INTEGRATION OF MULTIFUNCTIONS WITH RESPECT TO A MULTIMEASURE

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Dissertation presented for the Degree of Doctor of Philosophy

in

Mathematics

at the

University of Stellenbosch

Promoter: Professor P Maritz

December 1999

DECLARATION

I, the undersigned, hereby declare that the work contained in this dissertation is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

SUMMARY

The main objective of this thesis is to define and investigate the properties of the integral of a multifunction F (where F is from a point set T into a Banach space X) with respect to a multimeasure M (where M is defined on a ring $\mathcal{R}$ and with values in a Banach space Y). Integration of multifunctions with respect to a vector measure has been studied extensively because of its applications in mathematical economics. On the other hand, Papageorgiou [55], and later on Kandilakis [44], considered integration of a function with respect to a multimeasure. We define our integral in terms of the selectors of the multifunction F and the selectors of the multimeasure M so that both the above two integrals are special cases of our integral.

The first two chapters serve as an introduction and will provide the foundation for work done in the chapters that follow. In the first chapter we recall some of the basic definitions and results of the subject of vector measures and measurable functions. In particular, we give a brief overview of the procedure of extending a vector measure m, defined originally on a ring $\mathcal{R}$ of subsets of a point set T, to a δ-ring containing $\mathcal{R}$.

Chapter 2 is devoted to the basic theory of multifunctions and multimeasures. The standard reference for the section on measurable multifunctions is Maritz [51], who defined measurability of the multifunction (Definition 2.1.2) as the set-valued version of the measurability of a function (Definition 1.3.5). We start by discussing Maritz's [51] exposition of the characterization of measurability of a multifunction in terms of its graph, its inverse and its Castaing representation. Finally, we consider the measurability of some special multifunctions, namely the extreme points multifunction and the closed convex hull multifunction. The better part of Chapter 2 is devoted to the subject of multimeasures. Following Godet-Thobie [36] we define three different types of multimeasures and then discuss the logical implications among them. Next we give an outline on the existence of selectors of a multimeasure M and we discuss the topological properties of S_M, the class of all selectors of M. In particular, we investigate the conditions which will guarantee that $S_M \neq \emptyset$ and such that $M(A) = \{m(A) \mid m \in S_M\}$. Finally, we study transition multimeasures, that is multimeasures parametrized by the elements of a measurable space.

In Chapter 3 we are concerned with extension results for multimeasures and transition multimeasures. We start by extending additive set-valued set functions. Our results are along the extension procedure for a vector measure as was discussed in Chapter 1. In the main result of this chapter (Theorem 3.1.12) we prove the set-valued version of the Carathéodory-Hahn-Kluvanek theorem. In the process we extend the corresponding result (Theorem 3.1.7) of Kandilakis [44] to additive set-valued set functions. Finally, we prove extension results for normal multimeasures and transition multimeasures.

In the first section of Chapter 4 we review the bilinear integral $\int f(t)m(dt)$ of a function $f : T \to X$ with respect to a vector measure $m : \mathcal{R} \to Y$ as developed by Dinculeanu [27]. The integral, $\int F(t)M(dt)$ of a multifunction F with respect to a multimeasure M is then defined in terms of $\int f(t)m(dt)$. We continue by investigating the convexity and compactness of our integral and in the process we also establish Radon-Nikodým-type theorems for our integral. Finally, we discuss the commutativity of the closed convex hull operator and the extreme points operator with the integral operator.

Finally, in the first part of Chapter 5 we study the properties of the space of integrably bounded measurable multifunctions. In particular, we prove that the space of integrably bounded, measurable and compact- and convex-valued multifunctions is separable. In addition we also prove the equivalence of our integral and the integral of Debreu [24]. Finally, we investigate the properties of multimeasures defined by densities and we prove the set-valued version of the Lebesgue decomposition theorem.

OPSOMMING

Die hoofdoel van hierdie tesis is om die integraal van 'n multifunksie F (waar F vanaf 'n puntversameling T na 'n Banach ruimte X gedefinieer is) met betrekking tot 'n multimaat M (waar M op 'n ring $\mathcal{R}$ gedefinieer is en met waardes in 'n Banach ruimte Y) te definieer en dan die eienskappe te ondersoek. Die integrasie van multifunksies met betrekking tot 'n vektormaat is omvattend bestudeer as gevolg van die toepassings wat dit in wiskundige ekonomie het. Daarenteen het Papageorgiou [55], en later Kandilakis [44], integrasie van 'n funksie met betrekking tot 'n multimaat bestudeer. Ons definieer ons integraal in terme van die selektors van die multifunksie F en die selektors van die multimaat M sodat beide bostaande integrale spesiale gevalle is van ons integraal.

Die eerste twee hoofstukke dien as 'n inleiding en vorm die grondslag van die werk in die daaropvolgende hoofstukke. In die eerste hoofstuk hersien ons sommige van die basiese definisies en resultate van die teorie van vektormate en meetbare funksies. In die besonder gee ons 'n kort oorsig van die proses waarvolgens 'n vektormaat m, gedefinieer op 'n ring $\mathcal{R}$ van deelversamelings van 'n puntversameling T, uitgebrei word na 'n δ-ring wat vir $\mathcal{R}$ bevat.

Hoofstuk 2 word gewy aan die basiese teorie van multifunksies en multimate. Die standaard verwysing vir die gedeelte oor meetbare multifunksies is Maritz [51], wat meetbaarheid van die multifunksie (Definisie 2.1.2) gedefinieer het as die versamelingswaardige weergawe van die meetbaarheid van 'n funksie (Definisie 1.3.5). Ons begin met 'n bespreking van Maritz [51] se uiteensetting van die karakterisering van meetbaarheid van 'n multifunksie in terme van sy grafiek, sy inverse en sy Castaing-voorstelling. Laastens ondersoek ons die meetbaarheid van sekere spesiale multifunksies, naamlik die ekstreempuntmultifunksie en die geslote konvekse omhulsel multifunksie. Die grootste gedeelte van Hoofstuk 2 word gewy aan die teorie van multimate. Deur gebruik te maak van Godet-Thobie [36] definieer ons drie verskillende tipes multimate en bespreek dan die logiese implikasies tussen hulle. Verder skets ons dan ook die bestaan van selektors van 'n multimaat M en bespreek vervolgens die topologiese eienskappe van S_M, die klas van alle selektors van M. In die besonder ondersoek ons die voorwaardes wat sal waarborg dat $S_M \neq \emptyset$ en $M(A) = \{m(A) \mid m \in S_M\}$. Laastens bestudeer ons oorgangsmultimate, met ander woorde multimate wat geparametriseer word deur elemente van 'n meetbare ruimte.

In Hoofstuk 3 bewys ons uitbreidingsresultate vir multimate en oorgangsmultimate. Ons begin deur additiewe versamelingswaardige funksies uit te brei. Ons resultate is volgens die uitbreidingsproses vir vektormate soos in Hoofstuk 1 bespreek. In die hoofresultaat (Stelling 3.1.12) van hierdie hoofstuk bewys ons die versamelingswaardige weergawe van die Carathéodory-Hahn-Kluvanek stelling. In die proses brei ons die ooreenkomstige resultaat (Stelling 3.1.7) van Kandilakis [44] uit na additiewe versamelingswaardige

funksies. Ons sluit die hoofstuk af met uitbreidingsresultate vir normale multimate en oorgangsmultimate.

In die eerste gedeelte van Hoofstuk 4 hersien ons die bilineêre integraal $\int f(t)m(dt)$ van 'n funksie $f : T \to X$ met betrekking tot 'n vektormaat $m : \mathcal{R} \to Y$ soos ontwikkel deur Dinculeanu [27]. Die integral $\int F(t)M(dt)$ van 'n multifunksie F met betrekking tot 'n multimaat M word dan gedefinieer in terme van $\int f(t)m(dt)$. Ons ondersoek dan verder die konveksiteit en kompaktheid van ons integraal en terselfdertyd bewys ons Radon-Nikodým-tipe stellings vir hierdie integraal. Laastens bespreek ons die kommutatiwiteit van die geslote konvekse omhulsel operator en die ekstreempuntoperator met die integraaloperator.

Laastens, in die eerste gedeelte van Hoofstuk 5 bestudeer ons die eienskappe van die ruimte van integreerbaar-begrensde meetbare multifunksies. In die besonder bewys ons dat die ruimte van alle integreerbaar-begrensde, meetbare en konveks- en kompakwaardige multifunksies separabel is. Ons bewys ook die ekwivalensie van ons integraal met dié van Debreu [24]. Ons sluit dan die hoofstuk af met 'n ondersoek na die eienskappe van multimate wat gedefinieer word deur digthede en ons bewys die versamelingswaardige weergawe van die Lebesgue-ontbindingstelling.

ACKNOWLEDGEMENTS

I gratefully acknowledge the help and support of the following people:

My promoter, Professor P Maritz, for his guidance, advice and motivation.

My wife, Sonja, for always being by my side.

My family and friends for their encouragement.

The FRD for their financial support.

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INTRODUCTION

Multifunctions (set-valued functions) have been of interest for about seventy years now. For instance, in 1926 Wilson [68] introduced the notion of a multifunction in order to generalize the concepts of limit inferior and limit superior of a sequence of subsets of a topological space. The initial development has been slow. Even after World War II there has been a reluctance in mathematical sciences to deal with sequences of sets and set-valued maps. Most mathematicians, amongst them the Bourbakis, chose to restrict their study to single-valued maps, while they regarded set-valued maps as single-valued maps from a set to the power set of another set. This point of view misled many of the mathematicians into unneccessary detours and the whole study of set-valued analysis inherited the undeserved image of being something difficult and mysterious. However, as it turned out, the need for set-valued analysis in other fields of study was pressing enough to help mathematicians overcome this kind of opposition towards set-valued analysis. Since then there has been increasing interest in multifunctions because of their importance in several applied areas of research, such as mathematical economics (see [3,6,41,65]), optimization and control (see [13,63]), statistics [60], control theory (see [37,38]) and game theory [28]. Some of the recent texts like [5,46] display the applicability of multifunctions in excellent ways.

Various developments in mathematical economics and optimal control have led to the study of the measurability of multifunctions. Also, integrals of multifunctions have been studied in connection with statistical problems (see Kudō [48] and Richter [60]). Accordingly, many papers dealt with the basic theory of integration of multifunctions and several approaches were established. A beginning of what might be called a calculus of multifunctions can be found in [7]. In [7], Aumann considered integration of selectors of the multifunction and his integral turned out to be the appropriate analytic tool in the applied fields mentioned before.

On the other hand, the theory of multimeasures (set-valued measures) has its origins in mathematical economics and in particular in equilibrium theory for exchange economies with production, in which the coalitions and not the individual agents are the basic economic units (see Hildenbrand [42] and Vind [65]).

The traditional economic concept of a set of agents, each of which cannot influence the outcome of their collective activity but certain coalitions of which can influence that outcome has received a proper mathematical formulation by means of measure theory. In [53] Milnoz and Shapley considered a game with measure space of players, while Aumann [6] showed how the two basic concepts for an economy, namely the set of competitive allocations and the core, coincide when the set of consumers is an atomless positive finite measure space. In fact, the theory of general equilibrium for economies with a continuum of agents was inaugurated by Aumann [6,7]. Another solution of this equivalence problem was given by Vind [65], who was the first to introduce the concept of a multimeasure with values in $\mathbb{R}^n$.

Multimeasures in a functional analytical setting appear to have originated with Brooks' [10] work on a finitely additive function defined on a σ-algebra into the family of bounded convex subsets of a real Banach space. From this point of departure, Godet-Thobie has developed the subject of multimeasures extensively during 1970 to 1975 in a series of papers [31,32,33,34,35], culminating in her thesis [36]. Loosely speaking, one calls M a multimeasure if the range space Y is (at least) a commutative topological group and M is suitably countably additive. Central to the approaches that have been taken appear to be the definitions of convergence of an infinite sum of subsets of Y. In the papers on multimeasures different types of approaches can be distinguished according to the range space of the multimeasures. Significant contributions to the study of multimeasures were made by Artstein [4], Debreu and Schmeidler [25], Schmeidler [63] and Wenxiu, Jifeng and Aijie [67] for $\mathbb{R}^n$-valued multimeasures, by Aló, de Korvin and Roberts [1,2], Costé [17,18,19], Hiai [39], Papageorgiou [54,55,56] and Kandilakis [44] for Banach space-valued multimeasures and by Castaing [12], Costé and Pallu de la Barriére [20,21] and Godet-Thobie [34,36] for multimeasures in a locally convex vector space.

The main aim in this thesis is to define and investigate the properties of the integral of a multifunction with respect to a multimeasure. The first two chapters serve as an introduction, providing the foundation for work done in subsequent chapters. In the first chapter we study the subject of vector measures and measurable functions, while in Chapter 2 we are concerned with measurable multifunctions and multimeasures. Chapter 3 is devoted to the extension of multimeasures. In Chapter 4 we define the integral of a multifunction with respect to a multimeasure and we investigate the properties of this integral. In Chapter 5 we study spaces of integrably bounded multifunctions and we end the chapter by discussing multimeasures defined by densities.

CHAPTER 1

VECTOR MEASURES AND MEASURABLE FUNCTIONS

In this chapter we recall some of the basic definitions and results of the theory of vector measures and measurable functions. We will refer to the book of Dinculeanu [27] for most of the definitions and proofs. However, some of the shorter proofs will be included.

Throughout this chapter, and in all subsequent chapters, T will denote a non-empty point set on which no topological structure is required and X and Y are arbitrary vector spaces. In particular, X or Y can be the space of real numbers or the space of complex numbers. Furthermore, if $\mathbb{R}$ denote the set of real numbers, then we denote by $\mathbb{R}_+$ the set of non-negative real numbers and by $\overline{\mathbb{R}}_+$ the set $\mathbb{R}_+ \cup \{\infty\}$. Unless otherwise stated, $\mathcal{A}$ will always denote an arbitrary non-empty class of subsets of T.

If A and B are subsets of a given set, then set-theoretic inclusion, proper inclusion and subtraction will be denoted by $A \subseteq B$, $A \subset B$ and $A \backslash B$, respectively. Also, by $A \Delta B$ we will denote the symmetric difference between A and B. If A is a subset of a topological space, then $\overline{A}$ will denote the closure of the set A. Finally, the symbol $\mathbb{N}$ will denote the set of natural numbers and the symbol ■ will indicate the end of the proof of a specific result.

1.1 Set functions and measures

Definition 1.1.1 *A **set function** is a function defined on $\mathcal{A}$ and with values in Y or in $\overline{\mathbb{R}}_+$. The set functions with values in $\overline{\mathbb{R}}_+$ having at least one finite value will be called **positive set functions**. A positive set function μ is **finite on $\mathcal{A}$** if $\mu(A) < \infty$ for every $A \in \mathcal{A}$, and is **σ-finite on $\mathcal{A}$** if every set $A \in \mathcal{A}$ is the union of a sequence $(A_k) \subseteq \mathcal{A}$ such that $\mu(A_k) < \infty$ for every $k \in \mathbb{N}$.*

Definition 1.1.2 *A set function m, defined on $\mathcal{A}$ and with values in Y or in $\overline{\mathbb{R}}_+$, is said to be **additive** if*

$$m(A \cup B) = m(A) + m(B)$$

for every pair $A, B \in \mathcal{A}$ of disjoint sets such that $A \cup B \in \mathcal{A}$.

Definition 1.1.3 *A set function m, defined on $\mathcal{A}$ and with values in a Hausdorff topological vector space Y or in $\overline{I\!R}_+$, is said to be* ***countably additive*** *if*

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m(A_k)$$

for every sequence $(A_k) \subseteq \mathcal{A}$ of mutually disjoint sets such that $\cup_{k=1}^{\infty} A_k \in \mathcal{A}$.

Note that if m is a countably additive set function on $\mathcal{A}$ and with values in a Hausdorff topological vector space Y, and if $\emptyset \in \mathcal{A}$, then $m(\emptyset) = 0$ and m is additive.

Definition 1.1.4 *Let m be a set function, defined on $\mathcal{A}$ and with values in a normed space Y or in $\overline{I\!R}_+$, such that $m(\emptyset) = 0$ if $\emptyset \in \mathcal{A}$. For every $A \subseteq T$ we define the* ***variation of m on A****, denoted by $v(m, A)$, by*

$$v(m, A) = \sup_I \sum_{i \in I} \| m(A_i) \|,$$

where the supremum is taken for all the families of mutually disjoint sets $(A_i)_{i \in I} \subseteq \mathcal{A}$ contained in A. The set function $v(m)$ is called the ***variation of m****.*

Remark 1.1.5

(i) In the above definition the supremum may be taken for all the finite families $(A_i)_{i \in J}$ of mutually disjoint sets of $\mathcal{A}$ contained in A (see Proposition 1 on page 32 of [27]).

(ii) If $\mathcal{A}$ is a ring of subsets of T, then the supremum in the above definition may be taken for all the finite families $(A_i)_{i \in J}$ of mutually disjoint sets of $\mathcal{A}$ such that $\cup_{i \in J} A_i = A$ (see the Corollary on page 32 of [27]).

Definition 1.1.6 *Let m be a set function, defined on $\mathcal{A}$ and with values in a normed space Y or in $\overline{I\!R}_+$, such that $m(\emptyset) = 0$ if $\emptyset \in \mathcal{A}$. We say that m is with* ***finite variation*** *(with respect to $\mathcal{A}$) if $v(m, A) < +\infty$ for every $A \in \mathcal{A}$.*

The restriction of the variation $v(m)$ to the class $\mathcal{A}$ will again be denoted by $v(m)$. Observe that to say that a set function m is with finite variation $v(m)$ is the same as to say that the positive set function $v(m)$ is finite. For most of the properties of the variation we will refer to [27].

For the rest of this chapter we will let $\mathcal{R}$ denote a ring of subsets of T. The next result relates the additivity (countably additivity) of a set function m with the additivity (countably additivity) of its variation $v(m)$.

Theorem 1.1.7 *If Y is a normed space and $m : \mathcal{R} \to Y$ is an additive (countably additive) set function such that $m(\emptyset) = 0$, then $v(m)$ is an additive (countably additive) set function. Conversely, if $m : \mathcal{R} \to Y$ is an additive set function with finite variation $v(m)$ and $v(m)$ is countably additive, then m is also countably additive.*

PROOF: If we denote by $\tau(\mathcal{A})$ the class of all the subsets $B \subseteq T$ such that $A \cap B \in \mathcal{A}$ for every $A \in \mathcal{A}$, then from Proposition 18 on page 12 of [27] follows that $\tau(\mathcal{R})$ is a ring containing $\mathcal{R}$. Furthermore, since $v(m)$ is additive (countably additive) on $\tau(\mathcal{R})$ (by property 9 on page 35 of [27]), it is also additive (countably additive) on $\mathcal{R}$.

Conversely, let $(A_k) \subseteq \mathcal{R}$ be a sequence of mutually disjoint sets such that $A = \cup_{k=1}^{\infty} A_k \in \mathcal{R}$. For every $n \in \mathbb{N}$ we then have that

$$\begin{aligned} \|m(A) - \sum_{k=1}^{n} m(A_k)\| &= \|m(A) - m(\bigcup_{k=1}^{n} A_k)\| \\ &= \|m(\bigcup_{k=n+1}^{\infty} A_k)\| \\ &\leq v(m, \bigcup_{k=n+1}^{\infty} A_k). \end{aligned}$$

From the countably additivity of $v(m)$, we have that

$$\lim_{n\to\infty} v(m, \bigcup_{k=n+1}^{\infty} A_k) = \lim_{j\to\infty} \left(v(m, A) - \sum_{k=1}^{j} v(m, A_k) \right) = 0.$$

Hence $\lim_{j\to\infty} \|m(A) - \sum_{k=1}^{j} m(A_k)\| = 0$ and consequently

$$m(A) = \sum_{k=1}^{\infty} m(A_k).$$

■

Definition 1.1.8 *Let μ be a positive (finite or infinite) set function defined on $\mathcal{R}$. Then we say that a set $A \in \mathcal{R}$*

*(a) is an **atom** (with respect to μ) if $\mu(A) > 0$ and if for every set $B \in \mathcal{R}$ with $B \subseteq A$ we have that $\mu(B) = 0$ or $\mu(B) = \mu(A)$. We say that μ is **atomic** if there exists at least one atom in $\mathcal{R}$, and that μ is **non-atomic** if there exists no atom in $\mathcal{R}$.*

*(b) has the **Darboux property** (with respect to μ) if for every $\alpha \in \mathbb{R}$ such that $0 \leq \alpha \leq \mu(A)$ there exists a set $B \in \mathcal{R}$ with $B \subseteq A$ and $\mu(B) = \alpha$. We say that μ has the Darboux property if every set $A \in \mathcal{R}$ has the Darboux property.*

Definition 1.1.9 *A countably additive set function m, defined on $\mathcal{R}$ and with values in a normed space Y or in $\overline{\mathbb{R}}_+$, is called a **measure**. A measure with values in*

$\overline{I\!R}_+$ *is called a* ***positive measure****. If* μ *is a positive measure on* $\mathcal{R}$*, then we say that a set* $A \in \mathcal{R}$ *has* ***finite measure*** *if* $\mu(A) < \infty$*, and that* A *has* $\boldsymbol{\sigma}$***-finite measure*** *if* A *is the union of a sequence* (A_k) *of sets with finite measure. If every set* $A \in \mathcal{R}$ *has finite (respectively,* σ*-finite) measure, then we say that* μ *is a* ***finite*** *(respectively,* $\boldsymbol{\sigma}$***-finite****)* ***measure****.*

For the most important properties of measures we refer to page 18 of [27]. In addition, by the Corollary on page 28 of [27], note that every σ-finite non-atomic measure on a δ-ring has the Darboux property.

1.2 Extension of set functions

In this section we give a brief outline of the extension of a vector measure with finite variation. We first consider extension results for additive set functions and then study the extension of any vector measure of finite variation. Central to these results is the uniform continuity of the set functions.

We let $\mathcal{S}$ be a ring of subsets of T and μ a positive, finite, subadditive and increasing set function on $\mathcal{S}$. It is known that the function $\rho_\mu : \mathcal{S} \times \mathcal{S} \to I\!R$ defined by

$$\rho_\mu(A, B) = \mu(A \backslash B) + \mu(B \backslash A), \quad A, B \in \mathcal{S},$$

is a finite semi-distance on $\mathcal{S}$.

Proposition 1.2.1 ([27], p61, Lemma 1) *Let* Y *be a Banach space, suppose that* $\mathcal{R}$ *is a ring contained in* $\mathcal{S}$ *and let* $m : \mathcal{R} \to Y$ *be an additive set function. If for every* $A \in \mathcal{R}$

$$\|m(A)\| \leq \mu(A),$$

then m *is uniformly continuous on* $\mathcal{R}$*.*

PROOF: Since m is additive, for all $A, B \in \mathcal{R}$ we have that

$$\begin{aligned} m(A) - m(B) &= m[(A \backslash B) \cup (A \cap B)] - m[(B \backslash A) \cup (A \cap B)] \\ &= m(A \backslash B) - m(B \backslash A). \end{aligned}$$

Consequently, from

$$\begin{aligned} \|m(A) - m(B)\| &\leq \|m(A \backslash B)\| + \|m(B \backslash A)\| \\ &\leq \mu(A \backslash B) + \mu(B \backslash A) \\ &= \rho_\mu(A, B) \end{aligned}$$

follows then that m is uniformly continuous on $\mathcal{R}$. ■

Before we prove our first extension result, we will need the following result, the proof of which can be found in [29], page 23, Theorem 17.

Proposition 1.2.2 *Let U and V be metric spaces, with V complete. If A is a dense subset of U and if $f : A \to V$ is uniformly continuous on A, then f has a unique continuous extension $g : U \to V$. Moreover, g is uniformly continuous on U.*

Theorem 1.2.3 ([27], p62, Theorem 1) *Let $\mathcal{R}$ be dense in $\mathcal{S}$ for the topology defined by ρ_μ, let Y be a Banach space and $m : \mathcal{R} \to Y$ an additive set function such that*

$$\|m(A)\| \leq \mu(A), \quad A \in \mathcal{R}.$$

Then m can be extended to an additive set function $n : \mathcal{S} \to Y$ such that

$$\|n(A)\| \leq \mu(A), \quad A \in \mathcal{S}.$$

Furthermore, if μ is additive, then m has finite variation $v(m)$ on $\mathcal{R}$, n has finite variation $v(n)$ on $\mathcal{S}$ and $v(n)$ is an extension of $v(m)$. If μ is countably additive, then n is also countably additive.

PROOF: By Proposition 1.2.1 m is uniformly continuous on the dense set $\mathcal{R}$ and by Proposition 1.2.2 can be extended to a uniformly continuous set function $n : \mathcal{S} \to Y$. To prove that n is additive, let $A, B \in \mathcal{S}$ be such that $A \cap B = \emptyset$. Then there exist two sequences $(A_k), (B_k) \subseteq \mathcal{R}$ such that

$$\rho_\mu(A_k, A) \to 0 \text{ and } \rho_\mu(B_k, B) \to 0$$

as $k \to \infty$. Since the mappings $(A, B) \mapsto A \cup B$ and $(A, B) \mapsto A \backslash B$ are uniformly continuous on $\mathcal{S}$ (see Lemma 2 on page 61 of [27]), we deduce that

$$\rho_\mu(A_k \cup B_k, A \cup B) \to 0 \text{ and } \rho_\mu(A_k \backslash B_k, A \backslash B) = \rho_\mu(A_k \backslash B_k, A) \to 0$$

as $k \to \infty$. For the disjoint sets $A_k \backslash B_k$ and B_k we then have that

$$m(A_k \cup B_k) = m((A_k \backslash B_k) \cup B_k) = m(A_k \backslash B_k) + m(B_k)$$

so that

$$n(A \cup B) = n(A) + n(B)$$

after taking the limit.

Let now $A \in \mathcal{S}$ and let $(A_k) \subseteq \mathcal{R}$ be a sequence such that $\rho_\mu(A_k, A) \to 0$ as $k \to \infty$. Since m and μ are continuous, we have that

$$m(A_k) \to n(A) \text{ and } \mu(A_k) \to \mu(A)$$

as $k \to \infty$. From $\|m(A_k)\| \leq \mu(A_k)$ follows then that

$$\|n(A)\| \leq \mu(A).$$

Suppose now that μ is additive. Since $\|m(A)\| \leq \mu(A)$ for all $A \in \mathcal{R}$, we deduce that $v(m, A) \leq \mu(A)$; therefore m has finite variation $v(m)$. Consequently, $v(m)$ can be extended to an additive set function $v_1(m)$ on $\mathcal{S}$ such that $v_1(m) \leq \mu$. Furthermore, from the inequality $\|n(A)\| \leq \mu(A)$ follows that n has finite variation $v(n)$ on $\mathcal{S}$ and that $v(n) \leq \mu$.

To show that $v(n)$ is an extension of $v(m)$, first note that from

$$\|m(A)\| = \|n(A)\| \leq v(n, A); \quad A \in \mathcal{R}$$

follows that

$$v(m, A) \leq v(n, A)$$

for $A \in \mathcal{R}$. On the other hand, if $v_1(m)$ is the additive extension of $v(m)$ to $\mathcal{S}$, consider the semi-distance

$$\rho_1(A, B) = v_1(m, A \triangle B).$$

Then from the inequality $v_1(m) \leq \mu$ we deduce that $\rho_1 \leq \rho$; whence the topology defined on $\mathcal{S}$ by ρ_1 is weaker than the topology defined by ρ. This implies that $\mathcal{R}$ is dense in $\mathcal{S}$ for the topology defined by ρ_1, and from $\|m(A)\| \leq v_1(m, A)$ we deduce that m is uniformly continuous on $\mathcal{R}$ for this topology. Consequently, there exists an additive set function $n_1 : \mathcal{S} \to Y$ such that

$$\|n_1(A)\| \leq v_1(m, A)$$

for $A \in \mathcal{S}$. Therefore, n_1 is continuous on $\mathcal{S}$ for the semi-distance ρ_1 and hence also for ρ. Since n and n_1 are continuous on $\mathcal{S}$ for ρ and are equal on the dense set $\mathcal{R}$, it follows that $n = n_1$. Consequently,

$$\|n(A)\| \leq v_1(m, A); \quad A \in \mathcal{S}$$

so that $v(n) \leq v_1(m)$. In particular, $v(n, A) = v_1(m, A)$ for every $A \in \mathcal{R}$. Since $v(n)$ and $v_1(m)$ are continuous on $\mathcal{S}$ for ρ and are equal on $\mathcal{R}$, it follows that $v(n) = v_1(m)$.

Finally, if μ is countably additive, from the inequality $\|n(A)\| \leq \mu(A)$ we deduce that n is also countably additive. ∎

Corollary 1.2.4 *Let Y be a Banach space and $m : \mathcal{R} \to Y$ an additive set function with finite variation $v(m)$. If $v(m)$ can be extended to a positive, finite, additive set function ν on a ring $\mathcal{S} \supseteq \mathcal{R}$ and if $\mathcal{R}$ is dense in $\mathcal{S}$ for the semi-distance ρ_ν, then m can be extended to an additive set function $n : \mathcal{S} \to Y$ with finite variation $v(n)$ such that $v(n) = \nu$.*

We now study the extension of vector measures of bounded variation. For the rest of this section we let Y be a Banach space and we suppose that $\mu : \mathcal{R} \to \overline{I\!R}_+$ is a measure.

We denote by $\mathcal{H}(\mathcal{R})$ the class of all sets $A \subseteq T$ which can be covered by a sequence of sets of $\mathcal{R}$. Then $\mathcal{R} \subseteq \mathcal{H}(\mathcal{R})$ and $\mathcal{H}(\mathcal{R})$ is a *hereditary class*, that is, if $A \in \mathcal{H}(\mathcal{R})$, then $\mathcal{H}(\mathcal{R})$ contains all the subsets of A. Furthermore, $\mathcal{H}(\mathcal{R})$ is a σ-ring and we call $\mathcal{H}(\mathcal{R})$ the *hereditary σ-ring generated by $\mathcal{R}$.*

The extension of μ will be obtained in two steps: First μ will be extended to a set function μ^* on $\mathcal{H}(\mathcal{R})$. If we restrict μ^* to a certain σ-ring $\mathcal{T}(\mu)$, then μ^* becomes a measure. In the second step we extend μ^* from $\mathcal{T}(\mu)$ to the σ-ring $\mathcal{M}(\mu)$ of μ-measurable sets. The set function μ^* is then a measure on $\mathcal{M}(\mu)$.

Definition 1.2.5 *For every set $A \in \mathcal{H}(\mathcal{R})$ we define the* ***outer measure of A****, denoted by $\mu^*(A)$, by*

$$\mu^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \mu(A_k) \mid (A_k) \subseteq \mathcal{R},\ A \subseteq \bigcup_{k=1}^{\infty} A_k \right\}.$$

The set function μ^* defined on $\mathcal{H}(\mathcal{R})$ is called the *outer measure* induced by μ. For some of the properties of an outer measure we refer to page 64 of [27].

Definition 1.2.6 *We denote by $\mathcal{T}(\mu)$ the class of all sets $B \in \mathcal{H}(\mathcal{R})$ such that*

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \backslash B)$$

for every $A \in \mathcal{H}(\mathcal{R})$. We denote by $\mathcal{M}(\mu)$ the class of all sets $B \subseteq T$ such that $A \cap B \in \mathcal{T}(\mu)$ for every $A \in \mathcal{T}(\mu)$. The sets in $\mathcal{M}(\mu)$ are called ***μ-measurable sets****.*

The proofs of the following results about the classes $\mathcal{T}(\mu)$ and $\mathcal{M}(\mu)$ can all be found on pages 68-72 in [27].

Theorem 1.2.7

(a) The class $\mathcal{T}(\mu)$ is a σ-ring containing the ring $\mathcal{R}$ and μ^* is countably additive on $\mathcal{T}(\mu)$.

(b) The class $\mathcal{M}(\mu)$ is a σ-algebra containing $\mathcal{T}(\mu)$, and $A \in \mathcal{M}(\mu)$ if and only if $A \cap B \in \mathcal{T}(\mu)$ for every $B \in \mathcal{R}$.

(c) If μ is σ-finite on $\mathcal{R}$, then a set $A \subseteq T$ belongs to $\mathcal{T}(\mu)$ if and only if $A = B \backslash N$, where $B \subseteq T$ belongs to $\mathcal{S}(\mathcal{R})$, the σ-ring generated by $\mathcal{R}$, and $N \subseteq T$ is such that $\mu^*(N) = 0$.

Theorem 1.2.8 *If μ is σ-finite on $\mathcal{R}$, then μ^* is the unique countably additive positive extension of μ from $\mathcal{R}$ to $\mathcal{T}(\mu)$.*

We now extend μ^* from $\mathcal{T}(\mu)$ to $\mathcal{M}(\mu)$. But first we need to make the following

Definition 1.2.9 *For every set $A \in \mathcal{M}(\mu)$ we define the **outer-measure** $\mu^*(A)$ by*

$$\mu^*(A) = \sup\{\mu^*(B) \mid B \subseteq A, \ B \in \mathcal{T}(\mu)\}.$$

If we put $\nu(A) = \mu^*(A)$ for every $A \in \mathcal{T}(\mu)$, then from the countable additivity of ν we deduce that $v(\nu)$ is also countably additive. Then, since $v(\nu, A) = \mu^*(A)$ for every $A \in \mathcal{M}(\mu)$ (from the Corollary on page 32 of [27]), we have

Proposition 1.2.10 *The outer measure μ^* is positive and countably additive on $\mathcal{M}(\mu)$.*

Definition 1.2.11 *The sets $E \in \mathcal{M}(\mu)$ with $\mu^*(E) = 0$ are called **μ-negligible**. If a property $P(t)$, defined for all $t \in T$, is true for all the points of T except for a μ-negligible set, then we say that the property $P(t)$ is true **μ-almost everywhere**.*

Proposition 1.2.12 ([27], p74, Proposition 9 and 10)

(a) Every set $A \in \mathcal{H}(\mathcal{R})$ with $\mu^*(A) = 0$ is μ-negligible.

(b) A set $A \in \mathcal{M}(\mu)$ is μ-negligible if and only if every set $B \in \mathcal{T}(\mu)$ with $B \subseteq A$ is μ-negligible.

(c) If $A \cap B$ is μ-negligible for every set $A \in \mathcal{R}$, then B is μ-negligible.

(d) Every subset of a μ-negligible set is μ-negligible and the union of a sequence of μ-negligible sets is μ-negligible.

Proposition 1.2.13 ([27], p75, Proposition 11) *Let ν_1 and ν_2 be two countably additive, positive, σ- finite set functions on $\mathcal{R}$. If $\nu_1 \leq \nu_2$, then $\mathcal{M}(\nu_2) \subseteq \mathcal{M}(\nu_1)$ and*

$$\nu_1^*(A) \leq \nu_2^*(A)$$

for $A \in \mathcal{M}(\nu_2)$.

PROOF: Obviously, $\nu_1^*(A) \leq \nu_2^*(A)$ for $A \in \mathcal{H}(\mathcal{R})$. This means that every ν_2-negligible set $A \in \mathcal{H}(\mathcal{R})$ is also ν_1-negligible. From Corollary 1 on page 71 of [27] follows that $\mathcal{T}(\nu_2) \subseteq \mathcal{T}(\nu_1)$. As a consequence of Theorem 1.2.7(b) it follows that $\mathcal{M}(\nu_2) \subseteq \mathcal{M}(\nu_1)$. From the definition of the outer measure we deduce that

$$\nu_1^*(A) \leq \nu_2^*(A)$$

for all $A \in \mathcal{M}(\nu_2)$. ■

For an application of Proposition 1.2.13, see Proposition 5.2.4.

Definition 1.2.14 *If $A \in \mathcal{M}(\mu)$ and $\mu^*(A) < \infty$, then we say that A is* ***μ-integrable****. We denote the class of the μ-integrable sets by $\Sigma(\mu)$. For every μ-integrable set $A \subseteq T$ we define the measure $\mu(A)$ by*

$$\mu(A) = \mu^*(A).$$

We say that a set $A \in \mathcal{M}(\mu)$ has ***σ-finite measure*** *if A is the union of a sequence of μ-integrable sets.*

We note that every μ-negligible set is μ-integrable, and μ is a finite and complete measure on $\Sigma(\mu)$, that is, if A is a μ-negligible set of $\Sigma(\mu)$, then every set $B \subseteq A$ belongs to $\Sigma(\mu)$.

Theorem 1.2.15 ([27], p76, Theorem 3) *Let $m : \mathcal{R} \to Y$ be a measure with finite variation $v(m)$ and $\mathcal{S}$ a ring such that $\mathcal{R} \subseteq \mathcal{S} \subseteq \Sigma(v(m))$. Then m can be extended to a measure $n : \mathcal{S} \to Y$, with finite variation $v(n)$, such that $\mathcal{M}(v(m)) = \mathcal{M}(v(n))$ and $v^*(m) = v^*(n)$.*

1.3 Measurable functions

As in the previous sections, we let X be any vector space, T is a non-empty point set and $\mathcal{A}$ is a non-empty class of subsets of T.

Definition 1.3.1 *A function $f : T \to X$ is called an* ***$\mathcal{A}$-step function*** *if it is of the form*

$$f = \sum_{i \in I} x_i \chi_{A_i},$$

where I is a finite index set, χ_{A_i} is the characteristic function of the set A_i, $A_i \in \mathcal{A}$ and $x_i \in X$ for every $i \in I$. The set of $\mathcal{A}$-step functions $f : T \to X$ will be denoted by $\mathcal{E}_X(\mathcal{A})$.

Remark 1.3.2

(i) The set $\mathcal{E}_X(\mathcal{A})$ is a vector space.

(ii) We will simply write $\mathcal{E}_X(\mu)$ to denote the space of all $\Sigma(\mu)$-step functions.

(iii) If $\mathcal{A}$ is a ring of subsets of T, then the sets A_i in the above definition can be taken to be mutually disjoint (Proposition 1 on page 82 of [27]).

(iv) If $\mathcal{R}$ is a ring of subsets of T and the function $f \in \mathcal{E}_X(\mathcal{R})$ is not identically null, then we can write f uniquely in the form

$$f = \sum_{j \in J} y_j \chi_{B_j},$$

where J is a finite index set, $B_j \in \mathcal{R}$ are mutually disjoint and $y_j \in X$ are distinct from each other and from 0. Such a representation of f will be called the *standard representation* of f.

(v) If X is any arbitrary set, then a function $f : T \to X$ is called an $\mathcal{A}$-step function if the value $f(X)$ is finite.

Let $\mathcal{S}$ be a σ-algebra of subsets of T. Then any set which belongs to $\mathcal{S}$ will be called an $\mathcal{S}$-*measurable* set. If X is a topological space, then we denote the Borel σ-algebra of X by $\mathcal{B}_X$.

Definition 1.3.3 *A function $f : T \to X$ is said to be* ***$\mathcal{S}$-measurable*** *if $f^{-1}(B) \in \mathcal{S}$ for every $B \in \mathcal{B}_X$.*

Remark 1.3.4

(i) If we denote by $\mathcal{G}$ the class of all subsets of X generating the σ-ring $\mathcal{B}_X$, that is if $\mathcal{S}(\mathcal{G}) = \mathcal{B}_X$, then $f : T \to X$ is $\mathcal{S}$-measurable whenever $f^{-1}(A) \in \mathcal{S}$ for every $A \in \mathcal{G}$. Indeed, let $\mathcal{M}$ be the class of all sets $A \subseteq X$ such that $f^{-1}(A) \in \mathcal{S}$. Then clearly $\mathcal{M}$ is a σ-ring containing $\mathcal{G}$, hence containing $\mathcal{B}_X$ also; therefore f is $\mathcal{S}$-measurable.

(ii) From (i) follows that a function $f : T \to X$ is $\mathcal{S}$-measurable if and only if the set $f^{-1}(C)$ ($f^{-1}(O)$, respectively) is $\mathcal{S}$-measurable for every closed (open, respectively) subset C (O, respectively) of X.

For the rest of this section we let X be a Hausdorff topological space and μ a positive measure on the ring $\mathcal{R}$.

Definition 1.3.5 *A function $f : T \to X$ is said to be* ***μ-measurable*** *if and only if*

(a) for every closed set $C \subseteq X$, the set $f^{-1}(C)$ is μ-measurable;

(b) for every μ-integrable set $A \subseteq T$ there exists a μ-negligible set $N \subseteq A$ and a countable set $H \subseteq X$ such that $f(A \backslash N) \subseteq \overline{H}$.

Remark 1.3.6

(i) It follows from Remark 1.3.4 that condition (a) in the above definition may be replaced by any one of the following equivalent statements:
(a') For every open subset O of X, the set $f^{-1}(O)$ is μ-measurable.
(a'') For every Borel subset B of X, the set $f^{-1}(B)$ is μ-measurable.

(ii) Condition (b) in the above definition can be replaced by any one of the following equivalent statements:
(b') for every set $A \in \mathcal{R}$ there exists a μ-negligible set $N \subseteq A$ and a countable set $H \subseteq X$ such that $f(A \backslash N) \subseteq \overline{H}$.
(b'') for every μ-measurable set $A \subseteq T$ with σ-finite measure, there exists a μ-negligible set $N \subseteq A$ and a countable set $H \subseteq X$ such that $f(A \backslash N) \subseteq \overline{H}$.

(iii) If X is a separable space (in particular, if $X = \overline{\mathbb{R}}$), then condition (b) in the above definition is superfluous.

Example 1.3.7

1. A set $A \subseteq T$ is μ-measurable if and only if the characteristic function χ_A of A is μ-measurable.

2. A function $f : T \to X$ taking on a finite set of different values $a_1, a_2, \ldots, a_n$ is μ-measurable if and only if the set $f^{-1}(\{a_k\})$ is μ-measurable for $k = 1, 2, 3, \ldots, n$.

3. A function $f : T \to X$ taking on a countable set of different values $a_1, a_2, \ldots$ is μ-measurable if and only if the set $f^{-1}(\{a_k\})$ is μ-measurable for $k \in \mathbb{N}$.

4. If X is a Hausdorff topological vector space, then every $f \in \mathcal{E}_X(\mathcal{M}(\mu))$ is μ-measurable.

5. If X is a Hausdorff topological vector space, then every μ-negligible function $f : T \to X$ is μ-measurable.

We now list a few results (which will be needed in the sequel) on measurable functions. Some of the shorter proofs will be included and we refer to [27] for the rest of the results. Our first result follows immediately from the definition.

Proposition 1.3.8 *If $f : T \to X$ is a μ-measurable function and if $g : T \to X$ is a function such that $f(t) = g(t)$ μ-almost everywhere on T, then g is μ-measurable.*

Proposition 1.3.9 ([27], p91, Proposition 10) *If $f : T \to X$ is a μ-measurable function and if the function $g : T \to X$ is equal to f on a μ-measurable set $A \subseteq T$ and constant on $T \backslash A$, then g is μ-measurable.*

PROOF: We only verify condition (a) in Definition 1.3.5; condition (b) follows easily. Let O be any open subset of X. If $g(T\backslash A) \notin O$, then $g^{-1}(O) = f^{-1}(O) \cap A \in \mathcal{M}(\mu)$. On the other hand, if $g(T\backslash A) \in O$, then $g^{-1}(O) = f^{-1}(O) \cup T\backslash A \in \mathcal{M}(\mu)$, which concludes the proof. ■

Proposition 1.3.10 ([27], p93, Proposition 12) *If X is a normed space with topological dual space X', then a function $f : T \to X$ is μ-measurable if and only if*

(a) for every $x' \in X'$, the function $t \mapsto (x', f(t))$ is μ-measurable.

(b) for every μ-integrable set $A \subseteq T$ there exists a μ-negligible set $N \subseteq A$ and a countable set $H \subseteq X$ such that $f(A\backslash N) \subseteq \overline{H}$.

Theorem 1.3.11 (Egorov, [27], p94, Theorem 1) *Let X be a metric space and (f_k) a sequence of μ-measurable functions defined on T and with values in X. If (f_k) converges μ-almost everywhere to a function $f : T \to X$ then*

(a) f is μ-measurable;

(b) for every μ-integrable set $A \subseteq T$ and every $\epsilon > 0$, there exists a set $B \in \mathcal{D}(\mathcal{R})$, the δ-ring generated by $\mathcal{R}$, with $B \subseteq A$ and $\mu(A\backslash B) < \epsilon$, such that (f_k) converges uniformly to f on B.

The following two results are corollaries of Egorov's theorem and we include the proofs for completeness.

Corollary 1.3.12 ([27], p96, Corollary 2) *Let X be a metric space, (f_k) a sequence of μ-measurable functions defined on T and with values in X and let A be a μ-measurable set such that (f_k) converges μ-almost everywhere on A. If $f : T \to X$ is a function equal μ-almost everywhere to the limit of (f_k) on A and constant on $T\backslash A$, then $f : T \to X$ is μ-measurable.*

PROOF: Let $f(t) = a \in X$ for $t \in T\backslash A$. For $k \in \mathbb{N}$ define the function

$$g_k(t) = \begin{cases} f_k(t) & \text{if } t \in A \\ a & \text{if } t \in T\backslash A. \end{cases}$$

Then each g_k is μ-measurable (by Proposition 1.3.9). Since $g_k \to f$ μ-almost everywhere on T, it follows from Egorov's theorem that f is μ-measurable. ■

Corollary 1.3.13 ([27], p96, Corollary 3) *Let X be a metric space and $f : T \to X$ a function. If for every set $A \in \mathcal{R}$ there exists a sequence (f_k) of μ-measurable functions converging to $f : T \to X$ μ-almost everywhere on A, then $f : T \to X$ is μ-measurable.*

PROOF: Let $A \in \mathcal{R}$ and $a \in X$ a constant. Define the function $f_A : T \to X$ by

$$f_A(t) = \begin{cases} f(t) & \text{if } t \in A \\ a & \text{if } t \in T \backslash A. \end{cases}$$

Then by Corollary 1.3.12 it follows that f_A is μ-measurable. From Remark 1.3.6(ii) we obtain a μ-negligible set $N \subseteq A$ and a countable set $H \subseteq X$ such that $f_A(A \backslash N) \subseteq \overline{H}$. Consequently, $f(A \backslash N) \subseteq \overline{H}$, and condition (b) of Definition 1.3.5 is satisfied.

To prove condition (a) of Definition 1.3.5, let O be any open subset of X. If $O = X$, then $f^{-1}(O) = T \in \mathcal{M}(\mu)$. On the other hand, if $O \subset X$, then for every $A \in \mathcal{R}$ we choose f_A such that $f_A(t) = a \notin O$ for every $t \in T \backslash A$. From the μ-measurability of f_A follows that the set

$$f^{-1}(O) \cap A = \{t \in A \mid f(t) \in O\} = \{t \in T \mid f_A(t) \in O\} = f_A^{-1}(O)$$

is μ-measurable, that is, $f^{-1}(O) \cap A \in \mathcal{M}(\mu)$. From Theorem 1.2.7(b) we then have that

$$f^{-1}(O) \cap A = (f^{-1}(O) \cap A) \cap A \in \mathcal{T}(\mu).$$

By applying Theorem 1.2.7(b) once more, it follows that $f^{-1}(O) \in \mathcal{M}(\mu)$. ■

Proposition 1.3.14 ([27], p97, Proposition 13) *If $f : T \to X$ is a μ-measurable function taking on a countable set of values, then there exists a sequence (f_k) of μ-measurable step functions converging to $f : T \to X$ on T.*

PROOF: Let $a_1, a_2, \ldots$ be the values of f. Then for $k \in \mathbb{N}$, the set $A_k = f^{-1}(\{a_k\})$ is μ-measurable. If $a \in X$ is a constant, then the step function $f_k : T \to X$, defined by

$$f_k(t) = \begin{cases} a_i & \text{if } t \in A_i, 1 \leq i \leq k \\ a & \text{if } t \in T \backslash \cup_{i=1}^{k} A_i, \end{cases}$$

is also μ-measurable, and the sequence (f_k) converges to f on T. ■

Proposition 1.3.15 ([27], p97, Proposition 14) *Let X be a metric space and $f : T \to X$ a μ-measurable function. For every μ-measurable set $A \subseteq T$ with σ-finite measure, there exists a μ-negligible set $N \subseteq A$ and a sequence (f_k) of μ-measurable functions (with each of them taking on a countable set of values) such that (f_k) converges uniformly to $f : T \to X$ on $A \backslash N$. If X is a normed space, we can choose the sequence (f_k) such that $\|f_k(t)\| \leq \|f(t)\|$ for every $k \in \mathbb{N}$ and $t \in T$.*

From Proposition 1.3.9 and Corollary 1.3.13 of this thesis, and Theorem 2 on page 99 of [27] we have the following

Corollary 1.3.16 *Let X be a normed space. A function $f : T \to X$ is μ-measurable if and only if $f\chi_A$ is μ-measurable for every set $A \in \mathcal{R}$.*

Theorem 1.3.17 ([27], p100, Theorem 3) *Let (X_k) be a sequence of metric spaces, $X = \Pi_k X_k$ their cartesian product and Y a metric space. For $k \in \mathbb{N}$ let $f_k : T \to X_k$ be a μ-measurable function and let $f : T \to X$ be the function defined by the equality*

$$f(t) = f_k(t) \ \textit{for} \ t \in T.$$

For every continuous mapping $g : X \to Y$, the function $g \circ f : T \to Y$ is μ-measurable.

PROOF: Let $A \subseteq T$ be a μ-integrable set. Then for $k \in \mathbb{N}$ there exists a μ-negligible set $N_k \subseteq A$ and a sequence $(f_{k,p})_p$ of μ-measurable step functions converging to f_k on $A \backslash N_k$. The set $N = \cup_{k=1}^{\infty} N_k$ is then μ-negligible and for every $t \in A \backslash N$ and $n \in \mathbb{N}$ we then have $\lim_{p\to\infty} f_{k,p}(t) = f_k(t)$. For each $p \in \mathbb{N}$, the functions $f_p = (f_{k,p}) : T \to X$ has a countable set of values and is μ-measurable. Consequently,

$$\lim_{p\to\infty} f_p(t) = f(t) \ \text{ for } \ t \in A \backslash N$$

so that

$$\lim_{p\to\infty} (g \circ f_p)(t) = \lim_{p\to\infty} g(f_p(t)) = g(f(t)) \ \text{ for } \ t \in A \backslash N.$$

The functions $g \circ f_p$ are μ-measurable and have a countable set of values. From Corollary 1.3.13 it follows that $g \circ f$ is μ-measurable. ∎

Corollary 1.3.18 *Let X be a normed space, $f, g : T \to X$ two μ-measurable functions and c a scalar. Then the functions $f + g$, cf and $\|f\|$ are μ-measurable. If $X = \mathbb{R}$, then the function fg is also μ-measurable.*

If X and Z are Banach spaces, then by $\mathcal{L}^*(X, Z)$ ($\mathcal{L}(X, Z)$, respectively) we denote the vector space of linear (respectively, linear continuous) mappings of X into Z. For every $\alpha \in \mathcal{L}^*(X, Z)$ we put

$$\|\alpha\| = \sup\{\|\alpha(x)\| : x \in X, \ \|x\| \leq 1\}.$$

Definition 1.3.19 *We say that a function $U : T \to \mathcal{L}^*(X, Z)$ is* ***simply μ-measurable*** *if for every $x \in X$ the function $f_x : T \to Z$, defined by $f_x(t) = U(t)x$, is μ-measurable.*

Remark 1.3.20

(i) If m is a measure with finite variation $v(m)$, then we say that U is simply m-measurable if U is simply $v(m)$- measurable.

(ii) If X is the space of all scalars, then $\mathcal{L}^*(X, Z) = Z$, and U is simply μ-measurable if and only if U is μ-measurable.

(iii) Every μ-measurable function $U : T \to \mathcal{L}(X, Z)$ is also simply μ-measurable (see Corollary 3 on page 101 of [27]), but the converse is not true in general.

(iv) If $U, V : T \to \mathcal{L}^*(X, Z)$ are simply μ-measurable functions, then $U + V$ and cU are also simply μ-measurable.

For the rest of this section we will suppose that W is a *norming subspace* of Z', that is,

$$\|z\| = \sup\left\{\frac{|(z, w)|}{\|w\|} : w \in W,\ w \neq 0\right\}$$

for every $z \in Z$.

Definition 1.3.21 *We say that a function $U : T \to \mathcal{L}^*(X, Z)$ is **W-weakly μ-measurable** if for every $x \in X$ and for every $w \in W$, the function $f_{x,w} : T \to \mathbb{R}$, defined by $f_{x,w}(t) = (U(t)x, w)$, is μ-measurable.*

Remark 1.3.22

(i) To say that U is W-weakly μ-measurable means that for every $w \in W$ the function $U \circ w : T \to \mathcal{L}(X, \mathbb{C})$ is simply μ-measurable.

(ii) We say that a function $f : T \to Z$ is W-weakly μ-measurable if, considered with values in $\mathcal{L}(\mathbb{R}, Z)$, it is W-weakly μ-measurable, that is, for every $w \in W$ the function (f, w) is μ-measurable.

(iii) To say that a function $U : T \to \mathcal{L}^*(X, Z)$ is W-weakly μ-measurable means that for every $x \in X$ the function $f_x : T \to Z$, defined by $f_x(t) = U(t)x$, is W-weakly μ-measurable.

(iv) If Z is the space of scalars, then to say that a function $U : T \to \mathcal{L}(X, \mathbb{C}) = X'$ is X-weakly μ-measurable means that U is simply μ-measurable, that is, for every $x \in X$ the function Ux is μ-measurable.

(v) If $U, V : T \to \mathcal{L}^*(X, Z)$ are W-weakly μ-measurable, then $U + V$ and cU are also W-weakly μ-measurable.

We will refer to [27], pages 101-106, for the properties of simply and W-weakly μ-measurable functions.

CHAPTER 2

MULTIFUNCTIONS AND MULTIMEASURES

This chapter is devoted to the basic theory of multifunctions and multimeasures. The standard reference for the section on measurable multifunctions will be [51] and most of the results on multimeasures can be found in [36].

Throughout this chapter we shall employ our standard notations concerning the non-empty point set T, the ring $\mathcal{R}$ of subsets of T, the positive measure μ on $\mathcal{R}$, the δ-ring $\Sigma(\mu)$ of all μ-integrable subsets of T and the σ-ring $\mathcal{M}(\mu)$ of all μ-measurable subsets of T.

2.1 Measurable multifunctions

We let X be a non-empty point set and let $\mathcal{P}_0(X)$ denote the class of all subsets of X. If with each element t of T we associate the subset $F(t)$ of X, then we say that the mapping $t \mapsto F(t)$ is a *multifunction* of T into X, sometimes denoted by $F : T \to X$. A multifunction F can also be regarded as a single-valued function from T into $\mathcal{P}_0(X)$, and in this case we write $F : T \to \mathcal{P}_0(X)$. We shall employ the latter notation throughout.

Let $F : T \to \mathcal{P}_0(X)$ be a multifunction. Then we define the *domain* of F, denoted by D_F, by

$$D_F = \{t \in T \mid F(t) \neq \emptyset\},$$

and the *range* of F, denoted by R_F, by

$$R_F = \bigcup_{t \in T} F(t).$$

Furthermore, if $A \subseteq T$, then we put

$$F(A) = \bigcup_{t \in A} F(t)$$

and we call $F(A)$ the *image* of A under F. If $\mathcal{P}(X)$ denotes the class of all non-empty subsets of X and if $F : T \to \mathcal{P}(X)$ is a multifunction, then the *graph* of F, denoted by Gr_F, is defined by

$$Gr_F = \{(t, x) \in T \times X \mid x \in F(t)\}.$$

Definition 2.1.1 *Suppose that $F : T \to \mathcal{P}_0(X)$ is a multifunction and let $A \in \mathcal{P}_0(X)$. Then*

(a) the ***upper inverse*** *of F, denoted by F^+, is defined by*

$$F^+(A) = \{t \in D_F \mid F(t) \subseteq A\};$$

(b) the ***lower inverse*** *of F, denoted by F^-, is defined by*

$$F^-(A) = \{t \in T \mid F(t) \cap A \neq \emptyset\}.$$

Definition 2.1.2 *A multifunction $F : T \to \mathcal{P}_0(X)$ is said to be* ***μ-measurable*** *if and only if*

(a) for every closed subset C of X, the set $F^-(C)$ is μ-measurable;

(b) for every μ-integrable set $A \subseteq T$ there exists a μ-negligible set $N \subseteq A$ and a countable set $H \subseteq X$ such that $F(A \backslash N) \subseteq \overline{H}$.

Remark 2.1.3

(i) The above definition of the measurability of a multifunction is more restrictive than the original one that appeared in [14]. This definition is in fact only the set-valued version of Definition 1.3.5.

(ii) Referring to Remark 1.3.6(i), it is no longer true that the set-valued analogues of (a') and (a'') in Remark 1.3.6(i) are equivalent to condition (a) in Definition 2.1.2. In fact, as we will see in Proposition 2.1.6 and Corollary 2.1.7, some additional requirements on the multifunction F and the sets T and X will be needed.

(iii) If X is separable, then condition (b) in Definition 2.1.2 is superfluous.

Proposition 2.1.4 ([51], p33, Lemma 4.1) *A function $f : T \to X$ is μ-measurable if and only if the multifunction $F : T \to \mathcal{P}_0(X)$, defined by*

$$F(t) = \{f(t)\} \text{ for all } t \in T$$

is μ-measurable.

PROOF: Let C be a closed subset of X. Then condition (a) in Definition 2.1.2 follows from

$$f^{-1}(C) = \{t \in T \mid f(t) \in C\} = \{t \in T \mid F(t) \subseteq C\} = \{t \in T \mid F(t) \cap C \neq \emptyset\} = F^-(C).$$

For condition (b), let $A \in \Sigma(\mu)$, $N \subseteq A$, $\mu(N) = 0$ and H a countable subset of X. Then

$$F(A\backslash N) = \bigcup_{t \in A\backslash N} F(t) = \bigcup_{t \in A\backslash N} \{f(t)\} = \{f(t) \mid t \in A\backslash N\} = f(A\backslash N) \subseteq \overline{H}.$$

■

Proposition 2.1.5 ([51], p36, Lemma 4.3) *If each $F_k : T \to \mathcal{P}_0(X)$ is a μ-measurable multifunction, then the multifunction $F : T \to \mathcal{P}_0(X)$, defined for all $t \in T$ by*

$$F(t) = \bigcup_{k=1}^{\infty} F_k(t),$$

is μ-measurable.

PROOF: (a) Let C be a closed subset of X. The implications

$$\begin{aligned} t_0 \in F^-(C) &\iff F(t_0) \cap C \neq \emptyset \\ &\iff F_{k_0}(t_0) \cap C \neq \emptyset \text{ for some } k_0 \in I\!N \\ &\iff t_0 \in \{t \in T \mid F_{k_0}(t) \cap C \neq \emptyset\} \text{ for some } k_0 \in I\!N \\ &\iff t_0 \in \bigcup_{k=1}^{\infty} F_k^-(C) \end{aligned}$$

show that $F^-(C) = \cup_{k=1}^{\infty} F_k^-(C)$. Since $F_k^-(C) \in \mathcal{M}(\mu)$ for $k \in I\!N$, it follows that $\cup_{k=1}^{\infty} F_k^-(C)$, and hence $F^-(C)$, belongs to $\mathcal{M}(\mu)$.

To prove condition (b) in Definition 2.1.2, let A be any μ-integrable set. For $k \in I\!N$ there exists a μ-negligible set N_k, with $N_k \subseteq A$, and a countable set $H_k \subseteq X$ such that $F_k(A \backslash N_k) \subseteq \overline{H}_k$. If we put $N = \cup_{k=1}^{\infty} N_k$, then

$$F(A\backslash N) = \bigcup_{t \in A\backslash N} F(t) \subseteq \bigcup_{k=1}^{\infty} \bigcup_{t \in A\backslash N_k} F_k(t) = \bigcup_{k=1}^{\infty} F_k(A\backslash N_k) \subseteq \bigcup_{k=1}^{\infty} \overline{H}_k \subseteq \overline{\bigcup_{k=1}^{\infty} H_k},$$

and the proof is finished. ■

If we recall that the union (intersection, respectively) of a countable number of closed (open, respectively) subsets of a topological space is called an F_σ-set (a G_δ-set, respectively), we have the following result.

Proposition 2.1.6 *Suppose that every open subset of X is an F_σ-set and let $F : T \to \mathcal{P}_0(X)$ be a multifunction. If $F^-(C) \in \mathcal{M}(\mu)$ for every closed subset C of X, then $F^-(O) \in \mathcal{M}(\mu)$ for all open subsets O of X.*

Corollary 2.1.7 *If every open subset of X is an F_σ-set and if $F : T \to \mathcal{P}_0(X)$ is a multifunction such that $F^-(C) \in \mathcal{M}(\mu)$ for every closed subset C of X, then*

(a) for every closed subset C of X, the set $\{t \in T \mid F(t) \subseteq C\}$ is μ-measurable;

(b) for every closed subset C of X, the set $F^+(C)$ is μ-measurable;

(c) for every open subset O of X, the set $F^-(O)$ is μ-measurable.

PROOF: (a) Let C be any closed subset of X. If $C = X$, then $\{t \in T \mid F(t) \subseteq C\} = T \in \mathcal{M}(\mu)$. So suppose that $C \subset X$ and put $O = X \backslash C$. Then, by Proposition 2.1.6, $F^-(O) \in \mathcal{M}(\mu)$. But

$$F^-(O) = \{t \in T \mid F(t) \cap O \neq \emptyset\} = T \backslash \{t \in T \mid F(t) \subseteq C\},$$

so that $\{t \in T \mid F(t) \subseteq C\} = T \backslash F^-(O) \in \mathcal{M}(\mu)$.

(b) Let C be any closed subset of X. The set $D_F = \{t \in T \mid F(t) \neq \emptyset\} = \{t \in T \mid F(t) \cap X \neq \emptyset\} = F^-(X)$ is μ-measurable. Hence, $T \backslash D_F \in \mathcal{M}(\mu)$. From $\{t \in T \mid F(t) \subseteq C\} = \{t \in T \mid F(t) = \emptyset\} \cup \{t \in D_F \mid F(t) \subseteq C\}$ and (a) above follows that

$$\begin{aligned}\{t \in D_F \mid F(t) \subseteq C\} &= \{t \in T \mid F(t) \subseteq C\} \backslash \{t \in T \mid F(t) = \emptyset\} \\ &= \{t \in T \mid F(t) \subseteq C\} \backslash (T \backslash D_F) \in \mathcal{M}(\mu).\end{aligned}$$

(c) Let O be any open subset of X. Then from Proposition 2.1.6 we have

$$\{t \in D_F \mid F(t) \cap O \neq \emptyset\} = \{t \in T \mid F(t) \cap O \neq \emptyset\} \in \mathcal{M}(\mu).$$

■

If X is a topological vector space with topological dual X', then by $\mathcal{P}_{f(b)}(X)$ (respectively, $\mathcal{P}_k(X)$) we will denote the closed (bounded) (respectively, compact) sets in $\mathcal{P}(X)$. A c after $f(b)$ or k will mean that the set is in addition convex. A w in front of $f(b)$ (respectively, k) means that the closedness (respectively, compactness) is with respect to the weak topology $w(X, X')$.

We now let (X, d) be a metric space. Then the distance between a point $x \in X$ and a non-empty set $A \subseteq X$ is defined as

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

Furthermore, for any $A, B \in \mathcal{P}_k(X)$, we define their *Hausdorff semi-metric* by

$$d(A, B) = \sup\{d(a, B) \mid a \in A\},$$

and their *Hausdorff metric* by

$$H(A, B) = \max\{d(A, B), d(B, A)\}.$$

Whenever we refer to the metric space $\mathcal{P}_k(X)$, it must be understood that $\mathcal{P}_k(X)$ is equipped with the Hausdorff metric H. The following result (the proof of which can be found on page 354 of [24]) shows that the properties of completeness, compactness and separability carry over from X to $\mathcal{P}_k(X)$.

Theorem 2.1.8

(a) If (X, d) is a complete metric space, then $(\mathcal{P}_k(X), H)$ is a complete metric space.

(b) If (X, d) is a compact metric space, then $(\mathcal{P}_k(X), H)$ is a compact metric space.

(c) If (X, d) is a separable metric space, then $(\mathcal{P}_k(X), H)$ is a separable metric space.

Proposition 2.1.9 ([51], p40, Lemma 3.2) *Let X be a metric space and suppose that $F : T \to \mathcal{P}_k(X)$ is a multifunction. If $F^-(O) \in \mathcal{M}(\mu)$ for every open subset O of X, then $F^-(C) \in \mathcal{M}(\mu)$ for every closed subset C of X.*

PROOF: First note that $D_F = \{t \in T \mid F(t) \cap X \neq \emptyset\} = F^-(X) \in \mathcal{M}(\mu)$. Let C be any non-empty closed subset of X. For $k \in I\!N$ define the open sets O_k by $O_k = \{x \in X \mid d(x, C) < \frac{1}{k}\}$. Since $C = \{x \in X \mid d(x, C) = 0\}$, we have that $C \subseteq O_k$ and therefore $F^-(C) \subseteq F^-(O_k)$ for $k \in I\!N$. Consequently, we have that $F^-(C) \subseteq \cap_{k=1}^{\infty} F^-(O_k)$. For the inverse inclusion, let $t_0 \in \cap_{k=1}^{\infty} F^-(O_k)$. Then $F(t_0) \cap O_k \neq \emptyset$ for $k \in I\!N$. If $x_k \in F(t_0) \cap O_k$, then $d(x_k, C) < \frac{1}{k}$ for $k \in I\!N$. Furthermore, from the compactness of $F(t_0)$ we obtain a subsequence (x_{k_n}) of (x_k) such that $x_{k_n} \to x \in F(t_0)$ as $n \to \infty$. Consequently, $\lim_{n\to\infty} d(x_{k_n}, C) = d(x, C) = 0$. The closedness of C implies that $x \in C$. We then have that $F(t_0) \cap C \neq \emptyset$, that is, $t_0 \in F^-(C)$. Therefore the inverse inclusion follows, and

$$F^-(C) = \bigcap_{k=1}^{\infty} F^-(O_k) \in \mathcal{M}(\mu).$$

■

Proposition 2.1.10 ([51], p71, Proposition 7.11) *Let X be a separable locally compact metric space and suppose that $F : T \to \mathcal{P}_0(X)$ is a multifunction. Then $F : T \to \mathcal{P}_0(X)$ is μ-measurable if and only if $F^{-1}(K) \in \mathcal{M}(\mu)$ for every compact subset K of X.*

PROOF: Suppose that F is μ-measurable and let K be any compact subset of X. Since K is also closed, it follows that $F^-(K) \in \mathcal{M}(\mu)$.

Conversely, suppose that $F^-(K) \in \mathcal{M}(\mu)$ for every compact subset K of X. From page 51 of [49] follows that the separable locally compact metric space X may be written in the form $X = \cup_{i\in I} K_i$, where K_i is a compact subset of X and I is a countable index set. Let C be any closed subset of X. Then $C = \cup_{i\in I} C \cap K_i$, with $C \cap K_i$ a compact subset of C. The μ-measurability of F now follows from $F^-(C) = \cup_{i\in I} F^-(C \cap K_i)$ and from the separability of X. ■

For the rest of this section we discuss the generalization given in [51] for Aumann's definition of measurability of a multifunction. We also discuss the equivalence between this generalization and our definition of measurability.

Before we give Aumann's definition, we need to introduce some further notations. If (Ω_1, Σ_1) and (Ω_2, Σ_2) are two measurable spaces with Σ_1 and Σ_2 σ-rings of subsets of the sets Ω_1 and Ω_2, respectively, then we write

$$\Sigma_1 \times \Sigma_2 = \{A \times B \subseteq \Omega_1 \times \Omega_2 \mid A \in \Sigma_1, B \in \Sigma_2\},$$

and we denote by $\mathcal{S}(\Sigma_1 \times \Sigma_2)$ the σ-ring generated by $\Sigma_1 \times \Sigma_2$.

Definition 2.1.11 *A multifunction $F : [0,1] \to \mathcal{P}(I\!R^n)$ is called* ***Borel-measurable*** *if $Gr_F \in \mathcal{B}_{[0,1]\times I\!R^n}$.*

Maritz's [51] generalization of the above definition consists of [0,1] being replaced by the non-empty point set T, the Lebesgue σ-algebra being replaced by the σ-algebra $\mathcal{M}(\mu)$, $I\!R^n$ by a Polish space X (recall that a Polish space is a separable topological space that can be metrized by means of a complete metric) and we take $F : T \to \mathcal{P}_f(X)$. We then have the following

Theorem 2.1.12 ([51], p65, Theorem 6.38) *If T is a countable union of sets of the ring $\mathcal{R}$, X is a Polish space and $F : T \to \mathcal{P}_f(X)$ is a multifunction, then the following statements are equivalent:*

(a) F is a μ-measurable multifunction.

(b) $Gr_F \in \mathcal{S}(\mathcal{M}(\mu) \times \mathcal{B}_X)$.

(c) $F^-(B), F^+(B) \in \mathcal{M}(\mu)$ for every $B \in \mathcal{B}_X$.

Proposition 2.1.13 *If T is a countable union of sets of the ring $\mathcal{R}$, X is a Polish space and $F : T \to \mathcal{P}_k(X)$ is a multifunction, then the following statements are equivalent:*

(a) $F^-(C) \in \mathcal{M}(\mu)$ for all closed subsets C of X.

(b) $F^-(O) \in \mathcal{M}(\mu)$ for all open subsets O of X.

(c) $F^-(B) \in \mathcal{M}(\mu)$ for all Borel subsets B of X.

PROOF: (a) $\Leftrightarrow$ (b): Proposition 2.1.6 and Proposition 2.1.9;
(a) $\Leftrightarrow$ (c): Theorem 2.1.12.

■

2.2 Selectors of multifunctions

Definition 2.2.1 *A μ-measurable function $f : T \to X$ is called a **(μ-measurable) selector** of a multifunction $F : T \to \mathcal{P}(X)$ if $f(t) \in F(t)$ for every $t \in T$. We denote by S_F the class of all μ-measurable selectors of F.*

In this section our main concern will be the existence of measurable selectors of multifunctions. The first result (albeit not in the current form) in this regard was originally outlined by Rohlin [62]. The same result(in a stronger form) was also obtained in 1965 by Kuratowski and Ryll-Nardzewski [50] and in a restrictive form by Castaing [14].

Theorem 2.2.2 *If X is a Polish space and if $F : T \to \mathcal{P}_f(X)$ is a multifunction such that $F^-(O) \in \mathcal{M}(\mu)$ for every open subset O of X, then $S_F \neq \emptyset$.*

From Proposition 2.1.6 and Theorem 2.2.2 we have that

Corollary 2.2.3 *If X is a Polish space and if $F : T \to \mathcal{P}_f(X)$ is a μ-measurable multifunction, then $S_F \neq \emptyset$.*

Definition 2.2.4 *If $F : T \to \mathcal{P}_f(X)$ is a multifunction, then we say that F has a **Castaing representation** if there exists a denumerable set $\{f_i \mid i \in I\} \subseteq S_F$ such that*

$$F(t) = \overline{\{f_i(t) \mid i \in I\}}$$

for every $t \in T$.

Theorem 2.2.5 ([51], p69, Theorem 7.8) *If X is a Polish space and if $F : T \to \mathcal{P}_f(X)$ is a μ-measurable multifunction, then F has a Castaing representation.*

PROOF: Let d be a metric in X compatible with the given topology of X and let $H = \{x_j \mid j \in I\!N\}$ be a countable dense subset of X. For $j, k \in I\!N$ consider the closed spheres $S_{\frac{1}{2^k}}[x_j]$ with centre x_j and radius $\frac{1}{2^k}$. For $j, k \in I\!N$ the set $T_{j,k} = \{t \in T \mid F(t) \cap S_{\frac{1}{2^k}}[x_j] \neq \emptyset\}$ is μ-measurable. For $j, k \in I\!N$ define the multifunction $F_{j,k} : T \to \mathcal{P}_f(X)$ by

$$F_{j,k}(t) = \begin{cases} F(t) \cap S_{\frac{1}{2^k}}[x_j] & \text{if } t \in T_{j,k} \\ F(t) & \text{if } t \in T \backslash T_{j,k}. \end{cases}$$

Let C be now any closed subset of X. From the μ-measurability of F follows that

$$\begin{aligned}
F_{j,k}^{-}(C) &= \{t \in T \mid F_{j,k}(t) \cap C \neq \emptyset\} \\
&= \{t \in T_{j,k} \mid F_{j,k}(t) \cap C \neq \emptyset\} \cup \{t \in T \backslash T_{j,k} \mid F_{j,k}(t) \cap C \neq \emptyset\} \\
&= \{t \in T \mid F(t) \cap C \cap S_{\frac{1}{2^k}}[x_j] \neq \emptyset\} \cup \{t \in T \backslash T_{j,k} \mid F(t) \cap C \neq \emptyset\} \in \mathcal{M}(\mu).
\end{aligned}$$

Since X is separable, it follows that each $F_{j,k}$ is μ-measurable. From Corollary 2.2.3 we obtain a μ-measurable selector $f_{j,k} : T \to X$ of $F_{j,k}$. Let $I = \{(j,k) \mid j,k \in I\!N\}$ and put $M = \{f_i(t) \mid i \in I\}$. Then $f_i(t) \in F(t)$ for every $i \in I$ and $t \in T$. We now only need to show that the denumerable set $M(t)$ is dense in $F(t)$ for every $t \in T$. So, for $t \in T$, let $x \in F(t)$. For $k \in I\!N$ there exists a $x_j \in H$ such that $d(x, x_j) \leq \frac{1}{2^{k+1}}$. Consequently, $F(t) \cap S_{\frac{1}{2^{k+1}}}[x_j] \neq \emptyset$. From the construction of $M(t)$ we deduce that there is an $i \in I$ such that $d(x_j, f_i(t)) \leq \frac{1}{2^{k+1}}$. Hence

$$d(x, f_i(t)) \leq d(x, x_j) + d(x_j, f_i(t)) \leq \frac{1}{2^k}.$$

This shows that $F(t) = \overline{M(t)}$, and the proof is complete. ■

Theorem 2.2.6 ([51], p71, Theorem 7.12) *Let X be a separable locally compact metric space and $F : T \to \mathcal{P}_f(X)$ a multifunction. If F has a Castaing representation, then F is μ-measurable.*

Taking into account that every separable locally compact metric space is Polish ([9], page 122), we then have

Theorem 2.2.7 *Let X be a separable locally compact metric space and $F : T \to \mathcal{P}_f(X)$ a multifunction. Then F is μ-measurable if and only if F has a Castaing representation.*

The above theorem remains valid if we replace the locally compactness of X by completeness and if we suppose that T is a countable union of sets of the ring $\mathcal{R}$. In fact, we have the following

Theorem 2.2.8 ([51], p73, Theorem 7.15) *Let T be a countable union of sets of the ring $\mathcal{R}$, X a Polish space and $F : T \to \mathcal{P}_f(X)$ a multifunction. Then the following statements are equivalent:*

(a) F has a Castaing representation.

(b) For every $(x,t) \in X \times T$ the mapping $(x,t) \to d(x, F(t))$ is $\mathcal{S}(\mathcal{M}(\mu) \times \mathcal{B}_X)$-measurable.

(c) $Gr_F \in \mathcal{S}(\mathcal{M}(\mu) \times \mathcal{B}_X)$.

(d) F is μ-measurable.

For the rest of this section we will consider some special multifunctions and their measurability. We start with the closed convex hull multifunction.

Definition 2.2.9 *If X is a linear space and $A \subseteq X$, then we define the **convex hull** of A, denoted by $\operatorname{co} A$, as the intersection of all convex subsets of X containing A. If X is a linear topological space and $A \subseteq X$, then the set $\overline{\operatorname{co}}\, A$, called the **closed convex hull** of A, is the intersection of all closed convex subsets of X containing A.*

Proposition 2.2.10 ([51], p79, Lemma 8.3) *If $F : T \to \mathcal{P}_f(\mathbb{R}^n)$ is a μ-measurable multifunction, then the multifunction $\operatorname{co} F : T \to \mathcal{P}_{fc}(\mathbb{R}^n)$ is μ-measurable.*

PROOF: Put

$$\Lambda = \{(\mu_1, \mu_2, \dots, \mu_{n+1}) \mid \mu_i \geq 0,\ 1 \leq i \leq n+1;\ \sum_{i=1}^{n+1} \mu_i = 1\}.$$

Then the set Λ may be considered as the compact simplex in $\mathbb{R}^{n+1}$ with vertices $e_1 = (1,0,0,\dots,0), e_2 = (0,1,0,\dots,0), \dots, e_{n+1} = (0,0,0,\dots,1)$. Define the multifunction $G : T \to \mathcal{P}_k(\mathbb{R}^{n+1})$ by $G(t) = \Lambda$ for every $t \in T$. Furthermore, define the multifunction $K : T \to \mathcal{P}_f(\mathbb{R}^{(n+1)^2})$ by

$$K(t) = G(t) \times (F(t))^{n+1}.$$

To show that K is μ-measurable, let M_1 be the Castaing representation of G and let M_2 be the Castaing representation of $(F(t))^{n+1}$. If we put $M = M_1 \times M_2$, then M is denumerable. Furthermore,

$$K(t) = G(t) \times (F(t))^{n+1} = \overline{M_1(t)} \times \overline{M_2(t)} = \overline{(M_1 \times M_2)(t)} = \overline{M(t)};$$

therefore, K is μ-measurable. Consider now the continuous function $f : \mathbb{R}^{(n+1)^2} \to \mathbb{R}^n$ defined by

$$f(\lambda_1, \lambda_2, \dots, \lambda_{n+1}, y_1, y_2, \dots, y_{n+1}) = \sum_{i=1}^{n+1} \lambda_i y_i,$$

where $\lambda_i \in \mathbb{R}^{n+1}$, $y_i \in \mathbb{R}^n$, $1 \leq i \leq n+1$. Then

$$(f \circ K)(t) = \left\{ \sum_{i=1}^{n+1} \mu_i y_i \mid \mu_i \geq 0, y_i \in F(t), 1 \leq i \leq n+1; \sum_{i=1}^{n+1} \mu_i = 1 \right\}.$$

Consequently, $(f \circ K)(t) = (\operatorname{co} F)(t)$ for every $t \in T$. The μ-measurability of $\operatorname{co} F$ then follows from Theorem 7.20 of [51]. ■

Corollary 2.2.11 *If T is a countable union of sets of the ring $\mathcal{R}$, X is a Banach space with X' separable and if $F : T \to \mathcal{P}_{f(b)}(X)$ is a μ-measurable multifunction, then $\overline{co}\, F : T \to \mathcal{P}_{f(b)c}(X)$ is a μ-measurable multifunction.*

Definition 2.2.12 *Let A be a non-empty subset of a linear space X. A non-empty subset B of A is called an **extreme subset** of A if a proper convex combination $\alpha a_1 + (1-\alpha)a_2, 0 < \alpha < 1$, of two points $a_1, a_2 \in A$ is in B only if a_1 and a_2 are in B. An extreme subset of A consisting of only one point is called an **extreme point** of A. We denote by $ext\, A$ the set of all extreme points of A.*

Proposition 2.2.13 ([29], p439, Lemma 2) *If A is a non-empty compact subset of a locally convex linear topological Hausdorff space X, then $ext\, A \neq \emptyset$.*

Theorem 2.2.14 (Krein-Milman, [29], p440, Theorem 4) *If A is a compact subset of a locally convex linear topological Hausdorff space X, then $A \subseteq \overline{co}\, ext\, A$, that is $\overline{co}\, A = \overline{co}\, ext\, A$. If A is in addition convex, then every closed extreme subset of A contains an extreme point of A and $A = \overline{co}\, ext\, A$.*

Definition 2.2.15 *If X is a real locally convex linear topological Hausdorff space such that $X \neq \{0\}$ and if $F : T \to \mathcal{P}_0(X)$ is a multifunction, then we define the **extreme points multifunction** $ext\, F : T \to \mathcal{P}_0(X)$ by*

$$(ext\, F)(t) = \{x \in F(t) \mid x \text{ is an extreme point of } F(t)\}, \quad t \in T.$$

Proposition 2.2.16 ([51], p84, Proposition 8.17) *Let U be a non-empty compact convex and metrizable subset of X and suppose that X' is separable. If $F : T \to \mathcal{P}_{kc}(U)$ is a μ-measurable multifunction, then $Gr_{ext\, F} \in \mathcal{S}(\mathcal{M}(\mu) \times \mathcal{B}_U)$. If T is in addition a countable union of sets of the ring $\mathcal{R}$, then $ext\, F$ is a μ-measurable multifunction.*

Theorem 2.2.17 ([51], p77, Theorem 7.24) *Let T be a countable union of sets of the ring $\mathcal{R}$, X a Suslin space and $F : T \to \mathcal{P}_0(X)$ a multifunction. If $Gr_F \in \mathcal{S}(\mathcal{M}(\mu) \times \mathcal{B}_X)$, then $S_F \neq \emptyset$.*

Theorem 2.2.18 ([51], p85, Theorem 8.18) *Let T be a countable union of sets of the ring $\mathcal{R}$, X is a real locally convex linear topological Hausdorff space, U is a non-empty compact convex and metrizable subset of X and X' is separable. If $F : T \to \mathcal{P}_{kc}(U)$ is a μ-measurable multifunction, then*

$$ext\, S_F = S_{ext\, F}.$$

Theorem 2.2.19 ([51], p89, Theorem 8.21) *If $F : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ is a μ-measurable multifunction, then $Gr_{ext\,F} \in \mathcal{S}(\mathcal{M}(\mu)\times\mathcal{B}_{\mathbb{R}^n})$. If T is in addition a countable union of sets of the ring $\mathcal{R}$, then $ext\,F$ is a μ-measurable multifunction.*

PROOF: The space $\mathbb{R}^n$ may be written as the union of an increasing sequence (A_k) of non-empty compact convex subsets of X. Define the multifunctions $G_k : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ and $F_k : T \to \mathcal{P}_{kc}(A_k)$ by

$$G_k(t) = A_k \text{ and } F_k(t) = F(t) \cap A_k.$$

Then the multifunctions G_k and F_k are all μ-measurable, and consequently

$$Gr_{ext\,F_k} \in \mathcal{S}(\mathcal{M}(\mu) \times \mathcal{B}_{A_k}) \subseteq \mathcal{S}(\mathcal{M}(\mu) \times \mathcal{B}_{\mathbb{R}^n})$$

for $k \in \mathbb{N}$. Define the multifunction $E : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ by

$$E(t) = \bigcup_{k=1}^{\infty} \bigcap_{p=0}^{\infty} ext\,F_{k+p}(t).$$

Then $Gr_E \in \mathcal{S}(\mathcal{M}(\mu) \times \mathcal{B}_{\mathbb{R}^n})$. Furthermore, from Proposition 3 on page 725 of [41] we have that $(ext\,F)(t) = E(t)$ for every $t \in T$. Hence,

$$Gr_{ext\,F} = Gr_E \in \mathcal{S}(\mathcal{M}(\mu) \times \mathcal{B}_{\mathbb{R}^n}),$$

and the result then follows from Theorem 2.1.12. ■

Theorem 2.2.20 ([51], p90, Theorem 8.22) *If T is a countable union of sets of the ring $\mathcal{R}$ and $F : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ is a μ-measurable multifunction, then*

$$ext\,S_F = S_{ext\,F}.$$

2.3 Multimeasures

In this section we start by establishing the notations and terminology that go along with the subject of multimeasures.

As before we let T be any non-empty point set on which no topological structure is required and we let Y be a topological vector space with topological dual Y'. By a *set-valued set function* we mean a relation defined on a nonempty class $\mathcal{A}$ of subsets of T with values in $\mathcal{P}(Y)$, the class of all nonempty subsets of Y. Furthermore, if $A, B \in \mathcal{P}(Y)$, then we put

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

In this and in the next section we refer to Godet-Thobie [36] for some of our definitions and results about multimeasures. Due to the possible unavailibity of this reference, the definitions and results will be formulated and, for the sake of completeness of our development, some of the proofs from [36] will be included.

Definition 2.3.1 *If Y is a linear topological space, then a set-valued set function $M : \mathcal{A} \to \mathcal{P}(Y)$ is said to be* ***punctually additive*** *if*

$$M(A \cup B) = M(A) + M(B)$$

for every pair $A, B \in \mathcal{A}$ of disjoint sets such that $A \cup B \in \mathcal{A}$.

Definition 2.3.2 *If Y is a linear topological space, then a set-valued set function $M : \mathcal{A} \to \mathcal{P}_f(Y)$ is said to be* ***additive*** *if*

$$M(A \cup B) = \overline{M(A) + M(B)}$$

for every pair $A, B \in \mathcal{A}$ of disjoint sets such that $A \cup B \in \mathcal{A}$.

As for single-valued set functions we have the concept of countable additivity: We say that a set-valued set function $M : \mathcal{A} \to \mathcal{P}(Y)$ is *countably additive* if

$$M(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} M(A_k),$$

for every sequence $(A_k) \subseteq \mathcal{A}$ of mutually disjoint sets such that $\cup_{k=1}^{\infty} A_k \in \mathcal{A}$. Depending on how we define the above infinite sum we obtain different notions of countably additivity, as will be seen below.

For the rest of this section we consider $(T, \mathcal{S})$, where $\mathcal{S}$ is a σ-ring of subsets of T.

By a *multimeasure* we mean a countably additive set-valued set function $M : \mathcal{S} \to \mathcal{P}(Y)$ such that $M(\emptyset) = \{0\}$. In particular, we will differentiate between the following types of multimeasures:

Definition 2.3.3 *If Y is a linear topological space, then a set-valued set function $M : \mathcal{S} \to \mathcal{P}(Y)$ is called a* ***strong multimeasure*** *if and only if*

(a) $M(\emptyset) = \{0\}$ and M is punctually additive;

(b) for every $y_k \in M(A_k)$ the series $\Sigma_{k=1}^{\infty} y_k$ is unconditionally convergent and

$$M\left(\bigcup_{k=1}^{\infty} A_k\right) = \left\{y \in Y \mid y = \sum_{k=1}^{\infty} y_k,\ y_k \in M(A_k)\right\}.$$

Definition 2.3.4 *If Y is a linear topological space, then a set-valued set function $M : \mathcal{S} \to \mathcal{P}_f(Y)$ is called a* ***normal multimeasure*** *if and only if*

(a) *$M(\emptyset) = \{0\}$ and M is additive;*

(b) *for every sequence $(A_k) \subseteq \mathcal{S}$ of mutually disjoint sets such that $A = \cup_{k=1}^{\infty} A_k$, we have that*

$$\lim_{n\to\infty} H\left(M(A), \overline{\sum_{k=1}^{n} M(A_k)}\right) = 0.$$

Remark 2.3.5

(i) Note that if the set-valued set function $M : \mathcal{S} \to \mathcal{P}(Y)$ satisfies condition (b) in Definition 2.3.3, then either $M(\emptyset) = \{0\}$ or $M(\emptyset)$ is an unbounded set. Consequently, if $M : \mathcal{S} \to \mathcal{P}_b(Y)$ is countably additive in the sense of Definition 2.3.3, then $M(\emptyset) = \{0\}$.

(ii) If Y is finite-dimensional, then we can take the series $\sum_{i=1}^{\infty} y_i$ to be absolutely convergent (see pages 750 and 92 of [29]). In this case our Definition 2.3.3 coincides with that of [4].

(iii) Note that from Proposition 6 on page 57 of [36] follows that if Y is sequentially complete and if the set-valued set function $M : \mathcal{S} \to \mathcal{P}_{fb}(Y)$ satisfies condition (b) in Definition 2.3.4, then for every $y_k \in M(A_k)$ the series $\Sigma_{k=1}^{\infty} y_k$ is unconditionally convergent and

$$M\left(\bigcup_{k=1}^{\infty} A_k\right) = \overline{\left\{\{y \in Y \mid y = \sum_{k=1}^{\infty} y_k,\, y_k \in M(A_k)\right\}}.$$

Definition 2.3.6 *If Y is a linear topological space, then a set-valued set function $M : \mathcal{S} \to \mathcal{P}_f(Y)$ is called a **weak multimeasure** if and only if*

(a) *$M(\emptyset) = \{0\}$;*

(b) *for every $y' \in Y'$ the set function $A \mapsto \sigma(y', M(A)) = \sup_{y\in M(A)}(y', y)$ is a signed measure with values in $I\!R \cup \{+\infty\}$.*

As for single-valued measures we have the notion of total variation of a set-valued set function. In what follows $\mathcal{A}$ will denote a non-empty class of subsets of T. Also recall that $\|A\| = \sup\{\|a\| \mid a \in A\}$ for $A \in \mathcal{P}(Y)$.

Definition 2.3.7 *Let Y be a normed space and suppose that $M : \mathcal{A} \to \mathcal{P}(Y)$ is a set-valued set function such that $M(\emptyset) = \{0\}$ if $\emptyset \in \mathcal{A}$. For every $A \subseteq T$ we define the **variation of M on A**, denoted by $v(M, A)$, by*

$$v(M, A) = \sup_I \sum_{i\in 1} \|M(A_i)\|,$$

where the supremum is taken for all the families $(A_i)_{i\in I} \subseteq \mathcal{A}$ of mutually disjoint sets contained in A.

The set function $v(M)$ is called the *variation of* M and the restriction of $v(M)$ to the class $\mathcal{A}$ will again be denoted by $v(M)$.

Definition 2.3.8 *Let Y be a normed space and suppose that $M : \mathcal{A} \to \mathcal{P}(Y)$ is a set-valued set function such that $M(\emptyset) = \{0\}$ if $\emptyset \in \mathcal{A}$. Then we say that M is of* ***bounded variation*** *(with respect to $\mathcal{A}$) if $v(M, A) < \infty$ for every $A \in \mathcal{A}$.*

Remark 2.3.9

(i) Every set-valued set function of bounded variation is bounded. To say that a set-valued set function $M : \mathcal{S} \to \mathcal{P}(Y)$ is of bounded variation $v(M)$ is the same as to say that the set function $v(M)$ is finite.

(ii) If $M : \mathcal{A} \to \mathcal{P}(Y)$ is a set-valued set function of bounded variation, then $\sum_{k=1}^{\infty} y_k$ is absolutely convergent for each $y_k \in M(A_k)$, where (A_k) is a sequence of mutually disjoint elements of $\mathcal{A}$. Indeed, for all $n \in I\!N$ we have that

$$\|\sum_{k=1}^{n} y_k\| \;\leq\; \sum_{k=1}^{n} \|y_k\| \;\leq\; \sum_{k=1}^{n} \|M(A_k)\| \;<\; \infty.$$

The following result about the total variation of a multimeasure is the set-valued analogue for the total variation of a single-valued measure and the proof can be carried out in the same way.

Proposition 2.3.10 *Let Y be a normed space. If $M : \mathcal{S} \to \mathcal{P}(Y)$ is a strong multimeasure, then the variation $v(M)$ of M is a positive measure.*

PROOF: Evidently, from the definition, $v(M)$ is $I\!R_+$-valued. Also, since every family of disjoint sets of $\mathcal{S}$ contained in $\emptyset$ consists only of empty sets, we have that $v(M, \emptyset) = \{0\}$.

To show the countable additivity of $v(M)$, let (A_k) be a sequence of mutually disjoint elements of $\mathcal{S}$ and let $\{B_1, B_2 ..., B_j\}$ be a finite partition of $\cup_{k=1}^{\infty} A_k$. For $k \in I\!N$ we have that $\{A_k \cap B_1, A_k \cap B_2, ..., A_k \cap B_j\}$ is a finite partition of A_k. On the other hand, $\{A_1 \cap B_i, A_2 \cap B_i, \ldots\}$ consists of disjoint elements of $\mathcal{S}$ and $\cup_{k=1}^{\infty}(A_k \cap B_i) = B_i$, $i = 1, 2, \ldots, j$. Hence

$$\begin{aligned}
\sum_{i=1}^{j} \|M(B_i)\| &= \sum_{i=1}^{j} \|M\left(\bigcup_{k=1}^{\infty}(A_k \cap B_i)\right)\| \\
&= \sum_{i=1}^{j} \|\sum_{k=1}^{\infty} M(A_k \cap B_i)\| \\
&\leq \sum_{k=1}^{\infty}\sum_{i=1}^{j} \|M(A_k \cap B_i)\| \\
&\leq \sum_{k=1}^{\infty} v(M, A_k).
\end{aligned}$$

Since $B_i \subset \cup_{k=1}^{\infty} A_k$ was arbitrary, we deduce that

$$v(M, \bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} v(M, A_k).$$

For the inverse inequality, let $A \cap B = \emptyset$ with $A, B \in \mathcal{S}$. Choose an arbitrary number θ such that $\theta < v(M, A) + v(M, B)$. Then there are two numbers α and β such that $\alpha < v(M, A)$, $\beta < v(M, B)$ and $\alpha + \beta = \theta$. Consequently, we can find finite partitions $\{A_1, A_2, ..., A_n\}$ and $\{B_1, B_2, ..., B_m\}$ of A and B, respectively, such that

$$\alpha < \sum_{k=1}^{n} \|M(A_k)\| \text{ and } \beta < \sum_{k=1}^{m} \|M(B_k\|.$$

The sets $A_1, A_2, ..., A_n, B_1, B_2, ..., B_m$ are disjoint, belong to $\mathcal{S}$ and are contained in $A \cup B$. Therefore

$$\theta = \alpha + \beta < \sum_{k=1}^{n} \|M(A_k)\| + \sum_{k=1}^{m} \|M(B_k)\| \leq v(M, A \cup B).$$

It then follows that $v(M, A) + v(M, B) \leq v(M, A \cup B)$. By induction we deduce that

$$\sum_{k=1}^{\infty} v(M, A_k) \leq v(M, \bigcup_{k=1}^{\infty} A_k).$$

■

Definition 2.3.11 *If Y is a linear topological space, then a set $A \in \mathcal{S}$ is said to be an* ***atom*** *for a multimeasure $M : \mathcal{S} \to \mathcal{P}(Y)$ if $M(A) \neq \{0\}$ and if either $M(B) = \{0\}$ or $M(A \backslash B) = \{0\}$ holds for every $B \subseteq A, B \in \mathcal{S}$. We say that M is* ***atomic*** *if there exists at least one atom in $\mathcal{S}$, and that M is* ***non-atomic*** *if there are no atoms in $\mathcal{S}$.*

Definition 2.3.12 *Let Y be a linear topological space. If $\mu : \mathcal{S} \to Y$ is a positive measure on $\mathcal{S}$ and $M : \mathcal{S} \to \mathcal{P}(Y)$ is a multimeasure, then we say that M is* ***μ-continuous*** *on $\mathcal{S}$ if and only if for any $A \in \mathcal{S}$ with $\mu(A) = 0$ we have that $M(A) = \{0\}$.*

Proposition 2.3.13 *Let Y be a normed space and suppose that $M : \mathcal{S} \to \mathcal{P}_f(Y)$ is a normal multimeasure. If $\mu : \mathcal{S} \to Y$ is a positive measure, then*

(a) M is atomic (non-atomic, respectively) if and only if $v(M)$ is atomic (non-atomic, respectively).

(b) M is μ-continuous on $\mathcal{S}$ if and only if $v(M)$ is μ-continuous on $\mathcal{S}$.

PROOF: We will only prove (a). The proof of (b) follows trivially from the definition. Suppose that M is non-atomic and let $A \in \mathcal{S}$ be an atom of $v(M)$. Then $v(M, A) \neq 0$ and there exists a set $B \subseteq A$ such that $M(B) \neq \{0\}$. Furthermore, for every $C \subseteq B$, we have that $M(C) = \{0\}$ if $v(M, C) = 0$. Then, from

$$\overline{M(B \backslash C) + M(A \backslash B)} = M(A \backslash C),$$

we have that $M(B\backslash C) = \{0\}$ if $M(A\backslash C) = \{0\}$. But then it means that B is an atom of M, which is a contradiction. ■

Theorem 2.3.14 ([36], p59, Theorem 2) *Let Y be a Banach space and suppose that $M : \mathcal{S} \to \mathcal{P}_{fbc}(Y)$ is a normal multimeasure. If μ is a positive finite measure on $\mathcal{S}$ such that M is μ-continuous on $\mathcal{S}$, then*

$$\lim_{\mu(A)\to 0} M(A) = \{0\}, \quad A \in \mathcal{S},$$

or equivalently,

$$\lim_{\mu(A)\to 0} \|M(A)\| \;=\; 0, \quad A \in \mathcal{S}.$$

Corollary 2.3.15 ([36], p59, Theorem 2′) *Let Y be a Banach space and suppose that $M : \mathcal{S} \to \mathcal{P}_{fbc}(Y)$ is a strong multimeasure. If μ is a positive finite measure on $\mathcal{S}$ such that M is μ-continuous on $\mathcal{S}$, then*

$$\lim_{\mu(A)\to 0} \|M(A)\| \;=\; 0, \quad A \in \mathcal{S}.$$

PROOF: First note that by Proposition 2 on page 52 of [32] and Proposition 3 on page 53 of [32] we have that the set-valued set function $\overline{co}\, M : \mathcal{S} \to \mathcal{P}_{fbc}(Y)$ is a normal multimeasure. The corollary then follows from the equality

$$\|\overline{co}\, M(A)\| = H(\overline{co}\, M(A), \{0\}) = H(M(A), \{0\}) = \|M(A)\|$$

and Theorem 2.3.14. ■

Before discussing the relationships between the different types of multimeasures, we first look at some examples of multimeasures. Example 2 is due to Hiai [39], page 100, while the other two were taken from [36], page 57 and [34], page 114.

Example 2.3.16

1. Let $(T, \mathcal{S}, \mu)$ be a positive measure space, Y is a sequentially complete locally convex topological vector space and let B be a bounded subset of Y. If we put

$$\mathcal{M} = \{m : \mathcal{S} \to Y \mid m \text{ is a measure and } m(A) \in \mu(A)B,\ A \in \mathcal{S}\},$$

then $M : \mathcal{S} \to \mathcal{P}(Y)$ defined by

$$M(A) = \left\{ \sum_{k=1}^{n} m_k(A_k) \mid m_k \in \mathcal{M} \text{ and } \{A_1, A_2, ..., A_n\} \text{ is a finite } \mathcal{S}\text{-measurable partition of } A \right\}$$

is a strong multimeasure. For the punctual additivity, note that

$$\sum_{k=1}^{n} m_k(A_k) \in \sum_{k=1}^{n} \mu(A_k)B = \mu(\bigcup_{k=1}^{n} A_k)B = \mu(A)B,$$

which implies that $M(A) \subseteq \mu(A)B$ for every $A \in \mathcal{S}$.

If p is a seminorm and $\epsilon > 0$, then we put

$$S_\epsilon[p] = \{x \in X \mid p(x) \leq \epsilon\}.$$

Let $(A_r) \subseteq \mathcal{S}$ be a sequence of mutually disjoint sets with $A = \cup_{r=1}^\infty A_r$. Then for every $a \geq 0$ there exists a number N such that if $m > n \geq N$ we have that $\sum_{i=n}^s \mu(A_{\sigma(i)}) \leq a$. Since B is bounded, for every seminorm p and $\epsilon > 0$ there exists an a such that $aB \subseteq S_\epsilon[p]$. If $y_r \in M(A_r)$, where $A_r = \cup_{k=1}^{n(r)} A_k^r$, $A_i^{(r)} \cap A_j^{(r)} = \emptyset$, $i \neq j$, then $y_r = \sum_{k=1}^{n(r)} m_k(A_k^r)$, so that $y_r \in \sum_{k=1}^{n(r)} \mu(A_k^r)B = \mu(A_r)B$. Hence

$$\sum_{i=n}^s y_{\sigma(i)} \in \sigma_{i=n}^s \mu(A_{\sigma(i)})B \subseteq S_\epsilon[p].$$

Consequently, $\sum_{r=1}^\infty y_r$ is unconditionally convergent. Furthermore, if $y \in M(A)$, then there exists a finite partition $\{C_1, C_2, ..., C_j\}$ of A and measures $\{m_1, m_2, ..., m_j\}$ such that $y = \sum_{i=1}^j m_i(C_i)$. Hence, as

$$m_i(C_i) = m_i\left(\bigcup_{k=1}^\infty (C_i \cap A_k)\right),$$

we have that

$$y = \sum_{k=1}^\infty \sum_{i=1}^j m_i(C_i \cap A_k) = \sum_{k=1}^\infty y_k,$$

where $y_k \in M(A_k)$, from the definition of $M(A_k)$.

2. Let $m : \mathcal{S} \to Y$ be a vector measure. If we define $M : \mathcal{S} \to \mathcal{P}(Y)$ by

$$M(A) = \{m(B) \mid B \subseteq A,\ B \in \mathcal{S}\},$$

then M is a strong multimeasure. Note that M is non-atomic if and only if m is non-atomic and $v(M, A) = v(m, A)$ for all $A \in \mathcal{S}$.

3. Let $(T, \mathcal{S}, \mu)$ be a measure space and suppose that $F : T \to \mathcal{P}_{fbc}(Y)$ is a measurable multifunction such that $\sigma(y', F(\cdot))$ is μ-integrable for each $y' \in Y'$. If every $f \in S_F$ is Pettis-integrable and if we put

$$M(A) = \overline{\{\int_A f(t)\,\mu(dt) \mid f \in S_F\}}, \quad A \in \mathcal{S},$$

then $M : \mathcal{S} \to \mathcal{P}_{fbc}(Y)$ is a weak multimeasure. To see this, note that for all $y' \in Y'$ we have that

$$\sigma(y', M(A)) = \int_A \sigma(y', F(t))\,\mu(dt).$$

Consequently, if $(A_k) \subseteq \mathcal{S}$ is a sequence of mutually disjoint sets, then

$$\begin{aligned}
\sum_{k=1}^\infty \sigma(y', M(A_k)) &= \sum_{k=1}^\infty \int_{A_k} \sigma(y', F(t))\,\mu(dt) \\
&= \int_{\cup_{k=1}^\infty A_k} \sigma(y', F(t))\,\mu(dt) \\
&= \sigma(y', M(\bigcup_{k=1}^\infty A_k)).
\end{aligned}$$

For a strong multimeasure $M : \mathcal{S} \to \mathcal{P}(Y)$ the set-valued set function on $\mathcal{S}$ induced by taking the closure or the closed convex hull of M is not always a strong multimeasure. However, we have the following results:

Theorem 2.3.17 ([36], p57, Proposition 5) *Let Y be a linear topological space. If $M : \mathcal{S} \to \mathcal{P}(Y)$ is a strong multimeasure, then the set-valued set function $\overline{M} : \mathcal{S} \to \mathcal{P}_f(Y)$, defined by $\overline{M}(A) = \overline{M(A)}$, is a normal multimeasure.*

PROOF: First note that $M(A \cup B) = M(A) + M(B)$ for all $A, B \in \mathcal{S}$ with $A \cap B = \emptyset$. Then we must show that $\overline{M}(A \cup B) = \overline{\overline{M}(A) + \overline{M}(B)}$. Since $M(A) \subseteq \overline{M(A)}$ and $M(B) \subseteq \overline{M(B)}$, we have that

$$\overline{M}(A \cup B) = \overline{M(A) + M(B)} \subseteq \overline{\overline{M(A)} + \overline{M(B)}} = \overline{\overline{M}(A) + \overline{M}(B)}.$$

The converse inclusion follows from the fact that $\overline{M(A)} + \overline{M(B)} \subseteq \overline{M(A) + M(B)}$ and that $\overline{M(A) + M(B)}$ is closed.

For the countable additivity, note that for every $y_k \in M(A_k)$, where (A_k) is a sequence of mutually disjoint elements of $\mathcal{S}$ with $A = \cup_{k=1}^{\infty} A_k$, we have that $\Sigma_{k=1}^{\infty} y_k$ is unconditionally convergent and

$$\overline{M}(A) = \overline{M(A)} = \overline{\left\{ y \in Y \mid y = \sum_{k=1}^{\infty} y_k,\ y_k \in M(A_k) \right\}}.$$

■

Theorem 2.3.18 ([39], p99, Theorem 1.3) *If Y is a Banach space and if $M : \mathcal{S} \to \mathcal{P}(Y)$ is a strong multimeasure of bounded variation $v(M)$ such that $M(T)$ is relatively weakly compact, then $\overline{M}$ and $\overline{co}\, M$ are strong multimeasures on $\mathcal{S}$ and*

$$v(M, A) = v(\overline{M}, A) = v(\overline{co}\, M, A)$$

for all $A \in \mathcal{S}$.

PROOF: We only prove that $\overline{co}\, M$ is a strong multimeasure. To show that $\overline{M}$ is also a strong multimeasure is quite similar. First note that $M(\emptyset) = \{0\}$ implies that $\overline{co}\, M(\emptyset) = \{0\}$. Also, since $M(T) = M(A) + M(T \backslash A)$ for every $A \in \mathcal{S}$, $M(A)$ is relatively $w(Y, Y')$-compact. The Krein-Šmulian theorem ([29], page 434, Theorem 4) implies that $\overline{co}\, M(A)$ is $w(Y, Y')$-compact.

To prove the countable additivity, let $(A_k) \subseteq \mathcal{S}$ be a sequence of mutually disjoint sets. Then

$$M\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{n} M(A_k) + M\left(\bigcup_{k=n+1}^{\infty} A_k\right)$$

implies that

$$\overline{co}\, M\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{n} \overline{co}\, M(A_k) + \overline{co}\, M\left(\bigcup_{k=n+1}^{\infty} A_k\right).$$

Since M is of bounded variation, it follows that both $\overline{co}\, M(\cup_{k=n+1}^{\infty} A_k)$ and $\sum_{k=1}^{\infty} \overline{co}\, M(A_k)$ are $w(Y, Y')$-compact. Consequently, we have that

$$\begin{aligned}
& H\left(\overline{co}\, M\left(\bigcup_{k=1}^{\infty} A_k\right), \sum_{k=1}^{\infty} \overline{co}\, M(A_k)\right) \\
= \; & H\left(\sum_{k=1}^{n} \overline{co}\, M(A_k) + \overline{co}\, M\left(\bigcup_{k=n+1}^{\infty} A_k\right), \sum_{k=1}^{n} \overline{co}\, M(A_k) + \sum_{k=n+1}^{\infty} \overline{co}\, M(A_k)\right) \\
= \; & H\left(\overline{co}\, M\left(\bigcup_{k=n+1}^{\infty} A_k\right), \sum_{k=n+1}^{\infty} \overline{co}\, M(A_k)\right) \\
\leq \; & \|\overline{co}\, M\left(\bigcup_{k=n+1}^{\infty} A_k\right)\| + \|\sum_{k=n+1}^{\infty} \overline{co}\, M(A_k)\| \\
\leq \; & 2 \sum_{k=n+1}^{\infty} v(M, A_k) \to 0
\end{aligned}$$

as $n \to \infty$, which means that $\overline{co}\, M(\cup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \overline{co}\, M(A_k)$.

Finally, it is obvious that

$$v(M, A) = v(\overline{M}, A) = v(\overline{co}\, M, A)$$

for all $A \in \mathcal{S}$. ■

The following theorem (Propositions 8 and 9 on page 60 of [36]) shows that if the multimeasure takes on weakly-compact and convex values, then the different types of multimeasures coincide.

Theorem 2.3.19 *If Y is a locally convex topological vector space, then a set-valued set function $M : \mathcal{S} \to \mathcal{P}_{wkc}(Y)$ is a normal multimeasure if and only if M is a weak multimeasure.*

PROOF: Suppose that M is a normal multimeasure. If $(A_k) \subseteq \mathcal{S}$ is a sequence of mutually disjoint sets and if $A = \cup_{k=1}^{\infty} A_k$, then

$$\lim_{n\to\infty} H\left(M(A), \overline{\sum_{k=1}^{n} M(A_k)}\right) = 0.$$

Since $M(A) \in \mathcal{P}_{wkc}(Y)$ for all $A \in \mathcal{S}$, we have that $\overline{\sum_{k=1}^{n} M(A_k)} = \sum_{k=1}^{n} M(A_k)$. From Hormander's theorem [13, Theorem II-18] we then have that

$$H\left(M(A), \sum_{k=1}^{n} M(A_k)\right) = \sup\left\{|\sigma(y', M(A)) - \sigma(y', \sum_{k=1}^{n} M(A_k))| : \|y'\| \leq 1,\ y' \in Y'\right\}$$

$$= \sup\left\{ |\sigma(y', M(A)) - \sum_{k=1}^{n} \sigma(y', M(A_k))| : \|y'\| \leq 1,\ y' \in Y' \right\}.$$

Consequently, for each $y' \in Y'$ with $\|y'\| \leq 1$, we have that $\sum_{k=1}^{n} \sigma(y', M(A_k)) \to \sigma(y', M(A))$ as $n \to \infty$. The positive homogeneity of the support functionals implies that $\sum_{k=1}^{n} \sigma(y', M(A_k)) \to \sigma(y', M(A))$ as $n \to \infty$ for each $y' \in Y'$. Finally, since $\sigma(y', M(\emptyset)) = 0$ for all $y' \in Y'$, we conclude that $M(\emptyset) = \{0\}$.

Conversely, suppose that $\sigma(y', M(\cdot))$ is a real-valued measure for all $y' \in Y'$ and put $R_M = \bigcup_{A \in \mathcal{S}} M(A)$. Then the boundedness of M implies that $\sigma(y', R_M)$ is finite, which implies that R_M is bounded in Y. To prove the additivity of M, let $A, B \in \mathcal{S}$ be such that $A \cap B = \emptyset$. Then it follows that

$$\sigma(y', M(A \cup B)) = \sigma(y', M(A)) + \sigma(y', M(B)) = \sigma(y', \overline{M(A) + M(B)}).$$

Let now $(A_k) \subseteq \mathcal{S}$ be a sequence of mutually disjoint sets and let $A = \bigcup_{k=1}^{\infty} A_k$. If for $n \in I\!N$ we put $B_n = \bigcup_{k=1}^{n} A_k$, then we must show that $M(B_n)$ converges to $M(A)$ with respect to the Hausdorff uniformity. If $T = I\!N \cup \{\omega\}$ is the Alexandroff compactification of $I\!N$, then we define the multifunction $F : T \to \mathcal{P}_{wkc}(Y)$ by

$$F(n) = M(B_n) \text{ and } F(\omega) = M(A).$$

Then from the Corollary of Theorem 2, Chapter 0 of [36] we know that F is continuous in ω, and since the Vietoris topology with respect to the weak topology and the topology associated with the Hausdorff uniformity coincide on the family of weakly compact subsets of X, $M(B_n)$ converges to $M(A)$; therefore M is a normal multimeasure. ■

We now suppose that $M : \mathcal{S} \to \mathcal{P}_0(I\!R^n)$ is a multimeasure of bounded variation. Let $e_1, e_2, \ldots, e_{2n}$ be the $2n$-vectors $(0, \ldots, \pm 1, \ldots, 0)$, that is $e_1 = (1, 0, \ldots, 0)$, $e_{-1} = (-1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0)$, $e_{-2} = (0, -1, \ldots, 0)$, etc. If we put

$$\nu(A) = \sum_{k=1}^{2n} v(\sigma(e_k, M(A))),$$

then we call ν the *tight control measure* of M. The measure $\nu : \mathcal{S} \to I\!R$ has the following properties:

Proposition 2.3.20 *If $M : \mathcal{S} \to \mathcal{P}_0(I\!R^n)$ is a multimeasure of bounded variation $v(M)$, then*

(a) ν is a finite and nonnegative measure;

(b) $\nu(A) = 0$ if and only if $M(A) = \{0\}$;

(c) $\|M(A)\| \leq \nu(A) \leq \nu(T) < \infty$ for every $A \in \mathcal{S}$;

(d) ν is nonatomic if and only if M is nonatomic.

In [67], page 36, Theorem 1, Wenxiu, Jifeng and Aijie gave the following characterization of a finite-dimensional strong multimeasure.

Theorem 2.3.21 *A set-valued set function $M : \mathcal{S} \to \mathcal{P}_k(\mathbb{R}^n)$ is a strong multimeasure if and only if*

(a) M is punctually additive;

(b) $H(M(A_k), M(A)) \to 0$ as $k \to \infty$, where $(A_k) \subseteq \mathcal{S}$ is an increasing sequence of sets such that $\lim_{k\to\infty} A_k = A$.

PROOF: Suppose that $M : \mathcal{S} \to \mathcal{P}_k(\mathbb{R}^n)$ is a strong multimeasure. Since $M(\emptyset) = \{0\}$, we conclude that M is punctually additive. Consequently, $M(A) = M(A\backslash A_k) + M(A_k)$ where (A_k) is an increasing sequence of elements of $\mathcal{S}$ such that $\lim_{k\to\infty} A_k = A$. Hence

$$\begin{aligned} H(M(A), M(A_k)) &= H(M(A\backslash A_k) + M(A_k), M(A_k)) \\ &\leq \| M(A\backslash A_k) \| \\ &\leq \nu(A\backslash A_k) \to 0 \end{aligned}$$

as $k \to \infty$.

Conversely, let (A_k) be a sequence of mutually disjoint elements in $\mathcal{S}$ and put $A = \cup_{k=1}^{\infty} A_k$. Then

$$\lim_{n\to\infty} H\left(M(A), \sum_{k=1}^{n} M(A_k)\right) = \lim_{n\to\infty} H\left(M(A), M\left(\bigcup_{k=1}^{n} A_k\right)\right) = 0.$$

Also, for $n \in \mathbb{N}$, we have that

$$\| \sum_{k=1}^{n} M(A_k) \| \leq \sum_{k=1}^{n} \| M(A_k) \| \leq \sum_{k=1}^{n} \nu(A_k) = \nu(\bigcup_{k=1}^{n} A_k) \leq \nu(T) < \infty,$$

where the finite additivity of ν follows from the finite additivity of M. This means that $\sum_{k=1}^{\infty} M(A_k)$ is a nonempty bounded set. Therefore

$$\begin{aligned} & H\left(M(A), \sum_{k=1}^{\infty} M(A_k)\right) \\ \leq\ & H\left(M(A), \sum_{k=1}^{n} M(A_k)\right) + H\left(\sum_{k=1}^{n} M(A_k), \sum_{k=1}^{\infty} M(A_k)\right) \\ =\ & H\left(M(A), \sum_{k=1}^{n} M(A_k)\right) + H\left(\sum_{k=1}^{n} M(A_k), \sum_{k=1}^{n} M(A_k) + \sum_{k=n+1}^{\infty} M(A_k)\right) \\ \leq\ & H\left(M(A), \sum_{k=1}^{n} M(A_k)\right) + \sum_{k=n+1}^{\infty} \| M(A_k) \| \to 0 \end{aligned}$$

as $n \to \infty$. Consequently, $M(A) = M(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} M(A_k)$, which means that M is a strong multimeasure. ■

2.4 Selectors of multimeasures

Definition 2.4.1 *Let Y be a linear topological space. If $M : \mathcal{S} \to \mathcal{P}(Y)$ is a multimeasure, then we call a measure $m : \mathcal{S} \to Y$ a **selector** of M if $m(A) \in M(A)$ for all $A \in \mathcal{S}$. We denote by S_M the set of selectors of M.*

In this section we investigate the existence of selectors of multimeasures and study their topological properties. In particular, we study the relationship between the selectors of the multimeasure and certain points of the multimeasure. We also study the conditions which will guarantee the existence of a selector m of a multimeasure M such that

$$M(A) = \overline{\{m(A) \mid A \in \mathcal{S}\}}$$

for every $A \in \mathcal{S}$.

Our first result in this section is due to Hiai [39] and it relates the set of all exposed points of the multimeasure with the set of all selectors of the multimeasure. We first recall the notion of an exposed point.

Definition 2.4.2 *If Y is a Banach space, then a point y of a set $K \in \mathcal{P}(Y)$ is called **exposed** if there exists a $y' \in Y'$ such that $(y', y) > (y', z)$ for all $z \in K$ for which $z \neq y$. We denote by $exp\, K$ the set of all exposed points of the set K.*

Theorem 2.4.3 ([39], p101, Proposition 2.1) *Let Y be a Banach space and let $M : \mathcal{S} \to \mathcal{P}(Y)$ be a strong multimeasure of bounded variation $v(M)$. If $y \in exp\, M(T)$, then there exists an $m \in S_M$ such that $m(T) = y$.*

PROOF: Let $y' \in Y'$ be such that $(y', y) > (y', z)$ for all $z \in M(T \backslash \{y\})$. From the punctual additivity of M follows that for each $A \in \mathcal{S}$ we have that $y = y_1 + y_2$ with $y_1 \in M(A)$ and $y_2 \in M(T \backslash A)$. Since

$$\begin{aligned}(y', y) &= \sup\{(y', z) \mid z \in M(T)\} \\ &= \sup\{(y', z) \mid z \in M(A)\} + \sup\{(y', z) \mid z \in M(T \backslash A)\},\end{aligned}$$

it is easy to see that $(y', y_1) > (y', z)$ for all $z \in M(A) \backslash \{y_1\}$. Indeed, $(y', y_1) \leq (y', z)$ is impossible because $(y', y_1) + (y', y_2) \leq (y', y)$. Thus we showed that given $A \in \mathcal{S}$, there exists a point $m(A)$ of $M(A)$ which is exposed by y'.

It remains to show that m is a measure. Let $(A_k) \subseteq \mathcal{S}$ be a sequence of mutually disjoint sets such that $A = \bigcup_{k=1}^{\infty} A_k$. Since $\sum_{k=1}^{\infty} m(A_k)$ is absolutely convergent to an element in $M(A)$ and

$$\begin{aligned}(y', \sum_{k=1}^{\infty} m(A_k)) &= \sum_{k=1}^{\infty} \sup\{(y', y) \mid y \in M(A_k)\} \\ &= \sup\{(y', y) \mid y \in M(A)\} \\ &= (y', m(A)),\end{aligned}$$

we deduce that $m(A) = \sum_{k=1}^{\infty} m(A_k)$. ■

Corollary 2.4.4 ([55], p221, Theorem 5.4) *Suppose that Y is a separable and reflexive Banach space and let $M : \mathcal{S} \to \mathcal{P}_f(Y)$ be a nonatomic strong multimeasure of bounded variation $v(M)$. If $y \in ext\, M(T)$, then there exists an $m \in S_M$ such that $m(T) = y$.*

We now turn to the finite-dimensional case. We first prove the following proposition concerning the atomic properties of multimeasures and their selectors.

Proposition 2.4.5 *If $M : \mathcal{S} \to \mathcal{P}(\mathbb{R}^n)$ is a strong multimeasure and μ is a finite nonatomic measure such that M is μ-continuous, then M is nonatomic.*

PROOF: Recall that M is nonatomic if for every $A \in \mathcal{S}$ with $M(A) \neq \{0\}$, there exists a set $B \subseteq A$ such that $M(B) \neq \{0\}$ and $M(A \backslash B) \neq \{0\}$. To the contrary, suppose that A is an atom of M. Then $M(A) \neq \{0\}$. Since M is μ-continuous, it follows that $\mu(A) > 0$. Put $\mu(A) = \epsilon$. Since μ is nonatomic, and thus has the Darboux property, there is a set $A^* \subset A$ such that $\mu(A^*) = \frac{\epsilon}{2}$. Hence, $\mu(A \backslash A^*) = \frac{\epsilon}{2}$. Since M is atomic, it follows that either $M(A^*) = \{0\}$ or $M(A \backslash A^*) = \{0\}$. Define the set $A_1 \in \Sigma(A, \mu)$ by

$$A_1 = \begin{cases} A^* & \text{if } M(A^*) = \{0\} \\ A \backslash A^* & \text{if } M(A \backslash A^*) = \{0\}. \end{cases}$$

Then $A_1 \subseteq A$, $M(A_1) = \{0\}$ and $\mu(A_1) = \frac{\epsilon}{2}$. Since $0 < \mu(A \backslash A_1) = \frac{\epsilon}{2}$, there exists a set $A^{**} \subseteq A \backslash A_1 \subset A$ such that $\mu(A^{**}) = \frac{\epsilon}{2^2}$ and either $M(A^{**}) = \{0\}$ or $M(A \backslash A^{**}) = \{0\}$. Define the set $A_2 \in \Sigma(A, \mu)$ by

$$A_2 = \begin{cases} A^{**} & \text{if } M(A^{**}) = \{0\} \\ A \backslash (A_1 \cup A^{**}) & \text{if } M(A \backslash A^{**}) = \{0\}. \end{cases}$$

If $M(A^{**}) = \{0\}$, then $M(A_2) = \{0\}$. If $M(A \backslash A^{**}) = \{0\}$, then

$$H(M(A \backslash (A_1 \cup A^{**})), \{0\}) \leq \sup_{m \in S_M} \|m(A \backslash (A_1 \cup A^{**}))\| = 0$$

so that $M(A\backslash(A_1 \cup A^{**})) = \{0\}$. Consequently, $A_2 \subseteq A\backslash A_1$ and $\mu(A_2) = \frac{\epsilon}{2^2}$. In general, there exists a set $A_{k+1} \subseteq A\backslash \cup_{j=1}^{k} A_j$ such that

$$\mu(A_{k+1}) = \frac{\epsilon}{2^{k+1}} \text{ and } M(A_{k+1}) = \{0\}$$

for $k = 2, 3, \ldots$. If we put $A_0 = A\backslash \cup_{k=1}^{\infty} A_k$, then the sets A_k are mutually disjoint and $\mu(A_0) = 0$. This means that $M(A_0) = \{0\}$ because M is μ-continuous. Consequently,

$$M(A) = M\left(\bigcup_{k=0}^{\infty} A_k\right) = \sum_{k=0}^{\infty} M(A_k) = \{0\},$$

a contradiction. ■

Note that if m is a selector of the strong multimeasure M and M is μ-continuous, where μ is non-atomic, then m is μ-continuous and m is also non-atomic.

Theorem 2.4.6 ([4], p118, Theorem 8.1) *Suppose that $M : \mathcal{S} \to \mathcal{P}_c(\mathbb{R}^n)$ is a strong multimeasure and let μ be a finite and nonnegative measure on $\mathcal{S}$. If M is μ-continuous, then for every $y \in M(A)$ there exists an $m \in S_M$ such that $m(A) = y$.*

The following examples show that neither the convexity of M nor the μ-continuity of M can be omitted in the previous theorem. The first example is due to W. Hildenbrand [41]. We consider $(I,\mathcal{B},\lambda)$, where I is the closed unit interval $[0,1]$, $\mathcal{B}$ is the Borel σ-algebra of [0,1] and λ is the Lebesgue measure on $\mathcal{B}$.

Example 2.4.7

1. For $A \in \mathcal{B}$ we put

$$M(A) = \begin{cases} \{0\} & \text{if } \lambda(A) = 0 \\ \{1, 2, \ldots\} & \text{if } \lambda(A) > 0. \end{cases}$$

Then M is λ-continuous, from the definition of M. If m is a selector of M, then m is λ-continuous, and m is non-atomic (the Lebesgue measure λ being non-atomic). If also $m(I) = 1$, then the fact that m has the Darboux property implies that the whole interval [0,1] is in the range of m, which is a contradiction. Consequently, the only selector of M is the measure which is identically zero.

2. For $A \in \mathcal{B}$ we put

$$M(A) = \begin{cases} \{0\} & \text{if } A \text{ is a denumerable set} \\ (0, \infty) & \text{otherwise.} \end{cases}$$

For any finite measure μ on $(I, \mathcal{B})$ one can construct an uncountable measurable μ-null set (for example the Cantor set). Then it does not follow that $M(A) = \{0\}$ if $\mu(A) = 0$ for all $A \in \mathcal{B}$. Thus M does not admit a selector at all.

Theorem 2.4.8 (Lyapunov, [26], p264, p266) *Let (Ω, Σ) be a measurable space with Σ a σ-algebra of subsets of the set Ω.*

(a) If Y is a finite-dimensional Banach space and $m : \Sigma \to Y$ is a bounded measure, then the range $R(m)$ of m is compact. If m is in addition non-atomic, then $R(m)$ is convex.

(b) If Y has the Radon-Nikodým property and $m : \Sigma \to Y$ is a non-atomic measure with finite variation $v(m)$, then $\overline{R(m)}$ is norm-compact and convex in Y.

The convexity condition on the values of M in Theorem 2.4.6 can be replaced by the condition of nonatomicity to give the following version of the Lyapunov convexity theorem.

Theorem 2.4.9 ([4], p118, Theorem 8.2) *Suppose that $M : \mathcal{S} \to \mathcal{P}(\mathbb{R}^n)$ is a strong multimeasure such that M is μ-continuous, where μ is a finite and nonnegative measure on $\mathcal{S}$. Then, for every $A \in \mathcal{S}$ and $y \in M(A)$, there exists an $m \in S_M$ such that $m(A) = y$ if and only if M, restricted to the non-atomic part of μ, has only convex values.*

PROOF: Since the measure μ is finite, it has an at most countable number of atoms; therefore the expressions atomic and non-atomic part of μ have meaning. Let $A_1, A_2, \ldots$ be a finite or countable collection of mutually disjoint atoms of $\mathcal{S}$ and let $A = \cup_{k=1}^{\infty} A_k$. If $y \in M(A)$, then $y = \sum_{k=1}^{\infty} y_k$, where $y_k \in M(A_k)$ for $k \in \mathbb{N}$. For $k \in \mathbb{N}$, put

$$m(A_k) = \begin{cases} y_k & \text{if } A_k \text{ is an atom of } m \\ 0 & \text{otherwise.} \end{cases}$$

Then it follows that m is a selector of M, and $m(A) = \sum_{k=1}^{\infty} m(A_k) = \sum_{k=1}^{\infty} y_k = y$.

Conversely, let $y_1, y_2 \in M(A)$ and $0 < \alpha < 1$. Suppose also that m_1 and m_2 are two selectors of M such that $m_1(A) = y_1$ and $m_2(A) = y_2$. Since m_1 and m_2 are both μ-continuous, it follows that they are both non-atomic. Consider the $2n$-dimensional vector-valued measure (m_1, m_2). From the Lyapunov convexity theorem we obtain a set $B \subseteq A$ such that

$$m_1(B) = \alpha m_1(A) = \alpha y_1 \text{ and } m_2(B) = \alpha m_2(A) = \alpha y_2.$$

Since $m_2(A \backslash B) = m_2(A) - m_2(B) = (1 - \alpha) y_2$, it follows that

$$\alpha y_1 + (1 - \alpha) y_2 \in M(B) + M(A \backslash B) = M(A)$$

so that $M(A)$ is convex. ■

Theorem 2.4.10 ([4], p119, Theorem 8.3) *Suppose that $M : \mathcal{S} \to \mathcal{P}_b(\mathbb{R}^n)$ is a strong multimeasure of bounded variation. Then for every $A \in \mathcal{S}$ and $y \in M(A)$ there is an $m \in S_M$ such that $m(A) = y$.*

PROOF: We first show that the set function $\sigma(p, M(\cdot))$ is a finite and nonnegative measure for every $p \in I\!R^n$. The finiteness of $\sigma(p, M(\cdot))$ follows from the boundedness of M. The boundedness of M also implies that $M(\emptyset) = \{0\}$, from which we deduce that $\sigma(p, M(\emptyset)) = 0$. In addition, note that $\sigma(p, M(\cdot))$ is finitely additive (M being punctually additive).

It remains to show that if (A_k) is a sequence of mutually disjoint elements of $\mathcal{S}$, with $A = \cup_{k=1}^{\infty} A_k$, then $\sigma(p, M(A)) = \sum_{k=1}^{\infty} \sigma(p, M(A_k))$. Given $\epsilon > 0$, there is a $y \in M(A)$ such that

$$\sigma(p, M(A)) - \epsilon \;\leq\; p \cdot y \;\leq\; \sigma(p, M(A)).$$

By the countable additivity of M there is a sequence (y_k) in $M(A_k)$ such that $y = \sum_{k=1}^{\infty} y_k$. Hence

$$\liminf_n \sum_{k=1}^{n} \sigma(p, M(A_k)) \;\geq\; \sum_{k=1}^{\infty} p \cdot y_k \;=\; p \cdot y \;\geq\; \sigma(p, M(A)) - \epsilon.$$

On the other hand,

$$\sigma(p, M(A)) \;\geq\; p \cdot y \;=\; \sum_{k=1}^{\infty} p \cdot y_k.$$

To complete the proof we only need to show that

$$\sum_{k=1}^{\infty} p \cdot y_k + \epsilon \;\geq\; \limsup_n \sum_{k=1}^{n} \sigma(p, M(A_k)).$$

If not, then

$$\limsup_n \sum_{k=1}^{n} (\sigma(p, M(A_k)) - p \cdot y_k) \;=\; \limsup_n \sum_{k=1}^{n} \sigma(p, M(A_k)) - \sum_{k=1}^{\infty} p \cdot y_k > \epsilon.$$

Therefore, there is an integer k_0 such that $\sum_{k=1}^{k_0} (\sigma(p, M(A_k)) - p \cdot y_k) > \epsilon$. Consequently, there are $x_1, x_2, \ldots, x_{k_0}$ such that $\sum_{k=1}^{k_0} (p \cdot x_k - p \cdot y_k) > \epsilon$ and $x_k \in M(A_k)$ for $k = 1, 2, \ldots, k_0$. The series $x_1, x_2, \ldots, x_{k_0}, y_{k_0+1}, \ldots$ is convergent and by the countable additivity of M we get that $\sum_{k=1}^{k_0} x_k + \sum_{k=k_0+1}^{\infty} y_k \in M(A)$. But

$$\begin{aligned}
p \cdot \left(\sum_{k=1}^{k_0} x_k + \sum_{k=k_0+1}^{\infty} y_k \right) \;&=\; p \cdot \sum_{k=1}^{k_0} x_k + p \cdot \sum_{k=k_0+1}^{\infty} y_k \\
&>\; \epsilon + p \cdot \sum_{k=1}^{k_0} y_k + p \cdot \sum_{k=k_0+1}^{\infty} y_k \\
&=\; \epsilon + p \cdot \sum_{k=1}^{\infty} y_k \\
&=\; \epsilon + p \cdot y \;\geq\; \sigma(p, M(A)),
\end{aligned}$$

a contradiction.

Lastly, since M is $\sigma(p, M(\cdot))$-continuous and since M is nonatomic on the nonatomic part of $\sigma(p, M(\cdot))$, Theorem 4.2 of [4] implies that the values of M, on the nonatomic part of $\sigma(p, M(\cdot))$, are convex. The result then follows from Theorem 2.4.9. ∎

Definition 2.4.11 *Let Y be a normed space and suppose that $m : \mathcal{S} \to Y$ and $m_i : \mathcal{S} \to Y$ $(i \in I)$ are vector measures. Then we say that*

*(a) m is **strongly additive** if*

$$\| \sum_{k=1}^{\infty} m(A_k) \| < \infty$$

for every sequence $(A_k) \subseteq \mathcal{S}$ of mutually disjoint sets;

*(b) $\{m_i \mid i \in I\}$ is **uniformly strongly additive** if each m_i is strongly additive and*

$$\lim_{n\to\infty} \sup_{i \in I} \| \sum_{k=n}^{\infty} m_i(A_k) \| = 0$$

for every sequence $(A_k) \subseteq \mathcal{S}$ of mutually disjoint sets;

*(c) $\{m_i \mid i \in I\}$ is **uniformly bounded** if for each $A \in \mathcal{S}$ we have that*

$$\sup_{i \in I} \| m_i(A) \| < \infty.$$

Theorem 2.4.12 ([67], p37, Theorem 2) *A set-valued set function $M : \mathcal{S} \to \mathcal{P}(\mathbb{R}^n)$ is a compact-convex-valued strong multimeasure if and only if M is punctually additive and there is a sequence (m_k) of uniformly bounded and uniformly strongly additive measures on $\mathcal{S}$ such that, for every $A \in \mathcal{S}$,*

$$M(A) = \overline{co}\,\{m_k(A) \mid k \in \mathbb{N}\}.$$

PROOF: Suppose that M is punctually additive and let $M(A) = \overline{co}\,\{m_k(A) \mid k \in \mathbb{N}\}$, where (m_k) is a sequence of uniformly bounded and uniformly strongly additive measures. It immediately follows that M is closed- and convex-valued. Then

$$\| M(A) \| = \| \overline{co}\,\{m_k(A) \mid k \in \mathbb{N}\} \| \leq \sup_{k \in \mathbb{N}} \| m_k(A) \| < \infty;$$

and M is bounded. Also, let $(A_j) \subseteq \mathcal{S}$ be an increasing sequence such that $\lim_{j\to\infty} A_j = A$. Since (m_k) is uniformly strongly additive and M is punctually additive, we have that

$$\begin{aligned} H(M(A_j), M(A)) &= H(M(A_j), M(A \backslash A_j) + M(A_j)) \\ &\leq \| M(A \backslash A_j) \| \\ &\leq \sup_{k \in \mathbb{N}} \| m_k(A \backslash A_j) \| \to 0 \end{aligned}$$

as $j \to \infty$. It follows from Theorem 2.3.21 that M is a strong multimeasure.

Conversely, if M is a compact- and convex-valued strong multimeasure, then M is punctually additive. Futhermore, there is a countable subset $E = \{y_1, y_2, \dots\}$ of $M(T)$ such that $M(T) = \overline{E}$. From Theorem 2.4.9 follows that for each $y_k \in E$ there is an $m_k \in S_M$ such that $m_k(T) = y_k$ for $k \in I\!N$. Consequently,

$$M(T) = \overline{E} = \overline{\{m_k(T) \mid k \in I\!N\}}.$$

The convexity of $M(T)$ yields $M(T) = \overline{co}\,\{m_k(T) \mid k \in I\!N\}$. But for $A \in \mathcal{S}$ we have that

$$\begin{aligned} M(T) = \overline{co}\,\{m_k(T) \mid k \in I\!N\} &\subseteq \overline{co}\,\{m_k(A) \mid k \in I\!N\} + \overline{co}\,\{m_k(T\backslash A) \mid k \in I\!N\} \\ &\subseteq M(A) + M(T\backslash A) = M(T) \end{aligned}$$

so that

$$M(A) + M(T\backslash A) = \overline{co}\,\{m_k(A) \mid k \in I\!N\} + \overline{co}\,\{m_k(T\backslash A) \mid k \in I\!N\}.$$

Since $\overline{co}\,\{m_k(A) \mid k \in I\!N\} \subseteq M(A)$ and $\overline{co}\,\{m_k(T\backslash A) \mid k \in I\!N\} \subseteq M(T\backslash A)$, it follows that

$$M(A) = \overline{co}\,\{m_k(A) \mid k \in I\!N\}.$$

From $\| m_k(A) \| \leq \| M(A) \| < \infty$ follows that

$$\sup_{k \in I\!N} \| m_k(A) \| \leq \| M(A) \| < \infty,$$

which means that the family $\{m_k \mid k \in I\!N\}$ is uniformly bounded. Also, for any sequence (A_j) of mutually disjoint elements of $\mathcal{S}$, we have that

$$\| \sum_{j=1}^{\infty} m_k(A_j) \| \leq \sum_{j=1}^{\infty} \| m_k(A_j) \| \leq \sum_{j=1}^{\infty} \nu(A_j) \leq \nu(T) < \infty,$$

whence each (m_k) is strongly additive. Finally, if $(A_j) \subseteq \mathcal{S}$ is a decreasing sequence of sets such that $\lim_{j \to \infty} A_j = \emptyset$, then

$$\sup_{k \in I\!N} \| m_k(A_j) \| \leq \| M(A_j) \| \leq \nu(A_j) \to 0$$

as $j \to \infty$, and hence (m_k) is uniformly strongly additive. ■

Theorem 2.4.13 ([36], p63, Theorem 3) *Let Y be a linear topological space. If $M : \mathcal{S} \to \mathcal{P}_k(Y)$ is a set-valued set function such that*

(a) M is punctually additive;

(b) for every sequence $(A_k) \subseteq \mathcal{S}$ of mutually disjoint sets and for every $y_k \in M(A_k)$ the series $\sum_{k=1}^{\infty} y_k$ is unconditionally convergent and

$$M\left(\bigcup_{k=1}^{\infty} A_k\right) = \overline{\left\{y \in Y \mid y = \sum_{k=1}^{\infty} y_k,\ y_k \in M(A_k)\right\}},$$

then $S_M \neq \emptyset$.

PROOF: If $(A_k) \subseteq \mathcal{S}$ is a sequence of mutually disjoint sets such that $A = \bigcup_{k=1}^{\infty} A_k$, put

$$S(A) = \{y \in Y \mid y = \sum_{k=1}^{\infty} y_k,\ y_k \in M(A_k)\}.$$

We first show that

$$\sigma(y', S(A)) = \sum_{k=1}^{\infty} \sigma(y', M(A_k))$$

for all $y' \in Y'$. For this purpose, let $y \in S(A)$. Then for $k \in \mathbb{N}$ there exists a $y_k \in M(A_k)$ such that $y = \sum_{k=1}^{\infty} y_k$. Since

$$(y', y) = (y', \sum_{k=1}^{\infty} y_k) = \sum_{k=1}^{\infty} (y', y_k) \leq \sum_{k=1}^{\infty} \sigma(y', M(A_k)),$$

we deduce that $\sigma(y', S(A)) \leq \sum_{k=1}^{\infty} \sigma(y', M(A_k))$. For the inverse inequality, take $\epsilon > 0$. Then for $k \in \mathbb{N}$ there exists an element $y_k \in M(A_k)$ such that

$$\sigma(y', M(A_k)) \leq (y', y_k) + \frac{\epsilon}{2^k}.$$

By virtue of condition (b), the series $\sum_{k=1}^{\infty} y_k$ is unconditionally convergent with sum, say y. Then $y \in S(A)$ and

$$\begin{aligned}
\sum_{k=1}^{\infty} \sigma(y', M(A_k)) &\leq \sum_{k=1}^{\infty} ((y', y_k) + \frac{\epsilon}{2^k}) \\
&= \sum_{k=1}^{\infty} (y', y_k) + \epsilon \\
&= (y', \sum_{k=1}^{\infty} y_k) + \epsilon \\
&= (y', y) + \epsilon \\
&\leq \sigma(y', S(A)) + \epsilon.
\end{aligned}$$

Since ϵ was arbitrary, we conclude that $\sigma(y', S(A)) = \sum_{k=1}^{\infty} \sigma(y', M(A_k))$.

We now proceed by proving that $\sigma(y', M(A)) = \sigma(y', S(A))$ for all $y' \in Y'$ and $A \in \mathcal{S}$. Note that since $M(A) = \overline{S(A)}$, we have that $\sigma(y', M(A)) \geq \sigma(y', S(A))$. For the inverse inequality, let $\sigma(y', M(A)) > \sigma(y', S(A))$ and put $\beta = \sigma(y', M(A)) - \sigma(y', S(A))$. Then $\beta > 0$ and there is an element y_β of $\overline{S(A)}$ such that

$$\sigma(y', \overline{S(A)}) - \frac{\beta}{2} < (y', y_\beta) \leq \sigma(y', \overline{S(A)}).$$

From $y_\beta \in \overline{S(A)}$ we obtain a sequence (y_k) in $S(A)$ such that $y_k \to y_\beta$ as $k \to \infty$. From $(y', y_k) \to (y', y_\beta)$ as $k \to \infty$, we deduce that there is a $k_0 \in \mathbb{N}$ such that if $k \geq k_0$, then

$$|\,(y', y_k) - (y', y_\beta)\,| < \frac{\beta}{4}.$$

Therefore, if $k \geq k_0$, then

$$(y', y_\beta) - \frac{\beta}{4} < (y', y_k) < (y', y_\beta) + \frac{\beta}{4}.$$

But

$$(y', y_\beta) - \frac{\beta}{4} > \sigma(y', \overline{S(A)}) - \frac{\beta}{2} - \frac{\beta}{4} = \sigma(y', \overline{S(A)}) - \frac{3}{4}\beta > \sigma(y', S(A)),$$

which in turn implies that $(y', y_k) > \sigma(y', S(A))$ for all $k \geq k_0$, contradicting the fact that $(y', y_k) \leq \sigma(y', S(A))$ for $k \geq 1$. Hence, $\sigma(y', M(A)) \leq \sigma(y', S(A))$ and the result follows.

We now proceed by establishing the existence of a selector of M. Consider a well order on Y' (whose existence is guarenteed by the well ordering principle) and give Y the corresponding lexicographical ordering. Since by hypothesis M is compact-valued, we can find a lexicographic maximum $m(A)$ of $M(A)$ for every $A \in \mathcal{S}$. We show that m is a selector of M. For the additivity of m, let $A, B \in \mathcal{S}$ be such that $A \cap B = \emptyset$. If we denote by $\prec$ the lexicographical ordering, then for every $a \in M(A)$ and $b \in M(B)$ we have that $a \prec m(A)$ and $b \prec m(B)$. Since $M(A \cup B) = M(A) + M(B)$, every $c \in M(A \cup B)$ can be written in the form $c = a + b$, where $a \in M(A)$ and $b \in M(B)$. Since the lexicographical ordering is compatible with vector addition, it follows that $c \prec m(A) + m(B)$. This means that $m(A) + m(B)$ is the lexicographic maximum of $M(A \cup B)$ and therefore $m(A \cup B) = m(A) + m(B)$. Then we only need to show that for every $y' \in Y'$ the set function $(y', m(\cdot))$ is a real-valued measure. Indeed, since

$$-\sigma(-y', M(A)) \leq (y', m(A)) \leq \sigma(y', M(A))$$

for every $y' \in Y'$ and every $A \in \mathcal{S}$, and since $\sigma(y', M(A)) = \sum_{k=1}^{\infty} \sigma(y', M(A_k))$, it follows that $(y', m(\cdot))$ is a measure for every $y' \in Y'$. ■

We showed in the first part of the proof of the above theorem that if the set-valued set function $M : \mathcal{S} \to \mathcal{P}_k(Y)$ satisfies condition (b), then

(c) for every sequence $(A_k) \subseteq \Sigma$ of mutually disjoint sets with $A = \bigcup_{k=1}^{\infty} A_k$, we have that
$$\sigma(y', M(A)) = \sum_{k=1}^{\infty} \sigma(y', M(A_k))$$
for every $y' \in Y'$.

We needed condition (a) to show the existence of a selector of M. However, if $M : \mathcal{S} \to \mathcal{P}_{kc}(X)$ is a weak multimeasure, then M satisfies conditions (a) and (c) (for (a) see [29], page 414, Lemma 3). Consequently,

Corollary 2.4.14 *If Y is a linear topological space and if $M : \mathcal{S} \to \mathcal{P}_{kc}(Y)$ is a weak multimeasure, then $S_M \neq \emptyset$.*

Denote by $ca(Y)$ the space of all Y-valued measures on $\mathcal{S}$ and let τ denote the topology of pointwise convergence for $ca(Y)$. Then S_M has the following topological property:

Theorem 2.4.15 ([36], p66, Theorem 5) *If Y is a linear topological space and if $M : \mathcal{S} \to \mathcal{P}_{kc}(Y)$ is a weak multimeasure, then S_M is τ-compact and convex.*

PROOF: For every $A \in \mathcal{S}$ the set

$$H(A) = \{m(A) \mid m \in S_M\}$$

is relatively compact in Y and is contained in $M(A)$. By virtue of [45], page 218, we only need to show that S_M is τ-closed. Indeed, if $m(A) \in \overline{H(A)}$, then there exists a net $(m_i)_{i \in I}$ in S_M such that $\lim_{i \in I} m_i(A) = m(A)$ for all $A \in \mathcal{S}$. Since $M(A)$ is compact, it follows that $m(A) \in M(A)$.

It only remains to show that m is a measure. The additivity of the m_i's implies the additivity of m. As a consequence of the proof of Theorem 2.4.13 it follows that m is a measure. We conclude that S_M is τ-closed and the theorem follows. ∎

In the case that M is a weak multimeasure with nonempty compact and convex values, the last two theorems show that S_M is compact and convex relative to τ. By the Krein-Milman theorem it follows that $\overline{co}ext\, S_M = S_M$. As a consequence, we have that

Theorem 2.4.16 ([36], p67, Theorem 6) *If Y is a linear topological space and if $M : \mathcal{S} \to \mathcal{P}_{kc}(Y)$ is a weak multimeasure, then*

$$M(A) = \{m(A) \mid m \in S_M\} = \overline{co}\{m(A) \mid m \in extS_M\}$$

for every $A \in \mathcal{S}$.

PROOF: Since S_M is τ-compact and convex, S_M is the closed convex hull of its lexicographic maximum. By virtue of the proof of Theorem 2.4.13 every lexicographic maximum of $M(A)$ belongs to $H(A) = \{m(A) \mid m \in S_M\}$. The linearity and continuity of the mapping $m \mapsto m(A)$ from $ca(Y)$ into Y implies that $H(A)$ is compact and convex. As a result we have that $M(A) \subseteq H(A)$ for every $A \in \mathcal{S}$, while the inverse inclusion follows trivially.

Lastly, for the second equality, it follows immediately that $\overline{co}\{m(A) \mid m \in ext\, S_M\} \subseteq M(A)$. For the inverse inclusion observe that $\overline{co}ext\, S_M = S_M$ and that the mapping $m \mapsto m(A)$ is linear and continuous. ∎

The following result is a consequence of the first equality in the previous theorem.

Theorem 2.4.17 *Let Y be a linear topological space and let $M : \mathcal{S} \to \mathcal{P}_{kc}(Y)$ be a weak multimeasure. Then for every $A \in \mathcal{S}$ and $y \in M(A)$ there exists an $m \in S_M$ such that $m(A) = y$.*

Theorem 2.4.18 ([31], p154, Theorem 3) *If Y is a separable Banach space and if $M : \mathcal{S} \to \mathcal{P}_{wk}(Y)$ is an additive set function (normal multimeasure), then there exists a sequence $(m_k) \subseteq S_M$ of additive set functions (vector measures) such that*

$$M(A) = \overline{\{m_k(A) \mid k \in \mathbb{N}\}}$$

for every $A \in \mathcal{S}$.

Let $(I, \preceq)$ be a preordered set and let $(E_\alpha)_{\alpha \in I}$ be a family of sets indexed by I. For each pair (α, β) of elements of I such that $\alpha \preceq \beta$, let $f_{\alpha\beta}$ be a mapping of E_β into E_α such that the relation $\alpha \preceq \beta \preceq \gamma$ implies $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$ and $f_{\alpha\alpha}$ is the identity mapping of E_α. If we put $G = \prod_{\alpha \in I} E_\alpha$, let

$$E = \{x \in G \mid pr_\alpha x = f_{\alpha\beta}(pr_\beta x),\ \alpha \leq \beta\}.$$

Then E is said to be the *inverse limit* of the family $(E_\alpha)_{\alpha \in I}$ with respect to the family of mappings $(f_{\alpha\beta})$ and we write $E = \lim_{\leftarrow}(E_\alpha, f_{\alpha\beta})$. The pair $(E_\alpha, f_{\alpha\beta})$ is called an *inverse system of sets* relative to the index set I. The restriction f_α of the projection pr_α to E is called the *canonical mapping* of E into E_α and we have that $f_\alpha = f_{\alpha\beta} \circ f_\beta$ whenever $\alpha \preceq \beta$.

The rest of the results on selectors of multimeasures arise mainly from the following theorem:

Theorem 2.4.19 (Mittag-Leffler) *Let $(Y_\alpha, y_{\alpha\beta})$ be an inverse system of metrizable complete Hausdorff uniform spaces, indexed by a preordered set I which has a countable cofinal subset. Let $Y = \lim_{\leftarrow} Y_\alpha$ and let y_α be the canonical mapping from Y into Y_α. If, for each $\alpha \in I$, there is an index $\beta \succeq \alpha$ such that $y_{\alpha\beta}(Y_\beta)$ is dense in Y_α, then y_α is dense in Y_α for all $\alpha \in I$.*

The next five theorems from [19], pages III-8 - III-18, formulated here (without proofs) in our terminology, are important for the development to follow.

Theorem 2.4.20 *If Y is a linear topological space and if $M : \mathcal{S} \to \mathcal{P}_k(Y)$ is an additive set-valued set function, then for every $A \in \mathcal{S}$ we have that*

$$M(A) = \{m(A) \mid A \in \mathcal{S}\}.$$

Theorem 2.4.21 *Let Y be a linear topological space. If the ring $\mathcal{R}$ is countable and if the set-valued set function $M : \mathcal{R} \to \mathcal{P}_{fb}(Y)$ is additive, then for every $A \in \mathcal{R}$ we have that*

$$M(A) = \overline{\{m(A) \mid A \in \mathcal{R}\}}.$$

Theorem 2.4.22 *Let $(T, \mathcal{S}, \mu)$ be a finite positive measure space with its quotient ring separable and let Y be a linear topological space. If $M : \mathcal{S} \to \mathcal{P}_{fb}(Y)$ is a μ-continuous normal multimeasure, then for every $A \in \mathcal{S}$ we have that*

$$M(A) = \overline{\{m(A) \mid A \in \mathcal{S}\}}.$$

Theorem 2.4.23 *If Y is a separable Banach space and $M : \mathcal{S} \to \mathcal{P}_{fbc}(Y)$ is a normal multimeasure, then for every $A \in \mathcal{S}$ we have that*

$$M(A) = \overline{\{m(A) \mid A \in \mathcal{S}\}}.$$

Theorem 2.4.24 *If the Banach space Y has the Radon-Nikodým Property and if $M : \mathcal{S} \to \mathcal{P}_{fb}(Y)$ is a normal multimeasure, then for every $A \in \mathcal{S}$ we have that*

$$M(A) = \overline{\{m(A) \mid A \in \mathcal{S}\}}.$$

Theorem 2.4.25 *If the Banach space Y has the Radon-Nikodým Property and if $M : \mathcal{S} \to \mathcal{P}_{fb}(Y)$ is a non-atomic normal multimeasure of bounded variation $v(M)$, then*

(a) $M(A)$ is convex for all $A \in \mathcal{S}$;

(b) $\overline{\bigcup_{A \in \mathcal{S}} M(A)}$ is convex.

PROOF: (a) Since $v(M)$ is non-atomic, it has the Darboux property; therefore, for $k \in \mathbb{N}$, we obtain a partition P_k of T consisting of 2^k elements of $\mathcal{S}$ such that

$$v(M, A) = \frac{v(M, T)}{2^k}$$

for all $A \in P_k$. We may choose P_k in such a way that that P_{k+1} is finer than P_k for $k \in \mathbb{N}$. Put $\mathcal{R} = \bigcup_{k=1}^{\infty} P_k$ and denote by Σ the σ-ring generated by $\mathcal{R}$. Then it follows that the restriction of $v(M)$ to Σ is also non-atomic. Let now $y_1, y_2 \in M(T)$, $\alpha \in (0, 1)$ and $\epsilon > 0$. Since Σ is countably generated, we obtain selectors m_1 and m_2 of the restriction of M to Σ such that

$$\|m_1(T) - y_1\| < \epsilon \text{ and } \|m_2(T) - y_2\| < \epsilon.$$

Consequently,

$$\|\alpha y_1 + (1 - \alpha) y_2 - \alpha m_1(T) - (1 - \alpha) m_2(T)\| < \epsilon.$$

If we define the measure $\mu : \Sigma \to \mathbb{R} \times Y \times Y$ by

$$\mu(A) = (v(M, A), m_1(A), m_2(A)),$$

then μ is non-atomic and with finite measure. Since the space $\mathbb{R} \times Y \times Y$ has the Radon-Nikodým Property, we obtain a set $A \in \Sigma$ such that

$$\|m_1(A) - \alpha m_1(T)\| < \epsilon \text{ and } \|m_2(A) - \alpha m_2(T)\| < \epsilon.$$

Then, since $m_2(A) = m_2(T) - m_2(T\backslash A)$, we have that

$$\|\alpha y_1 + (1-\alpha)y_2 - (m_1(A) + m_2(T\backslash A))\| < 3\epsilon.$$

Since $m_1(A) + m_2(T\backslash A) \in M(A) + M(T\backslash A) \subseteq M(T)$, it follows that $\alpha y_1 + (1-\alpha)y_2 \in \overline{M(T)} = M(T)$.

To prove (b), let $y_1 \in M(A_1), y_2 \in M(A_2)$, $\alpha \in (0,1)$ and $\epsilon > 0$. If we put $A = A_1 \cap A_2$, then by the additivity of M, for $y_1', y_2' \in M(A)$, $y_1'' \in M(A_1\backslash A)$ and $y_2'' \in M(A_2\backslash A)$, we have that

$$\|y_1' + y_1'' - y_1\| < \epsilon \text{ and } \|y_2' + y_2'' - y_2\| < \epsilon.$$

By (a) follows that if we put $z = \alpha y_1' + (1-\alpha)y_2'$, then $z \in M(A)$. Furthermore,

$$\|\alpha y_1 + (1-\alpha)y_2 - z - \alpha y_1'' - (1-\alpha)y_2''\| < \epsilon.$$

Also, there exist $B_1 \subseteq A_1\backslash A$, $B_2 \subseteq A_2\backslash A$, $z_1 \in M(B_1)$ and $z_2 \in M(B_2)$ such that

$$\|\alpha y_1'' - z_1\| < \epsilon \text{ and } \|(1-\alpha)y_2'' - z_2\| < \epsilon.$$

Consequently,

$$\|\alpha y_1 + (1-\alpha)y_2 - (z_1 + z_2 + z)\| < 3\epsilon.$$

The additivity of M implies that $z_1 + z_2 + z \in M(B_1 \cup B_2 \cup A)$ so that $\alpha y_1 + (1-\alpha)y_2$ belongs to the range of M. ■

For the rest of this section we will study transition multimeasures and their selectors. We refer to [36] for some of our definitions and results. We consider the measurable spaces $(T, \mathcal{S})$ and $(\Omega, \mathcal{T})$. Unless otherwise stated, Y will be a locally convex vector space.

Definition 2.4.26 *A set-valued set function $M : \Omega \times \mathcal{S} \to \mathcal{P}_f(Y)$ is said to be a* ***transition multimeasure*** *if and only if*

(a) for all $A \in \mathcal{S}$, $\omega \mapsto M(\omega, A)$ is an $\mathcal{S}$-measurable multifunction;

(b) for all $\omega \in \Omega$, $A \mapsto M(\omega, A)$ is a multimeasure.

We will distinguish between strong, normal and weak transition multimeasures.

Definition 2.4.27 *A* ***selector transition measure*** *(or simply a* ***transition selector****) of a transition multimeasure $M : \Omega \times \mathcal{S} \to \mathcal{P}_f(Y)$ is a set function $m : \Omega \times \mathcal{S} \to Y$ such that*

(a) for all $A \in \mathcal{S}$, $\omega \mapsto m(\omega, A)$ is an $\mathcal{S}$-measurable function;

(b) for all $\omega \in \Omega$, $A \mapsto m(\omega, A)$ is a measure;

(c) *for all* $(\omega, A) \in \Omega \times \mathcal{S}$, $m(\omega, A) \in M(\omega, A)$.

The set of all transition selectors of M *will be denoted by* TS_M.

Theorem 2.4.28 ([36], p91, Theorem 1) *Let* Y' *be a separable Frechet space, let* $(\Omega, \mathcal{T})$ *be complete and suppose that* $M : \Omega \times \mathcal{S} \to \mathcal{P}_{kc}(Y)$ *is a weak transition multimeasure. If* $f : \Omega \to Y$ *is a measurable selector of the multifunction* $\omega \mapsto ext\, M(\omega, T)$, *then there exists a transition selector* $m : \Omega \times \mathcal{S} \to Y$ *of* M *such that*

(a) *for all* $\omega \in \Omega$, $m(\omega, T) = f(\omega)$.

(b) *for all* $(\omega, A) \in \Omega \times \mathcal{S}$, $m(\omega, A) \in ext\, M(\omega, A)$.

PROOF: First note that for every $A \in \mathcal{S}$ we have that

$$M(\omega, T) = M(\omega, A) + M(\omega, T\backslash A), \;\; \omega \in \Omega.$$

Since $f(\omega) \in ext\, M(\omega, T)$ for every $\omega \in \Omega$, from Lemma 2 on page 88 of [36] follows that there exist measurable functions $\omega \mapsto m(\omega, A)$ and $\omega \mapsto m(\omega, T\backslash A)$ such that

$$f(\omega) = m(\omega, A) + m(\omega, T\backslash A),$$

with $m(\omega, A) \in ext\, M(\omega, A)$ and $m(\omega, T\backslash A) \in ext\, M(\omega, T\backslash A)$. Then the mapping $(\omega, A) \mapsto m(\omega, A)$ is the required transition selector of M. ∎

Theorem 2.4.29 ([36], p92, Theorem 2) *Let* $(\Omega, \mathcal{T})$ *be complete and suppose that* $M : \Omega \times \mathcal{S} \to \mathcal{P}_{kc}(\mathbb{R}^n)$ *is a transition multimeasure. If* $f : \Omega \to \mathbb{R}^n$ *is a measurable function such that* $f(\omega) \in M(\omega, T)$ *for all* $\omega \in \Omega$, *then there exists a transition selector* $m : \Omega \times \mathcal{S} \to \mathbb{R}^n$ *of* M *such that*

$$m(\omega, T) = f(\omega)$$

for all $\omega \in \Omega$.

PROOF: For $n \in \mathbb{N}$, denote by Λ_{n+1} the simplex in $\mathbb{R}^{n+1}$ defined by

$$\Lambda_{n+1} = \{(\mu_1, \mu_2, \ldots, \mu_{n+1}) \mid \mu_i \geq 0, \, 1 \leq i \leq n; \, \sum_{i=1}^{n+1} \mu_i = 1\}.$$

Consider the continuous mapping $h : \mathbb{R}^{(n+1)^2} \to \mathbb{R}^n$ defined by the equality

$$h(\mu_1, \mu_2, \ldots, \mu_{n+1}, x_1, x_2, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} \mu_i y_i$$

for $\mu_i \in \mathbb{R}^{n+1}$ and $y_i \in \mathbb{R}^n$, $1 \leq i \leq n$. Since $M(\omega, T)$ is a compact and convex subset of $\mathbb{R}^n$, we have that

$$M(\omega, T) = h \circ \left(\Lambda_{n+1} \times (ext\, M(\omega, T))^{n+1}\right).$$

Let

$$\Phi(\omega) = \left\{ y \in \Lambda_{n+1} \times (ext\, M(\omega, T))^{n+1} \mid h(y) = f(\omega) \right\}$$

for all $\omega \in \Omega$. Then $\Phi(\omega) \neq \emptyset$ for all $\omega \in \Omega$ and Φ is $\mathcal{S}(\mathcal{T} \times \mathcal{S}(\mathcal{B}_{\mathbb{R}^n}))$-measurable. Then, for $i = 1, 2, 3, \ldots, n+1$, there exist mappings $y_i : \Omega \to \mathbb{R}^n$ and $\mu_i : \Omega \to \mathbb{R}$ such that

$$f(\omega) = \sum_{i=1}^{n+1} \mu_i(\omega) y_i(\omega),$$

with $y_i \in ext\, M(\omega, T)$, $\mu_i(\omega) \geq 0$, $\sum_{i=1}^{n+1} \mu_i(\omega) = 1$. Hence, for $i = 1, 2, 3, \ldots, n+1$, there exists an $m_i \in TS_M$ such that $m_i(\omega, T) = y_i(\omega)$ for all $\omega \in \Omega$. The mapping

$$(\omega, A) \mapsto m(\omega, A) = \sum_{i=1}^{n+1} \mu_i(\omega) m_i(\omega, A)$$

is then the desired transition selector of M. ■

Corollary 2.4.30 ([36], p93, Corollary 1) *Let $(\Omega, \mathcal{T})$ be complete and suppose that $M : \Omega \times \mathcal{S} \to \mathcal{P}_{kc}(\mathbb{R}^n)$ is a transition multimeasure. If $f : \Omega \to \mathbb{R}^n$ is a measurable function such that $f(\omega) \in M(\omega, A)$ for all $\omega \in \Omega$, then there exists a transition selector $m : \Omega \times \mathcal{S} \to \mathbb{R}^n$ of M such that*

$$m(\omega, A) = f(\omega), \quad A \in \mathcal{S}$$

for all $\omega \in \Omega$.

PROOF: The multifunction $\omega \mapsto M(\omega, T\backslash A)$ is $\mathcal{S}$-measurable and admits a measurable selector g. Consider the restriction M_A of M to $\Omega \times \mathcal{R}_A$, where $\mathcal{R}_A$ is the ring of all subsets of A. From the previous theorem there exists an $m_A \in TS_{M_A}$ such that $m_A(\omega, A) = f(\omega)$ for all $\omega \in \Omega$. Also, there exists an $m_{T\backslash A} \in TS_{M_{T\backslash A}}$ such that $m_{T\backslash A}(\omega, T\backslash A) = g(\omega)$ for all $\omega \in \Omega$. Since for all $(\omega, C) \in \Omega \times \mathcal{S}$,

$$M(\omega, C) = M(\omega, C \cap A) + M(\omega, C \cap T\backslash A),$$

it follows that the set function $m : \Omega \times \mathcal{S} \to \mathbb{R}^n$ defined by

$$m(\omega, C) = m_A(\omega, C \cap A) + m_{T\backslash A}(\omega, C \cap T\backslash A)$$

is the desired transition selector of M. ■

Theorem 2.4.31 ([36], p95, Proposition 1) *Let Y' be a separable Frechet space, let $(\Omega, \mathcal{T})$ be complete and suppose that $M : \Omega \times \mathcal{S} \to \mathcal{P}_{kc}(Y)$ is a weak transition multimeasure. If $f : \Omega \to Y$ is a measurable selector of the multifunction $\omega \mapsto ext\, M(\omega, A)$, $A \in \mathcal{S}$, then there exists a transition selector $m : \Omega \times \mathcal{S} \to Y$ of M such that*

$$m(\omega, A) = f(\omega)$$

for all $\omega \in \Omega$.

PROOF: For all $A \in \mathcal{S}$, the multifunction $ext\, M(\cdot, T\backslash A)$ is measurable and admits a measurable selector g. Using the same notation as in the proof of the previous corollary, the multimeasures $(ext\, M)_A$ and $(ext\, M)_{T\backslash A}$ admit transition selectors m_A and $m_{T\backslash A}$, respectively. If we define $m : \Omega \times \mathcal{S} \to Y$ by

$$m(\omega, C) = m_A(\omega, C \cap A) + m_{T\backslash A}(\omega, C \cap T\backslash A),$$

then the result follows. ∎

In the next result, we suppose that the σ-ring $\mathcal{S}$ is generated by the ring $\mathcal{R}$.

Proposition 2.4.32 ([36], p96, Proposition 2)

(a) If μ is a finite positive measure on $\mathcal{S}$ and $m : \mathcal{R} \to Y$ is a measure such that $\lim_{\mu(A)\to 0} m(A) = 0$ for all $A \in \mathcal{R}$, then the extension $n : \mathcal{S} \to Y$ of m to $\mathcal{S}$ is a unique measure.

(b) Let Y be a separable Frechet space, let $(\Omega, \mathcal{T})$ be complete and let $\mu : \Omega \times \mathcal{S} \to I\!R_+$ be a transition measure. If $m : \Omega \times \mathcal{R} \to Y$ is a transition measure such that $\lim_{\mu(\omega,A)\to 0} m(\omega, A) = 0$ for all $(\omega, A) \in \Omega \times \mathcal{R}$, then the extension $n : \Omega \times \mathcal{S} \to Y$ of m to $\Omega \times \mathcal{S}$ is a unique transition measure.

PROOF: To prove (a), note that from Lemma 1 on page 158 of [29] follows that the quotient set $\Theta = \mathcal{T}/\mathcal{N}(\mu)$, where $\mathcal{N}(\mu)$ is the class of all μ-negligible sets in $\mathcal{T}$, is a complete metric space under the metric

$$\rho(A^*, B^*) = \arctan \mu(A \triangle B), \quad A^*, B^* \in \Theta; A, B \in \mathcal{S}.$$

If we put $\Delta = \overline{\mathcal{R}}$, then Δ is dense in Θ. The set function $m^* : \Theta \to Y$, defined by

$$m^*(A^*) = m(A), \quad A \in \mathcal{S},$$

is uniformly continuous on Δ. This means that m^* can be extended to a unique set function n^*.

To show that n^* is additive, let $A, B \in \mathcal{S}$ be such that $A \cap B = \emptyset$. Then there exist sequences $(A_k), (B_k) \subseteq \mathcal{R}$ such that $\rho(A_k^*, A^*) \to 0$ and $\rho(B_k^*, B^*) \to 0$. Since the operations $A^* \cup B^*$ and $A^*\backslash B^*$ are uniformly continuous, we deduce that

$$\rho(A_k^* \cup B_k^*, A^* \cup B^*) \to 0 \text{ and } \rho(A_k^*\backslash B_k^*, A^*\backslash B^*)) = \rho(A_k^*\backslash B_k^*, A^*) \to 0$$

as $k \to \infty$. Since

$$m^*(A_k^* \cup B_k^*) = m((A_k\backslash B_k) \cup B_k) = m(A_k\backslash B_k) + m(B_k),$$

we get that

$$n^*(A^* \cup B^*) = n^*(A^*) + n^*(B^*)$$

after passing to the limit. If we define

$$n(A) = n^*(A^*), \quad A \in \mathcal{S},$$

then it follows that n is an additive set function on $\mathcal{S}$. The countable additivity of n follows from the μ-continuity of n^*.

To prove (b), first note that for all $\omega \in \Omega$, the set function $n(\omega, \cdot)$ is a measure on $\mathcal{S}$, where $n(\omega, \cdot)$ denotes the extension of $m(\omega, \cdot)$ to $\mathcal{S}$. Furthermore, if $A \in \mathcal{S}$, then there is a sequence $(A_k) \subseteq \mathcal{R}$ such that

$$n(\omega, A) = \lim_{k \to \infty} m(\omega, A_k)$$

for all $\omega \in \Omega$. Then it follows easily that n is the desired transition measure. ■

Theorem 2.4.33 ([36], p102, Theorem 4) *Let $(\Omega, \mathcal{T})$ be complete, $(T, \mathcal{S})$ countably generated and let μ be a positive finite measure on $\mathcal{S}$. Suppose that $M : \Omega \times \mathcal{S} \to \mathcal{P}_{fc}(\mathbb{R}^n)$ is a strong transition multimeasure such that $M(\omega, \cdot)$ is μ-continuous for all $\omega \in \Omega$ and such that $Gr_{M(\cdot, A)}$ is $\mathcal{S}(\mathcal{T} \times \mathcal{B}_{\mathbb{R}^n})$-measurable. If $f : \Omega \to \mathbb{R}^n$ is a measurable function such that $f(\omega) \in M(\omega, T)$ for all $\omega \in \Omega$, then there exists a transition selector $m : \Omega \times \mathcal{S} \to \mathbb{R}^n$ of M such that*

$$m(\omega, T) = f(\omega), \quad \omega \in \Omega.$$

Theorem 2.4.34 ([36], p107, Corollary 3) *Let Y be a separable Frechet space, let $(\Omega, \mathcal{T})$ be complete and $(T, \mathcal{S})$ countably generated and let μ be a positive finite measure on $\mathcal{S}$. Suppose that $M : \Omega \times \mathcal{S} \to \mathcal{P}_{wkc}(Y)$ is a transition multimeasure such that $\lim_{\mu(A) \to 0} \|M(\omega, A)\| = 0$ for all $(\omega, A) \in \Omega \times \mathcal{S}$ and let $f : \Omega \to \mathbb{R}$ be a measurable function. If we define the transition multimeasure $N : \Omega \times \mathcal{S} \to \mathcal{P}_{wkc}(Y)$ by*

$$N(\omega, A) = f(\omega).M(\omega, A),$$

then

$$TS_N = fTS_M = \{f(\omega)m(\omega, A) \mid m \in TS_M\}.$$

PROOF: Note that if $m \in TS_M$, then the mapping $(\omega, A) \mapsto f(\omega)m(\omega, A)$ belongs to TS_N. If we put $n(\omega, A) = f(\omega)m(\omega, A)$ and $\Omega_0 = \{\omega \in \Omega \mid f(\omega) \neq 0\}$, then $\Omega_0 \in \mathcal{T}$ and the mapping $(\omega, A) \mapsto \frac{n(\omega, A)}{f(\omega)}$ is a transition selector of M on $\Omega_0 \times \mathcal{S}$. The result then follows from Theorem 2.4.33. ■

CHAPTER 3

EXTENSION OF SET-VALUED SET FUNCTIONS

The extension problem for countably additive scalar measures has its roots in integration theory. To apply the Lebesgue construction it was necessary to extend scalar set functions, usually defined explicitly only on a ring, to the sigma-algebra of measurable sets. However, the extension problem for vector measures has had a more difficult development. The most inclusive statement about the extension theorem for vector measures has been given by Kluvanek [47]. On the other hand, only two approaches on the extension of multimeasures were thus far established. Kandilakis [44] and Xiaoping et al [69] considered the extension of Banach space-valued multimeasures, while in [67] extension results for multimeasures with values in a finite-dimensional space were given by Wenxiu et al.

In this chapter it is our purpose to study the extension of additive set-valued set functions and multimeasures in general. We also give extension results for transition multimeasures.

3.1 Extension of additive set-valued set functions

In this section we extend additive set-valued set functions and normal multimeasures. Our first set of results are along the lines of Theorem 1.2.3. Central to our proofs are the existence of selectors of the set-valued set functions and the uniform continuity of these selectors. We also prove the set-valued analogue of the Carathéodory-Hahn-Kluvanek theorem for additive set-valued set functions, thereby extending the corresponding result of Kandilakis [44] to additive set-valued set functions.

We let $\mathcal{S}$ be a ring of subsets of T and μ a positive, finite, subadditive and increasing set function on $\mathcal{S}$. If we consider the finite semi-distance ρ_μ as defined just before Proposition 1.2.1, then our first result is the set-valued analogue of Proposition 1.2.1.

Proposition 3.1.1 *Suppose that Y is a Banach space and let $\mathcal{R}$ be a ring contained in $\mathcal{S}$. If $M : \mathcal{R} \to \mathcal{P}_f(Y)$ is an additive set-valued set function such that*

$$\|M(A)\| \leq \mu(A)$$

for $A \in \mathcal{R}$, then M is a uniformly continuous mapping from $(\mathcal{R}, \rho_\mu)$ into $(\mathcal{P}_f(Y), H)$.

PROOF: For $A, B \in \mathcal{R}$ we have that

$$\begin{aligned}
H(M(A), M(B)) &= H(M(A \backslash B) + M(A \cap B), M(B \backslash A) + M(A \cap B)) \\
&\leq H(M(A \backslash B), M(B \backslash A)) + H(M(A \cap B), M(A \cap B)) \\
&\leq H(M(A \backslash B), \{0\}) + H(M(B \backslash A), \{0\}) \\
&= \|M(A \backslash B)\| + \|M(B \backslash A)\| \\
&\leq \mu(A \backslash B) + \mu(B \backslash A) \\
&= \rho_\mu(A, B).
\end{aligned}$$

■

For the rest of this section we suppose that $\mathcal{R}$ is a ring dense in $\mathcal{S}$ for the topology defined by ρ_μ.

Proposition 3.1.2 *Suppose that Y is a separable Banach space. If $M : \mathcal{R} \to \mathcal{P}_k(Y)$ is an additive set-valued set function such that*

$$\|M(A)\| \leq \mu(A)$$

for all $A \in \mathcal{R}$, then M can be extended to an additive set-valued set function $N : \mathcal{S} \to \mathcal{P}_k(Y)$ such that

$$\|N(A)\| \leq \mu(A)$$

for all $A \in \mathcal{S}$. If μ is additive, then $v(N)$ is an extension of $v(M)$.

PROOF: From Theorem 2.4.18 follows that $S_M \neq \emptyset$. Since $\|M(A)\| \leq \mu(A)$ for each $A \in \mathcal{R}$, we infer that each $m \in S_M$ is uniformly continuous on the dense class $\mathcal{R}$. By Theorem 1.2.3 follows that each $m \in S_M$ can be extended to a uniformly continuous finitely additive set function $n : \mathcal{S} \to Y$ such that $\|n(A)\| \leq \mu(A)$ for all $A \in \mathcal{S}$. For $A \in \mathcal{S}$, put

$$N(A) = \overline{\{n(B) \mid B \subseteq A, B \in \mathcal{S}\}}.$$

If $A, B \in \mathcal{S}$ with $A \cap B = \emptyset$, then

$$\begin{aligned}
\overline{N(A)+N(B)} &= \overline{\{n(C)+n(D) \mid C \subseteq A, D \subseteq B, C, D \in \mathcal{S}\}} \\
&= \overline{\{n(C \cup D) \mid C \cup D \subseteq A \cup B\}} \\
&= N(A \cup B);
\end{aligned}$$

whence N is an additive set-valued set function. Clearly we have that $\|N(A)\| \leq \mu(A)$ for all $A \in \mathcal{S}$. We now want to show that

$$|d(n(A), N(A)) - d(n(B), N(B))| < \|n(A) - n(B)\| + H(N(A), N(B)),$$

because by the uniform continuity of N and n it will then follow that the set function $A \mapsto d(n(A), N(A))$ is uniformly continuous. Indeed, from

$$d(n(A), N(A)) \leq d(n(B), N(A)) + \|n(A) - n(B)\|,$$

follows that we only need to prove that

$$d(n(B), N(A)) \leq d(n(B), N(B)) + H(N(A), N(B)).$$

For all $\epsilon > 0$ we can choose $x \in N(A)$ and $y \in N(B)$ such that

$$d(n(B), y) \leq d(n(B), N(B)) + \frac{\epsilon}{2} \text{ and } d(y, x) \leq d(y, N(A)) + \frac{\epsilon}{2}.$$

Consequently, for all $\epsilon > 0$,

$$\begin{aligned}
d(n(B), x) &\leq d(n(B), y) + d(y, x) \\
&\leq d(y, N(A)) + d(n(B), y) + \frac{\epsilon}{2} \\
&\leq d(y, N(A)) + d(n(B), N(B)) + \epsilon;
\end{aligned}$$

therefore

$$d(n(B), N(A)) \leq d(n(B), N(B)) + H(N(A), N(B)) + \epsilon.$$

Since the set function $A \mapsto d(n(A), N(A))$ is identically null on the dense class $\mathcal{R}$, and since N is closed-valued, we deduce that $n(A) \in N(A)$ and consequently

$$N(A) = \overline{\{n(A) \mid n \in S_N\}}$$

for all $A \in \mathcal{S}$. Also, by Theorem 2.4.21 we infer that $M(A) = N(A)$ for all $A \in \mathcal{R}$. Lastly, if μ is additive, then m has finite variation $v(m)$ on $\mathcal{R}$, n has finite variation $v(n)$ on $\mathcal{S}$ and $v(n)$ is an extension of $v(m)$. Since $v(N) = v(n)$, it follows that $v(N)$ is an extension of $v(M)$. ■

Proposition 3.1.3 *Let Y be a separable Banach space. If $M : \mathcal{R} \to \mathcal{P}_{fbc}(Y)$ is an additive set-valued set function such that*

$$\|M(A)\| \leq \mu(A)$$

for all $A \in \mathcal{R}$, then M can be extended to an additive set-valued set function $N : \mathcal{S} \to \mathcal{P}_{fb}(Y)$ such that

$$\|N(A)\| \leq \mu(A)$$

for all $A \in \mathcal{S}$. If μ is additive, then $v(N)$ is an extension of $v(M)$.

PROOF: By Theorem 2.4.18 follows that there is a sequence $(m_k) \subseteq S_M$ of finitely additive set functions from $\mathcal{R}$ into Y such that

$$M(A) = \overline{\{m_k(A) \mid k \in \mathbb{N}\}}$$

for all $A \in \mathcal{R}$. Since $\|M(A)\| \leq \mu(A)$ we have that $\| m_k(A) \| \leq \mu(A)$ for all $A \in \mathcal{R}$ so that each m_k is uniformly continuous on the dense class $\mathcal{R}$. For $k \in \mathbb{N}$, let n_k denote the extension of m_k to $\mathcal{S}$ and put

$$N(A) = \overline{\{n_k(A) \mid k \in \mathbb{N}\}}$$

for all $A \in \mathcal{S}$. Clearly, N is an additive $\mathcal{P}_{fbc}(Y)$-valued set function. Also, since $\|n_k(A)\| \leq \mu(A)$ for all $A \in \mathcal{S}$, we have that $\|N(A)\| \leq \mu(A)$. Lastly, since $v(N) = v(n_k)$ on $\mathcal{S}$ and $v(M) = v(m_k)$ on $\mathcal{R}$, the conclusion follows from Theorem 1.2.3. ■

We now discuss the set-valued analogue of the Carathéodory-Hahn-Kluvanek theorem for additive set-valued set functions. The set-valued Carathéodory-Hahn-Kluvanek theorem has been given by Kandilakis [44, page 88, Theorem 2.6] for countably additive set-valued set functions. The same type of results were also obtained in [69].

First we give an example of a punctually additive set-valued set function which is not a strong multimeasure.

Example 3.1.4

Consider the semiring $\mathcal{R} = \{A \subseteq \mathbb{R} \mid A \text{ is at most countable}\}$ and define the set-valued set function $M : \mathcal{R} \to [0, \infty]$ by

$$M(A) = \begin{cases} \{0\} & \text{if } A \text{ is finite} \\ \{\infty\} & \text{if } A \text{ is countable.} \end{cases}$$

To see that M is punctually additive, let $A, B \in \mathcal{R}$ be such that $A \cap B = \emptyset$. If both A and B are finite, then $A \cup B$ is finite so that

$$M(A \cup B) = \{0\} = M(A) + M(B).$$

On the other hand, if either A or B is countable, then $A \cup B$ is countable and

$$M(A \cup B) = \{\infty\} = M(A) + M(B).$$

$A \cup N = T$, such that A is positive and B is negative with respect to $\sigma(y', N(\cdot))$. Since $N(A) \in \mathcal{P}_{wkc}(Y)$ for every $A \in \mathcal{S}$, there is a $y_0 \in N(A)$ such that $(y', y_0) = \sigma(y', N(A))$. But then $(y', y_0) = \sigma(y', N(\mathcal{S}))$, and from the James theorem follows that $N(\mathcal{S})$ is relatively $w(Y, Y')$-compact. Since $M(\mathcal{A}) = N(\mathcal{A}) \subseteq N(\mathcal{S})$, we have that $M(\mathcal{A})$ is also relatively $w(Y, Y')$-compact. ■

The next result is due to Aló, de Korvin and Roberts [2]. We give the proof for completeness.

Proposition 3.1.8 *Let Y be a separable Banach space and suppose that $M : \mathcal{A} \to \mathcal{P}_{fbc}(Y)$ is an additive set-valued set function. If there exists a finitely additive nonnegative finite set function μ on $\mathcal{A}$ such that M is μ-continuous, then there exists a σ-algebra $\mathcal{S}$, a normal multimeasure $N : \mathcal{S} \to \mathcal{P}_{fbc}(Y)$ and a Boolean isomorphism $i : \mathcal{A} \to \mathcal{S}$ such that $M(A) = N(i(A))$ for all $A \in \mathcal{A}$.*

PROOF: By the Stone Representation Theorem there exists a compact, Hausdorff and totally disconnected topological space $\widetilde{Y}$ such that $\mathcal{A}$ is isomorphic (as a Boolean algebra) with the algebra $\widetilde{\mathcal{A}}$ of all clopen subsets of $\widetilde{Y}$. Let i be the isomorphism of $\mathcal{A}$ into $\widetilde{\mathcal{A}}$. From Theorem 2.4.18 we obtain a sequence $(m_k) \subseteq S_M$ of finitely additive set functions from $\mathcal{A}$ into Y such that

$$M(A) = \overline{\{m_k(A) \mid k \in \mathbb{N}\}}$$

for all $A \in \mathcal{A}$. Define $\widetilde{m_k}(i(A)) = m_k(A)$ for all $A \in \mathcal{A}$. Also let $\widetilde{\mu}(i(A)) = \mu(A)$ for all $A \in \mathcal{A}$. Then, for each $y' \in Y'$, the set function $(y', \widetilde{\mu})$ is a countably additive measure on $\widetilde{\mathcal{A}}$, that is, $\widetilde{\mu}$ is weakly countably additive on $\widetilde{\mathcal{A}}$. Consequently $\widetilde{\mu}$ has a countably additive extension to $\mathcal{S}(\widetilde{\mathcal{A}})$, the σ-algebra generated by $\widetilde{\mathcal{A}}$. If we put $\mathcal{S} = \mathcal{S}(\widetilde{\mathcal{A}})$, then $\widetilde{\mathcal{A}}$ is dense in $\mathcal{S}$ in the metric induced by $\widetilde{\mu}$. Furthermore, since each $\widetilde{m_k}$ is $\widetilde{\mu}$-continuous, each $\widetilde{m_k}$ can be extended to $\widetilde{n_k} : \mathcal{S} \to Y$. If we put

$$N(A) = \overline{\{\widetilde{n_k}(A) \mid k \in \mathbb{N}\}}$$

for every $A \in \mathcal{S}$, then clearly $N(i(A)) = M(A)$ for all $A \in \mathcal{A}$. Also, since we have that $\widetilde{n_k}(A) = \lim \widetilde{m_k}(i(A_n))$ uniformly in k as $i(A_n) \to A$ in the metric induced by $\widetilde{\mu}$, it follows that

$$H(N(A), N(i(A_n))) \to 0;$$

therefore $N(A) \in \mathcal{P}_{fbc}(Y)$ for all $A \in \mathcal{S}$ (since the metric space $(\mathcal{P}_{fbc}(Y), H)$ is complete).

It only remains to show that N is a normal multimeasure. We first show that N is additive. So, let $A, B \in \mathcal{S}$ with $A \cap B = \emptyset$. Then there are mutually disjoint sequences $(A_k), (B_k) \subseteq \mathcal{A}$ such that $i(A_k) \to A$ and $i(B_k) \to B$. Hence, from a result by [24, page 4], we have that

$$
\begin{aligned}
&H(N(A\cup B),\overline{N(A)+N(B)})\\
&\quad\le H(N(A\cup B),N(A_k\cup B_k))+H(N(A_k\cup B_k),\overline{N(A)+N(B)})\\
&\quad= H(N(A\cup B),N(A_k\cup B_k))+H(\overline{N(A_k)+N(B_k)},\overline{N(A)+N(B)})\\
&\quad\le H(N(A\cup B),N(A_k\cup B_k))+H(N(A_k),N(A))+H(N(B_k),N(B))\longrightarrow 0
\end{aligned}
$$

as $k\to\infty$; therefore $N(A\cup B)=\overline{N(A)+N(B)}$. Consequently, if we put $A=\cup_{k=1}^{\infty}A_k$, then

$$
\begin{aligned}
H\left(N(A),\overline{\sum_{k=1}^{n}N(A_k)}\right) &= H\left(\overline{\sum_{k=1}^{n}N(A_k)+N\left(\bigcup_{k=n+1}^{\infty}A_k\right)},\overline{\sum_{k=1}^{n}N(A_k)}\right)\\
&\le \|N\left(\bigcup_{k=n+1}^{\infty}A_k\right)\|\\
&\le \tilde{\mu}\left(\bigcup_{k=n+1}^{\infty}A_k\right)\longrightarrow 0
\end{aligned}
$$

as $n\to\infty$. Hence N is a normal multimeasure. ∎

Proposition 3.1.9 *Let Y be a Banach space and suppose that $M:\mathcal{A}\to\mathcal{P}_{fb}(Y)$ is a strongly additive set-valued set function such that the set function $A\to\sigma(y',M(A))$ is a finitely additive measure on $\mathcal{A}$ for every $y'\in Y'$. Then there exists a finitely additive nonnegative real-valued measure μ on $\mathcal{A}$ such that M is μ-continuous on $\mathcal{A}$.*

PROOF: Let $\tilde{\mathcal{A}}$ be the Stone representation algebra for $\mathcal{A}$ and let $i:\mathcal{A}\to\tilde{\mathcal{A}}$ be a Boolean isomorphism. Define $\tilde{M}:\tilde{\mathcal{A}}\to\mathcal{P}_{fb}(Y)$ by $\tilde{M}(i(A))=M(A)$ for all $A\in\mathcal{A}$. Since $i(A)\to\sigma(y',\tilde{M}(i(A)))$ is a finitely additive measure on $\tilde{\mathcal{A}}$ for every $y'\in Y'$, it follows that $i(A)\to\sigma(y',\tilde{M}(i(A)))$ is countably additive. By Theorem 3.1.7 there is a nonnegative and real-valued countably additive measure $\tilde{\mu}$ on $\tilde{\mathcal{A}}$ such that $\tilde{M}$ is $\tilde{\mu}$-continuous on $\tilde{\mathcal{A}}$. If we define $\mu(A)=\tilde{\mu}(i(A))$ for $A\in\mathcal{A}$, then the result follows. ∎

Proposition 3.1.10 *Let Y be a Banach space and suppose that $\mathcal{S}$ is a σ-algebra of subsets of the set T and let $N:\mathcal{S}\to\mathcal{P}_{wk}(Y)$ be a set-valued set function such that for every $y'\in Y'$ the set function $A\mapsto\sigma(y',N(A))$ admits a Hahn decomposition. Then $N(\mathcal{S})$ is a relatively $w(Y,Y')$-compact subset of Y.*

PROOF: Let $y'\in Y'$ and let (H^+,H^-) be a Hahn decomposition for the signed measure $\sigma(y',N(\cdot))$. Then we have that

$$\sigma(y',\overline{N(\mathcal{S})})=\sup_{A\in\mathcal{S}}\sigma(y',N(A))=\sup_{A\in\mathcal{S}}\sigma(y',N(A\cap H^+))=\sigma(y',N(H^+)).$$

But $N(H^+) \in \mathcal{P}_{wk}(Y)$, so we can find a $y_0 \in N(H^+)$, depending on y', such that $\sigma(y', \overline{N(H^+)}) = (y', y_0)$ and hence $\sigma(y', \overline{N(\mathcal{S})}) = (y', y_0)$. By James' theorem we conclude that $\overline{N(\mathcal{S})}$ is a $w(Y, Y')$-compact subset of Y. ■

Proposition 3.1.11 *Let Y be a Banach space. If $M : \mathcal{A} \to \mathcal{P}_{fb}(Y)$ is a set-valued set function such that $M(\mathcal{A})$ is a relatively weakly compact subset of Y and the set function $A \mapsto \sigma(y', M(A))$ is a finitely additive measure on $\mathcal{A}$ for every $y' \in Y'$, then M is strongly additive.*

PROOF: For every $y' \in Y'$ we have that

$$|\sum_{k=1}^{n}(y', y_k)| = |(y', \sum_{k=1}^{n} y_k)| \leq \sum_{k=1}^{n} |\sigma(y', M(A_k))|$$

for all $y_k \in M(A_k)$ with $A_k \in \mathcal{A}$, $k = 1, 2, \ldots, n$. But since the set function $A \mapsto \sigma(y', M(A))$ is of finite variation, we have that $\lim_{n\to\infty} \sum_{k=1}^{n} |\sigma(y', M(A_k))| < \infty$. Hence $\sum_{k=1}^{\infty} y_k$ is weakly unconditionally convergent and thus strongly unconditionally convergent (from Day [23]). ■

Summarizing the previous four results, we have:

Theorem 3.1.12 *If Y is a separable Banach space and if $M : \mathcal{A} \to \mathcal{P}_{wkc}(Y)$ is a set-valued set function such that the set function $\sigma(y', M(\cdot))$ is a finitely additive measure on $\mathcal{A}$ for every $y' \in Y'$, then the following are equivalent :*

(a) There exists a σ-algebra $\mathcal{S}$, a multimeasure $N : \mathcal{S} \to \mathcal{P}_{wkc}(Y)$ and a Boolean isomorphism $i : \mathcal{A} \to \mathcal{S}$ such that $M(A) = N(i(A))$ for all $A \in \mathcal{A}$.

(b) There exists a finitely additive nonnegative real-valued measure μ on $\mathcal{A}$ such that M is μ-continuous on $\mathcal{A}$.

(c) M is strongly additive.

(d) $M(\mathcal{A})$ is a relatively $w(Y, Y')$-compact subset of Y.

PROOF: We only need to show that (a) implies (d). But $M(\mathcal{A}) = N(i(\mathcal{A})) \subseteq N(\mathcal{S})$, which is a relatively weakly compact subset of Y (by Proposition 3.1.10). ■

3.2 Extension of multimeasures

We start this section with two results by Wenxiu, Jifeng and Aijie [67] on the extension of strong multimeasures. Unless otherwise stated, throughout this section we will suppose that $\mathcal{A}$ is an algebra of subsets of the set T and we let $\mathcal{S}$ be the σ-algebra generated by $\mathcal{A}$.

Theorem 3.2.1 ([67], p38, Theorem 3) *If $M : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}^n)$ is a strong multimeasure, then there exists a unique strong multimeasure $N : \mathcal{S} \to \mathcal{P}_{kc}(\mathbb{R}^n)$ such that $M(A) = N(A)$ for all $A \in \mathcal{A}$.*

PROOF: By Theorem 2.4.12 there exists a sequence (m_k) of uniformly bounded and uniform strongly additive measures on $\mathcal{A}$ such that

$$M(A) = \overline{co}\,\{m_k(A) \mid k \in \mathbb{N}\}, \quad A \in \mathcal{A}.$$

Let n_k be the extension of m_k to $\mathcal{S}$ and put

$$N(A) = \overline{co}\,\{n_k(A) \mid k \in \mathbb{N}\}.$$

Then clearly $M(A) = N(A)$ for $A \in \mathcal{A}$.

If $\overline{\nu}$ denotes the extension of the tight control measure ν, put

$$\mathcal{M} = \{A \in \mathcal{A} : \|n_k(A)\| \leq \overline{\nu}(A)\}.$$

Then $\mathcal{A} \subseteq \mathcal{M}$. We now proceed to show that $\mathcal{M}$ is a monotone class. So let $(A_j) \subseteq \mathcal{M}$ be any increasing or decreasing sequence such that $\lim_{j\to\infty} A_j = A$. Then

$$\|n_k(A)\| = \|\lim_{j\to\infty} n_k(A_j)\| \leq \lim_{j\to\infty} \nu(A_k) = \nu(A).$$

This shows that $A \in \mathcal{M}$, and thus $\mathcal{M}$ is a monotone class. By the monotone class theorem we have that $\mathcal{S} \subseteq \mathcal{M}$, and therefore

$$\|n_k(A)\| \leq \overline{\nu}(A)$$

for every $A \in \mathcal{S}$. Also, $\|N(A)\| \leq \overline{\nu}(A)$, and hence $N(A) \in \mathcal{P}_{kc}(\mathbb{R}^n)$.

Note that (n_k) is uniformly bounded on $\mathcal{S}$. To prove that (n_k) is uniformly strongly additive, observe that if $(A_j) \subseteq \mathcal{A}$ is a sequence of mutually disjoint sets, then

$$\|\sum_{j=1}^{\infty} n_k(A_j)\| \leq \sum_{j=1}^{\infty} \nu(A_j) \leq \nu(T) < \infty.$$

This means that (n_k) is strongly additive. Lastly,

$$\sup_{k\in\mathbb{N}} \|n_k(A_j)\| \leq \|M(A_j)\| \leq \nu(A_j) \to 0$$

as $j \to \infty$.

To prove that N is a multimeasure, we only need to prove that N is punctually additive. Firstly, since (n_k) is uniformly strongly additive, for any increasing or decreasing sequence $(A_j) \subseteq \mathcal{S}$ such that $\lim_{j\to\infty} A_j = A$, we have that

$$\begin{aligned}
H(N(A), N(A_j)) &= H(\overline{co}\,\{n_k(A) \mid k \in \mathbb{N}\}, \overline{co}\,\{n_k(A_j) \mid k \in \mathbb{N}\}) \\
&= H(\overline{co}\,\{n_k(A_j) + n_k(A \backslash A_j) \mid k \in \mathbb{N}\}, \overline{co}\,\{n_k(A_j) \mid k \in \mathbb{N}\}) \\
&\leq \sup_{k\in\mathbb{N}} \|n_k(A \backslash A_j)\| \to 0
\end{aligned}$$

as $j \to \infty$. Consequently, $N(A) = \lim_{j\to\infty} N(A_j)$. For $A \in \mathcal{A}$, put

$$\mathcal{M}_1 = \{B \in \mathcal{A} \mid N(A \cup B) = N(A) + N(B), A \cap B = \emptyset\}.$$

Then for any increasing or decreasing sequence $(B_j) \subseteq \mathcal{M}_1$ such that $\lim_{j\to\infty} B_j = B$, it it obvious that $B \cap A = \emptyset$ and

$$\begin{aligned} N(B \cup A) = N\left(\bigcup_{j=1}^{\infty}(B_j \cup A)\right) &= \lim_{j\to\infty} N(B_j \cup A) \\ &= \lim_{j\to\infty} N(B_j) + N(A) \\ &= N(B) + N(A). \end{aligned}$$

This means that $B \in \mathcal{M}_1$, and hence $\mathcal{M}_1$ is a monotone class. It then follows that for any $A \in \mathcal{A}$ and $B \in \mathcal{S}$, with $A \cap B = \emptyset$, $N(A \cup B) = N(A) + N(B)$. For $B \in \mathcal{S}$, put

$$\mathcal{M}_2 = \{A \in \mathcal{S} \mid N(A \cup B) = N(A) + N(B), A \cap B = \emptyset\}.$$

Just like before, we can prove that $\mathcal{M}_2$ is a monotone class and

$$N(A \cup B) = N(A) + N(B)$$

for $A, B \in \mathcal{S}$ with $A \cap B = \emptyset$.

To prove the uniqueness of N, suppose that $N^* : \mathcal{S} \to \mathcal{P}_{kc}(\mathbb{R}^n)$ is a strong multimeasure such that $N^*(A) = M(A)$ for all $A \in \mathcal{A}$. If we put

$$\mathcal{M}_3 = \{A \in \mathcal{A} \mid N(A) = N^*(A)\},$$

then $\mathcal{A} \subseteq \mathcal{M}_3$ and $\mathcal{M}_3$ is a monotone class. Consequently, $\mathcal{S} \subseteq \mathcal{M}_3$ so that $N(A) = N^*(A)$ for all $A \in \mathcal{S}$. ■

Theorem 3.2.2 ([67], p41, Theorem 8) *If $M : \mathcal{A} \to \mathcal{P}_k(\mathbb{R}^n)$ is a strong multimeasure, then there exists a unique strong multimeasure $N : \mathcal{S} \to \mathcal{P}_k(\mathbb{R}^n)$ such that $M(A) = N(A)$ for all $A \in \mathcal{A}$.*

PROOF: By Theorem 6 of [67],

$$M(A) = M_1(A) + M_2(A),$$

where

$$M_1(A) = \sum_{k=1}^{\infty} M(A \cap B_k) \text{ and } M_2(A) = \lim_{n\to\infty} M(A \cap T \setminus \cup_{k=1}^{n} B_k),$$

with $\{B_1, B_2, \ldots\}$ an at most countable set of atoms of M. Hence, by the previous theorem, there are strong multimeasures $N_1 : \mathcal{S} \to \mathcal{P}_{kc}(\mathbb{R}^n)$ and $N_2 : \mathcal{S} \to \mathcal{P}_k(\mathbb{R}^n)$ such that

$$M_1(A) = N_1(A) \text{ and } M_2(A) = N_2(A),$$

respectively. If for all $A \in \mathcal{S}$ we put

$$N(A) = N_1(A) + N_2(A), \quad A \in \mathcal{S},$$

then N is the required strong multimeasure. ■

Proposition 3.2.3 *Let Y be a Banach space and suppose that the σ-algebra $\mathcal{S}$ is countably generated and let $\mu : \mathcal{S} \to Y$ be a positive measure. If $M : \mathcal{A} \to \mathcal{P}_{fb}(Y)$ is a normal multimeasure such that M is μ-continuous on $\mathcal{A}$, then M can be extended to a normal multimeasure $N : \mathcal{S} \to \mathcal{P}_{fb}(Y)$ such that $\|N(A)\| \le \mu(A)$ for all $A \in \mathcal{S}$.*

PROOF: Since M is additive and the algebra $\mathcal{A}$ is countable, it follows from Theorem 2.4.21 that for all $A \in \mathcal{A}$

$$M(A) = \overline{\{m(A) \mid m \in S_M\}}.$$

Furthermore, by Theorem 2.3.14, it follows that $\lim_{\mu(A)\to 0} \|M(A)\| = 0$ so that $\lim_{\mu(A)\to 0} \| m(A) \| = 0$ for all $m \in S_M$ and $A \in \mathcal{A}$. Consequently, each $m \in S_M$ is uniformly continuous on the dense class $\mathcal{A}$ and thus may be extended to a uniformly continuous set function $n : \mathcal{S} \to Y$. If, for all $A \in \mathcal{S}$, we put

$$N(A) = \overline{\{n(B) \mid B \subseteq A, B \in \mathcal{S}\}},$$

then again we can prove that

$$N(A) = \overline{\{n(A) \mid n \in S_N\}}$$

for $A \in \mathcal{S}$. Clearly N extends M to $\mathcal{S}$.

It only remains to show that N is a normal multimeasure. Put $N'(A) = \{n(B) : B \subseteq A, B \in \mathcal{S}\}$ and let (A_k) be a sequence of mutually disjoint sets in $\mathcal{S}$. Then

$$\begin{aligned}
H\left(N'\left(\bigcup_{k=1}^{\infty} A_k\right), \sum_{k=1}^{\infty} N'(A_k)\right) &= H\left(\sum_{k=1}^{n} N'(A_k) + N'\left(\bigcup_{k=n+1}^{\infty} A_k\right), \sum_{k=1}^{n} N'(A_k) + \sum_{k=n+1}^{\infty} N'(A_k)\right) \\
&\le H\left(N'\left(\bigcup_{k=n+1}^{\infty} A_i\right), \sum_{k=n+1}^{\infty} N'(A_k)\right) \\
&\le \|N'\left(\bigcup_{k=n+1}^{\infty} A_k\right)\| + \| \sum_{k=n+1}^{\infty} N'(A_k) \| \\
&\le 2 \sum_{k=n+1}^{\infty} v(N', A_k) \longrightarrow 0
\end{aligned}$$

as $n \to \infty$. This shows that N' is a strong multimeasure. By Theorem 2.3.17 it follows that N is a $\mathcal{P}_{fb}(Y)$-valued normal multimeasure. Lastly, since $\|n(A)\| \le \mu(A)$ for all $A \in \mathcal{S}$, we conclude that $\|N(A)\| \le \mu(A)$. ■

Proposition 3.2.4 *Let Y be a separable Banach space and let $\mu : \mathcal{S} \to Y$ be a positive measure. If $M : \mathcal{A} \to \mathcal{P}_{kc}(Y)$ is a μ-continuous normal multimeasure, then M can be extended to a normal multimeasure $N : \mathcal{S} \to \mathcal{P}_{kc}(Y)$ such that $\|N(A)\| \leq \mu(A)$ for every $A \in \mathcal{S}$.*

PROOF: Since $M(A) \in \mathcal{P}_{kc}(Y)$ for all $A \in \mathcal{A}$ and Y is separable, there is a countable set $\{y_1, y_2, \ldots\}$ which is dense in $M(A)$. By Theorem 2.4.18 there exists a sequence $(m_k) \subseteq S_M$ such that $m_k(A) = y_k$ for all $A \in \mathcal{A}$ and by the convexity of M we have that

$$M(A) = \overline{co}\{m_k(A) \mid k \in I\!N\}$$

for all $A \in \mathcal{S}$. Let n_k be the extension of m_k to $\mathcal{S}$ and for every $A \in \mathcal{S}$ put

$$N(A) = \overline{co}\{n_k(A) \mid k \in I\!N\}.$$

If we put $N'(A) = \{n_k(A) \mid k \in I\!N\}$, then by Theorem 2.3.17 we only need to show that N' is a strong multimeasure. Let (A_k) be a sequence of mutually disjoint sets in $\mathcal{S}$ and let $A = \cup_{k=1}^{\infty} A_k$. Then

$$\begin{aligned} H\left(N'(A), \sum_{k=1}^{\infty} N'(A_k)\right) &\leq H\left(N'(A), \sum_{k=1}^{n} N'(A_k)\right) + \sum_{k=n+1}^{\infty} \|N'(A_k)\| \\ &= H\left(N'(A), N'(\bigcup_{k=1}^{n} A_k)\right) + \sum_{k=n+1}^{\infty} v(N', A_k) \longrightarrow 0 \end{aligned}$$

as $n \to \infty$. Hence $N'(A) = \sum_{k=1}^{\infty} N'(A_k)$ and therefore N' is a strong multimeasure. ∎

3.3 Extension of transition multimeasures

In this last section of this chapter we suppose that $(\Omega, \mathcal{T})$ is a complete measurable space and $\mathcal{R}$ is a ring of subsets of T. Let $\mathcal{S}$ be the σ-ring generated by $\mathcal{R}$ and let $\lambda : \Omega \times \mathcal{S} \to I\!R_+$ be a transition measure.

Theorem 3.3.1 *If $M : \Omega \times \mathcal{R} \to \mathcal{P}_{kc}(I\!R^n)$ is a strong transition multimeasure of bounded variation such that $\lim_{\lambda(\omega,A)\to 0} M(\omega, A) = 0$ for $(\omega, A) \in (\Omega, \mathcal{R})$, then M can be extended to a strong transition multimeasure $N : \Omega \times \mathcal{S} \to \mathcal{P}_{kc}(I\!R^n)$ of bounded variation such that $\lim_{\lambda(\omega,A)\to 0} N(\omega, A) = 0$ for $(\omega, A) \in (\Omega, \mathcal{S})$.*

PROOF: For all $A \in \mathcal{R}$, define $F(\omega) = M(\omega, A)$ for $\omega \in \Omega$. Since F is a measurable multifunction, there is a sequence $(f_k) \subseteq S_F$ of measurable functions $f_k : \Omega \to I\!R^n$ such that

$$F(\omega) = \overline{\{f_k(\omega) \mid k \in I\!N\}}$$

for all $\omega \in \Omega$ (see Theorem 2.2.5). By Theorem 2.4.29 follows that there is a sequence $(m_k) \subseteq TS_M$ such that $f_k(\omega) = m_k(\omega, A)$ for all $\omega \in \Omega$, and by the convexity of M follows that

$$M(\omega, A) = \overline{co}\,\{m_k(\omega, A) \mid k \in I\!N\}$$

for every $(\omega, A) \in \Omega \times \mathcal{R}$. Furthermore, $\lim_{\lambda(\omega,A)\to 0} M(\omega, A) = 0$ so that $\lim_{\lambda(\omega,A)\to 0} m_k(\omega, A)$ $= 0$ for all $(\omega, A) \in \Omega \times \mathcal{R}$. By Proposition 2.4.32(b) we may extend each m_k to a unique transition measure $n_k : \Omega \times \mathcal{S} \to \mathbb{R}^n$. For all $(\omega, A) \in \Omega \times \mathcal{S}$ put

$$N(\omega, A) = \overline{co}\{n_k(\omega, A) \mid k \in \mathbb{N}\}.$$

Since, for all $A \in \mathcal{S}$, we have that $\omega \mapsto \{n_k(\omega, A) \mid k \in \mathbb{N}\}$ is a measurable multifunction, it follows that N is also a measurable multifunction. Clearly $A \mapsto N(\omega, A)$ is a $\mathcal{P}_{kc}(\mathbb{R}^n)$-valued strong multimeasure. ■

Theorem 3.3.2 *Let Y be a Hausdorff locally convex real vector space and Y' a separable Fréchet space. If $M : \Omega \times \mathcal{R} \to \mathcal{P}_{kc}(Y)$ is a weak transition multimeasure of bounded variation such that $\lim_{\lambda(\omega,A)\to 0} M(\omega, A) = 0$ for $(\omega, A) \in (\Omega, \mathcal{R})$, then M can be extended to a weak transition multimeasure $N : \Omega \times \mathcal{S} \to \mathcal{P}_{kc}(Y)$ of bounded variation such that $\lim_{\lambda(\omega,A)\to 0} N(\omega, A) = 0$ for $(\omega, A) \in (\Omega, \mathcal{S})$.*

PROOF: If, for all $A \in \mathcal{R}$, we define $F(\omega) = M(\omega, A)$, then F is a measurable multifunction. Since $(\Omega, \mathcal{T})$ is complete, it is a Souslin family. Hence (see Theorem 8.4 of [66]), there is a sequence $(f_k) \subseteq S_{extF}$ of measurable functions $f_k : \Omega \to Y$ such that

$$F(\omega) = \overline{co}\{f_k(\omega) \mid k \in \mathbb{N}\}.$$

By Proposition 2.4.31 there is a sequence $(m_k) \subseteq TS_M$ such that $f_k(\omega) = m_k(\omega, A)$ for all $\omega \in \Omega$, and hence

$$M(\omega, A) = \overline{co}\{m_k(\omega, A) \mid k \in \mathbb{N}\}$$

for every $(\omega, A) \in \Omega \times \mathcal{R}$. Again, as before, $\lim_{\lambda(\omega,A)\to 0} M(\omega, A) = 0$ so that $\lim_{\lambda(\omega,A)\to 0} m_k(\omega, A)$ $= 0$ for all $(\omega, A) \in \Omega \times \mathcal{R}$. By Proposition 2.4.32(b) we may extend each m_k to a unique transition measure $n_k : \Omega \times \mathcal{S} \to Y$. For all $(\omega, A) \in \Omega \times \mathcal{S}$ put

$$N(\omega, A) = \overline{co}\{n_k(\omega, A) \mid k \in \mathbb{N}\}.$$

Since $A \mapsto N(\omega, A)$ is a strong multimeasure, it follows from Theorem 5.1 of [54] that $A \mapsto N(\omega, A)$ is a weak multimeasure. ■

CHAPTER 4

INTEGRATION

In the first section of this chapter we give a short outline of the integration of point-valued functions with respect to a vector measure. The standard reference for this section is Chapter II of the book of Dinculeanu [27]. The last section deals with the integration of multifunctions with respect to a multimeasure and we study some of the properties of the resulting set-valued bilinear integral.

Throughout this chapter we will assume that X, Y and Z are Banach spaces. As introduced in the previous chapters, T will denote a non-empty point set on which no topological structure is required and $\mathcal{R}$ is a ring of subsets of T. Furthermore, we let $m : \mathcal{R} \to Y$ be a measure of finite variation $v(m)$, $\mathcal{M}(v(m))$ is the σ-ring of $v(m)$-measurable subsets of T and $\Sigma(v(m))$ is the δ-ring of $v(m)$-integrable subsets of T. The extensions of m and $v(m)$ to $\Sigma(v(m)$ will again be denoted by m and $v(m)$, respectively. Finally, we suppose that there is a bilinear mapping $(x, y) \mapsto xy$ of $X \times Y$ into Z such that $\| xy \| \leq \| x \| \| y \|$ for every $(x, y) \in X \times Y$.

4.1 Integration of functions

If U and V are normed linear spaces, then $\mathcal{L}(U, V)$ will denote the space of all continuous linear transformations $\alpha : U \to V$ equipped with the norm $\|\alpha\| = \sup\{\|\alpha(u)\| \mid u \in U, \|u\| \leq 1\}$. We recall that $\mathcal{L}^*(U, V)$ denotes the space of all linear transformations from U into V. We also recall that

$$\mathcal{E}_X(\mathcal{R}) = \{f : T \to X \mid f = \sum_{k=1}^{n} x_k \chi_{A_k}, x_k \in X, A_k \in \mathcal{R}\}$$

and

$$\mathcal{E}_X(v(m)) = \{f : T \to X \mid f = \sum_{k=1}^{n} x_k \chi_{A_k}, x_k \in X, A_k \in \Sigma(v(m))\}.$$

Definition 4.1.1 *For every $f = \sum_{k=1}^{n} x_k \chi_{A_k} \in \mathcal{E}_X(v(m))$ we define the* ***integral of f with respect to m****, denoted by $\int f(t)\, m(dt)$, by*

$$\int f(t)\, m(dt) = \sum_{k=1}^{n} x_k m(A_k).$$

Remark 4.1.2

(i) The corollary on page 108 of [27] implies that the integral of f with respect to m depends only on f and not on the way in which f is written as a step function. From the definition of the integral follows immediately that

$$\int \chi_A(t)\, m(dt) = m(A)$$

for every $A \in \mathcal{R}$. If $f = \sum_{k=1}^{n} x_k \chi_{A_k} \in \mathcal{E}_X(v(m))$ and if $A \in \tau(\mathcal{R})$, then it follows that $f\chi_A = \sum_{k=1}^{n} x_k \chi_{A_k \cap A} \in \mathcal{E}_X(v(m))$ and in this case we write

$$\int_A f(t)\, m(dt) = \int (f\chi_A)(t)\, m(dt).$$

(ii) If $f = \sum_{k=1}^{n} x_k \chi_{A_k} \in \mathcal{E}_X(v(m))$, then $x_k \in X$ and $m(A_k) \in Y$ for $k = 1, 2, \ldots, n$. The existence of the bilinear transformation $(x, y) \mapsto xy$ from $X \times Y$ into Z then implies that $\sum_{k=1}^{n} x_k m(A_k) \in Z$ so that $\int f(t)\, m(dt) \in Z$.

(iii) We can take $Y \subseteq \mathcal{L}^*(X, Z)$ and the natural bilinear mapping $(x, y) \mapsto xy$ with $(x, y) \in X \times Y$. The general situation of a bilinear mapping xy of $X \times Y$ into Z can always be reduced to this case by identifying an element $y \in Y$ with the continuous linear mapping $x \mapsto xy$ of X into Z. It then follows that if $f \in \mathcal{E}_X(v(m))$ and $m : \Sigma(v(m)) \to Y \subseteq \mathcal{L}^*(X, Z)$, then $\int f(t)\, m(dt) \in Z$.

For every $f \in \mathcal{E}_X(v(m))$ put

$$N_1(f) = N_1(f, m) = N_1(f, v(m)) = \int \|f(t)\|\, v(m, dt).$$

It then follows that N_1 is a semi-norm on the space $\mathcal{E}_X(v(m))$. Furthermore, N_1 defines on $\mathcal{E}_X(v(m))$ a topology called the *topology of the convergence in mean.* Also, if $f = \sum_{k=1}^{n} x_k \chi_{A_k} \in \mathcal{E}_X(v(m))$, then

$$\begin{aligned}
\left\| \int f(t)\, m(dt) \right\| = \left\| \sum_{k=1}^{n} x_k m(A_k) \right\| &\le \sum_{k=1}^{n} \|x_k m(A_k)\| \\
&\le \sum_{k=1}^{n} \|x_k\|\, \|m(A_k)\| \\
&\le \sum_{k=1}^{n} \|x_k\| v(m, A_k) = \int \|f(t)\| v(m, dt) = N_1(f).
\end{aligned}$$

Definition 4.1.3 *A function $f : T \to X$ is said to be* ***m-integrable*** *if there exists a Cauchy sequence $(f_k) \subseteq \mathcal{E}_X(v(m))$ which converges to f $v(m)$-almost everywhere on T. The* ***integral of f with respect to m*** *is that element of Z, denoted by $\int f(t)\, m(dt)$, defined by*

$$\int f(t)\, m(dt) = \lim_{k \to \infty} \int f_k(t)\, m(dt).$$

We will denote by $\mathcal{L}^1_X(m)$ the set of all m-integrable functions $f : T \to X$.

Remark 4.1.4

(i) As for the simple functions, the integral $\int f(t)\, m(dt)$ does not depend on the Cauchy sequence $(f_k) \subseteq \mathcal{E}_X(v(m))$ (see Proposition 9 on page 119 of [27]). Furthermore, every step function $f \in \mathcal{E}_X(v(m))$ is m-integrable, that is, $\mathcal{E}_X(v(m)) \subseteq \mathcal{L}^1_X(m)$.

(ii) If f is m-integrable, then f is measurable. Note that if $f \in \mathcal{L}^1_X(m)$ and $A \in \mathcal{M}(v(m))$, then $f\chi_A \in \mathcal{L}^1_X(m)$ and we write

$$\int_A f(t)\, m(dt) = \int (f\chi_A)(t)\, m(dt).$$

Proposition 4.1.5 ([27], p120, Proposition 2; p122, Proposition 4) *Let $f : T \to X$ and $g : T \to X$ be two functions.*

(a) If $f(t) = g(t)$ $v(m)$-almost everywhere on T and if $f \in \mathcal{L}^1_X(m)$, then $g \in \mathcal{L}^1_X(m)$ and

$$\int f(t)\, m(dt) = \int g(t)\, m(dt).$$

(b) If $f \in \mathcal{L}^1_X(m)$, then $\|f\| \in \mathcal{L}^1_{\mathbb{R}}(v(m))$ and

$$\| \int f(t)\, m(dt)\| \leq \int \|f(t)\|\, v(m, dt).$$

Proposition 4.1.6 ([27], p125, Corollary 2) *If $f, g \in \mathcal{L}^1_X(m)$, then $\int \|f(t) - g(t)\|\, v(m, dt) = 0$ if and only if $f(t) = g(t)$ $v(m)$-almost everywhere on T. In this case we have that $\int f(t)\, m(dt) = \int g(t)\, m(dt)$.*

Just like for the step function, for every m-integrable function $f \in \mathcal{L}^1_X(m)$ we put

$$N_1(f) = N_1(f, m) = N_1(f, v(m)) = \int \|f(t)\|\, v(m, dt).$$

Then N_1 is a semi-norm on the linear space $\mathcal{L}^1_X(m)$ and the topology defined on $\mathcal{L}^1_X(m)$ by N_1 will again be called the *topology of the convergence in mean.* From the inequality

$$\| \int f(t)\, m(dt)\| \leq \int \|f(t)\|\, v(m, dt) = N_1(f)$$

follows that the mapping $\psi : \mathcal{L}^1_X(m) \to Z$, defined by $\psi(f) = \int f(t)\, m(dt)$, is linear and continuous for N_1. Also, from the inequality $| N_1(f) - N_1(g) | \leq N_1(f - g)$ we deduce that N_1 is also continuous on $\mathcal{L}^1_X(m)$.

We say that a sequence $(f_k) \subseteq \mathcal{L}^1_X(m)$ *converges in mean* to a function $f \in \mathcal{L}^1_X(m)$ if

$$\lim_{k\to\infty} N_1(f_k - f) = \lim_{n\to\infty} \int \|f_n(t) - f(t)\|\, v(m, dt) = 0.$$

Note that if (f_k) converges in mean to f, then $\lim_{k\to\infty} N_1(f_k) = N_1(f)$ because N_1 is continuous, and $\lim_{k\to\infty} \int f_k(t)\, m(dt) = \int f(t)\, m(dt)$ because the integral is continuous.

Proposition 4.1.7 ([27], p132, Proposition 16) *Let μ and ν be two positive measures on the ring $\mathcal{R}$. If $\mu \le \nu$, then $\mathcal{L}_X^1(\nu) \subseteq \mathcal{L}_X^1(\mu)$ and*

$$\int \|f(t)\|\,\mu(dt) \;\le\; \int \|f(t)\|\,\nu(dt)$$

for $f \in \mathcal{L}_X^1(\nu)$.

Let

$$\mathcal{N}_X^\infty(m) = \{f : T \to X \mid f(t) = 0 \; v(m)-\text{a.e on } T\}.$$

Denote the quotient space $\mathcal{L}_X^1(m)/\mathcal{N}_X^\infty(m)$ by $Q_X^1(m)$ and let $[f]$ denote the equivalence classes determined by $f \in \mathcal{L}_X^1(m)$, that is

$$[f] = \{g : T \to X \mid g(t) = f(t)\; v(m) - \text{a.e on } T\}.$$

Then it follows that if we put $\|[f]\|_1 = N_1(f)$, then $\|\cdot\|_1$ is a norm on $Q_X^1(m)$ and $(Q_X^1(m),\ \|\cdot\|_1)$ is a Banach space. We denote by $\mathcal{L}_X^\infty(v(m))$ the space of all $v(m)$-measurable functions $f : T \to X$ for which

$$N_\infty(f) = \inf\{a \le \infty \mid \|f(t)\| \;\le\; a \;\; v(m)-\text{a.e}\} < \infty.$$

Then N_∞ is a semi-norm on the linear space $\mathcal{L}_X^\infty(v(m))$. It then follows that the quotient space $\mathcal{L}_X^\infty(v(m))/\mathcal{N}_X^\infty(v(m))$, which we denote by $Q_X^\infty(v(m))$, is a Banach space under the norm $N_\infty([f]) = N_\infty(f)$, where $[f] \in Q_X^\infty(v(m))$ is the equivalence class modulo $\mathcal{N}_X^\infty(v(m))$ of the function $f \in \mathcal{L}_X^\infty(v(m))$.

Definition 4.1.8 *If p is a real number such that $0 < p < \infty$, then we denote by $\mathcal{L}_X^p(v(m))$ the set of all $v(m)$-measurable functions $f : T \to X$ for which $\|f\|^p \in \mathcal{L}_{\mathbb{R}}^1(v(m))$. For every $f \in \mathcal{L}_X^p(v(m))$ we put*

$$N_p(f) = \left(\int \|f(t)\|^p v(m, dt)\right)^{\frac{1}{p}}.$$

From the above definition then follows immediately that

Proposition 4.1.9 *We have that $f \in \mathcal{L}_X^p(v(m))$ if and only if f is $v(m)$-measurable and $\|f\| \in \mathcal{L}_{\mathbb{R}}^p(v(m))$.*

Proposition 4.1.10 *If $f : T \to X$ is $v(m)$-measurable and if there exists a positive function $g \in \mathcal{L}_{\mathbb{R}}^p(v(m))$ such that $\|f(t)\| \le g(t)$ $v(m)$-almost everywhere on T, then $f \in \mathcal{L}_X^p(v(m))$.*

PROOF: First note that $\|f\|^p$ is $v(m)$-measurable and $\|f\|^p \le g^p$. Since $g^p \in \mathcal{L}^1_{\mathbb{R}}(v(m))$, it then follows from Proposition 19, page 136 of [27] that $\|f\|^p \in \mathcal{L}^1_{\mathbb{R}}(v(m))$; therefore $f \in \mathcal{L}^p_X(v(m))$. ■

Proposition 4.1.11 (Hölder) *Let $1 \le p \le \infty$ and $1 \le q \le \infty$ be two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in \mathcal{L}^p_X(v(m))$ and $g \in \mathcal{L}^q_Y(v(m))$, then $fg \in \mathcal{L}^1_Z(m)$ and*

$$\| \int f(t)g(t)\, v(m,dt)\| \le \int \|f(t)\|\, \|g(t)\|\, v(m,dt) \le N_p(f)N_q(g).$$

PROOF: Since f and g are $v(m)$-measurable and the mapping $(x,y) \mapsto xy$ is continuous, it follows that fg is also $v(m)$-measurable. Also, since $\|f\| \in \mathcal{L}^p_{\mathbb{R}}(v(m))$ and $\|g\| \in \mathcal{L}^q_{\mathbb{R}}(v(m))$, it follows that $\|f\|\,\|g\| \in \mathcal{L}^1_{\mathbb{R}}(v(m))$. From the inequality $\|fg\| \le \|f\|\,\|g\|$ we then deduce that $fg \in \mathcal{L}^1_Z(m)$ and

$$\| \int f(t)g(t)\, v(m,dt)\| \le \int \|f(t)\|\, \|g(t)\|\, v(m,dt) \le N_p(f)N_q(g).$$

■

Proposition 4.1.12 (Minkowski, [27], p221, Proposition 11) *If $1 \le p \le \infty$ and $f, g \in \mathcal{L}^p_X(v(m))$, then*

$$N_p(f+g) \le N_p(f) + N_p(g).$$

The topology induced by the semi-norm N_p on $\mathcal{L}^p_X(v(m))$ is called the *topology of the convergence in mean of order p*. To say that a sequence $(f_k) \subseteq \mathcal{L}^p_X(v(m))$ *converges in* $\mathcal{L}^p_X(v(m))$ to a function $f \in \mathcal{L}^p_X(v(m))$ means that

$$\lim_{k\to\infty} N_p(f_k - f) = 0.$$

The quotient space $\mathcal{L}^p_X(v(m))/\mathcal{N}^\infty_X(v(m)) = Q^p_X(v(m))$ is a Banach space under the norm $\|[f]\|_p = N_p(f)$, $f \in [f] \in Q^p_X(v(m))$.

Proposition 4.1.13 ([27], p226, Theorem 3; p227, Corollary 1 & 2)

(a) The space $\mathcal{L}^p_X(v(m))$ is complete.

(b) The space $\mathcal{E}_X(\mathcal{R})$ is dense in $\mathcal{L}^p_X(v(m))$ and if $\mathcal{R}$ is countable and X is separable, then $\mathcal{L}^p_X(v(m))$ is separable.

To end this section, we now discuss the Radon-Nikodým theorem. But first we will need the notion of locally integrability and the concept of a measure with the direct sum property.

Definition 4.1.14 *A function $f : T \to X$ is said to be **locally $v(m)$-integrable** if for every set $A \in \mathcal{R}$ the function $f\chi_A$ is $v(m)$-integrable.*

Remark 4.1.15

(i) If m is a vector measure with finite variation $v(m)$, then we say that a function is locally m-integrable if it is locally $v(m)$-integrable. It follows immediately that every $v(m)$-integrable function is locally $v(m)$-integrable.

(ii) If a function f is $v(m)$-measurable and bounded on every set $A \in \mathcal{R}$, then f is locally $v(m)$-integrable. Conversely, every locally $v(m)$-integrable function is $v(m)$-measurable.

Definition 4.1.16 *Let m be a vector measure with finite variation $v(m)$. We denote by $\mathcal{D}(v(m))$ the set of all families $(A_i)_{i\in I}$ of mutually disjoint $v(m)$-integrable sets such that $T\setminus \cup_{i\in I} A_i$ is $v(m)$-negligible and such that for every set $A \in \mathcal{R}$ there exists a $v(m)$-negligible set $N \subseteq A$ and an at most countable set $J \subseteq I$ with $A\setminus N = \cup_{i\in J}(A \cap A_i)$. We then say that m has the* ***direct sum property*** *if $\mathcal{D}(v(m)) \neq \emptyset$.*

Remark 4.1.17

(i) Every bounded measure has the direct sum property.

(ii) If $T \in \tau(\mathcal{R})$, then every measure on $\mathcal{R}$ has the direct sum property.

(iii) Let μ and ν be two measures on $\mathcal{R}$ and suppose that ν is μ-continuous. If μ has the direct sum property, then ν has the direct sum property.

Let f be a scalar locally $v(m)$-integrable function. Then from Proposition 5 on page 122 of [27] follows that the scalar set function ν, defined on $\mathcal{R}$ by

$$\nu(A) = \int_A f(t)\, v(m, dt)$$

is a $v(m)$-continuous measure. Conversely, the Radon-Nikodým theorem states that if m has the direct sum property, then every m-continuous scalar measure ν on $\mathcal{R}$ is of the preceding form.

Theorem 4.1.18 (Radon-Nikodým, [27], p182, Theorem 5) *Let $m : \mathcal{R} \to Y$ be a measure with finite variation $v(m)$. If m has the direct sum property and if ν is an m-continuous scalar measure on $\mathcal{R}$, then there exists a scalar locally $v(m)$-integrable function f_ν such that*

$$\nu(A) = \int_A f_\nu(t)\, v(m, (dt)),\ A \in \mathcal{R}.$$

Moreover, if $T \in \mathcal{R}$, then we can take the function f_ν to be measurable with respect to the σ-ring generated by $\mathcal{R}$.

We now discuss the generalized Radon-Nikodým theorem. For the rest of this section we will suppose that W is a norming subspace of Z', that is

$$\|z\| = \sup\left\{\frac{|(z,w)|}{\|w\|} \mid w \in W, w \neq 0\right\}$$

for every $z \in Z$. But first we need the following:

Theorem 4.1.19 ([27], p263, Theorem 4) *Let $m : \mathcal{R} \to \mathcal{L}(X,Z)$ be a measure with finite variation $v(m)$. If m has the direct sum property, then there exists a function $U_m : T \to \mathcal{L}(X, W')$ such that*

(a) $\|U_m(t)\| = 1$ $v(m)$-almost everywhere on T;

(b) $(U_m f, w)$ is $v(m)$-integrable and

$$\left(\int f(t)\, m(dt), w\right) = \int (U_m(t)f(t), w)\; v(m, dt)$$

for $f \in \mathcal{L}^1_X(m)$ and $w \in W$;

(c) $U_m(t) \in \mathcal{L}(X,Z)$ for every $t \in T$ if $Z = W'$.

Remark 4.1.20

(i) In the proof of the above theorem the function U_m is defined in such a way that for every $x \in X$ and for every $w \in W$, the function $\phi_{x,w} : T \to \mathbb{R}$, defined by $\phi_{x,w}(t) = (U_m(t)x, w)$, is locally $v(m)$-integrable. Remark 4.1.4(ii) then implies that $\phi_{x,w}\chi_A$ is $v(m)$-measurable for every $A \in \mathcal{R}$ so that $\phi_{x,w}$ is also $v(m)$-measurable.

(ii) Since $U_m(t) \in \mathcal{L}(X,Z)$ if $Z = W'$, it follows that U_m is W-weakly $v(m)$-measurable. Furthermore, if W', and hence Z, is separable, then U_m is simply $v(m)$-measurable. If $f : T \to X$ is $v(m)$-measurable, then the function $g : T \to W' = Z$ defined by $g(t) = U_m(t)f(t)$ is $v(m)$- measurable. Furthermore, since

$$\|U_m(t)f(t)\| \;\leq\; \|U_m(t)\|\,\|f(t)\| \;=\; \|f(t)\|\;\; v(m)-\text{a.e on } T,$$

it follows that $U_m f \in \mathcal{L}^1_Z(m)$ whenever $f \in \mathcal{L}^1_X(m)$ (from Propositions 4.1.9 and 4.1.10). Consequently, if $f \in \mathcal{L}^1_X(m)$, then for every $w \in W$ we have that $(\int f(t)\, m(dt), w) = \int (U_m(t)f(t), w)\, v(m,dt) = (\int U_m(t)f(t)\, v(m,dt), w)$ so that

$$\int f(t)\, m(dt) = \int U_m(t)f(t)\, v(m, dt).$$

(iii) If there exists a countable set $H \subseteq \mathcal{L}(X, W')$ such that $U_m(t) \in H$ $v(m)$-almost everywhere (in particular, if $\mathcal{L}(X, W')$ is separable), then U_m is $v(m)$-measurable (from Proposition 24 on page 106 of [27]). In this case, if we put $Y = \mathcal{L}(X, W')$, then $U_m \in \mathcal{L}^\infty_Y(v(m))$ and

$$\int f(t)\, m(dt) = \int U_m(t)f(t)\, v(m, dt)$$

for $f \in \mathcal{L}^1_{\mathbb{R}}(v(m))$. In particular, if we put $f = \chi_A$, then

$$m(A) = \int_A U_m(t)\, v(m, dt), \quad A \in \Sigma(v(m)).$$

We now state the generalized Radon-Nikodým theorem.

Theorem 4.1.21 ([27], p269, Theorem 5) *Let ν be a scalar measure on $\mathcal{R}$ and $m : \mathcal{R} \to \mathcal{L}(X, Z)$ a measure with finite variation $v(m)$. If ν has the direct sum property and if m is ν-continuous, then there exists a function $V_m : T \to \mathcal{L}(X, W')$ such that*

(a) $\|V_m\|$ is locally ν-integrable and

$$\int f(t)\, v(m, dt) = \int \|V_m(t)\| f(t)\, v(\nu, dt)$$

for $f \in \mathcal{L}^1_{\mathbb{R}}(v(m))$;

(b) $(V_m f, w)$ is ν-integrable and

$$\left(\int f(t)\, m(dt), w\right) = \int (V_m(t) f(t), w)\, \nu(dt)$$

for $f \in \mathcal{L}^1_X(v(m))$ and $w \in W$;

(c) $V_m(t) \in \mathcal{L}(X, Z)$ for every $t \in T$ if $Z = W'$.

Remark 4.1.22

(i) If $W' = Z$ and W', and thus Z, is separable, then for every $f \in \mathcal{L}^1_X(v(m))$ the function $g : T \to W' = Z$, defined by $g(t) = V_m(t) f(t)$, is ν-integrable and

$$\int f(t)\, m(dt) = \int V_m(t) f(t)\, \nu(dt).$$

(ii) If there exists a countable set $H \subseteq \mathcal{L}(X, W')$ with $V_m(t) \in \overline{H}$ ν-almost everywhere (in particular, if $\mathcal{L}(X, W')$ is separable), then V_m is ν-measurable and

$$\int f(t)\, m(dt) = \int V_m(t) f(t)\, \nu(dt), \quad f \in \mathcal{L}^1_{\mathbb{R}}(v(m)).$$

In particular, V_m is locally ν-integrable and $m(A) = \int_A V_m(t)\, \nu(dt)$, $A \in \Sigma(v(m))$.

Theorem 4.1.23 ([27], p282, Corollary 1) *Let X' be separable and let $m : \mathcal{R} \to Y$ be a measure with finite variation $v(m)$. If m has the direct sum property, then the conjugate space of $\mathcal{L}_X^p(v(m))$, $1 \le p < \infty$, is isomorphic and isometric to the space $Q_{X'}^q(v(m))$, $\frac{1}{p} + \frac{1}{q} = 1$.*

Corollary 4.1.24 ([27], p282, Corollary 2) *Let X be a separable and reflexive Banach space and let $1 < p < \infty$. If $m : \mathcal{R} \to Y$ is a measure with finite variation $v(m)$ and m has the direct sum property, then $Q_X^p(v(m))$ is reflexive.*

Corollary 4.1.25 ([27], p282, Corollary 3) *Let $1 < p < \infty$ and $1 < q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $m : \mathcal{R} \to Y$ is a measure with finite variation $v(m)$ and m has the direct sum property, then*

$$(Q_{\mathbb{R}}^1(v(m)))' = Q_{\mathbb{R}}^{\infty}(v(m))$$

and

$$(Q_{\mathbb{R}}^p(v(m)))' = Q_{\mathbb{R}}^q(v(m)) \text{ and } (Q_{\mathbb{R}}^q(v(m)))' = Q_{\mathbb{R}}^p(v(m)).$$

4.2 Integration of multifunctions

In this section we define and investigate some of the properties of the bilinear integral of a multifunction with respect to a multimeasure.

Unless otherwise stated, $M : \mathcal{R} \to \mathcal{P}(Y)$ will be a multimeasure of bounded variation $v(M)$. We will assume that $S_M \neq \emptyset$. Indeed, by Theorem 2.5 of [39] follows that this will be the case if M is a closed-valued strong multimeasure of bounded variation.

Definition 4.2.1 *If $1 \le p < \infty$, then a multifunction $F : T \to \mathcal{P}_0(X)$ is said to be* ***p-integrably bounded*** *if there exists a $k \in \mathcal{L}_{\mathbb{R}}^p(v(M))$ such that*

$$\| F(t) \| \le k \;\; v(M)-\textit{almost everywhere on } T.$$

If $F : T \to \mathcal{P}_0(X)$ is 1-integrably bounded by $k \in \mathcal{L}_{\mathbb{R}}^1(v(M))$, then we say that F is ***integrably bounded*** *by k.*

Let $F : T \to \mathcal{P}_0(X)$ be a multifunction and suppose that $1 \le p \le \infty$. Then we denote by $S_F^p(v(m))$ the set of all selectors of F which belong to $\mathcal{L}_X^p(v(m))$, that is,

$$S_F^p(v(m)) = \{f \in \mathcal{L}_X^p(v(m)) \mid f \in S_F\}.$$

It follows that if $S_F^p(v(m))$ is nonempty and if F is closed-valued, then $S_F^p(v(m))$ is a closed subset of $\mathcal{L}_X^p(v(m))$. Obviously, $S_F^1(v(m))$ denotes the set of all $v(m)$-integrable selectors of F.

We now discuss some results (that we will need in the sequel) about the set $S^p_F(v(m))$. The first result is due to Hiai and Umegaki [40, Lemma 1.1 and Corollary 1.2], while the second result concerns the weak compactness of $S^1_F(m)$. This result was first given by Castaing [15], modified by Hiai and Umegaki in [40] and later on generalized by Papageorgiou [54]. We include the proof for completeness. The last result characterizes $S^p_F(v(m))$ in terms of decomposability.

Proposition 4.2.2 *Let $F, F_i : T \to \mathcal{P}_{wkc}(X), i = 1, 2,$ be $v(m)$-measurable multifunctions such that $S^p_F(v(m))$ and $S^p_{F_i}(v(m)), i = 1, 2,$ are nonempty for $1 \leq p \leq \infty$.*

(a) Then there exists a sequence $(f_k) \subseteq S^p_F(v(m))$ such that $F(t) = \overline{\{f_k(t) \mid k \in I\!N\}}$ $v(m)$-almost everywhere on T.

(b) If $S^p_{F_1}(v(m)) = S^p_{F_2}(v(m))$, then $F_1(t) = F_2(t)$ $v(m)$-almost everywhere on T.

Proof: (a) By Theorem 2.2.5 we obtain a sequence (g_j) of $v(m)$-measurable functions such that

$$F(t) = \overline{\{g_j(t) \mid j \in I\!N\}}$$

for every $t \in T$. Let $\{A_1, A_2, \ldots, A_n, \ldots\}$ be a countable measurable partition of T such that $v(m, A_n) < \infty$. If $f \in \mathcal{L}^p_X(v(m))$, then we define for $j, l, n \in I\!N$

$$B_{jln} = \{t \in T \mid l - 1 \ \leq \ \|g_j(t)\| \ < \ l\} \cap A_n$$

and

$$f_{jln} = \chi_{B_{jln}} g_j + \chi_{T \setminus B_{jln}} f.$$

If we put $f_k = f_{jln}$, then (f_k) is the desired sequence.

(b) This follows from (a). ∎

Before we proceed to prove our second result about $S^p_F(v(m))$, we need the following:

Proposition 4.2.3 ([51], p105, Lemma 10.11) *Let X be a Banach space and suppose that $F : T \to \mathcal{P}_{kc}(X)$ is a $v(m)$-measurable multifunction. Then, for every $x' \in X'$, the function $\sigma(x', F(\cdot)) : T \to I\!R$ is m-integrable.*

Proof: We first show that the function $\sigma(x', F(\cdot))$ is $v(m)$-measurable for each $x' \in X'$. So let $a \in I\!R$ and define

$$A_a = \{x \in X \mid (x', x) \leq a\} \text{ and } B_a = \{x \in X \mid (x', x) \geq a\}.$$

Then A_a and B_a are closed subsets of X and the sets $\{t \in T \mid F(t) \cap A_a \neq \emptyset\}$, $\{t \in T \mid F(t) \cap B_a \neq \emptyset\}$, $\{t \in T \mid F(t) \cap A_a = \emptyset\}$ and $\{t \in T \mid F(t) \cap B_a = \emptyset\}$ are all

$v(m)$-measurable subsets of T. If we let $\alpha \in I\!R$, then it follows that the set

$$\begin{aligned} C(\alpha) &= \{t \in T \mid \sigma(x', F(t)) = \alpha\} \\ &= \{t \in T \mid F(t) \cap A_\alpha \neq \emptyset\} \cap \left(\bigcap_{k=1}^{\infty} \{t \in T \mid F(t) \cap B_{\alpha+\frac{1}{k}} = \emptyset\} \right) \\ &\cap \left(\bigcap_{k=1}^{\infty} \{t \in T \mid F(t) \cap B_{\alpha-\frac{1}{k}} \neq \emptyset\} \right) \end{aligned}$$

is $v(m)$-measurable. Consequently, the set

$$\{t \in T \mid \sigma(x', F(t)) < \alpha\} = (\{t \in T \mid F(t) \cap A_\alpha \neq \emptyset\} \cap \{t \in T \mid F(t) \cap B_\alpha = \emptyset\}) \backslash C(\alpha)$$

is also $v(m)$-measurable; hence the function $\sigma(x', F(\cdot))$ is $v(m)$-measurable.

To prove the integrability of $\sigma(x', F(\cdot))$, note that since F is integrably bounded by k (say), there exists a $v(m)$-negligible set $N \subseteq T$ such that $\|F(t)\| \leq k(t)$ for all $t \in T\backslash N$. It then follows that for any $t \in T\backslash N$ and $x \in F(t)$ we have that

$$|\sigma(x', F(\cdot))| \leq |(x', x)| \leq \|x'\| k(t)$$

$v(m)$-almost everywhere on T. This proves that $\sigma(x', F(\cdot)) \in \mathcal{L}^1_{I\!R}(v(m))$. ■

Proposition 4.2.4 ([54], p187, Proposition 3.1) *Let X' be a separable Banach space and suppose that the measure $m : \Sigma(v(m)) \to Y$ has the direct sum property. If $F : T \to \mathcal{P}_{wkc}(X)$ is an integrably bounded $v(m)$-measurable multifunction, then $S^1_F(m)$ is a non-empty, convex and $w(Q^1_X(m), Q^\infty_{X'}(m))$-compact subset of $Q^1_X(m)$.*

PROOF: By the integrably boundedness of F, there exists a $k \in \mathcal{L}^1_{I\!R}(v(m))$ such that $\|F(t)\| \leq k$ $v(m)$-almost everywhere on T. Corollary 2.2.3 then provides F with a $v(m)$-measurable selector $f : T \to X$. Since $\|f(t)\| \leq k(t)$ $v(m)$-almost everywhere on T, Proposition 4.1.10 implies that $f \in \mathcal{L}^1_X(m)$ so that $S^1_F(m) \neq \emptyset$. Furthermore, note that by Corollary 1.6 of [40] follows that $S^1_F(m)$ is convex.

To show that $S^1_F(m)$ is $w(Q^1_X(m), Q^\infty_{X'}(m))$-compact, first observe that since $S^1_F(m)$ is closed in $Q^1_X(m)$, it follows that it is also weakly closed and bounded. By Theorem 4.1.23 we have that $(Q^1_X(m))' = Q^\infty_{X'}(m)$. If we let $x' \in Q^\infty_{X'}(m)$, then

$$\begin{aligned} \sup_{f \in S^1_F(m)} (x', f(t)) &= \sup_{f \in S^1_F(m)} \int (x', f(t))\, v(m, dt) \\ &= \int \sigma(x', F(t))\, v(m, dt), \end{aligned}$$

where the last equality follows from Theorem 2.2 of [40]. Define

$$H(t) = \{x \in F(t) \mid (x', x) = \sigma(x', F(t))\}.$$

Then since $F(t) \in \mathcal{P}_{wkc}(X)$ it follows immediately that $H(t)$ is non-empty. To show that H is $v(m)$-measurable, note that $Gr_H = \{(t, x) \in T \times X \mid (x', x) - \sigma(x', F(t)) =$

$0\} \cap Gr_F$. Since the function $(t,x) \mapsto (x',x) - \sigma(x',F(t))$ is jointly measurable and $Gr_F \in \mathcal{S}(\mathcal{M}(v(m)) \times \mathcal{B}_X)$, we have that $Gr_H \in \mathcal{S}(\mathcal{M}(v(m)) \times \mathcal{B}_X)$. Applying Theorem 2.2.8 we obtain a $v(m)$-measurable function $h : T \to X$ such that $h(t) \in H(t)$ $v(m)$-almost everywhere on T. Consequently, $h \in S_F^1(m)$ and

$$\sup_{f \in S_F^1(m)} (x', f(t)) = \int (x', h(t))\, v(m, dt) = (x', h(t)).$$

Since $x' \in Q_{X'}^{\infty}(m)$ was arbitrary we conclude that every element of $(Q_X^1(m))' = L_{Q'}^{\infty}(m)$ attains its supremum on $S_F^1(m)$, and by James' theorem it follows that $S_F^1(m)$ is weakly compact in $Q_X^1(m)$. ■

Let K be a set of $v(m)$-measurable functions $f : T \to X$. Then we say that K is *decomposable* if $f, g \in K$ and $A \in \Sigma(v(m))$ imply that $f\chi_A + g\chi_{T\setminus A} \in K$. It then follows easily that if K is decomposable, then $\sum_{i=1}^{n} f_i \chi_{A_i} \in K$ for each finite measurable partition $\{A_1, \ldots, A_n\}$ of T and $f_i \in K$, $i = 1, 2, \ldots, n$.

Proposition 4.2.5 ([40], p158, Theorem 3.1) *Let K be a nonempty closed subset of $\mathcal{L}_X^p(v(m))$, with $1 \le p < \infty$. Then K is decomposable if and only if there exists a $v(m)$-measurable multifunction $F : T \to \mathcal{P}_f(X)$ such that $K = S_F^p(v(m))$.*

Definition 4.2.6 *If $M : \mathcal{R} \to \mathcal{P}(Y)$ is a multimeasure and $F : T \to \mathcal{P}_0(X)$ is a multifunction, then for every set $A \in \mathcal{M}(v(M))$ we define the **integral of F with respect to M**, denoted by $\int_A F(t)\, M(dt)$, by the equality*

$$\int_A F(t) M(dt) = \left\{ \int_A f(t)\, m(dt) \mid f \in S_F^1(m),\ \ m \in S_M \right\}.$$

We note that the integral of F with respect to M will always exist, even if F is not $v(M)$-measurable. Moreover, if $S_F^1(m) = \emptyset$ for all $m \in S_M$, then $\int_A F(t) M(dt) = \emptyset$. Also, if $v(M, A) = 0$ for $A \in \mathcal{M}(v(M))$ and $S_F^1(m) \neq \emptyset$ for $m \in S_M$, then $\int_A F(t) M(dt) = \{0\}$.

Example 4.2.7

Let $T = [0,1]$, Σ is the Lebesgue σ-algebra of subsets of T and λ is the Lebesgue measure on Σ. Define $F : T \to I\!R$ by $F(t) = [0,1]$ and $M : \Sigma \to I\!R$ by $M(A) = [0, \infty)$. If we define $m : \Sigma \to I\!R$ by $m(A) = \chi_A$, then $m \in S_M$ and $\int_A F(t)\, M(dt) = [0, \infty)$.

Theorem 4.2.8 *If X is a separable Banach space, $M : \mathcal{R} \to \mathcal{P}_f(Y)$ is a strong multimeasure of bounded variation $v(M)$ and if $F : T \to \mathcal{P}_f(X)$ is an integrably bounded $v(M)$-measurable multifunction, then $\int_A F(t) M(dt) \neq \emptyset$ for every $A \in \mathcal{M}(v(M))$.*

PROOF: From Theorem 2.5 of [39] we obtain a selector $m : \mathcal{R} \to Y$ of M. From the first part of the proof of Proposition 4.2.4 we deduce that $S_F^1(m) \neq \emptyset$ for every $m \in S_M$; therefore $\int_A F(t)M(dt) \neq \emptyset$ for every $A \in \mathcal{M}(v(M))$. ∎

The proof of the next theorem is identical to Theorem 10.5 on page 99 of [51] and so will be omitted.

Theorem 4.2.9 *Let X be a separable Banach space, $M : \mathcal{R} \to \mathcal{P}_f(Y)$ a strong multimeasure of bounded variation and let $F : T \to \mathcal{P}_0(X)$ be a p-integrably bounded multifunction such that $Gr_F \in \mathcal{S}(\mathcal{M}(v(M)) \times \mathcal{S}(\mathcal{B}_X))$.*

(a) If T is a countable union of sets of $\mathcal{R}$ and if the bounding function k belongs to $\mathcal{L}^1_{\mathbb{R}}(v(M))$, then $\int_A F(t)M(dt) \neq \emptyset$ for every $A \in \mathcal{M}(v(M))$.

(b) If $T \in \mathcal{R}$ and if the bounding function k belongs to $\mathcal{L}^p_{\mathbb{R}}(v(M))$, $1 \leq p < \infty$, then $\int_A F(t)M(dt) \neq \emptyset$ for every $A \in \mathcal{M}(v(M))$.

In our next two results we list some useful properties of the bilinear integral of a multifunction F with respect to a multimeasure M. The first theorem is the set-valued version of the results of [27] on page 109. The second result shows that if F and M are both positive, then the integral of F with respect to M will also be positive, and vice versa.

If X, Y and Z are Banach lattices, then we denote by X_+, Y_+ and Z_+ the positive cones of X, Y and Z respectively.

Theorem 4.2.10 *Suppose that X, Y and Z are Banach lattices, let $M : \Sigma(v(M)) \to \mathcal{P}(Y)$ be a multimeasure of bounded variation $v(M)$ and let $F : T \to \mathcal{P}_0(X)$ be an integrably bounded $v(M)$-measurable multifunction.*

(a) If $Y = \mathcal{L}(X, Z)$ and if $M(A) \subseteq Y_+$ for all $A \in \Sigma(v(M))$, then for all $A \in \Sigma(v(M))$ the mapping $F \mapsto \int_A F(t)\, M(dt)$ of T into Z is increasing.

(b) If $M(A) \subseteq Y_+$ for all $A \in \Sigma(v(M))$ and if $F(t) \subseteq X_+$ $v(M)$-almost everywhere on T, then
$$\int_E F(t)\, M(dt) \subseteq \int_F F(t)\, M(dt),$$
for all $E, F \in \Sigma(v(M))$ with $E \subseteq F$.

(c) If $N : \Sigma(v(N)) \to \mathcal{P}(Y)$ is a multimeasure of bounded variation $v(N)$ such that $M(A) \subseteq N(A)$ for all $A \in \Sigma(v(N))$ and if $F(t) \subseteq X_+$ $v(N)$-almost everywhere on T, then
$$\int_A F(t)\, M(dt) \subseteq \int_A F(t)\, N(dt).$$

(d) *For all $A \in \Sigma(v(M))$ we have that*

$$\| \int_A F(t)\, M(dt) \| \leq \int_A \|F(t)\| v(M, dt).$$

Theorem 4.2.11 *Suppose that X, Y and Z are Banach lattices, let $M : \Sigma(v(M)) \to \mathcal{P}_0(Y)$ be a multimeasure of bounded variation $v(M)$ and let $F : T \to \mathcal{P}_0(X)$ be an integrably bounded $v(M)$-measurable multifunction. If $Y = \mathcal{L}(X, Z)$, $M(A) \subseteq Y_+$ for all $A \in \Sigma(v(M))$ and if $F(t) \subseteq X_+$ $v(M)$-almost everywhere on T, then $\int_A F(t)\, M(dt) \subseteq Z_+$. Conversely, if $X = \mathcal{L}(Y, Z)$, $M(A) \subseteq Y_+$ for all $A \in \Sigma(v(M))$ and if $\int_A F(t)\, M(dt) \subseteq Z_+$, then $F(t) \subseteq X_+$ $v(M)$-almost everywhere on T.*

PROOF: Let $M(A) \subseteq Y_+$ for all $A \in \Sigma(v(M))$ and let $F(t) \subseteq X_+$ $v(M)$-almost everywhere on T. From

$$M(A) = \{m(A) \mid m \in S_M\},$$

follows that $(y', m(A)) \geq 0$ for every $y' \in Y'_+$ and $m \in S_M$. Consequently, for $y' \in Y'_+$, $m \in S_M$ and $f \in S^1_F(m)$ we have that

$$(y', \int_A f(t)\, m(dt)) = \int_A f(t)(y', m(dt)) \geq 0$$

so that $\int_A F(t)\, M(dt) \subseteq Z_+$.

Conversely, by Proposition 4.2.2(a) we obtain a sequence $(f_k) \subseteq S^1_F(v(M))$ such that

$$F(t) = \overline{\{f_k(t) \mid k \in \mathbb{N}\}}$$

$v(M)$-almost everywhere on T. Since $\int_A F(t)\, M(dt) \subseteq Z_+$, it then follows that $\int_A f_k(t)\, m(dt) \in Z_+$ for all $m \in S_M$ and $k \in \mathbb{N}$. Consequently, for all $z' \in Z'_+$ and all $A \in \Sigma(v(M))$,

$$0 \leq \left(z', \int_A f_k(t)\, m(dt)\right) = \int_A (z', f_k(t))\, m(dt).$$

Since $m(A) \in M(A) \subseteq Y_+$ it then follows that $0 \leq (z', f_k(t))$ and hence $f_k(t) \in X_+$ for each $k \in \mathbb{N}$. We then conclude that $F(t) \subseteq X_+$ $v(M)$-almost everywhere on T. ■

The next theorem shows that the bilinear integral of a multifunction with respect to a multimeasure is in fact a multimeasure.

Theorem 4.2.12 *Let $M : \mathcal{R} \to \mathcal{P}_f(Y)$ be a strong multimeasure of bounded variation $v(M)$ and let $F : T \to \mathcal{P}_f(X)$ be an integrably bounded $v(M)$-measurable multifunction. If for each $A \in \Sigma(v(M))$ we define $N(A) = \int_A F(t) M(dt)$, then $N : \Sigma(v(M)) \to \mathcal{P}(Z)$ is a strong multimeasure of bounded variation.*

PROOF: We first show that N is of bounded variation. So let $(A_k) \subseteq T$ be a sequence of mutually disjoint sets of $\Sigma(v(M))$. From

$$\|N(A_k)\| \leq \| \int_{A_k} F(t)\, M(dt) \| \leq \int_T \|F(t)\| \;\, v(M, dt),$$

follows immediately that N is indeed of bounded variation.

To show that N is a strong multimeasure, let (A_k) be a sequence of mutually disjoint sets in $\Sigma(v(M))$ and let $A = \cup_{k=1}^{\infty} A_k$. Then we need to prove that

$$N(A) = \sum_{k=1}^{\infty} N(A_k).$$

For this purpose, let $z_k \in N(A_k)$ for $k \in I\!N$. Then there exist sequences $(m_k) \subseteq S_M$ and $(f_k) \subseteq S_F^1(m_k)$ such that $z_k = \int_{A_k} f_k(t)\, m_k(dt)$ for $k \in I\!N$. Define $f : T \to X$ by

$$f(t) = \begin{cases} f_k(t) & \text{if } t \in A_k \\ f_1(t) & \text{if } t \in T \backslash A \end{cases}$$

and $m : \Sigma(v(m)) \to Y$ by

$$m = \chi_{A_1} m_1 + \chi_{A_2} m_2 + \ldots + \chi_{T \backslash \cup_{i=1}^{n-1} A_i} m_n,$$

where $\chi_A m_k(B) = m_k(A \cap B)$ for $k = 1, 2, \ldots, n$. By the decomposability of S_F and S_M we then have that $f \in S_F^1(m)$ and $m \in S_M$, respectively. Consequently, for $z' \in Z'$, we have that

$$\begin{aligned} \left(z', \sum_{k=1}^{n} z_k\right) &= \left(z', \sum_{k=1}^{n} \int_{A_k} f(t)\, m_k(dt)\right) \\ &= \left(z', \int_{\cup_{k=1}^{n} A_k} f(t)\, m(dt)\right) \to \left(z', \int_A f(t)\, m(dt)\right) \end{aligned}$$

as $n \to \infty$. This means that the series $\sum_{k=1}^{\infty} z_k$ converges weakly to $z = \int_A f(t)\, m(dt)$ and a similar property holds for every subseries of $\sum_{k=1}^{\infty} z_k$. By the Orlicz-Pettis theorem follows that the series $\sum_{k=1}^{\infty} z_k$ converges unconditionally to $z \in N(A)$. This means that the series $\sum_{k=1}^{\infty} N(A_k)$ is unconditionally convergent and is contained in $N(A)$.

To prove the inverse inclusion, let $z \in N(A)$ with $A \in \Sigma(v(M))$. Then $z = \int_A f(t)\, m(dt)$ for some $m \in S_M$ and $f \in S_F^1(m)$. Then, as before, the series $\sum_{k=1}^{\infty} \int_{A_k} f(t)\, m(dt)$ converges to z. This shows that $z \in \Sigma_{k=1}^{\infty} N(A_k)$, which concludes the proof. ■

We have seen from the previous theorem that if $N(A) = \int_A F(t)\, M(dt)$, where M is a closed-valued multimeasure of bounded variation $v(M)$ and F is an integrably bounded $v(M)$-measurable multifunction with closed values, then N is a multimeasure. We now investigate the relationship between S_M, the selectors of M, and S_N, the selectors of N.

Proposition 4.2.13 *Let $M : \Sigma(v(M)) \to \mathcal{P}_k(Y)$ be a strong multimeasure of bounded variation $v(M)$, let $F : T \to \mathcal{P}_k(X)$ be an integrably bounded $v(M)$-measurable multifunction and for each $A \in \Sigma(v(M))$ let $N(A) = \int_A F(t)M(dt)$.*

(a) If $m \in S_M$ and $f \in S_F^1(m)$, then the measure defined by $n(A) = \int_A f(t)\, m(dt)$ is a selector of N.

(b) If $n \in S_N$, then there exist an $m \in S_M$ and an $f \in S_F^1(m)$ such that $n(A) = \int_A f(t)\, m(dt)$, $A \in \Sigma(v(M))$.

PROOF: (a) Let $m \in S_M$ (which exists by Theorem 2.5 of [39]) and let $f \in S_F^1(m)$ (whose existence is guaranteed by Theorem 4.2.4). Then the measure $n : \Sigma(v(m)) \to Y$ defined by $n(A) = \int_A f(t)\, m(dt)$ is clearly a selector of N.

(b) Since N is a compact-valued strong multimeasure of bounded variation (by Theorem 4.2.12), it follows from Theorem 2.5 of [39] that $S_N \neq \emptyset$. Let $n \in S_N$. From Theorem 1 of [34] follows that $N(A) = \{n(A) \mid n \in S_N\}$. But Theorem 4.2.4 implies that $\int_A F(t)\, M(dt) \neq \emptyset$, that is, there exist an $m \in S_M$ and an $f \in S_F^1(m)$ such that $\int_A f(t)\, m(dt) \in \int_A F(t)\, M(dt) = N(A)$. Consequently, if $n \in S_N$, then $n(A) = \int_A f(t)\, m(dt)$, $A \in \Sigma(v(M))$. ∎

Let $ca(Y)$ denote the space of all Y-valued measures on $\Sigma(v(m))$. We now discuss the topology of pointwise weak convergence on $ca(Y)$. If we consider $\mathcal{E}_{\mathbb{R}}(v(m)) \otimes Y'$, then $\mathcal{E}_{\mathbb{R}}(v(m)) \otimes Y'$ and $ca(Y)$ can be put into duality as follows:

$$(m, y) = \left(m, \sum_{k=1}^{n} \chi_{A_k} \otimes y_k'\right) = \sum_{k=1}^{n} (y_k', m(A_k)),$$

where $\{A_1, \ldots, A_n\}$ is a finite $v(m)$-measurable partition of T and $y_k' \in Y'$, $k = 1, 2, \ldots, n$. Then it follows that the topology of pointwise weak convergence on $ca(Y)$ is in fact the $w(ca(Y), \mathcal{E}_{\mathbb{R}}(v(m)) \otimes Y')$-topology.

Theorem 4.2.14 *Suppose that X is a separable Banach space and Z is finite-dimensional. Let $M : \Sigma(v(M)) \to \mathcal{P}_{wk}(Y)$ be a strong multimeasure of bounded variation $v(M)$ and suppose that $F : T \to \mathcal{P}_{wk}(X)$ is an integrably bounded $v(M)$-measurable multifunction. If for each $A \in \Sigma(v(M))$ we define $N(A) = \int_A F(t)M(dt)$, then $N : \Sigma(v(M)) \to \mathcal{P}_{wk}(Z)$ is a strong multimeasure of bounded variation.*

PROOF: The fact that N is of bounded variation follows just like before. To show that N is closed-valued, let $(\int_A f_k(t)\, m_k(dt)) \subseteq N(A)$ for $A \in \Sigma(v(M))$, where $(m_k) \subseteq S_M$ and $(f_k) \subseteq S_F^1(m_k)$. Since S_M is $w(ca(Y), \mathcal{E}_{\mathbb{R}}(v(m)) \otimes Y')$-compact and since $S_F^1(m_k)$ is weakly compact in $\mathcal{L}_X^1(v(m))$, there exist sequences $(m_{k_j}) \subseteq (m_k)$ and $(f_{k_j}) \subseteq (f_k)$ such that $m_{k_j} \to^w m \in S_M$ and $f_{k_j} \to^w f \in S_F^1(m)$. Then, for each $p \in Z$, we have

$$\| \left(p, \int_A f_{k_j}(t)\, m_{k_j}(dt)\right) - \left(p, \int_A f(t)\, m(dt)\right) \|$$

$$\leq \left\| \left(p, \int_A f_{k_j}(t)\, m(dt)\right) - \left(p, \int_A f_{k_j}(t)\, m_{k_j}(dt)\right) \right\| +$$

$$\left\| \left(p, \int_A f_{k_j}(t)\, m(dt)\right) - \left(p, \int_A f(t)\, m(dt)\right) \right\|$$

so that $(p, \int_A f_{k_j}(t) m_{k_j}(dt)) \to (p, \int_A f(t) m(dt))$ as $j \to \infty$.

We will now make use of Theorem 2.3.21 in order to show that N is a strong multimeasure. Let $A, B \in \Sigma(v(M))$ with $A \cap B = \emptyset$. To prove that $N(A \cup B) = N(A) + N(B)$, we only need to show that $N(A)+N(B) \subseteq N(A \cup B)$ because the inverse inclusion follows from the definition of N. So, if $z \in N(A) + N(B)$, then $z = \int_A f_1 m_1(dt) + \int_B f_2 m_2(dt)$, where $f_i \in S^1_F(m_i)$ and $m_i \in S_M$, for $i = 1, 2$. Put $f = \chi_A f_1 + \chi_B f_2$ and $m = \chi_A m_1 + \chi_B m_2$. Then $f \in S^1_F(m)$ and $m \in S_M$ because both S_F and S_M are decomposable, and therefore $z = \int_{A \cup B} f\, dm \in N(A \cup B)$.

Finally, let (A_k) be an increasing sequence in $\Sigma(v(M))$ and put $A = \cup_{k=1}^{\infty} A_k$. Then

$$\begin{aligned} H(N(A), N(A_k)) &= H(N(A_k) + N(A \backslash A_k), N(A_k)) \\ &\leq \|N(A \backslash A_k)\| \\ &\leq \int_{A \backslash A_k} \| F(t) \|\, v(M, dt) \longrightarrow 0 \end{aligned}$$

as $k \to \infty$. This shows that N is indeed a strong multimeasure. ■

We now investigate the convexity of $\int_A F(t)\, M(dt)$. In particular, we will see that if Z is finite dimensional, then $\int_A F(t)\, M(dt)$ is convex. The convexity fails in the infinite dimensional case; in fact, as it turns out, the closure of the integral will be convex (see Example 4.2.16 and Theorem 4.2.17 below). For results on the convexity of the integral of a multifunction with respect to a vector measure, see [13], [40] and [7]. Central to our proofs is the Lyapunov convexity theorem.

Theorem 4.2.15 *Suppose that X is a separable Banach space and Z is finite-dimensional. If $M : \Sigma(v(M)) \to \mathcal{P}_f(Y)$ is a non-atomic strong multimeasure of bounded variation $v(M)$ and $F : T \to \mathcal{P}_{wf}(X)$ is an integrably bounded $v(M)$-measurable multifunction, then $\int_A F(t)\, M(dt)$ is a convex set for each $A \in \Sigma(v(M))$.*

PROOF: If for $A \in \Sigma(v(M))$ we put $N(A) = \int_A F(t)\, M(dt)$, then from Theorem 4.2 of [4] follows that we only need to show that N is a bounded non-atomic strong multimeasure. The fact that N is a strong multimeasure of bounded variation follows from Theorem 4.2.12. Therefore it only remains to show that N is non-atomic. For this purpose, if $m \in S_M$, let $f \in S^1_F(m)$ and define the set functions $n : \Sigma(v(m)) \to Z$ and $\nu : \Sigma(v(M)) \to I\!R_+$ by

$$n(A) = \int_A f(t)\, m(dt) \text{ and } \nu(A) = \int_A \|f(t)\|\, v(M, dt)$$

for each $A \in \Sigma(v(M))$, respectively. By Proposition 4.2.13(a) we have that $n \in S_N$, and ν is $v(M)$-continuous. We now proceed by showing that n is ν-continuous, because then N will be $v(M)$-continuous, and hence non-atomic (by Proposition 2.4.5). Indeed, let $A \in \Sigma(v(M))$ and let $\{A_j : j \in J\}$ be an arbitrary finite partition of A into mutually disjoint sets $A_j \in \Sigma(v(M))$. Then

$$\begin{aligned}
\sum_{j\in J} \|n(A_j)\| &= \sum_{j\in J} \| \int_{A_j} f(t)\, m(dt))\| \\
&\leq \sum_{j\in J} \int_{A_j} \|f(t)\|\, v(M, dt) \\
&= \sum_{j\in J} \nu(A_j) \\
&= \nu(A).
\end{aligned}$$

Then, since $v(n, A) = \sup_J \sum_{j\in J} \|n(A_j)\|$, it follows that $v(n) \leq \nu$ and consequently n is ν-continuous; therefore n is $v(M)$-continuous. But from $N(A) = \{n(A) \mid n \in S_N\}$ follows immediately that N is $v(M)$-continuous. ■

Example 4.2.16

(i) The following example shows that Theorem 4.2.15 fails if Z is infinite-dimensional: Let $T = [0, 1]$, Σ is the Lebesgue σ-algebra of subsets of T and λ is the Lebesgue measure on Σ. Put $Z = L^1_{I\!R}(T, \Sigma, \lambda)$ and define $F : T \to I\!R$ by $F(t) = \{0, 1\}$ and $M : \Sigma \to L^1_{I\!R}(T, \Sigma, \lambda)$ by $M(A) = \{\chi_A\}$. Then

$$\int_A F(t)\, M(dt) = \left\{ \int_A F(t)\, m(dt) \mid m \in S_M \right\} = \{R(m) \mid m \in S_M\} = \{0, 1\},$$

which is not convex.

(ii) The following example shows that Theorem 4.2.15 fails if the multimeasure M is atomic: Let $T = \{t_0\}$ and $\Sigma = \{\emptyset, T\}$. Define $F : T \to I\!R$ by $F(t) = \{0, 1\}$ and $M : \Sigma \to I\!R$ by $M(\emptyset) = \{0\}$ and $M(T) = \{1\}$. Then M is atomic and $\int_A F(t)\, M(dt) = \{0, 1\}$, which is not convex.

Theorem 4.2.17 *Suppose that X and Z are separable Banach spaces. If $M : \Sigma(v(M)) \to \mathcal{P}_f(Y)$ is a non-atomic strong multimeasure of bounded variation $v(M)$ and $F : T \to \mathcal{P}_f(X)$ is an integrably bounded $v(M)$-measurable multifunction, then $\overline{\int_A F(t)\, M(dt)}$ is a convex subset of Z for each $A \in \Sigma(v(M))$.*

Proof: Let $z_1, z_2 \in \overline{\int_A F(t)\, M(dt)}$ for $A \in \Sigma(v(M))$, let $\epsilon > 0$ and let $\alpha \in [0, 1]$. Then we need to establish the existence of an $m \in S_M$ and an $f \in S^1_F(m)$ such that

$$\|\alpha z_1 + (1 - \alpha) z_2 - \int_A f(t)\, m(dt)\| < \epsilon.$$

Since $z_1, z_2 \in \overline{\int_A F(t)\, M(dt)}$, there exist $m_i \in S_M$ and $f_i \in S_F^1(m_i)$, $i = 1, 2$, such that

$$\|z_1 - \int_A f_1(t)\, m_1(dt)\| < \frac{\epsilon}{2} \text{ and } \|z_2 - \int_A f_2(t)\, m_2(dt)\| < \frac{\epsilon}{2}.$$

Define the set functions $n_i : \Sigma(v(M)) \to Z$ $(i = 1, 2)$ by

$$n_i(A) = \int_A f_i(t)\, m_i(dt), i = 1, 2.$$

In the same way as in the proof of the previous theorem we can prove that each n_i is a non-atomic measure. Consider now the Banach space $Z \times Z$ with norm defined by $\|(x_1, x_2)\| = \sqrt{\|x_1\|^2 + \|x_2\|^2}$ and define the set function $n : \Sigma(v(M)) \to Z \times Z$ by

$$n(A) = (n_1(A), n_2(A)) = \left(\int_A f_1(t)\, m_1(dt), \int_A f_2(t)\, m_2(dt)\right).$$

Then n is a non-atomic measure with finite variation. Indeed, suppose, on the contrary that E is a atom for n. Then, for all $E' \subset E$, $E' \in \Sigma(A, v(m))$, we have that either $n(E') = 0$ or $n(E \backslash E') = 0$. This in turn implies that either $n_1(E') = 0 = n_2(E')$ or $n_1(E \backslash E') = 0 = n_2(E \backslash E')$, contradicting the fact that both n_1 and n_2 is nonatomic. From the Lyapunov convexity theorem, it follows that the closure of the range $R(n)$ of n is convex in $Z \times Z$. Consequently, if $A \in \Sigma(v(M))$, then

$$\alpha n(A) + (1 - \alpha) n(\emptyset) = \alpha n(A) \in \overline{R(n)}.$$

This means that there exists a set $A_\alpha \subseteq A$ such that

$$\|\alpha n(A_\alpha) - n(A_\alpha)\| < \frac{\epsilon}{4} \text{ and } \|(1 - \alpha) n(A) - n(A \backslash A_\alpha)\| < \frac{\epsilon}{4},$$

that is

$$\|\alpha \int_A f_i(t)\, m_i(dt) - \int_{A_\alpha} f_i(t)\, m_i(dt)\| < \frac{\epsilon}{4}$$

and

$$\|(1 - \alpha) \int_A f_i(t)\, m_i(dt) - \int_{A \backslash A_\alpha} f_i(t)\, m_i(dt)\| < \frac{\epsilon}{4}$$

for $i = 1, 2$. If we put

$$f = f_1 \chi_{A_\alpha} + f_2 \chi_{T \backslash A_\alpha} \text{ and } m = \chi_{A_\alpha} m_1 + \chi_{T \backslash A_\alpha} m_2,$$

then $m \in S_M$, $f \in S_F^1(m)$ and $m(A) = m_1(A_\alpha) + m_2(A \backslash A_\alpha)$ for all $A \in \Sigma(v(M))$. The result then follows from the fact that

$$\begin{aligned}
&\|\alpha z_1 + (1 - \alpha) z_2 - \int_A f(t)\, m(dt)\| \\
\le\ & \|\alpha z_1 - \alpha \int_A f_1(t)\, m_1(dt)\| + \|\alpha \int_A f_1(t)\, m_1(dt) - \int_{A_\alpha} f_1(t)\, m_1(dt)\| + \\
& \|(1 - \alpha) z_2 - (1 - \alpha) \int_A f_2(t)\, m_2(dt)\| + \|(1 - \alpha) \int_A f_2(t)\, m_2(dt) - \int_{A \backslash A_\alpha} f_2(t)\, m_2(dt)\| \\
<\ & \alpha \frac{\epsilon}{2} + \frac{\epsilon}{4} + (1 - \alpha) \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon
\end{aligned}$$

■

Theorem 4.2.18 *Let T be a countable union of sets of the ring $\mathcal{R}$ and suppose that $Y = Z$ is a separable reflexive Banach space. If $F : T \to \mathcal{P}_{kc}(I\!R^n)$ is an integrably bounded $v(M)$-measurable multifunction and $M : \Sigma(v(M)) \to \mathcal{P}_{wkc}(Y)$ is a multimeasure of bounded variation $v(M)$, then for each $A \in \Sigma(v(M))$ the set $\int_A F(t)\, M(dt)$ is a convex and $w(Y, Y')$-compact subset of Y.*

PROOF: Obviously, for each $A \in \Sigma(v(M))$, $\int_A F(t)\, M(dt)$ is convex. Furthermore, note that $\int_A F(t)\, M(dt)$ is of bounded variation and thus bounded. To show that $\int_A F(t)\, M(dt)$ is a weakly compact subset of Y, let $(\int_A f_k(t)\, m_k(dt)) \subseteq \int_A F(t)\, M(dt)$ for all $A \in \Sigma(v(M))$, $(m_k) \subseteq S_M$ and $(f_k) \subseteq S^1_F(m_k)$. Since S_M is compact for the topology of simple weak pointwise convergence, there exists a sequence $(m_{k_j}) \subseteq (m_k)$ such that $m_{k_j} \to^w m \in S_M$. Also, since for each $m \in S_M$, the set $S^1_F(m)$ is compact in $Q^1_{I\!R^n}(v(m))$, there exists a sequence $(f_{k_j}) \subseteq (f_k)$ such that $f_{k_j} \to^w f \in S^1_F(m)$. Then, if $y \in Y$, the result then follows from

$$\begin{aligned} &\| \left(y, \int_A f_{k_j}(t)\, m_{k_j}(dt)\right) - \left(y, \int_A f(t)\, m(dt)\right) \| \\ \leq\ &\| \left(y, \int_A f_{k_j}(t)\, m(dt)\right) - \left(y, \int_A f_{k_j}(t)\, m_{k_j}(dt)\right) \| + \\ &\| \left(y, \int_A f_{k_j}(t)\, m(dt)\right) - \left(y, \int_A f(t)\, m(dt)\right) \|. \end{aligned}$$

■

In [4] Artstein discussed Radon-Nikodým derivatives of multimeasures whose values are convex sets in $I\!R^n$ while Castaing [12] and Godet-Thobie [35] gave Radon-Nikodým theorems for multimeasures with compact and convex values in a locally convex topological space. Note that Theorem 9.1 on page 120 in [4] has been shown in [22, pp. 305, 308] to be false. Costé [18] and Hiai [39] discussed Radon-Nikodým theorems for multimeasures whose values are closed, bounded and convex sets in a separable Banach space. Papageorgiou [57] proved two set-valued Radon-Nikodým theorems for transition multimeasures, and the results were recently ([58]) extended to the case where the dominating control measure is a transition measure. We now continue by establishing Radon-Nikodým-type theorems for our bilinear set-valued integral. In our first result the range spaces of the multimeasure and multifunction are finite-dimensional while in the results thereafter we take the range spaces to be arbitrary Banach spaces.

Theorem 4.2.19 (Radon-Nikodým) *Let T be a countable union of sets of the ring $\mathcal{R}$, μ is a scalar measure on $\mathcal{R}$ and let $M : \Sigma(v(M)) \to \mathcal{P}_{kc}(I\!R^n)$ be a multimeasure of bounded variation $v(M)$. If M is μ-continuous on $\mathcal{R}$, then there exists an integrably bounded $v(M)$-measurable multifunction $F : T \to \mathcal{P}_{kc}(I\!R^n)$ such that*

$$M(A) = \int_A F(t)\, \mu(dt)$$

for each $A \in \Sigma(v(M))$.

PROOF: From Theorem 2.4.16 follows that

$$M(A) = \{m(A) \mid m \in S_M\}$$

for all $A \in \mathcal{R}$. This means that every $m \in S_M$ is μ-continuous on $\mathcal{R}$. Since μ has the direct sum property, from Theorem 4.1.18 follows that for each $m \in S_M$ there exists a locally μ-integrable function $f_m : T \to I\!R^n$ such that

$$m(A) = \int_A f_m(t)\,\mu(dt), \quad A \in \mathcal{R}.$$

Put

$$K = \{f_m \chi_A \in \mathcal{L}^1_{I\!R^n}(\mu) \mid A \in \mathcal{R}, m \in S_M\}.$$

We first show that K is a closed subset of $\mathcal{L}^1_{I\!R^n}(\mu)$. So let $f_{m_k}\chi_A \in K$ and let f be such that $f_{m_k}\chi_A \to f$ in $\mathcal{L}^1_{I\!R^n}(\mu)$. Since S_M is compact for the topology of pointwise convergence, there exists an $m' \in S_M$ such that $m_k(A) \to m'(A)$ for all $A \in \mathcal{R}$. But

$$\begin{aligned}
\lim_{k\to\infty} \int (f_{m_k}\chi_A)(t)\,\mu(dt) &= \lim_{k\to\infty} \int_A f_{m_k}(t)\,\mu(dt) \\
&= \int_A f_{m'}(t)\,\mu(dt) \\
&= \int (f_{m'}\chi_A)(t)\,\mu(dt)
\end{aligned}$$

so that $f = f_{m'}\chi_A \in K$.

Evidently, K is convex, and by Proposition 4.2.5 there exists an integrably bounded μ-measurable multifunction $F : T \to \mathcal{P}_{kc}(I\!R^n)$ such that $K = S^1_F(\mu)$. Then, for each $A \in \mathcal{R}$,

$$\begin{aligned}
M(A) = \{m(A) \mid m \in S_M\} &= \left\{\int_A f_m(t)\,\mu(dt) \mid m \in S_M\right\} \\
&= \left\{\int (f_m\chi_A)(t)\,\mu(dt) \mid f_m\chi_A \in K\right\} \\
&= \left\{\int_A f_m(t)\,\mu(dt) \mid f_m \in S^1_F(\mu)\right\} \\
&= \int_A F(t)\,\mu(dt).
\end{aligned}$$

■

Corollary 4.2.20 *Under the hypotheses of Theorem 4.2.19 follows that there exists a unique integrably bounded $v(M)$-measurable multifunction $F : T \to \mathcal{P}_{kc}(I\!R^n)$ such that*

$$M(A) = \int_A F(t)\,\mu(dt)$$

for each $A \in \Sigma(v(M))$.

PROOF: Let $G : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ be a multifunction such that $M(A) = \int_A G(t)\,\mu(dt)$ for each $A \in \Sigma(v(M))$. Define $\phi : S_M \to S_F^1(\mu)$ by

$$\phi(m) = \frac{m(dt)}{\mu(dt)}.$$

Then it follows that ϕ is a linear isometric bijection and

$$S_F^1(\mu) = \phi(S_M) = S_G^1(\mu).$$

From Proposition 4.2.2(b) follows then that $F(t) = G(t)$ $v(M)$-almost everywhere on T. ■

Corollary 4.2.21 *Let T be a countable union of sets of the ring $\mathcal{R}$ and let μ be a scalar measure on $\mathcal{R}$. If $M : \Sigma(v(M)) \to \mathcal{P}_{kc}(\mathbb{R}^n)$ is a μ-continuous multimeasure of bounded variation $v(M)$, then*

$$S_M = \left\{ \int_{(\cdot)} f(t)\,\mu(dt) \mid f \in S_F^1(\mu), M(dt) = F(t)\mu(dt) \right\}.$$

PROOF: Let $m \in S_M$. Then for all $A \in \Sigma(v(M))$ we have that $m(A) \in M(A)$ so that m is also μ-continuous. From Theorem 4.1.18 we obtain an $f \in \mathcal{L}^1_{\mathbb{R}}(\mu)$ such that $m(A) = \int_A f(t)\,\mu(dt)$ for every $A \in \Sigma(\mu)$. Then $\int_A f(t)\,\mu(dt) \in \int_A F(t)\,\mu(dt)$ for all $A \in \Sigma(\mu)$. This shows that $f \in S_F^1(\mu)$ and consequently

$$S_M \subseteq \left\{ \int_{(\cdot)} f(t)\,\mu(dt) \mid f \in S_F^1(\mu), M(dt) = F(t)\mu(dt) \right\}.$$

For the inverse inclusion, let $f \in S_F^1(\mu)$ and consider $m(A) = \int_A f(t)\,\mu(dt)$, $A \in \Sigma(\mu)$. Then Proposition 4.2.12(a) implies that $m \in S_M$, and

$$\left\{ \int_{(\cdot)} f(t)\,\mu(dt) \mid f \in S_F^1(\mu), M(dt) = F(t)\mu(dt) \right\} \subseteq S_M.$$

■

Corollary 4.2.22 *Let T be a countable union of sets of the ring $\mathcal{R}$, μ is a scalar measure on $\mathcal{R}$ and for $i = 1, 2$ let $M_i : \Sigma(v(M_i)) \to \mathcal{P}_{kc}(\mathbb{R}^n)$ be a μ-continuous multimeasure of bounded variation $v(M_i)$. If $M_1(A) \subseteq M_2(A)$ for every $A \in \Sigma(v(M_2))$, then for $i = 1, 2$ there exists an integrably bounded $v(M_i)$-measurable multifunction $F_i : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ such that $F_1(t) \subseteq F_2(t)$.*

PROOF: From Theorem 4.2.19 we obtain an integrably bounded $v(M_i)$-measurable multifunction $F_i : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ such that

$$M(A_i) = \int_A F_i(t)\,\mu(dt), i = 1, 2.$$

Since $M_1(A) \subseteq M_2(A)$ for every $A \in \Sigma(v(M_2))$, it follows that $\sigma(p, M_1(A)) \leq \sigma(p, M_2(A))$ for $p \in \mathbb{R}^n$. Consequently, for $p \in \mathbb{R}^n$,

$$\int_A \sigma(p, F_1(t))\,\mu(dt) = \sigma(p, \int_A F_1(t)\,\mu(dt)) \leq \sigma(p, \int_A F_2(t)\,\mu(dt)) = \int_A \sigma(p, F_2(t))\,\mu(dt)$$

and we deduce that $\sigma(p, F_1(t)) \leq \sigma(p, F_2(t))$. Since both F_1 and F_2 are convex and compact-valued, it then follows that $F_1(t) \subseteq F_2(t)$. ■

Corollary 4.2.23 *Let T be a countable union of sets of the ring $\mathcal{R}$ and μ is a non-atomic scalar measure on $\mathcal{R}$. If $M : \Sigma(v(M)) \to \mathcal{P}_{kc}(\mathbb{R}^n)$ is a μ-continuous multimeasure of bounded variation $v(M)$, then there exists a strong multimeasure $N : \Sigma(v(M)) \to \mathcal{P}_{kc}(\mathbb{R}^n)$ of bounded variation such that $co\, M(A) = N(A)$ for each $A \in \Sigma(v(M))$.*

PROOF: From Theorem 4.2.19 follows that there exists an integrably bounded $v(M)$-measurable multifunction $F : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ such that $M(A) = \int_A F(t)\,\mu(dt)$ for each $A \in \Sigma(v(M))$. But then

$$co\, M(A) = co \int_A F(t)\,\mu(dt) = \int_A F(t)\,\mu(dt) = \int_A co\, F(t)\,\mu(dt),$$

where the last equality follows from the fact that $\int_A F(t)\,\mu(dt)$ is a convex set, and

$$\begin{aligned} \sigma(p, \int_A F(t)\mu(dt)) &= \int_A \sigma(p, F(t))\,\mu(dt) \\ &= \int_A \sigma(p, co\, F(t))\,\mu(dt) \\ &= \sigma(p, \int_A co\, F(t)\mu(dt)) \end{aligned}$$

for every $p \in \mathbb{R}^n$. If we put $N(A) = \int_A co\, F(t)\,\mu(dt)$, then N is the desired multimeasure. Indeed, by Proposition 2.2.10 follows that $co\, F$ is μ-measurable. According to Theorem 4.2.12 we then only need to show that $co\, F$ is integrably bounded. To start with, first note that from the integrably boundedness of F we obtain a $k \in \mathcal{L}^1_{\mathbb{R}}(\mu)$ such that $\|F(t)\| \leq k(t)$ for every $t \in T \backslash N$, where N is some μ-negligible subset of T. Let $x(t) \in co\, F(t)$ for $t \in T$. Then $x(t) = \sum_{j=1}^{n+1} \alpha_j(t) x_j(t)$, where $x_j(t) \geq 0$, $\sum_{j=1}^{n+1} \alpha_j(t) = 1$ and $x_j(t) \in F(t)$ for $j = 1, 2, \ldots, n+1$. If $t \in T \backslash N$, then it follows easily that $\|x(t)\| \leq k(t)$ so that $co\, F$ is indeed integrably bounded. ■

Theorem 4.2.24 *Let T be a countable union of sets of the ring $\mathcal{R}$ and let X and Z be separable Banach spaces such that $Y \subseteq \mathcal{L}(X, Z)$ and $Z = W'$, where W is a norming subspace of Z'. If $M : \Sigma(v(M)) \to \mathcal{P}_k(Y)$ is a strong multimeasure of bounded variation $v(M)$ and $F : T \to \mathcal{P}_f(X)$ is an integrably bounded $v(M)$-measurable multifunction, then there exists an integrably bounded $v(M)$-measurable multifunction $G : T \to \mathcal{P}_k(Z)$ such that*

$$\int_A F(t)\, M(dt) = \int_A G(t)\, v(M, dt)$$

for each $A \in \Sigma(v(M))$.

PROOF: From Theorem 4.2.8 we have that $\int_A F(t)\, M(dt) \neq \emptyset$ for $A \in \Sigma(v(M))$. Then

$$\begin{aligned}\int_A F(t)\, M(dt) &= \left\{ \int_A f(t)\, m(dt) \mid f \in S_F^1(m), m \in S_M \right\} \\ &= \left\{ \int_A U_m f(t)\, v(m, dt) \mid f \in S_F^1(m), m \in S_M \right\},\end{aligned}$$

where $U_m : T \to \mathcal{L}(Y, Z)$ is the function whose existence is guaranteed by Theorem 4.1.19. If, for each $t \in T$, we define

$$G(t) = \{(U_m f)(t) \mid f \in S_F^1(m), m \in S_M\},$$

then G is the desired multifunction. Indeed, first note that $G(t) \in \mathcal{P}_k(Z)$ because both S_M and S_F^1 are compact for the topology of pointwise convergence. To prove the integrably boundedness of G, let $z \in G(t)$ for all $t \in T$. Then there exist $m' \in S_M$ and $f' \in S_F^1(m')$ such that $z = U_{m'}(t) f'(t)$ for all $t \in T$. Therefore

$$\|z\| = \|U_{m'}(t) f'(t)\| \leq \|U_{m'}(t)\| \, \|f'(t)\| = \|f'(t)\|,$$

which implies that G is indeed integrably bounded.

Furthermore, since the mapping $t \mapsto (U_m f)(t)$ is $v(m)$-measurable for each $m \in S_M$ and $f \in S_F^1(m)$, it follows immediately that G is also $v(M)$-measurable. Obviously, for each $A \in \Sigma(v(M))$ we have that

$$\int_A F(t)\, M(dt) = \int_A G(t)\, v(M, dt).$$

■

By making use of Theorem 4.1.21 and Remark 4.1.22 we now have the following corollary, the proof of which is similar to the previous theorem.

Corollary 4.2.25 *Let T be a countable union of sets of the ring $\mathcal{R}$ and let X and Z be separable Banach spaces such that $Y \subseteq \mathcal{L}(X, Z)$ and $Z = W'$, where W is a norming subspace of Z'. Suppose that $M : \Sigma(v(M)) \to \mathcal{P}_k(Y)$ is a strong multimeasure of bounded variation $v(M)$ and let $F : T \to \mathcal{P}_f(X)$ be an integrably bounded $v(M)$-measurable multifunction. If μ is a scalar measure on $\mathcal{R}$ with the direct sum property such that M is μ-continuous, then there exists an integrably bounded $v(M)$-measurable multifunction $G : T \to \mathcal{P}_k(Z)$ such that*

$$\int_A F(t)\, M(dt) = \int_A G(t)\, \mu(dt)$$

for each $A \in \Sigma(v(M))$.

Theorem 4.2.26 *Let T be a countable union of sets of the ring $\mathcal{R}$ and let μ be a scalar measure on $\mathcal{R}$. If $M : \Sigma(v(M)) \to \mathcal{P}_{kc}(I\!R^n)$ is a multimeasure of bounded variation $v(M)$ such that M is μ-continuous and if $F : T \to \mathcal{P}_f(X)$ is an integrably bounded $v(M)$- measurable multifunction, then there exists an integrably bounded $v(M)$-measurable multifunction $G : T \to \mathcal{P}_{kc}(I\!R^n)$ such that*

$$\int_A F(t)\, M(dt) = \int_A F(t)G(t)\, \mu(dt)$$

for all $A \in \Sigma(v(M))$.

PROOF: By Theorem 4.2.19 we obtain an integrably bounded $v(M)$-measurable multifunction $G : T \to \mathcal{P}_{kc}(I\!R^n)$ such that $M(A) = \int_A G(t)\, \mu(dt)$ for all $A \in \Sigma(v(M))$. Therefore, for $m \in S_M$, there exists a $g \in S^1_G(\mu)$ such that

$$m(A) = \int_A g(t)\, \mu(dt).$$

Then, since $m(dt) = g(t)\mu(dt)$, we have that

$$\int_A f(t)\, m(dt) = \int_A f(t)g(t)\, \mu(dt) \in \int_A F(t)G(t)\, \mu(dt),$$

for every $f \in S^1_F(m)$ and therefore $\int_A F(t)\, M(dt) \subseteq \int_A F(t)G(t)\, \mu(dt)$ for all $A \in \Sigma(v(M))$. The inverse inclusion follows similarly. ∎

Theorem 4.2.27 *Let T be a countable union of sets of the ring $\mathcal{R}$ and let X and Z be separable Banach spaces such that $Y \subseteq \mathcal{L}(X, Z)$ and $Z = W'$, where W is a norming subspace of Z'. Suppose that $M : \Sigma(v(M)) \to \mathcal{P}_{wkc}(Y)$ is a strong multimeasure of bounded variation $v(M)$ and let $F : T \to \mathcal{P}_{wkc}(X)$ be an integrably bounded $v(M)$-measurable multifunction. If μ is a scalar measure on $\mathcal{R}$ with the direct sum property such that M is μ-continuous, then $\int_A F(t)\, M(dt)$ is a convex and $w(Z, Z')$-compact subset of Z for every $A \in \Sigma(v(M))$.*

PROOF: From Theorem 4.2.24 we obtain an integrably bounded $v(M)$-measurable multifunction $G : T \to \mathcal{P}_{wkc}(Z)$ such that $\int_A F(t)\, M(dt) = \int_A G(t)\, \mu(dt)$ for each $A \in \Sigma(v(M))$. If we put $N(A) = \int_A G(t)\, \mu(dt)$, then we know that $N : \Sigma(v(M)) \to \mathcal{P}_{wkc}(Z)$ is a multimeasure of bounded variation and $S_N \neq \emptyset$. Let $n \in S_N$ and define $\psi : L^1_Z(\mu) \to Z$ and $\phi : S_N \to Z$ by

$$\psi([g]) = \int g(t)\, \mu(dt), g \in [g] \text{ and } \phi(n) = \frac{n(dt)}{\mu(dt)}.$$

Then ψ is continuous with respect to the norm topologies of the spaces $L^1_Z(\mu)$ and Z. Also, since ϕ is a linear isometric bijection, it is continuous with respect to the norm topologies of the spaces $ca(Z)$ and $L^1_Z(\mu)$. Theorem 15 on page 422 of [29] asserts that ψ is continuous with respect to the topologies $w(L^1_Z(\mu), L^\infty_{Z'}(\mu))$ and $w(Z, Z')$ of the spaces

$L^1_Z(\mu)$ and Z, while ϕ is continuous with respect to the topologies $w(ca(Z), \mathcal{E}_{I\!R}(\mu) \otimes Z')$ and $w(L^1_Z(\mu), L^\infty_{Z'}(\mu))$ of the spaces $ca(Z)$ and $L^1_Z(\mu)$. We then have that

$$(\psi \circ \phi)(S_N) = \psi(\phi(S_N)) = \psi(S^1_G(\mu)) = \int_A G(t)\,\mu(dt) = \int_A F(t)\,M(dt).$$

Since $\psi \circ \phi$ is continuous with respect the topologies $w(ca(Z), \mathcal{E}_{I\!R}(\mu) \otimes Z')$ and $w(Z, Z')$ of $ca(Z)$ and Z, and since S_N is a convex and $w(ca(Z), \mathcal{E}_{I\!R}(\mu) \otimes Z')$-compact subset of $ca(Z)$, we then have that $\int_A F(t)\,M(dt)$ is a convex and $w(Z, Z')$-compact subset of Z.■

Corollary 4.2.28 *Under the conditions of the previous theorem we have that $\int_A F(t)\,M(dt)$ is a convex and closed subset of Z for every $A \in \Sigma(v(M))$.*

PROOF: By the previous theorem, $\int_A F(t)\,M(dt)$ is a convex and $w(Z, Z')$-compact subset of Z; therefore $\int_A F(t)\,M(dt)$ is also $w(Z, Z')$-closed. The result then follows from page 422 of [29]. ■

The following corollary is a result of the fact that the weak and strong topologies coincide on finite-dimensional spaces.

Corollary 4.2.29 *Let T be a countable union of sets of the ring $\mathcal{R}$ and let μ be a scalar measure on $\mathcal{R}$ with the direct sum property. If $M : \Sigma(v(M)) \to \mathcal{P}_{kc}(I\!R^n)$ is a μ-continuous multimeasure of bounded variation $v(M)$ and $F : T \to \mathcal{P}_{kc}(I\!R^m)$ is an integrably bounded $v(M)$-measurable multifunction, then $\int_A F(t)\,M(dt)$ is a convex and compact subset of $I\!R^{nm}$ for every $A \in \Sigma(v(M))$.*

PROOF: If we put $Z = I\!R^{nm}$, $W = I\!R^{nm}$ and consider $I\!R^n \subseteq \mathcal{L}(I\!R^m, I\!R^{nm})$, then it follows immediately that $Z = W'$ and W is a norming subspace of Z'. By Theorem 4.2.27 follows then that $\int_A F(t)\,M(dt)$ is convex and $w(I\!R^{nm}, (I\!R^{nm})')$-compact, and therefore convex and compact in $I\!R^{nm}$. ■

Theorem 4.2.30 *If $M : \Sigma(v(M)) \to \mathcal{P}(Y)$ is a strong multimeasure of bounded variation such that $M(T)$ is relatively weakly compact, then $\overline{ext\,M} : \Sigma(v(M)) \to \mathcal{P}_{wk}(Y)$ is a normal multimeasure.*

PROOF: Since $ext\,M(A) \subseteq M(T)$ for all $A \in \Sigma(v(M))$, it follows immediately that $ext\,M(A)$ is a relatively weakly compact subset of X. The result then follows from the fact that $ext\,M(A) \subseteq M(A)$, $A \in \Sigma(v(M))$ and that $ext\,M$ is an additive set function (see Proposition 2 of [34]). ■

Theorem 4.2.31 *Let T be a countable union of sets of the ring $\mathcal{R}$ and let μ be a scalar measure on $\mathcal{R}$. If $M : \Sigma(v(M)) \to \mathcal{P}_{kc}(I\!R^n)$ is a multimeasure of bounded variation $v(M)$ such that M is μ-continuous, then*

$$ext\,S_M = S_{ext\,M}.$$

PROOF: From the Krein-Milman theorem follows that $ext\, M \neq \emptyset$ and $ext\, S_M \neq \emptyset$. To show that $S_{ext\, M} \subseteq ext\, S_M$, let $m' \in S_{ext\, M}$. Then, since $m'(A) \in ext\, M(A)$ for all $A \in \Sigma(v(M))$, there are no distinct points $x_1, x_2 \in M(A)$ such that

$$m'(A) = \alpha x_1 + (1-\alpha)x_2, \alpha \in (0,1).$$

But then from Theorem 2.4.6 follows that there are $m_1, m_2 \in S_M$ such that

$$x_1 = m_1(A) \text{ and } x_2 = m_2(A)$$

for all $A \in \Sigma(v(M))$. Therefore

$$m'(A) = \alpha m_1 + (1-\alpha)m_2, \alpha \in (0,1),$$

which implies that $m'(A) \in ext\, S_M$ and hence

$$S_{ext\, M} \subseteq ext\, S_M.$$

To prove the inverse inclusion, suppose the above inclusion is strict, that is, there is an $m' \in ext\, S_M$ such that $m' \notin S_{ext\, M}$. Then, since $M(A) = \int_A F(t)\, \mu(dt)$, where $F : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ is an integrably bounded $v(M)$-measurable multifunction, it follows from Theorem 5.2 of [56] that $\frac{m'(dt)}{\mu(dt)} = f_m \in S^1_{ext\, F}$. Consequently

$$m'(A) = \int_A f_m(t)\, \mu(dt) \in \int_A ext\, F(t)\, \mu(dt).$$

Then there exist no distinct functions $f_1, f_2 \in S^1_F$ such that

$$m'(A) = \alpha \int_A f_1(t)\, \mu(dt) + (1-\alpha) \int_A f_2(t)\, \mu(dt), \alpha \in (0,1),$$

which is in contradiction with the fact that $m' \notin S_{ext\, M}$. ■

Theorem 4.2.32 *Let T be a countable union of sets of the ring $\mathcal{R}$ and let μ be a non-atomic scalar measure on $\mathcal{R}$. If $M : \Sigma(v(M)) \to \mathcal{P}_{kc}(\mathbb{R}^n)$ is a multimeasure of bounded variation $v(M)$ such that M is μ-continuous, and if $F : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ is an integrably bounded $v(M)$-measurable multifunction, then*

$$\int_A F(t)\, M(dt) = \int_A F(t)\, ext\, M(dt)$$

for all $A \in \Sigma(v(M))$.

PROOF: We only need to prove that $\int_A F(t)\, M(dt) \subseteq \int_A F(t)\, ext\, M(dt)$ because the inverse inclusion follows obviously. By Theorem 4.2.26 we have that $\int_A F(t)\, M(dt) = \int_A F(t)G(t)\, \mu(dt)$, where $G : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ is an integrably bounded $v(M)$-measurable multifunction. But since $F(t)G(t) = co\, ext\, F(t)G(t)$, we then have that

$$\int_A F(t)G(t)\, \mu(dt) = \int_A co\, ext\, F(t)G(t)\, \mu(dt) = \int_A ext\, F(t)G(t)\, \mu(dt).$$

Then we only need to show that $\int_A ext\, F(t)G(t)\,\mu(dt) \subseteq \int_A F(t) ext\, M(dt)$. For this purpose, let $h \in S^1_{FG}(\mu)$ and $m \in S_M$. Then the proof will be complete if we can show that $\int_A h(t)\,\mu(dt) = \int_A f(t)\,m(dt)$ for $f \in S^1_F(m)$ because then

$$\int_A h(t)\,\mu(dt) = \int_A f(t)\,m(dt) \in co \int_A f(t)\, ext\, M(dt) \in \int_A F(t)\, ext\, M(dt).$$

But if $g \in S^1_G(\mu)$ and $f \in S^1_F(m)$, then $\int_A h(t)\,\mu(dt) = \int_A f(t)g(t)\,\mu(dt) = \int_A f(t)\,m(dt)$ and the proof is complete. ■

Theorem 4.2.33 *Let T be a countable union of sets of the ring $\mathcal{R}$, and suppose that $M : \Sigma(v(M)) \to \mathcal{P}_{kc}(\mathbb{R}^p)$ is a multimeasure of bounded variation $v(M)$ and $F : T \to \mathcal{P}_f(\mathbb{R}^n)$ is an integrably bounded $v(M)$-measurable multifunction. If μ is a non-atomic scalar measure on $\mathcal{R}$ such that M is μ-continuous and if $Z = \mathbb{R}^{np}$, then*

$$\int_A F(t)\,M(dt) = \int_A F(t)\, co\, M(dt)$$

for all $A \in \Sigma(v(M))$.

PROOF: By Theorem 4.2.26 follows that there is an integrably bounded $v(M)$-measurable multifunction $G : T \to \mathcal{P}_{kc}(\mathbb{R}^p)$ such that $\int_A F(t)\,M(dt) = \int_A F(t)G(t)\,\mu(dt)$ for all $A \in \Sigma(v(M))$. Since $\int_A F(t)G(t)\,\mu(dt) = \int_A co\, F(t)G(t)\,\mu(dt)$ for all $A \in \Sigma(v(M))$, we only need to prove that

$$\int_A co\, F(t)G(t)\,\mu(dt) = \int_A F(t) co\, M(dt)$$

for all $A \in \Sigma(v(M))$. So let $m \in S_{co\, M}$. Then $m(A) \in co\, M(A)$ for all $A \in \Sigma(v(M))$. But

$$co\, M(A) = \int_A F(t)\,\mu(dt) = \int_A co\, F(t)\,\mu(dt),$$

that is, there is a $g \in S^1_{co\, F}(\mu)$ such that $m(A) = \int_A g(t)\,\mu(dt)$. Then since $m(dt) = g(t)\mu(dt)$, it follows immediately that $\int_A f(t)\,m(dt) = \int_A f(t)g(t)\,\mu(dt)$ and the proof is complete. ■

The next result, which gives the relationship between $\overline{co} S_M$ and $S_{\overline{co} M}$, has been given by Papageorgiou [56]. We include the proof for completeness.

Theorem 4.2.34 *Let T be a countable union of sets of the ring $\mathcal{R}$ and let μ be a scalar measure on $\mathcal{R}$. If $M : \Sigma(v(M)) \to \mathcal{P}_0(Y)$ is a μ-continuous strong multimeasure of bounded variation $v(M)$ and $F : T \to \mathcal{P}_f(X)$ is an integrably bounded $v(M)$-measurable multifunction, then*

$$\overline{co} S_M = S_{\overline{co} M}.$$

PROOF: Evidently, $S_{\overline{co}M}$ is a convex and $w(ca(Y), \mathcal{E}_X(v(m)) \otimes Y')$-closed subset of $ca(Y)$. It then follows that $\overline{co}S_M \subseteq S_{\overline{co}M}$.

Conversely, suppose that the inclusion $\overline{co}S_M \subsetneq S_{\overline{co}M}$ is strict. Then there exists an $m' \in S_{\overline{co}M}$ such that $m' \notin \overline{co}S_M$. From the separation theorem we then obtain a $y \in \mathcal{E}_X(v(m)) \otimes Y'$, $y = \sum_{k=1}^n \chi_{A_k} \otimes y'_k$ such that $\sigma(y, S_M) < (y, m')$. However,

$$(y, m') = \sum_{k=1}^{n} (y'_k, m'(A_k)) \leq \sum_{k=1}^{n} \sigma(y'_k, M(A_k))$$

and

$$\sigma(y, S_M) = \sup_{m \in S_M} (y, m) = \sup_{m \in S_M} \sum_{k=1}^{n} (y'_k, m(A_k)).$$

Since S_M is decomposable, it follows that for any $m_k \in S_M$, $k = 1, 2, \ldots, n$, we have that $m = \sum_{k=1}^n \chi_{A_k} m_k \in S_M$. Then, since $\sum_{k=1}^n (y'_k, m(A_k)) = \sum_{k=1}^n (y'_k, m_k(A_k))$, we then have that

$$\begin{aligned}
\sigma(y, S_M) &= \sup_{m_k \in S_M} \sum_{k=1}^{n} (y'_k, m_k(A_k)) \\
&= \sum_{k=1}^{n} \sup_{m_k \in S_M} (y'_k, m_k(A_k)) \\
&= \sum_{k=1}^{n} \sup_{m \in S_M} (y'_k, m(A_k)) \\
&= \sum_{k=1}^{n} \sigma(y'_k, M(A_k)).
\end{aligned}$$

But this is a contradiction and the result follows. ∎

Theorem 4.2.35 *Let T be a countable union of sets of the ring $\mathcal{R}$ and let X and Y be separable Banach spaces. Suppose that $M : \Sigma(v(M)) \to \mathcal{P}_f(Y)$ is a strong multimeasure of bounded variation $v(M)$ and $F : T \to \mathcal{P}_f(X)$ is an integrably bounded $v(M)$-measurable multifunction. Then*

(a) $\overline{\int_A F(t)\, \overline{co}\, M(dt)} = \overline{co} \int_A F(t)\, M(dt)$ for all $A \in \Sigma(v(M))$.

(b) If M is in addition non-atomic, then $\overline{\int_A F(t)\, \overline{co}\, M(dt)} = \overline{\int_A F(t)\, M(dt)}$ for all $A \in \Sigma(v(M))$.

PROOF: Since $S_{\overline{co}M} = \overline{co}S_M$, statement (a) then follows from the fact that

$$\begin{aligned}
\overline{\int_A F(t)\, \overline{co}\, M(dt)} &= \overline{\left\{ \int_A f(t)\, m(dt) \mid f \in S_F^1(m), m \in S_{\overline{co}M} \right\}} \\
&= \overline{co} \left\{ \int_A f(t)\, m(dt) \mid f \in S_F^1(m), m \in S_M \right\} \\
&= \overline{co} \int_A F(t)\, M(dt)
\end{aligned}$$

for all $A \in \Sigma(v(M))$. To prove the second statement, assume that M is non-atomic. Since $\overline{\int_A F(t)\, M(dt)}$ is convex, we have that $\overline{\int_A F(t)\, \overline{co}\, M(dt)} = \overline{co} \int_A F(t)\, M(dt) = \overline{\int_A F(t)\, M(dt)}$ for all $A \in \Sigma(v(M))$. ■

Note that instead of assuming that M is non-atomic in the second statement of the above theorem, we may let M be convex-valued. Indeed, if this is the case, then $\overline{S}_M = \overline{co} S_M = S_{\overline{co}M}$ so that $\overline{\int_A F(t)\, \overline{co}\, M(dt)} = \overline{\int_A F(t)\, M(dt)}$ for all $A \in \Sigma(v(M))$.

Theorem 4.2.36 *Let T be a countable union of sets of the ring $\mathcal{R}$ and let X and Y be separable Banach spaces. Suppose that $M : \Sigma(v(M)) \to \mathcal{P}_k(Y)$ is a strong multimeasure of bounded variation $v(M)$ and $F : T \to \mathcal{P}_k(X)$ is an integrably bounded $v(M)$-measurable multifunction. Then*

(a) $\overline{\int_A F(t)\, M(dt)} = \overline{\int_A F(t)\, ext\, M(dt)}$ *for all* $A \in \Sigma(v(M))$.

(b) If M is in addition convex, then $\int_A F(t)\, M(dt) = \overline{\int_A F(t)\, ext\, M(dt)}$ *for all* $A \in \Sigma(v(M))$.

PROOF: (a) By the Krein-Milman theorem (Theorem 2.2.14) follows that $\overline{co}\, M = \overline{co}\, ext\, M$. Consequently, by applying the previous theorem twice, we have

$$\begin{aligned} \overline{\int_A F(t)\, M(dt)} &= \overline{\int_A F(t)\, \overline{co}\, M(dt)} \\ &= \overline{\int_A F(t)\, \overline{co}\, ext\, M(dt)} \\ &= \overline{\int_A F(t)\, ext\, M(dt)} \end{aligned}$$

for all $A \in \Sigma(v(M))$. For statement (b), note that since M is convex-valued, we have that $S_M = \overline{co}\, ext\, S_M$ and the result follows immediately. ■

CHAPTER 5

INTEGRABLE MULTIFUNCTIONS AND MULTIMEASURES DEFINED BY DENSITIES

5.1 Introduction

Throughout this chapter, we will employ the notations that were used in the previous chapters: T is a non-empty point set on which no topological structure is required and $\mathcal{R}$ is a ring of subsets of T. We also consider Banach spaces X, Y and Z and a bilinear mapping $(x,y) \mapsto xy$ of $X \times Y$ into Z such that $\| xy \| \leq \| x \| \| y \|$. Unless otherwise stated, $M : \mathcal{R} \to \mathcal{P}_f(Y)$ is a strong multimeasure of bounded variation $v(M)$. Also, we will consider a real number p, with $0 < p < \infty$.

We recall (Definition 4.2.1) that a multifunction $F : T \to \mathcal{P}_0(X)$ is said to be *p-integrably bounded* if there exists a $k \in \mathcal{L}^p_{\mathbb{R}}(v(M))$ such that $\|F(t)\| \leq k(t)$ $v(M)$-almost everywhere on T. Furthermore, if $F : T \to \mathcal{P}_f(X)$ is a $v(M)$-measurable multifunction, then the mapping $h : T \to \mathbb{R}_+$, defined by

$$h(t) = \|F(t)\|,$$

is $v(M)$-measurable, and F is p-integrably bounded if and only if $\|F(\cdot)\|^p \in \mathcal{L}^1_{\mathbb{R}}(M)$.

Let $F, G : T \to \mathcal{P}_f(X)$ be two p-integrably bounded $v(M)$-measurable multifunctions. If we can show that $H(F(t), G(t)) \leq \|F(t) - G(t)\|$ $v(M)$-almost everywhere on T, then from the inequality

$$\int (H(F(t), G(t))^p \, v(M, dt) \;\leq\; \int (\| F(t) \| \,+\, \| G(t) \|)^p \, v(M, dt)$$

will follow immediately that the mapping $t \mapsto H(F(t), G(t))$ belongs to $\mathcal{L}^p_{\mathbb{R}}(v(M))$. So let $x_1 \in F(t)$ and $x_2 \in G(t)$. Then from $\|x_1 - G(t)\| \leq \|x_1 - x_2\|$ we deduce immediately that

$$\begin{aligned} \sup_{x_1 \in F(t)} \|x_1 - G(t)\| \;&\leq\; \sup_{x_1 \in F(t), x_2 \in G(t)} \|x_1 - x_2\| \\ &=\; \|F(t) - G(t)\|. \end{aligned}$$

Similarly, we can prove that

$$\sup_{x_2 \in G(t)} \|x_2 - F(t)\| \leq \|F(t) - G(t)\|.$$

Consequently,

$$\begin{aligned} H(F(t), G(t)) &= \max\left[\sup_{x_1 \in F(t)} \|x_1 - G(t)\|, \sup_{x_2 \in G(t)} \|x_2 - F(t)\|\right] \\ &\leq \|F(t) - G(t)\|. \end{aligned}$$

Lastly, note that if $S^1_F(v(M)) \neq \emptyset$, then

$$\int \|F(t)\|\, v(M, dt) = \sup_{f \in S^1_F(v(M))} \int \|f(t)\|\, v(M, dt),$$

which implies that a $v(M)$-measurable multifunction $F : T \to \mathcal{P}_f(X)$ is integrably bounded if and only if $S^1_F(v(M))$ is non-empty and bounded in $\mathcal{L}^1_X(v(M))$.

5.2 The spaces $L^1_X(v(M))$ and $L^\infty_X(v(M))$

Definition 5.2.1 *A multifunction $F : T \to \mathcal{P}_0(X)$ is called an **$\mathcal{R}$-step multifunction** if it is of the form*

$$F = \sum_{i \in I} X_i \chi_{A_i},$$

where I is a finite index set, $A_i \in \mathcal{R}$ and $X_i \in \mathcal{P}_0(X)$ for every $i \in I$.

Remark 5.2.2

(i) If we take the sets $A_i \in \Sigma(v(M))$ in the above definition, then we call F a *step multifunction.*

(ii) If $F : T \to \mathcal{P}_{f(c)}(X)$ ($F : T \to \mathcal{P}_{k(c)}(X)$, respectively) is an $\mathcal{R}$-step multifunction, or a step multifunction, then we take $X_i \in \mathcal{P}_{f(c)}(X)$ ($X_i \in \mathcal{P}_{k(c)}(X)$, repectively).

Definition 5.2.3 *We denote by $L^1_X(M)$ the space of all integrably bounded $v(M)$-measurable multifunctions $F : T \to \mathcal{P}_0(X)$. Furthermore, we put*

$$L^1_{f(c)(X)}(M) = \{F \in L^1_X(M) \mid F(t) \in \mathcal{P}_{f(c)}(X) \;\; v(M) - a.e\}$$

and

$$L^1_{k(c)(X)}(M) = \{F \in L^1_X(M) \mid F(t) \in \mathcal{P}_{k(c)}(X) \;\; v(M) - a.e\}.$$

Proposition 5.2.4 *Let $M, N : \mathcal{R} \to \mathcal{P}_f(Y)$ be two strong multimeasures of bounded variation $v(M)$ and $v(N)$, respectively. If $M(A) \subseteq N(A)$ for every $A \in \mathcal{R}$, then $L^1_{f(X)}(N) \subseteq L^1_{f(X)}(M)$ and*

$$\int \| F(t) \| \ v(M, dt) \ \leq \ \int \| F(t) \| \ v(N, dt)$$

for $F \in L^1_{f(X)}(N)$.

PROOF: Since $v(M, A) \leq v(N, A)$ for every $A \in \mathcal{R}$, it follows from Proposition 1.2.13 that $\mathcal{M}(v(N)) \subseteq \mathcal{M}(v(M))$. Let now $F \in L^1_{f(X)}(N)$. Then F is $v(N)$-measurable and integrably bounded by $k \in \mathcal{L}^1_{\mathbb{R}}(N)$. Hence F is $v(M)$-measurable and integrably bounded by $k \in \mathcal{L}^1_{\mathbb{R}}(N) \subseteq \mathcal{L}^1_{\mathbb{R}}(M)$. Consequently, $F \in L^1_{f(X)}(M)$. Clearly, for $F \in L^1_{f(X)}(N)$,

$$\int \|F(t)\| \, v(M, dt) \ \leq \ \int \|F(t)\| \, v(N, dt).$$

■

Corollary 5.2.5 *If the multifunction $F : T \to \mathcal{P}_f(X)$ is $v(M)$-measurable and if there exists a $G \in L^1_{f(X)}(M)$ such that $\|F(t)\| \leq \|G(t)\|$ $v(M)$-almost everywhere on T, then $F \in L^1_{f(X)}(M)$.*

PROOF: Since $G \in L^1_{f(X)}(M)$, there exists a $k \in \mathcal{L}^1_{\mathbb{R}}(M)$ such that $\|G(t)\| \leq k(t)$ $v(M)$-almost everywhere on T. Hence $\|F(t)\| \leq k(t)$ $v(M)$-almost everywhere on T. ■

Corollary 5.2.6 *A multifunction $F : T \to \mathcal{P}_f(X)$ belongs to $L^1_{f(X)}(M)$ if and only if F is $v(M)$-measurable and $\|F(\cdot)\| \in \mathcal{L}^1_{\mathbb{R}}(v(M))$.*

PROOF: By definition, $F \in L^1_{f(X)}(M)$ if and only if F is $v(M)$-measurable and F is integrably bounded. But F is integrably bounded if and only if $\int \| F(t) \| \ v((M, dt) < \infty$, that is $\| F \| \in \mathcal{L}^1_{\mathbb{R}}(v(M))$. ■

Following our discussion in section 5.1, we have that the mapping $t \mapsto H(F(t), G(t))$ belongs to $\mathcal{L}^1_{\mathbb{R}}(v(M))$ whenever $F, G : T \to \mathcal{P}_f(X)$ are two integrably bounded $v(M)$-measurable multifunctions. We are now in a position to make the following definition.

Definition 5.2.7 *If $F, G \in L^1_{f(X)}(M)$, then we put*

$$d_1(F, G) = \int H(F(t), G(t)) v(M, dt).$$

It then follows immediately that d_1 is a semi-metric on the space $L^1_{f(X)}(M)$. On the space $L^1_X(M)$ we now define an equivalence relation $\approx$ as follows: For two multifunctions

$F, G \in L^1_X(M)$ we write $F \approx G$ if and only if $\overline{F(t)} = \overline{G(t)}$ $v(M)$-almost everywhere on T. If the equivalence class of a multifunction $F \in L^1_X(M)$ is denoted by $\tilde{F}$ and we put $\tilde{d_1}(\tilde{F}, \tilde{G}) = d_1(F, G)$, where $F \in \tilde{F}, G \in \tilde{G}$, then

Proposition 5.2.8 *$(L^1_{f(X)}(M), \tilde{d_1})$ is a metric space.*

Definition 5.2.9 *We say that a sequence $(F_k) \subseteq L^1_{f(X)}(M)$* ***converges*** *in $L^1_{f(X)}(M)$ to F if and only if $\lim\limits_{k\to\infty} d_1(F_k, F) = 0$. Furthermore, to say that a sequence $(F_k) \subseteq L^1_{f(X)}(M)$ is a* ***Cauchy sequence*** *in $L^1_{f(X)}(M)$ means that $\lim\limits_{j,k\to\infty} d_1(F_j, F_k) = 0$.*

The next result shows that $L^1_{f(X)}(M)$ is a closed subspace of $\mathcal{P}(X)$.

Proposition 5.2.10 *If X is a separable Banach space, $(F_k) \subseteq L^1_{f(X)}(M)$ and $d_1(F_k, F) \to 0$ as $k \to \infty$, then $F \in L^1_{f(X)}(M)$.*

PROOF: Let $(F_k) \subseteq L^1_{f(X)}(M)$ and suppose that $d_1(F_k, F) \to 0$ as $k \to \infty$. Then

$$\lim_{k\to\infty} \int H(F_k(t), F(t)) v(M, dt) = 0,$$

and by the usual arguments we obtain a subsequence (k_j) of (k) such that $H(F_{k_j}(t), F(t)) \to 0$ as $j \to \infty$. But since $(\mathcal{P}_f(X), H)$ is a complete metric space and $F_{k_j} \in \mathcal{P}_f(X)$, it follows that $F \in \mathcal{P}_f(X)$.

It only remains to show that F is integrably bounded. But this follows immediately from the fact that

$$\| F(t) \| \leq H(F_k(t), F(t)) + \| F_k(t) \|$$

$v(M)$-almost everywhere on T. ■

Proposition 5.2.11 *If X is a separable Banach space, then the class of all step multifunctions in $L^1_{kc(X)}(M)$ is dense in $L^1_{kc(X)}(M)$.*

PROOF: Let $F \in L^1_{kc(X)}(M)$ be a step multifunction. Then $F(t) \in \mathcal{P}_{kc}(X)$ $v(M)$-a.e on T, and since $(\mathcal{P}_{kc}(X), H)$ is a separable metric space, we obtain a multifunction $G \in \mathcal{P}_{kc}(X)$ such that $H(F(t), G(t)) < \frac{\epsilon}{v(M)}$ for every $\epsilon > 0$. Consequently, for $\epsilon > 0$, we have that $d_1(F, G) < \epsilon$. Lastly, G is integrably bounded because

$$\| G(t) \| \leq H(F(t), G(t)) + \| F(t) \|$$

$v(M)$-almost everywhere on T. ■

Corollary 5.2.12 *If X is a separable Banach space, then the class of all $\mathcal{R}$-step multifunctions in $L^1_{kc(X)}(M)$ is dense in $L^1_{kc(X)}(M)$.*

PROOF: Let $F \in L^1_{kc(X)}(M)$ and $\epsilon > 0$. Then from the previous proposition we obtain a step multifunction $G : T \to \mathcal{P}_{kc}(X)$ such that $d_1(F,G) < \frac{\epsilon}{2}$. Put

$$G = \sum_{k=1}^{n} X_k \chi_{A_k}, \quad A_k \in \Sigma(v(M)), X_k \neq \{0\}.$$

From Proposition 13 on page 76 of [27] follows that for every set $A_k \in \Sigma(v(M))$ there exists a set $B_k \in \mathcal{R}$ such that

$$v(M, A_k \triangle B_k) < \frac{\epsilon}{2n \parallel X_k \parallel}.$$

If we put $K = \sum_{k=1}^{n} X_k \chi_{B_k}$, then $K : T \to \mathcal{P}_{kc}(X)$ is an $\mathcal{R}$-step multifunction. Furthermore, since

$$\|G - K\| = \| \sum_{k=1}^{n} X_k(\chi_{A_k} - \chi_{B_k})\| \le \sum_{k=1}^{n} \|X_k\| \, \|\chi_{A_k} - \chi_{B_k}\| = \sum_{k=1}^{n} \| X_k \| \, \chi_{A_k \triangle B_k},$$

we have that

$$\begin{aligned} d_1(G,K) = \int H(G(t), K(t))\, v(M,dt) &\le \int \| G(t) - K(t) \| \; v(M,dt) \\ &= \sum_{k=1}^{n} v(M, A_k \triangle B_k) \, \| X_k \| \\ &< \frac{\epsilon}{2} \end{aligned}$$

so that

$$d_1(F,K) \le d_1(F,G) + d_1(G,K) < \epsilon.$$

■

Corollary 5.2.13 *If X is a separable Banach space, then for every multifunction $F \in L^1_{kc(X)}(M)$ there exists a Cauchy sequence (F_m) of $\mathcal{R}$-step multifunctions $F_m : T \to \mathcal{P}_{kc}(X)$ such that $H(F_m(t), F(t)) \to 0$ as $m \to \infty$ for $v(M)$-almost all $t \in T$.*

PROOF: Since the set of $\mathcal{R}$-step multifunctions $F : T \to \mathcal{P}_{kc}(X)$ is dense in $L^1_{kc(X)}(M)$, there exists a sequence $F_k : T \to \mathcal{P}_{kc}(X)$ of $\mathcal{R}$-step multifunctions such that $\lim_{k\to\infty} d_1(F_k, F) = 0$. Then (F_k) is a Cauchy sequence in $\mathcal{P}_{kc}(X)$. Since $(\mathcal{P}_{kc}(X), H)$ is a complete metric space, there exists a subsequence $(F_{k_j}) \subseteq (F_k)$ such that $H(F_{k_j}(t), G(t)) \to 0$ as $j \to \infty$, where $G : T \to \mathcal{P}_{kc}(X)$. From the inequality

$$d_1(F,G) \le d_1(F, F_{k_j}) + d_1(G, F_{k_j})$$

we deduce that $d_1(F,G) = 0$; therefore $F(t) = G(t)$ $v(M)$-almost everywhere on T. If we put $F_m = F_{k_j}$, then (F_m) is the desired Cauchy sequence. ■

Theorem 5.2.14 *If X is a separable Banach space, then $(L^1_{f(X)}(M), d_1)$ is a complete metric space, and $L^1_{kc(X)}(M)$ and $L^1_{k(X)}(M)$ are closed subspaces of $L^1_{f(X)}(M)$.*

PROOF: Let (F_k) be a Cauchy sequence in $L^1_{f(X)}(M)$, that is

$$\lim_{j,k\to\infty} \int H(F_j(t), F_k(t))v(M, dt) = 0.$$

Since (F_k) is a Cauchy sequence in $\mathcal{P}_f(X)$ and since $(\mathcal{P}_f(X), H)$ is a complete metric space, there exists a multifunction $F \in \mathcal{P}_f(X)$ such that $\lim_{k\to\infty} H(F_k(t), F(t)) = 0$. Consequently, $d_1(F_k, F) \to 0$ as $k \to \infty$. The fact that F is integrably bounded follows from

$$\int \| F(t) \| \; v(M, dt) \;\leq\; \int H(F_k(t), F(t))\, v(M, dt) + \int \| F_k(t) \| \; v(M, dt) \;<\; \infty.$$

The last assertion of the theorem follows from Proposition 5.2.10. ■

Definition 5.2.15 *If $F \in L^1_{f(X)}(M)$, then we put*

$$N_1(F) = d_1(F, \{0\}) = \int \|F(t)\| \, v(M, dt).$$

It follows easily that

Proposition 5.2.16 *N_1 is a semi-norm on the space $L^1_{f(X)}(M)$.*

In a way similar to the single-valued case, we will say that a multifunction $F : T \to \mathcal{P}_0(X)$ is *M-negligible* if $F(t) = \{0\}$ $v(M)$-almost everywhere on T. We denote by $N^\infty_X(M)$ the space of all M-negligible multifunctions. We will also denote the quotient space $L^1_X(M)/N^\infty_X(M)$ by $Q^1_X(M)$. If we put $\widetilde{N_1}\,(\widetilde{F}) = N_1(F)$, where $F \in \widetilde{F}$, then the mapping $\widetilde{N_1}\,(\cdot)$ is a norm on $Q^1_{f(X)}(M)$. Indeed, from Proposition 5.2.16 we have that $\widetilde{N_1}\,(\cdot)$ is a semi-norm on $Q^1_{f(X)}(M)$. Also, note that

$$\widetilde{N_1}\,(\widetilde{F}) = 0 \Leftrightarrow \widetilde{F} = \{0\} \;\; v(m) - \text{a.e on } T.$$

Hence, from Theorem 5.2.14,

Proposition 5.2.17 *The space $Q^1_{f(X)}(M)$ is a Banach space.*

Definition 5.2.18 *If $a \in \mathbb{R}$, then we say that a multifunction $F : T \to \mathcal{P}_0(X)$ is **$v(M)$-essentially bounded** if there is a $v(M)$-negligible set $N \subseteq T$ such that $\| F(t) \| \leq a$ for all $t \in T \backslash N$.*

Definition 5.2.19 *We denote by $L^\infty_X(v(M))$ the space of all $v(M)$-essentially bounded $v(M)$-measurable multifunctions. Furthermore, for $F, G \in L^\infty_X(v(M))$, we put*

$$d_\infty(F, G) = \inf\{a \leq +\infty \mid H(F(t), G(t)) \leq a\}$$

Definition 5.2.20 *We say that a sequence $(F_k) \subseteq L^\infty_X(v(M))$* ***converges*** *in $L^\infty_X(v(M))$ to F if there exists a $v(M)$-negligible set $N \subseteq T$ such that*

$$\lim_{k\to\infty} H(F_k(t), F(t)) = 0$$

uniformly for all $t \in T \backslash N$. Furthermore, (F_k) is a ***Cauchy sequence*** *in $L^\infty_X(v(M))$ if*

$$\lim_{j,k\to\infty} d_\infty(F_j, F_k) = 0.$$

Proposition 5.2.21 *The space $(L^\infty_{f(X)}(v(M)), d_\infty)$ is a complete metric space.*

PROOF: Let (F_k) be a Cauchy sequence in $L^\infty_{f(X)}(v(M))$. Then for every $k \in I\!N$ there exists an integer n_k such that for $r, s \geq n_k$ we have that

$$d_\infty(F_r, F_s) \leq \frac{1}{k}.$$

Then there exists a $v(M)$-negligible set A_{rs} such that

$$H(F_r(t), F_s(t)) \leq \frac{1}{k}$$

for $t \notin A_{rs}$. If A_k is the union of the sets A_{rs}, with $r, s \geq n_k$, then A_k is $v(M)$-negligible and for $t \notin A_k$ we have

$$H(F_r(t), F_s(t)) \leq \frac{1}{k}$$

for $r, s \geq n_k$. If we put $A = \cup_{k=1}^\infty A_k$, then A is $v(M)$-negligible and

$$H(F_r(t), F_s(t)) \leq \frac{1}{k}$$

for $t \notin A$. This means that $F_k(t))$ is a Cauchy sequence for each $t \notin A$. Since $(\mathcal{P}_f(X), H)$ is complete, there exists an $F \in \mathcal{P}_f(X)$ such that $H(F_k(t), F(t)) \to 0$ for all $t \notin A$ and hence $d_\infty(F_n, F) \to 0$. Lastly, F is $v(M)$-essentially bounded because there exists a $v(M)$-negligible set A such that

$$\| F(t) \| \leq H(F_k(t), F(t)) + \| F_k(t) \| \leq a$$

for all $t \notin A$. ∎

Definition 5.2.22 *For* $F \in L^{\infty}_{f(X)}(v(M))$ *we put*

$$N_\infty(F) = d_\infty(F, \{0\}) = \inf\{a \leq +\infty \mid \| F(t) \| \leq a\}.$$

Proposition 5.2.23 N_∞ *is a semi-norm on the space* $L^{\infty}_{f(X)}(v(M))$.

If we put $\widetilde{N}_\infty(\widetilde{F}) = N_\infty(F)$, where $\widetilde{F}$ is the equivalence class determined by the multifunction $F \in L^{\infty}_{f(X)}(v(M))$, then $\widetilde{N}_\infty$ is a norm on the quotient space $Q^{\infty}_{f(X)}(v(M)) = L^{\infty}_{f(X)}(v(M))/N^{\infty}_{X}(v(M))$. Consequently, the space $Q^{\infty}_{f(X)}(v(M))$ is a Banach space.

If $F, G \in \mathcal{P}_f(X)$, then we define their product FG by

$$(FG)(t) = \{x_1x_2 \mid x_1 \in F(t), x_2 \in G(t)\}.$$

Proposition 5.2.24 *If* $F \in L^1_{f(X)}(M)$ *and* $G \in L^{\infty}_{f(Y)}(v(M))$, *then* $FG \in L^1_{f(Z)}(M)$ *and*

$$\| \int F(t)G(t)\, v(M, dt) \| \leq \int \| F(t) \| \| G(t) \| v(M, dt) \leq N_1(F)N_\infty(G).$$

PROOF: Since the mapping $(x, y) \mapsto xy$ is a continuous mapping from $X \times Y$ into Z, it follows that $FG \in \mathcal{P}_f(Z)$. Furthermore, if $\{f_k \mid k \in \mathbb{N}\}$ and $\{g_k \mid k \in \mathbb{N}\}$ are Castaing representations for F and G respectively, we have that

$$(FG)(t) = \overline{\{(f_k g_k)(t) \mid k \in \mathbb{N}\}}$$

so that FG is also $v(M)$-measurable. To see that FG is integrably bounded, note that

$$\begin{aligned} \| (FG)(t) \| &= \sup_{x \in F(t), y \in G(t)} \| xy \| \\ &\leq \sup_{x \in F(t), y \in G(t)} \| x \| \| y \| \\ &\leq \| F(t) \| \, N_\infty(G) \leq k, \end{aligned}$$

where $k \in \mathcal{L}^1_{\mathbb{R}}(v(M))$. Lastly, we have that

$$\begin{aligned} \| \int F(t)G(t)\, v(M, dt) \| &\leq \int \| F(t)G(t) \| v(M, dt) \\ &\leq N_\infty(G) \int \| F(t) \| v(M, dt) \\ &= N_\infty(G)N_1(F). \end{aligned}$$

■

5.3 The space $L^p_X(v(M)), 0 < p < \infty$

Definition 5.3.1 *If $0 < p < \infty$, then we denote by $L^p_X(v(M))$ the space of all p-integrably bounded $v(M)$-measurable multifunctions. Also, we put*

$$L^p_{f(c)(X)}(v(M)) = \{F \in L^p_X(v(M)) \mid F(t) \in \mathcal{P}_{f(c)}(X)\ v(M) - a.e\}$$

and

$$L^p_{k(c)(X)}(v(M)) = \{F \in L^p_X(v(M)) \mid F(t) \in \mathcal{P}_{k(c)}(X)\ v(M) - a.e\}.$$

Definition 5.3.2 *If $F, G \in L^p_{f(X)}(v(M))$, then we put*

$$d_p(F,G) = \left(\int (H(F(t),G(t)))^p v(M,dt)\right)^{\frac{1}{p}}.$$

Since $(\mathcal{P}_f(X), H)$ is a metric space, it follows that

Proposition 5.3.3 *$(L^p_{f(X)}(v(M)), d_p)$ is a metric space.*

Definition 5.3.4 *We say that a sequence $(F_k) \subseteq L^p_{f(X)}(v(M))$ **converges** in $L^p_{f(X)}(v(M))$ to F if and only if $d_p(F_k, F) \to 0$ as $k \to \infty$. Furthermore, we say that a sequence (F_k) in $L^p_{f(X)}(v(M))$ is a **Cauchy sequence** in $L^p_{f(X)}(v(M))$ if $\lim\limits_{j,k\to\infty} d_p(F_j, F_k) = 0$*

Theorem 5.3.5 *$(L^p_{f(X)}(v(M)), d_p)$ is a complete metric space, and $L^p_{kc(X)}(v(M))$ and $L^p_{k(X)}(v(M))$ are closed subspaces of $L^p_{f(X)}(v(M))$.*

PROOF: To show that $(L^p_{f(X)}(v(M)), d_p)$ is complete, let (F_k) be a Cauchy sequence in $L^p_{f(X)}(v(M))$. Then

$$\lim_{j,k\to\infty} \int (H(F_j(t), F_k(t)))^p\ v(M,dt) = 0.$$

Since $(\mathcal{P}_f(X), H)$ is a complete metric space, there exists a multifunction $F : T \to \mathcal{P}_f(X)$ such that $\lim\limits_{k\to\infty} H(F_k(t), F(t)) = 0$, and $d_p(F_k, F) \to 0$ as $k \to \infty$. Furthermore, since $\| F(t) \| \leq H(F_k(t), F(t)) + \| F_k(t) \|$, it follows that F is p-integrably bounded. Let $(F_k) \subseteq L^p_{k(X)}(v(M))$ be such that $d_p(F_k, F) \to 0$ as $k \to \infty$. Then

$$\lim_{k\to\infty} \int (H(F_k(t), F(t))^p\ v(M,dt) = 0.$$

This means that there exists a subsequence $(k_j) \subseteq (k)$ such that $H(F_{k_j}(t), F(t)) \to 0$ as $j \to \infty$. But since $(\mathcal{P}_k(X), H)$ is a complete metric space, we have that $F(t) \in \mathcal{P}_k(X)$.

Finally, $L^p_{k(c)(X)}(v(M))$ is closed because $\mathcal{P}_{k(c)}(X)$ is a closed subspace of $\mathcal{P}(X)$. ■

Proposition 5.3.6 *The space of all $\mathcal{R}$-step multifunctions in $L^p_{kc(X)}(v(M))$ is dense in $L^p_{kc(X)}(v(M))$.*

From Proposition 5.3.6 and since $\mathcal{P}_{kc}(X)$ is separable whenever X is separable, we have that

Corollary 5.3.7 *If the ring $\mathcal{R}$ is countable and X is a separable Banach space, then $(L^p_{kc(X)}(v(M)), d_p)$ is a separable metric space.*

Corollary 5.3.8 *If the ring $\mathcal{R}$ is countable, then $(L^p_{kc(\mathbb{R})}(v(M)), d_p)$ is a separable metric space.*

Proposition 5.3.9 *If $1 \leq r < p < s < \infty$, then $L^r_{f(X)}(v(M)) \cap L^s_{f(X)}(v(M)) \subseteq L^p_{f(X)}v(M))$.*

PROOF: Let $F \in L^r_{f(X)}(v(M)) \cap L^s_{f(X)}(v(M))$. Then there exists a $k \in \mathcal{L}^r_{\mathbb{R}}(v(M)) \cap \mathcal{L}^s_{\mathbb{R}}(v(M))$ such that

$$\|F(t)\| \leq k(t) \;\; v(M) - \text{a.e on} T.$$

But then $k \in \mathcal{L}^p_{\mathbb{R}}(v(M))$ (from Proposition 21 on page 237 of [27]), and consequently $F \in L^p_{f(X)}v(M))$. ■

Definition 5.3.10 *If $F \in L^p_{f(X)}(v(M))$, then we put*

$$N_p(F) = d_p(F, \{0\}) = \left(\int \| F(t) \|^p \, v(M, dt)\right)^{\frac{1}{p}}.$$

Proposition 5.3.11 *N_p is a semi-norm on $L^p_{f(X)}(v(M))$.*

If $F \in L^p_X(v(M))$ and we put $\widetilde{N}_p\,(\widetilde{F}) = N_p(F)$, where $\widetilde{F}$ is the equivalence class determined by the multifunction $F \in L^p_X(v(M))$, then the mapping $\widetilde{N}_p\,(\cdot)$ is a norm on $Q^p_{f(X)}(M) = L^p_{f(X)}(v(M))/N^p_X(v(M))$. It then follows that the space $Q^p_{f(X)}(v(M))$ is a Banach space.

Theorem 5.3.12 *Let p and q, with $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $F \in L^p_{f(X)}(v(M))$ and $G \in L^q_{f(Y)}(v(M))$, then $FG \in L^1_{f(Z)}(v(M))$ and*

$$N_1(FG) \leq N_p(F)N_q(G).$$

PROOF: Since $(x,y) \mapsto xy$ is a continuous mapping, we have that $FG \in \mathcal{P}_f(Z)$. Furthermore, if $\{f_k \mid k \in \mathbb{N}\}$ and $\{g_k \mid k \in \mathbb{N}\}$ are Castaing representations of F and G respectively, we have that

$$(FG)(t) = \overline{\{f_k(t)g_k(t) \mid k \in \mathbb{N}\}}$$

so that FG is also $v(M)$-measurable. Since F is p-integrably bounded, there exists a $k_1 \in \mathcal{L}^p_{\mathbb{R}}(v(M))$ such that $\|F(t)\| \le k_1(t)\ v(M)-$almost everywhere, and since G is q-integrably bounded, there exists a $k_2 \in \mathcal{L}^q_{\mathbb{R}}(v(M))$ such that $\|G(t)\| \le k_2(t)\ v(M)-$almost everywhere. Hence $k_1k_2 \in \mathcal{L}^1_{\mathbb{R}}(v(M))$ and from

$$\|(FG)(t)\| \;=\; \|F(t)G(t)\| \;\le\; \|F(t)\|\,\|G(t)\| \;\le\; k_1(t)k_2(t)$$

follows that FG is integrably bounded. Furthermore, taking into account that $N_p(F) = \| H(F(t),\{0\}) \|_p$, we have that

$$\begin{aligned}
\| \int (FG)(t)M(dt) \| &\le \int \| F(t)G(t) \| v(M,dt) \\
&\le \int \| F(t) \| \| G(t) \| v(M,dt) \\
&\le \| H(F(t),\{0\}) \|_p \| H(G(t),\{0\}) \|_q \\
&= N_p(F)N_q(G).
\end{aligned}$$

■

Theorem 5.3.13 *If $1 \le p \le \infty$ and $F, G \in L^p_{f(X)}(v(M))$, then*

$$N_p(F+G) \;\le\; N_p(F) + N_p(G).$$

PROOF: If $p = 1$ or $p = \infty$, the inequality follows then immediately. Let $1 < p < \infty$. Then we have that

$$\begin{aligned}
N_p(F+G) &= \| H(F(t)+G(t),\{0\}) \|_p \\
&\le \| H(F(t),\{0\}) + H(G(t),\{0\}) \|_p \\
&\le \| H(F(t),\{0\}) \|_p + \| H(G(t),\{0\}) \|_p \\
&= N_p(F) + N_p(G).
\end{aligned}$$

■

For the rest of this section we will study the relationship between our integral and the Debreu integral. Debreu [24] made use of an embedding theorem [59, Theorem 2] in order to treat compact-convex-valued multifunctions as functions, and then developed the integral of a multifunction as a case within the theory of the integral of a function. First we state the embedding theorem due to Rådström [59].

Theorem 5.3.14 *If X is a real normed vector space, then the space $(\mathcal{P}_{kc}(X), H)$ can be embedded as a convex cone in a normed real vector space U such that*

(a) the embedding is isometric;

(b) addition in U induces addition in $\mathcal{P}_{kc}(X)$;

(c) multiplication by non-negative real numbers in U induces the corresponding operation in $\mathcal{P}_{kc}(X)$;

(d) $\mathcal{P}_{kc}(X)$ spans U;

(e) the greatest subspace of U contained in the cone $\mathcal{P}_{kc}(X)$ is the set of the one-element subsets of $\mathcal{P}_{kc}(X)$.

From Theorem 2.1.8 we know that $\mathcal{P}_{kc}(X)$ is complete whenever X is complete, and that $\mathcal{P}_{kc}(X)$ (and hence U) is separable whenever X is. By the theorem on page 89 of [29] we may embed U as a dense subspace of a real Banach space U^*, the completion of U. Obviously, if U is a real Banach space, then $U = U^*$. Consequently, we see that multifunctions in $L^p_{kc(X)}(v(M))$ can be regarded as usual Banach space-valued integrable functions:

Theorem 5.3.15 *If X is a separable Banach space, then there exists a separable Banach space U such that $L^p_{kc(X)}(v(M))$ can be embedded as a convex cone in $\mathcal{L}^p_U(v(M))$ such that*

(a) the embedding is isometric;

(b) addition in $\mathcal{L}^p_U(v(M))$ induces addition in $L^p_{kc(X)}(v(M))$;

(c) multiplication by non-negative real functions in $\mathcal{L}^p_U(v(M))$ induces the corresponding operation in $L^p_{kc(X)}(v(M))$.

Making use of Theorem 5.3.14, if Y is a Banach space, we embed $(\mathcal{P}_{kc}(Y), H)$ as a convex cone in a Banach space V^*. For the rest of this section we suppose that $X = Z$ is a real Banach space with $X = \mathcal{L}(\mathbb{R}^n, X)$ and $Y = \mathbb{R}^n$. If $f, g : T \to U^*$ are two $\mathcal{R}$-step functions and $m : \mathcal{R} \to V^*$ is a vector measure, then we put

$$\Delta(f, g) = \int \|f(t) - g(t)\| \, m(dt).$$

We say that a sequence (f_k) of $\mathcal{R}$-step functions from T to U^* (that is, $(f_k) \subseteq \mathcal{E}_{U^*}(\mathcal{R})$) is Δ-*Cauchy* if $\Delta(f_j, f_k) \to 0$ as $j, k \to \infty$. If $A \in \mathcal{P}_{kc}(X)$, then we denote by A^* the image of A in U^* under the embedding of $\mathcal{P}_{kc}(X)$ in U^*. In particular, if $F : T \to \mathcal{P}_{kc}(X)$ is a multifunction, then we write F^* to denote the function $t \mapsto (F(t))^*$. Similarly, if $M : \mathcal{R} \to \mathcal{P}_{kc}(Y)$ is a multimeasure, then we will write M^* to denote the element $(M(A))^*$ in V^*.

Definition 5.3.16 *We say that a function $f : T \to U^*$ is m-integrable if there exists a Δ-Cauchy sequence $(f_k) \subseteq \mathcal{E}_{U^*}(\mathcal{R})$ converging in measure to f. The sequence (f_k) is said to determine f. A multifunction $F : T \to \mathcal{P}_{kc}(X)$ is* ***Debreu-integrable*** *if the function $F^* : T \to U^*$ is M^*-integrable. We denote the Debreu integral of F by $\oint F(t)\, M(dt)$.*

The above definition reduces the theory of integration of compact-and convex-valued multifunctions to the standard theory of integration of functions. The next result shows that the determining sequence of $\mathcal{R}$-step functions might as well assume their values in $\mathcal{P}_{kc}(X)$.

Proposition 5.3.17 *If $F : T \to \mathcal{P}_{kc}(X)$ is Debreu-integrable, then there exists a sequence (F_k) of $\mathcal{R}$-step functions from T into $\mathcal{P}_{kc}(X)$ which determines F.*

Debreu [24] proved the first result ([24], page 367, 6.5) about the equivalence of the Debreu and the Aumann integrals. However, Debreu's result is valid under the assumption that the space X is a reflexive Banach space. Extension of this result to the nonreflexive case was given by Byrne [11]. The main result in Byrne's extension is the following:

Proposition 5.3.18 *Let $F : T \to \mathcal{P}_{kc}(X)$ be Debreu-integrable and let (F_k) be a sequence of $\mathcal{R}$-step functions from T into $\mathcal{P}_{kc}(X)$ converging pointwise to F. Then the set $S_F \cup (\cup_{k=1}^{\infty} S_{F_k})$ is relatively weakly compact in $\mathcal{L}^1_X(v(M))$.*

Theorem 5.3.19 *Suppose that $X = Z$ is a real Banach space with $X = \mathcal{L}(I\!R^n, X)$ and let $Y = I\!R^n$. If $M : \mathcal{R} \to \mathcal{P}_{kc}(I\!R^n)$ is a multimeasure of bounded variation $v(M)$ and if $F : T \to \mathcal{P}_{kc}(X)$ is a Debreu-integrable multifunction, then*

$$\int F(t)\, M(dt) = \oint F(t)\, M(dt).$$

PROOF: We first prove that $\int F(t)\, M(dt) \subseteq \oint F(t)\, M(dt)$. So let $x \in \int F(t)\, M(dt)$. Then there exists a $m \in S_M$ and a $f \in S^1_F(m)$ such that $x = \int f(t)\, m(dt)$. Clearly, $d(f(t), F(t)) = 0$ $v(m)$-almost everywhere in T. Since

$$d\left(\int f(t)\, m(dt), \oint F(t)\, M(dt)\right) \leq \sup_{m \in S_M} \int d(f(t), F(t))\, m(dt) = 0,$$

(see 6.2 on page 366 of [24]) we have that $x = \int f(t)\, m(dt) \in \oint F(t)\, M(dt)$.

Conversely, let $x \in \oint F(t)\, M(dt)$. Then there is a sequence (F_k) of $\mathcal{R}$-step functions from T to $\mathcal{P}_{kc}(X)$ determining F. Hence, $H(\oint F_k(t)\, M(dt), \oint F(t)\, M(dt)) \to 0$ if $k \to \infty$. Hence, for $k \in \mathbb{N}$, there exists a $x_k \in \oint F_k(t)\, M(dt)$ such that $x_k \to x$. Furthermore, $x_k = \oint g_k(t)\, m(dt)$, where $g_k \in S_{F_k}$ and $m \in S_M$. Using then Proposition 5.3.18 we may now assume (perhaps after reindexing) that the sequence (g_k) converges weakly to a function $g \in \mathcal{L}^1_X(m)$. Consequently,

$$x_k = \int g_k(t)\, m(dt) \to \int g(t)\, m(dt)$$

as $k \to \infty$, and hence $x = \int g(t)\, m(dt)$. We only need to show that $g \in S_F$.

Let ϵ be a positive real number and choose a k such that $\int \|f_k(t) - f(t)\|\, m(dt) \leq \epsilon$ whenever $n \geq k$. By [29, page 422, Corollary 14] there exists a convex combination ϕ of elements of g_n (with $n \geq k$) of the weakly converging sequence (ϕ_k), with $\phi = \sum_{j=1}^{l} \lambda_j g_{i_j}$, where $\sum_{j=1}^{l} \lambda_j = 1$, and for every $j \in \{1, 2, \ldots, l\}$, $\lambda_j \geq 0$, $i_j \geq k$, such that

$$\int \|\phi(t) - g(t)\|\, m(dt) \leq \epsilon.$$

Furthermore, the convexity of $F(t)$ for every $t \in T$ implies that

$$d(\phi(t), F(t)) \leq \sum_{j=1}^{l} \lambda_j d(g_{i_j}(t), F(t)).$$

Since $g_n(t) \in F_n(t)$, we have that $d(g_n(t), F(t)) \leq d(F_n(t), F(t))$, and consequently, for every $n \geq k$,

$$\int d(g_n, F(t))\, m(dt) \leq \int d(F_n(t), F(t))\, m(dt) \leq \int H(F_n(t), F(t))\, m(dt) \leq \epsilon.$$

Thus, $\int d(\phi(t), F(t))\, m(dt) \leq \epsilon$. Since also $\int d(\phi(t), g(t))\, m(dt) \leq \epsilon$, from the triangle inequality for the function d we obtain

$$\int d(g(t), F(t))\, m(dt) \leq 2\epsilon,$$

and hence $\int d(g(t), F(t))\, m(dt) = 0$. We deduce that $d(g(t), F(t)) = 0$ $v(m)$-almost everywhere in T and thus $g(t) \in F(t)$. ■

5.4 Multimeasures defined by densities

Let $F : T \to \mathcal{P}_f(X)$ be an integrably bounded $v(M)$-measurable multifunction and put

$$N(A) = \int_A F(t)\, M(dt), A \in \mathcal{R}.$$

By Theorem 4.2.12 follows that the set-valued set function $N : \mathcal{R} \to \mathcal{P}(Z)$ is a strong multimeasure of bounded variation $v(N)$. If we put

$$\nu(A) = \int_A \|F(t)\|\, v(M, dt), A \in \mathcal{R},$$

then $\nu : \mathcal{R} \to \mathbb{R}$ is a positive measure and $v(N) \leq \nu$.

Definition 5.4.1 *If the multifunction F and the multimeasures M and N satisfy $N(A) = \int_A F(t)\, M(dt)$, $A \in \mathcal{R}$, then we say that N is the product of the multimeasure M by the multifunction F, or that N is the multimeasure with density F and base M. The multimeasure N is denoted by FM.*

Proposition 5.4.2 *If $F, G : T \to \mathcal{P}_f(X)$ are two integrably bounded $v(M)$-measurable multifunctions and $\alpha \in \mathbb{R}$, then*

$$(F+G)\, v(M) = Fv(M) + Gv(M) \text{ and } (\alpha F)v(M) = \alpha(Fv(M)).$$

Proposition 5.4.3 *Let $M, N : \mathcal{R} \to \mathcal{P}_f(Y)$ be two strong multimeasures of bounded variation $v(M)$ and $v(N)$, respectively and suppose that $F : T \to \mathcal{P}_f(X)$ is an integrably bounded $v(M)$-measurable and $v(N)$-measurable multifunction. If α and β are scalars, then*

$$F(\alpha v(M) + \beta v(N)) = \alpha(Fv(M)) + \beta(Fv(N)).$$

Proposition 5.4.4 *Suppose that X, Y and Z are Banach lattices, let $M : \mathcal{R} \to \mathcal{P}_0(Y)$ be a multimeasure of bounded variation $v(M)$ and let $F : T \to \mathcal{P}_0(X)$ be an integrably bounded $v(M)$-measurable multifunction. If $Y = \mathcal{L}(X,Z)$, $M(A) \subseteq Y_+$ for all $A \in \mathcal{R}$ and if $F(t) \subseteq X_+$ $v(M)$-almost everywhere on T, then $FM \subseteq Z_+$. Conversely, if $X = \mathcal{L}(Y,Z)$, $M(A) \subseteq Y_+$ for all $A \in \mathcal{R}$ and if $FM \subseteq Z_+$, then $F(t) \subseteq X_+$ $v(M)$-almost on T.*

PROOF: See the proof of Theorem 4.2.11. ■

Proposition 5.4.5 *Suppose that X, Y and Z are Banach lattices and let $M, N : \mathcal{R} \to \mathcal{P}(Y)$ be two strong multimeasures of bounded variations $v(M)$ and $v(N)$. If $M(A) \subseteq N(A)$ for all $A \in \mathcal{R}$ and if $F(t) \subseteq X_+$ $v(N)$-almost everywhere on T, then*

$$FM \subseteq FN.$$

Proposition 5.4.6 *Let $M : \mathcal{R} \to \mathcal{P}_{kc}(\mathbb{R}^n)$ be a multimeasure of bounded variation $v(M)$ and suppose that $F_1, F_2 : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ are two integrably bounded $v(M)$-measurable multifunctions. Then*

$$F_1 v(M) = F_2 v(M) \text{ if and only if } F_1(t) = F_2(t)\ \ v(M) - a.e \text{ on } T.$$

PROOF: From $F_1 v(M) = F_2 v(M)$ we have that

$$\sigma(p, \int_A F_1(t) v(M, dt)) = \sigma(p, \int_A F_2(t) v(M, dt))$$

for $p \in \mathbb{R}^n$. Consequently, for $p \in \mathbb{R}^n$,

$$\begin{aligned}\int_A \sigma(p, F_1(t))v(M, dt) &= \sigma(p, \int_A F_2(t)v(M, dt)) \\ &= \sigma(p, \int_A F_2(t)v(M, dt)) \\ &= \int_A \sigma(p, F_2(t))v(M, dt),\end{aligned}$$

and therefore $\sigma(p, F_1(t)) = \sigma(p, F_2(t))$ so that $F_1(t) = F_2(t)$ $v(M)$-almost everywhere on T. ∎

Proposition 5.4.7 *If $F : T \to \mathcal{P}_f(X)$ is an integrably bounded $v(M)$-measurable multifunction, then*

$$Fv(M) = \{0\} \text{ if and only if } F(t) = \{0\} \ v(M) - a.e \text{ on } T.$$

Proposition 5.4.8 *Let μ be a positive scalar measure and let $G : T \to \mathcal{P}_f(X)$ be an integrably bounded μ-measurable multifunction. If $M = G\mu$ and $F : T \to \mathcal{P}_f(X)$ is an integrably bounded $v(M)$-measurable multifunction, then $FM = (FG)\mu$.*

PROOF: Since M is a strong multimeasure of bounded variation $v(M)$, by Theorem 2.5 of [39] we have that $S_M \neq \emptyset$. Let $m \in S_M$. Then $m(A) \in M(A)$ for every $A \in \mathcal{R}$. Consequently, there exists a $g \in S^1_G(\mu)$ such that $m(A) = \int_A g(t)\mu(dt)$ for every $A \in \mathcal{R}$. Furthermore, if $z \in FM$, then $z = \int_A f(t)\, m(dt)$ where $f \in S^1_F(m)$. But from $m(dt) = g(t)\mu(dt)$ we have that $z = \int_A f(t)g(t)\mu(dt)$, and consequently, $z \in (FG)\mu$. The inverse inclusion follows similarly. ∎

Definition 5.4.9 *Suppose that $M, N : \mathcal{R} \to \mathcal{P}_f(Y)$ are two strong multimeasures of bounded variation $v(M)$ and $v(N)$, respectively. Then we say that M and N are **singular** if for every positive measure μ with $\mu \leq v(M)$ and $\mu \leq v(N)$ we have that $\mu \equiv 0$.*

Theorem 5.4.10 *Suppose that Y is a separable Banach space and let μ be a positive measure on $\mathcal{R}$ and suppose that $M : \mathcal{R} \to \mathcal{P}_{fb}(Y)$ is a strong multimeasure of bounded variation $v(M)$. If $v(M) + \mu$ has the direct sum property, then there exists multimeasures $M_1, M_2 : \mathcal{R} \to \mathcal{P}_{fb}(Y)$, of bounded variations, such that*

$$M = M_1 + M_2$$

and such that M_1 is μ-continuous and M_2 is μ-singular.

PROOF: By Theorem 2.5 of [39] we have that $S_M \neq \emptyset$. So let $m \in S_M$. Then $m(A) \in M(A)$ for every $A \in \mathcal{R}$. But from Theorem 2.4.24 we have that

$$M(A) = \overline{\{m(A) \mid A \in \mathcal{R}\}},\ A \in \mathcal{R}.$$

Since $v(M) = v(m)$, it then follows that $v(m) + \mu$ has the direct sum property. By Theorem 7 on page 189 of [27] we then obtain measures $m_1, m_2 : \mathcal{R} \to Y$, with m_1 μ-continuous and m_2 μ-singular, such that $m = m_1 + m_2$. If we put

$$M_1(A) = \overline{\{m_1(A) \mid A \in \mathcal{R}\}}$$

and

$$M_2(A) = \overline{\{m_2(A) \mid A \in \mathcal{R}\}},$$

then M_1 and M_2 are the desired multimeasures. ■

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