Binary closure operators

by

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Abdurahman Masoud Abdalla Date

ABSTRACT

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In this thesis we provide a new foundation to categorical closure operators, using more elementary binary closure operators on posets. The original goal of the thesis was to study a categorical closure operator in terms of the family of closure operators on the posets of subobjects. However, this does not allow to express hereditariness, which is an important property of a categorical closure operator. Representing instead a categorical closure operator in terms of the family of binary closure operators on the posets of subobjects, fixes this problem. Moreover, the structure of a binary closure operator on a poset is self-dual, unlike that of a unary closure operator or that of a categorical closure operator, and this duality has a useful application in the study of properties of closure operators on categories, where it groups properties of categorical closure operators in dual pairs, and allows to unify results which relate these properties to each other.

Opsomming

Binêre afsluitingsoperatore

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In hierdie tesis verskaf ons, deur gebruik te maak van meer elementêre binêre afsluitingsoperatore op parsiële geordende versamelings, 'n nuwe grondslag tot kategoriese afsluitingsoperatore. Die aanvanklike doel van die tesis was om 'n kategoriese afsluitingsoperator in terme van die familie van afsluitingsoperatore op parsiële die geordende versamelings van subobjekte te bestudeer. Dit laat egter nie toe om oorerflikheid, wat 'n belangrike eienskap van kategoriese operatore is, uit te druk nie. Hierdie probleem word opgelos deur 'n kategoriese operator in terme van die familie van binêre afsluitingsoperatore op parsiële die geordende versamelings van subobjekte te verteenwoordig. Bykomend is die struktuur van 'n binêre afsluitingsoperator op 'n parsiële geordende versameling self-duaal, in teenstelling met diè van 'n unêre of kategoriese afsluitingsoperator. Hierdie dualiteit het 'n nuttige toepassing in die studie van eienskappe van afsluitingsoperatore op kategorieë, waar dit eienskappe van kategoriese afsluitingsoperatore in duale pare groepeer en toelaat dat resultate, wat hierdie eienskappe in verband hou met mekaar, verenig word.

To my beloved mother

The pain of your passing before completion of my work, I will forever carry with me.

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Secondly, I would like to express a heartfelt and overwhelming appreciation for my wife Wafa Gwili for the many immense personal sacrifices she has had to make throughout our journey together. The extraordinary strength and grace with which she has carried our family through difficult and trying times has been an inspiration to observe and experience — and for which I will forever to her be indebted.

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Contents

Introduction

Categorical closure operators were introduced by D. Dikranjan and E. Giuli in [5]. A categorical closure operator is a structure on a category which makes the category resemble the category of topological spaces, where every embedding of topological spaces can be *closed* by considering the embedding of the topological closure of the image of the given embedding. It turns out that not just the category of topological spaces, but also many other categories, including those arising in algebra, have interesting closure operators (see e.g. [6]). Part of the theory of categorical closure operators is to identify principal properties of concrete categorical closure operators, and establish links between them at the level of general categories. In this thesis we show that the properties of categorical closure operators studied in [6] reduce to properties of less complex structures, and namely, that of what we call *binary closure operators* on posets. It then becomes possible to establish similar links between those properties of binary closure operators, and to deduce links between properties of categorical closure operators from these. Moreover, the context of a poset equipped with a binary closure operator, is self-dual, and duality can be used here to unify results on categorical closure operators. Thus, binary closure operators provide a simplified basis to the theory of categorical closure operators.

Adopting the more general definition of a categorical closure operator given in [15], it is not difficult to check that binary closure operators are in fact particular instances of categorical closure operators. So conversely, we could take known results on categorical closure operators and apply them to get some of our results on binary closure operators. However, since these results are developed in the literature for a more restricted notion of a categorical closure operator, this would mean first confirming that they carry over to the more general notion. We do not take this approach and rather take the opportunity to demonstrate how simple it is to work directly with binary closure operators. Furthermore, the language of binary closure operators naturally leads us to certain results which cannot be deduced from the existing results on categorical closure operators.

The aim of this thesis is to make a first step towards building the theory of binary closure operators. The thesis is divided in three chapters. In the first chapter we give preliminary material: definitions of basic mathematical structures encountered in the thesis (poset, lattice, monoid, etc.), basic notions from category theory (categories, functors, adjunctions, etc.), and an introduction to categorical closure operators. The second chapter is the core of the thesis. In it we develop basic theory of binary closure operators and towards the end apply it to categorical closure operators. We also remark that a special type of binary closure operators, and namely, *weakly hereditary idempotent binary closure operators*, can be seen as Eilenberg-Moore algebras for a suitable monad on the category of posets. This is the same monad as the one described in [8], but restricted to the subcategory of the category of categories consisting of posets. In view of the main result in [8], this shows that such binary closure operators are the same as factorization systems in the sense of P. Freyd and G. M. Kelly [7] on a poset regarded as a category (as conjectured by Thomas Weighill at the end of one of my talks on the subject, and as suggested, although only partially, by Theorem 2.4 in [6]). However, further than this, we do not develop the connection with the theory of factorization systems in this thesis. In the third chapter we analyze the structure of binary closure operators; among other things, we define and study composition and cocomposition of binary closure operators.

The main results of the thesis are as follows:

- The theorems in Section 2.2, which characterize and establish connections between various properties of binary closure operators. Many of these connections specialize to similar well known connections for categorical closure operators, as explained in Section 2.5. The characterizations, however, are entirely new. For example, such is Theorem 2.2.14, in which it is established that weakly hereditary idempotent binary closure operators are the same as associative binary closure operators.

- Theorem 2.4.3, which establishes that weakly hereditary idempotent binary closure operators are the same as algebras for a suitable monad over the category of posets.

- Theorems 2.5.6-2.5.14, which show how properties of binary closure operators specialize to familiar properties of categorical closure operators.

- Theorems 3.1.3, 3.1.4, 3.1.6 and 3.7.6, which characterize modularity and distributivity of lattices (bounded lattices in the first case, and complete lattices in the last case), via conditions on binary closures operators defined on them. In particular, Theorem 3.7.6 asserts that for a complete bounded lattice L, the operations of taking minimal core and hereditary hull of a closure operator commute with each other if and only if L is a modular lattice.

Chapter 1

Preliminaries

In this chapter, we introduce terminologies and preliminary results that we require in the next chapters. The definitions and concepts presented here are standard in texts of topology, lattice theory and category theory (see e.g. [1], [2], [3], [4], [?], [9], [10], [12], [13] and [14]). In Section 1.3 we recall basic material on categorical closure operators from [6].

1.1 Definitions of basic structures

Definition 1.1.1. *A* poset *is a pair $(O, \leq)$ consisting of a set (or more generally, a class) O and a binary relation $\leq$ on O such that the following conditions hold:*

(1) (Reflexivity) $a \leq a$, for all $a \in O$;

(2) (Antisymmetry) if $a \leq b$ and $b \leq a$, then $a = b$, for all $a, b \in O$;

(3) (Transitivity) if $a \leq b$ and $b \leq c$, then $a \leq c$, for all $a, b, c \in O$.

A pair $(O, \leq)$ is called a preordered set when $\leq$ is a reflexive and transitive relation.

If $(O, \leq)$ is a poset, the *dual* poset of $(O, \leq)$ is the poset $(O, \geq)$ with the order

$$a \geq b \quad \Leftrightarrow \quad b \leq a.$$

Definition 1.1.2. *A* bottom element *in a poset $(O, \leq)$ is an element of O, which we denote by $\perp$, having the property that $\perp \leq x$ for all $x \in O$. Dually, a* top element *is an element of O, denoted by $\top$, having the property that $x \leq \top$ for all $x \in O$.*

Definition 1.1.3. *Let* $(O_1, \leq_1)$ *and* $(O_2, \leq_2)$ *be posets and let* $f\colon O_1 \to O_2$ *be a function. Then* f *is said to be* order preserving *if for all* $a, b \in O_1$ *we have*

$$a \leq_1 b \quad \Rightarrow \quad f(a) \leq_2 f(b).$$

Definition 1.1.4. *We say that two posets* $(O_1, \leq_1)$ *and* $(O_2, \leq_2)$ *are* order-isomorphic*, if there is an onto function* $f\colon O_1 \to O_2$ *such that*

$$a \leq_1 b \quad \Leftrightarrow \quad f(a) \leq_2 f(b).$$

Definition 1.1.5. *Let* $(O, \leq)$ *be a poset and let* $T \subseteq O$*. An element* $x \in O$ *is said to be an* upper bound *of* T *if* $t \leq x$ *for all* $t \in T$*. An element* $s \in O$ *is said to be the* supremum *of* T *when the following two conditions hold:*

(1) s *is an upper bound of* T*.*

(2) $s \leq d$ *for all upper bounds* d *of* T*.*

Dually, $x \in O$ *is lower bound of* T *if* $x \leq t$ *for all* $t \in T$*. An element* $l \in O$ *is the infimum of* T *in* O *iff:*

(1) l *is a lower bound of* T *in* O*.*

(2) $d \leq l$ *for all lower bounds* d *of* T *in* O*.*

Definition 1.1.6. *A* lattice *is a poset* $(O, \leq)$ *in which any two-element set* $\{a, b\}$ *has a supremum and an infimum, which is written as* $a \vee b$ *and* $a \wedge b$*, called the* join *and the* meet *of* a *and* b*, respectively. We say that the lattice is a* complete lattice *when any subset of* O *has a supremum.*

Throughout the thesis, when we speak of a lattice L, we mean a lattice $(L, \leq)$.

Definition 1.1.7. *A lattice* L *is said to be*

(1) modular *if it satisfies the* modular law

$$x \leq z \quad \Rightarrow \quad x \vee (y \wedge z) = (x \vee y) \wedge z$$

for all $x, y, z \in L$*.*

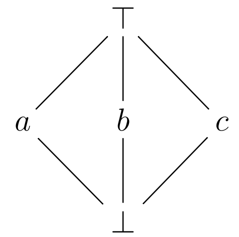
(2) distributive *if it satisfies the* distributive law

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

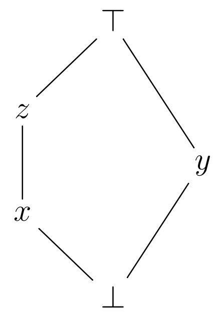
for all $x, y, z \in L$*.*

It is clear that distributivity implies modularity, but in general the converse is not true.

Example 1.1.8. *The following diagram displays an example of a modular lattice which is not distributive:*



The following diagram displays an example of a lattice which is not modular:



Proposition 1.1.9. *A lattice L is distributive if and only if*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all $x, y, z \in L$.

Definition 1.1.10. *Let L and K be lattices. A function $f\colon L \longrightarrow K$ is said to be a* lattice homomorphism *if f is* join-preserving *and* meet-preserving*, that is, for all $a, b \in L$,*

$$f(a \vee b) = f(a) \vee f(b) \quad \textit{and} \quad f(a \wedge b) = f(a) \wedge f(b).$$

Note that every lattice homomorphism $f\colon L \longrightarrow K$ is order-preserving, that is for all $a, b \in L$ we have $a \leq b \;\Rightarrow\; f(a) \leq f(b)$.

The distributive law can be described in terms of the *join maps* $j_a\colon L \to L$ and the *meet maps* $m_a\colon L \to L$, defined by

$$j_a(x) = a \vee x \quad \textit{and} \quad m_a(x) = a \wedge x,$$

as follows:

Lemma 1.1.11. *Let L be a lattice. Then*

(1) L is distributive if and only if the join map j_a is a lattice homomorphism for all $a \in L$.

(2) L is distributive if and only if the meet map is a lattice homomorphism for all $a \in L$.

Definition 1.1.12. *A* frame *is a complete lattice L with*

$$a \wedge \left(\bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \wedge b_i),$$

for any $a \in L$ and any family $(b_i)_{i \in I}$ of elements of L.

For frames F and G, a function $f: F \to G$ is said to be a frame homomorphism *if f preserves arbitrary joins and finite meet, that is*

$$f(a \wedge b) = f(a \wedge b)$$

for all $a, b \in F$, and

$$f \left(\bigvee_{i \in I} a_i \right) = \bigvee_{i \in I} f(a_i)$$

for each family $(b_i)_{i \in I}$ and $i \in I$.

Definition 1.1.13. *A* monoid *is a triple $(M, \Box, u)$, consisting of a set M, a binary operation*

$$\Box: M \times M \to M$$

which is associative, i.e

$$a \Box (b \Box c) = (a \Box b) \Box c$$

for all $a, b, c \in M$, and with an element $u \in M$ which is a unit for $\Box$, i.e

$$a \Box u = a = u \Box a$$

for all $a \in M$.

Definition 1.1.14. *A* topological space *is a pair (X, τ) consisting of a set X and a class τ of subsets of X satisfying the following conditions hold:*

(1) The empty set $\varnothing$ and the set X are elements of τ,

(2) The union of any family of sets in τ is a set in τ,

(3) The intersection of the collection of finitely many sets in τ is a set in τ.

Let (X, τ) be a topological space and $A \subseteq X$. A is said to be an *open set* if $A \in \tau$ and a *closed set* if $X \backslash A \in \tau$.

The *closure* of a set $A \subseteq X$, denoted by $\overline{A}$, is the intersection of all closed sets containing A, i.e

$$\overline{A} = \bigcap \{F \subseteq X |\ A \subseteq F,\ X \backslash F \in \tau\}.$$

The *interior* of a set $A \subseteq X$, denoted by A°, is the union of all open sets that are contained in A, i.e

$$A^\circ = \bigcup \{O \subseteq X |\ O \subseteq A,\ O \in \tau\}.$$

1.2 Basic concepts of category theory

A *category* $\mathbb{X}$ consists of the following:

- A class $|\mathbb{X}|$, whose elements will be called objects;
- For every pair X, Y of objects, a set $hom(X, Y)$, whose elements will be called morphisms (or arrows) from X to Y;
- For every triple X, Y, Z of objects, a composition law; i.e a map

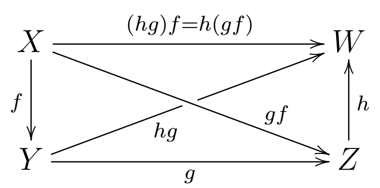
$$hom(X, Y) \times hom(Y, Z) \to hom(X, Z);$$

  the composite of pair (f, g) will be written $g \circ f$ or just gf
- For every object X a morphism $1_X \in hom(X, X)$, called the identity on X.

These data are subject to the following axioms:

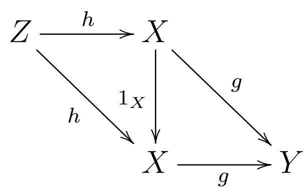
1. Associativity axiom: for any given X, Y, Z and W and morphisms $f \in hom(X, Y)$, $g \in hom(Y, Z)$ and $h \in hom(Z, W)$ the following equality holds:

$$(h \circ g) \circ f = h \circ (g \circ f).$$



2. Identity axiom: for any given X, Y and Z and morphisms $h \in hom(Z, X)$, $g \in hom(X, Y)$ the following equalities hold:

$$g \circ 1_X = g, \quad 1_X \circ h = h.$$



Example 1.2.1. *Any preordered set $(A, \leq)$ can be considered as a category. The elements of A are the objects and there is a unique morphism from an object a to an object b when $a \leq b$. Posets and order-preserving maps between them constitute a category, denoted by* **Ord** *and called the category of posets, where composition of order-preserving maps is defined in the usual way.*

Definition 1.2.2. *A category $\mathbb{C}$ is said to be a* small *category when its class of objects is a set.*

An arrow $m\colon X \to Y$ is *monic* in a category $\mathbb{X}$ when for any two arrows $f, g\colon Z \to X$,

$$m \circ f = m \circ g \quad \Longrightarrow \quad f = g.$$

An arrow $h\colon X \to Y$ is *epi* in $\mathbb{X}$ when for any two arrows $f, g\colon Y \to Z$,

$$f \circ h = g \circ h \quad \Longrightarrow \quad f = g.$$

For an arrow $h\colon A \to B$, a *section* of h is an arrow $s\colon B \to A$ with $hs = 1_B$. Similarly, for an arrow $h\colon A \to B$, a *retraction* for h is an arrow $r\colon B \to A$ with $rh = 1_A$.

Proposition 1.2.3. *If $h\colon A \to B$ has a section, then h is epi. If it has retraction, then h is monic.*

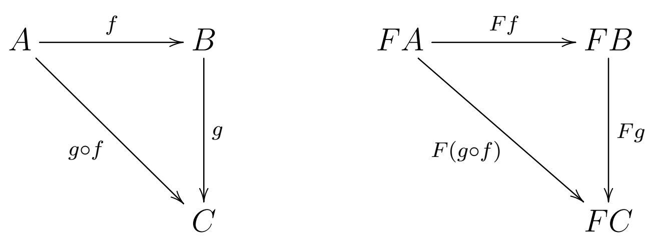
Proof. Let $h\colon A \to B$ have a section and suppose that $f, g\colon B \to C$ are two arrows such that $f \circ h = g \circ h$. Suppose that $s\colon B \to A$ is a section of h, then

$$f = f \circ 1_B = f \circ (h \circ s) = (f \circ h) \circ s = (g \circ h) \circ s = g \circ (h \circ s) = g \circ 1_B = g.$$

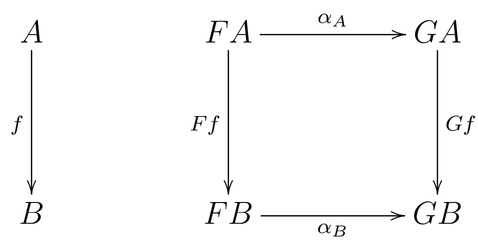
The second part of the proposition is the dual of the first part. $\square$

Consider the categories $\mathbb{X}, \mathbb{Y}$. A *functor* $F\colon \mathbb{X} \to \mathbb{Y}$ is a morphism of categories. It consists of a function F from the class of objects of $\mathbb{X}$ to the class of objects of $\mathbb{Y}$, and a function written again as F, from the class of morphisms of $\mathbb{X}$ to the class of morphisms of $\mathbb{Y}$ such that:

- ► For any morphism $f\colon A \to B$ in $\mathbb{X}$, the morphism Ff is in $hom(FA, FB)$.
- ► $F1_A = 1_{FA}$ for any object A of $\mathbb{X}$.
- ► $F(g \circ f) = Fg \circ Ff$ for any morphisms f and g for which the composite $g \circ f$ is defined in $\mathbb{X}$.



Let $F, G\colon \mathbb{X} \to \mathbb{Y}$ be two functors. A *natural transformation* $\alpha\colon F \to G$ is a function which assigns to each object X of $\mathbb{X}$ an arrow $\alpha_X\colon FX \to GX$ of $\mathbb{Y}$ such that for every arrow $f\colon A \to B$ in $\mathbb{X}$, the diagram



is commutative.

Proposition 1.2.4. *Let F, G and H be functors from a category $\mathbb{A}$ to a category $\mathbb{B}$ and let $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$ be natural transformations, then the formula*

$$(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$$

defines a new natural transformation $\beta \circ \alpha\colon F \Rightarrow H$.

Proof. We want to prove that for any morphism $f\colon A \to B$ in the category $\mathbb{A}$, the following diagram is commutative:

$$\begin{array}{ccc} FA & \xrightarrow{(\beta\circ\alpha)_A} & HA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Hf} \\ FB & \xrightarrow[(\beta\circ\alpha)_B]{} & HB \end{array}$$

This holds since

$$\begin{aligned} Hf \circ (\beta \circ \alpha)_A &= (Hf \circ \beta_A) \circ \alpha_A \\ &= (\beta_B \circ Gf) \circ \alpha_A \\ &= \beta_B \circ (Gf \circ \alpha_A) \\ &= \beta_B \circ (\alpha_B \circ Ff) \\ &= (\beta_B \circ \alpha_B) \circ Ff \\ &= (\beta \circ \alpha)_B \circ Ff. \end{aligned}$$

□

Proposition 1.2.5. *Consider the diagram*

$$\mathbb{A} \underset{G}{\overset{F}{\underset{\Downarrow\alpha}{\rightrightarrows}}} \mathbb{B} \underset{K}{\overset{H}{\underset{\Downarrow\beta}{\rightrightarrows}}} \mathbb{C}$$

where $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$ are categories, F,G, H and K are functors, and α and β are natural transformations. For every $A \in \mathbb{A}$, the formula

$$(\beta * \alpha)_A = \beta_{GA} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{FA}$$

(where the second equality follows from naturality of β) defines a natural transformation

$$\beta * \alpha \colon H \circ F \Rightarrow K \circ G.$$

Proof. For any object $A \in \mathbb{A}$, by naturality of α, β and functoriality of H and K we have the following commutative diagram:

$$\begin{array}{ccccc} FA & & HFA & \xrightarrow{\beta_{FA}} & KFA \\ {\scriptstyle \alpha_A}\downarrow & & {\scriptstyle H(\alpha_A)}\downarrow & & \downarrow{\scriptstyle K(\alpha_A)} \\ GA & & HGA & \xrightarrow[\beta_{GA}]{} & KGA \end{array}$$

Now, we want to show that for any morphism $f \colon A \to B$ in $\mathbb{A}$ one has a commutative diagram

$$\begin{array}{ccc} HFA & \xrightarrow{(\beta*\alpha)_A} & KGA \\ {\scriptstyle HFf}\big\downarrow & & \big\downarrow{\scriptstyle KGf} \\ HFB & \xrightarrow[(\beta*\alpha)_B]{} & KGB. \end{array}$$

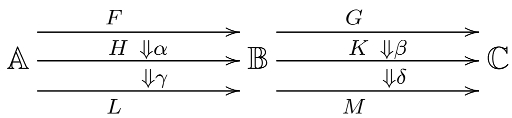
To see why this is so note that

$$\begin{aligned} KGf \circ (\beta * \alpha)_A &= KGf \circ (\beta_{GA} \circ H(\alpha_A)) \\ &= (KGf \circ \beta_{GA}) \circ H(\alpha_A) \\ &= (\beta_{GB} \circ HGf) \circ H(\alpha_A) \\ &= \beta_{GB} \circ H(Gf \circ \alpha_A) \\ &= (\beta_{GB} \circ H(\alpha_B)) \circ HFf \\ &= (\beta * \alpha)_B \circ HFf. \end{aligned}$$

□

It is not difficult to show that $*$ is associative. Furthermore, we have:

Proposition 1.2.6. *Consider the following situation:*



where $\mathbb{A}, \mathbb{B}$ *and* $\mathbb{C}$ *are categories,* F, G, H, K, L *and* M *are functors, and* α, β, γ *and* δ *are natural transformations. The following equality holds:*

$$(\delta * \gamma) \circ (\beta * \alpha) = (\delta \circ \beta) * (\gamma \circ \alpha).$$

Proof. For any object $A \in \mathbb{A}$, by naturality of α, β, γ and δ and functoriality of F, G, H, K, L

and M we have the following commutative diagram:

$$
\begin{array}{ccccc}
GFA & \xrightarrow{\beta_{FA}} & KFA & \xrightarrow{\delta_{FA}} & MFA \\
{\scriptstyle G(\alpha_A)}\downarrow & & \downarrow{\scriptstyle K(\alpha_A)} & & \downarrow{\scriptstyle M(\alpha_A)} \\
GHA & \xrightarrow{\beta_{HA}} & KHA & \xrightarrow{\delta_{HA}} & MHA \\
{\scriptstyle G(\gamma_A)}\downarrow & & \downarrow{\scriptstyle K(\gamma_A)} & & \downarrow{\scriptstyle M(\gamma_A)} \\
GLA & \xrightarrow[\beta_{LA}]{} & KLA & \xrightarrow[\delta_{LA}]{} & MLA.
\end{array}
$$

Fristly, we have

$$
\begin{aligned}
M(\gamma_A) \circ M(\alpha_A) \circ \delta_{FA} \circ \beta_{FA} &= M(\gamma_A) \circ \delta_{HA} \circ K(\alpha_A) \circ \beta_{FA} \\
&= (M(\gamma_A) \circ \delta_{HA}) \circ (K(\alpha_A) \circ \beta_{FA}) \\
&= (\delta * \gamma)_A \circ (\beta * \alpha)_A \\
&= ((\delta * \gamma) \circ (\beta * \alpha))_A.
\end{aligned}
$$

Secondly, we have

$$
\begin{aligned}
\delta_{LA} \circ \beta_{LA} \circ G(\gamma_A) \circ G(\alpha_A) &= (\delta \circ \beta)_{LA} \circ G(\gamma \circ \alpha)_A \\
&= ((\delta \circ \beta) * (\gamma \circ \alpha))_A.
\end{aligned}
$$

□

Definition 1.2.7. *An* adjunction *consists of*

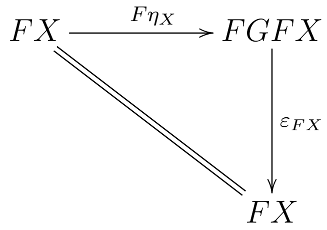
(1) two functors

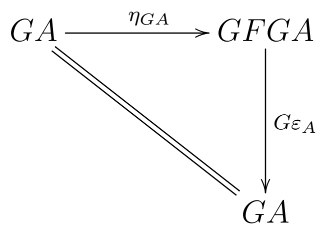
$$
\mathbb{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{A}
$$

(2) and two natural transformations

$$
\varepsilon\colon FG \Longrightarrow I_{\mathbb{A}}, \quad \eta\colon I_{\mathbb{X}} \Longrightarrow GF
$$

such that, for all $X \in \mathbb{X}$ and $A \in \mathbb{A}$ the diagrams





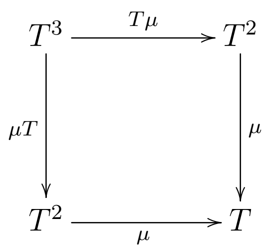
commute.

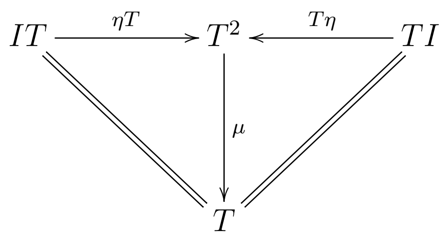
The natural transformations η and ε are called the *unit* and the *counit*, respectively, of the adjunction.

Definition 1.2.8. *A* monad $T = \langle T, \eta, \mu \rangle$ *on a category* $\mathbb{X}$ *consists of a functor* $T \colon \mathbb{X} \to \mathbb{X}$ *and two natural transformations*

$$\eta \colon I_{\mathbb{X}} \Rightarrow T, \quad \mu \colon T^2 \Rightarrow T$$

such that diagrams



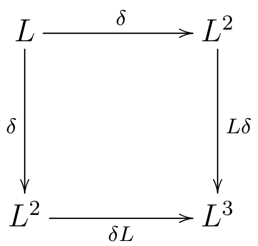


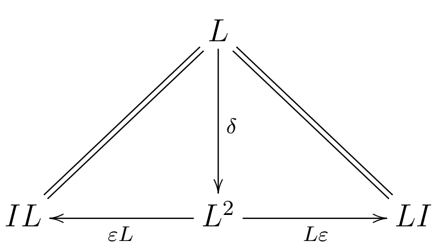
commute.

Dually, a comonad on a category $\mathbb{A}$ consists of a functor $L \colon \mathbb{A} \to \mathbb{A}$ and natural transformations

$$\varepsilon \colon L \to I_{\mathbb{A}}, \quad \delta \colon L \to L^2$$

such that diagrams





commute.

Proposition 1.2.9. *Let* $\mathbb{X} \underset{L}{\overset{R}{\rightleftarrows}} \mathbb{E}$ *be adjoint functors, with L left adjoint to R. Let us write* $\alpha\colon I_{\mathbb{E}} \Rightarrow RL$ *and* $\beta\colon LR \Rightarrow I_{\mathbb{X}}$ *for the unit and counit of this adjunction. If*

$$T = RL\colon \mathbb{E} \to \mathbb{E},\ \eta = \alpha\colon I_{\mathbb{E}} \Rightarrow T,\ \mu = I_R * \beta * I_L\colon T^2 \Rightarrow T$$

then $T = \langle T, \eta, \mu\rangle$ *is a monoad on* $\mathbb{E}$.

Proof. The naturality of β implies commutativity of the following diagram for every object $X \in \mathbb{X}$

$$\begin{array}{ccc} LRLRX & \xrightarrow{\beta_{LRX}} & LRX \\ {\scriptstyle LR\beta_X}\downarrow & & \downarrow{\scriptstyle \beta_X} \\ LRX & \xrightarrow[\beta_X]{} & X. \end{array}$$

Now, using the triangular identities for an adjunction, the following equalities hold:

$$\begin{aligned} \mu \circ \eta T = \mu \circ (\eta * I_T) &= (I_R * \beta * I_L) \circ (\alpha * I_R * I_L) \\ &= ((I_R * \beta) \circ (\alpha * I_R)) * I_L \\ &= I_R * I_L = I_T. \end{aligned}$$

$$\begin{aligned} \mu \circ \mu T &= \mu \circ (\mu * I_T) \\ &= (I_R * \beta * I_L) \circ (I_R * \beta * I_L * I_R * I_L) \\ &= I_R * [\beta \circ (\beta * I_L * I_R)] * I_L \\ &= I_R * [\beta \circ (I_L * I_R * \beta)] * I_L \\ &= (I_R * \beta * I_L) \circ (I_R * I_L * I_R * \beta * I_L) \\ &= \mu \circ (I_T * \mu) \\ &= \mu \circ T\mu. \end{aligned}$$

$$\begin{aligned} \mu \circ T\eta &= \mu \circ (I_T * \eta) \\ &= (I_R * \beta * I_L) \circ (I_R * I_L * \alpha) \\ &= I_R * [(\beta * I_L) \circ (I_L * \alpha)] \\ &= I_R * I_L = I_T. \end{aligned}$$

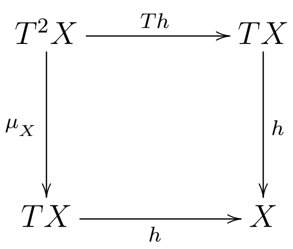
□

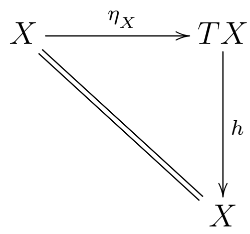
Definition 1.2.10. *Let* $\langle T, \eta, \mu\rangle$ *be a monad on a category* $\mathbb{X}$. *A* T-algebra *is a pair* $\langle X, h\rangle$

where

$$X \in \mathbb{X}, \quad h \colon TX \longrightarrow X \textit{in } \mathbb{X}$$

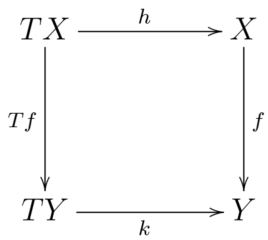
such that the diagrams





commute.

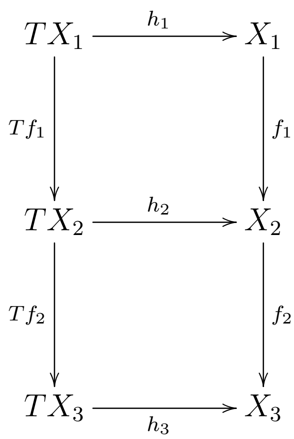
A morphism $f \colon \langle X, h \rangle \longrightarrow \langle Y, k \rangle$ of T-algebras is an arrow $f \colon X \longrightarrow Y$ of $\mathbb{X}$ which makes the diagram



commute.

Proposition 1.2.11. *Let $\langle T, \eta, \mu \rangle$ be a monad on a category $\mathbb{X}$. The T-algebras and their morphisms constitute a category, written as $\mathbb{X}^T$ and called the "category of Eilenberg-Moore algebras" of the monad.*

Proof. Suppose that $f_1 \colon \langle X_1, h_1 \rangle \to \langle X_2, h_2 \rangle$ and $f_2 \colon \langle X_2, h_2 \rangle \to \langle X_3, h_3 \rangle$ are morphisms of T-algebras, so we have the following commutative diagram:



Then:

$$
\begin{aligned}
(f_2 \circ f_1) \circ h_1 &= f_2 \circ (f_1 \circ h_1) \\
&= f_2 \circ (h_2 \circ Tf_1) \\
&= (f_2 \circ h_2) \circ Tf_1 \\
&= (h_3 \circ Tf_2) \circ Tf_1 \\
&= h_3 \circ T(f_2 \circ f_1).
\end{aligned}
$$

This shows that the composite $f_2 \circ f_1$ is a morphism of T-algebras. With this composition of arrows, the T-algebras evidently form a category $\mathbb{X}^T$. □

1.3 Subobjects, Images and Inverse Images

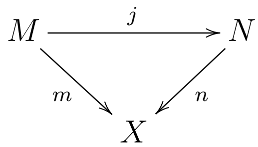
Consider a category $\mathbb{X}$ and a fixed class $\mathcal{M}$ of monomorphisms in $\mathbb{X}$. We assume that

- $\mathcal{M}$ is closed under composition, and that
- $\mathcal{M}$ contains all identity morphisms.

For every object X of $\mathbb{X}$, let $\mathcal{M}/X$ be the class of all $\mathcal{M}$-morphisms with codomain X. The relation given by

$$m \leq n \Leftrightarrow \exists j(n \circ j = m)$$

makes $\mathcal{M}/X$ into a preordered class. The diagram



illustrates the condition $\exists j(n \circ j = m)$. As explained on pages 1 and 2 in [6]; since n is monic, the morphism j is uniquely determined, and it is an isomorphism in $\mathbb{X}$ if and only if $n \leq m$ holds; in this case m and n are called *isomorphic*, and one writes $m \cong n$. It is easy to see that, " $\cong$ " is an equivalence relation and $\mathcal{M}/X$ modulo " $\cong$ " is a poset for which we can use all lattice-theoretic terminology and notations, such as $\wedge, \vee, \bigvee, \bigwedge$, etc. In fact, we shall use these notations for elements of $\mathcal{M}/X$ rather than for their $\cong$-equivalence classes both of which we refer to as $\mathcal{M}$-subobjects of X; the prefix $\mathcal{M}$ is often omitted. This means that, for $m, n \in \mathcal{M}/X$, $m \wedge n$ denotes a representative in $\mathcal{M}/X$ of the meet of the corresponding $\cong$-equivalence classes (whenever the meet exists). In other words, with $[m]$ denoting the $\cong$-equivalence class of m , we have the equivalences

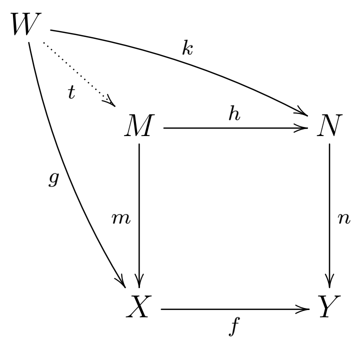
$$m \leq n \;\Leftrightarrow\; [m] \leq [n]$$

$$m \cong n \;\Leftrightarrow\; [m] = [n]$$

$$k \cong m \wedge n \;\Leftrightarrow\; [k] = [m] \wedge [n],$$

and analogously for $\vee, \bigvee, \bigwedge$. We will exclusively use the notation given by the left-hand sides of these equivalences. Furthermore, we will often not distinguish between the preordered class $\mathcal{M}/X$ and the corresponding poset of $\cong$-equivalence classes, where the order is defined by $[m] \leq [n] \;\Leftrightarrow\; m \leq n$.

Definition 1.3.1. *For a category $\mathbb{X}$ and a class $\mathcal{M}$ as above, one says that $\mathbb{X}$ has* $\mathcal{M}$-pullbacks, *if for every morphism $f\colon X \to Y$ and every $n \in \mathcal{M}/Y$ a pullback diagram*



exists in $\mathbb{X}$ with $m \in \mathcal{M}/X$. This means that $n \circ h = f \circ m$, and whenever $f \circ g = n \circ k$ holds in $\mathbb{X}$, then there is a unique morphism t with $m \circ t = g$ and $h \circ t = k$.

The morphism m is uniquely determined up to isomorphism; it is called the inverse image of n under f and denoted by

$$f^{-1}(n) \colon f^{-1}(N) \longrightarrow X.$$

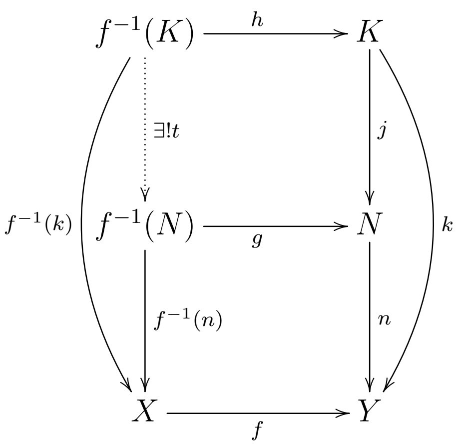
The pullback property of the previous definition yields

Proposition 1.3.2. *If $\mathbb{X}$ has $\mathcal{M}$-pullbacks, then for each $f \colon X \longrightarrow Y$ the map*

$$f^{-1}(-) \colon \mathcal{M}/Y \longrightarrow \mathcal{M}/X$$

is an order preserving map.

Proof. Let $k \colon K \longrightarrow Y$, $n \colon N \longrightarrow Y$ be two morphisms in $\mathcal{M}/Y$ with $k \leq n$. This means that there exists a morphism j such that $k = n \circ j$. From the bottom square of the diagram



we have

$$n \circ g = f \circ f^{-1}(n).$$

Also we have $n \circ (j \circ h) = f \circ f^{-1}(k)$, so from pullback property there is a unique morphism

$$t \colon f^{-1}(K) \longrightarrow f^{-1}(N)$$

such that $f^{-1}(k) = f^{-1}(n) \circ t$, which means that $f^{-1}(k) \leq f^{-1}(n)$. □

The following notion is a special case of the notion of an adjunction between categories:

Definition 1.3.3. *A pair of mappings* $\varphi\colon P \to Q$, $\psi\colon Q \to P$ *between preordered classes* P, Q *is said to be* adjoint *if*

$$m \le \psi(n) \quad \Leftrightarrow \quad \varphi(m) \le n \qquad (*)$$

holds for all $m \in P$ *and* $n \in Q$. *One says that* φ *is* left adjoint *to* ψ *or* ψ *is* right adjoint *to* φ *and writes* $\varphi \dashv \psi$.

Proposition 1.3.4. *For any pair of mappings* $\varphi\colon P \to Q$, $\psi\colon Q \to P$ *of preordered classes, the following are equivalent:*

(1) $\varphi \dashv \psi$ *;*

(2) ψ *is order-preserving, and* $\varphi(m) \cong \min\{n \in Q \mid m \le \psi(n)\}$ *holds for all* $m \in P$*;*

(3) φ *is order-preserving, and* $\psi(n) \cong \max\{m \in P \mid \varphi(m) \le n\}$ *holds for all* $n \in Q$*;*

(4) φ *and* ψ *are order-preserving, and*

$$m \le \psi(\varphi(m)) \quad \text{and} \quad \varphi(\psi(n)) \le n$$

hold for all $m \in P$ *and* $n \in Q$.

Proof. (1) $\Rightarrow$ (2). Put $n = \varphi(m)$ in $(*)$, we obtain $m \le \psi(\varphi(m))$. Now let

$$Q_m = \{n \in Q \mid m \le \psi(n)\},$$

then $\varphi(m) \in Q_m$. Moreover, for all $n \in Q_m$, $(*)$ gives $\varphi(m) \le n$, therefore

$$\varphi(m) \cong \min Q_m.$$

This formula implies that φ is order-preserving. Since for any $m \le n$ in P, we have that $m \le \psi(\varphi(n))$. Hence, by $(*)$ we obtain $\phi(m) \le \phi(n)$.

(1) $\Rightarrow$ (3) is dual to (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (4). φ is monotone as mentioned before. Since $\varphi(m) \in Q_m$, we have

$$m \le \psi(\varphi(m)) \le n \text{ for all } m \in P.$$

Similarly, we have $\varphi(\psi(n)) \le n$ for all $n \in Q$.

(3) $\Rightarrow$ (4) follows dually.

(4) $\Rightarrow$ (1).

$$m \le \psi(n) \quad \Rightarrow \quad \varphi(m) \le \varphi(\psi(n)) \le n,$$

$$\varphi(m) \leq n \quad \Rightarrow \quad m \leq \psi(\varphi(m)) \leq \psi(n).$$

□

Proposition 1.3.5. *For any pair of mappings $\varphi\colon P \to Q$ and $\psi\colon Q \to P$ of preordered classes, if $\varphi \dashv \psi$, then*

$$\varphi\left(\bigvee_{i\in I} m_i\right) \cong \bigvee_{i\in I} \varphi(m_i) \quad \textit{and} \quad \psi\left(\bigwedge_{i\in I} n_i\right) \cong \bigwedge_{i\in I} \psi(n_i).$$

Proof. Let $(m_i)_{i\in I}$ be a family of elements of P. Suppose that $m \cong \bigvee_{i\in I} m_i$. Since φ is monotone, we have $\varphi(m)$ is an upper bound of $\{\varphi(m_i) : i \in I\}$. Now, for any other upper bound n, we have $m_i \leq \psi(n)$ for all $i \in I$, therefore $m \leq \psi(n)$, and so $\varphi(m) \leq n$. This shows that φ preserves joins. That ψ preserves all existing meets follows dually. □

Let $\mathbb{X}$ have $\mathcal{M}$-pullbacks and for every $f\colon X \to Y$ in $\mathbb{X}$, let $f^{-1}(-)\colon \mathcal{M}/Y \to \mathcal{M}/X$ have a left adjoint $f(-)\colon \mathcal{M}/X \to \mathcal{M}/Y$. For $m\colon M \to X$ in $\mathcal{M}/X$, the morphism $f(m)\colon f(M) \to Y$ in $\mathcal{M}/Y$ is called the *image of m under f*; it is uniquely determined, up to isomorphism, by the following property:

$$m \leq f^{-1}(n) \Leftrightarrow f(m) \leq n$$

for all $n \in \mathcal{M}/Y$. Furthermore, according to Proposition 1.3.4, we have the following formulas:

(1) $m \leq k \ \Rightarrow \ f(m) \leq f(k)$;

(2) $m \leq f^{-1}(f(m))$ and $f(f^{-1}(n)) \leq n$;

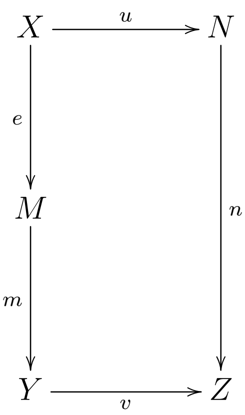
(3) $f\left(\bigvee_{i\in I} m_i\right) \cong \bigvee_{i\in I} f(m_i)$;

(4) $f^{-1}\left(\bigwedge_{i\in I} n_i\right) \cong \bigwedge_{i\in I} f^{-1}(n_i)$.

Proposition 1.3.6. *Let $\mathbb{X}$ have $\mathcal{M}$-pullbacks, and for $f\colon X \to Y$ in $\mathbb{X}$, let $f^{-1}(-)$ have a left adjoint $f(-)$. Then there are morphisms e, m in $\mathbb{X}$ such that*

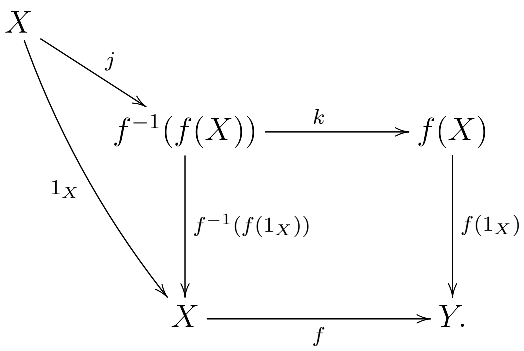
(1) $f = m \circ e$ with $m\colon M \to Y$ in $\mathcal{M}$, and

(2) (diagonalization property) *whenever one has a commutative diagram*

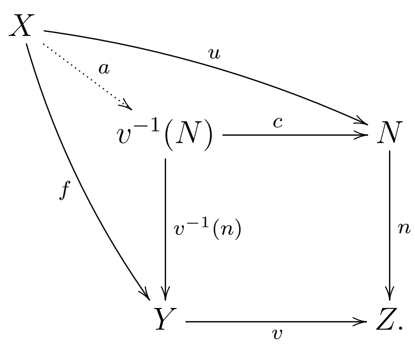


in $\mathbb{X}$ *with* $n \in \mathcal{M}$*, then there is a uniquely determined morphism* $w\colon M \to N$ *with* $n \circ w = v \circ m$ *and* $w \circ e = u$*.*

Proof. Let $f\colon X \to Y$ be a morphism in $\mathbb{X}$. Since $\mathbb{X}$ has pullbacks and $f^{-1}(-)$ has a left adjoint $f(-)$, we obtain the following commutative diagram:



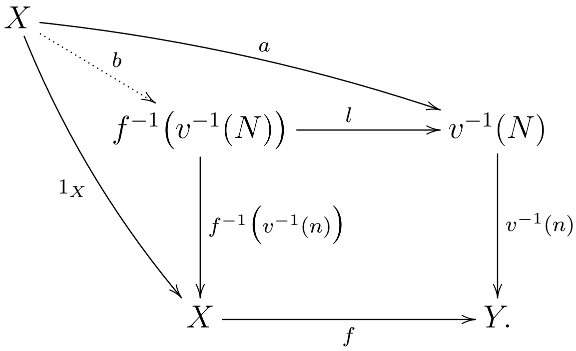
Let $e = k \circ j$ and $m = f(1_X)$, so we obtain (1). Consider a commutative diagram as in (2) with $n \in \mathcal{M}$. For morphisms $v\colon Y \to Z$ and $n\colon N \to Z$, we have the following commutative diagram:



Hence, by the pullback property, we obtain the morphism

$$a \colon X \to v^{-1}(N) \text{ with } f = v^{-1}(n) \circ a.$$

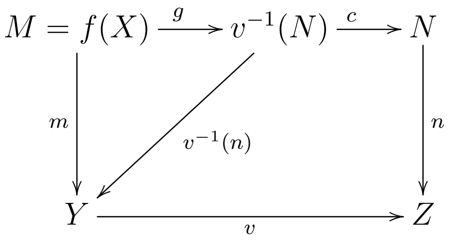
Again, for morphisms $f \colon X \to Y$ and $v^{-1}(n) \colon v^{-1}(N) \to Y$ we have the following commutative diagram:



Hence, by the pullback property, we have the morphism $b \colon X \to f^{-1}\big(v^{-1}(N)\big)$ with

$$1_X = f^{-1}\big(v^{-1}(n)\big) \circ b.$$

Accordingly, $m = f(1_X) \leq v^{-1}(n)$, by adjointness. Now we have the following commutative diagram

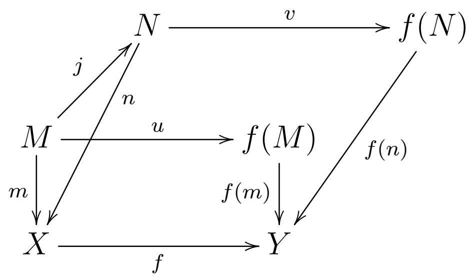


Let $w = c \circ g$. Therefore, $n \circ w = v \circ m$. Since the morphism n is monic, w is uniquely determined by $w = c \circ g$, and $w \circ e = u$ follows from $n \circ w \circ e = v \circ m \circ e = n \circ u$. ☐

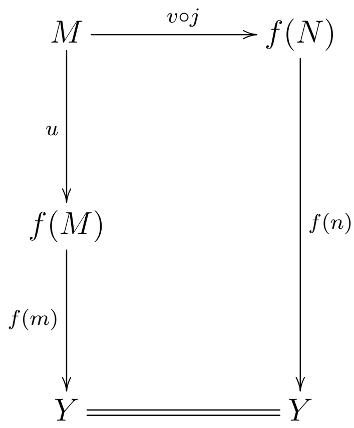
Any factorization $f = m \circ e$ of f such that the diagonilization property of the previous proposition holds is called *the right $\mathcal{M}$-factorization* of f.

Proposition 1.3.7. *Let every morphism in $\mathbb{X}$ have a right $\mathcal{M}$-factorization. For a morphism $f \colon X \to Y$ in $\mathbb{X}$ and $m \colon M \to X$ in $\mathcal{M}$, one defines $f(m) \colon f(M) \to Y$ to be any chosen $\mathcal{M}$-part of the composite $f \circ m$. Then the map $f(-) \colon \mathcal{M}/X \longrightarrow \mathcal{M}/Y$ is order-preserving*

Proof. Consider morphisms $m \leq n$ in $\mathcal{M}/X$ and a morphism $f \colon X \to Y$. From the following commutative diagram



we obtain the following diagram



which is commute. Since $f(m)$ and $f(n) \in \mathcal{M}$, by the diagonalization property, we obtain a morphism $w\colon f(M) \to f(N)$, which means that $f(m) \leq f(n)$. □

Theorem 1.3.8. *The following assertions are equivalent:*

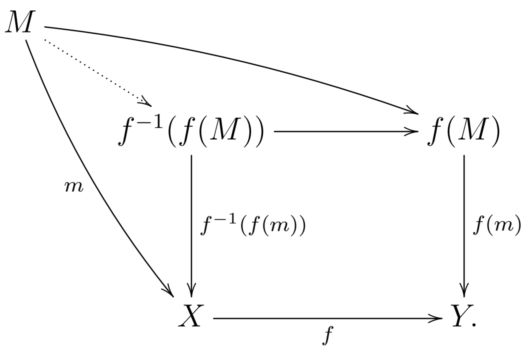
(1) $\mathbb{X}$ *has* $\mathcal{M}$*-pullbacks, and every morphism has a right* $\mathcal{M}$*-factorization;*

(2) $\mathbb{X}$ *has* $\mathcal{M}$*-pullbacks, and* $f^{-1}(-)$ *has a left-adjoint for every morphism* f*;*

(3) every morphism has a right $\mathcal{M}$*-factorization, and* $f(-)$ *has a right-adjoint for every morphism* f*.*

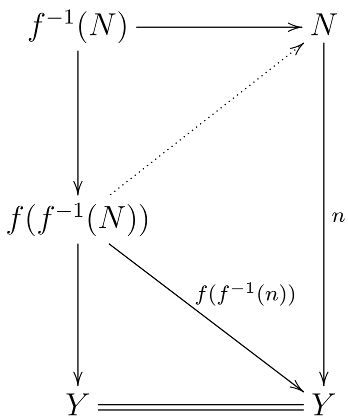
Proof. ((1) ⇒ (2)) & ((1) ⇒ (3)) Let $m \in \mathcal{M}/X$ and $f\colon X \to Y$ be a morphism in $\mathbb{X}$.

Then we have the following commutative diagram



From the pullback property we obtain $m \leq f^{-1}(f(m))$. Now let $n \in \mathcal{M}/Y$. By $\mathcal{M}$-pullbacks we obtain a morphism $f^{-1}(n)\colon f^{-1}(N) \to X$ and a right $\mathcal{M}$-factorization morphism to the morphism $f \circ f^{-1}(n)$ is $f(f^{-1}(n))$.

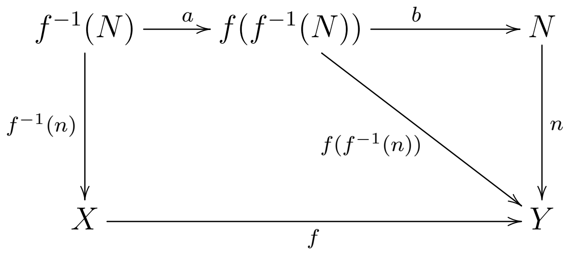
From the following diagram



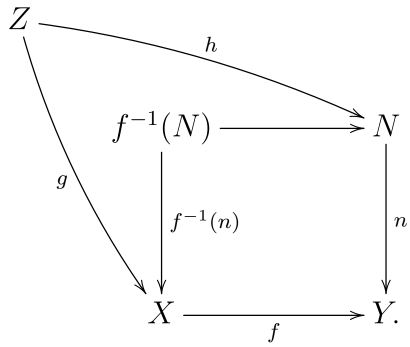
we obtain, by the diagonalization property, $f(f^{-1}(n)) \leq n$. Since both $f(-)$ and $f^{-1}(-)$ are order-preserving (see Propositions 1.3.2 and 1.3.7 above), Proposition 1.3.4 gives adjointness.

(2) $\Rightarrow$ (1) By Proposition 1.3.6.

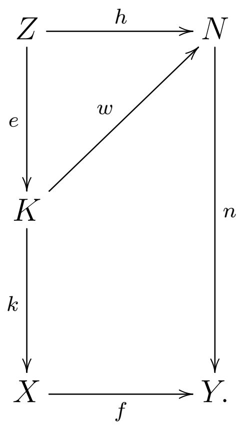
(3) $\Rightarrow$ (1) Suppose that any morphism has a right $\mathcal{M}$-factorization and denote the right adjoint of $f(-)$ by $f^{-1}(-)$. For any $n \in \mathcal{M}/Y$, the following diagram



commutes. Suppose we have the following commutative diagram



Using the right $\mathcal{M}$-factorization of $g = k \circ e$ and the diagonalization property we obtain the following commutative diagram



Again by the diagonalization property we obtain $f(k) \leq n$, and by adjointness we have $k \leq f^{-1}(n)$. Therefore, there is a morphism

$$j\colon K \to f^{-1}(N) \text{ with } f^{-1}(n) \circ j = k.$$

Let $t = j \circ e\colon Z \to f^{-1}(N)$, then we have

$$f^{-1}(n) \circ t = g.$$

Since both n and $f^{-1}(n)$ are monic, t is uniquely determined and $b \circ a \circ t = h$. □

One calls $\mathbb{X}$ *finitely* $\mathcal{M}$*-complete* if one and hence all of the assertions of Theorem 1.3.8 hold.

Chapter 2

The basic theory of binary closure operators

2.1 Definitions

Given a poset $(O, \leqslant)$, we denote by α the poset

$$\alpha = \{(a, b) \in O \times O \mid a \leqslant b\},$$

with the order given by

$$(a, b) \leqslant (c, d) \quad \Leftrightarrow \quad a \leqslant c \quad \& \quad b \leqslant d.$$

Definition 2.1.1. *A* binary closure operator *on a poset is an order-preserving map* $\alpha \xrightarrow{\bullet} O$ *such that*

$$a \leqslant \bullet(a, b) \leqslant b$$

for all $(a, b) \in \alpha$.

We will write $a \bullet b$ for $\bullet(a, b)$, and we call it the "closure of a in b".

Definition 2.1.2. *Two subrelations of an order relation, given by a binary closure operator, are defined as follows:*

a is closed in $b \quad \Leftrightarrow \quad (a, b) \in \alpha$ and $a \bullet b = a$.

a *is* dense in $b \quad \Leftrightarrow \quad (a, b) \in \alpha$ and $a \bullet b = b$.

In a diagram, we write $a \to b$ to mean $a \leqslant b$. For $(a, b) \leqslant (c, d)$ in α, we have the following diagram:

$$\begin{array}{ccccc} a & \longrightarrow & a \bullet b & \longrightarrow & b \\ \downarrow & & \downarrow & & \downarrow \\ c & \longrightarrow & c \bullet d & \longrightarrow & d \end{array}$$

Note thus that:

(1) If a is dense in b, then $b \leqslant c \bullet d$.

(2) If c is closed in d, then $a \bullet b \leqslant c$.

(3) If a is dense in b and c is closed in d, then $b \leqslant c$.

A binary closure operator on a poset

- is said to be *idempotent* (ID) if $a \bullet b$ is closed in b, i.e $(a \bullet b) \bullet b = a \bullet b$, for all $a \leq b$.
- is said to be *weakly hereditary* (WH) if a is dense in $a \bullet b$, i.e $a \bullet (a \bullet b) = a \bullet b$, for all $a \leq b$.
- satisfies (CT) if the "is closed" relation is transitive, i.e if $a \bullet b = a$ and $b \bullet c = b$, then $a \bullet c = a$ for all $a \leq b \leq c$.
- satisfies (DT) if the "is dense" relation is transitive, i.e if $a \bullet b = b$ and $b \bullet c = c$, then $a \bullet c = c$ for all $a \leq b \leq c$.
- is said to be *hereditary* (HE) when, if $a \leq b \leq c$, then $a \bullet b$ is the meet of $a \bullet c$ and b, i.e $(a \bullet c) \wedge b$ exists and is equal to $a \bullet b$.
- is said to be *minimal* (MI) when, if $a \leq b \leq c$, then $b \bullet c$ is the join of $a \bullet c$ and b, i.e $(a \bullet c) \vee b$ exists and is equal to $b \bullet c$.
- satisfies *the left-cancellation property of dense pairs* (LD) when for all $a \leq b \leq c$, if $a \bullet c = c$, then $a \bullet b = b$.
- satisfies *the right cancellation property of closed pairs* (RC) when for all $a \leq b \leq c$, if $a \bullet c = a$, then $b \bullet c = b$.

In the following conditions we assume the existence of joins or meets as required.

- ▶ is said to be *additive* (AD) if $(a \vee b) \bullet c = (a \bullet c) \vee (b \bullet c)$ for all $a \leq c$ and $b \leq c$.
- ▶ is said to be *fully additive* (FA) if for any family $(a_i)_{i \in I}$ we have

$$\left(\bigvee_{i \in I} a_i \right) \bullet b = \bigvee_{i \in I} (a_i \bullet b)$$

  when $a_i \leq b$ for all $i \in I$.

- ▶ is said to be *multiplicative* (MU) if $a \bullet (b \wedge c) = (a \bullet b) \wedge (a \bullet c)$ for all a, b, c such that $a \leq b$ and $a \leq c$.
- ▶ is said to be *fully multiplicative* (FM) if for any family $(b_i)_{i \in I}$ we have

$$a \bullet \left(\bigwedge_{i \in I} b_i \right) = \bigwedge_{i \in I} (a \bullet b_i)$$

  for all a such that $a \leq b_i, i \in I$.

- ▶ is said to be grounded (GR) if there is a bottom element 0 and $0 \bullet a = 0$ for all a.

Given a binary closure operator on a poset, the equality

$$b \circ a = a \bullet b$$

defines a binary closure operator on the dual poset, which we call the dual of the original binary closure operator. Note that the dual of the dual is the original binary closure operator. Each property in the left column of the table below is equivalent to the property in the right column of the same row, of the dual closure operator, and vice versa.

Property	Dual
(ID)	(WH)
(DT)	(CT)
(HE)	(MI)
(RC)	(LD)
(AD)	(MU)
(FA)	(FM)

2.2 Theorems

Throughout this section we will assume that $\bullet$ is a binary closure operator on a fixed poset O.

Theorem 2.2.1. *(WH) ⇒ (CT).*

Proof. For $a \leq b \leq c$ in the poset O, suppose that $a \bullet b = a$ and $b \bullet c = b$. Taking the closure of a and b in c, we have $a \bullet c \leq b \bullet c$, and hence by assumption $a \bullet c \leq b$. Hence we get $a \bullet (a \bullet c) \leq a \bullet b$. Now by using (WH) on the left side and using one of our assumptions on the right side, we obtain $a \bullet c \leq a$, but $a \leq a \bullet c$ is always true. Therefore $a \bullet c = a$. □

The following example shows that $(CT) \nRightarrow (WH)$.

Example 2.2.2. *(CT) but not (WH). Let $O = \{1, 2, 3, 4\}$ with the usual order. Define a binary closure operator on O by the following table:*

$\bullet$	*1*	*2*	*3*	*4*
1	*1*	*1*	*1*	*1*
2	–	*2*	*2*	*3*
3	–	–	*3*	*4*
4	–	–	–	*4*

The binary closure operator satisfies (CT) for any subset of three elements of O. However, this binary closure operator is not (WH).

$$2 \bullet (2 \bullet 4) = 2 \bullet 3 = 2 \neq 3 = 2 \bullet 4$$

Theorem 2.2.3. *(ID & CT) ⇒ (WH).*

Proof. For $a \leq b \in O$, we know that $a \bullet (a \bullet b) \leq a \bullet b \leq b$, so by (ID) we obtain

$$\Big(a \bullet (a \bullet b)\Big) \bullet (a \bullet b) = a \bullet (a \bullet b) \quad \text{and} \quad (a \bullet b) \bullet b = a \bullet b.$$

It follows that

$$\Big(a \bullet (a \bullet b)\Big) \bullet b = a \bullet (a \bullet b)$$

by using (CT). Now from $a \leq a \bullet (a \bullet b) \leq b$, we obtain

$$a \bullet b \leq \Big(a \bullet (a \bullet b)\Big) \bullet b = a \bullet (a \bullet b).$$

But since $a \bullet (a \bullet b) \leq a \bullet b$ is always true, we have that $a \bullet (a \bullet b) = a \bullet b$. □

Dualizing the previous theorems, we obtain

Theorem 2.2.4. *(WH & DT) ⇒ (ID) ⇒ (DT).*

Example 2.2.5. *$(CT$ & $DT)$ but neither (ID) nor (WH). Let $O = \{1, 2, 3, 4\}$. Define a binary closure operator on O by the following table:*

•	1	2	3	4
1	1	1	2	4
2	–	2	3	4
3	–	–	3	4
4	–	–	–	4

This binary closure operator satisfies both (DT) and (CT), but is neither (ID) nor (WH), because

$$(1 \bullet 3) \bullet 3 = 2 \bullet 3 = 3 \neq 2 = 1 \bullet 3,$$

$$1 \bullet (1 \bullet 3) = 1 \bullet 2 = 1 \neq 2 = 1 \bullet 3.$$

Corollary 2.2.6. $(ID$ & $CT) \quad \Leftrightarrow \quad (WH$ & $DT)$.

Theorem 2.2.7. *The binary closure operator • is idempotent if and only if*

$$a \bullet b = \bigwedge \{n \in O \mid a \leq n \leq b, n \bullet b = n\}$$

for all $a \leq b \in O$.

Proof. ($\Rightarrow$) Let $X = \{n \in O \mid a \leq n \leq b, n \bullet b = n\}$. Since the binary closure operator is idempotent, we have that $a \bullet b \in X$. Now for any $n \in X$, we have $a \leq n$, so $a \bullet b \leq n \bullet b = n$. This means $a \bullet b$ is the meet of the set X.

($\Leftarrow$) Suppose that $a \bullet b = \bigwedge\{n \in O \mid a \leq n \leq b, n \bullet b = n\}$. Since $a \bullet b \leq n$ for all $n \in X$, we obtain $(a \bullet b) \bullet b \leq n \bullet b = n$. It follows that, $(a \bullet b) \bullet b \leq a \bullet b$, but it is always true that $a \bullet b \leq (a \bullet b) \bullet b$. So the binary closure operator is idempotent. □

Dually, we have:

Theorem 2.2.8. *The binary closure operator $\bullet$ is weakly hereditary if and only if*

$$a \bullet b = \bigvee\{m \in O \mid a \leq m \leq b, a \bullet m = m\}$$

for all $a \leq b \in O$.

Theorem 2.2.9. *If O is a complete lattice, the the binary closure operator $\bullet$ is (ID) if and only if*

$$\left(\bigvee_{i \in I} a_i\right) \bullet b = \left(\bigvee_{i \in I} (a_i \bullet b)\right) \bullet b$$

for all b in O and any family $(a_i)_{i \in I}$ in O such that $a_i \leqslant b$ for all $i \in I$.

Proof. ($\Rightarrow$) We have $a_i \leqslant a_i \bullet b$ for all $i \in I$, so

$$\bigvee_{i \in I} a_i \leqslant \bigvee_{i \in I} (a_i \bullet b).$$

Now by taking closure of both sides in b, we obtain

$$\left(\bigvee_{i \in I} a_i\right) \bullet b \leqslant \left(\bigvee_{i \in I} (a_i \bullet b)\right) \bullet b.$$

For the reverse inequality, we have

$$a_j \leqslant \bigvee_{i \in I} a_i \quad \text{for all} \quad j \in I.$$

Now by taking closure of both sides in b, we obtain

$$a_j \bullet b \leqslant \left(\bigvee_{i \in I} a_i\right) \bullet b.$$

It follows that

$$\bigvee_{i\in I}(a_i \bullet b) \leqslant \left(\bigvee_{i\in I} a_i\right) \bullet b.$$

Now again by taking closure of both sides in b and using (ID) on the right side, we obtain,

$$\left(\bigvee_{i\in I}(a_i \bullet b)\right) \bullet b \leqslant \left(\left(\bigvee_{i\in I} a_i\right) \bullet b\right) \bullet b = \left(\bigvee_{i\in I} a_i\right) \bullet b.$$

The reverse implication follows trivially from the fact that (ID) is a special case of the above condition where I is one element set. □

The dual of the previous theorem is:

Theorem 2.2.10. *For a complete lattice, a binary closure operator is (WH) if and only if*

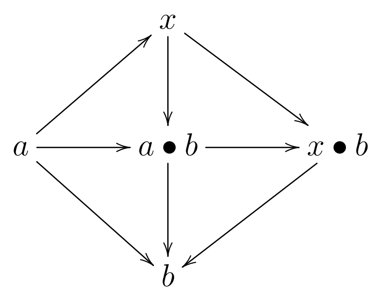
$$a \bullet \left(\bigwedge_{i\in I} b_i\right) = a \bullet \left(\bigwedge_{i\in I}(a \bullet b_i)\right)$$

for all a and any family $(b_i)_{i\in I}$ such that $a \leq b_i$ for all $i \in I$.

Theorem 2.2.11. *The binary closure operator $\bullet$ is (ID) if and only if*

$$x \bullet b = a \bullet b$$

for all $a \leq x \leq a \bullet b$.



Proof. (⇒) Suppose that $a \leq x \leq a \bullet b$. Then:

$$a \bullet b \leq x \bullet b \leq (a \bullet b) \bullet b = a \bullet b.$$

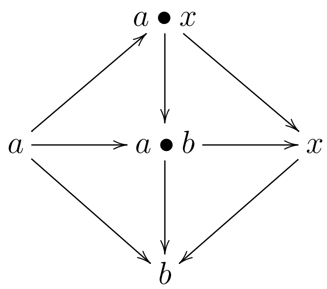
(⇐) Since $a \leq a \bullet b \leq b$ it follows by assumption that $(a \bullet b) \bullet b = a \bullet b$. □

We give the dual of the last theorem:

Theorem 2.2.12. *The binary closure operator* $\bullet$ *is (WH) if and only if*

$$a \bullet x = a \bullet b$$

for all $a \bullet b \leq x \leq b$.



Corollary 2.2.13. *The binary closure operator* $\bullet$ *is (ID & WH) if and only if*

$$x \bullet y = a \bullet b$$

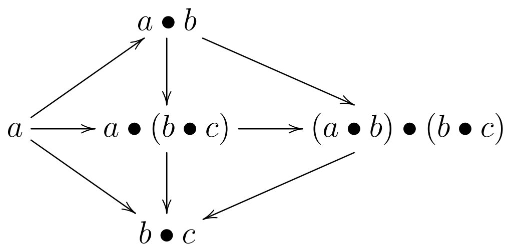
for all $a \leq x \leq a \bullet b \leq y \leq b$.

Theorem 2.2.14. *The binary closure operator* $\bullet$ *is (ID & WH) if and only if*

$$a \bullet (b \bullet c) = (a \bullet b) \bullet c$$

for all $a \leq b \leq c$.

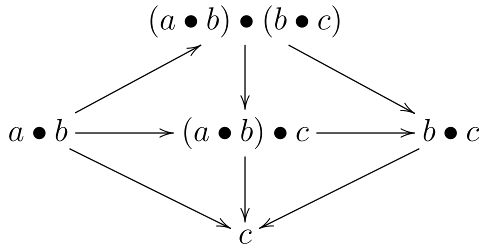
Proof. $(\Rightarrow)$ Let $a \leq b \leq c \in O$. By considering the diagram



we see that $a \bullet b \leq a \bullet (b \bullet c) \leq b \bullet c$ and hence by Theorem 2.2.11 we obtain

$$a \bullet (b \bullet c) = (a \bullet b) \bullet (b \bullet c).$$

By considering the diagram



we see that $(a \bullet b) \bullet c \leq b \bullet c \leq c$ and hence by Theorem 2.2.12 we obtain

$$(a \bullet b) \bullet c = (a \bullet b) \bullet (b \bullet c).$$

($\Leftarrow$) Let $a \leq b$. Then by the given condition we have that $a \bullet b = (a \bullet a) \bullet b = a \bullet (a \bullet b)$, and dually $a \bullet b = a \bullet (b \bullet b) = (a \bullet b) \bullet b$. □

Corollary 2.2.15. *The binary closure operator $\bullet$ is (ID & WH) if and only if*

$$(a \bullet b) \bullet (b \bullet c) = a \bullet c$$

for all $a \leq b \leq c$.

Proof. Since the binary closure operator $\bullet$ is (WH) and $a \bullet b \leq (a \bullet b) \bullet c \leq b \bullet c \leq c$, by Theorem 2.2.12 we obtain

$$(a \bullet b) \bullet (b \bullet c) = (a \bullet b) \bullet c.$$

Again, since $\bullet$ is (WH) and $a \leq a \bullet c \leq b \bullet c \leq c$, by Theorem 2.2.12 we obtain

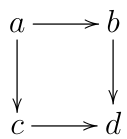
$$a \bullet c = a \bullet (b \bullet c).$$

Therefore by Theorem 2.2.14 we obtain $(a \bullet b) \bullet (b \bullet c) = a \bullet c$. □

Theorem 2.2.16. *The binary closure operator $\bullet$ is (ID & WH) if and only if*

$$(a \bullet b) \bullet (c \bullet d) = (a \bullet c) \bullet (b \bullet d),$$

when a, b, c and d as in the following diagram:



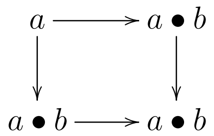
Proof. ($\Rightarrow$) Suppose that the binary closure operator is $(WH$ & $ID)$, so

$$a \bullet d = a \bullet (a \bullet d) \leq a \bullet (b \bullet d) \leq (a \bullet c) \bullet (b \bullet d) \leq (a \bullet c) \bullet d \leq (a \bullet d) \bullet d = a \bullet d.$$

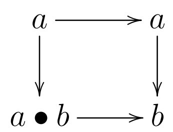
Now, the same calculation for the left side of the equation gives

$$a \bullet d = a \bullet (a \bullet d) \leq a \bullet (c \bullet d) \leq (a \bullet b) \bullet (c \bullet d) \leq (a \bullet b) \bullet d \leq (a \bullet d) \bullet d = a \bullet d.$$

($\Leftarrow$) From the diagram

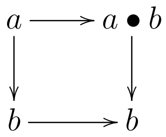


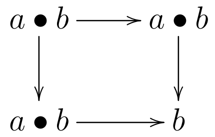
we obtain $\Big(a \bullet (a \bullet b)\Big) \bullet (a \bullet b) = a \bullet (a \bullet b)$. Now, from the diagram



we obtain $\Big(a \bullet (a \bullet b)\Big) \bullet (a \bullet b) = a \bullet b$. That is $a \bullet (a \bullet b) = a \bullet b$.

From the diagrams





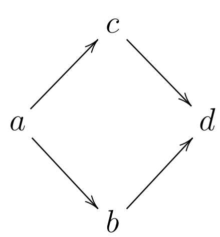
we obtain $(a \bullet b) \bullet b = a \bullet b$. □

The following result gives an example of a binary closure operator which satisfies the equation $(a \bullet b) \bullet (c \bullet d) = (a \bullet c) \bullet (b \bullet d)$, but $(a \bullet b) \bullet (c \bullet d) \neq a \bullet d$ and the binary closure operator is neither (WH) nor (ID).

Example 2.2.17. *For the poset* $(\mathbb{R}, \leq)$*, define a binary closure operator as follows*

$$a \bullet b = \frac{a+b}{2}.$$

Then for any diagram



we have

$$(a \bullet b) \bullet (c \bullet d) = \frac{a+b+c+d}{4} = (a \bullet c) \bullet (b \bullet d).$$

On the other hand, in general

$$a \bullet (a \bullet b) = \frac{a + \frac{a+b}{2}}{2} = \frac{3a+b}{4} \neq \frac{a+b}{2} = a \bullet b.$$

$$(a \bullet b) \bullet b = \frac{\frac{a+b}{2} + b}{2} = \frac{a+3b}{4} \neq \frac{a+b}{2} = a \bullet b.$$

Also,

$$(a \bullet b) \bullet (c \bullet d) = a \bullet d \quad \Leftrightarrow \quad \frac{a+b+c+d}{4} = \frac{a+d}{2} \quad \Leftrightarrow \quad b+c = a+d$$

which does not hold for example $a = 1,\ b = 2 = c,\ d = 4.$

Theorem 2.2.18.

$$(HE) \quad \Leftrightarrow \quad (WH \,\&\, LD).$$

Proof. $(HE) \Rightarrow (WH)$ Consider $x \leq y$. By using (HE) on $x \leq x \bullet y \leq y$, we obtain

$$x \bullet (x \bullet y) = (x \bullet y) \wedge (x \bullet y) = (x \bullet y).$$

$(HE) \Rightarrow (LD)$ Let $a \leq b \leq c$, suppose that $a \bullet c = c$, so that (HE) gives

$$a \bullet b = (a \bullet c) \wedge b = c \wedge b = b.$$

$(WH \,\&\, LD) \Rightarrow (HE)$ Let $a \leq b \leq c$. To show that $a \bullet b$ is a maximal element beneath both b and $a \bullet c$, suppose that

$$a \bullet b \leq x \leq b \quad \text{and} \quad x \leq a \bullet c.$$

Since $\bullet$ is (WH), we obtain

$$a \bullet b = a \bullet (a \bullet b) \leq a \bullet x \leq a \bullet b.$$

Therefore, $a \bullet x = a \bullet b$. Now, since $\bullet$ is (LD) and $a \leq x \leq a \bullet c$ and $a \bullet (a \bullet c) = a \bullet c$, hence $a \bullet x = x$. Therefore, $x = a \bullet b$. □

Dualizing the previous theorem, we obtain:

Theorem 2.2.19. $(MI) \quad \Leftrightarrow \quad (ID \,\&\, RC).$

By Theorem 2.2.4 and Theorem 2.2.18 we obtain

Corollary 2.2.20. $(ID \ \& \ HE) \quad \Leftrightarrow \quad (WH \ \& \ DT \ \& \ LD).$

Dually, we have:

Corollary 2.2.21. $(WH \ \& \ MI) \quad \Leftrightarrow \quad (ID \ \& \ CT \ \& \ RC).$

Theorem 2.2.22. *If O is a complete lattice, then*

$$(MI) \ \Rightarrow \ (FA) \ \Rightarrow \ (AD).$$

Proof. Let $(a_i)_{i\in I}$ be a non empty family such that $a_i \leq b$ for all $i \in I$. We have that

$$a_j \leq \bigvee_{i\in I} a_i \leq b$$

for all $j \in I$. Since $\bullet$ is (MI), we have

$$\left(\bigvee_{i\in I} a_i\right) \bullet b = (a_j \bullet b) \vee \left(\bigvee_{i\in I} a_i\right)$$

for all $j \in I$. Hence we have that

$$\left(\bigvee_{i\in I} a_i\right) \bullet b = \bigvee_{j\in I}\left((a_j \bullet b) \vee \left(\bigvee_{i\in I} a_i\right)\right) = \left(\bigvee_{j\in I}(a_j \bullet b)\right) \vee \left(\bigvee_{i\in I} a_i\right).$$

But

$$\left(\bigvee_{i\in I} a_i\right) \leq \bigvee_{i\in I}(a_i \bullet b).$$

Therefore,

$$\left(\bigvee_{i\in I} a_i\right) \bullet b = \bigvee_{i\in I}(a_i \bullet b).$$

Clearly, (FA)$\Rightarrow$ (AD). □

Dualizing the theorem, we obtain

Theorem 2.2.23. *If O is a complete lattice, then*

$$(HE) \ \Rightarrow \ (FM) \ \Rightarrow \ (MU).$$

2.3 Examples

Example 2.3.1. *For a topological space X, let $A \bullet B = A \cup B^\circ$ for all $A \subseteq B \subseteq X$. Then this is an order-preserving map and $A \subseteq A \bullet B \subseteq B$. This binary closure operator satisfies:*

- *(WH), since for any $A \subseteq B \subseteq X$ we have*

$$(A \cup B^\circ)^\circ \subseteq A \cup B^\circ$$

$$\Rightarrow A \cup (A \cup B^\circ)^\circ \subseteq A \cup B^\circ.$$

  Also we have

$$B^\circ \subseteq A \cup B^\circ \;\Rightarrow\; B^\circ \subseteq (A \cup B^\circ)^\circ \;\Rightarrow\; A \cup B^\circ \subseteq A \cup (A \cup B^\circ)^\circ$$

  This shows that $A \bullet (A \bullet B) = A \bullet B$.

- *(MI), that is, for any $A \subseteq B \subseteq C \subseteq X$, we have*

$$(A \bullet C) \cup B = (A \cup C^\circ) \cup B = B \cup C^\circ = B \bullet C.$$

But this binary closure operator fails to be hereditary. Indeed, for a space $X = \{1, 2, 3\}$ with $\phi, X, \{1\}, \{1, 2\}$ open sets, let $A = \{3\}$, $B = \{2, 3\}$ and $C = X$. Then one has

$$(A \bullet C) \cap B = (A \cup C^\circ) \cap B = X \cap B = B,$$

while

$$A \bullet B = A \cup B^\circ = A \cup \phi = A.$$

This shows that $(A \bullet C) \cap B \neq A \bullet B$.

The binary closure operator, which is defined by $A \bullet B = \overline{A} \cap B$ for all $A \subseteq B \subseteq X$, is (ID) and (HE), but not (MI).

Example 2.3.2. *Let R be a commutative ring, O the set of all ideals of R and*

$$\alpha = \{(I, J) \mid I, J \in O \;\text{ and }\; I \subseteq J\}.$$

The radical *of an ideal I of R is defined by*

$$r(I) = \{r \in R \mid \text{ there exists } n \in \mathbb{N} \text{ with } r^n \in I\}.$$

We define a binary closure operator $\alpha \xrightarrow{\bullet} O$ by,

$$I \bullet J = r(I) \cap J.$$

This binary closure operator is

- *(ID): To prove that* $(I \bullet J) \bullet J \subseteq I \bullet J$*, let* $a \in (I \bullet J) \bullet J = r(r(I) \cap J) \cap J$

$$\Rightarrow \exists n \in \mathbb{N} \text{ with } a^n \in r(I) \cap J$$
$$\Rightarrow \exists m \in \mathbb{N} \text{ with } (a^n)^m = a^{nm} \in I$$
$$\Rightarrow a \in r(I) \cap J = I \bullet J.$$

- *(HE): To prove this, suppose that* $I \subseteq J \subseteq K$ *are ideals. Then it is clear that*

$$I \bullet J = r(I) \cap J = r(I) \cap (K \cap J) = (r(I) \cap K) \cap J = (I \bullet K) \cap J.$$

Example 2.3.3. $(FA) \nRightarrow$ *(MI). Define the "point" binary closure operator of subsets* $A \subseteq B$ *of a topological space* X *by*

$$A \bullet B = \bigcup_{a \in A} (\overline{\{a\}} \cap B).$$

It is clear that

$$A = \bigcup_{a \in A} (\{a\} \cap B) \subseteq \bigcup_{a \in A} (\overline{\{a\}} \cap B) \subseteq B.$$

Indeed, for a topological space $X = \{1, 3, 4\}$ *with* $\phi, X, \{1, 3\}, \{4\}$ *open sets. Let* $A = \{4\}$*,* $B = \{1, 4\}$*. Now we have* $A \subseteq B \subseteq X$ *and*

$$A \bullet X = A,$$

$$B \bullet X = (\overline{\{1\}} \cap X) \cup (\overline{\{4\}} \cap X) = X.$$

Which means $B \bullet X = X \neq B = (A \bullet X) \cup B$*. i.e, this binary closure operator is not minimal. On the other hand, this binary closure operator is fully additive, because for any family of subsets* $(A_i)_{i \in I}$ *of a subset* B $(I \neq \emptyset)$ *we have:*

$$\left(\bigcup_{i \in I} A_i\right) \bullet B = \bigcup_{a \in \bigcup A_i} (\overline{\{a\}} \cap B) = \bigcup_{i \in I} \left(\bigcup_{a \in A_i} (\overline{\{a\}} \cap B)\right) = \bigcup_{i \in I} (A_i \bullet B).$$

Example 2.3.4. $(RC) \nLeftrightarrow (ID)$*. The binary closure operator in Example 2.2.5 satisfies (RC), but it is not (ID).*

For the other direction, consider a topological space $X = \{1, 2\}$ *with* $\emptyset, X$ *as open sets. Define a binary closure operator as follows:* $A \bullet B = \overline{A} \cap B$*. This binary closure operator is (ID), but it does not satisfy (RC). Indeed, from* $\emptyset \subset \{1\} \subset X$ *we have*

$$\emptyset \bullet X = \emptyset, \text{ but } \{1\} \bullet X = X \neq \{1\}.$$

Now we give an example which satisfies both of the properties (MI) and (HE).

Example 2.3.5. *Let $O = \{a, b, c\}$, with a binary relation $\leq$ on O as $a \leq b \leq c$. We define a binary closure operator on O by the following table:*

$\bullet$	a	b	c
a	a	b	b
b	$-$	b	b
c	$-$	$-$	c

This binary closure operator is $(MI$ & $HE)$.

The last few examples gave an idea of how we can get a closure of an element and its dual from a given binary closure operator.

Definition 2.3.6. *Let O be a poset with a top element 1, and let $\alpha \xrightarrow{\bullet} O$ be a hereditary binary closure operator, then we define the* closure *of an element $a \in O$ by*

$$\overline{a} = a \bullet 1.$$

Note that, $a \leq \overline{a}$ for all $a \in O$, and if $a \leq b$, then $a \bullet 1 \leq b \bullet 1$, and this implies that $\overline{a} \leq \overline{b}$.

For $a \leq b \in O$, by (HE) we obtain $a \bullet b = (a \bullet 1) \wedge b$, so that

$$a \bullet b = \overline{a} \wedge b.$$

In this case, every $a \in O$ is dense in $\overline{a}$.

Proposition 2.3.7. *Let O be a lattice with a top element 1 and let $\alpha = \{(a, b) \mid a, b \in O, a \leq b\}$. Let $f \colon \alpha \to O$ be a morphism such that $a \leq f(a, 1)$ for all $a \in O$. Then we can define a hereditary binary closure operator as follows:*

$$a \bullet b = \inf\{f(a, 1), b\}$$

for all $a \leq b$ in O.

Proof. For every $a \leq b$ in O, we have that $a \leq \inf\{f(a, 1), b\} \leq b$, also for every $(a_1, b_1) \leq (a_2, b_2)$ in α we have $a_1 \bullet b_1 \leq a_2 \bullet b_2$.

For (HE), suppose that $a \leq b \leq c$ in O. Therefore, $(a \bullet c) \wedge b = \inf\{\inf\{f(a, 1), c\}, b\}$.

If $f(a,1) \leq c$, then $(a \bullet c) \wedge b = \inf\{f(a,1), b\} = a \bullet b$.

If $c \leq f(a,1)$, then $(a \bullet c) \wedge b = \inf\{c, b\} = b = \inf\{f(a,1), b\} = a \bullet b$.

Consequently, the binary closure operator is hereditary. □

Definition 2.3.8. *Let O be a poset with a bottom element 0, and let $\alpha \xrightarrow{\bullet} O$ be a minimal binary closure operator. Then we define the* interior *of an element $a \in O$ by*

$$a^\circ = 0 \bullet a.$$

So, for $0 \leq a \leq b$, and by (MI) we obtain $a \bullet b = a \vee b^\circ$. In this case every pair (a°, a) is closed.

Dualizing Proposition 2.3.7, we have:

Proposition 2.3.9. *Let O be a lattice with a bottom element 0 and $\alpha = \{(a,b) : a, b \in O, a \leq b\}$. Let $f : \alpha \to O$ be a morphism such that $f(0,b) \leq b$ for all $b \in O$. Then we can define a minimal binary closure operator as follows:*

$$a \bullet b = \sup\{f(0,b), a\}$$

for all $a \leq b$ in O.

In this thesis, by a *reflexive graph* we mean a pair (X, E) where X is a set and $E \subseteq X \times X$ is a reflexive relation on X. In a reflexive graph, we write $x \to y$ when $(x, y) \in E$.

Example 2.3.10. *Every preordered $(X, \leq)$ is a reflexive graph.*

Definition 2.3.11. *For a reflexive graph (X, E) and a subset $M \subseteq X$ one defines the* up-closure *of M by*

$$\uparrow_X M = \{x \in X \mid \exists a \in M \ (x \to a)\},$$

and the down-closure *of M by*

$$\downarrow_X M = \{x \in X \mid \exists a \in M \ (a \to x)\}.$$

Proposition 2.3.12. *Let O be a poset and let $\alpha = \{(a,b) \mid a, b \in O, a \leq b\}$. Then (O, α) is a reflexive graph, and $\uparrow_O$ and $\downarrow_O$ are idempotent.*

Proof. First, it is clear by definition that (O, α) is a reflexive graph. Now we show that

$$\uparrow_O (\uparrow_O M) = \uparrow_O M, \quad \downarrow_O (\downarrow_O M) = \downarrow_O M$$

for every $M \subseteq O$.

From the definitions of up-closure and down-closure we have that

$$\uparrow_O M \subseteq \uparrow_O (\uparrow_O M), \quad \downarrow_O M \subseteq \downarrow_O (\downarrow_O M).$$

Let $x \in \uparrow_O (\uparrow_O M)$. Then there exists $b \in \uparrow_O M$ and $x \leq b$. Since $b \in \uparrow_O M$, there exists $a \in M$ with $b \leq a$. Now since α is transitive, $x \leq a$. Therefore $x \in \uparrow_O M$, so $\uparrow_O (\uparrow_O M) \subseteq \uparrow_O M$.

Similarly, $\downarrow_O$ is idempotent. □

Proposition 2.3.13. *Let O be a poset. For any $X \subseteq Y \subseteq O$, let*

$$X \bullet Y = \uparrow_Y X.$$

Then, this defines a binary closure operator on $P(O)$, where $P(O)$ is the power set of the set O and $\alpha = \{(X, Y) \in P^2(O) \mid X \subseteq Y\}$, and

$$\uparrow_Y X = (\uparrow_O X) \cap Y.$$

Proof. We want to prove the following:

(1) For any $X \subseteq Y$ we have $X \subseteq X \bullet Y \subseteq Y$,

(2) For any $(X_1, Y_1) \leq (X_2, Y_2)$ in α, we have that $X_1 \bullet Y_1 \subseteq X_2 \bullet Y_2$.

If $X = \emptyset$, then $\uparrow_Y X = \emptyset$. If X is not the empty set, then for any $x \in X$ we have that $x \to x$. Since $X \subseteq Y$, we obtain $X \subseteq X \bullet Y \subseteq Y$. Now for order preservation, suppose that $(X_1, Y_1) \leq (X_2, Y_2)$.

$$\begin{aligned} \text{Let } x \in X_1 \bullet Y_1 &\Rightarrow x \in Y_1 \text{ and } \exists\, a \in X_1 \text{ with } x \leq a \\ &\Rightarrow x \in Y_2,\ a \in X_2 \text{ and } x \leq a \\ &\Rightarrow x \in X_2 \bullet Y_2. \end{aligned}$$

□

Similarly, we have that $\downarrow_Y X$ is also a binary closure operator.

Proposition 2.3.14. *The binary closure operators $\uparrow_O$ and $\downarrow_O$ on a poset O are hereditary, grounded, fully additive, idempotent but not minimal.*

Proof. Suppose that $X \subseteq Y \subseteq Z$ of a given poset O. First, let us show that

$$X \bullet Y = \uparrow_Y X = (\uparrow_O X) \cap Y.$$

$$\begin{aligned} x \in \uparrow_X Y &\Leftrightarrow x \in Y \text{ and } \exists a \in X(x \leq a) \\ &\Leftrightarrow x \in Y \subseteq O \text{ and } \exists a \in X(x \leq a) \\ &\Leftrightarrow x \in \uparrow_O X \cap Y. \end{aligned}$$

Now

$$(X \bullet Z) \cap Y = (\uparrow_Z X) \cap Y = ((\uparrow_O X) \cap Z) \cap Y = (\uparrow_O X) \cap Y = X \bullet Y.$$

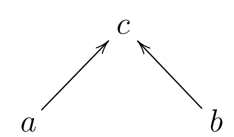
For (FA), let $\left(X_i\right)_{i\in I}$ be a family of subsets of a subset $Y \subseteq O$, then

$$\begin{aligned} x \in \uparrow_Y \left(\bigcup_{i\in I} X_i\right) &\Leftrightarrow x \in Y \text{ and } \exists a \in \bigcup_{i\in I} X_i(x \leq a) \\ &\Leftrightarrow x \in Y \text{ and } \exists a \in X_j(x \leq a) \text{ for some } j \in I \\ &\Leftrightarrow x \in X_j \bullet Y \text{ for some } j \in I \\ &\Leftrightarrow x \in \bigcup_{i\in I}(\uparrow_Y X_i). \end{aligned}$$

Dually, for $\downarrow_O$. □

Now we give an example to show that $\downarrow_O$ and $\uparrow_O$ in general are not minimal.

Example 2.3.15. *Let $O = \{a, b, c\}$ be a poset with the following diagram:*



Let $A = \{a\}$ and $B = \{a, c\}$. Then $\uparrow_O A = A$ and $\uparrow_O B = O$. Therefore,

$$\uparrow_O B \neq (\uparrow_O A) \cup B.$$

On the other hand, let $A = \emptyset$, $B = \{b\}$ and $C = \{b, c\}$. Then

$$\downarrow_C B = (\downarrow_O B) \cap C = \{b, c\} \cap C = C,$$

while $\downarrow_O A = \emptyset$. Therefore,

$$\downarrow_B C \neq (\downarrow_O A) \cup B.$$

2.4 Weakly hereditary idempotent binary closure operators as Eilenberg-Moore algebras

Definition 2.4.1. *A* whidset *is an ordered pair* $(O, \bullet)$ *where is* O *a poset* $O = (O, \leq)$ *and* $\bullet$ *is a weakly hereditary idempotent binary closure operator on* O.

Proposition 2.4.2. *For any poset* $(X, \leq)$*, the poset* $(TX, \leq^{'})$ *where*

$$TX = \{(x, y) \in X^2 \mid x \leq y\}$$

and

$$(x_1, y_1) \leq^{'} (x_2, y_2) \quad \Leftrightarrow \quad x_1 \leq x_2 \text{ and } y_1 \leq y_2$$

defines a functor $\mathbf{Ord} \to \mathbf{Ord}$.

Proof. For any morphism $f\colon X \longrightarrow Y$, we define $T(f)\colon TX \longrightarrow TY$ as follows:

$$T(f)(x, y) = (fx, fy).$$

So, for the identity morphism $1_X\colon X \longrightarrow X$ we have

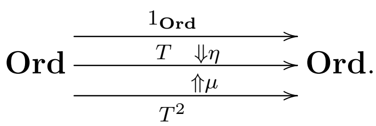
$$T(1_X)(x, y) = (1_X(x), 1_X(y)) = (x, y) = 1_{TX}(x, y).$$

Furthermore, it is clear that $T(g) \circ T(f) = T(g \circ f)$, for any morphisms $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$. □

Note that

$$(T \circ T)X = T^2X = \{((x_1, y_1), (x_2, y_2)) \in (TX)^2 \mid (x_1, y_1) \leq (x_2, y_2)\}.$$

Theorem 2.4.3. *Consider the following situation:*



Where T *and* T^2 *are functors as in the previous proposition,* $1_{\mathbf{Ord}}$ *is the identity functor, and* η *and* μ *are natural transformations with the components at an object* X *defined by*

$$\eta_X(x) = (x, x),$$

and

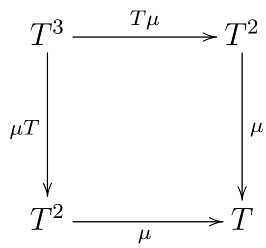
$$\mu_X((x_1, y_1), (x_2, y_2)) = (x_1, y_2).$$

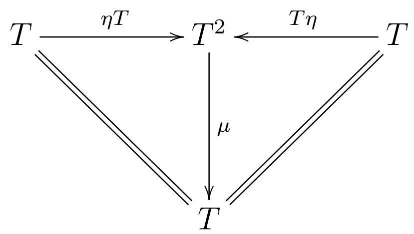
Then $\langle T, \eta, \mu \rangle$ *is a monad on the category* **Ord** *and for each order-preserving morphism* $h\colon TX \longrightarrow X$ *the following assertions are equivalent:*

(1) (X, h) *is a whidset,*

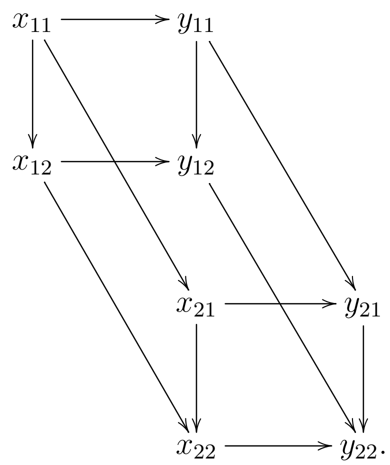
(2) $\langle X, h \rangle$ *is a* T*-algebra.*

Proof. $\langle T, \eta, \mu \rangle$ is a monad if the following diagrams



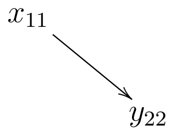


commute. Any element $(((x_{11}, y_{11}), (x_{12}, y_{12})), ((x_{21}, y_{21}), (x_{22}, y_{22})))$ in T^3X can be represented by a diagram



Applying $T\mu_X$ and μ_{TX} respectively to the previous element we obtain $((x_{11}, y_{12}), (x_{21}, y_{22}))$ and $((x_{11}, y_{11}), (x_{22}, y_{22}))$ which can be represented by

and which are in T^2X. Applying the natural transformation μ_X to the previous elements we obtain (x_{11}, y_{22}) which can be represented by

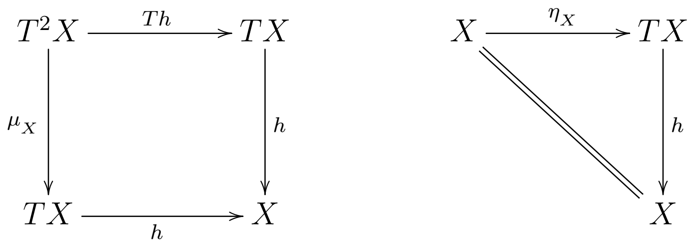


and which is in TX. This shows that $\mu \circ T\mu = \mu \circ \mu T$.

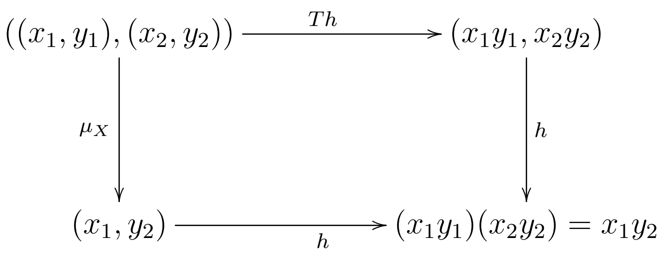
For the triangles, applying $T\eta_X$ and η_{TX} to any element $(x, y) \in TX$, we obtain $T\eta_X(x, y) = ((x, x), (y, y))$ and $\eta_{TX}(x, y) = (x, y), (x, y))$. Hence we have

$$\mu((x, y), (x, y)) = (x, y) = \mu((x, x), (y, y).$$

(1) $\Rightarrow$ (2) $\langle X, h\rangle$ is a T-algebra if the diagrams

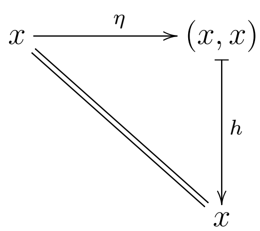


commute. For the first diagram, any element $((x_1, y_1), (x_2, y_2)) \in T^2X$, we have



$(x_1y_1)(x_2y_2) = x_1y_2$ comes from the fact that h is (ID) and (WH).

For the second diagram and for any $x \in X$ we have



(2) $\Rightarrow$ (1) We are going to show that the map $h\colon TX \longrightarrow X$ is a weakly hereditary idempotent closure operator. For any $x \in X$, since $h\eta(x) = x$ for all $x \in X$, we obtain $h(x,x) = x$. Let $x \leq y$. Since $(x,x) \leq (x,y) \leq (y,y)$ and h is order preserving, we obtain

$$x \leq h(x,y) \leq y.$$

That is h is a binary closure operator. The following diagram

$$\begin{array}{ccc} ((x,y),(y,y)) & \xrightarrow{Th} & (xy,y) \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle h} \\ (x,y) & \xrightarrow[h]{} & (xy)y = xy \end{array}$$

shows that h is (ID). The follows diagram

$$\begin{array}{ccc} ((x,x),(x,y)) & \xrightarrow{Th} & (x,xy) \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle h} \\ (x,y) & \xrightarrow[h]{} & x(xy) = xy \end{array}$$

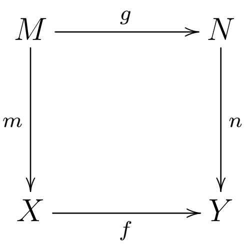
shows that h is (WH). □

2.5 Application to categorical closure operators

Consider a category $\mathbb{X}$ and a class $\mathcal{M}$ of monomorphisms. We assume that

- $\mathcal{M}$ is closed under composition,
- $\mathbb{X}$ is finitely $\mathcal{M}$-complete.

A part of the definition of $\mathbb{X}$ is finitely $\mathcal{M}$-complete is that $\mathcal{M}$ is stable under pullback, that is for each pullback diagram



$n \in \mathcal{M}$ implies $m \in \mathcal{M}$.

Definition 2.5.1. *A* closure operator *c is a family $c = (c_X)_{X\in\mathbb{X}}$ of maps*

$$c_X \colon \mathcal{M}/X \longrightarrow \mathcal{M}/X$$

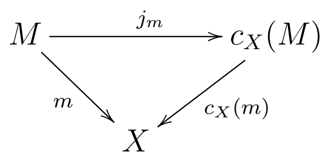
such that:

(1) $m \leq c_X(m)$ for all $m \in \mathcal{M}/X$,

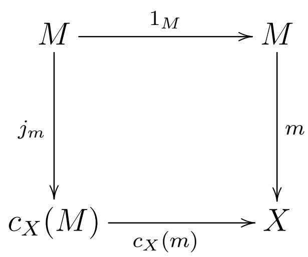
(2) If $m \leq n$ in $\mathcal{M}/X$, then $c_X(m) \leq c_X(n)$,

(3) $f(c_X(m)) \leq c_Y(f(m))$ for all $f \colon X \to Y$ in $\mathbb{X}$ and $m \in \mathcal{M}/X$.

From (1) we have a commutative diagram



with a uniquely determined morphism j_m. Since $m \in \mathcal{M}$ and $c_X(m)$ is monic, then we have a pullback diagram

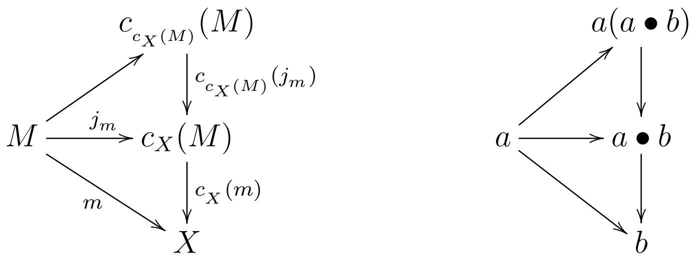


Hence, $j_m \in \mathcal{M}$.

In the following we give and compare the definitions of properties of categorical closure operators and binary closure operators, and show the similarity between diagrams of binary closure operators and diagrams of categorical closure operators.

Suppose we have a poset O, a category $\mathbb{X}$ and a pre-ordered class $\mathcal{M}/X$ of all $\mathcal{M}$-morphisms with co-domain X of $\mathbb{X}$. Now for $a \leq b$ in O, and $m \in \mathcal{M}/X$, $a \bullet b$ is the closure of a in b for binary closure operators, while $c_X(m)$ is the closure of m for categorical closure operators.

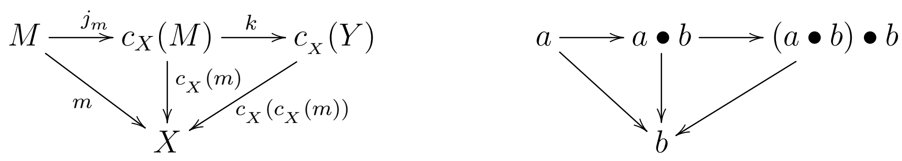
1. Weak hereditariness: when we take the closure of a in $a \bullet b$ for binary closure operators, we get $a \bullet b(a \bullet b)$. Similarly, for categorical closure operators, as in the following diagram, one takes the closure of j_m in $c_X(M)$ to be $c_{c_X(M)}(j_m)$.



Now, the binary closure operator is (WH) if $a \bullet (a \bullet b) = a \bullet b$ for all $a \leq b$ in O. For the categorical closure operator to be (WH), the morphism $c_{c_X(M)}(j_m)$ should be an isomorphism, that is $c_Y(j_m) \cong 1_Y$, with $Y = c_X(M)$ for all $m \colon M \longrightarrow X$ in $\mathcal{M}$.

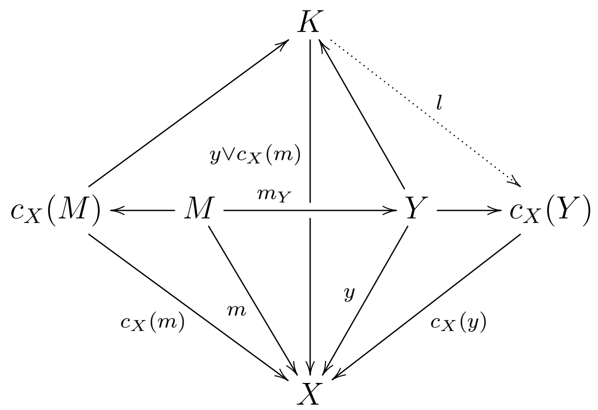
In other words, a binary closure operator is (WH) if a is dense in $a \bullet b$ for all $a \leq b$, and a closure operator c is (WH) if any $\mathcal{M}$-subobject of X is c-dense in its c-closure.

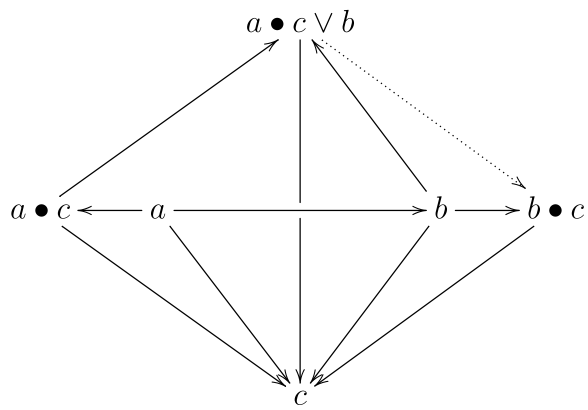
2. Idempotency: a binary closure is (ID) if $(a \bullet b) \bullet b = a \bullet b$, which means $a \bullet b$ is closed in b for all $a \leq b$. By similarity of the following diagrams



the closure operator c is (ID) if k is an isomorphism, i.e $c_X(c_X(m)) \cong c_X(m)$ for all $m \colon M \longrightarrow X$ in $\mathcal{M}$.

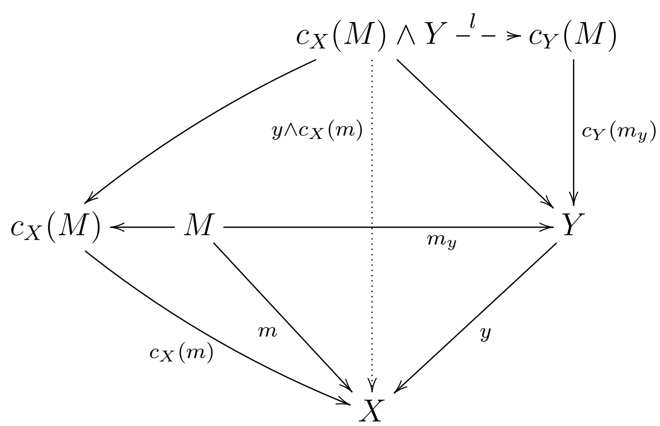
3. Minimality: a binary closure operator is (MI) if for any $a \leq b \leq c$, we have $b \bullet c = a \bullet c \vee b$. Again, by comparison between the following diagrams

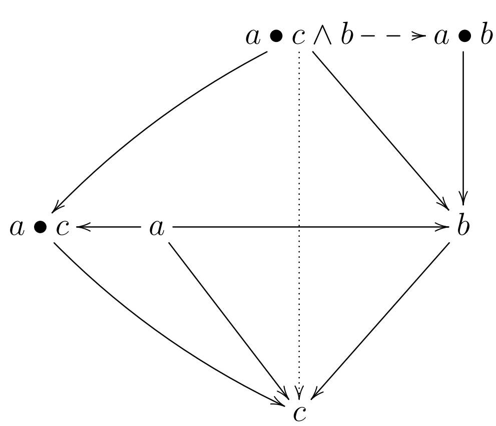




a closure operator c is (MI) when the morphism l is an isomorphism, i.e $c_X(y) \cong y \vee c_X(m)$.

4. Hereditariness: a binary closure operator is (HE), when $a \bullet b = a \bullet c \wedge b$ for every $a \leq b \leq c$. By comparing the following diagrams, we can better understand what (HE) is for categorical closure operators:

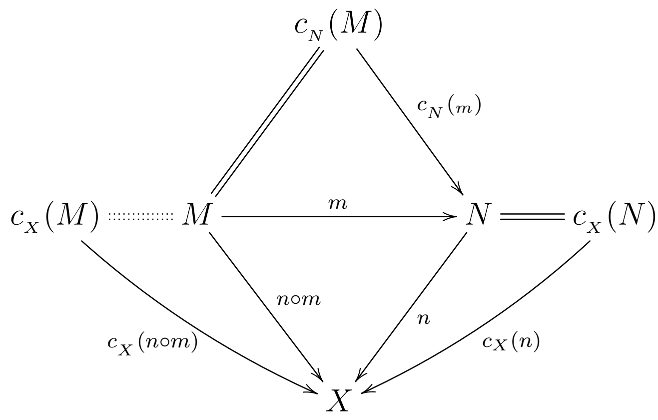


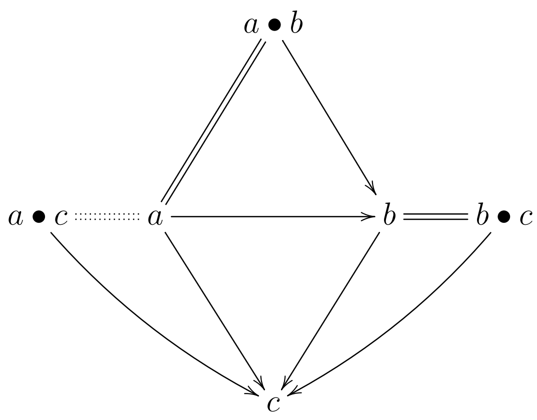


a closure operator c is (HE) if, for every $m \leq y$ we have $y \circ c_Y(m_y) \cong y \wedge c_X(m)$. Hence,

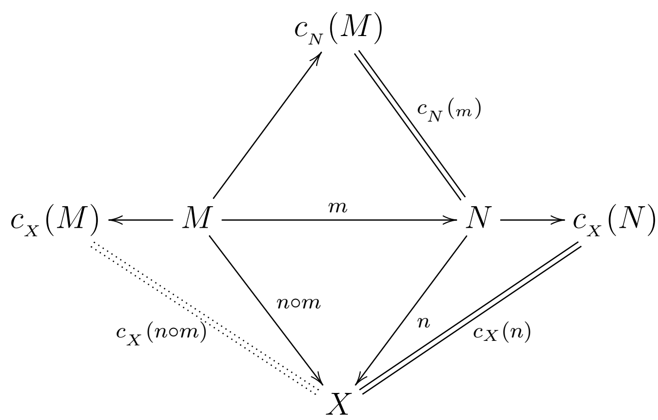
$$c_Y(m_y) \cong y^{-1}(y \wedge c_X(m)) \cong I_Y \wedge y^{-1}(c_X(m)) = y^{-1}(c_X(m)).$$

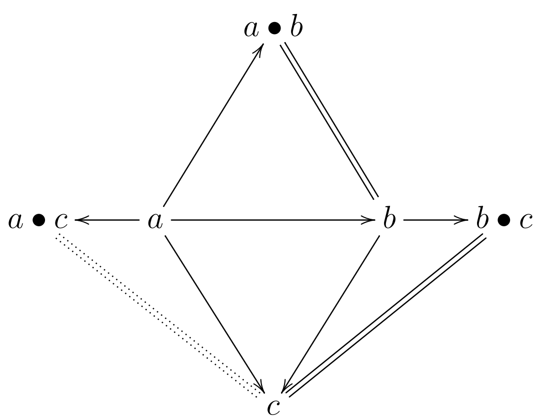
5. A binary closure operator satisfies (CT) if the "is closed" relation is transitive. a categorical closure operator satisfies (CC) if composites of c-closed $\mathcal{M}$-subobjects are c-closed, i.e for $m\colon M \to N$ and $n\colon N \to X$ in $\mathcal{M}$, if $c_N(m) = m$ and $c_X(n) = n$, then $c_X(n \circ m) = n \circ m$. We can see the similarity of the following diagrams:





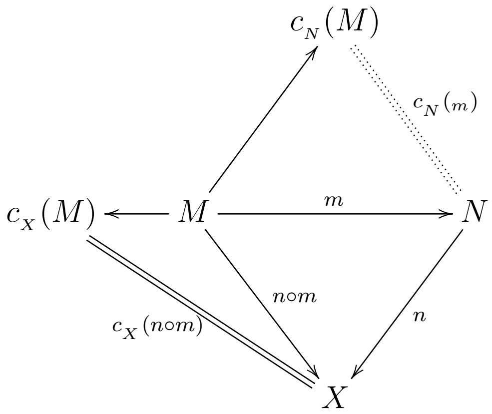
6. A binary closure operator satisfies (DT) when for every $a \leq b \leq c$, if a is dense in b (i.e $a \bullet b = b$) and b is dense in c, then a is dense in c. A categorical closure operator satisfies (CD) when composites of c-dense $\mathcal{M}$-subobjects are c-dense, i.e for $m\colon M \to N$ and $n\colon N \to X$ in $\mathcal{M}$, if $c_N(m) \cong 1_N$ and $c_X(n) \cong 1_X$, then $c_X(n \circ m) \cong 1_X$. Both (DT) in binary closure operators and (CD) in categorical closure operators have similar diagrams:

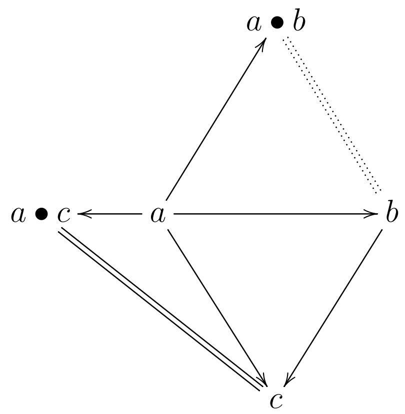




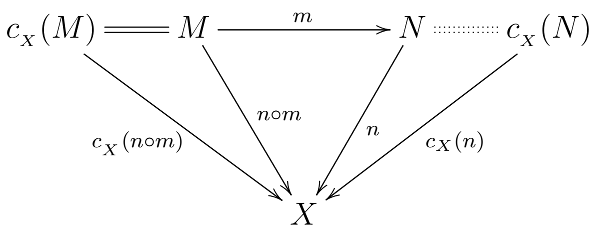
7. Left cancellation property (LD): a binary closure operator satisfies (LD) when for every $a \leq b \leq c$, if $a \bullet c = c$, then $a \bullet b = b$. While a categorical closure operator satisfies (LD)

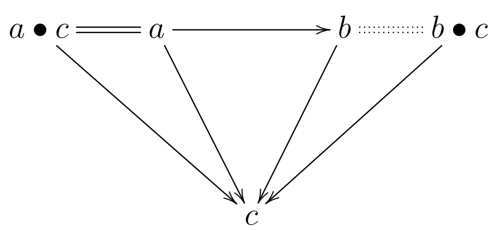
when for all $m\colon M \to N$ and $n\colon N \to X$ in $\mathcal{M}$, if $c_X(n \circ m) \cong 1_X$, then $c_N(m) \cong 1_N$. We can see the similarity of the following diagrams





8. Right cancellation property (RC): a binary closure operator satisfies (RC) when for every $a \leq b \leq c$, if $a \bullet c = a$, then $b \bullet c = b$. While a categorical closure operator satisfies (RC) when for all $m\colon M \to N$ and $n\colon N \to X$ in $\mathcal{M}$, if $c_X(n \circ m) \cong n \circ m$, then $c_X(n) \cong n$. We can see the similarity in the following diagrams:





The analogue theorem of Theorem 2.2.11 for categorical closure operators is the following:

Theorem 2.5.2. *A categorical closure operator c is (ID) if and only if*

$$c_X(n) \cong c_X(m)$$

for all $m \leq n \leq c_X(m)$ in $\mathcal{M}/X$.

Proof. ($\Rightarrow$) Suppose that $m \leq n \leq c_X(m)$. Then

$$c_X(m) \leq c_X(n) \leq c_X(c_X(m)) \cong c_X(m).$$

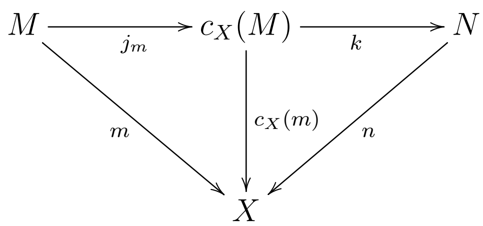
($\Leftarrow$) Let $n \cong c_X(m)$. Then c is (ID). □

The analogue theorem of Theorem 2.2.12 for categorical closure operators is the following:

Theorem 2.5.3. *A categorical closure operator c is (WH) if and only if*

$$c_X(m) \cong n \circ c_N(k \circ j_m)$$

for all m in $\mathcal{M}/X$ such that $c_X(m) \leq n$, where k and j_m are the unique morphisms making the diagram



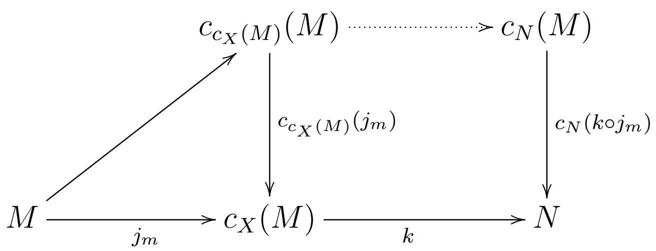
commute.

Proof. ($\Leftarrow$) Let $n \cong c_X(m)$. In this case, $k = 1_{c_X(M)}$. Hence,

$$c_X(m) \cong c_X(m) \circ c_{c_X(M)}(j_m).$$

Since $c_X(m)$ is monic, $c_{c_X(M)}(j_m) \cong 1_{c_X(M)}$. Therefore c is (WH).

($\Rightarrow$) From the following commutative diagram



we obtain

$$k \circ c_{c_X(M)}(j_m) \leq c_N(k \circ j_m).$$

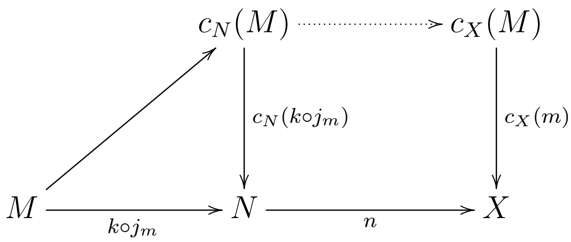
So,

$$n \circ k \circ c_{c_X(M)}(j_m) \leq n \circ c_N(k \circ j_m),$$

Therefore,

$$c_X(m) \leq n \circ c_N(k \circ j_m).$$

Now, from the following commutative diagram



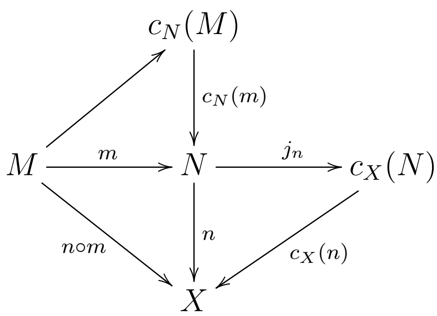
we obtain $n \circ c_N(k \circ j_m) \leq c_X(m)$. □

The analogue theorem of Theorem 2.2.14 for categorical closure operators is the following:

Theorem 2.5.4. *A categorical closure operator is idempotent and weakly hereditary if and only if*

$$c_X(n \circ c_N(m)) \cong c_X(n) \circ c_{c_X(N)}(j_n \circ m) \tag{2.1}$$

for all $m, n \in \mathcal{M}$, where j_n is the unique morphism making the diagram



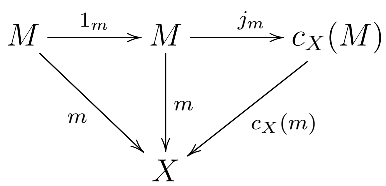
commute.

Proof. (⇐) For (ID), put $n = 1_X$ in (2.1), we obtain

$$c_X(1_X \circ c_N(m)) \cong c_X(1_X) \circ c_X(1_X \circ m).$$

Therefore, $c_X(c_X(m) \cong c_X(m)$.

For (WH), consider the following commutative diagram:

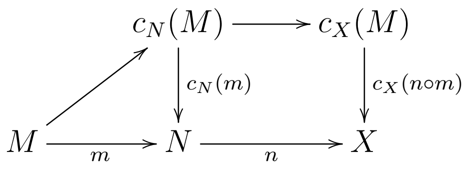


By using (2.1) we obtain

$$c_X(m) \cong c_X(m) \circ c_{c_X(M)}(j_m).$$

Since $c_X(m) \in \mathcal{M}$, we obtain $c_{c_X(M)}(j_m) \cong 1_{c_X(M)}$.

($\Rightarrow$) Consider morphisms m, n as given. Then from the following commutative diagram



we have

$$n \circ m \leq n \circ c_N(m) \leq c_X(n \circ m).$$

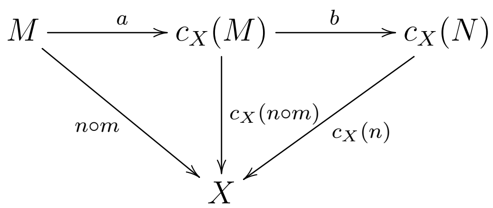
By taking the closure of each term in the inequalities above, we get:

$$c_X(n \circ m) \leq c_X(n \circ c_N(m)) \leq c_X(c_X(n \circ m)).$$

Since c is (ID), we obtain

$$c_X(n \circ m) \leq c_X(n \circ c_N(m)). \tag{2.2}$$

By using Theorem 2.5.3 on the following commutative diagram



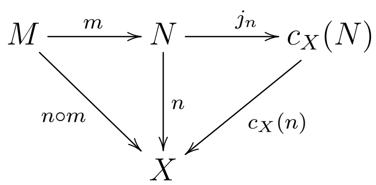
we obtain

$$c_X(n \circ m) \cong c_X(n) \circ c_{c_X(N)}(b \circ a). \tag{2.3}$$

From the diagram above, we have

$$n \circ m \cong c_X(n) \circ (b \circ a). \tag{2.4}$$

Also, from the diagram



we have

$$n \circ m \cong c_X(n) \circ (j_n \circ m). \tag{2.5}$$

From (2.4) and (2.5) and the fact that $c_X(n)$ is monic, we obtain $b \circ a \cong j_n \circ m$. Therefore,

$$c_{c_X(N)}(b \circ a) \cong c_{c_X(N)}(j_n \circ m). \tag{2.6}$$

Substituting (2.6) into (2.3), we obtain

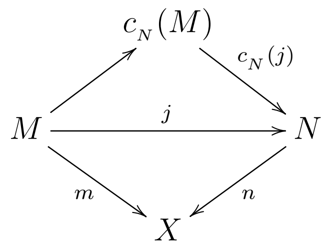
$$c_X(n \circ m) \cong c_X(n) \circ c_{c_X(N)}(j_n \circ m). \tag{2.7}$$

So, from (2.2) and (2.7) we get (2.1). □

Definition 2.5.5. *For any categorical closure operator $c = (c_X)_{X \in \mathbb{X}}$, we define a binary closure operator $\tilde{c}$ on $\mathcal{M}/X$, for each object X, by*

$$\tilde{c}(m, n) \cong n \circ c_N(j),$$

where $c_N(M)$ is the domain of c-closure $c_N(j)$ as in the following diagram:

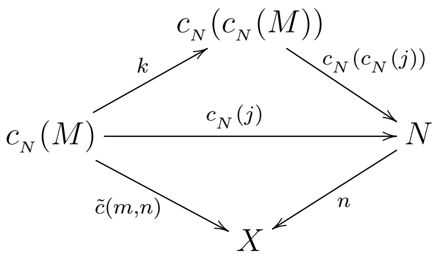


It is clear that $\tilde{c}(m, n) \in \mathcal{M}/X$, since both n and $c_N(j)$ belong to $\mathcal{M}$, and $\mathcal{M}$ is closed under composition.

Next theorems show that, for a given categorical closure operator c, c satisfies a given property if and only if the binary closure operator $\tilde{c}$ satisfies the corresponding property from the comparison at the beginning of the chapter.

Theorem 2.5.6. *$\tilde{c}$ is (ID) if and only if c is (ID).*

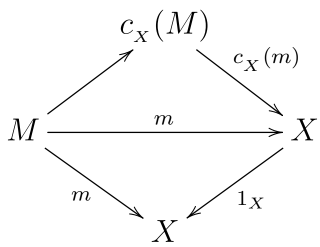
Proof. Consider the diagram:



($\Leftarrow$) Suppose that c is (ID). Then,

$$\tilde{c}(\tilde{c}(m,n),n)) \cong n \circ c_N(c_N(j)) \cong n \circ c_N(j) \cong \tilde{c}(m,n).$$

($\Rightarrow$) Suppose that $\tilde{c}$ is (ID). From the following diagram:



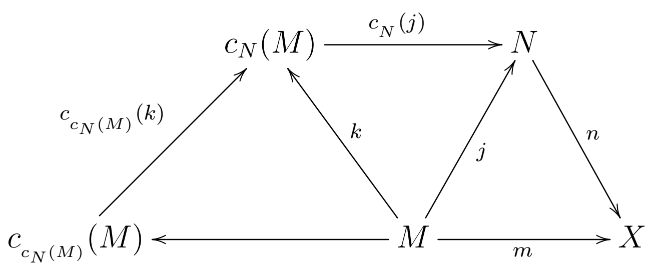
we have, $\tilde{c}(m, 1_X) \cong 1_X \circ c_X(m)$. Therefore,

$$c_X(m) \cong \tilde{c}(m, 1_X) \cong \tilde{c}(\tilde{c}(m, 1_X), 1_X) \cong c_X(\tilde{c}(m, 1_X)) \cong c_X(c_X(m)).$$

□

Theorem 2.5.7. *$\tilde{c}$ is (WH) if and only if c is (WH).*

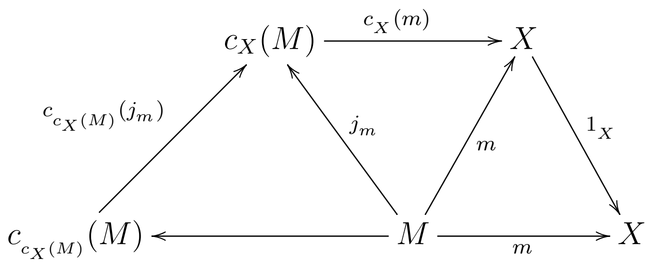
Proof. ($\Leftarrow$) Suppose that c is (WH) and $m \leq n$ in $\mathcal{M}/X$. From the commutative diagram



we have:

$$\tilde{c}(m, \tilde{c}(m,n)) \cong n \circ c_N(j) \circ c_Y(k) \cong n \circ c_N(j) \circ 1_Y \cong n \circ c_N(j) \cong \tilde{c}(m,n).$$

(⇒) Suppose that the binary closure operator $\tilde{c}$ is (WH). From the diagram



we have:

$$c_X(m) \cong \tilde{c}(m, 1_X) \cong \tilde{c}(m, \tilde{c}(m, 1_X)) \cong c_X(m) \circ c_Y(j_m).$$

Since $c_X(m)$ is monic, we obtain $c_Y(j_m) \cong 1_Y$. □

Theorem 2.5.8. *$\tilde{c}$ is (MI) if and only if c is (MI).*

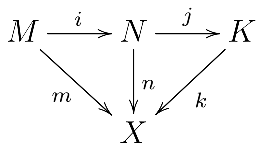
Proof. (⇒) Let $m \leq n$ in $\mathcal{M}/X$. Since $\tilde{c}$ is (MI) and $m \leq n \leq 1_X$, we obtain

$$\tilde{c}(n, 1_X) \cong \tilde{c}(m, 1_X) \vee n.$$

Therefore,

$$c_X(n) \cong c_X(m) \vee n.$$

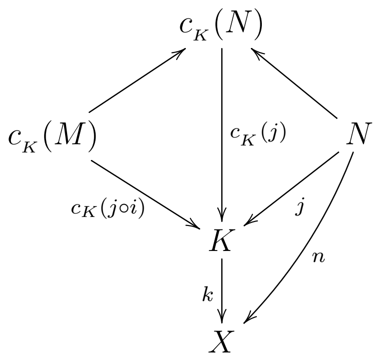
(⇐) Suppose that c is (MI). For any $m \leq n \leq k$ in $\mathcal{M}/X$



we want to prove $\tilde{c}(n, k) \cong \tilde{c}(m, k) \vee n$. Since c is (MI) and $j \circ i \leq j \leq c_K(j)$, we obtain

$$c_K(j) \cong j \vee c_K(j \circ i).$$

Therefore, one obtains a commutative diagram:



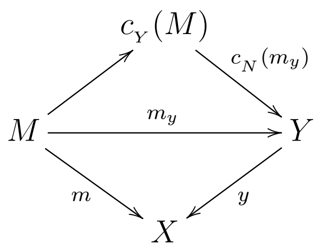
Hence, we obtain

$$k \circ c_K(j) \cong n \vee (k \circ c_K(j \circ i)).$$

Therefore, $\tilde{c}(n, k) \cong n \vee \tilde{c}(m, k)$. □

Theorem 2.5.9. *$\tilde{c}$ is (HE) if and only if c is (HE).*

Proof. ($\Rightarrow$) Let $m \leq y$ in $\mathcal{M}/X$, such that the diagram



commute.

Since $\tilde{c}$ is (HE) and $m \leq y \leq 1_X$ it follows that

$$\tilde{c}(m, y) \cong \tilde{c}(m, 1_X) \wedge y.$$

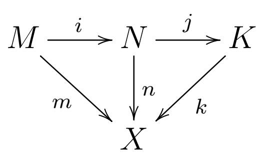
Therefore,

$$y \circ c_Y(m_Y) \cong c_X(m) \wedge y.$$

Hence,

$$c_Y(m_y) \cong y^{-1}(c_X(m) \wedge y) \cong y^{-1}(c_X(m)).$$

($\Leftarrow$) Suppose c is (HE). Consider a commutative diagram:



Since c is (HE), it follows that $c_N(i) \cong j^{-1}(c_K(j \circ i))$. Therefore, we have

$$\begin{aligned}\tilde{c}(m,k) \wedge n &\cong (k \circ c_K(j \circ i)) \wedge n \\ &\cong (k \circ j \circ c_N(i)) \wedge n \\ &\cong (n \circ c_N(i)) \wedge n \\ &\cong n \circ c_N(i) \\ &\cong \tilde{c}(m,n).\end{aligned}$$

□

Theorem 2.5.10. *$\tilde{c}$ satisfies (CT) if and only if c satisfies (CC).*

Proof. ($\Rightarrow$) Consider morphisms $m\colon M \to N$ and $n\colon N \to X$ such that $c_N(m) \cong m$ and $c_X(n) \cong n$. Since $n \circ m \leq n \leq 1_X$ we have

$$\tilde{c}(n \circ m, n) \cong n \circ c_N(m) \cong n \circ m,$$

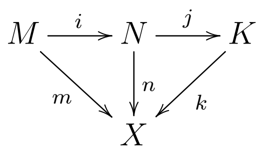
and in addition we have

$$\tilde{c}(n, 1_X) \cong c_X(n) \cong n.$$

Therefore, since $\tilde{c}$ satisfies (CT) we obtain

$$c_X(n \circ m) \cong \tilde{c}(n \circ m, 1_X) \cong n \circ m.$$

($\Leftarrow$) Let m, n and k be objects in $\mathcal{M}/X$ as shown in the diagram



such that $\tilde{c}(m,n) \cong m$ and $\tilde{c}(n,k) \cong n$. Therefore

$$n \circ i \cong m \cong \tilde{c}(m,n) \cong n \circ c_N(i).$$

Since n is monic, $c_N(i) \cong i$. Also, from

$$k \circ j \cong n \cong \tilde{c}(n,k) \cong k \circ c_K(j)$$

we obtain $j \cong c_K(j)$. Since c satisfies (CC), $c_K(j \circ i) \cong j \circ i$. Therefore,

$$\tilde{c}(m,k) \cong k \circ c_K(j \circ i) \cong k \circ j \circ i \cong m.$$

□

Theorem 2.5.11. *$\tilde{c}$ satisfies (DT) if and only if c satisfies (CD).*

Proof. ($\Rightarrow$) Consider morphisms $m\colon M \to N$ and $n\colon N \to X$ in $\mathcal{M}$ such that $c_N(m) \cong 1_N$ and $c_X(n) \cong 1_X$. Since $n \circ m \leq n \leq 1_X$ we have

$$\tilde{c}(n \circ m, n) \cong n \circ c_N(m) \cong n,$$

and in addition we have

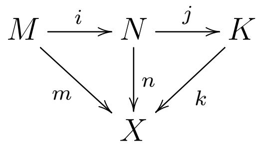
$$\tilde{c}(n, 1_X) \cong c_X(n) \cong 1_X.$$

Since $\tilde{c}$ satisfies (DT) we obtain

$$c_X(n \circ m) \cong \tilde{c}(n \circ m, 1_X) \cong 1_X.$$

This shows that c satisfies (CD).

($\Leftarrow$) Consider the following commutative diagram:



Suppose that $\tilde{c}(m, n) \cong n$ and $\tilde{c}(n, k) \cong k$. We want to show $\tilde{c}(m, k) \cong k$. From one of our assumption, we have

$$n \cong \tilde{c}(m, n) \cong n \circ c_N(i).$$

Since n is monic, we obtain $c_N(i) \cong 1_N$. Also, since

$$k \cong \tilde{c}(n, k) \cong k \circ c_K(j)$$

and k is monic, it follows that $c_K(j) \cong 1_K$. From the fact that c satisfies (CD), we obtain

$$c_K(j \circ i) \cong 1_K.$$

Therefore, we obtain $\tilde{c}(m, k) \cong k \circ c_K(j \circ i) \cong k$. □

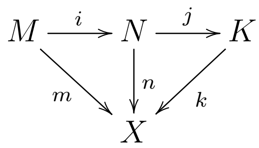
Theorem 2.5.12. *$\tilde{c}$ satisfies (LD) if and only if c satisfies (LD).*

Proof. ($\Rightarrow$) Let $m : M \to N$ and $n : N \to X$ be morphisms in $\mathcal{M}$. Suppose that $c_X(n \circ m) \cong 1_X$. Since $n \circ m \leq n \leq 1_X$, $\tilde{c}(n \circ m, 1_X) \cong 1_X$ and $\tilde{c}$ satisfies (LD) we obtain

$$\tilde{c}(n \circ m, n) \cong n.$$

Therefore, $n \circ c_N(m) \cong n$. Since n is monic, $c_N(m) \cong 1_N$.

($\Leftarrow$) Let m, n and k be objects in $\mathcal{M}/X$ as shown in the diagram



such that $\tilde{c}(m, k) \cong k$. We want to show that $\tilde{c}(m, n) \cong n$. By our assumption and the definition of $\tilde{c}$ we obtain

$$k \cong \tilde{c}(m, k) \cong k \circ c_K(j \circ i).$$

Since k is monic, $c_K(j \circ i) \cong 1_K$. Now since c satisfies (LD), $c_N(i) \cong 1_N$. Therefore, we obtain

$$\tilde{c}(m, n) \cong n \circ c_N(i) \cong n.$$

□

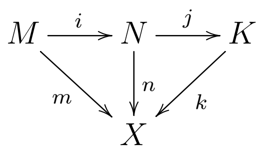
Theorem 2.5.13. *$\tilde{c}$ satisfies (RC) if and only if c satisfies (RC).*

Proof. ($\Rightarrow$) Let $m : M \to N$ and $n : N \to X$ be morphisms in $\mathcal{M}$ such that $c_X(n \circ m) \cong n \circ m$. Now from the fact that $n \circ m \leq n \leq 1_X$, we obtain

$$\tilde{c}(n \circ m, 1_X) \cong c_X(n \circ m) \cong n \circ m.$$

Since $\tilde{c}$ satisfies (RC), $c_X(n) \cong \tilde{c}(n, 1_X) \cong n$.

($\Leftarrow$) Consider the following commutative diagram:



Suppose that $\tilde{c}(m, k) \cong m$. We want to show that $\tilde{c}(n, k) \cong n$.

$$k \circ (j \circ i) \cong m \cong \tilde{c}(m, k) \cong k \circ c_K(j \circ i).$$

Therefore,

$$c_K(j \circ i) \cong j \circ i.$$

Since c satisfies (RC),

$$c_K(j) \cong j.$$

Hence, we obtain $\tilde{c}(n, k) \cong k \circ c_K(j) \cong k \circ j \cong n$. □

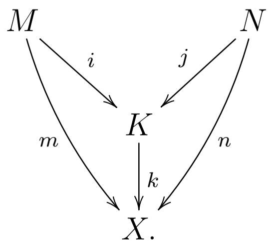
Theorem 2.5.14. *$\tilde{c}$ is (AD) if and only if c is (AD).*

Proof. Let $m : M \to X$ and $n : N \to X$ be morphisms in $\mathcal{M}$. Since $\tilde{c}$ is (AD), we obtain

$$\tilde{c}(m \vee n, 1_X) \cong \tilde{c}(m, 1_X) \vee \tilde{c}(n, 1_X).$$

Therefore, we obtain $c_X(m \vee n) \cong c_X(m) \vee c_X(n)$.

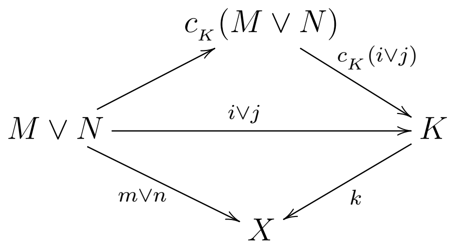
($\Leftarrow$) Let m, n and k be morphisms in $\mathcal{M}$ as shown in the diagram



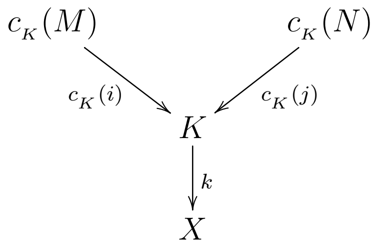
We want to show

$$\tilde{c}(m \vee n, k) \cong \tilde{c}(m, k) \vee \tilde{c}(n, k).$$

Since c is (AD), we have $c_K(i \vee j) \cong c_K(i) \vee c_K(j)$.



Hence, from the diagram



we obtain

$$k \circ c_K(i \vee j) \cong k \circ \Big(c_K(i) \vee c_K(j) \Big) \cong \Big(k \circ c_K(i) \Big) \vee \Big(k \circ c_K(i) \Big).$$

It follows that

$$\tilde{c}(m \vee n, k) \cong \tilde{c}(m, k) \vee \tilde{c}(n, k).$$

$\square$

The following display of logical connections between conditions on binary closure operators is analogous to a similar display given at the end of [6]:

$$(ID \ \& \ CT) \ \Rightarrow \ (WH) \ \Rightarrow \ (CT) \quad \text{and} \quad (WH \ \& \ DT) \ \Rightarrow \ (ID) \ \Rightarrow \ (DT)$$

$$(CT \ \& \ DT) \ \not\Rightarrow \ (ID) \quad \text{and} \quad (CT \ \& \ DT) \ \not\Rightarrow \ (WH)$$

$$(HE) \ \Leftrightarrow \ (WH \ \& \ LD) \quad \text{and} \quad (MI) \ \Leftrightarrow \ (ID \ \& \ RC)$$

$$(MI) \ \Rightarrow \ (FA) \ \Rightarrow \ (AD)$$

$$(MI) \ \not\Rightarrow \ (GR).$$

The following table shows a comparison between the properties of categorical closure operators and the corresponding properties of binary closure operators.

Property	Categorical closure operators	Binary closure operators
(ID)	$c_X(c_X(m)) \cong c_X(m)$	$(a \bullet b) \bullet b = a \bullet b$
(WH)	$c_Y(j_m) \approxeq 1_Y,$ with $Y = c_X(M)$	$a \bullet (a \bullet b) = a \bullet b$
(MI)	$c_X(y) \cong y \vee c_X(m)$	$b \bullet c = a \bullet c \vee b$
(HE)	$c_Y(m_Y) \cong y^{-1}(c_X(m))$	$a \bullet b = a \bullet c \wedge b$
(CT)	$c_N(m) \cong m \;\&\; c_X(n) \cong n \;\Rightarrow\; c_X(n \circ m) \cong n \circ m$	$a \bullet b = a \;\&\; b \bullet c = b \;\Rightarrow\; a \bullet c = a$
(DT)	$c_N(m) \cong 1_N \;\&\; c_X(n) \cong 1_X \;\Rightarrow\; c_X(n \circ m) \cong 1_X$	$a \bullet b = b \;\&\; b \bullet c = c \;\Rightarrow\; a \bullet c = c$
(LD)	$c_X(n \circ m) \cong 1_X \;\Rightarrow\; c_N(m) \cong 1_N$	$a \bullet c = c \;\Rightarrow\; a \bullet b = b$
(RC)	$c_X(n \circ m) \cong n \circ m \;\Rightarrow\; c_X(n) \cong n$	$a \bullet c = a \;\Rightarrow\; b \bullet c = b$
(AD)	$c_X(m \vee n) \cong c_X(m) \vee c_X(n)$	$(a \vee b) \bullet c = a \bullet c \vee b \bullet c$
(FA)	$c_X(\bigvee m_i) \cong \bigvee c_X(m_i)$	$(\bigvee a_i) \bullet b = \bigvee (a_i \bullet b)$

Chapter 3

Structure of binary closure operators

In this chapter, most results in Sections 3.2, 3.3, 3.5 and 3.6 are adapted from results on categorical closure operators obtained in [6] to binary closure operators. Proposition 3.3.5 and Corollaries 3.3.6-8 are based on results obtained in [15].

3.1 The lattice structure of all binary closure operators

Consider a lattice O, and let $\alpha = \{(a, b) \in O^2 \mid a \leq b\}$. We define the lattice $\mathsf{Biclo}(O)$ of all binary closure operators on O. It is ordered by

$$f \leq g \quad \Leftrightarrow \quad f(a, b) \leq g(a, b) \text{ for all } (a, b) \in \alpha.$$

To see that $\mathsf{Biclo}(O)$ is a lattice note that the meet and join are defined pointwise, while the largest element is defined by $\top(a, b) = b$ for all $(a, b) \in \alpha$, and the least element is defined by $\bot(a, b) = a$, for all $(a, b) \in \alpha$.

Proposition 3.1.1. *Let O be a complete lattice, then $\mathsf{Biclo}(O)$ is a complete lattice.*

Proof. Let $(f_i)_{i \in I}$ by a family of elements of $\mathsf{Biclo}(O)$. Since O is a complete lattice and $f_i(x, y) \in O$ for all $(x, y) \in \alpha$, $\bigvee_{i \in I} f_i$ and $\bigwedge_{i \in I} f_i$ exists and are defined pointwise. Let $f = \bigvee f_i$ and $g = \bigwedge f_i$. Indeed, this easily follows from the fact that since for each $x \leq y$ in O we have that $x \leq f_i(x, y) \leq y$ it follows that $x \leq \bigvee f_i(x, y) \leq y$.

This means that $f \in \mathsf{Biclo}(O)$. By duality one obtains $g \in \mathsf{Biclo}(O)$. $\square$

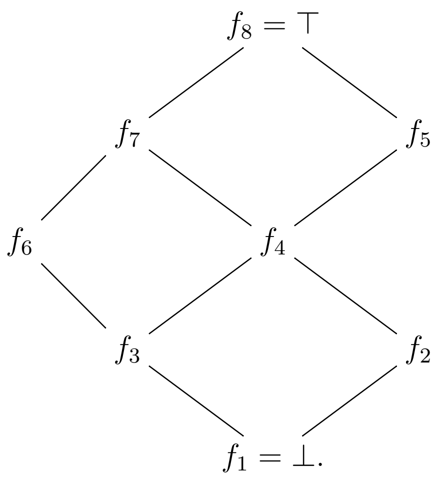
For a lattice O, by $\mathsf{Hemi}(O)$ we denote the set of all elements of $\mathsf{Biclo}(O)$ which are both (HE) and (MI).

Remark: it is clear that the largest and the smallest elements of $\mathsf{Biclo}(O)$ always belong to $\mathsf{Hemi}(O)$ for any given ordered set O.

Example 3.1.2. *Let $O = \{a, b, c\}$ be a poset with $a \leq b \leq c$. Then* $\mathsf{Biclo}(O)$ *has 8 different elements, as the following table illustrates:*

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
(a,b)	a	a	a	a	a	b	b	b
(a,c)	a	a	b	b	c	b	b	c
(b,c)	b	c	b	c	c	b	c	c

The lattice $\mathsf{Biclo}(O)$ *can be represented by the diagram*



In this example we find that f_2 and f_7 are (HE), f_3 and f_5 are (MI). While f_4 is not (ID) nor (WH) and $\mathsf{Hemi}(O) = \{f_1, f_6, f_8\}$.

Theorem 3.1.3. *Let L be a bounded lattice. Then there is a specified bijection (described in the proof of the theorem) between L and* Hemi*(L) if and only if L is modular.*

Proof. ($\Leftarrow$) First step: let $\alpha = \{(x, y) \in L^2 | x \leq y\}$ and for any $a \in L$ define a map $f_a \colon \alpha \longrightarrow L$ as follows:

$$f_a(x, y) = (a \vee x) \wedge y.$$

It is clear that $x \leq (a \vee x) \wedge y \leq y$. Also, f_a is an ordered preserving map. That is, f_a is a binary closure operator.

Second step: we want to prove that f_a is an element of $\mathsf{Hemi}(L)$.

For any $x \leq y \leq z$ in L, we will show that $f_a(x, y) = f_a(x, z) \wedge y$ and $f_a(y, z) = f_a(x, z) \vee y$. By the definition of f_a and since L is modular, we obtain

$$f_a(x, z) \wedge y = [(a \vee x) \wedge z] \wedge y = (a \vee x) \wedge y = f_a(x, y).$$

Now since $x \leq z$ and L is modular, we obtain

$$f_a(x, z) \vee y = [(a \vee x) \wedge z] \vee y = (a \wedge z) \vee x \vee y = (a \vee y) \wedge z = f_a(y, z).$$

Third step: define a map $\varphi \colon L \longrightarrow \mathsf{Hemi}(L)$ by $\varphi(a) = f_a$ for all $a \in L$. To prove φ is bijective, suppose that 0 and 1 are the bottom element and top element of L, respectively. Now for any $a \neq b$ in L we have

$$f_a(0, 1) = a \neq b = f_b(0, 1),$$

i.e φ is injective.

On the other hand, let $g \in \mathsf{Hemi}(L)$, and suppose that $g(0, 1) = a$. Now, for any $x \leq y$ in L we have

$$\begin{aligned}
g(x, y) &= g(0, y) \vee x && \text{(Since } g \text{ is (MI) and } 0 \leq x \leq y) \\
&= (g(0, 1) \wedge y) \vee x && \text{(Since } g \text{ is (HE) and } 0 \leq y \leq 1) \\
&= (a \wedge y) \vee x \\
&= x \vee (a \wedge y) \\
&= (x \vee a) \wedge y && \text{(by modularity)} \\
&= (a \vee x) \wedge y \\
&= f_a(x, y).
\end{aligned}$$

This shows that φ is onto.

($\Rightarrow$) For any $a, z \in L$, since f_z is (HE) we have

$$f_z(0, a) = f_z(0, 1) \wedge a = z \wedge a,$$

and since f_z is (MI) and $0 \leq a \leq 1$, we obtain

$$f_z(a, 1) = f_z(0, 1) \vee a = z \vee a.$$

Now, for any $x \leq y \in L$, and since f_z is (MI) we obtain

$$f_z(x, y) = f_z(0, y) \vee x = (z \wedge y) \vee x = x \vee (z \wedge y),$$

for all $z \in L$.

Since f_z is (HE) and $x \leq y \leq 1$ we obtain

$$f_z(x, y) = f_z(x, 1) \wedge y = (z \vee x) \wedge y = (x \vee z) \wedge y,$$

for all $z \in L$. Therefore,

$$x \leq y \quad \Rightarrow \quad (x \vee z) \wedge y = x \vee (z \wedge y).$$

□

Theorem 3.1.4. *Let L be a lattice and for any $x \in L$ define a binary closure operator as follows*

$$f_x(a, b) = a \vee (x \wedge b)$$

for all $a \leq b$ in L. Then L is distributive if and only if

$$f_{x \vee y} = f_x \vee f_y$$

for all $x, y \in L$.

Proof. ($\Rightarrow$) For any elements $x, y \in L$, we have

$$\begin{aligned} f_x(a, b) \vee f_y(a, b) &= \big(a \vee (x \wedge b)\big) \vee \big(a \vee (y \wedge b)\big) \\ &= a \vee \big((x \wedge b) \vee (y \wedge b)\big) \\ &= a \vee \big((x \vee y) \wedge b\big) \\ &= f_{x \vee y}(a, b). \end{aligned}$$

($\Leftarrow$) Let x, y, z be any elements of the lattice L. Then

$$f_{x \vee y}[(x \wedge y \wedge z), z] = (x \wedge y \wedge z) \vee [(x \vee y) \wedge z] = (x \vee y) \wedge z,$$

and

$$f_x[(x \wedge y \wedge z), z] \vee f_y[(x \wedge y \wedge z), z] = (x \wedge z) \vee (y \wedge z).$$

Consequently, $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. □

Dually, we have:

Corollary 3.1.5. *Let L be a lattice. Consider the binary closure operators*

$$f_x(a, b) = b \wedge (x \vee a)$$

for all $a \leq b$ in L and each $x \in L$. The following are equivalent:

(1) L is distributive;

(2) $f_{x \wedge y} = f_x \wedge f_y$ *for all* $x, y \in L$.

Theorem 3.1.6. *Let* L *be a lattice and consider the binary closure operators* $f_x(a, b) = (a \vee x) \wedge b$. *Then* L *is distributive if and only if* L *is modular and*

$$f_c(a, a \vee b) = f_{c \wedge b}(a, a \vee b)$$

for any a, b, c *in* L.

Proof. ($\Rightarrow$) From Corollary 3.1.5 we have that

$$\begin{aligned} f_{c \wedge b}(a, a \vee b) &= f_c(a, a \vee b) \wedge f_b(a, a \vee b) \\ &= [(a \vee c) \wedge (a \vee b)] \wedge [(a \vee b) \wedge (a \vee b)] \\ &= [(a \vee c) \wedge (a \vee b)] \wedge (a \vee b) \\ &= (a \vee c) \wedge (a \vee b) \\ &= f_c(a, a \vee b). \end{aligned}$$

($\Leftarrow$) For any a, b, c in L, we have

$$\begin{aligned} a \vee (c \wedge b) &= a \vee [(c \wedge b) \wedge (a \vee b)] \\ &= [a \vee (c \wedge b)] \wedge (a \vee b) \\ &= f_{c \wedge b}(a, a \vee b) \\ &= f_c(a, a \vee b) \\ &= (a \vee c) \wedge (a \vee b). \end{aligned}$$

□

Dualizing this theorem, we obtain:

Corollary 3.1.7. *Let* L *be a lattice and consider the binary closure operators* $f_x(a, b) = a \vee (x \wedge b)$. *Then* L *is distributive if and only if* L *is modular and*

$$f_c(a \wedge b, a) = f_{c \vee b}(a \wedge b, a)$$

for all a, b, c *in* L.

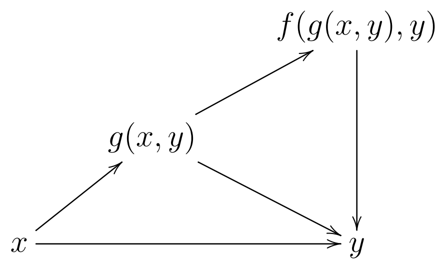
3.2 Composition and cocomposition of binary closure operators

In this section we show that the lattice structure of $\mathsf{Biclo}(O)$, for a given lattice O, and the algebraic structure given by composition and cocomposition of binary closure operators defined below, are compatible.

Definition 3.2.1. *The* composite *of two binary closure operators f and g is defined by setting*

$$fg(x,y) = f(g(x,y),y)$$

for each $x \leq y$ in O.



Lemma 3.2.2. *Let f, g be binary closure operators. Then fg is a binary closure operator. The composition supplies $\mathsf{Biclo}(O)$ with the structure of a monoid with zero and it is compatible with the lattice structure. More specifically, we have the following rules*

(1) $f \vee g \leq fg$,

(2) $(fg)h = f(gh)$,

(3) $\top f = \top = f\top$ $\quad$ ($\top$ *is the top element of* $\mathsf{Biclo}(O)$),

(4) $\bot f = f = f\bot$ $\quad$ ($\bot$ *is the bottom element of* $\mathsf{Biclo}(O)$),

(5) $f \leq g \Rightarrow fh \leq gh$ and $hf \leq hg$,

(6) $\left(\bigwedge_{i\in I} f_i\right) g = \bigwedge_{i\in I}(f_i g)$ and $\left(\bigvee_{i\in I} f_i\right) g = \bigvee_{i\in I}(f_i g)$ *(for $I \neq \emptyset$)*,

(7) $g(\bigwedge_{i\in I} f_i) \leq \bigwedge_{i\in I}(gf_i)$ and $g(\bigvee_{i\in I} f_i) \geq \bigvee_{i\in I}(gf_i)$.

Proof. (1) It is clear that $f(x,y) \leq f(g(x,y),y)$ and $g(x,y) \leq f(g(x,y),y)$. Therefore, $f \vee g \leq fg$.

(2) For any $f, g, h \in \mathsf{Biclo}(O)$, we have

$$\begin{aligned}((fg)h)(x,y) &= (fg)(h(x,y),y)\\ &= f(g(h(x,y),y))\\ &= f(gh(x,y),y)\\ &= (f(gh))(x,y).\end{aligned}$$

(3) $\top f(x,y) = \top(f(x,y),y) = y = \top(x,y) = y = f(y,y) = f(\top(x,y),y) = f\top(x,y)$.

(4) $\perp f(x,y) = \perp(f(x,y),y) = f(x,y) = f(\perp(x,y),y) = f\perp(x,y)$.

(5) Suppose that $f \leq g$. Then $fh(x,y) = f(h(x,y),y) \leq g(h(x,y),y) = gh(x,y)$.

To prove (6) & (7) consider a family $(f_i)_{i\in I}$ of binary closure operator.

(6) For all $i \in I$, we have that $(f_i g)(x,y) = f_i(g(x,y),y)$, so that

$$\bigwedge_{i\in I}(f_i g)(x,y) = \bigwedge_{i\in I} f_i(g(x,y),y) = ((\bigwedge_{i\in I} f_i)g)(x,y).$$

(7) From the fact $\bigwedge_{i\in I} f_i \leq f_j$ and by (5) we obtain $g(\bigwedge_{i\in I} f_i) \leq gf_j$ for all $j \in I$, so that

$$g\left(\bigwedge_{i\in I} f_i\right) \leq \bigwedge_{i\in I}(gf_i).$$

On the other hand we have $\bigvee_{i\in I} f_i \geq f_j$ for all $j \in I$. By the second inequality of (5) we obtain $g\left(\bigvee_{i\in I} f_i\right) \geq gf_j$ for all $j \in I$. Hence we obtain

$$g\left(\bigvee_{i\in I} f_i\right) \geq \bigvee_{i\in I}(gf_i).$$

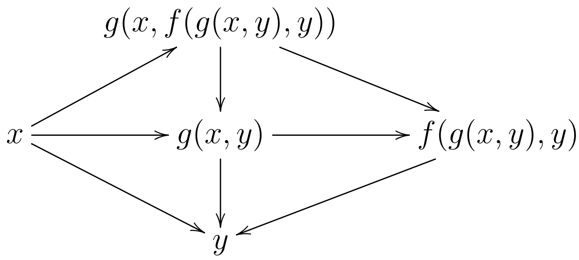
□

Proposition 3.2.3. *A binary closure operator f is*

$$(ID) \quad \Leftrightarrow \quad ff \leq f$$

Proposition 3.2.4. *If f and g are weakly hereditary, then fg is also weakly hereditary.*

Proof. We want to show that $fg(x, fg(x, y)) = fg(x, y)$. From the following diagram



and since g is (WH), hence by Theorem 2.2.12 we obtain $g(x, f(g(x, y), y)) = g(x, y)$. Also, since f is (WH) and $fg(x, y) \leq fg(x, y) \leq y$, hence by using Theorem 2.2.12 we obtain $f(g(x, y), fg(x, y)) = fg(x, y)$. Therefore, we have

$$\begin{aligned} fg(x, fg(x, y)) &= f(g(x, fg(x, y)), fg(x, y)) \\ &= f(g(x, y), fg(x, y))) \\ &= fg(x, y). \end{aligned}$$

□

Proposition 3.2.5. *If both f and g are minimal, then fg is also minimal.*

Proof. Suppose that f and g are minimal. Therefore we have

$$\begin{aligned} (fg)(b, c) &= f(g(b, c), c) \\ &= f(g(a, c), c) \vee g(b, c) && \text{(since } g(a, c) \leq g(b, c) \leq c \text{ and } f \text{ is (MI))} \\ &= (fg)(a, c) \vee (g(a, c) \vee b) && \text{(since } g \text{ is (MI))} \\ &= (fg)(a, c) \vee b. && \text{(since } g \leq fg) \end{aligned}$$

Therefore, fg is minimal. □

Proposition 3.2.6. *Let f be a minimal binary closure operator. Then for any binary closure operator g,*

$$fg = f \vee g.$$

Proof. For $a \leq b$ we have $a \leq g(a, b) \leq b$. Since f is (MI), we obtain

$$f(g(a, b), b) = f(a, b) \vee g(a, b).$$

That is $fg(a, b) = (f \vee g)(a, b)$. □

Corollary 3.2.7. *Let f be a minimal binary closure operator. Then*

$$f(g \vee h) = fg \vee fh.$$

Proposition 3.2.8. *If f and g are additive, then fg is additive.*

Proof. Suppose that f and g are additive and $a \leq c$ and $b \leq c$. Then

$$\begin{aligned}(fg)((a \vee b), c) &= f(g((a \vee b), c), c) \\ &= f((g(a, c) \vee g(b, c)), c) \\ &= f(g(a, c), c) \vee f(g(b, c), c) \\ &= fg(a, c) \vee fg(b, c).\end{aligned}$$

□

Definition 3.2.9. *For a binary closure operator f, we say that f satisfies the distributive law if*

$$f(g \vee h) = (fg) \vee (fh)$$

for any binary closure operators g, h.

Proposition 3.2.10. *If f is additive, then f satisfies the distributive law.*

Proof.

$$\begin{aligned}(f(g \vee h))(a, b) &= f(g(a, b) \vee h(a, b), b) \\ &= f(g(a, b), b)) \vee f(h(a, b), b) \\ &= fg(a, b) \vee fh(a, b).\end{aligned}$$

□

Example 3.2.11. *Let L be a lattice and $x \in L$. Then the binary closure operator*

$$f_x(a, b) = a \vee (x \wedge b)$$

is additive, because for any $a, b \leq c$ we have

$$\begin{aligned}f_x(a \vee b, c) &= (a \vee b) \vee (x \wedge c) \\ &= (a \vee (x \wedge c)) \vee (b \vee (x \wedge c)) \\ &= f_x(a, c) \vee f_x(b, c).\end{aligned}$$

Consequently, f_x satisfies the distributive law.

Example 3.2.12. *For a topological space X and $A \subseteq X$, the* Kuatowski closure *of A is defined by $\overline{A} = \bigcap\{F \subseteq X |\ A \subseteq F,\ F \text{ closed}\}$, while $\widehat{A} = \bigcap\{U \subseteq X |\ A \subseteq U,\ U \text{ open}\}$*

is called the inverse Kuratowski closure. *Now we define two binary closure operators on the power set of* X *as follows:* $f(A,B) = \overline{A} \cap B$, $g(A,B) = \widehat{A} \cap B$. *Both* f *and* g *are idempotent. The composite* gf *fails in general to be idempotent. Indeed, let* $X = \{a,b,c\}$ *with* $\emptyset, \{a\}, \{a,b\}, \{a,c\}, X$ *open. Let* $A = \{b\}$, $B = X$. *So,* $f(A,B) = A$, $g(A,B) = \widehat{A}$. *Now*

$$gf(A,B) = g(f(A,B),B) = g(A,B) = \widehat{A} = \{a,b\},$$

while

$$gf(gf(A,B),B) = gf(\widehat{A},B) = g(\overline{\widehat{A}},B) = B.$$

This gives $((gf)(gf))(A,B) \neq gf(A,B)$. *Furthermore,* f *and* g *as defined before are hereditary, but in general* gf *fails to be hereditary.*

Indeed, let $X = \{a,b,c\}$ *with* $\emptyset, \{b\}, \{c\}, \{b,c\}, X$ *open. Let* $A = \{b\}$, $B = \{b,c\}$, $C = X$. *Now we have*

$$gf(A,B) = g(f(A,B),B) = g(A,B) = \widehat{A} \cap B = A,$$

and

$$gf(A,C) = g(f(A,C),C) = g(\overline{A},C) = X.$$

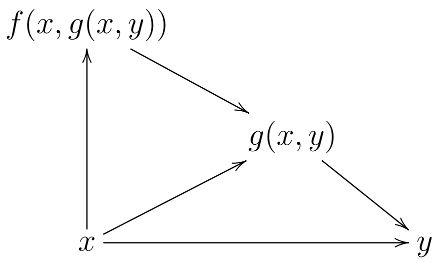
Hence,

$$gf(A,C) \cap B = B \neq A = gf(A,B).$$

The composite fg of two binary closure operators f and g is defined by mapping (a,b), where $a \leq b$, to f-closure of $g(a,b)$ in b; similarly, the cocomposite of two binary closure operators f and g is defined by mapping (a,b), where $a \leq b$, to f-closure of a in $g(a,b)$.

Definition 3.2.13. *The* cocomposite *of two binary closure operators* f *and* g *is given by:*

$$(f * g)(x,y) = f(x, g(x,y)).$$



It is clear that $f * g$ is an element of $\mathsf{Biclo}(O)$.

For binary closure operators f and g on a poset O, writing f^{op} for the induced binary closure operator on O^{op}, we have that $f * g = (f^{op} g^{op})^{op}$. Therefore, dualizing Lemma 3.2.2, we obtain:

Lemma 3.2.14. *Let O be a poset, then for any elements f,g and h of* $\mathsf{Biclo}(O)$ *one has the following rules:*

(1) $f * g \leq f \wedge g$,

(2) $(f * g) * h = f * (g * h)$,

(3) $\top * f = f = f * \top \quad (\top$ *is the top element of* $\mathsf{Biclo}(O))$,

(4) $\bot * f = \bot = f * \bot \quad (\bot$ *is the bottom element of* $\mathsf{Biclo}(O))$,

(5) $f \leq g \;\Rightarrow\; f * h \leq g * h$ and $h * f \leq h * g$ *(monotonicity),*

(6) $(\bigwedge_{i\in I} f_i) * g = \bigwedge_{i\in I}(f_i * g)$ *and* $(\bigvee_{i\in I} f_i) * g = \bigvee_{i\in I}(f_i * g)$ *(for* $I \neq \emptyset$*),*

(7) $g * (\bigwedge_{i\in I} f_i) \leq \bigwedge_{i\in I}(g * f_i)$ *and* $g * (\bigvee_{i\in I} f_i) \geq \bigvee_{i\in I}(g * f_i)$.

That is the cocomposition gives $\mathsf{Biclo}(O)$ *the structure of a monoid with zero which is compatible with its lattice structure.*

Proposition 3.2.15. *A binary closure operator f is*

$$(WH) \quad \Leftrightarrow \quad f \leq f * f$$

Dualizing Proposition 3.2.4, we have:

Proposition 3.2.16. *If both f and g are idempotent, then $f * g$ is also idempotent.*

Dualizing Proposition 3.2.5, we have:

Proposition 3.2.17. *If both f and g are hereditary, then also $f * g$ is hereditary.*

Dualizing Proposition 3.2.6, we have:

Proposition 3.2.18. *If f is (HE), then*

$$f * g = f \wedge g.$$

Corollary 3.2.19 (Cocomposite distributes over meet)**.** *If f is (HE), then*

$$f * (g \wedge h) = (f * g) \wedge (f * h)$$

for binary closure operators f, g, h.

Dualizing Proposition 3.2.8, we have:

Proposition 3.2.20. *If f and g are multiplicative, then $f * g$ is multiplicative. That is,*

$$(f * g)(a, b \wedge c) = (f * g)(a, b) \wedge (f * g)(a, c).$$

The next example shows that in general, if f and g are additive binary closure operators, then $f * g$ is not necessarily additive.

Example 3.2.21. *Let f and g two binary operators as defined in Example 3.2.12. Let $X = [0, 1] \cup \infty$ with the following topology: the unit interval $[0, 1]$ with its natural topology is an open subspace of X, and only neighbourhood of ∞ in X is X. Let $A = [0, \frac{1}{2})$ and $B = \{\infty\}$. Since $f(\{\frac{1}{2}\}, X) = \{\frac{1}{2}, \infty\}$, we obtain*

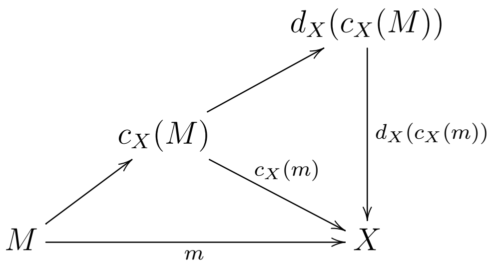
$$\frac{1}{2} \in (f * g)(A \cup B, X).$$

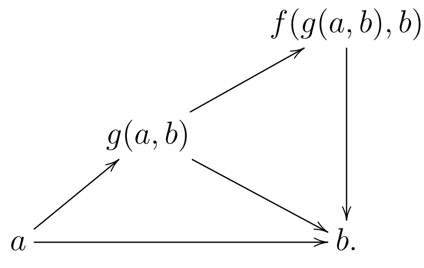
On the other hand, we have

$$(f * g)(A, X) = [o, \frac{1}{2}),$$
$$(f, g) = (B, X) = \{\infty\}.$$

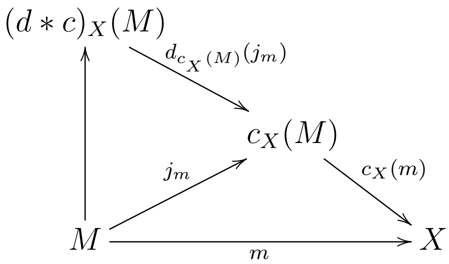
Therefore, $(f * g)(A \cup B, X) \neq (f * g)(A, X) \cup (f * g)(B, X)$.

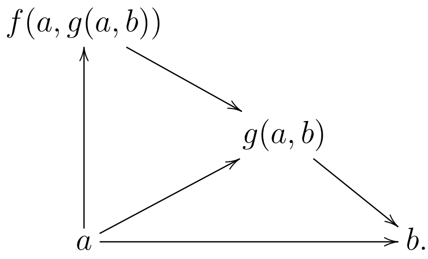
Now, we display the similarity between composition of two closure operators c, d and two binary closure operators f, g by the following diagrams:





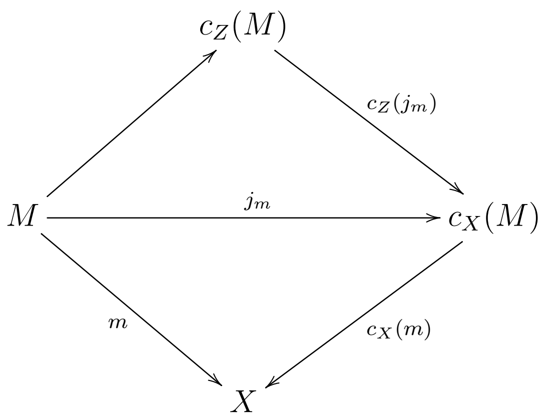
Again,we display the similarity between co-composition of two closure operators c, d and two binary closure operators f, g by the following diagrams:





Here $(d * c)_X(m) \cong c_X(m) \circ d_{c_X(M)}(j_m)$, whereas $(f * g)(a, b) = f(a, g(a, b))$.

Now we find a relation between binary closure operators and cocomposition of closure operators. For a morphism $m \in \mathcal{M}/X$ consider the diagram



We have:

$$\tilde{c}(m, c_X(m)) \cong c_X(m) \circ c_z(j_m) \cong (c * c)_X(m),$$

where $Z = c_X(M)$. Furthermore, we have

$$\tilde{c}(c_X(m), 1_X) \cong c_X(c_X(m))$$

for each element $m \in \mathcal{M}/X$ is exactly a closure of $c_X(m)$.

3.3 Combining composition and cocomposition

Combining Lemmas 3.2.2 (1) and 3.2.14 (1) we obtain:

Lemma 3.3.1. *For any two binary closure operators f, g on a poset, we have:*

$$f * g \leq f \wedge g \leq f \vee g \leq fg.$$

Proposition 3.3.2. *If g is idempotent, then $g(f * g) = g$ and (dually) if g is weakly hereditary, then $g * (fg) = g$.*

Proof. The following inequality is always true:

$$a \leq (f * g)(a, b) \leq g(a, b) \leq fg(a, b) \leq b.$$

Now, since g is (ID) we have:

$$g(a, b) \leq (g(f * g))(a, b) = g((f * g)(a, b), b) \leq g(g(a, b), b) = g(a, b).$$

On the other hand, g is (WH), so that we have:

$$g(a, b) = g(a, g(a, b)) \leq (g * fg)(a, b) = g(a, fg(a, b)) \leq g(a, b).$$

□

Proposition 3.3.3. *Let O be a distributive lattice. If f, g are minimal (hereditary), then $f \wedge g$ is minimal ($f \vee g$ is hereditary).*

Proof. Suppose $a \leq b \leq c$ for $a, b, c \in O$ and f, g are (MI):

$$\begin{aligned}(f \wedge g)(b, c) &= f(b, c) \wedge g(b, c) \\ &= [f(a, c) \vee b] \wedge [g(a, c) \vee b] \\ &= [f(a, c) \wedge g(a, c)] \vee b \qquad \text{(since } O \text{ is distributive.)} \\ &= (f \wedge g)(a, c) \vee b.\end{aligned}$$

□

Theorem 3.3.4. *For any binary closure operators f, g, h, k we have the inequality*

$$(f * g)(h * k) \leq (fh) * (gk).$$

Proof. Since $k \leq gk$, we obtain

$$h * k \leq h * (gk).$$

Also since $h * k \leq k$, we obtain

$$g(h * k) \leq gk.$$

So, for any $x \leq y$ of a given poset O we have

$$f[(h * k)(x, y), g(h * k)(x, y)] \leq f[(h * gk)(x, y), gk(x, y)].$$

The left side of the above inequality is

$$\begin{aligned} f[(h*k)(x,y), g(h*k)(x,y)] &= f[(h*k)(x,y), g((h*k)(x,y),y)] \\ &= (f*g)[(h*k)(x,y),y] \\ &= ((f*g)(h*k))(x,y). \end{aligned}$$

While the right side is

$$\begin{aligned} f[(h*gk)(x,y), gk(x,y)] &= f[(h(x,gk(x,y)), gk(x,y)] \\ &= fh[x, gk(x,y)] \\ &= (fh*gk)(x,y). \end{aligned}$$

That is $(f*g)(h*k) \leq (fh)*(gk)$. □

For categorical closure operators, the inequality

$$(C*D)(E*F) \leq (CE)*(DF)$$

is called a *lax middle-interchange law*. Theorem 3.4 in [15] explores situations in which the law holds strictly. We have analogous results for binary closure operators.

Proposition 3.3.5. *A binary closure operator f is (ID) and (WH) if and only if $(f*g)(h*k) = (fh)*(gk)$ for all binary closure operators g, h, k with $h \leq f \leq g$.*

Proof. ($\Rightarrow$) Suppose that f is (ID) and (WH). So, by Proposition 3.2.3 and 3.2.15, we have

$$fh \leq ff \leq f \leq f*f \leq f*g.$$

Consequently,

$$\begin{aligned} (fh)*(gk) &\leq fh \\ &\leq f \\ &\leq f(h*k) \\ &\leq (f*g)(h*k). \end{aligned}$$

($\Leftarrow$) Assume that $(f*g)(h*k) = (fh)*(gk)$ for all binary closure operators g, h, k with $h \leq f \leq g$. Put $g = f,\ h = \bot$ and $k = \top$. Consequently,

$$(f*f)(\bot*\top) = (f\bot)*(f\top),$$

so one has $f * f = f$, i.e f is (WH). Now put $h = f,\ g = \top$ and $k = \bot$. We obtain

$$(f * \top)(f * \bot) = (ff) * (\top\bot),$$

it follows that $f = ff$. □

Corollary 3.3.6. *Let f, g, h and k be binary closure operators. Then*

(1) g is (ID & WH) iff $(f * g)(h * k) = (fh) * (gk)$ for all f, h, k with $k \leq g \leq f$.

(2) h is (ID & WH) iff $(f * g)(h * k) = (fh) * (gk)$ for all f, g, k with $f \leq h \leq k$.

(3) k is (ID & WH) iff $(f * g)(h * k) = (fh) * (gk)$ for all f, g, h with $g \leq k \leq h$.

Proof. The proof is similar to the proof of Proposition 3.3.5 □

Now Proposition 3.3.2 becomes a part of the following corollary which is a special case of Proposition 3.3.5 and Corollary 3.3.6 .

Corollary 3.3.7. *Let f and g be a binary closure operators. Then the following are equivalent:*

(1) f is (ID) and (WH);

(2) for all g, $f(f * g) = f * fg$;

(3) for all g, $f(f * g) = fg * f$;

(4) for all g, $(f * g)f = f * gf$;

(5) for all g, $(g * f)f = gf * f$.

If any (and hence all) of the conditions (2)-(5) hold, then each of the composites which appears in of the identities of (2)-(5) is equal to f.

Proof. The condition (1) is a sufficient condition for (2)-(5) when we choose the comparable binary closure operators of Proposition 3.3.5 and Corollary 3.3.6 to be equal. Therefore in Proposition 3.3.5 we choose $h = g = f$, and so on.

On the other hand we can see that, each of the conditions (2)-(5) is a sufficient condition for (1) by choosing $g = \bot$ or $g = \top$. □

Corollary 3.3.8. *Let f, g and h be any binary closure operators.*

(1) If f is (WH), then $f(g*h) \leq (fg)*(fh)$.

(2) If f is (ID), then $(f*g)(f*h) \leq f*gh$.

Proof. (1) Suppose f is (WH). Since by Proposition 3.2.15 $f \leq f*f$, it follows by Theorem 3.3.4 that

$$f(g*h) \leq (f*f)(g*h) \leq (fg)*(fh).$$

(2) Immediate by duality of (1). ☐

3.4 Composite and cocomposite identities

In this section, we define non-trivial composite and co-composite identity binary closure operators for a given binary closure operator.

Definition 3.4.1. *Let f, g be binary closure operators. We call g a* composite identity *of f whenever*

$$fg = f = gf$$

and call g a cocomposite identity *of f whenever*

$$f*g = f = g*f.$$

Proposition 3.4.2. *For binary closure operators f, g, h. If g and h are composite (co-composite) identities of f, then gh and hg ($g*h$ and $h*g$) are also composite (co-composite) identities of f.*

Proof. By Lemma 3.2.2 (2) and Definition 3.4.1 we have

$$(gh)f = g(hf) = gf = f = fh = (fg)h = f(gh).$$

☐

Proposition 3.4.3. *Let f be an idempotent binary closure operator. Then for any binary closure operator g, such that $g \leq f$ we have*

$$gf = f = fg.$$

Proof. By Proposition 3.2.3 we obtain $f \leq gf \leq ff \leq f$. That is $gf = f$. Similarly, $f = fg$. ☐

Corollary 3.4.4. *Let f be an idempotent binary closure operator. Then for any binary closure operator g, we have that $g*f$ is a composite identity of f.*

Proof. This follows from the fact that $g * f \leq f$. □

Dualizing the previous proposition, we obtain:

Proposition 3.4.5. *Let f be a weakly hereditary binary closure operator. Then for any binary closure operator g, such that $f \leq g$, we have*

$$f * g = f = g * f.$$

Proposition 3.4.6 (Distributive laws)**.** *For any binary closure operators f,g,h, we have*

(1) $(gh) * f = (g * f)(h * f)$ if f is (ID).

(2) $(g * h)f = (gf) * (hf)$ if f is (WH).

Proof. (1) Since f is (ID), by Corollary 3.4.4 we have that $f(h * f) = f$. Now

$$\begin{aligned}((g * f)(h * f))(a, b) &= (g * f)((h * f)(a, b), b) \\ &= g((h * f)(a, b), f((h * f)(a, b), b)) \\ &= g((h * f)(a, b), f(h * f)(a, b)) \\ &= g((h * f)(a, b), f(a, b)) \\ &= g(h(a, f(a, b)), f(a, b)) \\ &= gh(a, f(a, b)) \\ &= (gh * f)(a, b).\end{aligned}$$

(2) is dual of (1). □

Definition 3.4.7. *Let f, g be binary closure operators. We say that f and g* commute under composition *if $fg = gf$, and say they* commute under co-composition *if $f * g = g * f$.*

Example 3.4.8. *For the poset $(\mathbb{R}, \leq)$, and for any $a, b \in \mathbb{R}$ such that $a \leq b$, define two binary closure operators on $\mathbb{R}$ as follows:*

$$f(a, b) = \frac{ma + nb}{n + m} \quad \text{and} \quad g(a, b) = \frac{va + ub}{u + v}$$

for fixed numbers $u, v, m, n \in \mathbb{N}$ such that $n + m \neq 0$ and $u + v \neq 0$. Now we show that f

and g commute under co-composition

$$\begin{aligned}(g * f)(a,b) &= g(a, f(a,b))\\ &= \frac{va + uf(a,b)}{u+v}\\ &= \frac{va + \frac{u(ma+nb)}{n+m}}{u+v}\\ &= \frac{v(n+m)a + uma + unb}{(u+v)(n+m)}\\ &= \frac{(u+v)ma + n(va+ub)}{(u+v)(n+m)}\\ &= \frac{ma + \frac{n(va+ub)}{u+v}}{n+m} = (f * g)(a,b).\end{aligned}$$

In the same way, we find that

$$fg(a,b) = \frac{\frac{m(va+ub)}{u+v} + nb}{n+m} = gf(a,b).$$

In the next example, we give two binary closure operators which commute under co-composition but do not commute under composition.

Example 3.4.9. *For a topological space X, and for any subsets $A \subseteq B \subseteq X$, one can define two comparable binary closure operators as follows:*

$$f(A,B) = \overline{A} \cap B \quad \text{and} \quad g(A,B) = (\overline{A} \cap B) \cup B^{\circ}.$$

Since f is (WH) and $f \leq g$, we have $f * g = g = g * f$.

However, $fg \neq gf$. Indeed, let $X = \{1,2,3\}$ be a topological space with the open sets $\phi,\ X,\ \{1\},\ \{1,2\}$. Let $A = \phi,\ B = \{1,3\}$.

$$\begin{aligned}g(f(A,B),B) &= [\overline{f(A,B)} \cap B] \cup B^{\circ}\\ &= [\overline{(\overline{A} \cap B)} \cap B] \cup B^{\circ} = B^{\circ} = \{1\},\end{aligned}$$

$$\begin{aligned}f(g(A,B),B) &= \overline{[(\overline{A} \cap B) \cup B^{\circ}]} \cap B\\ &= \overline{B^{\circ}} \cap B = \{1,3\}.\end{aligned}$$

Proposition 3.4.10. *If f, g are hereditary (minimal) binary closure operators, then f and g commute under co-composition (composition).*

Proof. Let f, g be hereditary binary closure operators. Then by Proposition 3.2.18, we have

$$f * g = f \wedge g = g \wedge f = g * f.$$

We obtain the assertion involving minimality by duality. □

Example 3.4.11. *Let O be a poset, $f(X,Y) = \uparrow_Y X$ and $g(X,Y) = \downarrow_Y X$. Then by Propositions 2.3.14 and 3.4.10 we have*

$$(f * g)(X,Y) = \uparrow_{(\downarrow_Y X)} X = (\uparrow_Y X) \cap (\downarrow_Y X) = \downarrow_{(\uparrow_Y X)} X = (g * f)(X,Y).$$

Proposition 3.4.12. *Let L be a lattice and $g, h \in \mathsf{Hemi}(L)$. Then*

(1) $(g \vee h) * f = (g \wedge f)(h \wedge f)$ if f is (ID),

(2) $(g \wedge h) f = (g \vee f) * (h \vee f)$ if f is (WH).

Proof. (1) Since f is (ID), by Proposition 3.4.6 we obtain $(gh) * f = (g * f)(h * f)$. Now since g is (MI), $gh = g \vee h$. Since g and h are (HE), we obtain $(g * f)(h * f) = (g \wedge f)(h \wedge f)$.

(2) is the dual of (1). □

Proposition 3.4.13. *Let f be a weakly hereditary binary closure operator. If f commutes under composition with both g and h, then*

$$f(g * h) \leq (g * h) f.$$

Proof. Suppose f is (WH). Then by Corollary 3.3.8 and Proposition 3.4.6 we obtain

$$f(g * h) \leq (fg) * (fh) = (gf) * (hf) = (g * h) f.$$

□

Dualizing the previous proposition, we obtain:

Proposition 3.4.14. *If f is an idempotent binary closure operator and commutes under co-composition with both g and h, then*

$$(gh) * f \leq f * (gh).$$

3.5 Properties stable under joins or meets

We say a property P for binary closure operators is *stable under joins* when, if P holds for each element of a family $(f_i)_{i \in I}$ of binary closure operators (and the join $\bigvee_{i \in I} f_i$ exists), then the property P holds for the join $\bigvee_{i\in I} f_i$. A property which is stable under meets is defined in a similar way.

Proposition 3.5.1. *The following properties*

(1) Weak hereditaness, groundedness, additivity and minimality are stable under joins.

(2) Idempotency, hereditariness and multiplicativily are stable under meets.

Proof. (1) A binary closure operator g is (WH) iff $g \leq g * g$. Now by (7) of Lemma 3.2.14 we have

$$g * \left(\bigvee_{i\in I} f_i\right) \geq \bigvee_{i\in I}(g * f_i).$$

Hence for $g = \bigvee_{i\in I} f_i$, we obtain

$$\begin{aligned} g * g &\geq \bigvee_{i\in I}\left(\left(\bigvee_{i\in I} f_i\right) * f_i\right) \\ &\geq \bigvee_{i\in I}(f_i * f_i) \\ &\geq \bigvee_{i\in I} f_i = g. \end{aligned}$$

Whenever each f_i is (WH).

For groundedness, suppose that 0 is the bottom element of a given lattice O and each element of the family $(f_i)_{i\in I}$ is grounded, i.e $f_i(0, a) = 0$ for all $a \in O$. Hence $\bigvee_{i\in I} f_i(0, a) = 0$.

For additivity, for any $a, b \leq c$ in the given lattice O, suppose that f_i is an additive binary closure operator for all $i \in I$, i.e $f_i(a \vee b, c) = f_i(a, c) \vee f_i(b, c)$. Then

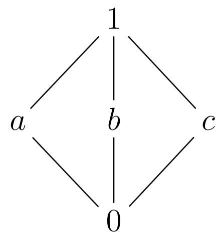
$$\begin{aligned} \bigvee_{i\in I} f_i(a \vee b, c) &= \bigvee_{i\in I} (f_i(a, c) \vee f_i(b, c)) \\ &= \left(\bigvee_{i\in I} f_i(a, c)\right) \vee \left(\bigvee_{i\in I} f_i(b, c)\right). \end{aligned}$$

For minimality, suppose that every element in the family $(f_i)_{i\in I}$ is minimal, that is for every $a \leq b \leq c$ in O, we have $f_i(b,c) = f_i(a,c) \vee b$. Then

$$\bigvee_{i \in I} f_i(b,c) = \bigvee_{i \in I} \big(f_i(a,c) \vee b\big) = \left(\bigvee_{i \in I} f_i(a,c)\right) \vee b.$$

(2) is dual of (1). □

Example 3.5.2. *Here we show that hereditariness is not in general stable under joins. Let L be a lattice represented by the following diagram:*



Now, define two binary closure operators, $f_a(x,y) = (a \vee x) \wedge y$ and $f_b(x,y) = (b \vee x) \wedge y$ for every $x \leq y$ in L. This lattice is modular, so by Theorem 3.1.3 we have that f_a and f_b are hereditary. Now from $0 \leq c \leq 1$, we show that $(f_a \vee f_b)(0,c) \neq (f_a \vee f_b)(0,1) \wedge c$.

$$(f_a \vee f_b)(0,c) = (a \wedge c) \vee (b \wedge c) = 0.$$

While

$$(f_a \vee f_b)(0,1) \wedge c = (a \vee b) \wedge c = 1 \wedge c = c.$$

This means $f_a \vee f_b$ is not (HE). Also, f_a and f_b are (MI), but $(f_a \wedge f_b)(c,1) = 1$ and $(f_a \wedge f_b)(0,1) \vee c = c$, i.e minimality is not stable under meet.

For a family $(f_i)_{i \in I}$ of hereditary binary closure operators, Proposition 3.5.2 shows that $\bigwedge_{i \in I} f_i$ is (HE), but in general $\bigvee_{i \in I} f_i$ is not (HE) as in Example 3.5.2. In the following proposition we give conditions under which the join (meet) of $(f_i)_{i \in I}$ is (HE) (respectively (MI)) whenever each f_i is (HE) (respectively (MI)).

Proposition 3.5.3. *Let L be a frame with a bottom element 0 and top element 1. Let $\mathsf{Heclo}(L)$ and $\mathsf{Miclo}(L)$ be sets of hereditary and minimal binary closure operators, respectively. Then $\mathsf{Heclo}(L)$ and $\mathsf{Miclo}(L)$ have the structure of a large complete lattice. In this case, $\mathsf{Heclo}(L) \cap \mathsf{Miclo}(L) = \mathsf{Hemi}(L)$ is also a complete lattice and there is an order preserving map $\varphi\colon L \to \mathsf{Hemi}(L)$, defined by $\varphi(a) = f_a$, when $f_a(x,y) = x \vee (a \wedge y)$, for all $a, x, y \in L$ with $x \leq y$.*

Proof. Let $(f_i)_{i\in I}$ be a family of hereditary binary closure operators. If $I = \emptyset$, then $\bigvee_{i\in I} f_i = \bot$ which is (HE). Since L is frame, we obtain

$$\bigvee_{i\in I} f_i(a,b) = \bigvee_{i\in I} \big(f_i(a,c) \wedge b\big) = \bigvee_{i\in I} f_i(a,c) \wedge b$$

for every $a \leq b \leq c$ in L. Consequently, $\mathsf{Hecol}(L)$ is a complete lattice. Dually for $\mathsf{Miclo}(L)$. □

Proposition 3.5.4. *Let $a \leq b$ in a given poset O. For arbitrary binary closure operators f and g,*

- *if a is closed in b under fg, then a is closed in b under both f and g.*
- *if a is dense in b under fg, then a is dense in b under both f and g.*

Proof. Suppose $fg(a,b) = a$. We know that $f \leq fg$ and $a \leq f(a,b)$. Therefore,

$$a \leq f(a,b) \leq fg(a,b) = a,$$

which means a is closed under f. Similarly, a is closed under g.

Dually, we get that if a is dense in b under $f * g$, then a is dense also under both f and g. □

In the next example we show that if an element a is dense in b under fg, then a is not necessarily dense in b under f.

Example 3.5.5. *Consider a topological space $X = \{a,b,c\}$ with $\emptyset, \{a\}, \{b\}, \{a,b\}, X$ open. Let $A = \{c\}$, $B = \{a,c\}$, $f(A,B) = \bar{A} \cap B$ and $g(A,B) = \widehat{A} \cap B$, where $\widehat{A}$ is defined as in Example 3.2.12. So we have*

$$fg(A,B) = f(g(A,B),B) = \bar{B} \cap B = B \neq f(A,B).$$

In addition, A is closed in B under $g * f$, but it is not under g.

Proposition 3.5.6. *Consider a complete lattice. For a given binary closure operator f, there is*

(1) a least idempotent binary closure operator $\hat{f}$ such that $f \leq \hat{f}$;

(2) a largest weakly hereditary binary closure operator $\check{f}$ such that $\check{f} \leq f$.

Proof. (1) For a given binary closure operator f, let $(f_i)_{i\in I}$ be a family of all idempotent closure operators such that $f \leq f_i$ for all $i \in I$. Since idempotency is stable under meets, we obtain that

$$\hat{f} = \bigwedge_{i\in I} f_i$$

is the least idempotent binary closure operator such that $f \leq \hat{f}$. If I is empty, then $\hat{f} = \top$.

(2) is dual of (1). □

3.6 Largest grounded binary closure operator

For a non-empty family of binary closure operators $(f_i)_{i\in I}$, we have: $\bigwedge_{i\in I} f_i$ is grounded if at least one binary closure operator f_i is grounded. If $I = \emptyset$, then $\bigwedge_{i\in I} f_i = \top$ which is not grounded unless a given lattice has one element. In Proposition 3.5.2, groundedness is stable under arbitrary join.

Proposition 3.6.1. *Let O be a complete lattice with a bottom element* 0*, let* $\mathsf{Gbclo}(O)$ *be the set of grounded binary closure operators. Then* $\mathsf{Gbclo}(O)$ *has the structure of a complete lattice. Non-empty meets and arbitrary joins are formed as in* $\mathsf{Biclo}(O)$*. The largest grounded binary closure operator in* $\mathsf{Gbclo}(O)$ *called the* indiscrete binary closure operator *and is defined by*

$$g(a,b) = \begin{cases} 0 & if\ a = 0 \\ b & o.w. \end{cases}$$

for all $a \leq b$ in O.

Proof. It is clear that $a \leq g(a,b) \leq b$ for every $a \leq b$ in O. Let $f \in \mathsf{Gbiclo}(O)$. We want to show that $f \leq g$. For any $a \leq b$ if $a = 0$, then $f(a,b) = 0 = g(a,b)$. If $a \neq 0$, then $a \leq f(a,b) \leq b = g(a,b)$. □

It is clear that the largest binary closure operator as defined above is (HE) and (ID).

Example 3.6.2. *In Example 3.1.2,* $\mathsf{Gbclo}(O) = \{f_1, f_2\}$.

Since groundedness is stable under arbitrary joins, for every binary closure operator f there is a largest grounded binary closure operator f^g, called *the grounding of* f, defined by $f^g = f \wedge g$, when g is the largest grounded binary closure operator as defined in the previous proposition.

Corollary 3.6.3. *Consider a complete poset. Let g be the largest binary closure operator. Then for a binary closure operator f, $f^g = f \wedge g$ is the largest grounded binary operator less than or equal f. If f is (ID) or (HE), then f^g has the same property.*

Proof. Since g is grounded, f^g is grounded. For any grounded binary closure operator $h \leq f$ we have $h \leq g$, therefore $h \leq f^g$. Since g is (ID) and (HE) and these properties are stable under meet, f^g is (ID) or (HE) as long as f is (ID) or (HE). □

For a complete poset, we have

$$\bot \leq g \leq \top.$$

3.7 Minimal core and hereditary hull

Proposition 3.5.2 shows that minimality is stable under joins. Therefore, for every binary closure operator f, there is a largest minimal binary closure operator less than or equal f.

Consider a lattice O with a bottom element 0 and a binary closure operator f. Then, for every minimal binary closure operator g such that $g \leq f$ we have

$$g(a, b) = g(0, b) \vee a \leq f(0, b) \vee a.$$

It is clear that

$$f^{mi}(a, b) = f(0, b) \vee a$$

is the largest binary closure operator less than or equal f. It is called the *minimal core* of f.

Proposition 3.7.1. *Let L be a complete lattice. Then for any $f \in \mathsf{Biclo}(L)$, f^{mi} is fully additive.*

Proof. For each $f \in \mathsf{Biclo}(L)$, we have

$$\begin{aligned} f^{mi}\left(\bigvee_{i\in I} a_i, b\right) &= f(0,b) \vee \left(\bigvee_{i\in I} a_i\right) \\ &= \bigvee_{i\in I} \left(f(0,b) \vee a_i\right) \\ &= \bigvee_{i\in I} f^{mi}(a_i, b). \end{aligned}$$

□

Proposition 3.7.2. *Consider a lattice L with a bottom element 0. If a binary closure operator f is weakly hereditary, then f^{mi} is weakly hereditary.*

Proof. Suppose that f is weakly hereditary. For any $a \leq b$ we have

$$f(0,b) \leq f(0,b) \vee a \leq b.$$

By taking closure of the bottom element 0 in every part in the above inequality we obtain

$$f(0, f(0,b)) \leq f(0, f(0,b) \vee a) \leq f(0,b).$$

Therefore,

$$f(0, f(0,b) \vee a) = f(0,b).$$

Now we want to show that f^{mi} is (WH). For any $a \leq b$ we have

$$\begin{aligned} \left(f^{mi} * f^{mi}\right)(a,b) &= f^{mi}(a, f^{mi}(a,b)) \\ &= f(0, f^{mi}(a,b)) \vee a \\ &= f(0, f(0,b) \vee a) \vee a \\ &= f(0,b) \vee a \\ &= f^{mi}(a,b). \end{aligned}$$

□

Proposition 3.7.3. *Let L be a modular lattice with a bottom element. If a closure operator f is hereditary, then f^{mi} is hereditary.*

Proof. For any $a \leq b \leq c$ in L we have

$$\begin{aligned} f^{mi}(a,c) \wedge b &= (f(0,c) \vee a) \wedge b \\ &= (f(0,c) \wedge b) \vee a \\ &= f(0,b) \vee a \\ &= f^{mi}(a,b). \end{aligned}$$

□

Proposition 3.5.2 shows that hereditariness is stable under meets, that is, for any binary closure operator f there is a least hereditary binary closure operator which is greater than or equal f. It is called *the hereditary hull* of f.

Consider a lattice L with top element 1, and a binary closure operator f. Then any hereditary binary closure operator g such that $g \geq f$ must satisfy

$$g(a,b) = g(a,1) \wedge b \geq f(a,1) \wedge b.$$

So,

$$f^{he}(a,b) = f(a,1) \wedge b$$

is the hereditary hull of f.

Dualizing Proposition 3.7.3, we have:

Proposition 3.7.4. *Let L be a modular lattice with a top element. If a closure operator f is minimal, then f^{he} is minimal.*

Dualizing Proposition 3.7.2, we have:

Proposition 3.7.5. *Consider a lattice L with a top element. Then a binary closure operator f is idempotent if and only if f^{he} is idempotent.*

Theorem 3.7.6. *Let L be a complete lattice. Then L is modular if and only if*

$$\left(f^{mi}\right)^{he} = \left(f^{he}\right)^{mi}$$

for all $f \in \mathsf{Biclo}(L)$.

Proof. (⇒) Suppose L is modular

$$\begin{aligned}\left(f^{mi}\right)^{he}(a,b) &= f^{mi}(a,1)\wedge b\\ &= (f(0,1)\vee a)\wedge b\\ &= (f(0,1)\wedge b)\vee a\\ &= f^{he}(0,b)\vee a\\ &= \left(f^{he}\right)^{mi}(a,b).\end{aligned}$$

(⇐) For any $c \in L$ we can define a binary closure operator f such that $f(0,1) = c$. That is,

$$a \leq b \quad\Rightarrow\quad \begin{aligned}[t] a\vee(c\wedge b) &= a\vee(f(0,1)\wedge b)\\ &= a\vee f^{he}(0,b)\\ &= \left(f^{he}\right)^{mi}(a,b)\\ &= \left(f^{mi}\right)^{he}(a,b)\\ &= f^{mi}(a,1)\wedge b\\ &= (f(0,1)\vee a)\wedge b\\ &= (a\vee c)\wedge b.\end{aligned}$$

□

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