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Article

A Case Study of Prospective Teachers Engaged in Professional Noticing of their Students' Mathematical Thinking

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Abstract: In this study, prospective teachers engaged in professional noticing of their students' mathematical thinking captured in pieces of written work. Researchers then worked to characterize the prospective teachers' professional noticing using a lens of responsive teaching. Results indicate that prospective teacher decisions about how to respond to their students' mathematical thinking fall on a responsiveness continuum, often shifting in responsiveness across pieces of student written work. The findings of this study provide guidance for teacher educators who work to develop K–12 educators' responsive teaching practices.

Keywords: professional noticing; responsive teaching; teacher education; mathematics education

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1. Introduction

There is a practice-based movement in mathematics teacher education that is focused on examining what teachers need to be able to do to be effective mathematics educators [1–3]. This practice-based movement shifts the focus in mathematics teacher education from what teachers need to know, often conceptualized according to Ball et al. as mathematical knowledge for teaching, with domains including common and specialized content knowledge, and knowledge of content and students, teaching, and curriculum [4]. Rather than focus on what teachers need to know, shifting to practice has led some scholars, including us, in the field to organize the work and scholarship of teacher education around high-leverage core instructional practices [1–3]. One high-leverage core instructional practice, teacher noticing, emphasizes the importance of eliciting and using student thinking in the teaching and learning of mathematics [2]. Embedded within the core instructional practice of teacher noticing is the construct of professional noticing of children's mathematical thinking [5]. Professional noticing of children's mathematical thinking is a set of interrelated skills that include (a) attending, (b) interpreting, and (c) deciding how to respond based on students' mathematical understandings [5]. The way a teacher engages in professional noticing of children's mathematical thinking can be responsive, or not, depending on if a teacher understands students' mathematical thinking and then uses that thinking to drive mathematics instruction [6,7]. Studying prospective teachers' engagement in high-level core instructional practices like professional noticing of children's mathematical thinking can provide teacher educators insight into what prospective know and are able to do during mathematics instruction.

Learning to notice children's mathematical thinking and responding to that thinking in ways that are responsive is difficult for novice teachers [8]. Several studies have identified strategies for supporting prospective teachers in developing this practice [9,10]. Other studies focus specifically on teacher decisions about how to respond and identify decide actions or teacher moves that work to support responsive teaching [11,12]. This study captures a unique snapshot of eight prospective teachers' professional noticing of children's mathematical thinking just prior to student teaching. In this study, we use a semistructured interview to facilitate prospective teachers' professional noticing of children's mathematical thinking in pieces of written work created in an after-school program. We then analyze prospective teachers' professional noticing using a responsive teaching lens. The research questions for this study are: (1) If a prospective teacher sufficiently attends to and interprets their students' mathematical thinking, what are their decide actions and their purposes for those actions? (2) Which decide actions and purposes for those actions support responsive, developing responsive, or unresponsive professional noticing?

2. Literature Review

2.1. Core Practice of Teacher Noticing

While there are many aspects that a teacher could notice in a classroom, the most critical is to actively notice student thinking [5]. Sherin and colleagues [13] articulate teacher noticing as involving two main components: (1) attending to and (2) interpreting student thinking. Jacobs and colleagues [5] have extended this framework to include a third element—deciding how to respond to student thinking. Emphasizing the importance of attending and interpreting student thinking prior to deciding how to respond Jacobs et al. [5] introduced the construct of professional noticing of children's mathematical thinking, hereafter referred to as professional noticing.

Research investigating professional noticing have examined various aspects of this construct, from what is noticed (e.g., students' words, characteristics, general observations), to why something is noticed (e.g., correct/incorrect, interesting) in a variety of contexts (e.g., video, student written work, scripting dialogues) [14–21]. Findings from prior research specific to prospective teacher professional noticing skills suggest that when supported (i.e., explicit questioning) prospective teachers can develop the ability to attend to and interpret contextual and disciplinary aspects of student thinking [10,15,19,20]. Research examining the interrelated nature of professional noticing skills often consider attending and interpreting together and explore the relationship of these two skills with the deciding how to respond skill [5,11,12]. Studies examining the deciding how to respond skill often focus on decide actions or teacher moves in response to noticed student thinking [11,12]. In this study, we examine prospective teacher decide actions and their purposes for decide actions using a responsive teaching lens.

2.2. Responsive Teaching

Responsive teaching is both a teaching stance and a practice that emphasizes the importance of using the substance of student mathematical thinking to guide instructional decisions rather than attempting to "fix" or "correct" student thinking [7,22–24]. Research on responsive teaching include studies that theoretically conceptualize this teaching stance [6] and studies that identify decide actions that facilitate responsive teaching [22–25]. For example, Dyer and Sherin [23] identified three decide actions that result in responsive teaching during live classroom discussions that involve: (1) a substantive probe of student ideas; (2) an invitation for student comment; and (3) a teacher uptake of student ideas. A responsive teaching lens allowed us to identify the ways prospective teachers responded to their students' thinking to determine if their professional noticing was rooted in the substance of student thinking, or not.

In this study, we worked to first determine if the prospective teachers understood their students' mathematical thinking, as this is an important precursor to teaching responsively [7]. Then, if a prospective teacher demonstrated an understanding of their students' mathematical thinking, we examined their proposed decide actions and related purposes to determine if they supported responsive teaching. To do this, we considered if each decide action was focused on taking up and pursuing student thinking rather than fixing student thinking [7,23]. For example, consider a teacher deciding to respond to student thinking by giving a student a new task. This decide action appears responsive as it could work to pursue student thinking. However, if the teacher explains that the "reason" they gave the student a new task is to see if the student will fix a calculation error, the decide action and the related purpose becomes unresponsive because it is focused on fixing rather than pursuing student thinking.

2.3. Authentic Teaching Spaces

Prior research also suggest that prospective teachers need authentic teaching spaces to develop their professional noticing skills in ways that are responsive to student thinking [3]. One such authentic teaching spaces are community-based educational spaces. Several studies point to the benefits of engaging prospective teachers in developing their teaching in community-based settings such as after-school programs because they are removed from curricular, time-based, and administrative contexts [3,26]. The prospective teachers in this study participated in an afterschool mathematics club located at a local primary school and worked with small groups of students to develop their mathematical understandings for solving multidigit computation problems. Using pieces of student work created in the after-school math club, we identify and analyze the professional noticing of prospective teachers just prior to student teaching. In the next section, we detail our methods for conducting this study.

3. Methods

3.1. Study Context

After the end of apartheid in South Africa, the government put in place a policy to transform teacher education to promote equitable education for all children [27]. The new framework encouraged teacher education programs to work with prospective teachers to take up and use student thinking in the teaching and learning of mathematics. Each of the participants were enrolled in the fourth and final year of a Bachelor of Education initial teacher certification program in a city in South Africa that allowed them to teach mathematics in primary school (Grades 4 to 7). The participants had nine weeks of practicum experience in their second and third years. During their final year, the participants worked with primary school students who attended a mathematics club at a local primary school as part of their mathematics method course. The participants also engaged in a two-hour weekly class session in which they examined learning theory and practiced professional noticing using student work created in the mathematics club.

Teacher educators emphasized shifting from traditional teaching methods that involved teachers telling, explaining, and showing students how to do mathematics toward student centered instruction. The participants had not been exposed specifically to the construct of responsive teaching as we used this as an analytical, not pedagogical, tool. Each student completed an assignment that asked them to express their beliefs about the importance of using student thinking in the teaching and learning of mathematics. Twelve students expressed strong beliefs about the importance of using student thinking in the teaching and learning of mathematics and they were asked to participate in the study. Eight out of these 12 students in the fourth-year mathematics methods class agreed to participate in the study.

3.2. Research Design

We used a single illustrative case study design [28] to provide a rich description of prospective teachers engaged in professional noticing as they examined student written work. Each participant engaged in a semistructured interview in which they examine two pieces of student written work that contained evidence of student thinking they considered worthy of further analysis. The interviewer asked explicit questions for each piece of written work that prompted the participants to engage in professional noticing: (1) How would you describe the student work? (2) What does that tell you about student thinking? (3) How would you respond? (4) Why did you make that decision?

The data set included eight interview transcripts and 16 pieces of student written work. First, we coded the transcripts by identifying and separating out evidence of each professional noticing skill, attend, interpret, or decide. We coded each attend or interpret segment as either written work or not written work. The attend and interpret segments coded as written work and the decide segments served as the data for this study. We then analyzed each piece of student written work to identify the important mathematical elements and created a checklist to be able to determine if participants also noticed the important mathematical elements in the pieces of written work.

3.3. Data Analysis

3.3.1. Students' Mathematical Thinking

The attend and interpret transcript segments related to each piece of written work were compared to the related checklists. Pieces of written work for which a participant noticed the important mathematical elements (<70%) entered the next stage of analysis. This analytical process resulted in 46 decide transcript segments entering the next stage of analysis. We then identified each participant's decide actions and purposes for those actions.

3.3.2. Decide Actions

Participants often shared more than one decision about how to respond to student thinking noticed in the pieces of student written work. At times, participants shared decide actions that they had enacted during a live teaching event. At other times, they shared decide actions they planned to enact in a future teaching event. Using open coding we identified six codes to capture the different decide actions posed by the participants. We refined these codes using prior studies [11,12,23]. We then determined if a decide action was responsive to student thinking by examining if each decide action was focused on taking-up and pursuing student thinking rather than fixing student thinking [7,23]. The responsive decide actions included the teacher asks the student: (1) an open question; (2) to use a different strategy; or (3) to work on a new task. The remaining decide actions, considered not responsive, included the teacher: (1) tells, instructs, and/or explains a strategy or concept to a student; (2) asks the student a funneling question; and (3) asks the student to rewrite their work (see Table 1).

Table 1. Decide actions.

Decide Actions The Teacher ...	**Illustrative Transcript Excerpts**	**Responsive**
... tells, instructs, and/or explains a strategy or concept to a student	I would tell her with two different methods on how to multiply.	N
... asks the student a funneling question(s)	Are you sure that 10 is added to your last number?	N
... asks the student to rewrite their work	She kind of confused herself with all of the erasing, so I would actually ask her to write it over.	N
... asks the student an open question(s)	I asked him to explain it [student's solution].	Y
... asks the student to use a different strategy	I would ask him to use a different method.	Y
... asks the student to work on a new task	I would give him a more challenging question with larger number like 27.	Y

3.3.3. Purposes

We used open coding to identify the purposes for decide actions resulting in eight codes. The first four codes were identified as not responsive and included the teacher wants: (1) the work to be error free, (2) the work to be neat, (3) to share their own mathematical understandings with the student, and (4) to understand student thinking confusing to the teacher. The remaining four purpose codes were identified as responsive and included the teacher wants: (1) the student to understand the problem context, (2) the student to be mathematically challenged, (3) the student to understand their own thinking (metacognition), and (4) to support the student in their chosen strategy (see Table 2).

Table 2. Purpose codes.

Purpose Codes The Teacher Wants . . .	Illustrative Transcript Excerpts	Responsive
. . . the work to be error free	I told her that she needs . . . write down the correct answer because she will not get her marks.	N
. . . the work to be neat	I want her to make it [the student work] cleaner.	N
. . . to share their mathematical understandings with the student	I wanted to show her how I would do it . . .	N
. . . to understand student thinking that is confusing to the teacher	I couldn't make sense of what he did and that is what made it interesting for me.	N
. . . the student to understand the problem context	Then maybe he would realize that the problem means multiplication.	Y
. . . the student to be mathematically challenged	I want to challenge the learner.	Y
. . . the student to understand their own thinking (metacognition)	I want to know he really understands it.	Y
. . . to support a student in their chosen strategy	I really worked with him [asked him a series of open questions] because he understood it, he just lacked those basic addition skills to get him to the answer.	Y

Once the final codes were developed, the researchers coded the remaining data independently and then met to discuss discrepancies and modify the codes until 100% consensus across all data points was achieved.

3.3.4. Responsiveness in Professional Noticing

Each piece of student work served as the unit of analysis. We identified three categories of responsiveness: (1) responsive professionally noticing, (2) developing responsive professional noticing, and (3) unresponsive professional noticing. If a participant noticed the important mathematical elements in a piece of written work and all the decide actions and purposes were considered responsive for that piece of written work, we considered that participant as engaging in responsive professional noticing. However, if a participant noticed the important mathematical elements in a piece of written work but only some decide actions and decide purposes were responsive, we considered that participant as engaging in developing responsive professional noticing. If none of the decide actions or purposes were responsive, then that participant was designated as engaging in unresponsive professional noticing.

4. Results

There were eight participants in this study: (1) Anne; (2) Barbara; (3) Charles; (4) David; (5) Elizabeth; (6) Fran; (7) Grace; and (8) Heather (pseudonyms). All participants sufficiently (>70%) attended to and/or interpreted the mathematical elements identified in all sixteen pieces of student work. We first present results for research question 1: If a prospective teacher sufficiently attends to and interprets their students' mathematical thinking, what are their decide actions and their purposes for those actions? We then present results for research question 2: Which prospective teachers' decisions about how to respond exhibited responsive, developing responsive, or unresponsive professional noticing? We then provide illustrative examples for each type of responsive professional noticing

4.1. Decide Actions

Across the eight participants and the 16 pieces of student written work, there were 46 decide actions and purposes. The most common decide action involved a teacher asking a student an open-ended question (14 instances). Six of the eight participants made decide actions that involved asking a student to work on a new task (11 instances).

Five participants made decide actions that involved the teacher telling, instructing, and/or explaining a strategy or concept to a student (10 instances). Three participants made decide actions that involved asking a funneling question(s) (6 instances). Three participants made decide actions that involved the teacher asking a student to use a different strategy (3 instances). The remaining two decide actions involved asking a student to rewrite their work (2 instances).

4.2. Purposes

There were 46 purposes—one for each decide action shared across the eight participants. The most common purpose for a decide action was the teacher wanting the student work to be error free (20 instances). The second most common purpose was the teacher wanting a student to be mathematically challenged (8 instances). There were five instances that involved the teacher wanting to share their own mathematical understandings with a student. There were four instances of the teacher wanting the student to understand the problem context and three instances of the teacher wanting to understand student thinking that was confusing to the teacher. There were two instances of the teacher wanting student work to be neat, two instances of the teacher wanting the student to understand their own thinking (metacognition) and two instances of the teacher wanting to support a student in their chosen strategy. There was not a one-to-one correspondence between decide actions and related purposes.

In summary, all prospective teachers noticed the mathematical elements in the written work, the most common decide action was for the teacher to ask an open-ended question and the most common purpose was the teacher wants for the student work to error free. Additionally, there was not a one-to-one correspondence between decide actions and related purposes.

Next, we examine the decide actions and purposes using responsive teaching as a lens to characterize prospective teacher professional noticing.

4.3. Responsiveness in Professional Noticing

Recall, a participant's decide actions and purposes, for a piece of student written work were individually identified as responsive or not. Then, if all the decide actions and purposes were considered responsive for that piece of written work, we considered that participant as engaging in responsive professional noticing. If only some decide actions and decide purposes were responsive, we considered that participant as engaging in developing responsive professional noticing. If none of the decide actions or purposes were responsive, then that participant was designated as engaging in unresponsive professional noticing.

David was the only participant who engaged in responsive professional noticing for both pieces of student written work while Anne, Charles, and Fran were designated as unresponsive professional noticing for both pieces of student written work. Only Grace was designated as developing responsive professional noticing for both pieces of student written work. Barbara and Elizabeth were designated as responsive professional noticing for their first piece of student written work but developing responsive professional noticing for their second piece of student written work. Heather was designated as developing responsive professional noticing for her first piece of student written work but unresponsive professional noticing for her second piece of student written work (see Table 3).

Next, we provide illustrative examples for responsive, developing, and unresponsive professional noticing.

Table 3. Participant responsiveness.

Participant	First Piece of Student Written Work	Second Piece of Student Written Work
Anne	Unresponsive Professional Noticing	Unresponsive Professional Noticing
Barbara	Responsive Professional Noticing	Developing Responsive Professional Noticing
Charles	Unresponsive Professional Noticing	Unresponsive Professional Noticing
David	Responsive Professional Noticing	Responsive Professional Noticing
Elizabeth	Responsive Professional Noticing	Developing Responsive Professional Noticing
Fran	Unresponsive Professional Noticing	Unresponsive Professional Noticing
Grace	Developing Responsive Professional Noticing	Developing Responsive Professional Noticing
Heather	Developing Responsive Professional Noticing	Unresponsive Professional Noticing

4.3.1. Responsive Professional Noticing

Barbara and Elizabeth engaged in responsive professional noticing for one out of the two pieces of student written work. David engaged in responsive professional noticing for both pieces of student written work, and combined, they posed eight total decide actions and related purposes. All three participants posed the decide action of providing a student with a new task (five out of eight instances). Interestingly, all participants had one or more instances in which their purposes for providing a new task involved wanting to challenge students mathematically. Only Barbara shared two different purposes for providing a new task. In one instance, she wanted the student to be mathematically challenged and in the other instance she wanted the student to understand the problem context. David had two different purposes for asking an open-ended question: (1) for the student to understand their own thinking (metacognition) and (2) to be mathematically challenged. Next, we provide an illustrative example of responsive professionally noticing.

Barbara noticed all the important mathematical elements in the piece of student written work that involved a word problem requiring students to determine the total number of bottles of milk for 21 boxes of milk that each contains six bottles of milk (see Figure 1).

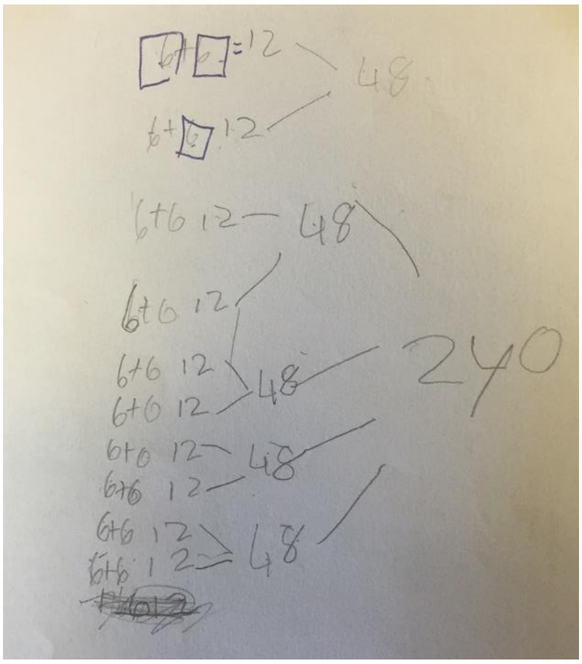


Figure 1. Barbara's shared piece of student work for a word problem that requires students to determine the total number of bottles of mild for 21 boxes of milk that each contains six bottles of milk.

Direct quotes from the transcript were used to determine that she had sufficiently attended and/or interpreted the student written work in ways that matched with four of the four (100%) established mathematical elements (see Table 4). This included her noticing that the student added six and six a total of ten times (20 total sixes), resulting in 20 and not 21 total sixes, as the problem indicated. She also noticed that the student found 12 + 12 to be equal to 48 (not 24). Finally, she noticed that the student added 48, five times, and found the correct solution of 240.

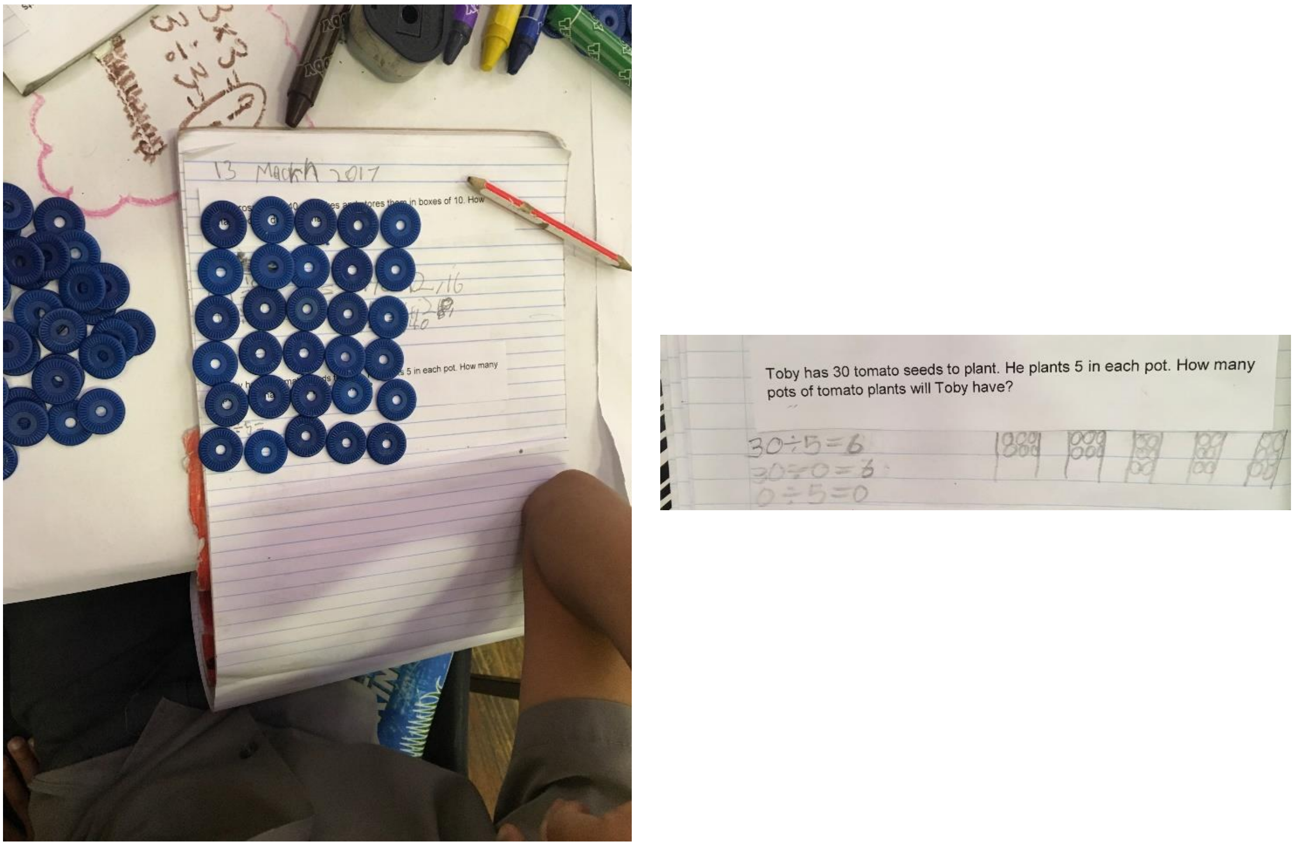
Table 4. Mathematical elements in Barbara's shared piece of student written work.

Mathematical Element	Transcript Quotes
The student uses repeated addition (6 + 6) ten times, and doesn't write remaining 6 down	The last 6 has not another 6 to add it up with, so he just scratched it out, and he did not even think about there were 21 boxes, so there must be one 6 which does not have another 6 next to it.
The student uses repeated addition (12 + 12) five times, resulting in the incorrect answer of 48.	Adding the 12 s for 48, and then he adds here, but he doesn't even add them up correctly. Well, I think he saw the 48 here, and then he didn't realize, oh, 12 + 12 is 24.
The student adds or multiplies (operation not shown) 5 instances of 48.	Instead of breaking them up to add 48 and 48 or something, he also here just tried to add them all up, but I'm not sure how . . .
The student finds an answer of 240, the correct answer for 48×5 but not the correct answer for the problem of 21×6	I think he saw the 48 here, and then he didn't realize, oh, 12 + 12 is actually 24, so he immediately went further and added the 48 s, although he didn't understand the in-between with the lines, but he added it up correctly.

While Barbara noticed the important mathematical elements in the piece of student work, she focused on the student adding 20 instead of 21 sixes, interpreting that the student did not understand what the problem was asking. She focused her professional noticing on this aspect of student thinking by stating, "there is one 6 which does not have another 6 next to it . . . he didn't understand that you had to have 21 bottles". She shared two decide actions and purposes for this piece of student work. First, she stated, "I asked him to use different methods . . . so that he actually can picture what the problem is asking". This first decide action was coded as the teacher asks the student to use a different strategy for the purpose of making sure the student understood the problem context. The second decide action involved providing the student with a new task for the purpose of making sure the student understood the problem context. She explains, "I would give him a bit smaller one maybe that would help him to draw it and realize, okay, that is what the problem is asking". Because Barbara noticed the important mathematical elements in the piece of written work and made decide actions with purposes for those decide actions that work to pursue the substance of student thinking, her professional noticing for this piece of written work was designated as responsive.

4.3.2. Developing Responsive Noticing

Barbara, Elizabeth, and Heather were identified as engaging in developing responsive professional noticing for one piece of their student written work. Grace was identified as engaging in developing responsive professional noticing for both pieces of student work. Across these four participants and five pieces of student written work, there were a total of 16 decide actions and related purposes. Interestingly, these participants all first shared a responsive decide action that involved the teacher asking an open-ended question (8/16 total instances). However, Barbara and Grace both shifted to the unresponsive decide actions of telling instructing and/or explaining a strategy or concept to a student (2/16 instances) or asking funneling question(s) (2/16 instances). Additionally, although Barbara, Grace, and Heather primarily posed responsive decide actions, they had nonresponsive purposes that involved wanting work to be error free (4/16 instances). Elizabeth posed a decide action that involved asking open-ended questions for the purpose of understanding student thinking that was confusing to the teacher (1/16 instances). Because teachers cannot be responsive to what they do not understand, these purposes were designated as unresponsive. Grace primarily asks open-ended questions for the second piece of student written work but her purpose for asking these questions was to share her own mathematical understanding rather than for probing student thinking (3/16 instances). Overall, since the participants' professional noticing for each piece of written work contained both responsive and unresponsive decide actions and purposes, their professional noticing was designated as developing responsive. Next, we share an illustrative example of developing responsive professional noticing by Grace in her second piece of student written work (see Figure 2).



(**a**) (**b**)

Figure 2. Grace's piece of student work involving a word problem that requires students to determine the number of pots needed to plant 30 tomato seeds with 5 seeds in each pot. (**a**) shows Grace using counters to find a solution (**b**) shows Grace drawing a picture to find a solution.

Grace noticed all the important mathematical elements in the piece of student written work (see Figure 2).

These mathematical elements included the student using chips to represent the 30 tomato seeds and placed them into six rows of five to count out thirty (see Table 5). The student did not see that the five rows of six could represent a solution and instead took these chips and redistributed them into five rows of six.

Grace shared four decide actions and related purposes for this piece of written work. First, she asked the student an open question, "you've got 30 seeds, and you need to put them into pots, what should you do?" This decide action was coded as the teachers asks the student an open question for the purpose of the student understanding the problem context, which is considered responsive. The student then counted out the 30 counters and placed them in six rows of five; she then asked another open question, "how are you going to use the counters to find your answer?" This decide action was also coded as the teacher asks the student an open question but for the purpose of supporting the student in their chosen strategy; both the action and purpose are considered responsive.

Table 5. Mathematical elements in Grace's shared piece of written work.

Mathematical Element	Transcript Quotes
Problem situation is division. 30 tomato seeds to plant, he plants 5 in each pot. How many pots.	You have 30 tomato seeds, and you need to put them into five pots. How many seeds does each pot need?
Student takes 30 chips and places in five columns. Six in each column.	He takes his 30 seeds, he takes the counters, and he puts them in rows of 5, and eventually he has six rows down.
Student doesn't see counters arranged in six rows and five columns as showing an answer.	He understood that they needed to be put into five pots, but he couldn't take what he had in front of him, this complete thing, and go, "Okay, well, actually, this makes sense".
Student writes $30 \div 5 = 6$, $30 \div 0 = 6$, $0 \div 5 = 0$	He could write $30 \div 5$, and then eventually after doing this then he got to six. The rest are wrong.
Student draws five pots and draws six circles in each one.	Then, he draws a picture and says each pot gets six.

However, Grace became fixated on the idea that the student had modeled the answer by arranging 30 chips into six rows of five but did not recognize the model as showing a solution. She explains that the student does not realize that he could use the counters to find a solution. "He can understand how to group things into rows but didn't know that he could find the answer using rows and columns". It was at this point that Grace shifted in her decide actions and began telling, instructing, and explaining, "I told him that he needed to look at the way (the chips in 6 rows of 5) that he had done it, because he really had the answer right in front of him". This decide action was coded as the teacher tells, instructs, and explains for the purpose of sharing their own mathematical understandings; both action and purpose are considered unresponsive.

Graces then states "I would give him another problem and I would use 5 again, say, 50, and 5 to get him to count the 50 out and see if he does put them into 10 rows of 5 to see an answer. Grace was still fixated on the student "seeing" an answer when they counted out chips. This decide action was coded as the teacher gives the student a new task for the purpose of the teacher explaining their own mathematical understandings. Grace's professional noticing shifts from responsive to unresponsive and was designated as "developing responsiveness".

Next, we provide a second illustrative example from Elizabeth, who was identified as developing responsive professional noticing for her second piece of student written work (see Figure 3).

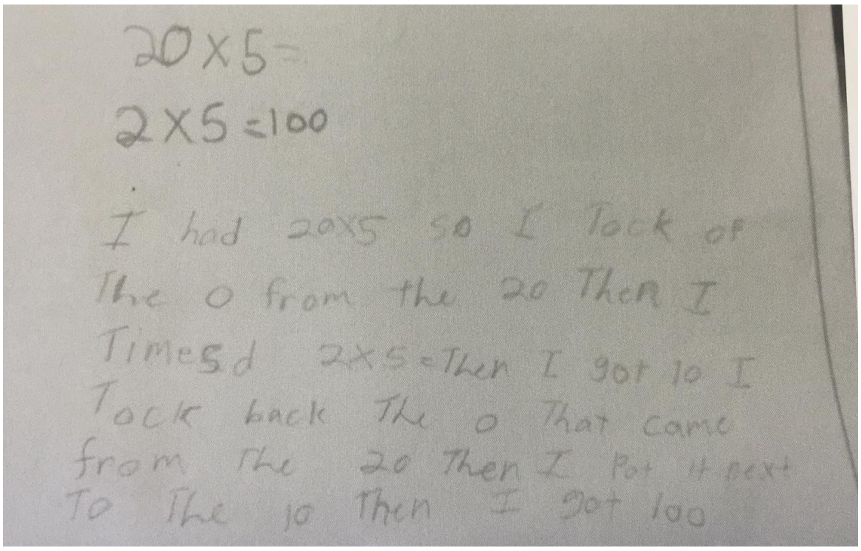


Figure 3. Elizabeth's piece of student work for a word problem that requires students to determine the number of trees that were planted in a school garden if each learner needed to plant 5 trees, and there were 20 learners in the class.

Elizabeth noticed all the important mathematical elements in the piece of student written work (see Table 6).

Table 6. Mathematical elements in Elizabeth's shared piece of written work.

Mathematical Element	Transcript Quotes
Problem situation is multiplication. 5 plants for each learner and 20 learners in the class. How many pots.	So, the question was, the Grade 6 class needs to plant trees in their school garden. Each learner needs to plant 5 trees around the school, and there are 20 learners in the class. How many trees will be planted?
Student writes $20 \times 5 =$	So, this learner, he first wrote out, he read the question and then he wrote 20 multiplied by 5,
Student writes $2 \times 5 = 100$	then he went back again and then he wrote $2 \times 5 = 100$,
Student describes in writing that they had 20×5 so they took one of the zeros from the 20 and had 2×5 which they knew was 10.	He explained it, he said he had 20×5, and then he took out the 0 from the 20, and then he timed or multiplied 2 by 5, and he got 10,
They then put the zero they had back and got 100	and then he put the 100 there again because he took it away, so now, he has to add it again, and then he got to 100 as the answer. That's what he did.

Elizabeth shared one decide action and a related purpose for this piece of written work. She asked the student an open question, "Would this have worked if it was 27×5". Elizabeth explained that her purpose for asking this open question was to determine if the student taking out and putting back a zero would work for any numbers or only for benchmark numbers (e.g., 10, 20, 30, 40, etc.). Elizabeth stated, "it looks simple, but it's really complicated. It was really a lot to think about, and that's what made it interesting for me". Elizabeth went on to explain that while she knows that the student had found the correct sum for 20×5, "it was difficult for me to exactly interpret if this would work for any numbers and so I wanted to see what he said". This decide action was coded as the teacher asks the student an open question for the purpose of the teacher understanding student thinking that is confusing to them, a purpose considered not responsive. We suspect that Elizabeth interpreted that this strategy worked for 20×5 but did not understand that the associative property of multiplication could be used to explain that 20×5 is the same as $10 \times 2 \times 5$, and the taking out and adding of the zero demonstrates multiplying by 10. Similarly, 27 could be decomposed into factors (e.g., $3 \times 3 \times 3$) and a student could multiple $3 \times 3 \times 3 \times 5$ in any order. This highlights an important element of teaching responsively—at times, a teacher needs to not only interpret what a student has done or said but also be able to connect or generalize to other mathematical ideas. While her current response is not considered responsive because its purpose was rooted in her own confusion, her willingness to share her confusion indicates that Elizabeth is working toward being able to make mathematical connections for herself that will in turn help her be more responsive to her students.

4.3.3. Unresponsive Noticing

Anne, Charles, and Fran engaged in unresponsive professional noticing for both pieces of written work. Heather demonstrated unresponsive professional noticing for her second piece of written work. These participants first noticed the important mathematical elements in each of the seven pieces of written work, but they then went on to primarily share decide actions that involved telling, instructing, and explaining (8/22 instances) or asking funneling questions (4/22 instances). The most common purpose for given decide actions was for student work to be error-free (16/22 instances). Two participants had idiosyncratic actions and purposes for those actions. For example, Anne wanted a student to rewrite a problem for the purpose of wanting student work to be neat, and Charles engaged in telling, instructing, and explaining for the purpose of sharing his own mathematical understandings. Next, we share Anne's professionally noticing for her second piece of student written work (see Figure 4) to exemplify unresponsive professional noticing.

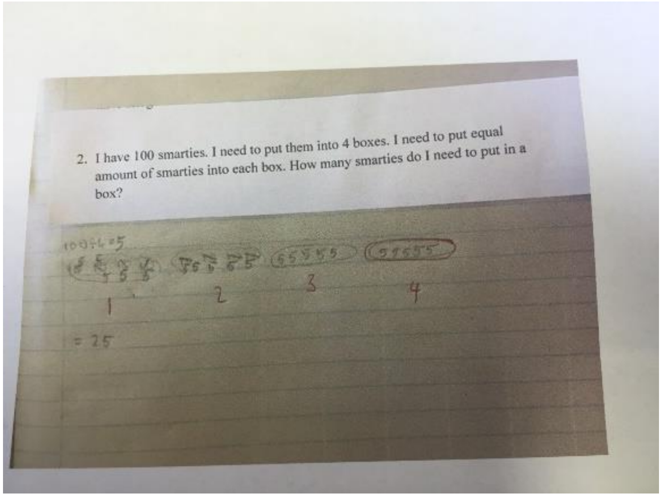


Figure 4. Anne's shared piece of student work for a word problem that requires students to determine if I have 100 smarties that needed equal amounts to be put into four boxes, how many smarties would in each of the four boxes. The student starts by writing 10 s and then changes to using 5 s, making four boxes of smarties with five 5 s in each box.

Anne noticed all the important mathematical elements in the piece of student written work that involved a word problem asking if I have 100 smarties that needed equal amounts to be put into four boxes, how many smarties would in each of the four boxes.

Anne attended and interpreted the student written work in ways that matched with 3 of the 3 (100%) of the established mathematical elements (see Table 7).

Table 7. Mathematical elements in Anne's shared piece of written work.

Mathematical Element	Transcript Quotes
Student (incorrectly) writes 100 ÷ 4 = 5	At the top of the paper, she wrote 100 ÷ 4 = 5. And then she told me she was so busy with 5 s that she kind of forgot the question.
Student draws four groups and starts by putting 10 in each group.	Then this learner immediately started writing 10, 10, 10, 10, so four 10 s and then they carried on, 10, 10, 10, 10. I think is she realized that this wasn't going to work.
Student changes to putting five in each group.	She scratched them out and then she replaced them with 5 s. She made four boxes of smarties with five 5 s and then obviously she counted and that's how she got to 25.

Anne shared two decide actions and related purposes for this piece of written work. First, she stated, "I would like her to write it over. Because it is messy with all the erasing and scratching things out". This decide action was coded as the teacher asks the student to rewrite student work for the purpose of the teacher wants the work to be neat. Anne went on to share, "I'd respond by telling her that she needs to go back and read the sum again and write down the correct answer. Because, everything is perfect, but at the top it does say that 100 ÷ 4 = 5 so, she will not get the marks, because it is not 5, it is 25". This decide action was coded as the teacher tells, instructs, and explains for the purpose of the work being error-free. Both decide actions and related purposes were designated as un-responsive, and, therefore, Anne's professional noticing for this piece of written work was designated as unresponsive.

5. Discussion

5.1. Pieces of Written Work

Our data indicates that our participants noticed the mathematical elements in self-selected pieces of student written work in all instances. This finding is not entirely surprising and suggests that analyzing student written work created in their own classroom provides an opportunity for prospective teachers practice professional noticing in their zone of proximal development [29] This result points to a benefit of using authentic teaching spaces to engage and study prospective teachers in the development of responsive professional noticing skills [30]. Furthermore, the ways that our participants noticed the mathematical elements in self-selected pieces of student written work in all instances provide insights into what prospective teachers need to know. For example, our participants needed to know how to describe and interpret their students' mathematical thinking, as this is an important precursor to engaging in responsive decisions about how to respond [7].

5.2. Categories of Decide Actions and Purposes

Our findings suggest that each participant had a go-to repertoire of decide actions and related purposes. The three most frequent decide actions posed by participants included: (1) asks the student an open question (14 instances); (2) asks the student work on a new task (11 instances); or (3) tells, instructs, or explains to the student (10 instances). The three most frequent purposes given for these actions included the teacher wants: (1) the student work to be error-free (20 instances); (2) the student to be mathematically challenged (8 instances); or (3) to share their own mathematical understandings with a student (5 instances). Additionally, there was not a one-to-one correspondence between decide actions and related purposes. Our study contributes to existing research focused on decide actions by identifying teachers' purposes for given decide actions. These results have important implications for teacher education.

First, at times, our participants shared responsive decide actions with unresponsive purposes or vice versa. These findings add to existing literature that emphasize the importance of studying decide actions in ways connected to how teachers are thinking about those actions [23]. Second, while our participants frequently shared responsive decide actions focused on asking students open questions and providing students with a new task, these are broad categories. Future researchers should conduct a more fine-grained analysis to identify the specific types of open-ended question prospective teachers pose. For example, are prospective teachers posing open-ended question that prompt their students to consider similarities and differences in their chosen strategies or do they pose questions that prompt students to explain their thinking? Additionally, teacher educators may need to work with prospective teachers in developing tasks that work to challenge students' conceptual understandings rather than attempting to make a problem more difficult using bigger numbers. Teacher educators could draw on the work of Lee et al. [31], who designed the Noticing-Oriented Task Modification Activity to support prospective teachers' ability to pose challenging tasks.

Third, at times, our participants shared unresponsive purposes for their decide actions. For example, the most common purpose for a decide action was the teacher wants the student work to be error-free. This supports prior research that student errors and misconceptions can be difficult for prospective teacher to respond to responsively [19,32]. There should be more research focused on ways to effectively scaffolding prospective teachers in responsively responding to students' mathematical errors or misconceptions. It is important to provide prospective teacher with opportunities to practice responding to student thinking in ways that work to develop student thinking rather than focusing on fixing procedural errors.

5.3. The Continuum of Responsiveness in Professional Noticing

Our findings show that prospective teachers' professional noticing skills fall on a continuum of responsiveness, both across participants and for individual participants across pieces of their shared written work. Three participants demonstrated responsive professional noticing for at least one piece of student written work. These participants consistently made decide actions that worked to uncover their student thinking and had purposes for those actions that focused on student thinking. These participants provided their students with new tasks to ensure their students were mathematically challenged and asked open-ended to questions to ensure their students understood the mathematical ideas embedded in different mathematical tasks. These participants demonstrate that beginning teachers can develop responsive professional noticing skills. Future research should expand in breadth of participants and depth of contexts to examine prospective teacher professional noticing in ways that capture the interrelated nature of the three professional noticing skills to expand our understandings about how teachers respond to what they notice in ways that are rooted in the pursuit of student thinking.

Four participants demonstrated developing professional noticing. These participants posed a mix of responsive and unresponsive decide actions and related purposes. These findings demonstrate that the overall responsiveness of a prospective teacher's professional noticing includes shifts that can occur through visible decide actions and/or a teacher's often hidden purpose for a decide action. Recall Grace's example of developing professional noticing. She was determined that her student use a teacher preferred strategy that involved counting chips. Consider if Grace was able to shift this focus to asking an open-ended question for the purpose of supporting her student in their chosen strategy. We suspect that all teaching professionals often experience shifts in responsiveness. This supports prior research that suggests there are various pathways toward teaching responsively and that teaching responsively is not an all-or-nothing endeavor [7]. Our work demonstrates that these shifts can occur through visible decide actions and through a teacher's purpose for these actions. In this example, a teacher educator might simply ask Grace to consider a shift in her focus on the student seeing the answer in the chips toward an action and related purpose that creates a space for the student to follow their own pathway to a solution, rather than being told these mathematical connections. In this slight shift, a teacher might become responsive to students' thinking. Future research could focus on how, when, and why these shifts occur.

Participants demonstrating unresponsive professional noticing often stated purposes for error-free work and posed decide actions that involved telling, instructing, explaining, and asking funneling questions. These findings again confirm prior research that suggests mathematical errors are particularly difficult for prospective teachers to respond to in ways that are responsive [19,32], making it an interesting area for future research.

6. Conclusions

In this study, we examined eight prospective teachers' professional noticing of their students' thinking in pieces of written work. The participants' noticing of important mathematical elements, decide actions, and purposes were analyzed using the lens of responsive teaching. As teacher educators facilitate these learning spaces, our results inform practice in several ways. First, a prospective teacher needs to be able to notice their students' mathematical thinking as a necessary precursor to teaching responsibly. While our findings suggest that using self-selected pieces of student work from a teaching event of their own making is an effective scaffold to support the development of professional noticing.

A prospective teacher's ability to move from unresponsive to responsive professional noticing involves shifts in both teaching actions and purposes for those actions. We suggest future studies should focus on capturing and analyzing teachers' professional noticing skills to determine both visible teaching actions and the purposes behind those actions to better understand the development of novice to expert responsive teaching practices. To support prospective teachers in these shifts, their thinking must be made explicit so that teacher educators can understand prospective teachers thinking just as teacher educators ask prospective teachers to understand their own students thinking. The participants' responsive, developing responsive, and unresponsive professional noticing described in this study may support teacher educators in developing learning experiences that support the development of responsive teaching along this continuum in targeted and individualized ways. Overall, these practice-based findings inform our understandings about what participants need to be able to do and know to be an effective mathematics teacher who asks probing questions, designs rigorous mathematical tasks, and responds to student errors in ways that support the development of students mathematical thinking.

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