Set-based Particle Swarm Optimization for Portfolio Optimization

by

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*Thesis presented in partial fulfilment of the requirements for
the degree of Master of Computer Science in the Faculty of
Science at Stellenbosch University*

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December 2021

Declaration

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Date: December 2021

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Abstract

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Portfolio optimization is a complex problem, not only in the depth of the topics it covers but also in breadth. It is the process of determining which assets to include in a portfolio while simultaneously maximizing profit and minimizing risk. Portfolio optimization is rich with interesting research not only by researchers in finance, but also in computer science. The overlap of these fields has lead to an increase in the use of meta-heuristics to make intelligent investment decisions. This thesis conducts a thorough investigation into the current state of evolutionary and swarm intelligence algorithms for portfolio optimization. The investigation showed that these algorithms suffer from stability issues for larger portfolio optimization problems. A new approach using set-based particle swarm optimization (SBPSO) is proposed to reduce the dimensionality, and therefore complexity, of portfolio optimization problems. The results show that SBPSO is capable of obtaining good-quality solutions while being relatively fast. New set-based diversity measures are developed in order to better understand the exploration and exploitation behaviour of SBPSO, and set-based algorithms in general. It is shown that SBPSO fails to converge to a single solution and uses an inadequate process to determine the contribution of each asset to the portfolio. Based on these findings, improvements are made to the proposed SBPSO approach that yield significant gains in performance. The first multi-objective adaptation of SBPSO is also developed and is shown to scale to larger portfolio problems better than the multi-guided particle swarm optimization (MGPSO) algorithm, with lower levels of risk.

Uittreksel

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Portefeulje-optimering is 'n ingewikkelde probleem, nie net in die diepte van die onderwerp wat oorgesien word nie maar ook in geheel. Dit is die proses om vas te stel watter bates om in 'n portefeulje op te neem, terwyl gelyktydig die wins te maksimeer en risiko te minimeer. Portefeulje-optimalisering is ryk aan interessante navorsing nie net deur finansiële navorsers nie, maar ook in rekenaar-wetenskap. Die oorvleueling van hierdie velde het gelei tot 'n toename in die gebruik van meta-heuristieke om intelligente beleggingsbesluite te neem. Hierdie tesis sluit 'n deeglike ondersoek in na die huidige stand van evolusie en swerm intelligensie algoritmes vir portefeulje-optimalisering. Die ondersoek het getoon dat hierdie algoritmes aan stabiliteitskwessies ly vir groter portefeulje-optimalisering probleme. 'n Nuwe benadering met behulp van versameling gebaseerde deeltjieswerm optimering (SBPSO) word voorgestel om die dimensionaliteit, en dus die kompleksiteit, van portefeulje-optimaliseringsprobleme te verminder. Die resultate toon aan dat SBPSO in staat is om oplossings van goeie gehalte te verkry, terwyl dit relatief vinnig is. Nuwe versameling gebaseerde diversiteitsmaatreëls word ontwikkel vir die verkenning en ontginning van SBPSO, en versameling gebaseerde algoritmes in die algemeen. Daar word aangetoon dat SBPSO nie daarin slaag om op 'n enkele oplossing te konvergeer nie, en gebruik 'n onvoldoende proses om die bydrae van elke bate tot die portefeulje te bepaal. Op grond van hierdie bevindings word verbeteringe aan die voorgestelde SBPSO-benadering aangebring wat beduidende winste in prestasies lewer. Die eerste meervoudige doelwit aanpassing van SBPSO word ook ontwikkel, en daar word gewys dat dit beter skaal na groter portefeulje-probleme as die meervoudige-gids deeltjieswerm optimering (MGPSO) algoritme, met laer risiko vlakke.

Acknowledgements

Firstly, I would like to thank Prof Andries Engelbrecht for the support and guidance he has provided. I'm eternally grateful for all the opportunities you have given me. I would also like to thank my mom, sister, and girlfriend for their love and support. Without them, none of this work would have been possible. Additional thanks to my sister and girlfriend for taking the time to proofread my work. Lastly, thank you to my friends for their patience and for taking an interest in my work. A special thanks to those who kept me company while I worked late.

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Chapter 1

Introduction

An investment in knowledge always pays the best interest

—Benjamin Franklin,

The Way to Wealth: Ben Franklin on Money and Success

The practice of investing has allowed many ordinary people to build long-term wealth. These investments are typically made with an objective in mind, such as savings for retirement, a rainy day, or to fund expensive purchases. Not only can investing be good for the individual, but also for society as it can lead to job creation, increased spending and overall economic growth. Put simply, investing is the commitment to increase prosperity over time through the transfer of current resources. There are many advantages to investing and because of these advantages, people want to make the best investment decisions as possible.

Investors purchase financial resources, referred to as assets, such as shares in a company or property, speculating that over time the monetary value of the asset will increase and can then be sold for a profit. However, it is not a guarantee that an asset's value will rise, as there is always some level of risk associated with the asset. There is no such thing as a free lunch. To illustrate the investment process, consider the game of Monopoly. In the game, players purchase property. When a player lands on another player's property, that player must then pay the owner a fee. The more expensive the property purchased, the higher the fees. Players must therefore strategize which properties to buy. In some cases, a player might take a risk and purchase an expensive property, but unfortunately, it is hardly ever visited by other players. In this scenario, the risk does not pay off and possibly jeopardises the player's position in the game. Players must determine what the best possible way to spend their money would be, in order to maximize their profits without going bankrupt.

The real world is much more complex with many moving parts, but it is similar to Monopoly in that investing can be a risky venture. However, it is possible that an asset's value increases significantly, making it a worthwhile

investment. Identifying which asset, or collection of assets known as a portfolio, would yield an optimal balance between risk and reward is not an easy task. If it was, people would need to be akin to fortune-tellers.

Portfolio optimization is the process of determining well-performing assets, and their corresponding contribution to a portfolio, that will maximize profits while mitigating risk. Portfolio optimization is a complex, multifaceted problem where computational intelligence techniques have found great success over traditional methods. However, the performance of these aforementioned computational intelligence techniques has been known to deteriorate for larger portfolio optimization problems [183]. This thesis redefines portfolio optimization as a set-based selection problem and proposes set-based algorithms for said problem.

The motivation for this study is discussed in Section 1.1. Section 1.2 lists the objectives of this study. The contributions made throughout this study are listed in Section 1.3. Finally, an outline of the remainder of this thesis is given in Section 1.4.

1.1 Motivation

There is no single solution to the portfolio optimization problem, but rather a set of optimal trade-offs between return and risk. Knowledge of, or an approximation of these trade-offs is valuable information that could aid in the decision-making process behind investments. Portfolio optimization has been studied extensively by researchers in computer science and finance, with new and novel work frequently published [86]. Traditional approaches to portfolio optimization, for example quadratic programming, can correctly identify solutions for simple portfolio models. However, investors often face constraints and multiple, sometimes conflicting objectives that render traditional methods computationally ineffective. As a result, researchers have turned to meta-heuristic approaches to approximate optimal solutions efficiently.

Meta-heuristic algorithms have been shown to efficiently identify optimal solutions for a variety of complex and unique portfolio models [86], [133]. Yet, these approaches are also not without their flaws. When the number of assets under consideration increases, that is, the dimensionality of the problem increases, performance diminishes [183]. This deterioration in performance is commonly referred to as the curse of dimensionality.

This study presents a new approach to portfolio optimization, set-based particle swarm optimization (SBPSO), to address the stability issues of previous algorithms. SBPSO performs a bi-stage search process where assets are first selected to reduce the dimensionality of the problem and then optimal weighting for each asset in the portfolio is determined. It is believed that this approach will result in more finely-tuned portfolios that are more profitable and risk-averse than those obtained by other algorithms.

1.2 Objectives

The objective of this study is to provide investors and researchers with new algorithms that make use of SBPSO for single and multi-objective portfolio optimization while overcoming the stability issues of previous approaches. Note that this thesis does not make any grandiose claims about the proposed algorithms being the be-all and end-all of investment approaches. Rather the proposed algorithms should be seen as another tool in an investor's arsenal to aid him in his investment decisions. After all, in a 2009 letter to shareholders, the legendary investor Warren Buffett said "beware of geeks bearing formulas".

These algorithms need to be able to identify a diverse set of optimal and practical solutions efficiently without succumbing to the curse of dimensionality. In working towards this goal, the following sub-objectives have been identified:

- to provide an overview of existing meta-heuristic techniques, namely swarm intelligence and evolutionary algorithms, that have been used to solve the portfolio-optimization problems.
- to investigate qualitative measures for calculating the diversity of a collection of sets.
- to propose an SBPSO for single-objective portfolio optimization problem that addresses scalability issues of previous approaches.
- to develop a multi-objective adaptation of SBPSO for multi-objective portfolio optimization problems.
- to investigate the performance of the above-mentioned algorithms for solving portfolio optimization problems.
- to compare the performance of the above-mentioned algorithms with state-of-the-art algorithms.
- to identify key areas for future work based on the findings of this study.

1.3 Contributions

The main contributions of this study are:

- A thorough investigation into current state of portfolio optimization with respect to evolutionary and swarm intelligence algorithms.
- A proposed method of solving the portfolio optimization problem using SBPSO.

- The finding that SBPSO addresses the stability issues of previous approaches and can find good-quality solutions quickly.
- Methods for calculating set-based diversity using the Jaccard distance and the Hamming distance measures.
- Several improvements to the proposed SBPSO approach that increase performance.
- A variant of the proposed SBPSO approach that uses adaptive coordinate descent (ACD) that is shown to be good at identifying high quality portfolios but at the cost of a longer search time.
- The finding of an alternative approach to parameter tuning of the multi-guide particle swarm optimization (MGPSO) algorithm that randomly samples control parameter values from the theoretically derived stability region.
- A multi-objective adaptation of SBPSO for multi-objective portfolio optimization that incorporates elements from MGPSO, namely multi-guide set-based particle swarm optimization (MGSBPSO).
- The finding that the MGSBPSO approach can scale to larger portfolio problems better than MGPSO.

1.4 Thesis Outline

- **Chapter 2** provides the necessary background on optimization, and particle swarm optimization (PSO) and its concepts. MGPSO and SBPSO are also discussed in detail.
- **Chapter 3** presents the necessary background on portfolio optimization and a thorough review and analysis of existing evolutionary and intelligence algorithms for portfolio optimization.
- **Chapter 4** proposes SBPSO for portfolio optimization and investigates its performance.
- **Chapter 5** investigates two approaches to measuring the diversity of sets using the Jaccard distance and the Hamming distance measures.
- **Chapter 6** conducts a diversity and performance analysis of SBPSO, from which several improvements are made.
- **Chapter 7** proposes MGSBPSO for multi-objective portfolio optimization and investigates its performance. An alternative strategy to parameter tuning for MGPSO is also investigated.

- **Chapter 8** concludes the thesis and provides a summary of all the findings of the presented work. Suggestions for future research are also given.
- **Appendix A** provides a list of the important acronyms used or newly defined in the course of this work, as well as their associated definition.
- **Appendix B** provides a list of the important symbols used or newly defined in the course of this work, as well as their associated definition.
- **Appendix C** provides a list of the derived publications from this work.
- **Appendix D** provides the tuned control parameters for each algorithm for portfolio optimization.
- **Appendix E** details the Walking Fish Group (WFG) toolkit used to benchmark the performance of MGPSON and the proposed approach to randomly sample control parameter values from the theoretically derived stability region.

Chapter 2

Particle Swarm Optimization

In 1995, Eberhart and Kennedy presented PSO, a stochastic population-based search algorithm inspired by the social behaviour of birds within a flock that converge to a single point [49]. PSO has since become an extremely popular choice for optimization problems, with many subsequent adaptations and improvements proposed by other researchers. The objective of this chapter is to provide the reader with the necessary background on PSO and its concepts. Two other PSO algorithms are also discussed, each developed for a specific type of optimization problem. Furthermore, all of the PSO algorithms presented in this chapter are used in later chapters.

An introduction to optimization problems is given in Section 2.1. Section 2.2 introduces the original PSO algorithm. A multi-objective adaptation of PSO, MGPSO, is presented in Section 2.3. SBPSO is discussed in Section 2.4. Section 2.5 summarises the chapter.

2.1 Optimization

The goal of an optimization algorithm is to find a solution to an optimization problem, such that a given quantity is optimized, possibly subject to a set of constraints [52]. Portfolio optimization is such a problem. In order to better understand portfolio optimization, single and multi-objective optimization are briefly discussed in Sections 2.1.1 and 2.1.2, respectively.

2.1.1 Single-objective Optimization

Formally, a single-objective optimization problem, f , assuming minimization, is defined as

$$\begin{aligned} \text{minimize } f(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2, \dots, x_n) \\ \mathbf{x} \in \Omega \end{aligned} \tag{2.1}$$

where $\mathbf{x}$ is a n -dimensional decision vector within the search space, Ω [35], [52]. Each component of the decision vector corresponds to a decision variable in f . Solutions to f are constrained to the bounds of Ω .

2.1.2 Multi-objective Optimization

A multi-objective optimization problem (MOOP) is the simultaneous optimization of two or three conflicting objectives [35], [52], [118]. Problems that consist of four or more objectives are commonly referred to as many-objective optimization problems [82]. Assuming minimization, multi-objective and many-objective optimization problems are defined as

$$\begin{aligned} &\text{minimize } \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x})) \\ &\mathbf{x} \in \Omega \end{aligned} \tag{2.2}$$

where m is the number of objectives.

The solution to a MOOP is not a single solution, but rather a set of optimal trade-offs, referred to as the Pareto-optimal solutions (POS) [35], [52]. The POS contains the vectors in the decision space that best balance the conflicting objectives. The objective function maps these decision vectors to the objective space, which forms the Pareto-optimal front (POF). The POF contains non-dominated solutions, which are objective vectors that are not dominated by any other feasible solution found. Formally, a decision vector $\mathbf{x}_1$ dominates a decision vector $\mathbf{x}_2$, denoted by $\mathbf{x}_1 < \mathbf{x}_2$, if and only if $f_k(\mathbf{x}_1) \leq f_k(\mathbf{x}_2) \forall k \in \{1, \dots, m\}$ and $\exists k \in \{1, \dots, m\}$ such that $f_k(\mathbf{x}_1) < f_k(\mathbf{x}_2)$, assuming a minimization problem. The true POF is the set of all feasible non-dominated solutions that exist within the objective space. The goal of a multi-objective algorithm is to obtain a diverse spread of non-dominated solutions that are as close to the true POF as possible [52].

2.2 Original Particle Swarm Optimization

PSO is a single-objective optimization algorithm that simulates the flocking behaviour of birds [49], [142]. PSO uses a population, referred to as a swarm, which is a collection of randomly initialized particles within the bounds of the optimization problem under consideration. Each particle's position represents a candidate solution to the optimization problem. The particles cooperate to solve the optimization problem by exchanging locally available information about their positions through network topologies. Particles move through the search space by means of velocity updates that guide the particles to more desirable solutions. The velocity of each particle, which is initially zero, is influenced by social and cognitive guides, as well as the previous velocity of the particle. The cognitive guide is a particle's personal best solution found

throughout the duration of the search. The social guide is the best-known solution within a network, also referred to as a neighbourhood, of particles.

Section 2.2.1 describes the PSO search process. Inertia, used to control the velocity of particles, is discussed in Section 2.2.2. Section 2.2.3 presents two commonly used network topologies in PSO. The stability conditions on PSO control parameters are discussed in Section 2.2.4.

2.2.1 Search Process

Particles iteratively move through the search space by updating their position with their velocity at each time step. Velocity determines the step size and search direction of the particle. The original PSO proposed by Eberhart and Kennedy updates a particle's velocity using [49]

$$\mathbf{v}_i(t+1) = \mathbf{v}_i(t) + c_1 \mathbf{r}_1(t)(\mathbf{y}_i(t) - \mathbf{x}_i(t)) + c_2 \mathbf{r}_2(t)(\hat{\mathbf{y}}_i(t) - \mathbf{x}_i(t)) \quad (2.3)$$

where $\mathbf{v}_i$ is the velocity of particle i ; $\mathbf{x}_i$ is the position of particle i ; $\mathbf{r}_1$ and $\mathbf{r}_2$ are vectors of random values sampled from a uniform distribution in the range $[0,1]$; c_1 and c_2 are positive constants referred to as the cognitive and social acceleration coefficients, respectively; $\mathbf{y}_i$ is the personal best position of particle i and $\hat{\mathbf{y}}_i$ is the best position within the particle's neighbourhood. Constants c_1 and c_2 are responsible for controlling the amount of influence that the cognitive and social guides have. A particle's position is updated using

$$\mathbf{x}_i(t+1) = \mathbf{x}_i(t) + \mathbf{v}_i(t+1) \quad (2.4)$$

Algorithm 1 contains pseudo code for PSO.

2.2.2 Inertia

The search behaviour of PSO should ideally balance the time spent between exploring the search space and exploiting promising areas. In other words, balance the time spent between global exploration and local exploration of the search space. Shi and Eberhart introduced an inertia weight, w , to the velocity update function to regulate the trade-off between exploitation and exploration [153]. The inertia-weighted velocity function is defined as

$$\mathbf{v}_i(t+1) = w\mathbf{v}_i(t) + c_1 \mathbf{r}_1(t)(\mathbf{y}_i(t) - \mathbf{x}_i(t)) + c_2 \mathbf{r}_2(t)(\hat{\mathbf{y}}_i(t) - \mathbf{x}_i(t)) \quad (2.5)$$

An inertia weight less than 1 reduces the influence of the previous velocity component, and depending on the values of c_1 and c_2 , the velocity of each particle will eventually reach zero [32]. Conversely, an inertia weight greater than 1 increases the influence of the previous velocity component and results in an explosion in a particle's velocity, leading to divergent particle behaviour. The inertia PSO is similar to Algorithm 1, but with velocities updated using equation (2.5).

Algorithm 1 Particle Swarm Optimization (Synchronous Version)

```

 $t = 0$ ;
Let  $f$  be the objective function;
Create and initialize a swarm,  $S$ , of  $n$  particles uniformly within a predefined
hypercube of dimension  $d$ ;
for each particle  $i = 1, \dots, n$  do
    Let  $\mathbf{y}_i$  represent the personal best position of particle  $\mathbf{x}_i$ , initialized to
     $\mathbf{x}_i(t)$ ;
    Let  $\hat{\mathbf{y}}_i$  represent the neighbourhood best position of particle  $\mathbf{x}_i$ , initialized
    to the best  $\mathbf{x}_i$  in  $i$ 's neighbourhood;
    Initialize  $\mathbf{v}_i(t)$  to  $\mathbf{0}$ ;
while stopping condition is not true do
    for each particle  $i = 1, \dots, n$  do
        if  $f(\mathbf{x}_i(t)) < f(\mathbf{y}_i(t))$  then
             $\mathbf{y}_i(t+1) = \mathbf{x}_i(t)$ ;
        for particles  $\hat{i}$  with particle  $i$  in their neighbourhood do
            if  $f(\mathbf{x}_i(t)) < f(\hat{\mathbf{y}}_i(t))$  then
                 $\hat{\mathbf{y}}_i(t+1) = \mathbf{x}_i(t)$ ;
    for each particle  $i = 1, \dots, n$  do
        Update particle  $i$ 's velocity using equation (2.3);
        Update particle  $i$ 's position using equation (2.4);
     $t = t + 1$ ;

```

2.2.3 Network Topologies

As mentioned above, particles exchange information about their best positions through neighbourhoods formed by network topologies. A local network topology, such as the one in Figure 2.1a, partitions the swarm into smaller, local neighbourhoods. In this case, each particle has two neighbours. The star topology, Figure 2.1b, creates a global neighbourhood where each particle is connected to all other particles. Engelbrecht showed, through empirical evidence, that there is no significant difference in performance between PSO over a large number of benchmark problems that uses the star topology (i.e. the global-best PSO) and PSO that uses the ring topology ring (i.e. the local-best PSO). The best topology to use is, however, sometimes problem-dependent [51].

Figure 2.1: Network topologies

2.2.4 Stability Conditions

Cleghorn and Engelbrecht derived theoretical stability conditions on the PSO control parameters to guarantee order-1 and order-2 stability [33]:

$$0 < c_1 + c_2 < \frac{24(1 - w^2)}{7 - 5w} \text{ and } |w| < 1 \quad (2.6)$$

Note that stability refers to, in the case of PSO, that the velocity of each particle becomes zero.

2.3 Multi-guide Particle Swarm Optimization

PSO was initially developed to solve single-objective optimization problems, but over the years researchers have progressively adapted PSO to solve MOOPs. As a result, many multi-objective PSO algorithms have been published [1], [34], [36], [78], [106], [125], [143], [145], [152], [156], [169], [170], [190]. One such multi-objective algorithm is the MGPSO algorithm recently proposed by Scheepers *et al* [152]. The MGPSO was inspired by vector evaluated particle swarm optimization (VEPSO) in that MGPSO also utilizes multiple swarms to find solutions to MOOPs [143]. However, unlike the VEPSO, the MGPSO uses an archive to share non-dominated solutions between sub-swarms. MGPSO is highly competitive to the current state of the art PSO-based and evolutionary-based multiple objective optimization algorithms [152]; namely, speed-constrained multi-objective particle swarm optimization (SMPSO), optimized multi-objective particle swarm optimization (OMOPSO), VEPSO, non-dominated sorting genetic algorithm II (NSGA II), strength Pareto evolutionary algorithm 2 (SPEA2), Pareto envelope-based selection algorithm II (PESA-II) and multi-objective evolutionary algorithm based on decomposition (MOEA/D).

Section 2.3.1 describes the MGPSO search process. The archive management strategy used is discussed in Section 2.3.2. Section 2.3.3 presents the stability conditions on the MGPSO control parameters.

2.3.1 Search Process

For an m -objective optimization problem, the MGPSO uses m sub-swarms, where each sub-swarm optimizes one of the objective functions. Non-dominated solutions found by the sub-swarms are added to a shared archive. Similar to the cognitive and social guides, MGPSO adds an archive guide to the velocity update equation thereby attracting particles to non-dominated solutions found within the archive. Velocity is updated using

$$\begin{aligned} \mathbf{v}_i(t+1) = & w\mathbf{v}_i(t) + c_1\mathbf{r}_1(t)(\mathbf{y}_i(t) - \mathbf{x}_i(t)) \\ & + \lambda_i c_2 \mathbf{r}_2(t)(\hat{\mathbf{y}}_i(t) - \mathbf{x}_i(t)) \\ & + (1 - \lambda_i) c_3 \mathbf{r}_3(t)(\hat{\mathbf{a}}_i(t) - \mathbf{x}_i(t)) \end{aligned} \quad (2.7)$$

where $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are vectors of random values sampled from a uniform distribution in $[0,1]$; c_1 , c_2 and c_3 are positive constants referred to as the cognitive, social and archive acceleration coefficients respectively; $\mathbf{y}_i$ is the personal best position of particle i ; $\hat{\mathbf{y}}_i$ is the best position within the particle's neighbourhood; $\hat{\mathbf{a}}_i$ is the archive guide for particle i and λ_i is the archive balance coefficient.

The archive balance coefficient balances the influence of the archive and that of the social component. Smaller values of λ_i will exploit the archive guide while proportionally decreasing the influence of the social guide. Similarly, larger values of λ_i encourage the social guide while discouraging the archive guide. The MGPSO initializes λ_i for each particle by randomly selecting a value from a uniform distribution in the range $[0,1]$ [152]. The sampled λ_i values remain static throughout the duration of the search. Random λ_i values have been shown to increase convergence to the POF while maintaining diversity in non-dominated solutions [152].

The archive guide is the least crowded non-dominated solution selected from a randomly created tournament of archive solutions. Crowding distance is used to determine the least crowded solution [41]. Crowding distance is the average distance between a point in the objective space and its nearest neighbours. Scheepers *et al.* found through empirical analysis that tournament sizes of two and three yielded good results [152]. The least crowded solution will facilitate a more diverse Pareto optimal front, because the MGPSO will focus more on sparsely populated areas of the objective space. Algorithm 2 contains pseudo code for the MGPSO.

Algorithm 2 Multi-guide Particle Swarm Optimization

```

 $t = 0;$ 
for each objective  $k = 1, \dots, m$  do
    Create and initialize a swarm,  $S_k$ , of  $n_{s_k}$  particles uniformly within a pre-
    defined hypercube of dimension  $d$ ;
    Let  $f_k$  be the objective function;
    for each particle  $i = 1, \dots, S_k \cdot n_{s_k}$  do
        Let  $S_k \cdot \mathbf{y}_i$  represent the personal best position of particle  $S_k \cdot \mathbf{x}_i$ , ini-
        tialized to  $S_k \cdot \mathbf{x}_i(t)$ ;
        Let  $S_k \cdot \hat{\mathbf{y}}_i$  represent the neighbourhood best position of particle  $S_k \cdot \mathbf{x}_i$ ,
        initialized to the best  $S_k \cdot \mathbf{x}_i$  in  $i$ 's neighbourhood;
        Initialize  $S_k \cdot \mathbf{v}_i(t)$  to  $\mathbf{0}$ ;
        Initialize  $S_k \cdot \lambda_i \sim U(0, 1)$ ;
    while stopping condition is not true do
        for each objective  $k = 1, \dots, m$  do
            for each particle  $i = 1, \dots, S_k \cdot n_{s_k}$  do
                if  $f_k(S_k \cdot \mathbf{x}_i(t)) < f_k(S_k \cdot \mathbf{y}_i(t))$  then
                     $S_k \cdot \mathbf{y}_i(t+1) = S_k \cdot \mathbf{x}_i(t)$ ;
                for particles  $\hat{i}$  with particle  $i$  in their neighbourhood do
                    if  $f_k(S_k \cdot \mathbf{x}_i(t)) < f_k(S_k \cdot \hat{\mathbf{y}}_i(t))$  then
                         $S_k \cdot \hat{\mathbf{y}}_i(t+1) = S_k \cdot \mathbf{x}_i(t)$ ;
                Update the archive with solution  $S_k \cdot \mathbf{x}_i(t)$  using Algorithm 3;
            for each objective  $k = 1, \dots, m$  do
                for each particle  $i = 1, \dots, S_k \cdot n_{s_k}$  do
                    Select a solution,  $S_k \cdot \hat{\mathbf{a}}_i(t)$ , from the archive using tournament se-
                    lection;
                    Update particle  $S_k \cdot i$ 's velocity using equation (2.7);
                    Update particle  $S_k \cdot i$ 's position using equation (2.4);
         $t = t + 1;$ 

```

2.3.2 Archive Management

The original MGPSO was implemented by setting the archive's maximum capacity to the number of particles within the search space. However, any archive size - even unbounded, can be used. This number does not change over time. Once a non-dominated solution is found, and it is not dominated by any other solution in the archive using the dominance relation, it is inserted into the archive. If any solutions in the archive became dominated by the newly inserted solution, then they are removed. If the archive is full, crowding distance is used to identify the most crowded solution in the archive [41]. The most crowded solution is then removed due to the fact that the surrounding non-

dominated solutions are similar. Algorithm 3 summarises the insert policy.

Algorithm 3 Archive Insert Policy

Let A represent the archive;

Let n_A represent the archive size;

Let c represent the maximum capacity of the archive Let $\mathbf{y}$ be the solution to insert into the archive;

if for all solutions $\mathbf{x}_i, \dots, \mathbf{x}_{n_A}$ in A there is no solution that dominates $\mathbf{y}$ **then**

if $n_A + 1 \leq c$ **then**

 Insert into A ;

else

 Delete most crowded solution;

 Insert into A ;

Remove any solutions in A that are dominated by $\mathbf{y}$;

2.3.3 Stability Conditions

Scheepers *et al.* derived theoretical stability conditions on the MGPSO control parameters to guarantee order-1 and order-2 stability [152]:

$$0 < c_1 + \lambda c_2 + (1 - \lambda)c_3 < \frac{4(1-w^2)}{1-w + \frac{(c_1^2 + \lambda^2 c_2^2 + (1-\lambda)^2 c_3^2)(1+w)}{3(c_1 + \lambda c_2 + (1-\lambda)c_3)^2}}, |w| < 1 \quad (2.8)$$

If control parameter values satisfy the above stability conditions, then the swarm is guaranteed to reach an equilibrium state.

2.4 Set-based Particle Swarm Optimization

PSO and MGPSO were designed to solve continuous-valued optimization problems. However, there are many real-world optimization problems that do not have continuous-valued decision variables, e.g. feature selection problems, assignment problems and scheduling problems. The set-based PSO (SBPSO) algorithm combines PSO and set theory to find solutions to combinatorial optimization problems where solutions can be represented as sets [104]. SBPSO uses sets to represent particle positions, which allows for positions (i.e. solutions) of varying size. This aids in addressing the curse of dimensionality problem.

Section 2.4.1 discusses the motivation for a generally applicable, functioning set-based PSO algorithm and introduces SBPSO. The set-based operators used to implement SBPSO are described in Section 2.4.2, and Section 2.4.3 describes the SBPSO search process.

2.4.1 Introduction

Several variations of PSO have been developed to solve discrete-valued optimization problems. The first such algorithm was proposed by Kennedy and Eberhart, namely binary particle swarm optimization (BPSO) [95]. A particle's position is represented as a binary string, while its velocity exists in continuous space. Velocities are transformed to respect the probability that a decision variable is either 1 or 0. Shen *et al.* proposed a modified binary particle swarm optimization (MBPSO) algorithm which introduced velocity update rules designed to increase the algorithm's exploitation ability at the beginning of the search as well as the exploration of the local area around a particle's position [25]. To avoid convergence to a local optimum, a fraction (that decreases over time) of the swarm was randomly reinitialized. Wang *et al.* proposed probability binary particle swarm optimization (PBPSO), which used a linear function to transform particles' positions to bit-valued vectors [179]. Chuang *et al.* introduced the catfish particle swarm optimization (CFPSO) algorithm, which was inspired by the practice of Norwegian fishermen to introduce catfish predators into tanks of captured sardines in order keep the sardines moving [30]. CFPSO replaces the worst performing particles in the swarm with new randomly generated particles, provided that the global best position has not changed for a given time. In a later publication, the authors applied the catfish effect to BPSO, creating a new algorithm referred to as catfish binary particle swarm optimization (CFBPSO) [31].

PSO algorithms using sets have also been previously proposed for discrete optimization problems. However, these algorithms are problem-specific, for example the algorithms proposed by Correa *et al.* [39], and Bock and Hettenhausen [13]. A generic set-based implementation of PSO was proposed by Veenhuis, where the positions and velocities of particles were represented as sets [177]. The algorithm used a clustering mechanism to avoid the particle positions from becoming too large, requiring a set-based distance metric. However, mathematical sets do not support the concept of distance as they lack spatial structure. Thus, Veenhuis used a problem-specific distance measure, erasing generality of the algorithm. Neethling and Engelbrecht proposed SetPSO for RNA structure prediction [127]. The algorithm represented a particle's position as a set and used probabilities to determine which elements to add to or remove from a particle. However, SetPSO was shown to perform worse for the RNA problem than other PSO approaches to discrete optimization problems, and could not be considered generally applicable [105]. Khan and Engelbrecht proposed fuzzy particle swarm optimization (FPSO) to optimize the topology design of distributed local area networks [97]. The algorithm used a fuzzy operator, the *unified And-Or operator*, to aggregate the multiple objectives in topology design. However, the fixed size of the FPSO particle positions meant that FPSO was not generally applicable.

Thus, there was a clear need for a truly problem-independent set-based

PSO algorithm. Langeveld and Engelbrecht created a generally applicable, functioning set-based PSO, namely set-based particle swarm optimization (SBPSO) [104]. Research has shown that SBPSO is a highly competitive algorithm for discrete optimization problems, such as the multi-dimensional knapsack problem and the feature selection problem [104], [105]. Furthermore, SBPSO has successfully been applied to training support vector machines [128] and polynomial approximation [196].

2.4.2 Set-based Operators

Formally, a set-particle's position, X_i , is defined as $X_i \in \mathcal{P}(U)$, where $\mathcal{P}$ is the power set of U and U is the universe of all elements with regard to a specific problem domain. A set-particle's velocity is represented as a set of operator pairs that either add or remove elements to or from a set-particle's position. An operation pair is denoted as $(\pm, e)$, where $(+, e)$ denotes the addition of element $e \in U$ to a particle position, and $(-, e)$ represents the removal of element e . The following set-operators are used to implement the velocity and position update equations of SBPSO:

Addition of two velocities, $V_1 \oplus V_2$, is implemented as the union of the two sets of operation pairs:

$$\begin{aligned} \oplus : \mathcal{P}(\{+, -\} \times U)^2 &\rightarrow \mathcal{P}(\{+, -\} \times U) \\ V_1 \oplus V_2 &= V_1 \cup V_2 \end{aligned} \quad (2.9)$$

Difference between two positions, $X_1 \ominus X_2$, is defined as a set of operation pairs that indicate the steps required to convert X_2 into X_1 :

$$\begin{aligned} \ominus : \mathcal{P}(U)^2 &\rightarrow \mathcal{P}(\{+, -\} \times U) \\ X_1 \ominus X_2 &= (\{+\} \times (X_1 \setminus X_2)) \cup (\{-\} \times (X_2 \setminus X_1)) \end{aligned} \quad (2.10)$$

Multiplication of a velocity by a scalar is implemented as

$$\begin{aligned} \otimes : [0, 1] \times \mathcal{P}(\{+, -\} \times U) &\rightarrow \mathcal{P}(\{+, -\} \times U) \\ \eta \otimes V &= B \subseteq V \end{aligned} \quad (2.11)$$

where B is a set of $\lfloor \eta \times |V| \rfloor$ elements randomly selected from V .

Addition of a velocity and a position is defined as the application of the velocity function V to the position X :

$$\begin{aligned} X \boxplus V : \mathcal{P}(U) \times \mathcal{P}(\{+, -\} \times U) &\rightarrow \mathcal{P}(U) \\ X \boxplus V &= V(X) \end{aligned} \quad (2.12)$$

Removal operator:

$$\begin{aligned}\odot^- : [0, |S|] \times \mathcal{P}(U) &\rightarrow \mathcal{P}(\{+, -\} \times U) \\ \beta \odot^- S &= \{-\} \times \left(\frac{N_{\beta, S}}{|S|} \otimes S \right)\end{aligned}\quad (2.13)$$

where S is shorthand for $X(t) \cup Y(t) \cup \hat{Y}(t)$, and $Y(t)$ and $\hat{Y}(t)$ are the cognitive and social guides, respectively. $N_{\beta, S}$ is defined as

$$N_{\beta, S} = \min \{|X|, \lfloor \beta \rfloor + \mathbb{I}_\beta\} \quad (2.14)$$

with

$$\mathbb{I}_\beta = \begin{cases} 1 & \text{if } r < \beta - \lfloor \beta \rfloor \\ 0 & \text{otherwise} \end{cases} \quad (2.15)$$

where β is an element of $\mathbb{R}^+$ and r is a uniformly randomly sampled value from $(0, 1)$.

Addition operator to facilitate exploration:

$$\begin{aligned}\odot^+ : [0, |A|] \times \mathcal{P}(U) &\rightarrow \mathcal{P}(\{+, -\} \times U) \\ \beta \odot_\tau^+ A &= \{+\} \times \tau\text{-Tournament Selection}(A, N_{\beta, A})\end{aligned}\quad (2.16)$$

where A is shorthand for $U \setminus (X(t) \cup Y(t) \cup \hat{Y}(t))$ and τ is the number of elements in the tournament. The tournament selection is repeated $N_{\beta, A}$ times. Thus, $N_{\beta, A}$ elements are added to $X(t)$. The element that yields the best objective function value is selected as the winner of the tournament.

2.4.3 Search Process

SBPSO uses the set-operators defined above to move particles through the search space. Formally, the velocity update of SBPSO is expressed as

$$\begin{aligned}V_i(t+1) &= c_1 r_1(t) \otimes (Y_i(t) \ominus X_i(t)) \\ &\oplus c_2 r_2(t) \otimes (\hat{Y}_i(t) \ominus X_i(t)) \\ &\oplus c_3 r_3(t) \odot_k^+ A_i(t) \\ &\oplus c_4 r_4(t) \odot^- S_i(t)\end{aligned}\quad (2.17)$$

where r_1 , r_2 , r_3 , and r_4 are random values, each sampled from a uniform distribution in the range $[0, 1]$; c_1 and c_2 are positive constants referred to as the cognitive and social acceleration coefficients, respectively; and c_3 and c_4 are positive acceleration constants for the addition and removal of elements, respectively. The position update is then the application of the velocity to the position, as shown in the following equation:

$$X_i(t+1) = X_i(t) \boxplus V_i(t+1) \quad (2.18)$$

Algorithm 4 contains pseudo code for SBPSO.

Algorithm 4 Set-based Particle Swarm Optimization

```

 $t = 0$ ;
Let  $f$  be the objective function;
Create and initialize a swarm,  $S$ , of  $n$  particles uniformly from set universe  $U$ ;
for each particle  $i = 1, \dots, n$  do
    Let  $Y_i$  represent the personal best position of particle  $X_i$ , initialized to  $X_i(t)$ ;
    Let  $\hat{Y}_i$  represent the neighbourhood best position of particle  $X_i$ , initialized to the best  $X_i$  in  $i$ 's neighbourhood;
    Initialize  $V_i(t)$  to the empty set;
while stopping condition is not true do
    for each particle  $i = 1, \dots, n$  do
        if  $f(X_i(t)) < f(Y_i(t))$  then
             $Y_i(t+1) = X_i(t)$ ;
        for particles  $\hat{i}$  with particle  $i$  in their neighbourhood do
            if  $f(X_i(t)) < f(\hat{Y}_i(t))$  then
                 $\hat{Y}_i(t+1) = X_i(t)$ ;
    for each particle  $i = 1, \dots, n$  do
        Update particle  $i$ 's velocity using equation (2.17);
        Update particle  $i$ 's position using equation (2.18);
     $t = t + 1$ ;

```

2.5 Summary

This chapter gave formal definitions of the optimization problems considered in this thesis. Furthermore, the necessary background on particle swarm optimization (PSO), multi-guide particle swarm optimization (MGPSO) and set-based particle swarm optimization (SBPSO) was also given. PSO and SBPSO are used in Chapters 4 and 6 to find solutions to the single-objective portfolio optimization problem. Multi-objective portfolio optimization is considered in Chapter 7, where MGPSO and SBPSO are used to obtain sets of non-dominated solutions. An investigation into the performance of MGPSO using random control parameter values that satisfy stability conditions in comparison with optimized control parameters is also undertaken in Chapter 7.

Chapter 3

Portfolio Optimization

Portfolio optimization is the process of determining which assets to invest in and how much should be invested in each asset, while considering various constraints and competing objectives. Due to the complexity of this problem, researchers have turned to meta-heuristic approaches, like evolutionary and swarm intelligence, for their simplicity and ability to approximate optimal solutions efficiently. This chapter¹ provides a comprehensive review of over 100 publications that have applied evolutionary and swarm intelligence algorithms to portfolio optimization. The publications discussed span over decades, from 1993 to 2019, and are categorized by whether constraints are considered, as well as if a single or multi-objective algorithm was used. This also includes a variety of different sub-classifications of evolutionary and swarm intelligence algorithms. Moreover, all publications considered portfolio optimization as a static optimization problem. Other characteristics of the publications such as commonly considered constraints, unique constraints, objectives, portfolio models, benchmarks and performance measures are also given.

The remainder of this chapter is organized as follows: Section 3.1 familiarises the reader with portfolio optimization and its concepts. Section 3.2 introduces the evolutionary and swarm intelligence algorithms found in the reviewed publications. Sections 3.3 and 3.4 discuss unconstrained and constrained portfolio optimization, respectively. The reviewed literature is analyzed in Section 3.5. Section 3.6 summarises the findings of the analysis and concludes the chapter.

3.1 Introduction

Investing is an effective way to put money to work and potentially build wealth. An investment is the exchange of currently available resources for assets in the expectation of a future gain. However, investing does not guarantee that one's

¹This chapter is currently being reviewed for the journal *Applied Soft Computing* under the title "Evolutionary and swarm intelligence algorithms for portfolio optimization" [58].

wealth will simply accumulate over time since the value of an investment often rises and falls. It is near impossible to predict exactly when these rises and falls in value will occur and as such, there is a level of risk associated with each asset. Formally, risk is the degree of uncertainty regarding the evaluation of an asset either increasing or decreasing in value. Naturally, the riskiness of an asset will change over time as it is affected by ongoing internal and external factors, such as unfavorable news and bear markets, respectively [68]. The risk, in some cases, might be worthwhile as an asset's value increases, exceeding expectations. It is very common for investors to have a stake in multiple assets. These assets are collectively referred to as a portfolio. Investors are, thus, continuously tasked with making investment decisions that maximize the profitability of their portfolio while simultaneously minimizing its risk exposure.

Risk can be mitigated through the strategic diversification of a portfolio through the selection of uncorrelated assets [68], [69]. The premise of the strategy is that the riskiness of an asset imposed on the portfolio can be negated by selecting an unrelated asset. For example, if one asset experiences losses, the gains of another will compensate. As the range of assets within a portfolio widens, so does the effectiveness of diversification. However, the riskiness and profitability of a portfolio are not set in stone, and as time passes these aspects of the portfolio will shift. For example, an asset in what was an equally weighted portfolio may have increased significantly, resulting in the other assets contributing less to the overall value of the portfolio. An investor will then have to re-balance the portfolio in order to realign it with their goals.

Choosing which assets to include in a portfolio that will result in an optimal trade-off between return and risk is a non-trivial task. This section presents the necessary background for the work presented in this chapter as well as for concepts discussed in other chapters. Section 3.1.1 introduces one of the most prominent portfolio optimization models, the mean-variance model. Additional constraints and objectives are presented in Section 3.1.2 and Section 3.1.3, respectively. Section 3.1.4 briefly discusses alternative portfolio optimization models.

3.1.1 History

In 1952, Markowitz, one of the founders of modern portfolio theory, presented a portfolio optimization model, known as the mean-variance model, in an effort to construct portfolios in such a way that the total profit of the portfolio is maximized subject to an acceptable level of risk [115], [164]. Markowitz's model would go on to earn him a Nobel Prize in Economic Sciences. Mean-variance is used to determine a portfolio that represents an optimal trade-off between the risk and returns. The quadratic programming model is formulated as follows:

$$\text{minimize } \lambda \bar{\sigma} - (1 - \lambda)R \quad (3.1)$$

where $\bar{\sigma}$ is the total risk of the portfolio, R is the total return of the portfolio, and λ represents an investor's risk tolerance in the range $[0, 1]$. Return is maximized irrespective of the risk involved when λ is equal to zero. A λ value of one minimizes risk regardless of profits. An explicit trade-off between risk and return is represented by a λ value less than 0 and greater than 1. A diverse set of optimal portfolios can, therefore, be obtained by solving equation (3.1) for varying values of λ . Risk is calculated using

$$\bar{\sigma} = \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij} \quad (3.2)$$

where n is the number of assets, w_i and w_j is a weighting of assets i and j , respectively, and σ_{ij} is the covariance between assets i and j . R is calculated using

$$R = \sum_{i=1}^n R_i w_i \quad (3.3)$$

where R_i is the return of asset i . Equation (3.1) requires that the total budget of the portfolio is distributed according to each asset. This constraint, referred to as the budget constraint, is expressed as

$$\sum_{i=1}^n w_i = 1 \quad (3.4)$$

Furthermore, the weight of each asset must be non-negative, i.e.

$$w_i \geq 0 \quad (3.5)$$

3.1.2 Constraints

The traditional mean-variance model does not take into account many of the constraints that portfolio managers must adhere to. As a result, many constrained adaptations have been proposed. These constraints significantly increase the complexity of portfolio management. The most commonly considered constraints for portfolio optimization are discussed below.

Cardinality restrains a portfolio by limiting the total number of assets included. The cardinality constraint is given as

$$\sum_{i=1}^n z_i = K \quad (3.6)$$

where z_i is equal to 1 if asset i is included in the portfolio, and K represents the desired number of assets to be included [18]. Alternatively, the constraint can be represented as an inequality, i.e.

$$K_{min} \leq \sum_{i=1}^n z_i \leq K_{max} \quad (3.7)$$

where K_{min} and K_{max} represent the minimum and maximum number of assets, respectively [26], [27], [42], [108], [122], [158].

Floor and ceiling, also known as the boundary constraints, represent the upper and lower bounds on the asset weights, respectively. This guarantees that an asset does not become too risky by over-representation or insignificant, by under-representation. The floor and ceiling constraints are defined as

$$0 \leq \varepsilon \leq w_i \leq \delta \leq 1 \quad (3.8)$$

where ε and δ are the upper and lower bounds, respectively [18]. When floor, ceiling and cardinality constraints are imposed on equation (3.1), it is referred to as cardinality-constrained portfolio optimization [18].

The minimum turnover constraint, which ensures that a portfolio must meet a given level of profitability, is enforced as

$$\sum_{i=1}^n w_i R_i \geq R^* \quad (3.9)$$

where R^* is the minimum turnover [19], [151], [159], [183].

Liquidity represents how easily an asset can be converted into other types of equity, such as cash. The total liquidity of a portfolio can be constrained to meet a minimum liquidity threshold. The liquidity constraint is defined as

$$\sum_{i=1}^n w_i e_i \geq e^* \quad (3.10)$$

where e^* is the minimum amount of liquidity and e_i is the liquidity of asset i [60], [131].

Transaction lots refer to the number of units of an asset that can be traded at a time. The number of units held of an asset must be a multiple of the specified transaction lot. Thus, the contribution weight of an asset is determined by the number of lots for an asset held in proportion the total number of lots held. Formally, the constraint is defined as

$$w_i = \frac{l_i q_i}{\sum_{j=1}^n l_j q_j} \quad (3.11)$$

where l and q are the transaction lot and quantity of lots held, respectively, for assets i and j [67], [70], [159].

Transaction costs are paid to an entity, such as an exchange, when buying or selling an asset. Transaction costs typically occur when rebalancing a

portfolio. Transaction costs are imposed as

$$\sum_{i=1}^n w_i c_i \leq B \quad (3.12)$$

where c_i is the cost of asset i and B is the budget [23], [24], [28], [70], [147], [159].

Capitalization refers to the total market value of a company's outstanding shares. Thus, sector capitalization is the capitalization of a sector, where a sector represents a part of the economy containing assets which offer similar goods or services. The sector capitalization constraint ensures that if an asset or assets representing a sector is included, then that sector should occupy a portion of the portfolio larger than a sector with a lower capitalization ranking. Formally, the constraint is defined as

$$\sum_{i=1}^{n_{s'}} w_{s',i} < \sum_{i=1}^{n_s} w_{s,i} \quad (3.13)$$

where $n_{s'}$ and n_s are the number of assets in sectors s' and s , respectively, and $w_{s',i}$ and $w_{s,i}$ are the asset weights of sectors s' and s , respectively. [67], [159].

3.1.3 Objectives

Many extensions to the mean-variance model have been proposed since its initial outline. Often these extensions include constraints, but also objectives. For example, maximizing the diversity of the types of assets included in the portfolio. Furthermore, the previously mentioned constraints can be reformulated as objectives, such as minimizing cardinality [2] and maximizing liquidity [192]. Portfolio optimization can also be viewed as a multi-objective optimization problem. Quite simply, equation (3.1) can be reformulated where $\bar{\sigma}$ and R are opposing objectives to be minimized and maximized, respectively. Thus, a set of non-dominated solutions can be obtained as opposed to optimizing for a given λ value.

3.1.4 Alternative models

Despite the success of the mean-variance model, it is not without its flaws. It has been criticized as being unrealistic, often leading to unsatisfactory predictions of investor behaviour [164], particularly in cases where the distribution of returns is asymmetric [66], [164]. This has led to the formulation of alternative portfolio optimization models. Often these alternative models make use of different definitions of $\bar{\sigma}$ and R or completely different objectives. One such approach by Markowitz is to use semi-variance, which measures the downside part of the return distribution as risk [164]. Semi-variance has been regarded

as a more appropriate $\bar{\sigma}$ measure than covariance, in the sense that it reflects an investor's intuition about risk. Using semi-variance, $\bar{\sigma}$ is redefined as

$$\bar{\sigma} = \frac{1}{n} \sum_{i=1, R_i < C}^n (R_i - C)^2 \quad (3.14)$$

where C is a benchmark, i.e. the average return over a period of time. However, deciding which benchmark to use is a common problem as different benchmarks could lead to different outcomes in results. R is then formulated as

$$R = \sum_{t=1}^T \sum_{i=1}^n w_i \frac{R_i}{T} \quad (3.15)$$

where T is the time period considered. Following equation (3.1), these definitions are used to create the mean-semivariance model. Other portfolio optimization models include, but is not limited to, value-at-risk (VaR) [108], [116], conditional value-at-risk (CVaR) [108], [116], prospect theory [110], [129], Sharpe ratio [5], [194], and mean absolute deviation (MAD) [19].

3.2 Evolutionary and Swarm Intelligence Algorithms

Evolutionary and swarm intelligence algorithms are stochastic population-based search algorithms inspired by naturally occurring phenomena. Many adaptations of such algorithms have been applied to single and multi-objective portfolio optimization. The original algorithms which inspired these adaptations are presented below to convey a general understanding of the ideas expressed by each algorithm. Section 3.2.1 discusses evolutionary algorithms (EAs) and 3.2.2 discusses swarm intelligence (SI).

3.2.1 Evolutionary Algorithms

Biological evolution is the optimization of an organism's ability to survive in dynamically changing and competitive environments [52]. EAs are inspired by this process and mimic its characteristics in order to seek global optima for computational optimization problems. The survival strength of individuals within a population is determined by the objective function to be optimized. Individuals are then selected based on their survival strength for reproduction, which produces offspring for the next generation of individuals. The following algorithms belong to the EA paradigm:

Differential evolution (DE) is an EA that uses distance and direction information from the current population to guide the search process [161].

Genetic algorithms (GAs) are inspired by Darwin's theory of natural selection [52], [85]. GAs simulate genetic evolution by representing individuals as collections of genotypes, where each genotype corresponds to a characteristic of the objective function.

3.2.2 Swarm Intelligence

SI is defined as the cooperation of individuals within a group to achieve an objective by exchanging locally available information [52]. PSO and its adaptations seen in Chapter 2 are examples of SI algorithms. The following are also SI based algorithms:

Ant colony optimization (ACO) refers to a collection of SI algorithms that model the foraging behaviour of ants [47]. Ants communicate indirectly with each other by releasing pheromones that slowly evaporate. Over time, all ants follow the trail/path with highest pheromone concentration.

Artificial bee colony (ABC) is an optimization algorithm based on the foraging behaviour of honey bees [92]. The colony is guided towards global optima based on the quantity of nectar found at various food sources located throughout the search space. Three types of artificial bees are used to perform the search process, each type having a particular role. Employed bees exploit the aforementioned food sources and share this information with the colony via a waggle dance. Onlooker bees watch this waggle dance and adjust their direction according to the quality of nectar. When all known food sources have been adequately exploited, scout bees search for new sources of nectar.

Bacterial foraging algorithm (BFA) mimics the foraging behaviour of *Escherichia coli* (*E.coli*) by implementing chemotaxis, reproduction, swarming and elimination-dispersal processes to search for optima [111]. *E.coli* are cooperative-foraging bacteria present in human intestines that form highly structured colonies around nutrient-rich environments. Furthermore, *E.coli* abandon nutrient-poor regions and actively avoid harm.

Bat algorithm (BA) mimics the foraging behaviour of bats, which use echolocation for avoidance of obstacles and detection of potential prey [186].

Cat swarm optimization (CSO) models the hunting behaviour of felines to find optimal solutions [29]. During the search, cats switch between two modes, seeking and tracing. Cats in seeking mode rest while analyzing their environment for the next best move. Tracing cats move in the direction of their velocity and are influenced by the position of their leader.

Chemical reaction optimization (CRO) models the phenomenon in which

products produced by the microscopic process of molecules undergoing chemical reaction are always more stable than the reactants [103].

Firefly algorithm (FA) models the flashing-based interactions between fireflies to seek global optima [187]. The following principles are used to guide the search process. Firstly, the brightness of a firefly represents the attractiveness of a solution. Secondly, a less bright firefly moves towards a brighter one. Lastly, brightness increases as the distance between fireflies decreases.

Fireworks algorithm (FWA) mimics the explosion of fireworks and the subsequent sparks produced by the fireworks [168]. The quantity and amplitude of sparks produced is determined by the closeness of a firework to the worst and best location (firework) found thus far, respectively. Fireworks closer to the worst performing firework produce less sparks. Fireworks closer to the best firework produce small explosions to act as a form of local search. Furthermore, sparks are randomly generated as a means of maintaining diversity.

Krill Herd (KH) is a biologically-inspired algorithm based on the simulation of the social and foraging behaviour of krill. The movement of krill is influenced by the density of the herd, distance from food, and random diffusion [65].

Invasive weed optimization (IWO) utilizes artificial colonies of weeds, that are in competition for survival, to seek global optima. Weeds are aggressive and invasive with respect to their growth behaviour. Furthermore, weeds have been shown to be robust and capable of adapting to change in the environment. These properties form the basis of IWO [117]. The colonies move through the search space by spreading seeds, where fitter individuals produce more seeds.

3.3 Unconstrained Portfolio Optimization

Chang *et al.* introduced the concept of unconstrained portfolio optimization which considered the mean-variance model [115], [164] without any additional constraints [18]. Thus, unconstrained portfolio optimization must only adhere to the budget and positive weight constraints defined in equations (3.4) and (3.5), respectively. The term unconstrained portfolio optimization has since grown to include other portfolio models without any additional constraints. This section presents single-objective algorithms for unconstrained portfolio optimization in Section 3.3.1 and multi-objective algorithms for unconstrained multi-objective portfolio optimization in Section 3.3.2.

3.3.1 Single-objective Algorithms

Ant Colony Optimization

Zhu *et al.* showed that ACO is capable of delivering similar performance to that of PSO for the unconstrained portfolio optimization problem with varying sized portfolios [193].

Bacterial Foraging Optimization

Niu *et al.* developed a variant of BFO with a linearly decreasing chemotaxis step length that was shown to be highly competitive amongst other algorithms, namely PSO and GA [131].

Genetic Algorithms

GAs have been widely used for unconstrained portfolio optimization. Arnone *et al.* applied a multi-species GA to a semi-variance trade-off based portfolio model where each species optimized the objective function for a different λ value [7]. Shoaf and Foster showed that a GA, as well as a multi-species implementation of a GA, found more good-quality portfolios than quadratic programming [154], [155]. Chang *et al.* showed that a GA was capable of finding higher quality portfolios than simulated annealing (SA) and tabu search (TS) [18]. Buseti confirmed GA's superiority over TS, especially for larger problems [15]. The performance of the GA was further improved by utilizing a two-step procedure that identified optimal assets beforehand [17]. The choice of control parameter values for GA, specifically applied to portfolio optimization, was empirically shown to have an effect on performance [76]. Likewise, the encoding of solutions has also been shown to have an impact on the ability of heuristics to identify optimal solutions. An example of an encoding which leads to superior solutions than that of an array-based encoding is to use a tree-based representation [5].

Particle Swarm Optimization

PSO has been shown to be very competitive amongst other approaches to unconstrained portfolio optimization. Kamali showed that PSO was able to obtain higher quality portfolios in less time than a GA [88]. PSO, compared with a GA and an industry tool, proved to be a superior algorithm in finding solutions to portfolio optimization [194]. PSO was shown to outperform a GA in terms of consistency and speed of convergence towards an optimal solution for a behavioural portfolio model [129]. Liu *et al.* presented a novel adaptation of PSO that was shown to be effective in finding portfolios for a multi-period portfolio model that considered behavioural aspects of investors [110]. Chen *et al.* found that the use of velocity clamping resulted in better portfolios [23]. Niu *et al.* proposed a symbiotic multi-swarm PSO model that was shown to yield superior portfolios, both in quality and diversity, than a previously proposed multi-swarm cooperative PSO [132].

3.3.2 Multi-objective Algorithms

Ant Colony Optimization

Khalidji *et al.* proposed a multi-objective ACO algorithm that was able to obtain solutions closer to the true POF than the NSGA II [96].

Genetic Algorithms

Multi-objective GAs have also been shown to be successful approaches to unconstrained portfolio optimization. Computational analysis found the Pareto envelope-based selection algorithm (PESA) to be superior to NSGA II and SPEA2 for minimising risk (equation (3.2)) and maximizing return (equation (3.3).) [45]. Chiam *et al.* suggested an order-based representation that was shown to lead to better performance for algorithms [26].

Particle Swarm Optimization

In a comparative study of four multi-objective optimization algorithms, namely the parallel single front genetic algorithm (PSFGA), multi-objective particle swarm optimization (MOPSO), SPEA2 and NSGA-II. MOPSO found a superior and more diverse set of portfolios [119]. In contrast, PSFGA was the worst performing algorithm. Wang and Watada made improvements to the MOPSO algorithm that were shown to be advantageous in finding a diverse POF [178]. A dynamic multi-swarm PSO consistently outperformed multi-objective differential evolution (MODE), MOPSO and NSGA-II for large scale portfolio optimization that considered 200 and 500 assets [109].

3.4 Constrained Portfolio Optimization

Constrained portfolio optimization was pioneered by Chang *et al.*, laying the foundation for future work [18]. Chang *et al.* proposed the cardinality constrained portfolio problem (CCPP), in which the cardinality and the floor and ceiling constraints were introduced to the classic mean-variance problem. Thereafter, extended problems and other models that include further constraints have also been studied. Section 3.4.1 discusses results of single-objective algorithms for constrained portfolio optimization and multi-objective algorithms for constrained multi-objective portfolio optimization are discussed in Section 3.4.2.

3.4.1 Single-objective Algorithms

Ant Colony Optimization

Deng and Lin found ACO to be more robust and effective than PSO in finding solutions to CCPPs, especially for low risk investment portfolios [43].

Artificial Bee Colony

ABC has been shown to be a highly competitive algorithm for the constrained portfolio optimization problem. ABC significantly outperformed local search techniques, such as SA and TS [20]. Furthermore, experimental results by Wang *et al.* showed that ABC is capable of finding higher quality solutions than PSO [181]. Chen *et al.* proposed an improved ABC algorithm that generates scout bees by selecting previously found solutions from an efficiency frontier archive [21]. The proposed improved ABC algorithm consistently outperformed the original ABC algorithm. A global best guided ABC model, inspired by PSO, found a more diverse set of non-dominated solutions than other algorithms, namely GA, ABC and FA [66], [167]. Several techniques used by GAs were applied to ABC to produce a very competitive and efficient approach to finding optimal portfolios [87]. Hong-mei *et al.* showed that ABC obtained more portfolios in fewer iterations than GA [77]. ABC continuously outperformed FA and GA for portfolio optimization subject to many constraints [10]. A hybrid ABC algorithm, in which employed bees used a FA search procedure, was shown to be highly competitive amongst previously tested algorithms [18], [40], [172].

Bacterial Foraging Algorithm

Experimental results confirmed BFO to be the superior choice for portfolio optimization over other algorithms, namely GA, TS, SA [18] and PSO [40], especially for portfolios that considered a large number of assets [91].

Bat Algorithm

Computational results showed that BA, PSO and CSO yielded similar performance for the constrained portfolio optimization problem [89] [163]. Strumberger *et al.* showed that the performance of BA was significantly improved by introducing scout behaviour adapted from ABC.

Cat Swarm Optimization

Kamili and Riffi found CSO to be a good approach to for the constrained portfolio selection problem [90].

Differential Evolution

Lwin and Qu integrated population based incremental learning with differential evolution and found that the proposed hybrid demonstrated superior performance over the individual algorithms [114]. Pai and Michel used DE to solve a unique portfolio optimization problem that involved long and short positions inspired by an equity market neutral portfolio strategy [137]. A long position asset, the most familiar kind of asset, yields returns as its value increases. A short position, in contrast, yield returns expecting a fall in the asset's value. Thus, a combination of long and short assets aims to minimize risk exposure. Mathematically, short positions are represented as negative weights. Furthermore, Pai and Michel defined a risk budget, where the riskiness of each asset

in the portfolio had to be under a given limit. Pai and Michel also applied DE, as well as a GA, to a portfolio model based on an investment strategy known as the 130-30 strategy [139]. This strategy aims to yield greater returns by short-selling poor-performing assets, 30% of the portfolio value. Then, 130% of the portfolio value is used to purchase assets, which investors feel that would outperform the market.

Chemical Reaction Optimization

Xu *et al.* evaluated the performance of CRO for constrained portfolio optimization and proposed an improved model that found higher quality solutions than the original CRO [184].

Krill Herd

Tabu *et al.* found that KH underperformed GA and FA for constrained portfolio optimization [174].

Firefly Algorithm

Bacanin and Tabu presented several modifications to enhance the exploratory and exploitative behaviour of FA [8], [173]. The proposed FAs were shown to be very competitive amongst previously proposed GA, TS, SA [18] and PSO [40].

Fireworks Algorithm

Tabu *et al.* [9] concluded that FWA performed significantly better than GA, ABC, KH and FA. FWA found higher quality solutions in very few iterations which continued to improve throughout the search. The authors concluded that the algorithm has great potential for solving the portfolio optimization problem and that further work should be done to further evaluate the performance.

Invasive Weed Optimization

Pouya *et al.* proposed a constrained portfolio model where risk, market expectations and experts recommendations were considered as weighted objectives [144]. The authors found that, in general, IWO and PSO demonstrated similar performance.

Genetic Algorithm

Chang *et al.* compared the performance of GA, SA, and TS. They found that GA only slightly outperformed SA and TS for cardinality-constrained portfolio optimization [18]. Furthermore, GA was consistently slower than the other approaches. Ehrgott *et al.* found similar results where GA, SA, TS and local search were used to optimize a constrained portfolio model which considered five equally-weighted sub-objectives [50]. Woodside-Oriakhi *et al.* used a set-based encoding which improved the performance of a GA, SA, and

TS. Furthermore, the improved algorithms were shown to scale well to larger test problems that included up to 1318 assets [183]. Chang *et al.* demonstrated the robustness of a GA in its ability to solve a variety of cardinality-constrained problems [19]. Similarly, Soleimani *et al.* confirmed GA to be a reliable approach to optimizing small and large sized portfolios under several constraints [159]. Jalot and Thakur designed an effective GA for cardinality-constrained portfolio optimization which utilized an efficient repair mechanism [18], as well as constraint-aware crossover and mutation operators [84]. Guo *et al.* discussed the difficulties in optimizing a fuzzy multi-period portfolio selection model with different investment horizons [70]. As a solution to the complex portfolio model, the authors proposed a fuzzy-based GA that was shown to be efficient and effective. Lin provided empirical results that demonstrated the applicability of GA for various constrained portfolio models [28]. Pai and Michel developed a GA which used *k-means* clustering to eliminate the cardinality constraint [138]. Huang and Shen used a GA to demonstrate the practicality of a multi-period portfolio model with transaction costs and fuzzy returns [79]. Huang showed that the GA was capable of finding optimal portfolios for fuzzy-constrained mean-variance and semi-variance models [80]. Ackora *et al.* showed that the GA's ability to find optimal portfolios is significantly impacted by the chosen crossover operator [71]. Drezewski and Doroz developed a co-evolutionary GA for constrained portfolio optimization [48]. Yu *et al.* presented a two-stage GA which first identified good-quality assets before optimizing the asset weights [189]. The two-stage procedure significantly improved performance in determining the most valuable portfolio. Integration of quadratic programming with GAs has shown to increase performance for constrained portfolio optimization [123], [147]–[149]. Evolutionary techniques are used to satisfy constraints while quadratic programming is used to determine optimal weights. Liu and Zhang applied a GA to a heavily constrained portfolio model that defined multiple fuzzy objectives, return, risk and liquidity, as a fuzzy single-objective optimization problem [112].

Particle Swarm Optimization

A large amount of work has considered PSO as an approach to portfolio optimization. Xu *et al.* were the first to evaluate the performance of PSO for constrained portfolio optimization [23], [61]. The authors concluded that the use of velocity clamping improved performance. PSO has been shown to achieve similar performance to those of previously tested GA, SA, and TS [18], [40]. In an effort to improve performance, significant work has been done on improving the velocity control of PSO. Koshino *et al.* studied several velocity management strategies and found that the use of a combination of techniques improved performance [99]. Likewise, Chen and Ben-yan modified the exploration and exploitation behaviour of PSO which resulted in superior performance over that of GA [22]. Feng *et al.* found that the use of dynamic coefficients, particle mutation and velocity clamping made PSO a more effective

tive algorithm for finding optimal portfolios than previously tested algorithms [44]. Wang *et al.* incorporated evolutionary attributes into PSO to maintain diversity and to avoid premature convergence [180]. The Hybrid PSO was found to be very competitive, producing higher quality solutions while being computationally efficient. Yaakob and Watada found that the use of mutation and velocity control bested a GA for constrained portfolio optimization problems [185]. Chen and Zhang demonstrated the versatility of PSO in finding solutions to a fuzzy variant of the mean-variance model [24]. Work by Corazza *et al.* found PSO to be a viable approach to constrained portfolio optimization [38]. Farzi *et al.* concluded that PSO under quantum motion was better suited to constrained portfolio optimization than use of classical Newtonian movements [60]. Ni *et al.* proposed several dynamic and random topology strategies for PSO [130]. Computational results showed that topologies which progressed from small ring structures to a fully connected network resulted in higher quality portfolios. Sadigh *et al.* proposed a PSO and neural network hybrid that outperformed the individual algorithms, as well as previously evaluated GA, TS and SA strategies [18], [150]. Sub *et al.* proposed a variation of PSO which mimicked the thermal and drift motion of electrons in a conduct under an electrical field [166]. Results showed that the proposed drift-based PSO was statistically significantly better than other optimization approaches, in both computational time and quality of solutions found. The combination of PSO [191] and binary PSO [94] successfully outperformed a GA [159], especially for larger cardinality constrained portfolios. In order to prevent PSO from being trapped in local optima, Mozafari *et al.* [126] introduced SA behaviour to PSO [99] that found better portfolios than GA, TS, SA [18] and PSO. Work done by Corazza *et al.* showed PSO to be a viable approach to constrained portfolio optimization [38]. However, Salahi *et al.* found that PSO [40] was bested by a local harmony search technique, both in time and accuracy [151]. Moreover, the performance difference was most notable on problem instances with larger data sets.

3.4.2 Multi-objective Algorithms

Ant Colony Optimization

Armananzas and Lozano proposed a multi-objective ACO and greedy search hybrid which demonstrated outstanding performance over standalone greedy search [6]. Doerner *et al.* proposed an enriched Pareto ACO algorithm that was shown to outperform Pareto SA and NSGA-II for randomly generated portfolio problems, as well as a real world problem [46].

Artificial Bee Colony

Kumar and Mishra proposed a multi-objective ABC algorithm which used covariance principles to approximate the gradient of the objective function [100]. Empirical results showed that the proposed ABC algorithm was able to pro-

duce a diverse POF.

Bacterial Foraging Algorithm

Mishra *et al.* presented a multi-objective BFO algorithm which was capable of identifying a diverse POF [121], [122].

Differential Evolution

Unlike other papers, Pai and Michel defined a portfolio optimization problem that considered the futures market, which is fraught with risks [140]. This portfolio model defined three objectives to minimize exposure to risk in future bonds, indices, and currency markets. MODE was shown to successfully obtain practical solutions. In the same line of work, Pai introduced transaction costs and risk budgets to further mimic the real world [136]. Most recently, Pai examined the ongoing effects of rebalancing a portfolio as a multi-objective optimization problem [135].

Particle Swarm Optimization

Zhou and Li presented an improved MOPSO algorithm that was shown to find a diverse set of solutions close to the POF for constrained portfolio optimization which considered return, variance and fuzzy-liquidity as objectives [192]. Mishra *et al.* developed a MOPSO algorithm which used a self-regulating velocity strategy that was found to be highly competitive against evolutionary algorithms, namely NSGA-II and SPEA-2 [120]. Kaucic presented a novel hybridized MOPSO algorithm to solve the constrained portfolio optimization problem [93]. The hybridized algorithm used mutation to improve the exploratory capabilities of MOPSO. Computational results confirmed the effectiveness of introducing mutation to aid MOPSO in finding optimal portfolios. Silva *et al.* presented a comprehensive study in which an adaptive ranking MOPSO was shown to be highly competitive against past approaches, most notably outperforming previously dominant MOEAs such as SPEA2 and NSGA-II [157].

Genetic Algorithm

SPEA2 consistently outperformed other state of the art multi-objective evolutionary algorithms (MOEAs), namely vector evaluated genetic algorithm (VEGA), multi-objective genetic algorithm (MOGA), and NSGA-II, in finding solutions close to the true POF [158]. Chiam *et al.* found that an order-based representation of solutions lead to improved performance [26], [27]. A comparative study by Anagnostopoulos and Mamanis found that state of the art MOEAs, in general, are efficient and reliable strategies for cardinality-constrained portfolio optimization [3]. Specifically, PESA obtained on average solutions slightly closer to the POF, while NSGA-II and SPEA2, on average, found more diverse solutions. Similar work was performed where the cardinality constraint was treated as a third objective and represented as an

inequality [2]. Computational analysis found SPEA2 to be the best performing algorithm whereas PESA, which placed second, was the fastest approach. Branke *et al.* proposed an integrated active set MOEA, referred to as the envelope multi-objective genetic algorithm (E-MOEA) [14]. E-MOEA was shown to find significantly better frontiers than NSGA-II. A study of the state of the art MOEAs as well as GA [18] applied to cardinality constrained portfolio optimization found that all MOEAs outperformed the single-objective GA [4]. Furthermore, SPEA2 provided the most reliable performance, however, at the cost of being computationally expensive. Further experiments confirmed the ability of best performing algorithms, i.e. SPEA2 and NSGA-II, to scale to larger search spaces of 2196 assets. Deb *et al.* proposed a hybridized NSGA-II procedure which used clustering and a local search to obtain more accurate solutions [42]. The authors demonstrated the effectiveness of the algorithm and concluded that it was a reliable approach to cardinality-constrained portfolio optimization. Liagkouras significantly improved the performance of the state of the art MOEAs by using a three-dimensional encoding that made processing time invariant to the size of the portfolio problem [108]. Liagkouras and Metaxiotis proposed a mutation operator that aggressively probes the search space to guide offspring to optimal solutions [107]. The mutation operator was shown to increase the effectiveness of NSGA-II and SPEA2 for cardinality constrained portfolio optimization, however, at the expense of being computationally expensive. Meghwani and Thakur presented an extended three-objective portfolio model that considered minimizing transaction costs as an additional objective [116]. A novel repair algorithm was proposed to handle the newly introduced transactional constraints, specifically equality constraints. MOEAs with the repair algorithm outperformed their standard counterparts. Streichert and Tanaka-Yamawaki proposed a hybridization of NSGA-II and a quadratic programming local search which demonstrated improved solution quality for some problem instances, however, suffered from premature convergence on others [162]. Subbu *et al.* proposed an architectural solution to the constrained portfolio optimization problem which sequences multiple techniques to produce Pareto optimal solutions [165]. Linear programming was used to generate the initial population for PESA. The efficient frontier obtained by PESA is enhanced by a GA which identifies possible gaps in the POF. Lastly, constraints are then applied to the enhanced efficient frontier to identify realistic portfolios. It was shown that the architecture could obtain solutions to a high dimensionality portfolio optimization problem. However, the authors noted further experiments were required to prove effectiveness of the proposed architecture.

3.5 Analysis

This section analyzes the characteristics of the aforementioned papers and discusses the findings. Section 3.5.1 compares the representation of evolutionary and swarm intelligence algorithms for various portfolio optimization problems. Section 3.5.2 discusses portfolio models, objectives and constraints which appeared in the reviewed literature. Section 3.5.3 presents solution encoding strategies and related constraint handling techniques. Section 3.5.4 provides an overview of portfolio optimization benchmarks. Lastly, Section 3.5.5 describes various performance measures used to evaluate the performance of evolutionary and swarm intelligence algorithms.

3.5.1 Representation of Algorithms

A variety of evolutionary and swarm intelligence algorithms have been proposed to solve the portfolio optimization problem. Figure 3.1 shows that the interest in evolutionary and swarm intelligence algorithms as portfolio optimizers has increased over time, specifically 2000-2010. In general, 68.87% of the surveyed literature made use of EAs while 54.72% used SI algorithms. There exists an overlap of 23.5% where publications made use of algorithms belonging to both paradigms. Table 3.1 shows that both paradigms are represented similarly across unconstrained and constrained portfolio optimization as a one, two, and three-objective optimization problem.

Figure 3.1: Number of publications per year

Table 3.1: Representation of evolutionary and swarm intelligence paradigms for unconstrained and constrained portfolio optimization

n_c	m	Paradigm	%	Reference(s)
0	1	EA	14.15%	[5], [7], [15], [17], [46], [48], [75], [76], [79], [80], [88], [129], [166], [189], [194]
		SI	8.49%	[43], [46], [88], [110], [129], [132], [166], [193], [194]
		Overall	17.92%	
	2	EA	4.72%	[45], [96], [109], [119], [146]
		SI	2.83%	[96], [109], [119]
		Overall	4.72%	
≥ 1	1	EA	31.13%	[9], [10], [18], [19], [22], [28], [40], [44], [50], [67], [70]–[72], [84], [87], [91], [99], [114], [123], [138], [139], [147]–[149], [154], [155], [159], [163], [173], [174], [181], [183], [185]
		SI	35.85%	[8]–[10], [20]–[24], [38], [40], [44], [60], [61], [66], [67], [77], [87], [89]–[91], [99], [114], [126], [130], [131], [144], [150], [151], [163], [167], [172]–[174], [180], [181], [184], [185], [188]
		Overall	52.83%	
	2	EA	12.26%	[3], [4], [14], [26], [27], [42], [107], [108], [120], [122], [136], [137], [162]
		SI	5.66%	[6], [93], [100], [120], [122], [157]
		Overall	16.04%	
	3	EA	4.72%	[2], [112], [135], [140], [158]
		SI	0.94%	[192]
		Overall	5.66%	

* For convenience, n_c is the number of constraints and m is the number of objectives.

Single-objective portfolio optimization accounts for 70.75% of the discussed literature, whereas multi-objective optimization represents 29.25%. Portfolio optimization as a single-objective problem is, thus, overwhelmingly represented. It is crucial to obtain a diverse set of optimal portfolios when making informed investment decisions. Single-objective optimization algorithms converge to a single solution, requiring multiple runs to obtain a set of solutions. In contrast, multi-objective optimization algorithms obtain many solutions, and are thus better suited for portfolio optimization problems.

Table 3.2 shows that GAs are more popular than DE with respect to evolutionary approaches for portfolio optimization. Likewise, PSO is the prominent approach to portfolio optimization amongst SI algorithms, as shown in Table 3.3. GA is, overall, the most popular approach by a margin of 30.19%. However, there is a wider range of SI algorithms than EAs. Furthermore, the first GA was published in the 1950s [63], [64], making it comparatively more well-established than the relatively new SI paradigm, which was first seen in the 1990s [49]. With regards to multi-objective optimization, NSGA-II, PESA, SPEA2 and MOPSO are the most popular approaches and have been shown to obtain diverse non-dominated solutions [2]–[4], [42], [108], [120], [158]. A

Table 3.2: Evolutionary algorithms for portfolio optimization

Algorithm	m	n_c	%	Reference(s)
DE	1	2	1.89%	[87], [114]
		3	0.94%	[139]
	2	0	1.89%	[109], [146]
		2	0.94%	[136]
		3	0.94%	[137]
	3	0	0.94%	[116]
		2	1.89%	[135], [140]
	Overall		9.43%	
GA	1	0	14.15%	[5], [7], [15], [17], [46], [48], [75], [76], [79], [80], [88], [129], [166], [189], [194]
		1	3.77%	[28], [99], [154], [155]
		2	15.09%	[9], [10], [18], [22], [40], [44], [50], [84], [87], [91], [114], [163], [173], [174], [181], [185]
		3	8.49%	[19], [70], [71], [123], [139], [147]–[149], [183]
		4	1.89%	[72], [138]
		5	0.94%	[67]
		6	0.94%	[159]
	2	0	4.72%	[45], [96], [109], [119], [146]
		2	7.55%	[3], [4], [42], [107], [108], [120], [122], [162]
		3	0.94%	[14]
		4	1.89%	[26], [27]
	3	0	1.89%	[116], [165]
		2	2.83%	[2], [135], [140]
		3	0.94%	[112]
		4	0.94%	[158]
	Overall		66.98%	

* For convenience, n_c is the number of constraints and m is the number of objectives.

limitation of the current research is that the performance of other approaches is largely unexplored to the same extent as GAs and PSO. Investing in these algorithms could be a potentially fruitful area of future work since initial results are promising. Furthermore, the hybridization of these algorithms have been shown to yield superior results over their individual counterparts [93], [126], [172].

Table 3.3: Swarm intelligence algorithms for portfolio optimization

Algorithm	m	n_c	%	Reference(s)
ABC	1	1	0.94%	[77]
		2	11.32%	[9], [10], [20], [21], [66], [87], [163], [167], [172], [181]
	2	2	0.94%	[100]
	Overall		13.2%	
ACO	1	0	2.83%	[43], [46], [193]
		2	0.94%	[96]
	2	2	0.94%	[6]
	Overall		4.72%	
BA	1	2	1.89%	[89], [163]
BFO	1	1	0.94%	[131]
		2	0.94%	[91]
	2	2	0.94%	[122]
	Overall		2.83%	
CRO	1	1	0.94%	[184]
CSO	1	2	1.89%	[89], [90]
FA	1	2	6.6%	[8]–[10], [163], [172]–[174]
FWA	1	2	1.89%	[9], [163]
IWO	1	1	0.94%	[144]
KH	1	2	2.83%	[9], [163], [174]
PSO	1	0	7.55%	[43], [88], [110], [129], [132], [166], [193], [194]
		1	4.72%	[61], [99], [131], [144], [180]
		2	13.21%	[22], [23], [40], [44], [87], [89], [91], [114], [126], [130], [150], [181], [185], [188]
		3	3.77%	[24], [38], [60], [151]
		5	0.94%	[67]
	2	0	1.89%	[109], [119]
		2	0.94%	[120]
		3	1.89%	[93], [157]
	3	0	0.94%	[178]
		2	0.94%	[192]
Overall		36.79%		

* For convenience, n_c is the number of constraints and m is the number of objectives.

Although a variety of algorithms have been proposed for portfolio optimization, it is important to note that if an algorithm excels at one optimization problem, it may not necessarily be suitable for another [182]. Furthermore, deriving algorithms from natural phenomena for the sake of being new and novel may stray research away from scientific rigor [160].

3.5.2 Portfolio Models

The mean-variance model is the most widely considered portfolio model for portfolio optimization, represented in the majority of the considered work. Mean-semivariance, considered by 4.72% of publications [7], [19], [66], [80],

[108], has been studied considerably less and the remaining proportion of work, that considered alternative portfolios models and $\bar{\sigma}$ measures, even more so. Example models are VaR [108], [116], CVaR [108], [116], prospect theory [110], [129], Sharpe ratio [5], [194], and MAD [19].

Single-objective portfolio optimization related work considered a trade-off between R and $\bar{\sigma}$, as expressed in equation (3.1), as an objective function [9], [18], [60], [66], [89], [174], [184]. Minimization of $\bar{\sigma}$ or maximization of R with a minimum R or maximum $\bar{\sigma}$ constraint, respectively, has also been considered [10], [129], [154], [159], [166].

Optimization of R and $\bar{\sigma}$ are the most widely considered objectives for multi-objective portfolio optimization. Additional objectives, such as liquidity [192], future risk [165], [178], and transaction costs [116] are largely under represented yet vital when considering portfolios. Cardinality, as an additional objective to be minimized, is a novel approach to obtaining a diverse POF of differently size portfolios [2]. Similarly, diversity as an objective to be maximized has been shown to guide algorithms to more diverse solutions [158]. Table 3.1 shows that only 5.66% of papers considered portfolio optimization as a three-objective optimization problem.

Portfolio optimization as a fuzzy optimization problem has also been studied [144], [178]. Fuzzy logic is introduced to represent the uncertainty of future returns. Multi-period portfolio models consider the optimization of a portfolio at several time periods [135]. However, this is different from a dynamic problem with a changing search space in the sense that the expected returns for each time period are known up front. Practically, this shows an algorithm's ability to optimize a portfolio subject to rebalancing. Research regarding futures markets, although practically useful, is largely unexplored [136], [140]. Likewise, there is very little research based on real-world investment strategies, like the 130-30 strategy [135], risk budgeting [135]–[137], and equity market neutral portfolio strategy [137]. Current research has not yet investigated the use of evolutionary and swarm intelligence algorithms as prediction methods for future prices and then making investment decisions based on these predicted prices.

A significant amount of research has treated portfolio optimization as a constrained optimization problem. This is done by adding additional constraints to the considered portfolio model, such as those shown in Table 3.4. The floor and ceiling, and cardinality constraints of equations (3.6) and (3.8), respectively, are the most considered constraints, across all m values, as a result of the popularity of the CCPP. However, Chang *et al.* noted that single-objective optimization algorithms that optimized the CCPP would effectivity be blind to solutions that could not be obtained by varying λ . Table 3.5 shows, that in general, publications considered two constraints for constrained portfolio optimization. The popularity of these constraints is attributed to the novel work done by Chang *et al.* [18]. Considering return as a constraint is particularly useful when trying to find portfolios that would outperform competitors.

Transaction lots, transaction costs, and sector capitalization have also been studied, albeit to a lesser extent. Such constraints make portfolio optimization more realistic. Other constraints identified, categorized as pre-assignment constraints, are not related to the objective function but rather the search space. Examples of pre-assignment constraints are to remove assets from the search space if they do not meet a given level of risk, return or liquidity [14], [72], [79], [157]. Heavily constrained portfolio optimization problems have been proposed, but are not as popular. Work done by Soleimani *et al.* [159] considered minimum turnover, asset boundaries, cardinality, sector capitalization, transaction costs, and transaction lots.

Table 3.4: Commonly considered constraints for portfolio optimization

m	Constraint	%	Reference(s)
1	Floor and ceiling	40.57%	[8]–[10], [18]–[23], [38], [40], [44], [50], [60], [66], [67], [71], [84], [87], [89]–[91], [114], [123], [126], [130], [138], [139], [147]–[151], [159], [163], [167], [172]–[174], [181], [183], [185], [188]
	Cardinality	36.79%	[8], [18]–[22], [38], [40], [44], [50], [60], [66], [67], [71], [84], [87], [89]–[91], [114], [123], [126], [130], [138], [139], [147]–[151], [159], [167], [172], [173], [181], [183]–[185], [188]
	Liquidity	1.89%	[60], [131]
	Turnover	21.7%	[9], [10], [19], [24], [38], [61], [67], [70], [71], [77], [99], [138], [144], [148], [149], [151], [154], [155], [159], [163], [174], [180], [183]
	Sector capitalization	1.89%	[67], [159]
	Transaction costs	5.66%	[23], [24], [28], [70], [147], [159]
	Transaction lots	2.83%	[67], [70], [159]
	Overall	2.83%	
2	Floor and ceiling	16.04%	[3], [4], [6], [14], [26], [27], [42], [93], [100], [107], [108], [120], [122], [136], [137], [157], [162]
	Cardinality	14.15%	[3], [4], [6], [14], [26], [27], [42], [93], [100], [107], [108], [120], [122], [157], [162]
	Turnover	1.89%	[26], [27]
	Transaction lots	1.89%	[26], [27]
	Overall	1.89%	
3	Floor and ceiling	5.66%	[2], [112], [135], [140], [158], [192]
	Cardinality	2.83%	[2], [112], [158]
	Turnover	0.94%	[158]
	Transaction lots	2.83%	[112], [158], [192]
	Overall	2.83%	

* For convenience, m is the number of objectives.

Table 3.5: The number of constraints considered for portfolio optimization per number of objectives

m	n_c	%	Reference(s)
1	1	9.43%	[28], [61], [77], [99], [131], [144], [154], [155], [180], [184]
	2	27.36%	[8]–[10], [18], [20]–[23], [40], [44], [50], [66], [84], [87], [89]–[91], [114], [126], [130], [150], [163], [167], [172]–[174], [181], [185], [188]
	3	12.26%	[19], [24], [38], [60], [70], [71], [123], [139], [147]–[149], [151], [183]
	4	1.89%	[72], [138]
	5	0.94%	[67]
	6	0.94%	[159]
Overall		52.83%	
2	2	10.38%	[3], [4], [6], [42], [100], [107], [108], [120], [122], [136], [162]
	3	3.77%	[14], [93], [137], [157]
	4	1.89%	[26], [27]
Overall		16.04%	
3	2	3.77%	[2], [135], [140], [192]
	4	0.94%	[158]
Overall		5.66%	

* For convenience, n_c is the number of constraints and m is the number of objectives.

3.5.3 Solution Encoding and Quality

The encoding scheme used to represent solutions can have a direct impact on the performance of an algorithm [5], [26], [27], [108]. Furthermore, there exists a variety of methods to ensure feasibility of solutions. The most prominent encoding is a direct representation of a portfolio as a collection of asset weights. The popularity of this encoding can largely be attributed to work which considered the unconstrained portfolio problem. However, when constraints are considered and no constraint handling techniques are used, a direct representation frequently leads to infeasible solutions [154]. A simple approach to ensuring that the budget constraint of equation (3.4) is satisfied is to normalize the asset weights.

Popularized by Chang *et al.* [18], binary decision variables to indicate the inclusion of an asset in the portfolio is commonly used for handling the cardinality constraint. When there are too many or too few assets included, a decision process is then used to remove or add assets, respectively. Such an approach is to randomly select assets to either be included or excluded from the portfolio until the constraint is satisfied [188].

The most commonly used constraint handling technique for the boundary constraint is the free portfolio proportion method, originally presented by Chang *et al* [18]. Formally, the technique is described as follows:

$$\begin{aligned}
 F &= 1 - \sum_{i=1}^n \varepsilon_i \\
 w_i &= \varepsilon_i + \frac{w_i}{\sum_{i=1}^n w_i} F
 \end{aligned} \tag{3.16}$$

where F is the free remaining proportion of the portfolio, and w_i is the portion of asset i . So far, this satisfies the lower boundary and budget constraints. An iterative procedure that recalculates the free portion, based on the process above, is performed on portions which violate the upper boundary constraint, if any. Furthermore, by initializing individuals of the initial population within $[\varepsilon, \delta]$, the population is guaranteed to satisfy the boundary constraint.

In a constrained search space, optima may exist separated or surrounded by infeasible regions. Under these circumstances, individuals will be discouraged by the infeasible regions from finding the aforementioned optima. Thus, to allow individuals to fully explore the search space, individuals may be allowed to move through infeasible regions subject to a given tolerance threshold [8]–[10], [87], [163], [172], [174]. Individuals that violate constraints or have exceeded a tolerance threshold can be *removed* by re-initialization of the individual within the bounds of the search space. A similar but more relaxed approach is to allow individuals to violate constraints. However, such individuals are penalized on how severely the constraints are violated [22], [23], [66], [67], [120], [144].

k-means clustering has been used to group similar performing assets into k groups [138]. The objective function is then optimized by finding optimal combinations of assets from each cluster. This strategy can be used to with the cardinality constraint where the number of groups is equal to the cardinality constraint. However, the quality of solutions found is dependent on the clusters identified. For example, high performing assets could be grouped into a single cluster, thus requiring less desirable assets to be included in the portfolio.

A hybrid encoding of asset weights and indices has been used to represent solutions to the cardinality constrained portfolio optimization problem [15], [20], [21], [60], [159]. This approach works well for problems that consider the cardinality constraint as an equality. By setting the size of a solution equal to K , the cardinality constraint is eliminated. Furthermore, the algorithm benefits from computational efficiency as the encoding is invariant to the size of the portfolio problem and the algorithm only needs to evaluate a subset of assets. If the cardinality constraint is treated as an inequality, a third vector of decision variables, representing the inclusion of assets in the portfolio, is introduced [108]. In this encoding strategy, the size of each vector is bound to the cardinality bounds. Thus, the complexity of the cardinality constraint is significantly reduced, while being extremely computationally efficient.

An order-based variation of the asset weights and indices encoding, proposed by Chiam *et al.* [26], [27], evaluates a solution in a left to right order, similar to that of a bin packing problem. Shoaf and Foster proposed the use of portion indices to represent portfolios [154]. Set-based encodings have also been shown to be effective representations [123], [148], [183]. Lastly, a tree-based representation where leaf nodes represented the portion allocated to an asset demonstrated superior performance to that of an asset weights representation for GA [5].

3.5.4 Benchmark Problems

A variety of data sets have been used to benchmark the performance of evolutionary and swarm intelligence algorithms for portfolio optimization. A data set, for example, might contain the expected return and risk for each asset, the correlation value between each asset pair and known optimal solutions.

The OR Library² was used in approximately 37% of publications, making it the most widely used data set for portfolio optimization [11], [18]. Summarised in Table 3.6, the OR Library contains five portfolio optimization benchmark problems, each containing real-world data from different stock markets with varying numbers of assets. The benchmark problems were initially published alongside the CCPP by Chang *et al.* [18]. The benchmark problems test the scalability and accuracy of an algorithm using data from various real world stock markets. Each benchmark problem contains the following: the number of assets, the mean return, the standard deviation of returns for each asset and the correlation value for all possible pairs of assets. The covariance between the returns of assets i and j is calculated as the product of the standard deviation of assets the returns of i and j and the correlation value of i and j . A POF containing 2000 pairs of R and $\bar{\sigma}$ values is also provided for each benchmark problem.

Table 3.6: Summary of the OR Library data sets for portfolio optimization

Stock Market	Region	Number of Assets
Hang Seng	Hong Kong	31
DAX 100	Germany	85
FTSE 100	UK	89
S&P 100	USA	98
Nikkei 225	Japan	225

However, the OR Library is opinionated in the sense that it uses mean-variance to model the expected returns and risk of assets. The majority of publications used independent data sets. In some cases, data was randomly generated to evaluate algorithms' ability to scale to extremely large asset spaces, i.e. more than 1000 assets [46], [50]. It has been shown that algorithms suffer from the curse of dimensionality problem [183]. Twenty eight percent of papers considered only one market with an average of approximately 150 assets. Some considered as few as 4, 5, 7, 8 or 10 assets [9], [61], [70], [75], [77], [174]. The remaining portion of independent data sets considered, on average, three markets ranging from 70 to 350 assets. This is a possible flaw in the surveyed literature as different data sets may be easier to optimize than others resulting in a false sense of confidence in the tested algorithms. This makes the cross comparison of algorithms difficult. Furthermore, due to the lack of availability of the data sets used, work cannot successfully be reproduced.

²<http://people.brunel.ac.uk/mastjjb/jeb/orlib/portinfo.html>

3.5.5 Performance Measures

The following performance measures, presented below, have been used to evaluate the usefulness of evolutionary and swarm intelligence algorithms in the context of portfolio optimization. Naturally, statistical measures such as mean, median, minimum, maximum and standard deviation, can be applied to the results of these performance measures, where applicable.

3.5.5.1 General Measures

The simplest performance measure is to compare the fitness scores obtained by the competing algorithms, where the fitness score is defined by the given objective function. However, a more accurate approach is to compare R and $\bar{\sigma}$ values. This indicates whether an algorithm is capable of obtaining superior solutions, than a competing approach. Some benchmark problems include data from a later date to assets how the obtained portfolios would have performed. Being able to obtain good-quality solutions in an efficient manner is also important as investment decisions are often subject to time constraints, particularly in volatile markets. Therefore, the time taken for the algorithm to reach the stopping condition should be minimized. Similarly, the number of evaluations required by each algorithm to obtain a specific or best objective value can also be compared.

3.5.5.2 Pareto Optimality Related Measures

Generational distance (GD) quantifies an algorithm's ability to approximate the Pareto-optimal front from the obtained POF to the true POF [175], [176]. Formally, GD is defined as

$$GD = \frac{\sqrt{\sum_{i=1}^{|Q|} d_i^2}}{|Q|} \quad (3.17)$$

where Q is the obtained POF and d_i is the Euclidean distance between the i 'th solution in the obtained POF and the nearest solution in the true POF, Q_{true} . The GD is to be minimized.

The inverted generational distance (IGD), like GD, measures the closeness of an obtained POF to the true POF, and also the diversity of the obtained POF [37], [171]. IGD, to be minimized, is formulated as

$$IGD = \frac{\sqrt{\sum_{i=1}^{|Q_{true}|} d_i^2}}{|Q_{true}|} \quad (3.18)$$

The spacing metric, which measures the diversity of an obtained POF, is defined as

$$\bar{S} = \sqrt{\frac{1}{|Q| - 1} \sum_{i=1}^{|Q|} (\bar{d} - d_i)^2} \quad (3.19)$$

with

$$d_i = \min_{x_j \in Q \wedge j \neq i} \sum_{k=1}^n |x_{i,k} - x_{j,k}| \quad (3.20)$$

and

$$\bar{d} = \sum_{i=1}^{|Q|} \frac{d_i}{|Q|} \quad (3.21)$$

where d_i is the Euclidean distance in objective space between solution i and its nearest neighbor in Q .

The error rate, ER , is the proportion of solutions within an obtained POF that are not members of the true POF. This metric, which is to be minimized, is an indicator of how well an algorithm is able to find the true POF. ER is formally defined as

$$ER = \frac{\sum_{i=1}^{|Q|} e_i}{|Q|} \quad (3.22)$$

where e_i is a binary variable indicating solution i 's absence from the Q_{true} .

The convergence metric, C , does not require the use of the true POF. C , given two sets of non-dominated solutions A and B , indicates the percentage of solutions in B which are dominated by A . The performance measure is expressed as

$$C(A, B) = \frac{|\{b \in B | \exists a \in A : a < b\}|}{|B|} \quad (3.23)$$

Hypervolume (HV) quantifies the volume of the objective space dominated by an obtained POF bounded by a reference point [195]. HV, to be maximized, is calculated using

$$HV = volume\left(\sum_{k=1}^{|Q|} V_k\right) \quad (3.24)$$

where V_k is the hypercube constructed between the reference point and solution $q_k \in Q$.

Algorithm effort, AE , is the ratio of computational time to the number of solutions in the obtained POF. Formally, AE is expressed as

$$AE = \frac{T}{|Q|} \quad (3.25)$$

where T is the computational time to obtain Q .

The spread measure, referred to as Δ , evaluates the diversity of the obtained POF [41]. A larger Δ value indicates that the approximated Pareto front may not be evenly distributed in the objective function space, and the search may be stuck only in certain regions. The spread measure is defined as

$$\Delta = \frac{d_f + d_l + \sum_{k=1}^{|Q|} (\bar{d} - d_i)}{d_f + d_l + (|Q| - 1)\bar{d}} \quad (3.26)$$

where d_f and d_l are the Euclidean distances between the extreme points in the true POF and the extreme points in the obtained POF.

3.6 Summary

This chapter conducted a thorough review of the current state of portfolio optimization with respect to evolutionary and swarm intelligence algorithms. The necessary background on portfolio optimization concepts, such as objectives, constraints and portfolio models, was provided. Furthermore, a brief introduction to the types of algorithms examined in this chapter was also given. The discussion and analysis that followed showed that evolutionary and swarm intelligence algorithms are well-established approaches to portfolio optimization. Decades of work have produced a variety of unique and successful approaches to identifying optimal portfolios. Particularly, genetic algorithms (GAs) and particle swarm optimization (PSO) are the most popular approaches. The performance of other algorithms is largely an unexplored area, however, initial results are promising. The majority of work has considered portfolio optimization as a single-objective optimization problem, although, multi-objective approaches have shown to be capable of obtaining higher quality solutions. Constrained portfolio optimization typically included a limit on the number of assets included in the portfolio and boundaries on how much an asset can contribute to the overall portfolio. A variety of other constraints have also been studied, however, to a lesser extent. For large scale portfolio optimization problems, a degradation in performance was generally reported. No work has formulated portfolio optimization as a dynamic optimization problem, which is more indicative of real world daily trading. Furthermore, real world investing is a complex multifaceted problem, and by learning from existing investment strategies, better and more problem-specific algorithms can be designed.

Chapter 4

Set-based Particle Swarm Optimization for Portfolio Optimization

The previous chapter showed that portfolio optimization is rich with interesting and highly competitive algorithms. It also shed light on the fact that existing approaches to portfolio optimization suffer from stability issues [183]: As the number of assets under consideration increases, solution quality deteriorates and the duration of the search is longer. Redefinition of the portfolio optimization problem as a set-based optimization problem has been shown to address the aforementioned scalability issues [124], [148], [183]. A solution to the set-based portfolio optimization problem represents a subset of assets. A set-based solution can then be used to form the search space for a secondary optimization task, where the weighting of each asset is optimized. Set-based algorithms have lead to improved performance in comparison to their non-set based counterparts, although they are dependent on quadratic programming to optimize the asset weights. This chapter¹ presents a new approach to the set-based portfolio optimization problem that uses SBPSO, and unlike previously proposed set-based approaches, a meta-heuristic is used as the weight optimizer for the selected assets.

The remainder of this chapter is organised as follows: Four popular algorithms for portfolio optimization are discussed in Section 4.1. Section 4.2 presents the proposed approach to portfolio optimization and gives an example scenario. The constraint handling technique used by the algorithms is described in Section 4.3. The empirical process used to assess the proposed approach is given in Section 4.4. Section 4.5 presents and discusses the results.

¹This chapter was published in the proceedings of the ANTS conference; and in lecture notes in computer science under the title "Set-based particle swarm optimization for portfolio optimization" [58]. This chapter is also currently being reviewed for the journal Applied Soft Computing under the title "Set-based particle swarm optimization for portfolio optimization: Empirical analysis" [102].

The findings are summarised in Section 4.6.

4.1 Algorithms for Portfolio Optimization

This section discusses four state-of-the-art algorithms (identified in the previous chapter) for the mean-variance portfolio optimization problem of equation (3.1). The algorithms, ABC, FA, GA and PSO, were selected based on their popularity and competitiveness, and are used later in this Chapter to compare the performance of the proposed SBPSO approach. Sections 4.1.1, 4.1.2, 4.1.3 and 4.1.4 present ABC, FA, GA and PSO for portfolio optimization, respectively.

4.1.1 Artificial Bee Colony for Portfolio Optimization

ABC is a highly competitive and popular algorithm for portfolio optimization. This section outlines the original ABC algorithm which was first used by Chen *et al.* to optimize the mean-variance model of equation (3.1) [20]. The colony consists of three types of bees: employed bees, onlooker bees and scout bees. The bees are attracted to food sources, which are randomly initialized throughout the search space. The number of employed bees, onlooker bees and scout bees are equal to the number of food sources. Each employed bee moves towards a food source, $\mathbf{x}_i$, and determines a neighboring food source, $\mathbf{o}_i$, using

$$\mathbf{o}_i = \mathbf{x}_i + \boldsymbol{\varphi}_i(\mathbf{x}_i - \mathbf{x}_k) \quad (4.1)$$

where $\boldsymbol{\varphi}_i$ is a vector of uniform random values in the range $[-1,1]$, and $\mathbf{x}_k$ is a randomly chosen food source. Then, the employed bee selects the most abundant food source between $\mathbf{x}_i$ and $\mathbf{o}_i$, and dances. In the case of maximization problems, the amount of nectar available at each food source is determined by the objective function value. However, in the case of minimization problems, e.g. the mean-variance portfolio optimization problem, the nectar of $\mathbf{x}_i$ is determined by

$$\kappa_i = \begin{cases} \frac{1}{1+f(\mathbf{x}_i)} & \text{if } f(\mathbf{x}_i) \geq 0 \\ 1 + |f(\mathbf{x}_i)| & \text{if } f(\mathbf{x}_i) < 0 \end{cases} \quad (4.2)$$

where κ_i is the nectar of food source i , and f is the objective function.

Next, each onlooker bee chooses a food source based on a probability determined by the food source's nectar relative to all other food sources. The probability, p_i , of selecting food source $\mathbf{x}_i$ is

$$p_i = \frac{\kappa_i}{\sum_{j=1}^n \kappa_j} \quad (4.3)$$

where n is the number of food sources. The onlooker bees exploit the selected food source by visiting $\mathbf{o}_i$.

If a food source is not updated with a better solution over a fixed number of iterations, scout bees are dispatched to explore new food sources. The number of iterations is determined by the *limit* control parameter. If the new food source is better than the old food source, it replaces the old food source. The new food source is a randomly generated position within the bounds of the search space. Algorithm 5 contains pseudo code for ABC.

Algorithm 5 Artificial Bee Colony

```

 $t = 0$ ;
Let  $f$  be the objective function;
Let  $limit$  be the food source trial limit;
Let  $\mathbf{x}_{i,t}$  be the trial counter for food source  $i$ , initially zero;
Create and initialize  $n$  food sources uniformly within a predefined hypercube
of dimension  $d$ ;
while stopping condition is not true do
  for each employed bee  $i = 1, \dots, n$  do
    Create neighbor food source  $\mathbf{v}_i$  using equation (4.1);
    if  $f(\mathbf{x}_i) < f(\mathbf{o}_i)$  then
      |  $\mathbf{x}_i = \mathbf{o}_i$ 
    else
      |  $\mathbf{x}_{i,t} = \mathbf{x}_{i,t} + 1$ ;
  for each onlooker bee  $i = 1, \dots, n$  do
    Choose food source  $\mathbf{x}_i$  based on probabilities calculated using equation
    (4.3);
    Create neighbor food source  $\mathbf{o}_i$  using equation (4.1);
    if  $f(\mathbf{x}_i) < f(\mathbf{o}_i)$  then
      |  $\mathbf{x}_i = \mathbf{o}_i$ 
    else
      |  $\mathbf{x}_{i,t} = \mathbf{x}_{i,t} + 1$ ;
  for each scout bee  $i = 1, \dots, n$  do
    if  $\mathbf{x}_{i,t} > limit$  then
      | Create and initialize food source  $\mathbf{x}'_i$  uniformly within the predefined
      | hypercube;
      | if  $f(\mathbf{x}_i) < f(\mathbf{x}'_i)$  then
      | |  $\mathbf{x}_i = \mathbf{x}'_i$ ;
      | |  $\mathbf{x}_{i,t} = 0$ ;
   $t = t + 1$ 

```

4.1.2 Firefly Algorithm

Much of the work centered around FAs for portfolio optimization considered portfolio optimization as a constrained optimization problem [8], [173]. Since this thesis regards portfolio optimization as an unconstrained optimization

problem, the FA is presented without any of the previously proposed modifications to handle constraints.

Fireflies generate flashes of light by a process of bioluminescence to facilitate communication. Algorithmically, the light intensity of a firefly is implemented as the objective function value. Like ABC, a transformation function can be used to accommodate minimization problems. That transformation function is

$$I(x_i) = \begin{cases} \frac{1}{1+f(x_i)} & \text{if } f(x_i) \geq 0 \\ 1 + |f(x_i)| & \text{if } f(x_i) < 0 \end{cases} \quad (4.4)$$

where I is the light intensity of firefly x_i and f is the objective function.

The attractiveness, β , of a firefly is relative to the distance between it and the firefly looking at it. The attractiveness between firefly i and firefly j is calculated as

$$\beta(r)_{ij} = \beta_0 e^{-\gamma r_{i,j}^2} \quad (4.5)$$

where r is the Euclidean distance between firefly i and firefly j , β_0 is the attractiveness when $r = 0$ and γ is the light absorption coefficient that models light absorbed by the air. The distance between firefly i and firefly j is calculated as

$$r_{ij} = \sqrt{\sum_{k=1}^d (x_{i,k} - x_{j,k})^2} \quad (4.6)$$

where d is the number of dimensions.

A firefly moves toward another brighter firefly if the move would increase the brightness of the firefly; otherwise, the firefly stays where it is. A firefly updates its position using

$$x_i(t+1) = x_i(t) + \beta(r)_{ij}(x_j - x_i) + \alpha(g - 0.5) \quad (4.7)$$

where α controls the influence of g , a random number sampled from a uniform distribution in $[0,1]$. Algorithm 6 presents the FA for portfolio optimization.

Algorithm 6 Firefly Algorithm

```

 $t = 0$ ;
Let  $f$  be the objective function;
Create and initialize  $n$  fireflies uniformly within a predefined hypercube of
dimension  $d$ ;
while stopping condition is not true do
    for each firefly  $i = 1, \dots, n$  do
        for each firefly  $j = 1, \dots, n$  do
            if  $f(\mathbf{x}_j) < f(\mathbf{x}_i)$  then
                Move firefly  $i$  towards firefly  $j$  using equation (4.7) to create  $\mathbf{x}'_i$ ;
                if  $f(\mathbf{x}'_i) < f(\mathbf{x}_i)$  then
                     $\mathbf{x}_i = \mathbf{x}'_i$ 
        end for
    end for
     $t = t + 1$ 

```

4.1.3 Genetic Algorithm

One of the most popular GAs for portfolio optimization was that proposed by Chang *et al* [18]. The GA which was used to solve the mean-variance portfolio optimization problem is summarised in Algorithm 7. Each individual represents a candidate solution to the mean-variance portfolio optimization problem. The genes of an individual represent the asset weights. The following operators are used to evolve the candidate solutions to the portfolio optimization problem: Tournament selection is used to select parents that recombine to create new individuals. The breeding process, referred to as uniform crossover, selects genes from two parents, where each gene from either parent has an equal chance of being selected. The result of the breeding process is a new individual, referred to as an offspring. Lastly, there is a chance that a randomly chosen gene of the offspring is mutated. The severity of the mutation is determined by the control parameter, S_m .

4.1.4 Particle Swarm Optimization

PSO, specifically inertia-weighted gbest PSO (refer to Section 2.2), has been widely studied as an approach to portfolio optimization [12], [24], [38], [44], [67], [130], [132], [150], [151], [166], [193]. PSO is selected as an algorithm to compare SBPSO because of its popularity amongst the reviewed literature and its ability to obtain good-quality solutions. Algorithm 1 describes PSO, however the velocities are updated using equation 2.5

Algorithm 7 Genetic Algorithm

```

 $t = 0$ ;
Let  $f$  be the objective function;
Let  $P_m$  represent the mutation probability of 0.5;
Let  $S_m$  represent the mutation severity;
Let  $\tau$  represent the size of the tournament;
Create and initialize the population of  $n$  individuals uniformly within a pre-
defined hypercube of dimension  $d$ ;
while stopping condition is not true do
    for each individual  $i = 1, \dots, n$  do
        Select parents  $\mathbf{x}_{i_1}, \mathbf{x}_{i_2} \in P$  using  $\tau$ -tournament selection;
        Apply uniform crossover on  $\mathbf{x}_{i_1}, \mathbf{x}_{i_2}$  to create  $\mathbf{x}'_i$ ;
        Randomly sample gene  $j$  from  $\{1, \dots, d\}$ ;
        if  $U[0, 1] \leq P_m$  then
             $\mathbf{x}'_{i_j} = (1.0 - S_m) * \mathbf{x}'_{i_j}$ ;
        else
             $\mathbf{x}'_{i_j} = (1.0 + S_m) * \mathbf{x}'_{i_j}$ ;
        Evaluate  $\mathbf{x}$  using  $f$ ;
     $t = t + 1$ ;

```

4.2 A New Approach to Set-based Portfolio Optimization

The universe for the SBPSO is the set of all assets. The objective of the SBPSO is to select a subset of assets that form the optimal portfolio for the mean-variance portfolio model of equation (3.1). Each particle represents a candidate portfolio of sets, which are randomly initialized. For each of these particles, a secondary optimization problem is solved, i.e to find optimal values for the weights of the assets that minimize the mean-variance portfolio model such that the total budget constraint of equation (3.4) and the non-negative weight constraint of equation (3.5) are satisfied. Algorithm 8 presents the pseudo code for SBPSO for portfolio optimization.

Algorithm 8 Set-based Particle Swarm Optimization for Portfolio Optimization

```

 $t = 0;$ 
Let  $f$  be the function the described in Algorithm 9;
Create and initialize a swarm,  $S$ , of  $n$  particles uniformly within a predefined
set universe  $U$ ;
for each particle  $i = 1, \dots, n$  do
    Let  $Y_i$  represent the personal best position of particle  $X_i$ , initialized to
     $X_i(t)$ ;
    Let  $\hat{Y}_i$  represent the global best position, initialized to the best  $X_i$ ;
    Initialize  $V_i(t)$  to  $\{\}$ , i.e the empty set;
while stopping condition is not true do
    for each particle  $i = 1, \dots, n$  do
        if  $f(X_i(t)) < f(Y_i(t))$  then
             $Y_i(t+1) = X_i(t)$ ;
        Update  $\hat{Y}_i(t+1)$  to the best  $X_i(t)$ ;
        for each particle  $i = 1, \dots, n$  do
            Update particle  $i$ 's velocity using equation (2.17);
            Update particle  $i$ 's position using equation (2.18);
         $t = t + 1$ ;

```

The weights represent the contribution of each asset to the portfolio. In other words, the weights represent how much of the total budget should be allocated to purchasing the corresponding asset. An inertia-weighted gbest PSO (refer to Section 2.2) is used as the weight optimizer. Thus, the search space of the inertia-weighted gbest PSO is determined by the assets of the corresponding set-particle. Furthermore, the best objective function value at the end of PSO search is then assigned to the corresponding set-particle. Algorithm 9 summarizes the weight determination stage.

Algorithm 9 Weight Determination for Set-based Portfolio Optimization

```

Let  $t$  represent the current iteration;
Let  $f$  be the objective function;
Let  $X_i$  represent set-particle  $i$ ;
Minimize  $f$  using Algorithm 1 for  $t_w$  iterations with assets in  $X_i$ ;
 $t = t + t_w$ ;
Return the best objective function value, and corresponding weights, found by
Algorithm 1;

```

To better illustrate the proposed approach, consider the following example. There are 50 assets in the universe. Initially, a particle randomly selects a subset of assets, say $\{5, 12, 23, 26, 31\}$ from the universe. These assets are then

used to create a continuous search space for the inner PSO. Each dimension in the search space of the PSO represents the weight of an asset. Then the PSO optimizes the asset weights for a fixed duration. The table below illustrates example results obtained by the weight optimizer.

Assets	5	12	23	26	31
Weights	0.31	0.16	0.11	0.05	0.37

The combination of the assets and weightings is a candidate portfolio. Continuing with the example, the resulting portfolio is visualised below.

Lastly, the $\odot^+$ operator for SBPSO is re-implemented as the random selection of $N_{\beta,A}$ elements from A and not using tournament selection. The purpose of this is twofold: $\odot^+$ implemented using random selection reduces computational complexity of the proposed approach since it would require a weight optimization process for each candidate in the tournament; and removes the selection bias of the tournament selection. The effects of re-implementing the $\odot^+$ operator is further studied in Chapter 6 and it is shown that there is no effect on performance.

The proposed SBPSO is expected to perform similarly to the other algorithms for portfolio optimization mentioned in Section 4.1, because the SBPSO uses the inertia gbest PSO as the weight optimizer. Furthermore, the SBPSO reduces the complexity of portfolio optimization by optimizing over subsets of the search space at a time. In comparison, the other algorithms optimize the entire search. Thus, the SBPSO should find better quality solutions for larger portfolio optimization problems than the other algorithms.

4.3 Constraint Handling

This section details the constraint handling techniques used by the algorithms in this chapter. In the case of the SBPSO, the constraint handling techniques are applied in the asset weight optimizer PSO. A position is transformed such that any negative number is treated as zero, satisfying equation (3.5). Then, the position is normalised by dividing each weight in the position by the sum of weights to ensure that the constraint of equation (3.4) is met. Thus, the position now represents the percentages of the total value to be invested for each asset. For example, the position $(2.34, -3.12, 0.95, 1.84, 5.33)$ violates both constraints. Using the described constraint handling technique, the position then becomes $(0.22, 0.0, 0.09, 0.18, 0.51)$. For the case that all asset weights are assigned a weight of zero, a sufficiently large objective function value is given to deter the algorithm from that area of the search space.

4.4 Empirical Process

This section describes the empirical process used to assess the performance of the proposed SBPSO approach and the other algorithms mentioned in Section 4.1. Note that the same empirical process is followed in subsequent chapters. Section 4.4.1 describes the implementation of the algorithms. The benchmark problems are discussed in Section 4.4.2. Section 4.4.3 describes the parameter tuning process used to optimize the control parameters of each algorithm for portfolio optimization. The performance measures used are listed in Section 4.4.4.

4.4.1 Implementation of Algorithms

The algorithms were implemented using the Computational Intelligence library², a highly concurrent functional programming library with composable algorithmic abstractions.

4.4.2 Benchmark Problems

The OR Library³ benchmark problems, summarised in Table 3.6, were used to evaluate the ability of the algorithms to optimize the mean-variance model of equation (3.1) for 50 evenly spaced λ values in the range $[0,1]$. The resulting objective function is optimized (minimized) by finding the asset weights that strike the best balance between risk and return. For each λ value, the algorithms were given 5000 iterations to optimise the mean-variance model. For the proposed SBPSO approach, SBPSO and the weight optimizer PSO

²<https://github.com/ciren/cilib>

³<http://people.brunel.ac.uk/mastjjb/jeb/orlib/portinfo.html>

interleaved every 10 iterations. Furthermore, each algorithm had a population size of 20, or in the case of the ABC, 20 food sources.

4.4.3 Control Parameter Tuning

The control parameters of each algorithm were optimized for each benchmark problem so that the algorithms could be compared fairly. To do so, parameter sets were generated using sequences of Sobol pseudo-random numbers that spanned the parameter space each of each algorithm [62]. The parameter space for each algorithm is summarised in Table 4.1. The parameter ranges were taken from the reviewed literature [18], [20], [52], [104], [173]. For each algorithm, 128 parameters sets were evaluated. However, in the case of the PSO, only the parameter sets (of the 128 parameters sets) that satisfied the stability conditions of equation (2.6) were considered. The control parameters for the weight optimizer PSO for the SBPSO were not tuned since the portfolio problems it solves vary due to the changes in the positions of set-particles. Thus, recommended parameters for the weight optimizer PSO that satisfy stability conditions and guarantee that an equilibrium state will be reached were used. These are $w = 0.729844$ and $c_1 = c_2 = 1.496180$ [32]. The parameter sets were then ranked according to their average objective function value for each of the λ values over 30 independent runs. The parameter set with the best ranking was selected as the optimal control parameter values for each algorithm, and for each benchmark problem. The optimal control parameter values are given in Appendix D.

Table 4.1: Parameter ranges used in the tuning process

Algorithm	Parameter	Range
ABC	<i>limit</i>	[0,100]
FA	β_0	[0.00,1.00]
	γ	[0.00,10.00]
	α	[0.00,1.00]
GA	S_m	[0.01,0.50]
	τ	[2,20]
PSO	w	[0.00,1.00]
	c_1	[0.00,2.00]
	c_2	[0.00,2.00]
SBPSO	c_1	[0.00,1.00]
	c_2	[0.00,1.00]
	c_3	[0.50,5.00]
	c_4	[0.50,5.00]

4.4.4 Performance Measures

The following performance measures were used to assess the quality, and efficiency of the obtained solutions: The number of optimal solutions (N), the cardinality/size of the solutions (K), return (R), risk ($\bar{\sigma}$), GD, IGD, HV (refer to Section 3.5.5) and the time in seconds to reach the stopping condition. Time is a useful performance measure since some investment decisions need to happen quickly and frequently. The results were recorded after each iteration and averaged over the 30 independent runs. The size of the solutions was measured as the number of non-zero weighted assets. Furthermore, the best objective function value at the end of every search process for a given λ was considered the best solution, and the corresponding return and risk values formed a solution on the obtained POF. Thus, each independent run of the λ values resulted in a POF. The mean and standard deviation for each performance measure was recorded and tabulated. Furthermore, to test for any meaningful statistical difference between two algorithms, one-tailed Mann Whitney U tests with a level of significance of 95% were used. The results of the statistical significance tests were used to rank the algorithms. All performance measures were equally weighted in the ranking process.

4.5 Results

This section discusses the results of each algorithm for each benchmark problem. The results are also given in the tables and figures below. Section 4.5.1 presents the Hang Seng results. The DAX 100 results are given in Section 4.5.2. Section 4.5.3 discusses the FTSE 100 results. Section 4.5.4 examines the S&P 100 results, and the Nikkei 225 results are discussed in Section 4.5.5.

4.5.1 Hang Seng

Table 4.2 shows that, on average, ABC obtained higher return values than all other algorithms. The rankings of the statistical significance tests in Table 4.3 confirm that ABC was the best performing algorithm with regard to return. However, the profitability of ABC was at the cost of more risk, with ABC ranking last with respect to risk. On the other hand, FA obtained the worst return value, but the best value for risk. The GA, PSO and SBPSO ranked similarly to each other with regard to return and risk, and their standard deviations for return and risk were significantly smaller than those of the ABC and FA - indicating that the GA, PSO and SBPSO were more consistent in their performance. The similarity in performance of the GA, PSO and SBPSO can also be seen in the size of the portfolios they obtained. On average all three algorithms obtained portfolios that contained three assets. Conversely, the portfolios obtained by ABC were much larger with an average of 15 assets

per portfolio. With regard to the number of optimal portfolios obtained, ABC and FA obtained slightly more solutions than the other algorithms.

Table 4.2: Hang Seng results for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
ABC	$\bar{x}$	37	14	0.007848	0.002110	0.000176	0.000210	0.814630	11.04
	σ	2	2	0.000142	0.000125	0.000009	0.000000	0.000000	0.41
FA	$\bar{x}$	38	5	0.006084	0.001783	0.003959	0.000556	0.748081	38.95
	σ	4	1	0.000592	0.000268	0.000485	0.000085	0.028972	3.71
GA	$\bar{x}$	32	3	0.007576	0.001884	0.000279	0.000222	0.779399	18.10
	σ	1	0	0.000052	0.000042	0.000065	0.000005	0.025320	0.64
PSO	$\bar{x}$	33	3	0.007607	0.001897	0.000219	0.000212	0.786254	15.02
	σ	1	0	0.000003	0.000001	0.000049	0.000002	0.026135	0.69
SBPSO	$\bar{x}$	34	3	0.007614	0.001908	0.000463	0.000236	0.776926	9.37
	σ	1	0	0.000022	0.000009	0.000141	0.000010	0.023747	0.53

Table 4.3: Hang Seng rankings for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time	Overall
ABC	Wins	3	0	4	0	4	4	4	3	22
	Losses	0	4	0	4	0	0	0	1	9
	Draws	1	0	0	0	0	0	0	0	1
	Difference	3	-4	4	-4	4	4	4	2	13
	Rank	1	5	1	5	1	1	1	2	1
FA	Wins	3	1	0	3	0	0	0	0	7
	Losses	0	3	4	0	4	4	4	4	23
	Draws	1	0	0	1	0	0	0	0	2
	Difference	3	-2	-4	3	-4	-4	-4	-4	-16
	Rank	1	4	4	1	5	5	5	5	5
GA	Wins	0	2	1	2	2	2	1	1	11
	Losses	4	2	3	0	2	2	2	3	18
	Draws	0	0	0	2	0	0	1	0	3
	Difference	-4	0	-2	2	0	0	-1	-2	-7
	Rank	3	3	3	2	3	3	4	4	4
PSO	Wins	1	3	2	2	3	3	2	2	18
	Losses	2	1	1	1	1	1	1	2	10
	Draws	1	0	1	1	0	0	1	0	4
	Difference	-1	2	1	1	2	2	1	0	8
	Rank	2	2	2	3	2	2	2	3	2
SBPSO	Wins	1	4	2	1	1	1	1	4	15
	Losses	2	0	1	3	3	3	1	0	13
	Draws	1	0	1	0	0	0	2	0	4
	Difference	-1	4	1	-2	-2	-2	0	4	2
	Rank	2	1	2	4	4	4	3	1	3

Figure 4.1 shows that all algorithms were able to approximate the POF, except FA. The GD, IGD and HV rankings in Table 4.3 provide further evidence that FA was unable to approximate the POF as it ranked last in all three measures. ABC performed the best out of all the algorithms with regard

to GD, IGD and HV. Furthermore, the standard deviations of ABC for the aforementioned measures were the lowest. The PSO, which obtained similar results to the ABC, ranked second for GD, IGD and HV. The GA and SBPSO significantly outperformed the FA for the same performance measures, but performed worse than the PSO.

Figure 4.1: Obtained Pareto-optimal fronts for Hang Seng

SBPSO was statistically significantly faster than all the other algorithms in reaching the stopping condition of 5000 iterations, as shown in Table 4.3. SBPSO was approximately four times faster than FA, the slowest algorithm. The second fastest algorithm was ABC, about two seconds slower than SBPSO. The PSO and GA were also quick to reach the stopping condition, despite ranking third and fourth, respectively.

The ABC was the highest ranked algorithm for the Hang Seng benchmark problem, overall. The PSO ranked second and the newly proposed SBPSO ranked third. The lowest ranking algorithms overall were the GA and FA, respectively.

4.5.2 DAX 100

Like the previous results, ABC ranked first for return but last for risk, and the FA ranked last in return but first for risk (refer to Table 4.5). Table 4.4 shows

that the profitability of the portfolios obtained by ABC were on average twice that of the FA. Conversely, the average risk value for the FA was almost half that of ABC. The PSO and GA performed similarly with regard to return and risk. The SBPSO, however, underperformed in return.

Table 4.4: DAX 100 results for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
ABC	$\bar{x}$	49	14	0.008633	0.001219	0.000228	0.000304	0.787923	43.68
	σ	1	1	0.000021	0.000029	0.000033	0.000000	0.000000	0.56
FA	$\bar{x}$	34	11	0.004485	0.000768	0.003702	0.000568	0.844798	166.26
	σ	7	2	0.000606	0.000143	0.000571	0.000103	0.033878	21.28
GA	$\bar{x}$	39	10	0.008387	0.000953	0.000295	0.000307	0.893540	50.86
	σ	1	0	0.000034	0.000025	0.000152	0.000004	0.057716	2.02
PSO	$\bar{x}$	42	5	0.008456	0.000967	0.000222	0.000304	0.816574	52.40
	σ	1	0	0.000017	0.000013	0.000053	0.000001	0.057681	1.73
SBPSO	$\bar{x}$	34	6	0.007670	0.000960	0.002334	0.000357	0.826267	26.87
	σ	5	1	0.000319	0.000105	0.000527	0.000019	0.054478	1.63

Table 4.5: DAX 100 rankings for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time	Overall
ABC	Wins	4	0	4	0	3	4	0	3	18
	Losses	0	4	0	4	0	0	4	1	13
	Draws	0	0	0	0	1	0	0	0	1
	Difference	4	-4	4	-4	3	4	-4	2	5
	Rank	1	5	1	5	1	1	5	2	2
FA	Wins	0	1	0	4	0	0	2	0	7
	Losses	3	3	4	0	4	4	1	4	23
	Draws	1	0	0	0	0	0	1	0	2
	Difference	-3	-2	-4	4	-4	-4	1	-4	-16
	Rank	4	4	5	1	4	5	2	5	4
GA	Wins	2	2	2	2	2	2	4	2	18
	Losses	2	2	2	1	2	2	0	2	13
	Draws	0	0	0	1	0	0	0	0	1
	Difference	0	0	0	1	0	0	4	0	5
	Rank	3	3	3	2	2	3	1	3	2
PSO	Wins	3	4	3	1	3	3	1	1	19
	Losses	1	0	1	2	0	1	2	3	10
	Draws	0	0	0	1	1	0	1	0	3
	Difference	2	4	2	-1	3	2	-1	-2	9
	Rank	2	1	2	4	1	2	4	4	1
SBPSO	Wins	0	3	1	1	1	1	1	4	12
	Losses	3	1	3	1	3	3	1	0	15
	Draws	1	0	0	2	0	0	2	0	5
	Difference	-3	2	-2	0	-2	-2	0	4	-3
	Rank	4	2	4	3	3	4	3	1	3

ABC found statistically significantly more unique portfolios than the other algorithms, with an average of 49 portfolios - seven more than the closest algo-

rithm (i.e. PSO). Furthermore, the portfolios obtained by ABC contained statistically significantly more assets than the other algorithms, but by a smaller margin than for the Hang Seng benchmark problem. The GA and FA found portfolios with an average of 11 and 10 assets, respectively, which was roughly twice that of SBPSO and PSO. Although SBPSO was able to find small portfolios, it did not find as many optimal portfolios as ABC, PSO and the GA. Hence, SBPSO ranked last for the number of optimal portfolios obtained alongside the FA.

ABC, PSO, the GA were able to approximate the true POF, as seen in Figure 4.2. The points on the POF obtained by PSO were more centered along the curve than the points obtained by ABC and the GA, which were more evenly distributed by comparison. Nonetheless, the PSO was as competitive as ABC, and better than the GA, with regard to GD. Furthermore, the PSO ranked second in the IGD measure, outperforming the GA but losing to ABC. The FA failed to approximate the true POF as indicated by its scattered POF and low rankings for GD and IGD. The SBPSO was better than the FA at approximating the true POF, but was not as good as ABC, PSO and the GA. Furthermore, the average GD scores for FA and SBPSO were an order of magnitude worse than the other algorithms.

Figure 4.2: Obtained Pareto-optimal fronts for DAX 100

SBPSO reached the stopping condition in 26.87 seconds, which was statis-

tically significantly faster than the other algorithms. The ABC ranked second in time and reached the stopping condition in 43.68 seconds. The PSO and GA took around 50 seconds to reach the stopping condition. Lastly, the FA was the slowest algorithm with an average time of 166.26 seconds - at least three times slower than the other algorithms.

Overall, the PSO was the highest ranked algorithm for the DAX 100 benchmark problem. The ABC and GA tied for second place, while the SBPSO and FA ranked third and last, respectively.

4.5.3 FTSE 100

The performance of SBPSO, PSO, and the GA were very similar for portfolio-related performance measures, as seen in Table 4.6. The return and risk results for the aforementioned algorithms were much the same. Furthermore, Table 4.7 shows that those algorithms ranked similarly for return and risk. ABC again ranked first for return and last for risk, and the FA ranked last for return and first for risk.

Table 4.6: FTSE 100 results for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
ABC	$\bar{x}$	49	16	0.007277	0.001073	0.000166	0.000232	0.679689	47.09
	σ	1	1	0.000011	0.000005	0.000020	0.000000	0.020228	1.02
FA	$\bar{x}$	36	12	0.004745	0.000779	0.003347	0.000467	0.714994	186.78
	σ	4	2	0.000445	0.000098	0.000356	0.000087	0.013245	25.15
GA	$\bar{x}$	29	12	0.006635	0.000778	0.000408	0.000238	0.766213	58.45
	σ	1	1	0.000163	0.000029	0.000321	0.000024	0.035444	1.29
PSO	$\bar{x}$	29	6	0.006667	0.000782	0.000190	0.000232	0.704657	59.86
	σ	0	0	0.000008	0.000001	0.000039	0.000001	0.049243	0.95
SBPSO	$\bar{x}$	33	6	0.006605	0.000822	0.001206	0.000264	0.725351	13.81
	σ	6	2	0.000176	0.000061	0.000472	0.000017	0.037266	5.43

Table 4.7: FTSE 100 rankings for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time	Overall
ABC	Wins	4	0	4	0	4	3	0	3	18
	Losses	0	4	0	4	0	0	3	1	12
	Draws	0	0	0	0	0	1	1	0	2
	Difference	4	-4	4	-4	4	3	-3	2	6
	Rank	1	4	1	3	1	1	5	2	1
FA	Wins	3	1	0	2	0	0	2	0	8
	Losses	1	2	4	0	4	4	2	4	21
	Draws	0	1	0	2	0	0	0	0	3
	Difference	2	-1	-4	2	-4	-4	0	-4	-13
	Rank	2	3	3	1	5	4	3	5	4
GA	Wins	0	1	1	2	2	2	4	2	14
	Losses	3	2	1	0	2	2	0	2	12
	Draws	1	1	2	2	0	0	0	0	6
	Difference	-3	-1	0	2	0	0	4	0	2
	Rank	4	3	2	1	3	2	1	3	3
PSO	Wins	0	3	1	2	3	3	0	1	13
	Losses	3	1	1	0	1	0	2	3	11
	Draws	1	0	2	2	0	1	2	0	8
	Difference	-3	2	0	2	2	3	-2	-2	2
	Rank	4	2	2	1	2	1	4	4	3
SBPSO	Wins	2	4	1	1	1	1	2	4	16
	Losses	2	0	1	3	3	3	1	0	13
	Draws	0	0	2	0	0	0	1	0	3
	Difference	0	4	0	-2	-2	-2	1	4	3
	Rank	3	1	2	2	4	3	2	1	2

As for the previous benchmarks, SBPSO and PSO obtained smaller portfolios than the other algorithms with fewer of the number of assets. The results show that ABC found 49 optimal portfolios on average, whereas the next best algorithm, the FA, found 13 fewer.

Figure 4.3 shows that ABC and PSO were able to approximate the true POF. The GA also performed well in approximating the true POF, but its obtained POF contains solutions that are dominated by the true POF. Further visual analysis shows that points of the POF obtained the FA were scattered and do not follow the trend of the true POF. The POF obtained by the SBPSO somewhat resembled the true POF, but it was not as close as ABC, PSO or the GA. The average GD and IGD scores for ABC were the best out of all algorithms, however, its average HV score was the worst. The poor HV score coincides with the concentration of points on the upper part of its obtained POF.

Figure 4.3: Obtained Pareto-optimal fronts for FTSE 100

The fastest algorithm to reach the stopping condition was SBPSO, with an average time of 13.81 seconds. Furthermore, the search time of SBPSO decreased from the previous benchmark problem, despite the increase in problem size. Also, the SBPSO was approximately 13 times faster than the FA, the slowest algorithm. ABC was the second fastest algorithm with an average time to reach the stopping condition of 47.09 seconds, while PSO and the GA were approximately 10 seconds slower.

For the FTSE 100 benchmark problem, ABC was the best performing algorithm overall. The proposed SBPSO approach was the second best performing algorithm, while the GA and PSO tied for third place. The FA ranked last.

4.5.4 S&P 100

Table 4.8 shows that ABC and PSO had the highest average return values for the S&P benchmark problem, with ABC narrowly beating PSO. The rankings in Table 4.9 confirm this assessment. Furthermore, the average risk value of the portfolios found by PSO was lower than that of ABC. Thus, the PSO was more competitive for this benchmark problem than ABC. The rankings and average values for return and risk show that SBPSO and the GA performed similarly. Both SBPSO and the GA were more profitable than the FA while being less risky than PSO and ABC.

Table 4.8: S&P 100 results for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
ABC	$\bar{x}$	49	24	0.007843	0.001398	0.000236	0.000241	0.841527	56.54
	σ	1	3	0.000031	0.000035	0.000031	0.000000	0.018891	1.21
FA	$\bar{x}$	38	18	0.005203	0.000891	0.003536	0.000535	0.825091	230.75
	σ	5	2	0.000691	0.000246	0.000453	0.000093	0.016919	24.39
GA	$\bar{x}$	35	6	0.007430	0.001177	0.000810	0.000260	0.881914	54.52
	σ	2	0	0.000088	0.000052	0.000265	0.000007	0.042699	1.31
PSO	$\bar{x}$	44	8	0.007705	0.001245	0.000269	0.000241	0.892730	64.55
	σ	1	0	0.000010	0.000012	0.000096	0.000001	0.052169	1.26
SBPSO	$\bar{x}$	43	9	0.007269	0.001219	0.001775	0.000301	0.890120	35.69
	σ	2	3	0.000428	0.000174	0.000620	0.000036	0.031804	5.56

Table 4.9: S&P 100 rankings for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time	Overall
ABC	Wins	4	0	4	0	3	3	1	2	17
	Losses	0	4	0	4	0	0	3	2	13
	Draws	0	0	0	0	1	1	0	0	2
	Difference	4	-4	4	-4	3	3	-2	0	4
	Rank	1	5	1	5	1	1	2	3	3
FA	Wins	1	1	0	4	0	0	0	0	6
	Losses	3	3	4	0	4	4	4	4	26
	Draws	0	0	0	0	0	0	0	0	0
	Difference	-2	-2	-4	4	-4	-4	-4	-4	-20
	Rank	4	4	4	1	4	4	3	5	5
GA	Wins	0	4	1	3	2	2	2	3	17
	Losses	4	0	2	1	2	2	0	1	12
	Draws	0	0	1	0	0	0	2	0	3
	Difference	-4	4	-1	2	0	0	2	2	5
	Rank	5	1	3	2	2	2	1	2	2
PSO	Wins	3	3	3	1	3	3	2	1	19
	Losses	1	1	1	3	0	0	0	3	9
	Draws	0	0	0	0	1	1	2	0	4
	Difference	2	2	2	-2	3	3	2	-2	10
	Rank	2	2	2	4	1	1	1	4	1
SBPSO	Wins	2	2	1	2	1	1	2	4	15
	Losses	2	2	2	2	3	3	0	0	14
	Draws	0	0	1	0	0	0	2	0	3
	Difference	0	0	-1	0	-2	-2	2	4	1
	Rank	3	3	3	3	3	3	1	1	4

The ABC and FA obtained much larger portfolios than the other algorithms with an average of 24 and 18 assets per portfolio, respectively. In comparison, the other algorithms had smaller portfolios with fewer than 10 assets. The portfolios obtained by the GA had an average of 6 assets, while the portfolios obtained by PSO and SBPSO had 8 and 9 assets, respectively. ABC found the highest number of optimal portfolios, i.e. 49, which was consistent with previous results. SBPSO and PSO found a similar number of optimal portfolios with averages of 44 and 43, respectively. The GA ranked fourth for the number

of optimal portfolios found with an average of 38. Lastly, the FA algorithm found 35 optimal portfolios, the least of all algorithms.

Both ABC and PSO ranked first in the GD and IGD performance measures. This tie for first place is consistent with the very close average GD scores between ABC and PSO, and their identical IGD scores. The PSO, however, ranked higher than ABC, with regard to the HV measure. In fact, PSO, SBPSO and the GA all ranked first for the HV measure, while ABC ranked second and the FA ranked last. SBPSO failed to approximate the POF as well as the other algorithms, except the FA, as seen in Figure 4.4. Furthermore, the low GD and IGD rankings for SBPSO quantify its inability to approximate the true POF for the S&P 100 benchmark problem. Yet again the FA produced a scattered POF that had little resemblance to the true POF.

Figure 4.4: Obtained Pareto-optimal fronts for the S&P 100 benchmark

ABC, PSO and the GA took approximately a minute to reach the stopping condition, while the FA took almost four times that. SBPSO, the fastest algorithm thus far, reached the stopping condition in 35.69 seconds.

Overall, the PSO was the best performing algorithm for the S&P benchmark problem. The second best performing algorithm was the GA, and the third was ABC. SBPSO ranked fourth overall and the FA ranked last.

4.5.5 Nikkei 225

Table 4.11 shows that ABC ranked second for risk, while the FA ranked third - the first time that ABC ranked higher for risk than the FA. Furthermore, ABC ranked first with regard to return, making it a very competitive algorithm for this benchmark problem. Another first was that the FA ranked last for risk, whereas the FA ranked first for risk on previous benchmark problems. Both the SBPSO and PSO also ranked first for risk. However, the PSO was statistically significantly more profitable than SBPSO. To elaborate, the portfolios obtained by ABC, PSO, and the GA were roughly 1.3 times more profitable than those obtained by SBPSO, while being relatively as risky, as shown in Table 4.10.

Table 4.10: Nikkei 225 results for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
ABC	$\bar{x}$	49	24	0.003348	0.000858	0.000470	0.000210	0.924789	254.99
	σ	0	6	0.000022	0.000015	0.000221	0.000001	0.046806	2.91
FA	$\bar{x}$	28	12	0.001086	0.000962	0.004191	0.000490	0.734680	1087.28
	σ	6	7	0.000484	0.000114	0.000561	0.000056	0.064953	117.14
GA	$\bar{x}$	46	22	0.003185	0.000858	0.001702	0.000276	0.906150	248.82
	σ	4	3	0.000105	0.000068	0.000344	0.000019	0.043527	2.82
PSO	$\bar{x}$	44	7	0.003264	0.000782	0.000703	0.000218	0.957354	266.89
	σ	2	0	0.000023	0.000022	0.000126	0.000008	0.037560	4.40
SBPSO	$\bar{x}$	42	10	0.002454	0.000800	0.002396	0.000353	0.826649	29.22
	σ	4	5	0.000592	0.000065	0.000535	0.000065	0.067572	4.19

All algorithms found a high number of optimal portfolios in comparison with previous benchmark problems, except for the FA. The portfolios obtained by the ABC and FA contained statistically significantly more assets than the GA, PSO and SBPSO - with portfolios which contained fewer than 10 assets.

Figure 4.5 shows that all algorithms struggled to approximate the true POF, although PSO and ABC were the closest. Rankings for the Pareto optimality measures support this statement, with ABC ranking first for GD and IGD, and second for HV. The PSO ranked second for GD and IGD, and first for HV. The GA performed relatively well, ranking third for all the Pareto optimality measures. SBPSO, however, ranked fourth for all Pareto optimality measures and was only statistically significantly better than the FA.

Table 4.11: Nikkei 225 rankings for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time	Overall
ABC	Wins	4	0	4	1	4	4	2	2	21
	Losses	0	4	0	2	0	0	0	2	8
	Draws	0	0	0	1	0	0	2	0	3
	Difference	4	-4	4	-1	4	4	2	0	13
	Rank	1	4	1	2	1	1	2	3	1
FA	Wins	0	2	0	0	0	0	0	0	2
	Losses	4	1	4	4	4	4	4	4	29
	Draws	0	1	0	0	0	0	0	0	1
	Difference	-4	1	-4	-4	-4	-4	-4	-4	-27
	Rank	4	2	5	3	5	5	5	5	4
GA	Wins	3	1	2	1	2	2	2	3	16
	Losses	1	3	2	2	2	2	1	1	14
	Draws	0	0	0	1	0	0	1	0	2
	Difference	2	-2	0	-1	0	0	1	2	2
	Rank	2	3	3	2	3	3	3	2	2
PSO	Wins	1	4	3	3	3	3	3	1	21
	Losses	2	0	1	0	1	1	0	3	8
	Draws	1	0	0	1	0	0	1	0	3
	Difference	-1	4	2	3	2	2	3	-2	13
	Rank	3	1	2	1	2	2	1	4	1
SBPSO	Wins	1	2	1	3	1	1	1	4	14
	Losses	2	1	3	0	3	3	3	0	15
	Draws	1	1	0	1	0	0	0	0	3
	Difference	-1	1	-2	3	-2	-2	-2	4	-1
	Rank	3	2	4	1	4	4	4	1	3

For Nikkei 225, the largest benchmark problem, the FA took approximately 1000 seconds to reach the stopping condition - an order of magnitude worse than ABC, PSO and the GA, and two orders of magnitude worse than SBPSO. In contrast, ABC, PSO and the GA took around 250 seconds to reach the stopping condition.

PSO was the highest ranked algorithm overall for the Nikkei 225 benchmark problem. The GA ranked second overall and the ABC third. However, the difference in performance by the GA and ABC is small as the ABC incurred only one more loss than the GA. The SBPSO ranked fourth overall, and the FA ranked last.

Figure 4.5: Obtained Pareto-optimal fronts for the Nikkei 225 benchmark

4.6 Summary

This chapter proposed a set-based particle swarm optimization (SBPSO) for the mean-variance portfolio optimization problem. The SBPSO is used for asset selection while a particle swarm optimization (PSO) algorithm is used for asset weight determination. A thorough investigation into the performance of the SBPSO performance was undertaken. This included comparing the SBPSO with four known algorithms for portfolio optimization, namely artificial bee colony (ABC), firefly algorithm (FA), PSO and a genetic algorithm (GA), on five benchmark problems. ABC was the best performing algorithm overall. Furthermore, it was fast, had higher return values and obtained statistically significantly more optimal portfolios than the other algorithms. The performance of the PSO was close to that of ABC and the PSO found smaller sized portfolios than the other algorithms. The GA was the third best performing algorithm overall. The performance of the proposed SBPSO was better than the FA and close to the GA. Furthermore, the ability of SBPSO to approximate the true POF deteriorated as the dimensionality of the portfolio problems increased. However, the SBPSO was statistically significantly faster than the other algorithms, and was consistent in its time to reach the stopping condition, despite the dimensionality of the problems increasing. The FA was

the worst performing algorithm, and failed to approximate the true POF for all portfolio problems.

The SBPSO results show that it is possible to obtain good quality portfolios quickly by treating portfolio optimization as a set-based problem. However, SBPSO did not perform as well as PSO and ABC. It is hypothesized that this gap in performance is due to the inadequate time SBPSO allocates to optimize the weights for a candidate solution. Furthermore, the performance of SBPSO can be better understood by examining its exploration and exploitation behaviour. Chapter 6 further investigates the performance of SBPSO for portfolio optimization and suggests improvements.

Chapter 5

Set-based Diversity Measures

Chapter 4 showed that SBPSO obtains good-quality solutions while being relatively fast. However, SBPSO performed worse than the ABC, PSO and GA in its ability to approximate the POF. Insight into the exploration and exploitation behaviour of SBPSO will help us understand its performance for portfolio optimization. This insight can be gained by measuring the diversity of the particles moving through a search space. Much of the work done on quantifying diversity has focused on heuristics for continuous-valued search spaces [134]. Due to the fact that sets are not of the same size, and since all existing diversity measures assume same size candidate solutions, these measures are not applicable to set-based algorithms. Furthermore, sets lack spatial structure in that distance, in the traditional sense, cannot be measured between two sets. Therefore, there is a need for set-based diversity measures that give researchers the same insights for set-based heuristics as the diversity measures for heuristics that require candidate solutions to be of the same size. The objective of this chapter¹ is to propose diversity measures for set-based optimization algorithms. To do this, a thorough investigation into the use of the Jaccard distance and the Hamming distance as methods for calculating the diversity of set-based populations is conducted.

The rest of this chapter is organized as follows: Section 5.1 introduces the reader to diversity concepts and existing diversity measures. Section 5.2 formally defines the Jaccard distance and the Hamming distance. Numerical examples demonstrating these distance measures as set-based diversity measures are given in Section 5.3. Section 5.4 describes the empirical process used to validate the two set-based diversity measures. Section 5.5 presents and discusses the results. Section 5.6 summarises the findings.

¹This chapter was published in the 7th International Conference on Soft Computing Machine Intelligence under the title "Diversity measures for set-based meta-heuristics" [53].

5.1 Introduction

The diversity of a population of agents over time allows a researcher to assess the ability of the heuristic to sufficiently explore the search space (initially), later to refine (exploit) potentially good solutions. Low diversification indicates that the agents are refining a potentially good solution, while high diversification shows that the agents are exploring the search space. If diversity is low throughout the search, modifications can then be made to the heuristic to influence the agents to spend more time exploring other areas of the search space. Likewise, if the diversity is constantly high, changes can then be made to promote exploitation of fruitful areas. If the distance measures are normalised by search space bounds, then a diversity value of zero indicates that all agents in the population have converged to the same point in the search space. Conversely, a diversity value of one means that the agents are totally different from one other. If the distance measures are not normalised by search space bounds, then the maximum diversity is a function of the largest distance between extreme points in the search space. A researcher expects that the value of a diversity measure to be at a maximum when the agents differ completely and that the value decreases as the agents become more similar. Furthermore, a researcher expects that when the agents are identical the value of a diversity measure is zero.

The diversity of population-based algorithms, like particle swarm optimization (PSO) [49], [52] and genetic algorithms (GAs) [52], [85], can be quantified using various methods typically associated with the Euclidean distance [134]. For example, the average Euclidean distance between each individual or the average distance from each individual to the epicenter of the population can be considered the diversity of the population [134]. However, no work has been done on measuring the diversity of set-based algorithms.

Existing diversity measures are not suitable for set-based populations, because sets can differ in size and there is no concept of distance between elements in the sets. This chapter proposes two set-based diversity measures; one based on the Jaccard distance [83] and the other based on the Hamming distance [73].

5.2 Set-based Distance Measures

This section defines the Jaccard distance and Hamming distance, both of which are well-established non-Euclidean distance measures for discrete-valued representations. Section 5.2.1 defines the Jaccard distance and Section 5.2.2 defines the Hamming distance.

5.2.1 Jaccard Distance

The Jaccard index measures the similarity between two sets as the cardinality of their intersection divided by the cardinality of their union [83]. Formally, the Jaccard index, also known as the Jaccard similarity coefficient, is

$$J(A, B) = \frac{A \cap B}{A \cup B} \quad (5.1)$$

where A and B are two sets.

The Jaccard distance is simply the complement of the Jaccard index and is calculated as

$$d_J(A, B) = 1 - \frac{A \cap B}{A \cup B} \quad (5.2)$$

The Jaccard distance is equal to 1.0 when A and B are disjoint. Moreover, when A and B are identical, d_J is equal to 0.0. The Jaccard distance is suited for measuring diversity, since sets in a given collection that are disjoint will have an average d_J of 1.0. As these sets become more similar to one another, the average d_J value will approach 0.0.

5.2.2 Hamming Distance

The Hamming distance is an indication of the minimum number of substitutions required to change one string into another string of equal length, by counting the number of positions where corresponding symbols differ [73]. This can generally be applied to vectors of equal length, so long as the vectors are bit vectors. The Hamming distance between vectors $\mathbf{u}$ and $\mathbf{v}$, each of length n , is calculated as

$$d_H(\mathbf{u}, \mathbf{v}) = \frac{d'_H(\mathbf{u}, \mathbf{v})}{n_x} \quad (5.3)$$

where

$$d'_H(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{n_x} \iota(u_j, v_j) \quad (5.4)$$

and

$$\iota(u, v) = \begin{cases} 1 & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases} \quad (5.5)$$

The Hamming distance produces a maximum value of 1.0 when all components between $\mathbf{u}$ and $\mathbf{v}$ are in disagreement. The Hamming distance is 0.0 when all the components are in agreement.

5.3 Diversity Measures for Sets

This section proposes the Jaccard and Hamming diversity measures for set-based heuristics, and showcases the difference between the two. Jaccard diversity is defined in Section 5.3.1 and Hamming diversity is defined in Section

5.3.2. Section 5.3.3 gives a series of examples of calculating the Jaccard and Hamming diversity measures.

5.3.1 Jaccard Diversity

Jaccard diversity is the average Jaccard distance between all set pairs in a given population, S . Formally, Jaccard diversity is calculated as

$$\bar{d}_J = \frac{\sum_i^{n-1} \sum_{j=i+1}^n d_J(S_i, S_j)}{\sum_i^{n-1} \sum_{j=i+1}^n 1} \quad (5.6)$$

where n is the cardinality of S , and S_i and S_j are the sets corresponding to indices i and j , respectively.

5.3.2 Hamming Diversity

Hamming diversity is the average Hamming distance between all set pairs in S . Note that the Hamming distance is defined for vectors of the same length of discrete valued elements. In order to apply the Hamming distance to sets, set-based representations have to be converted to vector-based representations. This is done by applying a mapping function, ω , to the set.

$$\omega : S \subseteq U \rightarrow B^n \quad (5.7)$$

where $B = \{0, 1\}$.

Given $S = \{u_1, u_2, \dots, u_n\}$, the corresponding bit vector is

$$\omega(S) = (V_1, V_2, \dots, V_n) \quad (5.8)$$

where

$$V_j = \begin{cases} 1 & \text{if } U_j \in S \\ 0 & \text{otherwise} \end{cases} \quad (5.9)$$

The following illustrates the use of ω to convert the set $\{1, 2, 5\}$ in the universe of the first five natural numbers to a bit vector. The length of the bit vector is the size of the universe. Each index in the bit vector corresponds to an element in the universe. For each element in the set $\{1, 2, 5\}$, the corresponding bit vector values are set to 1 and all other values are set to 0, since they are not present in the set. Therefore, the bit vector representation of set $\{1, 2, 5\}$ is

$$\omega(\{1, 2, 5\}) = (1, 1, 0, 0, 1)$$

Diversity using Hamming distance, or simply, Hamming diversity is

$$\bar{d}_H = \frac{\sum_i^{n-1} \sum_{j=i+1}^n d_H(\omega(S_i), \omega(S_j))}{\sum_i^{n-1} \sum_{j=i+1}^n 1} \quad (5.10)$$

One immediate downside to the Hamming diversity is that each set has to be converted to a larger $|U|$ -dimensional vector where U is the set universe.

5.3.3 Examples

The purpose of this section is to illustrate the application of the diversity measures on special cases. The first two examples demonstrate how to calculate the Jaccard and Hamming diversity of sets. The remaining examples show sets that start at maximum diversity progressively moving to a minimum diversity. Furthermore, the results of the examples are discussed.

5.3.3.1 Example 1 - Jaccard Diversity

Consider the collection of sets, $S = (\{1\}, \{3\}, \{1, 2\})$. The Jaccard diversity, $\bar{d}_J$, of S is calculated as follows:

$$\begin{aligned} d_J(\{1, 2\}, \{1\}) &= 1 - \frac{|\{1, 2\} \cap \{1\}|}{|\{1, 2\} \cup \{1\}|} = 0.5 \\ d_J(\{1, 2\}, \{3\}) &= 1 - \frac{|\{1, 2\} \cap \{3\}|}{|\{1, 2\} \cup \{3\}|} = 1.0 \\ d_J(\{1\}, \{3\}) &= 1 - \frac{|\{1\} \cap \{3\}|}{|\{1\} \cup \{3\}|} = 1.0 \\ \therefore \bar{d}_J &= \frac{0.5 + 1.0 + 1.0}{3} = 0.8334 \end{aligned}$$

This can be interpreted as the sets within S are 83.34% dissimilar, or, simply, 16.67% similar. This outcome makes sense as there is only one element that is shared and it is only shared between two of the three sets.

5.3.3.2 Example 2 - Hamming Diversity

The Hamming diversity, $\bar{d}_H$, of S , where S contains $(\{1\}, \{3\}, \{1, 2\})$, is calculated as

$$\begin{aligned} U &= \{1, 2, 3\} \\ \omega(\{1\}) &= (1, 0, 0) \\ \omega(\{3\}) &= (0, 0, 1) \\ \omega(\{1, 2\}) &= (1, 2, 0) \\ d_H((1, 0, 0), (1, 1, 0)) &= 0.3\dot{3} \\ d_H((1, 1, 0), (0, 0, 1)) &= 1.0 \\ d_H((1, 0, 0), (0, 0, 1)) &= 0.6\dot{6} \\ \therefore \bar{d}_H &= \frac{0.3\dot{3} + 1.0 + 0.6\dot{6}}{3} = 0.6667 \end{aligned}$$

Here the sets are 66.67% dissimilar, or 33.33% similar. This is an over representation of how similar the sets are, since there is no common element shared by all three sets.

5.3.3.3 Example 3 - Sets at Maximum Diversity

The rest of the examples in this section are all based on the scenario in which three set-based agents attempt to converge to a known optimum of $\{1, 2\}$ in a set universe that contains the first 10 natural numbers, i.e. $U = \{1, 2, \dots, 10\}$. The search space consists of all permutations of set elements of all possible cardinalities. Each agent is initialized such that the diversity of the population S is at a maximum by ensuring the elements of each agent is unique. The agents then move progressively closer towards the optimal set by changing one element at a time to better resemble the optimal set. The purpose of these examples is to mimic a scenario wherein diverse agents move over a number of iterations to collapse on a target set, i.e. $\{1, 2\}$. Furthermore, these examples serve to establish a sense of the proposed measures.

The first example in this scenario uses

$$S_0 = (\{1, 3, 10\}, \{4, 6, 8\}, \{2, 5, 7, 9\})$$

where S_t is the collection of sets at time step t . Each set in S_t corresponds to an agent. Through inspection, the reader can deduce that the contents of each set differs from one another. Furthermore, it is expected that proposed set-based diversity measures will yield a value of 1.0, since the sets are completely different. The diversity measures are as follows

$$\bar{d}_J(S_0) = 1.0$$

$$\bar{d}_H(S_0) = 0.6667$$

This suggests that the Hamming diversity underrepresents diversity, because a value of 1.0 was expected.

5.3.3.4 Example 4 - Sets That Are Slightly Similar

This example shows the set-based agents from the previous example beginning to converge to the optimal set. Here, S_1 is $(\{1, 2, 10\}, \{1, 6\}, \{2, 5, 7\})$. The diversity measures are

$$\bar{d}_J(S_1) = 0.85$$

$$\bar{d}_H(S_1) = 0.4$$

Both the Jaccard diversity and Hamming diversity values decreased from the previous time step, as expected, because it is assumed that the diversity values will decrease as the sets become more similar.

5.3.3.5 Example 5 - Sets That Are Extremely Similar

Two of the set-based agents are shown to have converged on the optimal set, with $S_2 = (\{1, 2\}, \{1, 2\}, \{1, 2, 7\})$. The values for the diversity measures are

$$\bar{d}_J(S_2) = 0.2221$$

$$\bar{d}_H(S_2) = 0.0667$$

Alternatively, this can be read as the sets within S_2 are 22.24% dissimilar according to $\bar{d}_J$ or 6.67% according to $\bar{d}_H$. The decrease in diversity values is expected since the sets have become more similar.

5.3.3.6 Example 6 - Sets at Minimum Diversity

The last example shows that all the set-based agents have converged on the optimal set. Here, S_3 contains $(\{1, 2\}, \{1, 2\}, \{1, 2\})$. The Jaccard diversity and the Hamming diversity values are

$$\bar{d}_J(S_3) = 0.0$$

$$\bar{d}_H(S_3) = 0.0$$

Both values are in agreement that S_3 has converged, as expected.

5.4 Empirical Process

An empirical process that mimics set-based agents moving through a search is used to compare the proposed diversity measures. The empirical process involves two experiments, both of which move sets closer to a randomly generated optimal set. Furthermore, the first experiment focuses on investigating the effect of agent size with respect to diversity. The second experiment focuses on the size of the search space. The diversity values for each time step are recorded over 30 independent runs and are graphed.

5.4.1 Experiment 1 - Agent Size

In this experiment, set-based agents are randomly initialized such that each element only appears once in any set, so that the intersection between all sets is empty. This maximises the diversity of the population. An additional set, referred to as the optimal set, is also randomly initialized. Agents are updated at each time step by randomly selecting one of two options: The first is to randomly remove an element not within the optimal set. The second option is to add a randomly chosen element from the optimal set to the agent if the element is not already present. If one of these options is not applicable, then the other is used. If neither of the options are applicable, then the agent is equal

to the optimal set. Thus, agents move closer, or in other words, become more similar, to the optimal set after each time step. This process is repeated until all sets are the same, giving a minimum diversity. It is therefore expected that the diversity decreases from the maximum diversity to the minimum diversity.

Three different agent sizes were used, i.e. small, large, and random (from a uniform distribution in $[1,10]$). For small, a size of 3 was used and for large, 10. Furthermore, each of the agent sizes, referred to as n_A where A is the agent, is paired with three different optimal set sizes. The sizes for the optimal set, Y^* , were 1, 5 and 10. The optimal set size is denoted as n_{Y^*} . Thus, a total of nine artificial scenarios were tested. For each scenario, a population of 10 was used, and a universe of 100 elements, i.e. $U = \{x \in \mathbb{R} \mid 1 \leq x \leq 100\}$.

5.4.2 Experiment 2 - Universe Size

Experiment 2 follows the same procedure as experiment 1. However, the effect of universe size is investigated instead of agent size. To accommodate this, elements can appear more than once during the initialisation process and each set is randomly sized. Thus, the initial agents can, to some degree, be similar. Three different universe sizes, n_U , are paired with the same n_{Y^*} configurations from the previous experiment. The universe sizes considered were 10, 50, and 100.

5.5 Results

The diversity profiles for all experiment 1 scenarios in Figure 5.1 show that Hamming diversity under-represents diversity, despite the contents of the sets on initialization being entirely different. The diversity, as a researcher would expect, should be at a maximum at the beginning of the search. The under-representation was caused by taking the size of the set universe into account when calculating diversity. For example, sets $\{1\}$ and $\{2\}$ in the universe of the first 10 natural numbers, i.e. $\{1, 2, \dots, 10\}$, were different in the sense that the contents were different. However, they were both similar in the sense that each excludes nine other elements. On the other hand, the Jaccard diversity was 1.0 for all initial sets in Figure 5.1. Each figure shows a smooth curve for Jaccard diversity, except for Figures 5.1(a) and 5.1(d) which show sharp drops in diversity as the population collapses to the single element optimal set. The sharp drops show that changes in populations with relatively small sized agents can drastically affect Jaccard diversity values. In comparison, the trends for the Jaccard diversity values of larger sets were much smoother. Hamming diversity shows small decreases in diversity at each time step, but maintained a comparatively low diversity throughout the search. The diversity trends for the Jaccard diversity were therefore as expected, while the Hamming diversity underestimated the diversity of the population.

The second experiment shows similar results, as seen in Figure 5.2. The diversity values based on the Hamming distance were larger than for experiment 1, yet still smaller than the values based on the Jaccard distance. Furthermore, the trends for Hamming diversity were very similar across all figures. Jaccard diversity generally produced smooth trends with varying maximum diversity values. Figure 5.2(b) shows somewhat larger jumps in diversity than the other graphs for this experiment. However, the jumps were not as drastic as the worst cases of experiment 1. There is an initial uptick in diversity in the second experiment, unlike in the second experiment where diversity started at a maximum and then decreased. This uptick is due to the fact that the agents in the second experiment share similarity on initialization and then briefly become dissimilar as elements not within the optimal set are removed. Overall, the trends for each diversity measure capture the changes in diversity as the agents collapse on the optimal set, regardless of the size of the search space.

Figure 5.1: Jaccard Diversity (blue) and Hamming Diversity (orange) for Experiment 1

Figure 5.2: Jaccard Diversity (blue) and Hamming Diversity (orange) for Experiment 2

5.6 Summary

This chapter proposed two methods for calculating the diversity of set-based populations, namely Jaccard diversity and Hamming diversity based on their respective distance measures. It was shown that the diversity reflects the correct trends as expected as a population moves from maximum diversity to minimum diversity. Jaccard diversity resulted in big changes in diversity when small-sized sets were considered. Large, oscillating changes in Jaccard diversity can give the false impression that the population is switching back and forth between states of exploration and exploitation. Hamming distance produced more subtle changes in diversity. Hamming diversity regards that sets are similar in regard to the elements they commonly exclude. Thus, in cases where the elements of sets differ completely, Hamming diversity still regards these sets as similar. As a result of this similarity between completely different sets, the Hamming diversity is shown to under-represent diversity, especially in cases where diversity is intuitively at a maximum. Therefore, the

author believes that Jaccard diversity better represents about set diversity.

Chapter 6

Set-based Particle Swarm Optimization Improvement and Diversity Analysis

The focus of this chapter¹ is to address the gap in performance between SBPSO and the two top performing algorithms, namely ABC and PSO, as reported in Chapter 4. Several design problems of the proposed approach to portfolio optimization are identified, from which possible improvements are derived and investigated. Furthermore, the newly defined Jaccard diversity measure is used to better understand the exploration and exploitation behaviour of SBPSO.

The rest of this chapter is organized as follows: The design problems are discussed in Section 6.1, followed by an analysis of the Jaccard diversity values of SBPSO for portfolio optimization in Section 6.2. Modifications to improve the performance of SBPSO are given in Section 6.3. Section 6.4 proposes an alternative weight optimizer using ACD. Section 6.5 evaluates the performance between SBPSO with modifications and SBPSO and the other algorithms in Chapter 4. The findings are summarised in Section 6.6.

6.1 Design Problems

The bi-stage search process of SBPSO demonstrated good performance for portfolio optimization problems, better than the FA and comparable to that of the GA, but not as good as ABC and PSO. The advantage of SBPSO over the other algorithms was that it was significantly faster, and the search time was invariant to the number of assets considered, albeit at the cost of

¹This chapter was published in the proceedings of the 2020 IEEE Swarm Intelligence Symposium under the title "Improved Set-based Particle Swarm optimization for Portfolio optimization" [54]. This chapter is also currently being reviewed for the IEEE Swarm Intelligence Symposium under the title "Set-based Particle Swarm Optimization for Portfolio Optimization with Adaptive Coordinate Descent Weight Optimization" [59].

a slight degradation in solution quality. The degradation in performance is, in general, hypothesized to be a result of an inadequate weight determination process. The following outlines, in hindsight, the design problems of SBPSO:

- The weight-determination stage was limited to 10 iterations, which is not sufficient to finely tune the weights of a candidate solution. Furthermore, when a set-particle contains only one asset, such as in the case of strictly maximizing profit, the next 10 iterations are spent determining the asset's weight. However, considering equation (3.4), the weight of an asset in a single-asset portfolio can only ever have a value of one. Thus, 10 iterations would then be unnecessarily wasted and increase the time to reach the stopping condition.
- After the weight determination stage, it may be the case that the weights of certain assets are zero, meaning that they have no impact on the value of a portfolio. Yet, these assets remain in the position of a particle, making them likely to re-appear in the future. The existence of the zero-weighted assets unnecessarily increases the dimensionality in the weight determination stage and adds noise. Furthermore, the ability of SBPSO to converge to a single set-solution is also affected. For example, if the weight determination process for three candidate solutions, $\{5, 1, 2\}$, $\{5, 3, 4\}$ and $\{5, 6, 7\}$, assigns a weight of one to asset 5, while all other assets have a weight of zero, all candidate solutions will have the same objective function value. In light of this, the proposed SBPSO approach will converge to similar weights, but not necessarily a set of common elements.

6.2 Diversity Analysis

Figure 6.1 shows that the SBPSO for portfolio optimization is unable to converge to a single solution. The Jaccard diversity for SBPSO generally stagnates around 60% dissimilarity throughout the search, except for FTSE 100 where it dips to between 40% and 60%. Nikkei 225 also shows a slight downwards trend, but quickly stagnates. Hence, SBPSO spends too much time exploring the search space and not enough refining solutions. The Jaccard diversity values in Figure 6.1 are used as a baseline which the proposed modifications are compared to. The proposed modifications are expected to balance the exploration and exploitation behavior of the SBPSO for portfolio optimization, while also improving solution quality.

Figure 6.1: Average Jaccard diversity over time for SBPSO

6.3 Modifications

This section iteratively suggests and investigates modifications of SBPSO, based on the aforementioned issues, to create a more problem-specific version for portfolio optimization by leveraging the relationship with the weight optimizer. Each modification builds upon the previous, unless the modification severely impacts performance. Each modified version of SBPSO, of which there are six, is referred to as SBPSO_{m_x} , where x is the corresponding modification number. For example, the first modified version of SBPSO is referred to as SBPSO_{m1} , the second is SBPSO_{m2} , and so on.

Section 6.3.1 proposes a modification to remove zero-weighted assets from the set-particles. A more efficient approach to optimizing single-asset portfolios is given in Section 6.3.2. Section 6.3.3 proposes a modification that allows the weight optimizer to execute until it converges. A modification to only allow assets to be removed from set-particle positions if the assets are zero-weighted is proposed in Section 6.3.4. Section 6.3.5 reintroduces tournament selection to the SBPSO velocity update equation. A modification to balance the exploration and exploitation behaviour is presented in Section 6.3.6.

6.3.1 Removal of Zero-weighted Assets

The first of the suggested modifications is quite simple. Any assets that have a weighting of zero after the weight determination process are removed from the position of the set-particle. Thus, set-positions become smaller and reduce the dimensionality of future PSO searches, thereby making it easier and faster for

the PSO to identify good-quality weights. It is expected that this modification will significantly decrease the time to obtain good solutions.

Figure 6.2 shows that the average Jaccard diversity of SBPSO_{m1} for all portfolio optimization problems was much higher than that of SBPSO. The particles in SBPSO_{m1} are 80% to 100% dissimilar, whereas the particles in SBPSO were 20% less dissimilar. Thus, the Jaccard diversity for SBPSO_{m1} was worse than SBPSO for portfolio optimization. However, this is not to say that the convergence of SBPSO was good, as both versions were shown to be in a constant state of exploration. The addition of this modification sheds light on the fact that the portfolios obtained by SBPSO contain common zero-weighted assets, leading to lower Jaccard diversity values. Table 6.1 shows the SBPSO set-particles contained a high number of common zero-weight assets for all benchmark problems.

Figure 6.2: Average Jaccard Diversity over Time for SBPSO_{m1}

Table 6.1: SBPSO versus SBPSO_{m1}: Number of zero-weight assets for each benchmark problem

	Algorithm	Result	$K_{\lambda=0.0}$	$K_{\lambda=1.0}$	K	K_0	$K_{0,\forall x_i}$
Hang Seng	SBPSO	$\bar{x}$	13.50	17.44	14.86	11.66	12.87
		σ	1.68	1.69	2.27	1.65	1.95
	SBPSO _{m1}	$\bar{x}$	1.19	5.23	2.43	0.00	0.00
		σ	0.14	0.33	1.36	0.00	0.00
DAX 100	SBPSO	$\bar{x}$	37.68	50.33	40.11	28.03	38.76
		σ	5.44	4.42	5.70	3.03	5.49
	SBPSO _{m1}	$\bar{x}$	1.28	11.76	3.19	0.00	0.00
		σ	0.24	0.64	2.72	0.00	0.00
FTSE 100	SBPSO	$\bar{x}$	23.97	42.06	26.83	19.63	23.97
		σ	10.62	11.36	12.00	7.27	12.16
	SBPSO _{m1}	$\bar{x}$	1.02	11.14	2.82	0.00	0.00
		σ	0.04	1.25	2.72	0.00	0.00
S&P 100	SBPSO	$\bar{x}$	32.60	56.06	38.49	26.08	35.91
		σ	12.36	10.27	13.54	6.99	13.74
	SBPSO _{m1}	$\bar{x}$	1.01	13.56	3.99	0.00	0.00
		σ	0.02	0.89	3.91	0.00	0.00
Nikkei 225	SBPSO	$\bar{x}$	47.54	59.54	50.55	35.18	44.42
		σ	17.75	21.33	18.37	10.54	18.44
	SBPSO _{m1}	$\bar{x}$	1.01	5.01	2.22	0.00	0.00
		σ	0.02	0.34	1.19	0.00	0.00

* For convenience, $K_{\lambda=0.0}$ is the size of the set-particle's position when optimizing the mean-variance model for return, $K_{\lambda=1.0}$ is the size of the set-particle's position when optimizing the mean-variance model for risk, K is the average size of the set-particle's position, K_0 is the number of zero-weight assets in a set-particle's position, and $K_{0,\forall x_i}$ is the number of common zero-weighted assets across all set-particle positions. A common zero-weighted asset refers to a zero-weighted asset that appears in more than one set-particle position.

Although the modification to remove zero-weighted assets did not improve the diversity of the SBPSO, it did significantly improve its time to reach the stopping condition, as expected. Table 6.2 shows that the time results of SBPSO_{m1} for all benchmark problems was better than that of SBPSO. Performance gains can also be seen across other performance measures. Most notable, the size of the solutions obtained and the profitability of said solutions were statistically significantly better. Likewise, the GD, IGD and HV values were generally statistically significantly better. However, the riskiness of the portfolios obtained by SBPSO_{m1} for Hang Seng, DAX 100 and S&P 100 was slightly worse than those obtained by SBPSO. Nevertheless, the gains in performance by simply removing zero-weighted assets from the set-particle's position are worthwhile and makes this modification a good first step towards improving the performance of SBPSO.

Table 6.2: SBPSO versus SBPSO_{m1}: Mean and standard deviations for each performance measure

	Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
Hang Seng	SBPSO	$\bar{x}$	34	3	0.007614	0.001908	0.000463	0.000236	0.776926	9.37
		σ	1	0	0.000022	0.000009	0.000141	0.000010	0.023747	0.53
	SBPSO _{m1}	$\bar{x}$	33	3	0.007629	0.001937	0.000816	0.000253	0.768025	4.70
		σ	1	0	0.000071	0.000047	0.000133	0.000014	0.027005	0.41
DAX 100	SBPSO	$\bar{x}$	34	6	0.007670	0.000960	0.002334	0.000357	0.826267	26.87
		σ	5	1	0.000319	0.000105	0.000527	0.000019	0.054478	1.63
	SBPSO _{m1}	$\bar{x}$	38	3	0.008359	0.001013	0.001258	0.000358	0.840054	6.15
		σ	3	0	0.000105	0.000056	0.000402	0.000026	0.064622	0.91
FTSE 100	SBPSO	$\bar{x}$	33	6	0.006605	0.000822	0.001206	0.000264	0.725351	13.81
		σ	6	2	0.000176	0.000061	0.000472	0.000017	0.037266	5.43
	SBPSO _{m1}	$\bar{x}$	29	3	0.006693	0.000797	0.000596	0.000247	0.738303	6.13
		σ	1	0	0.000036	0.000018	0.000163	0.000010	0.035322	1.17
S&P 100	SBPSO	$\bar{x}$	43	9	0.007269	0.001219	0.001775	0.000301	0.890120	35.69
		σ	2	3	0.000428	0.000174	0.000620	0.000036	0.031804	5.56
	SBPSO _{m1}	$\bar{x}$	44	5	0.007746	0.001331	0.000690	0.000272	0.875669	7.34
		σ	1	0	0.000037	0.000027	0.000096	0.000012	0.039881	1.86
Nikkei 225	SBPSO	$\bar{x}$	42	10	0.002454	0.000800	0.002396	0.000353	0.826649	29.22
		σ	4	5	0.000592	0.000065	0.000535	0.000065	0.067572	4.19
	SBPSO _{m1}	$\bar{x}$	41	3	0.003264	0.000792	0.000788	0.000264	0.911819	8.55
		σ	2	0	0.000033	0.000022	0.000101	0.000019	0.051362	0.40

6.3.2 Single Asset Portfolios

The second proposed modification addresses the needless optimization of single asset portfolios. To reiterate, SBPSO uses 10 iterations to optimize the weight of a single asset portfolio. Given equation (3.4), the weight can only have a value of 1.0. Thus, the second modification is to assign a contribution weight of 1.0 if the position of a set-particle contains exactly one asset, and to then immediately calculate the objective function. Thus, no weight determination process is required. It is expected that this modification will not impact the Pareto results obtained, but rather decrease the time to reach the stopping condition.

Table 6.3 shows that performance of the SBPSO_{m1} and SBPSO_{m2} is identical. As expected, the search time of SBPSO_{m2} was slightly better than that of SBPSO_{m1}. It should be noted that the algorithms used the same set of seeds for random number generation which is why the results (other than time) are identical. The average Jaccard diversity for SBPSO_{m2}, shown in Figure 6.3, is also identical to that of SBPSO_{m1} in Figure 6.2. Nevertheless, it is a modification that provides good housekeeping, so to speak, at no cost to performance.

Table 6.3: SBPSO_{m1} versus SBPSO_{m2}: Mean and standard deviations for each performance measure

	Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
Hang Seng	SBPSO _{m1}	$\bar{x}$	33	3	0.007629	0.001937	0.000816	0.000253	0.768025	4.70
		σ	1	0	0.000071	0.000047	0.000133	0.000014	0.027005	0.41
	SBPSO _{m2}	$\bar{x}$	33	3	0.007629	0.001937	0.000816	0.000253	0.768025	4.13
		σ	1	0	0.000071	0.000047	0.000133	0.000014	0.027005	0.65
DAX 100	SBPSO _{m1}	$\bar{x}$	38	3	0.008359	0.001013	0.001258	0.000358	0.840054	6.15
		σ	3	0	0.000105	0.000056	0.000402	0.000026	0.064622	0.91
	SBPSO _{m2}	$\bar{x}$	38	3	0.008359	0.001013	0.001258	0.000358	0.840054	5.94
		σ	3	0	0.000105	0.000056	0.000402	0.000026	0.064622	1.09
FTSE 100	SBPSO _{m1}	$\bar{x}$	29	3	0.006693	0.000797	0.000596	0.000247	0.738303	6.13
		σ	1	0	0.000036	0.000018	0.000163	0.000010	0.035322	1.17
	SBPSO _{m2}	$\bar{x}$	29	3	0.006693	0.000797	0.000596	0.000247	0.738303	5.29
		σ	1	0	0.000036	0.000018	0.000163	0.000010	0.035322	1.66
S&P 100	SBPSO _{m1}	$\bar{x}$	44	5	0.007746	0.001331	0.000690	0.000272	0.875669	7.34
		σ	1	0	0.000037	0.000027	0.000096	0.000012	0.039881	1.86
	SBPSO _{m2}	$\bar{x}$	44	5	0.007746	0.001331	0.000690	0.000272	0.875669	7.13
		σ	1	0	0.000037	0.000027	0.000096	0.000012	0.039881	1.88
Nikkei 225	SBPSO _{m1}	$\bar{x}$	41	3	0.003264	0.000792	0.000788	0.000264	0.911819	8.55
		σ	2	0	0.000033	0.000022	0.000101	0.000019	0.051362	0.40
	SBPSO _{ZS}	$\bar{x}$	41	3	0.003264	0.000792	0.000788	0.000264	0.911819	7.78
		σ	2	0	0.000033	0.000022	0.000101	0.000019	0.051362	0.62

Figure 6.3: Average Jaccard Diversity over Time for SBPSO_{m2}

6.3.3 Weight Determination Until Convergence

Instead of allocating 10 iterations to the weight optimization of assets, this modification allows the weight optimizer to execute until there is no improve-

ment in solution quality. Solution quality is measured as the best objective function value. If solution quality does not improve over three iterations, the weight optimization process stops. A limit of three iterations is chosen instead of larger limit to avoid too much time spent optimizing the weights and maintain the fast search time. Furthermore, a limit smaller than three could end the weight determination process too early and forsake potential performance gains. If the stopping condition is reached during the weight optimization process, then the weight optimizer stops and the SBPSO search terminates. It is expected that this modification will improve the Pareto-optimality related results because the weight optimizer has more time to refine solutions.

Table 6.4 shows that the average number of solutions obtained and profitability of SBPSO_{m3} was greater than of SBPSO_{m2}. Likewise, the average GD, IGD and HV scores were better with lower standard deviations. In other words, the performance of SBPSO_{m3} was better and more consistent. The increases in performance were verified by statistical significance tests that show for the aforementioned performances measures, that the performance of SBPSO_{m3} was generally better or the same as SBPSO_{m2}. SBPSO_{m3} was slower to reach the stopping condition than SBPSO_{m2}, but still better than or comparable to SBPSO.

Table 6.4: SBPSO_{m2} versus SBPSO_{m3}: Mean and standard deviations for each performance measure

	Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
Hang Seng	SBPSO _{m2}	$\bar{x}$	33	3	0.007637	0.001921	0.000590	0.000241	0.767049	5.33
		σ	0	0	0.000035	0.000024	0.000183	0.000011	0.028889	1.25
	SBPSO _{m3}	$\bar{x}$	33	3	0.007614	0.001900	0.000274	0.000217	0.785971	7.04
		σ	1	0	0.000007	0.000005	0.000088	0.000006	0.019312	1.89
DAX 100	SBPSO _{m2}	$\bar{x}$	42	4	0.008487	0.000997	0.000506	0.000328	0.850058	10.87
		σ	2	0	0.000056	0.000027	0.000250	0.000021	0.062547	2.47
	SBPSO _{m3}	$\bar{x}$	43	4	0.008498	0.000986	0.000413	0.000330	0.851289	12.60
		σ	1	0	0.000020	0.000013	0.000119	0.000017	0.068197	2.69
FTSE 100	SBPSO _{m2}	$\bar{x}$	30	4	0.006696	0.000795	0.000433	0.000240	0.747358	14.97
		σ	1	0	0.000021	0.000007	0.000090	0.000007	0.035120	3.80
	SBPSO _{m3}	$\bar{x}$	30	4	0.006684	0.000786	0.000282	0.000237	0.745490	16.97
		σ	0	0	0.000007	0.000001	0.000050	0.000006	0.045856	3.14
S&P 100	SBPSO _{m2}	$\bar{x}$	44	5	0.007745	0.001291	0.000510	0.000260	0.877128	16.94
		σ	0	0	0.000016	0.000032	0.000122	0.000008	0.038541	3.21
	SBPSO _{m3}	$\bar{x}$	44	6	0.007736	0.001275	0.000361	0.000253	0.865971	19.44
		σ	0	0	0.000013	0.000023	0.000069	0.000008	0.041210	3.15
Nikkei 225	SBPSO _{m2}	$\bar{x}$	43	3	0.003299	0.000794	0.000478	0.000240	0.919549	41.38
		σ	2	0	0.000034	0.000024	0.000141	0.000015	0.048358	3.20
	SBPSO _{m3}	$\bar{x}$	44	4	0.003305	0.000790	0.000323	0.000230	0.908527	49.23
		σ	1	0	0.000020	0.000017	0.000080	0.000008	0.036250	5.68

Unfortunately, the exploitation behaviour of SBPSO_{m3} was no better than that of SBPSO_{m2}, as seen in Figure 6.4. This suggests that the imbalance in the exploratory and exploitative behaviour is not a result of the weight

determination stage, but rather the formulation of SBPSO for portfolio optimization. Even so, allowing the weight determination to execute until there is no improvement in solution quality, yields significant gains in other performance metrics.

Figure 6.4: Average Jaccard Diversity over Time for $SBPSO_{m3}$

6.3.4 Asset Removal

This modification investigates the impact of the $\ominus$ operator on the performance of SBPSO for portfolio optimization. The $\ominus$ operator is responsible for removing elements in $X(t) \cup Y(t) \cup \hat{Y}(t)$ from $X(t)$. It is hypothesized that the $\ominus$ operator removes good-quality assets that would otherwise be useful in the weight determination stage. This modification excludes the $\ominus$ operator from the velocity update function. Therefore, assets can only be removed via the weight determination process, provided that the asset has a weighting of zero. This is arguably a more intuitive approach to asset removal, because it uses the information gained via weight determination to make intelligent decisions. The updated velocity equation is

$$\begin{aligned} V_i(t+1) = & c_1 r_1 \otimes (Y_i(t) \ominus X_i(t)) \\ & \oplus c_2 r_2 \otimes (\hat{Y}_i(t) \ominus X_i(t)) \\ & \oplus (c_3 r_3 \odot_k^+ A_i(t)) \end{aligned} \quad (6.1)$$

It is expected that this modification will improve the convergence behaviour of SBPSO, because the set-particles will, over time, become more similar due to fewer assets being removed. It is also expected that solution quality will improve as a result of set-particles retaining good-quality assets.

Figure 6.5 shows that $SBPSO_{m4}$ was, in some cases, better at reducing diversity over time with reference to the Jaccard diversity values of SBPSO. FTSE 100 shows a smooth downwards trend from 80% to 20% dissimilarity. In other words, the set-particles were approximately 60% more similar at the end of the search than at the start. Other benchmarks show a decrease from 80% to 60% dissimilarity over time, similar to that of SBPSO. However, for DAX 100, the average Jaccard diversity was worse for $SBPSO_{m4}$ at about 80% dissimilarity throughout the search.

Figure 6.5: Average Jaccard Diversity over Time for $SBPSO_{m4}$

The performance of $SBPSO_{m4}$ was very similar to that of $SBPSO_{m3}$, as seen in Table 6.5. However, the standard deviation of the performance scores for $SBPSO_{m4}$ was typically half of that of $SBPSO_{m3}$, and in some cases better by an order of magnitude. Statistical significance tests also show that the GD, IGD and HV scores of $SBPSO_{m4}$ were the same or better than those of $SBPSO_{m3}$ for all benchmark problems. Furthermore, the risk values of $SBPSO_{m4}$ were lower than those of $SBPSO_{m3}$. This modification is worth preserving, since it improved the results obtained and slightly improved the Jaccard diversity values of SBPSO.

Table 6.5: SBPSO_{m3} versus SBPSO_{m4}: Mean and standard deviations for each performance measure

	Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
Hang Seng	SBPSO _{m3}	$\bar{x}$	33	3	0.007637	0.001921	0.000590	0.000241	0.767049	5.33
		σ	0	0	0.000035	0.000024	0.000183	0.000011	0.028889	1.25
	SBPSO _{m4}	$\bar{x}$	33	3	0.007614	0.001900	0.000274	0.000217	0.785971	7.04
		σ	1	0	0.000007	0.000005	0.000088	0.000006	0.019312	1.89
DAX 100	SBPSO _{m3}	$\bar{x}$	42	4	0.008487	0.000997	0.000506	0.000328	0.850058	10.87
		σ	2	0	0.000056	0.000027	0.000250	0.000021	0.062547	2.47
	SBPSO _{m4}	$\bar{x}$	43	4	0.008498	0.000986	0.000413	0.000330	0.851289	12.60
		σ	1	0	0.000020	0.000013	0.000119	0.000017	0.068197	2.69
FTSE 100	SBPSO _{m3}	$\bar{x}$	30	4	0.006696	0.000795	0.000433	0.000240	0.747358	14.97
		σ	1	0	0.000021	0.000007	0.000090	0.000007	0.035120	3.80
	SBPSO _{m4}	$\bar{x}$	30	4	0.006684	0.000786	0.000282	0.000237	0.745490	16.97
		σ	0	0	0.000007	0.000001	0.000050	0.000006	0.045856	3.14
S&P 100	SBPSO _{m3}	$\bar{x}$	44	5	0.007745	0.001291	0.000510	0.000260	0.877128	16.94
		σ	0	0	0.000016	0.000032	0.000122	0.000008	0.038541	3.21
	SBPSO _{m4}	$\bar{x}$	44	6	0.007736	0.001275	0.000361	0.000253	0.865971	19.44
		σ	0	0	0.000013	0.000023	0.000069	0.000008	0.041210	3.15
Nikkei 225	SBPSO _{m3}	$\bar{x}$	43	3	0.003299	0.000794	0.000478	0.000240	0.919549	41.38
		σ	2	0	0.000034	0.000024	0.000141	0.000015	0.048358	3.20
	SBPSO _{m4}	$\bar{x}$	44	4	0.003305	0.000790	0.000323	0.000230	0.908527	49.23
		σ	1	0	0.000020	0.000017	0.000080	0.000008	0.036250	5.68

6.3.5 Tournament Selection

The original SBPSO proposed by Langeveld and Engelbrecht implemented the $\odot^+$ operator (refer to Section 2.4) using tournament selection. The purpose of the $\odot^+$ operator is to add good quality elements to the positions of set-particles and to increase diversity. The SBPSO for portfolio optimization proposed in Chapter 4 redesigned the $\odot^+$ operator to randomly select elements from A . This redesign was motivated by the removal of the selection bias of the tournament selection and to reduce computational complexity, since a weight determination process would be required for each element in the tournament. However, this section reintroduces tournament selection for the $\odot^+$ operator as a means to better control the diversity added to set-particles and to calculate its impact on performance. It is expected that set-particles will become more similar as good quality assets are introduced to the set-particle positions by tournament selection. Furthermore, larger tournament sizes should result in less diversity since competitive assets would be frequently selected. To test this modification, tournament sizes (τ) of three, five and ten were used.

Table 6.6 shows that across all performance measures, except for time, the values obtained by SBPSO_{m4} and SBPSO_{m5} were mostly similar, despite the different tournament sizes. Furthermore, statistical significance tests that show that there was no difference in the size, profitability and riskiness of the obtained portfolios. SBPSO_{m5} was statistically significantly slower than SBPSO_{m4} for all tournament sizes. Furthermore, the time to reach the stop-

ping condition worsens as the tournament size increases. The degradation in the time to reach the stopping condition was due to the weight determination required to calculate the objective function value for each asset in the tournament selection.

Table 6.6: SBPSO_{m4} versus SBPSO_{m5}: Mean and standard deviations for each performance measure

	Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
Hang Seng	SBPSO _{m4}	$\bar{x}$	33	3	0.007614	0.001900	0.000274	0.000217	0.785971	7.04
		σ	1	0	0.000007	0.000005	0.000088	0.000006	0.019312	1.89
	SBPSO _{m5, $\tau=3$}	$\bar{x}$	33	3	0.007614	0.001900	0.000333	0.000219	0.779533	24.33
		σ	0	0	0.000010	0.000003	0.000110	0.000007	0.019819	13.83
	SBPSO _{m5, $\tau=5$}	$\bar{x}$	34	3	0.007614	0.001900	0.000289	0.000216	0.791427	38.70
		σ	0	0	0.000009	0.000006	0.000092	0.000006	0.018159	22.35
	SBPSO _{m5, $\tau=10$}	$\bar{x}$	34	3	0.007614	0.001900	0.000272	0.000216	0.792602	71.37
		σ	0	0	0.000006	0.000002	0.000094	0.000005	0.019566	47.08
DAX 100	SBPSO _{m4}	$\bar{x}$	43	4	0.008498	0.000986	0.000413	0.000330	0.851289	12.60
		σ	1	0	0.000020	0.000013	0.000119	0.000017	0.068197	2.69
	SBPSO _{m5, $\tau=3$}	$\bar{x}$	43	4	0.008497	0.000986	0.000386	0.000322	0.853886	70.29
		σ	1	0	0.000019	0.000012	0.000124	0.000017	0.063812	19.43
	SBPSO _{m5, $\tau=5$}	$\bar{x}$	42	4	0.008493	0.000980	0.000384	0.000325	0.850306	82.99
		σ	1	0	0.000016	0.000007	0.000101	0.000013	0.063659	29.32
	SBPSO _{m5, $\tau=10$}	$\bar{x}$	43	4	0.008497	0.000985	0.000413	0.000326	0.858660	159.18
		σ	0	0	0.000013	0.000010	0.000108	0.000018	0.065767	62.19
FTSE 100	SBPSO _{m4}	$\bar{x}$	30	4	0.006684	0.000786	0.000282	0.000237	0.745490	16.97
		σ	0	0	0.000007	0.000001	0.000050	0.000006	0.045856	3.14
	SBPSO _{m5, $\tau=3$}	$\bar{x}$	30	4	0.006685	0.000786	0.000275	0.000239	0.746774	50.08
		σ	0	0	0.000007	0.000001	0.000040	0.000007	0.049173	19.30
	SBPSO _{m5, $\tau=5$}	$\bar{x}$	30	4	0.006683	0.000787	0.000284	0.000237	0.741699	69.80
		σ	0	0	0.000008	0.000001	0.000048	0.000007	0.048705	26.82
	SBPSO _{m5, $\tau=10$}	$\bar{x}$	30	4	0.006684	0.000785	0.000269	0.000236	0.758768	97.80
		σ	0	0	0.000012	0.000005	0.000040	0.000006	0.035666	53.43
S&P 100	SBPSO _{m4}	$\bar{x}$	44	6	0.007736	0.001275	0.000361	0.000253	0.865971	19.44
		σ	0	0	0.000013	0.000023	0.000069	0.000008	0.041210	3.15
	SBPSO _{m5, $\tau=3$}	$\bar{x}$	44	6	0.007736	0.001276	0.000363	0.000251	0.889834	64.89
		σ	1	0	0.000012	0.000021	0.000071	0.000007	0.044258	21.45
	SBPSO _{m5, $\tau=5$}	$\bar{x}$	44	6	0.007731	0.001268	0.000367	0.000252	0.869455	82.25
		σ	0	0	0.000011	0.000015	0.000063	0.000005	0.039291	32.86
	SBPSO _{m5, $\tau=10$}	$\bar{x}$	44	6	0.007733	0.001273	0.000366	0.000251	0.869151	135.20
		σ	0	0	0.000012	0.000017	0.000060	0.000005	0.038260	69.21
Nikkei 225	SBPSO _{m4}	$\bar{x}$	44	4	0.003305	0.000790	0.000323	0.000230	0.908527	49.23
		σ	1	0	0.000020	0.000017	0.000080	0.000008	0.036250	5.68
	SBPSO _{m5, $\tau=3$}	$\bar{x}$	43	4	0.003305	0.000788	0.000314	0.000229	0.908582	272.27
		σ	1	0	0.000023	0.000018	0.000066	0.000010	0.047458	18.88
	SBPSO _{m5, $\tau=5$}	$\bar{x}$	44	4	0.003306	0.000791	0.000306	0.000231	0.921503	393.48
		σ	1	0	0.000021	0.000014	0.000064	0.000008	0.037051	25.03
	SBPSO _{m5, $\tau=10$}	$\bar{x}$	44	4	0.003308	0.000792	0.000297	0.000226	0.927029	767.74
		σ	1	0	0.000014	0.000013	0.000065	0.000009	0.042172	54.28

Surprisingly, SBPSO_{m5} did not reduce the diversity of the set-particles despite the selection bias of the tournament as shown in Figures 6.6, 6.7 and 6.8. In fact, the diversity trends of SBPSO_{m5} for all tournament sizes were

identical to that of SBPSO_{m4}. Therefore, it can be said that the reintroduction of tournament selection was not a worthwhile modification and will not be used going forward.

Figure 6.6: Average Jaccard Diversity over Time for SBPSO_{m5, $\tau=3$}

Figure 6.7: Average Jaccard Diversity over Time for SBPSO_{m5, $\tau=5$}

Figure 6.8: Average Jaccard Diversity over Time for $\text{SBPSO}_{m5, \tau = 10}$

6.3.6 Exploration Balance Coefficient

The previous modification attempted to control the amount of diversity added to the set-particle's positions by means of tournament selection. However, there were no gains in performance and it severely impacted the time to reach the stopping condition. The exploration balance coefficient modification continues the endeavour to improve the balance between exploration and exploitation behaviour of SBPSO while simultaneously improving performance.

The cognitive, $(Y_i(t) \ominus X_i(t))$, and social, $(\hat{Y}_i(t) \ominus X_i(t))$, components of the SBPSO velocity update equation (6.1) guide set-particles towards previously found good areas of the search space. If not properly controlled, the cognitive and social components could lead towards premature convergence of the swarm since the set-particles would become more and more similar sooner. Conversely, the third component (the exploration component) of the SBPSO velocity update equation (6.1) injects diversity into the swarm by adding elements outside of $X(t) \cup Y(t) \cup \hat{Y}(t)$ to $X(t+1)$. Too much diversity would result in a constant state of exploration, and the swarm would be unable to converge to a single solution. Ideally, the time between these two stages should be balanced, allowing for time to explore the search space in the beginning of the search, and gradually exploiting fruitful areas over time. With this ideal scenario in mind, this modification adds a linearly increasing control parameter, λ_c , to moderate the exploitation of the cognitive and social components, and exploration caused by the exploration component. The linearly increasing control parameter, referred to as the exploration balance coefficient, initially

zero, is defined as

$$\lambda_c(t) = \frac{t}{n_t} \quad (6.2)$$

where n_t is the maximum number of iterations. At the start of the search, i.e. when $t = 0$, λ_c is set to zero. The updated velocity equation for this modification is

$$\begin{aligned} V_i(t+1) = & \lambda_c(t)r_1 \otimes (Y_i(t) \ominus X_i(t)) \\ & \oplus \lambda_c(t)r_2 \otimes (\hat{Y}_i(t) \ominus X_i(t)) \\ & \oplus (1 - \lambda_c(t))r_3 \otimes A_i(t) \end{aligned} \quad (6.3)$$

where r_1 , r_2 , and r_3 are random values, each sampled from a uniform distribution in the range $[0,2]$. The range $[0,2]$ is used instead of $[0,1]$ to ensure that each component is not too small as a result of $\lambda_c(t)$.

When λ_c is equal to zero, the social and cognitive guides have no influence on the set-particle's next position whereas the influence of the exploration component is at a maximum. Thereafter, the influence of the social and cognitive guides increases linearly over time while the exploration guide decreases proportionately. This means that the set-swarm will be encouraged to explore the asset search space at the beginning of the search. Then as time goes on the set-swarm will be encouraged to refine solutions as the influence of the social and cognitive guides increases while the exploration guide decreases. When λ_c is equal to one, the influence of the social and cognitive guides is at a maximum and the exploration guide has no influence. It is expected that this modification will not only improve the exploration and exploitation behaviour of SBPSO, but also the quality of solutions obtained.

Figure 6.9 shows smooth downwards trends from large Jaccard diversity values to Jaccard diversity values close to zero were obtained. In contrast, the Jaccard diversity values of SBPSO were consistently high and never reached zero. Thus, the introduction of λ_c allows the set-swarm to explore the search space early on and to refine solutions as time goes on, until eventually the swarm converges.

Figure 6.9: Average Jaccard Diversity over Time for SBPSO_{m6}

The average and standard deviations of the results obtained by SBPSO_{m6} were typically similar to, if not better than those obtained SBPSO_{m4}, as seen in Table 6.7. For example, the number of optimal solutions obtained by either algorithm was the same, and the average size of those optimal solutions was the same except for FTSE 100. Statistical significance tests show that profitability of SBPSO_{m6} was the same or worse than that of SBPSO_{m4}, although the average values were very close. However, statistical significance tests also show that SBPSO_{m6} was as or less risky than SBPSO_{m4}. Furthermore, SBPSO_{m6} was statistically significantly the same or better than SBPSO_{m4} for the Pareto optimality measures, namely GD, IGD and HV. Lastly, SBPSO_{m6} was faster to reach the stopping condition than SBPSO_{m4}, except for Nikkei 225 where SBPSO_{m6} was slower by approximately 14 seconds.

Table 6.7: SBPSO_{m4} versus SBPSO_{m6}: Mean and standard deviations for each performance measure

	Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
Hang Seng	SBPSO _{m4}	$\bar{x}$	33	3	0.007614	0.001900	0.000274	0.000217	0.785971	7.04
		σ	1	0	0.000007	0.000005	0.000088	0.000006	0.019312	1.89
	SBPSO _{m6}	$\bar{x}$	33	3	0.007607	0.001897	0.000218	0.000212	0.781949	4.52
		σ	0	0	0.000003	0.000000	0.000045	0.000002	0.026784	2.87
DAX 100	SBPSO _{m4}	$\bar{x}$	43	4	0.008498	0.000986	0.000413	0.000330	0.851289	12.60
		σ	1	0	0.000020	0.000013	0.000119	0.000017	0.068197	2.69
	SBPSO _{m6}	$\bar{x}$	43	4	0.008469	0.000972	0.000250	0.000306	0.878138	8.99
		σ	0	0	0.000006	0.000000	0.000061	0.000008	0.065429	3.19
FTSE 100	SBPSO _{m4}	$\bar{x}$	30	4	0.006684	0.000786	0.000282	0.000237	0.745490	16.97
		σ	0	0	0.000007	0.000001	0.000050	0.000006	0.045856	3.14
	SBPSO _{m6}	$\bar{x}$	30	5	0.006683	0.000785	0.000247	0.000235	0.741994	10.57
		σ	0	0	0.000007	0.000001	0.000038	0.000007	0.049170	3.55
S&P 100	SBPSO _{m4}	$\bar{x}$	44	6	0.007736	0.001275	0.000361	0.000253	0.865971	19.44
		σ	0	0	0.000013	0.000023	0.000069	0.000008	0.041210	3.15
	SBPSO _{m6}	$\bar{x}$	44	6	0.007716	0.001248	0.000275	0.000247	0.875057	16.45
		σ	0	0	0.000006	0.000001	0.000044	0.000003	0.044175	2.78
Nikkei 225	SBPSO _{m4}	$\bar{x}$	44	4	0.003305	0.000790	0.000323	0.000230	0.908527	49.23
		σ	1	0	0.000020	0.000017	0.000080	0.000008	0.036250	5.68
	SBPSO _{m6}	$\bar{x}$	44	4	0.003310	0.000794	0.000260	0.000223	0.916014	63.19
		σ	0	0	0.000007	0.000001	0.000060	0.000005	0.047097	6.13

The results from Table 6.7 and Figure 6.9 clearly show that this proposed modification to introduce a linearly increasing control parameter to the velocity update equation, was a significant improvement. As a consequence of this modification, parameter tuning is no longer necessary, which can often be a costly investigation in terms of computational resources and time. For these reasons, this proposed modification was certainly worth keeping.

6.4 Adaptive Coordinate Descent Weight Optimization

This section presents ACD as an alternative weight optimizer for the improved SBPSO_{m6} approach [113]. Section 6.4.1 gives the necessary background on ACD. The motivation for using ACD as a weight optimizer is given in Section 6.4.2. Section 6.4.3 discusses the results.

6.4.1 Adaptive Coordinate Descent

Loshchilov *et al.* developed ACD in 2011 and found it to be a good approach to several optimization problems [113]. ACD borrows and generalises the adaptive technique used in the covariance matrix adaptation evolution strategy (CMA-ES) [74], such that it can be applied to any optimization algorithm. Adaptive encoding (AE), given in Algorithm 10, iteratively transforms the

search system of the objective function so that it is as similar as possible to the Sphere function, which is known to be simple for analysis and optimization [113]. This process results in a very efficient search process for non-separable problems e.g. portfolio optimization.

Algorithm 10 Adaptive Encoding

Input: $\mathbf{x}_1, \dots, \mathbf{x}_n$;

if *initialize* **then**

 Let d represent the number of dimensions in the search space;

$w_i = \frac{1}{n}$; $c_p = \frac{1}{\sqrt{d}}$; $c_1 = \frac{0.5}{d}$; $c_n = \frac{0.5}{d}$;

$\mathbf{p} = \mathbf{0}$;

$\mathbf{C} = \mathbf{I}$;

$\mathbf{B} = \mathbf{I}$;

$\mathbf{m} = \sum_{i=1}^n w_i \mathbf{x}_i$;

$\mathbf{m}^- = \mathbf{m}$;

$\mathbf{m} = \sum_{i=1}^n w_i \mathbf{x}_i$;

$\mathbf{z}_0 = \frac{\sqrt{d}}{\|\mathbf{B}^{-1}(\mathbf{m} - \mathbf{m}^-)\|} (\mathbf{m} - \mathbf{m}^-)$;

$\mathbf{z}_i = \frac{\sqrt{d}}{\|\mathbf{B}^{-1}(\mathbf{x}_i - \mathbf{m}^-)\|} (\mathbf{x}_i - \mathbf{m}^-)$; $i = 1, \dots, n$;

$\mathbf{p} = (1 - c_p)\mathbf{p} + \sqrt{c_p(2 - c_p)}\mathbf{z}_0$;

$\mathbf{C}_n = \sum_{i=1}^n w_i \mathbf{z}_i \mathbf{z}_i^T$;

$\mathbf{C} = (1 - c_1 - c_n)\mathbf{C} + c_1 \mathbf{p} \mathbf{p}^T + c_n \mathbf{C}_n$;

$\mathbf{B}^o \mathbf{D} \mathbf{D} \mathbf{B}^o = \text{eigendecomposition}(\mathbf{C})$;

$\mathbf{B} = \mathbf{B}^o \mathbf{D}$;

Output: $\mathbf{B}$

AE is similar to principal component analysis (PCA) where the goal is to convert a set of possibly correlated variables into a set of uncorrelated variables, called principal components. The principal components are the eigenvectors of the covariance matrix, $\mathbf{C}$. The principal components are determined at each iteration by the eigendecomposition of the current covariance matrix $\mathbf{C}$.

Coordinate descent (CD) is based on the idea that an n -dimensional separable optimization problem can be broken up into n one-dimensional optimization problems. CD iteratively optimizes each independent variable while keeping the other decision variables fixed.

ACD, summarised in Algorithm 11, is CD that makes use of AE. Thus, ACD is an improvement over CD as it can handle non-separable problems. ACD has been shown to be competitive with CMA-ES algorithms, and in some cases even up to two times faster [113]. Furthermore, ACD has also been shown to be more advantageous than traditional CD. Algorithm 11 describes the ACD algorithm in which adaptive encoding is applied to coordinate descent.

The lower and upper bounds for the search are given by $\mathbf{a}$ and $\mathbf{b}$, respectively. The parameters k_{shrink} and k_{grow} determine the size of the interval along

which the ACD algorithm should search for the next best solution. When a solution better than the current best is found on either end of the search interval, the interval size is expanded by a factor of k_{grow} in that coordinate. Otherwise, the search interval is symmetrically decreased in size by a factor of k_{shrink} . These parameters therefore control the exploration-exploitation trade-off the search algorithm and should be tuned per problem.

Algorithm 11 Adaptive Coordinate Descent

Let $\mathbf{a}$ represent the lower bounds of the search space;
 Let $\mathbf{b}$ represent the upper bounds of the search space;
 Let $\mathbf{m}$ represent the midpoint which is randomly initialized within the bounds of the search space;
 Set $\boldsymbol{\sigma}$ to $(\mathbf{a}/\mathbf{b})/4$;
 Let f_{best} represent the best solution found thus far;
while *stopping condition is not true* **do**
 for $i = 1, \dots, d$ **do**
 $\mathbf{x}' = \mathbf{0}$;
 $\mathbf{x}'_i = -\boldsymbol{\sigma}_i$; $\mathbf{x}_1 = \mathbf{B}\mathbf{x}'$; $f_1 = f(\mathbf{x}_1)$;
 $\mathbf{x}'_i = \boldsymbol{\sigma}_i$; $\mathbf{x}_2 = \mathbf{B}\mathbf{x}'$; $f_2 = f(\mathbf{x}_2)$;
 succ = False;
 if $f_1 < f_{best}$ **then**
 $f_{best} = f_1$; $\mathbf{m} = \mathbf{x}_1$;
 succ = True;
 if $f_2 < f_{best}$ **then**
 $f_{best} = f_2$; $\mathbf{m} = \mathbf{x}_2$;
 succ = True;
 if $succ = True$ **then**
 $\boldsymbol{\sigma}_i = k_{grow}\boldsymbol{\sigma}_i$;
 succ = True;
 else
 $\boldsymbol{\sigma}_i = k_{shrink}\boldsymbol{\sigma}_i$;
 $\mathbf{a}_{2i-1}^x = \mathbf{x}_1$; $\mathbf{a}_{2i-1}^f = f_1$;
 $\mathbf{a}_{2i}^x = \mathbf{x}_2$; $\mathbf{a}_{2i}^f = f_2$;
 Sort $\mathbf{a}_x$ and $\mathbf{a}_f$ according to $\mathbf{a}_f$;
 Adaptive encoding with $(\mathbf{a}_{x,1}, \dots, \mathbf{a}_{x,d})$;

6.4.2 Motivation

Portfolio optimization is a non-separable problem since risk (equation (3.2)) is calculated using variance which is not separable. Additionally, portfolio optimization is made further non-separable because the asset weights are dependent on each other. Considering equation (3.4), as one weight increases, the other weights decrease. The inverse is also true. ACD has been shown to

be an efficient optimizer for single-objective optimization problems [113]. ACD is capable of optimizing non-separable problems as it iteratively transforms the search space to maximize the decorrelation of decision variables with respect to the objective function. ACD is also a completely deterministic approach, unlike PSO. Deterministic methods ensure, after a finite time, an approximation of a global optimum will be found [101]. The decision to use ACD as weight optimizer is motivated by the aforementioned characteristics. It is expected that ACD will improve the quality of the solutions obtained by SBPSO. For convenience purposes, the ACD variant of SBPSO is referred to as SBPSO_{ACD}.

6.4.3 Results

Like SBPSO_{m6}, SBPSO_{ACD} balances time between exploration and exploitation well as seen in Figure 6.10. In fact, the Jaccard diversity values at the start of the search for both algorithms are very similar, but SBPSO_{ACD} ends the search with Jaccard diversity values much closer to zero than SBPSO_{m6}. Thus, the convergence behaviour of SBPSO_{ACD} was better than that of SBPSO_{m6}. Table 6.8 shows that the average risk, GD and IGD results obtained by SBPSO_{ACD} were better than those of SBPSO_{m6}, except for DAX 100 where SBPSO_{m6} had a lower IGD score on average. SBPSO_{ACD} performed particularly well for the Hang Seng benchmark problem where it was statistically significantly the same or better than SBPSO_{m6}. Statistical significance tests also show that SBPSO_{ACD} was better than SBPSO_{m6} at obtaining solutions closer to the true POF than SBPSO_{m6}. However, SBPSO_{m6} was statistically significantly faster than SBPSO_{ACD} for all benchmark problems, except for Hang Seng. The time that SBPSO_{ACD} took to reach the stopping condition increased as the dimensionality of the problem increased. One of the motivations for SBPSO for portfolio optimization was to address the scalability problems of other algorithms for portfolio optimization. Although SBPSO_{ACD} was scalable in that it found good quality results despite the dimensionality of the problem increasing, it did not find solutions to larger problems in a timely manner. For example, SBPSO_{ACD} took approximately 60 seconds to find solutions to the FTSE problem, whereas SBPSO_{m6} took 10 seconds. For Nikkei, the largest problem, SBPSO_{ACD} took about 340 seconds while SBPSO_{m6} took 60 seconds. Overall the results show that SBPSO_{ACD} is a competitive approach to SBPSO_{m6}.

Figure 6.10: Average Jaccard Diversity over Time for $SBPSO_{ACD}$ **Table 6.8:** $SBPSO_{m6}$ versus $SBPSO_{ACD}$: Mean and standard deviations for each performance measure

	Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
Hang Seng	$SBPSO_{m6}$	$\bar{x}$	33	3	0.007607	0.001897	0.000218	0.000212	0.781949	4.52
		σ	0	0	0.000003	0.000000	0.000045	0.000002	0.026784	2.87
	$SBPSO_{ACD}$	$\bar{x}$	34	3	0.007600	0.001891	0.000190	0.000210	0.814630	2.17
		σ	0	0	0.000025	0.000021	0.000003	0.000000	0.000000	2.31
DAX 100	$SBPSO_{m6}$	$\bar{x}$	43	4	0.008474	0.000973	0.000299	0.000311	0.863570	8.99
		σ	0	0	0.000010	0.000002	0.000090	0.000010	0.067062	3.19
	$SBPSO_{ACD}$	$\bar{x}$	42	4	0.008460	0.000971	0.000209	0.000346	0.804378	28.09
		σ	9	0	0.000001	0.000000	0.000000	0.000000	0.165428	45.21
FTSE 100	$SBPSO_{m6}$	$\bar{x}$	30	5	0.006683	0.000785	0.000247	0.000235	0.741994	10.57
		σ	0	0	0.000007	0.000001	0.000038	0.000007	0.049170	3.55
	$SBPSO_{ACD}$	$\bar{x}$	29	5	0.006655	0.000775	0.000169	0.000232	0.711504	35.75
		σ	1	0	0.000024	0.000011	0.000019	0.000001	0.051801	61.04
S&P 100	$SBPSO_{m6}$	$\bar{x}$	44	6	0.007716	0.001248	0.000275	0.000247	0.875057	16.45
		σ	0	0	0.000006	0.000001	0.000044	0.000003	0.044175	2.78
	$SBPSO_{ACD}$	$\bar{x}$	44	6	0.007708	0.001244	0.000205	0.000242	0.872137	91.53
		σ	0	0	0.000001	0.000000	0.000022	0.000001	0.049381	166.15
Nikkei 225	$SBPSO_{m6}$	$\bar{x}$	44	4	0.003310	0.000794	0.000260	0.000223	0.916014	63.19
		σ	0	0	0.000007	0.000001	0.000060	0.000005	0.047097	6.13
	$SBPSO_{ACD}$	$\bar{x}$	44	4	0.003305	0.000791	0.000139	0.000209	0.898166	342.28
		σ	0	0	0.000001	0.000001	0.000027	0.000001	0.024326	59.95

If one is willing to sacrifice time in order to obtain higher quality solutions, then $SBPSO_{ACD}$ is a compelling option. The performance of $SBPSO_{ACD}$ for portfolio optimization is compared with other algorithms in the next section.

6.5 Modified Set-based Particle Swarm Optimization for Portfolio Optimization in Comparison with Other Algorithms

This section compares the performance of SBPSO_{m6} and SBPSO_{ACD} against that of ABC, FA, GA, PSO and original SBPSO for portfolio optimization. Both SBPSO_{m6} and SBPSO_{ACD} are used in this comparison as they represent the accumulative gain in performance over the original approach. SBPSO_{m6} has been shown to deliver good quality results quickly, while SBPSO_{ACD} obtained better results than SBPSO_{m6}, but was slower.

The Hang Seng results are given in Section 6.5.1. Section 6.5.2 discusses the DAX 100 results. Sections 6.5.3 and 6.5.4 present the FTSE and S&P results respectively. The Nikkei 225 results are discussed in Section 6.5.5.

6.5.1 Hang Seng

Table 6.9 shows that the average results for risk, GD, IGD and HV of SBPSO_{m6} and SBPSO_{ACD} were better than those of SBPSO. Furthermore, both of the modified SBPSOs were faster than the original SBPSO, which was already faster than the other algorithms. The rankings in Table 6.10 show that, overall, SBPSO_{ACD} outperformed all other algorithms, a significant improvement over the original SBPSO. SBPSO_{m6} ranked third overall, also an improvement over the original approach. With regard to risk and return, SBPSO_{m6} and SBPSO_{ACD} ranked similarly to the PSO. The original SBPSO on the other hand ranked second last for risk and return. Lastly, Figure 6.11 shows that SBPSO_{m6} and SBPSO_{ACD} were also able to approximate the true POF as well as the other algorithms.

Table 6.9: Hang Seng results for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
ABC	$\bar{x}$	37	14	0.007848	0.002110	0.000176	0.000210	0.814630	11.04
	σ	2	2	0.000142	0.000125	0.000009	0.000000	0.000000	0.41
FA	$\bar{x}$	38	5	0.006084	0.001783	0.003959	0.000556	0.748081	38.95
	σ	4	1	0.000592	0.000268	0.000485	0.000085	0.028972	3.71
GA	$\bar{x}$	32	3	0.007576	0.001884	0.000279	0.000222	0.779399	18.10
	σ	1	0	0.000052	0.000042	0.000065	0.000005	0.025320	0.64
PSO	$\bar{x}$	33	3	0.007607	0.001897	0.000219	0.000212	0.786254	15.02
	σ	1	0	0.000003	0.000001	0.000049	0.000002	0.026135	0.69
SBPSO	$\bar{x}$	34	3	0.007614	0.001908	0.000463	0.000236	0.776926	9.37
	σ	1	0	0.000022	0.000009	0.000141	0.000010	0.023747	0.53
SBPSO _{m6}	$\bar{x}$	33	3	0.007607	0.001897	0.000218	0.000212	0.781949	4.52
	σ	0	0	0.000003	0.000000	0.000045	0.000002	0.026784	2.87
SBPSO _{ACD}	$\bar{x}$	34	3	0.007600	0.001891	0.000190	0.000210	0.814630	2.17
	σ	0	0	0.000025	0.000021	0.000003	0.000000	0.000000	2.31

Table 6.10: Hang Seng rankings for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time	Overall
ABC	Wins	5	0	6	0	6	5	5	3	30
	Losses	0	6	0	6	0	0	0	3	15
	Draws	1	0	0	0	0	1	1	0	3
	Difference	5	-6	6	-6	6	5	5	0	15
	Rank	1	6	1	6	1	1	1	4	2
FA	Wins	5	1	0	5	0	0	0	0	11
	Losses	0	5	6	0	6	6	6	6	35
	Draws	1	0	0	1	0	0	0	0	2
	Difference	5	-4	-6	5	-6	-6	-6	-6	-24
	Rank	1	5	6	1	6	5	5	7	7
GA	Wins	0	2	1	2	2	2	1	1	11
	Losses	6	4	4	0	4	4	3	5	30
	Draws	0	0	1	4	0	0	2	0	7
	Difference	-6	-2	-3	2	-2	-2	-2	-4	-19
	Rank	4	4	5	3	4	3	4	6	6
PSO	Wins	1	3	2	2	3	3	2	2	18
	Losses	3	3	1	2	2	2	2	4	19
	Draws	2	0	3	2	1	1	2	0	11
	Difference	-2	0	1	0	1	1	0	-2	-1
	Rank	3	3	3	4	3	2	2	5	4
SBPSO	Wins	1	6	3	1	1	1	1	4	18
	Losses	3	0	1	5	5	5	2	2	23
	Draws	2	0	2	0	0	0	3	0	7
	Difference	-2	6	2	-4	-4	-4	-1	2	-5
	Rank	3	1	2	5	5	4	3	3	5
SBPSO _{m6}	Wins	1	4	2	2	3	3	1	5	21
	Losses	3	1	1	2	2	2	2	1	14
	Draws	2	1	3	2	1	1	3	0	13
	Difference	-2	3	1	0	1	1	-1	4	7
	Rank	3	2	3	4	3	2	3	2	3
SBPSO _{ACD}	Wins	4	4	1	4	5	5	5	6	34
	Losses	2	1	2	1	1	0	0	0	7
	Draws	0	1	3	1	0	1	1	0	7
	Difference	2	3	-1	3	4	5	5	6	27
	Rank	2	2	4	2	2	1	1	1	1

Figure 6.11: Obtained Pareto-optimal fronts for Hang Seng

6.5.2 DAX 100

For the DAX 100 problem, SBPSO_{m6} and SBPSO_{ACD} were the best and second best performing algorithms, respectively, as shown in Table 6.12. SBPSO_{m6} ranked second in the IGD and HV measures, beating all other algorithms except for ABC. Similarly, SBPSO_{ACD} ranked first for the GD measure. In comparison, the original SBPSO ranked second last for all the Pareto optimality measures. Table 6.11 shows that SBPSO_{m6} was extremely fast in comparison with the other algorithms. SBPSO_{ACD} was slightly slower than SBPSO, but still faster than ABC, FA, PSO and GA. Table 6.11 also shows that the average GD scores of SBPSO_{m6} and SBPSO_{ACD} were an order of magnitude better

than SBPSO. A visual analysis of the obtained POFs in Figure 6.12 shows that SBPSO_{m6} and SBPSO_{ACD} were able to approximate the true POF as well as ABC and better than the PSO, FA, GA and original SBPSO.

Table 6.11: DAX 100 results for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
ABC	$\bar{x}$	49	14	0.008633	0.001219	0.000228	0.000304	0.787923	43.68
	σ	1	1	0.000021	0.000029	0.000033	0.000000	0.000000	0.56
FA	$\bar{x}$	34	11	0.004485	0.000768	0.003702	0.000568	0.844798	166.26
	σ	7	2	0.000606	0.000143	0.000571	0.000103	0.033878	21.28
GA	$\bar{x}$	39	10	0.008387	0.000953	0.000295	0.000307	0.893540	50.86
	σ	1	0	0.000034	0.000025	0.000152	0.000004	0.057716	2.02
PSO	$\bar{x}$	42	5	0.008456	0.000967	0.000222	0.000304	0.816574	52.40
	σ	1	0	0.000017	0.000013	0.000053	0.000001	0.057681	1.73
SBPSO	$\bar{x}$	34	6	0.007670	0.000960	0.002334	0.000357	0.826267	26.87
	σ	5	1	0.000319	0.000105	0.000527	0.000019	0.054478	1.63
SBPSO _{m6}	$\bar{x}$	43	4	0.008469	0.000972	0.000250	0.000306	0.878138	8.99
	σ	0	0	0.000006	0.000000	0.000061	0.000008	0.065429	3.19
SBPSO _{ACD}	$\bar{x}$	42	4	0.008460	0.000971	0.000209	0.000346	0.804378	28.09
	σ	9	0	0.000001	0.000000	0.000000	0.000000	0.165428	45.21

Table 6.12: DAX 100 rankings for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time	Overall
ABC	Wins	6	0	6	0	3	5	0	3	23
	Losses	0	6	0	6	1	0	6	3	22
	Draws	0	0	0	0	2	1	0	0	3
	Difference	6	-6	6	-6	2	5	-6	0	1
	Rank	1	6	1	7	3	1	5	4	4
FA	Wins	0	1	0	6	0	0	3	0	10
	Losses	5	5	6	0	6	6	2	6	36
	Draws	1	0	0	0	0	0	1	0	2
	Difference	-5	-4	-6	6	-6	-6	1	-6	-26
	Rank	5	5	6	1	7	6	3	7	7
GA	Wins	2	2	2	4	2	3	6	2	23
	Losses	4	4	4	1	4	2	0	4	23
	Draws	0	0	0	1	0	1	0	0	2
	Difference	-2	-2	-2	3	-2	1	6	-2	0
	Rank	4	4	4	2	5	3	1	5	5
PSO	Wins	3	4	3	3	4	4	1	1	23
	Losses	2	2	2	2	0	1	2	5	16
	Draws	1	0	1	1	2	1	3	0	9
	Difference	1	2	1	1	4	3	-1	-4	7
	Rank	3	2	3	3	2	2	4	6	3
SBPSO	Wins	0	3	1	1	1	1	1	5	13
	Losses	5	3	5	1	5	5	2	1	27
	Draws	1	0	0	4	0	0	3	0	8
	Difference	-5	0	-4	0	-4	-4	-1	4	-14
	Rank	5	3	5	4	6	5	4	2	6
SBPSO _{m6}	Wins	5	5	5	1	3	3	3	6	31
	Losses	1	0	1	4	2	0	1	0	9
	Draws	0	1	0	1	1	3	2	0	8
	Difference	4	5	4	-3	1	3	2	6	22
	Rank	2	1	2	6	4	2	2	1	1
SBPSO _{ACD}	Wins	3	5	3	2	5	2	1	4	25
	Losses	2	0	2	3	0	4	2	2	15
	Draws	1	1	1	1	1	0	3	0	8
	Difference	1	5	1	-1	5	-2	-1	2	10
	Rank	3	1	3	5	1	4	4	3	2

Figure 6.12: Obtained Pareto-optimal fronts for DAX 100

6.5.3 FTSE 100

Table 6.13 show that the averages and standard deviations of the results obtained by SBPSO_{m6} were better than those of SBPSO. Likewise, SBPSO_{ACD} also obtained results that were on average better than SBPSO with lower standard deviations, except for HV and time. SBPSO_{ACD}, on average, obtained portfolios that were slightly more profitable than PSO with slightly less risk. SBPSO_{m6} was also competitive with regard to risk and return as it obtained portfolios that were on average more profitable than FA, GA, PSO, SBPSO and SBPSO_{ACD} at a similar level of risk. Table 6.14 shows that, as for the previous benchmark problem, SBPSO_{m6} ranked first overall and SBPSO_{ACD}

ranked second overall. However, SBPSO_{ACD} ranked first in risk, GD and IGD. In Chapter 4 the FA ranked first for risk and last for GD and IGD. Therefore, SBPSO_{ACD} is very competitive in that it can obtain solutions as close to the true POF that are extremely risk averse. SBPSO_{m6} generally ranked second and third for the performance measures, but ranked first for time and the size of the solutions obtained. The improvements in performance made by SBPSO_{ACD} and SBPSO_{m6} over SBPSO can also be seen in the obtained POFs shown in Figure 6.13. SBPSO_{ACD} and SBPSO_{m6} obtained POFs that closely resemble the true POF, outperforming the FA, GA, PSO and SBPSO. In comparison, the original SBPSO was the second-worst algorithm with regard to approximating the true POF.

Table 6.13: FTSE 100 results for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
ABC	$\bar{x}$	49	16	0.007277	0.001073	0.000166	0.000232	0.679689	47.09
	σ	1	1	0.000011	0.000005	0.000020	0.000000	0.020228	1.02
FA	$\bar{x}$	36	12	0.004745	0.000779	0.003347	0.000467	0.714994	186.79
	σ	4	2	0.000445	0.000098	0.000356	0.000087	0.013245	25.15
GA	$\bar{x}$	29	12	0.006635	0.000778	0.000408	0.000238	0.766213	58.45
	σ	1	1	0.000163	0.000029	0.000321	0.000024	0.035444	1.29
PSO	$\bar{x}$	29	6	0.006667	0.000782	0.000190	0.000232	0.704657	59.86
	σ	0	0	0.000008	0.000001	0.000039	0.000001	0.049243	0.95
SBPSO	$\bar{x}$	33	6	0.006605	0.000822	0.001206	0.000264	0.725351	13.81
	σ	6	2	0.000176	0.000061	0.000472	0.000017	0.037266	5.43
SBPSO _{m6}	$\bar{x}$	30	5	0.006683	0.000785	0.000247	0.000235	0.741994	10.57
	σ	0	0	0.000007	0.000001	0.000038	0.000007	0.049170	3.55
SBPSO _{ACD}	$\bar{x}$	29	5	0.006655	0.000775	0.000169	0.000232	0.711504	35.75
	σ	1	0	0.000024	0.000011	0.000019	0.000001	0.051801	61.04

Table 6.14: FTSE 100 rankings for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time	Overall
ABC	Wins	6	0	6	0	5	4	0	3	24
	Losses	0	6	0	6	0	1	5	3	21
	Draws	0	0	0	0	1	1	1	0	3
	Difference	6	-6	6	-6	5	3	-5	0	3
	Rank	1	4	1	5	1	2	6	4	3
FA	Wins	5	1	0	2	0	0	3	0	11
	Losses	1	4	6	0	6	6	3	6	32
	Draws	0	1	0	4	0	0	0	0	5
	Difference	4	-3	-6	2	-6	-6	0	-6	-21
	Rank	2	3	6	2	6	5	3	7	7
GA	Wins	0	1	1	2	2	2	6	2	16
	Losses	4	4	2	0	4	3	0	4	21
	Draws	2	1	3	4	0	1	0	0	11
	Difference	-4	-3	-1	2	-2	-1	6	-2	-5
	Rank	4	3	4	2	4	3	1	5	6
PSO	Wins	0	3	2	3	4	4	0	1	17
	Losses	4	3	2	1	2	1	2	5	20
	Draws	2	0	2	2	0	1	4	0	11
	Difference	-4	0	0	2	2	3	-2	-4	-3
	Rank	4	2	3	2	2	2	5	6	4
SBPSO	Wins	3	4	1	1	1	1	2	5	18
	Losses	2	0	2	5	5	5	2	1	22
	Draws	1	2	3	0	0	0	2	0	8
	Difference	1	4	-1	-4	-4	-4	0	4	-4
	Rank	3	1	4	4	5	4	3	2	5
SBPSO _{m6}	Wins	3	4	5	2	3	2	3	6	28
	Losses	2	0	1	2	3	3	1	0	12
	Draws	1	2	0	2	0	1	2	0	8
	Difference	1	4	4	0	0	-1	2	6	16
	Rank	3	1	2	3	3	3	2	1	1
SBPSO _{ACD}	Wins	0	4	1	4	5	6	1	4	25
	Losses	4	0	3	0	0	0	2	2	11
	Draws	2	2	2	2	1	0	3	0	12
	Difference	-4	4	-2	4	5	6	-1	2	14
	Rank	4	1	5	1	1	1	4	3	2

Figure 6.13: Obtained Pareto-optimal fronts for FTSE 100

6.5.4 S&P 100

This benchmark problem was particularly difficult for the original SBPSO as discussed in Section 6.5.3. Table 6.15 shows that SBPSO_{m6} and SBPSO_{ACD} made significant gains in performance over the original SBPSO. Specifically, SBPSO_{m6} and SBPSO_{ACD} on average obtained more profitable portfolios than SBPSO at a similar level of risk. Furthermore, these portfolios were significantly closer to the true POF, as indicated by the lower GD and IGD scores. Table 6.16 shows that SBPSO_{ACD} ranked first for all three Pareto optimality measures and ranked first overall. However, the increase in performance made by SBPSO_{ACD} is at the cost of a longer time to reach the stopping condi-

tion. SBPSO_{ACD} ranked second last, behind the FA, with regard to time to reach the stopping condition. In contrast, the original SBPSO ranked second in time. SBPSO_{m6} did not rank as well as SBPSO_{ACD}, ABC or PSO for GD and IGD, but did obtain similar average values while being statistically significantly faster. Furthermore, SBPSO_{m6} ranked second overall. Lastly, Figure 6.14 shows that SBPSO_{m6} and SBPSO_{ACD} were able to approximate the true POF much better than the original SBPSO and better than the GA, FA and PSO.

6.15

Table 6.15: S&P 100 results for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
ABC	$\bar{x}$	49	24	0.007843	0.001398	0.000236	0.000241	0.841527	56.54
	σ	1	3	0.000031	0.000035	0.000031	0.000000	0.018891	1.21
FA	$\bar{x}$	38	18	0.005203	0.000891	0.003536	0.000535	0.825091	230.75
	σ	5	2	0.000691	0.000246	0.000453	0.000093	0.016919	24.39
GA	$\bar{x}$	35	6	0.007430	0.001177	0.000810	0.000260	0.881914	54.52
	σ	2	0	0.000088	0.000052	0.000265	0.000007	0.042699	1.31
PSO	$\bar{x}$	44	8	0.007705	0.001245	0.000269	0.000241	0.892730	64.55
	σ	1	0	0.000010	0.000012	0.000096	0.000001	0.052169	1.26
SBPSO	$\bar{x}$	43	9	0.007269	0.001219	0.001775	0.000301	0.890120	35.69
	σ	2	3	0.000428	0.000174	0.000620	0.000036	0.031804	5.56
SBPSO _{m6}	$\bar{x}$	44	6	0.007716	0.001248	0.000275	0.000247	0.875057	16.45
	σ	0	0	0.000006	0.000001	0.000044	0.000003	0.044175	2.78
SBPSO _{ACD}	$\bar{x}$	44	6	0.007708	0.001244	0.000205	0.000242	0.872137	91.53
	σ	0	0	0.000001	0.000000	0.000022	0.000001	0.049381	166.15

Table 6.16: S&P 100 rankings for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time	Overall
ABC	Wins	6	0	6	0	4	4	1	3	24
	Losses	0	6	0	6	1	0	5	3	21
	Draws	0	0	0	0	1	2	0	0	3
	Difference	6	-6	6	-6	3	4	-4	0	3
	Rank	1	6	1	7	2	1	2	4	4
FA	Wins	1	1	0	6	0	0	0	0	8
	Losses	5	5	6	0	6	6	6	6	40
	Draws	0	0	0	0	0	0	0	0	0
	Difference	-4	-4	-6	6	-6	-6	-6	-6	-32
	Rank	4	5	5	1	6	5	3	7	7
GA	Wins	0	6	1	5	2	2	2	4	22
	Losses	6	0	4	1	4	4	0	2	21
	Draws	0	0	1	0	0	0	4	0	5
	Difference	-6	6	-3	4	-2	-2	2	2	1
	Rank	5	1	4	2	4	3	1	3	5
PSO	Wins	3	3	3	2	4	4	2	2	23
	Losses	1	3	2	4	1	0	0	4	15
	Draws	2	0	1	0	1	2	4	0	10
	Difference	2	0	1	-2	3	4	2	-2	8
	Rank	2	3	3	5	2	1	1	5	3
SBPSO	Wins	2	2	1	4	1	1	2	5	18
	Losses	4	4	4	2	5	5	0	1	25
	Draws	0	0	1	0	0	0	4	0	5
	Difference	-2	-2	-3	2	-4	-4	2	4	-7
	Rank	3	4	4	3	5	4	1	2	6
SBPSO _{m6}	Wins	3	4	5	1	3	3	2	6	27
	Losses	1	1	1	5	3	3	0	0	14
	Draws	2	1	0	0	0	0	4	0	7
	Difference	2	3	4	-4	0	0	2	6	13
	Rank	2	2	2	6	3	2	1	1	2
SBPSO _{ACD}	Wins	3	4	3	3	6	4	2	1	26
	Losses	1	1	2	3	0	0	0	5	12
	Draws	2	1	1	0	0	2	4	0	10
	Difference	2	3	1	0	6	4	2	-4	14
	Rank	2	2	3	4	1	1	1	6	1

Figure 6.14: Obtained Pareto-optimal fronts for S&P 100

6.5.5 Nikkei 225

Table 6.17 shows that SBPSO_{m6} and SBPSO_{ACD} were very competitive with regard to risk and return. The portfolios obtained by SBPSO_{m6} and SBPSO_{ACD} were more profitable than the FA, GA, PSO and SBPSO, while being less risky. Furthermore, SBPSO_{m6} and SBPSO_{ACD} ranked first and second overall, respectively, as shown in Table 6.18. In comparison, the original SBPSO, despite being the fastest algorithm to reach the stopping condition, ranked second last. SBPSO_{ACD} was statistically significantly better than the other algorithms for the GD and IGD measure. However, SBPSO_{ACD} was statistically significantly slower than the other algorithms to reach the stopping condition

except for the FA. In contrast, SBPSO_{m6} was the second fastest algorithm to reach the stopping condition and ranked second and fourth for the GD and IGD measures, respectively. In addition to the high rankings of SBPSO_{m6} and SBPSO_{ACD}, Figure 6.15 shows that SBPSO_{m6} and SBPSO_{ACD} were the only algorithms to approximate the true POF - a significant gain in performance over the original approach.

Table 6.17: Nikkei 225 results for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
ABC	$\bar{x}$	49	24	0.003348	0.000858	0.000470	0.000210	0.924789	254.99
	σ	0	6	0.000022	0.000015	0.000221	0.000001	0.046806	2.91
FA	$\bar{x}$	28	12	0.001086	0.000962	0.004191	0.000490	0.734680	1087.28
	σ	6	7	0.000484	0.000114	0.000561	0.000056	0.064953	117.14
GA	$\bar{x}$	46	22	0.003185	0.000858	0.001702	0.000276	0.906150	248.82
	σ	4	3	0.000105	0.000068	0.000344	0.000019	0.043527	2.82
PSO	$\bar{x}$	44	7	0.003264	0.000782	0.000703	0.000218	0.957354	266.89
	σ	2	0	0.000023	0.000022	0.000126	0.000008	0.037560	4.40
SBPSO	$\bar{x}$	42	10	0.002454	0.000800	0.002396	0.000353	0.826649	29.22
	σ	4	5	0.000592	0.000065	0.000535	0.000065	0.067572	4.19
SBPSO _{m6}	$\bar{x}$	44	4	0.003310	0.000794	0.000260	0.000223	0.916014	63.19
	σ	0	0	0.000007	0.000001	0.000060	0.000005	0.047097	6.13
SBPSO _{ACD}	$\bar{x}$	44	4	0.003305	0.000791	0.000139	0.000209	0.898166	342.28
	σ	0	0	0.000001	0.000001	0.000027	0.000001	0.024326	59.95

Table 6.18: Nikkei 225 rankings for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time	Overall
ABC	Wins	6	0	6	1	4	5	3	3	28
	Losses	0	6	0	4	2	1	0	3	16
	Draws	0	0	0	1	0	0	3	0	4
	Difference	6	-6	6	-3	2	4	3	0	12
	Rank	1	5	1	4	3	2	2	4	3
FA	Wins	0	2	0	0	0	0	0	0	2
	Losses	6	3	6	6	6	6	6	6	45
	Draws	0	1	0	0	0	0	0	0	1
	Difference	-6	-1	-6	-6	-6	-6	-6	-6	-43
	Rank	6	3	7	5	7	7	7	7	7
GA	Wins	5	1	2	1	2	2	3	4	20
	Losses	1	5	4	4	4	4	1	2	25
	Draws	0	0	0	1	0	0	2	0	3
	Difference	4	-4	-2	-3	-2	-2	2	2	-5
	Rank	2	4	5	4	5	5	3	3	5
PSO	Wins	1	4	3	5	3	4	5	2	27
	Losses	2	2	3	0	3	2	0	4	16
	Draws	3	0	0	1	0	0	1	0	5
	Difference	-1	2	0	5	0	2	5	-2	11
	Rank	4	2	4	1	4	3	1	5	4
SBPSO	Wins	1	2	1	3	1	1	1	6	16
	Losses	4	3	5	0	5	5	5	0	27
	Draws	1	1	0	3	0	0	0	0	5
	Difference	-3	-1	-4	3	-4	-4	-4	6	-11
	Rank	5	3	6	2	6	6	6	1	6
SBPSO _{m6}	Wins	3	5	5	3	5	3	2	5	31
	Losses	2	0	1	2	1	3	1	1	11
	Draws	1	1	0	1	0	0	3	0	6
	Difference	1	5	4	1	4	0	1	4	20
	Rank	3	1	2	3	2	4	4	2	1
SBPSO _{ACD}	Wins	2	5	4	4	6	6	2	1	30
	Losses	3	0	2	1	0	0	3	5	14
	Draws	1	1	0	1	0	0	1	0	4
	Difference	-1	5	2	3	6	6	-1	-4	16
	Rank	4	1	3	2	1	1	5	6	2

Figure 6.15: Obtained Pareto-optimal fronts for Nikkei 225

6.6 Summary

It was hypothesized at the end of Chapter 4 that the gap in performance between set-based particle swarm optimization (SBPSO) the other algorithms, namely artificial bee colony (ABC), genetic algorithm (GA), and particle swarm optimization (PSO), was due to an inadequate weight determinate stage and an imbalance in exploration and exploitation behaviour. An in depth look into the diversity and performance of SBPSO confirmed these hypotheses. In doing so, several SBPSO modifications were studied. The modification to allow the weight optimizer to execute until there was no improvement in solution quality, instead of a fixed number of time steps, resulted in significant gains

in performance. The imbalance in exploration and exploitation behaviour was corrected with the introduction of an exploration balance coefficient. The modified SBPSO with a completely deterministic weight optimizer using adaptive coordinate descent (ACD) was also investigated. Performance analysis showed that ACD variant improved the closeness of the found portfolios to the true POFs, but at the cost of a longer search time. Finally, the performance of SBPSO with the modifications and the ACD variant was evaluated against the ABC, firefly algorithm (FA), GA, PSO, and original SBPSO, as seen in Chapter 4. The results showed that new SBPSO approaches outperformed the other algorithms.

Chapter 7

Set-based Particle Swarm Optimization for Multi-objective Portfolio Optimization

Portfolio optimization, by definition, is a MOOP with risk and profit, or some form of the two, as competing objectives. Single-objective portfolio optimization requires a trade-off coefficient to be specified in order to balance the two objectives. A collection of equally good, but different, solutions can then be obtained by solving the single-objective optimization problem for various trade-off coefficient values. A more sophisticated and appropriate approach would be to use a MOOP solving algorithm to identify an equally spread set of non-dominated solutions. This chapter¹ proposes the first multi-objective optimization (MOO) approach to SBPSO. Other approaches to MOOPs for sets by Khan and Engelbrecht used a fuzzy aggregation operator, namely the unified And-Or operator, to convert a multi-objective problem formulation into a fuzzy single-objective formulation [98]. The proposed MOO approach to SBPSO does not require the use of an aggregation operator, but rather optimizes the multiple objectives independently. Furthermore, the proposed approach makes use of the improvements to SBPSO for portfolio optimization identified in the previous chapter. Thus, assets are selected that maximize return and minimize risk. Weight determination for the assets is also a MOOP for which MGPSO (refer to Chapter 2.3) is used. For optimal performance, the parameters of MGPSO need to be tuned for portfolio optimization. However, parameter tuning is often an expensive and time-consuming process. An approach to randomly sample control parameter values for MGPSO that satisfy stability conditions is also investigated in this chapter.

The remainder of this chapter is organized as follows: Section 7.1 investi-

¹This chapter is currently being reviewed for the IEEE Swarm Intelligence Symposium under the title "A tuning free approach to multi-guide particle swarm optimization" [55] and the journal IEEE Transactions on Evolutionary Computation under the title "Multi-guide set-based particle swarm optimization for portfolio optimization" [57].

gates the approach of randomly sampling control parameter values that satisfy stability conditions for MGPSPSO. Section 7.2 proposes multi-guide set-based particle swarm optimization (MGSBPSO) for portfolio optimization. The empirical process used to assess the proposed approach is given in Section 7.3. The results are discussed in Section 7.4 and the findings are summarised in Section 7.5.

7.1 Tuning-free Approach to Multi-guide Particle Swarm Optimization

This section proposes a tuning-free approach to MGPSPSO that uses control parameter values sampled from the stability region defined in equation (2.8). Section 7.1.1 proposes the tuning-free approach to MGPSPSO. The experimental process and results empirically validating the tuning-free approach are presented in Sections 7.1.2 and 7.1.3 respectively.

7.1.1 Proposal

A formal control parameter performance study for MGPSPSO showed that its performance is sensitive to its control parameters [152]. Furthermore, Carolus and Engelbrecht showed that the inertia component value has the greatest sensitivity and is the most important control parameter to tune when optimizing the MGPSPSO [16]. Equation (2.8) defines the theoretical stability conditions of the MGPSPSO control parameters to guarantee order-1 and order-2 stability. If the control parameter values satisfy the stability conditions, the swarm is guaranteed to reach a state of equilibrium where the velocity of each particle is zero. The proposed approach is to then randomly sample control parameter values for the MGPSPSO from the stability region at every iteration for every particle. Since each particle samples its own values for the control parameters, a mix of search behaviours will be used by the swarms. If the randomly generated control parameter values do not satisfy the MGPSPSO stability conditions of equation (2.8), then they are discarded and new control parameter values are generated. This process is repeated until control parameter values that satisfy equation (2.8) are generated. However, to avoid MGPSPSO from spending too much time on generating parameter values, the process is limited to ten attempts. If no parameter values generated satisfy the stability conditions, the current parameters are reused. Therefore, this is a tuning-free approach to MGPSPSO with a guarantee that the swarm will reach a state of equilibrium. It is expected that the tuning-free approach, referred to as stable MGPSPSO or MGPSPSO_s for short, will perform similarly to the MGPSPSO using tuned control parameter values.

7.1.2 Empirical Process

The WFG tool kit (described in Appendix E) was used to test if MGPSO_s performs similarly to MGPSO using tuned parameter values, listed in Table 7.1. The WFG tool kit can be used to generate scalable MOOPs. In this study, two-objective and three-objective benchmark problems were used with 10, 30 and 50 dimensions. The jMetal framework² provided POFs used to measure the GD, IGD and HV scores for each algorithm. These scores were recorded over 30 independent runs, where each independent run lasted 2000 iterations. The archive of each MGPSO algorithm at the end of each independent run was regarded as the obtained POF for that independent run. The MGPSO algorithms were implemented and executed using the Cilib framework [141]. Lastly, MGPSO_s used the same tournament sizes as MGPSO to ensure that any differences in performance between the two were a direct result of the parameter value generation.

Table 7.1: MGPSO control parameters used for each WFG test problem

3	0.275	1.65	1.80	1.75	Problem	Objectives	$ S_1 $	$ S_2 $	$ S_3 $	T	w	c_1	c_2	c_3
WFG1						2	45	5		3	0.275	1.65	1.75	1.05
WFG2						2	24	26		2	0.750	1.15	1.70	1.05
WFG3						2	31	19		2	0.600	1.60	1.85	0.95
WFG4						2	2	48		2	0.100	0.80	1.65	1.70
WFG5						2	50	0		2	0.600	0.80	1.60	1.85
WFG6						2	19	31		2	0.525	0.65	0.60	1.65
WFG7						2	29	21		2	0.450	1.20	1.85	1.55
WFG8						2	37	13		3	0.750	1.00	1.65	1.05
WFG9						2	13	37		2	0.275	1.00	0.50	1.70
WFG1						3	37	4	9	2	0.125	1.20	1.30	1.75
WFG2						3	24	25	1	2	0.275	1.25	1.40	1.70
WFG3						3	29	10	11	2	0.525	1.65	1.75	0.75
WFG4						3	29	21	0	2	0.275	1.75	0.50	1.05
WFG5						3	2	48	0	3	0.575	0.60	1.85	1.75
WFG6						3	5	30	15	3	0.300	0.90	0.90	1.90
WFG7						3	10	22	18	2	0.425	1.45	1.50	1.40
WFG8						3	4	23	23	3	0.425	0.95	1.75	1.85
WFG9						3	4	45	1	2	0.275	1.25	0.75	1.50

7.1.3 Results

Table 7.2 shows that the average scores between the two approaches for all two-objective WFG problems were very similar, with marginal differences. Likewise, Table 7.3 shows that the algorithms performed very similarly. There was no case where the performance between either algorithm differ by an order of magnitude. Furthermore, performance was consistent as the dimensionality of the problems increased. Table 7.4 shows that there was no statistical significant difference between the MGPSO with tuned parameters and MGPSO_s

²<https://github.com/jMetal/jMetal>

with regard to GD and IGD for all two-objective WFG problems. A similar result was observed in Table 7.5 where there was no statistical significant difference between the IGD scores of the two algorithms for three-objective WFG problems. MGPSO performed better on three-objective WFG3 with regard to GD for all dimensions, otherwise there was no statistically meaningful difference in the three-objective WFG GD results. The HV scores, however, were not as cut and dry. For the 10 dimensional two-objective WFG problems, MGPSO_s scored more wins than the tuned MGPSO. However, the number of wins by MGPSO_s deteriorated as the dimensionality increased. Nevertheless, there were cases where MGPSO_s scored higher HV values on average, such as the 50 dimensional WFG2 and WFG6 problems. For the three-objective WFG HV results, the algorithms reached a draw for more problems than for the two-objective WFG HV results, but MGPSO still has the advantage. Overall, the results show that the performance of MGPSO with control parameter values randomly sampled from the stability region defined in equation (2.8) was similar to that of the MGPSO using tuned parameters. Furthermore, MGPSO_s did not require computationally expensive control parameter tuning unlike MGPSO.

Table 7.2: MGPSO versus MGPSO_s: Mean results for 2-objective WFG

		10d			30d			50d		
		GD	IGD	HV	GD	IGD	HV	GD	IGD	HV
WFG1	MGPSO	0.4874	0.0930	4.7760	0.4912	0.0930	1.9958	0.4912	0.0930	1.8051
	MGPSO _s	0.4866	0.0930	5.8049	0.4915	0.0930	2.0912	0.4908	0.0930	1.9070
WFG2	MGPSO	0.4844	0.1200	6.1056	0.4849	0.1199	5.6518	0.4849	0.1199	5.3721
	MGPSO _s	0.4845	0.1200	6.0987	0.4846	0.1200	6.0069	0.4848	0.1200	5.8025
WFG3	MGPSO	0.4860	0.0927	5.5900	0.4863	0.0927	5.1302	0.4865	0.0927	4.7155
	MGPSO _s	0.4860	0.0927	5.5930	0.4862	0.0927	5.3485	0.4863	0.0927	5.0490
WFG4	MGPSO	0.4847	0.0923	3.0780	0.4848	0.0922	2.7094	0.4849	0.0922	2.5551
	MGPSO _s	0.4847	0.0922	3.0084	0.4850	0.0922	2.5778	0.4850	0.0922	2.3641
WFG5	MGPSO	0.4846	0.0922	2.9002	0.4846	0.0922	2.8856	0.4846	0.0922	2.8741
	MGPSO _s	0.4846	0.0922	2.9177	0.4847	0.0922	2.9076	0.4845	0.0923	2.8987
WFG6	MGPSO	0.4846	0.0922	3.0825	0.4848	0.0922	2.7441	0.4849	0.0922	2.5826
	MGPSO _s	0.4846	0.0922	3.1527	0.4847	0.0922	3.0301	0.4847	0.0922	2.8607
WFG7	MGPSO	0.4846	0.0923	3.3350	0.4846	0.0923	3.2523	0.4847	0.0922	3.0524
	MGPSO _s	0.4845	0.0923	3.3359	0.4846	0.0923	3.2933	0.4847	0.0922	3.1298
WFG8	MGPSO	0.4863	0.0919	1.9608	0.4850	0.0921	2.2243	0.4850	0.0921	2.1365
	MGPSO _s	0.4900	0.0918	1.8354	0.4849	0.0921	2.2662	0.4849	0.0921	2.3031
WFG9	MGPSO	0.4846	0.0922	3.0987	0.4848	0.0922	2.7293	0.4849	0.0922	2.6676
	MGPSO _s	0.4846	0.0922	3.1148	0.4849	0.0922	2.7660	0.4849	0.0922	2.6310

Table 7.3: MGPSO versus MGPSO_s: Mean results for 3-objective WFG

		10d			30d			50d		
		GD	IGD	HV	GD	IGD	HV	GD	IGD	HV
WFG1	MGPSO	0.4978	0.0821	21.2980	0.4992	0.0822	18.9183	0.4989	0.0822	18.8586
	MGPSO _s	0.4943	0.0822	27.0259	0.4986	0.0822	19.1579	0.4987	0.0822	18.7677
WFG2	MGPSO	0.4890	0.0692	54.6878	0.4886	0.0694	51.6535	0.4893	0.0693	49.4731
	MGPSO _s	0.4889	0.0693	53.6745	0.4884	0.0694	50.0681	0.4885	0.0695	46.9867
WFG3	MGPSO	0.5125	0.0371	6.1599	0.5116	0.0369	5.0631	0.5117	0.0368	4.6463
	MGPSO _s	0.5130	0.0372	6.4064	0.5125	0.0371	5.7365	0.5125	0.0371	5.2145
WFG4	MGPSO	0.4869	0.0371	29.0554	0.4871	0.0371	26.8826	0.4871	0.0370	25.7538
	MGPSO _s	0.4869	0.0371	26.8352	0.4870	0.0371	25.0841	0.4871	0.0371	24.8821
WFG5	MGPSO	0.4866	0.0364	27.8030	0.4867	0.0364	26.7786	0.4867	0.0363	26.4431
	MGPSO _s	0.4865	0.0364	28.6180	0.4866	0.0364	27.4528	0.4867	0.0364	27.0387
WFG6	MGPSO	0.4868	0.0365	30.2637	0.4870	0.0365	26.0443	0.4873	0.0365	24.6185
	MGPSO _s	0.4868	0.0365	30.4891	0.4868	0.0365	30.0679	0.4871	0.0365	27.6585
WFG7	MGPSO	0.4868	0.0365	31.7465	0.4869	0.0365	30.2471	0.4870	0.0365	28.4909
	MGPSO _s	0.4868	0.0365	31.9518	0.4869	0.0365	30.6151	0.4871	0.0365	29.4147
WFG8	MGPSO	0.4881	0.0362	21.4920	0.4876	0.0363	22.5010	0.4875	0.0363	22.0185
	MGPSO _s	0.4880	0.0362	21.2494	0.4875	0.0363	22.7724	0.4874	0.0363	22.2965
WFG9	MGPSO	0.4867	0.0363	27.9181	0.4868	0.0363	25.3823	0.4871	0.0363	24.6624
	MGPSO _s	0.4866	0.0363	27.4443	0.4867	0.0363	26.1049	0.4870	0.0363	25.4001

Table 7.4: 2-objective WFG rankings for each performance measure

		10d			30d			50d			Overall
Algorithm	Result	GD	IGD	HV	GD	IGD	HV	GD	IGD	HV	
MGPSO	Wins	0	0	3	0	0	6	0	0	7	16
	Losses	0	0	3	0	0	1	0	0	1	5
	Draws	9	9	3	9	9	2	9	9	1	60
	Difference	0	0	0	0	0	5	0	0	6	11
	Rank	1	1	1	1	1	1	1	1	1	1
MGPSO _s	Wins	0	0	3	0	0	1	0	0	1	5
	Losses	0	0	3	0	0	6	0	0	7	16
	Draws	9	9	3	9	9	2	9	9	1	60
	Difference	0	0	0	0	0	-5	0	0	-6	-11
	Rank	1	1	1	1	1	2	1	1	2	2

Table 7.5: 3-objective WFG rankings for each performance measure

Algorithm	Result	10d			30d			50d			Overall
		GD	IGD	HV	GD	IGD	HV	GD	IGD	HV	
MGPSO	Wins	0	0	4	0	0	6	0	0	4	14
	Losses	0	0	2	0	0	2	0	0	1	5
	Draws	9	9	3	9	9	1	9	9	4	62
	Difference	0	0	2	0	0	4	0	0	3	9
	Rank	1	1	1	1	1	1	1	1	1	1
MGPSO _s	Wins	0	0	2	0	0	2	0	0	1	5
	Losses	0	0	4	0	0	6	0	0	4	14
	Draws	9	9	3	9	9	1	9	9	4	62
	Difference	0	0	-2	0	0	-4	0	0	-3	-9
	Rank	1	1	2	1	1	2	1	1	2	2

7.2 A New Approach to Multi-objective Set-based Portfolio Optimization

This section proposes a multi-objective set-based algorithm for multi-objective portfolio optimization. Elements from MGPSO are incorporated into SBPSO to enable SBPSO to solve multiple objectives simultaneously. The proposed approach, referred to as MGSBPSO uses multiple swarms, where each swarm optimizes one of the multiple objectives in the asset space. Thus, there is a swarm to select assets that minimize risk and a swarm to select assets that maximize profit. As for MGPSO, non-dominated solutions found by the swarms are stored in an archive (initially empty) of a fixed size. The archive management process as described in Section 2.3.2 is also used in the MGSBPSO. However, the crowding distance of the non-dominated solutions in the archive is calculated with respect to their objective function values instead of their set-based positions, as the set-based positions lack distance in the traditional sense. Non-dominated solutions are selected from the archive using tournament selection and are used to guide the particles to non-dominated regions of the search space. Like the archive management strategy, the crowding distance of the non-dominated solutions in the tournament is calculated with respect to their objective function values. Furthermore, the successful modifications identified in Chapter 6 are also included in the MGSBPSO, namely the removal of zero-weighted assets; the immediate calculation of the objective function for single-asset portfolios; the decision to allow the weight optimizer to execute until it converges; only allowing assets to be removed via the weight determination stage and the exploration balance coefficient for improved convergence behaviour. Taking these improvements into account as well as the

Figure 7.1: MGSBPSO Structure

archive guide, the velocity equation is

$$\begin{aligned}
 V_i(t+1) = & \lambda_c(t)r_1(t) \otimes (Y_i(t) \ominus X_i(t)) \\
 & \oplus \lambda_i\lambda_c(t)r_2(t) \otimes (\hat{Y}_i(t) \ominus X_i(t)) \\
 & \oplus (1 - \lambda_i)\lambda_cr_3(t) \otimes (\hat{A}_i(t) \ominus X_i(t)) \\
 & \oplus (1 - \lambda_c(t))r_4(t) \otimes A_i(t)
 \end{aligned} \tag{7.1}$$

where r_1, r_2, r_3 and r_4 are random values, each sampled from a uniform distribution in the range $[0,1]$; λ_c is the linearly increasing exploration balance coefficient; X_i is the position of particle i ; Y_i is the best position found by particle i ; $\hat{Y}_i$ is the best position within particle i 's neighbourhood; $\hat{A}_i$ is the archive guide for particle i ; the influence of the archive guide is controlled by the archive coefficient λ_i ; A_i is shorthand for $U \setminus (X(t) \cup Y(t) \cup \hat{Y}(t))$ where U is the set universe, and $\otimes$ and $\oplus$ are the set-based operators defined in Section 2.4.

For the purpose of weight determination, the newly proposed MGPSO_s is used. Hence, asset weight determination is also a multi-objective optimization task. For each set-particle, a MGPSO_s is instantiated to optimize the corresponding asset weights with regard to risk and return. Figure 7.1 illustrates the overall structure of the swarms in MGSBPSO and their objective.

Figure 7.1 shows that there are two MGPSO_s algorithms, one to find continuous valued solutions for each set-swarm in MGSBPSO. Each MGPSO_s has its own archive. Specifically, the MGPSO_s for MGSBPSO s_1 (that maximizes return) has its own archive and the MGPSO_s for MGSBPSO s_2 (that minimizes risk) has its own archive. There is also a global archive, the MGSBPSO archive, which is used to store the best solutions found by the MGPSO_s algorithms. An MGPSO_s terminates when no non-dominated solutions are added to their archive over three iterations. The non-dominated solutions in the archive of an MGPSO_s are then inserted into the global archive along with the corresponding set-position. Lastly, the best objective function value of the

non-dominated solutions in an MGPSO_s archive is assigned to the corresponding set-particle with regard to the objective of the swarm the set-particle is in. For example, if the set-particle is in the swarm to minimize risk, then the best risk value of the non-dominated solutions in the MGPSO_s archive is used. Algorithm 12 presents pseudocode for the multi-objective weight optimization process and how the MGPSO_s archives interact with the global archive. The pseudocode for MGSBPSO is given in Algorithm 13.

Algorithm 12 Multi-objective Weight Determination for Set-based Portfolio Optimization

Let t represent the current iteration;

Let $S_k \cdot X_i$ represent set-particle i in MGSBPSO swarm k ;

Minimize risk and maximize return using MGPSO_s with assets in X_i until no solutions are inserted into the MGPSO_s archive;

$t = t + t_w$;

Insert the non-dominated solutions in the MGPSO_s archive with assets in $S_k \cdot X_i$ into the global archive;

if $k = 1$ **then**

 Return the best risk value, and corresponding weights, found by the MGPSO_s;

else

 Return the best return value, and corresponding weights, found by the MGPSO_s;

Algorithm 13 Multi-guide Set-based Particle Swarm Optimization

```

 $t = 0;$ 
for each objective  $k = 1, \dots, m$  do
    Create and initialize a swarm,  $S_k$ , of  $n_{s_k}$  particles uniformly within a pre-
    defined set universe  $U$ ;
    Let  $f_k$  be the function described in Algorithm 12; for each particle  $i =$ 
     $1, \dots, S_k \cdot n_{s_k}$  do
        Let  $S_k \cdot Y_i$  represent the personal best position of particle  $S_k \cdot X_i$ , ini-
        tialized to  $S_k \cdot X_i(t)$ ;
        Let  $S_k \cdot \hat{Y}_i$  represent the neighbourhood best position of particle  $S_k \cdot X_i$ ,
        initialized to the best  $S_k \cdot X_i$  in particle  $i$ 's neighbourhood;
        Initialize  $V_i(t)$  to  $\{\}$ , i.e the empty set;
while stopping condition is not true do
    for each objective  $k = 1, \dots, m$  do
        for each particle  $i = 1, \dots, S_k \cdot n_{s_k}$  do
            if  $f_k(S_k \cdot X_i(t)) < f_k(S_k \cdot Y_i(t))$  then
                 $S_k \cdot Y_i(t+1) = S_k \cdot X_i(t);$ 
            for particles  $\hat{i}$  with particle  $i$  in their neighbourhood do
                if  $f_k(S_k \cdot X_i(t)) < f_k(S_k \cdot \hat{Y}_i(t))$  then
                     $S_k \cdot \hat{Y}_i(t+1) = S_k \cdot X_i(t);$ 
            Update the global archive with solution  $S_k \cdot X_i(t)$ ;
        for each objective  $k = 1, \dots, m$  do
            for each particle  $i = 1, \dots, S_k \cdot n_{s_k}$  do
                Select a solution,  $S_k \cdot \hat{A}_i(t)$ , from the global archive using tournament
                selection;
                Update particle  $i$ 's velocity using equation (7.1);
                Update particle  $i$ 's position using equation (2.18);
     $t = t + 1;$ 

```

The proposed MGSBPSO is expected to perform similarly to the MGPSO_s for multi-objective portfolio optimization since MGSBPSO makes use of MGPSO_s for asset weight optimization. It is also expected that the reduction in dimensionality by MGSBPSO will result in higher-quality solutions than MGPSO_s for larger portfolio problems.

7.3 Empirical Process

The empirical process described in Chapter 4 is also used here to gauge the performance of MGSBPSO with the caveat that instead of using the mean-variance model of equation (3.1), risk (equation (3.2)) and return (equation

(3.3)) are optimized independently. The performance of MGSBPSO is compared with that of MGPSO_s as a standalone approach where solution representation is a fixed-length vector of floats. For each algorithm, a total swarm size of 50 is used with 25 particles allocated to each swarm. The archive size for each algorithm, as well as the global archive for MGSBPSO, is set to 50. Furthermore, all swarms makes use of a gbest topology. The algorithms are allocated 5000 iterations per independent run. The performance of SBPSO_{m6} is also included in the performance assessment of MGSBPSO since it was shown to outperform other algorithms for single-objective portfolio optimization in the previous chapter. It should be noted that, for this experiment, the time performance measure refers to the time in seconds to obtain the POF, or in the case of SBPSO_{m6}, the time in seconds to optimize equation (3.1) for 50 evenly spaced λ values.

7.4 Results

This section discusses the results of MGPSO, MGSBPSO and SBPSO_{m6} for each benchmark problem. Section 7.4.1 examines the Hang Seng results. The DAX 100 and FTSE 100 results are discussed in Sections 7.4.2 and 7.4.3, respectively. Section 7.4.4 discusses the S&P 100 results, and the Nikkei 225 results are discussed in Section 7.4.5.

7.4.1 Hang Seng

Table 7.6 shows that MGPSO_s and MGSBPSO performed similarly. Both algorithms obtained optimal solutions that were close to the true POF, as indicated by the GD and IGD scores, in a reasonable time. However, MGSBPSO was slightly more competitive than MGPSO_s with regard to risk and return. MGSBPSO obtained portfolios that on average had a higher return value and lower risk value than MGPSO_s. SBPSO_{m6} by comparison obtained fewer portfolios that were much more profitable at a similar level of risk. Table 7.7 shows that the SBPSO_{m6} generally ranked first across the performance measures. However, SBPSO_{m6} ranked second overall as a result of ranking last in the number of optimal solutions obtained, HV and time to reach the stopping condition. MGPSO_s and MGSBPSO ranked first and second overall, respectively. MGSBPSO ranked first in time to reach the stopping condition. Figure 7.2 shows that MGSBPSO was able to approximate the true POF slightly better than MGPSO_s.

Table 7.6: Hang Seng results for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
MGPSO _s	$\bar{x}$	43	14	0.007261	0.001967	0.000723	0.000218	1.192743	114.67
	σ	0	2	0.000705	0.000407	0.000262	0.000025	0.002658	4.95
MGSBPSO	$\bar{x}$	47	5	0.007382	0.001814	0.000761	0.000252	1.190891	100.50
	σ	0	1	0.000552	0.000326	0.000077	0.000029	0.001930	2.69
SBPSO _{m6}	$\bar{x}$	33	3	0.007607	0.001897	0.000218	0.000212	0.781949	226.05
	σ	0	0	0.000003	0.000000	0.000045	0.000002	0.026784	143.66

Table 7.7: Hang Seng rankings for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time	Overall
MGPSO _s	Wins	1	0	0	0	1	1	2	1	6
	Losses	0	2	0	0	1	0	0	1	4
	Draws	1	0	2	2	0	1	0	0	6
	Difference	1	-2	0	0	0	1	2	0	2
	Rank	1	3	2	1	2	1	1	2	1
MGSBPSO	Wins	1	1	0	0	0	0	1	2	5
	Losses	0	1	1	0	2	2	1	0	7
	Draws	1	0	1	2	0	0	0	0	4
	Difference	1	0	-1	0	-2	-2	0	2	-2
	Rank	1	2	3	1	3	2	2	1	3
SBPSO _{m6}	Wins	0	2	1	0	2	1	0	0	6
	Losses	2	0	0	0	0	0	2	2	6
	Draws	0	0	1	2	0	1	0	0	4
	Difference	-2	2	1	0	2	1	-2	-2	0
	Rank	2	1	1	1	1	1	3	3	2

Figure 7.2: Obtained Pareto-optimal fronts for Hang Seng

7.4.2 DAX 100

Table 7.8 shows that the MGSBPSO did not perform as well as MGPSO_s and SBPSO_{m6} for the DAX 100 benchmark problem. Specifically, the MGSBPSO had the lowest average return value and an average risk value that was an order of magnitude worse than the risk values of the other two algorithms. Furthermore, MGSBPSO had the worst average GD value. However, the average values and the rankings in Table 7.9 show that the MGSBPSO performed well for IGD, HV and time to reach the stopping condition. As for the previous benchmark problem, SBPSO_{m6} took statistically significantly longer to find fewer optimal portfolios. SBPSO_{m6} also obtained a much lower HV. The MGPSO_s ranked second for GD and first for IGD, HV, risk and the number of optimal solutions obtained. As a result, the MGPSO_s ranked first overall. However, MGPSO_s obtained significantly larger portfolios - an order of magnitude larger than the portfolios of the other two algorithms. The obtained POFs in Figure 7.3 show that the MGSBPSO was able to approximate the true POF better than the MGPSO_s. The POF obtained by SBPSO_{m6} resembles the true POF the best.

Table 7.8: DAX 100 results for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
MGPSO _s	$\bar{x}$	46	41	0.005960	0.000688	0.001058	0.000264	1.195303	266.54
	σ	1	4	0.000459	0.000267	0.000490	0.000053	0.003118	5.26
MGSBPSO	$\bar{x}$	45	13	0.005838	0.005388	0.003119	0.000259	1.192224	203.25
	σ	2	1	0.000388	0.026832	0.011012	0.000042	0.003413	9.77
SBPSO _{m6}	$\bar{x}$	43	4	0.008469	0.000972	0.000250	0.000306	0.878138	449.54
	σ	0	0	0.000006	0.000000	0.000061	0.000008	0.065429	159.43

Table 7.9: DAX 100 rankings for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time	Overall
MGPSO _s	Wins	1	0	0	2	1	1	2	1	8
	Losses	0	2	1	0	1	0	0	1	5
	Draws	1	0	1	0	0	1	0	0	3
	Difference	1	-2	-1	2	0	1	2	0	3
	Rank	1	3	2	1	2	1	1	2	1
MGSBPSO	Wins	1	1	0	0	0	1	1	2	6
	Losses	0	1	1	2	2	0	1	0	7
	Draws	1	0	1	0	0	1	0	0	3
	Difference	1	0	-1	-2	-2	1	0	2	-1
	Rank	1	2	2	3	3	1	2	1	2
SBPSO _{m6}	Wins	0	2	2	1	2	0	0	0	7
	Losses	2	0	0	1	0	2	2	2	9
	Draws	0	0	0	0	0	0	0	0	0
	Difference	-2	2	2	0	2	-2	-2	-2	-2
	Rank	2	1	1	2	1	2	3	3	3

Figure 7.3: Obtained Pareto-optimal fronts for DAX 100

7.4.3 FTSE 100

Table 7.14 shows that the MGSBPSO performed better than the MGPSO_s for the size of the obtained portfolios, risk, GD, and IGD. Although MGSBPSO did take less time on average than the MGPSO_s to reach the stopping condition, the standard deviation of the time taken by MGSBPSO was much larger than that of MGPSO_s. Table 7.15 shows that overall the MGSBPSO ranked first, SBPSO_{m6} ranked second and MGPSO_s ranked third. The SBPSO_{m6} obtained smaller sized portfolios that were more profitable than those obtained by the other two algorithms, but with higher risk. However, the SBPSO_{m6} was much slower to reach the stopping condition than MGPSO_s and MGSBPSO. A visual analysis of the obtained POFs in Figure 7.6 shows that MGPSO_s and MGSBPSO were somewhat able to approximate the true POF. The POFs of both algorithms contained non-dominated points that were present in the true POF, but also points that were dominated by the true POF.

Table 7.10: FTSE 100 results for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
MGPSO _s	$\bar{x}$	44	47	0.005239	0.000570	0.001111	0.000258	1.189333	264.95
	σ	2	8	0.000508	0.000097	0.000606	0.000127	0.003496	4.08
MGSBPSO	$\bar{x}$	47	14	0.005079	0.000438	0.001003	0.000241	1.186224	233.95
	σ	0	1	0.000249	0.000048	0.000076	0.000026	0.001914	68.16
SBPSO _{m6}	$\bar{x}$	30	5	0.006683	0.000785	0.000247	0.000235	0.741994	528.62
	σ	0	0	0.000007	0.000001	0.000038	0.000007	0.049170	177.55

Table 7.11: FTSE 100 rankings for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time	Overall
MGPSO _s	Wins	1	0	1	1	0	0	2	1	6
	Losses	1	2	1	1	2	2	0	1	10
	Draws	0	0	0	0	0	0	0	0	0
	Difference	0	-2	0	0	-2	-2	2	0	-4
	Rank	2	3	2	2	3	2	1	2	3
MGSBPSO	Wins	2	1	0	2	1	1	1	2	10
	Losses	0	1	2	0	1	0	1	0	5
	Draws	0	0	0	0	0	1	0	0	1
	Difference	2	0	-2	2	0	1	0	2	5
	Rank	1	2	3	1	2	1	2	1	1
SBPSO _{m6}	Wins	0	2	2	0	2	1	0	0	7
	Losses	2	0	0	2	0	0	2	2	8
	Draws	0	0	0	0	0	1	0	0	1
	Difference	-2	2	2	-2	2	1	-2	-2	-1
	Rank	3	1	1	3	1	1	3	3	2

Figure 7.4: Obtained Pareto-optimal fronts for FTSE 100

7.4.4 S&P 100

The average results in Table 7.12 shows that the MGSBPSO performed well for this benchmark problem. Specifically, the MGSBPSO performed better than MGPSO_s for the GD measure and better than SBPSO_{m6} for the HV measure. Furthermore, MGSBPSO obtained portfolios with the least amount of risk and it was the fastest algorithm to reach the stopping condition. The MGPSO_s also performed well, particularly for HV, but it obtained much larger portfolios than MGSBPSO and SBPSO_{m6}. The rankings in Table 7.13 show that the MGSBPSO was overall the best performing algorithm. The MGPSO_s ranked last overall, but ranked higher than MGSBPSO for the Pareto optimality measures. Furthermore, MGPSO_s ranked higher than SBPSO_{m6} for the HV measure. However, Figure 7.5 shows that the MGPSO_s did not approximate the true POF as well as SBPSO_{m6} or MGSBPSO. Furthermore, the POFs obtained by MGPSO_s and MGSBPSO contains points that are dominated by the true POF.

Table 7.12: S&P 100 results for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
MGPSO _s	$\bar{x}$	48	65	0.004993	0.000579	0.001344	0.000276	1.195282	281.01
	σ	0	4	0.000359	0.000150	0.000235	0.000037	0.001887	4.30
MGSBPSO	$\bar{x}$	48	21	0.004819	0.000388	0.001263	0.000330	1.190151	220.53
	σ	0	2	0.000195	0.000053	0.000099	0.000038	0.001808	9.91
SBPSO _{m6}	$\bar{x}$	44	6	0.007716	0.001248	0.000275	0.000247	0.875057	822.26
	σ	0	0	0.000006	0.000001	0.000044	0.000003	0.044175	139.06

Table 7.13: S&P 100 rankings for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time	Overall
MGPSO _s	Wins	1	0	1	1	0	1	2	1	7
	Losses	1	2	1	1	1	1	0	1	8
	Draws	0	0	0	0	1	0	0	0	1
	Difference	0	-2	0	0	-1	0	2	0	-1
	Rank	2	3	2	2	2	2	1	2	3
MGSBPSO	Wins	2	1	0	2	0	0	1	2	8
	Losses	0	1	2	0	1	2	1	0	7
	Draws	0	0	0	0	1	0	0	0	1
	Difference	2	0	-2	2	-1	-2	0	2	1
	Rank	1	2	3	1	2	3	2	1	1
SBPSO _{m6}	Wins	0	2	2	0	2	2	0	0	8
	Losses	2	0	0	2	0	0	2	2	8
	Draws	0	0	0	0	0	0	0	0	0
	Difference	-2	2	2	-2	2	2	-2	-2	0
	Rank	3	1	1	3	1	1	3	3	2

Figure 7.5: Obtained Pareto-optimal fronts for S&P 100

7.4.5 Nikkei 225

The MGSBPSO was more competitive than MGPSO_s with regard to risk and return for this benchmark problem, as shown in Table 7.14. MGSBPSO obtained more profitable and less risky portfolios than MGPSO_s while being faster to reach the stopping condition. Furthermore, the MGSBPSO obtained more optimal portfolios than MGPSO_s and portfolios with fewer assets than MGPSO_s. Table 7.15 shows that MGSBPSO ranked higher than MGPSO_s with regard to GD and IGD, and ranked first overall. SBPSO_{m6} was approximately three times slower than the MGPSO_s to reach the stopping condition and four and a half times slower than MGSBPSO. The SBPSO_{m6} ranked second overall and the MGPSO_s ranked last. Figure 7.6 shows that the MGPSO_s was unable to approximate the true POF, unlike MGSBPSO and SBPSO_{m6}.

Table 7.14: Nikkei 225 results for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time
MGPSO _s	$\bar{x}$	45	22	0.002193	0.000650	0.001631	0.000294	1.190877	996.02
	σ	0	5	0.000338	0.000122	0.000393	0.000051	0.003675	11.07
MGSBPSO	$\bar{x}$	49	5	0.002437	0.000633	0.001242	0.000242	1.191968	679.75
	σ	0	1	0.000272	0.000113	0.000114	0.000030	0.003169	69.26
SBPSO _{m6}	$\bar{x}$	44	4	0.003310	0.000794	0.000260	0.000223	0.916014	3159.48
	σ	0	0	0.000007	0.000001	0.000060	0.000005	0.047097	306.54

Table 7.15: Nikkei 225 rankings for each performance measure

Algorithm	Result	N	K	R	$\bar{\sigma}$	GD	IGD	HV	Time	Overall
MGPSO _s	Wins	1	0	0	1	0	0	1	1	4
	Losses	1	2	2	0	2	2	0	1	10
	Draws	0	0	0	1	0	0	1	0	2
	Difference	0	-2	-2	1	-2	-2	1	0	-6
	Rank	2	3	3	1	3	3	1	2	3
MGSBPSO	Wins	2	1	1	1	1	1	1	2	10
	Losses	0	1	1	0	1	1	0	0	4
	Draws	0	0	0	1	0	0	1	0	2
	Difference	2	0	0	1	0	0	1	2	6
	Rank	1	2	2	1	2	2	1	1	1
SBPSO _{m6}	Wins	0	2	2	0	2	2	0	0	8
	Losses	2	0	0	2	0	0	2	2	8
	Draws	0	0	0	0	0	0	0	0	0
	Difference	-2	2	2	-2	2	2	-2	-2	0
	Rank	3	1	1	2	1	1	2	3	2

Figure 7.6: Obtained Pareto-optimal fronts for Nikkei 225

7.5 Summary

This chapter proposed the multi-guide set-based particle swarm optimization (MGSBPSO) algorithm, a multi-objective adaptation of the set-based particle swarm optimization (SBPSO) algorithm that incorporates elements from the multi-guide particle swarm optimization (MGPSO) algorithm. For the purpose of optimising the asset weights, MGPSO was used. To guarantee that MGPSO finds high-quality solutions, its control parameters have to be tuned. Since parameter tuning is an expensive process, an alternative method to randomly sample control parameters values from the stability region defined in equation (2.8) was investigated. It was found that the accuracy of MGPSO using this approach, i.e. MGPSO_s, was similar to that of MGPSO using tuned control parameter values. Thus, MGPSO_s was used as the weight optimizer for MGSBPSO, as well as a standalone algorithm to compare the results of MGSBPSO with. Furthermore, SBPSO with the modifications presented in Chapter 6 was also used to assess the performance of MGSBPSO. The results showed that both the MGSBPSO and MGPSO_s performed well for multi-objective portfolio optimization. However, the MGPSO_s did not scale to larger portfolio problems as well as MGSBPSO and the SBPSO. The search time of MGSBPSO was faster than that of MGPSO_s and the SBPSO with modifications for all benchmark problems. Furthermore, the ability of

MGPSO_s to approximate the true POF deteriorated as the size of the portfolio problems increased whereas the performance of MGSBPSO was somewhat consistent across benchmark problems. The SBPSO with modifications also performed well. However, it was optimising the mean-variance portfolio model of equation (3.1) for a given risk-return tradeoff value, which does require that risk and return are optimised independently - which is what MGSBPSO and MGPSO_s did. Furthermore, the SBPSO with modifications only found a single solution after each independent run, whereas MGSBPSO and MGPSO_s found many non-dominated solutions. Overall, the results show that the benefits of redefining the portfolio optimization problem as a set-based problem are also applicable to multi-objective portfolio optimization.

Chapter 8

Conclusions

This study proposed a set-based particle swarm optimization (SBPSO) for the portfolio optimization problem, set-based diversity measures, improvements to the SBPSO for portfolio optimization, a version of multi-guide particle swarm optimization (MGPSO) that does not require parameter tuning, and the first multi-objective SBPSO. The purpose of this chapter is to summarise the major findings of this study and to provide avenues for future research.

Section 8.1 presents a summary of the major findings and conclusions of this thesis, followed by suggestions for future work in Section 8.2.

8.1 Major Findings and Conclusions

A literature review of over 100 publications from 1993 to 2019 found that a variety of evolutionary and swarm intelligence algorithms have been applied to portfolio optimization. The most popular of which are particle swarm optimization (PSO) and genetic algorithms (GAs). Lesser known algorithms, such as cat swarm optimization (CSO), chemical reaction optimization (CRO), firefly algorithm (FA), fireworks algorithm (FWA), krill Herd (KH) and invasive weed optimization (IWO) have also been studied, albeit to a lesser degree than PSO and GAs. In fact, many of these lesser known algorithms have only been applied to portfolio optimization once or twice. Researchers should ask themselves if the work being proposed is truly novel or to simply *cash* in on a trend of deriving meta-heuristics from nature.

The review also found that the majority of the work considered portfolio optimization as a single-objective optimization problem. This is done by introducing a trade-off coefficient to balance the risk and profit objectives. Thus, by optimizing for varying values of the trade-off coefficient, a set of optimal solutions can be obtained. However, once constraints are introduced, these approaches are effectively blind to solutions that cannot be obtained by optimizing for a given trade-off weighting. Multi-objective approaches have been shown to be capable of obtaining higher quality solutions while under

constraints.

Since its inception in 2000, the cardinality constrained portfolio problem (CCPP) was the most widely considered constrained portfolio optimization problem. It places a limit on the number of assets included in the portfolio and bounds on how much an asset can contribute to the overall portfolio. The data proposed alongside the CCPP has also become extremely popular. A variety of other constrained portfolio problems and data sets have been studied, although they have not gained as much traction. The success of CCPP is largely due to it being well researched and clearly defined. Furthermore, the corresponding data set is publicly available online – the unconstrained version is used in this study.

No work has yet considered evolutionary and swarm intelligence algorithms for dynamic portfolio optimization. Dynamic portfolio optimization mimics high-frequency trading, such as daily trading. However, the lack of interest in this field is possibly due to the complexity it introduces.

The last key finding of the literature was that existing approaches suffer from stability issues when the size of the portfolio optimization problem increases, providing the motivation for this study: to propose a new scalable approach to portfolio optimization using SBPSO. The proposed approach makes use of a bi-stage search process that first selects assets and then determines the weights for said assets. The key idea behind this search process is that by selectively choosing which assets to optimize, the complexity of the problem reduces through a reduction in dimensionality. Set-based approaches have been used before, although with quadratic programming. The proposed approach uses a traditional PSO for weight optimization. An empirical analysis showed that SBPSO was capable of identifying good-quality solutions while being relatively fast. It performed better than the firefly algorithm (FA), but worse than artificial bee colony (ABC), a GA and PSO. Nevertheless, it was concluded that redefining the portfolio optimization problem as a set-based asset selection problem is an effective method for reducing the duration of the search, while still obtaining acceptable levels of performance.

Set-based diversity measures were proposed in order to investigate the performance and convergence behaviour of SBPSO, namely Jaccard diversity and Hamming diversity based on their respective distance measures. Both measures were able to represent the diversity of sets over time. It was found that Jaccard diversity better represents a researcher's intuition about the diversity of sets.

Using the newly defined Jaccard diversity, the effects of several SBPSO modifications were studied. In particular, the introduction of a linearly increasing control parameter that balanced exploratory and exploitative behaviour significantly improved the ability of SBPSO to converge to a single solution. Likewise, allowing the weight optimizer to execute until there is no improvement in solution quality, instead of a fixed number of time steps, resulted in large gains in performance. Adaptive coordinate descent (ACD) was used as

a completely deterministic approach to weight determination. Performance analysis showed that SBPSO using ACD found higher-quality portfolios, but at the cost of a longer search time. Hence, PSO is still the preferred weight determination approach. The performance of SBPSO with the modifications and the ACD variant was evaluated against the aforementioned ABC, FA, GA and PSO as well as the original SBPSO for portfolio optimization. It was found that SBPSO with modifications and the ACD variant were statistically significantly better than the other algorithms in many ways.

A method to randomly sample control parameter values for MGPSO that satisfy stability conditions was investigated, with the motivation that it would allow researchers to easily use MGPSO to find optimal solutions to other problems without the need for parameter tuning. It was shown through empirical analysis that randomly sampling control parameter values that satisfy stability conditions yields similar performance to that of MGPSO using tuned parameters, and therefore is a viable alternative to parameter tuning.

A multi-objective adaptation of SBPSO was proposed, namely multi-guide set-based particle swarm optimization (MGSBPSO). MGSBPSO is a combination of SBPSO with the multi-objective components of MGPSO. MGSBPSO also makes use of the successful modifications mentioned earlier, and uses the MGPSO with the approach of randomly sampling control parameter values from the stability region as the weight optimizer. It was shown that MGSBPSO scaled to larger portfolio optimization problems better than MGPSO.

8.2 Future Work

This study concludes with the following ideas for future research:

Maintaining Weights

SBPSO for portfolio optimization uses a weight determination process to calculate the objective function and to identify good-performing assets. However, the asset weights are forgotten between weight determination stages. Thus, future work should explore strategies for maintaining and sharing good-quality weights between weight optimization searches. A potential solution is to archive the asset weights, which are updated over time. A population-based weight optimizer can then make use of the archive when initialising its population to make it partly aware of previously found fruitful areas. Because of this, care needs to be taken to avoid the weight optimizer prematurely converging on a solution.

Alternative Weight Optimizers

This study examined one alternative weight optimizer, ACD, which helped SBPSO to find low risk portfolios. There is potentially performance to be gained by investigating other alternative weight optimizers. For example, GAs and ABC which were shown to be well studied approaches to portfolio optimization in the Chapter 3. Likewise, these algorithms could also benefit from the reduction in dimensionality that SBPSO provides.

Set-based Particle Swarm Optimization for Constrained Portfolio Optimization

SBPSO could easily be adapted to handle several constraints. For example, cardinality constraints and asset class constraints can be handled by SBPSO, creating an unconstrained weight determination environment. The research done here could also be applied to other constrained set-based optimization problems.

Set-based Particle Swarm Optimization for Dynamic Portfolio Optimization

A study of dynamic portfolio optimization requires a review of existing approaches to dynamic combinatorial optimization problems, from which a new dynamic version of SBPSO can be developed. Likewise, current literature on dynamic evolutionary algorithms (EAs) and SI algorithms can be applied to dynamic portfolio optimization, which would be a first. The proposed algorithm can later be used as a dynamic weight optimizer. How the dynamic SBPSO and weight optimizer would interact with each other is not immediately obvious and should be investigated. For example, would the dimensionality of the weight determination search space change as assets are added or removed from corresponding set-particles? If so, how would that affect the current solutions of the weight optimizer? Also, if each set-particle spawns a weight determination process, and these are evaluated sequentially, how would that process be adapted to dynamic environments? Perhaps concurrently? These questions would need to be investigated as a part of this work.

Multi-guide Set-based Particle Swarm Optimization for Multi-objective Combinatorial Optimization Problems

MGSBPSO demonstrated good performance for multi-objective portfolio optimization. Future work should investigate the performance of MGSBPSO for traditional multi-objective combinatorial problems, such as multi-objective knapsack problems. A part of this investigation would be to study the diver-

sity of sub-swarms over time, as well as the effects of various parameter sets for different problems.

A Modern Portfolio Optimization Benchmark Library

This area of future work should attempt to propose a successor to the extremely popular portfolio optimization benchmarks of the OR Library. The benchmarks are well-researched and can be used for unconstrained portfolio optimization and the cardinality constrained portfolio problem. However, these benchmarks were first proposed 20 years ago, and since then there have been many constraints and objectives put forward. A modern portfolio optimization library should provide the necessary data to be compatible with most portfolio models and allow for additional constraints. Furthermore, the data sets should include in-sample and out-of-sample data to evaluate performance of the obtained portfolios in the real world. The data selected should be indicative of real world scenarios, such as market crashes and booms. Ideally, stock market indices should also be included to see if the algorithms are able to beat the market. An opinionated constrained portfolio model that extends the CCPP should also be given. Hopefully, this will encourage users to investigate more difficult problems. Lastly, the data sets should be publicly accessible and easy to use.

Prediction Models

There has yet to be any work done on using evolutionary and swarm intelligence algorithms to predict the prices of assets and make investment decisions based on these predictions. Investigation of such an approach will require researching and reviewing prediction models which can be optimized by an algorithm. The knowledge gained from the prediction models can then be used by the same algorithm, or perhaps a different one, to optimize a portfolio model. This area of work potentially overlaps with the need for a modern portfolio optimization library due to the data sets required. The work done here can also be extended to dynamic environments.

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Appendices

Appendix A

Acronyms

ABC	artificial bee colony
ACD	adaptive coordinate descent
ACO	ant colony optimization
AE	adaptive encoding
BA	bat algorithm
BFA	bacterial foraging algorithm
BPSO	binary particle swarm optimization
CCPP	cardinality constrained portfolio problem
CD	coordinate descent
CFBPSO	catfish binary particle swarm optimization
CFPSO	catfish particle swarm optimization
CMA-ES	covariance matrix adaptation evolution strategy
CRO	chemical reaction optimization
CSO	cat swarm optimization
CVaR	conditional value-at-risk
DE	differential evolution
E-MOEA	envelope multi-objective genetic algorithm
EA	evolutionary algorithm
FA	firefly algorithm
FPSO	fuzzy particle swarm optimization
FWA	fireworks algorithm
GA	genetic algorithm
GD	generational distance
HV	hypervolume
IGD	inverted generational distance
IWO	invasive weed optimization
KH	krill Herd
MAD	mean absolute deviation
MBPSO	modified binary particle swarm optimization
MGPSO	multi-guide particle swarm optimization

MGSBPSO	multi-guide set-based particle swarm optimization
MODE	multi-objective differential evolution
MOEA	multi-objective evolutionary algorithm
MOEA/D	multi-objective evolutionary algorithm based on decomposition
MOGA	multi-objective genetic algorithm
MOO	multi-objective optimization
MOOP	multi-objective optimization problem
MOPSO	multi-objective particle swarm optimization
NSGA II	non-dominated sorting genetic algorithm II
OMOPSO	optimized multi-objective particle swarm optimization
PBPSO	probability binary particle swarm optimization
PCA	principal component analysis
PESA	Pareto envelope-based selection algorithm
PESA-II	Pareto envelope-based selection algorithm II
POF	Pareto-optimal front
POS	Pareto-optimal solutions
PSFGA	parallel single front genetic algorithm
PSO	particle swarm optimization
SA	simulated annealing
SBPSO	set-based particle swarm optimization
SI	swarm intelligence
SMPSO	speed-constrained multi-objective particle swarm optimization
SPEA2	strength Pareto evolutionary algorithm 2
TS	tabu search
VaR	value-at-risk
VEGA	vector evaluated genetic algorithm
VEPSO	vector evaluated particle swarm optimization
WFG	Walking Fish Group

Appendix B

List of Symbols

α	controls a randomized move
$\bar{\sigma}$	total risk
$\bar{d}_H$	Hamming diversity
$\bar{d}_J$	Jaccard diversity
β_i	attractiveness of firefly i
$\hat{\mathbf{a}}_i$	archive guide of particle i
$\hat{\mathbf{y}}_i$	social guide of particle i
$\mathbf{a}, \mathbf{b}$	lower and upper bounds of the ACD search
$\mathbf{o}_i$	neighboring food source to food source i
$\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4$	uniform vectors with components in the range $[0,1]$
$\mathbf{v}_i$	velocity of particle i
$\mathbf{x}$	decision vector
$\mathbf{x}_i(t)$	decision vector i at time step t
$\mathbf{y}_i$	cognitive guide of particle i
$\boxplus$	set-based operator for addition of a set-based velocity and a set-based position
γ_i	light absorption coefficient
$\hat{A}_i$	archive guide of set-particle i
$\hat{Y}_i$	social guide of set-particle i

κ_i	amount of nectar at food source i
λ	risk-return trade-off
λ_c	exploration balance coefficient of particle i
λ_i	archive balance coefficient of particle i
Ω	search space of an optimization problem
ω	set to bit vector transformation function
$\ominus$	set-based operator for the difference between two set-based positions
$\oplus$	set-based operator for the addition of two set-based velocities
$\otimes$	set-based operator for the multiplication of a set-based velocity by a scalar value
$<$	Pareto-dominance relation
τ	tournament size
ε, δ	lower and upper bounds of the asset weights
φ	uniform vector with components in the range $[-1,1]$
B	total portfolio budget
C	benchmark to compare portfolio performance
c_1, c_2, c_3, c_4	positive acceleration coefficients
c_i	cost of asset i
d	number of dimensions in a search space
d_H	Hamming distance
d_J	Jaccard distance
e^*	minimum amount of liquidity
F	remaining free portion of the portfolio
f	objective function
g	uniform value in the range $[0,1]$
I_i	light intensity of firefly i

J	Jaccard index
K	number of assets to be included in a portfolio
k	number of clusters
K_{min}, K_{max}	lower and upper bounds of the number of assets to be included in a portfolio
k_{shrink}, k_{grow}	intervals along which the ACD algorithm should search for the next solution
l_i	transaction lot for asset i
m	number of objectives
n	number of candidate solutions
n_s	number of sectors
p_i	probability of selecting food source i
P_m	mutation rate
Q	Pareto-optimal front
q_i	quantity of transaction lot held for asset i
Q_{true}	true Pareto-optimal front
R	total return
R^*	minimum total return
r_1, r_2, r_3, r_4	uniform values in the range $[0,1]$
R_i	return of asset i
S	swarm of particles
S_m	mutation severity
T	investment time period
t	time step
U	set search space of an optimization problem
V_i	velocity of set-particle i
w	inertia weight

w_i	weighting of asset i
$X(t)$	position of set-particle i at time step t
Y_i	cognitive guide of set-particle i
z_i	bit value indicating whether asset i is included in the portfolio

Appendix C

Derived Publications

This appendix lists publications derived from the work presented in this study in order of publication date:

- K. Erwin and A. P. Engelbrecht, “Set-Based particle swarm optimization for portfolio optimization”, in *Proceedings of the 12th International Swarm Intelligence Conference (ANTS)*; and in *Lecture Notes in Computer Science*, Springer International Publishing, 2020, pp. 333–339, ISBN: 978-3-030-60376-2
- K. Erwin and A. P. Engelbrecht, “Diversity measures for set-based metaheuristics”, in *Proceedings of the 7th International Conference on Soft Computing Machine Intelligence*, 2020, pp. 45–50
- K. Erwin and A. P. Engelbrecht, “Improved set-based particle swarm optimization for portfolio optimization”, in *Proceedings of the IEEE Swarm Intelligence Symposium*, 2020, pp. 1573–1580
- K. Erwin and A. P. Engelbrecht, “Evolutionary and swarm intelligence algorithms for portfolio optimization”, *Applied Soft Computing (under review)*, 2021
- Kyle and A. P. Engelbrecht, “Set-based particle swarm optimization for portfolio optimization: Empirical analysis.”, *Applied Soft Computing (under review)*, 2021
- K. Erwin and A. P. Engelbrecht, “Set-based particle swarm optimization for portfolio optimization with adaptive coordinate descent weight optimization”, *IEEE Swarm Intelligence Symposium (under review)*, 2021
- K. Erwin and A. P. Engelbrecht, “A tuning free approach to multi-guide particle swarm optimization”, *IEEE Swarm Intelligence Symposium (under review)*, 2021

- K. Erwin and A. P. Engelbrecht, “Multi-guide set-based particle swarm optimization for portfolio optimization”, *IEEE Transactions on Evolutionary Computation* (under review), 2021

Appendix D

Control Parameter Values

Table D.1: Tuned control parameters for the Hang Seng benchmark problem for each algorithm

Algorithm	Parameter	Value
ABC	<i>limit</i>	97
FA	β_0	4.765625
	γ	0.476563
	α	0.992188
GA	S_m	0.155469
	τ	11
PSO	w	0.937500
	c_1	0.476563
	c_2	0.992188
SBPSO	c_1	0.375000
	c_2	0.375000
	c_3	3.312500
	c_4	4.437500

Table D.2: Tuned control parameters for the DAX 100 benchmark problem for each algorithm

Algorithm	Parameter	Value
ABC	$limit$	98
FA	β_0	5.234375
	γ	0.523438
	α	0.992188
GA	S_m	0.052109
	τ	13
PSO	w	0.851563
	c_1	1.140625
	c_2	1.234375
SBPSO	c_1	0.117188
	c_2	0.117188
	c_3	3.488281
	c_4	2.714844

Table D.3: Tuned control parameters for the FTSE 100 benchmark problem for each algorithm

Algorithm	Parameter	Value
ABC	$limit$	91
FA	β_0	8.593750
	γ	0.859375
	α	0.921875
GA	S_m	0.017656
	τ	16
PSO	w	0.781250
	c_1	1.562500
	c_2	1.312500
SBPSO	c_1	0.367188
	c_2	0.867188
	c_3	2.363281
	c_4	3.839844

Table D.4: Tuned control parameters for the S&P 100 benchmark problem for each algorithm

Algorithm	Parameter	Value
ABC	$limit$	96
FA	β_0	8.203125
	γ	0.820313
	α	0.960938
GA	S_m	0.270313
	τ	3
PSO	w	0.781250
	c_1	1.562500
	c_2	1.312500
SBPSO	c_1	0.828125
	c_2	0.484375
	c_3	2.960938
	c_4	2.257813

Table D.5: Tuned control parameters for the Nikkei 225 benchmark problem for each algorithm

Algorithm	Parameter	Value
ABC	$limit$	97
FA	β_0	3.984375
	γ	0.398438
	α	0.820313
GA	S_m	0.036797
	τ	19
PSO	w	0.851563
	c_1	1.140625
	c_2	1.234375
SBPSO	c_1	0.828125
	c_2	0.484375
	c_3	2.960938
	c_4	2.257813

Appendix E

Walking Fish Group Toolkit

Huband *et al.* proposed a sophisticated toolkit, referred to as the WFG toolkit, that contains nine scalable benchmarks with varying properties [81]. All problems created by the WFG toolkit conform to the following format:

m is the number of objectives, $\mathbf{x}$ is a set of underlying parameters, where $\mathbf{x}_m$ is an underlying distance parameter and $x_{1:m-1}$ are underlying position parameters;

$\mathbf{z}$ is a set of $j + 1 = n \geq m$ working parameters, where the first j and the last l working parameters are position- and distance-related parameters, respectively;

$D > 0$ is a distance scaling constant equal to one;

$A_{1:m-1} \in \{0, 1\}$ are degeneracy constants, for each $A_i = 0$, the dimensionality of the POF is reduced by one;

$h_1 : m$ are shape functions;

$S_{1:m} > 0$ are scaling constants and

$\mathbf{t}^{1:p}$ are transition vectors, where " $\Leftarrow$ " indicates that each transition vector is created from another vector via transformation functions;

The domain of all $z_i \in \mathbf{z}$ is $[0, z_{i,\max}]$, where all $z_{i,\max} > 0$. Note that all $x_i \in \mathbf{x}$ will have domain $[0, 1]$.

E.1 Shape functions

The following five shape functions determine the nature of the POF, and map parameters with domain $[0, 1]$ onto the range $[0, 1]$. Furthermore, for all shape functions $x_1, \dots, x_{m-1} \in [0, 1]$, A , α and β are constants. The shape functions are

Linear

$$\begin{aligned}
\text{linear}_1(x_1, \dots, x_{m-1}) &= \prod_{i=1}^{m-1} x_i \\
\text{linear}_{k=2:m-1}(x_1, \dots, x_{m-1}) &= \left(\prod_{i=1}^{m-k} x_i \right) (1 - x_{m-k+1}) \\
\text{linear}_m(x_1, \dots, x_{m-1}) &= 1 - x_1
\end{aligned} \tag{E.2}$$

When $h_{k=1:m} = \text{linear}_j$, the POF is a linear hyperplane, where $\sum_{k=1}^m = 1$.

Convex

$$\begin{aligned}
\text{convex}_1(x_1, \dots, x_{m-1}) &= \prod_{i=1}^{m-1} \left(1 - \cos\left(x_i \frac{\pi}{2}\right) \right) \\
\text{convex}_{k=2:m-1}(x_1, \dots, x_{m-1}) &= \left(\prod_{i=1}^{m-k} \left(1 - \cos\left(x_i \frac{\pi}{2}\right) \right) \right) \left(1 - \sin\left(x_{m-k+1} \frac{\pi}{2}\right) \right) \\
\text{convex}_m(x_1, \dots, x_{m-1}) &= 1 - \sin\left(x_1 \frac{\pi}{2}\right)
\end{aligned} \tag{E.3}$$

When $h_{k=1:m} = \text{convex}_j$, the POF is a purely convex.

Concave

$$\begin{aligned}
\text{concave}_1(x_1, \dots, x_{m-1}) &= \prod_{i=1}^{m-1} \left(1 - \sin\left(x_i \frac{\pi}{2}\right) \right) \\
\text{concave}_{k=2:m-1}(x_1, \dots, x_{m-1}) &= \left(\prod_{i=1}^{m-k} \left(\sin\left(x_i \frac{\pi}{2}\right) \right) \right) \cos\left(x_{m-k+1} \frac{\pi}{2}\right) \\
\text{concave}_m(x_1, \dots, x_{m-1}) &= \cos\left(x_1 \frac{\pi}{2}\right)
\end{aligned} \tag{E.4}$$

When $h_{k=1:m} = \text{concave}_j$, the POF is a purely concave, and a region of the hyper-sphere of radius one centred at the origin, where $\sum_{k=1}^m = h_k^2 = 1$.

mixed convex/concave ($\alpha > 0, A \in \{1, 2, \dots\}$)

$$\text{mixed}_m(x_1, \dots, x_{m-1}) = \left(1 - x_1 - \frac{\cos\left(2A\pi x_1 + \frac{\pi}{2}\right)}{2A\pi} \right)^\alpha \tag{E.5}$$

Equation (E.5) causes the POF to contain both convex and concave segments, the number of which is controlled by A . The overall shape is controlled by α : when $\alpha > 1$ or when $\alpha < 1$, the overall shape is convex or concave, respectively. When $\alpha = 1$, the overall shape is linear.

Disconnected ($\alpha, \beta > 0, A \in \{1, 2, \dots\}$)

$$\text{disc}_m(x_1, \dots, x_{m-1}) = 1 - (x_1)^\alpha \cos^2(A(x_1)^\beta \pi) \quad (\text{E.6})$$

Equation (E.6) to have disconnected regions, the number of which is controlled by A . The overall shape is controlled by α : when $\alpha > 1$ or when $\alpha < 1$, the overall shape is convex or concave, respectively. When $\alpha = 1$, the overall shape is linear. The location of the disconnected regions is controlled by β : larger values of β push the location of the disconnected regions towards larger values of x_1 , and vice versa.

E.2 Transformation functions

Transformation functions map input parameters with domain $[0, 1]$ onto the range $[0, 1]$. A total of eight transformation functions are provided by the WFG toolkit and are categorised as bias, shift and reduction functions. The primary parameters y and $y, \dots, y_{|y|_3}$ always have domain of $[0, 1]$. A, B, C, α and β are constants.

E.2.1 Bias Functions

Bias transformations impact the search process by biasing the fitness landscape. The following bias transformations have been defined:

Polynomial ($\alpha > 0, \alpha \neq 1$)

$$\text{b_poly}(y, \alpha) = y^\alpha \quad (\text{E.7})$$

When $\alpha > 1$, y is biased towards zero, and when $\alpha < 1$, y is biased towards one.

Flat Region ($A, B, C \in [0, 1], B < C, B = 0 \Rightarrow A = 0 \wedge C \neq 1, C = 1 \Rightarrow A = 1 \wedge B \neq 0$)

$$\begin{aligned} \text{b_flat}(y, A, B, C) = & A + \min(0, \lfloor y - B \rfloor) \frac{A(B - y)}{B} \\ & - \min(0, \lfloor C - y \rfloor) \frac{(1 - A)(y - C)}{(1 - C)} \end{aligned} \quad (\text{E.8})$$

Values of y between B and C , the area of the flat region, are all mapped to the value of A .

Parameter Dependent ($A \in (0, 1), 0 < B < C$)

$$\begin{aligned} b_param(y, \mathbf{y}', A, B, C) &= y^{B+(C-B)v(u(\mathbf{y}'))} \\ v(u(\mathbf{y}')) &= A - (1 - 2u(\mathbf{y}'))|[0.5 - u(\mathbf{y}')] + A| \end{aligned} \quad (E.9)$$

where $\mathbf{y}'$ is a secondary parameter vector, of domain $[0, 1]$ and u is a reduction function. A , B , C , and $\mathbf{y}'$ together determine the degree to which y is biased by being raised to an associated power, values of $u(\mathbf{y}') \in [0, 0.5]$ are mapped linearly onto $[B, B + (C - B)A]$, and values of $u(\mathbf{y}') \in [0.5, 1]$ are mapped linearly onto $[B + (C - B)A, C]$.

E.2.2 Shift

Shift transformations move the location of optima. In the absence of any shift, all distance-related parameters would be extremal parameters, with optimal value at zero. Shift transformations can be used to set the location of parameter optima (subject to skewing by bias transformations), which is useful if medial and extremal parameters are to be avoided. It is recommended that all distance-related parameters be subjected to at least one shift transformation. The deceptive and multimodal shift transformations, defined below, make the corresponding problem deceptive and multimodal, respectively.

Linear ($A \in (0, 1)$)

$$s_linear(y, A) = \frac{y - A}{|[A - y] + A|} \quad (E.10)$$

A is the value for which y is mapped to zero.

Deceptive ($A \in (0, 1), 0 < B \ll 1, 0 < C \ll 1, A - B > 0, A + B < 1$)

$$\begin{aligned} s_decept(y, A, B, C) &= 1 + (|y - A| - B) \left(\frac{[y - A + B](1 - C + \frac{A-B}{B})}{A - B} \right. \\ &\quad \left. + \frac{[A + B - y](1 - C + \frac{1-A-B}{B})}{1 - A - B} + \frac{1}{B} \right) \end{aligned} \quad (E.11)$$

A is the value for which y is mapped to zero, and is the global minimum of the transformation. B is the “aperture” size of the well/basin leading to the global minimum at A , and C is the value of the deceptive minima - there are always two deceptive minima.

multimodal ($A \in \{1, 2, \dots\}, B \geq 0, (4A + 2)\pi \geq 4B, C \in (0, 1)$)

$$s_multi(y, A, B, C) = \frac{1 + \cos((4A + 2)\pi(0.5 - \frac{|y-C|}{2([C-y]+C)})) + 4B(\frac{|y-C|}{2([C-y]+C)})^2}{B + 2} \quad (E.12)$$

A controls the number of minima, B controls the magnitude of the “hill sizes” of the multi-kodality, and C is the value for which y is mapped to zero. When $B = 0$, $2A + 1$ values of y , one at C , are mapped to zero, and when $B \neq 0$, there are $2A$ local minima, and one global minimum at C . Larger values of A and smaller values of B create more difficult problems.

E.2.3 Reduction

The reduction functions are

Weighted Sum ($|\mathbf{w}| = |\mathbf{y}|, w_1 \dots, w_{|\mathbf{y}|} > 0$)

$$\text{r_sum}(\mathbf{y}, \mathbf{w}) = \frac{\sum_{i=1}^{|\mathbf{y}|} w_i y_i}{\sum_{i=1}^{|\mathbf{y}|} w_i} \quad (\text{E.13})$$

The constant weight vector $\mathbf{w}$ forces an algorithm to treat the parameter vector, $\mathbf{y}$, differently.

Non-separable ($A \in \{1, \dots, |\mathbf{y}|\}, |\mathbf{y}| \bmod A = 0$)

$$\text{r_nonsep}(\mathbf{y}, A) = \frac{\sum_{i=1}^{|\mathbf{y}|} (y_i + \sum_{j=0}^{A-2} |y_i + y_1 + (i + j) \bmod |\mathbf{y}||)}{\frac{|\mathbf{y}|}{A} \lceil \frac{A}{2} \rceil (1 + 2A - 2 \lceil \frac{A}{2} \rceil)} \quad (\text{E.14})$$

A controls the degree of non-separability, noting that $\text{r_nonsep}(\mathbf{y}, 1) = \text{r_sum}(\mathbf{y}, 1)$

E.3 Restrictions

To ensure problems created using the WFG toolkit are well-designed, the following restrictions apply:

Primary Parameters

For any given transition vector, all the parameters of the originating transition vector must be employed exactly once as a primary parameter, counting parameters that appear independently as primary parameters, and in the same order in which the parameters appear in the originating transition vector.

Secondary Parameters

Care must be taken to avoid cyclical dependencies in $\mathbf{b_param}$. If a is a primary parameter of $\mathbf{b}$, and b is a secondary parameter, then a depends on b . If b likewise depends on c , then a depends, indirectly, on c . To prevent cyclical dependencies, no two parameters should be dependent on one another. A parameter should also not depend on itself.

Shifts

Parameters should only be subjected to a maximum of one shift transformation.

Reductions

Reduction transformation should belong to transition vectors that are closer to the underlying parameter vector than any shift transformations.

b_flat

When $A = 0$, b_flat should only belong to transition vectors that are further away from the underlying parameter vector than any shift or reduction transformation.

Constants

Constants must be fixed values and cannot be tied to the value of any parameter. The values and domains of the working parameters for the nine WFG test problems are defined as follows:

$$\begin{aligned} S_k &= 1 : m = 2k \\ A_1 &= 1 \\ A_{2:m-1} &= \begin{cases} 0 & \text{for WFG3} \\ 1 & \text{otherwise} \end{cases} \end{aligned} \tag{E.15}$$

The settings for $S_{1:m}$ ensure that the POFs have dissimilar trade-off magnitudes, and the settings for $A_{1:m-1}$ ensure that the POFs are not degenerate, except in the case of WFG3, which has a one-dimensional POF.

Domains

The domains for the working parameters, defined below, are of dissimilar magnitude:

$$z_{i=1:n,\max} = 2i \tag{E.16}$$

E.4 Test Problems

The nine scalable test problems, summarised in Table E.1 are defined as follows:

WFG1

$$\begin{aligned}
\text{shape} \quad h_{k=1:m-1} &= \text{convex}_j \\
h_m &= \text{mixed}_j, \text{ with } \alpha = 1 \text{ and } A = 5 \\
t^1 \quad t_{i=1:j}^1 &= y_i \\
t_{i=j+1:n}^1 &= \text{s_linear}(y_i, 0.35) \\
t^2 \quad t_{i=1:j}^2 &= y_i \\
t_{i=j+1:n}^2 &= \text{b_flat}(y_i, 0.8, 0.75, 0.85) \\
t^3 \quad t_{i=1:n}^3 &= \text{b_poly}(y_i, 0.02) \\
t^4 \quad t_{i=1:m-1}^4 &= \text{r_sum}((y_{(i-1)j/(m-1)+1}, \dots, y_{ij/(j-1)}), \\
&\quad (2((i-1)j/(m-1) + 1), \dots, 2ij/(m-1))) \\
t_k^4 &= \text{r_sum}((y_{j+1}, \dots, y_n), (2k(j+1), \dots, 2n))
\end{aligned} \tag{E.17}$$

WFG2

$$\begin{aligned}
\text{shape} \quad h_{k=1:m-1} &= \text{convex}_j \\
h_m &= \text{disc}_j, \text{ with } \alpha = \beta = 1 \text{ and } A = 5 \\
t^1 \quad t_{i=1:j}^1 &= y_i \\
t_{i=j+1:n}^1 &= \text{s_linear}(y_i, 0.35) \\
t^2 \quad t_{i=1:j}^2 &= y_i \\
t_{i=j+1:n}^2 &= \text{r_nonsep}((y_{j+2(i-j)-1}, y_{j+2(i-j)}), 2) \\
t^3 \quad t_{1:m-1}^3 &= \text{r_sum}((y_{(i-1)j/(m-1)+1}, \dots, y_{ij/(j-1)}), (1, \dots, 1)) \\
t_m^3 &= \text{r_sum}((y_{j+1}, \dots, y_{j+l/2}), (1, \dots, 1))
\end{aligned} \tag{E.18}$$

WFG3

$$\begin{aligned}
\text{shape} \quad h_{k=1:m} &= \text{linea}_j \\
t^1 \quad t_{i=1:j}^1 &= y_i \\
t_{i=j+1:n}^1 &= \text{s_linear}(y_i, 0.35) \\
t^2 \quad t_{i=1:j}^2 &= y_i \\
t_{i=j+1:n}^2 &= \text{r_nonsep}((y_{j+2(i-j)-1}, y_{j+2(i-j)}), 2) \\
t^3 \quad t_{1:m-1}^3 &= \text{r_sum}((y_{(i-1)j/(m-1)+1}, \dots, y_{ij/(j-1)}), (1, \dots, 1)) \\
t_m^3 &= \text{r_sum}((y_{j+1}, \dots, y_{j+l/2}), (1, \dots, 1))
\end{aligned} \tag{E.19}$$

WFG4

$$\begin{aligned}
\text{shape} \quad h_{k=1:m} &= \text{concave}_j \\
t^1 \quad t_{i=1:n}^1 &= \text{s_multi}(y_i, 30, 10, 0.35) \\
t^2 \quad t_{i=1:m-1}^2 &= \text{r_sum}((y_{(i-1)j/(m-1)+1}, \dots, y_{ij/(j-1)}), (1, \dots, 1)) \\
t_m^2 &= \text{r_sum}((y_{j+1}, \dots, y_n), (1, \dots, 1))
\end{aligned} \tag{E.20}$$

WFG5

$$\begin{aligned}
\text{shape} \quad & h_{k=1:m} = \text{concave}_j \\
t^1 \quad & t_{i=1:n}^1 = \text{s_decept}(y_i, 0.35, 0.001, 0.05) \\
t^2 \quad & t_{i=1:m-1}^2 = \text{r_sum}((y_{(i-1)j/(m-1)+1}, \dots, y_{ij/(j-1)}), (1, \dots, 1)) \\
& t_m^2 = \text{r_sum}((y_{j+1}, \dots, y_n, (1, \dots, 1)))
\end{aligned} \tag{E.21}$$

WFG6

$$\begin{aligned}
\text{shape} \quad & h_{k=1:m} = \text{concave}_j \\
t^1 \quad & t_{i=1:j}^1 = y_i \\
& t_{i=j+1:n}^1 = \text{s_linear}(y_i, 0.35) \\
t^2 \quad & t_{i=1:m-1}^2 = \text{r_nonsep}((y_{(i-1)j/(m-1)+1}, \dots, y_{ij/(j-1)}), J/(m-1)) \\
& t_m^2 = \text{r_nonsep}((y_{j+1}, \dots, y_n), l)
\end{aligned} \tag{E.22}$$

WFG7

$$\begin{aligned}
\text{shape} \quad & h_{k=1:m} = \text{concave}_j \\
t^1 \quad & t_{i=1:j}^1 = \text{b_param}(y_i, \text{r_sum}((y_{i+1}, \dots, y_n), (1, \dots, 1)), \frac{0.98}{49.98}, 0.02, 50) \\
& t_{i=j+1:n}^1 = y_i \\
t^2 \quad & t_{i=1:j}^2 = y_i \\
& t_{i=j+1:n}^2 = \text{s_linear}(y_i, 0.35) \\
t^3 \quad & t_{i=1:m-1}^3 = \text{r_sum}((y_{(i-1)j/(m-1)+1}, \dots, y_{ij/(j-1)}), (1, \dots, 1)) \\
& t_m^3 = \text{r_sum}((y_{j+1}, \dots, y_n, (1, \dots, 1)))
\end{aligned} \tag{E.23}$$

WFG8

$$\begin{aligned}
\text{shape} \quad & h_{k=1:m} = \text{concave}_j \\
t^1 \quad & t_{i=1:j}^1 = y_i \\
& t_{i=j+1:n}^1 = \text{b_param}(y_i, \text{r_sum}((y_{i+1}, \dots, y_n), (1, \dots, 1)), \frac{0.98}{49.98}, 0.02, 50) \\
t^2 \quad & t_{i=1:j}^2 = y_i \\
& t_{i=j+1:n}^2 = \text{s_linear}(y_i, 0.35) \\
t^3 \quad & t_{i=1:m-1}^3 = \text{r_sum}((y_{(i-1)j/(m-1)+1}, \dots, y_{ij/(j-1)}), (1, \dots, 1)) \\
& t_m^3 = \text{r_sum}((y_{j+1}, \dots, y_n, (1, \dots, 1)))
\end{aligned} \tag{E.24}$$

WFG9

$$\begin{aligned}
\text{shape} \quad & h_{k=1:m} = \text{concave}_j \\
t^1 \quad & t_{1:n-1}^1 = \text{b_param}(y_i, \text{r_sum}((y_{i+1}, \dots, y_n), (1, \dots, 1)), \frac{0.98}{49.98}, 0.02, 50) \\
& t_n^1 = y_n \\
t^2 \quad & t_{i=1:j}^2 = \text{s_decep}(y_i, 0.35, 0.001, 0.05) \\
& t_{i=j+1:n}^2 = \text{s_multi}(y_i, 30, 95, 0.35) \\
t^2 \quad & t_{i=1:m-1}^2 = \text{r_nonsep}((y_{(i-1)j/(m-1)+1}, \dots, y_{ij/(j-1)}), J/(m-1)) \\
& t_m^2 = \text{r_nonsep}(y_{j+1}, \dots, y_n, l)
\end{aligned} \tag{E.25}$$

Table E.1: Properties of the WFG problems

Name	Separability	modality	Shape
WFG1	Separable	Unimodal	Convex, mixed
WFG2	Non-separable	Unimodal/multimodal	Convex, disconnected
WFG3	Non-separable	Unimodal	Linear, degenerate
WFG4	Separable	Multimodal	Concave
WFG5	Separable	Multimodal	Concave
WFG6	Non-separable	Unimodal	Concave
WFG7	Separable	Unimodal	Concave
WFG8	Non-separable	Unimodal	Concave
WFG9	Non-separable	Multimodal, deceptive	Concave