COMBINATORICS - PAST AND PRESENT

INAUGURAL ADDRESS: PROF HELMUT PRODINGER MAY 2006

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ABOUT THE AUTHOR

Helmut Prodinger was born in Mödling, Austria in 1954. He attended school there and studied Computer Science at the Technical University of Vienna from 1972. In 1973 he began to study Mathematics at the University of Vienna. In 1976 he was awarded the degree of a Diplom-Ingenieur (equivalent to a Master's degree), and obtained a Doctorate in 1978 with a thesis on formal languages and combinatorics on words. Inspired by Rainer Kemp and Philippe Flajolet he turned his attention to the analysis of algorithms, in the style of Donald Knuth. In 1981 he attained his *Habilitation* for discrete mathematics and theoretical computer science. (Wikipedia explains: A *Habilitation* qualifies one to be admitted as a professor at a university.)

In 1998 he decided to leave Austria and to accept the offer of a chair at the University of the Witwatersrand in Johannesburg; concurrently, in 2002, he became director of the John Knopfmacher Centre for applicable analysis and number theory. He joined the Mathematics Department at the Stellenbosch University in January 2005. He has published more than 200 papers always centring on combinatorics, but also covering various other topics, such as number theory, probability theory, asymptotic analysis and computer science. He is on the editorial board of four research journals and has co-edited several special issues on the mathematical analysis of algorithms. He has supervised 11 PhD students and approximately 20 MSc students.

Prodinger has received the following honours: a medal from the Austrian Mathematical Society in 1985, the Vice-Chancellor's research award in 2001, a gold medal from the South African Mathematical Society in 2001, and the award for research distinction in 2003. He was elected as a Fellow of the Royal Society of South Africa in 2003. In 2005 he became an honorary professor at the Technical University of Graz, Austria. He holds an A rating from the NRF. A two-day conference was organised in 2004 on the occasion of his 50th birthday.

Helmut Prodinger is unmarried and lives in Somerset West. He likes to do some home recording, playing all the instruments himself, as well as singing; he also enjoys reading old-fashioned books, mostly by German authors.

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Combinatorics — Past and Present

It is known that it is very difficult to speak about a mathematical subject to a non-specialist audience. What is even more difficult under such circumstances is to describe the research that one is doing oneself. While I feel that what I am doing is very exciting for me, it might not be that for everybody else. Thus, I decided to rather give a historical talk and to describe COMBINATORICS, *which is my area of research, by going quickly through the centuries.*

What is Combinatorics? MATHWORLD has this list:

Combinatorics

- Binomial Coefficients (47)
- Bracketing (3)
- Combinatorial Identities (2)
- Combinatorial Optimization (1)
- Configurations (22)
- Covers (4)
- Designs (55)
- Enumeration (17)
- General Combinatorics (42)
- Lattice Paths and Polygons (8)
- Partitions (66)
- Permutations (71)
- Weighing (1)

The AMERICAN MATHEMATICAL SOCIETY has this list (2000; before there were only three entries):

Combinatorics

- 05Axx Enumerative combinatorics
- 05Bxx Designs and configurations. For applications of design theory, see 94C30
- 05Cxx Graph theory. For applications of graphs, see 68R10, 90C35, 94C15
- 05Dxx Extremal combinatorics
- 05Exx Algebraic combinatorics

AN EARLY EXAMPLE: FIBONACCI NUMBERS

The original problem that **Fibonacci** [The "greatest European mathematician of the middle ages"; full name was Leonardo of Pisa] investigated (in the year 1202) was how fast rabbits could breed in ideal circumstances.

Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of one month, so that at the end of its second month a female can produce another pair of rabbits. Suppose that our rabbits never die and that the female always produces one new pair (one male, one female) every month from the second month on. The puzzle that Fibonacci posed was ...

How many pairs will there be in one year?

(1) At the end of the first month they mate, but there is still only 1 pair.
(2) At the end of the second month the female produces a new pair, so now there are 2 pairs of rabbits in the field.
(3) At the end of the third month the original female produces a second pair, making 3 pairs in all in the field.
(4) At the end of the fourth month the original female has produced yet another new pair, the female born two months ago also produces her first pair, making 5 pairs.

The Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ (add the last two to get the next)

Today – there is a journal **The Fibonacci Quarterly**!

Euler and The Königsberg bridge problem

The Königsberg bridge problem asks if the seven bridges of the city of Königsberg, formerly in Germany but now known as Kaliningrad and part of Russia, over the river Preger can all be traversed in a single trip without doubling back, with the additional requirement that the trip ends in the same place it began. This is equivalent to asking if the multigraph on four nodes and seven edges has an Eulerian circuit. This problem was answered in the negative by Euler (1736), and represented the beginning of **graph theory.**

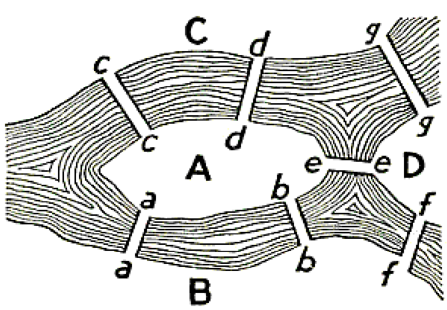
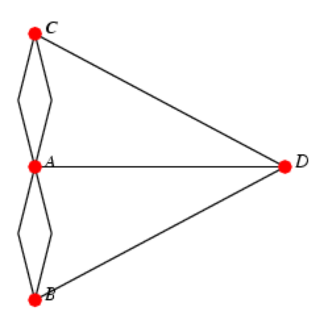


Figure 98. *Geographic Map: The Königsberg Bridges.*



Kirkman's Schoolgirl Problem

In a boarding school there are fifteen schoolgirls who always take their daily walks in rows of threes. How can it be arranged so that each schoolgirl walks in the same row with every other schoolgirl exactly once a week? Finding the solution of this problem is equivalent to constructing a *Kirkman triple system.* The following table gives one of the seven distinct solutions to the problem.

Sun	ABC	DEF	GHI	JKL	MNO
Mon	ADH	BEK	CIO	FLN	GJM
Tue	AEM	BHN	CGK	DIL	FJO
Wed	AFI	BLO	CHJ	DKM	EGN
Thu	AGL	BDJ	CFM	EHO	IKN
Fri	AJN	BIM	CEL	DOG	FHK
Sat	AKO	BFG	CDN	EIJ	HLM

Biographical note about Kirkman: *. . . in 1835 he entered the Church of England. He spent five years as a curate, first in Bury, then in Lymm. By 1839 he became vicar in the Parish of Southworth in Lancashire, a position he held for 52 years.*

Ballot Problem

Suppose A and B are candidates for office and there are $2n$ voters, n voting for A and n for B. In how many ways can the ballots be counted so that A is always ahead of or tied with B? The solution is a **Catalan number** C_n.

$$\boxed{1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots}$$

Catalan numbers appear in an amazing number of different contexts!
Richard Stanley keeps a list of them:
`http://www-math.mit.edu/~rstan/ec/`

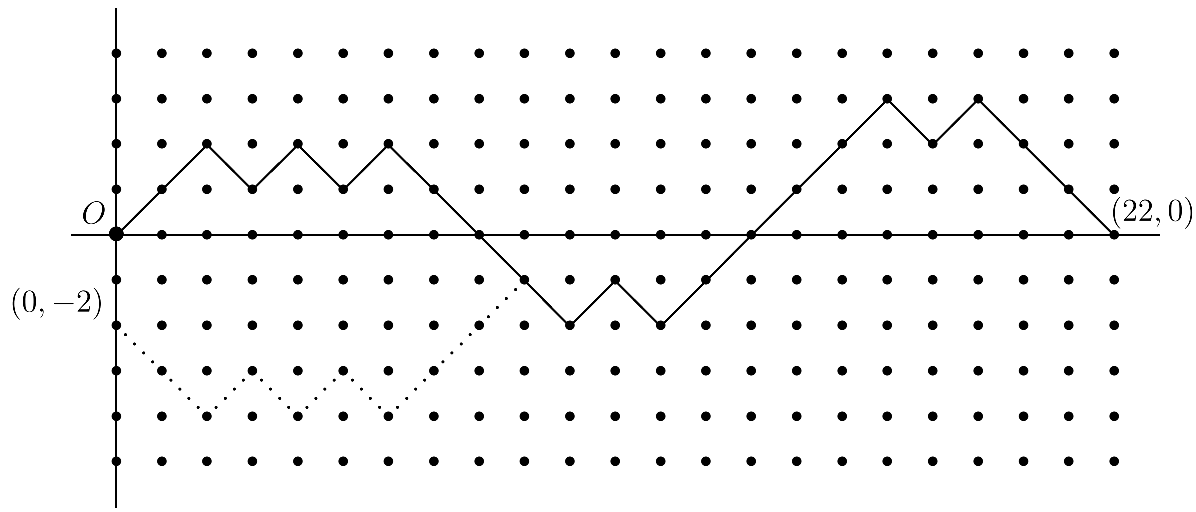
> "**Catalan addendum** (Postscript or PDF) (version of 30 October 2005; 53 pages).
>
> An addendum of new problems (and solutions) related to Catalan numbers.
>
> This addendum will be continually updated.
>
> Current number of combinatorial interpretations of C_n: **135**."

This solution was first shown by M. Bertrand. Another elegant solution was provided by André (1887) using the so-called **André's reflection method.**

This enumerates the sequences that are **NOT** wanted, i.e. where B leads the count at least once. The correct number comes out as

$$\text{number of ALL sequences} - \text{number of UNWANTED sequences} = \binom{2n}{n} - \binom{2n}{n+1}.$$

Let us consider as an example this sequence of votes: $AABABABBBBABAAAAABABBB$



So the enumeration of the unwanted sequences is reduced to the enumeration of the (unrestricted) sequences, starting in $(0, -2)$, ending in $(2n, 0)$, which is easy.

We have just seen **a bijective method!**

Desiré André was a mathematician, a professor in a Parisian lycée. He had written his thesis, but was never able to obtain a position at the University of Paris.

Until 1965, say, we had the following:

- Famous mathematicians like Euler, Cayley or Polya worked on combinatorial problems, but this was not their main activity.
- Talented amateurs worked on some problems, which often had a recreational character.
- There were some problems from probability theory which were of an intrinsically combinatorial nature.

Gian-Carlo Rota

Rota received the Steele Prize from the American Mathematical Society in 1988. The Prize citation singles out the 1964 paper **On the Foundations of Combinatorial Theory** as:

> ... the single paper most responsible for the revolution that incorporated combinatorics into the mainstream of modern mathematics.

This paper was the first of a series of ten papers with this main title; all ten have subtitles (for example, the first one was subtitled *Theory of Möbius functions*) and all the remaining nine have between one and three additional co-authors. Papers two to nine were all published between 1970 and 1974, with the tenth being published in 1992.

Symbolic methods

George Polya. He is also famous for his quotations. Here are two of them: *"I am too good for philosophy and not good enough for physics. Mathematics is in between."* — *"John von Neumann was the only student I was ever afraid of."*

In how many ways can an amount of 100 Rand be changed into coins of 1 Rand, 2 Rand, 5 Rand?

All *configurations* can be described as

$$\Big(\emptyset + \mathbf{1} + \mathbf{11} + \mathbf{111} + \cdots\Big)\Big(\emptyset + \mathbf{2} + \mathbf{22} + \mathbf{222} + \cdots\Big)\Big(\emptyset + \mathbf{5} + \mathbf{55} + \mathbf{555} + \cdots\Big)$$

This is translated into a *generating function,* taking the value of the coin into account:

$$\Big(1 + z^1 + z^2 + z^3 + \cdots\Big)\Big(1 + z^2 + z^4 + z^6 + \cdots\Big)\Big(1 + z^5 + z^{10} + z^{15} + \cdots\Big) =$$
$$= \frac{1}{(1-z)(1-z^2)(1-z^5)}$$

In this function, we must look at the *coefficient* of z^{100}, from which we get the answer 541.

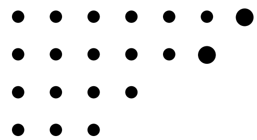
(This number is totally uninteresting, but the concept is important!)

If we live in an ideal country, where coins of any type $1, 2, 3, \ldots$ are available, then we get as a straight-forward generalisation the generating function

$$P(z) = \frac{1}{(1-z)(1-z^2)(1-z^3)\ldots} = \prod_{k\ge 1} \frac{1}{1-z^k}.$$

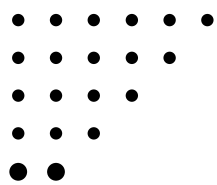
This is the celebrated *partition generating function.* The coefficient of z^n in $P(z)$ is called $p(n)$; it is the number of partitions of a natural number n.

Partitions can be graphically represented as *Ferrers diagrams.*



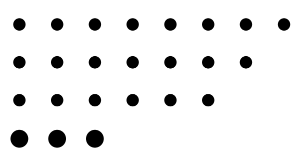
A partition of 20
with different parts $7, 6, 4, 3$
number of parts is even

Transformed into



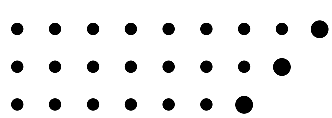
A partition of 20
with different parts $6, 5, 4, 3, 2$
number of parts is odd

Another instance:



A partition of 24
with different parts $8, 7, 6, 3$
number of parts is even

Transformed into



A partition of 24
with different parts $9, 8, 7$
number of parts is odd

For "almost" all partitions into distinct parts, exactly one of the two transformation rules is applicable. This leads to a combinatorial proof of *Euler's pentagonal theorem* (not stated here). This bijection is due to *Fabian Franklin,* and is the *first mathematical achievement of an American!*

Ramanujan

Srinivasa Aiyangar Ramanujan (1887–1920) was an Indian mathematician and one of the most esoteric mathematical geniuses in the twentieth century. Nicknamed **"the man who knew infinity,"** he had uncanny mathematical manipulative abilities. He excelled in number theory and modular functions. He also made significant contributions to the development of partition functions and summation formulas involving constants such as π. A child prodigy, he was largely self-taught in mathematics and had compiled over 3,000 theorems by the year 1914, when he moved to Cambridge. Often his formulæ were stated without proof and were only later proven to be true. His results have inspired a large amount of research and many mathematical papers. In 1997 the **Ramanujan Journal** was launched to publish work "in areas of mathematics influenced by Ramanujan."

Hardy and Ramanujan studied $p(n)$, the number of partitions of the natural number n. This was an ideal collaboration of the well-trained analyst Hardy and Ramanujan, with his incredible intuition.

Partitions belong to *number theory* and *combinatorics!*

Mathematical formulæ are not the best idea for a talk to non-specialists, but I cannot resist mentioning (one of the) **Rogers-Ramanujan identities**:

$$\sum_{n\ge 0} \frac{q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{k\ge 0} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})},$$

which could be interpreted as an identity for the functions, valied for $|q| < 1$, but a combinatorialist would rather see both of them representing the series $1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + 5q^9 + 6q^{10} + \cdots$ (and give a meaning to the numbers [coefficients] $1, 1, 1, 1, 2, 2, 3, 3, 4, 5, 6, \dots$ in two different ways).

Combinatorial Identities

Typical quantities in *combinatorics* are *factorials*

$$n! = 1 \times 2 \times 3 \times \cdots \times n$$

and *binomial coefficients* (aka *Pascal's triangle*)

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

They satisfy various relations; an example is *Vandermonde's convolution*

$$\binom{2n}{n} = \binom{n}{n}\binom{n}{0} + \binom{n}{n-1}\binom{n}{1} + \cdots + \binom{n}{0}\binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Proving such identities was considered to be an *art*, and the lists of formulæ by Gould and Riordan (independently) are worthwhile mentioning.

John Riordan: Combinatorial identities (1968)

He was a successful combinatorialist, and one of the rare cases of a researcher *without a PhD!* Along with Leonard Carlitz (*... certainly one of the most prolific mathematical researchers of all time*) Riordan was a pioneer in combinatorics in the 20th century. But his view was a bit extreme:

> *Combinatorialists use recurrence, generating functions, and such transformations as the Vandermonde convolution; others,* ***to my horror,*** *use contour integrals, differential equations, and other resources of mathematical analysis.*

Odlyzko wrote a monograph on “Asymptotic enumeration methods” (1995). He says:

> *Analytic methods are extremely powerful and, when they can be used, they often yield estimates of unparalleled precision.*

Asymptotics means the following: We want to replace an ungainly expression (such as $n! = 1 \times 2 \times \cdots \times n$) by something “simpler” that has the same “growth behaviour.”
For instance,

$$\begin{aligned} 50! &= 30414093201713378043612608166064768844377641568960512000000000000, \\ \text{approximation} &= 30363445939381558207983726752112093959052599802286296951906806786.885\ldots, \end{aligned}$$

which does not look too impressive, but is accurate to within 99%.

Hypergeometric functions. People like Askey, Andrews and others realised that *hypergeometric functions* are the right way of expressing combinatorial identities. The “classical” hypergeometric function is due to Gauss:

$$F(a,b;c;x) = 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{x^3}{3!} + \cdots$$

When translated into this “language,” most identities, which previously looked quite different, now turn out to be the same, and only a few identities (Euler, Gauss, Kummer, Dixon, Watson, ...) do the job!

Shalosh B. Ekhad
Short Proofs of Two Hypergeometric Summation Formulas of Karlsson
Proceedings of the American Mathematical Society, vol. 107, No. 4 (Dec. 1989), pp. 1143–1144
Abstract.
Karlsson [2] gave elegant proofs of two hypergeometric summation formulas conjectured by Gosper, which were mentioned in [1]. Here I give new proofs that are much shorter, but less elegant.

Shalosh B. Ekhad is Doron Zeilberger's computer!
Everybody can create (via Maple and Zeilberger's algorithm) a *mathematical paper.*

> zeilpap(binomial$(n,k)^2, k, n$);

A PROOF OF A RECURRENCE[a]
By Shalosh B. Ekhad, Temple University, ekhad@math.temple.edu
Theorem: Let, $F(n,k)$, be given by $\binom{n}{k}^2$
and let, SUM(n), be the sum of, $F(n,k)$, with respect to, k
SUM(n), satisfies the following linear recurrence equation

$$(-4n-2)\mathrm{SUM}(n) + (n+1)\mathrm{SUM}(n+1) = 0.$$

PROOF: We cleverly construct,

$$G(n,k) := \frac{(-3n-3+2k)k^2}{(-n-1+k)^2}\binom{n}{k}^2$$

with the motive that

$$(-4n-2)F(n,k) + (n+1)F(n+1,k) = G(n,k+1) - G(n,k),$$

(Check!)
and the theorem follows upon summing with respect to, k, . QED.

[a] Computer-generated text!

Shalosh B. Ekhad has published 27 papers and would qualify in this way for a *chair* in a mathematics department!

The On-Line Encyclopedia of Integer Sequences

You type in your (obscure?) sequence of numbers: $\boxed{1, 1, 2, 4, 9, 21, 51, 127, 323, 835}$
And you get

A001006 Motzkin numbers: number of ways of drawing any number of non-intersecting chords among n points on a circle. (Formerly M1184 N0456)

(Creative) Guessing is nowadays incorporated in computer algebra systems such as Maple or Mathematica. For instance, the program GFUN (Generating functions) does this for you:

$$> \text{guessgf}([1, 1, 2, 3, 5], z);$$

And your answer is

$$\frac{1}{1 - z - z^2},$$

as it should be.

References

[1] G. Andrews. *The Theory of Partitions*, vol. 2 of *Encyclopedia of Mathematics and its Applications.* Addison-Wesley, 1976.

[2] G. E. Andrews, R. Askey, and R. Roy. *Special functions*, vol. 71 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1999.

[3] P. Flajolet and R. Sedgewick. *Analytic Combinatorics.* Cambridge University Press, Cambridge, 2007.

[4] Henry W. Gould. *Combinatorial identities.* Henry W. Gould, Morgantown, W.Va., 1972. A standardized set of tables listing 500 binomial coefficient summations.

[5] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics (Second Edition).* Addison Wesley, 1994.

[6] G. H. Hardy. *Ramanujan. Twelve lectures on subjects suggested by his life and work.* Cambridge University Press, Cambridge, England, 1940.

[7] A. M. Odlyzko. Asymptotic enumeration methods. In *Handbook of combinatorics,* vols. 1 and 2, pages 1063–1229. Elsevier, Amsterdam, 1995.

[8] M. Petkovšek, H. Wilf, and D. Zeilberger. $A = B$. A.K. Peters, Ltd., 1996.

[9] J. Riordan. *Combinatorial Identities.* John Wiley, 1968.

[10] G.-C. Rota. On the foundations of combinatorial theory. I. Theory of Möbius functions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 2:340–368 (1964), 1964.

[11] B. Salvy and P. Zimmermann. GFUN: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Transactions on Mathematical Software*, 20:163–167, 1994.

[12] R. Stanley. *Enumerative combinatorics,* vols. 1 and 2. Cambridge University Press, Cambridge, 1997 and 1999.