The nonvanishing of almost-prime twists of modular L-functions

by

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Abstract

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L-functions are special types of Dirichlet series which often hold fundamental arithmetic information. Hence, they are among the most important objects in analytic number theory. In this thesis, we consider the so-called Hecke L-function $L(s, f, \chi_d)$ associated to a given normalized holomorphic newform f twisted by the Kronecker symbol χ_d. It is well known that the twisted $L(s, f, \chi_d)$ converges absolutely for $\mathrm{Re}(s) > 1$ and admits a functional equation which extends it analytically to the whole complex plane. The value of $L(s, f, \chi_d)$ at $s = 1/2$ is of special interest. For instance, if the form f parametrizes a twisted elliptic curve E of given rank $r \geq 0$, then the Birch-Swinnerton-Dyer conjecture asserts that r is precisely the order of vanishing of $L(s, f, \chi_d)$ at $s = 1/2$.

In this work, we fix a holomorphic newform f of weight at least 2, level N with trivial nebentype and consider the family of twisted L-functions $L(s, f, \chi_d)$ where d is any fundamental discriminant with $(d, N) = 1$. Using an adaptation of a method by Iwaniec, we prove that there are infinitely many fundamental discriminants d such that $L(1/2, f, \chi_d) \neq 0$. In addition, following an idea outlined by Hoffstein and Luo, using combinatorial sieve, we prove that the same holds for infinitely many almost-prime fundamental discriminants d with at most 84 prime factors. Further improvement of this result, which relies on properties of some multiple-Dirichlet series, is also discussed in this work. Under some assumptions on certain weight factors, it is possible to reduce the number 84 to just 4.

Uittreksel

Die nie-verdwynendheid van byna-prime draaie van modulêre L-funksies

("The nonvanishing of almost-prime twists of modular L-functions ")

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L-funksies is spesiale tipes Dirichlet-reekse wat dikwels fundamentele aritmetiese inligting bevat. Daarom is hulle een van die belangrikste objekte in analitiese getalteorie. In hierdie tesis ondersoek ons die sogenaamde Hecke L-funksie $L(s, f, \chi_d)$ wat geassosieer word met 'n gegewe genormaliseerde holomorfe nuwe vorm f wat deur die Kronecker-simbool χ_d verdraai is. Dit is algemeen bekend dat die verdraaide $L(s, f, \chi_d)$ absoluut konvergeer vir $\text{Re}(s) > 1$ en 'n funksionele vergelyking het wat dit analities tot die hele komplekse vlak uitbrei. Die waarde van $L(s, f, \chi_d)$ by $s = 1/2$ is van besondere belang. Byvoorbeeld, as die vorm f 'n verdraaide elliptiese kromme E van 'n gegewe rang $r \geq 0$ parametriseer, beweer die Birch-Swinnerton-Dyer-vermoede dat r presies die orde van nulstelling van $L(s, f, \chi_d)$ by $L(s, f, \chi_d)$ is.

In hierdie werk, bepaal ons 'n holomorfe nuwe vorm f van gewig van ten minste 2, met 'n vlak N met 'n triviale nebentipe, en ons oorweeg die familie van verdraaide L-funksies $L(s, f, \chi_d)$ waar d enige fundamentele diskriminant met $(d, N) = 1$ is. Deur 'n aanpassing van 'n metode deur Iwaniec, bewys ons dat daar oneindig baie fundamentele diskriminante d is sodat $L(1/2, f, chi_d) \neq 0$. Daarbenewens bewys ons, volgens 'n idee deur Hoffstein en Luo, deur gebruik te maak van 'n kombinatoriese sif, dat dieselfde waar is vir oneindig baie bykans-primêre fundamentele diskriminante d met hoogstens 84 primêre faktore. Verdere verbetering van hierdie resultaat, wat berus op eienskappe van sekere multiple-Dirichlet-reekse, word ook in hierdie werk bespreek. Onder sekere aanannames oor sekere gewigsfaktore, is dit moontlik om die getal 84 tot net 4 te verminder.

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Dedication

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Chapter 1

Introduction

1.1 Historical background and statement of the main result

Even though this work belongs entirely to the analytic world of L-functions, its motivation comes from the world of elliptic curves. So, let us start from the latter. The Birch-Swinnerton-Dyer (BSD) conjecture is one of the most profound unsolved problems in number theory. This conjecture digs into the deep domain of elliptic curves and was first put up in the early 1960s by Bryan Birch and Peter Swinnerton-Dyer. Briefly, an elliptic curve E over the rational field $\mathbb{Q}$ is a smooth curve given by an equation of the form

$$E : y^2 = x^3 + Ax + B. \tag{1.1}$$

With an additional point called "*point at infinity*" and a suitable operation, one can define an abelian group $E(\mathbb{Q})$, called the Mordell-Weil group, on the rational points of E; the reader can consult [1] for more background on this. Moreover, it well known that [2]

$$E(\mathbb{Q}) \simeq E(\mathbb{Q})_{Tor} \times \mathbb{Z}^r,$$

where $E(\mathbb{Q})_{Tor}$ is a finite group with order no greater than 12 and r is a finite positive integer called the rank of E. One can define the (normalized) L-function $L(s,E)$ associated to an elliptic E by the following Euler product:

$$L(s,E) = \prod_{p|N}(1-a_p p^{-1/2-s})^{-1}\prod_{p\nmid N}(1-a_p p^{-1/2-s}+p^{-2s})^{-1} = \sum_{n}^{\infty}\frac{A_n}{n^s}, \tag{1.2}$$

where N is the conductor of E and $a_p = p+1-\#E(\mathbb{F}_p)$. Then, the BSD conjecture claims that (see [3])

$$r = \operatorname*{Ord}_{s=1/2} L(s,E).$$

In particular, the conjecture claims that if the $L(s,E)$ vanishes at $s = 1/2$, then the curve has infinitely many rational points. This conjecture has profound implications for the understanding of Diophantine equations, which seek integer or rational solutions to polynomial equations, making it a central focus of research in number theory.

We can start to approach this problem by studying a specific family of elliptic curves. Let E be the elliptic curve defined in (1.1), D be a fundamental discriminant and χ_D be the Kronecker character for the field $\mathbb{Q}(\sqrt{D})$. Let us consider the *twisted* elliptic curve E_D given by

$$E_D: \quad y^2 = x^3 + D^2Ax + D^3B. \tag{1.3}$$

It is well known that the L-function associated to E_D is

$$L(s,E,\chi_D) = \sum_{n=1}^{\infty} \frac{A_n\chi_D(n)}{n^s}. \tag{1.4}$$

The study of such family of elliptic curves are important. For instance, they appear in the study of congruent numbers [4], where the elliptic curves of the form

$$y^2 = x^3 - n^2x$$

occur. Moreover, J. Silvermann also brings the following conjecture

Conjecture 1. *([5]) Let $E(\mathbb{Q})$ be an elliptic curve. Then, there exist infinitely many primes p such that $E_p(\mathbb{Q})$ or $E_{-p}(\mathbb{Q})$ has rank 0.*

For example, in [6], Ono and Skinner proved that if E has conductor less than 100 then Conjecture 1 is true.

On the other hand, the well-celebrated Modularity theorem , proved by Wiles, Breuil, Conrad, Diamond, and Taylor in [7], which also plays a crucial role in the proof of Fermat's last theorem, claims that each elliptic curve E corresponds to a modular form f_E such that

$$L(s,E) = L(s,f_E).$$

Hence, if the BSD conjecture holds for any elliptic curve, then Conjecture 1 can be studied through the non-vanishing of the central value $L(1/2, f_E, \chi_{\pm p})$. Therefore, it is natural to ask for any large positive number X, if the set

$$\{L(1/2, f_E, \chi_p) : |p| < X, \quad p \text{ prime}\}$$

contains some values that do not vanish. To approach this problem, one can compute the average

$$\sum_{\substack{|p|<X \\ |p| \text{ prime}}} L(1/2, f_E, \chi_p).$$

If this sum is nonzero, then there must be at least one nonzero term, and if the sum goes to ∞ as $X \to \infty$, then $L(1/2, f_E, \chi_p)$ does not vanish for infinitely many p such that $|p|$ is prime. The sum over primes is quite hard to estimate, but it turns out that it is possible to obtain an estimate of a similar sum over *almost-primes*, *i.e.,* integers with bounded prime factors. The method was outlined in 8-page long paper by Hoffstein and Luo [8] which combines a method of Iwaniec [9], the theory of multiple Dirichlet series, and the combinatorial sieve method. However, very little detail was given in [8]. So, our purpose in this work is to follow the idea outlined by Hoffstein and Luo as well as fill in the details that were left out.

In this thesis, we do not only consider modular forms that are associated to elliptic curves but any newform f of weight greater or equal to 2 (note that forms associated to elliptic curves are of weight 2). The problems stated above remain valid in this general case. Our main result states as follows:

Theorem 1.1. *Let f be a newform of weight k, level N and with trivial nebentype and S be a finite set of prime numbers. Then, there exist infinitely many odd fundamental discriminants d_0 such that*

$$L(1/2, f, \chi_{d_0}) \neq 0,$$

such that d_0 has at most 84 prime factors and $\chi_{d_0}(p) = 1$ for $p \in S$. Moreover, under some assumptions on certain weight factors described in Assumption A1 and Assumption A2, the same holds if d_0 is constrained to have at most 4 prime factors.

As mentioned previously, proving this result requires a combination of different techniques. Each of these techniques will be discussed in this thesis. We organize this thesis as follows:

- The rest of this chapter is devoted to a brief introduction to newforms and the Rankin-Selberg theory.

- In Chapter 2, we provide an adaptation of a method by Iwaniec [9] to estimate various sums similar to (1.1).

- In Chapter 3, we will consider a multiple Dirichlet series of the form

$$\sum \frac{L(s, f, \chi_d) P_d(s)}{d^w},$$

  and we prove the analytic continuation of such under certain conditions on the weight factor $P_d(s)$. The main ideas are taken from a work of [10] adapted to the case of $GL(2)$ forms.

- In Chapter 4, we use a sieve method and combine the relevant results from different chapters to finally prove Theorem 1.1.

- In Chapter 5, we discuss the possibility of removing the artificial assumptions that we have made on the weight factors. This would prove unconditionally that the number 84 in Theorem 1.1 can indeed be reduced to 4. But this is left as a future project as significant work still needs to be done to achieve such a result. Further, extensions are also discussed

1.2 Basic foundation of newforms

In this section, we will present all the properties of newforms that are necessary for this project. Newforms are normalized eigenforms that are orthogonal under the Petersson inner product to any $g(dz)$, where g is a cusp form of level $N'|N$, $N' \neq N$ and $d|N/N'$. Let us start by stating few key known properties without proof. Let $f(z) = \sum_{n\geq 1} \lambda_n n^{(k-1)/2} e(nz)$ be a newform of even weight $k \geq 2$, level N with trivial nebentype χ_0. Then, the coefficients $(\lambda_n)_n$ satisfy the following :

Property 1.2. *$\lambda_1 = 1$ and $\lambda_n \in \mathbb{R}$ for all n,*

Property 1.3. *for any n, m,*

$$\lambda_n \lambda_m = \sum_{d|(m,n)} \chi_0(d) \lambda_{nm/d^2}, \tag{1.5}$$

and in particular, we have the multiplicative property

$$\lambda_n \lambda_m = \lambda_{nm} \text{ if } (n,m) = 1, \tag{1.6}$$

and

$$\lambda_{p^\nu} = \lambda_{p^{\nu-1}} \lambda_p - \chi_0(p) \lambda_{p^{\nu-2}}, \tag{1.7}$$

for all prime p and $\nu \geq 2$,

Property 1.4. *(Deligne's bound)*

$$|\lambda_n| \leq \tau(n) \tag{1.8}$$

for all n where τ is the divisor function,

Property 1.5.

$$\sum_{n\leq X} |\lambda_n|^2 n^{k-1} \ll X^k, \tag{1.9}$$

Property 1.6.

$$\sum_{n\leq X} \lambda_n n^{(k-1)/2} e(\alpha n) \ll X^{k/2} \log X, \tag{1.10}$$

uniformly for $\alpha \in \mathbb{R}$.

More details on newforms can be found in [11]. We can then define the L-function associated to this newform by

$$L(s,f)=\sum_{n=1}^{\infty}\lambda_n n^{-s}.$$

The condition in (1.8) shows that $L(f,s)$ is absolutely convergent for $\mathrm{Re}(s)>1$, while (1.7) can be used to prove the following Euler product:

$$L(s,f)=\prod_p\left(1-\lambda_p p^{-s}+\chi_0(p)p^{-2s}\right)^{-1}=\prod_p\left(1-\alpha_p p^{-s}\right)^{-1}\left(1-\beta_p p^{-s}\right)^{-1}, \tag{1.11}$$

where $\alpha_p=\pm p^{-1/2},0$. $\beta_p=0$ if $p|N$ and $|\alpha_p|=|\beta_p|=1$ otherwise. A natural way to generalize these two arithmetic functions is to define $\alpha=(\alpha_n)_n$ and $\beta=(\beta_n)_n$ as follow:

$$\alpha_n=\prod_{p^\gamma||n}\alpha_p^\gamma, \quad \text{and} \quad \beta_n=\prod_{p^\gamma||n}\beta_p^\gamma. \tag{1.12}$$

Hence, we get

$$\lambda_n=\sum_{uv=n}\alpha_u\beta_v. \tag{1.13}$$

Another characteristic of this L-function, which we will make frequent use of in this thesis, is that $L(s,f)$ is entire and satisfies the following functional equation

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s\Gamma\left(s+\frac{k-1}{2}\right)L(s,f)=\omega\left(\frac{\sqrt{N}}{2\pi}\right)^{1-s}\Gamma\left(1-s+\frac{k-1}{2}\right)L(1-s,f), \tag{1.14}$$

where $\omega=\pm1$ and $\Gamma(s)$ is the usual gamma function. Furthermore, for any real primitive Dirichlet character χ_d of conductor $|d|$, we can define its twisted L-function by

$$L(s,f,\chi_d)=\sum_{n=1}^{\infty}\lambda_n\chi_d(n)n^{-s}.$$

This L-function is also entire and satisfies the functional equation

$$\begin{aligned}&\left(\frac{|d|\sqrt{N}}{2\pi}\right)^s\Gamma\left(s+\frac{k-1}{2}\right)L(s,f,\chi_d)\\&\quad=\omega\chi_d(-N)\left(\frac{|d|\sqrt{N}}{2\pi}\right)^{1-s}\Gamma\left(1-s+\frac{k-1}{2}\right)L(1-s,f,\chi_d).\end{aligned} \tag{1.15}$$

1.3 Introduction to Rankin-Selberg theory

The purpose of this section is to demonstrate how we can create new L-function from pre-existing L-functions. More precisely, if we have two L-functions

$$L_1(s) = \sum_n \frac{a_1(n)}{n^s},$$

and

$$L_2(s) = \sum_n \frac{a_2(n)}{n^s},$$

then it is natural for us to consider the L-function generated by the product of the coefficients of L_1 and L_2, or more precisely, the L-function

$$L(s) = \sum_n \frac{a_1(n)a_2(n)}{n^s}.$$

For modular L-functions, the Rankin-Selberg theory enables us to obtain the analytic properties of such function using Eisenstein series. In order to describe this theory, we need to begin by recalling some basic notations. For an integer N, we denote by $\Gamma_0(N)$, the group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\},$$

and by Γ_∞ the group

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\}.$$

Also define $\overline{\Gamma_0}(N) = \Gamma_0(N)\{\pm 1\}/\{\pm 1\}$ and $\overline{\Gamma_\infty} = \Gamma_\infty\{\pm 1\}/\{\pm 1\}$ the image of $\Gamma_0(N)$ and Γ_∞ in $PSL_2(\mathbb{Z})$. Let us define *non-holomorphic Eisenstein series* $E(z,s,N)$ of level N by

$$E(z,s,N) = \sum_{\gamma \in \overline{\Gamma_\infty} \backslash \overline{\Gamma_0}(N)} \operatorname{Im}(\gamma z)^s = \frac{1}{2} \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{y^s}{|cz+d|^{2s}}.$$

The second equality comes from the fact that

$$\operatorname{Im}(\gamma z) = \frac{\operatorname{Im}(z)}{|cz+d|^2},$$

for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. The following properties of $E(z,s,N)$ are given in [12]:

Property 1.7. *$E(z,.,N)$ is absolutely convergent for* $\mathrm{Re}(s) > 1$*,*

Property 1.8. *$E(.,s,N)$ is $\Gamma_0(N)$-invariant, i.e., $E(\gamma z,s,N) = E(z,s,N)$ for all $\gamma \in \Gamma_0(N)$,*

Property 1.9. *$E(z,.,N)$ is holomorphic in* $\mathrm{Re}(s) > 1/2$ *except for a simple pole at $s = 1$ with a residue V^{-1} where*

$$V = \int_{\mathbb{H}\backslash\Gamma_0(N)} \frac{dxdy}{y^2}, \tag{1.16}$$

and $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$,

Property 1.10. *$E(z,.,N)$ has a meromorphic continuation to $\mathbb{C}$,*

Property 1.11. *$E(.,s,N)$ is of polynomial growth when* $\mathrm{Im}(z)$ *tends to ∞.*

Definition 1.12. *As in Section 1.3, let $f_1 = \sum \lambda_1(n)n^{(k-1)/2}e(nz)$ and $f_2 = \sum \lambda_2(n)n^{(k-1)/2}e(nz)$ be two newforms of the same weight k, of level N_1 and N_2, and with nebentypes χ_1 and χ_2 respectively. The **Rankin-Selberg** L-function is defined by*

$$L(s, f_1 \otimes f_2) = L(2s, \chi_1\chi_2) \sum_{n=1}^{\infty} \frac{\lambda_1(n)\lambda_2(n)}{n^s},$$

where

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Since f_1 and f_2 are both newforms, the associated L-functions $L(s,f_1)$ and $L(s,f_2)$ have Euler products which are

$$L(s,f_1) = \prod_p \left(1 - \lambda_1(p)p^{-s} + \chi_1(p)p^{-2s}\right)^{-1} = \prod_p \left(1 - \alpha_1(p)p^{-s}\right)^{-1}\left(1 - \beta_1(p)p^{-s}\right)^{-1}, \tag{1.17}$$

and

$$L(s,f_2) = \prod_p \left(1 - \lambda_2(p)p^{-s} + \chi_2(p)p^{-2s}\right)^{-1} = \prod_p \left(1 - \alpha_2(p)p^{-s}\right)^{-1}\left(1 - \beta_2(p)p^{-s}\right)^{-1}. \tag{1.18}$$

The following gives the Euler product of the Rankin Selberg L-function.

Proposition 1.13. *For* $\mathrm{Re}(s) > 1$*, the L-function $L(s, f_1 \otimes f_2)$ also has an Euler product which is given by*

$$\begin{aligned} L(s, f_1 \otimes f_2) = \prod_p &\left(1 - \alpha_1(p)\alpha_2(p)p^{-s}\right)^{-1}\left(1 - \alpha_1(p)\beta_2(p)p^{-s}\right)^{-1} \\ &\times \left(1 - \beta_1(p)\alpha_2(p)p^{-s}\right)^{-1}\left(1 - \beta_1(p)\beta_2(p)p^{-s}\right)^{-1}. \end{aligned} \tag{1.19}$$

Proof. See Lemma in [13]. □

We now move on to one of the theorems that served as the basis for the Rankin-Selberg theory.

Theorem 1.14. *If f_1 and f_2 have the same level N and the same nebentype χ, then for* $\mathrm{Re}(s) > 1$, *we have*

$$\langle f_1 \cdot E(z,s,N), f_2\rangle = \frac{L(s, f_1 \otimes \overline{f}_2)}{L(2s,\chi_0)} \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}}, \tag{1.20}$$

where $\langle\cdot,\cdot\rangle$ is the Petersson inner product given by

$$\langle f_1, f_2\rangle = \int_{\Gamma_0(N)\backslash\mathbb{H}} f_1(z)\overline{f}_2(z) y^{k-2} dxdy,$$

χ_0 is the trivial character modulo N and

$$L(s, f_1 \otimes \overline{f}_2) = \sum_n \frac{\lambda_1(n)\overline{\lambda_2(n)}}{n^s}.$$

Proof. From Property 1.11, $E(z,.,N)$ is of polynomial growth when $\mathrm{Im}(z)$ goes to the infinity, hence $f_1(z)E(z,s,N)\overline{f_2}(z)$ decay exponentially as $\mathrm{Im}(z) \to \infty$. Let $\tilde{\alpha} = \tilde{a}/\tilde{b}$ be a cusp of $\Gamma_0(N)$ different from the infinity. Taking $y > 0$ small enough, we have

$$\begin{aligned} E(\tilde{a}/\tilde{b} + iy, s, N) &\leq y^s + \sum_{c,d\neq 0} \frac{y^s}{|c(\tilde{a}/\tilde{b} + iy) + d|^{2s}} \\ &= y^s + \sum_{c,d\neq 0} \frac{|\tilde{b}|^{2s} y^s}{(|c\tilde{a} + \tilde{b}d|^2 + |c\tilde{b}y|^2)^s} \\ &= y^s + \frac{1}{y^s} \sum_{c,d\neq 0} \frac{1}{|c|^{2s}(1 + \left|\frac{c\tilde{a}+\tilde{b}d}{c\tilde{b}y}\right|^2)^s} \\ &\ll \frac{1}{y^s} \sum_{c,d\neq 0} \frac{1}{|c|^{2s}(1 + \left|\frac{\tilde{a}}{\tilde{b}} + \frac{d}{c}\right|^2)^s}. \end{aligned}$$

This letter sum is convergent for $\mathrm{Re}(s) > 1$, hence, $E(z,s,N)$ has polynomial growth as $z \to \tilde{\alpha}$. Thus, the inner product $\langle f_1 \cdot E(z,s,N), f_2\rangle$ is well defined and we have

$$\begin{aligned} \langle f_1 \cdot E(z,s,N), f_2\rangle &= \int_{\Gamma_0(N)\backslash\mathbb{H}} f_1(z)E(z,s,N)\overline{f}_2(z) y^{k-2} dxdy \\ &= \int_{\Gamma_0(N)\backslash\mathbb{H}} f_1(z)\overline{f}_2(z) \sum_{\gamma\in\overline{\Gamma_\infty}\backslash\overline{\Gamma_0}(N)} \mathrm{Im}(\gamma z)^s y^{k-2} dxdy, \end{aligned}$$

Notice that $f_1(z)\overline{f}_2(z)y^k$ is $\Gamma_0(N)$-invariant. This is because

$$f_1(\gamma z) = (cz+d)^k f_1(z), \quad \overline{f}_2(\gamma z) = \overline{(cz+d)^k f_2(z)}, \text{ and } \quad \mathrm{Im}(\gamma z) = \frac{Im(\gamma z)}{|cz+d|^2},$$

for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Thus,

$$\begin{aligned}\langle f_1 \cdot E(z,s,N), f_2\rangle &= \int_{\Gamma_0(N)\backslash\mathbb{H}} \sum_{\gamma\in\overline{\Gamma_\infty}\backslash\overline{\Gamma_0}(N)} f_1(\gamma z)\overline{f}_2(\gamma z)\mathrm{Im}(\gamma z)^s \mathrm{Im}(\gamma z)^k y^{-2}dxdy \\ &= \sum_{\gamma\in\overline{\Gamma_\infty}\backslash\overline{\Gamma_0}(N)} \int_{\Gamma_0(N)\backslash\mathbb{H}} f_1(\gamma z)\overline{f}_2(\gamma z)\mathrm{Im}(\gamma z)^s \mathrm{Im}(\gamma z)^k y^{-2}dxdy.\end{aligned}$$

The last equality comes from the absolute convergence of the series for $\mathrm{Re}(s) > 1$. Recall that $\mathbb{H}$ is $\{\pm 1\}$-invariant, so $\Gamma_0(N)\backslash\mathbb{H} = \overline{\Gamma_0}(N)\backslash\mathbb{H}$. Hence, we have

$$\begin{aligned}\langle f_1 \cdot E(z,s,N), f_2\rangle &= \sum_{\gamma\in\overline{\Gamma_\infty}\backslash\overline{\Gamma_0}(N)} \int_{\overline{\Gamma_0}(N)\backslash\mathbb{H}} f_1(\gamma z)\overline{f}_2(\gamma z)\mathrm{Im}(\gamma z)^s \mathrm{Im}(\gamma z)^k y^{-2}dxdy \\ &= \int_{\overline{\Gamma_\infty}\backslash\mathbb{H}} f_1(z)\overline{f_2}(z)y^{s+k-2}dxdy.\end{aligned}$$

This is called the **unfolding method**. Moreover, we have

$$\overline{\Gamma_\infty}\backslash\mathbb{H} = \{z \in \mathbb{C} : 0 < x < 1, y > 0\}.$$

Therefore, we have

$$\begin{aligned}\langle f_1 \cdot E(z,s,N), f_2\rangle &= \int_0^\infty \int_0^1 f_1(z)\overline{f}_2(z)y^{s+k-2}dxdy \\ &= \int_0^\infty \int_0^1 \left(\sum_n \lambda_1(n)n^{(k-1)/2}e(nz)\right) \\ &\quad \times \left(\sum_n \bar{\lambda}_2(n)n^{(k-1)/2}e(-m\bar{z})\right) y^{s+k-2}dxdy \\ &= \int_0^\infty \int_0^1 \sum_{n,m} \lambda_1(n)\bar{\lambda}_2(m)(nm)^{(k-1)/2} \\ &\quad \times e((n-m)x)\exp(-2\pi(n+m)y)y^{s+k-2}dxdy.\end{aligned}$$

Since $\int_0^1 e((n-m)x)dx = \delta_{n,m}$, we have

$$\langle f_1 \cdot E(z,s,N), f_2\rangle = \int_0^\infty \sum_n \lambda_1(n)\bar{\lambda}_2(n)n^{k-1}\exp(-4\pi ny)y^{s+k-2}dy.$$

If we apply the change of variable $4\pi n y \mapsto y$, we obtain

$$\begin{aligned}\langle f_1 \cdot E(z,s,N), f_2\rangle &= \int_0^\infty \frac{1}{(4\pi)^{s-k+1}} \sum_n \frac{\lambda_1(n)\bar{\lambda}_2(n)}{n^s} \exp(-y) y^{s+k-2} dy \\ &= \sum_n \frac{\lambda_1(n)\bar{\lambda}_2(n)}{n^s} \frac{1}{(4\pi)^{s-k+1}} \int_0^\infty \exp(-y) y^{s+k-2} dy \\ &= \frac{L(s, f_1 \otimes \bar{f}_2)}{L(2s,\chi_0)} \frac{\Gamma(s+k-1)}{(4\pi)^{s-k+1}}.\end{aligned}$$

□

Corollary 1.15. *The L-function $L(s, f_1 \otimes \bar{f}_2)$ has a meromorphic continuation to $s \in \mathbb{C}$. Moreover, it has a pole at $s = 1$ if and only if $\langle f_1, f_2\rangle \neq 0$, and if it has a pole, then its residue is*

$$\operatorname*{Res}_{s=1} L(s, f_1 \otimes \overline{f}_2) = L(2,\chi_0) \frac{V \cdot (4\pi)^k}{\Gamma(k)} \langle f_1, f_2\rangle,$$

where V is give in (1.16)*.*

Proof. We know from Property 1.10 that $E(z,s,N)$ has a meromorphic continuation on $s \in \mathbb{C}$. Thus, $\langle f_1 \cdot E(z,s), f_2\rangle$ and $L(s, f_1 \otimes \overline{f}_2)$ can also be meromorphically continued on $\mathbb{C}$. As for the pole, from (1.16), we can write

$$E(z,s,N) = \frac{1}{V \cdot (s-1)} + \eta(z,s),$$

where $\eta(z,s)$ is holomorphic in a neighborhood of $s = 1$. Hence, we obtain

$$\langle f_1 \cdot E(z,s), f_2\rangle = \frac{\langle f_1, f_2\rangle}{V \cdot (s-1)} + \langle f_1 \cdot \eta(z,s), f_2\rangle.$$

Hence, using Theorem 1.14, we can obtain have the result. □

From now on, let us focus on the newform f as defined as in Section 1.2. Indeed, recall that all the coefficient λ_n are real, hence,

$$L(s, f \otimes \overline{f}) = L(s, f \otimes f).$$

It would then be obvious for us to study this latter. Indeed, we have the following functional equation:

Theorem 1.16. *([14]) There exist $\theta_p(s) = (1 - c_p p^{-s})^{-1}$ for $p|N$ with $|c_p| < p$, and two constants A_f and A'_f, such that if we define $\Lambda_f(s)$ by*

$$\Lambda_f(s) = \Gamma(s)\Gamma(s+k-1) \prod_{p|N} \theta_p(s) L(s, f \otimes f),$$

then we have

$$\Lambda_f(s) = A'_f A_f^s \Lambda_f(1-s).$$

Let us now introduce a new L-function derived from $L(s, f \otimes f)$.

Definition 1.17. *We define the symmetric square L-function of f by*

$$L(s, Sym^2 f) = \frac{L(s, f \otimes f)}{L(s, \chi_0)}.$$

Proposition 1.18. *For* $\mathrm{Re}(s) > 1$*, the L-function* $L(s, Sym^2 f)$ *has the following Euler product:*

$$L(s, Sym^2 f) = \prod_p \left(1 - \alpha_p^2 p^{-s}\right)^{-1} \left(1 - \chi_0(p) p^{-s}\right)^{-1} \left(1 - \beta_p^2 p^{-s}\right)^{-1}. \quad (1.21)$$

Moreover, we also have

$$L(s, Sym^2 f) = L(2s, \chi_0) \sum_{n=1}^{\infty} \frac{\lambda_{n^2}}{n^s}.$$

Proof. The Euler product can be deduced from Proposition 1.13, and using the fact that $\alpha_p \beta_p = \chi_0(p)$. For the second equality, we can use (1.5) to get

$$\begin{aligned}
\sum_n \frac{\lambda_n^2}{n^s} &= \sum_n \frac{1}{n^s} \sum_{d|n} \chi_0(d) \lambda_{n^2/d^2} = \sum_n \frac{1}{n^s} \sum_{d|n} \chi_0(n/d) \lambda(d^2) \\
&= \sum_d \lambda_{d^2} \sum_m \frac{\chi_0(m)}{(md)^s} = \sum_d \frac{\lambda_{d^2}}{d^s} \sum_m \frac{\chi_0(m)}{m^s} \\
&= L(s, \chi_0) \sum_d \frac{\lambda_{d^2}}{d^s}.
\end{aligned}$$

However, from Definition 1.12, we have

$$L(s, f \otimes f) = L(2s, \chi_0) \sum_{n=1}^{\infty} \frac{\lambda_n^2}{n^s}.$$

Therefore, we can obtain the result by dividing it with $L(s, \chi_0)$. □

The next result gives the functional equation for the symmetric square L-function $L(s, Sym^2 f)$.

Proposition 1.19. *The symmetric square L-function* $L(s, Sym^2 f)$ *has an analytic continuation to the whole plane* $\mathbb{C}$*. Moreover, using the same notation as in Theorem 1.16, if we define* $\theta_p'(s) = \theta_p(s)(1 - p^{s-1})$ *and*

$$\Lambda_f'(s) = \Gamma\left(\frac{s+1}{2}\right) \Gamma(s + k - 1) \prod_{p|N} \theta_p'(s) L(s, Sym^2 f),$$

then there exist two constants B_f and B'_f such that

$$\Lambda'_f(s) = B'_f B_f^s \Lambda'_f(1-s).$$

Proof. The analytic continuation of $L(s, Sym^2 f)$ is prove in [15]. As for the functional equation, We know that

$$\frac{\Gamma(s)}{\Gamma(s/2)} = 2^{1-s}\sqrt{\pi}\Gamma\Big(\frac{s+1}{2}\Big).$$

Then, using Definition 1.17, we can deduce that

$$\Lambda'_f(s) = 2^{s-1}\frac{\Lambda_f(s)}{\Gamma(s/2)\prod_{p|N}(1-p^{s-1})(1-p^{-s})\zeta(s)},$$

where $\zeta(s)$ is the usual Riemann zeta function. However, it is also well known that

$$\Gamma(s/2)\prod_{p|N}(1-p^{s-1})(1-p^{-s})\zeta(s) = \pi^{-1/2+s}\Gamma((1-s)/2)\prod_{p|N}(1-p^{s-1})(1-p^{-s})\zeta(1-s).$$

Therefore, using Theorem 1.16, we can obtain

$$\Lambda'_f(s) = B'_f B_f^s \Lambda'_f(1-s),$$

where $B'_f = \pi^{1/2}A'_f/2$ and $B_f = 2A_f/\pi$. □

Corollary 1.20. *The symmetric L-function $L(s, Sym^2 f)$ is of polynomial growth on every vertical strip.*

Proof. This results directly from the functional equation of $L(s, Sym^2 f)$ and the Phragmen-Lindelöf principles. □

Chapter 2

A method of Iwaniec

2.1 Introduction and statement of result

In this chapter, we will use the method of Iwaniec,in [9], to approach Theorem 1.1. Let us briefly explain what was achieved in [9]. Iwaniec considered an L-function $L(s,E)$ associated to an elliptic curve E such that $L(1/2,E,\chi_{d_0}) = 0$ for d_0 in a specific set of odd fundamental discriminants and studied the non-vanishing of their derivatives $L'(1/2,E,\chi_{d_0})$. In particular, it was proved in [9] that

$$\sum_{d_0}{}' L'(1/2,E,\chi_{d_0})F\Big(\frac{d_0}{Y}\Big) = A_f Y \log Y + B_f Y + o(Y),$$

where $\sum'$ means that the summation is over a particular set, F is some nonnegative smooth bump function with positive mean value, and A_f and B_f are real numbers such that $A_f \neq 0$.

In our work, we are interested in the central value of the twisted L-function $L(1/2,f,\chi_{d_0})$ associated to a newform f as described in Section 1.2. Recall that our project in this research is to determine whether there is an infinity of odd fundamental discriminant d_0 coprime with N such that $L(1/2,f,\chi_{d_0})$ do not vanish, we can begin by eliminating trivial cases. For example, if $\omega\chi_{d_0}(-N) = -1$, then from the Functional Equation (1.15),

$$L(1/2,f,\chi_{d_0}) = 0.$$

As a result, we can select a set of odd fundamental discriminant d_0 where

$$\omega\chi_{d_0}(-N) = 1.$$

Throughout, let M be a squarefree number such that M is divisible by 2 , by all prime numbers dividing N, and by all prime $p \in S$. Let us then consider the set

$$\mathcal{D} = \mathcal{D}_\omega = \{\omega d > 0 : |\mu(d)| = 1, d \equiv \nu^2 \mod 4M, \text{for } (\nu,4M) = 1\}. \quad (2.1)$$

Note that if $d_0 \in \mathcal{D}$, then

$$\chi_{d_0}(p) = \left(\frac{d_0}{p}\right) = 1, \tag{2.2}$$

for any $p|M$. Hence,

$$\omega\chi_{d_0}(-N) = \omega \cdot \operatorname{sign}(d_0) = 1.$$

Consequently, following Iwaniec ([9]), we achieved the following theorems, which are two of the three main results of this chapter:

Theorem 2.1. *For $\epsilon > 0$ and $Y > 0$, we have*

$$S_4(Y) = \sum_{\substack{d_0 \in \mathcal{D} \\ |d_0| < Y}} |L(1/2, f, \chi_{d_0})|^4 \ll Y^{2+\epsilon}. \tag{2.3}$$

Theorem 2.2. *Let M' be a positive squarefree number coprime to M and F be a bump function on $(0,1)$ with positive mean value. Then, for $\epsilon > 0$ and $Y > 0$, we have*

$$S(Y, M') = \sum_{\substack{d_0 \equiv 0[M'] \\ d_0 \in \mathcal{D}}} L(1/2, f, \chi_{d_0}) F\left(\frac{|d_0|}{Y}\right) = \rho(M')C_f Y + O(\tau(M')^2 M'^{1/2} Y^{\frac{13}{14}+\epsilon}), \tag{2.4}$$

where $C_f > 0$ and

$$\rho(M') = \prod_{p|M'} \frac{1}{p}\left[1 + \frac{1}{p} + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\right]^{-1}.$$

A direct implication of these two theorems will be:

Corollary 2.3. *Let $\epsilon > 0$ and $Y > 0$ large enough. Then, there exist at least $Y^{2/3-\epsilon}$ odd fundamental discriminants $d_0 \in \mathcal{D}$, $|d_0| < Y$ such that*

$$L(1/2, f, \chi_{d_0}) \neq 0.$$

2.2 Approximation of $L(1/2, f, \chi_{d_0})$

A naive way to obtain the central values $L(1/2, f, \chi_{d_0})$ is to compute the sum

$$\sum_{n=1}^{\infty} \frac{\lambda_n \chi_{d_0}(n)}{n^{1/2}}.$$

However, the fact that its absolute convergence is not assured prevents us from interchanging sums. Following Iwaniec, we need a function $V(x)$ such that $V(x)$ is continuous in $[0,\infty)$ and decreases exponentially as $x \to \infty$. Hence, let us define

$$\mathcal{A}(X,\chi_{d_0}) = \sum_{n=1}^{\infty} \frac{\lambda_n \chi_{d_0}(n)}{n^{1/2}} V\left(\frac{2\pi n}{X}\right), \tag{2.5}$$

which, with the factor $V(\frac{2\pi n}{X})$, is absolutely convergent for any $X > 0$. Now, we want to determine $V(x)$ in such a way that $L(1/2, f, \chi_{d_0})$ can be expressed as $\mathcal{A}(X,\chi_{d_0})$ for some X. First, we want the Mellin transform of $V(x)$ to be of the form $G(s)\Gamma(s)$ *,i.e.,*

$$V(x) = \frac{1}{2\pi i}\int_{(c)} G(s)\Gamma(s)x^{-s}ds, \tag{2.6}$$

for any $c > 0$, or equivalently,

$$G(s)\Gamma(s) = \int_0^{\infty} V(x)x^{s-1}dx, \tag{2.7}$$

for some holomorphic function $G(s)$ in $\mathbb{C}$. Consequently, $V(x)$ is only determined by $G(s)$. It then follows that

$$\mathcal{A}(X,\chi_{d_0}) = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{(c)} \frac{\lambda_n \chi_{d_0}(n)}{n^{1/2+s}} G(s)\Gamma(s)\left(\frac{2\pi}{X}\right)^{-s} ds. \tag{2.8}$$

If we choose $1/2 < c < 1$, then we have the absolute convergence of

$$\sum_{n=1}^{\infty} \frac{\lambda_n \chi_{d_0}(n)}{n^{1/2+s}}$$

allowing us to apply Fubini's theorem on $\mathcal{A}(X,\chi_d)$. Thus,

$$\mathcal{A}(X,\chi_d) = \frac{1}{2\pi i}\int_{(c)} L(1/2+s,f,\chi_d)G(s)\Gamma(s)\left(\frac{2\pi}{X}\right)^{-s} ds. \tag{2.9}$$

Now, let us move the integration line to $(-c)$. Since $c < 1$, then, we only pass through a simple pole of residue $G(0)L(1/2,f,\chi_{d_0})$ because $L(s,f,\chi_{d_0})$ is entire and $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1. Hence,

$$\mathcal{A}(X,\chi_{d_0}) = G(0)L(1/2,f,\chi_{d_0}) + \frac{1}{2\pi i}\int_{(-c)} L(1/2+s,f,\chi_{d_0})G(s)\Gamma(s)\left(\frac{2\pi}{X}\right)^{-s} ds. \tag{2.10}$$

However, we can use the functional equation in Equation (1.15) to get

$$L(1/2+s,f,\chi_{d_0}) = \omega\chi_{d_0}(-N)\left(\frac{|d_0|\sqrt{N}}{2\pi}\right)^{-2s} L(1/2-s,f,\chi_{d_0})\frac{\Gamma(k/2-s)}{\Gamma(k/2+s)}.$$

Since we only take $d_0 \in \mathcal{D}$, then $\omega\chi_{d_0}(-N) = 1$ and Equation (2.10) becomes

$$\begin{aligned} A(X,\chi_{d_0}) &= G(0)L(1/2,f,\chi_{d_0}) \\ &\quad + \frac{1}{2\pi i}\int_{(-c)} L(1/2-s,f,\chi_{d_0})\frac{\Gamma(k/2-s)}{\Gamma(k/2+s)}G(s)\Gamma(s)\left(\frac{|d_0|^2N}{2\pi X}\right)^{-s} ds \end{aligned}$$

Then, after the change of variable $s \mapsto -s$ and isolating $G(0)L(1/2,f,\chi_{d_0})$ we finally get

$$\begin{aligned} G(0)L(1/2,f,\chi_{d_0}) &= A(X,\chi_{d_0}) \\ &\quad - \frac{1}{2\pi i}\int_{(c)} L(1/2+s,f,\chi_{d_0})\frac{\Gamma(k/2+s)}{\Gamma(k/2-s)}G(-s)\Gamma(-s)\left(\frac{2\pi X}{|d_0|^2N}\right)^{-s} ds. \end{aligned} \tag{2.11}$$

Hence, we want $G(0) = 1$ so that we only have $L(1/2,f,\chi_{d_0})$ on the left-hand side. We also want the integral on the right-hand side to look like the integral in Equation (2.9). Thus, we need the following conditions for $G(s)$:

$$\begin{cases} G(0) = 1, \\ G(s)\Gamma(s) = -\frac{\Gamma(k/2+s)}{\Gamma(k/2-s)}G(-s)\Gamma(-s). \end{cases} \tag{2.12}$$

We know that for $s \notin \mathbb{Z}$,

$$\Gamma(k/2+s) = \prod_{j=0}^{k/2-1}(j+s)\Gamma(s).$$

Hence,

$$\frac{\Gamma(k/2+s)}{\Gamma(k/2-s)} = \prod_{j=0}^{k/2-1}\frac{(j+s)}{(j-s)}\frac{\Gamma(s)}{\Gamma(-s)} = -\prod_{j=1}^{k/2-1}\frac{(j+s)}{(j-s)}\frac{\Gamma(s)}{\Gamma(-s)}.$$

We can then change the second condition for $G(s)$ to

$$G(s)\Gamma(s) = \prod_{j=1}^{k/2-1}\frac{(j+s)}{(j-s)}G(-s)\Gamma(s),$$

or equivalently

$$\frac{G(s)}{G(-s)} = \prod_{j=1}^{k/2-1} \frac{(j+s)}{(j-s)}.$$

An obvious solution to this equation is

$$G(s) = \frac{1}{(k/2-1)!} \prod_{j=1}^{k/2-1} (j+s). \tag{2.13}$$

This implies that the Mellin transform (2.7) becomes

$$\begin{aligned}\int_0^\infty V(x)x^{s-1}dx = G(s)\Gamma(s) &= \frac{1}{(k/2-1)!}\prod_{j=1}^{k/2-1}(j+s)\Gamma(s)\\ &= \frac{1}{(k/2-1)!}\frac{\Gamma(s+k/2)}{s}\\ &= \frac{1}{(k/2-1)!}\int_0^\infty \frac{t^{s+k/2-1}}{s}e^{-t}dt.\end{aligned}$$

Since

$$\frac{t^s}{s} = \int_0^t x^{s-1}dx,$$

we have

$$\begin{aligned}\int_0^\infty V(x)x^{s-1}dx &= \frac{1}{(k/2-1)!}\int_0^\infty \left(\int_0^t x^{s-1}dx\right)t^{k/2-1}e^{-t}dt\\ &= \frac{1}{(k/2-1)!}\int_0^\infty \left(\int_x^\infty t^{k/2-1}e^{-t}dt\right)x^{s-1}dx.\end{aligned}$$

Therefore, from the Mellin inversion formula, we have

$$V(x) = \int_x^\infty \frac{t^{k/2-1}}{(k/2-1)!}e^{-t}dt.$$

As a result, from Equation (2.11), we get

$$\begin{aligned}L(1/2, f, \chi_{d_0}) &= A(X, \chi_{d_0})\\ &\quad + \frac{1}{2\pi i}\int_{(c)} L(1/2+s, f, \chi_{d_0})G(s)\Gamma(s)\left(\frac{2\pi X}{|d_0|^2 N}\right)^{-s} ds.\end{aligned} \tag{2.14}$$

Note that this last integral is $\mathcal{A}(|d_0|^2 N X^{-1})$ by (2.9). We then obtain the exact formula for $L(1/2, f, \chi_{d_0})$:

Proposition 2.4. *With the above definition of $\mathcal{A}(X,\chi_d)$ and $V(x)$, we have*

$$L(1/2,f,\chi_{d_0}) = \mathcal{A}(X,\chi_{d_0}) + \mathcal{A}(|d_0|^2NX^{-1}). \tag{2.15}$$

In particular, we get

$$L(1/2,f,\chi_{d_0}) = 2\mathcal{A}(|d_0|\sqrt{N},\chi_{d_0}). \tag{2.16}$$

We also obtain the following approximation:

Corollary 2.5. *We have*

$$L(1/2,f,\chi_{d_0}) = \mathcal{A}(X,\chi_{d_0}) + O(|d_0|X^{-1/2}). \tag{2.17}$$

Proof. If we can prove that

$$\mathcal{A}(X,\chi_{d_0}) \ll X^{1/2},$$

then our corollary comes directly from Proposition 2.4. Using the definition of $V(x)$, we have

$$\begin{aligned} A(X,\chi_{d_0}) &= \sum_{n=1}^{\infty} \frac{\lambda_n\chi_{d_0}(n)}{n^{1/2}} V\left(\frac{2\pi n}{X}\right) \\ &= \sum_{n=1}^{\infty} \frac{\lambda_n\chi_{d_0}(n)}{n^{1/2}} \int_{\frac{2\pi n}{X}}^{\infty} \frac{t^{k/2-1}}{(k/2-1)!} e^{-t}\,dt. \end{aligned} \tag{2.18}$$

As we did previously, we can interchange the sum and integration. Hence,

$$A(X,\chi_{d_0}) = \int_{\frac{2\pi}{X}}^{\infty} \frac{t^{k/2-1}}{(k/2-1)!} e^{-t} \sum_{n=1}^{Xt/2\pi} \frac{\lambda_n\chi_{d_0}(n)}{n^{1/2}}\,dt. \tag{2.19}$$

On the other hand, using Abel's summation formula, we have

$$\begin{aligned} \sum_{n\le M} \frac{\lambda_n}{n^{1/2}} &= \sum_{n\le M} \frac{\lambda_n n^{(k-1)/2}}{n^{k/2}} \\ &= \frac{1}{M^{k/2}} \sum_{n\le M} \lambda_n n^{(k-1)/2} + \frac{k}{2}\int_1^M \frac{1}{t^{k/2+1}} \sum_{n\le t} \lambda_n n^{(k-1)/2}\,dt. \end{aligned}$$

However, from (1.9), we have

$$\sum_{n\le M} |\lambda_n|^2 n^{k-1} \ll M^k,$$

and by using Hólder's inequality, we get

$$\sum_{n\leq M}|\lambda_n|n^{(k-1)/2}\ll\sqrt{\sum_{n\leq M}|\lambda_n|^2n^{k-1}}\sqrt{\sum_{n\leq M}1}\ll M^{(k+1)/2}.$$

Thus, we obtain

$$\sum_{n\leq M}\frac{\lambda_n}{n^{1/2}}\ll M^{1/2}+\int_1^M t^{-1/2}dt\ll M^{1/2}.$$

Importing this result in Equation (2.19), we get

$$A(X,\chi_{d_0})\ll\int_{\frac{2\pi}{X}}^{\infty}t^{k/2-1}e^{-t}\big(Xt\big)^{1/2}dt\ll X^{1/2}\int_{\frac{2\pi}{X}}^{\infty}t^{(k-1)/2}e^{-t}dt\ll X^{1/2}.$$

This completes proof. □

2.3 Large sieve and the fourth moment

The goal of this section is to prove Theorem 2.1. Let us recall its statement:

$$S_4(Y)=\sum_{\substack{d_0\in\mathcal{D}\\|d_0|<Y}}|L(1/2,f,\chi_{d_0})|^4\ll Y^{2+\epsilon},$$

for any $\epsilon>0$ and $Y>0$. To obtain this upper bound, we need an additional tool called *Large sieve inequality*.

Proof of Theorem 2.1. To begin with, note that we can replace the character χ_{d_0} in (2.5) with any Dirichlet character χ. Then, we still have

$$A(X,\chi)=\int_{\frac{2\pi}{X}}^{\infty}\frac{t^{k/2-1}}{(k/2-1)!}e^{-t}\sum_{n=1}^{Xt/2\pi}\frac{\lambda_n\chi(n)}{n^{1/2}}dt.$$

Using Jensen's inequality, we obtain

$$|\mathcal{A}(X,\chi)|^4\leq g(X)^3\int_{\frac{2\pi}{X}}^{\infty}\frac{t^{k/2-1}}{(k/2-1)!}e^{-t}\Big|\sum_{n=1}^{Xt/2\pi}\frac{\lambda_n\chi(n)}{n^{1/2}}\Big|^4dt,\qquad(2.20)$$

where

$$g(X)=\int_{\frac{2\pi}{X}}^{\infty}\frac{t^{k/2-1}}{(k/2-1)!}e^{-t}dt.$$

Now, let us use the Large sieve inequality Bombieri [16], which states that

$$\sum_{q\le Q}\ \sum_{\chi(mod\, q)}{}' \left|\sum_{n=1}^{T} b_n\chi(n)\right|^2 \le (Q^2+T-1)\sum_{n=1}^{T}|b_n|^2,$$

for any sequences $(b_n)_n$, where $\sum'$ means that the summation is over primitive character. If we square the inner sum of the left hand side, we get

$$\sum_{q\le Q}\ \sum_{\chi(mod\, q)}{}' \left|\sum_{n=1}^{T} b_n\chi(n)\right|^4 \le (Q^2+T^2-1)\sum_{n=1}^{T^2}\Big|\sum_{\substack{kl=n\\k,l\le N}} b_k b_l\Big|^2.$$

Using this latter inequality in Equation (2.20), we get

$$\begin{aligned}\sum_{D\le Y}\sum_{\chi(mod\, D)}{}' |\mathcal{A}(X,\chi)|^4 &\le g(X)^3\int_{\frac{2\pi}{X}}^{\infty}\frac{t^{k/2-1}}{(k/2-1)!}e^{-t}\sum_{D\le Y}\sum_{\chi(mod\, D)}{}' \Big|\sum_{n=1}^{Xt/2\pi}\frac{\lambda_n\chi(n)}{n^{1/2}}\Big|^4 dt\\ &\le g(X)^3\int_{2\pi/X}^{\infty}\frac{t^{k/2-1}}{(k/2-1)!}e^{-t}(Y^2+X^2t^2)\sum_{n=1}^{(Xt/2\pi)^2}\Big|\sum_{\substack{kl=n\\k,l\le Xt/2\pi}}\frac{\lambda_k\lambda_l}{n^{1/2}}\Big|^2 dt.\end{aligned}$$

Let us fix $\epsilon > 0$. Since $|\lambda_n| \le \tau(n)$ and $\tau(n) \ll n^{\epsilon}$, we have

$$\Big|\sum_{\substack{kl=n\\k,l\le N}}\lambda_k\lambda_l\Big| \le \sum_{\substack{kl=n\\k,l\le N}}\tau(k)\tau(l) \ll \sum_{kl=n}k^{\epsilon}l^{\epsilon} \ll n^{\epsilon}\tau(n) \ll n^{2\epsilon}.$$

Hence,

$$\Big|\sum_{\substack{kl=n\\k,l\le Xt/2\pi}}\frac{\lambda_k\lambda_l}{n^{1/2}}\Big| \ll n^{2\epsilon-1/2}.$$

Thus, after squaring the latter, we obtain

$$\begin{aligned}\sum_{D\le Y}\sum_{\chi(mod\, D)}{}' |\mathcal{A}(X,\chi)|^4 &\ll g(X)^3\int_{\frac{2\pi}{X}}^{\infty}t^{k/2-1}e^{-t}\sum_{n=1}^{(Xt/2\pi)^2}n^{4\epsilon-1}dt\\ &\ll g(X)^3\int_{2\pi/X}^{\infty}t^{k/2-1}e^{-t}(Y^2+X^2t^2)(Xt)^{8\epsilon}dt.\end{aligned}$$

We know that $\int_0^\infty t^u e^{-t} dt$ is always bounded for any $u > 0$. Thus, $g(X) \ll 1$ and

$$\sum_{D \leq Y} \sum_{\chi (mod\, D)}{}' |\mathcal{A}(X,\chi)|^4 \ll Y^2 X^{8\epsilon} + X^{2+8\epsilon}. \tag{2.21}$$

On the other hand, from Proposition 2.4, we have

$$\int_{|d_0|}^{|d_0|N} L(1/2, f, \chi_{d_0}) X^{-1} dX$$
$$= \int_{|d_0|}^{|d_0|N} \mathcal{A}(X, \chi_{d_0}) X^{-1} dX + \int_{|d_0|}^{|d_0|N} \mathcal{A}(|d_0|^2 N X^{-1}, \chi_{d_0}) X^{-1} dX \tag{2.22}$$

Let us compute each element of this equation separately. First, we have

$$\int_{|d_0|}^{|d_0|N} L(1/2, f, \chi_{d_0}) X^{-1} dX = \log(N) \cdot L(1/2, f, \chi_{d_0}).$$

Secondly, using the transformation $X \mapsto |d_0|^2 N X^{-1}$, we can see that

$$\int_{|d_0|}^{|d_0|N} \mathcal{A}(|d_0|^2 N X^{-1}, \chi_{d_0}) X^{-1} dX = \int_{|d_0|}^{|d_0|N} \mathcal{A}(X, \chi_{d_0}) X^{-1} dX.$$

As a result, if we put these together, we obtain

$$L(1/2, f, \chi_{d_0}) = (2/\log N) \int_{|d_0|}^{|d_0|N} \mathcal{A}(X, \chi_{d_0}) X^{-1} dX.$$

Using the Jensen's inequality again, it follows that

$$|L(1/2, f, \chi_{d_0})|^4 \ll \int_{|d_0|}^{|d_0|N} |\mathcal{A}(X, \chi_{d_0})|^4 X^{-1} dX.$$

If we take $d_0 \in \mathcal{D}$ and $|d_0| < Y$, then we have

$$|L(1/2, f, \chi_{d_0})|^4 \ll \int_1^{YN} |\mathcal{A}(X, \chi_{d_0})|^4 X^{-1} dX.$$

Let us now sum $|L(1/2, f, \chi_{d_0})|^4$ over $d_0 \in \mathcal{D}$, $|d_0| < Y$. With the inequality in

Equation (2.21), we obtain

$$\begin{aligned}\sum_{d_0\in\mathcal{D},|d_0|<Y}|L(1/2,f,\chi_{d_0})|^4 &\ll \int_1^{YN}\sum_{d_0\in\mathcal{D},|d_0|<Y}|\mathcal{A}(X,\chi_{d_0})|^4X^{-1}dX\\ &\ll \int_1^{YN}\sum_{D\le Y}\sideset{}{'}\sum_{\chi(mod\ D)}\left|\mathcal{A}(X,\chi)\right|^4X^{-1}dX\\ &\ll \int_1^{YN}(Y^2X^{8\epsilon}+X^{2+8\epsilon})X^{-1}dX\\ &\ll Y^{2+8\epsilon}.\end{aligned} \tag{2.23}$$

Therefore, can get the result by choosing the right $\epsilon > 0$. □

Remark 2.6. *If we use the same method to get an upper bound for the second moment, we still have the same result, i.e.,*

$$\sum_{d\in\mathcal{D},|d|<Y}|L(1/2,f,\chi_d)|^2 \ll Y^{2+\epsilon}.$$

Corollary 2.7. *For any $\epsilon > 0$ and $Y > 1$, we have*

$$\sum_{d\in\mathcal{D},|d|<Y}|L(1/2,f,\chi_d)| \ll Y^{5/4+\epsilon}.$$

Proof. We begin by applying Holder's inequality on the average of central value. Thus,

$$\sum_{d\in\mathcal{D},d<Y}|L(1/2,f,\chi_d)| \le \left(\sum_{d\in\mathcal{D},d<Y}|L(1/2,f,\chi_d)|^4\right)^{1/4}\left(\sum_{d\in\mathcal{D},d<Y}1\right)^{3/4}.$$

Therefore, from Equation (2.3), we obtain

$$\sum_{d\in\mathcal{D},d<Y}|L(1/2,f,\chi_d)| \ll Y^{1/2+\epsilon}Y^{3/4}.$$

and we have the result. □

2.4 Estimation of $S(Y,M')$

The purpose of this section is to demonstrate Theorem 2.2 which asserts that

$$S(Y,M') = \sum_{\substack{d_0\equiv 0[M']\\ d_0\in\mathcal{D}}} L(1/2,f,\chi_{d_0})F\Big(\frac{d_0}{Y}\Big) = \rho(M')C_fY + O(\tau(M')M'^{1/2}Y^{\frac{13}{14}+\epsilon}),$$

for some multiplicative function ρ and $C_f > 0$. To achieve this result, we need the following lemma

Lemma 2.8. *Let $(c_n)_n$ be any sequence such that the series $\sum_{n=1}^{\infty} c_n$ converges absolutely. Thus,*

$$\sum_{n\equiv 0[M']} c_n = \sum_{d'|M'} \mu(d') \sum_{(n,d')=1} c_n. \tag{2.24}$$

Proof. Let us point out that the absolute convergence of $\sum_{n=1}^{\infty} c_n$ allows us to manipulate the sum as we want. Using the Mobius function, we have

$$\begin{aligned}\sum_{d'|M'} \mu(d') \sum_{(n,d')=1} c_n &= \sum_{d'|M'} \mu(d') \sum_{n} c_n \sum_{\substack{k|n \\ k|d'}} \mu(k) \\ &= \sum_{n} c_n \sum_{k|n} \mu(k) \sum_{\substack{d'|M' \\ d'\equiv 0[k]}} \mu(d').\end{aligned}$$

However, M' is squarefree, and since $d'|M'$, we can write $d' \equiv 0[k]$ as $kd'|M'$, such that $(k,d') = 1$. Hence,

$$\begin{aligned}\sum_{d'|M'} \mu(d') \sum_{(n,d')=1} c_n &= \sum_{n} c_n \sum_{k|n} \mu(k) \sum_{d'|(M'/k)} \mu(kd') \\ &= \sum_{n} c_n \sum_{k|n} \mu(k)^2 \sum_{d'|(M'/k)} \mu(d')\end{aligned}$$

The last sum is only equal to 1 if $(M'/k) = 1$ and 0 otherwise. Thus k must be equal to M' and since $k|n$, we have

$$\sum_{d'|M'} \mu(d') \sum_{(n,d')=1} a_n = \sum_{\substack{n \\ M'|n}} a_n$$

□

If we apply Lemma 2.8 on $S(Y, M')$, we obtain

$$S(Y, M') = \sum_{d'|M'} \mu(d') S_{d'}(Y), \tag{2.25}$$

where

$$S_{d'}(Y) = \sum_{\substack{(d_0,d')=1 \\ d_0\in\mathcal{D}}} L(1/2, f, \chi_{d_0}) F\Big(\frac{|d_0|}{Y}\Big). \tag{2.26}$$

Therefore, we can approximate $S(Y, M')$ by making an estimation of $S_{d'}(Y)$. Indeed, we will prove the following proposition

Proposition 2.9. *For $\epsilon > 0$ and $Y > 0$, we have*

$$S_{d'}(Y) = \prod_{p|d'} \left[1 + \frac{1}{p}\left(1 + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]^{-1} C_f Y + O(\tau(d') d'^{1/2} Y^{\frac{13}{14}+\epsilon}). \quad (2.27)$$

To establish this proposition, we start by using (2.16) in Equation (2.26). Hence,

$$S_{d'}(Y) = 2 \sum_{\substack{(d_0,d')=1 \\ d_0 \in \mathcal{D}}} \mathcal{A}(|d_0|\sqrt{N}, \chi_{d_0}) F\left(\frac{|d_0|}{Y}\right). \quad (2.28)$$

Now, let us consider a new set

$$\mathcal{D}' = \{\omega d_0' > 0 : d_0' \equiv \nu^2 \mod 4M, \text{for } (\nu, 4M) = 1\}, \quad (2.29)$$

which is similar to $\mathcal{D}$ but includes non-squarefree numbers. For $d_0' \in \mathcal{D}'$, we take $\chi_{d_0'}$ to be the Kronecker symbol, *i.e.*,

$$\chi_{d_0'}(\cdot) = \left(\frac{d_0'}{\cdot}\right).$$

Then, we remove the squarefree condition by introducing the factor $\sum_{a^2|d_0'} \mu(a)$, and we obtain

$$S_{d'}(Y) = 2 \sum_{\substack{(d_0',d')=1 \\ d_0' \in \mathcal{D}'}} \mathcal{A}(|d_0'|\sqrt{N}, \chi_{d_0'}) F\left(\frac{|d_0'|}{Y}\right) \sum_{a^2|d} \mu(a). \quad (2.30)$$

Now, let us separate this sum to $a \leq A$ and $a > A$, where A will be determined later. Hence, we obtain

$$S_{d'}(Y) = S_{d'}(Y, A) + R_{d'}(Y, A), \quad (2.31)$$

where

$$S_{d'}(Y, A) = 2 \sum_{a \leq A, (a, 4Md')=1} \mu(a) \sum_{\substack{(d_0',d')=1 \\ d_0' \in \mathcal{D}'}} \mathcal{A}(a^2|d_0'|\sqrt{N}, \chi_{a^2 d_0'}) F\left(\frac{a^2|d_0'|}{Y}\right), \quad (2.32)$$

and

$$R_{d'}(Y, A) = 2 \sum_{d_0 \in \mathcal{D}'} \mathcal{A}(|d_0'|\sqrt{N}, \chi_{d_0'}) F\left(\frac{a^2|d_0'|}{Y}\right) \sum_{\substack{a^2|d_0' \\ a > A}} \mu(a). \quad (2.33)$$

Note that the condition $(a, 4Md') = 1$ was added in (2.32). This comes from the fact that $a^2|d_0'$ and that $(d_0, 4Md') = 1$.

2.4.1 Upper bound for $R_{d'}(Y, A)$

In this subsection, we will establish an upper bound for $R_{d'}(Y, A)$.

Proposition 2.10. *For $\epsilon > 0$ and $1 < A < Y$, we have*

$$R_{d'}(Y, A) \ll (Y^{5/4}A^{-3/2} + A^{-3}Y^{3/2})Y^{\epsilon}.$$

Proof. We start by writing d_0' in Equation (2.33) as $b^2 d_0$ where d_0 is squarefree. This is possible because $b^2 d_0 \in \mathcal{D}'$ and $(b^2 d_0, d') = 1$ are exactly equivalent to $b^2 d_0$, where $d_0 \in \mathcal{D}$ and b coprime with $4Md'$. Thus,

$$R_{d'}(Y, A) = 2 \sum_{(b,4Md')=1} \left(\sum_{a>A,\, a|b} \mu(a) \right) \sum_{d_0 \in \mathcal{D}} \mathcal{A}(b^2|d_0|\sqrt{N}, \chi_{b^2 d_0}) F\Big(\frac{b^2|d_0|}{Y}\Big). \tag{2.34}$$

However, we have

$$\mathcal{A}(X, \chi_{b^2 d_0}) = \sum_{(n,b)=1} \frac{\lambda_n \chi_{d_0}(n)}{n^{1/2}} V\left(\frac{2\pi n}{X}\right).$$

Then, using the definition of α and β in Equation (1.11), we have

$$\mathcal{A}(X, \chi_{b^2 d_0}) = \sum_{(u,b)=1} \sum_{(v,b)=1} \alpha_u \beta_v \chi_{d_0}(uv) u^{-1/2} v^{-1/2} V\left(\frac{2\pi uv}{X}\right).$$

We can replace the condition $(\cdot, b) = 1$ with the factor $\sum_{k|(\cdot,b)} \mu(k)$. Hence,

$$\begin{aligned}
\mathcal{A}(X, \chi_{b^2 d_0}) &= \sum_{u} \sum_{v} \alpha_u \beta_v \chi_{d_0}(uv) u^{-1/2} v^{-1/2} V\left(\frac{2\pi uv}{X}\right) \sum_{\substack{k|u \\ k|b}} \sum_{\substack{l|v \\ l|b}} \mu(k)\mu(l) \\
&= \sum_{k|b} \sum_{l|b} \sum_{u} \sum_{v} \alpha_{uk} \beta_{vl} \chi_{d_0}(uv) \chi_{d_0}(kl) \\
&\qquad \times \mu(k)\mu(l) u^{-1/2} v^{-1/2} (kl)^{-1/2} V\left(\frac{2\pi uv}{X}\right) \\
&= \sum_{k|b} \sum_{l|b} \alpha_k \alpha_k \chi_{d_0}(kl) \mu(k)\mu(l)(kl)^{-1/2} \\
&\qquad \times \sum_{n} \left(\sum_{n=uv} \alpha_u \alpha_v \right) \chi_{d_0}(n) n^{-1/2} V\left(\frac{2\pi nkl}{X}\right) \\
&= \sum_{k|b} \sum_{l|b} \alpha_k \alpha_k \chi_{d_0}(kl) \mu(k)\mu(l)(kl)^{-1/2} \mathcal{A}(X/kl, \chi_{d_0})
\end{aligned}$$

Since $d_0 \in \mathcal{D}$, we can use Corollary 2.5 on $\mathcal{A}(X/kl, \chi_{d_0})$, and we get

$$A(X, \chi_{b^2 d_0}) = L(1/2, f, \chi_{d_0}) \left(\sum_{k|b} \mu(k) \chi_{d_0}(k) \frac{\alpha_k}{k^{1/2}} \right) \left(\sum_{l|b} \mu(l) \chi_{d_0}(l) \frac{\beta_l}{l^{1/2}} \right) + O\left[|d_0| X^{-1/2} \left(\sum_{k|b} |\alpha_k| \right) \left(\sum_{l|b} |\beta_l| \right) \right].$$

Recall that $|\alpha_k| = |\beta_l| = 1$ because l and k are coprime with N, thus each sum can be bounded by $\sum_{k|b} = \tau(b)$. Hence, by taking $X = b^2 |d_0| \sqrt{N}$, we have

$$\mathcal{A}(b^2 |d_0| \sqrt{N}, \chi_{b^2 d_0}) \ll (|L(1/2, f, \chi_{d_0})| + b^{-1} |d_0|^{1/2}) \tau(b).$$

Putting this upper bound in (2.34), we obtain

$$R_{d'}(Y, A) \ll \sum_{\substack{(b,4Md')=1 \\ b<Y}} \left(\sum_{A<a|b} 1 \right) \sum_{\substack{d_0 \in \mathcal{D}_0 \\ |d_0| < Y b^{-2}}} (|L(1/2, f, \chi_{d_0})| + b^{-1} |d_0|^{1/2}) \tau(b). \tag{2.35}$$

The condition $b < Y$ and $|d_0| < Y b^{-2}$ comes from the bump function $F(b^2 |d_0| / Y)$. We know $\tau(b) \ll b^{\epsilon}$ for any $\epsilon > 0$, and since $b < Y$, we have $\tau(b) \ll Y^{\epsilon}$. Moreover we can also use Corollary 2.7 because $d_0 \in \mathcal{D}$. Hence,

$$R_{d'}(Y, A) \ll \sum_{\substack{(b,4Md')=1 \\ b<Y}} \left(\sum_{A<a|b} 1 \right) (Y^{5/4} b^{-5/2} + b^{-1} Y^{3/2} b^{-3}) Y^{\epsilon}.$$

However, using the same $\epsilon > 0$, we have

$$\sum_{A<a|b} 1 \leq \tau(b) \ll Y^{\epsilon}$$

because $b < Y$. We also have the condition $b > A$ since $a|b$ and $a > A$. Thus, (2.35) becomes

$$R_{d'}(Y, A) \ll \sum_{\substack{(b,4Md')=1 \\ A<b<Y}} (Y^{5/4} b^{-5/2} + Y^{3/2} b^{-4}) Y^{2\epsilon}.$$

We can now ignore the condition $(b, 4Md') = 1$ and $b < Y$, and then, we will just sum the upper bound over $b > A$. Therefore, we have

$$R_{d'}(Y, A) \ll (Y^{5/4} A^{-3/2} + A^{-3} Y^{3/2}) Y^{2\epsilon}.$$

□

2.4.2 Estimation of $S_{d'}(Y, A)$

As in the previous subsection, we begin by looking at the components of the sum $S_{d'}(Y, A)$ which are of the form $\mathcal{A}(a^2|d_0'|\sqrt{N}, \chi_{a^2 d_0})$. Indeed, for $(a, 4Md') = 1$, we have

$$\begin{aligned}\mathcal{A}(a^2|d_0'|\sqrt{N}, \chi_{a^2 d_0'}) &= \sum_n \frac{\lambda_n \chi_{a^2 d_0'}(n)}{n^{1/2}} V\left(\frac{2\pi n}{a^2|d_0'|\sqrt{N}}\right) \\ &= \sum_{(n,a)=1} \frac{\lambda_n \chi_{d_0'}(n)}{n^{1/2}} V\left(\frac{2\pi n}{a^2|d_0'|\sqrt{N}}\right).\end{aligned}$$

Using this latter in Equation (2.32), we obtain

$$S = 2 \sum_{a\leq A,(a,4Md')=1} \mu(a) \sum_{\substack{(d_0',d')=1 \\ d_0'\in\mathcal{D}'}} \sum_{(n,a)=1} \frac{\lambda_n \chi_{d_0'}(n)}{n^{1/2}} V\left(\frac{2\pi n}{a^2|d_0'|\sqrt{N}}\right) F\left(\frac{a^2|d_0'|}{Y}\right).$$

Let us notice that the first two sums are both finite because of the bump function F. We also have an absolute convergence for the last sum, hence, we can interchange the sums as we want. Now, let us write the n in the sum as $n = hl^2m$ where all prime factors of h divide $4M$, $(l^2m, 4M) = 1$ and m is squarefree. Thus,

$$\chi_{d_0'}(n) = \chi_{d_0'}(h)\chi_{d_0'}(l^2)\chi_{d_0'}(m).$$

Notice that from definition of $\mathcal{D}'$ in (2.29), $d_0' \equiv v^2 \mod 4M$ for some v coprime with $4M$. Using the same argument as in Equation (2.2), we can then deduce that $\chi_{d_0'}(h) = 1$. It follows that $\chi_{d_0'}(n) = \chi_{d_0'}(m)$ if $(d_0', l) = 1$ and 0 otherwise. Consequently,

$$\begin{aligned}S_{d'}(Y, A) = 2 &\sum_{a\leq A,(a,4Md')=1} \mu(a) \sum_{\substack{(d_0',d')=1 \\ d_0'\in\mathcal{D}'}} \\ &\times \sum_{\substack{n=hl^2m \\ (n,a)=1 \\ (d_0',l)=1}} \frac{\lambda_n \chi_{d_0'}(m)}{n^{1/2}} V\left(\frac{2\pi n}{a^2|d_0'|\sqrt{N}}\right) F\left(\frac{a^2|d_0'|}{Y}\right).\end{aligned}$$

However, the condition $(d_0', l) = 1$ and $(d_0', d') = 1$ can be replaced by introducing again the factor $\sum_{q|(d_0', ld')} \mu(l)$. Hence, if we rearrange the sum, then we get

$$
\begin{aligned}
S_{d'}(Y,A) &= 2 \sum_{\substack{a \le A \\ (a,4Md')=1}} \mu(a) \sum_{d_0' \in \mathcal{D}'} \sum_{\substack{n=hl^2m \\ (n,a)=1}} \\
&\quad \times \sum_{\substack{q|d_0' \\ q|ld'}} \mu(q) \frac{\lambda_n \chi_{d_0'}(m)}{n^{1/2}} V\left(\frac{2\pi n}{a^2|d_0'|\sqrt{N}}\right) F\left(\frac{a^2|d_0'|}{Y}\right) \\
&= 2 \sum_{\substack{a \le A \\ (a,4Md')=1}} \mu(a) \sum_{\substack{n=hl^2m \\ (n,a)=1}} \frac{\lambda_n}{n^{1/2}} \\
&\quad \times \sum_{q|ld'} \mu(q) \sum_{qd_0' \in \mathcal{D}'} \chi_{qd_0'}(m) V\left(\frac{2\pi n}{a^2 q|d_0'|\sqrt{N}}\right) F\left(\frac{a^2 q|d_0'|}{Y}\right).
\end{aligned}
$$

Now, let us use the following formula ([9]):

$$
\chi_{d_0'}(m) = \frac{\overline{\epsilon_m}}{m^{1/2}} \sum_{2|r|<m} \chi_{Mr}(m) e\left(\frac{\overline{4M}rd}{m}\right), \tag{2.36}
$$

where

$$
\epsilon_m = \begin{cases} 1 & \text{if} \quad m \equiv 1 \mod 4 \\ i & \text{if} \quad m \equiv 3 \mod 4, \end{cases} \tag{2.37}
$$

and $\overline{4M}4M \equiv 1 \mod m$. Hence,

$$
S_{d'}(Y,A) = 2 \sum_{\substack{a \le A \\ (a,4Md')=1}} \mu(a) \sum_{\substack{n=hl^2m \\ (n,a)=1}} \frac{\lambda_n}{n^{1/2}} \frac{\overline{\epsilon_m}}{m^{1/2}} \sum_{q|ld'} \mu(q) \sum_{2|r|<m} \chi_{Mrq}(m) \sum_{d_0'}, \tag{2.38}
$$

where

$$
\sum_{d_0'} = \sum_{qd_0' \in \mathcal{D}'} V\left(\frac{2\pi n}{a^2 q|d_0'|\sqrt{N}}\right) F\left(\frac{a^2 q|d_0'|}{Y}\right) e\left(\frac{\overline{4M}rd_0'}{m}\right). \tag{2.39}
$$

Let us consider $\Delta = \min(1/2, a^2 q Y^{-1+\epsilon})$. Following that, we will split S over r in the following way:

- $S_{0,d'}(Y,A)$ where we sum over $r = 0$,
- $S_{1,d'}(Y,A)$ where we sum over $0 < |r| < \Delta m$ and
- $S_{2,d'}(Y,A)$ where we sum over $\Delta m < |r| < m/2$.

2.4.3 Upper bound for $S_{1,d'}(Y,A)$

We start first by getting an upper bound for the sum $S_{1,d'}(Y,A)$. Indeed, we have

Proposition 2.11.

$$S_{1,d'}(Y,A) \ll \tau(d')d'^{\frac{1}{2}}A^2Y^{\frac{1}{2}+\epsilon}.$$

In order to achieve this, let us assert the following lemma:

Lemma 2.12. *Let $\alpha \in \mathbb{R}$, A a positive integer and ϕ be a periodic function of period r such that $|\phi| \leq 1$. Then, we have*

$$\sum_{\substack{m\leq X \\ (m,A)=1 \\ m \text{ squarefree}}} \lambda_m \phi(m) e(\alpha m) \ll \tau(A) r^{\frac{1}{2}} X^{\frac{1}{2}} (\log X)^7.$$

Proof. Use the same proof as Lemma 1 in [9] using the bound

$$\sum_{m\leq X} \lambda_m e(\alpha m) \ll X^{\frac{1}{2}} \log X.$$

□

Proof of Proposition 2.11. We begin by summing S_1 over m. Hence, get

$$\sum_{\substack{\frac{r}{\Delta}<m \\ (m,a4M)=1}} \lambda_n n^{-1/2} \overline{\epsilon}_m m^{-1/2} \chi_{Mrq}(m) V\left(\frac{2\pi h l^2 m}{a^2 q |d_0'| \sqrt{N}}\right) e\left(\frac{\overline{4Mr} d_0'}{m}\right)$$

$$= \lambda_h h^{-1/2} \sum_{\substack{\frac{r}{\Delta}<m \\ (m,a4M)=1}} \lambda_{l^2 m} (l^2 m)^{-1/2} \overline{\epsilon}_m m^{-1/2} \chi_{Mrq}(m) V\left(\frac{2\pi h l^2 m}{a^2 q |d_0'| \sqrt{N}}\right) e\left(\frac{\overline{4Mr} d_0'}{m}\right).$$

To separate l^2 and n in $\lambda_{l^2 m}$, we are going to use the following formula ([11])

$$\lambda_{l^2 m} = \sum_{u|(l^2,m)} \mu(u) \lambda_{\frac{l^2}{u}} \lambda_{\frac{m}{u}}.$$

Since m and u are squarefree, we can write m as um with $(u,m) = 1$, and we obtain

$$\frac{\lambda_k}{\sqrt{k}} \sum_{u|l^2} \mu(u) \frac{\lambda_{l^2/u} \lambda_u}{lu} \chi_{Mrq}(u)$$
$$\times \sum_{\substack{\frac{r}{u\Delta}<m \\ (m,au4M)=1}} \frac{\lambda_m \overline{\epsilon}_{um} \chi_{Mrq}(m)}{m} V\left(\frac{2\pi h l^2 u m}{a^2 q |d_0'| \sqrt{N}}\right) e\left(\frac{\overline{4Mr} d_0'}{um}\right). \quad (2.40)$$

Let us now focus on the inner sum. We have

$$e\left(\frac{\overline{4M}rd}{um}\right) = e\left(\frac{rd}{4Mum} - \frac{\overline{um}rd}{4M}\right),$$

where $\overline{um}um \equiv 1 \mod 4M$. Thus, the inner sum of (2.40) becomes Let us set

$$\begin{aligned}\sum_{\substack{\frac{r}{u\Delta}<m\\(m,au4M)=1}} & \frac{\lambda_m \overline{\epsilon}_{um}\chi_{Nrq}(m)}{m} V\left(\frac{2\pi hl^2um}{a^2q|d_0'|\sqrt{N}}\right) e\left(\frac{rd}{4Mum} - \frac{\overline{um}rd}{4M}\right) \\ &= \sum_{\substack{\frac{r}{u\Delta}<m\\(m,au4M)=1}} \lambda_m \phi(m)\frac{1}{m}V\left(\frac{2\pi hl^2um}{a^2q|d_0'|\sqrt{N}}\right)e\left(\frac{rd}{4M}\right),\end{aligned} \tag{2.41}$$

where

$$\phi(m) = \overline{\epsilon}_{um}\chi_{Nrq}(m)e\left(-\frac{\overline{um}rd}{4M}\right)$$

is a periodic function of period $4Mrq$ satisfying $|\phi(m)| < 1$. Let us now apply Abel's partial summation on (2.41) with

$$g(x) = \frac{1}{x}V\left(\frac{2\pi hl^2ux}{a^2q|d_0'|\sqrt{N}}\right)e\left(\frac{rd}{x4Mu}\right), \tag{2.42}$$

and

$$A(x) = \sum_{\substack{(m,au4M)=1\\ m\le x\\ m \text{ squarefree}}} \lambda_m \phi(m). \tag{2.43}$$

Hence, in a similar way as in [9], we have

$$\begin{aligned}\sum_{0<|r|<\Delta m}\sum & \frac{\lambda_n\overline{\epsilon_m}}{(nm)^{1/2}}\chi_{Mrq}(m)\, V\left(\frac{2\pi hl^2m}{a^2q|d_0'|\sqrt{N}}\right)e\left(\frac{\overline{4M}rd_0'}{m}\right) \\ &\ll d'^{1/2}h^{-3/2}l^{-3}a^3q^2Y^{-1/2+\epsilon}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}S_{1,d'}(Y,A) &\ll \sum_{a\le A}\sum_{hl^2}\sum_{q|ld'}\sum_{d_0} F\left(\frac{a^2q|d_0'|}{Y}\right)d'^{1/2}h^{-3/2}l^{-3}a^3q^2Y^{-1/2+\epsilon} \\ &\ll \tau(d')d'^{1/2}A^2Y^{1/2+\epsilon}.\end{aligned}$$

□

2.4.4 Upper bound for $S_{2,d'}(Y,A)$

The purpose of this subsection is to establish the following upper bound for $S_{2,d'}(Y,A)$.

Proposition 2.13. *We have*

$$S_{2,d'}(Y,A) \ll \tau(d'),$$

where the implied constant depends only on f and F.

However, in order to demonstrate this proposition, we require the following lemma.

Lemma 2.14. *Let $g(x)$ be a smooth integrable function on $\mathbb{R}$ which satisfies $g^{(j)} \ll_j B^j(|x|+X)^{-j}$ for all $j \geq 1$ and some X and B different from 0. Let also $\alpha \in \mathbb{R}$ such that $4M\alpha$ is not an integer. Then*

$$\sum_{n \equiv \nu[4M]} g(n)e(\alpha n) \ll_j B^j X \left(\frac{1}{X\|4M\alpha\|}\right)^j,$$

for any $j \geq 2$ and ν, where $\|4M\alpha\|$ is the closest integer to $4M\alpha$.

Proof. similar to Lemma 2 in [9]. □

Proof of Proposition 2.13. In order to apply Lemma 2.14, we are going to take

$$g(x) = V\left(\frac{2\pi n}{a^2 qx\sqrt{N}}\right) F\left(\frac{a^2 qx}{Y}\right).$$

To simplify the computations, let us take $A = a^2 q$, $H = \frac{2\pi n}{\sqrt{N}}$, and we have

$$g(x) = V\left(\frac{H}{Ax}\right) F\left(\frac{Ax}{Y}\right).$$

We can prove by induction that, for any $j > 1$, $g^{(j)}$ has the following form:

$$g^{(j)}(x) = \left(\frac{A}{Y}\right)^j F^{(j)}\left(\frac{Ax}{Y}\right) V\left(\frac{H}{Ax}\right)$$
$$+ \sum_{\substack{i_1+i_2+i_3=j \\ i_1 \neq j}} c_{i_1,i_2,i_3} \left(\frac{A}{Y}\right)^{i_1} \left(\frac{H}{Ax^2}\right)^{i_2} \left(\frac{H}{Ax}\right)^{\frac{k}{2}-1} \left(\frac{1}{x}\right)^{i_3} F^{(i_1)}\left(\frac{Ax}{Y}\right) \exp\left(-\frac{H}{Ax}\right), \tag{2.44}$$

for some integers c_{i_1,i_2,i_3}. Indeed, we have

$$g'(x) = \left(\frac{A}{Y}\right) F'\left(\frac{Ax}{Y}\right) V\left(\frac{H}{Ax}\right) + \left(\frac{H}{Ax^2}\right)\left(\frac{H}{Ax}\right)^{\frac{k}{2}-1} F\left(\frac{Ax}{Y}\right)\exp\left(-\frac{H}{Ax}\right).$$

For $j \geq 2$, we can see that the derivative of each element in $g^{(j-1)}$ can be written as linear combination fo the elements in $g^{(j)}$. Let us now get an upper bound of $g^{(j)}(x)$. It is well-known that $\exp(-x) \ll x^{-c}$ for any $c > 0$ and $x > 0$. In particular, we can choose

$$\exp\left(\frac{-H}{Ax}\right) \ll \left(\frac{Ax}{H}\right)^{\frac{k}{2}-1+i_2+j\epsilon/2}.$$

Hence, we get

$$\left(\frac{H}{Ax^2}\right)^{i_2}\left(\frac{H}{Ax}\right)^{\frac{k}{2}-1}\left(\frac{1}{x}\right)^{i_3}\exp\left(\frac{-H}{Ax}\right) \ll \left(\frac{1}{x}\right)^{i_2+i_3}\left(\frac{Ax}{H}\right)^{j\epsilon/2}.$$

Due to the fact that F is a bump function on $(0, 1)$, then there exists $0 < c' < C' < 1$ such that $c' < Ax/Y < C'$. Thus, we have $1/x \ll A/Y$ and

$$\left(\frac{A}{Y}\right)^{i_1}\left(\frac{H}{Ax^2}\right)^{i_2}\left(\frac{H}{Ax}\right)^{\frac{k}{2}-1}\left(\frac{1}{x}\right)^{i_3}\exp\left(\frac{-H}{Ax}\right) \ll \left(\frac{A}{Y}\right)^{i_1+i_2+i_3}\left(\frac{Ax}{H}\right)^{j\epsilon/2}.$$

On the other hand, we have

$$V(x) = \int_x^\infty \frac{t^{\frac{k}{2}-1}}{(k/2-1)!}e^{-t}dt \ll \int_x^\infty t^{-1-j\epsilon/2}dt \ll x^{-j\epsilon/2}.$$

Thus,

$$\left(\frac{A}{Y}\right)^j F^{(j)}\left(\frac{Ax}{Y}\right) V\left(\frac{H}{Ax}\right) \ll \left(\frac{A}{Y}\right)^j\left(\frac{Ax}{H}\right)^{j\epsilon/2}$$

Since $i_1 + i_2 + i_3 = j$, we obtain the same upper bound for all the elements of $g^{(j)}(x)$. Consequently,

$$g^{(j)}(x) \ll \left(\frac{A}{Y}\right)^j\left(\frac{Ax}{H}\right)^{j\epsilon/2} \max_{0\leq i_1\leq j} F^{(i_1)}\left(\frac{Ax}{Y}\right).$$

Let us recall that for any bump function $h(x)$ on $(0, 1)$ and for $j \geq 1$, we have

$$h(x) \ll (|x| + 1)^{-j}. \tag{2.45}$$

Hence, we get

$$F^{(i_1)}\left(\frac{Ax}{Y}\right) \ll \frac{Y^j}{(A|x|+Y)^j}.$$

Thus,

$$\begin{aligned} g^{(j)}(x) &\ll \left(\frac{A}{Y}\right)^j \left(\frac{Ax}{H}\right)^{j\epsilon/2} \frac{Y^j}{(A|x|+Y)^j} \\ &\ll \left(\frac{Y}{H}\right)^{j\epsilon/2} \frac{1}{(|x|+Y/A)^j}, \end{aligned} \tag{2.46}$$

where the last inequality comes from the fact that $Ax/Y < 1$. We can then apply Lemma 2.14 with $\alpha = \overline{4M}r/m$. Indeed, if we write $\overline{4M}4M = 1 + Km$, for some integer K, then

$$\alpha 4M = 4M\frac{\overline{4M}r}{m} = K + \frac{r}{m}.$$

Moreover, since $a^2qY^{-1+\epsilon} < |r|/m < 1/2$, $4M\alpha$ is different from zero and

$$\frac{1}{\|4M\alpha\|} \leq \frac{Y}{a^2qY^\epsilon} = \frac{Y}{AY^\epsilon}.$$

Therefore, using Lemma 2.14 we obtain

$$\begin{aligned} \sum_{d_0'} &\ll \left(\frac{Y}{H}\right)^{j\epsilon/2} \left(\frac{Y}{A}\right)\left(\frac{A}{Y}\frac{Y}{AY^\epsilon}\right)^j \\ &\ll H^{-j\epsilon/2}Y^{1-j\epsilon/2}. \end{aligned}$$

Now, if we come back to $S_{2,d'}(Y,A)$ and change H to $2\pi n/\sqrt{N}$, then we obtain

$$\begin{aligned} S_{2,d'}(Y,A) &\ll_j \sum_{a\leq A,(a,4Nd')=1} \sum_{\substack{n=hl^2m \\ (n,a)=1}} |\lambda_n| n^{-1/2}m^{-1/2} \sum_{q|ld'} \mu(q) \sum_{\Delta m<|r|<m/2} n^{-j\epsilon/2}Y^{1-j\epsilon/2} \\ &\ll_j Y^{1-j\epsilon/2}\left(\sum_{a\leq A}\right)\left(\sum_{n=hl^2m} |\lambda_n| n^{-1/2}m^{-1/2} \sum_{q|ld'} \mu(q) \sum_{\Delta m<|r|<m/2} n^{-j\epsilon/2}\right) \\ &\ll_j AY^{1-j\epsilon/2}\tau(d') \sum_{n=hl^2m} |\lambda_n| n^{-1/2}m^{-1/2}\tau(l) \sum_{\Delta m<|r|<m/2} n^{-j\epsilon/2}. \end{aligned}$$

Since $A < Y$, by taking a sufficiently large j, we can see that the inner sum is convergent, hence, bounded. Therefore, we only have $\tau(d')$ left in the upper bound, and we get the result. □

2.4.5 Estimation of $S_{0,d'}(Y,A)$

We are now approaching the last part of this section. Indeed, we will apply the Rankin-Selberg theory to obtain the main part of the sum $S_{M'}(Y)$ via $S_{0,d'}(Y,A)$. Let us then present our proposition:

Proposition 2.15. *We have*

$$S_{0,d'}(Y,A) = \prod_{p|d'}\left[1+\frac{1}{p}\left(1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]^{-1} C_f Y + O(\tau(d')(AY^{1/2}+A^{-1}Y)Y^{\epsilon})$$

for some constant $C_f \neq 0$.

Proof. Because r is equal to 0, $\chi_r(m) = 0$ for all m except for $m = 1$. Hence, from the definition of $S_{0,d'}(Y,A)$ in Subsection 2.4.2, we have

$$S_{0,d'}(Y,A) = 2\sum_{a\leq A,(a,4Md')=1}\mu(a)\sum_{\substack{n=hl^2\\(n,a)=1}}\frac{\lambda_n}{n^{1/2}}\sum_{q|ld'}\mu(q)\sum_{d'_0}, \tag{2.47}$$

where

$$\sum_{d'_0} = \sum_{qd'_0\in\mathcal{D}'} V\left(\frac{2\pi n}{a^2q|d'_0|\sqrt{N}}\right)F\left(\frac{a^2q|d'_0|}{Y}\right). \tag{2.48}$$

Moreover, using the same argument as in [17], then, for any continuous function g on $[x,y]$ with a continuous derivative g', we have

$$\sum_{\substack{d\equiv v[4M]\\y<d\leq x}} g(d) = \frac{1}{4M}\int_y^x g(t)dt + O\left(|g(x)|+|g(y)|+\int_x^y |g'(t)|dt\right).$$

Let us consider

$$g(x) = V\left(\frac{2\pi n}{a^2qx\sqrt{N}}\right)F\left(\frac{a^2qx}{Y}\right) \tag{2.49}$$

Hence, our sum is over $|d'_0| \equiv \omega v^2\overline{q} \mod 4M$ for any v coprime with $4M$ and $|d'_0| > 0$. Moreover, in a similar way as for (2.46), we have

$$\int_0^{\infty} g'(t)dt \ll \left(\frac{Y}{a^2q}\right)^{j-1}\left(\frac{Y}{n}\right)^{\epsilon}\int_0^{\infty}\frac{1}{(t+Y/a^2q)^j}dt \ll Y^{\epsilon}, \tag{2.50}$$

for any $j \geq 2$. Hence, (2.48) becomes

$$\begin{aligned}
\sum_{d_0'} &= \sum_{\nu^2 \in (\mathbb{Z}/4M\mathbb{Z})^*} \sum_{\substack{|d_0'| \equiv \omega \nu^2 \bar{q}[4N] \\ |d|>0}} V\left(\frac{2\pi n}{a^2 q |d_0'| \sqrt{N}}\right) F\left(\frac{a^2 q |d_0'|}{Y}\right) \\
&= \sum_{\nu^2 \in (\mathbb{Z}/4M\mathbb{Z})^*} \frac{1}{4M} \int_0^\infty V\left(\frac{2\pi n}{a^2 q t \sqrt{N}}\right) F\left(\frac{a^2 q t}{Y}\right) dt + O(Y^\epsilon) \\
&= \sum_{\nu^2 \in (\mathbb{Z}/4M\mathbb{Z})^*} \frac{Y}{a^2 q 4M} \int_0^\infty V\left(\frac{2\pi n}{t Y \sqrt{N}}\right) F(t) dt + O(Y^\epsilon) \\
&= \frac{\gamma(4M) Y}{a^2 q 4M} \int_0^\infty V\left(\frac{2\pi n}{t Y \sqrt{N}}\right) F(t) dt + O(Y^\epsilon),
\end{aligned}$$

where

$$\gamma(4M) = \#\{\nu^2 \mod 4M : (\nu, 4M) = 1\}.$$

Substituting $\sum_{d_0'}$ in Equation (2.47) with this later, we get

$$\begin{aligned}
S_{0,d'}(Y,A) = 2 &\sum_{a \leq A, (a, 4Md')=1} \mu(a) \sum_{\substack{n = hl^2 \\ (n,a)=1}} \frac{\lambda_n}{n^{1/2}} \sum_{q|ld'} \mu(q) \frac{\gamma(4M)\phi(d_0) Y}{a^2 q 4M} \\
&\times \int_0^\infty V\left(\frac{2\pi n}{t Y \sqrt{N}}\right) F(t) dt + O(\tau(d') A Y^{1/2+\epsilon}) \\
= \frac{\gamma(4M) Y}{2M} &\sum_{n = hl^2} \frac{\lambda_n}{n^{1/2}} \left(\sum_{\substack{a \leq A \\ (a, 4Mld')=1}} \frac{\mu(a)}{a^2} \right) \left(\sum_{q|ld'} \frac{\mu(q)}{q} \right) \\
&\times \int_0^\infty V\left(\frac{2\pi n}{t Y \sqrt{N}}\right) F(t) dt + O(\tau(d') A Y^{1/2+\epsilon}).
\end{aligned}$$

Note that the condition $(a, 4Mld') = 1$ follows from $(a, 4Md') = 1$ and $(a, n) = 1$, as well as the fact that all prime factors of $4M$ in $n = hl^2$ are all in h. On the other hand, we also have

$$\begin{aligned}
\sum_{\substack{a \leq A \\ (a, 4Mld')=1}} \frac{\mu(a)}{a^2} &= \sum_{(a, 4Mld')=1} \frac{\mu(a)}{a^2} - \sum_{\substack{a > A \\ (a, 4Mld')=1}} \frac{\mu(a)}{a^2} \\
&= \frac{1}{\zeta(2)} \prod_{p | 4Mld'} \left(1 - \frac{1}{p^2}\right)^{-1} + O(A^{-1}).,
\end{aligned}$$

and

$$\sum_{q|ld'} \frac{\mu(q)}{q} = \prod_{p|ld'} \left(1 - \frac{1}{p}\right)$$

Since $\zeta(2) = \pi^2/6$, we get

$$S_{0,d'}(Y,A) = \frac{3\gamma(4M)Y}{\pi^2 M} \prod_{p|4M} \left(1 - \frac{1}{p^2}\right)^{-1} \prod_{p|d'} \left(1 + \frac{1}{p}\right)^{-1}$$

$$\times \sum_{n=hl^2} \frac{\lambda_n}{n^{1/2}} \prod_{\substack{p|l \\ p\nmid d'}} \left(1 + \frac{1}{p}\right)^{-1} \int_0^\infty V\left(\frac{2\pi n}{tY\sqrt{N}}\right) F(t)dt + O(\tau(d')(A^{-1}Y + AY^{1/2})Y^\epsilon)$$

$$= \prod_{p|d'} \left(1 + \frac{1}{p}\right)^{-1} \frac{3\gamma(4M)Y}{\pi^2 M} \prod_{p|4M} \left(1 - \frac{1}{p^2}\right)^{-1}$$

$$\times \int_0^\infty \left(\sum_{n=hl^2} \frac{\lambda_n}{n^{1/2}} \prod_{\substack{p|l \\ p\nmid d'}} \left(1 + \frac{1}{p}\right)^{-1} V\left(\frac{2\pi n}{tY\sqrt{N}}\right)\right) F(t)dt + O(\tau(d')(A^{-1}Y + AY^{1/2})Y^\epsilon).$$

Therefore,

$$S_{0,d'}(Y,A) = \prod_{p|d'} \left(1 + \frac{1}{p}\right)^{-1} C_M Y \int_0^\infty \mathcal{B}_{d'}(tY\sqrt{N}) F(t)dt + O(\tau(d')(A^{-1}Y + AY^{1/2})Y^\epsilon), \tag{2.51}$$

where

$$C_M = \frac{3\gamma(4M)}{\pi^2 M} \prod_{p|4M} \left(1 - \frac{1}{p^2}\right)^{-1}$$

and

$$\mathcal{B}_{d'}(X) = \sum_{n=hl^2} \frac{b_{n,d'}}{n^{1/2}} V\left(\frac{2\pi n}{X}\right),$$

with

$$b_{n,d'} = \lambda_n \prod_{\substack{p|n \\ p\nmid 4Md'}} \left(1 + \frac{1}{p}\right)^{-1}.$$

Hence, the estimation of $S_{0,d'}(Y,A)$ depends on the function $\mathcal{B}_{d'}(X)$. Indeed, if we use the Mellin transformation of $V(x)$ as we did in Section 2.2, we can obtain

$$\mathcal{B}_{d'}(X) = \int_{(c)} L_{d'}(1/2 + s) \prod_{j=1}^{k/2-1} \left(1 + \frac{s}{j}\right) \Gamma(s) \left(\frac{2\pi}{X}\right)^{-s} ds, \tag{2.52}$$

for any $c > 0$, where

$$L_{d'}(s) = \sum_{n=hl^2} \frac{b_{n,d'}}{n^s}. \tag{2.53}$$

We begin by noticing that the coefficients $(b_{n,d'})_n$ are multiplicative as products of two multiplicative functions. Thus, for Re(s) large enough, we can write

$$L_{d'}(s) = P_{4M}(s)L_{4M,d'}(s)\tilde{L}_{4M,d'}(s), \tag{2.54}$$

where

$$P_{4M}(s) = \sum_{\substack{k \\ p|k \Rightarrow p|4M}} \frac{\lambda_n}{n^s}, \tag{2.55}$$

$$L_{4M,d'}(s) = \sum_{\substack{l^2 \\ p|l \Rightarrow d'}} \frac{b_{n,d'}}{n^s} = \sum_{\substack{l \\ p|l \Rightarrow p|d'}} \frac{\lambda_{l^2}}{l^{2s}}, \tag{2.56}$$

and

$$\tilde{L}_{4M,d'} = \sum_{l,(l,4Md')=1} \frac{b_{n,d'}}{n^s} = \sum_{\substack{l \\ (l,4Md')=1}} \frac{\lambda_{l^2}}{l^{2s}} \prod_{p|l} \left(1 + \frac{1}{p}\right)^{-1}. \tag{2.57}$$

Before we proceed, let us point out that we always take Re(s) large enough when we want to find the Euler product of $L_{d'}(s)$. First, the Euler products of P_{4M} and $L_{4M,d'}(s)$ can be deduced from the Euler product of $L(s,f)$ in Equation (1.14) and $L(s, Sym^2 f)$ Proposition 1.18 respectively. Hence, we have

$$P_{4M}(s) = \prod_{p|4N} (1 - \alpha_p p^{-s})^{-1}(1 - \beta_p - s)^{-1} =: \prod_{p|4M} P_p(s)^{-1}, \tag{2.58}$$

and

$$\begin{aligned} L_{4M,d'}(s) &= \sum_{\substack{l \\ p|l \Rightarrow p|d'}} \frac{\lambda_{l^2}}{l^{2s}} = \prod_{p|d'} \left[1 + \sum_{j \geq} \frac{\lambda_{p^{2j}}}{p^{j2s}}\right] \\ &= \prod_{p|d'} \frac{(1 - \chi_0(p)p^{-4s})}{(1 - \alpha_p^2 p^{-2s})(1 - \chi_0(p)p^{-2s})(1 - \beta_p^2 p^{-2s})} \\ &= \prod_{p|d'} \frac{(1 + \chi_0(p)p^{-2s})}{(1 - \alpha_p^2 p^{-2s})(1 - \beta_p^2 p^{-2s})} =: \prod_{p|d'} \frac{1}{Q_p(s)}. \end{aligned} \tag{2.59}$$

Note that the factor $(1 - \chi_0(p)p^{-4s})$ comes from $L(4s, \chi_0)$ in Proposition 1.18.

Secondly, for the Euler product of $L_{4M,d'}(s)$, we have

$$\begin{aligned}\tilde{L}_{4M,d'}(s) &= \sum_{(l,4Nd')=1} \frac{\lambda_{l^2}}{l^{2s}} \prod_{p|l}\left(1+\frac{1}{p}\right)^{-1} = \prod_{p\nmid 4Md'}\left[1+\left(1+\frac{1}{p}\right)^{-1}\sum_{j\geq 1}\frac{\lambda_{p^{2j}}}{p^{j2s}}\right] \\ &= \prod_{p\nmid 4Md'}\left(1+\frac{1}{p}\right)^{-1}\left[\frac{1}{p}+1+\sum_{j\geq 1}\frac{\lambda_{p^{2j}}}{p^{j2s}}\right] = \prod_{p\nmid 4Md'}\left(1+\frac{1}{p}\right)^{-1}\left[\frac{1}{p}+\frac{1}{Q_p(s)}\right] \\ &= \prod_{p\nmid 4Md'}\frac{1}{Q_p(s)}\left(1+\frac{1}{p}\right)^{-1}\left[\frac{Q_p(s)}{p}+1\right] = \prod_{p\nmid Md'}\frac{1}{Q_p(s)}T_p(s),\end{aligned}$$

where

$$T_p(s) = \left(1+\frac{1}{p}\right)^{-1}\left[\frac{Q_p(s)}{p}+1\right]. \tag{2.60}$$

Therefore, for $\mathrm{Re}(s)$ large enough, we have

$$L_{d'}(s) = \prod_{p|4M}\frac{1}{P_p(s)}\prod_{p\nmid 4Md'}T_p(s)\prod_{p\nmid 4M}\frac{1}{Q_p(s)}. \tag{2.61}$$

Now, let us check if $L_{d'}(s)$ can have an analytic continuation. Using Proposition 1.18 again, we know that for $\mathrm{Re}(s) > 1$,

$$\prod_{p}\frac{1}{Q_p(s)} = \frac{L(2s, sym^2 f)}{L(4s,\chi_0)},$$

and according to G. Shimura [15], $L(2s, Sym^2 f)$ has an analytic continuation on $\mathbb{C}$. It is also well known that $L(4s,\chi_0)$ has an analytic continuation on $\mathbb{C}$ as well, with a simple pole at $s = 1/4$ and does not have any zero in $\mathrm{Re}(s) > 1/4$. Thus, (2.61) becomes

$$L_{d'}(s) = \frac{L(2s, sym^2 f)}{L(4s,\chi_0)}\prod_{p|4M}\frac{1}{P_p(s)}\prod_{p|4M}Q_p(s)\prod_{p\nmid 4Md'}T_p(s). \tag{2.62}$$

We know that

$$\prod_{p|4M}P_p(s)^{-1} \quad \text{and} \quad \prod_{p|4M}Q_p(s)$$

are both finite, hence, they are holomorphic in $\mathbb{C}$. Moreover, $|\alpha_p|$ and $|\beta_p|$ are both less or equal than 1, thus, from the definition of $L_p(s)$ and $Q_P(s)$ in (2.58) and (2.59),

$$\prod_{p|4M}P_p(s)^{-1}\prod_{p|4M}Q_p(s) \neq 0$$

for all $\mathrm{Re}(s) > 0$. Next, we have

$$\begin{aligned} T_p(s) &= \left(1 + \frac{1}{p}\right)^{-1} \left[\frac{Q_p(s)}{p} + 1\right] = \left[\frac{Q_p(s) + p}{p+1}\right] \\ &= \left[1 + \frac{Q_p(s) - 1}{p+1}\right]. \end{aligned}$$

Thus, in order to search for the domain of absolute convergence of $\prod_{p \nmid 4Md'} T_p(s)$, we just need to study the convergence of

$$\sum_{p \nmid 4Md'} \left|\frac{Q_p(s) - 1}{p+1}\right|.$$

For $p \nmid 4M$, we have

$$\begin{aligned} Q_p(s) - 1 &= \frac{(1 - \alpha_p^2 p^{-2s})(1 - \beta_p^2 p^{-2s}) - (1 + p^{-2s})}{1 + p^{-2s}} \\ &= \frac{-(1 + \alpha_p^2 + \beta_p^2) p^{-2s} + \alpha_p^2 \beta_p^2 p^{-4s}}{1 + p^{-2s}}. \end{aligned}$$

However $|\alpha_p| = |\beta_p| = 1$ for $p \nmid 4Md'$. It follows that

$$\sum_{p \nmid 4Md'} \left|\frac{Q_p(s) - 1}{p+1}\right| \leq \sum_{p \nmid 4Md'} \frac{3p^{-2\mathrm{Re}(s)} + p^{-4\mathrm{Re}(s)}}{(p+1)(1 - p^{-2\mathrm{Re}(s)})} = \sum_{p} \frac{3 + p^{-2\mathrm{Re}(s)}}{(p+1)(p^{\mathrm{Re}(s)} - 1)},$$

which is convergent for $\mathrm{Re}(s) > 0$. Hence,

$$\prod_{p|4M} \frac{1}{P_p(s)} \prod_{p|4M} Q_p(s) \prod_{p \nmid 4Md'} T_p(s) \tag{2.63}$$

converges absolutely for $\mathrm{Re}(s) > 0$. As a result, the L-function $L_{d'}(s)$ has an analytic continuation on $\mathrm{Re}(s) > 1/4$. We can then move the integration line of $\mathcal{B}_{d'}(X)$ in Equation (2.52) to $(-1/4)$. Consequently, we obtain

$$\mathcal{B}_{d'}(X) = L_{d'}(1/2) + \frac{1}{2\pi i} \int_{(-1/4)} L_{d'}(1/2 + s)\Gamma(s) \left(\frac{X}{2\pi}\right)^s ds. \tag{2.64}$$

Let us now compute $L_{d'}(1/2)$. For this, we need to come back to the Rankin-Selberg L-function

$$L(s, f \otimes f) = L(2s, \chi_0) \sum_n \frac{\lambda_n^2}{n^s}.$$

From Corollary 1.15, $L(s, f \otimes f)$ has a simple pole at $s = 1$ with residue

$$\operatorname*{res}_{s=1} L(s, f \otimes f) = H > 0.$$

However, using Definition 1.17, we have

$$L(s, Sym^2 f) = \frac{L(s, f \otimes f)}{L(s, \chi_0)} = \frac{L(s, f \otimes f)}{\zeta(s) \prod_{p|N}(1 - p^{-s})},$$

where $\zeta(s)$ is the usual zeta function. We know that $\zeta(s)$ has a simple pole with residue 1 at $s = 1$. It then follows that this the pole of $L(s, f \otimes f)$ and $\zeta(s)$ are canceling each other, and we have

$$L(1, Sym^2 f) = \frac{H}{\prod_{p|N}(1 - p^{-1})} > 0,$$

and

$$\frac{L(1, Sym^2 f)}{L(2, \chi_0)} = \frac{6H \prod_{p|N}(1 - p^{-2})}{\pi^2 \prod_{p|N}(1 - p^{-1})} = \frac{6}{\pi^2} H \prod_{p|N}(1 + p^{-1}) > 0.$$

Therefore, we obtain

$$L_{d'}(1/2) = \frac{6}{\pi^2} \prod_{p|N}(1 + p^{-1}) \prod_{p|4M} \frac{1}{P_p(1/2)} \prod_{p|4M} Q_p(1/2) \prod_{p \nmid 4Md'} T_p(1/2).$$

We can now remove the factor $\prod_{p|d'} T_P(1/2)$ because $(d', 4M) = 1$. Thus, using the absolute convergence of (2.63) for $\mathrm{Re}(s) > 0$, we have

$$L_{d'}(1/2) = \prod_{p|d'} \frac{1}{T_p(1/2)} C_f'', \tag{2.65}$$

where

$$C_f'' = \frac{6}{\pi^2} H \prod_{p|N}(1 + p^{-1}) \prod_{p|4M} \frac{1}{P_p(1/2)} \prod_{p|4M} Q_p(1/2) \prod_{p \nmid 4M} T_p(1/2) > 0,$$

and

$$T_p(1/2) = \left(1 + \frac{1}{p}\right)^{-1} \left[1 + \frac{1}{p}\left(1 + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]$$

Finally, we need an upper bound for the integral in (2.64), *i.e.,*

$$\int_{(-1/4)} L_{d'}(1/2+s)\prod_{j=1}^{k/2-1}\left(1+\frac{s}{j}\right)\Gamma(s)\left(\frac{X}{2\pi}\right)^{s} ds$$
$$= \int_{(-1/4)} \frac{L(1+2s, Sym^2 f)}{L(2+4s,\chi_0)}\prod_{p|4M}\frac{1}{P_p(1/2+s)}\prod_{p|4M}Q_p(1/2+s)$$
$$\times \prod_{p\nmid 4Md'} T_p(1/2+s)\prod_{j=1}^{k/2-1}\left(1+\frac{s}{j}\right)\Gamma(s)\left(\frac{X}{2\pi}\right)^{s} ds.$$

We know that

$$\prod_{p|4M}\frac{1}{P_p(1/2+s)}\prod_{p|4M}Q_p(1/2+s)\prod_{p\nmid 4Md'}T_p(1/2+s)$$

is absolutely convergent on $\mathrm{Re}(s) > -1/2$, hence, bounded on the vertical line $(-1/4)$. On the other hand

$$\prod_{j=1}^{k/2-1}\left(1+\frac{s}{j}\right)$$

is of polynomial growth and $\Gamma(s)$ decay exponentially as $\mathrm{Im}(s)$ tends to ∞. As for

$$\frac{L(1+2s, Sym^2 f)}{L(2+4s,\chi_0(s))}$$

, we know from Corollary 1.20 that $L(1+2s, Sym^2 f)$ is of polynomial growth on every vertical strip. Moreover, it is well known that $L(1+2s,\chi_0)$ is also of polynomial growth. We can then deduce that

$$L_{d'}(1/4+it)\prod_{j=1}^{k/2-1}\left(1+\frac{1/4+it}{j}\right) \ll (|t|+1)^K,$$

for some K. Consequently,

$$\int_{(-1/4)} L_{d'}(1/2+s)\prod_{j=1}^{k/2-1}\left(1+\frac{s}{j}\right)\Gamma(s)\left(\frac{X}{2\pi}\right)^{s} ds \ll X^{-1/4},$$

and inserting this last upper bound and (2.65) into Equation (2.64), we

$$\mathcal{B}_{d'}(X) = \prod_{p|d'}\frac{1}{T_p(1/2)}C_f'' + O\left(\int_0^\infty F(t)X^{-1/4}dt\right). \tag{2.66}$$

Now that we get an approximation for $\mathcal{B}_{d'}(X)$, we take $X = tT\sqrt{N}$ and bring (2.66) in Equation (2.51). Therefore, if we replace $T_p(1/2)$ with (2.60), then we finally obtain

$$S_{0,d'}(Y,A) = \prod_{p|d'}\left[1+\frac{1}{p}\left(1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]C_fY + O(\tau(d')(A^{-1}Y + AY^{1/2}Y^{\epsilon})), \tag{2.67}$$

where

$$C_f = C_M C_f'' \int_0^{\infty} F(t)dt > 0.$$

□

Proof of Proposition 2.9. Gathering Proposition 2.10, 2.11, 2.13 and 2.15, we have

$$S_{d'}(Y) = \prod_{p|d'}\left[1+\frac{1}{p}\left(1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]^{-1} C_f$$
$$+ O(\tau(d')(AY^{1/2} + A^{-1}Y + d'^{1/2}A^2Y^{1/2} + A^{-3/2}Y^{5/4} + A^{-3}Y^{5/2})Y^{\epsilon}).$$

If we choose $A = Y^{3/14}$, then we have the result. □

Proof of Theorem 2.2. We have

$$S(Y,M') = \sum_{d'|M'} \mu(d')S_{d'}(Y).$$

Using Proposition 2.9, we have

$$S(Y,M') = \sum_{d'|M'} \mu(d')\prod_{p|d'}\left[1+\frac{1}{p}\left(1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]^{-1} C_fY + O\left(\sum_{d'|M'} \tau(d')d'^{1/2}Y^{\frac{13}{14}+\epsilon}\right). \tag{2.68}$$

However, we have

$$\sum_{d'|M'} \mu(d')\prod_{p|d'}\left[1+\frac{1}{p}\left(1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]^{-1} = \prod_{p|M'}\left[1-\left(1+\frac{1}{p}\left(1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right)^{-1}\right]$$
$$= \prod_{p|M'}\frac{1}{p}\left[1+\frac{1}{p}+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right]^{-1}.$$

Therefore, we obtain

$$S(Y,M') = \prod_{p|M'} \frac{1}{p}\left[1+\frac{1}{p}+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right]^{-1} C_f Y + O(\tau(M')^2 M'^{1/2} Y^{\frac{13}{14}+\epsilon}.)$$

□

Proof of Corollary 2.3. Using Hölder's inequality, we have

$$\left|\sum_{\substack{d_0\in\mathcal{D}\\|d_0|<Y}} L(1/2,f,\chi)\right| \leq \left(\sum_{\substack{d_0\in\mathcal{D}\\|d_0|<Y}} |L(1/2,f,\chi_{d_0})|^4\right)^{1/4}\left({\sum_{d_0}}' 1\right)^{3/4}, \tag{2.69}$$

where ${\sum_{d_0}}'$ means that the summation is over $d_0 \in \mathcal{D}$, $|d_0| < Y$ such that $L(1/2,f,\chi_{d_0}) \neq 0$. If we take $M' = 1$ in Theorem 2.2, we have

$$\left|\sum_{d_0} L(1/2,f,\chi)\right| \asymp Y.$$

Hence, from Theorem 2.1, (2.69) becomes

$$Y \ll Y^{1/2+\epsilon}\left({\sum_{d_0}}' 1\right)^{3/4}.$$

Thus,

$${\sum_{d_0}}' 1 \gg Y^{2/3-\epsilon}.$$

□

2.5 Estimation of the weighted sum $S^{'}(Y,M^{'})$

In the previous section, we already have an approximation of the sum

$$\sum_{\substack{d_0\in\mathcal{D}\\d_0\equiv 0[M']}} L(1/2,f,\chi_{d_0})F\left(\frac{|d_0|}{Y}\right). \tag{2.70}$$

We will see in Chapter 4, that the error term will play a crucial role in the application of the sieve method, hence, we want to minimize it. One way to achieve this is to consider analytic methods using multiple Dirichlet series. However, as we have seen in [10], [18], [19], and also in Chapter 3, the multiple Dirichlet series with the desired analytic properties contain additional weighted factors. The addition of these weight factors however alters the main term in the estimation above. Hence, in this section we give an estimation of a sum analog to (2.70) but with the addition of weight factors.

In the rest of this chapter, we will write d in an unique way $d = d_0 d_1^2$ such that and d_0 is squarefree. We will then assume the existence of the Dirichlet Polynomials $P_{d_0,d_1}(s)$ defined by

$$P_{d_0,d_1}(s) = \prod_{p^\gamma || d_1} \left(1 + A^{(\gamma)}_{d_0,p} p^{-s} + \cdots + A^{(\gamma)}_{d_0,p^{4\gamma}} p^{-4\gamma s}\right) = \sum_{g | d_1^4} \frac{A(d_1, d_0, g)}{g^s}. \tag{2.71}$$

for d_0 and d_1 coprime with M. We will study more about these Dirichlet Polynomials in Chapter 3, but now, let us state some of their properties ([10]):

1. for $\epsilon > 0$ and $\mathrm{Re}(s) \geq 1/2$, we have

$$P_{d_0,d_1}(s) \ll |d_0 d_1^2|^\epsilon, \tag{2.72}$$

2. for all γ, $0 \leq e \leq 4\gamma$ and prime p,

$$A_{d_0,p^e} = \begin{cases} \chi_{d_0}(p^e) A^{(\gamma)}_{1,p^e} & \text{if } p \nmid d_0 \\ \chi_{\frac{d_0}{p}}(p^e) A^{(\gamma)}_{p,p^e} & \text{if } p | d_0, \end{cases} \tag{2.73}$$

3. for all γ, $0 \leq e \leq 2\gamma - 1$ and prime p,

$$A_{p,p^{2e+1}} = 0. \tag{2.74}$$

With this two (2.73) and (2.74), we can deduce that

$$A^{(\gamma)}_{d_0,p^e} = \chi_{d_0}(p^e)\left(A^{(\gamma)}_{1,p^e} - A^{(\gamma)}_{p,p^e}\right) + A^{(\gamma)}_{p,p^e}. \tag{2.75}$$

Hence,

$$\begin{aligned} A(d_1, d_0, g) &= \prod_{p^e || g} A^{(v_p(d_1))}_{d_0,p^e} \\ &= \prod_{p^e || g} \left(\chi_{d_0}(p^e)\left(A^{(v_p(d_1))}_{1,p^e} - A^{(v_p(d_1))}_{p,p^e}\right) + A^{(v_p(d_1))}_{p,p^e}\right) \\ &= \sum_{\substack{\delta | g \\ p|\delta \Rightarrow p\delta \nmid g}} \chi_{d_0}(\delta) C(d_1, g, \delta), \end{aligned} \tag{2.76}$$

where $v_p(\cdot)$ is the p-adic valuation and

$$C(d_1, g, \delta) = \prod_{p^e || \delta} \left(A_{1,p^e}^{(v_p(d_1))} - A_{p,p^e}^{(v_p(d_1))}\right) \prod_{p^e || (g/\delta)} A_{p,p^e}^{(v_p(d_1))}. \tag{2.77}$$

Thus, we obtain

$$P_{d_0,d_1}(s) = \sum_{g | d_1^4} \frac{1}{g^s} \sum_{\substack{\delta | g \\ p|\delta \Rightarrow p\delta \nmid g}} \chi_{d_0}(\delta) C(d_1, g, \delta). \tag{2.78}$$

Therefore, our goal in this section is to prove the following theorem:

Theorem 2.16. *Let M' be a positive number coprime to M, F be a bump function on $(0,1)$ and $P_{d_0,d_1}(s)$ be the Dirichlet polynomials defined in Equation (2.71). Then, for $\epsilon > 0$ and $Y > 0$, we have*

$$\begin{aligned} S'(Y, M') &= \sum_{\substack{(d_0 d_1, M)=1 \\ d_0 d_1^2 \equiv 0 [M'] \\ d_0 \in \mathcal{D}}} \sum L(1/2, f, \chi_{d_0}) P_{d_0,d_1}(1/2) F\left(\frac{d_1^2 |d_0|}{Y}\right) \\ &= \rho'(M') C_f' Y + O(\tau(M')^2 M'^{1/2} Y^{13/14+\epsilon}), \end{aligned} \tag{2.79}$$

where

$$\rho'(M') = \prod_{p | M'} \frac{1 + \sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma}} + \left(\sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma-1}}\right)\left(1 + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\right)}{p\left(1 + \sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma}}\right)\left[1 + \frac{1}{p} + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\right]}, \tag{2.80}$$

$$C_f' = C_f \prod_{p \nmid M} \left[1 + \sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma}}\right], \tag{2.81}$$

$$\rho'''(p^\gamma) = \left(1 + \rho''(p^\gamma, p^2) p^{-1} + \cdots + \rho''(p^\gamma, p^{4\gamma}) p^{-2\gamma}\right), \tag{2.82}$$

and

$$\rho''(p^\gamma, p^{2e}) = \left[A_{p,p^{2e}}^{(\gamma)} + \left(A_{1,p^{2e}}^{(\gamma)} - A_{p,p^{2e}}^{(\gamma)}\right)\left(1 + \frac{1}{p}\left(1 + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right)^{-1}\right]. \tag{2.83}$$

Remark 2.17. *We can see that the error term in Theorem 2.16 is the same as in Theorem 2.2. However, we will see in Chapter 4 that it can be reduced using the analytic properties of $P_{d_0,d_1}(s)$ described in Chapter 3.*

Before proving Theorem 2.16, let us first begin by applying Lemma 2.8 on $S(Y,M')$ to deal with the condition $d_0 d_1^2 \equiv 0[M']$. Hence, we obtain

$$S'(Y,M') = \sum_{d'|M'} \mu(d') S'_{d'}(Y), \tag{2.84}$$

where

$$S_{d'}(Y) = \sum_{\substack{(d_0 d_1, d')=1 \\ d_0 \in \mathcal{D}}} L(1/2, f, \chi_{d_0}) P_{d_0,d_1}(1/2) F\Big(\frac{d_1^2|d_0|}{Y}\Big). \tag{2.85}$$

With Equation (2.78), we then obtain

$$S_{d'}(Y) = \sum_{(d_1,d')=1} \sum_{g|d_1^4} \frac{1}{g^{1/2}} S_{d'}(Y, d_1, g), \tag{2.86}$$

where

$$S_{d'}(Y,d_1,g) = \sum_{\substack{\delta|g \\ p|\delta \Rightarrow p\delta\nmid g}} C(d_1,g,\delta) \sum_{\substack{(d_0,d')=1 \\ d_0 \in \mathcal{D}}} \chi_{d_0}(\delta) L(1/2, f, \chi_{d_0}) F\Big(\frac{d_1^2|d_0|}{Y}\Big). \tag{2.87}$$

Thus, let us study $S_{d'}(Y,d_1,g)$:

Proposition 2.18. *Let* $\epsilon > 0$ *and* $Y > 0$*. Then, we have*

$$S_{d'}(Y,d_1,g) = \rho''(d_1,g) \frac{C_f Y}{d_1^2} \prod_{p|d'} \Big[1 + \frac{1}{p} \Big(1 + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j} \Big)^{-1} \Big]^{-1} + O(\tau(d') d'^{1/2} (d_1^{-2} Y)^{13/14} Y^{\epsilon}), \tag{2.88}$$

where $\rho''(d_1,g) = 0$ *if* g *is not a perfect square, otherwise*

$$\rho''(d_1, g = g_1^2) = \prod_{p^e || g_1} \Big[A_{p,p^{2e}}^{(v_p(d_1))} + \Big(A_{1,p^{2e}}^{(v_p(d_1))} - A_{p,p^{2e}}^{(v_p(d_1))} \Big) \Big(1 + \frac{1}{p} \Big(1 + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j} \Big)^{-1} \Big)^{-1} \Big]. \tag{2.89}$$

Proof. We begin by writing δ as $\delta = \delta_0 \delta_1^2$ such that δ_1 is squarefree. Let us notice that δ and M are coprime since $\delta | d_1$. Hence, for any $d_0' \in \mathcal{D}'$, we have $\chi_{d_0'}(\delta) = \chi_{d_0'}(\delta_0)$ if $(\delta_1, d_0') = 1$. Thus, using the same technique as in Section 2.4, we can write $S'_{d'0}(Y, d_1, g)$ as

$$S'_{d'}(Y,d_1,g) = S'_{d'}(Y,d_1,g,A) + R'_{d'}(Y,d_1,g,A), \tag{2.90}$$

where

$$S'_{d'}(Y,d_1,g,A) = 2\sum_{\substack{\delta|g\\ \delta=\delta_0\delta_1^2\\ p|\delta\Rightarrow p\delta\nmid g}} C(d_1,g,\delta)\sum_{a\le A,(a,4Md'\delta_1)=1}\mu(a)$$
$$\times \sum_{\substack{(d_0',d')=1\\ d_0'\in\mathcal{D}'}} \chi_{d_0'}(\delta)A(a^2|d_0'|\sqrt{N},\chi_{a^2d_0'})F\left(\frac{a^2d_1^2|d_0'|}{Y}\right), \tag{2.91}$$

and

$$R'_{d'}(Y,d_1,g,A) = 2\sum_{\substack{\delta|g\\ \delta=\delta_0\delta_1^2\\ p|\delta\Rightarrow p\delta\nmid g}} C(d_1,g,\delta)\sum_{(b,4Md'\delta_1)=1}\left(\sum_{a>A,\,a|b}\mu(a)\right) \tag{2.92}$$
$$\times \sum_{d_0\in\mathcal{D}} \chi_{d_0}(\delta)\mathcal{A}(b^2|d_0|\sqrt{N},\chi_{b^2d_0})F\Big(\frac{b^2d_1^2|d_0|}{Y}\Big).$$

Using the fact that

$$C(d_1,g,\delta) \ll |d_0d_1^2|^{\epsilon} \ll Y^{\epsilon},$$

and the same argument as in Subsection 2.4.1, we obtain

$$R'_{d'}(Y,d_1,g,A) \ll ((d_1^{-2}Y)^{5/4}A^{-3/2} + A^{-3}(d_1^{-2}Y)^{3/2})Y^{\epsilon}. \tag{2.93}$$

Thus, as in Subsection 2.4.2, we have

$$S_{d'}(Y,d_1,g,A) = 2\sum_{\substack{\delta|g\\ \delta=\delta_0\delta_1^2\\ p|\delta\Rightarrow p\delta\nmid g}} C(d_1,g,\delta)\sum_{a\le A,(a,4Md'\delta_1)=1}\mu(a)\sum_{\substack{(d_0',\delta_1d')=1\\ d_0'\in\mathcal{D}'}}$$
$$\times \sum_{\substack{n=hl^2m\\ (n,a)=1\\ (d_0',l)=1}} \frac{\lambda_n\chi_{d_0'}(m\delta_0)}{n^{1/2}}V\left(\frac{2\pi n}{a^2|d_0'|\sqrt{N}}\right)F\left(\frac{a^2|d_0'|}{Y}\right)$$
$$= 2\sum_{\substack{\delta|g\\ \delta=\delta_0\delta_1^2\\ p|\delta\Rightarrow p\delta\nmid g}} C(d_1,g,\delta)\sum_{a\le A,(a,4Md'\delta_1)=1}\mu(a)\sum_{\substack{n=hl^2m\\ (n,a)=1}}\frac{\lambda_n}{n^{1/2}}$$
$$\times \sum_{q|ld'\delta_1}\mu(q)\sum_{qd_0'\in\mathcal{D}'}\chi_{qd_0'}(m\delta_0)V\left(\frac{2\pi n}{a^2q|d_0'|\sqrt{N}}\right)F\left(\frac{a^2qd_1^2|d_0'|}{Y}\right). \tag{2.94}$$

Using the same formula as in (2.36) on $\chi_{d_0'}(m\delta_0)$, we obtain

$$S_{d'}(Y,d_1,g,A) = 2\sum_{\substack{\delta|g\\ \delta=\delta_0\delta_1^2\\ p|\delta\Rightarrow p\delta\nmid g}} C(d_1,g,\delta)\sum_{a\le A,(a,4Md'\delta)=1}\mu(a)$$
$$\times\sum_{\substack{n=hl^2m\\(n,a)=1}}\frac{\lambda_n}{n^{1/2}}\frac{\overline{\epsilon_{m\delta_0}}}{(m\delta_0)^{1/2}}\sum_{q|ld'\delta_1}\mu(q)\sum_{2|r|<m}\chi_{Nrq}(m\delta_0)\sum_{d_0'},\tag{2.95}$$

where

$$\sum_{d_0'} = \sum_{qd_0'\in\mathcal{D}'} V\left(\frac{2\pi n}{a^2q|d_0'|\sqrt{N}}\right)F\left(\frac{a^2qd_1^2|d_0'|}{Y}\right)e\left(\frac{\overline{4M}rd}{m\delta_0}\right),$$

with $\overline{4M}4M \equiv 1 \mod m\delta_1$. With $\Delta = \min(1/2, a^2qY^{\epsilon-1})$ as in Subsection 2.4.2, we are going to split $S_{d'}(Y,d_1,g,A)$ over r follows

- $S'_{0,d'}(Y,d_1,g,A)$ where we sum over $r=0$,
- $S'_{1,d'}(Y,d_1,g,A)$ where we sum over $0<|r|<\Delta m\delta_0$ and
- $S'_{2,d'}(Y,d_1,g,A)$ where we sum over $\Delta m\delta_1<|r|<m\delta_0/2$.

The upper bound of $S'_{1,d'}(Y,d_1,g,A)$ and $S'_{2,d'}(Y,d_1,g,A)$ can be obtained in a similar way as Proposition 2.11 and 2.13. Indeed, we can have

$$S'_{1,d'}(Y,d_1,g,A) \ll \tau(d')d'^{1/2}A^2(d_1^{-2}Y)^{1/2}Y^{\epsilon},\tag{2.96}$$

and

$$S'_{2,d'}(Y,d_1,g,A) \ll \tau(d')Y^{\epsilon}.\tag{2.97}$$

As for $S'_{0,d'}(Y,d_1,g,A)$, we have $m\delta_0 = 1$

$$S'_{0,d'}(Y,d_1,g,A) = 2\sum_{\substack{\delta|g\\ \delta=\delta_1^2\\ p|\delta\Rightarrow p\delta\nmid g}} C(d_1,g,\delta)\sum_{a\le A,(a,4Md'\delta_1)=1}\mu(a)\sum_{\substack{n=hl^2\\(n,a)=1}}\frac{\lambda_n}{n^{1/2}}$$
$$\times\sum_{q|ld'\delta_1}\mu(q)\sum_{qd_0'\in\mathcal{D}'}V\left(\frac{2\pi n}{a^2q|d_0'|\sqrt{N}}\right)F\left(\frac{a^2qd_1^2|d_0'|}{Y}\right)$$
$$= \sum_{\substack{\delta|g\\ \delta=\delta_1^2\\ p|\delta\Rightarrow p\delta\nmid g}} C(d_1,g,\delta)S_{0,d'\delta_1}\left(\frac{Y}{d_1^2},A\right),\tag{2.98}$$

where $S_{0,d'}(Y,A)$ is defined in Equation (2.47). Hence, using Proposition 2.15, we obtain

$$S'_{0,d'}(Y,d_1,g,A) = \frac{1}{d_1^2} \sum_{\substack{\delta|g \\ \delta=\delta_1^2 \\ p|\delta \Rightarrow p\delta\nmid g}} C(d_1,g,\delta) \prod_{p|d'\delta_1} \left[1+\frac{1}{p}\left(1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]^{-1} C_f Y$$
$$+ O(\tau(M')(A(d_1^{-2}Y)^{1/2} + A^{-1}(d_1^{-2}Y))Y^{\epsilon}). \tag{2.99}$$

Therefore, putting all the upper bounds (2.93), (2.96), (2.97) and (2.99) in Equation (2.90), and choosing $A = (d_1^{-2}Y)^{3/14}$, we obtain

$$S_{d'}(Y,d_1,g) = \frac{1}{d_1^2} \sum_{\substack{\delta|g \\ \delta=\delta_1^2 \\ p|\delta \Rightarrow p\delta\nmid g}} C(d_1,g,\delta) \prod_{p|d'\delta_1} \left[1+\frac{1}{p}\left(1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]^{-1} C_f Y$$
$$+ O(\tau(d')d'^{1/2}(d_1^{-2}Y)^{13/14}Y^{\epsilon}). \tag{2.100}$$

We can sharpen the result by using the formula of $C(d_1,g,\delta)$ in (2.77). Then, after isolating the primes $p|d'$, which is possible because d' and δ_1 are coprime, we have

$$\sum_{\substack{\delta|g \\ \delta=\delta_1^2 \\ p|\delta \Rightarrow p\delta\nmid g}} C(d_1,g,\delta) \prod_{p|\delta_1} \left[1+\frac{1}{p}\left(1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]^{-1}$$
$$= \sum_{\substack{\delta|g \\ \delta=\delta_1^2 \\ p|\delta \Rightarrow p\delta\nmid g}} \prod_{p^e||\delta} \left(A_{1,p^e}^{(v_p(d_1))} - A_{p,p^e}^{(v_p(d_1))}\right) \prod_{p^e||(g/\delta)} A_{p,p^e}^{(v_p(d_1))} \prod_{p|\delta_1} \left[1+\frac{1}{p}\left(1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]^{-1} \tag{2.101}$$

Let us notice that here, g must be a perfect square, otherwise, there must exist an odd number e such that $p^e||(g/\delta_1^2)$, in which case from (2.74), we have

$A_{p,p^e}^{v_p(d_1)} = 0$. Thus,

$$\sum_{\substack{\delta|g \\ \delta=\delta_1^2 \\ p|\delta \Rightarrow p\delta\nmid g}} C(d_1,g,\delta)\prod_{p|\delta_1}\left[1+\frac{1}{p}\left(1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]^{-1}$$

$$= \prod_{p^{2e}||g}\left[A_{p,p^{2e}}^{(v_p(d_1))} + \left(A_{1,p^{2e}}^{(v_p(d_1))} - A_{p,p^{2e}}^{(v_p(d_1))}\right)\left(1+\frac{1}{p}\left(1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right)^{-1}\right]. \tag{2.102}$$

This completes proof. □

Corollary 2.19. *For $\epsilon > 0$ and $Y > 0$, we have*

$$S'_{d'}(Y) = \rho_0(d')C'_f Y + O(\tau(d')d'^{1/2}Y^{13/14+\epsilon}), \tag{2.103}$$

for some $R'_{d'}(Y)$, where

$$C'_f = C_f \prod_{p\nmid M}\left[1+\sum_{\gamma=1}^{\infty}\frac{\rho'''(p^\gamma)}{p^{2\gamma}}\right], \tag{2.104}$$

$$\rho_0(d') = \prod_{p|d'}\left[1+\sum_{\gamma=1}^{\infty}\frac{\rho'''(p^\gamma)}{p^{2\gamma}}\right]^{-1}\left[1+\frac{1}{p}\left(1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]^{-1}, \tag{2.105}$$

$$\rho'''(p^\gamma) = \left(1+\rho''(p^\gamma,p^2)p^{-1}+\cdots+\rho''(p^\gamma,p^{4\gamma})p^{-2\gamma}\right), \tag{2.106}$$

and

$$\rho''(p^\gamma,p^{2e}) = \left[A_{p,p^{2e}}^{(\gamma)} + \left(A_{1,p^{2e}}^{(\gamma)} - A_{p,p^{2e}}^{(\gamma)}\right)\left(1+\frac{1}{p}\left(1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right)^{-1}\right]. \tag{2.107}$$

Proof. With (2.86), we will define $R'_{d'}(Y)$ in the following way:

$$S'_{d'}(Y) = C_f Y\prod_{p|d'}\left[1+\frac{1}{p}\left(1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]^{-1}\sum_{(d_1,d')=1}\frac{1}{d_1^2}\sum_{g_1|d_1^2}\frac{\rho''(d_1,g_1^2)}{g_1}$$
$$+ O\left(\tau(d')d'^{1/2}Y^{13/14+\epsilon}\sum_{d_1}\frac{1}{d^{13/7}}\right). \tag{2.108}$$

Let us notice that, here, we write g as g_1^2. However, we have

$$\sum_{(d_1,d')=1} \frac{1}{d_1^2} \sum_{g_1|d_1^2} \frac{\rho''(g_1^2)}{g_1} = \sum_{(d_1,d')=1} \frac{1}{d_1^2} \sum_{g_1|d_1^2} \prod_{p^e||g_1} \frac{\rho''(d_1,p^{2e})}{p^e}$$
$$= \sum_{(d_1,d')=1} \frac{1}{d_1^2} \prod_{p^\gamma||d_1} \left(1+\rho''(p^\gamma,p^2)p^{-1}+\cdots+\rho''(p^\gamma,p^{4\gamma})p^{-2\gamma}\right)$$
$$= \prod_{p\nmid Md'} \left[1+\sum_{\gamma=1}^{\infty}\frac{\rho'''(p^\gamma)}{p^{2\gamma}}\right],$$

where

$$\rho'''(p^\gamma) = \left(1+\rho''(p^\gamma,p^2)p^{-1}+\cdots+\rho''(p^\gamma,p^{4\gamma})p^{-2\gamma}\right). \tag{2.109}$$

Using the fact that $\rho'''(p^\gamma) \ll (p^\gamma)^\epsilon$ for any $p \nmid M, \gamma$ and $\epsilon > 0$, we can conclude that

$$\prod_{p\nmid M} \left[1+\sum_{\gamma=1}^{\infty}\frac{\rho'''(p^\gamma)}{p^{2\gamma}}\right]$$

is absolutely convergent and we have

$$S'_{d'}(Y) = \prod_{p|d'} \left[1+\sum_{\gamma=1}^{\infty}\frac{\rho'''(p^\gamma)}{p^{2\gamma}}\right]^{-1} \prod_{p|d'} \left[1+\frac{1}{p}\left(1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]^{-1} C'_f(Y)$$
$$+ O(\tau(d')d'^{1/2}Y^{13/14+\epsilon}), \tag{2.110}$$

where

$$C'_f = C_f \prod_{p\nmid M} \left[1+\sum_{\gamma=1}^{\infty}\frac{\rho'''(p^\gamma)}{p^{2\gamma}}\right]$$

□

Now, let us conclude this chapter with the following proof.

Proof of Theorem 2.16. Using Corollary 2.19 in Equation (2.84), we obtain

$$S'(Y,M') = \sum_{d'|M'} \mu(d')\rho_0(d')C'_f Y + O(\tau(M')^2 M'^{1/2} Y^{13/14+\epsilon}), \tag{2.111}$$

Focusing on the main term, we have

$$\sum_{d'|M'} \mu(d')\rho_0(d') = \sum_{d'|M'} \mu(d') \prod_{p|d'} \left[1 + \sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma}}\right]^{-1} \left[1 + \frac{1}{p}\left(1 + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]^{-1}$$

$$= \prod_{p|M'} \left[1 - \left(1 + \sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma}}\right)^{-1} \left[1 + \frac{1}{p}\left(1 + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]^{-1}\right]. \tag{2.112}$$

However, we have

$$1 - \left(1 + \sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma}}\right)^{-1} \left[1 + \frac{1}{p}\left(1 + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right]^{-1}$$

$$= \frac{1 + \sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma}} + \left(\sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma-1}}\right)\left(1 + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\right)}{p\left(1 + \sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma}}\right)\left[1 + \frac{1}{p} + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\right]}. \tag{2.113}$$

Hence, we have our result. □

Chapter 3

Multiple Dirichlet series

It is in this chapter that we give details on the Dirichlet polynomials $P_{d_0,d_1}(s)$ introduced in Section 2.5 and why they are needed. Briefly, in order to obtain estimates of sums similar to those in the previous chapter by means of analytic techniques, one naturally needs to consider the Dirichlet series

$$\sum_{d} \frac{L(1/2, f, \chi_d)}{d^w},$$

and hope that it admits the desired analytic properties. That is not necessarily true, but it turns out that one needs to add the weight factors and consider the following double Dirichlet series instead:

$$\sum_{d} \frac{L(s, f, \chi_{d_0}) P_{d_0,d_1}(s)}{d^w}, \tag{3.1}$$

(we are still using the notation $d = d_0 d_1^2$). This double Dirichlet series has nice analytic properties, and specifically, a functional equation. This latter will allow us to analyze such double Dirichlet series quite accurately, and get a sharpened upper bound of the error term in Theorem 2.16 as we will see in Chapter 4.

3.1 Basic concept

Let f be a newform as in Section 1.2 and M be the squarefree number as in (2.1). For any Dirichlet character χ, we are going to define the L-functions $L_M(s, f, \chi)$ and $L_M(w, \chi)$ by

$$\begin{aligned} L_M(s, f, \chi) &= \sum_{(m,M)=1} \frac{\lambda_m \chi(n)}{n^s} \\ &= L(s, f, \chi) \prod_{p|M} \left(1 - \chi(p)\alpha_p p^{-s}\right)\left(1 - \chi(p)\beta_p p^{-s}\right), \end{aligned}$$

and

$$L_M(w,\chi) = \sum_{(m,M)=1} \frac{\chi(m)}{m^w} = L(w,\chi)\prod_{p|M}\left(1-\chi(p)p^{-w}\right).$$

We will extend the definition of the character χ_d by

$$\chi_d(n) = \left(\frac{D}{n}\right),$$

where

$$D = \begin{cases} d & \text{if } d \equiv 1 \mod 4 \\ 4d & \text{if } d \equiv 2,3 \mod 4. \end{cases} \tag{3.2}$$

If d is squarefree, then the functional equations of $L_M(s,f,\chi_d)$ and $L_M(w,\chi_d)$ may be found, respectively, by using Equation 1.15 and the functional equation of $L(w,\chi_d)$:

$$\left(\frac{|D|}{\pi}\right)^{\frac{w}{2}} \Gamma\left(\frac{w+a}{2}\right) L(w,\chi_d) = \left(\frac{|D|}{\pi}\right)^{\frac{1-w}{2}} \Gamma\left(\frac{1-w+a}{2}\right) L(1-w,\chi_d), \tag{3.3}$$

where $a = 0,1$ is given by $\chi_d(-1) = (-1)^a$. We will then assume the following proposition, which will be the basis of this chapter:

Proposition 3.1. *For positive integer n and d, let us write $d = d_0 d_1^2$ and $n = n_0 n_1^2$, such that d_0 and n_0 are both squarefree. Then, for $l_1, l_2 | M$ positive, $a_1, a_2 \in \{\pm 1\}$, there exist two unique Dirichlet polynomials $P_{a_1 l_1 \cdot d_0, d_1}(s)$ and $Q_{a_2 l_2 \cdot n_0, n_1}(w)$ of the form*

$$P_{a_1 l_1 \cdot d_0, d_1}(s) = \prod_{p^\gamma || d_1} \left(1 + A^{(\gamma)}_{a_1 l_1 \cdot d_0, p} p^{-s} + \cdots + A^{(\gamma)}_{a_1 l_1 \cdot d_0, p^{4\gamma}} p^{-4\gamma s}\right), \tag{3.4}$$

and

$$Q_{a_2 l_2 \cdot n_0, n_1}(w) = \prod_{p^\gamma || n_1} \left(1 + B^{(\gamma)}_{a_2 l_2 \cdot n_0, p} p^{-w} + \cdots + B^{(\gamma)}_{a_2 l_2 \cdot n_0, p^{2\gamma}} p^{-2\gamma w}\right), \tag{3.5}$$

for some real numbers $A^{(\gamma)}_{a_1 l_1 \cdot d_0, p^e}$ and $B^{(\gamma)}_{a_2 l_2 \cdot n_0, p^e}$, such that these two Dirichlet polynomials satisfy the following functional equations

$$P_{a_1 l_1 d_0, d_1}(s) = d_1^{2(1-2s)} P_{a_1 l_1 d_0, d_1}(1-s), \tag{3.6}$$

and

$$Q_{a_2 l_2 n_0, n_1}(w) = n_1^{1-2w} Q_{a_2 l_2 n_0, n_1}(1-w), \tag{3.7}$$

and the following interchange of summation

$$\sum_{(d,M)=1} \frac{L_M(s,f,\chi_{d_0}\chi_{a_1l_1})\chi_{a_2l_2}(d_0)P_{a_1l_1d_0,d_1}(s)}{d^w} = \sum_{(n,M)=1} \frac{L_M(w,\tilde{\chi}_{n_0}\chi_{a_2l_2})\chi_{a_1l_1}(n_0)\lambda_n Q_{a_2l_2n_0,n_1}(w)}{n^s}, \tag{3.8}$$

for sufficiently large $\mathrm{Re}(s)$, $\mathrm{Re}(w)$, *where* $\tilde{\chi}_{n_0}=\left(\frac{\cdot}{n_0}\right)$. *Moreover, they have the following upper bound*

$$P_{a_1l_1d_0,d_1}(s) \ll |d|^{\epsilon}, \qquad \lambda_n Q_{a_2l_2n_0,n_1}(w) \ll |n|^{\frac{1}{2}+\epsilon}, \tag{3.9}$$

for any $\epsilon > 0$, *and* $\mathrm{Re}(s) \geq 1/2$ *and* $\mathrm{Re}(w) \geq 3/2$.

Proof. The existence and the upper bound of $P_{a_1l_1\cdot d_0,d_1}(s)$ and $Q_{a_2l_2\cdot n_0,n_1}(w)$ are proven in [20], while their construction is detailed in [10]. □

Therefore, let us consider the double Dirichlet series

$$Z_M(s,w,\chi_{a_2l_2},\chi_{a_1l_1}) = \sum_{(d,M)=1} \frac{L_M(s,f,\chi_{d_0}\chi_{a_1l_1})\chi_{a_2l_2}(d_0)P_{a_1l_1d_0,d_1}(s)}{d^w}, \tag{3.10}$$

or equivalently

$$Z_M(s,w,\chi_{a_2l_2},\chi_{a_1l_1}) = \sum_{(n,M)=1} \frac{L_M(w,\tilde{\chi}_{n_0}\chi_{a_2l_2})\chi_{a_1l_1}(n_0)\lambda_n Q_{a_2l_2n_0,n_1}(w)}{n^s}. \tag{3.11}$$

3.2 Systems of functional equations

The properties of this double Dirichlet series are primarily determined by its functional equations and its domain of convergence. In this section, we will provide a proof for its two functional equations.

Before we proceed, let us consider the set

$$Div(M) = \{al : 0 < l|M, a = \pm 1\},$$

and take $a_1l_1, a_2l_2 \in Div(M)$. We can then define the $2\tau(M)$ column vector

$$\overrightarrow{Z_M}(s,w,\chi_{a_2l_2},\chi_{Div(M)})$$

whose component is of the form $Z_M(s,w,\chi_{a_2l_2},\chi_{al})$ for all $al \in Div(M)$.

Proposition 3.2. *For $w \neq 1$, there exists a matrix $\Phi^{(a_2 l_2)}(w)$ of dimension $2\tau(M) \times 2\tau(M)$ such that for any s where the series*

$$Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1}) = \sum_{(n,M)=1} \frac{L_M(w, \tilde{\chi}_{n_0}\chi_{a_2 l_2})\chi_{a_1 l_1}(n_0)\lambda_n Q_{a_2 l_2 n_0, n_1}(w)}{n^s}$$

converges absolutely, we have

$$\prod_{p|(M/l_2)} (1 - p^{-2+2w}) \cdot \overrightarrow{Z_M}(s, w, \chi_{a_2 l_2}, \chi_{Div(M)}) = \Phi^{(a_2 l_2)}(w)\overrightarrow{Z_M}(s + w - 1/2, 1 - w, \chi_{a_2 l_2, \chi_{Div(M)}}). \tag{3.12}$$

Moreover, the components of the matrix $\Phi^{(a_2 l_2)}(w)$ are meromorphic on $\mathbb{C}$.

Proof. We start by writing $L_M(w, \tilde{\chi}_{n_0}\chi_{a_2 l_2})$ as

$$L_M(w, \tilde{\chi}_{n_0}\chi_{a_2 l_2}) = L(w, \tilde{\chi}_{n_0}\chi_{a_2 l_2}) \cdot \prod_{p|(M/l_2)} (1 - \tilde{\chi}_{n_0}\chi_{a_2 l_2}(p)p^{-w}). \tag{3.13}$$

Indeed, we can ignore the prime $p|l_2$ because $\chi_{a_2 l_2}(p) = 0$ for any $p|l_2$. Since n_0 is an odd positive squarefree number, then using the quadratic reciprocity,

$$\tilde{\chi}_{n_0}(m) = \left(\frac{\chi_{-1}(n_0)n_0}{m}\right) = \chi_{\chi_{-1}(n_0)n_0}(m)$$

which is exactly the real primitive character of conductor n_0. Moreover, let us notice that in $L(w, \tilde{\chi}_{n_0}\chi_{a_2 l_2})$, we sum over positive m. Thus, using the fact that $(n_0, a_2 l_2) = 1$ and n_0 is odd, we get

$$\chi_{\chi_{-1} n_0}\chi_{a_2 l_2}(m) = \left(\frac{\chi_{-1}(n_0)n_0}{m}\right)\left(\frac{D_{a_2 l_2}}{m}\right) = \left(\frac{\chi_{-1}(n_0)n_0 D_{a_2 l_2}}{m}\right) = \chi_{\chi_{-1}(n_0)n_0 \cdot a_2 l_2},$$

where

$$D_{a_2 l_2} = \begin{cases} a_2 l_2 & \text{if} \quad a_2 l_2 \equiv 1 \mod 4 \\ 4a_2 l_2 & \text{otherwise} \end{cases}.$$

Hence, using Equation 3.3, we have the functional equation

$$G_\epsilon(w)(n_0|D_{a_2 l_2}|)^{w/2} L(w, \tilde{\chi}_{n_0}\chi_{a_2 l_2}) = G_\epsilon(1-w)(n_0|D_{a_2 l_2}|)^{(1-w)/2} L(1-w, \tilde{\chi}_{n_0}\chi_{a_2 l_2}), \tag{3.14}$$

where $\epsilon = \tilde{\chi}_{n_0}\chi_{a_2 l_2}(-1)$ and

$$G_\epsilon(w) = \begin{cases} \pi^{-\frac{w}{2}}\Gamma(\frac{w}{2}) & \text{if} \quad \epsilon = 1 \\ \pi^{-\frac{w+1}{2}}\Gamma(\frac{w+1}{2}) & \text{if} \quad \epsilon = -1 \end{cases}.$$

By definition, we have $\chi_{a_2 l_2}(-1) = \text{sign}(a_2 l_2) = a_2$. As a result, we have

$$\tilde{\chi}_{n_0}\chi_{a_2 l_2}(-1) = a_2\chi_{-1}(n_0) = \begin{cases} a_2 & \text{if } n_0 \equiv 1 \mod 4 \\ -a_2 & \text{if } n_0 \equiv -1 \mod 4. \end{cases} \tag{3.15}$$

Again, recall that n_0 is odd, that is why we have $\tilde{\chi}_{n_0}(-1) = \chi_{-1}(n_0)$. Then, by applying (3.7), (3.13), (3.14) and (3.15) in Equation (3.11), we get

$$\begin{aligned} Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1}) = \sum_{a_3=\pm 1} \sum_{\substack{(n,M)=1 \\ n_0 \equiv a_3 [4]}} & \frac{G_{a_2 a_3}(1-w)(|D_{a_2 l_2}|)^{1/2-w}}{G_{a_2 a_3}(w) n^{s+w-1/2}} \\ & \times L_M(1-w, \tilde{\chi}_{n_0}\chi_{a_2 l_2})\chi_{a_1 l_1}(n_0)\lambda_n Q_{a_2 l_2 n_0, n_1}(1-w) \\ & \times \prod_{p|(M/l_2)} (1 - \tilde{\chi}_{n_0}\chi_{a_2 l_2}(p)p^{-w})(1 - \tilde{\chi}_{n_0}\chi_{a_2 l_2}(p)p^{-1+w})^{-1}. \end{aligned} \tag{3.16}$$

Since

$$1 - p^{-2+2w} = (1 - \tilde{\chi}_{n_0}\chi_{a_2 l_2}(p)p^{-1+w})(1 + \tilde{\chi}_{n_0}\chi_{a_2 l_2}(p)p^{-1+w}),$$

for all $p|(M/l_2)$ different from 2, we can multiply (3.16) by

$$\prod_{p|(M/l_2)} (1 - p^{-2+2w})$$

to cancel all the factors $\prod_{p|(M/l_2)}(1 - \tilde{\chi}_{n_0}\chi_{a_2 l_2}(p)p^{-1+w})^{-1}$ except for $p = 2$ that we will deal separately. Hence, we have

$$\begin{aligned} & \prod_{p|(M/l_2)} (1 - p^{-2+2w}) \cdot Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1}) \\ & \quad = \sum_{a_3=\pm 1} \frac{G_{a_2 a_3}(1-w)(|D_{a_2 l_2}|)^{1/2-w}}{G_{a_2 a_3}(w)} \\ & \quad \times \sum_{\substack{(n,M)=1 \\ n_0 \equiv a_3 [4]}} \frac{L_M(1-w, \tilde{\chi}_{n_0}\chi_{a_2 l_2})\chi_{a_1 l_1}(n_0)\lambda_n Q_{a_2 l_2 n_0, n_1}(1-w)}{n^{s+w-1/2}} \\ & \quad \times \prod_{p|(M/l_2)} (1 - \tilde{\chi}_{n_0}\chi_{a_2 l_2}(p)p^{-w}) \prod_{\substack{p|(M/l_2) \\ p\nmid 2}} (1 + \tilde{\chi}_{n_0}\chi_{a_2 l_2}(p)p^{-1+w}) \\ & \quad \times A_2(1-w, \tilde{\chi}_{n_0}\chi_{a_2 l_2}). \end{aligned} \tag{3.17}$$

where

$$A_2(w, \tilde{\chi}_{n_0}\chi_{a_2 l_2}) = \begin{cases} (1-2^{-2w})(1-\tilde{\chi}_{n_0}\chi_{a_2 l_2}(2)2^{-w})^{-1}, & \text{if } 2|l_2 \\ 1 & \text{otherwise} \end{cases} \tag{3.18}$$

However, we have

$$\prod_{p|(M/l_2)} (1-\tilde{\chi}_{n_0}\chi_{a_2 l_2}(p)p^{-w}) \prod_{\substack{p|(M/l_2)\\ p\nmid 2}} (1+\tilde{\chi}_{n_0}\chi_{a_2 l_2}(p)p^{-1+w})$$
$$= \sum_{\substack{l_3,l_4|(M/l_2)\\ 2\nmid l_4}} \mu(l_3)\tilde{\chi}_{n_0}\chi_{a_2 l_2}(l_3 l_4) l_3^{-w} l_4^{-1+w}.$$

Let us notice that $\tilde{\chi}_{n_0}(l_3 l_4) = \chi_{l_3 l_4}(n_0)$, using the fact that $(n, M) = 1$ and $2|M$. Hence, from (3.17), we obtain

$$\prod_{p|(M/l_2)} (1-p^{-2+2w}) \cdot Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1})$$
$$= \sum_{a_3=\pm 1} \frac{G_{a_2 a_3}(1-w)(|D_{a_2 l_2}|)^{1/2-w}}{G_{a_2 a_3}(w)} \sum_{\substack{l_3,l_4|(M/l_2)\\ 2\nmid l_4}} \mu(l_3)\chi_{a_2 l_2}(l_3 l_4) l_3^{-w} l_4^{-1+w}$$
$$\times \sum_{\substack{(n,M)=1\\ n_0\equiv a_3[4]}} A_2(1-w, \tilde{\chi}_{n_0}\chi_{a_2 l_2}) \frac{L_M(1-w, \tilde{\chi}_{n_0}\chi_{a_2 l_2})\chi_{a_1 l_1}\chi_{l_3 l_4}(n_0)\lambda_n Q_{a_2 l_2 n_0, n_1}(1-w)}{n^{s+w-1/2}},$$

To deal with the last condition $n_0 \equiv a_3[4]$, we will use the following sieve

$$\frac{1+a_3\chi_{-1}(n_0)}{2} = \begin{cases} 1 & \text{if } n_0 \equiv a_3 \mod 4 \\ 0 & \text{if } n_0 \equiv -a_3 \mod 4. \end{cases} \tag{3.19}$$

Hence,

$$\prod_{p|(M/l_2)} (1-p^{-2+2w}) \cdot Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1})$$
$$= \frac{1}{2} \sum_{a_3=\pm 1} \frac{G_{a_2 a_3}(1-w)(|D_{a_2 l_2}|)^{1/2-w}}{G_{a_2 a_3}(w)} \sum_{\substack{l_3,l_4|(M/l_2)\\ 2\nmid l_4}} \mu(l_3)\chi_{a_2 l_2}(l_3 l_4) l_3^{-w} l_4^{-1+w}$$
$$\times \Bigg[\sum_{(n,M)=1} A_2(1-w, \tilde{\chi}_{n_0}\chi_{a_2 l_2}) \frac{L_M(1-w, \tilde{\chi}_{n_0}\chi_{a_2 l_2})\chi_{a_1 l_1}\chi_{l_3 l_4}(n_0)\lambda_n Q_{a_2 l_2 n_0, n_1}(1-w)}{n^{s+w-1/2}}$$
$$+ a_3 \sum_{(n,M)=1} A_2(1-w, \tilde{\chi}_{n_0}\chi_{a_2 l_2}) \frac{L_M(1-w, \tilde{\chi}_{n_0}\chi_{a_2 l_2})\chi_{-1}\chi_{a_1 l_1}\chi_{l_3 l_4}(n_0)\lambda_n Q_{a_2 l_2 n_0, n_1}(1-w)}{n^{s+w-1/2}} \Bigg].$$

Let us notice that

$$\chi_{a_1l_1}\chi_{l_3l_4}(n_0) = \chi_{a_1l_1l_3l_4}(n_0) = \chi_{al}(n_0)$$

and

$$\chi_{-1}\chi_{a_1l_1}\chi_{l_3l_4}(n_0) = \chi_{-a_1l_1l_3l_4}(n_0) = \chi_{-al}(n_0)$$

for some $al \in Div(M)$ because $(n, M) = 1$. On the other hand, using the definition of $A_2(w, \tilde{\chi}_{n_0}\chi_{a_2l_2})$ in (3.18), we have

$$A_2(w, \tilde{\chi}_{n_0}\chi_{a_2l_2}) = \begin{cases} 1 & \text{if} \quad 2|l_2 \\ 1 + \tilde{\chi}_{n_0}\chi_{a_2l_2}(2)2^{-w} & \text{if} \quad a_2l_2 \equiv 1 \mod 4, \\ 1 + 2^{-2w} & \text{if} \quad a_2l_2 \equiv -1 \mod 4. \end{cases} \tag{3.20}$$

Therefore, if $a_2l_2 \equiv 1 \mod 4$, we obtain

$$\begin{aligned} &\prod_{p|(M/l_2)} (1 - p^{-2+2w}) \cdot Z_M(s, w, \chi_{a_2l_2}, \chi_{a_1l_1}) \\ &= \frac{1}{2} l_2^{1/2-w} \sum_{l_3,l_4|(M/l_2)} \mu(l_3)\chi(l_3l_4) l_3^{-w} l_4^{-1+w} \sum_{a_3=\pm 1} \frac{G_{a_2a_3}(1-w)(|D_{a_2l_2}|)^{1/2-w}}{G_{a_2a_3}(w)} \\ &\times \left(Z_M(s + w - 1/2, 1 - w, \chi_{a_2l_2}, \chi_{a_1l_1l_3l_4}) + a_3 Z_M(s, w, \chi_{a_2l_2}, \chi_{-a_1l_1l_3l_4})\right), \end{aligned} \tag{3.21}$$

if $2|l_2$, we have

$$\begin{aligned} &\prod_{p|(M/l_2)} (1 - p^{-2+2w}) \cdot Z_M(s, w, \chi_{a_2l_2}, \chi_{a_1l_1}) \\ &= \frac{1}{2} (4l_2)^{1/2-w} \sum_{l_3,l_4|(M/l_2)} \mu(l_3)\chi(l_3l_4) l_3^{-w} l_4^{-1+w} \sum_{a_1=\pm 1} \frac{G_{a_2a_3}(1-w)(|D_{a_2l_2}|)^{1/2-w}}{G_{a_2a_3}(w)} \\ &\times \left(Z_M(s + w - 1/2, 1 - w, \chi_{a_2l_2}, \chi_{a_1l_1l_3l_4}) + a_3 Z_M(s, w, \chi_{a_2l_2}, \chi_{-a_1l_1l_3l_4})\right), \end{aligned} \tag{3.22}$$

and, if $a_2l_2 \equiv -1 \mod 4$, we get

$$\begin{aligned} &\prod_{p|(M/l_2)} (1 - p^{-2+2w}) \cdot Z_M(s, w, \chi_{a_2l_2}, \chi_{a_1l_1}) \\ &= \frac{1}{2} (4l_2)^{1/2-w} (1 + 2^{-2+2w}) \sum_{l_3,l_4|(M/l_2), 2\nmid l_4} \mu(l_3)\chi(l_3l_4) l_3^{-w} l_4^{-1+w} \\ &\times \sum_{a_1=\pm 1} \frac{G_{a_2a_3}(1-w)(|D_{a_2l_2}|)^{1/2-w}}{G_{a_2a_3}(w)} \\ &\times \left(Z_M(s + w - 1/2, 1 - w, \chi_{a_2l_2}, \chi_{a_1l_1l_3l_4}) + a_3 Z_M(s, w, \chi_{a_2l_2}, \chi_{-a_1l_1l_3l_4})\right). \end{aligned} \tag{3.23}$$

This completes proof. □

In the same way, we can define the $2\tau(M)$ column vector

$$\overrightarrow{Z_M}(s,w,\chi_{Div(M)},\chi_{a_1l_1})$$

whose component has the form $Z_M(s,w,\chi_{al},\chi_{a_1l_1})$ for all $al \in Div(M)$. Therefore, we have

Proposition 3.3. *For any s, there exists a matrix* $\Psi^{(a_2l_2)}(s)$ *of dimension* $2\tau(M)\times 2\tau(M)$ *such that for w where the series*

$$Z_M(s,w,\chi_{a_2l_2},\chi_{a_1l_1}) = \sum_{(d,M)=1} \frac{L_M(s,f,\chi_{d_0}\chi_{a_1l_1})\chi_{a_2l_2}(d_0)P_{a_1l_1d_0,d_1}(s)}{d^w}$$

converges absolutely, we have

$$\begin{aligned}\prod_{p|(M/l_1)} (1-\alpha_p^2 p^{-2+2s})(1-\beta_p^2 p^{-2+2s})\overrightarrow{Z_M}(s,w,\chi_{Div(M)},\chi_{a_1l_1}) = \\ \Psi^{(a_1l_1)}(s)\overrightarrow{Z_M}(1-s,w+2s-1,\chi_{Div(M)},\chi_{a_1l_1}).\end{aligned} \tag{3.24}$$

Moreover, the components of the matrix $\Psi^{(a_2l_2)}(s)$ *are meromorphic in* $\mathbb{C}$.

Proof. Let us begin by noticing that

$$\chi_{d_0}\chi_{a_1l_1}(m) = \chi_{a_1l_1d_0}(m)$$

for any odd positive integer m, and in particular, for $(m,M)=1$. Hence,

$$L_M(s,f,\chi_{d_0}\chi_{a_1l_1}) = L_M(s,f,\chi_{a_1d_0l_1}), \tag{3.25}$$

and we obtain

$$\begin{aligned} L_M(s,f,\chi_{d_0}\chi_{a_1l_1}) &= L(s,f,\chi_{a_1d_0l_1}) \prod_{p|(M/l_1)} (1-\alpha_p\chi_{a_1d_0l_1}(p)p^{-s})(1-\beta_p\chi_{a_1d_0l_1}(p)p^{-s}) \\ &= L(s,f,\chi_{a_1d_0l_1})\left(\sum_{l_\alpha|(M/l_1)} \mu(l_\alpha)\alpha_{l_\alpha}\chi_{a_1d_0l_1}(l_\alpha)l_\alpha^{-s}\right) \\ &\quad \times\left(\sum_{l_\beta|(M/l_1)} \mu(l_\beta)\beta_{l_\beta}\chi_{a_1d_0l_1}(l_\beta)l_\beta^{-s}\right). \end{aligned}$$

Therefore, using the functional equation 1.15, we obtain

$$\begin{aligned} L_M(s,f,\chi_{d_0}\chi_{a_1l_1}) &= \omega\chi_{a_1d_0l_1}(-N)(d_0|D_{a_1d_0l_1}|)^{1-2s}\frac{G(s)}{G(1-s)}L_M(1-s,f,\chi_{d_0}\chi_{a_1l_1}) \\ &\times\left(\sum_{l_\alpha|(M/l_1)}\mu(l_\alpha)\alpha_{l_\alpha}\chi_{a_1d_0l_1}(l_\alpha)l_\alpha^{-s}\right)\left(\sum_{l_\beta|(M/l_1)}\mu(l_\beta)\beta_{l_\beta}\chi_{a_1d_0l_1}(l_\beta)l_\beta^{-s}\right) \\ &\times\prod_{p|(M/l_1)}(1-\alpha_p\chi_{a_1d_0l_1}(p)p^{-1+s})^{-1}(1-\beta_p\chi_{a_1d_0l_1}(p)p^{-1+s})^{-1}, \end{aligned} \tag{3.26}$$

where

$$D_{a_1d_0l_1} = \begin{cases} a_1l_1 & \text{if } a_1d_0l_1 \equiv 1 \mod 4 \\ 4a_1l_1 & \text{otherwise,} \end{cases}$$

and

$$G(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s\Gamma\left(s+\frac{k-1}{2}\right).$$

In the same way as Proposition 3.2, we can insert the (3.6) and (3.26) in Equation (3.10). Thus,

$$\begin{aligned} &Z_M(s,w,\chi_{a_2l_2},\chi_{a_1l_1}) \\ &= \sum_{(d,M)=1}\omega\chi_{a_1d_0l_1}(-N)|D_{a_1d_0l_1}|^{1-2s}\frac{G(s)}{G(1-s)} \\ &\times\frac{L_M(1-s,f,\chi_{d_0}\chi_{a_1l_1})\chi_{a_2l_2}(d_0)P_{a_1d_0l_1}(1-s)}{d^{w+2s-1}} \\ &\times\left(\sum_{l_\alpha|(M/l_1)}\mu(l_\alpha)\alpha_{l_\alpha}\chi_{a_1d_0l_1}(l_\alpha)l_\alpha^{-s}\right)\left(\sum_{l_\beta|(M/l_1)}\mu(l_\beta)\beta_{l_\beta}\chi_{a_1d_0l_1}(l_\beta)l_\beta^{-s}\right) \\ &\times\prod_{p|(M/l_1)}(1-\alpha_p\chi_{a_1d_0l_1}(p)p^{-1+s})^{-1}(1-\beta_p\chi_{a_1d_0l_1}(p)p^{-1+s})^{-1}. \end{aligned} \tag{3.27}$$

Now, we let us use the formula

$$\begin{aligned} (1-\alpha_p^2p^{-2+2s})(1-\beta_p^2p^{-2+2s}) &= (1-\alpha_p\chi_{a_1d_0l_1}(p)p^{-1+s})(1-\beta_p\chi_{a_1d_0l_1}(p)p^{-1+s}) \\ &\times(1+\alpha_p\chi_{a_1d_0l_1}(p)p^{-1+s})(1+\beta_p\chi_{a_1d_0l_1}(p)p^{-1+s}), \end{aligned}$$

for $p|(M/l_1)$ different from 2 to eliminate the factor

$$\prod_{p|(M/l_1)}(1-\alpha_p\chi_{a_1d_0l_1}(p)p^{-1+s})^{-1}(1-\beta_p\chi_{a_1d_0l_1}(p)p^{-1+s})^{-1}.$$

As we have seen before, $p = 2$ is an exception, hence we will isolate it. Thus,

$$\prod_{p|(M/l_1)} (1-\alpha_p^2 p^{-2+2s})(1-\beta_p^2 p^{-2+2s}) \cdot Z_M(s,w,\chi_{a_2 l_2},\chi_{a_1 l_1})$$
$$= \sum_{(d,M)=1} \omega \chi_{a_1 d_0 l_1}(-N)|D_{a_1 d_0 l_1}|^{1-2s} \frac{G(s)}{G(1-s)}$$
$$\times \frac{L_M(1-s,f,\chi_{d_0}\chi_{a_1 l_1})\chi_{a_2 l_2}(d_0) P_{a_1 d_0 l_1}(1-s)}{d^{w+2s-1}}$$
$$\times \left(\sum_{l_\alpha|(M/l_1)} \mu(l_\alpha)\alpha_{l_\alpha}\chi_{a_1 d_0 l_1}(l_\alpha) l_\alpha^{-s} \right)\left(\sum_{l_\beta|(M/l_1)} \mu(l_\beta)\beta_{l_\beta}\chi_{a_1 d_0 l_1}(l_\beta) l_\beta^{-s} \right)$$
$$\times \left(\sum_{k_\alpha|(M/l_1),(k_\alpha,2)=1} \alpha_{k_\alpha}\chi_{a_1 d_0 l_1}(k_\alpha) k_\alpha^{-1+s} \right)\left(\sum_{k_\beta|(M/l_1),(k_\alpha,2)=1} \beta_{k_\beta}\chi_{a_1 d_0 l_1}(k_\beta) k_\beta^{-1+s} \right)$$
$$\times B_2(s,\chi_{a_1 d_0 l_1}), \tag{3.28}$$

where

$$B_2(s,\chi_{a_1 d_0 l_1}) = \begin{cases} (1-\alpha_2^2 2^{-2s})(1-\alpha_2\chi_{a_1 d_0 l_1}(2)2^{-w})^{-1} & \\ \times(1-\beta_2^2 2^{-2s})(1-\beta_2\chi_{a_1 d_0 l_1}(2)2^{-w})^{-1} & \text{if } 2|l_1 \\ 1 & \text{otherwise.} \end{cases}$$

More precisely, $B_2(s,\chi_{a_1 d_0 l_1})$ is equal to

$$\begin{cases} 1 & \text{if } \ 2|l_1 \\ (1+\alpha_2\chi_{a_1 d_0 l_1}(2)2^{-w})(1+\beta_2\chi_{a_1 d_0 l_1}(2)2^{-w}) & \text{if } \ a_1 d_0 l_1 \equiv 1 \mod 4, \\ (1-\alpha_2^2 2^{-2s})(1-\beta_2^2 2^{-2s}) & \text{if } \ a_1 d_0 l_1 \equiv -1 \mod 4. \end{cases} \tag{3.29}$$

In this scenario, it is crucial to understand each quadratic character. Since each case is different (depending on $a_1 l_1$ and N), we will demonstrate only one example and skip the others because they are very similar. Let us take the trivial case, *i.e,* $a_1 l_1 \equiv 1 \mod 4$ and $(N,2) = 1$. If $d_0 \equiv 1 \mod 4$, then,

$$\chi_{a_1 d_0 l_1}(-N) = \left(\frac{a_1 d_0 l_1}{-N}\right) = \left(\frac{a_1 l_1}{-N}\right)\left(\frac{d_0}{-N}\right) = \left(\frac{a_1 l_1}{-N}\right)\left(\frac{-N}{d_0}\right) = \chi_{a_1 l_1}(-N)\chi_{-N}(d_0)$$

and in the same way,

$$\chi_{a_1 d_0 l_1}(l) = \chi_{a_1 l_1}(l)\chi_l(d_0)$$

for all $1 \leq l|(M/l1)$. If $d_0 \equiv 3 \mod 4$, then,

$$\chi_{a_1 d_0 l_1}(-N) = \left(\frac{4a_1 d_0 l_1}{-N}\right) = \left(\frac{-4a_1 l_1}{-N}\right)\left(\frac{-d_0}{-N}\right)$$
$$= -\left(\frac{-4a_1 l_1}{-N}\right)\left(\frac{-N}{-d_0}\right) = \chi_{-a_1 l_1}(-N)\chi_{-N}(d_0),$$

and

$$\chi_{a_1 d_0 l_1}(l) = \left(\frac{4a_1 d_0 l_1}{l}\right) = \left(\frac{-4a_1 l_1}{l}\right)\left(\frac{-d_0}{l}\right)$$
$$= \left(\frac{-4a_1 l_1}{l}\right)\left(\frac{l}{-d_0}\right) = \chi_{a_1 l_1}(l)\chi_l(d_0)$$

We ca use χ_{-1} to sieve the congruence class of d_0 as in (3.19). Thus, we get

$$\prod_{p|(M/l_1)} (1 - \alpha_p^2 p^{-2+2s})(1 - \beta_p^2 p^{-2+2s}) \cdot Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1})$$
$$= \omega l_1^{1-2s} \frac{G(s)}{G(1-s)} \Bigg[\Bigg(\chi_{a_1 l_1}(-N) \sum_{l_\alpha|(M/l_1)} \mu(l_\alpha)\alpha_{l_\alpha}\chi_{l_\alpha}(a_1 l_1) l_\alpha^{-s}$$
$$\times \sum_{l_\beta|(M/l_1)} \mu(l_\beta)\beta_{l_\beta}\chi_{l_\beta}(a_1 l_1) l_\beta^{-s} \sum_{k_\alpha|(M/l_1)} \alpha_{k_\alpha}\chi_{k_\alpha}(a_1 l_1) k_\alpha^{-1+s}$$
$$\times \sum_{k_\beta|(M/l_1),} \beta_{k_\beta}\chi_{k_\beta}(a_1 l_1) k_\beta^{-1+s}(Z_M(1-s, w+2s+1, \chi_{-N}\chi_{a_2 l_2 l_\alpha l_\beta k_\alpha k_\beta}, \chi_{a_1 l_1})$$
$$+ Z_M(1-s, w+2s+1, \chi_{-1}\chi_{-N}\chi_{a_2 l_2 l_\alpha l_\beta k_\alpha k_\beta}, \chi_{a_1 l_1})\Bigg)$$
$$+ \Bigg(\chi_{-a_1 l_1}(-N) 4^{1-2s}(1 - \alpha_2^2 2^{-2+2s})(1 - \beta_2^2 2^{-2+2s}) \sum_{l_\alpha|(M/l_1),(l_\alpha,2)=1} \mu(l_\alpha)\alpha_{l_\alpha}\chi_{-a_1 l_1}(l_\alpha) l_\alpha^{-s}$$
$$\times \sum_{l_\beta|(M/l_1),(l_\beta,2)=1} \mu(l_\beta)\beta_{l_\beta}\chi_{-a_1 l_1}(l_\beta) l_\beta^{-s} \sum_{k_\alpha|(M/l_1),(k_\alpha,2)=1} \alpha_{k_\alpha}\chi_{-a_1 l_1}(k_\alpha) k_\alpha^{-1+s}$$
$$\times \sum_{k_\beta|(M/l_1),(k_\alpha,2)=1} \beta_{k_\beta}\chi_{-a_1 l_1}(k_\beta) k_\beta^{-1+s}(Z_M(1-s, w+2s+1, \chi_{-N}\chi_{a_2 l_2 l_\alpha l_\beta k_\alpha k_\beta}, \chi_{a_1 l_1})$$
$$- Z_M(1-s, w+2s+1, \chi_{-1}\chi_{-N}\chi_{a_2 l_2 l_\alpha l_\beta k_\alpha k_\beta}, \chi_{a_1 l_1})\Bigg)\Bigg]$$

We can obtain similar formula for $a_1 l_1 \not\equiv 1 \mod 4$, and we complete the proof of the proposition. □

3.3 Domain of holomorphy

In Section 3.2, two functional equations are provided for the double Dirichlet series $Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1})$. In this section, we will examine this double Dirichlet series as an analytic function in two variables. Indeed, we will start by exploring its domain of convergence and then determine whether it can be extended. Additionally, we will also analyze its poles and some of its interesting properties. Before we proceed, it is essential to review some notions related to the analytic function in several complex variables ([21],[18]):

Definition 3.4. *A domain or open set $U \subset \mathbb{C}^n$ is called a domain of holomorphy if for every connected open set R such that R is not contained in U and $R \cap U \neq \emptyset$, and for every component $R_1 \subset R \cap U$, there is a function f analytic on U whose restriction $f|_{R_1}$ has no analytic continuation to R.*

Definition 3.5. *A domain $U \subset \mathbb{C}^n$ is called a tube if there is an open set $B \subset \mathbb{R}^n$ such that $U = \{s : \mathrm{Re}(s) \in B\}$. The set B is called the base of U.*

Proposition 3.6. *If U is a connected tube, then every analytic function on U has an analytic continuation to the convex hull of U.*

Proposition 3.7. *Let U and U' be two domain of holomorphy in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, and let f ba an analytic function of U to $\mathbb{C}^m$. Then the set*

$$U_f = \{s \in U : f(s) \in U'\}$$

is also a domain of holomorphy.

Throughout, we will denote $\mathrm{Re}(s)$ by σ and $\mathrm{Re}(w)$ by v. Let us then state the following proposition:

Proposition 3.8. *The function $(w-1)Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1})$ is analytic in a region R_1 (see Figure 3.1) which is the convex hull of the set*

$$\{v > 2 - 2\sigma, \sigma \leq 0\} \cup \{\sigma > 2 - v, v \leq -1/2\}.$$

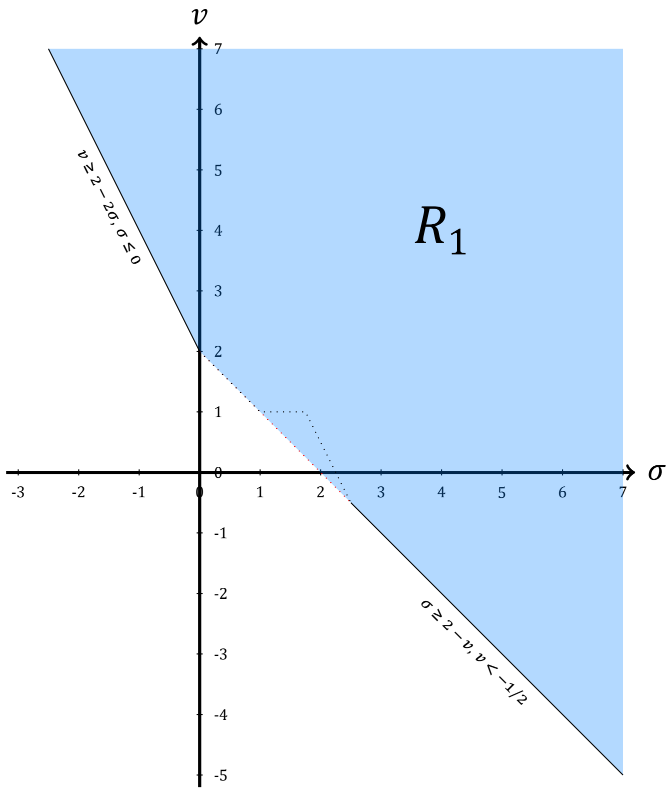
Proof. We begin with the upper bound of the Dirichlet Polynomials in Equation (3.9):

$$P_{a_1 l_1 d_1, d_0}(s) \ll |d|^{\epsilon}, \quad \lambda_n Q_{a_2 l_2 n_0, n_1}(w) \ll |n|^{1/2+\epsilon},$$

for $\sigma \geq 1/2$ and $v \geq 3/2$. Then, let us take the series in Equation (3.10):

$$Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1}) = \sum_{(d,M)=1} \frac{L_M(s, f, \chi_{d_0} \chi_{a_1 l_1}) \chi_{a_2 l_2}(d_0) P_{a_1 l_1 d_0, d_1}(s)}{d^w}.$$

Figure 3.1: The region R_1



We know that $L_M(s, f, \chi_{d_0}\chi_{a_1l_1})$ is absolutely convergent for $\sigma > 1$, hence,

$$|L_M(s, f, \chi_{d_0}\chi_{a_1l_1})P_{a_1l_1d_0,d_1}(s)| \ll |d|^{\epsilon}$$

for $\sigma \geq 1$. Consequently, we have absolute convergence of $Z_M(s, w, \chi_{a_1l_1}, \chi_{a_2l_2})$ in the set

$$\{v > 1, \sigma \geq 1\}.$$

Now, if we apply both of the functional equations of $L_M(s, f, \chi_{d_0}\chi_{a_1l_1})$ and $P_{a_1l_1d_0,d_1}(s)$ in (1.15) and (3.6) on $L_M(s, f, \chi_{d_0}\chi_{a_1l_1})P_{a_1l_1d_0,d_1}(s)$, then we obtain

$$|L_M(s, f, \chi_{d_0}\chi_{a_1l_1})P_{a_1l_1d_0,d_1}(s)| \ll d^{1-2\sigma+\epsilon}$$

for $\sigma \leq 0$. As a result, we have an absolute convergence of the series in the set

$$\{v > 2 - 2\sigma, \sigma \leq 0\}.$$

After that, we are going to apply the Phragmnen-Lindelöf principle in the region $\{0 \leq \sigma \leq 1\}$. Hence, we have

$$|L_M(s, f, \chi_{d_0}\chi_{a_1l_1})P_{a_1l_1d_0,d_1}(s)| \ll d^{1-\sigma+\epsilon}$$

for $0 \leq \sigma \leq 1$. It follows that the series is absolutely convergent in the region defined by

$$\{v > -\sigma + 2, 0 \leq \sigma \leq 1\}.$$

Now, let us use the second form of $Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1})$ as in Equation 3.11:

$$Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1}) = \sum_{(n,M)=1} \frac{L_M(w, \tilde{\chi}_{n_0} \chi_{a_2 l_2}) \chi_{a_1 l_1}(n_0) \lambda_n Q_{a_2 l_2 n_0, n_1}(w)}{n^s}.$$

In the same way, we know that $L_M(w, \tilde{\chi}_{n_0} \chi_{a_2 l_2})$ converges absolutely for $v > 1$. Thus, with the bound of $Q_{a_2 l_2 n_0, n_1}$, we have

$$|L_M(w, \tilde{\chi}_{n_0} \chi_{a_2 l_2}) \lambda_n Q_{a_2 l_2 n_0, n_1}(w)| \ll n^{1/2+\epsilon}$$

for $v \geq 3/2$. Thus, the series converges absolutely in the region

$$\{\sigma > 3/2, v \geq 3/2\}.$$

Again, using the functional equation of $L_M(w, \tilde{\chi}_{n_0} \chi_{a_2 l_2})$ and $Q_{a_2 l_2 n_0, n_1}(w)$ in (3.3) and (3.7), we have

$$|L_M(w, \tilde{\chi}_{n_0} \chi_{a_2 l_2}) \lambda_n Q_{a_2 l_2 n_0, n_1}(w)| \ll n^{1-v+\epsilon},$$

for $v \leq -1/2$. It implies that the series is absolutely convergent in

$$\{\sigma > 2 - v, v \leq -1/2\}.$$

Then, finally, with the Phragmnen-Lindelöf principle, we obtain

$$|(w-1) L_M(w, \tilde{\chi}_{n_0} \chi_{a_2 l_2}) \lambda_n Q_{a_2 l_2 n_0, n_1}(w)| \ll n^{-v/2+5/4+\epsilon},$$

for $-1/2 \leq v \leq 3/2$. Hence, we get an absolute convergence on the region defined by

$$\{\sigma > -v/2 + 9/4, -1/2 \leq v \leq 3/2\}.$$

Therefore, the function $Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1})$ is absolutely convergent, hence analytic, in the union of all this set. If we denote by R_1 the convex hull of this set. we can conclude that the double Dirichlet series is analytic in R_1 from Proposition 3.6. □

In order to have an analytic continuation of the function $Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1})$, we will apply the two functional equations from Section 3.2 on the region R_1. For this, let us define the following transformations:

$$\begin{aligned} \alpha &: (s, w) \mapsto (1 - s, w + 2s - 1), \\ \beta &: (s, w) \mapsto (s + w - 1/2, 1 - w). \end{aligned} \tag{3.30}$$

Let us point out that α and β correspond to the functional equations (3.24) and (3.12). Then, we first want to apply α on the region R_1. Using the form of R_1 in Proposition 3.8, we have

$$\begin{aligned}&\alpha(\{(\sigma,v) : v > 2-2\sigma, \sigma \le 0\})\\ &= \{(1-\sigma, v+2\sigma-1) : 1-\sigma \ge 1, v+2\sigma-1 > 2-2\sigma+2\sigma-1 = 1\}\\ &= \{(\sigma',v') : \sigma' \ge 1, v' > 1\},\end{aligned}$$

and,

$$\begin{aligned}&\alpha(\{(\sigma,v) : \sigma > 2-v, v < -1/2\}) = \alpha(\{(\sigma,v) : 2-\sigma < v < -1/2\})\\ &= \{(1-\sigma, v+2\sigma-1) : v+2\sigma-1 > 2-\sigma+2\sigma-1 = 2-(1-\sigma),\\ &\quad v+2\sigma-1 < -1/2+2\sigma-1 = 1/2-2(1-\sigma)\}\\ &= \{(\sigma',v') : 2-\sigma' < v' < 1/2-2\sigma'\}.\end{aligned}$$

Hence, $\alpha(R_1)$ is the convex hull of the set

$$\{(\sigma,v) : \sigma \ge 1, v > 1\} \cup \{(\sigma,v) : 2-\sigma < v < 1/2-2\sigma\}.$$

Consequently, if we define by R_2 the convex hull containing $\alpha(R_1) \cup R_1$, then

$$R_2 = \{v > 2-\sigma\},$$

which is a half plane as we can see in Figure 3.2. Secondly, we are going to apply β on R_2 and take once again the convex hull of $\beta(R_2) \cup R_2$. In a similar way, we will obtain all $\mathbb{C}^2$. In this section, we will occasionally refer to it as R_3.

Now, to simplify the computations, we are going to generalize functional equations in Section 3.2. For this, consider

$$A_M(s,w) = \prod_{p|M}(1-\alpha_p^2 p^{-2+2s})(1-\beta_p^2 p^{-2+2s}),$$

$$\tilde{\Psi}^{(a_1 l_1)}(s,w) = \Psi^{(a_1 l_1)}(s)\prod_{p|l_1}(1-\alpha_p^2 p^{-2+2s})(1-\beta_p^2 p^{-2+2s}),$$

$$B_M(s,w) = \prod_{p|M}(1-p^{-2+2w}),$$

and

$$\tilde{\Phi}^{(a_2 l_2)}(s,w) = \Phi^{(a_2 l_2)}(w)\prod_{p|l_2}(1-p^{-2+2w}).$$

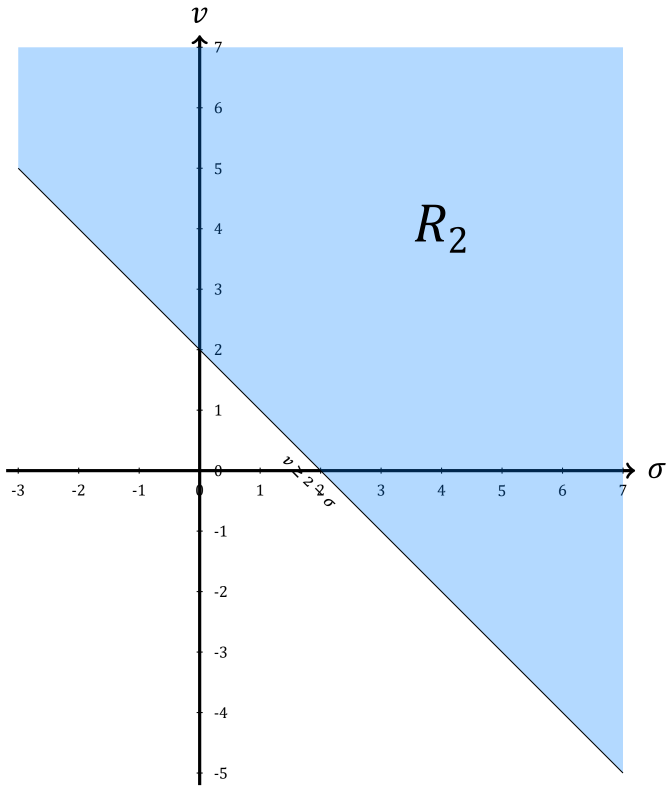
As a result, our functional equations become

$$A_M(s,w)\overrightarrow{Z_M}(s,w,\chi_{a_1 l_1},\chi_{Div(M)}) = \tilde{\Psi}^{(a_1 l_1)}(s,w)\overrightarrow{Z_M}(\alpha(s,w),\chi_{a_1 l_1},\chi_{Div(M)}) \tag{3.31}$$

$$B_M(s,w)\overrightarrow{Z_M}(s,w,\chi_{Div(M)},\chi_{a_2 l_2}) = \tilde{\Phi}^{(a_2 l_2)}(s,w)\overrightarrow{Z_M}(\beta(s,w),\chi_{Div(M)},\chi_{a_2 l_2}), \tag{3.32}$$

and we can go through the analytic continuation process.

Figure 3.2: The region R_2



Proposition 3.9. *Let $P(s,w) = (w-1)$. Define $\tilde{Z}_M(s,w,\chi_{a_2l_2},\chi_{a_1l_1})$ the function given by*

$$\tilde{Z}_M(s,w,\chi_{a_2l_2},\chi_{a_1l_1}) = A_M(s,w)A_M(\beta(s,w))B_M(s,w) \times P(s,w)P(\alpha(s,w))P(\beta(s,w))P(\alpha\beta(s,w))Z_M(s,w,\chi_{a_1l_1},\chi_{a_2l_2}). \tag{3.33}$$

Then $\tilde{Z}_M(s,w,\chi_{a_2l_2},\chi_{a_1l_1})$ is analytic over $\mathbb{C}^2$.

Proof. By multiplying the functional equation (3.31) with $P(s,w)P(\alpha(s,w))$, we obtain

$$A_M(s,w)P(\alpha(s,w))\left[P(s,w)\overrightarrow{Z_M}(s,w,\chi_{a_1l_1},\chi_{Div(M)})\right] = \tilde{\Phi}^{a_1l_1}(s,w)P(s,w)\left[P(\alpha(s,w))\overrightarrow{Z_M}(\alpha(s,w),\chi_{a_1l_1},\chi_{Div(M)})\right]$$

From Proposition 3.8, $P(s,w)\overrightarrow{Z_M}(s,w,\chi_{a_1l_1},\chi_{Div(M)})$ is analytic on R_1. On the other hand, since $\alpha^2 = 1$,

$$P(\alpha(s,w))\overrightarrow{Z_M}(\alpha(s,w),\chi_{a_1l_1},\chi_{Div(M)})$$

is analytic on $\alpha(R_1)$. Note that the poles in $\tilde{\Phi}^{a_1 l_1}(s,w)$ can be canceled by the zeroes of $L(1-s, f, \chi_d)$. Therefore, the function

$$A_M(s,w)P(s,w)P(\alpha(s,w))\overrightarrow{Z_M}(s,w,\chi_{a_1 l_1},\chi_{Div(M)})$$

is analytic on the convex hull of $\alpha(R_1)\cup R_1$ which is R_2. After that, we are going to multiply the functional equation (3.32) with

$$A_M(s,w)A_M(\beta(s,w))P(\alpha(s,w))P(\beta(s,w))P(\alpha\beta(s,w)).$$

Hence, we get

$$\begin{aligned}
&B_M(s,w)A_M(\beta(s,w))P(\beta(s,w))P(\alpha\beta(s,w))\\
&\times\left[A_M(s,w)P(s,w)P(\alpha(s,w))\overrightarrow{Z_M}(s,w,\chi_{Div(M)},\chi_{a_2 l_2})\right]=\\
&\qquad\tilde{\Phi}^{a_2 l_2}(s,w)A_M(s,w)P(s,w)P(\alpha(s,w))\\
&\qquad\times\left[A_M(\beta(s,w))P(\beta(s,w))P(\alpha\beta(s,w))\overrightarrow{Z_M}(s,w,\chi_{Div(M)},\chi_{a_2 l_2})\right].
\end{aligned}$$

We have seen that $A_M(s,w)P(s,w)P(\alpha(s,w))\overrightarrow{Z_M}(s,w,\chi_{Div(M)},\chi_{a_2 l_2})$ is analytic on R_2. For the right hand side, since $\beta^2=1$,

$$A_M(\beta(s,w))P(\beta(s,w))P(\alpha\beta(s,w))\overrightarrow{Z_M}(s,w,\chi_{Div(M)},\chi_{a_2 l_2})$$

is also analytic on $\beta(R_2)$. Again, all the poles of $\Phi^{a_2 l_2}(s,w)$ can be canceled by the zeroes of $L(1-w,\chi_{m_0})$ and $P(s,w)$. Hence, if R_3 is the convex hull generated by R_2 and $\beta(R_2)$, then

$$A(s,w)A(\beta(s,w))B(s,w)P(s,w)P(\alpha(s,w))P(\beta(s,w))P(\alpha\beta(s,w))Z_M(s,w,\chi_{a_1 l_1},\chi_{a_2 l_2})$$

is analytic on R_3. This completes proof. □

3.4 The function $Z_M(s,w,\chi_{a_1 l_1},\chi_{a_2 l_2})$ at $s=1/2$

Recall that our primary objective is to investigate the distribution of $L(1/2,f,\chi_{d_0})$ as mentioned in Chapter 1. Hence we can focus on the function $Z_M(1/2,w,\chi_{a_2 l_2},\chi_{a_1 l_1})$. In this section, we will study the poles of this latter and try to establish an upper bound on the critical strip, if possible. For this, let us define the $4\tau(M)^2$ column vectors $\overrightarrow{Z_M}(s,w)$ consisting of all the functions $Z_M(s,w,\chi_{a_2 l_2},\chi_{a_1 l_1})$, for all $a_1,a_2\in\{\pm1\}$ and $l_1,l_2|M$. Hence, from Proposition 3.2 and 3.3, we can have two matrices $\Phi(s,w)$ and $\Psi(s,w)$ such that

$$A_M(s,w)\overrightarrow{Z_M}(s,w)=\Psi(s,w)\overrightarrow{Z_M}(\alpha(s,w)) \tag{3.34}$$

and

$$B_M(s,w)\overrightarrow{Z_M}(s,w)=\Phi(s,w)\overrightarrow{Z_M}(\beta(s,w)). \tag{3.35}$$

Proposition 3.10. *For every $\epsilon > 0$ and every vertical strip K, we have*

$$Z_M(1/2, v + it, \chi_{a_1 l_1}, \chi_{a_2 l_2}) = O_K(e^{t+\epsilon}).$$

Proof. As in the proof of Proposition 3.8, we have

$$|L(1/2, f, \chi_{a_1 l_1}\chi_{d_0})P_{a_1 l_1 d_0, d_1}(1/2)| \ll d^{1+\epsilon}.$$

Thus, we have the upper bound

$$Z_M(1/2, v + it, \chi_{a_1 l_1}, \chi_{a_2 l_2}) \ll_\epsilon 1, \tag{3.36}$$

for $v > 2 + \epsilon$. Let us now notice that $\beta\alpha\beta(1/2, w) = (1/2, 1 - w)$. Hence, by applying the functional equation (3.34) and (3.35), we have

$$\begin{aligned}\overrightarrow{Z_M}(1/2, w) &= \Phi(1/2, w)B_M(1/2, w)^{-1}\Psi(\beta(1/2, w)) \\ &\times A_M(\beta(1/2, w))^{-1}\Phi(\alpha\beta(1/2, w))A_M(\alpha\beta(1/2, w))^{-1}\overrightarrow{Z_M}(1/2, 1 - w).\end{aligned} \tag{3.37}$$

Using Stirling's formula, we can see that all these factors are of polynomial growth on every vertical strips. Thus, from the upper bound in (3.36), we obtain

$$Z_M(1/2, w, \chi_{a_2 l_2}, \chi_{a_1 l_1}) \ll_\epsilon (1 + |t|)^C,$$

for $v < -1 - \epsilon$ for some constant $C > 0$.

For $-1 - \epsilon < w < 2 + \epsilon$, we are going to use the following technique:
In the proof of Proposition 3.8, we have a region where the function

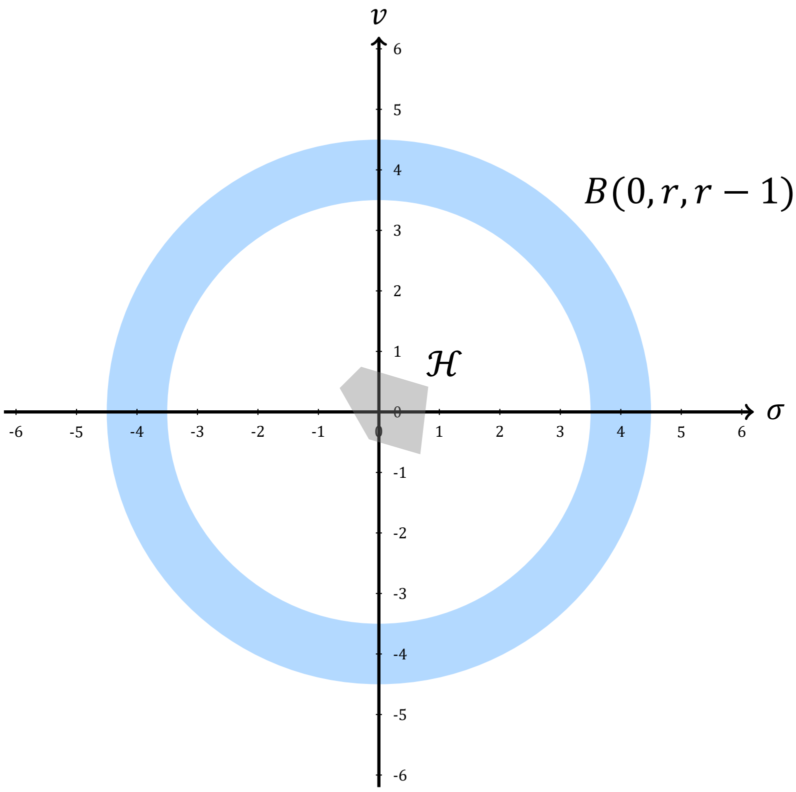
$$P(s, w)Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1})$$

is absolutely convergent. Let us denote this region by R. We should point out that this region is not R_1, but, R_1 is just the convex hull of R. Using repetitively the transformation α and β on R, we can have all the plan with a hole in the shape of a polygon, let us denote by $\mathcal{H}$ (see Figure 3.3). As we did previously, using the functional equation (3.34) and (3.35), we can see that the function $\tilde{Z}_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1})$is of polynomial growth on every tube with base outside $\mathcal{H}$. Let r be the radius such that $\mathcal{H}$ is inside $B(0, r - 1) \subset \mathbb{R}^2$, and let us take the function

$$f(s, w) = \Gamma(s + r)\Gamma(w + r)\tilde{Z}_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1})$$

defined over the tube generated by $B(0, r)$ which we can denote by B. Therefore, we know that f is holomorphic on B. Moreover, it is BOUNDED on the tube B' with the annulus $B(0, r, r - 1)$ as base (see Figure 3.3) because the function gamma tends exponentially to 0 one every vertical strip. Let us then take $\rho > 0$ a bound of f on B' and consider the ball $B(0, \rho) \subset \mathbb{C}$. We know that B and

Figure 3.3: The polygon $\mathcal{H}$ and the annulus $B(0, r, r-1)$



$B(0, \rho)$ are both domains of holomorphy. Hence, by applying Proposition 3.7, the set

$$B_f = \{(s, w) \in B : f(s, w) \in B(0, \rho)\}$$

is also a domain of holomorphy. However, since $B' \subset B_f$ and, according to Proposition 3.6, the smallest domain holomorphy containing B' is B, then we can deduce that $B \subset B_f$, which implies that f is bounded on B. Thus, we can conclude that the growth of $Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1})$ in B is at most exponential. This completes proof. □

For our methods to work, we must make this assumption on the Dirichlet polynomials:

Assumption A1. For all $d = d_0 d_1^2 > 0$ coprime with M and $a_1 l_1 \in Div(M)$

$$P_{a_1 l_1 \cdot d_0, d_1}(1/2) \geq 0, \tag{3.38}$$

However, we cannot prove this for now, due to the lack of information about the Dirichlet Polynomials. Hence, we now come to most crucial theorem of this chapter:

Theorem 3.11. *Let $\epsilon > 0$, $w = v + it$ and $a_1 l_1, a_2 l_2 | M$ with $a_1, a_2 \in \{\pm 1\}$. If $(l_1, l_2) = 1$ or 2, then for $-\epsilon < v$,*

$$Z_M(1/2, v + it, \chi_{a_2 l_2}, \chi_{a_1 l_1})$$

is analytic except for possible poles at $w = 0, 1$. In addition, assuming Assumption (A1), we have the following upper bounds:

$$Z_M(1/2, 1 + \epsilon + it, \chi_{a_2 l_2}, \chi_{a_1 l_1}) \ll_\epsilon 1,$$

and

$$Z_M(1/2, -\epsilon + it, \chi_{a_2 l_2}, \chi_{a_1 l_1}) \ll_\epsilon M^{1+8\epsilon} |t|^{2+4\epsilon},$$

for $|t| > 1$.

Proof. Let us first discuss about the poles. From Proposition 3.9, we know that

$$A_M(1/2, w) A_M(\beta(1/2, w)) B_M(1/2, w) P(1/2, w) P(\alpha(1/2, w)) P(\beta(1/2, w))$$
$$\times P(\alpha\beta(1/2, w)) Z_M(s, w, \chi_{a_1 l_1}, \chi_{a_2 l_2})$$

is analytic on $\mathbb{C}$. Therefore, the only possible poles for $Z_M(1/2, w, \chi_{a_2 l_2}, \chi_{a_1 l_1})$ are the zeros of

$$A_M(1/2, w) A_M(\beta(1/2, w)) B_M(1/2, w) \tag{3.39}$$

and

$$P(1/2, w) P(\alpha(1/2, w)) P(\beta(1/2, w)) P(\alpha\beta(1/2, w)). \tag{3.40}$$

Equation (3.40) has only two possible zeros which are $w = 1$ or 0. As for (3.39), its factors are of the form $1 - u_p p^{-2+2w}$ where $|u_p| = 1$. Therefore, the only possible poles are of the form $w = 1 + it_0$. We can suppose that $t_0 \neq 0$ because it is very likely that we have poles at $w = 1$. We will now demonstrate that it cannot be a pole for $Z_M(1/2, w, \chi_{a_2 l_2}, \chi_{a_1 l_1})$. For this, let us return $Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1})$. We recall again that the only poles occur at

$$A_M(s, w) A_M(\beta(s, w)) B_M(s, w) P(s, w) P(\alpha(s, w)) P(\beta(s, w)) P(\alpha\beta(s, w)).$$

In the same way as for (3.39), the poles of $A_M(s, w) A_M(\beta(s, w)) B_M(s, w)$ are of the form $(s - 1 - it_n)$, $(s + w - 3/2 - it'_n)$ and $(w - 1 - it''_n)$ for some discrete sequences $(t_n)_n$, $(t'_n)_n$ and $(t''_n)_n$. We also have

$$P(s, w) P(\alpha(s, w)) P(\beta(s, w)) P(\alpha\beta(s, w)) = w(w - 1)(2s + w - 2)^2.$$

Looking at the form of these poles, we can deduce that we do not have poles in the region $\mathrm{Re}(s) \geq 1/2$ and $\mathrm{Re}(w) > 1$ except for some discrete sequences on the line $\mathrm{Re}(s) = 1$, or equivalently, $Z_M(s, w, \chi_{a_2 l_2}, \chi_{a_1 l_1})$ is analytic on $Re(s) \geq 1/2$ and $\mathrm{Re}(w) > 1$ except for some discrete sequences on the line $\mathrm{Re}(s) = 1$.

Then, we take $\mathcal{P}(s,w)$ to be the the product of multiple polynomials such that $\mathcal{P}(s,w)Z_M(1/2,w,\chi_{a_2l_2},\chi_{a_1l_1})$ is analytic on a neighborhood of $(1/2,1+it_0)$ and

$$\lim_{(s,w)\to(1/2,1+it_0)} P(s,w)Z_M(s,w,\chi_{a_2l_2},\chi_{a_1l_1}) \neq 0. \tag{3.41}$$

Let us notice that the only polynomials which vanish on $(1/2,1+it_0)$ are

$$(w-1-it_0) \text{ and } (s+w-3/2-it_0)$$

corresponding to the factors $B_M(s,w)$ and $A_M(\beta(s,w))$. By the Hartog's theorem on separate analyticity,

$$\lim_{w\to 1+it_0} \mathcal{P}(s,w)Z_M(s,w,\chi_{a_2l_2},\chi_{a_1l_1}) \tag{3.42}$$

is analytic on a connected domain containing $s = 1/2$ and $\mathrm{Re}(s) > 1/2$ except for some discrete sequences on the line $\mathrm{Re}(s) = 1$. However, by Equation (3.11), $Z_M(s,1+it_0,\chi_{a_2l_2},\chi_{a_1l_1})$ is absolutely convergent for $\mathrm{Re}(s)$ large enough. Hence, if $\mathcal{P}(s,w)$ contains $(w-1-it_0)$ as factors, then

$$\lim_{w\to 1+it_0} \mathcal{P}(s,w)Z_M(s,w,\chi_{a_2l_2},\chi_{a_1l_1}) = 0. \tag{3.43}$$

Thus,

$$\lim_{(s,w)\to(1/2,1+it_0)} P(s,w)Z_M(s,w,\chi_{a_2l_2},\chi_{a_1l_1}) = \lim_{s\to 1/2}\lim_{w\to 1+it_0} P(s,w)Z_M(s,w,\chi_{a_2l_2},\chi_{a_1l_1}) = 0,$$

which contradicts Equation (3.41).

For the pole at $w = 3/2 + it_0 - s$, we can use the functional equation in (3.32) on $(s,w) = (1+it_0, s-1/2-it)$. Thus, we get

$$\prod_{p|M}(1-p^{-2(3/2+it_0-s)})\overrightarrow{Z_M}(1-it_0,s-1/2-it_0,\chi_{Div(M)},\chi_{a_2l_2}) = \tilde{\Phi}^{(a_2l_2)}(1-it_0,s-1/2-it_0)\overrightarrow{Z_M}(s,3/2+it_0-s,\chi_{Div(M)},\chi_{a_2l_2}).$$

Using Equation (3.11), the left hand side converges absolutely for $\mathrm{Re}(s)$ large enough. Thus, we can deduce that the right hand side is also analytic on the same region. Since

$$\tilde{\Phi}^{(a_2l_2)}(1-it_0,s-1/2-it_0)$$

does not vanish for $Re(s)$ large (but may have poles), we can conclude that $Z_M(s,3/2+it_0-s,\chi_{a_2l_2},\chi_{a_1l_1})$ is analytic on for $Re(s)$ very large. As a result, if $\mathcal{P}(s,w)$ contains polynomials of the form $s+w-3/2-it_0$, then

$$\mathcal{P}(s,3/2+it_0-s)Z_M(s,3/2+it_0-s,\chi_{a_2l_2},\chi_{a_1l_1}) = 0.$$

Hence, we obtain

$$\lim_{(s,w)\to(1/2,1+it_0)} P(s,w)Z_M(s,w,\chi_{a_2l_2},\chi_{a_1l_1}) = \lim_{s\to 1/2} \mathcal{P}(s,3/2+it_0-s)Z_M(s,3/2+it_0-s,\chi_{a_2l_2},\chi_{a_1l_1}) = 0,$$

which again contradicts Equation (3.41). Therefore, we proved that the only possible poles of $Z_M(1/2,w,\chi_{a_1l_1},\chi_{a_2l_2})$ are at $w = 0,1$.

Now, let us get into the upper bound. From [22] and using Assumption (A1), we have

$$L_M(1/2,f,\chi_{a_1l_1}\chi_{d_o})P_{a_1l_1d_0,d_1}(1/2) \geq 0$$

for all $a_1l_1|M$ and $(d,M) = 1$. However, we know that the only possible poles for $Z_M(1/2,w,\chi_{a_2l_2},\chi_{a_1l_1})$ are $w = 0,1$ and the pole at $w = 1$ is certain if $a_2l_2 = 1$. Hence, using Landau's theorem, we can get the absolutely convergence of the double Dirichlet series

$$\sum_{(d,M)=1} \frac{L_M(1/2,f,\chi_{d_0}\chi_{a_1l_1})P_{a_1l_1\cdot d_0,d_1}(1/2)}{d^w}$$

for $v > 1$. For the second upper bound, we can notice that if we take $(s,w) = (1/2,-\epsilon+it)$, then

$$\beta(s,w) = (-\epsilon+it,1+\epsilon-it), \qquad \alpha\beta(s,w) = (1+\epsilon-it,-\epsilon+it) \qquad \text{and}$$

$$\beta\alpha\beta(s,w) = (1/2,1+\epsilon-it).$$

The idea is to use the Stirling's formula and to apply successively the functional equation in Proposition 3.2 and 3.3 exactly in the following order : β, α and β. Hence, we get

$$\begin{aligned} Z_M(1/2,-\epsilon+it,\chi_{a_2l_2},\chi_{a_1l_1}) \ll_\epsilon l_2^{1/2+\epsilon} \sum_{l_3,l_4|(M/l_2)} M^\epsilon \sum_{a=\pm 1} |t|^{1/2+\epsilon} \\ \times \sum_{\substack{a_3=\pm 1, d_2|M \\ (d_2,l_2)=1 \text{ or } 2}} \left|Z_M(-\epsilon+it,1+\epsilon-it,\chi_{a_2l_2},\chi_{a_3d_2})\right|. \end{aligned} \tag{3.44}$$

Here, $a_1l_1l_3l_4$ is replaced by their squarefree part d_2. (It is important to notice that $(l_2,l_1l_3l_4) = 1$ or 2).

In the same way, we obtain

$$\begin{aligned} Z_M(-\epsilon+it,1+\epsilon-it,\chi_{a_2l_2},\chi_{a_3d_2}) \ll_\epsilon d_2^{1+2\epsilon}|t|^{1+2\epsilon} \sum_{l_\alpha,l_\beta,k_\alpha,k_\beta} M^{2\epsilon} \\ \times \sum_{\substack{a_4=\pm 1, d_3|M \\ (d_3,d_2)=1 \text{ or } 2}} \left|Z_M(1+\epsilon-it,-\epsilon+it,\chi_{a_4d_3},\chi_{a_3d_2})\right|, \end{aligned} \tag{3.45}$$

Indeed, $a_2 l_2 l_\alpha l_\beta k_\alpha k_\beta$ is again replaced by their squarefree part d_3. For the final transformation, we obtain

$$Z_M(1+\epsilon-it,-\epsilon+it,\chi_{a_4 d_3},\chi_{a_3 d_2}) \ll_\epsilon d_3^{1/2+\epsilon} \sum_{l_3',l_4'|(M/d_3)} M^\epsilon \sum_{a=\pm 1} |t|^{1/2+\epsilon}$$
$$\times \sum_{\substack{a5=\pm 1, d_4|M \\ (d_2,l_2)=1 \text{ or } 2}} \left| Z_M(-\epsilon+it, 1+\epsilon-it, \chi_{a_4 d_3}, \chi_{a_5 d_4}) \right|. \tag{3.46}$$

We can see that the all these terms do not depend on $l_3, l_4, l_\alpha, l_\beta, k_\alpha, k_\beta$ since they are replaced with d_i. Therefore, thee sums over l_i s are bounded by M^ϵ. Thus, putting (3.44), (3.45) and (3.46) together, we obtain

$$Z_M(1/2,-\epsilon+it,\chi_{a_2 l_2},\chi_{a_1 l_1}) \ll_\epsilon M^{5\epsilon}|t|^{2+4\epsilon} \sum l_2^{1/2+\epsilon} d_2^{1+2\epsilon} d_3^{1/2+\epsilon} \cdot S, \tag{3.47}$$

where the sum is over $d_2, d_3|M$ such that (l_2, d_2) and (d_2, d_3) are equal to 1 or 2, and

$$S = \sum_{a5=\pm} \sum_{d_4|M} \sum_{\substack{d=d_0 d_1^2 \\ (d,M)=1}} \frac{|L(1/2, f, \chi_{d_0}\chi_{a_5 d_4}) P^{(al)}_{d_0,d_1}(1/2)|}{d^{1+\epsilon}}.$$

Now, using the fact that(l_2, d_2) and (d_2, d_3) are equal to 1 or 2, we have

$$l_2^{1/2+\epsilon} d_2^{1+2\epsilon} d_3^{1/2+\epsilon} = (l_2 d_2)^{1/2+\epsilon} (d_2 d_3)^{1/2+\epsilon} \ll_\epsilon M^{1+2\epsilon}.$$

Therefore, we have

$$Z_M(1/2,-\epsilon+it,\chi_{a_2 l_2},\chi_{a_1 l_1}) \ll_\epsilon M^{1+8\epsilon}|t|^{2+4\epsilon}$$
$$\times \sum_{a=\pm 1} \sum_{l|M} \sum_{\substack{d=d_0 d_1^2 \\ (d,M)=1}} \frac{|L(1/2, f, \chi_{d_0}\chi_{al}) P^{(al)}_{d_0,d_1}(1/2)|}{d^{1+\epsilon}}.$$

We can therefore complete the proof using the first upper bound. □

Chapter 4

Sieve method

4.1 Introduction to sieve method

The sieve method is an elementary tool in number theory based on the inclusion-exclusion principle and is generally used to estimate sums over almost-prime numbers *i.e.,* numbers with at most r prime factors.

Let us begin by giving the general idea of the sieve method that use in this chapter. We are given $\mathfrak{A} = (a_d)_d$ a sequence of non-negative numbers and a set of prime numbers $\mathcal{P}$. Also, consider the number

$$P(z) = \prod_{\substack{p \in \mathcal{P} \\ p < z}} p.$$

The goal of the sieve method is to estimate the *sifting function* given by

$$\mathcal{S}(\mathfrak{A}, Y, z) = \sum_{\substack{d \leq Y \\ (d, P(z)) = 1}} a_d, \tag{4.1}$$

as $Y \to \infty$. Using the properties of Möbius function, we can write $\mathcal{A}(\mathfrak{A}, Y, z)$ as follows:

$$\mathcal{S}(\mathfrak{A}, Y, z) = \sum_{d \leq Y} a_d \sum_{\substack{M' | P(z) \\ M' | d}} \mu(M') = \sum_{M' | P(z)} \mu(M') A_{M'}(Y), \tag{4.2}$$

where

$$A_{M'}(Y) = \sum_{\substack{d \leq Y \\ d \equiv 0 [M']}} a_d. \tag{4.3}$$

In most of the cases, we obtain

$$A_{M'}(Y) = g(M') X + r_{M'}(Y), \tag{4.4}$$

where g is a multiplicative function and X is an approximation of $A_1(Y)$, *i.e.,*

$$\sum_{d\leq Y} a_d \sim X.$$

Inserting (4.4) in Equation (4.3), we obtain

$$\mathcal{S}(\mathfrak{A}, Y, z) = V(z)X + R(Y, z), \tag{4.5}$$

where

$$V(z) = \sum_{M'|P(z)} \mu(M')g(M') = \prod_{p|P(z)} (1 - g(p)), \tag{4.6}$$

and

$$R(Y, z) = \sum_{M'|P(z)} \mu(M')r_{M'}(Y). \tag{4.7}$$

The method depends on a good estimate of $R(Y, z)$. This is generally quite difficult to do, and one needs further information on the specific problem to obtain an estimate of $R(Y, z)$ that will produce nontrivial results.

4.2 Sifting weights and the beta-sieve

In this section, we will introduce new coefficient $(\lambda^{\pm}_{M'})_{M'}$ and $D < Y$ such that $\lambda^{\pm}_1 = 1$ and $\lambda^{\pm}_{M'} = 0$ for any $M' > D$, and

$$\mathcal{S}^-(\mathfrak{A}, Y, z, D) \leq \mathcal{S}(\mathfrak{A}, Y, z) \leq \mathcal{S}^+(\mathfrak{A}, Y, z, D), \tag{4.8}$$

where

$$\mathcal{S}^{\pm}(\mathfrak{A}, Y, z, D) = \sum_{M'|P(z)} \lambda^{\pm}_{M'} A_{M'}(Y) = \sum_{n\leq Y} a_n \sum_{M'|(n,P(z))} \lambda^{\pm}_{M'}. \tag{4.9}$$

Such $\lambda^{\pm}_{M'}$ are called “sifting weights” of level D. Let us notice that (4.8) can be satisfied with the following condition:

$$\sum_{M'|n} \lambda^{-}_{M'} \leq \sum_{M'|n} \mu(M') = 0 \leq \sum_{M'|n} \lambda^{+}_{M'}, \tag{4.10}$$

for any $n > 1$. Thus, one way to get these sifting weights is to look for a set $\mathcal{M}^{\pm}$, such that

$$\lambda^{\pm}_{M'} = \begin{cases} \mu(M') & \text{if } M' \in \mathcal{M}^{\pm} \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

In fact, we have

$$\sum_{M'|n} \mu(M') = 1 + \sum_{\substack{M'|n \\ \omega(M')\geq 1}} \mu(M'), \tag{4.12}$$

where $\omega(M')$ is the number of prime factors dividing M'. Hence, we know that every squarefree $M'|n$ with $\omega(M') \geq 1$ can be written in an unique way as $p_1 \cdots p_m$, with $p_1 > \cdots > p_m$. Hence,

$$\sum_{\substack{M'|n \\ \omega(M')\geq 1}} \mu(M') = \sum_{p_1} \mu(p_1) \sum_{M'|P(p_1)} \mu(M') = -\sum_{p_1} \delta_{1,P(p_1)},$$

where $\delta_{1,P(p_1)} = 1$ if $P(p_1) = 1$ and 0 otherwise. Equation (4.12) then becomes

$$\sum_{M'|n} \mu(M') = 1 - \sum_{p_1} \delta_{1,P(p_1)}. \tag{4.13}$$

Since $\delta_{1,P(p_1)} \geq 0$, we can have an upper bound of $\sum_{M'|n} \mu(M')$ by removing $p_1 \geq y_1$ for some y_1. By continuing the process, we have the following upper bound

$$\sum_{M'|n} \mu(M') \leq 1 - \sum_{p_1<y_1} 1 + \sum\sum_{p_2<p_1<y_1} 1 - \sum\sum\sum_{\substack{p_3<p_2<p_1<y_1 \\ p_3<y_3}} 1 + \cdots. \tag{4.14}$$

Consequently, we can get $(\lambda^+_{M'})_{M'}$ by taking

$$\mathcal{M}^+ = \{M' = 1 \text{ or } p_1 \cdots p_l : p_1 < \cdots < p_l,\ p_m < y_m \text{ for } 2 \nmid m\}, \tag{4.15}$$

for any $y_m = y_m(p_1, \cdots, p_m)$. In the same way, we can have the following lower bound

$$\sum_{M'|n} \mu(M') \geq 1 - \sum_{p_1} 1 + \sum\sum_{\substack{p_2<p_1 \\ p_2<y_2}} 1 - \sum\sum\sum_{\substack{p_3<p_2<p_1 \\ p_2<y_2}} 1 + \cdots. \tag{4.16}$$

Hence, $(\lambda^-_{M'})_{M'}$ can also be obtained from the set

$$\mathcal{M}^- = \{M' = 1 \text{ or } p_1 \cdots p_l : p_1 < \cdots < p_l,\ p_m < y_m \text{ for } 2|m\}, \tag{4.17}$$

for any $y_m = y_m(p_1, \cdots, p_m)$.

For beta-sieve, we are going to take $y_m = (D/p_1 \cdots p_m)^{\frac{1}{\beta}}$ for some $\beta \geq 1$. Then, our set becomes

$$\mathcal{M}^+ = \{M' = 1 \text{ or } p_1 \cdots p_l : p_1 < \cdots < p_l,\ p_1 \cdots p_m p_m^{\beta} < D \text{ for } 2 \nmid m\} \tag{4.18}$$

and

$$\mathcal{M}^- = \{M' = 1 \text{ or } p_1 \cdots p_l : p_1 < \cdots < p_l,\ p_1 \cdots p_m p_m^\beta < D \text{ for } 2|m\}. \quad (4.19)$$

With $\lambda_{M'}^{\pm}$ defined in (4.11), the condition (4.10) is satisfied, and hence, we have (4.8). Thus, using (4.4) , we have

$$\mathcal{S}^{\pm}(\mathfrak{A}, Y, z, D) = V^{\pm}(z, D)X + R^{\pm}(Y, z, D), \quad (4.20)$$

where

$$V^{\pm}(z, D) = \sum_{M'|P(z)} \lambda_{M'}^{\pm} g(M'), \quad (4.21)$$

and

$$R^{\pm}(Y, z, D) = \sum_{M'|P(z)} \lambda_{M'}^{\pm} r_{M'}(Y). \quad (4.22)$$

The interesting thing about the level D is the following upper bound

$$|R^{\pm}(Y, z, D)| \leq \sum_{M'<D} |r_{M'}(Y)|. \quad (4.23)$$

The following is one of the key theorems that we will use in our problem.

Theorem 4.1. *Suppose that g is a multiplicative function such that*

$$0 \leq g(p)) < 1, \quad (4.24)$$

for $p|P(z)$, and

$$\prod_{z_1 \leq p < z} (1 - g(p))^{-1} \leq \left(\frac{\log z}{\log z_1}\right)^{\kappa} \left(1 + \frac{L}{\log z_1}\right), \quad (4.25)$$

for $2 \leq z_1 < z$, $\kappa \geq 0$ and $L \geq 1$. Then, we have

$$V^+(z, D) \leq V(z)(F_0(s) + O((\log D)^{-1/6})) \qquad \text{if } s \geq \beta - 1 \quad (4.26)$$

$$V^-(z, D) \geq V(z)(f_0(s) + O((\log D)^{-1/6})) \qquad \text{if } s \geq \beta, \quad (4.27)$$

where $s = \log D / \log z$, $\beta = \beta(\kappa)$, $F_0(s)$ and $f_0(s)$ are two continuous functions which are strictly monotonously decreasing and increasing respectively, and satisfying $F_0(s) = 1 + O(e^{-s})$ and $f_0(s) = 1 + O(e^{-s})$ as $s \to \infty$. In particular, if $\kappa = 1$, then $\beta = 2$ and $f(2) = 0$.

Proof. See Theorem 11.12 in [23]. □

Corollary 4.2. *If g satisfy the condition* (4.24) *and* (4.25)*, then, we have*

$$\mathcal{S}(\mathfrak{A}, Y, z) \leq XV(z)(F_0(s) + O((\log D)^{-1/6})) + R^+(Y, z, D) \qquad \text{if } s \geq \beta - 1 \quad (4.28)$$

$$\mathcal{S}(\mathfrak{A}, Y, z) \geq XV(z)(f_0(s) + O((\log D)^{-1/6})) + R^-(Y, z, D) \qquad \text{if } s \geq \beta, \quad (4.29)$$

where s, β, $F_0(s)$ and $f_0(s)$ are defined as in Theorem 4.1.

Proof. We just need to apply Theorem 4.1 to (4.8). □

4.3 Non vanishing and sieve method

As we discussed in Chapter 1, we want to see if there exists an infinity of odd fundamental discriminant d_0 such that $L(1/2, f, \chi_{d_0}) \neq 0$ and the number of prime factors of d is at most r. For this, let us consider the set $\mathfrak{C} = (c_d)_{d>0}$ given by

$$c_d = \begin{cases} L(1/2, f, \chi_{\omega d}) F\left(\frac{d}{Y}\right) & \text{if } \omega d \in \mathcal{D}_\omega \\ 0 & \text{otherwise,} \end{cases} \tag{4.30}$$

and the set of primes

$$\mathcal{P}' = \{p : (p, 4M) = 1\}, \tag{4.31}$$

where $\mathcal{D}_\omega$ and M are given in Equation (2.1), and F is a non-negative bump function on $(0, 1)$ with positive mean value. We can now define

$$A(Y, r) = \sum_{\substack{d<Y \\ \omega(d)\le r}} c_d, \tag{4.32}$$

where $\omega(d)$ is the number of prime factors of d. From [22], we have the non-negativity of the coefficients c_d. Thus, if $d < Y$ and $\omega(d) > r$, then d must have a prime factor less than $Y^{\frac{1}{r+1}}$. And since $c_d = 0$ if $(d, 4M) > 1$, the previous statement is equivalent to $(d, P'(Y^{1/(r+1)})) > 1$ where

$$P'(Y^{1/(r+1)}) = \prod_{\substack{p\in\mathcal{P}' \\ p<Y^{1/(r+1)}}} p.$$

Hence, we obtain

$$A(Y, r) \ge \sum_{(d,P'(Y^{1/(r+1)}))=1} c_d = \mathcal{S}(\mathfrak{C}, Y, Y^{1/(r+1)}). \tag{4.33}$$

With $S(Y, M')$ given in (2.4),

$$\mathcal{S}(\mathfrak{C}, Y, Y^{1/(r+1)}) = \sum_{M'|P'(Y^{1/(r+1)})} \mu(M') S(Y, M'), \tag{4.34}$$

and from Theorem 2.2, we have

$$S(Y, M') = \rho(M') C_f Y + O(\tau(M') M'^{1/2} Y^{13/14+\epsilon}),$$

where $C_f > 0$ and

$$\rho(M') = \prod_{p|M'} \frac{1}{p} \left[1 + \frac{1}{p} + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\right]^{-1}. \tag{4.35}$$

First, we can see that $\rho(M')$ is multiplicative. Secondly, we can notice that from Proposition 1.18, we have

$$1+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}=\frac{(1+p^{-1})}{(1-\alpha_p^2p^{-1/2})(1-\beta_p^2p^{-1/2})}>0 \qquad (4.36)$$

for $p\nmid N$. This is because $|\alpha_p|=|\beta_p|=1$ and $\alpha_p\beta_p=1$. Thus, for $p\nmid 4M$,

$$0\le\frac{1}{p}\left[1+\frac{1}{p}+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right]^{-1}<1,$$

as product of positive numbers and by using the lower bound (4.36). Thirdly, using Deligne's bound, we have

$$\left[1+\frac{1}{p}+\sum_{j=1}^{\infty}\frac{\lambda_{p^{2j}}}{p^j}\right]^{-1}=1+O\Big(\frac{1}{p}\Big). \qquad (4.37)$$

Hence, for $2\le z_1<z$, we have

$$\begin{aligned}\prod_{\substack{p\in\mathcal{P}'\\ z_1\le p<z}}(1-\rho(p))^{-1}&=\exp\Big(-\sum_{\substack{p\in\mathcal{P}'\\ z_1\le p<z}}\log(1-\rho(p))\Big)\\ &=\exp\Big(\sum_{\substack{p\in\mathcal{P}'\\ z_1\le p<z}}\rho(p)+O\Big(\frac{1}{p^2}\Big)\Big)\\ &=\exp\Big(\sum_{\substack{p\in\mathcal{P}'\\ z_1\le p<z}}\frac{1}{p}+O\Big(\frac{1}{p^2}\Big)\Big).\end{aligned}$$

Then, using Mertens' formula for the sum of reciprocal primes, we obtain

$$\begin{aligned}\prod_{\substack{p\in\mathcal{P}'\\ z_1\le p<z}}(1-\rho(p))^{-1}&=\exp\Big(\log\log z-\log\log z_1+O\Big(\frac{1}{z_1}\Big)\Big)\\ &=\Big(\frac{\log z}{\log z_1}\Big)\exp\Big(O\Big(\frac{1}{z_1}\Big)\Big)\\ &\le\Big(\frac{\log z}{\log z_1}\Big)\Big(1+O\Big(\frac{1}{z_1}\Big)\Big)\\ &\le\Big(\frac{\log z}{\log z_1}\Big)\Big(1+\frac{L}{\log z_1}\Big),\end{aligned}$$

for some constant $L > 1$. It then follows that ρ satisfies the conditions of Theorem 4.1. As a result, from Corollary 4.2, we have

$$\mathcal{S}(\mathfrak{C}, Y, Y^{1/(r+1)}) \geq V(Y^{1/(r+1)})C_f Y(f_0(s) + O(\log D)^{-1/6}) + R^-(Y, Y^{1/(r+1)}, D), \tag{4.38}$$

for $s \geq 2$ and $f_0(s) = 1 + O(e^{-s})$ as $s \to \infty$ and

$$R^-(Y, Y^{1/(r+1)}, D) = \sum_{\substack{M' | P'(Y^{1/(r+1)}) \\ M' \in \mathcal{M}^-}} \mu(M')\Big[S(Y, M') - \rho(M')C_f Y\Big]. \tag{4.39}$$

We can now deduce our first result.

Theorem 4.3. *If we take* $r = 84$*, then*

$$A(Y, r) \gg \frac{Y}{\log Y}.$$

Proof. Let us begin by computing an upper bound for $R^-(z, D)$. For this, let us take $\epsilon > 0$. From (4.39), we have

$$\begin{aligned} |R^-(Y, z, D)| &\leq \sum_{\substack{M' | P(Y^{1/(r+1)}) \\ M' < D}} \tau(M')^2 M'^{1/2} Y^{13/14+\epsilon} \\ &\ll Y^{13/14+\epsilon} \sum_{M' < D} \tau(M')^2 M'^{1/2} \ll Y^{13/14+\epsilon} D^{3/2+\epsilon}. \end{aligned}$$

The last inequality comes from the fact that $\tau(M') \ll M'^{\epsilon/2}$. Hence, if we take $D = Y^{1/42}$, then

$$|R^-(Y, z, Y^{1/42})| \ll Y^{27/28+(43/42)\epsilon}, \tag{4.40}$$

where the implied constant depends only on f and F. On the other hand, if we take $z_1 = 2$ in (4.3), then

$$\prod_{p | P(z)} (1 - \rho(p))^{-1} \leq K \log(z),$$

for some $K > 0$. Hence, if we take $z = Y^{1/(r+1)}$, we obtain

$$V(Y^{1/(r+1)}) = \prod_{p | P(Y^{1/(r+1)})} (1 - \rho(p)) \gg_r \frac{1}{\log Y}. \tag{4.41}$$

Hence, let us choose $r = 84$, so that $s = \frac{\log D}{\log z} = \frac{85}{42} > 2$. Thus, Equation (4.38) becomes

$$\mathcal{S}(\mathfrak{C}, Y, Y^{\frac{1}{85}}) \gg \frac{Y}{\log Y}\left(f_0\left(\frac{85}{42}\right) + O((\log Y)^{-1/6})\right) + O(Y^{27/28+(43/42)\epsilon}).$$

Therefore, if we take $\epsilon > 0$ small enough, we obtain the result. □

4.4 Multiple Dirichlet series and sieve method

As in the previous section, let us consider $\mathfrak{C}' = (C_d)_d$ with

$$C_d = \begin{cases} L(1/2, f, \chi_{\omega d}) P_{\omega d_0, d_1}(1/2) F\left(\frac{d}{Y}\right) & \text{if } \omega d_0 \in \mathcal{D} \\ 0 & \text{otherwise.} \end{cases}$$

where $d > 0$ is written as $d_0 d_1^2$. Therefore, we have

$$\mathcal{S}(\mathfrak{C}', Y, Y^{1/(r+1)}) = \sum_{M' | P'(Y^{1/(r+1)})} S'(Y, M'), \tag{4.42}$$

where

$$S'(Y, M') = \sum_{d \equiv 0 [M']} C_d \tag{4.43}$$

is the same as the one defined in Equation (2.79). Using Theorem 2.16, we obtain

$$S'(Y, M') = \rho'(M') C_f' Y + O(\tau(M') M'^{1/2} Y^{13/14+\epsilon}), \tag{4.44}$$

where

$$C_f' = C_f \prod_{p \nmid M} \left[1 + \sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma}}\right] \tag{4.45}$$

for some $C_f > 0$,

$$\rho'(M') = \prod_{p | M'} \frac{1 + \sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma}} + \left(\sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma-1}}\right)\left(1 + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\right)}{p\left(1 + \sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma}}\right)\left[1 + \frac{1}{p} + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\right]}, \tag{4.46}$$

$$\rho'''(p^\gamma) = \left(1 + \rho''(p^\gamma, p^2) p^{-1} + \cdots + \rho''(p^\gamma, p^{4\gamma}) p^{-2\gamma}\right),$$

and

$$\rho''(p^\gamma, p^{2e}) = \left[A_{p,p^{2e}}^{(\gamma)} + \left(A_{1,p^{2e}}^{(\gamma)} - A_{p,p^{2e}}^{(\gamma)}\right)\left(1 + \frac{1}{p}\left(1 + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\right)^{-1}\right)^{-1}\right].$$

Now, for the sieve method to work, we have to make the following assumption

Assumption A2. For any $p \nmid M$,

$$1 + \sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma}} > 0. \tag{4.47}$$

This can be computed directly from the coefficients of the Dirichlet polynomials. However, we do not have sufficient information to prove this assumption.

We can notice that Assumption (A2) implies immediately that $C_f' > 0$. Moreover, together with Assumption (A1), we have the following proposition:

Proposition 4.4. *Assuming Assumption (A1) and (A2), we have*

$$0 \le \rho'(p) < 1, \text{ for } p \nmid 4M \tag{4.48}$$

and

$$\prod_{z_1 \le p < z} (1 - \rho(p))^{-1} \le \Big(\frac{\log z}{\log z_1}\Big)\Big(1 + \frac{L'}{\log z_1}\Big). \tag{4.49}$$

Proof. By Assumption (A1) and [22], $S'(Y,p)$ is a sum of non negative numbers, hence, $\rho'(p)$ must be non negative. If $\rho'(p) > 1$ for a fixed prime p, we have

$$\sum_{d \not\equiv 0[p]} C_d = \sum_d C_d - \sum_{d \not\equiv 0[p]} C_d = (1 - \rho'(p))C_f'Y + o(Y),$$

which is negative for Y large enough. This contradicts the fact that the coefficients C_d are non negative. Moreover, $\rho'(p) \neq 1$ from (2.113). As a result, the condition (4.48) is satisfied. Secondly, we have

$$\rho'(p) = \frac{1 + \sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma}} + \Big(\sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma-1}}\Big)\Big(1 + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\Big)}{p\Big(1 + \sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma}}\Big)\Big[1 + \frac{1}{p} + \sum_{j=1}^{\infty} \frac{\lambda_{p^{2j}}}{p^j}\Big]}.$$

Since

$$\sum_{\gamma=1}^{\infty} \frac{\rho'''(p^\gamma)}{p^{2\gamma}} = O\Big(\frac{1}{p^{2-\epsilon}}\Big),$$

for any $\epsilon > 0$, we obtain

$$\rho'(p) = \frac{1}{p} + O\Big(\frac{1}{p^{2-\epsilon}}\Big). \tag{4.50}$$

Consequently, in the same way as in Section 4.3, we can obtain

$$\prod_{z_1 \le p < z} (1 - \rho(p))^{-1} \le \Big(\frac{\log z}{\log z_1}\Big)\Big(1 + \frac{L'}{\log z_1}\Big). \tag{4.51}$$

□

We can then apply Corollary 4.2 with $\kappa = 1$ on $\mathfrak{C}'$. Hence,

$$\mathcal{S}(\mathfrak{C}', Y, Y^{1/(r+1)}) \geq V'(Y^{1/(r+1)})C_f'Y(f_0(s)+O((\log D)^{-1/6}))+R'^{-}(Y^{1/(r+1)}, D), \tag{4.52}$$

for $s \geq 2$ where

$$V'(z) = \prod_{p|P(z)} (1-\rho'(p)). \tag{4.53}$$

and

$$R'^{-}(Y, z, D) = \sum_{\substack{M'|P(z)\\ M'\in\mathcal{M}^-}} \mu(M')\Big[\rho'(M')C_f' - S'(Y, M')\Big]. \tag{4.54}$$

Let us claim the following proposition, which is the purpose of the weight factors $P_{\omega d_0, d_1}(1/2)$:

Proposition 4.5. *For $\epsilon > 0$, $Y > 0$ and $D \leq Y$, we have*

$$|R'^{-}(Y, z, D)| \ll Y^{1/2}D^{1+\epsilon}. \tag{4.55}$$

Proof. In this proof, we fix $a_1 = \omega$. Let us consider

$$\mathcal{S}^{-}(\mathfrak{C}', Y, z, D) = \sum_{\substack{M'|P'(z)\\ M'\in\mathcal{M}^-}} \mu(M') \sum_{\substack{d\equiv 0[M']\\ (d,M)=1\\ a_1\cdot d_0\in\mathcal{D}}} L(1/2, f, \chi_{a_1\cdot d_0})P_{a_1\cdot d_0, d_1}(1/2)F\Big(\frac{d}{Y}\Big). \tag{4.56}$$

Using the change of variable $x = Y^{-1}$, we have

$$W(x, z, D) = \sum_{\substack{M'|P'(z)\\ M'\in\mathcal{M}^-}} \mu(M') \sum_{\substack{d\equiv 0[M']\\ (d,M)=1\\ a_1\cdot d_0\in\mathcal{D}}} L(1/2, f, \chi_{a_1\cdot d_0})P_{a_1\cdot d_0, d_1}(1/2)F(dx) \tag{4.57}$$

Taking the Mellin transform of $W(x, z, D)$, we obtain

$$\begin{aligned}\tilde{W}(w, z, D) &= \int_0^\infty W(x, z, D)x^{w-1}dx\\ &= \sum_{\substack{M'|P'(z)\\ M'\in\mathcal{M}^-}} \mu(M') \sum_{\substack{d\equiv 0[M']\\ (d,M)=1\\ a_1\cdot d_0\in\mathcal{D}}} L(1/2, f, \chi_{a_1\cdot d_0})P_{a_1\cdot d_0, d_1}(1/2)\\ &\qquad \times \int_0^\infty F(dx)x^{w-1}dx\\ &= A(w)Z(w, z, D),\end{aligned} \tag{4.58}$$

where

$$A(w) = \int_0^\infty F(x)x^{w-1}dx, \tag{4.59}$$

and

$$Z(w,z,D) = \sum_{\substack{M'|P(z) \\ M'\in\mathcal{M}^-}} \mu(M') \sum_{\substack{d\equiv 0[M'] \\ (d,M)=1 \\ a_1\cdot d_0\in\mathcal{D}}} \frac{L(1/2,f,\chi_{a_1\cdot d_0})P_{a_1\cdot d_0,d_1}(1/2)}{d^w}. \tag{4.60}$$

Thus, using the Mellin inversion formula, we obtain

$$W(x,z,D) = \frac{1}{2\pi i}\int_{(c)} A(w)Z(w,z,D)x^{-w}dw, \tag{4.61}$$

for some large $c > 0$.

Now, we want to express $Z(w,z,D)$ as linear combination of $Z_M(1/2,w,a_2l_2,a_1l_1)$. First, let us notice that for $a_1d_0 \in \mathcal{D}$, we have

$$L(1/2,f,\chi_{a_1\cdot d_0}) = L_M(1/2,f,\chi_{d_0}\chi_{a_1})\prod_{p|M}(1-\lambda_p p^{-1/2}+\chi_0(p)p^{-1}). \tag{4.62}$$

Moreover, from [24], we know that $a_1d_0 \equiv \nu^2[4M]$ for any $(\nu,4M) = 1$ if and only if

$$\left(\frac{a_1d_0}{p}\right) = 1 \qquad \text{for any odd prime } p|M, \text{and} \tag{4.63}$$

$$a_1d_0 \equiv 1 \mod 8. \tag{4.64}$$

Since $(d,M) = 1$, then the first condition can be removed by introducing the the factor

$$\prod_{p|(M/2)} \frac{1}{2}\left(1+\left(\frac{a_1d_0}{p}\right)\right) = \frac{1}{2^{\omega(M)-1}}\sum_{l_2|(M/2)}\left(\frac{a_1d_0}{l_2}\right) = \frac{1}{2^{\omega(M)-1}}\sum_{l_2|(M/2)}\left(\frac{a_1}{l_2}\right)\chi_{\chi_{-1}l_2}(d_0),$$

where $\omega(M)$ is the number of prime factors of M. In the same way, we can remove the second condition with the following factor

$$\frac{1}{8}\big(1+\chi_{-1}(a_1d_0)\big)\big(1+\chi_2(a_1d_0)\big)\big(1+\chi_{-2}(a_1d_0)\big).$$

Now, let us set the coefficient $c_{a_2l_2}(a_1)$ given by

$$\sum_{a_2l_2\in Div(M)} c_{a_2l_2}(a_1)\chi_{a_2l_2}(d_0) = \frac{1}{2^{\omega(M)+2}}\prod_{p|(M/2)}\left(1+\left(\frac{a_1d_0}{p}\right)\right) \times \big(1+\chi_{-1}(a_1d_0)\big)\big(1+\chi_2(a_1d_0)\big)\big(1+\chi_{-2}(a_1d_0)\big) \tag{4.65}$$

Consequently, we have

$$Z(w,z,D) = \tilde{C}_M \sum_{\substack{M'|P(z)\\ M'\in\mathcal{M}^-}} \mu(M') \sum_{a_2 l_2} c_{a_2 l_2}(a_1) \times \sum_{\substack{d\equiv 0[M']\\ (d,M)=1}} \frac{L_M(1/2, f, \chi_{a_1\cdot d_0})\chi_{a_2 l_2}(d_0)P_{a_1\cdot d_0, d_1}(1/2)}{d^w}, \tag{4.66}$$

where

$$\tilde{C}_M = \frac{1}{2^{\omega(M)+2}} \prod_{p|M} (1 - \lambda_p p^{-1/2} + \chi_0(p)p^{-1}).$$

Next, to deal with the condition $d \equiv 0[M']$, we will use Lemma 2.8. Hence,

$$Z(w,z,D) = \tilde{C}_M \sum_{\substack{M'|P(z)\\ M'\in\mathcal{M}^-}} \mu(M') \sum_{a_2 l_2} c_{a_2 l_2}(a_1) \sum_{d'|M'} \mu(d') \times \sum_{(d,Md')=1} \frac{L_M(1/2, f, \chi_{a_1\cdot d_0})\chi_{a_2 l_2}(d_0)P_{a_1\cdot d_0, d_1}(1/2)}{d^w}. \tag{4.67}$$

Again, we have

$$L_M(1/2, f, \chi_{a_1\cdot d_0}) = L_{Md'}(1/2, f, \chi_{d_0}\chi_{a_1}) \prod_{p|d'} \left(1 - \chi_{a_1}\chi_{d_0}(p)\lambda_p p^{-1/2} + p^{-1}\right). \tag{4.68}$$

Let us notice that $\chi_{d_0}(p) = \chi_{\chi_{-1}(p)p}(d_0)$ for any prime $p|d'$, hence, we can write

$$\prod_{p|d'} \left(1 - \chi_{a_1}\chi_{d_0}(p)\lambda_p p^{-1/2} + p^{-1}\right) = \sum_{a_2' l_2' \in Div(d')} \tilde{c}_{a_2' l_2'}(a_1)\chi_{a_2' l_2'}(d_0), \tag{4.69}$$

where

$$Div(d') = \{a_2' l_2' : 0 < l_2' | d',\ a_2' \in \{\pm 1\}\}.$$

Therefore, (4.67) becomes

$$Z(w,z,D) = \tilde{C}_M \sum_{\substack{M'|P(z)\\ M'\in\mathcal{M}^-}} \mu(M') \sum_{a_2 l_2} c_{a_2 l_2}(a_1) \sum_{d'|M'} \mu(d') \sum_{a_2' l_2' \in Div(d')} \tilde{c}_{a_2' l_2'}(a_1) \times \sum_{(d,Md')=1} \frac{L_M(1/2, f, \chi_{a_1\cdot d_0})\chi_{a_2 l_2}(d_0)P_{a_1\cdot d_0, d_1}(1/2)}{d^w}. \tag{4.70}$$

Finally, if we rearrange the sum, we can obtain

$$\begin{aligned} Z(w,z,D) = \tilde{C}_M \sum_{\substack{M'|P(z)\\ M'\in\mathcal{M}^-}} \mu(M') \sum_{d'|M'} \mu(d') \\ \times \sum_{a_2l_2\in Div(Md')} \tilde{b}_{a_2l_2}(a_1) Z_{Md'}(1/2, w, \chi_{a_2l_2}, \chi_{a_1}). \end{aligned} \tag{4.71}$$

Moreover, with a trivial computation of the coefficient $\tilde{b}_{a_2l_2}(a_1)$, we can have

$$\tilde{b}_{a_2l_2}(a_1) \ll d'^{\epsilon}, \tag{4.72}$$

for any $\epsilon > 0$. Therefore, from Theorem 3.11, we can deduce that $Z(w,z,D)$ is analytic for $\mathrm{Re}(w) > 0$. Therefore, we can move the line of integration in (4.61) to (1/2) passing through a possible pole at $w = 1$. Hence, we obtain

$$W(x,z,D) = x^{-1}A(1) \operatorname*{Res}_{w=1} Z(w,z,D) + \frac{1}{2\pi i}\int_{(1/2)} A(w)Z(w,z,D)x^{-w}dw. \tag{4.73}$$

Note that using the same technique on $S(Y, M')$ and Theorem 2.16, we can get

$$A(1)\operatorname*{Res}_{w=1} \tilde{Z}(w,z,D) = C_f' \sum_{\substack{M'|P(z)\\ M'\in\mathcal{M}^-}} \mu(M')\rho'(M'). \tag{4.74}$$

Now, we want an upper bound for $Z(w,z,D)$ at the vertical line (1/2). From (4.66) and the fact that $|c_{a_2l_2}(a_1)| \leq 1$, we have

$$\begin{aligned} \left|Z(1+\epsilon+it,z,D)\right| &\leq \sum_{\substack{M'|P(z)\\ M'\in\mathcal{M}^-}} \sum_{a_2l_2} \sum_{\substack{d\equiv 0[M']\\ (d,M)=1}} \frac{L(1/2,f,\chi_{a_1\cdot d_0})P_{a_1\cdot d_0,d_1}(1/2)}{d^{1+\epsilon}} \\ &\leq \sum_{(d,M)=1} \frac{L(1/2,f,\chi_{a_1\cdot d_0})P_{a_1\cdot d_0,d_1}(1/2)}{d^{1+\epsilon}} \sum_{\substack{M'|d\\ M'<D}} \sum_{a_2l_2} 1 \\ &\ll D^{\epsilon} \sum_{(d,M)=1} \frac{L(1/2,f,\chi_{a_1\cdot d_0})P_{a_1\cdot d_0,d_1}(1/2)}{d^{1+\epsilon}}\tau(d). \end{aligned} \tag{4.75}$$

However, from Theorem 3.11, the series

$$\sum_{(d,M)=1} \frac{L(1/2,f,\chi_{a_1\cdot d_0})P_{a_1\cdot d_0,d_1}(1/2)}{d^{1+\epsilon}}\tau(d)$$

converges. Hence, we obtain

$$\left|Z(1+\epsilon+it,z,D)\right| \ll D^{\epsilon}. \tag{4.76}$$

On the other hand, from (4.71) and (4.72), we have

$$|Z(-\epsilon+it,z,D)| \ll \sum_{M'<D}\sum_{d'|M'}\sum_{a_2l_2\in Div(Md')} d'^{\epsilon}|Z_{Md'}(1/2,w,\chi_{a_2l_2},\chi_{a_1})| \tag{4.77}$$

From Theorem 3.11, we have

$$|Z_{Md'}(1/2,w,\chi_{a_2l_2},\chi_{a_1})| \ll d'^{1+\epsilon},$$

hence,

$$\begin{aligned}|Z(-\epsilon+it,z,D)| &\ll \sum_{M'<D}\sum_{d'|M'}\sum_{a_2l_2\in Div(Md')} l_2^{\epsilon}d^{1+\epsilon}\\ &\ll \sum_{M'<D} M'^{1+\epsilon} \ll D^{2+\epsilon}.\end{aligned} \tag{4.78}$$

Consequently, using the Phragmen-Lindelöf principle, we can get

$$|Z(1/2+it,z,D)| \ll D^{1+\epsilon}. \tag{4.79}$$

Since $A(1/2+it) \ll 1$, we have

$$\left|\int_{(1/2)} A(w)Z(w,z,D)x^{-w}dw\right| \ll x^{-1/2}D^{1+\epsilon}. \tag{4.80}$$

Therefore, replacing $x = Y^{-1}$ in (4.73), we got

$$\mathcal{S}^{-}(\mathfrak{C}',Y,z,D) = C_f'Y \sum_{\substack{M'|P(Y^{1/(r+1)})\\ M'\in\mathcal{M}^{-}}} \mu(M')\rho'''(M') + O(Y^{1/2}D^{1+\epsilon}). \tag{4.81}$$

□

We finally get the following theorem

Theorem 4.6. *Assuming Assumption (A1) and (A2), for $r = 4$, we obtain*

$$\mathcal{S}(\mathfrak{C}',Y,Y^{1/(r+1)}) \gg \frac{Y}{\log Y}. \tag{4.82}$$

Proof. From Proposition 4.5 and Equation (4.52), we have

$$\mathcal{S}(\mathfrak{C}', Y, Y^{1/(r+1)}) \geq V(Y^{1/(r+1)})C_f' Y(f_0(s) + O(\log D)^{-1/6}) + O(Y^{1/2}D^{1+\epsilon}), \tag{4.83}$$

for any $\epsilon > 0$. Now, let us take $D = Y^{(1-\epsilon)/2}$. Hence,

$$Y^{1/2}D^{1+\epsilon} \ll Y^{1-\epsilon^2/2}.$$

Therefore, using the same technique as in Section 4.3, we can choose ϵ small enough, so that

$$s = \frac{\log D}{\log z} = \frac{(1-\epsilon)(r+1)}{2} > 2.$$

This completes proof. □

Theorem 4.7. *Let S be a finite set of primes. Then, there exist infinitely many odd fundamental discriminant d_0 such that*

$$L(1/2, f, \chi_{d_0}) \neq 0,$$

d_0 has at most 84 primes factors and $\chi_{d_0}(p) = 1$ for all $p \in S$. Moreover, if we assume Assumption (A1) and (A2), then, we can reduce d_0 to almost prime, with at most 4 primes factors.

Proof. The first statement is a direct application of Theorem 4.3. Moreover, from (2.1), we have $\chi_{d_0}(p) = 1$ for all $d_0 \in \mathcal{D}$ and $p \in S$. For the second statement, from Theorem 4.6 and assuming Assumption (A1), we have

$$\begin{aligned}\sum_{\substack{d_0 \in \mathcal{D} \\ (d_0, Y^{1/5})=1}} L(1/2, f, \chi_{d_0}) \sum_{d_1} P_{\omega \cdot d_0, d_1}(1/2) F\Big(\frac{d_1^2|d_0|}{Y}\Big) \\ \geq \sum_{\substack{d_0 \in \mathcal{D} \\ (d_0, Y^{1/5})=1}} L(1/2, f, \chi_{d_0}) \sum_{(d_1, Y^{1/5})=1} P_{\omega d_0, d_1}(1/2) F\Big(\frac{d_1^2|d_0|}{Y}\Big) \\ \gg \frac{Y}{\log Y}.\end{aligned} \tag{4.84}$$

Clearly, this sum tends to ∞ as $Y \to \infty$. However, that does not necessarily mean that they will be infinitely many odd discriminant $d_0 \in \mathcal{D}$ whose number of prime factors are less than 4 such that $L(1/2, f, \chi_{d_0}) \neq 0$. So now, we assume by contradiction that there are only finitely many such d_0. Using (3.9), fixing d_0, we have

$$\sum_{d_1} P_{d_0, d_1}(1/2) F\Big(\frac{d_1^2|d_0|}{Y}\Big) \ll \sum_{d_1 \leq \sqrt{Y/|d_0|}} d^{\epsilon} \ll \frac{Y^{1/2+\epsilon/2}}{\sqrt{|d_0|}}$$

Since the sum over d_0 is finite, we have

$$\sum_{\substack{d_0 \in \mathcal{D} \\ (d_0, Y^{1/5})=1}} L(1/2, f, \chi_{d_0}) \sum_{d_1} P_{\omega \cdot d_0, d_1}(1/2) F\Big(\frac{d_1^2 |d_0|}{Y}\Big)$$

$$\ll Y^{1/2+\epsilon/2} \sum_{\substack{d_0 \in \mathcal{D} \\ (d_0, Y^{1/5})=1 \\ |d_0|<Y}} \frac{L(1/2, f, \chi_{d_0})}{\sqrt{|d_0|}} \ll Y^{1/2+\epsilon/2},$$

which contradicts with the lower bound in Equation (4.84). □

Chapter 5

Conclusion

Let us briefly summarize what was achieved in this project. In Chapter 2, we use an adaptation of a method by Iwaniec [9] to prove that

$$\sum_{\substack{d_0 \equiv 0[M'] \\ d_0 \in \mathcal{D}}} L(1/2, f, \chi_{d_0}) F\Big(\frac{|d_0|}{Y}\Big) = \rho(M') C_f Y + O(\tau(M') M'^{1/2} Y^{13/14+\epsilon}), \quad (5.1)$$

for some multiplicative function ρ and a constant $C_f \neq 0$. The properties of ρ and the error term played a crucial role in the sieve method described in Chapter 4 to demonstrate existence infinitely many discriminants d_0 with at most 84 prime factors such that $L(1/2, f, \chi_{d_0}) \neq 0$. The estimate in (5.1) can be generalized with the addition of the weight factors $P_{d_0,d_1}(1/2)$. Indeed, in a similar way as for (5.1), we have

$$\begin{aligned} &\sum\sum_{\substack{d_0 d_1^2 \equiv 0[M'] \\ d_0 \in \mathcal{D}}} L(1/2, f, \chi_{d_0}) P_{d_0,d_1}(1/2) F\Big(\frac{d_1^2 |d_0|}{Y}\Big) \\ &\qquad = \rho'(M') C'_f Y + O(\tau(M') M'^{1/2} Y^{13/14+\epsilon}), \end{aligned} \quad (5.2)$$

where ρ is a multiplicative function and

$$C'_f = C_f \prod_{p \nmid 4M} (1 + \rho_p^0) \quad (5.3)$$

for some real numbers ρ_p^0 and $C_f > 0$. Then, we used the analytic properties of $P_{d_0,d_1}(s)$ to considerably reduce the upper bound of the error term. With this latter, we expected to improve the result from 84 prime factors to 4 prime factors. However, the required conditions for ρ' and C'_f to apply the sieve method were not fully guaranteed. Hence, two assumptions are made:

1. $P_{d_0,d_1}(1/2) \geq 0$ for all $(d_0 d_1, M) = 1$: this assumption was first made in Chapter 3 to guarantee the absolute convergence of the double Dirichlet series
$$\sum_{(d,M)=1} \frac{L_M(1/2, f, \chi_{d_0}) P_{d_0,d_1}(1/2))}{d^w}, \tag{5.4}$$
for $\mathrm{Re}(w) > 1$. Secondly, it was needed so that the input of the sieve is non negative and $0 \leq \rho'(p) \leq 1$ for any $p \nmid 4M$.

2. $(1 + \rho_p^0) > 0$ for any $p \nmid 4M$: this allows us to prove that $C_f' \neq 0$ and $\rho'(p) \neq 1$.

Indeed, there are evidences for these assumptions. For the first assumption, Hoffstein and Luo seem to assume the non negativity of $P_{d_0,d_2}(1/2)$ in [8]. Additionally, the following formula for $P_{d_0,p}(s)$ was given in [20]:

$$P_{d_0,p}(s) = 1 - \chi_{d_0}(p)\lambda_p p^{-s} + p^{-2s} - p\chi_{d_0}(p)\lambda_p p^{-3s} + p^{2-4s}.$$

for any p. Hence, $P_{d_0,p}(1/2)$ is non negative for any p different from 2. For the second assumption, the infinite product in (5.3) is an absolutely convergent. Thus, we can expect C_f' to be different from 0.

If one of these two assumptions does not hold, adjustments can be made. If the second assumption is not met, we can remove the factors that are equal to zero in (5.3). Moreover, we have

$$\rho'(p) = \frac{1}{p} + O\Big(\frac{1}{p^{2-\epsilon}}\Big).$$

Hence, for p sufficiently large, $0 \leq \rho(p) < 1$. Thus, we can also exclude small primes p. Regarding the first assumption, we can slightly change the sieve method, so that if we remove the primes we have mentioned above, the technique is still valid. We can then deal with the absolute convergence of (5.4) separately.

Further work still need to be done to fully understand the Dirichlet polynomials $P_{d_0,d_1}(s)$. Both assumptions that we made depend on the properties of these polynomials. These polynomials can be obtained explicitly either by using metaplectic techniques as in [19], or by a more combinatorial approach as in [10] and [18]. It might even be possible to obtain the main term in (5.2) using the residue of the multiple Dirichlet series at $w = 1$. However, the methods in [10], [18] and [19] are still beyond the scope of this project.

There are a few ideas for future research. As suggested in [9] it might be possible to reduce the number of prime factors of d_0 in the main theorem further to 3 by considering weighted sieve. One could also ask questions about the case of automorphic L-functions on $GL(3)$ where the appropriate weight factors are known to exist, and so, the theory of multiple Dirichlet series applies. This is no longer the case for automorphic L-functions on $GL(4)$. The group of transformations induced by the corresponding systems of functional equations is infinite which prevents the analytic continuation of the double Dirichlet series that we have seen here in Chapter 3 for the case of $GL(2)$ (or [10] for $GL(3)$), so it is interesting know what happens in this case. This is, of course, a very ambitious plan.

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