual Banach spaces, equivalent to the RNP.

Chapter two deals with Banach spaces and their topological dual spaces lacking the RNP and conditions sufficient for a Banach space to lack the RNP. It gives an idea as to how to distinguish between Banach spaces with and those without the RNP, and properties equivalent to and those not equivalent to the RNP. Some examples are given which are really useful to a better understanding of the properties and confirmation of that theory.

In the third chapter, we introduce the KMP (Krein-Milman Property), which is a property generally implied by the RNP. The existence of extreme points is used to characterise those closed bounded and convex sets, and Banach spaces, with the KMP.

The fourth and fifth chapters are aimed at discussing the lack of the KMP and of both the KMP and the RNP, respectively. The theory of martingales and trees will be introduced to characterise Banach spaces failing both the RNP and the KMP.

The most important chapter of this thesis is the sixth, in which the equivalence of the RNP and the KMP is dealt with in depth, along with all other properties, that are equivalent to both these two. Restrictions and conditions are imposed on Banach spaces and their dual Banach spaces, so as to ensure an equivalence between the RNP and the KMP in these spaces. Those restrictions and conditions are discussed in chapter six as well.

We also consider some applications of the Radon-Nikodým theorem, in the last chapter of this thesis, namely chapter 7. We are looking at the applications to Business Mathematics, amongst others, in this chapter and few applications in Pure Mathematics itself.

Chapter 1

The Radon-Nikodým Property

In this chapter we discuss the Radon-Nikodým Property and properties equivalent to it. We shall first discuss Rieffel's approach to the general Radon-Nikodým theorem for the Bochner integral, dating from 1967. Subsequently, the concept of dentability will be introduced, and its role in the theory of the Radon-Nikodým Property (RNP) in real Banach spaces will be established.

Notations and terminology:

Throughout this thesis, Ω will denote a non-empty point set on which no topological structure is required. The symbol X will be used to denote a real Banach space, and X^* will denote the topological dual of X. If $A \subset X$, then $cl(A)$ will denote the strong (norm) closure of a set A, $co(A)$ the convex hull of A and $clco(A)$ its closure.

The basis for this material is a σ-finite positive measure space (Ω, Σ, μ), where Σ is a σ-algebra of subsets of Ω and $\mu : \Sigma \longrightarrow \mathbb{R}^+$ is a countably additive measure. A function $f : \Omega \rightarrow X$ is called a Σ-simple function if it is of the form $f = \sum_{i \in I} b_i \chi_{A_i}$ where I is a finite set, $A_i \in \Sigma$ and $b_i \in X$, for every $i \in I$. We suppose that every simple function is in standard form, that is, the vectors b_i are mutually distinct and the sets $A_i \in \Sigma$ are mutually disjoint, $i \in I$. The *Bochner integral* of such a simple function $f = \sum_{i \in I} b_i \chi_{A_i}$ is defined as $\int f \mathrm{d}\mu = \sum_{i \in I} b_i \mu(A_i)$. The linear space of all simple functions $f : \Omega \longrightarrow X$ will be denoted by $\Psi_X(\Sigma)$. A sequence $(f_n)_{n \geq 1}$ in $\Psi_X(\Sigma)$ is called a mean Cauchy sequence if $\lim_{m,n \to \infty} \int \| f_m - f_n \| \, d\mu = 0$.

For such a sequence $(f_n)_{n\geq 1}$ in $\Psi_X(\Sigma)$ it is easily seen that $(\int f_n d\mu)_{n\geq 1}$ is a Cauchy sequence in X. A function $f : \Omega \to X$ is called *Bochner integrable* on Ω if it is a limit μ-*a.e* on Ω of a mean Cauchy sequence $(f_n)_{n\geq 1}$ in $\Psi_X(\Sigma)$; that is, $\int f \mathrm{d}\mu = \lim_{n\to\infty} \int f_n d\mu$. The space of all Bochner integrable functions $f : \Omega \to X$ will be denoted by $L^1(\Omega, \Sigma, \mu, X)$. A function $f : \Omega \to X$ is called μ-measurable (or just measurable) if there exists a sequence $(f_n)_{n\geq 1}$ in $\Psi_X(\Sigma)$ such that $f_n \to f$ pointwise μ-*a.e* on Ω.

1.1 The Radon-Nikodým Theorem

This section is dedicated to the Radon-Nikodým Theorem as stated and proved by M. A. Rieffel in 1967.

We first formulate the standard Radon-Nikodým Theorem in the case of σ-finite measures and a non-negative measurable function.

Theorem 1.1.1 [2, **Theorem 8.9, p.85**]
Let (Ω, Σ, μ) be a σ-finite positive measure space, $\lambda : \Sigma \to \mathbb{R}$ a σ-finite positive measure which is μ-continuous. Then there exists a non-negative measurable function $f : \Omega \to \mathbb{R}$ such that $\lambda(E) = \int_E f d\mu$ for every $E \in \Sigma$.
Moreover, the function f is uniquely determined μ-almost everywhere.

Example 1.1.3 below shows that the Radon-Nikodým Theorem in the form of Theorem 1.1.1 cannot be carried over to the case of the Bochner integral. We need, however, the following result.

Theorem 1.1.2 [15, **Proposition 7, p.123**]
Let (Ω, Σ, μ) be a measure space and X be a Banach space, and let $f : \Omega \to X$ be integrable. Then $\int T \circ f d\mu = T(\int f d\mu)$ (1)
holds for each $T \in X^*$.

Proof

The Bochner integrability of $T \circ f$ follows from that of f. If f is a simple integrable function in standard form $f = \sum_{i=1}^{k} a_i \chi_{A_i}$, where $a_i \neq 0, i = 1, 2, ..., k$, then

$$\begin{aligned}\int T \circ f d\mu &= \int T(\textstyle\sum_{i=1}^{k} a_i \chi_{A_i}) d\mu \\ &= \int \textstyle\sum_{i=1}^{k} T(a_i) \chi_{A_i} d\mu \\ &= \textstyle\sum_{i=1}^{k} T(a_i) \mu(A_i)\end{aligned}$$

and

$$\begin{aligned}T(\int f d\mu) &= T(\int \textstyle\sum_{i=1}^{k} a_i \chi_{A_i} d\mu) \\ &= T(\textstyle\sum_{i=1}^{k} a_i \mu(A_i)) \\ &= \textstyle\sum_{i=1}^{k} T(a_i) \mu(A_i).\end{aligned}$$

Hence (1) holds for simple integrable functions. Next suppose that f is an arbitrary Bochner integrable function. If $(f_n)_{n \geq 1}$ is a Cauchy sequence in $\Psi_X(\Sigma)$ converging μ-a.e to f, then $(T \circ f_n)_{n \geq 1}$ is a Cauchy sequence in $\Psi_{\mathbb{R}}(\Sigma)$ converging μ-a.e to $T \circ f$, and then

$$\begin{aligned}\int (T \circ f) d\mu &= lim_{n \to \infty} \int (T \circ f_n) d\mu \\ &= T(lim_{n \to \infty} \int f_n d\mu) \\ &= T(\int f d\mu).\end{aligned}$$

This completes the proof. □

Example 1.1.3

Let $\Omega = [0, 1]$, and Σ be the σ-algebra of Lebesgue subsets of $[0, 1]$, and μ the Lebesgue measure on Σ. Take $X = L_1(\Omega, \Sigma, \mu, \mathbb{R})$ and define $m : \Sigma \to X$ by $m(A) = \chi_A$ for every $A \in \Sigma$. Then m is is an X-valued measure. Furthermore,

$$\begin{aligned}|m|(A) &= \sup_I \textstyle\sum_{i \in I} \|m(A_i)\| \\ &= \sup_I \textstyle\sum_{i \in I} \mu(A_i) \\ &= |\mu|(A) \\ &= \mu(A)\end{aligned}$$

where the summation is over all classes of mutually disjoint sets in Σ, with $\bigcup_{i \in I} A_i \subset A$ and I is a countable index set. Then $|m| = \mu$, and so $m \ll \mu$. Now suppose that a Radon-Nikodým derivative $f = \frac{dm}{d\mu}$ exists; then

$$m(A) = \int_A f(t) d\mu(t)$$

for each $A \in \Sigma$. Let $T \in L_\infty(\Omega, \Sigma, \mu, \mathbb{R}) = X^* = (L_1)^*$. The notation (T, h) below denotes the value $T(h)$, $h \in X$. Then,

$$\begin{aligned}
\int_A (T, f(t)d\mu(t) &= (T, \int_A f(t)d\mu(t)) \quad \text{(from Theorem 1.1.2)} \\
&= (T, m(A)) \\
&= (T, \chi_A) \\
&= \int_A T(t)d\mu(t).
\end{aligned}$$

Then $(T, f(t)) = T(t)$ for μ-a. all $t \in [0,1]$, that is, $(T, f(t)) = T(t)$ for all $t \in [0,1] \backslash A(T)$, where $\mu(A(T)) = 0$. Denote by $\{I_n : n \in \mathbb{N}\}$ the class of all sub-intervals of [0,1] with rational endpoints. For each $n \in \mathbb{N}$, let $T_n = \chi_{I_n} \in X^*$. Put $A = \cup_{n=1}^{\infty} A(T_n)$ and let $x \in [0,1] \backslash A$. Then,

$$\begin{aligned}
\int_{I_n} f(x)(s)d\mu(s) &= \int T_n(s) f(x)(s)d\mu(s) \\
&= T_n(x) \\
&= 0, \text{ for } x \notin I_n.
\end{aligned}$$

Then, $f(x)(s) = 0$ for μ-almost all $s \in [0,1]$ as long as $x \in [0,1] \backslash A$; whence $f : [0,1] \to X$ vanishes μ-a.e. But this contradicts the fact that $\int_B f d\mu = \chi_B \neq 0$ whenever $\mu(B) > 0$ for any $B \in \Sigma$. Consequently, the space $L_1(\Omega, \Sigma, \mu, \mathbb{R})$ does not satisfy the Radon-Nikodým Theorem in the form of Theorem 1.1.1.

We now state and prove a theorem due to M. A Rieffel, where the construction of a Bochner integrable Radon-Nikodým derivative is shown, under suitable assumptions.

Rieffel's Radon-Nikodým Theorem 1.1.4 [42, Main Theorem, p.466]
Let (Ω, Σ, μ) be a σ-finite positive measure space, and X be a real Banach space. Let $m : \Sigma \to X$ be a measure. Then m is the indefinite integral with respect to μ of a Bochner integrable function $f : \Omega \to X$ if and only if:

1. m is μ-continuous, that is, $m \ll \mu$.

2. the total variation $|m|$ of m is a finite measure on Σ.

3. locally m somewhere has compact average range, that is, given $E \in \Sigma$ with $0 < \mu(E) < \infty$, there is an $F \subseteq E$ such that $\mu(F) > 0$ and $A_F(m) = \{\frac{m(G)}{\mu(G)} : G \subseteq F, \mu(G) > 0\}$ is relatively norm compact, or *equivalently:*

3'. locally m somewhere has compact direction, that is, given $E \in \Sigma$ with $0 < \mu(E) < \infty$, there is an $F \subseteq E$ and a compact subset K of X not containing 0, such that $\mu(F) > 0$ and $m(G)$ is contained in the cone generated by K for all $G \subseteq F$.

Following Rieffel, we first establish the necessity part of the Theorem 1.1.4 above. We need the following notations:

If $f : \Omega \to X$ is a measurable function and $E \in \Sigma$, then the *essential range of f restricted to E*, is the set,

$er_E(f) = \{b \in X : \forall \varepsilon > 0,\ \mu(\{x \in E : \|f(x) - b\| < \varepsilon\}) > 0\}$.

Note that, where $S_\varepsilon(b)$ denotes the open ball with radius ε and centre b:

$$\begin{aligned} er_E(f) &= \{b \in X : \forall \varepsilon > 0,\ \mu(\{x \in E : f(x) \in S_\varepsilon(b)\}) > 0\} \\ &= \{b \in X : \forall \varepsilon > 0,\ \mu(E \cap (f^{-1}(S_\varepsilon(b)) > 0\}. \end{aligned}$$

Furthermore, if $D \subseteq X$, then $cone(D)$ will denote the cone with vertex 0 generated by D. Lastly, if $f : \Omega \to X$ is Bochner integrable then the indefinite integral of f is the X-valued measure μ_f defined by $\mu_f(E) = \int_E f d\mu$, $E \in \Sigma$. The *average range* of μ_f on $E \in \Sigma$ is defined by

$$A_E(\mu_f) = \{\frac{\mu_f(F)}{\mu(F)} : F \in \Sigma,\ F \subseteq E, 0 < \mu(F)\}.$$

Theorem 1.1.5 [17, Theorem 15, p.22]

If K is a subset of a Banach space X, the following are equivalent:

1. K is sequentially compact

2. $cl(K)$ is compact

3. K is totally bounded and $cl(K)$ is complete.

Theorem 1.1.6 [42, **Proposition 1.1, p.469**]

If $f : \Omega \to X$ is μ-measurable, then f is locally μ-almost compact valued, that is, given $E \in \Sigma$ with $\mu(E) < \infty$ and given $\varepsilon > 0$, there exists a set $F_\varepsilon \in \Sigma$, $F_\varepsilon \subset E$ such that $\mu(E \backslash F_\varepsilon) < \varepsilon$ and $f(F_\varepsilon)$ is (norm) relatively compact subset of X.

Proof

Since f is μ-measurable, there exists a sequence $(f_n)_{n\geq 1}$ in $\Psi_X(\Sigma)$ converging to f μ-a.e. on X. By Egoroff's Theorem, f_n converges almost uniformly to f on E, that is, for every $\varepsilon > 0$, there exists a set $E_\varepsilon \in \Sigma$, $E_\varepsilon \subset E$, with $\mu(E_\varepsilon) < \varepsilon$ such that $f_n \to f$ uniformly on $E \backslash E_\varepsilon$. Let $F_\varepsilon = E \backslash E_\varepsilon$. Then $\mu(E \backslash F_\varepsilon) = \mu(E_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on F_ε. We show that $f(F_\varepsilon) = \{f(x) : x \in F_\varepsilon\}$ is totally bounded:

Let $r > 0$ be arbitrarily chosen. Let $n_0 \in \mathbb{N}$ such that $\|f(x) - f_{n_0}(x)\| < r$ for every $x \in F_\varepsilon$. Let $R(f_{n_0}) = \{b_1, b_2, ..., b_k\}$. Then $R(f_{n_0}) \subset X$. Consider the class of open balls $\{B_r(b_i) : i = 1, ..., k\}$. Now,

$$
\begin{aligned}
x \in F_\varepsilon &\Rightarrow \|f(x) - f_{n_0}(x)\| < r \\
&\Rightarrow f(x) \in B_r(f_{n_0}(x)) \\
&\Rightarrow f(x) \in B_r(b_i), \text{ where } b_i = f_{n_0}(x) \\
&\Rightarrow f(F_\varepsilon) \subset \cup_{i=1}^{k} B_r(b_i) \\
&\Rightarrow f(F_\varepsilon) \text{ is totally bounded.}
\end{aligned}
$$

Then $cl(f(F_\varepsilon))$ is compact by Theorem 1.1.5, that is, $f(F_\varepsilon)$ is a norm relatively compact subset of X. □

Proposition 1.1.7 [42, **Proposition 1.3, p.469**]

If $f, g : \Omega \to X$ are μ-measurable, and if $E, F \in \Sigma$, then

1. $er_E(f)$ is a closed subset of X
2. $er_E(f)$ is contained in the closure of the range of f restricted to E
3. If $f = g$ μ-a.e. on E, then $er_E(f) = er_E(g)$, provided (Ω, Σ, μ) is complete
4. If $\mu(E) = 0$, then $er_E(f) = \emptyset$

5. If $F \subset E$, then $er_F(f) \subset er_E(f)$.

Corollary 1.1.8 [42, Corollary 1.4, p.469]

If $f : \Omega \to X$ is μ-measurable, then f is locally almost essentially compact valued, that is, given $E \in \Sigma$ with $\mu(E) < \infty$, and given $\varepsilon > 0$, there exists a set $F_\varepsilon \in \Sigma, F_\varepsilon \subset E$, such that $\mu(E \backslash F_\varepsilon) < \varepsilon$ and $er_{F_\varepsilon}(f)$ is compact.

Proof

From Theorem 1.1.6, there exists a set F_ε satisfying the hypothesis such that $f(F_\varepsilon)$ is norm relatively compact, that is, $clf(F_\varepsilon)$ is compact in X. We know that $er_{F_\varepsilon}(f)$ is closed (Proposition 1.1.7 (1)) and that $er_{F_\varepsilon}(f) \subset clf(F_\varepsilon)$ (Proposition 1.1.7 (2)). Because $er_{F_\varepsilon}(f)$ is then a closed subset of the compact set $clf(F_\varepsilon)$, the set $er_{F_\varepsilon}(f)$ is itself compact in X. $\square$

Theorem 1.1.9 [17, Theorem 6, p.416]

If X is a Banach space and A is a compact subset of X, then $clco(A)$ is compact.

Theorem 1.1.10 [42, Proposition 1.9, p.470]

If $f : \Omega \to X$ is an integrable function, and if $E \in \Sigma$ is such that $0 < \mu(E) < \infty$, then $\frac{\mu_f(E)}{\mu(E)} \in clco(er_E(f))$, so that $A_E(\mu_f) \subset clco(er_E(f))$, where
$A_E(\mu_f) = \{\frac{\mu_f(F)}{\mu(F)} : F \in \Sigma, F \subset E, 0 < \mu(F)\}$ is the avarage range of μ_f on E.

Proposition 1.1.11 [42, Proposition 1.12, p.472] Necessity:

If $f : \Omega \to X$ is a Bochner integrable function, then the measure $\mu_f : \Sigma \to X$ satisfies hypotheses 1,2,3, and 3′ of Rieffel's theorem. In fact, hypothesis 3 can be strengthened:
3a. locally m almost has compact average range, that is, given $E \in \Sigma$ and $\varepsilon > 0$ with $\mu(E) < \infty$, there exists an $F \subseteq E$ such that $\mu(E \backslash F) < \varepsilon$ and $A_F(m)$ is relatively compact.

Proof

The fact that μ_f satisfies hypotheses 1 and 2 of Theorem 1.1.4 follows from the fact that μ_f is an indefinite integral, by the definition of μ_f.

We now show that 3a *holds:*

If $E \in \Sigma$ with $\mu(E) > 0$ and if $\varepsilon > 0$ are given, there exists an $F \in \Sigma$, $F \subseteq E$ such that $\mu(E \backslash F) < \varepsilon$ and $er_F(f)$ is compact, by Corollary 1.1.8. Then $clco(er_F(f))$ is also compact, by Theorem 1.1.9. Since $A_F(\mu_f) \subseteq clco(er_F(f))$ (Theorem 1.1.10), so $A_F(\mu_f)$ is relatively compact.

We now show that $3'$ *holds, that is,* μ_f satisfies hypothesis $3'$ of Theorem 1.1.4*:*

Given $E \in \Sigma$ with $0 < \mu(E) < \infty$, choose $F_0 \subseteq E$ so that $er_{F_0}(f)$ is compact and $\mu(F_0) > 0$. If $er_{F_0}(f) = \{0\}$, the range of μ_f on F_0 is $\{0\}$ which is contained in a cone generated by any single point. If $er_{F_0}(f) \neq \{0\}$, then there exists $b \in er_{F_0}(f)$, with $b \neq 0$, and let $\delta = \frac{\|b\|}{2}$. Let $F = \{x \in F_0 : \|f(x) - b\| < \delta\}$, so that $\mu(F) > 0$ and let $K = clco(er_{F_0}(f))$ so that K is compact and convex, and does not contain 0. Then $\frac{\mu_f(G)}{\mu(G)}$ is in K for all $G \subseteq F$, $\mu(G) > 0$, and so $\mu_f(G)$ is in a $cone(K)$, for all $G \subseteq F$. $\square$

Note that 3a is a stronger version of 3, hence 3a implies 3. We shall return to this later.

In order to prove that the hypotheses of the Main Theorem 1.1.4 are sufficient, we introduce the following notations:

Let Π denote the set of all collections π, each consisting of a finite number of disjoint sets from Σ with strictly positive finite measure. Then Π is essentially a directed set, when $\pi_1 \geq \pi$ is defined to mean that every element of π is, except for a null set, the union of the elements of π_1. For each $\pi \in \Pi$, and each measurable function $f : \Omega \to X$, which is integrable on sets of finite measure, define a function f_π by

$$f_\pi = \sum_{E \in \pi} (\mu_f(E)/\mu(E))\chi_E,$$

where $\mu_f(E) = \int_E f d\mu$. Each f_π is a simple integrable function, and thus $f_\pi \in L^p(\Omega, \Sigma, \mu, X)$, $p \in [1, \infty)$.

Definition 1.1.12 [42, Definition 2.1, p.474]

Let (Ω, Σ, μ) be a σ-finite measure space, and let m be an X-valued measure on Σ. If K is a closed convex cone in X, with vertex 0, then a μ-measurable set E is called *K-pure* for $m : \Sigma \to X$ if $m(F) \in K$ for all $F \subseteq E$. If $m \ll \mu$ and if K is any closed convex subset of X, then a μ-measurable set E will be called *K-pure* for m relative to μ, denoted otherwise by *(K, μ)-pure*, if $m(F)/\mu(E) \in K$ for all $F \subseteq E$ with $0 < \mu(E) < \infty$, that is, if $A_E(m) \subseteq K$.

Decomposition Theorem 1.1.13 [42, p.475]

Let (Ω, Σ, μ) be a σ-finite measure space, and let $m : \Sigma \to X$ be a vector measure on Σ which is μ-continuous. Let $E \in \Sigma$ and suppose that $cl(A_E(m))$ is compact. Let $\{B_1, ..., B_n\}$ be a collection of open convex subsets of X which covers $cl(A_E(m))$. Then there exists a collection of measurable sets $\{E_1, ..., E_n\}$ whose union is E such that E_i is *$(cl(B_i), \mu)$-pure* for $1 \leq i \leq n$.

We are now set to prove the sufficiency part of the Main Theorem 1.4, and at the end of the proof we show that hypotheses 3, 3′ and 3a are equivalent.

Proposition 1.1.14 [42, p.477] Sufficiency:

Assume first that m satisfies hypotheses 1, 2 and 3a (see the necessity part).

We show that m is an indefinite integral with respect to μ, and our proof is divided into three parts as follows:

We prove that $\{f_\pi : \pi \in \Pi\}$ is a mean Cauchy net:

Let $\varepsilon > 0$ be given. Since $|m|$ is a finite measure, we can find $E \in \Sigma$ such that $\mu(E) < \infty$ and $|m|(\Omega \backslash E) < \frac{\varepsilon}{3}$. Since m is μ-continuous (hypothesis 1), so is $|m|$ since it is a finite measure (hypothesis 2). Then, since $|m|$ is a finite measure, there corresponds to every $\varepsilon > 0$ a $\delta > 0$ such that if $\mu(F) < \delta$ then $|m|(F) < \frac{\varepsilon}{6}$ for every set $F \in \Sigma$ since $|m| \ll \mu$. Choose $E_0 \subseteq E$ such that $\mu(E_0) < \delta$ and the average range $cl(A_{E \backslash E_0})$ is compact. Let $b_1, .., b_n$ be elements of X which are $(\frac{\varepsilon}{6}\mu(E))$*-dense* in $cl(A_{E \backslash E_0})$, that is, if B_i denotes an open ball about b_i with radius $\frac{\varepsilon}{6}\mu(E)$, then $cl(A_{E \backslash E_0}) \subseteq \cup_{i=1}^{n} B_i$. Then, by the Decomposition Theorem 1.1.13, we can find disjoint sets $E_1, ..., E_n$ whose union is $E \backslash E_0$, such that E_i is $(b_i, \frac{\varepsilon}{6}\mu(E))$-*pure* for each i. By eliminating those E_i which are null sets, by eliminating the corresponding b_i, and then adjusting the remaining E_i by null sets, we can assume that $\mu(E_i) > 0$ for each i because the remaining E_i are not null (but the b_i need no longer be $\frac{\varepsilon}{6}\mu(E)$-*dense*).

Let $\pi_0 = \{E_i : 0 \leq i \leq n\}$, or, if $\mu(E) = 0$, let $\pi_0 = \{E_i : 1 \leq i \leq n\}$, so that $\pi_0 \in \Pi$.

We show that, if $\pi \geq \pi_0$, *then* $\|f_\pi - f_{\pi_0}\|_1 < \varepsilon$:

Assume $\mu(E_0) > 0$, for it will be clear how the proof simplifies if $\mu(E_0) = 0$. If $\pi > \pi_0$, then, except for possible null sets,

$$\pi = \{F_1, ..., F_k\} \cup \{F_{ij} : 0 \leq i \leq n, \quad 1 \leq j \leq k_i\}$$

where the $F_i \cap E = \emptyset$ for $i = 1, ..., k$, and $E_i = \cup_{j=1}^{k_i} F_{ij}$ for $i = 1, ..., n$. This follows from the fact that $E_i \in \pi_0$ and $F_{ij} \in \pi$ and by assumption the elements of π_0 are unions of elements of π_0, for each $i = 1, ..., n$ and $1 \leq j \leq k_i$. The elements of π are mutually disjoint and have strictly positive measure. Then

$$\begin{aligned}
\|f_\pi - f_{\pi_0}\|_1 &= \int \| f_\pi(x) - f_{\pi_0}(x) \| \, d\mu(x) \\
&= \textstyle\sum_{i=1}^{k} \|m(F_i)\| + \sum_{j=1}^{k_0} \|m(F_{0j})/\mu(F_{0j}) - m(E_0)/\mu(E_0)\|\mu(F_{0j}) \\
&\quad + \textstyle\sum_{i=1}^{n}\{\sum_{j=1}^{k_i} \|m(F_{ij})/\mu(F_{ij}) - m(E_i)/\mu(E_i)\|\mu(F_{ij})\} \\
&\leq | \, m \, | \, (\Omega \backslash E) + \textstyle\sum_{j=0}^{k_0} \|m(F_{0j})\| + \|m(E_0)\| \\
&\quad + \textstyle\sum_{i=1}^{n}\{\sum_{j=1}^{k_i}(\|m(F_{ij})/\mu(F_{ij}) - b_i\| + \|b_i - m(E_i)/\mu(E_i)\|)\mu(F_{ij})\} \\
&\leq \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{6} + \tfrac{\varepsilon}{6} + (\tfrac{\varepsilon}{3}\mu(E))\mu(\cup_{i=1}^{n} E_i) \\
&\leq \varepsilon \, .
\end{aligned}$$

Thus the net $\{f_\pi : \pi \in \Pi\}$ is a mean Cauchy net, and so converges in mean to some element $f \in L^1(\Omega, \Sigma, \mu, X)$. In particular

$$\mu_f(E) = \int_E f d\mu = \lim_\pi \int_E f_\pi d\mu$$

for every $E \in \Sigma$.

We now show that $m(E) = \mu_f(E)$ for all $E \in \Sigma$:

Let $E \in \Sigma$. If $\mu(E) = 0$, the result follows from the μ-continuity of m, that is, since $\mu_f(E) = \int_E f d\mu$, if $\mu(E) = 0$ then $\mu_f(E) = \int_E f d\mu = 0$. If $0 < \mu(E) < \infty$, let $\pi_0 = \{E\}$. Then

$$\int_E f_\pi d\mu = m(E)$$

whenever $\pi \geq \pi_0$ and so

$$\mu_f(E) = \lim_\pi \int_E f_\pi d\mu = m(E).$$

The results when $\mu(E) = \infty$ then follows from the σ-finiteness of μ. This concludes the proof of the sufficiency part.

Lastly, we show that, the hypotheses 3, 3′, and 3a *are equivalent:*

Here our general assumption is that hypotheses 1 and 2 hold.

3a⇒3: This is obvious from the descriptions, that is, if locally m almost has compact average range, then it implies that m somewhere has compact average range.

3a⇒3′: If m satisfies hypothesis 3a, then since we have proved so far, that for some Bochner integrable f, we have $m = \mu_f$, which satisfy hypotheses 1, 2, 3′, hence m satisfies hypothesis 3′.

3′ ⇒3: Let m satisfy 3′. Given $E \in \Sigma$ with $0 < \mu(E) < \infty$, choose $F_0 \subseteq E$ and compact $K \subseteq X$ not containing 0 such that $\mu(F_0) > 0$ and the range of m on F_0 is contained in $cone(K)$. Choose a constant c large enough so that the measure $c\mu - |m|$ is not purely negative on F_0, and let F be the purely positive part of a Hahn decomposition of $c\mu - |m|$. Then $\mu(F) > 0$ and $\|m(G)/\mu(G)\| < c$ for $G \subseteq F$ and $\mu(G) > 0$. It follows that if $\frac{1}{k}$ is the distance from K to 0, then $m(G)/\mu(G)$ is in $clco(ckK \cup \{0\})$ which is compact and so 3 is satisfied.

3⇒3a First we make this observation: $A_{E \cup F} \subseteq co(A_E \cup A_F)$ for any $E, F \in \Sigma$, so that $cl(A_{E\cup F})$ is compact if $cl(A_E)$ and $cl(A_F)$ are compact. Now let m satisfy hypothesis 3, let $E \in \Sigma$ with $\mu(E) < \infty$ be given. Let $a = \sup\{\mu(F) : F \subseteq E$ and $cl(A_F)$ is compact}. It suffices to show that $a = \mu(E)$:

Let $(F_i)_{i\geq 1}$ be a sequence of subsets of E such that $\mu(F_i) \to a$ and $cl(A_{F_i})$ is compact. By the observation made above we can assume that the F_i are increasing. Let $E_0 = \cup_i F_i$ so that $\mu(E_0) = a$. If $a < \mu(E)$, then $\mu(E \backslash E_0) > 0$ so that by hypothesis 3 there exists an $F_0 \subseteq E \backslash E_0$ such that $\mu(F_0) > 0$ and $cl(A_{F_0})$ is compact. Then $cl(A_{F_i \cup F_0})$ is compact, and $\mu(F_i \cup F_0) = \mu(F_i) + \mu(F_0) \to \mu(F_0) + a > a$. But $\mu(F_i \cup F_0) \leq a$ for all $i \in \mathbb{N}$ since $F_i \cup F_0 \subseteq E$ for all $i \in \mathbb{N}$, hence $\mu(F_i \cup F_0) \to \mu(F_0) + a > a$ leads to a contradiction to a definition of a supremum. □

It is now clear that the space $L_1(\Omega, \Sigma, \mu, \mathbb{R})$ of Example 1.1.3 has no Radon-Nikodým derivative because 3 (and also 3′) in the statement of Theorem 1.1.4 does not hold.

Henceforth, (Ω, Σ, μ) denotes a finite positive measure space.

Rieffel [41] improved on his Radon-Nikodým Theorem, namely Theorem 1.1.4 above, by introducing the concept of dentability and attempted to characterise those subspaces of a Banach space that are dentable.

1.2 The RNP and Dentability

In this section we introduce the terms **dentable**, **c-dentable** and **s-dentable**. These concepts provide us with the first concrete evidence that the Radon-Nikodým Property (RNP), which stems from the Radon-Nikodým Theorem, is a geometric property of Banach spaces. Rieffel [41, p.71], in an attempt to give a new proof of Phillips' Radon-Nikodým Theorem [38, p.130], introduced the class of dentable subsets of a Banach space. In 1973, Maynard introduced the strictly larger class of s-dentable sets. There are different forms of dentability we will discuss and compare, that give different characterisations of Banach spaces with the RNP. For brevity, we write RNP instead of the phrase Radon-Nikodým Property, and the RN-theorem will be used, to mean the Radon-Nikodým Theorem.

Notations

$B_\varepsilon(x)$ will denote the closed ball with radius $\varepsilon > 0$ centered at $x \in X$, X a Banach space.
$S_\varepsilon(x)$ will denote the open ball with radius $\varepsilon > 0$ centered at $x \in X$.

Definition 1.2.1 [13, Definition 3, p.61]

A Banach space X has the *RNP with respect to* (Ω, Σ, μ) if for each μ-continuous vector measure $m : \Sigma \to X$ of bounded variation, there exists a function $f \in L^1(\Omega, \Sigma, \mu, X)$, such that $m(E) = \int_E f d\mu$ for all $E \in \Sigma$.
A Banach space X has the RNP if it has the RNP with respect to every finite measure space.

In this section, we deal exclusively with the RNP in Banach spaces, and observe different characterisations of this property.

Definition 1.2.2 [22] and [41, p.71]

A set A in a Banach space X is *dentable* if for any $\varepsilon > 0$ there exists $x \in A$ such that $x \notin clco(A \backslash B_\varepsilon(x))$. A point $x \in A$ is called a *denting point* of A if $x \notin clco(A \backslash B_\varepsilon(x)$ for any $\varepsilon > 0$.

It should be noted that the ball that is being used here is closed, even though some authors, such as Bourgin [7, p.18] and Diestel and Uhl [14, p.13], use the open ball.

Remark 1.2.3

Rieffel [41, p.75] remarked that all the geometric difficulties involved in obtaining the Radon-Nikodým theorem for the Bochner integral (of a function) with values in some Banach space are contained in the problem of determining which subsets of the Banach space are dentable. Rieffel [41] proved that any relatively norm compact convex subset K of a Banach space is dentable. He went about doing that using the following reasoning:

1. He showed that any extreme point of a relatively norm compact convex subset K is a denting point of K.

2. For the general case, Rieffel [41] showed that if K is any subset of a Banach space, and if $clco(K)$ is dentable, then so is K itself.

So, the extreme points of sets have entered into the theories of dentability and the RNP – dentability assumptions are in a sense extremal in character.
Rieffel [41, p.75] then turned to relatively weakly compact subsets of a Banach space and posed the question:

Question 1.2.4
Are relatively weakly compact subsets of a Banach space dentable?

He could not answer this question completely, but stated that an extreme point is not necessarily a denting point in a closed bounded and convex subset of a Banach space X.

Definition 1.2.5 [13, Theorem 10, p.138]
Let D be a bounded subset of a Banach space X.

1. A point $x \in D$ is called an *extreme point* of D if $x = \lambda y + (1-\lambda)z$, for some $\lambda \in [0,1]$ and for some $y, z \in D$, then either $y = x$ or $z = x$.

2. A point $x \in D$ is called an *exposed point* of D if there is a functional $f^* \in X^*$ such that $f^*(x) > f^*(y)$ for all $y \in D \backslash \{x\}$.

3. A point $x \in D$ is called a *strongly exposed point* of D if there is a functional $f^* \in X^*$ such that $f^*(x) > f^*(y)$ for all $y \in D \backslash \{x\}$, and such that $f^*(x_n) \to f^*(x)$ for $(x_n)_{n \geq 1}$ in D implies that $x_n \to x$.

As Rieffel observed, if x is a strongly exposed point, then x is a denting point. It follows from the Definitions 1.2.2 and 1.2.5 (1) that a denting point of a set A is an extreme point of A. Consequently, a strongly exposed point of a set is an extreme point of that set. The converse, however, is not true, as the following example shows.

In $\mathbb{R}^2$, let $K = co(\{(x;y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(1;-1)\})$. The point $(1;0)$ is an extreme point of K, but $(1;0)$ is not an exposed point of K, hence also not a strongly exposed point of K.

Hence a set of strongly exposed points a closed bounded convex subset of a Banach space is contained in the set of denting point of such a set, which in turn is properly contained in the set of extreme point of such a set.

Rieffel [41] posed three more questions:

Questions 1.2.6

1. Which are the dentable subsets of $C(T)$, the Banach space of all continuous functions on some compact Hausdorff space T?
2. Which Banach spaces have the property that all non-empty bounded subsets are dentable?
3. Does there exist a closed set which has no strongly exposed points?

In this thesis, some of these questions will be answered by means of known proofs from the literature. The answer to the second question is the core of this thesis.

Question 1.2.4 had been answered in the affirmative by Troyanski [51] and partially by Diestel and Uhl [14, p.14]. Furthermore, Edelstein [18] gave an example of the unit ball in the conjugate space m (also denoted by l_∞) which is weak*-compact, but not dentable.

It follows from the Definition 1.2.2 that a subset D of a Banach space X is *non-dentable* if there exists an $\varepsilon > 0$ such that for each $x \in D$, $x \in clco(D \backslash B_\varepsilon(x))$. We now present an example of a non-dentable set.

Example 1.2.7 [13, Example 5, p.135]
The closed unit ball D of $L_\infty[0,1]$ is non-dentable:

Let $f \in D$ and let $\varepsilon > 0$.

Case 1:

If $\|f\|_\infty > \varepsilon$, then for any $m \in \mathbb{N}$, there are disjoint Lebesgue measurable sets $E_1, E_2, ..., E_m$ such that $\|f\chi_{E_n}\|_\infty > \varepsilon$ for each $n = 1, ..., m$. Setting $f_n = f - f\chi_{E_n}$, one sees that $\|f - f_n\|_\infty = \|f\chi_{E_n}\|_\infty > \varepsilon$, $n = 1, ..., m$.

Furthermore, $\|f - \sum_{n=1}^{m} \frac{1}{m} f_n\|_\infty \leq \frac{1}{m}\|f\|_\infty$.

Since $\frac{1}{m}$ can be made as small as we please, we see that, for $0 < \varepsilon < 1$, $f \in clco(D\backslash B_\varepsilon(f))$ provided $\|f\|_\infty > \varepsilon$.

Case 2:

If $\|f\|_\infty \leq \varepsilon < \frac{1}{3}$, then $\|f + 2\varepsilon\chi_{[0,1]} - f\|_\infty = 2\varepsilon$ and $\|f - 2\varepsilon\chi_{[0,1]} - f\|_\infty = 2\varepsilon$. Setting $f_1 = f + 2\varepsilon\chi_{[0,1]}$ and $f_2 = f - 2\varepsilon\chi_{[0,1]}$, implies that $\|f_i\|_\infty \leq \varepsilon < 1$ and $\|f - f_i\|_\infty = 2\varepsilon$, $i = 1, 2$. Therefore $f_1, f_2 \in D\backslash B_\varepsilon(f)$ for $i = 1, 2$ but $f = \frac{1}{2}(f_1 + f_2)$. Thus, $f \in D$ implies $f \in clco(D\backslash B_\varepsilon(f))$ for any $f \in D$. Therefore D is non-dentable.

Definition 1.2.8 [10]

A bounded set A is *s-dentable* if for each $\varepsilon > 0$, there exists $x_\varepsilon \in A$ such that $x_\varepsilon \notin s(A\backslash B_\varepsilon(x_\varepsilon))$, where $s(B) = \{\sum_{i=1}^{\infty} \lambda_i x_i : \lambda_i \geq 0, \sum \lambda_i = 1, \{x_i\} \subset B\}$.

The concept **s-dentable** is sometimes called **σ-dentable**.

C. Stegall [49] calls $s(B)$ the sequential hull of B.

If x is a denting point of a set A, then from Definition 1.2.2, x is not the limit of any sequence in $co(A\backslash B_\varepsilon(x))$. But the limit of a sequence in $co(A\backslash B_\varepsilon(x))$ has the form $\sum_{i=1}^{\infty} \lambda_i x_i$, where $\lambda_i \geq 0$, $\sum \lambda_i = 1, \{x_i\} \subset A\backslash B_\varepsilon(x)$. It then follows from Definition 1.2.8 that x is an s-denting point of a set A. Consequently, **dentable sets are s-dentable**. However, Maynard [32] gave an example of a bounded set which is s-dentable, but not dentable.

Example 1.2.9 [7, Example 2.1.6, p.18]

Consider the Banach space $C[0, 1]$ of continuous functions on $[0, 1]$, with $\|f\| = \max\{|f(t)| :$

$t \in [0,1], \}$ for $f \in C[0,1]$.

The closed unit ball K of $C[0,1]$ is s-dentable, non-dentable and fails the KMP:

K has exactly two extreme points, namely, the functions $f_1, f_2 \in C[0,1]$ defined by $f_1(x) = 1, f_2(x) = -1$ for any x in $[0,1]$.

K is non-dentable:

Suppose $f \in K$ and for any $n \in \mathbb{Z}^+$ choose functions $f_1^n, ..., f_n^n$ in K so that $f_i^n(t) = f(t)$ for $t \notin [\frac{i-1}{n}, \frac{i}{n}]$ and $|f_i^n(t_i^n) - f(t_i^n)| > \frac{1}{2}$ for some $t_i^n \in (\frac{i-1}{n}, \frac{i}{n})$. Then $\|f_i^n - f\| > \frac{1}{2}$ for $i = 1, ..., n$ and yet $\| \sum_{i=1}^n \frac{1}{n} f_i^n - f \| \leq \frac{2}{n}$. It follows that $f \in clco(K \backslash B_{\frac{1}{2}}(f))$ since n was arbitrary. Hence K is not dentable (and thus $C[0,1]$ fails the RNP).

K is s-dentable:

This follows by taking either extreme points (f_1 or f_2) to be f_ε in Definition 1.2.8.

Definition 1.2.10 [32, p.497]

A Banach space X is said to be an *s-dentable* space if and only if every bounded set $K \subset X$ is s-dentable.

Maynard in [32] extended Rieffel's result and established the following theorem.

Theorem 1.2.11 [32, Theorem 3.1, p.497]

A Banach space X has the RNP if and only if X is an s-dentable space.

Maynard left open, however, the question as to whether in a space X with the RNP, every bounded non-empty set is dentable. Davis and Phelps [10] answered this question; we shall refer to this result (Theorem 1.2.16) as the Davis-Phelps-Rieffel Theorem.

In order to establish the Davis-Phelps-Rieffel Theorem, we must first prove three propositions.

Proposition 1.2.12 [10, Lemma 1, p.119]

A subset A of a Banach space X is non-dentable if and only if there exists an $\varepsilon > 0$ such that $A \subset clco(A \backslash B_\varepsilon(x))$ for each $x \in A$. If A is closed and convex, this is equivalent to $A = clco(A \backslash B_\varepsilon(x))$ for each $x \in A$.

Proof

If there exists a number $\varepsilon > 0$ such that $A \subset clco(A \backslash B_\varepsilon(x))$ for each $x \in A$, then A is non-dentable. For the converse, suppose that A is non-dentable. Then there exists $2\varepsilon > 0$ such that for each $y \in A$, $y \in clco(A \backslash B_{2\varepsilon}(y))$. Let $x, y \in A$ with $\|x - y\| > \varepsilon$. Then $y \in A \backslash B_\varepsilon(x) \subset clco(A \backslash B_\varepsilon(x))$. If $\|x - y\| \leq \varepsilon$, then $B_\varepsilon(x) \subset B_{2\varepsilon}(y)$, so that $A \backslash B_{2\varepsilon}(y) \subset A \backslash B_\varepsilon(x)$, and hence $clco(A \backslash B_{2\varepsilon}(y)) \subset clco(A \backslash B_\varepsilon(x))$. This completes the proof. □

Proposition 1.2.13 [10, Lemma 2, p.120]

Let C be a closed convex set in X with non-empty interior, denoted by $intC$, and suppose that C in non-dentable. Then there exists $\varepsilon > 0$ such that for each $x \in C$, $intC \subset co(int(C \backslash B_\varepsilon(x)))$. In particular, $intC$ is not s-dentable.

Proof

By Proposition 1.2.12, there exists a number $\varepsilon > 0$ such that $C = clco(C \backslash B_\varepsilon(x))$ for each $x \in C$. Put $A_x = C \backslash B_\varepsilon(x)$. Then $intA_x = int(C \backslash B_\varepsilon(x))$. For ε sufficiently small, it follows that $intA_x \neq \emptyset$ for each $x \in C$. Fix x and let $A = A_x$. Then $C = clco(A)$. We want to show that $int(clco(A)) \subset co(intA)$. We first show that $A \subset cl(intA)$:

If $y \in A$, then $y \in C \backslash B_\varepsilon(x)$ (for x fixed), so that $y \in C$ and $\|x - y\| > \varepsilon$. Let $z \in intC$. Then $[z, x) \subset intC$, and $[z, x) \cap B_\varepsilon(x) \neq \emptyset$. Let $u \in [z, x) \cap B_\varepsilon(x)$. Therefore $[u, y) \subset intC$. Then for some $v \in [u, y)$ we have that $[v, y) \subset int(C \backslash B_\varepsilon(x))$. Thus $y \in cl(intA))$. This now shows that $A \subset cl(intA)$. Then $A \subset cl(intA) \subset clco(intA)$, and so , $co(A) \subset clco(intA)$. Using the fact that the interior of a convex set coincides with the interior of its closure, we now have that,

$$\begin{aligned} intC &= int(clco(A)) \\ &= int(co(A)) \\ &\subset int(clco(intA)) \end{aligned}$$

$= co(intA)$

$= co(int(C\backslash B_\varepsilon(x)))$.

This completes the proof of the Proposition. $\square$

Propositon 1.2.14 [10, Proposition, p.120]

If a Banach space X contains a bounded non-empty non-dentable set, then it contains a bounded closed convex and symmetric set C which is non-dentable and which has non-empty interior. In particular, X can be renormed so that the new unit ball is non-dentable and the interior of the new unit ball is not s-dentable.

Proof

If A is a bounded non-empty non-dentable non-empty subset of X, then the same is true of the sets $A_1 = A \cup (-A)$ (by definition of non-dentability), $A_2 = clcoA_1$, (see [41, Proposition 2]) and $A_3 = B_1(0) + A_2$. Let $C = cl(A_3)$. Again, by [41, Proposition 2], C is non-dentable. By Proposition 1.2.13, $intC$ is not s-dentable. $\square$

What we have shown above is that every bounded subset of a Banach space X is dentable if and only if every bounded subset of X is s-dentable. This now yields the following result which is closely connected to Theorem 1.2.11.

Definition 1.2.15

A Banach space X is said to be a *dentable* space if and only if every bounded set $K \subset X$ is dentable.

The Davis-Phelps-Rieffel Theorem 1.2.16 [10, Corollary, p.121]

A Banach space X has the RNP if and only if X is a dentable space.

Hence, a Banach space X lacks the RNP if and only if there exists a bounded non-empty non-dentable set in X, see [23, p.160], [13, p.136]. This brings us closer to answering Question 1.2.6 (2).

Definition 1.2.17 [7, Definition 2.3.1, p.27]

Let D be a bounded subset of a Banach space X and let $f^* \in X^*$, $f^* \neq 0$. Let $M(D, f^*) = \sup\{f^*(x) : x \in D\}$. If $\alpha > 0$, then the set $S(D, f^*, \alpha) = \{x \in D : f^*(x) > M(D, f^*) - \alpha\}$ is called the *slice of D determined by f^* and α.*

Theorem 1.2.18 [7, Proposition 2.3.2, p.28]

A bounded subset D of a Banach space X is dentable if and only if D has slices of arbitrary small diameter (if and only if X has the RNP).

Proof

If D is dentable and $\varepsilon > 0$, choose $x \in D$ such that $x \notin clco(D \backslash B_\varepsilon(x))$. The Hahn-Banach theorem guarantees that there is an $f \in X^*$, $\|f\| = 1$, such that $f(x) > r > M(clco(D \backslash B_\varepsilon(x)), f)$, for some $r \in \mathbb{R}$, in the notation of Definition 1.2.17. Let $\alpha = M(D, f) - r$. Then $S(D, f, \alpha) \subset B_\varepsilon(x) \cap D$ and its diameter is at most 2ε.

Conversely, if $\varepsilon > 0$ is prescribed, and let $S(D, f, \alpha)$ be a slice of diameter less than ε. If $x \in S(D, f, \alpha)$, then $clco(D \backslash B_\varepsilon(x)) \subset clco(D \backslash S(D, f, \alpha)) \subset f^{-1}((-\infty, r])$ where $r = M(D, f) - \alpha$. Since $f(x) > r$, it follows that $x \notin clco(D \backslash B_\varepsilon(x))$. That completes the proof. $\square$

Theorem 1.2.19 [36, Lemma 4, p.81]

Suppose that every bounded subset of X is dentable (that is, suppose X has the RNP), and that $g \in X^*, \|g\| = 1$. If $\varepsilon > 0$ and if C denotes a non-empty bounded closed and convex subset of X with $C \backslash g^{-1}(0) \neq \emptyset$, then there exists a slice of C of diameter less than ε which misses the set $D = C \cap g^{-1}(0)$.

Thereom 1.2.20 [36, Theorem 5, p.82]

Suppose that every bounded subset of X is dentable (that is, X has the RNP), and that C is a bounded, closed and convex subset of X. Then C is the closed convex hull of its denting points.

Proof

By the separation theorem, it suffices to show that each slice $S(g, \beta, C)$ of C contains a denting point of C. By translation, we can assume that the origin is contained in the hyperplane

$$\{x \in E : g(x) = M(g, C) - \beta\},$$

that is, that this is the same as $g^{-1}(0)$. Let $C_1 = S(g, \beta, C)$ and apply Theorem 1.2.19 to get a slice of C_1 which misses $C \cap g^{-1}(0)$ and has diameter less than $\frac{1}{2}$. This slice is necessarily a slice of C and is contained in C_1. We can continue by induction to get a nested sequence of slices of C whose diameters converge to 0; their intersection is necessarily a denting point of C inside C_1. □

Phelps [36] managed to prove Theorem 1.2.20 with **denting points** being replaced by **strongly exposed points**.

Theorem 1.2.21 [36, Theorem 9, p.85], [13, Theorem 3, p.202]

Let X be a Banach space. Then every bounded subset of X is dentable (that is, X has the RNP) if and only if every bounded closed convex subset of X is the closed convex hull of its strongly exposed points.

Denote by $D(A)$, $SE(A)$ and $E(A)$ the sets of all denting, strongly exposed and extreme points of A, respectively.

Using the paragraph just below Definition 1.2.5 we can now formulate:

Proposition 1.2.22

If A is a non-empty bounded closed convex subset of a Banach space X, then $SE(A) \subset D(A) \subset E(A)$.

A proof of this Proposition will be given after Theorem 1.2.24.

Definition 1.2.23 [30, p.526]

Let A be a bounded closed and convex subset of a Banach space X. Then

1. x is a *point of continuity (PC)* for A if the identity mapping $id : (A, weak) \rightarrow (A, norm)$ is continuous at such x.

2. x is a *strongly extreme point* of A if for any sequences $(y_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ in A, $\lim_{n\to\infty} \|\frac{1}{2}(y_n + z_n) - x\| = 0$ implies $\lim_{n\to\infty} \|y_n - x\| = 0$.

3. x is a *weak*-extreme point* of A if x is an extreme point of $\bar{A}$, where $\bar{A}$ is a weak*-closure of A in X^{**}.

4. $x \in A$ is a *very strong extreme point* of A if for every sequence $(f_n)_{n\geq 1}$ of A-valued Bochner integrable functions on [0,1], the condition $\lim_{n\to\infty} \| \int_0^1 f_n(t)dt - x\| = 0$ implies $\lim_{n\to\infty} \int_0^1 \|f_n(t) - x\|dt = 0$.

Theorem 1.2.24 [30, p.526]

Let x be an element in a bounded closed convex set A of a Banach space. Then the following are equivalent:

1. x is a denting point of A.

2. x is a very strong extreme point of A.

3. x is a PC for A, and x is an extreme point of A, (respectively, strong extreme point, weak*-extreme point of A).

It is well-known that the closed unit ball B_c in the space c of all convergent sequences has extreme points, but no denting points, see [23, p.163]. Hence, if $x \in E(B_c)$, then such an x is not a PC of B_c.

Proof (of Proposition 1.2.22)

Firstly, $D(A) \subset E(A)$ follows from Theorem 1.2.24 above.

Secondly, we show that a strongly exposed point is a denting point:

Let x be any strongly exposed point of A, that is, $x \in SE(A)$. Then there exists a slice $S(A, f, \alpha) = \{x \in A : f(x) > M(A, f) - \alpha\}$, with $f \in X^*$, $A \subset X$, X Banach and A bounded, and $M(A, f) = \sup\{f(x) : x \in A\}$ [7, p.27]. Hence A is dentable and $x \in S(A, f, \alpha)$ with $S(A, f, \alpha) \cap clco(A \backslash B(x, \varepsilon) = \emptyset$ for some $\varepsilon > 0$ [7, p.28]. Hence $x \notin clco(A \backslash B(x, \varepsilon))$ and then x is a denting point of A. This thus completes the proof of Proposition 1.2.22. $\square$

Our important observation so far is the relation between the denting and extreme points in a bounded closed convex and dentable subset of a Banach space. Diestel and Uhl [14, p.14] state that in compact convex subset of a Banach space, denting points are all extreme points and extreme points are all denting points. That is, for B compact and convex in a Banach space X, $D(B) = E(B)$. This result was in fact proved by Rieffel [41, p.72, proposition 1]. We are also interested in finding out in which Banach space(s) X the equality $D(A) = E(A)$ holds for a bounded closed convex subset A of X.

Theorem 1.2.25 [36, Theorem 2, p.80]

If every bounded subset of the Banach space X is dentable, and if C is a bounded closed convex subset of X, then C is the closed convex hull of its extreme points.

This theorem can be rephrased as follows:

If a Banach space X has the RNP, then any closed bounded convex subset C of X equals $clco(E(C))$.

We give a proof using results discussed by Diestel and Uhl [13, pp 217,218].

Proof (of Theorem 1.2.25)

X has the RNP by Theorem 1.2.16. Let C be any bounded closed convex subset of X. Then C is dentable, and it follows that C has at least one denting point and thus an extreme point, see Theorem 1.2.24. Since C is a bounded closed and convex set with an extreme point, C is the closed convex hull of its extreme points by a result of Lindenstrauss, see Diestel and Uhl , see section 1.8 and also [13, Conditions 20 and 20a, p.218]. Furthermore, using the fact that such a closed bounded and convex set C in the hypothesis is the closed convex hull of its strongly exposed points (from Theorem 1.2.21 above), that is, $C = clco(SE(C))$ and the fact that a strongly exposed point is an extreme point, we have the following:

Since $SE(C) \subset E(C)$, then $clco(SE(C)) \subset clco(E(C))$, and since $C = clco(SE(C))$ then $C \subset clco(E(C))$.

We now show that $clco(E(C)) \subset C$:

Since C is dentable, it has a denting point and hence an extreme point and thus $E(C) \neq \emptyset$. But $E(C) \subset C$ and thus $clcoE(C)) \subset clco(C) = C$ since C is closed and convex. Hence $C = clco(E(C))$ and the proof is completed. □

Remark 1.2.26

It is important to note that the relationship between the denting points and the extreme points in very important in this discussion and we shall attempt and explore it extensively.

Theorem 1.2.27 [11, Theorem 2, p.171]

If X is a Banach space, then every weakly compact, convex subset A of X equals the closed convex hull of its strongly exposed points.

Remark 1.2.28

Huff and Morris [22] summarise some of the results we have observed thus far in the following manner:

The RNP in a Banach space X has been shown to be equivalent to:

1. Every non-empty closed, bounded and convex subset of X is the closed convex hull of its strongly exposed points (R.R. Phelps [36]).

2. Every bounded subset of X is dentable [22].

3. Every non-empty closed bounded convex subset of X is the closed convex hull of its extreme points (Lindenstrauss, see [36, p.80]).

Definition 1.2.29 [23, p.157]

A Banach space X has the *Strong Krein Milman Property* (SKMP) if every closed bounded (not necessarily convex) subset of X has an extreme point.

Alternatively, a Banach space X has the SKMP if every bounded closed subset B of X contains an extreme point of its closed convex hull (that is, $x \in B$ for some $x \in E(clco(B))$) [14, p.34].

Theorem 1.2.30 [23], [14, p.34]

Let X be a Banach space. X has the RNP if and only if X has the SKMP.

Hence this result implies that:

Every non-empty closed bounded subset of a Banach space X is dentable if and only if every non-empty closed bounded (not necessarily convex) subset of X has an extreme point.

Alternatively,

Every non-empty closed bounded subset of a Banach space X is dentable if and only if every non-empty closed bounded subset of X contains an extreme point of its closed convex hull.

Proof (of Theorem 1.2.30)

Let X have the RNP and C be any non-empty bounded closed subset of X. Then C is dentable and contains at least one denting point. Then C has an extreme point, see Theorem 1.1.24, hence X has the SKMP.

Conversely, let X have the SKMP and B be any closed convex bounded subset of X. Then B has an extreme point. Hence X has the RNP, see section 1.8 and also [13, condition 20, p.218]. $\square$

We now discuss the concept of c-dentability, introduced and discussed by Bourgin [7].

Definition 1.2.31 [7, Definition 2.1.5, p.18]

Let D be a bounded subset of X. Then D is *c-dentable* if for each $\varepsilon > 0$, there exists a point $x_\varepsilon \in D$ such that $x_\varepsilon \notin co(D \backslash B_\varepsilon(x_\varepsilon))$.

Proposition 1.2.32

Dentability implies c-dentability for any bounded subset of a Banach space X.

Proof

If a bounded subset D of X is dentable, then for each $\varepsilon > 0$ there exists $x_\varepsilon \in D$ such that $x_\varepsilon \notin clco(D \backslash B_\varepsilon(x_\varepsilon))$. It follows that $x_\varepsilon \notin co(D \backslash B_\varepsilon(x_\varepsilon))$. $\square$

Question 1.2.33

What is the relation (if any) between s-dentability and c-dentability?

We show that c-denting points are s-denting points:

Let $x \in D$ such that $x \notin co(D \backslash B_\varepsilon(x))$ for each $\varepsilon > 0$. Then $x \neq \sum_{i=1}^{n} x_i \lambda_i$, $\lambda_i \geq 0$, with $\sum_{i=1}^{n} \lambda_i = 1$ and $\{x_i\} \in D \backslash B_\varepsilon(x)$ for every $n \in \mathbb{N}$. Hence $x \neq \sum_{i=1}^{\infty} x_i \lambda_i$, $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$, $\{x_i\} \in D \backslash B_\varepsilon(x)$ and thus $x \notin s(D \backslash B_\varepsilon(x))$.

Thus we conclude that:

$D(C) \subset c\text{-}D(C) \subset s\text{-}D(C)$, where $D(C), c\text{-}D(C)$ and $s\text{-}D(C)$ denote the denting points, c-denting points and s-denting points of a bounded set C, respectively.

Remark 1.2.34

It is possible to carry the notion of RNP over from a Banach space X to a non-empty closed bounded convex subset of X. Subsequently, we discuss conditions under which a subset of a Banach space can have the RNP.

Definition 1.2.35 [7, Definition 2.1.1, p.15]

It is said that a (closed bounded and convex) set K in a Banach space X has the *RNP for* (Ω, Σ, μ) if for each measure $m : \Sigma \to X$ for which $m \ll \mu$ holds and for which its average range, $A(m) = \{m(A)/\mu(A) : A \in \Sigma, \mu(A) > 0\}$ is contained in K, there exists an $f \in L^1(\Omega, \Sigma, \mu, X)$ such that $m(A) = \int_A f d\mu$ for each $A \in \Sigma$. The set K is said to have the RNP if K has the RNP for each finite positive measure space. Moreover, suppose that C is a closed convex, possibly unbounded, subset of X. Then C has the RNP if each of its bounded convex subsets has the RNP.

Furthermore, a Banach space X has the RNP if every closed bounded and convex subset of X has the RNP, see [50, p.508].

Definition 1.2.36 [7, Definition 1.4.2, p.11]

Suppose that $(\Sigma_n : n \in \mathbb{N})$ is a sequence of sub-σ-algebras of Σ and that $\Sigma_n \subset \Sigma_m$ whenever $n \geq m$. Suppose that $f_n \in L^1(\Omega, \Sigma, \mu, X)$ for each $n \in \mathbb{N}$. If the sequence $(f_n)_{n\geq 1}$ satisfies the conditions

1. f_n is Σ_n-*measurable* for each $n \in \mathbb{N}$, and

2. $\int_A f_n d\mu = \int_A f_m d\mu$ whenever $n < m$ and $A \in \Sigma_n$,

then the sequence $(f_n, \Sigma_n)_{n\geq 1}$ is said to be an X-*valued martingale.*

If each Σ_n is generated by countably many atoms (since $\mu(\Sigma) < \infty$, μ can have at most countably many atoms) and if Σ is generated by $\cup_{n=1}^{\infty}\Sigma_n$, then $(f_n, \Sigma_n)_{n\geq 1}$ is called an elementary *martingale.* A closed bounded convex set $K \subset X$ has the *Martingale Convergence Property (MCP)* for (Ω, Σ, μ) if whenever $(f_n, \Sigma_n)_{n\geq 1}$ is a martingale for which $\cup_{n=1}^{\infty}\Sigma_n$ generates Σ and $f_n \in L^1(\Omega, \Sigma, \mu, X)$ for which $f_n(\omega) \in K$ μ-a.e on Ω for each $n \in \mathbb{N}$, then there exists an $f \in L^1(\Omega, \Sigma, \mu, X)$ for which $f(\omega) \in K$ μ-a.e on X such that $\lim_{n\to\infty} \|f_n(\omega) - f(\omega)\| = 0$ μ-a.e on X. The set K has the MCP if it has the MCP for each finite positive measure space (Ω, Σ, μ). The set K has the elementary MCP if the above conditions are required to hold only for elementary martingale $(f_n, \Sigma_n)_{n\geq 1}$ which are K-valued and such that $\cup_{n=1}^{\infty}\Sigma_n$ generates Σ.

Remark 1.2.37

In [7, p.12], it is shown that the unit ball of the space c_0, and consequently c_0 itself, lacks the MCP.

The following results are vital for the subsequent development:

Theorem 1.2.38 [7, Theorem 2.2.1-2.2.3, pp.19,20,21]

Suppose that K is a non-empty closed bounded convex subset of X. Then,

1. If K has the RNP for (Ω, Σ, μ), then K has the MCP for (Ω, Σ, μ).

2. If K has the elementary MCP for the measure space ([0,1), Borel σ-algebra, Lebesgue measure), then K is subset s-dentable, that is, each of its bounded non-empty subset of K is s-dentable.

3. If K is s-dentable, then K has the RNP.

Proof (of 1)

Assume that K has the RNP for (Ω, Σ, μ), that $(\Sigma_n)_{n\geq 1}$ is an increasing sequence of σ-algebras such that Σ is generated by $\bigcup_{n=1}^{\infty}\Sigma_n$, and that $(f_n, \Sigma_n)_{n\geq 1}$ is a martingale taking its values in K. Let $m_n(A) = \int_A f_n d\mu$ for each $A \in \Sigma$. Then m_n is an X-valued measure on Σ, and $m_n \ll \mu$. Also, $A_\Omega(m_n) \subset K$. Indeed, if $A \in \Sigma$, $\mu(A) > 0$, then we have for each $F \in X^*$ that

$$F(\tfrac{m_A(A)}{\mu(A)}) = \tfrac{1}{\mu(A)} \int_A F \circ f_n d\mu$$
$$\leq \sup\{F(x) : x \in K\}.$$

Since K is a weakly closed convex set, $\frac{m_n(A)}{\mu(A)} \in K$. Since $(f_n, \Sigma_n)_{n\geq 1}$ is a martingale, it follows that $\lim_n m_n(A)$ exits for each $A \in \bigcup_{n=1}^{\infty}\Sigma_n$. We now show that this limit exists for each $A \in \Sigma$. Let $M = \sup\{\|x\| : x \in K\}$. Given $\varepsilon > 0$ and $A \in \Sigma$, find $B \in \Sigma_N$ for some N such that $\mu(A\triangle B) < \frac{\varepsilon}{2M}$. If $n \geq N$, then

$$\| \int_A f_n d\mu - \int_A f_N d\mu\| \leq \| \int_B f_n d\mu - \int_B f_N d\mu\|$$

$$
\begin{aligned}
&+ \int_{A\triangle B} \|f_n\| d\mu + \int_{A\triangle B} \|f_N\| d\mu \\
&\leq 2\mu(A\triangle B)M \\
&< \varepsilon.
\end{aligned}
$$

Consequently, $(m_n(A))_{n\geq 1}$ is a Cauchy sequence and $m(A) = \lim_n m_n(A)$ exists for all $A \in \Sigma$. The Hahn-Vitali-Saks Theorem implies that m is a measure. Then $m \ll \mu$ and $A_\Omega(m) \subset cl(\bigcup_{n=1}^\infty A_\Omega(m)) \subset K$. Since K has the RNP for (Ω, Σ, μ) there is an $f \in L_1(\Omega, \Sigma, \mu, X)$ such that $\int_A f d\mu = m(A)$ for each $A \in \Sigma$. For $A \in \Sigma_n$ we have $\int_A f d\mu = m(A) = m_n(A) = \int_A f_n d\mu$. Then the conditional expectation of f gives $\Sigma_n = f_n$ μ-a.e. Then $\lim_n \|f_n(w) - f(w)\| = 0$ μ-a.e. That is, K has the MCP for (Ω, Σ, μ). □

The theorem below summarises the RNP in a subset of a Banach space.

Theorem 1.2.39 [7, **Theorem 2.3.6, p.31**]

Suppose that K is a non-empty closed bounded convex subset of a Banach space X. Then the following statements are equivalent:

1. K has the RNP.
2. Each closed bounded convex separable subset of K has the RNP.
3. K has the RNP for the measure space ([0,1), Borel σ-algebra, Lebesgue measure).
4. K has the MCP.
5. K has the MCP for the measure space ([0,1), Borel σ-algebra, Lebesgue measure).
6. K is subset s-dentable.
7. K is subset c-dentable.
8. Each countable subset of K is c-dentable.
9. K is subset dentable.
10. Each subset of K has a slice of arbitrary small diameter.

11. Each closed bounded convex subset of K is dentable.

Proof

1⇒4 Theorem 1.2.38, condition 1

4⇒5 Follows from Theorem 1.2.38 and Definition 1.2.36.

5⇒6 Theorem 1.2.38, condition 2.

6⇒1 Theorem 1.2.38, condition 3. It follows from Definition 1.2.10 and Theorem 1.2.11.

1⇒3 Follows from Theorem 1.2.38 and Definition 1.2.36.

3⇒5 Theorem 1.2.38, condition 1.

1⇒2 From Definition 1.2.35

2⇒8 Let D be a countable subset of K. Then $co(D)$ is countable and convex, so that $K_1 = clco(D)$ is a separable closed bounded convex set in K. By hypothesis K_1 has the RNP, and by the equivalence between 1 and 6, K_1 is subset s-dentable, and D is bounded in K_1. Since K_1 is bounded, D is s-dentable, and in particular, c-dentable.

9⇒8 Let K be subset dentable, hence K is subset c-dentable since dentability implies c-dentability. Since any countable subset C of K is also bounded since K is bounded, C is c-dentable.

6⇒7 Let N be any bounded subset of K, hence be c-dentable by hypothesis. Since c-dentability implies s-dentability, N is s-dentable. Hence K is subset s-dentable.

11⇒8 Let F be any bounded subset of K, then $clco(F)$ is closed convex and bounded, hence dentable by assumption. Then F is also dentable (see paragraph below, [14, p.14]) and hence c-dentable.

We have that 1, 3, 4, 5 and 6 are equivalent and that 1⇒2⇒8. It is evident that 9⇒8, 6⇒7 and 7⇒8.

11⇒9 Let D be any non-empty bounded subset of K. Then $clco(D)$ is a closed bounded convex subset of K, and it is dentable by hypothesis. Hence by D is dentable (see a paragraph below on facts refardinf dentability from [14,p.14]). Thus K is subset dentable.

9⇒11 Obviously, every closed convex subset of K is bounded, and also dentable by hypothesis.

9⇒10 Theorem 1.2.18.

8⇒11 See [7, pp.31,32]. Again, if C is countable in K then C is bounded since K is bounded. Hence C is dentable since K is dentable and has the RNP. Hence for any bounded closed and convex set N in K, N is dentable whether countable or not because K has the RNP. $\square$

Facts regarding the dentable sets and dentability [14, p.14]:
For any bounded set B is a Banach space X,

- If the closed convex hull of B is dentable, so is B.
- If B is compact convex set, then extreme points in B are all denting point (and, of course, conversely).
- Strongly exposed points of B are denting points.
- If B is a weakly compact set, then B is dentable.
- If every countable set in B is dentable, then so is B.

In addition, a Banach space X is dentable if and only if X has the RNP, by Theorem 1.2.16.

1.3 The RNP and the Bishop-Phelps Property

In this section, we introduce the Bishop-Phelps property (BPP), which originates from the Bishop-Phelps theorem, and discuss its relationship with the RNP.

For notational convenience define for each $x^* \in X^*$ and $M > 0$ a closed convex cone $K(x^*, M)$ by $K(x^*, M) = \{x \in X : \|x\| \leq Mx^*(x)\}$. If X and Y are Banach spaces, $L(X, Y)$ denotes the set of all continuous linear operators on X into Y.

In order to prove the Bishop-Phelps Theorem we need the following results.

Lemma 1.3.1 [13, Lemma 1, p.188]

Let C be a closed convex subset of X. If $x^* \in X^*$ and x^* is bounded on C and $M > 0$, then for each $y \in C$ there exists $x_0 \in C$ with $x_0 - y \in K(x^*, M)$ and such that $x_0 + K(x^*, M)$ supports C at x_0 in the sence that $C \cap (x_0 + K(x^*, M)) = \{x_0\}$.

Lemma 1.3.2 [13, Lemma 2, p.188]

Let $x^*,\ y^* \in X^*$ with $\|x^*\| = 1 = \|y^*\|$. If $\varepsilon > 0$ and $|y^*(x)| \leq \frac{\varepsilon}{2}$ whenever $\|x\| \leq 1$ and $x^*(x) = 0$, then either $\|x^* - y^*\| \leq \varepsilon$ or $\|x^* + y^*\| \leq \varepsilon$.

Proof

Restrict y^* to the null space of x^* and then let z^* be any Hahn-Banach (norm preserving) extension of this functional back to a member of X^*. Consequently, $\|z^*\| \leq \frac{\varepsilon}{2}$. Moreover, $y^* - z^*$ vanishes whenever x^* does hence $y^* - z^* = \alpha x^*$ for some α. Now,

$$|1 - |\alpha|| = |\|y^*\| - \|y^* - z^*\|| \leq \|z^*\| \leq \tfrac{\varepsilon}{2}.$$

Thus if, $\alpha \geq 0$, we have

$$\|x^* - y^*\| = \|(1-\alpha)x^* - z^*\| \leq |1 - \alpha| + \|z^*\| \leq \varepsilon;$$

but if $\alpha < 0$, we have

$$\|x^* + y^*\| = \|(1+\alpha)x^* + z^*\| \leq |1 + \alpha| + \|z^*\| \leq \varepsilon.$$

Hence for any such α the result holds, and the lemma is proved. $\square$

Lemma 1.3.3 [13, Lemma 3, p.188]

Let $x^*,\ y^* \in X^*$ with $\|x^*\| = 1 = \|y^*\|$. If $0 < \varepsilon < 1$ and $M > 1 + 2\varepsilon^{-1}$ then $\|x^* - y^*\| \leq \varepsilon$ whenever y^* is non-negative on $K(x^*, M)$.

Proof

Choose $x \in X$ such that $\|x\| = 1$ and $1 + 2\varepsilon^{-1} < Mx^*(x)$. If $y \in X, \|y\| < 2\varepsilon^{-1}$ and $x^*(y) = 0$, then we have

$$\|x \pm y\| \leq 1 + 2\varepsilon^{-1} < Mx^*(x) = Mx^*(x \pm y).$$

Accordingly, $x \pm y \in K(x^*, M)$. By hypothesis, $y^*(x \pm y) \geq 0$; so $|y^*(y)| \leq y^*(x) \leq \|x\| = 1$. Lemma 1.3.2 now ensures that either $\|x^* + y^*\| \leq \varepsilon$ or $\|x^* - y^*\| \leq \varepsilon$.

We show that $\|x^* + y^*\| \leq \varepsilon$ does not hold:

Since ε and $M^{-1} < 1$, there is $z \in X$ such that $\|z\| = 1$ and $\max(\varepsilon, M^{-1}) < x^*(z)$. But then $\|z\| \leq Mx^*(z)$ and $z \in K(x^*, M)$. Again $y^*(z) \geq 0$ and hence $\varepsilon < (x^* + y^*)(z) \leq \|x^* + y^*\|$. Hence only $\|x^* - y^*\| \leq \varepsilon$ holds and the proof is complete. □

The Bishop-Phelps Theorem 1.3.4 [13, Theorem 4, p.189]

Let C be a closed bounded convex subset of a Banach space X. The collection of linear functionals that achieve their maximum values on C is norm dense in X^*.

Proof

It is sufficient to approximate $x^* \in X^*$ with $\|x^*\| = 1$ by funtionals that achieve their maximum values on C. Furthermore it can be assumed that $0 \in C$. Let $0 < \varepsilon < 1$ and choose $M > 1 + 2\varepsilon^{-1}$. Since $M > 1$, $K(x^*, M)$ is a closed convex cone with non-empty interior (if $x_0 \in X$ is chosen so that $x^*(x_0\|x_0\|^{-1}) > M^{-1}$ then $K(x^*, M)$ contains an open ball centered at $x_0\|x_0\|^{-1}$). Apply Lemma 1.3.1 to C with $z = 0$ to obtain $x_0 \in C \cap (x_0 + K(x^*, M))$ such that $x_0 + K(x^*, M)$ supports C at x_0 in the sense of Lemma 1.3.1. Next, separate $x_0 + K(x^*, M)$ from C by $y^* \in Y^*$ chosen such that

$$\begin{aligned}\sup_{x \in C} y^*(x) &= y^*(x_0) \\ &= \inf_{x \in K(x^*, M)} y^*(x + x_0) \\ &= \inf_{x \in K(x^*, M)} y^*(x) + y^*(x_0).\end{aligned}$$

With this y^* we find that $y^*(x) \geq 0$ for $x \in K(x^*, M)$. It follows from Lemma 1.3.3 that $\|x^* - y^*\| \leq \varepsilon$. Since y^* achieves its maximum value at $x_0 \in C$, the proof is complete. □

Definition 1.3.5 [6, p.266]

A non-empty bounded and closed subset B of X is said to possess the *Bishop-Phelps Property* (BPP) whenever given any Banach space Y and any operator $T \in L(X, Y)$, there is an approximating sequence $(T_n)_{n \geq 1}$ in $L(X, Y)$, $(\|T_n - T\| \to 0)$, where each T_n achieves its maximum norm $N(T_n, B)$ on B where $N(T_m, B) = \sup\{\|T_m x\| : x \in B\}$.

Furthermore, every bounded closed and convex subset $B \subset X$ has the BPP, if and only if X has the BPP.

Examples 1.3.6 [13, p.216]

The space l_1 and all reflexive Banach spaces have the BPP, (closed unit balls of these spaces may have the BPP).

Proposition 1.3.7 [6, Proposition 1, p.266]

Let C be any non-empty separable bounded closed and convex subset of X. If C has the BPP, then C is dentable.

The similar result was stated by Bourgin in [7, p.388].

Remark 1.3.8

Suppose one can show that there exists an x in C (in Proposition 1.3.7 above), strongly exposed by at least one term T_n, of an aproximating sequence $(T_n)_{n\geq 1}$ in $L(X, Y)$. Such an element $x \in C$ would be a strongly exposed point, hence a denting point, hence C would be dentable.

From Proposition 1.3.7 above we realise that if any bounded closed and convex subset C of X has the BPP, then C is dentable and hence X has the RNP. This is formally stated as,

Corollary 1.3.9 [6, Corollary 2, p.266]

A Banach space with the BPP has the RNP.

Proof

Let X be a Banach space with the BPP. Then every bounded closed and convex subset of X has the BPP, see Definition 1.3.5. Let C be any non-empty separable bounded closed and convex set in X. By Definition 1.3.5, C has BPP. Then C is dentable by Proposition 1.3.7 above. Since dentability is separably determined, that is every separable subset of

X is dentable if and only if X is dentable, then X is dentable. A dentable Banach space has the RNP, see Theorem 1.2.16, hence X has the RNP. $\square$

Remark 1.3.10

In the proof above we used the fact that a dentable Banach space X has the RNP, see Definition 1.2.15 and Theorem 1.2.16. Hence a Banach space X has the RNP if and only if X is dentable if and only if any of its bounded subsets is dentable. This result was also mentioned by Bourgain [5, p.135], and it is vital for our subsequent discussion and developments.

Bourgain [6] extended Corollary 1.3.9 above to:

Theorem 1.3.11 [6, Theorem 7, p.269]

A Banach space X has the BPP if and only if it has the RNP.

Proof

We only need to establish the sufficiency part of this implication.

Let B be any non-empty bounded closed and convex set in X. Since X has the RNP, every bounded subset of X is dentable, including B and all its subsets, see Theorem 1.2.39. Then for any Banach space Y, the set of those operators $T \in L(X, Y)$ which attain their maximum norm $N(T, B)$ on B is dense in $L(X, Y)$ (by the BP-theorem 1.3.4). Hence B has the BPP. It follows that X has the BPP, see Definition 1.3.5. $\square$

The result of Theorem 1.3.11 above gives us an important characterisation of Banach spaces with the RNP. We shall explore in subsequent chapters if the BPP in X is equivalent to the property that, every bounded closed convex subset, say C, in X equals the closed convex hull of its extreme points.

1.4 The RNP, Decomposition and Bushes

In an attempt to find sufficient conditions for a Banach space to possess the RNP, we introduce the notion of bushes in, and decompositions of, Banach spaces. This section is dedicated to the partial progress that has been made on this subject.

Definition 1.4.1 [25, p.255]

A *bush* in a Banach space X is a bounded partially ordered subset B for which each member has at least two (finitely many) successors and is a convex combination of its successors, and there is a positive separation constant δ such that $\| v - u \| \geq \delta$ if v is a successor of u, and B has a first member to which each member of B can be joined by a linearly ordered chain of successive members of B. Such a bush is also called a δ-bush.

Proposition 1.4.2

A bush in a Banach space X is a bounded non-dentable subset of X.

Proof

Let B be any bush. Then B is bounded and each element $b \in B$ is a convex combination if its successors. Suppose B is dentable. Then there exists $b_o \in B$, which is a denting point in B, such that (by Definition 1.2.2) $b_o \notin clco(B \backslash B_\varepsilon(b_o))$, for any $\varepsilon > 0$. Then $b_o \notin co(B \backslash B_\varepsilon(b_o)) \subset clco(B \backslash B_\varepsilon(b_o))$, which means b_o cannot be written as a convex combination of elements in $B \backslash B_\varepsilon(b_o)$, hence elements of B, including its successors. This follows from the fact that the separation constant can be bigger than the radius of the arbitrary ball centered at any point in the bush, that is $\varepsilon < \delta$, for a separation constant δ. This contradicts the fact that B is a bush. Hence a bush is non-dentable. □

Also, Bourgin [7, p.34] remarked that a δ-bush is not c-dentable, hence non-dentable.

Lemma 1.4.3 [13, p.216]

A bounded infinite δ-bush can be found inside any non-s-dentable set.

Theorem 1.4.4 [7, p.31]

A Banach space X has the RNP if it does not contains a bush.

Proof

Suppose a Banach space X lacks the RNP. By Theorem 1.2.11 and Definition 1.2.10 X contains a bounded non-s-dentable set A. But then A contains a bush by Lemma 1.4.3, which contradicts the hypothesis that X does not contain a bush. Hence X has the RNP and the proof as complete. $\square$

See Proposition 2.3.3 for an extension of this result.

Definition 1.4.5 [31, Definition 1.g.1, p.47]

A *Schauder decomposition* of Banach space X is a sequence $(X_n)_{n\geq 1}$ of non-trivial closed subspaces of X such that every $x \in X$ can be expressed uniquely in the form $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in X_n$ for every $n \in \mathbb{N}$.

Moreover, if $\dim(X_n) < \infty$ for all $n \in \mathbb{N}$, then such a decomposition is said to be finite dimensional, and is denoted by **FDD**.

Definition 1.4.6 [43, p.160]

A Schauder decomposition $(X_n)_{n\geq 1}$ of X is called *boundedly complete* if whenever $(\sum_{n=1}^{m} x_n)_{m\geq 1}$ is a bounded sequence with $x_n \in X_n$ for every $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} x_n$ converges.

Definition 1.4.7 [31, p.1&18], [13, p.64]

A sequence $(x_n)_{n\geq 1}$ in a Banach space X is called a *Schauder basis* of X if for every $x \in X$, there exists a unique sequence of scalars $(a_n)_{n\geq 1}$ such that $x = \sum_{n=1}^{\infty} a_n x_n$.

A Schauder basis $(x_n)_{n\geq 1}$ of X is called *boundedly complete* if for each scalar sequence $(a_n)_{n\geq 1}$ such that $\sup_n \| \sum_{k=1}^{n} a_k x_k\| < \infty$, the series $\sum_{n=1}^{\infty} a_n x_n$ converges.

Theorem 1.4.8 [13, Theorem 6, p.64]

If a Banach space X has a boundedly complete Schauder basis, then X has the RNP.

Theorem 1.4.9 [5, p.135]

A Banach space X possesses the RNP if and only if every subspace with finite dimensional Schauder decomposition has the RNP.

Bourgain [5] proved this theorem in two cases, firstly, where every bounded closed convex subset of X has a PC and secondly, where X has a bounded closed convex set without a PC. Schachermayer [47, p.100] gives a simplified proof of the case where every bounded closed convex subset of X has a PC.

With respect to the above theorem by Bourgain [5], it is worth noticing that, if a Banach space X has the FDD, then the space X is reflexive (see, Lindenstrauss and Tzafriri [31, p.47]), and a reflexive space has the RNP (see [7, p.74]).

Definition 1.4.10 [43, p.167]

If $(X_i)_{i\geq 1}$ is a decomposition for a Banach space, with $X_i \subset X$, for each $i \geq 1$, then $(H_i)_{i\geq 1}$ is a *skipped-blocking* of $(X_i)_{i\geq 1}$ if there exist sequences of positive integers (m_k) and (n_k) so that $m_k < n_k + 1 < m_{k+1}$ and each H_k equals the closed linear subspace spanned by $(X_i)_{i=m_k}^{n_k}$.

Definition 1.4.11 [24, p.912]

A finite dimensional Schauder decomposition, FDD, is *basic* if and only if each member $x \in X$ has a unique representation as $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in X_n$ for all $n \in \mathbb{N}$, and convergence is **convergence in norm.** Furthermore, such a decomposition is *unconditionally basic* if the convergence is unconditional for each x (hence the series $\sum_{n=1}^{\infty} x_n$ converges unconditionally; for a definition of unconditional convergence visit Lindenstrauss and Tzafriri [31, p.15]). Henceforth, an unconditionally basic FDD is denoted by UBFDD, and whenever the UBFDD is *skipped-blocking*, it is then denoted by UBSBFDD.

Theorem 1.4.12 [24, p.913]

If a Banach space X has UBSBFDD, then X has the RNP if and only if X does not have a subspace isomorphic with c_0.

A Banach space, as we shall see in subsequent chapters, not having a subspace isomorphic with c_0, or not having an isomorphic copy of c_0, or not containing a subset of c_0, also has the property that every closed bounded convex subset, say M, in X is such that $M = clco(E(M))$. Theorem 1.4.12 above gives us a characterisation of those Banach spaces in which the RNP and this above-mentioned property are equivalent. Such spaces will be discussed thoroughly toward the end of this thesis.

1.5 The RNP and Dual spaces

This section is devoted to an exposition of the connection between dual spaces and the RNP. As will be seen, separability plays a vital role in this discussion.

Dunford-Pettis Theorem 1.5.1 [13, Theorem 1, p.79], [11, p.225]

If X is a Banach space and X^* is separable, then X^* has the RNP.

In order to prove this theorem we require the following result.

Lemma 1.5.2 [11, Lemma 5, p.225]

Let X be a Banach space with separable dual X^*. Let B be a non-empty closed bounded convex subset of X^*, let D be the weak*-closure of B, and let E denote the set of extreme points of D. Then $B \cap E$ is weak*-dense in E.

Proof

The set D is weak*-compact and convex. By [11, Lemma 4, p.222] the set Z of all points of continuity of the identity map on D between the weak* and norm topologies relativised to D intersects E in a weak*-dense G_δ-subset of E. Let $z \in Z$. Since B is weak*-dense in D, there is a net (f_α) of members of B converging weak* to z. But $z \in Z$, so the net (f_α) converges to z in the norm topology. Thus $z \in B$. It follows that $Z \subset B$, and hence by [11, lemma 4, p.222] we have $B \cap E$ is weak*-dense in E. □

Proof of Theorem 1.5.1

Due to Theorem 1.2.39 we need only to show that each non-empty norm-closed bounded convex subset B of X^* has a denting point. Using the notation of Lemma 1.5.2, for any such B, any extreme point z of D which belongs to Z are extreme points of B. Given $\varepsilon > 0$. Since $z \in Z$, there is a weak*-open subset W of X^* such that $z \in W \cap D$ and the norm-diameter of $W \cap D \leq \varepsilon$. Now, z is not an element of the weak*-closed convex hull of $D \backslash W$ (Milman's Theorem) and so z is not in the norm-closed convex hull of $D \backslash W$. Since $B \backslash \{f \in X^* : \|f - z\| \leq \varepsilon\} \subset D \backslash W$, it follows that z is not in the norm-closed convex hull of $B \backslash \{f \in X^* : \|f - z\| \leq \varepsilon\}$. Consequently, z is a denting point of B. This completes the proof. □

Definition 1.5.3

The RNP in a Banach space X is said to be *separably determined* if and only if each separable subspace of X has the RNP.

The next result will strengthen Theorem 1.5.1 to obtain a more general result.

Theorem 1.5.4 [28, p.497].

Let X be a Banach space. The following are equivalent:

1. X possesses the RNP.
2. Every closed subspace of X possesses the RNP.
3. Every closed separable subspace of X possesses the RNP.
4. X is separably determined.

Since every separable closed subspace Y of a reflexive Banach space X is a separable dual space, it follows from Theorem 1.5.4 that every such Y has the RNP, from which it follows that a reflexive Banach space X possesses the RNP.

We can now formulate a more general theorem.

Theorem 1.5.5 [49, p.218], [28, p.498]
A dual X^* of a Banach space X possesses the RNP if and only if every (closed) separable subspace Y of X has a separable dual Y^*.

Similar to the Dunford-Pettis Theorem (Theorem 1.5.1) above, Bourgin [7, p.75] stated that all separable dual Banach spaces have the RNP.

Remark 1.5.6
The Dunford-Pettis result can also be stated as follows:

1. If $D \subset X^*$ is a weak*-compact and separable set, then D is subset s-dentable [7, p.71].
2. If X^* is separable, then its closed unit ball B_{X^*} is weak*-compact and norm separable, and hence, (by 1 above), subset s-dentable. Thus, by Bourgin [7, p.31], B_{X^*} has the RNP and so has X^*.
3. A set D has the RNP.

Definition 1.5.7 [28, p.498]
A Banach space X is called *quasi-separable* if for each separable subspace Y of X, Y^* is separable ($\Leftrightarrow X^*$ possesses the RNP).

Hence, by Theorem 1.5.5, X is quasi-separable if and only if X^* has RNP.

Remark 1.5.8 [28, p.498]

If X is quasi-separable then every continuous linear closed image of X is also quasi-separable. For, if K is a continuous linear closed image of X, then K^* is isomorphic to a subspace of X^*, and then K^* has the RNP. Hence K is quasi-separable (see Kuo [28, p.498]).

There are cases where both X^* and X^{**} (and hence X) have the RNP. For this purpose, we turn to quotient spaces. We first recall some results from [28].
On account of Theorem 1.5.5, the quasi-separability concept is equivalent to the possession of the RNP by X^*. It is not known whether a Banach space X is quasi-separable if the closed unit ball $B_{X^{**}}$ of X^{**} is weak*-sequentially compact. This can be equivalently translated as whether a conjugate space X^* has the RNP if $B_{X^{**}}$ is weak*-sequentially compact.

Definition 1.5.9 [28, p.498]

A Banach space X is said to be *weakly compactly generated* (wcg) if it is the closed span of some weakly compact subset C of itself, that is, if the linear span of C is dense in X.

Weakly compactly generated Banach spaces will again be discussed Section 1.6.

Theorem 1.5.10 [9, Theorem 3.6, p.908]

Let X_1 and X_2 be Banach spaces where X_1 is quasi-reflexive. If X_1^* is isomorphic to X_2^*, then X_1 is insomorphic to X_2.

Theorem 1.5.11 [28, p.501]

Let X be a Banach space such that X^{**}/X^* is separable. Then both X^{**} and X^* have the RNP.

Proof

In view of Theorem 1.5.5 it suffices to show that every closed separable subspace of X

(respectively, X^*) has a separable dual. Let Y be a closed separable subspace of X. By Theorem 1.5.10, Y^{**}/Y is isomorphic to a subspace of X^{**}/X because $Y^{**} \subset X^{**}$. By hypothesis, X^{**}/X^* is separable, hence X^{**}/X is separable. It follows that, Y^{**}/Y is separable because $Y^{**}/Y \subset X^{**}/X$. Thus Y^{**} is separable, hence Y^* is separable. Hence by Theorem 1.5.5, X^* has the RNP.

Let K be a closed separable subspace of X^*. Then there exists a separable subspace W of X such that K is isometrically isomorphic to a subspace of W^* . K^* is thus a continuous linear image of the separable space W^{**}, and hence K^* is separable. Consequently, X^{**} has the RNP. This completes the proof. $\square$

Proposition 1.5.12 [28, p.501]

If both X^{**} and X^* have the RNP, then every closed separable subspace of X has a separable second dual.

Proof

If Y is a separable closed subspace of X, then Y^* is separable, by Theorem 1.5.5. That is, since X^* has the RNP, then every separable subspace Y of X has a separable dual. But Y^{**} is isometrically isomorphic to a subspace of X^{**}, and deducing from the result of Theorem 1.5.5, X^{**} has the RNP if and only if every separable subspace Y^* of X^* has a separable dual Y^{**}. Hence, since Y^* is separable, Y^{**} is separable. $\square$

We realise that the combined effect of the Theorem 1.5.11 and Proposition 1.5.12 above, gives the following:

If X is a Banach space such that X^{**}/X^* is separable, then every closed separable subspace of X has a separable second dual (and also a separable dual).

It should be noted that if given X^{**} and X^* have the RNP, it does not necessarily imply that X^{**}/X is separable. See [28, p.502] for a reference with counter-examples.

Corollary 1.5.13 [49, p.222]

1. If X^* has the RNP and the space Y is isomorphic to a quotient space of X, then Y^* has the RNP.

2. If there is a subspace Y of X such that Y^* and $(X/Y)^*$ have the RNP, then X^* has the RNP.

Proof

1. This follows from the fact that $Y \subset X$ and every separable subspace K of Y is also in X. Hence K^* is separable since X^* has the RNP, see Theorem 1.5.5. Consequently Y^* has the RNP.

2. Suppose $\mu : X \to X\backslash Y$ is the canonical quotient operator. Let M be a separable subspace of X. Since μ is onto, there exists a separable subspace W of X, $M \subset W$ and $\mu(W)$ is closed in $X\backslash Y$. Let $\psi : W \to \mu(W)$, $\psi = \mu|_W$, the restriction of μ to W. The kernel of the operator ψ is $W \cap Y$. Both $\mu(W)$ and $W \cap Y$ are separable and their duals have the RNP, so their duals are separable. Then W^* is separable, hence M^* is separable. Since M was arbitrarily chosen in X, X^* has the RNP. □

In the proof above the following were employed:

- If $A \subset B$ and B^* is separable, then A^* is separable.

- If W is separable and μ is onto and structure preserving map, then $\mu(W)$ is separable.

Corollary 1.5.14 [13, Corollary 10, p.198]
Let X be a Banach space and Y be a Banach space which is a continuous linear image of closed subspace of X. If X^* has the RNP, then Y^* has the RNP.

Proof

If K is a separable closed subspace of Y, then K is a continuous linear image of a closed subspace V of X. Since X^* has RNP and V is separable, then V^* is separable, by Theorem

1.5.5. Since K is a quotient of V, K^* is a subspace of V^*, and K^* is separable since V^* is separable. Hence K, a separable subspace of a Banach space Y has a separable dual K^*. Since K was arbitrarily chosen, then any such K in Y has a separable dual, hence Y^* has RNP. $\square$

Remark 1.5.15

In the proof above, we assume that there exists $f : X \to f(X)$, for f continuous and linear, and there exists V, a closed subset of X, such that $f(V) = Y$, with $V^* = Y$.
We then show that $Y^* = V^{**}$ has the RNP, that is, there exists a separable subspace K of $Y = V^*$ such that K^* is separable.

The RNP is a useful tool in studying spaces of operators. If X and Y are Banach spaces, then $L(X, Y)$ denotes the space of continuous linear operator from X to Y. The following results are known, see [14, p.39]:

1. If the Banach spaces X^* and Y have the RNP and if every continuous linear operator from X to Y is compact, then $L(X, Y)$ has the RNP.

2. If $L(X, Y)$ has the RNP and if Y has a complemented subspace with an unconditional basis, then every operator $T \in L(X, Y)$ is compact.

1.6 The RNP and Weakly Compactly Generated spaces

One way new classes of spaces have been shown to possess the RNP is by showing that separable subspaces are subspaces of separable duals. For this purpose, we investigate weakly compactly generated spaces. In this section, spaces that are generated by weakly compact sets are being discussed, and the existence of the RNP is investigated.

Notations

$B_X, B_{X^*}, B_{X^{**}}$ denote the closed unit balls of X, X^* and X^{**} respectively.

Definition 1.6.1 [28, p.498], [11, p.143]

A Banach space X is *weakly compactly generated* (wcg) if it is the closed linear span of some weakly compact subset C of itself, that is, if the linear span of C is dense in X.

Definition 1.6.2 [28, p.499]

A compact Hausdorff space S is *Eberlein compact* if it is homeomorphic to a weakly compact subset of some Banach space.

Kuo [28, p.499] asserts that, due to the Eberlein-Šmulian Theorem, S is sequentially compact if it is Eberlein compact, that is, if it is homeomorphic to a weakly compact subset of some Banach space. Futhermore, Kuo [28] asserts that if X^* is isomorphic to a subspace of a weakly compactly generated (wcg) space, then X^* has the RNP.

Remark 1.6.3 [28, p.498]

A Banach space X is wcg if and only if the unit ball B_{X^*} is Eberlein compact in its weak*-topology.

This is similar to Corollary 3 in [11, p.148]. In particular B_{X^*} is weak*-sequentially compact.

Theorem 1.6.4 [28, p.499]

Let X be a Banach space. Suppose that $B_{X^{**}}$ is Eberlein compact in the weak*-topology. Then X^* possesses the RNP.

Proof

In view of Theorem 1.5.5, it suffices to show that every closed separable subspace Y of X has a separable dual Y^*. For this purpose, let Y be a closed separable subspace of X. Then B_Y is weak*-dense in $B_{Y^{**}}$ (Goldstine's theorem). Thus, $B_{Y^{**}}$ is weak*-separable. Let $J : Y \to X$ be the inclusion map. Then $J^{**} : Y^{**} \to X^{**}$ is a weak*-isomorphism of

Y^{**} onto $Y^{\perp\perp} = \{f \in X^{**} : f(y) = 0 \text{ for all } y \in Y^{\perp}\}$, with $J^{**}(B_{Y^{**}}) = B_{Y^{\perp\perp}}$. Then $B_{Y^{\perp\perp}}$ is weak*-separable. Moreover, $B_{Y^{\perp\perp}}$ is weak*-closed in $B_{X^{**}}$, which is Eberlein compact by hypothesis, whence $B_{Y^{\perp\perp}}$ is itself Eberlein compact. Now $B_{Y^{\perp\perp}}$ is metrizable because a separable Eberlein compact space is metrizable. This then implies that $B_{Y^{**}}$ is metrizable. Therefore, Y^* is separable, which completes the proof. □

Theorem 1.6.5 [28, p.499], [11, p.226]

Suppose X^* is isomorphic to a subspace of a wcg Banach space W. Then X^* possesses the RNP.

Proof

In the view of Theorem 1.5.5, it suffices to show that every closed separable subspace Y of X has a separable dual Y^*. So, let Y be a closed separable subspace of X. Applying the same argument as in the proof of Theorem 1.6.4, we see that $B_{Y^{**}}$ is weak*-separable. Let $(x_n^{**})_{n\geq 1}$ be a weak*-dense sequence in $B_{Y^{\perp\perp}}$ and $J : X^* \to W$ be an isomorphism. Then $J^* : W^* \to X^{**}$ is surjective. By the Open Mapping Theorem, there exists a bounded sequence $(w_n^*)_{n\geq 1}$ in W^* such that $J^* w_n^* = x_n^{**}$. Denote by Z the weak*-closure of $(w_n^*)_{n\geq 1}$. By the hypothesis W is wcg, B_{W^*} is the Eberlein compact in the weak*-topology and hence Z is also Eberlein compact. This together with the separability of Z implies that Z is a compact metric space in the weak*-topology. $J^*(Z)$ is then weak*-compact and contains $(x_n^{**})_{n\geq 1} \subset B_{Y^{\perp\perp}}$. Hence, $J^*(Z) = B_{Y^{\perp\perp}}$. Moreover, being a continuous image of a compact metric space, $B_{Y^{\perp\perp}}$ is compactly metrizable. Therefore, $B_{Y^{**}}$ is metrizable and Y^* is separable. This completes the proof. □

Diestel and Uhl [14, p.7] show that weakly compactly generated (wcg) dual spaces possess the RNP. The same result was also stated by Kuo [28] and it is to be stated subsequently. The proof of this result as given by Diestel and Uhl [14] goes as follows:

If Y is a separable subspace of a wcg space X^*, then there exists a separable subspace S of X such that Y is a subspace of S^*.

Then S^* is separable:

Clearly S^* is a quotient of X^*, thus S^* is also wcg. Let K be a weakly compact convex subset of S^* that generates S^*. Then K is a weak*-separable. But S is separable so K is weak*-metrizable, hence K is weak*-separable. But since K is compact in both Hausdorff topologies (weak and norm), it follows that those topologies coincide on K, that is, K is weakly separable. Then K is norm separable and thus S^* is separable, and is the closed linear span of K. Hence X^* has the RNP.

Theorem 1.6.6 [11, p.226]

Suppose X is a Banach space whose dual is a wcg subspace of some Banach space Y. Then X^* has the RNP.

The method of the proof once again is to show that if A is an arbitrary closed separable subspace of X, then A^* is separable.

It follows immediately that all wcg dual spaces (and most particularly, all reflexive Banach space) possesses the RNP.

Corollary 1.6.7 [28, Corollary 2.3, p.500]

If X^* is wcg then X^* has the RNP.

Lemma 1.6.8 [28, Lemma 3.4, p.502]

If Z is a wcg subspace of a Banach space such that Y/Z is separable, then Y is wcg.

Proof

Because Y/Z is separable, there exists a separable subspace $W \subset Y$ such that $Z + W$ is dense in Y. Since both W and Z are wcg, it follows that Y is wcg. $\square$

Theorem 1.6.9 [28, p.502]

If X^{**}/X^* is separable, then X^* is wcg.

Proof

Under the given hypothesis, there exists a separable subspace Z such that X/Z is reflexive. Then $Z^{\perp}$ is reflexive and $X^*/Z^{\perp}$ is separable. It then follows from Lemma 1.6.8 that X^* is wcg. □

It then follows from Corollary 1.6.7 and Theorem 1.6.9 above that if X^{**}/X^* is separable X^* has the RNP.

Example 1.6.10 [11], **[14, p.7]**

All reflexive Banach spaces and $c_0(\Gamma)$ are wcg spaces. Reflexive Banach spaces are wcg dual spaces, hence they have the RNP. On the other hand, $c_0(\Gamma)$ is wcg but fails the RNP (see, Remark 1.7.14).

This follows from the fact that $c_0(\Gamma)$ is not a dual space. Note that wcg Banach spaces can possess the RNP only if they are dual spaces. The space $L^1(\Omega, \Sigma, \mu, \mathbb{R})$ is wcg, because the set $\{\chi_A : \mu(A) < \infty\}$ is relatively weakly compact by the classical Dunford-Pettis criterion and has a dense linear span.

For any set Γ, the space $c_0(\Gamma)$ is wcg because the set $\{e_\gamma : \gamma \in \Gamma\}$ of unit vectors is relatively weakly compact and has a dense linear span.

It is interesting that wcg Banach spaces can be characterised in terms of the ranges of linear operators on reflexive spaces as follows,

Theorem 1.6.11 [11, **p.163]**

A Banach space X is wcg if and only if there exists a reflexive Banach space Y and a one-to-one continuous linear operator $T \in L(Y, X)$ such that $T(Y)$ is dense in X.

Examples of wcg spaces [11, p.143]:

- $c_0(\Gamma)$, Γ any set.
- $L_1(\mu)$, μ any σ-finite measure (also stated in [28, p.500]).

- $l_1(\Gamma)$, Γ any countable set.

1.7 Asplund spaces, Dual spaces and the RNP

In this section we study the connection between Asplund spaces and dual spaces with the RNP. Separability has been used thus far to characterise dual Banach spaces with the RNP, and this section follows suit.

Notations

$C(X)$ denotes the collection of continuous complex-valued functions defined on a Banach space X, and $C(X)^*$ its dual.

Definition 1.7.1 [7, Definition 5.1.1, p.117], [11, p.247]

If X is a Banach space, $D \subset X$ and $F : X \to \mathbb{R}$ a function, then F is called *D-differentiable* at $x \in X$ if there is an $f \in X^*$ such that $\lim_{t\to 0^+} \sup_{d\in D} | \frac{F(x+td)-F(x)}{t} - f(d)| = 0$. When the set of directions D is the unit ball of X, F is *Fréchet defferentiable* at x.

A *convex function* is a function whose value at the midpoint of every interval in its domain, does not exceed the average of its values at the ends of the interval. Generally, suppose Z is a convex set in a Banach space X. Then for any function $f : Z \to X$, if for any $x, y \in Z$, with $x \neq y$, and any $\lambda \in (0,1)$, we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$, we say f is convex.

Definition 1.7.2 [45, p.105], [33, p.735]

A real Banach space X is called an *Asplund space* if every continuous convex real-valued function defined on a non-empty open convex subset D of X is Fréchet-differentiable on a dense G_δ*-subset* of D.

Remark 1.7.3 [33, p.735]

Asplund spaces are sometimes called the strongly differentiable spaces.

If X is a Banach space, then the dual space X^* is called a (DA)-space if it satisfies the following condition (*):

If K is a weak-compact convex subset of X^*, then K is the weak*-closed convex hull of those of its points strongly exposed by funtionals on X.*

Proposition 1.7.4 [13, p.213]

The dual X^* of an Asplund space X satisfies condition (*) above, and so it is a (DA)-space.

One of the main theorems by Namioka and Phelps [33] establishes the converse to Proposition 1.7.4, that: If X^* is a (DA)-space, then X is an Asplund space (see Theorem 1.7.10).

Proposition 1.7.5 [See 45, p.105]

A Banach space X is an Asplund space if and only if each separable subspace of X has a separable dual.

In view of Theorem 1.5.5, it follows that the interest in Asplund spaces is motivated by the result:

A Banach space X is an Asplund space if and only if its dual X^* has the RNP.

This result is influenced by the following proposition:

Proposition 1.7.6 [33, p.741]

Suppose that X is an Asplund space. Then the following hold:

1. X^* has the RNP.

2. Every separable subspace of X has a separable dual.

Does the converse hold?

If X^* has the RNP, does this imply that X is an Asplund space?

The affirmative answer follows from the following theorem by Stegall [50, p.515].

Theorem 1.7.7 [50, p.515], [7, p.132]

A Banach space X is an Asplund space if and only if X^* has the RNP.

Proof

Suppose X is an Asplund space. Then X^* has the RNP, by Proposition 1.7.6 above. *Conversely,* if X^* has the RNP, then every separable subspace of X has a separable dual (Theorem 1.5.5). Hence X is an Asplund space, by Proposition 1.7.5. □

Remark 1.7.8 [14, p.35]

If X is Asplund, then X^* has the RNP; if X^* is wcg, then X is Asplund. As a consequence of Theorems 1.7.7 and 1.5.5, and Proposition 1.7.6, we have:

X is an Asplund space if and only if X^* has the RNP if and only if every separable subspace Y of X has a separable dual Y^*.

Corollary 1.7.9 [33, p.742]

Let X be a separable Banach space. Then X is an Asplund space (that is, X^* has the RNP) if and only if X^* is separable.

Proof

Let X be Asplund, then by Proposition 1.7.6, every separable subspace of X has a separable dual. Since X is a separable subspace of itself, X^* is separable.

Conversely, let X^* be separable. Then X^* has the RNP, see Theorem 1.5.1, and X is separable. Thus X is an Asplund space, see Theorem 1.7.7. □

The following theorem is interesting in that it characterises those dual spaces with the RNP whose closed subspaces have the RNP.

Theorem 1.7.10 [33, p.742]

If M is a closed linear subspace of an Asplund space X, then M is an Asplund space.

In other words: If M is a closed linear subspace of a space X with X^* having the RNP, then M^* possesses the RNP.

Theorem 1.7.11 [33, p.739]

If X is a Banach space such that X^* is a (DA)-space, then X is an Asplund space.

Namioka and Phelps [33, pp.745, 746] mention two open questions and results related to these:

1. From Proposition 1.7.6: If X^* is a (DA)-space, then X^* has the RNP. Is the converse true? If so, it would provide a very interesting characterisation of Asplund spaces as precisely the preduals of spaces with the RNP.

2. Is the existence of an equivalent norm which is Fréchet differentiable on $X\backslash\{0\}$ either a necessary or a sufficient condition for X to be an Asplund space?

An affirmative answer to 1 would provide an affirmative answer to the sufficiency portion of 2. Indeed, if the norm in X is Fréchet differentiable, then the norm continuity of the differentiable map, together with the Bishop-Phelps density Theorem 1.3.4, see [13, p.189], show that every separable subspace of X has a separable dual, hence X^* has the RNP, this would then imply that X is an Asplund space.

Definition 1.7.12 [33, p.746]

A Banach space X is *dispersed* if every nonempty subset of X has a (relative) isolated point.

Theorem 1.7.13 [33, pp.746,747]

Let X be a compact Hausdorff space. Then $C(X)$ is an Asplund space if and only if X is dispersed.

We present a proof in one direction:

Proof

Let X be embedded in $C(X)^*$ with the relative weak*-topology. If $C(X)$ is an Asplund space, then $C(X)^*$ is a (DA)-space and if $A \subset X$ is non-empty, then there exists a relative weak*-open non-empty subset U of A of diameter less than 1. But if $x, y \in X$ such that $x \neq y$, then $\|x - y\| > 1$, say $\|x - y\| = 2$. Thus U consists of a single point, that is, an isolated point of A. Hence X is dispersed. $\square$

It is known (see [33, p.749]) that the space $C(X)^*$ has the RNP if and only if X is dispersed. Hence we have the following;

Corollary 1.7.14 [33, p.749]

Let X be compact Hausdorff space. Then $C(X)$ is an Asplund space if and only if $C(X)^*$ has the RNP.

This result follows from Theorem 1.7.7.

Remark 1.7.15

1. If X is a Banach space and if X^* is separable then, X^* has the RNP(see Theorem 1.5.1), and X is an Asplund space (see Proposition 1.7.6).

2. If X is a Banach space and if X is reflexive, then X is an Asplund space (see [33, p.735]); by Proposition 1.7.6, X^* has the RNP. All Hilbert spaces are thus Asplund spaces, and their duals have the RNP.

3. An example of an Asplund space which fails the RNP:

   If Γ is any set, then the space $c_0(\Gamma)$ is Asplund (see [33, p.741]) and it fails the RNP. But its dual $l_1(\Gamma)$ has the RNP, because $l_1(\Gamma)$ is a (DA)-space.

All reflexive Banach spaces (see [7, p.74]), and Hilbert spaces (see [27, p.241]) have the RNP.

Theorem 1.7.16 [7. p.74]
Let K be a weak*-compact convex set in X^* for which $E(K)$ is norm separable. Then K has RNP, K is itself norm separable and $K = clco(E(K))$.

Since strongly exposed points are extreme point in any closed bounded and convex subset of a Banach space, then Theorem 1.7.16 above is related to Remark 1.7.4 stated earlier. In this following section, we summarize properties that are equivalent to the RNP.

1.8 Overview of the RNP and equivalent properties

Each of the following conditions is sufficient and necessary for a Banach space X to have the RNP [13, p.217]:

1. Every closed linear subspace of X has the RNP.
2. Every separable closed linear subspace of X has the RNP.
3. Every function $f : [0, 1] \to X$ of bounded variation is differentiable a.e.
4. Every absolutely continuous function $f : [0, 1] \to X$ is differentiable a.e. In this case $f(b) - f(a) = \int_a^b f'(t)\, dt$ for any a and $b \in [0, 1]$.
5. Every bounded subset of X is dentable.
6. Every closed bounded convex subset of X is dentable.
7. Every bounded subset of X is *σ-dentable* (*s-dentable*).
8. Every non-empty closed bounded subset of X contains an extreme point of its closed convex hull.
9. Every non-empty closed bounded convex subset of X is the closed convex hull of its denting points.
10. Every non-empty closed bounded convex subset of X has a strongly exposed point.

11. Every non-empty closed bounded convex subset of X is the closed convex hull of its strongly exposed points.

    If X is isomorphic to a dual space of a Banach space Y, then the following are equivalent to those above:

12. Every separable subspace (of) Y has a separable dual.

13. Every separable subspace of X is isomorphic to a subspace of a separable dual.

14. Every non-empty closed bounded convex subspace of X has an extreme point.

15. Every non-empty closed bounded convex subset of X is a closed convex hull of its extreme points. (14)$\Rightarrow$15) is in [7, p.39]).

It is worth noting that, all conditions stated in Theorem 1.2.39 are sufficient for the existence of the RNP in a Banach space X. This follows from the fact that, if any bounded closed and convex subset of X has the RNP, X has the RNP as well [50, p.508].

Proposition 1.8.1

Each of the following conditions is sufficient for a dual Banach space X^* (exclusively) to possess the RNP,

1. Every separable subspace of X has a separable dual.

2. X is quasi-separable.

3. Every continuous linear closed image of X is quasi-separable.

4. X^* is isomorphic to a subspace of a wcg Banach space.

5. X^* is wcg (weakly compactly generated).

6. X is an Asplund space.

Proofs for these conditions follows from Theorem 1.5.5, Definition 1.5.7, Remark 1.5.8, Theorem 1.6.5, Corollary 1.6.7, and Theorem 1.7.6 respectively.

Corollary 1.8.2 [7, p.67]

Let $C \subset X$ be a closed convex set. If C has the RNP then each closed bounded subset of C contains an extreme point of its closed convex hull.

This statement is similar to 10)$\Rightarrow$8) above, where C is replaced by X, and X is a larger space.

Proof

Suppose C has the RNP, then by 10) above, every non-empty closed bounded convex subset of C has a strongly exposed point. Let D be any closed bounded subset of C, then $clco(D) \subset C$, say $K = clco(D)$, has a strongly exposed point, say x, which is in C since K is closed bounded and convex in C. Hence $E(clco(D)) \neq \emptyset \neq E(K)$. We need to show that there exists an element $y \in D$ such that $y \in E(K)$:

From the above, there exists x a strongly exposed point in K, that is, $x \in SE(K)$. Since strongly exposed points are extreme points then x is an extreme point in $clco(D)$. The above proof is slightly modified from the original proof. $\square$

Examples 1.8.3

Examples of spaces with RNP [13, p.218]

- reflexive spaces.
- separable (dual) spaces.
- Duals of Asplund space (denoted by (DA)-space) [14, p.31].
- weakly compactly generated (wcg) duals, and their dual subspaces.
- spaces with boundedly complete basis (Theorem 1.4.10)
- $l_1(\Gamma)$, Γ any non-empty set (proof [7, p.76], [13, p.8, Corollary 8] and [41, p.76]).
- the spaces of unconditionally convergent series in X if X has RNP.
- X^{**}, X^* when X^{**}/X^* is separable.

Almost each of the above-mentioned examples where proven in this chapter.

Examples of spaces without RNP [13, p.219]

- $L_1[0,1]$ (because it contains a δ-tree, which is a δ-bush, which in turn is a bounded and non-dentable set, see Proposition 1.4.2 and [7, p.15]).
- c_0 (because its closed unit ball have no extreme points [14, p.23], and it is non-dentable [41, p.77]).
- c (because the closed unit ball in c is non-dentable [23, p.163]).
- l_∞ (because it contains a copy of c_0).
- $L_\infty[0,1]$ (the closed unit ball of $L_\infty[0,1]$) is non-dentable, see Example 1.2.7).
- X^*, if X contains l_1 (because $(l_1)^* = l_\infty \subset X^*$).

Each of these examples of the spaces without the RNP stated above must negate at least one of those conditions stated under the overview heading. Which ones in particular? More on these will be discussed in the following chapter and reasons will be given as to why they lack the RNP.

Chapter 2

Spaces failing the RNP

In this chapter we discuss and characterise Banach spaces failing the RNP. The chapter is divided into three sections, namely, using dentability, decomposition, and separability (especially in dual spaces), as characterisations of Banach spaces failing the RNP, in sections one, two and three, respectively.

Examples of such spaces were already mentioned in previous sections. Some of the properties equivalent to a lack of RNP in Banach spaces will be compared to those discussed earlier.

2.1 Lack of the RNP and dentability

In this section we discuss the lack of the RNP in Banach spaces and characterise such spaces by dentability property or perhaps the lack of a dentable subset.

Notations

B_X denotes the closed unit ball of a Banach space X.

Theorem 2.1.1 [23, Theorem 2, p.160]

For a Banach space X, the following are equivalent:

1. X fails the RNP.

2. There exists a bounded non-dentable subset in X.

Proof

The result follows directly from Theorem 1.2.16. □

Remark 2.1.2

Considering the fact observed earlier , see Theorem 1.2.16, that, a Banach space possesses the RNP if and only if each of its non-empty bounded subsets is dentable, it is inevitable that the existence of a **bush** in any Banach space X would lead to the lack of the RNP in such a space, since a bush is non-empty bounded non-dentable subset, see Proposition 1.4.2. Bushes and the lack of the RNP shall be discussed in subsequent sections.

In fact, the equivalence of 1 and 2 in Theorem 2.1.1 (see [10, p.121]) had some interesting spin-offs:

- Lindenstrauss used it to show that the RNP implies the KMP.
- Huff and Morris [22] used it to show that KMP implies the RNP in dual spaces (this implication is still open for arbitrary non-dual Banach spaces).
- Edgar generalised Lindenstrauss' result by obtaining a Choquet-type representation theorem for bounded closed convex subsets of a Banach space with the RNP.

$L_\infty[0,1]$ is an example of a non-dentable Banach space, see Example 1.2.7, hence $L_\infty[0,1]$ fails the RNP.

Theorem 2.1.3 [23, Theorem 4, p.161]

X is a Banach space, the following are equivalent:

1. X does not have the RNP.
2. There exists a bounded closed convex subset in X which does not have an extreme point.
3. There exists a closed bounded convex set $A \subset X$ such that no nontrivial $f \in X^*$ attains its supremum on A (hence there exists no strongly exposed point in A).

The equivalence between 1 and 2 follows from Theorem 1.2.25 and Section 1.8 overview summary condition 14. The equivalence between 1 and 3 follows from Theorem 1.2.21. In the previous chapter we gave an overview of the conditions sufficient for a Banach space to have the RNP. Below we state few conditions sufficient for a Banach space to lack the RNP.

Proposition 2.1.4

A Banach space X fails the RNP if any one of the following holds:

1. There exists a closed linear subspace of X that fails the RNP.
2. There exists a separable closed linear subspace of X failing the RNP.
3. There exists a closed bounded convex non-dentable subset in X.
4. There exists a bounded non-dentable subset of X (see, [13, p.133]).
5. There exists a bounded non-σ-dentable subset in X (see, [13,p.132]).
6. There exists a non-empty closed bounded convex subset of X which is not a closed convex hull of its denting/strongly exposed/extreme points.
7. X has no BPP.
8. X has a bounded infinite δ-tree (i.e a sequence whose successive terms has a distance of at least $\delta > 0$ between them, see [13, Corollary 5, p.127]).

We need to comment only on 4, 5 and 8:

If a set is non-σ-dentable (similarly, non-s-dentable) then it is non-dentable. This follows from the fact dentable sets are s-dentable (see below Definition 1.2.8), and if a set is a bounded infinite δ-tree then it is non-σ-dentable because of the separation constant $\delta > 0$. The concept of δ-tree to be discussed thoroughly subsequently.

As a consequence of Theorem 1.2.39 we subsequently state conditions sufficient for a non-empty closed bounded and convex subset K of a Banach space X, to lack the RNP.

Proposition 2.1.5

For a non-empty bounded closed and convex subset K of a Banach space X to fail the RNP, the following are sufficient:

1. There exists a closed bounded convex separable subset of K failing the RNP.
2. K fails the MCP.
3. There exists a bounded non-s-dentable (hence non-c-dentable, hence non-dentable) set in K.
4. There exists a countable non-c-dentable subset in K.
5. There exists a closed bounded non-dentable convex subset of K.

2.2 Lack of the RNP in Dual spaces

This section is aimed at characterising those dual Banach spaces failing the RNP. Separability plays an important role in such a charaterisation.

Theorem 2.2.1 [13, **Theorem 6, p.195**]

If a Banach space X has a separable subspace whose dual is not separable, then X^* lacks the RNP.

The proof of Theorem 2.2.1 above follows from the result that if X^* has the RNP, then each separable subspace of X has a separable dual, see Theorem 1.5.5.

Corollary 2.2.2 [49, **Corollary 2, p.219**]

If X is a Banach space such that there exists a separable subspace Y of X, such that Y^* is non-separable (equivalently, X^* does not have RNP), then there exists a separable subspace Z of X^*, such that Z is not isomorphic to a subspace of a separable dual space.

Corollary 2.2.2 above follows from the fact that a Banach space X has a dual X^* with the RNP if and only if each of the separable subspaces of X has a separable dual (see Theorem 1.5.5). So, if we can find a separable subspace Z in X with non-separable dual, then we would have the lack of the RNP in X^*. Such Z is assumed to be separable in a dual space, but it is not a separable dual itself, and this is really interesting.

We have observed in the preceeding chapter, conditions sufficient for a dual space to possess the RNP. We now state conditions sufficient for a dual Banach space X^* to lack the RNP.

Proposition 2.2.3

Each of the following is sufficient for a dual Banach space X^* to lack the RNP:

1. X is not quasi-separable (that is, X has a separable subspace whose dual space is not separable).

2. X is not Asplund.

Proposition 2.2.3 follows from Definition 1.5.7 and Proposition 1.7.7:
X is an Asplund space if and only if X^* has the RNP if and only if X is quasi-separable.

2.3 Lack of the RNP, bushes and decompositions

In this section we discuss how the existence, or perhaps the lack of, bushes and decompositions in Banach spaces can influence the lack of the RNP in such spaces.

Definition 2.3.1 [24]

A bounded closed convex subset K of a Banach space X has the *point-of-continuity property (PCP)* if for each closed subset C of K, there exists a point $x \in C$ such that the weak and norm topologies (restricted to C) coincide at x.

Remark 2.3.2

It should be noted however that if every bounded closed convex subset of X has a PC then

X has the PCP but the converse does not hold in general. This follows from observing that not all closed bounded subsets C in the Definition 2.3.1 above are **convex**. If K in X has the PCP, and all such subset C in K are **convex**, then K would have the property called the convex-point-of continuity property, also known as the CPCP.

Proposition 2.3.3 [21, p.347]

A Banach space X fails the RNP if and only if it contains a bush.

Proof

Let X fail the RNP. Then X contains a bush (Theorem 1.4.4).

Conversely,

Let X contain a bush, then there exists a bounded non-dentable subset in X, see Proposition 1.4.2. Hence X fails the RNP, and this completes the proof. □

In the proof above we used the fact that a bush is a bounded non-dentable subset, proved in the preceeding chapter in which a bush was defined. As a consequence of Proposition 2.3.3 above, we believe the following Proposition is worth mentioning:

Proposition 2.3.4

A Banach space X has a bounded non-dentable subset if and only if X has a bush.

Proof

It follows from Theorem 2.1.1 and Proposition 2.3.3 that if X has a bounded non-dentable subset then X fails the RNP and hence contains a bush. Conversely, if X contains a bush then X has a bounded non-dentable set since a bush is bounded and non-dentable. □

Theorem 2.3.5 [25, p.256]

There is a Banach space X which is a subspace of a Banach space Z for which X and Z have the following properties:

1. Z has an UBFDD (hence is contained in a space with unconditional basis, [31,

p.51]).

2. X fails the RNP.

3. X has no subspace isomorphic with c_0.

4. X has PCP.

5. X fails KMP.

6. X does not have an UBFDD (hence $Z \backslash X$ has an UBFDD).

We need properties 1 and 2, that is, *the Banach subspace X of a Banach space Z with an UBFDD, fails RNP.*

Definition 2.3.6 [7, p.34]

A δ-bush (or simply a bush) which has exactly two branches at every point, each of which is the midpoint of its branching point, is called a *δ-tree*. Thus formally, a *δ-tree* is a sequence $(x_n)_{n\geq 1}$ such that $x_n = \frac{1}{2}x_{2n} + \frac{1}{2}x_{2n+1}$ and $\|x_{2n} - x_n\| = \|x_{2n+1} - x_n\| \geq \delta$ for each $n \in \mathbb{N}, \delta > 0$.

Remark 2.3.7

Bourgin [7, p.34] mentions that a closed bounded convex set K which contains a δ-tree for some $\delta > 0$, lacks the RNP. The reason is that, such K contains a non-dentable set if it contains a δ-tree for some $\delta > 0$, which means that K is not subset dentable. Thus K fails the RNP (see Theorem 1.2.39). The converse does not hold! That is, if K fails fails the RNP, it doesn't necessarily have a δ-tree for some $\delta > 0$: there exists a subspace B of $L_1[0,1]$ which lacks RNP, but which contains no bounded δ-tree for some $\delta > 0$, see [7, p.34].

We repeat the definition of an FDD in the following form:

Definition 2.3.8 [5, p.135]

$(P_n, M_n)_{n\geq 1}$ is a *finite dimensional Schauder decomposition* for a Banach space X if and only if each P_n is a continuous linear projection of X onto the finite dimensional M_n,

$P_nP_m = 0$ if $n \neq m$ and $x = \sum_{i=1}^{\infty} P_i(x)$ for each $x \in X$. The partial sum operators S_n are defined by $S_n = \sum_{i=1}^{n} P_i$.

Since $(S_n)_{n\geq 1}$ is pointwise convergent (that is, $S_n(y) \to S_1(y)$, for all $y \in X$), and S_1 an identity operator, it is uniformly bounded (that is, $\sup_n \| S_n(x)\| < \infty$ for all S_n in $(S_n)_{n\geq 1}$).

We denote by $G(M_n, n)$ the number $\sup_n \| S_n\|$, which is called the *Grynblum constant* of the decomposition.

Theorem 2.3.9 [5, **Theorem 2, p.136**]

If X fails the RNP, then for every $\lambda > 1$, there exists a closed subspace X_λ of X without the RNP and with a finite dimensional Schauder decomposition with Grynblum constant at most λ.

Proof

Indeed, X_λ fails the RNP by the Overview in section 1.8. If we take $P_1 = S_1$, $P_2 = S_2 - S_1$, $P_3 = S_3 - S_2, \ldots, P_{n+1} = S_{n+1} - S_n$. Then P_n are mutually disjoint for each n, and $(P_n, P_nX)_{n\geq 1}$ is a finite dimensional Schauder decomposition of X with $G(P_nX, n) \leq \lambda$. This completes the proof. $\square$

The reader is referred to [5] and references stated there, for a more elegant proof. Theorem 2.3.9 can be rephrased as follows: *If X fails the RNP, then there exists a subspace X_λ of X with the FDD failing the RNP.* This result follows from the fact that, a Banach space X has the RNP if and only if each of its subspaces with the FDD has the RNP as well [5, p.135].

Chapter 3

The Krein-Milman Property

In this chapter, we discuss the Krein-Milman Property (KMP) and properties that are equivalent to it. We first state the Krein-Milman theorem from which the KMP originates. Extreme points, dentability and separability are used to characterise Banach spaces with the KMP, as well as the existence of the RNP is such spaces.

3.1 The Krein-Milman Theorem

In this section we state the Krein-Milman Theorem and give its proof as it is important for the rest of this chapter. We use the formulation by Rudin [44]:

Krein-Milman Theorem 3.1.1 [44, p.75]
Let X be a Banach space such that $f : X \to \mathbb{C}$ is injective for all $f \in X^*$, (that is, X^* *separates* points of X). If K is a non-empty compact convex set in X, then K is the closed convex hull of its extreme points, that is, $K = clco(E(K))$, where $E(K)$ denotes the set of all extreme points of K.

The proof of the above theorem can be found in [44, p.75]. We shall instead give the proof by Diestel [12], which we feel is so elegant and easy to follow it is noteworthy. We also state the theorem the same way Diestel [12, p.148] did.

Definition 3.1.2 [12, p.148], [44], [23, p.157]

A subset A of a convex set B is *extremal (or an extreme set)* in B if A is a non-empty convex subset of B with the property that should $x, y \in B$ and $\lambda x + (1-\lambda)y \in A$, for some $\lambda \in (0,1)$, then $x, y \in A$.

The *extreme points* of a set K are the extreme sets that consist of just one point, and each extreme point in K cannot be written as a convex combination of two distinct points of K.

Theorem 3.1.3 [12, p.148]

Let K be a non-empty compact convex subset of a Banach space X. Then K has an extreme point and is in fact the closed convex hull of its extreme points.

Proof

Let Δ be the collection of all non-empty closed extremal subsets of K (hence $K \in \Delta$). Since K is convex and it is a convex subset of itself, then for any $x, y \in K$, $\lambda x + (1-\lambda)y \in K$ for some $\lambda \in (0,1)$. Hence K is an extremal subset of K, and since it is closed, it is in Δ. There is an ordering in Δ defined by $K_1 \leq K_2$ whenever $K_2 \subseteq K_1$. The compactness of K along with the classical Zorn's lemma, produce a maximal $K_0 \in \Delta$.

We claim that K_0 is a singleton:

Indeed if $x, y \in K_0$, such that $x \neq y$, there is a linear continuous function f on X with $f(x) < f(y)$. Then $K_0 \cap \{z : f(z) = \max f(K_0)\}$ is a proper closed extremal subset of K_0. Note that f is one-to-one, hence $\{z : f(z) = \max f(K_0)\}$ is a singleton. This implies that $K_0 \cap \{z : f(z) = \max f(K_0)\}$ is a singleton and a proper subset of K_0. Hence it is not possible that $K_0 \cap \{z : f(z) = \max f(K_0)\}$ is an extremal subset of K_0, and we have a contradiction.

Hence K has an extreme point.

We now show that $K = clco(E(K))$, $K \in \Delta$:

Let C be the closed convex hull of set of extreme points of K. We need to show that $C = K$:

Let however $x \in K \backslash C$:

Then there exists a linear continuous function f on X such that $\max f(C) < f(x)$. Looking at $\{z \in K : f(z) = \max f(K)\}$, we should see a closed extremal subset of K which entirely misses C, that is, $\{z \in K : f(z) = \max f(K)\} \cap C = \emptyset$. On the other hand, each closed extremal subset of K contains an extreme point on K. Hence there exists no extreme point outside C, hence $K \backslash C = \emptyset$. Then all extreme points are in C, hence $C = K$. □

3.2 The KMP and extreme points

In this section we discuss how the existence of extreme points in Banach spaces and/or in their bounded subspaces influence the existence of the KMP in such spaces.

Definition 3.2.1 [13, **Definition 6, p.190**]

A Banach space X has the *Krein-Milman Property (KMP)* if every closed bounded convex subset of X is the closed convex hull of its extreme points.

Theorem 3.2.2 [13, **Theorem 7, p.190**]

If each non-empty closed bounded convex subset of a Banach space X contains an extreme point then X has the KMP.

Proof

Let B be any non-empty closed bounded convex subset of X, and let $E = clco(E(B))$, where $E(B)$ denotes the set of extreme points of B.

We need to prove that $E = B$:

Suppose $E \neq B$, then by the separation property of X^* and the Bishop-Phelps theorem (Theorem 1.3.4), there exists $x^* \in X^*$ such that $\sup x^*(E) < \sup x^*(B) = x^*(b_0)$ for some $b_0 \in B$. $(E \subset B)$. By the choice of x^* and b_0, the set $C = \{b \in B : x^*(b) = \sup x^*(B)\}$ is non-empty closed bounded convex in X. By hypothesis, C has an extreme point and by the choice of x^*, its easy to see that an extreme point of C is an extreme point of B (that is, $C \cap B \neq \emptyset$ because they have an extreme point in common). This contradicts $C \cap E = \emptyset$, since $E \subset B$, from assumption. Hence $E = B$. □

This theorem above was also stated by Diestel [11, Proposition 1, p.230].

Theorem 3.2.3 [11, Theorem 1, p.231]

If a Banach space X possesses the RNP then X possesses the KMP.

Proof

If B be a non-empty closed bounded convex subset of X. Then B and all its subsets are dentable. Let $\varepsilon > 0$ be given. Since B is dentable, there exists an $x_1 \in B$ such that $x_1 \notin clco(B \backslash B_{\frac{\varepsilon}{2}}(x_1)) = C_1$. By the Hahn-Banach theorem, there exists $f \in X^*$ such that $\sup f(C_1) < f(x_1)$. By the Bishop-Phelps Theorem 1.3.4, we can select $f \in X^*$ such that $\sup f_1(C_1) < \sup f_1(B) = f_1(b_0)$ for some $b_0 \in B$.

Let $B_1 = \{b \in B : f_1(b) = f_1(b_0)\}$. Then B_1 is non-empty, closed and bounded convex subset of B. Since B is dentable, so is B_1 by assumption. Moreover B has a norm diameter $\leq \varepsilon$. Thus there exists an $x_2 \in B_1$ such that $x_2 \notin clco(B_1 \backslash B_{\frac{\varepsilon}{4}}(x_2)) = C_2$. By the Hahn-Banach theorem, there exists $f \in X^*$ which separates x_2 from $\sup f_2(C_2) < \sup f_2(B_1) = f_2(b_1)$ for some $b_1 \in B_1$.

Let $B_2 = \{b \in B_1 : f_2(b) = f_2(b_1)\}$. Then B_2 is a non-empty closed bounded convex subset of B_1. Moreover, B_2 has a norm diameter less than $\frac{\varepsilon}{2}$. By induction we obtain a sequence $(B_n)_{n\geq 1}$ of non-empty closed bounded convex subsets of B such that $B \supseteq B_1 \supseteq B_2 \supseteq ... \supseteq B_n \supseteq ...$ where the norm diameter of B_n is $\leq \frac{\varepsilon}{2^n}$ and B_{n+1} is a face of B_n. Let $\{x\} = \cap_n^\infty B_n$. Then x is an extreme point of B. In fact, if $x = \frac{1}{2}y + \frac{1}{2}z$ with $y, z \in B$, then for each n $f_n(x) = \frac{1}{2}f_n(y) + \frac{1}{2}f_n(z)$. Hence, if $x \in \cap_n^\infty B_n$, we have $y, z \in \cap_n^\infty B_n$ which means that $x = y = z$ and x is an extreme point. $\square$

Our main aim is to establish conditions under which the RNP and the KMP are equivalent. Since we have established thus far that the RNP implies the KMP (see Theorem 3.2.3), the conditions that are sufficient for a Banach space to have the RNP are also sufficient for a Banach space to have the KMP. We state some of them below.

Definition 3.2.4 [7, Definition 3.1.2, p.40]

A closed convex subset K of a Banach space X is said to have the *Krein-Milman Property (KMP)* if each closed bounded convex subset A of K satisfies $A = clco(E(A))$.

Proposition 3.2.5

Each of the following conditions is sufficient for a Banach space X to possess the KMP:

1. Every closed linear subspace of X has the RNP
2. Every bounded subset of X is dentable/σ-dentable/s-dentable
3. Every separable closed linear subspace has the RNP
4. Every non-empty closed bounded convex subset of X is a closed convex hull of its denting/strongly exposed/extreme points.

Proofs of these conditions follow from the fact that X has the KMP if it has the RNP, from Theorem 3.2.3 above. In addition, if any bounded closed and convex subset of X has the RNP (hence dentable), then X has the RNP, hence X has the KMP, see Theorem 1.2.39. Thus each of the conditions in Theorem 1.2.39, is sufficient for the existence of the RNP in X, hence of the KMP in a Banach space X.

3.3 The KMP and dual spaces

In this section we discuss the KMP in dual Banach spaces, and investigate what characteristics are needed for a dual of a Banach space to possess the KMP. The separability property is used mostly to determine such characteristics.

Lemma 3.3.1 [Stegall; see 13, p.194]

Let X be a separable Banach space whose dual X^* is non-separable. Then for each $\varepsilon > 0$ there is a non-empty weak*-compact subset Δ of the unit ball B_{X^*} of X^*, a Haar system of closed and open subsets $(C_n)_{n\geq 1}$ of Δ and a sequence $(x_n)_{n\geq 1}$ in X such that $\|x_n\| \leq 1 + \varepsilon$ for all $n \in \mathbb{N}$ and $|x^*(x_n) - \chi_{C_n}(x^*)| < \varepsilon 2^{-k}$ for all $k = 0, 1, 2, ...$, all n with

$2^k \leq n < 2^{k+1}$ and all $x^* \in \Delta$. In addition, the sequence $(C_n)_{n\geq 1}$ may be chosen so that the weak*-diameter of C_n tends to zero as n approaches infinity.

Theorem 3.3.2 [Stegall; see 13, p. 195]

If a Banach space X has a separable subspace Y whose dual is not separable, then there is a bounded infinite δ-tree in X^*. Consequently, if a Banach space X has a subspace whose dual is non-separable, then X^* lacks the RNP.

Proof

Employ Lemma 3.3.1 to produce a non-empty weak*-compact subset Δ of X^*, a Haar system $(C_n)_{n\geq 1}$ of subsets of Δ (with $C_1 = \Delta$) and a sequence $(y_n)_{n\geq 1}$ in Y such that $\|y_n\| \leq \frac{9}{8}$ for all $n \in \mathbb{N}$ and such that $|y^*(y_n) - \chi_{C_n}(y^*)| < 2^{-k-3}$ for all $k = 0, 1, 2, ...,$ all n with $2^k \leq n < 2^{k+1}$ and all $y^* \in \Delta$. Also, choose $(C_n)_{n\geq 1}$ so that the limit of the weak*-diameter of C_n tends to zero as n approaches ∞. Let Σ be the σ-field generated by $(C_n)_{n\geq 1}$ and let μ be the unique countably additive finite measure on Σ such that $\mu(C_n) = 2^{-k}$ for $2^k \leq n < 2^{k+1}$. Each $\phi \in C(\Delta)$ is Σ-measurable because $\delta(C_n)$ approaches zero as n approaches ∞. Then, if $y \in Y$, it follows that $(Ty)(y^*) = y^*(y)$, $y^* \in \Delta$, defines a linear operator $T : Y \to L_\infty(\mu)$ which is bounded. Since $L_\infty(\mu)$ is injective, T has a bounded linear extension, again denoted by T, to all of X. In this notation, the condition $|y^*(y_n) - \chi_{C_n}(y^*)| < 2^{-k-3}$ for all $k = 0, 1, 2, ...,$ all n with $2^k \leq n < 2^{k+1}$ and all $y^* \in \Delta$ translates into

$$\|T(y_n) - \chi_{C_n}\|_\infty \leq \frac{\mu(C_n)}{8} \tag{1}$$

for all n. By considering $L_1(\mu)$ as a subspace of $(L_\infty(\mu))^* = L_\infty^*(\mu)$, consider the sequence $(\frac{T^*(\chi_{C_n})}{\mu(C_n)})_{n\geq 1}$. This sequence is bounded, by the definition of μ and the fact that T is bounded. The fact that $(C_n)_{n\geq 1}$ is a Haar system, and the definition of μ guarantee that this sequence is a tree in X^*. We proceed by showing that this is a $\frac{7}{18}$-tree in X^*. To this end note that

$$\begin{aligned}
\|\frac{T^*(\chi_{C_j})}{\mu(C_j)}) - \frac{T^*(\chi_{C_{2j+1}})}{\mu(C_{2j+1})})\| &= \frac{1}{\mu(C_j)}\|T^*(\chi_{C_j}) - 2T^*(\chi_{C_{2j+1}})\| \\
&= \frac{1}{\mu(C_j)}\|T^*(\chi_{C_{2j}}) - T^*(\chi_{C_{2j+1}})\| \\
&\geq \frac{8}{9\mu(C_j)}|T^*(\chi_{C_{2j}} - \chi_{C_{2j+1}})(y_{2j})|
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{8}{9\mu(C_j)} \int_{C_j} T(y_{2j})(\chi_{C_{2j}} - \chi_{C_{2j+1}})\ \mathrm{d}\mu \\
&\geq \tfrac{8}{9\mu(C_j)} [\int_{C_j} \chi_{C_{2j}}(\chi_{C_{2j}} - \chi_{C_{2j+1}})\ \mathrm{d}\mu \\
&\quad - \int_{\Delta} |T(y_{2j}) - \chi_{C_{2j}}||\chi_{C_{2j}} - \chi_{C_{2j+1}}|\ \mathrm{d}\mu \\
&\geq \tfrac{8}{9\mu(C_j)} [\tfrac{\mu(C_j)}{2} - \tfrac{1}{2}\tfrac{\mu(C_j)}{8} \cdot \mu(C_j)] \\
&= \tfrac{4}{9} - \tfrac{1}{18}\mu(C_j) \\
&\geq \tfrac{7}{18}.
\end{aligned}
$$

A similar method can be used to show that $\|\frac{T^*(\chi_{C_j})}{\mu(C_j)} - \frac{T^*(\chi_{C_{2j}})}{\mu(C_{2j})})\| \geq \frac{7}{18}$. This proves the first assertion. The second assertion follows from the fact that if Y^* is not separable, then it (and hence X^*) contains a bounded infinite δ-tree, which is by definition a bush, and by Theorem 1.4.4, X^* lacks the RNP. □

Theorem 3.3.3 [13, p.196]

If a Banach space X has a separable subspace Y whose dual Y^* is non-separable, then X^* lacks the KMP.

Proof

We employ the following from the proof of Theorem 3.3.2: the sequences $(y_n)_{n\geq 1}$ and $(C_n)_{n\geq 1}$ and the operator $T : X \to L_\infty(\mu)$. The space $L_1(\mu)$ will be regarded as a subspace of $L_\infty^*(\mu)$, and $L_\infty^*(\mu)$ will be regarded as the space of all finitely additive measures on Σ that vanishes when μ vanishes, equipped with the variation norm. Let C be the weak*-closed convex hull of $\{\frac{\chi_{C_n}}{\mu(C_n)}\}$ in $L_\infty^*(\mu)$, let $x_n^* = T^*(\frac{\chi_{C_n}}{\mu(C_n)})$, let D be the weak*-closed convex hull of the set $\{x_n^*\} \in X^*$, and let $K = \{x^* \in D : \lim_{n\to\infty} x^*(y_n) = 0\}$. Then C and D are weak*-compact convex subsets of their ambient spaces. The set K is convex (by the definition of convexity) and K is bounded (being a subset of D).

K is non-empty:

$$
\begin{aligned}
|x_n^*(y_m)| &= |T(\tfrac{\chi_{C_n}}{\mu(C_n)})(y_m)| \\
&= \tfrac{1}{\mu(C_n)} |\int_{C_n} T(y_m)\mathrm{d}\mu| \\
&\leq |\tfrac{1}{\mu(C_n)} \int_{C_n} |T(y_m) - \chi_{C_m}|\mathrm{d}\mu + \tfrac{1}{\mu(C_n)} \int_{C_n} \chi_{C_m}\mathrm{d}\mu| \\
&\leq \tfrac{\mu(C_m)}{8} \cdot \tfrac{\mu(C_n)}{\mu(C_n)} + \tfrac{\mu(C_m \cap C_n)}{\mu(C_n)}
\end{aligned}
$$

from line (1) in Theorem 3.3.2. Now $\lim_{m\to\infty} \mu(C_m) = 0$ implies that $\lim_{n\to\infty} x_n^*(y_n) = 0$. This shows that $x_n^* \in K$; so $K \neq \emptyset$

K is norm closed:

Let x^* be a norm cluster point of K and let $\varepsilon > 0$. Then there exists $y^* \in K$ such that $\|x^* - y^*\| < \frac{\varepsilon}{2\|y_n\|}$ for all $n \in \mathbb{N}$. Since $y^* \in K$ there is a positive integer m such that $|y^*(y_j)| < \frac{\varepsilon}{2}$ for $j \geq m$. Thus, when $j \geq m$,

$$\begin{aligned}|x^*(y_j)| &\leq |x^*(y_j) - y^*(y_j)| + |y^*(y_j)| \\ &\leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Hence, $x^* \in K$ and K is norm closed.

$E(K) = \emptyset$:

Note that, from what we have above

$$\begin{aligned}x_n^*(y_m) &= \tfrac{1}{\mu(C_n)} \int_{C_n} T(y_m)\mathrm{d}\mu \\ &= \tfrac{1}{\mu(C_n)} [\int_{C_n} (T(y_m) - \chi_{C_m})\mathrm{d}\mu + \mu(C_m \cap C_n)] \\ &\geq \tfrac{-\mu(C_m)}{8}\end{aligned}$$

since $\|T(y_m) - \chi_{C_m}\|_\infty < \frac{\mu(C_m)}{8}$. Since $\lim\limits_{n\to\infty} \mu(C_n) = 0$, it follows that $\liminf\limits_{m} x^*(y_m) \geq 0$ for all $x^* \in D$. From this it follows that $E(K) \subset E(D)$. We now show that $E(D) \cap K = \emptyset$. Let $e^* \in E(D)$. Since $T^*(C) = D$, it follows that $C \cap (T^*)^{-1}(\{e^*\})$ is a non-empty convex weak*-closed subset of C. From the Krein-Milman Theorem we obtain an extreme point β of $A = C \cap (T^*)^{-1}(\{e^*\})$. If $p \in E(A)$, let $p = (1-\alpha)x + \alpha y$ with $\alpha \in [0, 1]$ and $x, y \in C$. Then $T^*(p) = e^* = (1-\alpha)T^*(x) + \alpha T^*(y)$ from which it follows that $T^*(p) = T^*(x) = T^*(y)$, so that $p \in E(C)$. The weak*-closure of $\{\frac{\chi_{C_n}}{\mu(C_n)}\}$ in $L_\infty^* \mu$ is a weak*-compact set whose closed convex hull is C. Since $\beta \in E(A)$, we have that $\beta \in E(C)$, it follows from Milman's Theorem that the finitely additive measure β is in the weak*-closure of $\{\frac{\chi_{C_n}}{\mu(C_n)}\}$. Hence there is a net $(\frac{\chi_{C_\alpha}}{\mu(C_\alpha)})_{\alpha\in\Lambda}$ in the set $\{\frac{\chi_{C_n}}{\mu(C_n)}\}$ such that $\beta(E) = \lim\limits_{\alpha} \int_E [\frac{\chi_{C_\alpha}}{\mu(C_\alpha)}]\mathrm{d}\mu$ for all $E \in \Sigma$. In particular, $\beta(C_m) = \lim\limits_{\alpha} \int_{C_m} [\frac{\chi_{C_\alpha}}{\mu(C_\alpha)}]\mathrm{d}\mu$ for all m. But since $\frac{\chi_{C_k}}{\mu(C_k)} = \frac{1}{2}[\frac{\chi_{C_{2k}}}{\mu(C_{2k})} + \frac{\chi_{C_{2k+1}}}{\mu(C_{2k+1})}]$ it follows that $\frac{\chi_{C_k}}{\mu(C_k)} \notin E(C)$, and so $\frac{\chi_{C_\alpha}}{\mu(C_\alpha)} \notin E(C)$. Thus, the net $(\int_{C_m} \frac{\chi_{C_\alpha}}{\mu(C_\alpha)}\mathrm{d}\mu)_{\alpha\in\Lambda}$ is a convergent net of 0's and 1's. Then, $\beta(C_m) = 0$ or 1 for all m. In addition, for all k, $\sum_{n=2^k}^{2^{k+1}-1} \beta(C_n) = \beta(\Delta) = \lim\limits_{\alpha} \int_\Delta \frac{\chi_{C_\alpha}}{\mu(C_\alpha)}\mathrm{d}\mu = 1$. Then, $\beta(C_m) = 1$ for infinitely many m. If $\beta(C_m) = 1$, then

$$|e^*(y_m) - 1| = |e^*(y_m) - \beta(C_m)|$$

$$= |T^*\beta(y_m) - \beta(C_m)|$$
$$= |\int_\Delta T(y_m) - \chi_{C_m} \, d\beta|$$
$$\le \|\beta\| \|T(y_m) - \chi_{C_m}\|$$
$$\le \tfrac{1}{8}$$

since $\|\beta\| \le 1$ and $\|T(y_m) - \chi_{C_m}\| \le \frac{\mu(C_m)}{8} \le \frac{1}{8}$.
Then, $e^*(y_m) \ge \frac{7}{8}$ for infinitely many m. Thus $\lim\limits_m e^*(x_m) \ne 0$, and $e^* \notin K$. This then shows that $E(D) \cap E(K) = \emptyset$, hence $E(K) = \emptyset$ since $E(K) \subset E(D)$. We deduce from Definition 3.2.1 that X^* lacks the KMP. □

Thus, if X^* has the KMP, every separable subspace Y of X has a separable dual Y^*.

This means that (by Theorem 1.5.5), if X^* lacks the RNP, then X^* lacks the KMP. We shall discuss more of the dual spaces lacking the KMP later in the subsequent sections. But then, if X^* has the RNP, X^* has the KMP.
Theorem 3.3.3 above can be reformulated as follows:
If a Banach space X is non-quasi-separable, then X^* lacks the KMP. That is, if X^* has the KMP, then X is quasi-separable (see Definition 1.5.7).

If $f \in X^*$, $f \ne 0$, let $M(D, f) = \sup\{f(x) : x \in D\}$.

Proposition 3.3.4 [7, Proposition 3.1.1, p.39]
Let C be a closed convex subset of a Banach space X. If each closed bounded convex subset of C has at least one extreme point, then each such set is the closed convex hull of its extreme points.

Proof
Let K be a closed bounded convex subset of C, let $K_1 = clco(E(K))$, and suppose that K_1 is a proper subset of K. By the separation theorem there is an $f \in X^*$ such that $M(K_1, f) < r < M(K, f)$ for some $r \in \mathbb{R}$. Let $K_2 = f^{-1}[r, \infty) \cap K$. Let $y \in E(K_2)$, where $E(K_2) \ne \emptyset$ by hypothesis. Since $y \notin K_1$, $y \notin E(K)$, and thus y is interior to the

line segment $[x, z]$ in K with $f(x) < r < f(z)$. Let w be the endpoint of the ray from x through z intersected with K. Then $f(w) > r$, so that $w \notin E(K)$. On the other hand, if w were interior to some non-trivial line segment in K it would follow that $y \notin E(K)$, a contradiction. $\square$

Proposition 3.3.5 [7, Proposition 3.1.3, p.40]

Reflexive Banach spaces have the KMP.

Proof

If we let X to be a reflexive Banach space, and K be any closed bounded convex subset of X, then K, being weakly compact and convex, has at least one extreme point, and hence $E(K)$ is non-empty. Hence K has at least one extreme point, and thus it is the closed convex hull of its extreme points, by Proposition 3.3.4, that is $K = clco(E(K))$. Since K was arbitrarily chosen, every such K in X is the closed convex hull of its extreme points. Thus X has the KMP. $\square$

Remark 3.3.6

Proposition 3.3.5 can also be established using our earlier result (see Example 1.6.9) that reflexive Banach spaces have the RNP. Since RNP implies the KMP, a reflexive Banach space has the KMP.

Subsequently we look at the subsets of a dual Banach space having the KMP, and then at the dual Banach spaces having the KMP.

Definition 3.3.7 [7, Definition 3.5.3, p.54]

A bounded subset $A \subset X^*$ is *weak*-dentable* if and only if for each $\varepsilon > 0$ there exists a point $x_\varepsilon \in A$ such that $x_\varepsilon \notin w^*\text{-}cl(co(A \backslash B_\varepsilon(x_\varepsilon)))$, where $w^*\text{-}cl(B)$ is the weak* closure of B.

Theorem 3.3.8 [7, Theorem 4.2.13, p.91]

Let C be a weak*-compact convex subset of a dual space X^*. Then the following are equivalent:

1. C has the RNP.

2. C has the KMP.

3. If Y is a separable subspace of X, then $C|_Y$ is separable, where $C|_Y = \{f|_Y : f \in C\} \subset Y^*$.

4. C contains no δ-tree for any $\delta > 0$.

5. C is subset weak*-dentable.

Proof

$1 \Rightarrow 2$ Since C is a weak*-compact convex subset of X^*, it follows that C is a closed bounded convex subset of X^* (see, Rudin [44, p.66]). Then C has the KMP if C has the RNP (by Lindenstrauss theorem , see [7, p.49]).

$1 \Leftrightarrow 4$ If C contains a bounded δ-tree for some $\delta > 0$, then C contains is a bounded non-dentable set. Thus, C is not subset dentable and fails the RNP. The converse also holds, see Proposition 2.1.4 (8). (A set C is subset dentable if every bounded subset of C is dentable.)

$3 \Rightarrow 5$ Let C have a subset, say F, which is not weak*-dentable. Since $F \subset C \subset X^*$, X^* lacks the RNP. Hence there exists a separable subset Y in X such that Y^* or $C|_Y$ is non-separable, see Theorem 1.5.5. In addition, see [7, p.94], where it is shown that the negation of 5 implies the negation of 3.

$5 \Rightarrow 1$ Since weak*-dentable sets are dentable, then the fact that C is subset weak*-dentable implies that it is subset dentable. Hence C has the RNP (see Theorem 1.2.39).

$3 \Rightarrow 1$ Let $B = \{h_i : i \in \mathbb{N}\}$ be a countable subset of C and let $A \subset B_X$, be a countable set with $\|h_i - h_j\| = \sup_{x\in A}(h_i(x) - h_i(x))$, for all $i, j \in \mathbb{N}$, where B_X is a closed unit ball in X. Let Y be the closed span of A; then Y is separable. By hypothesis $C|_Y$ is separable and since $C|_Y$ is weak*-compact and convex, $C|_Y$ is subset s-dentable (a weak*-compact separable subset of a dual space is subset s-dentable [7, p.71]). But for id:$Y \to X$, the inclusion map id*:$X^* \to Y^*$ is an affine isometry on B and id*$(B) = B|_Y \subset C|_Y$ is s-dentable. Hence B is itself s-dentable. Since B is an arbitrary countable subset in C which is s-dentable, C has the RNP, by Theorem 1.2.39.

$5 \Rightarrow 4$ Suppose C contains a δ-tree for some $\delta > 0$. Then C fails to be subset dentable and hence fails to be subset weak*-dentable. Thus C contains no δ-tree for some $\delta > 0$, if it is subset weak*-dentable.

$4 \Rightarrow 5$ If C contains no δ-tree for some $\delta > 0$, then it is subset dentable, hence subset weak*-dentable.

$4 \Rightarrow 3$ If C contains no δ-tree for some $\delta > 0$, then C is subset dentable and hence C has the RNP. Hence, for every separable subset Y of X, Y^* is in X^*. Hence $C|_Y$ is separable. □

Properties 1 and 2 are equivalent and this will be discussed and proved in chapter six. The following theorem is a global translate of more local theorems stated by Bourgin [7]:

Theorem 3.3.9 [7, Theorem 4.4.1, p.111]

For a dual space X^*, the following are equivalent:

1. X^* has the RNP.
2. X^* has the KMP.
3. If Y is a separable subspace of X, then Y^* is separable.
4. X^* contains no bounded δ-tree for any $\delta > 0$.
5. For each weak*-compact convex subset of K of X^*, $K = w^*\text{-}cl(co(w^*\text{-}SE(K)))$.
6. Each bounded subset of X^* is weak*-dentable.

The above Theorem 3.3.9 is an extension of results on a subset of a dual Banach space (Theorem 3.3.8 above) to a dual Banach space itself. In fact, $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5 \Leftrightarrow 6$ follows from Theorem 3.3.8 applied to the closed unit ball B_{X^*} of X^*.

Since all finite dimensional normed spaces and all Hilbert spaces have the RNP, all finite dimensional normed spaces and all Hilbert spaces have the KMP. A proof to these follows from the fact that the RNP implies the KMP.

As we have observed thus far that, in dual spaces the RNP implies the KMP, then conditions sufficient for a dual Banach space to possess the RNP are also sufficient for a dual Banach space to possess the the KMP.

Proposition 3.3.10

The following conditions are sufficient for a Banach space X^* to possess the KMP.

1. Every separable subspace of X has a separable dual (Theorem 1.5.5).

2. X is quasi-separable (Definition 1.5.7).

3. Every continuous linear closed image of X is quasi-separable (Remark 1.5.8).

4. X^* is isomorphic to a subspace of a wcg Banach space (Theorem 1.6.5).

5. X^* is wcg (weakly compactly generated) (Corollary 1.6.7).

6. X is an Asplund space (Proposition 1.7.6).

These conditions follow from the fact that the RNP implies the KMP in dual Banach space (see Theorems 1.5.5 and 3.3.9), and Proposition 1.8.1.

Examples 3.3.11

The following are examples of spaces with the KMP:

- Reflexive spaces (Proposition 3.3.5).

- weakly compactly generated (dual) spaces (e.g. reflexive Banach spaces).
- dual spaces with the RNP.
- l_1.
- All separable dual spaces.

Remark 3.3.12

It is noteworthy that in any Banach space X, the RNP implies the KMP. If a Banach space is a dual space, the KMP implies the RNP as well. It is then our intention to explore conditions under which the KMP would implies the RNP in any Banach space.

3.4 The KMP, decompositions and bushes

In this section we discuss how decompositions and bushes influence the existence of the KMP in Banach spaces.

As a consequence of Theorems 1.4.4 and 3.2.2, we have the following:

Proposition 3.4.1

If X contains no bush, then X has the KMP.

As a consequence of Theorems 1.4.8 and 3.2.2, we have the following:

Proposition 3.4.2

If X has a boundedly complete Schauder basis, then X has the KMP.

As a consequence of Theorem 1.4.9 and Theorem 3.2.2, we have:

Proposition 3.4.3

If every subspace of a Banach space X with FDD has the RNP, then X has the KMP.

Consequently, we formulate the following proposition as an overview of these above:

Proposition 3.4.4

Let X be a Banach space. Then X has the KMP if either one of the following holds,

1. X contains no bush.

2. X has a boundedly complete Schauder basis.

3. Every subspace of X with the FDD has the RNP.

The proof for this proposition follows from the preceeding propositions.

Chapter 4

Lack of the KMP in Banach spaces

In this chapter we deal with those Banach spaces that fail the Krein-Milman property. All these spaces fail the Radon-Nikodým property as well. Lack of extreme points, lack of separable dual subspace and the existence of non-separable dual spaces will be used to characterise Banach space without the KMP.

4.1 Lack of the KMP and extreme points

The aim of this section is to illustrate the lack of the KMP in Banach spaces by showing the lack of extreme points in such spaces.

Example 4.1.1 [7, p.38]

The Banach space c_0, of all sequences converging to zero, fails the KMP:

It is sufficient to show that, for K, the closed unit ball in c_0, $E(K) = \emptyset$:

It is clear that K is convex and that an interior point of K cannot be an extreme point of K. Consider then a boundary point $x = (x_i) \in K$. Then $\| x \| = 1$ and $\lim_{i\to\infty} x_i = 0$. Select a term x_n of x such that $|x_n| < \frac{1}{2}$. Let $\alpha = (y_i)$, $\beta = (z_i)$ be the points of K in c_0 defined by the following equations, $x_i = y_i = z_i$ for $i \neq n$, $y_n = x_n - \frac{1}{2}$ and $z_n = x_n + \frac{1}{2}$. Then $\alpha \neq \beta, \alpha \neq x \neq \beta$, $\| \alpha \| = \| \beta \| = 1$, $\lim_{i\to\infty} z_i = \lim_{i\to\infty} y_i$ and $x = \frac{1}{2}\alpha + \frac{1}{2}\beta$. Then for any arbitrary boundary point x in K, we have that $x \notin E(K)$, hence $E(K) = \emptyset$. By Definition 3.2.1, c_0 fails the KMP.

Example 4.1.2 [7, p.18]

Consider the Banach space $C[0,1]$ of continuous functions on $[0,1]$, with $\|f\| = \max\{|f(t)| : t \in [0,1],\}$ for $f \in C[0,1]$.

The closed unit ball K of $C[0,1]$ is s-dentable, non-dentable and fails the KMP:

K has exactly two extreme points, namely, the functions $f_1, f_2 \in C[0,1]$ defined by $f_1(x) = 1, f_2(x) = -1$ for any x in $[0,1]$.

K is non-dentable:

Suppose $f \in K$ and for any $n \in \mathbb{Z}^+$ choose functions $f_1^n, ..., f_n^n$ in K so that $f_i^n(t) = f(t)$ for $t \notin [\frac{i-1}{n}, \frac{i}{n}]$ and $|f_i^n(t_i^n) - f(t_i^n)| > \frac{1}{2}$ for some $t_i^n \in (\frac{i-1}{n}, \frac{i}{n})$. Then $\|f_i^n - f\| > \frac{1}{2}$ for $i = 1, ..., n$ and yet $\| \sum_{i=1}^n \frac{1}{n} f_i^n - f\| \leq \frac{2}{n}$. It follows that $f \in clco(K \backslash B_{\frac{1}{2}}(f))$ since n was arbitrary. Hence K is not dentable (and thus $C[0,1]$ fails the RNP).

K is s-dentable:

This follows by taking either extreme points ($f_1 or f_2$) to be f_ε in Definition 1.2.8.

K fails the KMP:

Let $C_1 = \{f \in K : f(0) = 0\}$. Clearly C_1 is a closed bounded convex subset of K, moreover $E(C_1) = \emptyset$. That is, for any $f \in C_1$, pick $a, \varepsilon \in \mathbb{R}^+$ such that $|f(t)| < 1 - \varepsilon$ for $t \in [0,a]$, let $g(t) = \varepsilon \sin(\frac{\pi t}{a})$, $t \in [0,a]$, and $g(t) = 0, t \notin [0,a]$, then $f \pm g \in C_1$, hence $\frac{1}{2}(f+g) + \frac{1}{2}(f-g) = f \notin E(C_1)$.

Since $f \in C_1$ was arbitrarily chosen, then $E(C_1) = \emptyset$, Thus K contains a closed bounded convex subset without extreme points, hence K does not have KMP. In fact, $C[0,1]$ fails the KMP, and thus fails the RNP.

We state below, conditions that are consequences of a lack of the KMP in a Banach space X.

That is, the lack of the KMP in a Banach space implies each of the following:

1. There exists a closed linear subspace failing the RNP.

2. There exists a separable closed linear subspace without the RNP.

3. There exists a bounded non-dentable/non-σ-dentable/non-s-dentable subset in X

(hence without an extreme point).

4. There exists a closed bounded convex subset of X, that cannot be written as a closed convex hull of its denting/strongly exposed or extreme points (hence contains no extreme points).

Proof

If X fails the KMP, then X fails the RNP. Thus, each of the conditions above is the consequence of X lacking the RNP. Suppose X lacks the KMP. Then X fails the RNP and thus there exists a closed linear subspace of X failing the RNP. We take X to be a closed linear subspace of itself, hence condition 1 is implied by a lacks the KMP in X. $\square$

The proofs for all others follow the similar line of reasoning.

4.2 Lack of the KMP in Dual spaces

The aim of this section is to show that the existence of non-separable dual spaces implies the lack of the KMP. A few examples of Banach spaces without the KMP are given.

Example 4.2.1 [7, p.50]

The Banach space l_∞ lacks the KMP:

The closed unit ball of c_0 (which is a closed bounded convex set) lacks extreme points and is a subset of l_∞. This follows from the fact that $c_0 \subset l_\infty$. Hence l_∞ lacks the KMP, and consequently the RNP.

The following theorem is a consequence of Theorem 3.3.9 (2 and 3).

Theorem 4.2.2 [13, Theorem 7, p.196]

If a Banach space X has a separable subspace Y whose dual Y^* is non-separable, then X^* lacks the KMP.

We observed in the preceeding sections that, if a Banach space X has a separable subspace

whose dual is not separable, then X^* lacks the RNP. Then, this fact and the results of Theorem 4.2.2 above lead us to the following:
X^* lacks the RNP implies X^* lacks the KMP.

Proposition 4.2.3

Each of the following conditions is sufficient for a dual Banach space X^* to lack the KMP:

1. There exists a separable subspace of X with a non-separable dual.
2. There exists a separable subspace of X which is not isomorphic to a subspace of a separable dual.
3. X is not quasi-separable.
4. X is not a wcg space, neither is it isomorphic to a subspace of a wcg Banach space.
5. X is not an Asplund space and is not reflexive.

The proofs of these conditions above follows from Proposition 3.3.10 and the fact that, if X^* lacks the RNP, then X^* lacks the KMP, see Theorem 3.3.9.

Examples 4.2.4

The following are examples of spaces lacking the KMP:

- dual spaces without the RNP (Theorem 3.3.9).
- c_0 (because its closed unit ball lacks extreme points, Example 4.1.1).
- l_∞ (proved earlier, Example 4.2.1).
- $L_1(0,1)$ (because it contains an infinite tree, see [13, pp.123,124], Theorems 2.2.1 and 4.2.2).

Chapter 5

Spaces without trees and failing the RNP

In this section, we characterise Banach spaces failing the RNP and lacking the bounded δ-tree, and in most cases failing the KMP as well. Any Banach space fails the RNP if it fail the KMP.

We know that if a Banach space X contains a bounded δ-tree, then X fails the RNP (see, Remark 2.1.4). For several years it was unknown whether the converse was true, that is, whether a Banach space had a bounded δ-tree precisely when it lacked the RNP. In 1979, Bourgin (see [7] for all the references) outlined a counterexample. Only an outline of the construction is given below, see [7, p.265].

It will be convenient to think of trees in terms of the range of a certain type of a martingale. For each $n \in \mathbb{N}$, let D_n denote the σ-algebra of subsets of [0,1] generated by the sets $\{[\frac{j-1}{2^n}, \frac{j}{2^n}) : 1 \leq j \leq 2^n\}$ and let λ denote the Lebesgue measure on $B([0,1])$, the Borel σ-algebra on [0,1]. Then a dyadic martingale is one based on ([0,1], $B([0,1])$, λ) in which the nth σ-algebra is D_n. Then the martingale is of the form (F_n, D_n). If X is a Banach space and (F_n, D_n) is an X-valued dyadic martingale, then F_{n+1} is constant on each atom of D_{n+1} and hence the (constant) value of F_n on $[\frac{j-1}{2^n}, \frac{j}{2^n})$ is the average of the value of F_{n+1} on $[\frac{2j-2}{2^{n+1}}, \frac{2j-1}{2^{n+1}})$ and the value of F_{n+1} on $[\frac{2j-1}{2^{n+1}}, \frac{2j}{2^{n+1}})$. Thus the range of a dyadic martingale may be viewed as a tree and evidently each tree in X arises from some dyadic martingale in this manner. A dyadic martingale (F_n, D_n) giving rise to a bounded δ-tree satisfies $\|F_{n+1}(t) - F_n(t)\| \geq \delta$ for each n and each $t \in [0,1)$, and hence it satisfies the inequality:

$$\int_{[0,1]} \|F_{n+1}(t) - F_n(t)\| \mathrm{d}\lambda(t) \geq \delta \text{ for each } n \in \mathbb{N}.$$

More generally, let $(F_n)_{n\geq 1}$ be an increasing sequence of finite sub-σ-algebras of $B([0,1])$. Then the range of an X-valued martingale (F_n, G_n) is a bush (and each bush in a Banach space appears as a range of some martingale (F_n, G_n) for an appropriate choice of finite σ-algebra G_n, $n \in \mathbb{N}$). Consequently in part 3 of the main theorem of this chapter, Theorem 5.2, the special case $G_n = D_n$ for each n gives a stronger conclusion that X has no bounded δ-tree for any $\delta > 0$.

The proof of Theorem 5.2 below is omitted because it spans the best of 32 pages in [7].

The following terminology will be used:

(a) For any X-valued martingale (F_n, G_n), the difference sequence associated with (F_n, G_n) is the sequence $(D_n)_{n\geq 1}$ of functions where $D_1 \equiv F_1$ and for each $n > 1$, $D_n \equiv F_n - F_{n-1}$.

(b) The norm on the Banach space $L^1_X(\lambda)$ will be denoted by $\|.\|_1$, and the norm on $L^1(\lambda)$ will be denoted by $|.|$.

Definition 5.1 [7, p.290]

A Banach space X satisfies the *Schur property* if each sequence in X which tends to 0 weakly, tends to 0 in norm.

Theorem 5.2 (Bourgin, Rosenthal) [7, Theorem 7.3.2, p.267]

Let $(q_n)_{n\geq 1}$ be an increasing sequence of positive integers. Then there is a subspace X of $L^1(\lambda)$ such that whenever $(G_n)_{n\geq 1}$ is an increasing sequence of finite sub-σ-algebras of $B([0,1])$ which is dominted by $(q_n)_{n\geq 1}$, then:

1. X fails the RNP.

2. the closed unit ball B_X of X is relatively compact in the topology of convergence in λ-measure.

3. $\lim_{n\to\infty} \inf \|D_n\|_1 = 0$ whenever (F_n, G_n) is a bounded X-valued martingale with associated difference sequence $(D_n)_{n\geq 1}$.

4. X has a strong Schur property. In particular, each weakly convergent sequence converges in norm.

5. X lacks the KMP.

The required Banach space X is constructed by the following theorem:

Theorem 5.3 [7, p. 281]

There is an increasing sequence $(X_m)_{m\geq 1}$ of finite dimensional subspaces of $L^1(\lambda)$ such that the Banach space $X \equiv L^1(\lambda)$ - closure of $\cup_{m=1}^{\infty} X_m$, has the following properties:

1. X fails the RNP.

2. For each n and each $f \in B_X$, $\mathrm{d}(f, B_{X_n})) < \delta(\frac{1}{n}, x_n)$. In particular, X has relatively compact unit ball in the topology of convergence in λ-measure.

The fact that the constructed space X also lacks the KMP is independently due to Bourgin and Elton, see [7, p.265] for the references.

Examples 5.3

The following are examples of spaces failing both the RNP and the KMP:

- Non-separable dual spaces.
- c_0.
- $C([0,1])$.
- l_∞.

- dual spaces without the RNP.
- $L_1(0,1)$.

It should be noted however that, dual Banach spaces fail both the RNP and the KMP, if they fail one of these properties, due to the established equivalence between these properties in dual Banach spaces.

On the other hand, non-dual Banach spaces lack both of these properties if they lack the KMP, since the existence of the RNP implies the existence of the KMP in non-dual Banach spaces. If we can show that non-dual Banach spaces fail the KMP if they fail the RNP, we would be a step closer to establishing the equivalence of these properties in non-dual spaces.

Chapter 6

Equivalence between the RNP and the KMP

To this point, we have gathered some information on the Banach spaces with the RNP and those with the KMP. Futhermore, we have observed conditions sufficient for a Banach space to possess either RNP or KMP and/or both. From this point, we will discuss conditions under which the RNP and the KMP are equivalent in Banach spaces.

6.1 Equivalence in Banach spaces

We have seen that, in general, if a Banach space possesses the RNP, then it also possesses the KMP. It has been left as an open question as to whether the converse holds. Reseachers on this topic have put restrictions and imposed conditions on Banach spaces, to effect the equivalence between the RNP and the KMP in such spaces. Those conditions will be discussed in this section.

We will show that in Banach spaces with unconditional bases, with strongly regular sets, with convex point-of-continuity property, with decompositions, and those isomorphic to their squares, the RNP and the KMP are equivalent.

6.1.1 Unconditional bases, the RNP and the KMP

Definition 6.1.1.1 [31, pp.1 and 18]

An unconditionally convergent Schauder basis $(x_n)_{n\geq 1}$ of a Banach space X is called an *unconditional basis* of X.

Huff and Morris ([22] and [23]) assert that in a Banach space with **unconditional basis**, the RNP and the KMP are equivalent. We rephrase this assertion as follows:

Proposition 6.1.1.2 [22, p.105]

Let X be a Banach space with unconditional basis. Then X has the RNP if and only if X has the KMP.

Proof

We know in general that a Banach space possesses the KMP if it possesses the RNP.
To prove the converse it is sufficient to show that if a Banach space X has an unconditional basis and if every separable subspace of X has the KMP, then every such subspace is isomorphic to a subspace of a separable dual. Since every separable subspace of X is contained in a separable subspace with an unconditional basis, we may assume that X itself is separable.
If X has the KMP, then X contains no isomorphic copy of c_0 (since c_0 lacks the KMP by Example 4.1.1) and the basis of X is boundedly complete[3], by the results of James and Karlin, see [22, p. 105]. By Theorem 1.4.8, X has the RNP. □

On the other hand, the above proof can be slightly modified, using the following useful assertion by Huff and Morris [22, p.105] that:
In a Banach space with unconditional basis, RNP and KMP are separably determined.

If X is a Banach space with unconditional basis and possessing the RNP, then by the above assertion, every separable subspace of X has the RNP. Since the RNP implies the KMP in any Banach space, then every separable subspace of X has the KMP. Hence X

has the KMP since KMP is separably determined.

For the converse suppose X have the KMP. Thus every separable subspace Y of X has the KMP, and Y is isomorphic to a subspace of a separable dual (as in a version of the proof of Proposition 6.1.1.2 above). Hence Y has the RNP. This implies that any separable subspace of X has the RNP, and by Huff and Morris's [22] assertion above, X has the RNP, and the result follows.

[3]Question 6.1.1.3

Is it accurate to assume or deduce that if a Banach space contains no copy of c_0, then it has a boundedly complete basis?

It is worth noting that a Banach space X with a boundedly complete basis has the RNP, by Theorem 1.4.8, hence does not contain a copy of c_0. The converse does not hold in general since $L_1(\mu)$ (μ non-atomic) contains no copy of c_0 and has no boundedly complete basis, because $L_1(\mu)$ fails the RNP. This answers the Question 6.1.1.3 in the negative.

Theorem 6.1.1.4 [48, p.683]

If D is a closed convex bounded subset of a Banach space X with an unconditional basis and D fails the RNP, then there exists a closed convex set $C \subset D$ with no extreme point.

Proof

If D fails the RNP, then, by Theorem 1.2.39, D is non-dentable. By the Overview in Section 1.8, X lacks the RNP, and also lacks the KMP, by Proposition 6.1.1.2. Hence there exists a closed convex set $C \subset D$ with no extreme points. $\square$

Corollary 6.1.1.5

If a bounded closed convex set D in X with an unconditional basis fails the RNP, then D fails the KMP.

This results follows directly from the proof of Theorem 6.1.1.4 and from Definition 3.2.4.

6.1.2 Strongly regular sets, the RNP and the KMP

In this section we investigate the equivalence of the RNP and the KMP for strongly regular sets in real Banach spaces.

Definition 6.1.2.1 [48, **Definition 1.2, p.674**]

A bounded convex (closed) subset D of a Banach space X is *strongly regular* if for every convex set C in D and $\varepsilon > 0$, there exists slices $S_1, S_2, ..., S_m$ of C such that the arithmetic mean of the diameters of these slices is less than ε.

If D is a bounded convex subset of a real Banach space X, a slice of D will be a set $S(x^*, \alpha) = \{x \in D : \langle x^*, x\rangle > M_{x^*} - \alpha\}$, where x^* is an element of the unit sphere of X^*, $\alpha > 0$, and $M_{x^*} = \sup\{\langle x^*, x\rangle : x \in D\}$. The definition of a slice was given in Definition 1.2.17.

Schachermayer [48] shows that for strongly regular sets, the RNP and the KMP are equivalent.

Theorem 6.1.2.2 [48, **p.674**]

Let X be a Banach space and D be a strongly regular convex bounded and closed subset of X. Then D has the RNP if and only if D has the KMP.

Proof

If D has the RNP, then D has the KMP (see Theorem 1.2.39 and Definition 3.2.4).
Now for the converse. By assumption, D has the KMP and it is strongly regular. It suffices to show that any bounded subset of D has a denting point, or that any convex closed and separable subset C in D which has an extreme point by assumption, also have a denting point:
Since D is strongly regular and C is convex in D, then there exists $\varepsilon > 0$ and slices $S_1, S_2, ..., S_m$ of C with $\operatorname{diam}(m^{-1}\sum_{j=1}^{m} S_j) < \varepsilon$. By Bourgin [7], if $\cap_{j=1}^{m} S_j \neq \emptyset$, then $x \in \cap_{j=1}^{m} S_j$ is a denting point. If $S_r \cap S_k = \emptyset$, $r, k \leq m, r \neq k$, there exists $S_l \neq \emptyset$ and

there exists $x \in S_l \subset C \subset D$, such that x is the denting point of C. In addition, since $\operatorname{diam}(m^{-1}\sum_{j=1}^{m} S_j) < \varepsilon$, then C has a slice of arbitrary small diameter, and by Theorem 1.2.39 (10), C is dentable. Thus D has the RNP, by Theorem 1.2.39 (9). □

Corollary 6.1.2.3

If we assume X fails the RNP, then D also fails the RNP and hence D fails the KMP, as proven above. Thus, there exists a bounded closed convex subset, say F, in D, hence in X, such that F cannot be expressed as a closed convex hull of its extreme points. Then if follows that X fails the KMP.

Corollary 6.1.2.4

If a convex bounded closed subset D of X is strongly regular and fails the RNP, then there exists a closed bounded convex and separable subset C of D which does not have an extreme point.

This then implies that the RNP and the KMP are equivalent in a Banach space X if each closed bounded and convex set D in X is strongly regular.

6.1.3 Convex-Point-of-Continuity Property, the RNP and the KMP

The aim of this section is to show that in Banach spaces with the CPCP, the RNP and the KMP are equivalent, firstly using strong regularity and secondly, using the relation between the denting and extreme points in bounded closed and convex subsets of Banach space with the CPCP.

Definition 6.1.3.1 [48], [47, p.96]

A closed bounded and convex subset C of a Banach space X has the *Convex Point-of-Continuity Property (CPCP)* if for every convex subset $D \subset C$ and $\varepsilon > 0$, there exists a relatively weakly open subset $U \subset D$ with $\operatorname{diam}(U) < \varepsilon$, alternatively, if for every closed convex bounded subset D of C, the map $(D, weak) \to (D, norm)$ has a point of continuity, see Definition 1.2.23. A Banach space X has the *CPCP* if every bounded closed and convex subset K of X has a PC (point-of-continuity).

Remark 6.1.3.2

An important observation by Bourgain (see Schachermayer [48, p.683]) is that, *a set D in a Banach space having the CPCP is strongly regular.* In light of this, we use the result in the preceeding Theorem 6.1.2.2 that, for D strongly regular, D has the RNP if and only if D has the KMP. The following result by James [24] is then immediate:

The proof of the Proposition below follows the same reasoning as the one provided earlier, where X (or D in that case) is strongly regular.

Proposition 6.1.3.3 [24, p.913]

Let X be a space with the CPCP. Then X has the RNP if and only if X has the KMP.

Proof

Let X have the RNP and the CPCP. Then X has the KMP. For the converse, each bounded closed convex subset K of X has the CPCP, hence each K is strongly regular by Remark 6.1.3.2. By the second part of the proof of Theorem 6.1.2.2, K is dentable. We deduce that X has the RNP. In addition, for the proof of the converse, any such K is the closed convex hull of its extreme points and K has a PC, by assumption. If x is a PC in K, it is **sufficient** to show that x is an extreme point in K. We know already that $x \in clco(E(K))$. It would follow from Lin, Lin and Troyanski [30, p.256] that, such $x \in K$ is a denting point in K, and thus K is dentable. Since K is an arbitrary bounded (closed and convex) subset in X, then X has the RNP. $\square$

6.1.4 Decomposition, the RNP and the KMP

In this section we investigate how decompositions in Banach space contribute to the equivalence between the RNP and the KMP.

James [25] has shown that if X is a space with UBFDD (Definition 1.4.11), then the RNP and the KMP are equivalent in X. James [24, p.917] mentions that for a bounded closed subset of Banach space with UBSBFDD, the RNP and the KMP are equivalent. We rephrase it as follows:

Proposition 6.1.4.1 [24, **p.917**]

Let X be a Banach space with UBSBFDD and B be a bounded closed convex subset of X. Then B has the RNP if and only if B has the KMP.

Proof

If X has the UBSBFDD, then X has the UBFDD. Let B be any bounded (convex) closed subset of X. If B has the RNP, then B is dentable in X and thus X has the RNP. From the above assertion by James [25], and the fact that X has the UBFDD, and X has the RNP, then X has the KMP. Since B has the RNP, then it has the KMP.

Conversely, if B has the KMP and X has the UBSBFDD, then X has the UBFDD and by James [25], B has the RNP. $\square$

6.1.5 A Banach space X isomorphic to its square X^2, the RNP and the KMP

In this section, we present another condition on a Banach space that guarantees the equivalence between the RNP and the KMP, namely, if a Banach space is isomorphic to its square.

Notations

X^2 denotes $X \times X$ and $A \simeq B$ denotes that A is isomorphic to B, for Banach spaces A and B.

We first introduce a notion intermediary to the RNP and the KMP.

Definition 6.1.5.1 [46, p.329]

A separable Banach space X has the *Integral Representation Property* (IRP) if for every bounded closed convex subset C of X and every $x \in C$ there is a probability measure μ on the extreme points of C such that x is the barycentre of μ, that is, $x = \int_C c d\mu(c)$, and μ(extreme points of C) $= 1$, see also [13, p.145] and [7, p.178].

It was shown by Edgar (see [46, p.329] and [13, p.145]), that the RNP implies the IRP in the case that C is separable. There still does not exist a general result on the extension to the non-separable case. It is also true that the IRP implies the KMP. The respective converses to these two implications are still open, see [46, p.329]. It has already been shown that a Banach space X has the RNP if and only if $l^2(X)$ (or any other appropriate space of sequences in X) has the IRP, see [46, p.329]. Also, if X is isomorphic to its square $X \times X$, denoted by X^2, then $l^2(X)$ has the IRP if and only if X has the IRP [46, p.329]. In this case, the IRP and the RNP are equivalent. Note that,
$l^2(X) = \{(x_n)_{n\geq 1} : x_n \in X, \|(x_n)\|^2 = \sum_{n=1}^{\infty} \|x_n\|^2 < \infty\}$.

Definition 6.1.5.2 [46, p.330]

Let (Ω, Σ, μ) be a probability space. A bounded linear operator $T : L_1(\Omega, \Sigma, \mu) \to X$, X a Banach space, is called *representable* if there is a function $F \in L_\infty(\Omega, \Sigma, P)$ such that for every $f \in L_1(\Omega, \Sigma, \mu)$, $Tf(w) = \int f(w)F(w)dP(w)$.

Theorem 6.1.5.3 [46, p.333]

A separable Banach space X has the RNP if and only if $l^2(X)$ has the KMP.

The lengthy proof of this theorem is bases upon several preliminary results on representability in [46].

Definition 6.1.5.4 [46, Definition 3.1, p.335]

A Banach space X *semi-embeds* into a Banach space Y if there is an injective continuous operator $j : X \to Y$ such that the image of the unit ball in X under j is closed in Y.

It is known that a separable Banach space X that semi-embeds into a Banach space Y which has the RNP, has already the RNP, see [46, p.335]. For the purpose of the KMP, a stronger notion than semi-embedding is needed.

Definition 6.1.5.5 [46, **Definition 3.2, p.335**]

A Banach space X $\frac{3}{4}$-*embeds* into a Banach space Y if there exists an injective operator $f : X \to Y$ which maps *closed bounded convex* sets into *closed bounded convex* sets.

Theorem 6.1.5.6 [46, **Proposition 3.3, p.335**]

If a Banach space Y has the KMP and the Banach space X $\frac{3}{4}$-embeds into Y, then X has the KMP.

Proof

If X fails the KMP, then there exists a closed bounded closed convex set C in X which has no extreme points. If $f : X \to Y$ is a $\frac{3}{4}$-embedding, then $f(C)$ is closed bounded and convex in Y and has no extreme points, hence contradicting the hypothesis that Y has the KMP. $\square$

Corollary 6.1.5.7 [46, **Corollary 3.4, p.335**]

If the Banach space X is separable and if $l^2(X)$ $\frac{3}{4}$-embeds into X, then X has the KMP if and only if X has the RNP.

The proof follows from Theorems 6.1.5.6 and 6.1.5.3.

Corollary 6.1.5.8 [46, **Proosition 3.5, p.335**]

If $X \times X$ $\frac{3}{4}$-embeds into X, then $l^2(X)$ $\frac{3}{4}$-embeds into X.

The main result in the paper [46] is:

Theorem 6.1.5.9 [46, **Corollary 3.7, p.335**]

If X is a separable and if $X \times X$ $\frac{3}{4}$-embeds into X (in particular, if X is isomorphic to

$X \times X$), then X has the KMP if and only if X has the RNP.

Proof

Suppose $X \times X$ $\frac{3}{4}$-embeds into X, and let X have the KMP. Then, by Corollary 6.1.5.8 above, $l^2(X)$ $\frac{3}{4}$-embeds into X. Thus X has the RNP, by Corollary 6.1.5.7.
For the converse, suppose $X \times X$ $\frac{3}{4}$-embeds into X, and let X have the RNP. Then, by Corollary 6.1.5.8 above, $l^2(X)$ $\frac{3}{4}$-embeds into X. Thus X has the KMP, by the above Corollary 6.1.5.7. $\square$

6.1.6 Banach lattices, the RNP and the KMP

In this section we introduce the concept of Banach lattices as Banach spaces in which the RNP and the KMP are equivalent.

Definition 6.1.6.1 [7, p.422]

A partially ordered Banach space $(X, \|.\|)$ is a *Banach lattice* if

- $x \leq y$ implies $x + z \leq y + z$ for each $x, y, z \in X$.
- $0 \leq tx$ whenever $0 \leq x \in X$ and $0 \leq t \in \mathbb{R}$.
- for each $x, y \in X$, there exists a least upper bound written $x \vee y$.
- $\|x\| \leq \|y\|$ provided that $|x| \leq |y|$ where $|x|$ means $x \vee (-x)$.

Theorem 6.1.6.2 [7, Theorem 7.13.1, p.423]

Let X be a Banach lattice. Then X has the RNP if and only if X has the KMP.

Theorem 6.1.6.3 [7, Theorem 7.13.2, p.423]

Let X be a separable Banach lattice. If X has the RNP then X is isometrically isomorphic to the dual of a Banach lattice.

Theorem 6.1.6.2 gives another characterisation of a Banach space in which the RNP and the KMP are equivalent. The proofs of both Theorems 6.1.6.2 and 6.1.6.3 involve the notion of order dentability [7, p.423].

6.2 Overview of conditions in a non-dual Banach space

The following are conditions sufficient for the RNP and the KMP to be equivalent in a Banach space X :

- X has an unconditional basis.
- X has strongly regular, closed convex bounded sets.
- X has the convex point-of-continuity property.
- X has Finite Dimensional Schauder Decomposition..
- X is isomorphic to its square, that is, $X \simeq X^2 = X \times X$.
- X is a Banach lattice.

6.3 Equivalence in Dual spaces

In this section, we show that in dual Banach spaces the RNP and the KMP are equivalent. We start with the general dual space and continue to specific dual spaces, namely separable dual and reflexive space.

6.3.1 Equivalence in general dual Banach spaces

Huff and Morris [22] state and prove that every dual Banach space with KMP has the RNP. We find the same result in [7, p.111].

Theorem 6.3.1.1 [7, Theorem 4.4.1, p.111]
Let X^* be a dual space, then X^* has the RNP if and only if X^* has the KMP.

The proof follows from the following results:

1. X^* has the RNP if and only if every separable subspace in X has a separable dual (Theorem 1.5.5).

2. X^* has the KMP if and only if every separable subspace of X has a separable dual (Theorems 3.3.3 and 3.3.9).

The alternative proof of Theorem 6.3.1.1 is given by Huff and Morris [22, p.104].

6.3.2 Separable duals, the RNP and the KMP

Separable dual Banach spaces are dual Banach spaces in any case, hence Davis and Phelps [10] state that, in separable dual spaces, the RNP and the KMP are equivalent, and we rephrase it as follows:

Proposition 6.3.2.1

Let X^* be a separable dual Banach space. Then X^* has both the RNP and the KMP.

The proof of this follows from Theorem 6.3.1.1 above. Furthermore, a separable dual space has both the RNP [7, p.74] and the KMP (Theorem 3.3.9).

6.3.3 Reflexive spaces, the RNP and the KMP

Davis and Phelps [10] state that, in reflexive Banach spaces, the RNP and the KMP are equivalent, and we state this formally as follows:

Proposition 6.3.3.1 [10, p.121]

Let X be a reflexive Banach space. Then X has both the RNP and the KMP.

This result has already been stated in Examples 1.8.3 and 3.3.11.

6.3.4 Summary on the equivalence in dual Banach spaces

This section is aimed at listing those conditions equivalent to both the RNP and the KMP in dual Banach space X^*, and they are as follows:

1. X^* has RNP.

2. X^* has KMP.

3. If Y is a separable subspace of X, then Y^* is separable.

4. X^* contains no bounded δ-tree for any $\delta > 0$.

5. For each weak*-compact convex subset K of X^*, $K = w^*cl(co(w^*\text{-}SE(K))$, that is, weak*-closure of the convex hull of weak*-strongly exposed points.

6. Each bounded subset of X^* is weak*-dentable.

7. Every separable subspace of X^* is a subspace of a separable dual (see [13, p.198])

Proofs can be found in [7, pp.91,111 and 112].

Chapter 7

Applications of the Radon-Nikodým theorem

The Radon-Nikodým theorem has many applications in more than few branches of mathematics, especially in the Business Mathematics. In this chapter we will try and explore, without diverting from our main discussion, these applications as briefly as possible.

7.1 Subalgebras

This section introduces the notion of a subalgebra and it is intended to introduce some notation for section 7.2.

Definition 7.1.1 [19, p.109]
Let Ω be a set and Σ be a σ-algebra of subsets of Ω. A *σ-subalgebra* of Σ is a σ-algebra T of subsets of Ω such that $T \subseteq \Sigma$. If (Ω, Σ, μ) is a measure space and T is a σ-subalgebra of Σ, then $(\Omega, T, \mu|_T)$ is again a measure space, where $\mu|_T$ denotes the restriction of μ to T.

Lemma 7.1.2 [19, p.109]
Let (Ω, Σ, μ) be a measure space and T a σ-subalgebra of Σ. A real-valued function f defined on a subset of Ω is $\mu|_T$-measurable if and only if:

1. f is μ-integrable.

2. dom f is $\mu|_T$-co-negligible.

3. f is $\mu|_T$-vitually measurable, and in this case $\int f \mathrm{d}(\mu|_T) = \int f d\mu$.

Proof

Note first that if f is $\mu|_T$-simple, that is, expressible as $\sum_{i=0}^{n} a_i \chi E_i$ where $a_i \in \mathbb{R}$, $E_i \in T$ and $\mu|_T(E_i) < \infty$ for each i, then $\int f d\mu = \sum_{i=0}^{n} a_i \mu(E_i) = \int f \mathrm{d}(\mu|_T)$.

Let U_μ (respectively $U_{\mu|_T}$) be a set of non-negative μ-integrable (respectively $\mu|_T$-integrable) functions.

Suppose $f \in U_{\mu|_T}$. Then there is a non-decreasing sequence $(f_n)_{n\geq 1}$ of $\mu|_T$-simple functions such that $f(x) = \lim_{n\to\infty} f_n$ $\mu|_T$-a.e and $\int f \mathrm{d}(\mu|_T) = \lim_{n\to\infty} \int f_n \mathrm{d}(\mu|_T)$. But every f_n is also μ-simple, and $\int f_n d\mu = \int f_n \mathrm{d}(\mu|_T)$ for every n, and $f = \lim_{n\to\infty} f_n$ μ-a.e. Hence we have,

$$\begin{aligned}\int f \mathrm{d}(\mu|_T) &= \lim_{n\to\infty} \int f_n \mathrm{d}(\mu|_T)\\ &= \lim_{n\to\infty} \int f_n \mathrm{d}\mu\\ &= \int \lim_{n\to\infty} f_n \mathrm{d}\mu\\ &= \int f d\mu.\end{aligned}$$

Hence $f \in U_\mu$.

Now suppose f is $\mu|_T$-integrable. Then f is the difference of two members of $U_{\mu|_T}$, so is μ-integrable, and $\int f \mathrm{d}\mu = \int f \mathrm{d}(\mu|_T)$.

Also conditions 2 and 3 are satisfied (see [19]).

Conversely,

Suppose f satisfies conditions 1 to 3. Then $|f| \in U_\mu$, and there is a co-negligible set $E \subseteq T$ and $f|_E$, the restriction of f to E, is T-measurable. Accordingly $(|f|)|_E$, the restriction of $|f|$ to E, is T-measurable. Now, if $\varepsilon > 0$, then

$$\begin{aligned}(\mu|_T)\{x:\ x \in E,\ |f|(x) \geq \varepsilon\} = \mu\{x:\ x \in E,\ |f|(x) \geq \varepsilon\}\\ &\leq \tfrac{1}{\varepsilon} \int |f| \mathrm{d}(\mu|_T)\\ &< \infty,\end{aligned}$$

moreover,

$\sup\{\int g \mathrm{d}(\mu|_T) : g$ is a $\mu|_T$-simple function, $g \leq |f|$ $\mu|_T$-a.e.$\}$

$= \sup\{\int g \mathrm{d}\mu : g$ is a $\mu|_T$-simple function, $g \leq |f|$ $\mu|_T$-a.e.$\}$

$\leq \sup\{\int g\mathrm{d}\mu : g$ is a μ-simple function, $g \leq |f|$ μ-a.e.$\}$

$\leq \int |f|\mathrm{d}\mu < \infty$.

Hence $|f| \in U_{\mu|_T}$. Consequently f, being $\mu|_T$-virtually T-measurable, is $\mu|_T$-integrable.

The essential point is that, while a $\mu|_T$-negligible set is always μ-negligible, a μ-negligible set need to be $\mu|_T$-negligible. □

Example 7.1.3 [19, p.110]

Let (Ω, Σ, μ) be $[0,1]^2$ with Lebesgue measure. Let T be the set of those members of Σ expressed as $F \times [0,1]$, for some $F \subseteq [0,1]$; it is easy to see that T is a σ-subalgebra of Σ. Consider $f, g : \Omega \to [0,1]$ defined by saying that,

$f(t,u) = 1$ if $u > 0$, 0 otherwise

$g(t,u) = 1$ if $t > 0$, 0 otherwise

Then both f and g are μ-integrable, being constant μ-a.e. But only g is $\mu|_T$- integrable, because any non-negligible $E \in T$ includes complete vertical section $\{t\}\times[0,1]$, so that f takes both values 0 and 1 on E. If we set,

$h(t,u) = 1$ if $u > 0$, undefined otherwise,

Then again, h is μ-integrable but not $\mu|_T$-integrable as there is no co-negligible member of T included in the domain of h.

7.2 Conditional Expectation and the Radon-Nikodým theorem

This section is devoted to an application of the Radon-Nikodým theorem, in abstract probability theory.

Remark 7.2.1 [19, p.110]

Let (Ω, Σ, μ) be a probability space. For any μ-integrable real-valued function f defined on a co-negligible subset of Ω, we have a corresponding indefinite integral $\nu_f : \Sigma \to \mathbb{R}$ given by the formula $\nu_f(E) = \int_E f d\mu$ for any $E \in \mathbb{R}$ (by the Radon-Nikodým theorem).

We know that ν_f is countably additive and truly continuous with respect to μ, which in the present context is the same as saying that it is absolutely continuous. Now consider the restrictions $\nu_f|_T$, $\mu|_T$ of ν_f and μ (respectively) to the σ-algebra T. It follows directly from the definition of 'countably additive' and 'absolutely continuous' that $\nu_f|_T$ is countably additive and absolutely continuous with respect to $\mu|_T$, therefore truly continuous with respect to $\mu|_T$.

Consequently, the *Radon Nikodým theorem* tells us that there is a $\mu|_T$-integrable function g such that $(\nu_f|_T)F = \int_F fd(\mu|_T)$ for every $F \in T$.

Definition 7.2.2 [19, p.111], [8, p.153]
A *conditional expectation* of f on T is a $\mu|_T$-integrable function g such that
$\int_F gd(\mu|_T) = \int_F fd\mu$ for every $F \in T$.
Hence, for such g we have,

$$\begin{aligned}\int_F g\mathrm{d}(\mu|_T) &= \int g \times \chi_F \mathrm{d}(\mu|_T)\\ &= \int g \times \chi_F \mathrm{d}(\mu|_T)\\ &= \int_F g\mathrm{d}\mu, \text{ for every } F \in T\end{aligned}$$

Also, g is almost everywhere equal to a T-measurable function defined everywhere on Ω which is also a conditional expectation of f on T.

Remark 7.2.3 [8, p.218]
With the Radon-Nikodým theorem at our disposal, we can verify the existence of conditional expectation for integrable function very simply:
The bounded measure $(\nu_f|_T)F = \int_F gd(\mu|_T)$ is absolutely continuous with respect to $\mu|_T$, whenever $\nu_f \ll \mu$. This means that the Radon-Nikodým theorem has been used to define conditional expectation of a function f on a σ-subalgebra T.

7.3 Additional Applications

There are some applications of the Radon-Nikodým theorem, in theories of *exchange economy* and the *non-atomic games.*

Remark 7.3.1

The pure exchange economy is simply the trades that are being made between consumers and producers or manufacturers, see [26, p.144]. A μ-mixing sequence $\{\theta_n\}$ is a sequence of μ-measure-preserving automorphisms of a measure space (A, Σ, μ) such that $\mu(S \cap \theta_n T) \to \mu(s)\mu(T)$, for S and T in Σ, see [1, p114].

Without sidetracking from the main theme below we cite instances where the Radon-Nikodým theorem has been applied.

- In proving the result of the theorem on [26, p.224], Klein used the Radon-Nikodým theorem where he defined the functions $\pi(\cdot, p)$ and $\xi(S, p)$ such that $\xi(S, p) = \int_S \pi(\cdot, p) d\nu$. Hence showing that $\pi(\cdot, p)$ is a ν-integrable function and $\xi(S, p)$ can be expressed as its integral any $S \in \Sigma$ with (A, Σ, ν) a (probability) measure space.

- Secondly, the Radon-Nikodým theorem is used in a proof the result in [1, p.121, Lemma 15]. Aumann and Shapley [1] show that for ψ non-negative measure with $\psi \ll \mu$ and $\{\theta_n\}$ a μ-mixing sequence, $\lim_{n\to\infty}\psi(\theta_n T) = \mu(T)\psi(I)$ for all measurable sets T. Aumann and Shapley [1] use the fact that $\psi(\theta_n T) = \int_T g(t)\mathrm{d}\mu(t)$ for some $g \geq 0$ an integrable (characteristic) function, see [1, p.121]. This is the application of the RN-theorem in that the image of a ψ is expressed as an integral of an integrable function g, where ψ and g have the same domain.

It is clear that the Radon-Nikodým theorem is an important theorem and it is useful not only if Pure Mathematics. However, there are several applications in Pure Mathematics itself. For example in [13, chapter IV] one finds applications of the Radon-Nikodým theorem to,

1. the isolation of the dual of $L_p(\mu, X)$, for $1 \leq p < \infty$,

2. the finding of the weakly compact subsets of $L_1(\mu, X)$,

3. a characterisation of a Gel'fand space in terms of the RNP with respect to Lebesgue measure on the Borel sets in [0,1],

4. a study of operators on $L_p(\mu, X)$ that are defined by means of Pettis and Bochner integrals,

5. show that if X is a complemented infinite dimensional subspace of $L_1(\mu, X)$ and if X has the RNP, then X is isomorphic to l_1.

CONCLUSION

Our main aim was to find those Banach spaces in which the RNP and the KMP are equivalent, hence explore their characteristics.

We started with the characterisation of Banach spaces with the RNP, using predominantly a geometric property called 'dentability'. Furthermore, we witnessed how the different forms of dentability, namely, s-dentability and c-dentability, contribute to the existence of the RNP in Banach spaces.

In dual Banach spaces, in particular, separability of Banach spaces and of their dual spaces leads to the existence of the RNP.

Banach spaces with the KMP, on the other hand, are characterised by the existence of extreme points, which in some cases are connected to denting points. The relation between denting points and extreme points in some cases, gave us the most important link between the RNP and the KMP. This is simply because the existence of denting points leads to the existence of the RNP, and the existence of extreme points leads to that of the KMP.

We mostly used the fact that the existence of the RNP implies the existence of the KMP, which leads to the fact that, conditions sufficient for existence of the RNP, are also sufficient for the existence of the KMP.

From this point on, our aim shifted to exploring the converse, that is, the conditions and restrictions imposed on the Banach spaces with the KMP to have the RNP, hence establishing the equivalence between these properties. Exploring this converse, we realised, it would be made much simplier if we first find out what conditions and characteristics sufficient for a Banach space to lack either the RNP, the KMP or both of them, and that is what we did in chapters 2, 4 and 5.

We feel that the core of this thesis is chapter 6, in which conditions sufficient for the RNP and the KMP to be equivalent in a Banach space, or simply conditions and characteristics making it possible for the KMP to imply the RNP, are being explored and discussed.

We firstly looked at this equivalence in Banach spaces (not necessarily dual) and found out that, existence of unconditional basis, bounded strongly regular set, convex-point-of-continuity property (or existence of a PC in every bounded closed convex subset), finite dimensional Schauder decomposition (FDD), isomorphism between the Banach space and its square, are all sufficient restrictions and characteristics of a Banach space in which the RNP and the KMP are equivalent.

Secondly, in dual Banach spaces, separable duals, reflexive spaces are examples of Banach spaces in which the RNP and the KMP are equivalent. The separable dual of a separable subspace of a Banach space, leads to the equivalence of the RNP and the KMP in such dual Banach space.

It is noteworthy that the relationship between the denting points and the extreme points, especially in a closed bounded (dentable) convex subset of a Banach space X, is the key to solving the equivalence between the RNP and the KMP. To prove the equivalence between the RNP and the KMP, we only need to prove that any Banach space X failing the RNP also fails the KMP. This means that if X has a bounded closed convex non-dentable set, say B, then it suffices to show that B is not equal to the closed convex hull of its extreme points.

Lastly, we looked at applications of the RN-theorem which makes for an interesting read. It is interesting to see that the RN-theorem is applicable in disciplines such as economy and probability analysis.

We hope you enjoyed reading through this thesis as much as we enjoyed putting it together!

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