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An equivalence of categories in algebraic geometry and some unlikely intersections in powers of elliptic curves

by

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Thesis presented in partial fulfilment of the requirements for the degree of Master of Mathematics in the Faculty of Science at Stellenbosch University

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Declaration

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Date: 2024/03/05 ....

Abstract

An equivalence of categories in algebraic geometry and some unlikely intersections in powers of elliptic curves

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Thesis: MSc Mathematics

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Let E be an elliptic curve defined over a number field and let $C_1, C_2 \subseteq E^N(\mathbb{C})$ be irreducible closed algebraic curves where $N \geqslant 3$. Suppose that C_1 is not contained in a 1-dimensional algebraic subgroup of $E^N(\mathbb{C})$ and $C_1 \cup C_2$ is not contained in a 2-dimensional algebraic subgroup of $E^N(\mathbb{C})$. Extending work of Boxall on the multiplicative group to elliptic curves, we prove that, if at least one of C_1 and C_2 is not defined over $\overline{\mathbb{Q}}$, then there are at most finitely many points $x \in C_1$ such that there exists an $n \in \mathbb{N}$ such that $nx \in C_2$ and that $[n]C_1 \nsubseteq C_2$ where $[n]C_1 = \{nx \mid x \in C_1\}$. Moreover, we consider a definition of affine varieties and prevarieties, in the classical sense, over an arbitrary field and provide expository development of many well-known properties of these classical affine varieties. Additionally, extending well-known definitions of functors in the algebraically closed field case, we rigorously construct functors in both directions, between the category of these prevarieties and the category of reduced schemes of finite type over the same arbitrary field, which we show to be quasi-inverse so that they give rise to an equivalence of categories. Finally, in an appendix, we include the well-known definition and some properties of schemes as well as some other basic topics for convenience.

Uittreksel

’n Ekwivalensie van kategorieë in algebraïese meetkunde en ’n paar onwaarskynlike kruisings in kragte van elliptiese kurwes

(“An equivalence of categories in algebraic geometry and some unlikely intersections in powers of elliptic curves”)

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Laat E ’n elliptiese kurwe wees wat oor ’n getalveld gedefinieer word en laat $C_1, C_2 \subseteq E^N(\mathbb{C})$ ‘n onherleibare geslote algebraïese kurwe wees waar $N \geqslant 3$. Gestel C_1 is nie vervat in ’n 1-dimensionele algebraïese subgroep van $E^N(\mathbb{C})$ nie en $C_1 \cup C_2$ is nie vervat in ’n 2-dimensionele algebraïese subgroep van $E^N(\mathbb{C})$ nie. Uitbreiding van werk van Boxall op die vermenigvuldigende groep tot elliptiese kurwes, bewys ons dat, as ten minste een van C_1 en C_2 nie oor $\overline{\mathbb{Q}}$ gedefinieer word nie, is daar hoogstens eindige hoeveelheid punte $x \in C_1$, sodanig dat daar ’n $n \in \mathbb{N}$ bestaan sodat $nx \in C_2$ en dat $[n]C_1 \nsubseteq C_2$ waar $[n]C_1 = \{nx \mid x \in C_1\}$. Verder beskou ons ’n definisie van verwante variëteite en voorvariëteite, in die klassieke sin, oor ’n arbitrêre veld en bied blootstellingsontwikkeling van baie bekende eienskappe van hierdie klassieke verwante variëteite. Daarbenewens, om bekende definisies te verleng van funktore in die algebraïes geslote veldsaak, konstrueer ons funktore streng in beide rigtings, tussen die kategorie van hierdie voorvariëteite en die kategorie van verminderde skemas van eindige tipe oor dieselfde arbitrêre veld, wat ons wys dat dit kwasi-omgekeerd is sodat dit aanleiding gee tot ’n ekwivalensie van kategorieë. Ten slotte, in ’n aanhangsel, sluit ons die bekende definisie en sommige eienskappe van skemas sowel as ander basiese onderwerpe vir gerief.

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Particularly, I would also like to thank Sophie for assisting me in the presentation of the statements and proofs of Definition 3.2.4, Theorem 3.1.7 and Lemmas 3.2.7, 3.2.8 and for allowing me to include her contributions being Definition-Lemma 2.1.5, Definition 2.4.2, Corollary 2.4.13 and most of the results in section 3.3 and I would like to thank Gareth for contributing an essential result being Lemma 6.2.4.

Dedication

To my dear grandfather Leslie and my grandmothers Aroma and Carol.

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List of symbols

$[[1, n]]$	denotes the set of natural numbers $\{1, \ldots, n\}$.
$\subseteq_{\mathrm{cl}}$	indicates an inclusion between topological spaces where the subspace is closed in the other.
$\subseteq_{\mathrm{op}}$	indicates an inclusion between topological spaces where the subspace is open in the other.
$\mathcal{C}_0$	the underlying objects of a category $\mathcal{C}$
$\mathcal{C}_1$	the underlying morphisms of a category $\mathcal{C}$
$\mathcal{F}_0$	the map sending objects to objects for some functor $\mathcal{F}$
$\mathcal{F}_1$	the map sending morphisms to morphisms for some functor $\mathcal{F}$
$G(F)$	the F-rational points for any prevariety G.
$\mathcal{O}_E$	the endomorphism ring of an elliptic curve E
$\mathcal{O}_X$	the associated sheaf to a scheme X
$\mathcal{O}_X\|_U$	restriction of the associated sheaf to a scheme X to an open set U of X
$\mathcal{O}_{X,x}$	the stalk of a scheme X at the point $x \in X$
$\mathfrak{m}_{X,x}$	the unique maximal ideal of the stalk of a scheme X at the point $x \in X$
$\mathcal{R}_X$	the associated sheaf to a prevariety X

Chapter 1

Introduction

Given two varieties of dimension m and r, respectively, in some ambient space of dimension n, one would expect their intersection to be empty given $m + r < n$. Such intersections have been subject to much study and consequently coined as "unlikely intersections". This area has been central within Diophantine geometry with many deep questions. One such question posed by A. Levin, as seen in [37, p.39], is:

Suppose that X, Y are curves in $\mathbb{G}_m^n$ where $\mathbb{G}_m$ is an algebraic torus. What can be said about the points $x \in X$ such that some power x^m lies in Y?

One could completely determine the answer to this question for all n by use of the following deep conjecture known as Zilber's conjecture on the intersection with tori (CIT) which Zilber originally stated in [38]:

Conjecture 1.0.1 ([9, p.309])**.** *Let K be a field of characteristic 0. For every variety V of dimension r in $\mathbb{G}_m^{r+s}$ defined over K and irreducible over its algebraic closure $\overline{K}$, there is a finite union $\mathcal{U} = \mathcal{U}(V)$ of proper algebraic subgroups of $\mathbb{G}_m^{r+s}$ such that $V \cap \mathcal{H}_{s-1}$ is contained in $\mathcal{U}$ where $\mathcal{H}_{s-1}$ is the union of all algebraic subgroups of dimension $s - 1$ in $\mathbb{G}_m^{r+s}$.*

To answer Levin's question in the case where at least one of the curves is not defined over $\overline{\mathbb{Q}}$, Boxall in [11] proved what was shown, by Zannier in [37], to be a consequence of Zilber's CIT which is stated as follows:

Theorem 1.0.2. *Let $C_1, C_2 \subseteq \mathbb{G}_m^N(\mathbb{C})$ be irreducible closed algebraic curves with $N \geqslant 3$. Further assume that at least one of C_1 and C_2 is not defined over $\overline{\mathbb{Q}}$. Suppose that there does not exist an algebraic subgroup $G \subseteq \mathbb{G}_m^N(\mathbb{C})$ of dimension one such that $C_1 \subseteq G$ and that there does not exist an algebraic subgroup $H \subseteq \mathbb{G}_m^N(\mathbb{C})$ of dimension two such that $C_1 \cup C_2 \subseteq H$. Let $\mathcal{N} = \{n \in \mathbb{N} \mid [n]C_1 \subseteq C_2\}$. Then $\bigcup_{n\in\mathbb{N}\setminus\mathcal{N}}\{x \in C_1 \mid x^n \in C_2\}$ is finite.*

Indeed, Levin's question does not make the assumption that at least one C_1 and C_2 is not defined over $\overline{\mathbb{Q}}$, however, Boxall's arguments could not be used when both curves are defined over $\overline{\mathbb{Q}}$. Now, as elliptic curves are abelian varieties and share many structural properties with algebraic tori, it was natural to question whether the following conjecture holds:

Conjecture 1.0.3. *Let E be an elliptic curve defined over $\overline{\mathbb{Q}}$ and let $C_1, C_2 \subseteq E^N(\mathbb{C})$ be irreducible closed algebraic curves with $N \geqslant 3$. Further assume that at least one of C_1 and C_2 is not defined over $\overline{\mathbb{Q}}$. Suppose that there does not exist an algebraic subgroup $G \subseteq E^N(\mathbb{C})$ of dimension one such that $C_1 \subseteq G$ and that there does not exist an algebraic subgroup $H \subseteq E^N(\mathbb{C})$ of dimension two such that $C_1 \cup C_2 \subseteq H$. Let $\mathcal{N} = \{n \in \mathbb{N} \mid [n]C_1 \subseteq C_2\}$. Then $\bigcup_{n \in \mathbb{N} \setminus \mathcal{N}} \{x \in C_1 \mid x^n \in C_2\}$ is finite.*

It follows from the work of Barroero and Dill in [5], that, if E is not defined over a number field, then the conjecture holds. Proving Conjecture 1.0.3, would then give the full picture and, indeed, we prove that it is true in chapter 6. In proving Theorem 1.0.2, Boxall used work of Bays, Kirby and Wilkie in [7] which gave an analogue of Schanuel's conjecture for raising to an exponentially transcendental power. This required the notion of partial exponential fields given in [26], which are fields equipped with a homomorphism from a divisible subgroup of the additive group to the multiplicative group, and a closure operator used to express exponential transcendence. Following a similar pattern, in chapter 5, we define an analogue to partial exponential fields, known as "Γ-fields" with respect to an elliptic curve E, that are fields equipped with a certain divisible $\mathcal{O}_E$-module where $\mathcal{O}_E$ is the endomorphism ring of E, which one may think roughly of as the graph of the exponential map associated to an elliptic curve, and define a closure operator to express a form of transcendence. Our definition of Γ-fields is a relaxed version of the definition of Γ-fields from Bays and Kirby in [6], as we do not require, as in their case, for a Γ-field to be the field generated by its $\mathcal{O}_E$-module, but, rather, contain the field generated by it. This allows us to work with a version of the usual exponential map associated to E where we restrict to the infinitesimal elements of a field arising as the algebraic closure of a certain κ-saturated extension analogously to that of [11]. Moreover, due to the different Γ-field definition, we introduce an additional concept known exponentially finitely generated Γ-field extensions in section 5.2 to give flexibility in following the presentation of [6]. In chapter 4, we prove an analogue of the Bays-Kirby-Wilkie result in [7] within the framework of Γ-fields that is used in proving Conjecture 1.0.3. Initially, this thesis had two goals. The first was to prove Conjecture 1.0.3 by generalizing Boxall's arguments and the other was to prove Theorem 1.0.2 without using model theory. We sought to understand the essence of the model theoretic techniques used by Boxall and understand how they might translate in the setting of algebraic geometry. A key concept was that of generic points which, model-theoretically, can be described as a point on an affine or projective variety V such that any point on V satisfies, at least, any polynomial equation that the generic point satisfies. Alternatively, if V has dimension d, it can be said that a point on V is generic if it has transcendence degree d over the field generated by the coordinates of the polynomial equations defining V. By this definition, when considering V over a sufficiently large algebraically closed field, there are infinitely many generic points on V. This is a stark contrast to the notion of generic points for schemes as, in that case, they are unique. As a start, Weil's original definitions of algebraic varieties in [34] gave a means to better understand the former definition of generic points. Another issue being that many recent and well-known books in algebraic geometry have presented algebraic geometry in either the classical sense or with schemes. This left a divide in which version one would consider best to use in particular contexts. Moreover, many texts in the classical sense only considered algebraic geometry over algebraically closed fields where one the strengths of schemes is that you could consider their construction over arbitrary rings and provide a much wider generalisation. Having noted this, we strived to pose classical algebraic geometry over arbitrary fields. This construction was briefly exposed upon in appendix of Bombieri and Gublers's [8, p. 516] which also, very briefly, described a functor between reduced schemes of finite type over k and his construction of classical affine varieties and prevarieties akin to Hartshorne's in [20, Proposition 2.6, p. 78].

Using inspiration from Weil's foundations, in chapter 2, we give a slight change to Bombieri and Gubler's definition of classical affine varieties and prevarieties and show that most well-known statements, such the Nullstellensatz, about classical varieties hold. This chapter is standard in the sense that many results are proven analogously to the algebraically closed case. We, however, present this in a slightly different manner using category theory which schemes are often presented with. To unify these approaches of algebraic geometry, we wanted prove a equivalence between the definitions of Weil, varieties in the classical sense and the more modern schemes. Although, in this thesis, we only managed to give an equivalence between the latter two approaches. Chapter 3 is dedicated to this and generalises the functor constructions of Hartshorne in [20, Proposition 2.6, p. 78] and that of Wedhorn and Görtz in [19, p. 81], which in their case only covered algebraically closed fields, to arbitrary fields. In particular, we give explicit constructions of functors in both directions between prevarieties, in the classical sense, over an arbitrary field to reduced schemes of finite type over the same field. In section 3.1, we define a functor which sends prevarieties to reduced schemes of finite type over arbitrary fields. This is expanded more upon that in the other direction in the literature and, as such, we fill the gaps of the arguments of Hartshorne, Wedhorn and Görtz. In section 3.3, we define the functor which sends reduced schemes of finite type over an arbitrary field to prevarieties over the same field which was briefly explained by Wedhorn and Görtz in [19, p. 81], for algebraically closed fields, with many details left out. In particular, we provide the full details. Finally, in section 3.5 of chapter 3, we give a few results, for completeness, that were contributed and written by one of the supervisors of this thesis, S. Marques, which made use of our constructions and portrays that the functors we define are quasi-inverse so we may deduce an equivalence of categories.

To be more self-contained, the appendix has stated and proven many respective definitions and results which serve as a background to thesis. In particular, basic topology, commutative algebra, category theory are given. The most significant part of the appendix is the chapters discussing sheaves, locally ringed spaces and schemes which give well-known definitions and results of scheme theory that are used in the main chapters of the thesis. Notably, chapters $A.7.2$ and $A.7.3$ give comprehensive details of well-known results of schemes and play an essential role within chapter 3. These appendices are given in a more encyclopedic format which states definitions and proves results from various sources without much discussion.

Lastly, we note that, stylistically, chapters 2 and 3 are written in a more expository format whilst chapters $4, 5$ and 6 are written more in the style of a publication with additional details.

Chapter 2

Classical Algebraic Geometry

Altering Bombieri and Gubler's definition of classical affine varieties in [8, p. 514-515] with inspiration from Weil's [34], we state and prove fundamental concepts of classical algebraic geometry in a categorical manner. This will lay the groundwork for future sections. This chapter is adapted mainly from Appendix A.2 of [8] and various sections of [17], however, we give explicit references due to the close nature of proofs between the arbitrary field case and the algebraically closed field case.

Throughout this chapter, we set M to be an algebraically closed field and k to be a subfield of M.

2.1 Categories of algebraic sets and their ideals

We begin by defining an essential concept known as affine n-space over M which will serve as the ambient space of algebraic sets.

Definition 2.1.1 ([8, p. 514]). *For any* $n \in \mathbb{N}$*, we define the* affine n-space in M *denoted by* $\mathbb{A}^n_M$ *to be the set*

$$\mathbb{A}^n_M := \{(a_1, ..., a_n) \mid a_i \in M, \ \forall i = 1, ..., n\}.$$

In the following definition, we introduce algebraic n-sets in M which are the zero loci of polynomial equations that are identified with subsets of $\mathbb{A}^n_M$. These will be the main objects of our discussion.

Definition 2.1.2 ([8, p. 515]). *For any* $n \in \mathbb{N}$ *and* $I \subseteq k[x_1, \ldots, x_n]$ *an ideal, we define the* zero locus of I in M *as*

$$\mathbb{V}_{k,M}(I) := \{a \in \mathbb{A}^n_M \mid f(a) = 0, \ \textit{for all } f \in I\}.$$

We call sets of this form algebraic n-sets over k in M. *When* k *and* M *are understood, we will usually drop the subscripts for* k *and* M *and refer to such* *sets as* algebraic n-sets*. We make the convention that an algebraic set over k in M is an algebraic n-set over k in M for some $n \in \mathbb{N}$. We denote $\mathbb{V}_{k,M}(T)$ for $\mathbb{V}_{k,M}(\langle T \rangle)$ where $T \subseteq k[x_1, \ldots, x_n]$ and $\mathbb{V}_{k,M}(f)$ for $\mathbb{V}_{k,M}(\langle f \rangle)$ where $f \in k[x_1, \ldots, x_n]$.*

We define the zero locus within a subset of affine space as follows.

Definition 2.1.3. *Let $S \subseteq \mathbb{A}^n_M$ and $I \subseteq k[x_1, \ldots, x_n]$ be an ideal for some $n \in \mathbb{N}$. We define the* zero locus of I in S *as*

$$\begin{aligned} \mathbb{V}_S(I) &:= \mathbb{V}(I) \cap S \\ &= \{a \in S \mid f(a) = 0, \ \forall f \in I\}. \end{aligned}$$

In particular, $\mathbb{V}_{\mathbb{A}^n_M}(I) = \mathbb{V}(I)$. We denote $\mathbb{V}_S(T)$ for $\mathbb{V}_S(\langle T \rangle)$ where $T \subseteq k[x_1, \ldots, x_n]$ and $\mathbb{V}_S(f)$ for $\mathbb{V}_S(\langle f \rangle)$ where $f \in k[x_1, \ldots, x_n]$.

The next lemma follows directly from the definitions.

Lemma 2.1.4 ([18, Lemma 1.4, p. 6])**.** *Let I, J be ideals in $k[x_1, \ldots, x_n]$ such that $I \subseteq J$, then $\mathbb{V}_S(J) \subseteq \mathbb{V}_S(I)$ for any $S \subseteq \mathbb{A}^n_M$.*

To formalize the previous constructions using category theory, in this next definition, we introduce certain categories and functors as a means to portray the relationship between algebraic sets and ideals that we can associate to them.

Definition-Lemma 2.1.5. *For some $n \in \mathbb{N}$, we define the following categories and functors:*

1. *The* category of algebraic sets over k in M*, denoted by $\mathbf{AS}_{k,M,n}$, is the category whose:*
   - *objects are algebraic n-sets over k in M.*
   - *morphisms are inclusions between algebraic n-sets over k in M.*
2. *The* category of ideals of $k[x_1, \ldots, x_n]$*, denoted by $\mathbf{I}_{k,n}$, is the category whose:*
   - *objects are ideals of $k[x_1, \ldots, x_n]$.*
   - *morphisms are inclusions between ideals.*
3. *We denote the full subcategory of $\mathbf{I}_{k,n}$ whose objects are radical ideals as $\mathbf{RI}_{k,n}$.*

4. *We define the functor $\mathcal{V}_{k,M,n}$ as a pair of maps:*

$$(\mathcal{V}_{k,M,n,0} : \mathbf{I}^{\text{op}}_{k,n,0} \to \mathbf{AS}_{k,M,n,0}, \quad \mathcal{V}_{k,M,n,1} : \mathbf{I}^{\text{op}}_{k,n,1} \to \mathbf{AS}_{k,M,n,1})$$

such that $\mathcal{V}_{k,M,n,0}(I) := \mathbb{V}_{k,M}(I)$ and $\mathcal{V}_{k,M,n,1}(\mathfrak{i}^{\text{op}}_{J,I}) := \mathfrak{i}_{\mathbb{V}_{k,M}(I),\mathbb{V}_{k,M}(J)}$, where $I, J \in \mathbf{I}_{k,n,0}$ with $J \subseteq I$ and where $\mathfrak{i}_{J,I}$ denotes the inclusion from J to I and $\mathfrak{i}_{\mathbb{V}_{k,M}(I),\mathbb{V}_{k,M}(J)}$ denotes the inclusion from $\mathbb{V}_{k,M}(I)$ to $\mathbb{V}_{k,M}(J)$. When the context is clear, we will remove k, M and n from the subscript to ease notation.

The following shows how we can associate an ideal to an algebraic set.

Definition 2.1.6 ([8, p. 515])**.** *Let $n \in \mathbb{N}$ and $S \subseteq \mathbb{A}^n_M$. We define the* ideal of polynomials vanishing on S over k *as*

$$\mathbb{I}_{k,M}(S) := \{f \in k[x_1, \ldots, x_n] \mid f(a) = 0, \ \forall a \in S\}.$$

When k and M are understood, we will typically drop the subscript. We also say ideal of S *to refer to the ideal of polynomials vanishing on S.*

We can also define the ideal of polynomials vanishing on the intersection of an algebraic n-set with a subset of the affine n-space.

Definition 2.1.7. *Let $S \subseteq \mathbb{A}^n_M$ for some $n \in \mathbb{N}$ and let X be an algebraic n-set over k in M. We define the* ideal of X in S *as*

$$\mathbb{I}_S(X) := \mathbb{I}(S) + \mathbb{I}(X).$$

In particular, $\mathbb{I}_{\mathbb{A}^n_M}(X) = \mathbb{I}(X)$. We denote $\mathbb{I}_S(a)$ for $\mathbb{I}_S(\{a\})$ where $a \in \mathbb{A}^n_M$.

In analogy to Lemma 2.1.4, we have the following.

Lemma 2.1.8 ([18, Remark 1.9, p. 8])**.** *If $S \subseteq \mathbb{A}^n_M$ and X, Y are algebraic n-sets over k such that $X \subseteq Y$ for some $n \in \mathbb{N}$, then $\mathbb{I}_S(Y) \subseteq \mathbb{I}_S(X)$.*

Proof. Let $f \in \mathbb{I}_S(Y)$. Then $f(a) = 0$ for all $a \in S \cap Y$. Since $X \subseteq Y$, it follows that $f(a) = 0$ for all $a \in S \cap X$. Thus, $f \in \mathbb{I}_S(X)$. ☐

Continuing our trend of categorical constructions, we now define a functor closely tied with $\mathcal{V}$.

Definition 2.1.9 (Lemma)**.** *For any $n \in \mathbb{N}$, we define the functor $\mathcal{I}_{k,M,n}$ from $\mathbf{AS}_{k,M,n}$ to $\mathbf{I}^{\text{op}}_{k,n}$ as the pair*

$$(\mathcal{I}_{k,M,n,0} : \mathbf{AS}_{k,M,n,0} \to \mathbf{I}^{\text{op}}_{k,n,0}, \mathcal{I}_{k,M,n,1} : \mathbf{AS}_{k,M,n,1} \to \mathbf{I}^{\text{op}}_{k,n,1})$$

such that $\mathcal{I}_{k,M,n,0}(X) := \mathbb{I}(X)$ and $\mathcal{I}_{k,M,n,1}(\mathfrak{i}_{X,Y}) = \mathfrak{i}^{\text{op}}_{\mathbb{I}(Y),\mathbb{I}(X)}$ where $X, Y \in \mathbf{AS}_{k,0}$ such that $X \subseteq Y$. Again, when the context is clear, we will remove k and M from the subscript to ease notation.

In this next lemma, we highlight how $\mathcal{I}$ and $\mathcal{V}$ are related by showing that $\mathcal{V}$ acts as a left inverse of $\mathcal{I}$.

Lemma 2.1.10 ([18, Proposition 1.10, p. 8]). *For any algebraic set X, we have $\mathbb{V}(\mathbb{I}(X)) = X$. In other words, $\mathcal{V}$ is a left inverse functor of $\mathcal{I}$.*

Proof. We begin by showing the backward inclusion. Let $a \in X$. We have that $f(a) = 0$ for all $f \in \mathbb{I}(X)$, so $a \in \mathbb{V}(\mathbb{I}(X))$ by definition.

The forward inclusion is as follows. Since X is an algebraic set, it is of the form $X = \mathbb{V}(I)$ for some ideal $I \subseteq k[x_1, \ldots, x_n]$. By definition, we have $I \subseteq \mathbb{I}(\mathbb{V}(I))$. Furthermore, since from the first part of the lemma $I \subseteq \mathbb{I}(\mathbb{V}(I))$, it follows from Lemma 2.1.4 that $\mathbb{V}(\mathbb{I}(\mathbb{V}(I))) \subseteq \mathbb{V}(I)$. Thus, $\mathbb{V}(\mathbb{I}(X)) \subseteq X$. ☐

However, $\mathcal{I}$ is not a left inverse of $\mathcal{V}$. Indeed, for $I \subseteq k[x_1, \ldots, x_n]$, $\mathbb{I}(\mathbb{V}(I)) = I$ may not hold. This can be seen by considering the example $I = \langle x^2 \rangle \subseteq k[x]$, $\mathbb{I}(\mathbb{V}(I)) = \mathbb{I}(\{0\}) = \langle x \rangle \neq I$. Restricting our functors will allow us to construct an isomorphism of categories which will be discussed later in the chapter.

2.2 Topological structure of algebraic sets

We now aim to show that we can consider algebraic sets as ringed topological spaces giving a richer structure to these constructions. We first introduce the topology on algebraic sets and assume $n \in \mathbb{N}$.

Proposition 2.2.1 ([18, Lemma 1.7, p. 7], [20, Lemma 2.1., p. 70]). *The following properties hold:*

1. *$\mathbb{V}(k[x_1, \ldots, x_n]) = \varnothing$ and $\mathbb{V}(0) = \mathbb{A}^n_M$.*
2. *For any ideals I and J in $k[x_1, \ldots, x_n]$, we have $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(IJ) = \mathbb{V}(I \cap J)$.*
3. *Let $(I_\alpha)_{\alpha \in A}$ be a family of ideals in $k[x_1, \ldots, x_n]$ with some index set A, then $\bigcap_{\alpha \in A} \mathbb{V}(I_\alpha) = \mathbb{V}(\sum_{\alpha \in A} I_a)$.*

Proof. Property 1. follows immediately from definitions. We now prove Property 2. Let $I, J \subseteq k[x_1, \ldots, x_n]$ be ideals. Since $I \cap J \subseteq I$ and $I \cap J \subseteq J$, it follows from Lemma 2.1.4 that $\mathbb{V}(I) \cup \mathbb{V}(J) \subseteq \mathbb{V}(I \cap J)$. Furthermore, by Lemma 2.1.4 and Lemma A.2.24, we have that $\mathbb{V}(I \cap J) \subseteq \mathbb{V}(IJ)$. Thus, it remains to show that $\mathbb{V}(IJ) \subseteq \mathbb{V}(I) \cup \mathbb{V}(J)$. We consider the contrapositive: Let $a \in \mathbb{A}^n_M$ such that $a \notin \mathbb{V}(I) \cup \mathbb{V}(J)$. Then, there exists $f \in I$ and $g \in J$ such that $f(a) \neq 0$ and $g(a) \neq 0$. Consequently, $(f \cdot g)(a) \neq 0$, implying $a \notin \mathbb{V}(IJ)$.

We conclude the proof by establishing Property 3. We observe that $a \in \bigcap_{\alpha \in A} \mathbb{V}(I_\alpha)$ if and only if, for all $\alpha \in A$, $f(a) = 0$ for all $f \in I_\alpha$. We claim that $a \in \mathbb{V}(\sum_{\alpha \in A} I_\alpha)$ if and only if $f(\alpha) = 0$ for all $f \in I_\alpha$ and $a \in A$. To demonstrate this, assume $a \in \mathbb{V}(\sum_{\alpha \in A} I_\alpha)$. Then $f(a) = 0$ for any $f \in \sum_{\alpha \in A} I_\alpha$. Since $I_\beta \in \sum_{\alpha \in A} I_\alpha$ for all $\beta \in A$, it follows that $f(a) = 0$ for all $f \in I_\alpha$ and $\alpha \in A$. Conversely, if $f(a) = 0$ for all $f \in I_\alpha$ and $\alpha \in A$, we observe that any element of $\sum_{\alpha \in A} I_\alpha$ can be expressed as a finite sum of terms of the form $f_{\alpha_1} + \ldots + f_{\alpha_m}$, for some $f_{\alpha_i} \in I_{\alpha_i}$ and $\alpha_i \in A$, where $i \in \{1, \cdots, m\}$ and $m \in \mathbb{N}$. Thus, by assumption, $(f_{\alpha_1} + \ldots + f_{\alpha_m})(a) = 0$ for all $f_{\alpha_i} \in I_{\alpha_i}$ and $\alpha_i \in A$, where $i \in \{1, \cdots, m\}$ and $m \in \mathbb{N}$, leading to $a \in \mathbb{V}(\sum_{\alpha \in A} I_\alpha)$. □

In particular, Proposition 2.2.1 demonstrates that affine n-space has a natural topology given by closed sets which are algebraic sets.

Definition 2.2.2 ([8, p. 515]). *We refer to the topology, defined by the complements of algebraic sets, on $\mathbb{A}^n_M$ as the* classical Zariski topology with respect to k and M. *We will often drop the k and M and refer to this topology simply as the* classical Zariski topology.

Using Definition 2.1.3, we may treat any subset of $\mathbb{A}^n_k$ as a topological space by endowing it with the subspace topology. This coincides with the closed sets defined by Definition 2.1.3. An interesting difference in our setting to the typical definition of the classical Zariski topology is that singletons in $\mathbb{A}^n_M$ may not necessarily be closed.

We will now consider a natural basis for the classical Zariski topology.

Definition 2.2.3 ([8, p. 517]). *Let X be an algebraic set and $f \in k[x_1, \ldots, x_n]$. We call*

$$\mathbb{D}(f) := X \backslash \mathbb{V}(f)$$

the distinguished open subset of f in X.

Lemma 2.2.4 ([18, Remark 3.7, p. 24]). *$\{\mathbb{D}(f)\}_{f \in k[x_1, \ldots, x_n]}$ forms a basis for the classical Zariski topology on affine n-space.*

Proof. Let $U \subseteq \mathbb{A}^n_k$ be an open set so that there exists an ideal $I \subseteq k[x_1, \ldots, x_n]$ such that $U = \mathbb{A}^n_k \backslash \mathbb{V}(I)$. Using Property 3 of Proposition 2.2.1, we have:

$$\begin{aligned} U &= \mathbb{A}^n_k \backslash \mathbb{V}(I) \\ &= \mathbb{A}^n_k \backslash \mathbb{V}\Big(\sum_{f \in I} \langle f \rangle\Big) \\ &= \mathbb{A}^n_k \backslash \bigcap_{f \in I} \mathbb{V}(f) \\ &= \bigcup_{f \in I} (\mathbb{A}^n_k \backslash \mathbb{V}(f)) \\ &= \bigcup_{f \in I} \mathbb{D}(f). \end{aligned}$$

This shows that U can be expressed as a union of distinguished open sets, demonstrating that $\{\mathbb{D}(f)\}_{f \in k[x_1,\dots,x_n]}$ forms a basis for the topology on affine n-space. $\square$

It turns out that this topology is also compact (see Corollary 2.3.6), however, to show this, we will need an additional fact discussed in the next section.

2.3 Hilbert's Nullstellensatz

We now possess the necessary tools which will allow us to prove one of the main results of the chapter and a fundamental result within algebraic geometry, namely Hilbert's Nullstellensatz. This result illuminates the exact relationship between algebraic sets and their associated ideals or, more generally, the geometric aspects of algebraic geometry to algebra. Particularly, it will also allow us to show that certain restrictions of $\mathcal{V}$ and $\mathcal{I}$ will be inverse.

The next 3 results give a proof of the Nullstellensatz following that of Fulton in [15, p. 10] where he makes use of a proof due to Rabinowitsch.

Proposition 2.3.1. *Suppose that k is an algebraically closed field, and let $\mathfrak{m} \subseteq k[x_1, \dots, x_n]$ be a maximal ideal. Then $\mathfrak{m} = \langle x_1 - a_1, \dots, x_n - a_i \rangle$ for some $a_1, \dots, a_n \in k$ and $\mathbb{V}(\mathfrak{m}) = \{(a_1, \dots, a_n)\}$.*

Proof. Let $L = k[x_1, \dots, x_n]/\mathfrak{m}$. Since L is finitely generated as a k-algebra, it follows [15, Proposition 4, p. 15] that L is an algebraic extension of k. Since k is algebraically closed, we deduce there exists an isomorphism $\psi : L \to k$. Denote the projection from $k[x_1, \dots, x_n]$ to L as π. We note that $\psi \circ \pi$ is surjective and has kernel $\mathfrak{m}$. Set $a_i = (\psi \circ \pi)(x_i)$ for all $i \in [[1, n]]$. Since π, ψ, and hence $\psi \circ \pi$ are k-algebra homomorphisms, we have that $a_i = (\psi \circ \pi)(a_i)$ for all $i \in [[1, n]]$. Thus, $(\psi \circ \pi)(a_i) = (\psi \circ \pi)(x_i)$ for all $i \in [[1, n]]$. As ψ is an isomorphism, taking ψ^{-1} on both sides gives that $\pi(a_i) = \pi(x_i)$ and $\pi(x_i - a_i) = 0$ for all $i \in [[1, n]]$. We deduce that $x_i - a_i \in \mathfrak{m}$ for all $i \in [[1, n]]$. So, $\langle x_1 - a_1, \dots x_n - a_n \rangle$ is an ideal contained in $\mathfrak{m}$. It is also maximal, by the first isomorphism theorem, since $k[x_1, ..., x_n]/\langle x_1 - a_1, \dots, x_n - a_n \rangle = k$. We conclude that $\mathfrak{m} = \langle x_1 - a_1, \dots, x_n - a_n \rangle$. $\square$

Proposition 2.3.2 (Weak Nullstellensatz)**.** *If $I \subseteq k[x_1, \dots, x_n]$ is a proper ideal, then $\mathbb{V}(I) \neq \varnothing$.*

Proof. Denote I^{ext} as the ideal generated by I in $M[x_1, \dots, x_n]$. That is, $I^{\text{ext}} = \langle I \rangle_M$. By Lemma A.2.23, I^{ext} is contained in a maximal ideal $\mathfrak{m}$ of $M[x_1, \dots, x_n]$, since I^{ext} is proper as I is proper. It follows from Proposition 2.3.1 that $\mathfrak{m} = \langle x_1 - a_1, \dots, x_n - a_n \rangle_M$ for some $a_1, \dots a_n \in M$.

As $I^{\text{ext}} \subseteq \mathfrak{m}$, it follows that $I \subseteq \mathfrak{m}$. Since $f(a_1, \dots, a_n) = 0$ for all $f \in \mathfrak{m}$, we deduce that $g(a_1, \dots, a_n) = 0$ for all $g \in I$. Hence, $\{(a_1, \dots, a_n)\} \subseteq \mathbb{V}(I)$, ensuring that $\mathbb{V}(I)$ is non-empty. $\square$

Proposition 2.3.3 (Hilbert's Nullstellensatz [15, p. 10])**.** *For any ideal* $I \subseteq k[x_1, \dots, x_n]$*, we have* $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$.

Proof. It is easy to see that $\sqrt{I} \subseteq \mathbb{I}(\mathbb{V}(I))$: Let $f \in \sqrt{I}$. By the definition of $\sqrt{I}$, there exists an $m \in \mathbb{N}$ such that $f^m \in I$. Then, f^m vanishes on all of $\mathbb{V}(I)$. This implies that f also vanishes on $\mathbb{V}(I)$, so $f \in \mathbb{I}(\mathbb{V}(I))$.

We now show that $\mathbb{I}(\mathbb{V}(I)) \subseteq \sqrt{I}$. As $k[x_1, \dots, x_n]$ is Noetherian, I is finitely generated as an ideal with generators $f_1, \dots, f_m \in k[x_1, \dots, x_n]$, for some $m \in \mathbb{N}$. Let $g \in \mathbb{I}(\mathbb{V}(f_1, \dots, f_m))$, and let

$$J = (f_1, \dots, f_m, x_{n+1} \cdot g - 1) \subseteq k[x_1, \dots, x_n, x_{n+1}].$$

In particular, $\mathbb{I}(\mathbb{V}(I)) = \mathbb{I}(\mathbb{V}(f_1, \dots, f_m))$ is equal to

$$\{f \in k[x_1, \dots, x_n] \mid f(a) = 0 \ \forall a \in \{b \in \mathbb{A}_M^n \mid f_i(b) = 0 \ \forall i \in [[1, m]]\}\}.$$

So that g vanishes whenever the $f_1, \dots, f_m$ vanish. It follows that $\mathbb{V}(J) = \varnothing$ since $(x_{n+1} \cdot g - 1)$ has value -1 whenever the $f_1, \dots, f_m$ vanish. Therefore, by the contrapositive of Proposition 2.3.2, $1 \in J$, so there exists a relation $1 = \sum_{i=1}^{m} (p_i \cdot f_i) + q \cdot (x_{n+1} \cdot g - 1)$ for some $p_1, \dots, p_m, q \in k[x_1, \dots, x_{n+1}]$. Taking the image of this last equation via the k-algebra homomorphism from $k[x_1, \dots, x_n, x_{n+1}]$ to $k(x_1, \dots, x_n)$ given by sending x_i to x_i for all $i \in \{1, \cdots, n\}$ and x_{n+1} to $\frac{1}{g}$, we obtain

$$1 = \sum_{i=1}^{m} p_i\left(x_1, \dots, x_n, \frac{1}{g}\right) \cdot f_i. \tag{2.1}$$

Multiplying through by g^r where r is the highest power of $\frac{1}{g}$ appearing in (2.1), the equation becomes

$$g^r = \sum_{i=1}^{m} p_i'(x_1, \dots, x_n) f_i$$

for some $p_1', \dots, p_m' \in k[x_1, \dots, x_n]$. It follows that $g \in \sqrt{I}$ as there exists $r \in \mathbb{N}$ such that $g^r \in I$. Since g was arbitrary, we conclude that $\mathbb{I}(\mathbb{V}(f_1, \dots, f_m)) \subseteq \sqrt{I}$. □

The following results are immediate consequences of the Nullstellensatz and Lemma 2.1.10.

Corollary 2.3.4 ([15, Corollary 1, p. 11])**.** *For any ideal* $I \subseteq k[x_1, \dots, x_n]$*, we have* $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$.

Corollary 2.3.5 ([15, Corollary 1, p. 11])**.** $\mathcal{V}_{k,M,n}|_{\mathbf{RI}_{k,n}} : \mathbf{RI}_{k,n}^{\mathrm{op}} \to \mathbf{AS}_{k,M,n}$ *and* $\mathcal{I}_{k,n}|^{\mathbf{RI}_{k,n}^{\mathrm{op}}} : \mathbf{AS}_{k,M,n} \to \mathbf{RI}_{k,n}^{\mathrm{op}}$ *are inverse to one another.*

As mentioned in the previous section, we are now in a position to show that the classical Zariski topology is compact.

Corollary 2.3.6 ([18, Lemma 2.12, p. 15])**.** *Algebraic sets form Noetherian topological spaces.*

Proof. Let X be an algebraic n-set over k for some $n \in \mathbb{N}$. If there existed an infinite decreasing chain:

$$X_1 \supset X_2 \supset \dots$$

of algebraic subsets of X, then, by Corollary 2.3.5, we would have the chain:

$$\mathbb{I}(X_1) \subset \mathbb{I}(X_2) \subset \dots .$$

of ideals in $k[x_1, \dots, x_n]$. Moreover, $X_1 \supsetneq X_2$ then $\mathbb{I}(X_1) \subsetneq \mathbb{I}(X_2)$, by Lemma 2.1.10. But then, this contradicts that $k[x_1, \dots, x_n]$ is a Noetherian ring. Thus, X is a Noetherian topological space. □

The following can be thought of as a form of the Nullstellensatz for algebraically closed fields stated in [15, Corollary 2, p. 11].

Corollary 2.3.7. *Let $I \subseteq k[x_1, \dots, x_n]$ be a radical ideal and denote the set of maximal ideals of $M[x_1, \dots, x_n]/I^{\text{ext}}$ as* $\operatorname{maxSpec}(M[x_1, \dots, x_n]/I^{\text{ext}})$*. The map $\phi : \mathbb{V}_{k,M}(I) \to \operatorname{maxSpec}(M[x_1, \dots, x_n]/I^{\text{ext}})$, which sends $a \in \mathbb{V}_{k,M}(I)$ to $[\mathbb{I}_{M,M}(a)]$ where $[\mathbb{I}_{M,M}(a)] = \{[b] \mid b \in \mathbb{I}_{M,M}(a)\}$ with $[b]$ the image of $b \in M[x_1, \dots, x_n]$ in the quotient $M[x_1, \dots, x_n]/I^{\text{ext}}$, is a bijection.*

Proof. We have, for all $a = (a_1, \dots, a_n) \in \mathbb{V}(I)$, that $\mathbb{I}_{M,M}(a) = \langle x_1 - a_1, \dots, x_n - a_1 \rangle$ is a maximal ideal. Therefore, so is $[\mathbb{I}_{M,M}(a)]$ proving that ϕ is well-defined. Applying Corollary 2.3.5 with $k = M$, it follows that ϕ is injective.

We need only show that ϕ is surjective. To that end, let $\mathfrak{m}$ be a maximal ideal in $M[x_1, \dots, x_n]/I^{\text{ext}}$. By Proposition 2.3.1, $\mathfrak{m} = [\mathbb{I}_{M,M}(b)]$ for some $b = (b_1, \dots, b_n) \in M^n$. Then $f(b) = 0$ for all $f \in \mathfrak{m}$ and $f(b) = 0$ for all $f \in I$. Indeed, By Proposition 2.3.3, $I = \mathbb{I}_{k,M}(\mathbb{V}(I)) \subseteq \mathbb{I}_{M,M}(b)$. So that $b \in \mathbb{V}(I)$ and f is surjective. □

As M is a k-algebra, as in [28, Example 1.15, p. 5], we have an isomorphism between $M[x_1, \dots, x_n]$ and $k[x_1, \dots, x_n] \otimes_k M$ which maps $a \in M$ to $1 \otimes_k a$ and $x_i \in k[x_1, \dots, x_n]$ to $x_i \otimes_k 1$ for all $i \in [[1, n]]$ and another isomorphism between $k[x_1, \dots, x_n]/I \otimes_k M$ and $M[x_1, \dots, x_n]/I^{\text{ext}}$ which follows from [28, Corollary 1.13, p. 5]. As a consequence of the previous result, we have the following corollary. We explicitly state these as the bijection given in this corollary will be extensively used in the subsequent sections of this thesis.

Corollary 2.3.8. *Let* $I \subset k[x_1, \ldots, x_n]$ *be a radical ideal and let* I^{ext} *denote the ideal generated by the image of* I *in* $k[x_1, \ldots, x_n] \otimes_k M$*. The map* $\Psi_{I,M} : \mathbb{V}(I) \to \operatorname{maxSpec}(k[x_1, \ldots, x_n]/I \otimes_k M)$ *which sends* $(a_1, \ldots, a_n) \in \mathbb{V}(I)$ *to* $\langle [x_1] \otimes_k 1 - [1] \otimes_k a_1, \ldots, [x_n] \otimes_k 1 - [1] \otimes_k a_n \rangle$ *is a bijection.*

2.4 Ringed space structure on algebraic sets

The coordinate ring of an algebraic set is another fundamental concept within algebraic geometry as it algebraically encodes geometric properties of an algebraic set once again illustrating ties between geometry and algebra. In our journey of algebraic geometry, this ring will play a crucial role in defining sheaves that we associate to algebraic sets giving them a ringed spaced structure. This sheaf of rings captures local and topological aspects of algebraic sets allowing us to study their properties and functions between them. We begin by introducing the coordinate ring and exploring its applications.

Definition 2.4.1 ([18, Definition 1.15, p. 10])**.** *Let* X *be an algebraic set. We call the ring*

$$k[X] := k[x_1, \ldots, x_n]/\mathbb{I}(X)$$

the coordinate ring of X.

This construction gives rise to a natural functor that establishes a link from the category of algebraic sets to the category of k-algebras.

Definition 2.4.2. *We introduce the* coordinate ring functor over k, *denoted as* $\mathcal{C}_k$*. This functor operates from the category of algebraic sets to the category of* k*-algebras and maps algebraic sets to their associated coordinate rings. Additionally, it maps an inclusion of algebraic sets to the natural projection induced by the reverse inclusion of the corresponding ideals.*

We now introduce a relative version of Definition 2.1.2 for ideals in $k[X]$ which will be useful in simplifying the arguments and statements leading up to the definition of the sheaf we will associate to algebraic sets.

Definition 2.4.3 ([18, Construction 1.17, p. 10])**.** *Let* X *be an algebraic set, and let* $I \subseteq k[X]$ *be an ideal. Let* $\pi_X : k[x_1, \ldots, x_n] \to k[X]$ *be the usual projection. We define*

$$\mathbb{V}(I) := \mathbb{V}_X(\pi_X^{-1}(I)).$$

In particular, for any $f \in k[X]$*, we define* $\mathbb{V}(f) := \mathbb{V}(\langle f \rangle)$*.*

Further illustrating the correspondence between ideals and geometry of algebraic sets discussed in the first sections, the following proposition gives a additional characterization of $\mathcal{C}_k$.

Proposition 2.4.4 ([18, Proposition 2.8, p. 14])**.** *Let X be an algebraic set. Then X is irreducible if and only if $k[X]$ is an integral domain.*

Proof. First, we will show the forward inclusion by means of contradiction. Assume that $k[X]$ is not an integral domain so that there exists non-zero $f, g \in k[X]$ such that $f \cdot g = 0$. It follows that $\mathbb{V}(f)$, $\mathbb{V}(g)$ are closed and are not equal to X and $\mathbb{V}(f) \cup \mathbb{V}(g) = \mathbb{V}(f \cdot g) = \mathbb{V}(0) = X$. Hence X is reducible.

We conclude the proof by showing the converse also by means of contradiction. Assume that X is reducible so that $X_1 \cup X_2 = X$ for non-empty closed $X_1, X_2 \subset X$. By Proposition 2.3.5, we have $\mathbb{I}(X_1), \mathbb{I}(X_2) \neq \{0\}$, so that there exists non-zero $f \in \mathbb{I}(X_1) \subset \mathbb{I}(X)$ and $g \in \mathbb{I}(X_2) \subset \mathbb{I}(X)$. It follows that $f(a)g(a) = 0$ for all $a \in X = X_1 \cup X_2$ so that $f \cdot g = 0 \in \mathbb{I}(X)$. Since $f, g \notin \mathbb{I}(X)$, it follows that $\mathbb{I}(X)$ is not prime. Hence, by definition, $k[X]$ is not an integral domain. □

We can now define the sheaf of so-called "regular functions" on any algebraic set. We first give a definition to ease the notation.

Definition 2.4.5 ([18, Definition 3.1, p. 22])**.** *Let X be an algebraic set over k and let $f \in k[X]$. For any $a \in X$, we define $f(a) := p(a)$ where $p \in \pi_X^{-1}(\{f\})$ and $\pi_X : k[x_1, \ldots, x_n] \to k[X]$ is the usual projection homomorphism.*

The above definition is well-defined. Indeed, let $f \in k[X]$ with liftings $p_1, p_2 \in \pi_{\mathbb{I(X)}}^{-1}(\{f\})$. Then $p_1 = p_2 + u$ for some $u \in \mathbb{I}(X)$. Since, for any $a \in X$, $u(a) = 0$, it follows that $f_1(a) = f_2(a)$.

Definition 2.4.6 ([18, Definition 3.6, p. 24])**.** *Let X be an algebraic set, and let U be an open subset of X. A* regular function on U to M *is a map $s : U \to M$ with the property that, for every $a \in U$, there exists an open $U_a \subseteq U$ with $a \in U_a$ and there exist polynomials $g_a, h_a \in k[X]$ such that*

$$s(b) = \frac{g_a(b)}{h_a(b)}$$

holds, where $h_a(b) \neq 0$ for all $b \in U_a$. We define the ring of regular functions on U to M *denoted by $\mathcal{R}_X(U)$ to be the ring consisting of regular functions on U to M with binary operations being point-wise addition and multiplication.*

The preceding construction can be viewed as establishing a connection between the category of algebraic sets and the category of k-algebras as follows.

Definition 2.4.7. *Let X be an algebraic set. We define a functor, denoted as $\mathcal{R}_X$, from the category of open sets of X, denoted* $\mathbf{Ops}_X$*, to the category of k-algebras, denoted k-**Alg**. This functor associates each open set $U \subseteq X$ with the ring of all regular functions on U to M, and contravariantly assigns, for any $U, V \subseteq_{\text{op}} X$ such that $V \subseteq U$, any inclusion morphism $\mathfrak{i}_{V,U}$ to the standard restriction map that restricts the domain of functions from U to V.*

The following result is also well-known, confirming that we have a ringed space $(X, \mathcal{R}_X)$ for any algebraic set X.

Lemma 2.4.8. *Let X be an algebraic set. Then $\mathcal{R}_X$ is a sheaf.*

Proof. It is easily seen that $\mathcal{R}_X$ is a presheaf. We will show that it is indeed a sheaf by confirming the glueing and uniqueness conditions. Let $\{U_i\}_{i \in I}$ be an open covering for some open $U \subseteq X$ and let $s_i \in \mathcal{R}_X(U_i)$ for all $i \in I$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$. We define $s \in \mathcal{R}_X(U)$ such that $s(a) := s_i(a)$ if $a \in U_i$ for all $i \in I$. Since s_i, s_j agree on $U_i \cap U_j$ for all $i, j \in I$, this is well defined. By definition of s, we have that, for all $a \in U$, there exists, for any $i \in I$, an open $U_{i,a} \subseteq U_i \subseteq U$ containing a and there exists $g_{i,a}, h_{i,a} \in k[X]$ such that

$$s(b) = s_i(b) = \frac{g_{i,a}(b)}{h_{i,a}(b)}$$

for all $b \in U_{i,a}$. We deduce that $s \in \mathcal{R}_X(U)$. Furthermore, since s is point-wise defined, it is necessarily unique. ☐

Before continuing, we introduce a relative version of Definition 2.2.3 to, again, ease notation and prove a result that will have utility later in the chapter.

Definition 2.4.9 ([18, Remark 3.7, p. 24])**.** *Let X be an algebraic set and let $f \in k[X]$. We define*

$$\mathbb{D}_X(f) := X \backslash \mathbb{V}(f)$$

which we call the distinguished open subset of f in X. Recall that here $\mathbb{V}(f) = \mathbb{V}_X(\pi_X^{-1}(\langle f \rangle))$ with $\pi_X : k[x_1, \ldots, x_n] \to k[X]$ is the projection.

Lemma 2.4.10. *If X be an algebraic set, then $\{\mathbb{D}(f)\}_{f \in k[X]}$ forms a basis for the classical Zariski topology on X.*

Proof. Let $I = \mathbb{I}(X)$, let $\pi : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/I$ be the usual projection and let $U \subseteq X$ be an open set. Then, there exists an ideal $J \subseteq k[x_1, \ldots, x_n]$ such that $I \subseteq J$ and $U = X \backslash \mathbb{V}(J)$. Let $\pi : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/J$ be the usual projection. Using Property 3 of Proposition 2.2.1, we have:

$$\begin{aligned} U &= X \backslash \mathbb{V}(J) \\ &= X \backslash \mathbb{V}\Big(\sum_{f \in J} \langle f \rangle \Big) \\ &= X \backslash \bigcap_{f \in J} \mathbb{V}(f) \\ &= \bigcup_{f \in J} (X \backslash \mathbb{V}(f)) \\ &= \bigcup_{f \in \pi(J)} (X \backslash \mathbb{V}(f)) \\ &= \bigcup_{f \in \pi(J)} \mathbb{D}(f). \end{aligned}$$

By the correspondence given in Corollary A.2.37, $\{X \backslash \mathbb{V}(J)\}_{\substack{J \subseteq k[x_1,\ldots,x_n] \text{ an ideal,} \\ I \subseteq J}}$ is the set of all open sets of X. This shows that any open set of X can be expressed as a union of distinguished open sets in $\{\mathbb{D}(f)\}_{f \in k[x_1,\ldots,x_n]/I}$. This shows that $\{\mathbb{D}(f)\}_{f \in k[x_1,\ldots,x_n]/I}$ forms a basis for the topology on X. ☐

Algebraic sets are, in fact, a more specific type of ringed space. To show this, we now begin expanding on the local properties of regular functions.

Proposition 2.4.11 ([20, Proposition 2.2 b., p. 71]). *Let X be an algebraic set and let $f \in k[X]$. Then, the canonical map $\Psi_{X,f} : k[X]_f \to \mathcal{R}_X(\mathbb{D}(f))$, defined by $\frac{g}{f^m} \mapsto \Psi_{X,f}(\frac{g}{f^m})$, where $\Psi_{X,f}(\frac{g}{f^m})(a) = \frac{g(a)}{f^m(a)}$ for $a \in \mathbb{D}(f)$ and some $m \in \mathbb{N}$, is a well-defined k-algebra isomorphism.*

Proof. We begin by showing that $\Psi_{X,f}$ is well-defined. Let $\frac{g}{f^m}$ and $\frac{h}{f^r}$ in $k[X]_f$ for some $m, r \in \mathbb{N}$ and $h, g \in k[X]$ such that $\frac{h}{f^r} = \frac{g}{f^m}$. Then, for some $s \in \mathbb{N}$, $f^s \cdot (h \cdot f^m) = f^s \cdot (g \cdot f^r)$. We have, for any $a \in \mathbb{D}(f)$,

$$\begin{aligned} & (f^s \cdot (h \cdot f^m))(a) = (f^s \cdot (g \cdot f^r))(a) \\ \Leftrightarrow\ & f^s(a) \cdot h(a) \cdot f^m(a) = f^s(a) \cdot g(a) \cdot f^r(a) \\ \Leftrightarrow\ & h(a) \cdot f^m(a) = g(a) \cdot f^r(a) \\ \Leftrightarrow\ & \frac{h(a)}{f^r(a)} = \frac{g(a)}{f^m(a)} && (M \text{ is a field and } a \in \mathbb{D}(f)) \end{aligned}$$

Thus, $\Psi_{X,f}$ is indeed well-defined. It is clear that $\Psi_{X,f}$ is a homomorphism. We now show that $\Psi_{X,f}$ is injective. Assume that $\Psi_{X,f}(\frac{g}{f^m}) = 0$ for some

$g \in k[X]$ and $m \in \mathbb{N}$. We then have

$$\begin{aligned} &\Psi_{X,f}(\frac{g}{f^m}) = 0 \\ \Leftrightarrow\ & \frac{g(a)}{f^m(a)} = 0\ \forall a \in \mathbb{D}(f) \\ \Leftrightarrow\ & g(a) = 0\ \forall a \in \mathbb{D}(f) \\ \Leftrightarrow\ & (f \cdot g)(a) = 0\ \forall a \in X \\ \Leftrightarrow\ & g \cdot 1 - f^m \cdot 0 = 0 \\ \Leftrightarrow\ & \frac{g}{f^m} = \frac{0}{1} \text{ in } k[X]_f. \end{aligned}$$

We now show that $\Psi_{X,f}$ is surjective. Let $s \in \mathcal{R}_X(\mathbb{D}(f))$. We want to find an element $\frac{r}{f^m} \in k[X]_f$ for some $m \in \mathbb{N}$ such that $\Psi_{X,f}(\frac{r}{f^m}) = s$. By definition, s is a map from $\mathbb{D}(f)$ to M such that, for every $a \in \mathbb{D}(f)$, there exists some open $U_a \subseteq \mathbb{D}(f)$ such that $a \in U_a$ and $g_a, h_a \in k[X]$ such that $h_a(b) \neq 0$ and $s(b) = \frac{g_a(b)}{h_a(b)}$ for all $b \in U_a$. Now, fix some $a \in \mathbb{D}(f)$. We may assume U_a is of the form $\mathbb{D}(p_a)$ for some $p_a \in k[X]$ as distinguished open sets give an open covering of open sets in affine space. Then $s(b) = \frac{g_a(b)}{h_a(b)} \cdot \frac{p_a(b)}{p_a(b)} = \frac{(g_a \cdot p_a)(b)}{(h_a \cdot p_a)(b)}$ for all $b \in \mathbb{D}(p_a)$. Indeed, we have that $h_a \cdot p_a$ vanishes where p_a vanishes and does not vanish where p_a does not vanish. We set $q_a := h_a \cdot p_a$ and $r_a := g_a \cdot p_a$. As a consequence, we have $s(b) = \frac{r_a(b)}{q_a(b)}$, for all $b \in \mathbb{D}(q_a)$. We have

$$\begin{aligned} &\mathbb{D}(f) \subseteq \bigcup_{a \in \mathbb{D}(f)} \mathbb{D}(q_a) \\ \Leftrightarrow\ & \bigcap_{a \in \mathbb{D}(f)} \mathbb{V}(q_a) \subseteq \mathbb{V}(f) \\ \Leftrightarrow\ & \mathbb{V}\Big(\sum_{a \in \mathbb{D}(f)} \langle q_a \rangle\Big) \subseteq \mathbb{V}(f) && \text{(3. of Proposition 2.2.1)} \\ \Rightarrow\ & \langle f \rangle \subseteq \sqrt{\sum_{a \in \mathbb{D}(f)} \langle q_a \rangle} && \text{(Proposition 2.3.3).} \end{aligned}$$

Hence, $f^m = \sum_{a \in I} \lambda_a q_a$ for some $m \in \mathbb{N}$ and some finite set $I \subseteq \mathbb{D}(f)$. It follows that $\mathbb{D}(f) \subseteq \bigcup_{a \in I} \mathbb{D}(q_a)$.

On $\mathbb{D}(q_a) \cap \mathbb{D}(q_b) = \mathbb{D}(q_a \cdot q_b)$ for any $a, b \in I$, we have two representatives $\frac{r_a}{q_a}$ and $\frac{r_b}{q_b}$. By the injectivity of $\Psi_{X,q_a \cdot q_b}$, it follows that $\frac{r_a(c)}{q_a(c)} = \frac{r_b(c)}{q_b(c)}$ for all $c \in \mathbb{D}(q_a \cdot q_b)$. Hence, $(r_a \cdot q_b)(c) = (r_b \cdot q_a)(c)$ for all $c \in \mathbb{D}(f)$. Now, we set $r := \sum_{a \in I} \lambda_a r_a$. For all $b \in I$ and all $c \in \mathbb{D}(f)$, we have

$$(q_b \cdot r)(c) = (\sum_{a \in I} \lambda_a r_a q_b)(c) = (\sum_{a \in I} \lambda_a q_a r_b)(c) = (f^m \cdot r_b)(c).$$

Hence, $\frac{r(c)}{f^m(c)} = \frac{r_b(c)}{q_b(c)}$ for all $c \in \mathbb{D}(q_b)$ and all $b \in I$. We deduce that $\Psi_{X,f}(\frac{r}{f^m}) = s$ so that $\Psi_{X,f}$ is surjective. $\square$

Taking $f = 1$, we obtain the following corollary.

Corollary 2.4.12 ([20, Proposition 2.2 c., p. 71])**.** *Let X be an algebraic set. Then, the canonical map $\Psi_X : k[X] \to \mathcal{R}_X(X)$, defined by $g \mapsto \Psi_X(g)$, where $\Psi_X(g)(a) = g(a)$ for $a \in X$, is a well-defined k-algebra isomorphism.*

From the work done above, it is difficult to deduce that the canonical isomorphism Ψ_X, where X is an algebraic set, induces a natural correspondence between the coordinate ring functor over k and the functor of regular functions over k.

Corollary 2.4.13. *The family of maps $\Psi = (\Psi_X)_{X \in \mathbf{AS}_k}$ establishes a natural correspondence between the coordinate ring functor over k and the functor of regular functions over k. More precisely, for any algebraic sets X and Y such that $Y \subseteq X$, $\Psi_Y \circ \pi_{X,Y} = r_{X,Y} \circ \Psi_Y$, where $\pi_{X,Y} : k[X] \to k[Y]$ is the natural projection induced by the inclusion $Y \subseteq X$, and $r_{X,Y} : \mathcal{R}_X(X) \to \mathcal{R}_Y(Y)$ is the natural restriction induced by the inclusion $Y \subseteq X$ that is $r_{X,Y} = - \circ \iota_{Y,X}$.*

In the following lemma, we establish that the stalk of the sheaf of regular functions forms a local ring.

Lemma 2.4.14 ([20, Proposition 2.2. a., p. 71])**.** *Let X be an algebraic set, $a \in X$, and $\mathfrak{m}_a$ be the image of $\mathbb{I}(a)$ in $k[X]$. The canonical map $\Psi_{X,a} : k[X]_{\mathfrak{m}_a} \to \mathcal{R}_{X,a}$, defined by sending any $\frac{f}{g}^{\mathfrak{m}_a} \in k[X]_{\mathfrak{m}_a}$ to the image of the regular function $s : \mathbb{D}(g) \to M$, which maps $y \in \mathbb{D}(g)$ to $\frac{f(y)}{g(y)} \in M$, in $\mathcal{R}_{X,a}$ via the stalk map, is an isomorphism.*

Proof. We begin by showing that the map $\Psi_{X,a}$ is well-defined. Suppose $\frac{f}{g}, \frac{f'}{g'} \in k[X]_{\mathfrak{m}_a}$ such that $\frac{f}{g}^{\mathfrak{m}_a} = \frac{f'}{g'}^{\mathfrak{m}_a}$ in $k[X]_{\mathfrak{m}_a}$. This implies that there exists an element $h \in k[X] \backslash \mathfrak{m}_a$ for which $h(gf' - g'f) = 0$. Consequently, we have $\frac{f(b)}{g(b)} = \frac{f'(b)}{g'(b)}$ for all $b \in \mathbb{D}(h) \cap \mathbb{D}(g) \cap \mathbb{D}(g')$. Thus, $\Psi_{X,a}$ is indeed well-defined. It is also clear that $\Psi_{X,a}$ is a homomorphism.

Next, we show surjectivity. Let $s_a \in \mathcal{R}_{X,a}$ be a regular function with lifting $s \in \mathcal{R}_X(U)$ via the stalk map for some open set $U \subseteq X$ containing a. By the definition of regular functions, there exist $f_a, g_a \in k[X]$ and an open set $U_a \subseteq U$ with $a \in U_a$ such that $g_a(b) \neq 0$ and $s(b) = \frac{f_a(b)}{g_a(b)}$ for all $b \in U_a$. Using Lemma 2.2.4, we can replace U_a with $\mathbb{D}(h_a)$ for some $h_a \in k[X]$ where $a \in \mathbb{D}(h_a)$ and $\mathbb{D}(h_a) \subseteq U_a$. Since $\mathbb{D}(h_a) \subseteq U_a \subseteq \mathbb{D}(g_a)$, the regular function $t : \mathbb{D}(g_a) \to M$, which maps $b \in \mathbb{D}(g_a)$ to $\frac{f_a(b)}{g_a(b)}$, has the same image via the stalk map as s in $\mathcal{R}_{X,a}$ since they coincide on $\mathbb{D}(h_a) \cap \mathbb{D}(g_a) = \mathbb{D}(h_a)$. We conclude that $\Psi_{X,a}(\frac{f_a}{g_a}^{\mathfrak{m}_a}) = s_a$.

Finally, we prove injectivity. Suppose that $\Psi_{X,a}(\frac{f}{g}^{\mathfrak{m}_a}) = 0$ for some $\frac{f}{g}^{\mathfrak{m}_a} \in k[X]_{\mathfrak{m}_a}$. This implies that $\frac{f}{g}^{\mathfrak{m}_a}$ vanishes in a neighborhood of a. Again, using Lemma 2.2.4, we can replace this neighborhood with a smaller open set $\mathbb{D}(h)$, where $h \in k[X]$, such that $a \in \mathbb{D}(h)$. It follows that $h(g \cdot 1 - 0 \cdot f)$ vanishes on all of X. Since $a \in \mathbb{D}(h)$, we have $h(a) \neq 0$, which implies that $h \notin \mathfrak{m}_a$. We deduce that $\frac{f}{g}^{\mathfrak{m}_a} = \frac{0}{1}^{\mathfrak{m}_a}$. □

Finally, based on the results we have derived thus far, we can conclude that we have induced a structure of a locally ringed space on any algebraic set.

Corollary 2.4.15. *For any algebraic set X, the pair $(X, \mathcal{R}_X)$ forms a locally ringed space.*

2.5 Several categories of classical varieties

To introduce the concept of classical affine varieties, we use a slight alteration of the framework of Wedhorn and Görtz in [19, Definition 1.46, p. 23] where we consider a classical affine variety as a certain ringed space which is isomorphic to an algebraic set. Namely, we introduce "spaces with functions" which are often used in the presentation of manifolds and gives a similar definition to that of schemes. Moreover, we define the notion of a prevariety which generalises that of a classical affine variety, will give some characterizations of morphisms of these constructions and provide some examples.

Definition 2.5.1 ([19, Definition 1.35, p. 19])**.** *We define*

1. *a* space with functions over k in M *as a pair $(X, \mathcal{R}_X)$ consisting of a topological space X with a sheaf with respect to X such that $\mathcal{R}_X(U) \subseteq$* $\mathrm{Hom}_{\mathbf{Sets}}(U, M)$ *for all open $U \subseteq X$,*

2. *a* morphism of spaces with functions over k in M from $(X, \mathcal{R}_X)$ to $(Y, \mathcal{R}_Y)$ *as continuous maps $f : X \to Y$ such that, for all open $V \subseteq Y$ and all $s \in \mathcal{R}_Y(V)$, the map $f^{\#}_V : \mathcal{R}_Y(V) \to f_*\mathcal{R}_X(V)$ which sends s to $s \circ f|_{f^{-1}(V)} : f^{-1}(V) \to M$ is well-defined and is a k-algebra homomorphism and*

3. *a morphism of spaces with functions over k in M as an* isomorphism *if it is bijective and its inverse is also a morphism of spaces with functions over k in M.*

Definition 2.5.2. *The category of spaces with functions over k in M, denoted by $\mathbf{SF}_M$, is the category whose:*

- *objects are spaces with functions over M and*

- *morphisms are morphisms of spaces with functions.*

As (X, R_X) is a space with functions over M where X is an algebraic set, we can give the following definition.

Definition 2.5.3 ([19, Definition 1.46, p. 23])**.** *We define*

1. *a* classical affine variety over k in M *as any space with functions over k in M isomorphic to $(X, \mathcal{R}_X)$, where X is an algebraic set over k in M, and*
2. *we will denote the full subcategory from* $\mathbf{SF}_M$ *to such sets as* $\mathbf{CAVar}_{k,M}$.

Moreover, we will often omit k and M and refer to such objects simply as classical affine varieties.

Readers familiar with manifolds and analytic spaces should notice similarities with our construction of algebraic sets. Indeed, regular functions behave similarly to how one might understand holomorphic maps as seen by the following two Lemmas. This was also a key contrast in Weil's original definition of algebraic varieties.

Lemma 2.5.4 ([18, Lemma 3.4, p. 23])**.** *Let U be an open subset of a classical affine variety X and let $\phi \in \mathcal{R}_X(U)$ for some open $U \subseteq X$. Then the set $\{x \in U \mid \phi(x) = 0\}$ is closed in U.*

Proof. By definition of classical affine varieties, there exists an algebraic set V over k in M such that there exists an isomorphism $f : X \to V$. We want to show that $\{x \in U \mid \phi(x) \neq 0\}$ is open. Since f is an isomorphism, there exists $s \in \mathcal{R}_V(f(U))$ such that $(s \circ f|_U)(x) = \phi(x)$ for all $x \in U$. It follows that

$$\begin{aligned}\{x \in U \mid \phi(x) \neq 0\} &= \{x \in U \mid (s \circ f|_U)(x) \neq 0\} \\ &= f^{-1}(\{y \in f(U) \mid s(y) \neq 0\}) \quad (f \text{ is an isomorphism}).\end{aligned}$$

We need only show that $\{y \in f(U) \mid s(y) \neq 0\}$ is open, since f is continuous. We have, by definition, that, for any $y \in f(U)$, there is an open neighbourhood $W_y \subseteq f(U)$ where $s = \frac{f_y}{g_y}$ for some $f_y, g_y \in k[V]$ such that g_y is non-zero on W_y. Thus

$$\{x \in W_y \mid s(x) \neq 0\} = W_y \backslash \{x \in W_y \mid s(x) = 0\} = W_y \backslash \mathbb{V}(f_y)$$

is open in V. It follows that

$$\begin{aligned}\{y \in f(U) \mid \phi(y) \neq 0\} &= f(U) \backslash \{y \in f(U) \mid \phi(y) = 0\} \\ &= \bigcup_{y \in f(U)} W_y \backslash \{x \in W_y \mid s(x) = 0\}\end{aligned}$$

is open in $f(U)$. So that $\{y \in f(U) \mid \phi(y) \neq 0\}$ is open in $f(U)$. □

Lemma 2.5.5 ([18, Remark 3.5, p. 23])**.** *Let X be an irreducible classical affine variety and let U, V be non-empty open subsets of X such that $U \subseteq V$. If $\phi_1 : V \to M$ and $\phi_2 : V \to M$ are regular functions such that $\phi_1(x) = \phi_2(x)$ for all $x \in U$, then $\phi_1 = \phi_2$.*

Proof. Set $\phi = \phi_1 - \phi_2$. We have that ϕ is a regular function and

$$S := \{x \in V \mid \phi_1(x) = \phi_2(x)\} = \{x \in V \mid \phi_1(x) - \phi_2(x) = 0\}$$

is closed by Lemma 2.5.4. Since $U \subseteq S$, it follows that $\overline{U} \subseteq S \subseteq V \subseteq X$. However, V is irreducible, by Lemma A.1.12, so that U is dense in V. We conclude that $U = S = V$. ☐

We now introduce the notion of a prevariety which is a more general construction than that of a classical affine variety which will allow us to extend the scope of algebraic geometry.

Definition 2.5.6 ([19, Definition 1.46, p. 23])**.** *We define*

1. *a* prevariety over k in M *as a space with functions over M, $(X, \mathcal{R}_X)$, such that there is a finite open covering $\{X_i\}_{i \in I}$ of X with $(X_i, \mathcal{R}_X|_{X_i})$ a classical affine variety over k in M for all $i \in I$ and*

2. *we will denote the full subcategory of spaces with functions over M to prevarieties over k in M as* $\mathbf{Prevar}_{k,M}$.

As in the case with classical affine varieties, we will often omit the k and M and refer to such objects simply as prevarieties.

The following is well-known and an immediate consequence of results in previous sections.

Lemma 2.5.7. *Any prevariety is a Noetherian topological space.*

Proof. Since, by Corollary 2.3.6, any algebraic set is Noetherian, by definition of classical affine varities, any classical affine variety is Noetherian and since any prevariety has a finite open covering by classical affine varieties, the result follows from Proposition A.1.10. ☐

We now give a fundamental example of affine varieties, namely, the distinguished open sets of an algebraic set.

Lemma 2.5.8 ([19, Lemma 1.50, p. 24])**.** *Let X be an algebraic set and let $f \in k[X]$. We have that $(\mathbb{D}(f), \mathcal{R}_X|_{\mathbb{D}(f)})$ is an affine variety.*

Proof. Let $I \subseteq k[x_1, \ldots, x_n]$ be a radical ideal such that $X = \mathbb{V}(I)$. Let Y be the algebraic set given by $\mathbb{V}(J)$ where $J \subseteq k[x_1, \ldots, x_n, x_{n+1}]$ is the ideal generated by $I \cup \{x_{n+1}f(x_1, \ldots, x_n) - 1\}$. Let $\pi : \mathbb{A}^{n+1}_M \to \mathbb{A}^n_M$ denote the projection. This map is continuous and, as $Y = X \times \mathbb{A}_M$ as sets, it induces the map $\pi|_Y : X \times \mathbb{A}^1_M \to \mathbb{D}(f)$ given by sending $(x_1, \ldots, x_n, \frac{1}{f(x_1,\ldots,x_n)})$ to $(x_1, \ldots, x_n)$. We want to show that $\pi|_Y$ is an isomorphism of spaces with functions. Clearly, $\pi|_Y$ is a bijection and is continuous as it is a restriction of a continuous map. To show that $\pi|_Y$ is a homeomorphism, we show that it is also an open mapping. Let $\phi : k[x_1, \ldots, x_n]/I \to k[x_1, \ldots, x_n, x_{n+1}]/J$ be the obvious inclusion. We observe that, as $[x_{n+1}]$ has inverse $\phi(f)$ in $k[Y]$, any element of $k[Y]$ is of the form $\frac{\phi(g)}{\phi(f)^m}$ for some $m \in \mathbb{N}$ and $g \in k[X]$. By Lemma 2.4.10, $\{\mathbb{D}(\frac{\phi(g)}{\phi(f)^m})\}_{\substack{g\in k[X],\\ m\in\mathbb{N}}}$ is a base for the topology on Y. Now, for some fixed $g \in k[X]$, we have

$$\pi|_Y\left(\mathbb{D}\left(\frac{\phi(g)}{\phi(f)^m}\right)\right) = \pi|_Y(\mathbb{D}(\phi(g)\phi(f))) = \pi|_Y(\mathbb{D}(\phi(gf))) = \mathbb{D}(gf).$$

As $\pi|_Y$ is continuous, we deduce that $\pi|_Y$ is an open mapping. We now need to show that $\pi|_Y$ and $\pi|_Y^{-1}$ are morphisms of spaces with functions. Let $s \in \mathcal{R}_X(\mathbb{D}(fg))$. By Proposition 2.4.11, any $s \in \mathcal{R}_X(\mathbb{D}(fg))$ sends $a \in X$ to $\frac{h(a)}{(fg)^m(a)}$ for some $m \in \mathbb{N}$ and some $h \in k[x_1, \ldots, x_n]/I$. Let $a' \in Y$ and let $\pi|_Y(a') = a$. We then have

$$\begin{aligned} s(\pi|_Y(a')) &= \frac{h(\pi|_Y(a'))}{(fg)^m(\pi|_Y(a'))} \\ &= \frac{h(a)}{(fg)^m(a)} \\ &= \frac{\frac{h(a)}{f^m(a)}}{g^m(a)} \\ &= \frac{\frac{\phi(h)(a)}{\phi(f)^m(a)}}{\phi(g)^m(a)} \end{aligned}$$

which clearly is a regular function in $\mathcal{R}_Y(\mathbb{D}(\phi(g)))$. It follows from Proposition A.5.11 that $\pi|_Y$ is a morphism. Since $\pi|_Y$ is a homeomorphism, to check that $\pi|_Y^{-1}$ is a morphism is analogous to reversing the above arguments. Hence, $\pi|_Y$ is an isomorphism. ☐

Combining Lemma 2.5.8, Lemma 2.4.10 and Corollary 2.3.6, we obtain the following result that gives us an example of a prevariety.

Corollary 2.5.9 ([19, Proposition 1.51, p. 24])**.** *Let X be an algebraic set and let $U \subseteq_{\text{op}} X$. Then $(U, \mathcal{R}_X|_U)$ is a prevariety.*

We now give a criterion to verify that a map between a prevariety and an a classical affine variety is a morphism of prevarities as outlined below.

Proposition 2.5.10 ([20, Lemma 3.6, p. 19])**.** *Let* $(X, \mathcal{R}_X)$ *be a prevariety and let* $(Y, \mathcal{R}_Y)$ *be a classical affine variety. A map* $f : X \to Y$ *is a morphism of prevarieties if and only if, for any* $g : Y \to V$ *an isomorphism of previarieties with* $V \subseteq \mathbb{A}_k^n$ *an algebraic set,* $\Psi_{\mathbb{A}_M^n}(x_i) \circ \iota_{V,\mathbb{A}_M^n} \circ g \circ f$ *is a regular function on* X *where* $\iota_{V,\mathbb{A}_M^n}$ *denotes the inclusion morphism from* V *into* $\mathbb{A}_M^n$*. We note that* $\Psi_{\mathbb{A}_M^n}(x_i) \circ \iota_{V,\mathbb{A}_M^n} = \Psi_V([x_i])$ *where* $[x_i]$ *is the class of* x_i *in* $k[V]$*.*

Proof. We first show the forward direction. Assume that $f : X \to Y$ is a morphism of prevarieties. It is immediate that $\iota_{V,\mathbb{A}_M^n} \circ g \circ f$ is a morphism of prevarieties as it is a composition of morphisms of prevarieties. By the definition of morphisms of prevarieties, it becomes evident that $\phi(x_i) \circ (\iota_{V,\mathbb{A}_M^n} \circ g \circ f)$ is a regular function on X for all $i \in [[1, n]]$, given that $\phi(x_i)$ is already a regular function on $\mathbb{A}_M^n$.

We now show the converse. Assume that $\phi(x_i) \circ (\iota_{V,\mathbb{A}_M^n} \circ g \circ f)$ is regular for all $i \in [[1, n]]$. First, we aim to show that $\iota_{V,\mathbb{A}_M^n} \circ g \circ f$ is continuous. To simplify the notation, we set $h := \iota_{V,\mathbb{A}_M^n} \circ g \circ f$. Let $Z \subseteq \mathbb{A}_k^n$ be a closed set such that there exists a radical ideal $I \subseteq k[x_1, \ldots, x_n]$ with $Z = \mathbb{V}(I)$. By the Hilbert basis theorem, there exist polynomials $p_1, \ldots, p_m \in k[x_1, \ldots, x_n]$ for some $m \in \mathbb{N}$ such that $I = \langle p_1, \ldots, p_m \rangle$, and hence $Z = \mathbb{V}(p_1, \ldots, p_m)$. We thus have

$$\begin{aligned} h^{-1}(Z) &= \{x \in X \mid h(x) \in Z\} \\ &= \{x \in X \mid p_i(h(x)) = 0 \ \forall\ i \in [[1, m]]\} \\ &= \{x \in X \mid p_i(\Psi_{\mathbb{A}_M^n}(x_1) \circ h, \ldots, \Psi_{\mathbb{A}_M^n}(x_n) \circ h)(x) = 0 \ \forall\ i \in [[1, m]]\}. \end{aligned}$$

Since each p_i is a polynomial, the function $p_i(\phi(x_1) \circ h, \ldots, \phi(x_n) \circ h)$ is regular. Consequently, by Lemma 2.5.4, we deduce that $h^{-1}(Z)$ is closed, implying that h is continuous. To show the continuity of f, we observe that, for a closed set $Z \subseteq Y$,

$$\begin{aligned} f^{-1}(Z) &= \{x \in X \mid f(x) \in Z\} \\ &= \{x \in X \mid (g \circ f)(x) \in g(Z)\} && (g \text{ is an isomorphism}) \\ &= \{x \in X \mid h(x) \in (\iota_{V,\mathbb{A}_M^n} \circ g)(Z)\} && (h = \iota_{V,\mathbb{A}_M^n} \circ g \circ f \\ & && \text{and } \iota_{V,\mathbb{A}_M^n} \text{ is injective}) \\ &= h^{-1}((\iota_{V,\mathbb{A}_M^n} \circ g)(Z)). \end{aligned}$$

Since $(\iota_{V,\mathbb{A}_M^n} \circ g)(Z)$ is closed (as $\iota_{V,\mathbb{A}_M^n}$ is a closed immersion and g is an isomorphism) and since h was previously shown to be continuous, it follows that f is continuous.

To conclude the proof, consider $s \in \mathcal{R}_Y(U)$ for some open set $U \subseteq Y$. We will show that $s \circ f|_{f^{-1}(U)} \in \mathcal{R}_X(f^{-1}(U))$. We proceed in several steps:

Since g is an isomorphism, there exists $r \in \mathcal{R}_V(g(U))$ such that $s(x) = (r \circ g|_U)(x)$ for all $x \in U$. Hence

$$(s \circ f|_{f^{-1}(U)})(x) = (r \circ g|_U \circ f|_{f^{-1}(U)})(x)$$

for all $x \in f^{-1}(U)$.

Next, since $\iota_{V,\mathbb{A}^n_M}$ is a morphism, and the topology of V is the subspace topology induced by $\mathbb{A}^n_k$, there exists $r' \in \mathcal{R}_{\mathbb{A}^n_k}(\iota_{V,\mathbb{A}^n_M}(g(U)))$ such that $(r' \circ \iota_{V,\mathbb{A}^n_M}|_{g(U)})(x) = r(x)$ for all $x \in g(U)$. Thus:

$$\begin{aligned}(r \circ g|_U \circ f|_{f^{-1}(U)})(x) &= (r' \circ \iota_{V,\mathbb{A}^n_M}|_{g(U)} \circ g|_U \circ f|_{f^{-1}(U)})(x) \\ &= (r' \circ h|_{f^{-1}(U)})(x)\end{aligned}$$

for all $x \in f^{-1}(U)$.

Since $h^{-1}(\iota_{V,\mathbb{A}^n_M}(g(U))) = f^{-1}(U)$ (as $h = \iota_{V,\mathbb{A}^n_M} \circ g \circ f$, $\iota_{V,\mathbb{A}^n_M}$ is injective and g is an isomorphism) and since h is a morphism of prevarieties, we conclude that $s \circ f|_{f^{-1}(U)} \in \mathcal{R}_X(f^{-1}(U))$ so that that f is a morphism of prevarieties.

☐

In the following result, we establish that it suffices to know the homomorphism between the global sections of the sheaves of regular functions to determine a morphism from any prevariety to a classical affine variety.

Proposition 2.5.11 ([20, Proposition 3.5., p. 19]). *Let $(X, \mathcal{R}_X)$ be a prevariety and let $(Y, \mathcal{R}_Y)$ be a classical affine variety over k. Then there is a natural bijection between $\mathrm{Hom}_{\mathbf{Prevar}_k}(X, Y)$ and $\mathrm{Hom}_{\mathbf{Alg_k}}(\mathcal{R}_Y(Y), \mathcal{R}_X(X))$ given by sending any $f \in \mathrm{Hom}_{\mathbf{Prevar}_k}(X, Y)$ to $f^{\#}_Y$.*

Proof. We first show the forward implication. Assume that $f : X \to Y$ is a morphism. Then, by Definition 2.5.1, f induces a k-algebra homomorphism $f^{\#}_Y : \mathcal{R}_Y(Y) \to \mathcal{R}_X(X)$. This defines our wanted algebra homomorphism.

Conversely, we now show that given a k-algebra homomorphism, we have an associated morphism of prevarieties over k. Let $\phi : \mathcal{R}_Y(Y) \to \mathcal{R}_X(X)$ be a k-algebra homomorphism. Since Y is a classical affine variety, there is an algebraic n-set V for some $n \in \mathbb{N}$ such that there exists an isomorphism $g : Y \to V$. In the case where Y is an algebraic set, we set $g = \mathrm{id}_Y$. We note that g induces an isomorphism $\sigma : k[x_1, \ldots, x_n]/I \to \mathcal{R}_Y(Y)$ for some fixed $n \in \mathbb{N}$ and radical ideal $I \subseteq k[x_1, \ldots, x_n]$. Let $\pi : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/I$ be the usual projection and let $\psi = \sigma \circ \pi$. Define the map $f : X \to \mathbb{V}(I)$ such that $x \in X$ is sent to $(\phi(\psi(x_1))(x), \ldots, \phi(\psi(x_n))(x))$. We want to show that the image of f is contained in $\mathbb{V}(I)$ so that f is well-defined. To do so, we need to any $p \in \mathbb{I}(\mathbb{V}(I))$ vanishes at show that any $p \in \mathbb{I}(\mathbb{V}(I))$ vanishes at $(\phi(\psi(x_1))(x), \ldots, \phi(\psi(x_n))(x))$. Fix some $p \in \mathbb{I}(\mathbb{V}(I))$. For all $x \in X$, we have

$$\phi(\psi(p(x_1, \ldots, x_n)))(x) = p(\phi(\psi(x_1))(x), \ldots, \phi(\psi(x_n))(x))$$

since ϕ and ψ are k-algebra homomorphisms. Now, as $p \in \mathbb{I}(\mathbb{V}(I))$, we have that $\pi(p(x_1, \ldots, x_n)) = 0$ in $k[x_1, \ldots, x_n]/I$, so

$$\phi(\psi(p(x_1, \ldots, x_n)))(x) = 0.$$

This shows that f is well-defined. Let $\iota : \mathbb{V}(I) \to \mathbb{A}^n_k$ denote the inclusion of $\mathbb{V}(I)$ into $\mathbb{A}^n_k$. Since $\psi(x_i) \circ \iota \circ f$ is regular for all $i \in [[1, n]]$, by Proposition 2.5.10, f is a morphism of prevarieties. Composing f with g^{-1} gives our wanted morphism. To conclude the proof, we need to check that $g^{-1} \circ f$ does not depend on the choice of the isomorphism g when Y is not an algebraic set. Assume that Y is not an algebraic set and assume $g_1, g_2 : Y \to V$ are distinct isomorphisms. Pick $h : Y \to V$ as some isomorphism with corresponding k-algebra homomorphism $\phi' : k[V] \to \mathcal{R}_Y(Y)$. Then, as $h \circ (g_1^{-1} \circ f)$ and $h \circ (g_2^{-1} \circ f)$ are both morphisms to algebraic sets given by $\phi' \circ \phi$, we must have, by the reasoning above when Y is an algebraic set, that $h \circ (g_1^{-1} \circ f) = h \circ (g_2^{-1} \circ f)$. Since g_1, g_2 are distinct, this is a contradiction.

□

Chapter 3

Equivalence of Categories of Varieties

We now begin one of the main chapters of the thesis which discusses an equivalence between our constructions in the first section to certain schemes. As schemes are historically known to be generalisations of our constructions, it is natural to expect that there is a clear and fundamental relationship between the two. Understanding this transition would allow for a deeper understanding of the mathematical landscape of algebraic geometry and would further allow results to be translated from one setting to the other. This not only improves our comprehension of algebraic geometry, but also illustrates the evolution of algebraic geometry by tracing its development. The chapter is given in three sections where the first and second expand on the constructions of two functors respectively and, in the final section, show that these functors, indeed, give an equivalence of categories.

3.1 Associating a scheme to a prevariety

We begin this chapter, and this section, by defining the functor from prevarieties to reduced schemes of finite type over k. This generalises work of Hartshorne in [20, Proposition 2.6, p. 78], of which Wedhorn and Görtz also similarly prove in [19, Theorem 3.37, p. 81], which we will follow closely adding details where appropriate. Particularly, this section will illustrate how one considers the relationship between prevarieties and schemes.

We now begin following Hartshorne, Wedhorn and Görtz by showing how to naturally construct a scheme from a prevariety where we first consider the topological space.

Definition 3.1.1 ([19, Remark 3.38, p. 82]). *Let X be a topological space. We denote $t(X)$ as the set of non-empty irreducible closed subsets of X.*

The next lemma illustrates how we can establish a topology on $t(X)$ where the closed sets are precisely of the form $t(Y)$ where Y is closed in X.

Lemma 3.1.2 ([19, Remark 3.38, p. 82])**.** *Let X be a topological space. The set $\{t(Y) \mid Y \subseteq_{\mathrm{cl}} X\}$ forms a closed set topology on $t(X)$, and $(t(X), \tau_{t(X)})$ is a topological space, where $\tau_{t(X)} := \{t(X) \backslash t(Y) \mid Y \subseteq_{\mathrm{cl}} X\}$. In other words, the following properties hold:*

1. *$t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$, for any closed subsets Y_1 and Y_2 of X;*
2. *$t\left(\bigcap_{i\in I} Y_i\right) = \bigcap_{i\in I} t(Y_i)$ for any family $\{Y_i\}_{i\in I}$ of closed sets of X where I is an index set.*

We also denote simply $t(X)$ for the topological space $(t(X), \tau_{t(X)})$.

Proof. We first show that $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$ for all closed subsets Y_1 and Y_2 of X. Let Y_1 and Y_2 be two closed subsets of X. Now, consider an irreducible closed subset Z of $Y_1 \cup Y_2$. We can represent Z as the union of two closed subsets: $Z = Y_1' \cup Y_2'$, where $Y_1' = Y_1 \cap Z$ is a closed subset of Y_1, and $Y_2' = Y_2 \cap Z$ is a closed subset of Y_2. Since Z is irreducible, either $Y_1' = Z$ or $Y_2' = Z$. This implies that Z is an irreducible closed subset of Y_1 or Y_2. We deduce that $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$ as the reverse inclusion is trivial.

Next, we prove that $t\left(\bigcap_{i\in I} Y_i\right) = \bigcap_{i\in I} t(Y_i)$ for any closed subsets Y_i of X with $i \in I$ for some index set I. As a closed subset Z is an irreducible closed subset of $\bigcap_{i\in I} Y_i$ if and only if Z is an irreducible closed subset of Y_i for every $i \in I$, the equality directly follows. ☐

The following lemma introduces a natural map from X to $t(X)$ which will allow us to construct a sheaf with respect to $t(X)$ induced by the sheaf on X.

Lemma 3.1.3 ([19, Remark 3.38, p. 82])**.** *Let X be a prevariety. The map $\alpha_X : X \to t(X)$, which assigns $x \in X$ to $\overline{\{x\}}$, is continuous. More precisely, for all closed $Y \subseteq X$, $\alpha_X^{-1}(t(Y)) = Y$.*

Proof. We begin by showing that α_X is well-defined. To do so, we claim that $\overline{\{x\}}$ is irreducible in $t(X)$ for any $x \in X$. To see this, suppose otherwise. We then have $\overline{\{x\}} = X_1 \cup X_2$ with X_1 and X_2 closed in X and not equal to $\overline{\{x\}}$. However, either X_1 or X_2 would contain x and therefore contain the closure of $\{x\}$ which is a contradiction. We now show the continuity of α_X. By Lemma 3.1.2, any closed set of $t(X)$ is of the form $t(Y)$ for $Y \subseteq_{\mathrm{cl}} X$. Let $Y \subseteq_{\mathrm{cl}} X$ so that

$$\begin{aligned}\alpha_X^{-1}(t(Y)) &= \{x \in X \mid \overline{\{x\}} \in t(Y)\}\\ &= \{x \in X \mid \overline{\{x\}} \subseteq Y\}\\ &= Y.\end{aligned}$$

Hence, α_X is continuous. ☐

We now possess all the necessary components to construct a locally ringed topological space from a prevariety. To prove the following, ideas were used from [36].

Proposition 3.1.4. *Let X be a prevariety. Then, $(t(X), \alpha_{X*}\mathcal{O}_X)$ is a locally ringed space.*

Proof. Since, by Lemma 3.1.3, α_X is continuous and $\mathcal{O}_X$ is a sheaf of rings on X, we obtain that $\alpha_{X*}\mathcal{O}_X$ is a sheaf of rings on $t(X)$.

We now prove that $(t(X), \alpha_{X*}\mathcal{O}_X)$ is a locally ringed space. We need only show that $(\alpha_{X*}\mathcal{O}_X)_Z$ is a local ring for all $Z \in t(X)$. Let $Z \in t(X)$. We denote $\mathrm{Ops}_{t(X),Z}$ the set of open sets of $t(X)$ that contain Z and partially ordered by set-inclusion. We have that $(\alpha_{X*}\mathcal{O}_X)_Z$ is the direct limit of the system

$$\left((\mathcal{O}_X(\alpha_X^{-1}(U)))_{U \in I}, \left(\rho^{\mathcal{O}_X}_{\alpha_X^{-1}(U), \alpha_X^{-1}(V)} \right)_{\substack{U,V \in Ops_{t(X),Z} \\ U \subseteq V}} \right).$$

To simplify the above direct system, we claim that $Z \in U \Leftrightarrow Z \cap \alpha_X^{-1}(U) \neq \varnothing$. To prove this assertion, let U be open in $t(X)$ such that $Z \in U$. We can then write $U = t(X) \backslash t(W)$ for some fixed closed subset $W \subseteq X$. We have $Z \in U \Leftrightarrow Z \notin t(W) \Leftrightarrow Z \cap (X \backslash W) \neq \varnothing$. Furthermore, by Lemma 3.1.3, $\alpha_X^{-1}(U) = (X \backslash W)$, so the claim holds.

Therefore, since the inverse image of α_X induces a one-to-one correspondence between the open sets of X and those of $t(X)$, we have

$$\begin{aligned} \{\alpha_X^{-1}(U) \mid U \in \mathrm{Ops}_{t(X),Z}\} &= \{\alpha_X^{-1}(U) \mid U \subseteq_{\mathrm{op}} t(X) \text{ and } Z \in U\} \\ &= \{\alpha_X^{-1}(U) \mid U \subseteq_{\mathrm{op}} t(X) \text{ and } Z \cap \alpha_X^{-1}(U) \neq \varnothing\} \\ &= \{V \mid V \subseteq_{\mathrm{op}} X \text{ and } Z \cap V \neq \varnothing\}. \end{aligned}$$

Setting $J := \{V \mid V \subseteq_{\mathrm{op}} X \text{ and } Z \cap V \neq \varnothing\}$, we observe that J is partially ordered by set-inclusion. It follows that $\alpha_{X*}\mathcal{O}_{X,Z}$ is the direct limit of $\left((\mathcal{O}_X(U))_{U \in J}, (\rho^{\mathcal{O}_X}_{U,V})_{\substack{U,V \in J \\ U \subseteq V}} \right)$.

Let $\mathfrak{m} = \{[U, f] \in \alpha_{X*}\mathcal{O}_{X,Z} \mid f(x) = 0 \ \forall x \in U \cap Z\}$. As $\alpha_{X*}\mathcal{O}_{X,Z}$ is a local ring, we claim that this is the unique maximal ideal of $\alpha_{X*}\mathcal{O}_{X,Z}$.

Let $[U, f] \in \alpha_{X*}\mathcal{O}_{X,Z} \backslash \mathfrak{m}$ so that $f \neq 0$ on $U \cap Z$. We claim that $[V, \frac{1}{f}]$ is the inverse of $[U, f]$, where $V = U \cap \{x \in X \mid f(x) \neq 0\}$ and $\frac{1}{f} : V \to M$ is such that $v \in V$ is sent to $\frac{1}{f(v)} \in M$. Indeed, $[V, \frac{1}{f}] \in \alpha_{X*}\mathcal{O}_{X,Z}$ since $Z \cap V = Z \cap U \cap \{x \in X \mid f(x) \neq 0\} \neq \varnothing$ and $[U, f] \cdot [V, \frac{1}{f}] = [V, f] \cdot [V, \frac{1}{f}] = [V, 1] = [X, 1]$. It follows that $\alpha_{X*}\mathcal{O}_{X,Z}$ is a local ring. $\square$

We have just constructed our candidate for a scheme and, in the next lemma, we give an association between morphisms of prevarieties and morphisms of their associated locally ringed spaces which turn out to be schemes (see Theorem 3.1.7). As before, we used ideas from [32] to prove the following.

Lemma 3.1.5. *Let* X, Y *be prevarieties, and let* $(f, f^\sharp) : X \to Y$ *be a morphism of prevarieties. Then the pair of maps* $(f_{\mathrm{sch}}, f^\sharp_{\mathrm{sch}})$ *is a morphism of locally ringed spaces where* $f_{\mathrm{sch}} : t(X) \to t(Y)$ *which sends* $Z \in t(X)$ *to* $\overline{f(Z)}$ *and* $f^\#_{\mathrm{sch}} : \alpha_{Y*}\mathcal{R}_Y \to f_{\mathrm{sch}*}\alpha_{X*}\mathcal{R}_X$ *such that* $f^\#_{\mathrm{sch},U}$ *is defined such that* $s \in \alpha_{Y*}\mathcal{R}_Y(U)$ *is sent to* $s \circ f|_{(\alpha_Y \circ f)^{-1}(U)}$*, for all open* $U \subseteq Y$*. Furthermore, if* f *is an isomorphism, then* f_{sch} *is an isomorphism of locally ringed spaces.*

Proof. If Z is irreducible, then from Lemma A.1.13 and Lemma A.1.14 that $\overline{f(Z)} \subseteq Y$ is irreducible. Therefore, f_{sch} is well defined. Let V be closed in Y. To prove that f_{sch} is continuous, it is enough to show that $f_{\mathrm{sch}}^{-1}(t(V)) = t(f^{-1}(V))$, by Lemma 3.1.2.

Let $Z \in t(f^{-1}(V))$. Since $\overline{f(Z)}$ is irreducible by the above and $\overline{f(Z)} \subseteq V$, we have $Z \in f_{\mathrm{sch}}^{-1}(t(V))$. Now, let $Z \in f_{\mathrm{sch}}^{-1}(t(V))$. Then $Z \in t(X)$ such that $\overline{f(Z)} \in t(V)$. We have $\overline{f(Z)} \subseteq V$, which implies that $f^{-1}(\overline{f(Z)}) \subseteq f^{-1}(V)$. Since $Z \subseteq f^{-1}(\overline{f(Z)})$, we conclude that $Z \in t(f^{-1}(V))$. Hence, $f_{\mathrm{sch}}^{-1}(t(V)) = t(f^{-1}(V))$.

Since, for some open $U \subseteq t(Y)$ and $s \in \alpha_{Y*}\mathcal{R}_Y(U)$, $s \circ f|_{(\alpha_Y \circ f)^{-1}(U)}$ as a priori, we need check that $f_{\mathrm{sch}*}\alpha_{X*}\mathcal{R}_X(U) = f_*\mathcal{R}_X(\alpha_Y^{-1}(U))$, so that $f^\#_{\mathrm{sch}U} := f^\#_{\alpha_Y^{-1}(U)}$ gives a well-defined morphism of sheaves $f^\#_{\mathrm{sch}}$. To that end, we check that $(\alpha_X^{-1} \circ f_{\mathrm{sch}}^{-1})(U) = (f^{-1} \circ \alpha_Y^{-1})(U)$. We have

$$
\begin{aligned}
(f^{-1} \circ \alpha_Y^{-1})(U) &= f^{-1}(\{x \in Y \mid \overline{\{x\}} \in U\}) \\
&= \{x \in X \mid \overline{\{f(x)\}} \in U\} \\
&= \{x \in X \mid \overline{f(\{x\})} \in U\} \\
&= \{x \in X \mid \overline{f(\overline{\{x\}})} \in U\} && (f \text{ is continuous}) \\
&= \alpha_X^{-1}(\{x \in t(X) \mid \overline{f(x)} \in U\}) \\
&= (\alpha_X^{-1} \circ f_{\mathrm{sch}}^{-1})(U).
\end{aligned}
$$

Moreover, assume that f is an isomorphism. Then f_{sch} is a bijection. It follows that $f(t(f^{-1}(V))) = t(V)$ for any closed $V \subseteq Y$. Since f is an isomorphism, any closed set of X is of the form $f^{-1}(V)$ for some closed $V \subseteq Y$. We deduce that f_{sch} is a homeomorphism. Since f is an isomorphism, $f^\#_{\alpha_Y^{-1}(U)}$ is an isomorphism which implies that $f^\#_{\mathrm{sch}}$ is an isomorphism of sheaves. We deduce that $f^\#_{\mathrm{sch}}$ is an isomorphism of ringed space so that by Lemma A.6.7. □

The following lemma gives a key idea of how one relates open sets of a prevariety X and the open sets of $t(X)$ which has not been well-expanded upon in the literature.

Lemma 3.1.6. *Let X be a prevariety. Let $U \subseteq_{\text{op}} X$, let $\iota_{U,X} : U \to X$ be the inclusion of prevarieties and let $\iota_{t(X)\backslash t(X\backslash U),t(X)}$ be the inclusion of schemes. There exists an isomorphism of ringed space*

$$\Xi_{U,X} : (t(X)\backslash t(X\backslash U), (\alpha_{X*}\mathcal{O}_X)|_{t(X)\backslash t(X\backslash U)}) \to (t(U), \alpha_{U*}\mathcal{O}_X|_U)$$

such that the diagram

$$\begin{tikzcd}
t(X)\backslash t(X\backslash U) \arrow[r, "\Xi_{U,X}"] \arrow[d, "\iota_{t(X)\backslash t(X\backslash U),t(X)}"] & t(U) \arrow[d, "{\iota_{U,X}}_{\text{sch}}"] \\
t(X) \arrow[r, "\text{id}_{t(X)}"] & t(X)
\end{tikzcd}$$

commutes.

Proof. To construct the isomorphism $\Xi_{U,X}$, we first need a homeomorphism $\Xi_{U,X,0} : t(X)\backslash t(X\backslash U) \to t(U)$. We define the map $\Xi_{U,X,0}$ such that $Z \in t(X)\backslash t(X\backslash U)$ is sent to $Z \cap U$. To see that $\Xi_{U,X,0}$ is well-defined, we note that if Z is an irreducible closed subset of X not contained in $X\backslash U$ then $Z \cap U$ is a non-empty irreducible closed set in U. We now define a map which will turn out to be the inverse of $\Xi_{U,X,0}$. Let $g : t(U) \to t(X)\backslash t(X\backslash U)$ which sends $Z \in t(U)$ to $\overline{Z}$ where $\overline{Z}$ denotes the closure of Z in X. To see that g is well-defined, we have that if $Z \in t(U)$, it is non-empty, so that $\overline{Z}$ cannot be contained in $X\backslash U$.

We now show that g and $\Xi_{U,X,0}$ are inverse as maps. Let $Z \in t(X)\backslash t(X\backslash U)$. To show that $(g \circ \Xi_{U,X,0})(Z) = Z$, we need only show that $Z = \overline{Z \cap U}$. As Z is irreducible, by Lemma A.1.11, $Z \cap U$ is dense in Z so that $\overline{Z \cap U} = Z$. Let $Z \in t(U)$. We now show that $(\Xi_{U,X,0} \circ g)(Z) = Z$ which is to show that $\overline{Z} \cap U = Z$. By definition of the subspace topology of U, we have that any closed subset of U is of the form $V \cap U$ for some $V \subseteq_{\text{cl}} X$. Let $Z' \subseteq_{\text{cl}} X$ such that $Z = Z' \cap U$. We have $\overline{Z} = \overline{Z' \cap U} \subseteq Z'$ so that $\overline{Z} \cap U \subseteq Z' \cap U = Z$. Conversely, we have $Z \subseteq \overline{Z}$ and $Z \subseteq U$ so that $Z \subseteq \overline{Z} \cap U$. Thus, $Z = \overline{Z} \cap U$. We deduce that $\Xi_{U,X,0}$ and g are bijections.

By Lemma 3.1.2, any closed subset of $t(U)$ is of the form $t(V)$ for some $V \subseteq_{\text{cl}} U$. To show that $\Xi_{U,X,0}$ is continuous, consider some $V \subseteq_{\text{cl}} U$. As before, we have $\overline{V} \cap U = V$. It follows that

$$\begin{aligned}
\Xi_{U,X,0}^{-1}(t(V)) &= \{Z \in t(X)\backslash t(X\backslash U) \mid Z \cap U \in t(V)\} \\
&= \{Z \in t(X)\backslash t(X\backslash U) \mid Z \cap U \in t(\overline{V} \cap U)\} \\
&= \{Z \in t(X)\backslash t(X\backslash U) \mid Z \in t(\overline{V})\} \\
&= t(\overline{V}).
\end{aligned}$$

Conversely, we have that any closed set of $t(X)\backslash t(X\backslash U)$ is of the form $t(V)$ for some $V \subseteq_{\text{cl}} X$ such that $V \cap U \neq \varnothing$. Letting $V \subseteq_{\text{cl}} X$ such that $V \cap U \neq \varnothing$, it is analogous to above to show that $g^{-1}(t(V)) = t(V \cap U)$. Thus, $\Xi_{U,X,0}$ and g are homeomorphisms.

To give an isomorphism of sheaves, we first observe the following: For any $V \subseteq_{\text{cl}} U$, we have

$$\begin{aligned}
&\alpha_{U*}\mathcal{O}_X|_U(t(U)\backslash t(V)) \\
&= \mathcal{O}_X(U\backslash V) && \text{(Lemma 3.1.3)}\\
&= \mathcal{O}_X(U\backslash(\overline{V} \cap U)) && (\overline{V} \cap U = V)\\
&= \mathcal{O}_X(U \cap (X\backslash\overline{V}))\\
&= \mathcal{O}_X(X\backslash((X\backslash U) \cup \overline{V}))\\
&= (\alpha_{X*}\mathcal{O}_X)(t(X)\backslash t((X\backslash U) \cup \overline{V})) && \text{(Lemma 3.1.3)}\\
&= (\alpha_{X*}\mathcal{O}_X)(t(X)\backslash(t(X\backslash U) \cup t(\overline{V})))\\
&= (\alpha_{X*}\mathcal{O}_X)((t(X)\backslash t(X\backslash U))\backslash t(\overline{V})) && \text{(1. of Lemma 3.1.2)}\\
&= (\alpha_{X*}\mathcal{O}_X)|_{t(X)\backslash t(X\backslash U)}((t(X)\backslash t(X\backslash U))\backslash t(\overline{V}))\\
&= \Xi_{U,X,0_*}(\alpha_{X*}\mathcal{O}_X)|_{t(X)\backslash t(X\backslash U)}((t(U)\backslash t(V))
\end{aligned}$$

We deduce that $\{\mathrm{id}_{\alpha_{U*}\mathcal{O}_X|_U(t(U)\backslash t(V))}\}_{V\subseteq_{\text{cl}}U}$ is our wanted isomorphism. ☐

We are now in a position to state the main result of the section where we establish that our constructions indeed give schemes. The statement of this theorem is the same as [20, Proposition 2.6, p. 78], however, the sheaf part is of the proof is proven differently.

Theorem 3.1.7. *Let X be a prevariety. If $\alpha_X : X \to t(X)$ is the continuous map described in Proposition 3.1.4, then $(t(X), \alpha_{X*}\mathcal{R}_X)$ is a scheme. Moreover, if X is a classical affine variety, then $(t(X), \alpha_{X*}\mathcal{R}_X)$ is an affine scheme. More precisely, if X is an algebraic set, then $(t(X), \alpha_{X*}\mathcal{R}_X)$ is an affine scheme and the pair of maps*

$$(\gamma_X, \gamma_X^\sharp) : (t(X), \alpha_{X*}\mathcal{R}_X) \to (\mathrm{Spec}(k[X]), \mathcal{O}_{\mathrm{Spec}(k[X])})$$

is an isomorphism of locally ringed spaces where:

- *The map $\gamma_X : t(X) \to \mathrm{Spec}(k[X])$, which sends $Y \subseteq X$ closed and irreducible to $\pi(\mathbb{I}(Y))$ where $\pi : k[x_1, \dots, x_n] \to k[X]$ is the canonical quotient map. Its inverse sends prime ideals $\mathfrak{p}$ in $k[X]$ to $\mathbb{V}_X(\mathfrak{p})$. For all $J \subseteq k[X]$, we have $\gamma_X^{-1}(\mathbb{V}_{\text{sch}}(J)) = t(\mathbb{V}(\sqrt{J}))$.*
- *$\gamma_X^\#: \mathcal{O}_{\mathrm{Spec}(k[X])} \to \gamma_{X*}(\alpha_{X*}\mathcal{R}_X)$ is the natural transformation defined as follows: for all $U \subseteq_{\text{op}} \mathrm{Spec}(k[X])$, $\gamma_{XU}^\# : \mathcal{O}_{\mathrm{Spec}(k[X])}(U) \to (\gamma_X \circ$*

$\alpha_X)_*\mathcal{R}_X(U)$ by sending $s \in \mathcal{O}_{\mathrm{Spec}(k[X])}(U)$ to the regular function $\gamma^{\#}_{XU}(s)$ defined pointwise for all $y \in (\gamma_X \circ \alpha_X)^{-1}(U)$ as:

$$\gamma^{\#}_U(s)(y) := s((\gamma_X \circ \alpha_X)(y))(y) = \frac{a_y(y)}{b_y(y)}$$

where $a_y, b_y \in k[X]$ such that $b_y \notin (\gamma_X \circ \alpha_X)(y)$.

Proof. Because X is a prevariety, it has a finite open covering by classical affine varieties, say $\{X_i\}_{i\in I}$ for some finite index set I. Since $t(X) = t(\bigcup_{i\in I} X_i) = \bigcup_{i\in I} t(X_i)$ by Lemma 3.1.2 and Lemma 3.1.6, it suffices to show, if X is a classical affine variety, that $(t(X), \alpha_{X*}\mathcal{R}_X)$ is an affine scheme. Using Lemma 3.1.5, we may assume that X is an algebraic set in $\mathbb{A}^n_k$, for some $n \in \mathbb{N}$, so that $X = \mathbb{V}(J)$ for some ideal $J \subseteq k[x_1, ..., x_n]$. We will show, in particular, that $(t(X), \alpha_{X*}\mathcal{R}_X)$ is isomorphic to $(\mathrm{Spec}(k[X]), \mathcal{O}_{\mathrm{Spec}(k[X])})$. By virtue of Lemma A.6.7, it suffices to only give a homeomorphism of the form $\gamma_X : t(X) \to \mathrm{Spec}(k[X])$ and an isomorphism of sheaves of the form $\gamma^{\#}_X : \mathcal{O}_{\mathrm{Spec}(k[X])} \to \gamma_{X*}(\alpha_{X*}\mathcal{R}_X)$. We begin by defining the map $\gamma_X : t(X) \to \mathrm{Spec}(k[X])$ which sends $Y \subseteq X$ closed and irreducible to $\pi(\mathbb{I}(Y))$ where $\pi : k[x_1, ..., x_n] \to k[X] = k[x_1, ..., x_n]/J$ is the natural projection. Since Y is irreducible, it follows from Proposition 2.4.4 that $\mathbb{I}(Y)$ is prime and hence γ_X is well-defined. We now show that this map is indeed a homeomorphism. To prove that γ_X is continuous, we need to show that $\gamma_X^{-1}(\mathbb{V}_{\mathrm{sch}}(J))$ is closed in $t(X)$ for some ideal $J \subseteq k[X]$. We have

$$\begin{aligned}
\gamma_X^{-1}(\mathbb{V}_{\mathrm{sch}}(J)) &= \gamma_X^{-1}(\mathbb{V}_{\mathrm{sch}}(\sqrt{J})) && \text{(Lemma A.4.3)}\\
&= \{Z \in t(X) \mid \pi(\mathbb{I}(Z)) \in \mathbb{V}_{\mathrm{sch}}(\sqrt{J})\}\\
&= \{Z \in t(X) \mid \sqrt{J} \subseteq \pi(\mathbb{I}(Z))\}\\
&= \{Z \in t(X) \mid \pi^{-1}(\sqrt{J}) \subseteq \mathbb{I}(Z)\}\\
&= \{Z \in t(X) \mid \mathbb{V}(\pi^{-1}(\sqrt{J})) \supseteq Z\} && \text{(Lemma 2.1.4)}\\
&= t(\mathbb{V}(\pi^{-1}(\sqrt{J})))\\
&= t(\mathbb{V}(\sqrt{J}))\\
&= t(\mathbb{V}(J)) && \text{(Corollary 2.3.4)}
\end{aligned}$$

Note that, by Lemma A.2.35, $\pi^{-1}(\sqrt{J})$ is an ideal. Since $\mathbb{V}(\sqrt{J})$ is a closed subset of X, we have that $t(\mathbb{V}(\sqrt{J}))$ is closed in $t(X)$ and hence γ_X is continuous. To show that γ_X is indeed a homeomorphism, we need only give an inverse map which is continuous. Define $\mu_X : \mathrm{Spec}(k[X]) \to t(X)$ which sends a prime ideal $\mathfrak{p} \in k[X]$ to $\mathbb{V}(\mathfrak{p})$. Since $\mathfrak{p}$ is prime, $\pi^{-1}(\mathfrak{p})$ is prime. It follows by Proposition 2.4.4 that $\mathbb{V}(\mathfrak{p})$ is irreducible so that μ_X is well-defined. Furthermore, by Corollary 2.3.5, μ_X is an inverse to γ_X so that γ_X is a bijection.

As γ_X is a bijection, we deduce that $\gamma_X(t(\mathbb{V}(J))) = \mathbb{V}_{\text{sch}}(J)$ for any $J \subseteq k[X]$ so that γ_X is an open mapping. Hence, γ_X is a homeomorphism.

We conclude the proof by constructing an isomorphism of sheaves $\gamma_X^{\#} : \mathcal{O}_{\text{Spec}(k[X])} \to \gamma_{X*}(\alpha_{X*}\mathcal{R}_X)$. To ease notation, let $\beta_X = \gamma_X \circ \alpha_X$ so that we need only give the isomorphism $\gamma_X^{\#} : \mathcal{O}_{\text{Spec}(k[X])} \to \beta_{X*}\mathcal{R}_X$. Let $U \subseteq_{\text{op}} \text{Spec}(k[X])$. We define $\gamma_{XU}^{\#} : \mathcal{O}_{\text{Spec}(k[X])}(U) \to \beta_{X*}\mathcal{R}_X(U)$ by sending $s \in \mathcal{O}_{\text{Spec}(k[X])}(U)$ to the regular function $\gamma_{XU}^{\#}(s)$ that we define pointwise, for all $y \in \beta_X^{-1}(U)$, as

$$\gamma_{XU}^{\#}(s)(y) := s(\beta_X(y))(y) = \frac{a_y(y)}{b_y(y)}$$

where $a_y, b_y \in k[X]$ such that $b_y \notin \beta_X(y)$. This map is indeed well-defined: By definition of s, for all $y \in \beta_X^{-1}(U)$, there is some open $V_{\beta_X(y)} \subseteq U$ such that $\beta_X(y) \in V_{\beta_X(y)}$, $s(\mathfrak{q}) = \frac{a_y}{b_y}{}_{\mathfrak{q}}$ and $b_y \notin \mathfrak{q}$ for all $\mathfrak{q} \in V_{\beta_X(y)}$. As $\beta_X(y) \in V_{\beta_X(y)}$, we have that $b_y \notin \beta_X(y)$ which implies that $b_y \notin \pi(\mathbb{I}(\{y\}))$ so that $b_y(y) \neq 0$ for all $y \in \beta_X^{-1}(V)$. Furthermore, $\beta_X^{-1}(V)$ is non-empty since $\beta_X(y) \in V$.

We now show that $\gamma_U^{\#}$ is an isomorphism. We first consider distinguished open subsets. Let $U = \mathbb{D}_{\text{sch}}(f)$ for some $f \in k[X]$. We observe that

$$\begin{aligned}\beta_X^{-1}(\mathbb{D}_{\text{sch}}(f)) &= \beta_X^{-1}(\text{Spec}(k[X])\backslash\mathbb{V}_{\text{sch}}(f)) \\ &= X\backslash\beta^{-1}(\mathbb{V}_{\text{sch}}(f)) \\ &= X\backslash\alpha_X^{-1}(t(\mathbb{V}(\sqrt{f}))) \\ &= X\backslash\mathbb{V}(\sqrt{\langle f\rangle}) \\ &= X\backslash\mathbb{V}(f) \\ &= \mathbb{D}(f)\end{aligned}$$

so that $\gamma_U^{\#}$ is a map from $\mathcal{O}_{\text{Spec}(k[X])}(\mathbb{D}_{\text{sch}}(f))$ to $\mathcal{R}_X(\mathbb{D}(f))$. We observe that composing the isomorphism described in A.7.5, that sends $\mathcal{O}_{\text{Spec}(k[X])}(\mathbb{D}_{\text{sch}}(f))$ to $k[X]_f$, with the isomorphism described in Proposition 2.4.11, which sends $k[X]_f$ to $\mathcal{R}_X(\mathbb{D}(f))$, is exactly $\gamma_U^{\#}$ so that $\gamma_U^{\#}$ is an isomorphism. Let $U \subseteq_{\text{op}} X$ and let $t \in \mathcal{R}_X(\beta_X^{-1}(U))$. Let $\{\mathbb{D}_{\text{sch}}(f_i)\}_{i\in I}$ be the open covering of U consisting of all distinguished open subsets and let $t_i = t|_{\beta_X^{-1}(\mathbb{D}_{\text{sch}}(f_i))}$ for all $i \in I$. Since $\gamma_{\mathbb{D}_{\text{sch}}(f_i)}^{\#}$ is an isomorphism, we may set $s_i = (\gamma_{\mathbb{D}_{\text{sch}}(f_i)}^{\#})^{-1}(t_i)$ for all $i \in I$. Since $t_i = t_j$ on $\mathbb{D}(f_i) \cap \mathbb{D}(f_j)$, it is clear that $s_i = s_j$ on $\mathbb{D}_{\text{sch}}(f_i) \cap \mathbb{D}_{\text{sch}}(f_j)$ for all $i, j \in I$. It follows from the glueing and uniqueness axioms of sheaves that there exists a unique $s \in \mathcal{O}_{\text{Spec}(k[X])}(U)$ such that $s|_{\mathbb{D}_{\text{sch}}(f_i)} = s_i$ for all $i \in I$. We deduce that $\gamma_U^{\#}$ is injective. Moreover, $\gamma_U^{\#}(s) = t$ so that $\gamma_U^{\#}$ is surjective. It follows that $\gamma^{\#} = \{\gamma_U^{\#}\}_{U\in\tau_{\text{Spec}(k[X])}}$ is an isomorphism of sheaves so that we are done. □

As a consequence of the previous theorem, we have the following.

Lemma 3.1.8. *If $(X, \mathcal{R}_X)$ is a prevariety over k, then its associated scheme $(t(X), \alpha_{X*}\mathcal{R}_X)$, given in Theorem 3.1.7, is a reduced scheme of finite type over k.*

Proof. To first see that $(t(X), \alpha_{X*}\mathcal{R}_X)$ is a k-scheme, we need a morphism $(f, f^{\#}) : (t(X), \alpha_{X*}\mathcal{R}_X) \to (\mathrm{Spec}(k), \mathcal{O}_{\mathrm{Spec}(k)})$. We know that $\alpha_{X*}\mathcal{O}_X(t(X))$ is a k-algebra, which induces a map $f^{\#}_{\mathrm{Spec}(k)} : \mathcal{O}_{\mathrm{Spec}(k)}(\mathrm{Spec}(k)) \to \alpha_{X*}\mathcal{O}_X(t(X))$. Using the correspondence of [28, Lemma 3.23, p. 48], $(f, f^{\#})$ exists.

We now show that X is of finite type over k. First, assume that X is affine with coordinate ring $k[X]$. Then $t(X) \cong \mathrm{Spec}(k[X])$. Since $k[X]$ is finitely generated k-algebra, it follows that X is of finite type over k. Using Proposition A.7.17, $t(X)$ being reduced follows easily.

Next, assume that X is non-affine, so that, by Definition A.7.19, there exists a finite open covering $\{X_i\}_{i \in I}$ of X such that $|I| \geqslant 2$ and X_i is an classical affine variety for all $i \in I$. Using our arguments above, $t(X_i)$ is a reduced affine scheme of finite type over k. By Definition A.7.19, it follows that $t(X)$ is a scheme of finite type over k as well. Again, by Proposition A.7.17, it follows that $t(X)$ is reduced. ☐

To state our wanted functors, we introduce the following categories where our choice of scheme was influenced by the discussion in [8, p. 516].

Definition 3.1.9. *An* affine algebraic variety *over k is an reduced affine scheme of finite type over k. An* algebraic variety *over k is a reduced scheme of finite type over k. We define the category of algebraic varieties over k, denoted by $\mathbf{Var}_k$, to be the full subcategory of $\mathbf{Sch}_k$. Furthermore, we will denote the full subcategory of affine varieties over k in $\mathbf{Var}_k$ as $\mathbf{AVar}_k$.*

Having laid the groundwork, we now put the pieces together to construct a functor between prevarieties and schemes.

Definition 3.1.10. *Define the maps*

$$(\mathcal{V}^{\mathrm{Sch}}_{k,M,0} : \mathbf{Prevar}_{k,M,0} \to \mathbf{Var}_{k,0}, \mathcal{V}^{\mathrm{Sch}}_{k,M,1} : \mathbf{Prevar}_{k,M,1} \to \mathbf{Var}_{k,1})$$

such that:

- *For any prevariety over k in M, $(X, \mathcal{R}_X)$, $\mathcal{V}^{\mathrm{Sch}}_{k,0}$ sends $(X, \mathcal{R}_X)$ to its associated scheme $(t(X), \alpha_{X*}\mathcal{R}_X)$ described in Theorem 3.1.7.*
- *For any morphism $(f : X \to Y, f^{\#} : \mathcal{R}_Y \to f_*\mathcal{R}_X)$ of prevarieties over k in M, $\mathcal{V}^{\mathrm{Sch}}_{k,1}$ sends $(f, f^{\#})$ to the morphism $(f_{\mathrm{sch}} : t(X) \to t(Y), f^{\#}_{\mathrm{sch}} : \alpha_{Y*}\mathcal{R}_Y \to f_*\alpha_{X*}\mathcal{R}_X)$ where $f_{\mathrm{sch}}(P) := \overline{f(P)}$ for all $P \in t(X)$ and $f^{\#}_{\mathrm{sch},U} : \mathcal{R}_Y(U) \to f_*\mathcal{R}_X(U)$ is defined such that $s \in \mathcal{R}_Y(U)$ is sent to $s \circ f|_{f^{-1}(U)} \in f_*\mathcal{R}(U)$ for all open $U \subseteq Y$.*

We define $\mathcal{A}^{\mathrm{sch}}_{k,M}$ *to be the restriction and corestriction of* $\mathcal{V}^{\mathrm{Sch}}_{k,M}$ *to* $\mathbf{CAVar}_{k,M}$ *and* $\mathbf{AVar}_k$*, respectively.*

Lemma 3.1.11. $\mathcal{A}^{\mathrm{sch}}_{k,M}$ *and* $\mathcal{V}^{\mathrm{Sch}}_{k,M}$ *are functors.*

Proof. First, $\mathcal{V}^{\mathrm{sch}}_{k,M,0}$ being well-defined follows from Theorem 3.1.7 and Lemma 3.1.8. The well-definedness of $\mathcal{V}^{\mathrm{sch}}_{k,M,1}$ is achieved in Lemma 3.1.5. Next, we check that $\mathcal{V}^{\mathrm{sch}}_{k,M}$ is indeed a functor. Let X, Y, Z be prevarieties with morphisms $f : X \to Y$ and $g : Y \to Z$. We have, for all $P \in t(X)$,

$$\begin{aligned}
\mathcal{V}^{\mathrm{sch}}_{k,M}(g \circ f)(P) &= (g \circ f)_{\mathrm{sch}}(P) \\
&= \overline{g(f(P))} \\
&= \overline{g(\overline{f(P)})} && \text{(g is continuous)} \\
&= g_{\mathrm{sch}}(f_{\mathrm{sch}}(P)) \\
&= (\mathcal{V}^{\mathrm{sch}}_{k,M}(g) \circ \mathcal{V}^{\mathrm{sch}}_{k,M}(f))(P)
\end{aligned}$$

and, we have, for all $U \subseteq_{\mathrm{op}} Z$ and $s \in \mathcal{R}_Z(U)$,

$$\begin{aligned}
\mathcal{V}^{\mathrm{sch}}_{k,M}(g \circ f)^{\#}_U(s) &= s \circ (g \circ f)|_{(g \circ f)^{-1}(U)} \\
&= (s \circ g|_{g^{-1}(U)}) \circ f|_{f^{-1}(g^{-1}(U))} \\
&= (\mathcal{V}^{\mathrm{sch}}_{k,M}(g) \circ \mathcal{V}^{\mathrm{sch}}_{k,M}(f))(s).
\end{aligned}$$

It is straightforward to show that $\mathcal{V}^{\mathrm{sch}}_{k,M}(\mathrm{id}_X) = \mathrm{id}_X$. As a consequence of Theorem 3.1.7 and Lemma 3.1.5, we conclude that $\mathcal{A}^{\mathrm{sch}}_{k,M}$ is also a functor. ☐

3.2 Associating a prevariety to a scheme

We will now work towards defining a functor from the varieties in the scheme sense to prevarieties. Wedhorn and Görtz in [19, Theorem 3.37, p. 82], indicated how to construct a prevariety from a variety over k, however, they (and other literature) have done so with little detail and have not explicitly elaborated on how to associate morphisms of varieties to morphisms of prevarieties. The aim of this section is to expand on Wedhorn and Görtz's constructions and provide full detail.

Henceforth, we will use the following notations for any k-scheme X:

- X_M denotes the fibred product $X \times_k M$,
- $\pi_{M,X}$ denotes the projection from X_M to X,
- $\iota_{M,X}$ denotes the inclusion map of closed points of X_M into X_M and,

- for any k-scheme Y and morphism $f : X \to Y$, $f_M : X_M \to Y_M$ will denote the induced morphism by the universal property of fibred products.

To construct the functor in the reverse direction, we make use of the closed points of schemes which will be used to construct a prevariety. Wedhorn and Görtz in [19, Theorem 3.37, p. 82] used the closed points of integral schemes of finite type over an algebraically closed field to construct a prevariety. We operate in the same manner for arbitrary fields by using a base change to M. We introduce some additional notation to establish our wanted correspondence effectively.

Definition 3.2.1. *Let X be a scheme. We will denote the set of closed points of X as X_{cl}. When X is a k-scheme, we will denote the closed points of X_M as $X_{M,cl}$.*

As in Lemma A.7.21, morphisms between schemes of finite type over a field naturally induce continuous mappings on their closed points restricting the domain and corestricting the codomain to the closed points. This allows us to define the following definition.

Definition 3.2.2. *Let X, Y be schemes of finite type over k, and let $f : X \to Y$ be a morphism of schemes over k. We denote f_{cl} the restriction of f to X_{cl} corestricted to Y_{cl}. Furthermore, we will denote $f_{M\,\mathrm{cl}}$ as $f_{M,\mathrm{cl}}$.*

When X is a scheme of finite type, we will equip $X_{M,\mathrm{cl}}$ with the topology induced by the composition of mappings $\pi_{M,X} \circ \iota_{M,X}$. As shown in the proof of Lemma A.7.24, we can identify U_M with $\pi^{-1}_{M,X}(U)$ for any k-scheme X and $U \subseteq_{\mathrm{op}} X$. Therefore, for any $U \subseteq_{\mathrm{op}} X$, we can deduce that $(\pi_{M,X} \circ \iota_{M,X})^{-1}(U) = \iota^{-1}_{M,X} \circ \pi^{-1}_{M,X}(U) = \iota^{-1}_{M,X}(U_M) = U_{M,\mathrm{cl}}$, as shown in Lemma A.7.21. We have thus shown that any open set for this topology is of the form $U_{M,\mathrm{cl}}$ for some $U \subseteq_{\mathrm{op}} X$. The next result naturally follows.

Lemma 3.2.3. *Consider a variety X over k. The following properties hold:*

1. *For any $Y_1, Y_2 \subseteq_{\mathrm{cl}} X$, we have $(Y_1 \cup Y_2)_{M,\mathrm{cl}} = Y_{1M,\mathrm{cl}} \cup Y_{2M,\mathrm{cl}}$.*
2. *For any family $\{Y_i\}_{i\in I}$ of closed subsets of X with an index set I, we have $(\bigcap_{i\in I} Y_i)_{M,\mathrm{cl}} = \bigcap_{i\in I}(Y_{iM,\mathrm{cl}})$.*

For any scheme X of finite type over M with a closed point $x \in X$, we know that there exists an open affine subscheme U of X that is also of finite type over M and contains x. In particular, $\mathcal{O}_{X,x}$ is isomorphic to $\mathcal{O}_{U,x}$.

Now, according to Proposition A.7.3, we have $\mathcal{O}_{U,x} \cong R_{\mathfrak{m}}$ for some finitely generated M-algebra R with a maximal ideal $\mathfrak{m} \subseteq R$ corresponding to the point x. Consequently, since M is algebraically closed, we can deduce that $\mathcal{O}_{U,x}/\mathfrak{m}_{U,x} \cong M$, which in turn leads to $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \cong M$. This next definition follows Part I of [19, Theorem 3.37p. 81-82] where we now factor in a base change with how we construct a sheaf with respect to $X_{M,\mathrm{cl}}$.

Definition 3.2.4. *Let X be a variety over k, and let $U \subseteq_{\text{op}} X$. For each $s \in \mathcal{O}_X(U)$, we can associate a map $s_{\mathrm{M,cl}} : U_{M,\mathrm{cl}} \to M$. To do so, we first set $\Theta_{X,U,M,x} := \phi_{X,M,x} \circ p_{X,M,x} \circ \psi_{X,U,M,x} \circ \pi^{\#}_{X,M,U}$ where*

- *$\psi_{X,U,M,x} : \mathcal{O}_{X_M}(\pi^{-1}_{M,X}(U)) \to \mathcal{O}_{X_M,x}$ denotes the canonical ring homomorphism for any $x \in U_{\mathrm{cl}}$.*
- *$p_{X,M,x} : \mathcal{O}_{X_M,x} \to \mathcal{O}_{X_M,x}/\mathfrak{m}_{X_M,x}$ denotes the canonical quotient map.*
- *$\phi_{X,M,x} : \mathcal{O}_{X_M,x}/\mathfrak{m}_{X_M,x} \to M$ denotes the isomorphism in the discussion above.*

We set, for all $x \in U_{M,\mathrm{cl}}$,

$$s_{M,\mathrm{cl}}(x) := \Theta_{X,U,M,x}(s).$$

We define the set $\mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}(U_{M,\mathrm{cl}})$ of all functions $s_{\mathrm{M,cl}}$ when s runs through $\mathcal{O}_X(U)$. The set $\mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}(U_{M,\mathrm{cl}})$ forms a ring when equipped with pointwise addition and pointwise multiplication, which are induced by the ring structure of M.

*We also introduce a functor, denoted as $\mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}$, from the category of open sets of $X_{M,\mathrm{cl}}$, denoted by $\mathrm{Ops}_{X_{M,\mathrm{cl}}}$, to the category k-**Alg**. This functor assigns to each $U_{M,\mathrm{cl}} \subseteq_{\text{op}} X_{M,\mathrm{cl}}$ the ring $\mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}(U_{M,\mathrm{cl}})$. It is contravariant and, for any $U_{M,\mathrm{cl}}, V_{M,\mathrm{cl}} \subseteq_{\text{op}} X_{M,\mathrm{cl}}$ where $V_{M,\mathrm{cl}} \subseteq U_{M,\mathrm{cl}}$, it assigns an inclusion morphism $\mathfrak{i}_{V_{M,\mathrm{cl}},U_{M,\mathrm{cl}}}$ to the standard restriction map that restricts functions from $U_{M,\mathrm{cl}}$ to $V_{M,\mathrm{cl}}$.*

Since $\Theta_{X,U,M,x}$ is a ring homomorphism for all open $U \subseteq X$ and $x \in U_{\mathrm{cl}}$, the following is immediate.

Lemma 3.2.5. *Let X be a variety over k. Then $(X_{M,\mathrm{cl}}, \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X})$ is a space with functions over k in M, where $X_{M,\mathrm{cl}}$ is equipped with the topology induced by the composition of mappings $\pi_{M,X} \circ \iota_{M,X}$.*

By Lemma 3.2.3 and Definition 3.2.4, the following is immediate.

Lemma 3.2.6. *Let X be a variety over k and let $U \subseteq_{\text{op}} X$. Then*

$$\mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X|_U} = \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}|_{U_{M,\mathrm{cl}}}.$$

We have now constructed a space with functions over k in M which is our candidate for a prevariety. In this next lemma, we canonically assign a morphism of varieties over k to a morphism of spaces of functions over k in M.

Lemma 3.2.7. *Let X and Y be varieties over k, and let $f : X \to Y$ be a morphism. Then the map $f_{M,\mathrm{cl}}$ (see Definition 3.2.2) is a morphism of spaces of functions over M. In particular, for all $U \subseteq_{\mathrm{op}} X$, $x \in U_{M,\mathrm{cl}}$ and $s \in \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_Y}(U)$, we have $s_{M,\mathrm{cl}}(f_{M,\mathrm{cl}}(x)) = f^{\sharp}_U(s)_{M,\mathrm{cl}}(x)$. Moreover, if f is an isomorphism, then $f_{M,\mathrm{cl}}$ is an isomorphism.*

Proof. We first show that $f_{M,\mathrm{cl}}$ is a continuous map. Let $U \subseteq_{\mathrm{op}} Y$. We have, by definition of the topology on $Y_{M,\mathrm{cl}}$, that any open set of $Y_{M,\mathrm{cl}}$ is of the form $U_{M,\mathrm{cl}} = (\pi_{M,Y} \circ \iota_{M,Y})^{-1}(U)$. Moreover, we have

$$\begin{aligned}
f^{-1}_{M,\mathrm{cl}}(U_{M,\mathrm{cl}}) &= \{x \in X_{M,\mathrm{cl}} \mid f_{M,\mathrm{cl}}(x) \in U_{M,\mathrm{cl}}\} \\
&= \{x \in X_{M,\mathrm{cl}} \mid f_{M,\mathrm{cl}}(x) \in (\pi_{M,Y} \circ \iota_{M,Y})^{-1}(U)\} \\
&= \{x \in X_{M,\mathrm{cl}} \mid \pi_{M,Y}(\iota_{M,Y}(f_{M,\mathrm{cl}}(x))) \in U\} \\
&= \{x \in X_{M,\mathrm{cl}} \mid (f \circ \pi_{M,X} \circ \iota_{M,X})(x) \in U\} \\
&= \{x \in X_{M,\mathrm{cl}} \mid \iota_{M,X}(\pi_{M,X}(x)) \in f^{-1}(U)\} \\
&= \{x \in X_{M,\mathrm{cl}} \mid x \in (\pi_{M,X} \circ \iota_{M,X})^{-1}(f^{-1}(U))\} \\
&= f^{-1}(U)_{M,\mathrm{cl}}.
\end{aligned}$$

Since $f^{-1}(U)$ is open, it follows that $f^{-1}(U)_{M,\mathrm{cl}}$ is open in $X_{M,\mathrm{cl}}$, so $f_{M,\mathrm{cl}}$ is continuous.

We now show that $f_{M,\mathrm{cl}}$ is a morphism of prevarieties. We pick an arbitrary open set of $Y_{M,\mathrm{cl}}$, denoted as $U_{\mathrm{M,cl}}$ for some $U \subseteq_{\mathrm{op}} X$, and an arbitrary element $s_{M,\mathrm{cl}}$ of $\mathcal{R}_{M,\mathrm{cl},\mathcal{O}_Y}(U_{M,\mathrm{cl}})$, which corresponds to $s \in \mathcal{O}_Y(U)$.

We want to show that $f^{\#}_{M,\mathrm{cl}_{U_{M,\mathrm{cl}}}} : \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_Y}(U_{\mathrm{M,cl}}) \to f_{M,\mathrm{cl}*}\mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}(U_{\mathrm{M,cl}})$ is well-defined and is a M-algebra homomorphism. To prove its well-definedness, it is enough to show that for all $x \in f^{-1}_{M,\mathrm{cl}}(U_{M,\mathrm{cl}})$, the following equation holds:

$$\Theta_{Y,U,M,f_{M,\mathrm{cl}}(x)}(s)(f_{M,\mathrm{cl}}|_{f^{-1}_{M,\mathrm{cl}}(U)}(x)) = \Theta_{X,f^{-1}(U),M,x}(f^{\#}_U(s))(x). \tag{3.1}$$

Since $f_{M,\mathrm{cl}}(x) = f_M(x)$ and f_M is a morphism of locally ringed spaces, by the universal property of the product, it follows that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_Y(U) \arrow[r, "\pi^{\sharp}_{M,Y,U}"] \arrow[d, "f^{\#}_U"] & \mathcal{O}_{Y_M}(\pi_Y^{-1}(U)) \arrow[r, "\psi_{Y,U,M,f_{M,\mathrm{cl}}(x)}"] \arrow[d, "f^{\#}_{M\pi_Y^{-1}(U)}"] & \mathcal{O}_{Y_M,f_{M,\mathrm{cl}}(x)} \arrow[r, "\phi_{Y,M,f_{M,\mathrm{cl}}(x)} \circ p_{Y,M,f_{M,\mathrm{cl}}(x)}"] \arrow[d, "\sigma_{X,Y,x}"] & M \\
\mathcal{O}_X(f^{-1}(U)) \arrow[r, "\pi^{\sharp}_{M,X,f^{-1}(U)}"'] & \mathcal{O}_{X_M}(\pi_X^{-1}(f^{-1}(U))) \arrow[r, "\psi_{X,f^{-1}(U),M,x}"'] & \mathcal{O}_{X_M,x} \arrow[ur, "\phi_{X,M,x} \circ p_{X,M,x}"'] &
\end{tikzcd}$$

where $\sigma_{X,Y,x}$ is the stalk morphism. The commutativity of this diagram precisely implies that equation 3.1 holds.

Finally, ${f^{\#}_{M,\mathrm{cl}}}_{U_{M,\mathrm{cl}}}$ is an M-algebra morphism as it is a composite of two morphisms of M-algebras, $\Theta_{X,f^{-1}(U),M,x}$ and $f^{\#}_{U}$.

Now, suppose that f is an isomorphism. It follows that f_M is also an isomorphism. Clearly, $f_{M,\mathrm{cl}}$ is also a bijection. Analogously to above, the inverse of $f_{M,\mathrm{cl}}$ is continuous. Hence, $f_{M,\mathrm{cl}}$ is an isomorphism. $\square$

In the next lemma, we give an essential description of the sheaf associated to $X_{M,\mathrm{cl}}$ when $X = \mathrm{Spec}(k[x_1, \ldots, x_n]/I)$ for some radical ideal $I \subseteq k[x_1, \ldots, x_n]$. This permits us to show that our previously constructed spaces with functions are prevarieties.

Lemma 3.2.8. *Let $X = \mathrm{Spec}(k[x_1, \ldots, x_n]/I)$ for some radical ideal $I \subseteq k[x_1, \ldots, x_n]$. For any closed point $\mathfrak{n} \in X_{M,\mathrm{cl}}$, the maximal ideal $\mathfrak{n}$ defines a point in M^n, as explained in Corollary 2.3.8, that we denote as $x_{\mathfrak{n}}$. Then, for all $U \subseteq_{\mathrm{op}} X$, $\mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}(U_{\mathrm{M,cl}})$ is the set of maps $r : U_{\mathrm{M,cl}} \to M$ such that, for all $\mathfrak{m} \in U_{M,\mathrm{cl}}$, there exists $U_{M,\mathrm{cl},\mathfrak{m}} \subseteq U_{M,\mathrm{cl}}$ with $\mathfrak{m} \in U_{M,\mathrm{cl},\mathfrak{m}}$ and there exists $g_{\mathfrak{m}}, h_{\mathfrak{m}} \in k[x_1, \ldots, x_n]/I$ such that $h_{\mathfrak{m}}(x_{\mathfrak{n}}) \neq 0$ and*

$$r(\mathfrak{n}) = \frac{g_{\mathfrak{m}}(x_{\mathfrak{n}})}{h_{\mathfrak{m}}(x_{\mathfrak{n}})}$$

for all $\mathfrak{n} \in U_{\mathrm{cl},\mathfrak{m}}$. Moreover, the map $\kappa_X : X_{M,\mathrm{cl}} \to \mathbb{V}(I)$ by sending $\mathfrak{m} \in X_{M,\mathrm{cl}}$ to $x_{\mathfrak{m}}$ is an isomorphism of classical affine varieties.

Proof. Denote $k[x_1, \ldots, x_n]/I$ as R, $M[x_1, \ldots, x_n]/I^{\mathrm{ext}}$ as R^{ext}, and $\mathrm{Spec}(R^{\mathrm{ext}})$ as X^{ext}. Consider the map $f : \mathrm{Spec}(R^{\mathrm{ext}}) \to \mathrm{Spec}(R)$ induced from the inclusion of R in R^{ext}, using Proposition A.7.12. Let $U \subseteq_{\mathrm{op}} \mathrm{Spec}(R)$ and $\mathfrak{m} \in f^{-1}(U)$ be a closed point, and hence a maximal ideal of R^{ext}. We pick $r \in \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}(U_{\mathrm{M,cl}})$, such that there exists $s \in \mathcal{O}_{\mathrm{Spec}(R)}(U)$ with $r = s_{M,\mathrm{cl}}$.

Using Proposition A.7.12 or the explicit description of the morphism correspondence in the proof of [20, Proposition 2.3 b., p. 73], there exists $V_{\mathfrak{m}} \subseteq_{\mathrm{op}} f^{-1}(U)$, and there exist $g_{\mathfrak{m}}, h_{\mathfrak{m}} \in R^{\mathrm{ext}}$ such that $h_{\mathfrak{m}} \notin \mathfrak{r}$ and

$$f^{\#}_{U}(s)(\mathfrak{r}) = \frac{g_{\mathfrak{n}}}{h_{\mathfrak{n}}}^{\mathfrak{r}},$$

for all $\mathfrak{r} \in V_{\mathfrak{m}}$. Now, consider the following morphisms:

- the canonical morphism to the stalk $\psi_{X^{\mathrm{ext}},f^{-1}(U),\mathfrak{n}} : \mathcal{O}_{X^{\mathrm{ext}}}(f^{-1}(U)) \to \mathcal{O}_{X^{\mathrm{ext}},\mathfrak{n}}$,
- the canonical quotient map $p_{X^{\mathrm{ext}},\mathfrak{n}} : \mathcal{O}_{X^{\mathrm{ext}},\mathfrak{n}} \to \mathcal{O}_{X^{\mathrm{ext}},\mathfrak{n}}/\mathfrak{m}_{X^{\mathrm{ext}},\mathfrak{n}}$ and
- the canonical morphism $\phi_{X^{\mathrm{ext}},\mathfrak{n}} : \mathcal{O}_{X^{\mathrm{ext}},\mathfrak{n}}/\mathfrak{m}_{X^{\mathrm{ext}},\mathfrak{n}} \to M$. This canonical morphism is an isomorphism, since X_M is of finite type over M, by Lemma A.7.22.

Let $\mathfrak{n} \in f^{-1}(U_{\mathfrak{m}})_{\text{cl}}$. Using Proposition A.7.3 and [17, Example 6.2, p. 59], there respectively exist canonical isomorphisms $\nu : \mathcal{O}_{R^{\text{ext}},\mathfrak{n}} \to R^{\text{ext}}_{\mathfrak{n}}$ and $\mu : R^{\text{ext}}_{\mathfrak{n}}/\mathfrak{n}R^{\text{ext}}_{\mathfrak{n}} \to M$ such that the diagram

$$\begin{tikzcd}
\mathcal{O}_{X^{\text{ext}},\mathfrak{n}} \arrow[r, "p_{X^{\text{ext}},\mathfrak{n}}"] \arrow[d, "\nu"] & \mathcal{O}_{X^{\text{ext}},\mathfrak{n}}/\mathfrak{m}_{X^{\text{ext}},\mathfrak{n}} \arrow[r, "\phi_{X^{\text{ext}},\mathfrak{n}}"] & M \\
R^{\text{ext}}_{\mathfrak{n}} \arrow[r, "\pi"] & R^{\text{ext}}_{\mathfrak{n}}/\mathfrak{n}R^{\text{ext}}_{\mathfrak{n}} \arrow[ur, "\mu"] &
\end{tikzcd}$$

commutes, where $\pi : R^{\text{ext}}_{\mathfrak{n}} \to R^{\text{ext}}_{\mathfrak{n}}/\mathfrak{n}R^{\text{ext}}_{\mathfrak{n}}$ denotes the canonical quotient map. Chasing the diagram and using Proposition A.7.12 or the explicit description of the morphism correspondence in the proof of b. of Proposition 2.3 in [20, p.73], we have that $h_{\mathfrak{m}}(x_{\mathfrak{n}}) \neq 0$ and

$$\begin{aligned}
&\phi_{X^{\text{ext}},\mathfrak{n}} \circ p_{X^{\text{ext}},\mathfrak{n}} \circ \psi_{X^{\text{ext}},f^{-1}(U),\mathfrak{n}}(f^{\sharp}_{U}(s)) \\
=&\mu \circ \pi \circ \nu \circ \psi_{X^{\text{ext}},f^{-1}(U),\mathfrak{n}}(f^{\sharp}_{U}(s)) \\
=&\frac{g_{\mathfrak{m}}(x_{\mathfrak{n}})}{h_{\mathfrak{m}}(x_{\mathfrak{n}})}
\end{aligned}$$

where $x_{\mathfrak{n}}$ is the point of R_M corresponding to the maximal ideal $\mathfrak{n}$ as explained by Proposition 2.3.1.

By [28, Example 1.15, p. 5], we have an isomorphism $\rho : R \otimes_k M \to R^{\text{ext}}$ such that

$$\begin{tikzcd}
R \arrow[r, "f^{\#}_{X}"] \arrow[dr, "\pi^{\#}_{M,X,\text{Spec}(R)}"'] & R^{\text{ext}} \arrow[d, "\rho"] \\
& R \otimes_k M
\end{tikzcd}$$

commutes so that, by Proposition A.7.12, there exists an isomorphism $g : X^{\text{ext}} \to X_M$ such that

$$\begin{tikzcd}
X & X^{\text{ext}} \arrow[l, "f"'] \arrow[d, "g"] \\
& X_M \arrow[ul, "\pi_{M,X}"]
\end{tikzcd}$$

commutes. We deduce that

$$\begin{aligned} r(\mathfrak{n}) &= s_{M,\mathrm{cl}}(\mathfrak{n}) = \Theta_{X,U,M,g(\mathfrak{n})}(s) \\ &= (\phi_{X,M,\mathfrak{n}} \circ p_{X,M,\mathfrak{n}} \circ \psi_{X,U,M,\mathfrak{n}} \circ \pi^{\#}_{X,M,U})(s) \\ &= \phi_{X^{\mathrm{ext}},\mathfrak{n}} \circ p_{X^{\mathrm{ext}},\mathfrak{n}} \circ \psi_{X^{\mathrm{ext}},f^{-1}(U),\mathfrak{n}}(f^{\sharp}_{U}(s)) \\ &= \frac{g_{\mathfrak{m}}(x_{\mathfrak{n}})}{h_{\mathfrak{m}}(x_{\mathfrak{n}})} \end{aligned}$$

Now, we want to show that κ_X is an isomorphism of prevarieties. We first show that κ_X is a homeomorphism. Corollary 2.3.7 implies that κ_X is a bijection, so that we need only show that κ_X is continuous and is an open mapping. We have, for some radical ideal $J \subseteq k[x_1, \ldots, x_n]/I$,

$$\begin{aligned} \kappa_X^{-1}(\mathbb{V}(J)) &= \{\mathfrak{m} \in X_{M,\mathrm{cl}} \mid \kappa_X(\mathfrak{m}) \in \mathbb{V}(J)\} \\ &= \{\mathfrak{m} \in X_{M,\mathrm{cl}} \mid \{\kappa_X(\mathfrak{m})\} \subseteq \mathbb{V}(J)\} \\ &= \{\mathfrak{m} \in X_{M,\mathrm{cl}} \mid J \subseteq \mathbb{I}(\kappa_X(\mathfrak{m}))\} \qquad \text{(Nullstellensatz)} \\ &= \iota_{M,X}^{-1}(\pi_{M,X}^{-1}(\mathbb{V}_{\mathrm{sch}}(J))). \end{aligned}$$

Since $\iota_{M,X}^{-1}(\pi_{M,X}^{-1}(\mathbb{V}_{\mathrm{sch}}(J)))$ is closed by definition of the topology on $X_{M,\mathrm{cl}}$, we deduce that κ_X is continuous.

Let $Y_{M,\mathrm{cl}} \subseteq_{\mathrm{cl}} X_{M,\mathrm{cl}}$. By definition of the topology on $X_{M,\mathrm{cl}}$, there exists an ideal $J \subseteq k[x_1, \ldots, x_n]/I$ such that $Y_{M,\mathrm{cl}} = \iota_{M,X}^{-1}(\pi_{M,X}^{-1}(\mathbb{V}_{\mathrm{sch}}(J)))$. Now, since κ_X is a bijection, $\kappa_X(Y_{M,\mathrm{cl}}) = \kappa_X(\kappa_X^{-1}(\mathbb{V}(J))) = \mathbb{V}(J)$. Hence, κ_X is a homeomorphism.

Let $\widetilde{\kappa_X}$ denote the inverse of κ_X. To conclude the proof, we need only show that, for all $U_{M,\mathrm{cl}} \subseteq_{\mathrm{op}} X_{M,\mathrm{cl}}$ the map $\widetilde{\kappa_X}^{\#}_{U_{M,\mathrm{cl}}} : \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}(U_{M,\mathrm{cl}}) \to \mathcal{R}_{\mathbb{V}(I)}(\widetilde{\kappa_X}^{-1}(U_{M,\mathrm{cl}}))$ given by sending $r \in \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}(U_{M,\mathrm{cl}})$ to $r \circ \widetilde{\kappa_X}|_{\widetilde{\kappa_X}^{-1}(U_{M,\mathrm{cl}})}$ is a well-defined R-algebra isomorphism. We first consider well-definedness. Let $r \in \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}(U_{M,\mathrm{cl}})$ and $\mathfrak{m} \in U_{M,\mathrm{cl}}$. From the arguments above, there exists $U_{M,\mathrm{cl},\mathfrak{m}} \subseteq U_{M,\mathrm{cl}}$ with $\mathfrak{m} \in U_{M,\mathrm{cl},\mathfrak{m}}$ and there exists $g_{\mathfrak{m}}, h_{\mathfrak{m}} \in k[x_1, \ldots, x_n]/I$ such that $h_{\mathfrak{m}}(x_{\mathfrak{n}}) \neq 0$ and

$$r(\mathfrak{n}) = \frac{g_{\mathfrak{m}}(x_{\mathfrak{n}})}{h_{\mathfrak{m}}(x_{\mathfrak{n}})}$$

for all $\mathfrak{n} \in U_{\mathrm{cl},\mathfrak{m}}$.

Since κ_X is a homeomorphism, any open set in $\mathbb{V}(I)$ is of the form $\kappa_X(U_{M,\mathrm{cl}})$ for some $U \subseteq_{\mathrm{op}} X$ and, for any point $b \in \kappa_X(U_{M,\mathrm{cl}})$, there exists an $\mathfrak{n} \in U_{M,\mathrm{cl}}$ such that $\kappa_X(\mathfrak{n}) = x_{\mathfrak{n}} = b$. Let $U_{M,\mathrm{cl}} \subseteq_{\mathrm{op}} X_{M,\mathrm{cl}}$ and consider the open set $\kappa_X(U_{M,\mathrm{cl}}) \subseteq \mathbb{V}(I)$ and some $a \in \kappa_X(U_{M,\mathrm{cl}})$. We have $\kappa_X(U_{M,\mathrm{cl},\widetilde{\kappa_X}(a)}) \subseteq_{\mathrm{op}}$

$\kappa_X(U_{M,\mathrm{cl}})$ with $a \in \kappa_X(U_{M,\mathrm{cl},\widetilde{\kappa_X}(a)})$ and $h_{\widetilde{\kappa_X}(a)}(b) \neq 0$ and

$$r(\widetilde{\kappa_X}|_{\widetilde{\kappa_X}^{-1}(U_{\mathrm{M,cl}})}(b)) = r(\mathfrak{n}) = \frac{g_{\widetilde{\kappa_X}(a)}(b)}{h_{\widetilde{\kappa_X}(a)}(b)}$$

for all $b \in \kappa_X(U_{M,\mathrm{cl},\widetilde{\kappa_X}(a)})$, where $\mathfrak{n} = \widetilde{\kappa_X}(b)$. We deduce that $s \circ \widetilde{\kappa_X}|_{\widetilde{\kappa_X}^{-1}(U_{M,\mathrm{cl}})} \in \mathcal{R}_{\mathbb{V}(I)}(\widetilde{\kappa_X}^{-1}(U_{M,\mathrm{cl}}))$. It is also easily shown that $\widetilde{\kappa_X}^{\#}_{U_{M,\mathrm{cl}}}$ is a k-algebra homomorphism. Similarly, we can prove that, for any open $U \subseteq \mathbb{V}(I)$, that $\kappa^{\#}_{XU} : \mathcal{R}_{\mathbb{V}(I)}(U) \to \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}(\kappa_X^{-1}(U))$ is a well-defined k-algebra morphism. □

We can now give the main result of this section which confirms that our constructed spaces with functions are indeed prevarieties.

Proposition 3.2.9. *Let $(X, \mathcal{O}_X)$ be a variety over k. Then $(X_{M,\mathrm{cl}}, \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X})$ is a prevariety over k in M.*

Proof. Since $(X, \mathcal{O}_X)$ has a finite open covering by affine schemes of finite type over k, by definition of a scheme of finite type, it suffices to show that if $(X, \mathcal{O}_X)$ is an affine variety over k, then $(X_{M,\mathrm{cl}}, \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X})$ is a classical affine variety over k in M. By Lemma 3.2.7, we may assume that $X = \mathrm{Spec}(k[x_1, \ldots, x_n]/I)$ for some $n \in \mathbb{N}$ and a radical ideal $I \subseteq k[x_1, \ldots, x_n]$. The result then follows from Lemma 3.2.8. □

As in the first section of this chapter, we have, again, laid the groundwork to construct a functor between varieties, in the scheme sense, and prevarities. The next definition and subsequent lemma put the pieces together.

Definition 3.2.10. *We define the maps $\mathcal{V}^{\mathrm{clsc}}_{k,M,0} : \mathbf{Var}_{k,0} \to \mathbf{Prevar}_{k,M,0}$ and $\mathcal{V}^{\mathrm{clsc}}_{k,M,1} : \mathbf{Var}_{k,1} \to \mathbf{Prev}_{k,M,1}$ such that:*

- *Given a variety $(X, \mathcal{O}_X)$ over k, $\mathcal{V}^{\mathrm{clsc}}_{k,M,0}$ sends $(X, \mathcal{O}_X)$ to $(X_{M,\mathrm{cl}}, \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X})$.*
- *Given any morphism $(f : X \to Y, f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X)$ of varieties over k, $\mathcal{V}^{\mathrm{clsc}}_{k,1}$ sends $(f, f^{\#})$ to the morphism $f_{M,\mathrm{cl}} : X_{M,\mathrm{cl}} \to Y_{M,\mathrm{cl}}$ of prevarieties over k in M.*

According to Lemma 3.2.8, we can define $\mathcal{A}^{\mathrm{clsc}}_{k,M}$ to be the restriction and corestriction of $\mathcal{V}^{\mathrm{clsc}}_{k,M}$ to **AVar**$_k$ *and* **CAVar**$_{k,M}$*, respectively.*

Lemma 3.2.11. *The maps $\mathcal{V}^{\mathrm{clsc}}_{k,M}$ and $\mathcal{A}^{\mathrm{clsc}}_{k,M}$ are functors.*

Proof. By Lemma 3.2.9 and Lemma 3.2.7, $\mathcal{A}^{\mathrm{clsc}}_{k}$ is well-defined. Clearly, we have that $\mathcal{V}^{\mathrm{clsc}}_{k}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{V}^{\mathrm{clsc}}_{k}(X)}$ for any $X \in \mathbf{Var}_k$. As restrictions of maps respect composition, it also follows, given any morphisms $f : X \to Y$ and $g : Y \to Z$ for some schemes X, Y, Z, that $\mathcal{V}^{\mathrm{clsc}}_{k,M}(g \circ f) = \mathcal{V}^{\mathrm{clsc}}_{k,M}(g) \circ \mathcal{V}^{\mathrm{clsc}}_{k,M}(f)$. Finally, as a consequence of Lemma 3.2.9 and Lemma 3.2.7, $\mathcal{A}^{\mathrm{clsc}}_{k,M}$ is also a functor. □

3.3 An equivalence of categories between prevarieties and varieties as schemes

Using work contributed by S. Marques, we are now able to show, using our previously constructed functors, that we indeed have an equivalence of categories. Notably, in this section, the general construction using glueing of Definition-Lemma 3.3.3 and Proposition 3.3.8 were contributed by D.J. Smith.

The next definition allows us to associate morphisms between the spectra of certain finitely generated k-algebras to morphisms of classical affine varieties.

Definition 3.3.1. *Let* $X = \mathrm{Spec}(k[x_1, \cdots, x_n]/I)$, $Y = \mathrm{Spec}(k[y_1, \cdots, y_m]/J)$ *and let* $f : X \to Y$ *be a morphism of* k*-schemes. We define the map* $\mathbb{V}(f) : \mathbb{V}(I) \to \mathbb{V}(J)$ *to be* $\kappa_Y \circ f_{M,\mathrm{cl}} \circ \kappa_X^{-1}$

We also show that any morphism between algebraic sets corresponds to a morphism between the spectra of certain certain finitely generated k-algebras.

Lemma 3.3.2. *Let* $I \subseteq k[x_1, \cdots, x_n]$ *be an ideal,* $J \subseteq k[x_1, \cdots, x_m]$ *an ideal and let* $v : \mathbb{V}(I) \to \mathbb{V}(J)$ *be a morphism of prevarieties. We have* $\mathbb{V}(f) = v$ *where* $f = \gamma_{\mathbb{V}(J)} \circ v_{\mathrm{sch}} \circ \gamma_{\mathbb{V}(I)}^{-1}$ *and* f *is a scheme morphism.*

Proof. We have that f is a morphism of k-schemes, as it is the composite of morphisms of k-schemes, as shown in Lemma 3.1.7 and Lemma 3.1.5. Let $a \in \mathbb{V}(I)$. We then have

$$\begin{aligned}
\mathbb{V}(f)(a) &= (\kappa_Y \circ f_{M,\mathrm{cl}} \circ \kappa_X^{-1})(a) \\
&= (\kappa_Y \circ \gamma_{\mathbb{V}(J)}{}_{M,\mathrm{cl}} \circ v_{\mathrm{sch}}{}_{M,\mathrm{cl}} \circ \gamma_{\mathbb{V}(I)}^{-1}{}_{M,\mathrm{cl}} \circ \kappa_X^{-1})(a) \\
&= (\kappa_Y \circ \gamma_{\mathbb{V}(J)}{}_{M,\mathrm{cl}} \circ v_{\mathrm{sch}}{}_{M,\mathrm{cl}})(\{a\}) \\
&= (\kappa_Y \circ \gamma_{\mathbb{V}(J)}{}_{M,\mathrm{cl}})(\{v(a)\}) \\
&= v(a)
\end{aligned}$$

□

The following definition constructs a canonical natural correspondence from algebraic sets to prevarieties which will allow us to establish the equivalence of categories.

Definition-Lemma 3.3.3. *We define* η *as the family* $(\eta_X)_{X \in \mathbf{CVar}_{k,M}}$*, where* η_X *can be constructed following the steps below:*

1. *For an algebraic set* X, $\eta_X = \kappa_{\mathrm{Spec}(k[X])} \circ \gamma_{X\,M,cl}$*, with* $\kappa_{\mathrm{Spec}(k[X])}$ *defined as in Lemma 3.2.8 and* γ_X *defined in Lemma 3.1.7.*

2. *For a classical affine variety* X *and an isomorphism of varieties* $\phi : X \to V$*, where* V *is an algebraic set,* $\eta_X = \phi^{-1} \circ \eta_V \circ \phi$.

3. *For a prevariety* X *and the open covering* $\{X_i\}_{i\in I}$ *of* X *by all classical open affine varieties, we define* η_X *as the composite morphism obtained by gluing together the morphisms* $\iota_{X_i,X} \circ \eta_{X_i} \circ {\Xi_{X_i,X}}_{M,\mathrm{cl}}$*, where* $\Xi_{X_i,X}$ *is defined in Lemma 3.1.6, for all* $i \in I$*.*

Then, we have the following properties:

1. η_X *does not depend on the choice of the isomorphism* ϕ*.*

2. η *forms a natural correspondence from* $\mathcal{V}^{\mathrm{clsc}}_{k,M} \circ \mathcal{V}^{\mathrm{sch}}_{k,M}$ *to* $\mathrm{id}_{\mathbf{CVar}_{k,M}}$*.*

Proof. To simplify the notation, we set $\mathscr{A} = \mathcal{A}^{\mathrm{clsc}}_{k,M} \circ \mathcal{A}^{\mathrm{sch}}_{k,M}$ and $\mathscr{V} = \mathcal{V}^{\mathrm{clsc}}_{k,M} \circ \mathcal{V}^{\mathrm{sch}}_{k,M}$. For all algebraic sets X, as η_X is a composition of morphisms of prevarieties given in Lemma 3.2.8 and Lemma 3.1.7, it is a morphism of prevarieties. To prove naturality over algebraic sets, let X and Y be algebraic sets and let $f : Y \to X$ be a morphism of prevarieties. We need to show that the diagram

$$\begin{tikzcd}
t(Y)_{M,\mathrm{cl}} \arrow[r, "\eta_Y"] \arrow[d, "\mathscr{A}(f)"] & Y \arrow[d, "f"] \\
t(X)_{M,\mathrm{cl}} \arrow[r, "\eta_X"] & X
\end{tikzcd}$$

commutes. This follows directly from Lemma 3.3.2. We now prove that, when X is a classical affine variety, the definition of η_X in the lemma does not depend on the choice of an isomorphism to some algebraic set. Suppose that there are two algebraic sets V and W such that there exist isomorphisms $\phi : V \to X$ and $\psi : W \to X$. We set $\eta_X = \phi^{-1} \circ \eta_V \circ \phi$. To prove that $\eta_X = \psi^{-1} \circ \eta_W \circ \psi$, consider the following diagram

$$\begin{tikzcd}
t(X)_{M,\mathrm{cl}} \arrow[r, "\eta_X"] \arrow[d, "\mathscr{A}(\phi)"] & X \arrow[d, "\phi"] \\
t(V)_{M,\mathrm{cl}} \arrow[r, "\eta_V"] \arrow[d, "\mathscr{A}(\psi\circ\phi^{-1})"] & V \arrow[d, "\psi\circ\phi^{-1}"] \\
t(W)_{M,\mathrm{cl}} \arrow[r, "\eta_W"] & W
\end{tikzcd}$$

From the naturality of η on algebraic sets proven earlier, the bottom square commutes. From our initial assumption, the top square also commutes. Hence, the outer diagram commutes, which shows that η_X does not depend on the choice of an isomorphism.

We now conclude the proof by proving the naturality of η over $\mathbf{CVar}_{k,M}$. Let X and Y be two classical affine varieties, $f : X \to Y$ be a morphism of classical varieties, and V and W be two algebraic sets such that there exist isomorphisms of classical varieties $\phi : X \to V$ and $\psi : Y \to W$. The naturality follows directly from the commutativity of the following diagram:

$$\begin{tikzcd}
t(X) \arrow[r, "\eta_X"] \arrow[d, "\mathscr{A}(\phi)"'] & X \arrow[d, "\phi"] \\
t(V)_{M,\mathrm{cl}} \arrow[r, "\eta_V"] \arrow[d, "\mathscr{A}(\psi \circ f \circ \phi^{-1})"'] & V \arrow[d, "\psi \circ f \circ \phi^{-1}"] \\
t(W)_{M,\mathrm{cl}} \arrow[r, "\eta_W"] & W \\
t(Y)_{M,\mathrm{cl}} \arrow[r, "\eta_Y"] \arrow[u, "\mathscr{A}(\psi)"] & Y \arrow[u, "\psi"']
\end{tikzcd}$$

Indeed, the top and bottom squares commute by definition of η_X and η_Y, respectively. The middle square commutes since we have proven naturality on algebraic sets. Therefore, the outer diagram commutes as desired.

We now consider the case when X is an arbitrary prevariety. Let $\{X_i\}_{i \in I}$ be the open covering of X consisting of all affine open sets. By Proposition A.7.10, there exists an open covering $\{V_{i,j,k}\}_{k \in J_{i,j}}$ for $X_i \cap X_j$ for all $i, j \in I$. Denote, for any classical affine variety X, η_X^{-1} as ξ_X. As shown above, for all $i, j \in I$ and $k \in J_{i,j}$ the diagram

$$\begin{tikzcd}
X_j \arrow[r, "\xi_{X_j}"] & t(X_j)_{M,\mathrm{cl}} \\
V_{i,j,k} \arrow[r, "\xi_{V_{i,j,k}}"] \arrow[u, "\iota_{V_{i,j,k},X_j}"] \arrow[d, "\iota_{V_{i,j,k},X_i}"'] & t(V_{i,j,k})_{M,\mathrm{cl}} \arrow[u, "\mathscr{A}(\iota_{V_{i,j,k},X_j})"'] \arrow[d, "\mathscr{A}(\iota_{V_{i,j,k},X_i})"] \\
X_i \arrow[r, "\xi_{X_i}"] & t(X_i)_{M,\mathrm{cl}}
\end{tikzcd}$$

commutes for all $i, j \in I$ and $k \in J_{i,j}$. As inclusions commutes, the following diagram

$$\begin{tikzcd}
X_j \arrow[r, "\xi_{X_j}"] & t(X_j)_{M,\mathrm{cl}} \arrow[dr, "\mathscr{V}(\iota_{X_j,X})"] & \\
V_{i,j,k} \arrow[u, "\iota_{V_{i,j,k},X_j}"] \arrow[d, "\iota_{V_{i,j,k},X_i}"'] \arrow[r, "\xi_{V_{i,j,k}}"] & t(V_{i,j,k})_{M,\mathrm{cl}} \arrow[u, "\mathscr{A}(\iota_{V_{i,j,k},X_j})"'] \arrow[d, "\mathscr{A}(\iota_{V_{i,j,k},X_i})"] \arrow[r, "\mathscr{V}(\iota_{V_{i,j,k},X})"] & t(X)_{M,\mathrm{cl}} \\
X_i \arrow[r, "\xi_{X_i}"] & t(X_i)_{M,\mathrm{cl}} \arrow[ur, "\mathscr{V}(\iota_{X_i,X})"'] &
\end{tikzcd} \tag{3.2}$$

commutes for all for all $i, j \in I$ and $k \in J_{i,j}$. Now, let

- $\widetilde{\xi_{V_{i,j,k}}} = \mathscr{V}(\iota_{V_{i,j,k},X}) \circ \xi_{V_{i,j,k}}$ and
- $\widetilde{\xi_{X_i}} = \mathscr{V}(\iota_{X_i,X}) \circ \xi_{X_i}$

for all $i, j \in I$ and $k \in J_{i,j}$. We would like to glue the $\widetilde{\xi_{X_i}}$ to give a morphism ξ_X. To do so, we need to check that $\widetilde{\xi_{X_i}}|_{X_i \cap X_j} = \widetilde{\xi_{X_j}}|_{X_i \cap X_j}$ for all $i, j \in I$. Using (3.2), we have that

$$\widetilde{\xi_{X_i}}|_{X_i \cap X_j}|_{V_{i,j,k}} = \widetilde{\xi_{X_i}}|_{V_{i,j,k}} = \widetilde{\xi_{V_{i,j,k}}} = \widetilde{\xi_{X_j}}|_{V_{i,j,k}} = \widetilde{\xi_{X_j}}|_{X_i \cap X_j}|_{V_{i,j,k}}$$

for all $i, j \in I$ and $k \in J_{i,j}$. Since for all $i, j \in I$, $\{V_{i,j,k}\}_{k \in J_{i,j}}$ is open covering of $X_i \cap X_j$, it follows from Proposition A.6.6 that $\widetilde{\xi_{X_i}}|_{X_i \cap X_j} = \widetilde{\xi_{X_j}}|_{X_i \cap X_j}$. Again, by Proposition A.6.6, there exists a unique locally ringed space morphism $\xi_X : X \to t(X)_{M,\mathrm{cl}}$ such that $\xi_X|_{X_i} = \widetilde{\xi_{X_i}}$ for all $i \in I$. We need to check that ξ_X is a morphism of spaces with functions. To that end, we need only check, for any $U \subseteq_{\mathrm{op}} t(X)_{M,\mathrm{cl}}$ and $s \in \mathcal{R}_{M,\mathrm{cl},\alpha_{X*}\mathcal{R}_X}(U)$, that $s \circ \xi_X|_{\xi_X^{-1}(U)} \in \mathcal{R}_X(\xi_X^{-1}(U))$. We have that

$$\begin{aligned}
(s \circ \xi_X|_{\xi_X^{-1}(U)})|_{\xi_X^{-1}(U \cap X_i)} &= (s \circ \xi_X)|_{\xi_X^{-1}(U \cap X_i)} \\
&= (s \circ \xi_{X_i})|_{\xi_X^{-1}(U \cap X_i)} \in \mathcal{R}_X(U \cap X_i)
\end{aligned}$$

for all $i \in I$. However, by the glueing axiom of sheaves, there exists a unique map $r \in \mathcal{R}_X(\xi_X^{-1}(U))$ such that $r|_{\xi_X^{-1}(U \cap X_i)} = (s \circ \xi_{X_i})|_{\xi_X^{-1}(U \cap X_i)}$ for all $i \in I$. As $(s \circ \xi_{X_i})|_{\xi_X^{-1}(U \cap X_i)}$ are also maps to M and $\{\xi_X^{-1}(U \cap X_i)\}_{i \in I}$ covers $\xi_X^{-1}(U)$, we deduce that r and $s \circ \xi_X|_{\xi_X^{-1}(U)}$ agree point-wise so that $r = s \circ \xi_X|_{\xi_X^{-1}(U)}$. Thus, ξ_X is a morphism of spaces with functions.

By Lemma 3.1.2 and Lemma 3.2.3, as $\{X_i\}_{i\in I}$ is an open covering of X, $\{(t(X)\backslash t(X\backslash X_i))_{M,\mathrm{cl}}\}_{i\in I}$ is an open covering for $t(X)_{M,\mathrm{cl}}$. We note that, for all $i, j \in I$,

$$\begin{aligned}
&(t(X)\backslash t(X\backslash X_i))_{M,\mathrm{cl}} \cap (t(X)\backslash t(X\backslash X_j))_{M,\mathrm{cl}} \\
={}& (t(X)\backslash t(X\backslash X_i) \cap t(X)\backslash t(X\backslash X_j))_{M,\mathrm{cl}} && \text{(Lemma 3.2.3)} \\
={}& (t(X)\backslash t(X\backslash X_i \cap X_j))_{M,\mathrm{cl}} && \text{(Lemma 3.1.2).}
\end{aligned}$$

Furthermore, we have that $\{(t(X)\backslash t(X\backslash V_{i,j,k}))_{M,\mathrm{cl}}\}_{k\in J_{i,j}}$ is an open covering of $(t(X)\backslash t(X\backslash X_i \cap X_j))_{M,\mathrm{cl}}$ for all $i, j \in I$. Using similar arguments as above, and that the diagram

$$\begin{tikzcd}
 & X_j \arrow[ld, "\iota_{X_j,X}"'] & (t(X)\backslash t(X\backslash X_j))_{M,\mathrm{cl}} \arrow[l, "\eta_{X_j}\circ\Xi_{X_j,X_{M,\mathrm{cl}}}"'] \\
X & V_{i,j,k} \arrow[l, "\iota_{V_{i,j,k},X}"'] \arrow[u, "\iota_{V_{i,j,k},X_j}"'] \arrow[d, "\iota_{V_{i,j,k},X_i}"] & (t(X)\backslash t(X\backslash V_{i,j,k}))_{M,\mathrm{cl}} \arrow[l, "\eta_{V_{i,j,k}}\circ\Xi_{V_{i,j,k},X_{M,\mathrm{cl}}}"'] \arrow[u] \arrow[d] \\
 & X_i \arrow[lu, "\iota_{X_i,X}"] & (t(X)\backslash t(X\backslash X_i))_{M,\mathrm{cl}}, \arrow[l, "\eta_{X_i}\circ\Xi_{X_i,X_{M,\mathrm{cl}}}"']
\end{tikzcd}$$

commutes, it follows that there exists a morphism $\eta_X : t(X)_{M,\mathrm{cl}} \to X$ such that $\eta_X|_{(t(X)\backslash t(X\backslash X_i))_{M,\mathrm{cl}}} = \iota_{X_i,X} \circ \eta_{X_i} \circ \Xi_{X_i,X_{M,\mathrm{cl}}}$ for all $i \in I$. To show that η_X and ξ_X are inverse, as they are morphisms of prevarieties, we need only show that they are inverse of each other as maps. Let $x \in t(X)_{M,\mathrm{cl}}$. There exists an $i \in I$ such that $x \in (t(X)\backslash t(X\backslash X_i))_{M,\mathrm{cl}}$ and so $\eta_X(x) \in X_i$, so that

$$\begin{aligned}
\xi_X(\eta_X(x)) &= \xi_X(\eta_X|_{(t(X)\backslash t(X\backslash X_i))_{M,\mathrm{cl}}}(x)) \\
&= \xi_X(\iota_{X_i,X} \circ \eta_{X_i} \circ \Xi_{X_i,X_{M,\mathrm{cl}}}(x)) \\
&= \mathscr{V}(\iota_{X_i,X}) \circ \xi_{X_i}(\eta_{X_i}(\Xi_{X_i,X_{M,\mathrm{cl}}}(x))) \\
&= \mathscr{V}(\iota_{X_i,X})(\Xi_{X_i,X_{M,\mathrm{cl}}}(x)) \\
&= x.
\end{aligned}$$

Let $x \in X$ so that there exists $i \in I$ such that $x \in X_i$. Then $\xi_X(x) \in$

$(t(X)\backslash t(X\backslash X_i))_{M,\mathrm{cl}}$, so that

$$\begin{aligned}\eta_X(\xi(x)) &= \eta_X(\xi_X|_{X_i}(x))\\ &= \eta_X(\mathscr{V}(\iota_{X_i,X})\circ\xi_{X_i}(x))\\ &= (\iota_{X_i,X}\circ\eta_{X_i}\circ\Xi_{X_i,X\,M,\mathrm{cl}}\circ\mathscr{V}(\iota_{X_i,X})\circ\xi_{X_i})(x)\\ &= (\iota_{X_i,X}\circ\eta_{X_i}\circ\xi_{X_i})(x)\\ &= \iota_{X_i,X}(x)\\ &= x\end{aligned}$$

We deduce that $\xi_X^{-1}=\eta_X$ so that η_X is an isomorphism. Let Y be a variety over k and let $f:X\to Y$ be a morphism. We now need to show that the diagram

$$\begin{tikzcd} X \arrow[r, "\xi_X"] \arrow[d, "f"] & t(X)_{M,\mathrm{cl}} \arrow[d, "\mathscr{V}(f)"] \\ Y \arrow[r, "\xi_Y"] & t(Y)_{M,\mathrm{cl}} \end{tikzcd}$$

commutes. Now, let $\{Y_j\}_{j\in J}$ be the open covering of all affine open varieties of Y and let $\{X_{i,j}\}_{i\in I_j}$ be an open covering of $f^{-1}(Y_j)$ for all $j\in J$. By our constructions above, we have that

$$\begin{tikzcd} X \arrow[r, "\xi_X"] & t(X)_{M,\mathrm{cl}} \\ X_{i,j} \arrow[u, "\iota_{X_{i,j},X}"] \arrow[r, "\xi_{X_{i,j}}"] & t(X_{i,j})_{M,\mathrm{cl}} \arrow[u, "\mathscr{V}(\iota_{X_{i,j},X})"'] \end{tikzcd} \tag{3.3}$$

commutes for all $j\in J$ and $i\in I_j$. Similarly, we have that

$$\begin{tikzcd} Y \arrow[r, "\xi_Y"] & t(Y)_{M,\mathrm{cl}} \\ Y_j \arrow[u, "\iota_{Y_j,Y}"] \arrow[r, "\xi_{Y_j}"] & t(Y_j)_{M,\mathrm{cl}} \arrow[u, "(\mathscr{V}(\iota_{Y_j,Y})"'] \end{tikzcd} \tag{3.4}$$

commutes for all $j\in J$. Let $f_{i,j}$ denote the restriction of f to $X_{i,j}$ and corestriction to Y_j. As shown above, the following diagram

$$\begin{tikzcd}
X_{i,j} \arrow[r, "\xi_{X_{i,j}}"] \arrow[d, "f_{i,j}"] & t(X_{i,j})_{M,\mathrm{cl}} \arrow[d, "\mathscr{V}(f_{i,j})"] \\
Y_j \arrow[r, "\xi_{Y_j}"] & t(Y_j)_{M,\mathrm{cl}}
\end{tikzcd} \tag{3.5}$$

also commutes. Combining (3.3), (3.4) and (3.5) with the fact that $\mathscr{V}$ is a functor, we have the commutative diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[rrr, bend left, "\xi_X"] & X_{i,j} \arrow[l, "\iota_{X_i,X}"'] \arrow[r, "\xi_{X_{i,j}}"'] \arrow[d, "f_{i,j}"'] & t(X_{i,j})_{M,\mathrm{cl}} \arrow[r, "{\Xi_{X_{i,j},X}}_{M,\mathrm{cl}}"'] \arrow[d, "\mathscr{V}(f_{i,j})"'] & t(X)_{M,\mathrm{cl}} \arrow[d, "\mathscr{V}(f)"'] \\
Y \arrow[rrr, bend right, "\xi_Y"'] & Y_j \arrow[l, "\iota_{Y_j,Y}"'] \arrow[r, "\xi_{Y_j}"'] & t(Y_j)_{M,\mathrm{cl}} \arrow[r, "{\Xi_{Y_j,Y}}_{M,\mathrm{cl}}"'] & t(Y)_{M,\mathrm{cl}}
\end{tikzcd} \tag{3.6}$$

for all $i \in I_j$ and all $j \in J$. Using (3.6), we have

$$\begin{aligned}
(\mathscr{V}(f) \circ \xi_X)|_{X_{i,j}} &= \mathscr{V}(f) \circ \xi_X|_{X_{i,j}} \\
&= \mathscr{V}(f) \circ {\Xi_{X_{i,j},X}}_{M,\mathrm{cl}} \circ \xi_{X_{i,j}} \\
&= {\Xi_{Y_j,Y}}_{M,\mathrm{cl}} \circ \xi_{Y_j} \circ f_{i,j} \\
&= \xi_Y \circ \iota_{Y_j,Y} \circ f_{i,j} \\
&= \xi_Y \circ f|_{X_{i,j}} \\
&= (\xi_Y \circ f)|_{X_{i,j}}
\end{aligned}$$

for all $i \in I_j$ and all $j \in J$. Since $\{X_{i,j}\}_{i \in J_i, j \in I}$ is an open covering for X, it follows from the uniqueness described in Proposition A.6.6 that

$$\mathscr{V}(f) \circ \xi_X = \xi_Y \circ f.$$

□

Similarly, we associate a canonical morphism between the spectra of certain finitely generated k-algebras to a morphism between algebraic sets as follows.

Definition 3.3.4. *Let I be an ideal of $k[x_1, \cdots, x_n]$, let J be an ideal of $k[y_1, \cdots, y_m]$ and let $v : \mathbb{V}(I) \to \mathbb{V}(J)$ be a classical variety morphism. We denote $X = \operatorname{Spec}(k[x_1, \cdots, x_n]/I)$ and $Y = \operatorname{Spec}(k[y_1, \cdots, y_m]/J)$. We define the map $\mathbb{S}(v) : X \to Y$ to be the composition $\gamma_{\mathbb{V}(J)} \circ v_{\mathrm{sch}} \circ \gamma_{\mathbb{V}(I)}^{-1}$.*

We now show that any morphism between the spectra of certain finitely generated k-algebras corresponds to a morphism between their corresponding algebraic sets.

Lemma 3.3.5. *Let $X = \operatorname{Spec}(k[x_1, \cdots, x_n]/I)$, $Y = \operatorname{Spec}(k[y_1, \cdots, y_m]/J)$ and $f : X \to Y$ be a morphism of schemes. We have $\mathbb{S}(v) = f$, where $v = \kappa_Y \circ f_{M,\mathrm{cl}} \circ \kappa_X^{-1}$ is a morphism of prevarieties from $\mathbb{V}(I)$ to $\mathbb{V}(J)$.*

Proof. By [28, Lemma 3.23, p. 48], it suffices to show that the following diagram

$$\begin{tikzcd}
\mathcal{O}_Y(Y) \arrow[r, "f_Y^\sharp"] \arrow[d, "\gamma_{\mathbb{V}(J)}^\#"'] & \mathcal{O}_X(X) \arrow[d, "\gamma_{\mathbb{V}(I)}^\#"'] \\
\mathcal{R}_{\mathbb{V}(J)}(\mathbb{V}(J)) \arrow[r, "-\circ\kappa_Y\circ f_{M,\mathrm{cl}}\circ\kappa_X^{-1}"'] & \mathcal{R}_{\mathbb{V}(I)}(\mathbb{V}(I))
\end{tikzcd}$$

commutes. We first introduce some notation where

- $R_I = k[x_1, \cdots, x_n]/I$ and $R_J = k[y_1, \cdots, y_m]/J$,
- $\Psi_{X,\mathfrak{p}}$ the canonical isomorphism from $\mathcal{O}_{X,\mathfrak{p}}$ to $R_{I\mathfrak{p}}$, for all $\mathfrak{p} \in X$,
- $\Psi_{Y,\mathfrak{q}}$ the canonical isomorphism from $\mathcal{O}_{Y,\mathfrak{q}}$ to $R_{J\mathfrak{q}}$, for all $\mathfrak{q} \in Y$,
- $\mathrm{ev}_{X,a}$ to be the evaluation map from $R_{I\mathfrak{p}}$ to M sending $\frac{g}{h}_{\mathfrak{p}}$ to $\frac{g(a)}{h(a)}$ for any $a \in M^n$ such that $h(a) \neq 0$, for all $\mathfrak{p} \in X$,
- $\mathrm{ev}_{Y,a}$ to be the evaluation map from $R_{J\mathfrak{q}}$ to M sending $\frac{g}{h}_{\mathfrak{q}}$ to $\frac{g(a)}{h(a)}$ for any $a \in M^n$ such that $h(a) \neq 0$, for all $\mathfrak{q} \in Y$ and
- $f_{\mathfrak{p}} : \mathcal{O}_{Y,f(\mathfrak{p})} \to \mathcal{O}_{X,\mathfrak{p}}$ to be the canonical stalk map induced by f.

Let $s \in \mathcal{O}_Y(Y)$. Let $b = (b_1, \ldots, b_n) \in \mathbb{V}(I)$ and recall, from Proposition 2.3.8, that we use $\mathfrak{m}_b$ to denote $\langle b_1 \otimes_k 1, \ldots, b_n \otimes_k 1\rangle$. Since the diagram

$$\begin{tikzcd}
X_M \arrow[r, "f_M"] \arrow[d, "\pi_{M,X}"] & Y_M \arrow[d, "\pi_{M,Y}"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

commutes, we have that

$$\begin{aligned} f(\mathfrak{m}_b \cap R_I) &= f(\pi_{M,X}(\mathfrak{m}_b)) \\ &= \pi_{M,X}(f_M(\mathfrak{m}_b)) \\ &= f_M(\mathfrak{m}_b) \cap R_J. \end{aligned}$$

For any $a \in \mathbb{V}(I)$, we then have

$$\begin{aligned} \gamma^{\#}_{\mathbb{V}(I)}(f^{\sharp}_Y(s))(a) =& \mathrm{ev}_{X,a}(\Psi_{X,\mathfrak{m}_a \cap R_J}(f^{\sharp}_Y(s)(\mathfrak{m}_a \cap R_I))) \\ =& \mathrm{ev}_{X,a}(\Psi_{X,\mathfrak{m}_a \cap R_J}(f_{\mathfrak{m}_a \cap R_J}(s(f_M(\mathfrak{m}_a) \cap R_J))) \end{aligned}$$

and we also have that

$$\begin{aligned} (\gamma^{\#}_{\mathbb{V}(J)}(s) \circ \kappa_Y \circ f_{M,\mathrm{cl}} \circ \kappa_X^{-1})(a) =& \gamma^{\#}_{\mathbb{V}(J)}(s)(x_{f_M(\mathfrak{m}_a)}) \\ =& \mathrm{ev}_{Y,x_{f_M(\mathfrak{m}_a)}}(\Psi_{Y,f_M(\mathfrak{m}_a) \cap R_J}(s(f_M(\mathfrak{m}_a) \cap R_J))). \end{aligned}$$

The equality $\gamma^{\#}_{\mathbb{V}(I)}(f^{\sharp}_Y(s))(a) = (\gamma^{\#}_{\mathbb{V}(J)}(s) \circ \kappa_Y \circ f_{M,\mathrm{cl}} \circ \kappa_X^{-1})(a)$ follows the commutativity of diagram

$$\begin{tikzcd}
\mathcal{O}_{Y,f_M(\mathfrak{m}_a) \cap R_J} \arrow[r, "\Psi_{Y,f_M(\mathfrak{m}_a) \cap R_J}"] \arrow[d, "f_{X,\mathfrak{m}_a \cap R_I}"'] & {R_J}_{f_M(\mathfrak{m}_a) \cap R_J} \arrow[r, "\mathrm{ev}_{Y,x_{f_M(\mathfrak{m}_a)}}"] \arrow[d] & M \\
\mathcal{O}_{X,\mathfrak{m}_a \cap R_I} \arrow[r, "\Psi_{X,\mathfrak{m}_a \cap R_I}"] & {R_I}_{\mathfrak{m}_a \cap R_I} \arrow[ur, "\mathrm{ev}_{X,a}"'] &
\end{tikzcd}$$

(see also [20, Proposition 2.3, p. 73]). This completes the proof. □

We are finally in a position to construct a natural correspondence from $\mathcal{V}^{\mathrm{sch}}_{k,M} \circ \mathcal{V}^{\mathrm{clsc}}_{k,M}$ to $\mathrm{id}_{\mathbf{Var}_k}$, the final component for a fully explicit equivalence of categories. We omit the proof as it is very similar that of Definition-Lemma 3.3.3.

Definition-Lemma 3.3.6. *We define μ as the family $(\mu_X)_{X \in \mathbf{Var}_k}$, where μ_X can be constructed following the steps below:*

1. *When $X = \mathrm{Spec}(k[x_1, \cdots, x_n])/I$, $\mu_X = \gamma_{\mathbb{V}(I)} \circ \kappa_{X\,sch}$, with κ_X defined as in Lemma 3.2.8 and $\gamma_{\mathbb{V}(I)}$ defined in Lemma 3.1.7.*

2. *For affine variety and an isomorphism of varieties $\phi : X \to V$, where V is an algebraic set, $\mu_X = \phi^{-1} \circ \mu_V \circ \phi$.*

3. *For a variety X and the covering $\{X_i\}_{i \in I}$ of X by all open affine varieties, we define μ_X as the composite morphism obtained by gluing together the morphisms $\iota_{X_i,X} \circ \mu_{X_i} \circ \Xi_{X_{iM,\mathrm{cl}},X}$, where $\Xi_{X_{iM,\mathrm{cl}},X}$ is defined in Lemma 3.1.6, for all $i \in I$.*

Then, we have the following properties:

1. *μ_X does not depend on the choice of the isomorphism ϕ.*
2. *μ forms a natural correspondence from $\mathcal{V}^{\mathrm{sch}}_{k,M} \circ \mathcal{V}^{\mathrm{clsc}}_{k,M}$ to $\mathrm{id}_{\mathbf{Var}_k}$.*

We can finally provide a complete description of the equivalence relation between classical varieties and varieties.

Theorem 3.3.7. *The two functors $\mathcal{V}^{\mathrm{sch}}_{k,M}$ and $\mathcal{V}^{\mathrm{clsc}}_{k,M}$ together with the two natural transformations η and μ form an equivalence of category between prevarieties over k in M and varieties over k.*

Proof. This follows directly from Definition-Lemma 3.3.3 and Definition-Lemma 3.3.6. □

To conclude this section, we prove that there is a natural isomorphism between the sheaf of a geometrically reduced variety over k to the sheaf of its associated prevariety. The proof of this result is very similar to the discussion at the bottom of the page in [19, Theorem 3.37, p. 81].

Proposition 3.3.8. *Let X be a geometrically reduced variety over k. For any $U \subseteq_{\mathrm{op}} X$, there exists a natural correspondence $\chi : \mathcal{O}_X \to (\pi_{M,X} \circ \iota_{M,X})_* \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}$, which is the family of maps $(\chi_U)_{U \in \mathbf{Op}_X}$ where $\chi_U : \mathcal{O}_X(U) \to \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}(U_{M,\mathrm{cl}})$ is defined by sending $s \in \mathcal{O}_X(U)$ to $s_{M,\mathrm{cl}} : U_{M,\mathrm{cl}} \to M$, for all $U \subseteq_{\mathrm{op}} X$.*

Proof. Fix some $U \subseteq_{\mathrm{op}} X$. By the definition of the topology on $X_{M,\mathrm{cl}}$, we have $U_{M,\mathrm{cl}} \subseteq_{\mathrm{op}} X_{M,\mathrm{cl}}$. According to Definition 3.2.4, there exists a well-defined and surjective map $\chi_U : \mathcal{O}_X(U) \to \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}(U_{M,\mathrm{cl}})$ defined by sending $s \in \mathcal{O}_X(U)$ to $s_{M,\mathrm{cl}} : U_{M,\mathrm{cl}} \to M$. To establish injectivity of this map, consider s and $t \in \mathcal{O}_X(U)$ such that $\chi_U(s) = \chi_U(t)$. In other words,

$$\Theta_{X,U,M,x}(s) = \Theta_{X,U,M,x}(t)$$

for all $x \in U_{M,\mathrm{cl}}$. Since U can be covered by open affine sets of X as per Proposition A.7.10, and considering the glueing and uniqueness axioms of sheaves, it suffices to prove the result when U is affine.

Thus, we may assume that U is affine. Furthermore, let us assume that $U = \mathrm{Spec}(R)$ where $R = k[x_1, \ldots, x_n]/I$ for some radical ideal $I \subseteq k[x_1, \ldots, x_n]$, which implies $\pi^{-1}_{M,X}(U) = \mathrm{Spec}(R \otimes_k M)$.

Let $x \in U_{M,\mathrm{cl}}$. We recall that $\Theta_{X,U,M,x} := \phi_{X,M,x} \circ p_{X,M,x} \circ \psi_{X,U,M,x} \circ \pi^{\#}_{X,M,U}$

- $\psi_{X,U,M,x} : \mathcal{O}_{X_M}(\pi^{-1}_{M,X}(U)) \to \mathcal{O}_{X_M,x}$ denotes the canonical ring homomorphism for any $x \in U_{\mathrm{cl}}$.
- $p_{X,M,x} : \mathcal{O}_{X_M,x} \to \mathcal{O}_{X_M,x}/\mathfrak{m}_{X_M,x}$ denotes the canonical quotient map.

- $\phi_{X,M,x} : \mathcal{O}_{X_M,x}/\mathfrak{m}_{X_M,x} \to M$ denotes the isomorphism discussed earlier.

Thus, we have $\psi_{X,U,M,x}(\pi^\sharp_{M,X,U}(s) - \pi^\sharp_{M,X,U}(t)) \in \mathfrak{m}_{X_M,x}$.
Therefore, we can conclude that

$$\pi^\sharp_{M,X,U}(s) - \pi^\sharp_{M,X,U}(t) \in \psi^{-1}_{X,U,M,x}(\mathfrak{m}_{X_M,x}).$$

As a consequence, we obtain

$$\pi^\sharp_{M,X,U}(s) - \pi^\sharp_{M,X,U}(t) \in \bigcap_{x \in U_{M,\mathrm{cl}}} \psi^{-1}_{X,U,M,x}(\mathfrak{m}_{X_M,x}) = \psi^{-1}_{X,U,M,x}\Big(\bigcap_{x \in U_{M,\mathrm{cl}}} \mathfrak{m}_{X_M,x}\Big).$$

Since any algebra of finite type over a field is Jacobson (see [19, Theorem 1.7, p. 10]), $R \otimes_k M$ is Jacobson. Furthermore, by Proposition A.7.6, $\mathcal{O}_{X_M}$ is also Jacobson as it is isomorphic to $R \otimes_k M$. Additionally, since we assume our variety to be geometrically reduced, $R \otimes_k M$ is reduced. Therefore, we have $\bigcap_{x \in U_{M,\mathrm{cl}}} \mathfrak{m}_{X_M,x} = \{0\}$, and this implies that

$$\pi^\sharp_{M,X,U}(s - t) = 0.$$

Now, $\pi^\sharp_{M,X,U}$ is injective, as it corresponds up to canonical isomorphisms to the injective morphism $M \to M[x_1, \cdots, x_n]/I^{\mathrm{ext}}$ (see [20, Exercise 2.18, p. 81]). The latter is clearly an injective morphism since it is an M-algebra morphism.

To prove naturality, we observe that for any $U, V \subseteq_{\mathrm{op}} X$, both $\rho^{\mathcal{O}_X}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ and $\rho^{\mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}}_{U_{M,\mathrm{cl}},V_{M,\mathrm{cl}}} : \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}(U_{M,\mathrm{cl}}) \to \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}(V_{M,\mathrm{cl}})$ correspond to the restriction of the domain of the functions. By the property of the stalk morphism, it follows that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_X(U) \arrow[r, "\chi_U"] \arrow[d, "\rho^{\mathcal{O}_X}_{U,V}"] & \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}(U_{M,\mathrm{cl}}) \arrow[d, "\rho^{\mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}}_{U_{M,\mathrm{cl}},V_{M,\mathrm{cl}}}"] \\
\mathcal{O}_X(V) \arrow[r, "\chi_V"] & \mathcal{R}_{M,\mathrm{cl},\mathcal{O}_X}(V_{M,\mathrm{cl}})
\end{tikzcd}$$

□

Chapter 4

Some Unlikely Intersections Results

With this chapter, we begin following [11] to generalise results of Boxall. In section 2 of [11], Boxall proved some algebro-geometric results that described useful properties of a certain family of curves in algebraic tori. In this chapter, we closely follow the presentation of section 2 in [11] supplying more details and making changes where appropriate.

For this chapter, we set E as some elliptic curve defined over $\mathbb{C}$. We will denote the endomorphism ring of E as $\mathcal{O}_E$ and, as $\mathcal{O}_E$ is an integral domain (see [31, Proposition 4.2.c, p. 68]), the field of fractions of $\mathcal{O}_E$ as $k_{\mathcal{O}_E}$ or simply $k_{\mathcal{O}}$. Furthermore, we will identify $\mathcal{O}_E$ with a subring of $\mathbb{C}$ and, hence, $k_{\mathcal{O}}$ with a subfield of $\mathbb{C}$ given in [31, Theorem 4.1, p. 171]. In particular, by [31, Corollary 9.4, p. 102], $k_{\mathcal{O}_E}$ is a number field.

Just as in the case of algebraic tori, we have a concrete characterization of algebraic subgroups of powers of elliptic curves by use of the endomorphism ring of the underlying elliptic curve. The following restatement of Theorem 4.8(a) in [24] serves as an indication of the similarity between powers of elliptic curves and algebraic tori.

Proposition 4.0.1. *Let* $N \geqslant 2$. *Then* $A \subseteq E^N(\mathbb{C})$ *is an algebraic subgroup of codimension* $r \leqslant N$ *if and only if* A *is the solution set of* r *equations of the form*

$$\sum_{j=1}^{N} \lambda_{i,j}(x_j) = 0$$

with the $\lambda_{i,j} \in \mathcal{O}_E$ *for all* $i \in [[1, r]]$ *such that* $(\lambda_{1,1}, ..., \lambda_{1,N}), ..., (\lambda_{r,1}, \dots, \lambda_{r,N})$ *are* $k_{\mathcal{O}_E}$*-linearly independent.*

We now consider the Mordell-Lang conjecture for curves, stated in the case of powers of elliptic curves, which was proven in a sequence of papers by Vojta, Faltings and Hindry as given in [12, p. 2]. In order to state this theorem, we first give a well-known definition.

Definition 4.0.2. *Let G be a group. A* finite rank subgroup *of G is any subgroup of G such that it is contained in the divisible hull of a finitely generated subgroup of G.*

Theorem 4.0.3. *Let $C \subseteq E^N(\mathbb{C})$ be an irreducible closed algebraic curve for some $N \in \mathbb{N}$ with $N \geqslant 2$. Let Γ be a finite rank subgroup of $E^N(\mathbb{C})$. If $C \cap \Gamma$ is infinite, then C is a translate of an algebraic subgroup of $E^N(\mathbb{C})$.*

The following corollary is a well-known consequence Theorem 4.0.3 with historical significance. Namely, the third point is otherwise known as the "Manin-mumford conjecture" in the case where the abelian variety is a power of an elliptic curve. Following that of Boxall, this result will be useful later.

Corollary 4.0.4. *Let $N \geqslant 2$ and let $C \subseteq E^N(\mathbb{C})$ be an irreducible closed algebraic curve. Assume that at least one of the following conditions is satisfied:*

1. *There is a non-torsion $x \in E^N(\mathbb{C})$ such that, for infinitely many $n \in \mathbb{N}$, $n \cdot x \in C$.*
2. *There is a non-torsion $y \in E^N(\mathbb{C})$ such that, for infinitely many $n \in \mathbb{N}$, there exists $x \in C$ such that $n \cdot x = y$.*
3. *C contains infinitely many torsion points of $E^N(\mathbb{C})$.*

Then C is contained in an algebraic subgroup of $E^N(\mathbb{C})$ of dimension 1.

Proof. Assume that 1., 2. or 3. holds. If 1. holds, let $x \in E^N(\mathbb{C})$ be the non-torsion point in the statement and let $\Gamma = \{nx \mid n \in \mathbb{Z}\}$. If 2. holds, let $y \in E^N(\mathbb{C})$ be the non-torsion point in the statement and let Γ be the group generated by $\{x \in E^N(\mathbb{C}) \mid \exists n \in \mathbb{Z} \text{ such that } nx = y\}$. If 3. holds, let Γ be the torsion subgroup of $E^N(\mathbb{C})$. In each case, Γ is a finite rank subgroup. Using Γ and Theorem 4.0.3, C is a translate of an algebraic subgroup of $E^N(\mathbb{C})$ of dimension 1. By Proposition 4.0.1, we have that C is contained in $N-1$ hypersurfaces given by an equations of the form

$$\sum_{j=1}^{N} \lambda_{i,j}\pi_j(x) = \alpha_j$$

with $\alpha_i \in E$ and the $\lambda_{i,j} \in \mathcal{O}_E$ for all $i \in [[1, r]]$ such that

$$(\lambda_{1,1}, \ldots, \lambda_{1,N}), \ldots, (\lambda_{r,1}, \ldots, \lambda_{r,N})$$

are $k_{\mathcal{O}_E}$-linearly independent. We need only show that α_i is a torsion point for any $i \in [[1, r]]$. Let $i \in [[1, r]]$. If 1. holds with $x \in E^N(\mathbb{C})$ as in the statement of 1., there exist two distinct $m, n \in \mathbb{N}$ such that, setting $\beta = \sum_{j=1}^{N} \lambda_{i,j}\pi_j(x)$, $n\beta = \alpha_i = m\beta$. If 2. holds with $y \in E^N(\mathbb{C})$ as in the statement of 2., then there exist distinct $n, m \in \mathbb{N}$ such that, setting $\beta = \sum_{j=1}^{N} \lambda_{i,j}\pi_j(y)$, $n\alpha_i = \beta = m\alpha_i$. If 3. holds, then α_i is an additive combination of torsion points. In each case, α_i is a torsion point. ☐

It should be noted that the conclusion of Corollary 4.0.4 could be restated by rather saying that C is a translate of an algebraic subgroup by a torsion element.

Now, using Corollary 4.0.4, we begin our generalization of results used by Boxall in section 2 of [11]. Throughout this chapter, given two irreducible curves $C_1, C_2 \subseteq E^N(\mathbb{C})$, we often consider $\mathcal{N} = \{n \in \mathbb{N} \mid [n]C_1 \subseteq C_2\}$, where $[n]C_1 = \{nx \mid x \in C_1\}$, as we will seek to understand how multiples of elements in a curve interact with other curves. This investigation becomes rather trivial if the $n \in \mathbb{N}$ we consider are such that $[n]C_1 \subseteq C_2$. Luckily, for the family of curves we consider, this is not the case as seen in the following corollary which, to prove, we follow the same reasoning as the algebraic torus case given in [2, Section 2.4].

Corollary 4.0.5. *Let $N \geqslant 2$ and $C_1, C_2 \subseteq E^N(\mathbb{C})$ be irreducible closed algebraic curves. Assume that there is no algebraic subgroup $H \subseteq E^N(\mathbb{C})$ of dimension 1 such that $C_1 \subseteq H$. Let $\mathcal{N} = \{n \in \mathbb{N} \mid [n]C_1 \subseteq C_2\}$. Then $\mathcal{N}$ is finite.*

Proof. Assume that $\mathcal{N}$ is infinite. Let $K \subseteq \mathbb{C}$ be a minimal field over which C_1, C_2 and E are defined such that all endomorphisms in $\mathcal{O}_E$ are defined over K and let $x \in C_1$ be a generic point over K. Since C_1 is of dimension 1, it follows that $x \in C_1$ is not a torsion point of $E^N(\mathbb{C})$. By assumption, $\mathcal{N}$ is infinite, so that there exist infinitely many $n \in \mathbb{N}$ such that $nx \in C_2$. It follows from Corollary 4.0.4 that C_2 is contained in a 1-dimensional algebraic subgroup of $E^N(\mathbb{C})$. By Proposition 4.0.1, any point in C_2 satisfies $N-1$ equations of the form $\sum_{j=1}^{N} \lambda_{i,j}(x_j) = 0$ where $\lambda_{i,j} \in \mathcal{O}_E$ for all $i \in [[1, N]]$ and all $j \in [[1, N-1]]$ such that $(\lambda_{1,1}, ..., \lambda_{1,N}), ..., (\lambda_{r,1}, \ldots, \lambda_{r,N})$ are $k_{\mathcal{O}_E}$-linearly independent. Identifying $n \in \mathcal{N}$ with its corresponding element in $\mathcal{O}_E$, which we will denote by λ_n, we have that $(\lambda_n \cdot \lambda_{1,1}, ..., \lambda_n \cdot \lambda_{1,N}), ..., (\lambda_n \cdot \lambda_{r,1}, \ldots, \lambda_n \cdot \lambda_{r,N})$ are $k_{\mathcal{O}_E}$-linearly independent. It follows from Proposition 4.0.1 that x is contained in a 1-dimensional algebraic subgroup of $E^N(\mathbb{C})$. Since x is a generic point of C_1 over K, it follows that C_1 is also contained in the same 1-dimensional algebraic subgroup of $E^N(\mathbb{C})$ which is a contradiction. ☐

Continuing our investigation of which $n \in \mathbb{N}$ would be of interest, we obtain a further result.

Lemma 4.0.6. *If $C_1, C_2 \subseteq E^N(\mathbb{C})$ are irreducible closed algebraic curves, with $N \geqslant 3$, and $\bigcup_{n\in\mathbb{N}\backslash\mathcal{N}}\{x \in C_1 \mid nx \in C_2\}$ infinite, where $\mathcal{N} = \{n \in \mathbb{N} \mid [n]C_1 \subseteq C_2\}$, then there exist infinitely many $n \in \mathbb{N}\backslash\mathcal{N}$ with the property that there exists $x \in C_1$ with $nx \in C_2$.*

Proof. Assume that there existed only finitely many $n \in \mathbb{N}\backslash\mathcal{N}$ with the property that there exists $x \in C_1$ with $nx \in C_2$. Then there exists $n \in \mathbb{N}\backslash\mathcal{N}$ such that there are infinitely many $x \in C_1$ with $nx \in C_2$. As multiplication by n is an endomorphism of $E^N(\mathbb{C})$, which we will denote by $\lambda_n : E^N(\mathbb{C}) \to E^N(\mathbb{C})$, it follows that $\lambda_n^{-1}(C_2)$ is a subvariety of $E^N(\mathbb{C})$. Hence, $\lambda_n^{-1}(C_2) \cap C_1$ is a subvariety of C_1. As there are infinitely many $x \in C_1$ such that $nx \in C_2$, we have that $\lambda_n^{-1}(C_2) \cap C_1$ must be a curve. Let C_3 be an irreducible component of $\lambda_n^{-1}(C_2) \cap C_1$. If C_3 were not the entirety of C_1, then we would have the chain of irreducible components $C_0 \subset C_3 \subset C_1$ of C_1 where C_0 consists of any single point of C_3. This contradicts the dimension of C_1 so that $\lambda_n^{-1}(C_2) \cap C_1 = C_1$. This again contradicts that $n \notin \mathcal{N}$. ☐

As in [11], to prove Conjecture 1.0.3, we will first deal with a special case that will be useful for later arguments.

Proposition 4.0.7. *Let $C_1, C_2 \subseteq E^N(\mathbb{C})$ be irreducible closed algebraic curves with $N \geqslant 3$. Assume that C_1 is not contained in a proper algebraic subgroup of $E^N(\mathbb{C})$. Let $\mathcal{N} = \{n \in \mathbb{N} \mid [n]C_1 \subseteq C_2\}$. If C_1 or C_2 is contained in a translate of an algebraic subgroup of $E^N(\mathbb{C})$ of dimension $N-2$, then $\bigcup_{n\in\mathbb{N}\backslash\mathcal{N}}\{x \in C_1 \mid nx \in C_2\}$ is finite.*

Proof. Assume that $\bigcup_{n\in\mathbb{N}\backslash\mathcal{N}}\{x \in C_1 \mid nx \in C_2\}$ is infinite and that C_1 is contained in a translate of an algebraic subgroup of $E^N(\mathbb{C})$ of dimension $N-2$. By change of coordinates, we may assume that there exist $a, b \in E$ such that $\pi_{1,2}(x) = (a, b)$ for all $x \in C_1$ where $\pi_{1,2} : E^N(\mathbb{C}) \to E^2(\mathbb{C})$ is the projection to the first two coordinates. If (a, b) was a torsion point, then C_1 would be contained in a proper algebraic subgroup of $E^N(\mathbb{C})$. Thus, (a, b) is not a torsion point. By Corollary 4.0.4 1., $\overline{\pi_{1,2}(C_2)}$ is contained in a 1-dimensional algebraic subgroup of $E^2(\mathbb{C})$, so that $\pi_{1,2}(C_2)$ is contained in the same algebraic subgroup, say G. Let $x \in C_1$ and $n \in \mathbb{N}\backslash\mathcal{N}$ such that $nx \in C_2$. We have $\pi_{1,2}(nx) = n\pi_{1,2}(x) = n(a, b)$. Now, since $n(a, b) \in \pi_{1,2}(C_2)$ so that $n(a, b) \in G$, (a, b) is contained in a 1-dimensional algebraic subgroup. Thus, a and b are additively dependent which contradicts our assumption that C_1 is not contained in a proper algebraic subgroup of $E^N(\mathbb{C})$.

Assume that $\bigcup_{n\in\mathbb{N}\backslash\mathcal{N}}\{x \in C_1 \mid nx \in C_2\}$ is infinite and that C_2 is contained in a translate of an algebraic subgroup of $E^N(\mathbb{C})$ of dimension $N-2$. Again, by change of coordinates, we may assume that there exist $a, b \in E$ such that $\pi_{1,2}(x) = (a, b)$ for all $x \in C_2$ where $\pi_{1,2} : E^N(\mathbb{C}) \to E^2(\mathbb{C})$ is the projection to the first two coordinates. Applying Corollary 4.0.4 2. to $\overline{\pi_{1,2}(C_1)}$, we must have that (a, b) is a torsion point. By Corollary 4.0.4 3., applied to $\overline{\pi_{1,2}(C_1)}$, it follows that C_1 is contained in a proper algebraic subgroup of $E^N(\mathbb{C})$ which, again, contradicts our assumptions. $\square$

We are now in a position to prove the main results of the section. These will allow us to reduce Conjecture 1.0.3 to a simpler case where we need only consider curves in $E^3(\mathbb{C})$.

Proposition 4.0.8. *Suppose it is true that, if $C_1', C_2' \subseteq E^3(\mathbb{C})$ are irreducible closed algebraic curves and C_1' is not contained in a proper algebraic subgroup of $E^3(\mathbb{C})$, then, with $\mathcal{N}' = \{n \in \mathbb{N} \mid [n]C_1' \subseteq C_2'\}$, the set $\bigcup_{n \in \mathbb{N} \setminus \mathcal{N}'} \{x \in C_1' \mid nx \in C_2'\}$ is finite. Then, it is also true that, if $C_1, C_2 \subseteq E^N(\mathbb{C})$ are irreducible closed algebraic curves, with $N \geqslant 3$, and there does not exist a 1-dimensional algebraic subgroup $G \subseteq E^N(\mathbb{C})$ such that $C_1 \subseteq G$ and there does not exist a 2-dimensional algebraic subgroup $H \subseteq E^N(\mathbb{C})$ such that $C_1 \cup C_2 \subseteq H$, then, with $\mathcal{N} = \{n \in \mathbb{N} \mid [n]C_1 \subseteq C_2\}$, the set $\bigcup_{n \in \mathbb{N} \setminus \mathcal{N}} \{x \in C_1 \mid nx \in C_2\}$ is finite.*

Proof. Let $N \geqslant 3$ and let $C_1, C_2 \subseteq E^N(\mathbb{C})$ be irreducible closed algebraic curves such that C_1 is not contained in a 1-dimensional algebraic subgroup of $E^N(\mathbb{C})$ and $C_1 \cup C_2$ is not contained in a 2-dimensional algebraic subgroup of $E^N(\mathbb{C})$. We will argue by contradiction by considering several cases. Assume that $\bigcup\limits_{n \in \mathbb{N} \setminus \mathcal{N}} \{x \in C_1 \mid nx \in C_2\}$ is infinite.

<u>Case 1</u>: Suppose that there is a 2-dimensional algebraic subgroup $H \subseteq E^N(\mathbb{C})$ such that $C_1 \subseteq H$. We break this case further into two subcases.

<u>Subcase 1.1</u>: Assume that there are infinitely many $y \in C_2$ such that there exist $x \in C_1$ and $n \in \mathbb{N} \setminus \mathcal{N}$ such that $nx = y$. Denote the set of all such $y \in C_2$ as S. Since $C_1 \subseteq H$ and H is an algebraic subgroup of $E^N(\mathbb{C})$ it follows that $S \subseteq H$. We have that $S \subseteq C_2 \cap H$ so that $C_2 \cap H$ is infinite. As argued before, we must have that $C_2 \cap H$ is a subvariety of C_2 of dimension 1. Since C_2 is irreducible and also of dimension 1, we deduce that $C_2 \cap H = C_2$ so that $C_2 \subseteq H$. This contradicts our assumption that $C_1 \cup C_2$ cannot be contained in a 2-dimensional algebraic subgroup of $E^N(\mathbb{C})$.

<u>Subcase 1.2</u>: Now, assume the contrary so that there are only finitely many $y \in C_2$ such that there exists $x \in C_1$ and $n \in \mathbb{N} \setminus \mathcal{N}$ such that $nx = y$. Then there exists a $y \in C_2$ such that, for infinitely many $x \in C_1$, there is some $n \in \mathbb{N} \setminus \mathcal{N}$ such that $nx = y$. If y is a torsion point, then C_1 would have infinitely many torsion points which, by Corollary 4.0.4, implies that C_1 is contained in a 1-dimensional algebraic subgroup. If y were non-torsion, then again, by Corollary 4.0.4, C_1 would be contained in a 1-dimensional algebraic subgroup. In either case, this is a contradiction to our assumptions on C_1.

Case 2: Assume that C_1 is not contained in a 2-dimensional algebraic subgroup. Then there exists a projection $\pi : E^N(\mathbb{C}) \to E^3(\mathbb{C})$ such that $\pi(C_1)$ is not contained in a proper algebraic subgroup of $E^3(\mathbb{C})$. Let C_1' and C_2' denote the Zariski closures in $E^3(\mathbb{C})$ of $\pi(C_1)$ and $\pi(C_2)$, respectively. By Lemma A.1.13 and Lemma A.1.14, C_1' and C_2' are both irreducible. We have that C_1' is a curve. Indeed, suppose that this were not true. We have that $\bigcup\limits_{n \in \mathbb{N}\backslash\mathcal{N}} \{x \in C_1' \mid nx \in C_2'\}$ is non-empty since $\bigcup\limits_{n \in \mathbb{N}\backslash\mathcal{N}} \{x \in C_1 \mid nx \in C_2\}$ is non-empty. Then C_1' would be a single point, say $x \in C_1'$, as it is irreducible and, by Lemma 4.0.6, $nx \in C_2'$ for infinitely many $n \in \mathbb{N}\backslash\mathcal{N}$. If x is not a torsion point, then Corollary 4.0.4 gives that C_2 is contained in an algebraic subgroup of dimension 1. This implies that C_1' is also contained in a proper algebraic subgroup which is a contradiction. If x were a torsion point, then it would be contained in the algebraic subgroup defined by $\sum\limits_{i=1}^{N} \lambda_m(x_i) = 0$ where λ_m is the endomorphism of E corresponding to some $m \in \mathbb{N}$. This is also a contradiction. We now consider various subcases as before. Let $\mathcal{N}' = \{n \in \mathbb{N} \mid [n]C_1' \subseteq C_2'\}$.

Subcase 2.1: Assume that C_2' is finite. Since C_2' is irreducible, it consists of a single point, say $y \in C_2'$. We consider two subcases.

Subcase 2.1.1: Assume that $\bigcup\limits_{n \in \mathbb{N}\backslash\mathcal{N}'} \{x \in C_1' \mid nx \in C_2'\}$ is finite. If $\bigcup\limits_{n \in \mathbb{N}\backslash\mathcal{N}'} \{x \in C_1 \mid nx \in C_2\}$ is infinite, then there is $x \in C_1'$ such that $\pi^{-1}(\{x\}) \cap C_1$ is infinite. Thus, $\pi^{-1}(\{x\}) \cap C_1$ is a curve in C_1. It follows from previous arguments that $\pi^{-1}(\{x\}) \cap C_1 = C_1$. This contradicts that C_1' is infinite. Thus, $\bigcup\limits_{n \in \mathbb{N}\backslash\mathcal{N}'} \{x \in C_1 \mid nx \in C_2\}$ is finite. By Corollary 4.0.5, $\mathcal{N}'$ is finite. We conclude that $\bigcup\limits_{n \in \mathbb{N}\backslash\mathcal{N}} \{x \in C_1 \mid nx \in C_2\}$ is finite.

Subcase 2.1.2: Assume that $\bigcup\limits_{n \in \mathbb{N}\backslash\mathcal{N}'} \{x \in C_1' \mid nx \in C_2'\}$ is infinite. If y is a non-torsion point then, by 2. of Corollary 4.0.4, C_1' is contained in a 1-dimensional algebraic subgroup which implies that C_1 is also contained in an algebraic subgroup of codimension 2. If y is a torsion point, then C_1' contains infinitely many torsion points so that, by 3. of Corollary 4.0.4, C_1' is contained a 1-dimensional algebraic subgroup. This, again, implies that C_1 is contained an algebraic subgroup of codimension 2. In both cases, using Proposition 4.0.7, this is a contradiction to our assumptions.

Subcase 2.2: Assume that C_2' is infinite. Then C_1' and C_2' are both curves. By assumption, $\bigcup\limits_{n \in \mathbb{N}\backslash\mathcal{N}'} \{x \in C_1' \mid nx \in C_2'\}$ is finite. We deduce that $\bigcup\limits_{n \in \mathbb{N}\backslash\mathcal{N}} \{x \in C_1 \mid nx \in C_2\}$ is finite by similar arguments to that of subcase 2.1.1. $\square$

We now cover the case where at least one of C_1 and C_2 is not defined over $\overline{\mathbb{Q}}$. We will make a further assumption that E is defined over $\overline{\mathbb{Q}}$.

Proposition 4.0.9. *Suppose that E is defined over $\overline{\mathbb{Q}}$ and that it is true that, if $C_1', C_2' \subseteq E^3(\mathbb{C})$ are irreducible closed algebraic curves with at least one of C_1' and C_2' not defined over $\overline{\mathbb{Q}}$ and C_1' is not contained in a proper algebraic subgroup of $E^3(\mathbb{C})$, then, with $\mathcal{N}' = \{n \in \mathbb{N} \mid [n]C_1' \subseteq C_2'\}$, the set $\bigcup_{n \in \mathbb{N} \setminus \mathcal{N}'} \{x \in C_1' \mid nx \in C_2'\}$ is finite. Then, it is also true that, if $C_1, C_2 \subseteq E^N(\mathbb{C})$ are irreducible closed algebraic curves, with $N \geqslant 3$ and at least one of C_1' and C_2' not defined over $\overline{\mathbb{Q}}$, and there does not exist a 1-dimensional algebraic subgroup $G \subseteq E^N(\mathbb{C})$ such that $C_1 \subseteq G$ and there does not exist a 2-dimensional algebraic subgroup $H \subseteq E^N(\mathbb{C})$ such that $C_1 \cup C_2 \subseteq H$, then, with $\mathcal{N} = \{n \in \mathbb{N} \mid [n]C_1 \subseteq C_2\}$, the set $\bigcup_{n \in \mathbb{N} \setminus \mathcal{N}} \{x \in C_1 \mid nx \in C_2\}$ is finite.*

Proof. We can use analogous arguments to those in the proof of Proposition 4.0.8. We need only check that, in Case 2 of the proof of Proposition 4.0.8, if C_1 and C_2 are not defined over $\overline{\mathbb{Q}}$, then we can choose the projection $\pi : E^N(\mathbb{C}) \to E^3(\mathbb{C})$ such that $C_1' = \overline{\pi(C_1)}$ or $C_2' = \overline{\pi(C_2)}$ is not defined over $\overline{\mathbb{Q}}$. Suppose we could not do so and suppose that C_1 is not contained in a 2-dimensional algebraic subgroup. Let $\pi' : E^N(\mathbb{C}) \to E^k(\mathbb{C})$ be a projection such that $\pi'(C_1)$ is not contained in a proper algebraic subgroup of $E^k(\mathbb{C})$. We may assume that k is maximal so that $k \geqslant 3$ as C_1 is not contained in a 2-dimensional algebraic subgroup. We now claim that $\bigcup\limits_{n \in \mathbb{N} \setminus \mathcal{N}} \{y \in C_2 \mid \text{ there exists } x \in C_1 \text{ such that } nx = y\}$ is infinite. To prove the claim, assume the contrary so that there exists $y \in C_2$ such that there are infinitely many $x \in C_1$ such that there exists an $n \in \mathbb{N} \setminus \mathcal{N}$ where $nx = y$. If y is not a torsion point, then, by 2. of Corollary 4.0.4, C_1 is contained in an algebraic subgroup of $E^N(\mathbb{C})$ of dimension 1. If y is a torsion point, then by 3. of Corollary 4.0.4, C_1 is again contained in an algebraic subgroup of $E^N(\mathbb{C})$ of dimension 1. In both cases, we reach a contradiction. So, $\bigcup\limits_{n \in \mathbb{N} \setminus \mathcal{N}} \{y \in C_2 \mid \text{ there exists } x \in C_1 \text{ such that } nx = y\}$ is infinite. We now claim that y has additive dimension zero over $\pi'(y)$ for all $y \in C_2$. If $k = N$, then the claim is immediate. Assume that $k < N$ and assume, for simplicity, that $\pi'(y_1, \ldots, y_N) = (y_1, \ldots, y_k)$ for any $(y_1, \ldots, y_N) \in E^N$. Let $j > k$ and let $\pi'' : E^N(\mathbb{C}) \to E^{k+1}(\mathbb{C})$ be the projection that, which we also may assume for simplicity, sends $(y_1, \ldots, y_N)$ to $(y_1, \ldots, y_k, y_j)$. By the maximality of k, $\pi''(C_1)$ is contained in a proper algebraic subgroup H of $E^{k+1}(\mathbb{C})$. It follows that there is a proper algebraic subgroup G of $E^N(\mathbb{C})$ such that $\pi''(G) = H$. Let $x \in C_1$ such that there exists an $n \in \mathbb{N} \setminus \mathcal{N}$ where $nx \in C_2$. As G is an algebraic group and $x \in G$, it follows that, for any $n \in \mathbb{N}$, $nx \in G$. As $\bigcup\limits_{n \in \mathbb{N} \setminus \mathcal{N}} \{y \in C_2 \mid \text{ there exists } x \in C_1 \text{ such that } nx = y\}$ is infinite, it follows that there are infinitely many points in $G \cap C_2$. As in the proof of Lemma 4.0.6, since C_2 is of dimension 1 and is irreducible, we deduce that $C_2 \subseteq G$ so that $\pi''(C_2) \subseteq H$. Hence, the additive dimension of $(y_1, \ldots, y_k, y_j)$ over $(y_1, \ldots, y_k)$ is zero. As j was arbitrary, we deduce that $(y_1, \ldots, y_N)$ has additive dimension zero over $(y_1, \ldots, y_k)$, thus proving our second claim. Since, by assumption, $\pi'(C_1)$ and $\pi'(C_2)$ are defined over $\overline{\mathbb{Q}}$, it follows that C_1 and C_2 are defined over $\overline{\mathbb{Q}}$. This is a contradiction.

$\square$

Chapter 5

Γ-Closure

A key ingredient in Boxall's arguments in [11] was the use of an analogue of Schanuel's conjecture given by Bays, Kirby and Wilkie in [7, Theorem 1.2, p. 2] for raising to an exponentially transcendental power. This relied on previous work of Kirby in [26] on fields equipped with an exponential function (in the sense of the usual exponential function on an algebraic torus) that made use of an exponential algebraic closure operator. In trying to generalize the arguments of Boxall in a naive manner, it was clear that this result of Kirby would not apply in the case of powers of elliptic curves due to the different nature of the exponential function associated to elliptic curves to the one associated to algebraic tori. Further work was done by Bays and Kirby in [6], with model-theoretic motivations to generalize work of Zilber on $\mathbb{C}_{\exp}$, which generalized exponential fields to a wider setting that considered exponential maps for simple abelian varieties by use of another closure operator. We make use of Bays and Kirby's work and will survey the results in [6] that will be applicable to elliptic curves. Furthermore, we will use Bays and Kirby's construction to prove an analogue of the Bays-Kirby-Wilkie result in [7, Theorem 1.2, p. 2]. As in the case of section 2.1 of this thesis, in this chapter we will state references in our statements and results when they are analogous to the relevant literature.

5.1 Finitary closure operators

We begin the chapter by first stating some basic definitions and results needed to understand closure operators, as well as some useful examples, which can be found at the bottom of page 165 and the top of page 166 in [27].

Definition 5.1.1. *Let S be a set. We say a map* $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$ *is a* finitary closure operator on S *if it satisfies the following properties:*

1. *For all $A \subseteq S$, $A \subseteq \mathrm{cl}(A)$ and $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.*

2. *For all* $A, B \subseteq S$*, if* $A \subseteq B$*, then* $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$*.*
3. *For all* $A \subseteq S$ *and* $a \in S$*, if* $a \in \mathrm{cl}(A)$*, then there is a finite subset* $A_0 \subseteq A$ *such that* $a \in \mathrm{cl}(A_0)$*.*

Definition 5.1.2. *A pregeometry is a pair* (S, cl) *where* S *is a set and* cl *is a finitary closure operator on* S *with the following additional property:*

4. *For all* $A \subseteq S$ *and* $a, b \in S$*, if* $a \in \mathrm{cl}(A \cup \{b\}) \backslash \mathrm{cl}(A)$*, then* $b \in \mathrm{cl}(A \cup \{a\})$*.*

There are two well-known closure operators which are considered often in general mathematics.

Definition 5.1.3. *Let* F *be a field and let* $A \subseteq F$*. We define* $\mathrm{acl}_F(A)$ *to be the relative algebraic closure of the subfield generated by* A *in* F*. When* F *is understood, we will omit the subscript.*

Definition 5.1.4. *Let* V *be a vector space and let* $A \subseteq V$*. We define* $\mathrm{span}_V(A)$ *to be the span of* A *in* V*. As in the case of* acl*, when* V *is understood, we will omit the subscript.*

The following is also well-known.

Lemma 5.1.5. *Let* F *be a field and* V *a vector space over* F*. Then* (F, acl) *and* (V, span) *are pregeometries.*

5.2 Γ-fields

We now focus our attention on the exponential field analogue in the case of an elliptic curve. The following section is a slightly adapted version from sections 3.1 and 3.2 in [6].

We will henceforth assume that all fields are of characteristic 0. Let E be an elliptic curve and let $\mathcal{K}_0$ be a number field over which E is defined. We will assume that $\mathcal{O}_E$ and $k_{\mathcal{O}_E}$ are both defined over $\mathcal{K}_0$. We set F as an algebraically closed field that is a field extension of $\mathcal{K}_0$. We consider $E(F)$ as an $\mathcal{O}_E$-module equipped with $E(F)$'s group operation as addition and $\mathcal{O}_E$ acting on $E(F)$ as scalar multiplication. We have that $\mathbb{G}_a(F)$ is also an $\mathcal{O}_E$-module equipped with the usual addition and scalar multiplication given by multiplication by elements in $\mathcal{O}_E$ (which we identify with a subring in $\mathcal{K}_0$). Set $H = \mathbb{G}_a \times E$ where we consider $H(F)$ with the natural $\mathcal{O}_E$-module structure.

Before proceeding, we will need the following concept.

Definition 5.2.1. *Let* V *be an algebraic variety and let* $A \subseteq V(F)$*. We call the smallest field* $K \subseteq F$ *such that* $A \subseteq V(K)$*, the field* generated by coordinates in A.

We now give a novel adaptation of a few definitions given in [6] related to "Γ-fields" which are an analogue of partial exponential fields and will be the main objects of consideration in this chapter. We begin with an adapted version of [6, Definition 3.8, p. 10].

Definition 5.2.2. *A* Γ-field *(with respect to the $\mathcal{O}_E$-module $H(F)$) is a pair (K, K_Γ) where K_Γ is a divisible $\mathcal{O}_E$-submodule of $H(F)$ and K is a field extension of $\mathcal{K}_0$ such that K contains the field generated by the coordinates of K_Γ. We will usually write (K, K_Γ) simply as K. Write $\pi_1 : H(K) \to \mathbb{G}_a(K)$ and $\pi_2 : H(K) \to E(K)$ as the natural projections and set $\Gamma_i(K) = \pi_i(K_\Gamma)$ for $i = 1, 2$.*

Definition 5.2.3 ([6, Definition 3.8, p. 10])**.** *The* kernels *of a Γ-field K are defined to be*

$$\ker_1(K) := \{x \in \mathbb{G}_a(K) \mid (x, 0) \in K_\Gamma\} \text{ and}$$
$$\ker_2(K) := \{y \in E(K) \mid (0, y) \in K_\Gamma\}.$$

We will, henceforth, consider F as a Γ-field with respect to $H(F)$ equipped with some divisible $\mathcal{O}_E$-submodule of $H(F)$.

Naturally, one would want to consider how Γ-fields might interact with one another and how to construct such fields. The next few definitions are dedicated to expanding on this and will be useful for later definitions in the following sections.

Definition 5.2.4. *An* extension *of a Γ-field K is a Γ-field L such that $K_\Gamma \subseteq L_\Gamma$, K is a subfield of L and K_Γ contains all torsion of L_Γ. We also say that K is a* Γ-subfield of L. *We say that an extension $K \subseteq L$ of Γ-fields* preserves the kernels *and that K and L* have the same kernels *if $\ker_i(K) = \ker_i(L)$ for $i = 1, 2$.*

Definition 5.2.5. *Let L be a Γ-field and let $\{K_i \mid i \in I\}$ be a family of Γ-subfields of L with the same kernels. We define $\bigwedge_{i\in I} K_i$ to be the Γ-field with underlying field $\bigcap_{i\in I} K_i$ and $\mathcal{O}_E$-module $\bigcap_{i\in I} K_{i\Gamma}$.*

Lemma 5.2.6 ([6, Lemma 3.14, p. 11])**.** *Let L be a Γ-field and let $\{K_i \mid i \in I\}$ be a family of Γ-subfields of L with the same kernels. Then $K = \bigwedge_{i\in I} K_i$ is a Γ-subfield of L.*

Proof. Since K_Γ is an intersection of a set of $\mathcal{O}_E$-submodules of L_Γ, it is also a $\mathcal{O}_E$-submodule of L_Γ. Since $K_\Gamma \subseteq L_\Gamma \subseteq H(F)$ and L_Γ is a $\mathcal{O}_E$-submodule of $H(F)$, it follows that K_Γ is an $\mathcal{O}_E$-submodule of $H(F)$. By definition, $K_{i\Gamma}$ contains all the torsion of L_Γ for all $i \in I$, so that K_Γ contains all the torsion of L_Γ. Therefore, since K_Γ is the intersection of divisible $\mathcal{O}_E$-submodules of L_Γ, we deduce that Γ is a divisible $\mathcal{O}_E$-submodule of $H(F)$. □

Definition 5.2.7. *Let L be a Γ-field, let $X \subseteq L$ and let $Y \subseteq L_\Gamma$. Let $\mathcal{B}_{X,Y}$ be the set of all Γ-subfields K of L with the same kernels such that $X \subseteq K$ and $Y \subseteq K_\Gamma$. We say that $\bigwedge \mathcal{B}_{X,Y}$ is the Γ-subfield* generated by X and Y *and denote it by $\langle X, Y \rangle_L$ where we usually omit the subscript L when it is understood.*

We now define a class of Γ-fields which will be of importance in the later sections.

Definition 5.2.8. *Let K be a Γ-field and let L be a Γ-field extension of K. We say that L is* finitely generated over K *if there exist a finite set $X \subseteq L$ and a finite set $Y \subseteq L_\Gamma$ such that $L = \langle K \cup X \cup Z, K_\Gamma \cup Y \rangle_L$ where Z is the field generated by the coordinates of Y. We say that L is* finitely generated *if K is $\mathbb{Q}$ with $K_\Gamma = \{(0, 0_H)\}$.*

Finitely generated Γ-field extensions can introduce elements in the field that are not generated by its associated $\mathcal{O}_E$-module, which gives a wide range of examples. We will want a more restrictive definition which considers Γ-fields generated only using elements from $H(F)$. As we consider F along with the graph of the exponential map associated to $E(F)$ as a typical example of a Γ-field, we introduce the naming convention of *exponentially finitely generated* Γ-fields, as follows, for finitely generated Γ-fields which are generated only by elements in their associated $\mathcal{O}_E$-module.

Definition 5.2.9. *Let K be a Γ-field and let L be a Γ-field extension of K. We say that L is* exponentially finitely generated over K *if there exists a finite set $Y \subseteq L_\Gamma$ such that $L = \langle K \cup X, K_\Gamma \cup Y \rangle_L$ where X is the field generated by the coordinates of Y. We say that L is* exponentially finitely generated *if K is $\mathbb{Q}$ with $K_\Gamma = \{(0, 0_H)\}$.*

5.3 Predimension

We now introduce the so-called "predimension" function and strong Γ-field extensions. The methods that we apply using these constructions were first introduced by Hrushovski in [21] and have been widely used since. They will play a crucial role in defining our wanted closure operator and proving the weak Schanuel result that we seek. This section follows the presentation of sections 4.1 and 4.2 in [6] and makes changes where appropriate. In particular, we have introduced exponentially finitely generated Γ-field extensions as well as finitely generated Γ-field extensions and often move between them. This contrasts the work of Bays and Kirby who only make use of finitely generated Γ-field extensions in the sense of [6, Section 3, p. 12].

Let K and L be Γ-fields such that L is a Γ-field extension of K. We can associate a natural vector space to this extension. Indeed, the quotient space

L_Γ/K_Γ is divisible and, by definition of Γ-field extensions, is torsion-free which implies that this is a $k_{\mathcal{O}_E}$-vector space. We now consider a lemma making use of this associated vector space which can be used to measure how "big" a finitely generated extension is.

Lemma 5.3.1. *Let K be a Γ-field and let L be a Γ-field extension of K. If L is finitely generated over K, then $\operatorname{ldim}_{k_{\mathcal{O}_E}}(L_\Gamma/K_\Gamma)$ is finite.*

Proof. Assume that L_Γ is finitely generated over K_Γ, so that there exists a finite set $Y \subseteq L_\Gamma$ such that L_Γ is the divisible $\mathcal{O}_E$-module generated by Y and K_Γ. It follows that $K_\Gamma \cup Y$ spans L_Γ as a $\mathcal{O}_E$-module. Using a linearly independent subset of $K_\Gamma \cup Y$ and embedding this in L_Γ/K_Γ, we deduce that $\operatorname{ldim}_{k_{\mathcal{O}_E}}(L_\Gamma/K_\Gamma)$ is finite. □

We now define the predimension function δ.

Definition 5.3.2 ([6, Definition 4.1, p. 15])**.** *Let K be a Γ-field. For any Γ-field L that is finitely generated over K, let*

$$\delta(L/K) := \operatorname{td}(L/K) - \operatorname{ldim}_{k_{\mathcal{O}_E}}(L_\Gamma/K_\Gamma).$$

By Definition 5.2.8, $\operatorname{td}(L/K)$ is finite and, by Lemma 5.3.1, $\operatorname{ldim}_{k_{\mathcal{O}_E}}(L_\Gamma/K_\Gamma)$ is finite, so that δ is well-defined. As a convention, for any $(a_1, \ldots, a_n) \in F_\Gamma^n$ for some $n \in \mathbb{N}$, we set

$$\delta(a_1, \ldots, a_n/K) := \delta(X/K),$$

where $X = \langle K \cup X_{\{a_1,\ldots,a_n\}}, K_\Gamma \cup \{a_1, \ldots, a_n\}\rangle_F$ with $X_{\{a_1,\ldots,a_n\}}$ the subfield of F generated by $\{a_1, \ldots, a_n\}$.

The predimension shares many properties with the dimension functions td and ldim associated to acl and span, respectively.

Lemma 5.3.3 ([6, Lemma 4.2, p. 15])**.** *Let K be a Γ-subfield of a Γ-field L.*

1. *(Finite character) If $a \in L_\Gamma^n$ for some $n \in \mathbb{N}$, then there is a finitely generated Γ-subfield K_0 of K such that, for any intermediate Γ-field $K_0 \subseteq K' \subseteq K$, we have $\delta(a/K) = \delta(a/K')$.*

2. *(Addition formula) If A, B are Γ-subfields of L finitely generated over K with $A \subseteq B$, then*

$$\delta(B/K) = \delta(B/A) + \delta(A/K).$$

3. *(Submodularity) If A, B are Γ-subfields of L with A finitely generated over $A \wedge B$, then, abbreviating $\langle A \cup B\rangle$ by AB, we have*

$$\delta(AB/B) \leqslant \delta(A/A \wedge B).$$

Proof. Since td and $\operatorname{ldim}_{k_{\mathcal{O}_E}}$ have finite character, 1. follows immediately.

Similarly for 2., the addition formula holds for both td and $\operatorname{ldim}_{k_{\mathcal{O}_E}}$, so that the result holds for δ.

To prove 3., we have that td is submodular so that

$$\operatorname{td}(AB/B) \leqslant \operatorname{td}(A/A \cap B).$$

Additionally, we have that ldim is modular so that

$$\operatorname{ldim}_{k_{\mathcal{O}_E}}(\Gamma(AB)/\Gamma(B)) = \operatorname{ldim}_{k_{\mathcal{O}_E}}(\Gamma(A)/\Gamma(A \wedge B)).$$

Subtracting from both sides gives the wanted result. ☐

The predimension function δ allows us to define the following class of Γ-fields which, again, differs to that of Bays and Kirby where we use exponentially finitely generated Γ-extensions rather than finitely generated ones.

Definition 5.3.4. *We say that extension $K \subseteq L$ of Γ-fields is a* strong extension, *denoted by $K \lhd L$, if*

1. *the extension preserves kernels and*
2. *for every Γ-subfield K' of L that is exponentially finitely generated over K, we have $\delta(K'/K) \geqslant 0$.*

We also say that K is a strong Γ-subfield of *L.*

The following lemma expresses that strong extensions defined using exponentially finitely generated extensions are equivalent to what we would have had if we had used finitely generated extensions instead.

Lemma 5.3.5. *Let K and L be Γ-fields such that $K \lhd L$. If A is a finitely generated Γ-field over K in L, then $\delta(A/K) \geqslant 0$.*

Proof. Let $A' = \langle K \cup X, A_\Gamma \rangle_L$ where X is the field generated by A_Γ. It follows by the addition formula that $\delta(A/K) = \delta(A/A') + \delta(A'/K)$. We have, by definition of A', that A' is exponentially finitely generated over K in L. Furthermore, since A is finitely generated over K, A has finite transcendence degree over A' as fields and $A_\Gamma = A'_\Gamma$. We deduce that $\delta(A/A') \geqslant 0$ and, since $K \lhd L$, we also have that $\delta(A'/K) \geqslant 0$. Thus $\delta(A/K) \geqslant 0$. ☐

We now prove two often used properties of strong extensions.

Lemma 5.3.6 ([6, Lemma 4.4, p. 16])**.** *Let K, K', L be Γ-fields such that $K \lhd K'$ and $K' \lhd L$. Then $K \lhd L$.*

Proof. It is clear that the kernel of K is the same as the kernel of L. We need only show condition 2. of Definition 5.3.4. Let A be exponentially finitely generated over K in L. We then have

$$\delta(A/K) = \delta(A/K' \wedge A) + \delta(K' \wedge A/K)$$

by the addition formula. As $K' \lhd L$, by Lemma 5.3.5, we then have that $\delta(AK'/K') \geqslant 0$. It follows that $\delta(A/K' \wedge A) \geqslant \delta(AK'/K') \geqslant 0$ by submodularity. Additionally, by Lemma 5.3.5, $\delta(A \wedge K'/K) \geqslant 0$ since $K \lhd K'$. Hence $\delta(A/K) \geqslant 0$. □

Lemma 5.3.7 ([6, Lemma 4.5, p. 16])**.** *Let K be a Γ-field and let $A_i \lhd K$ for all $i \in I$. Then $A = \bigwedge_{i\in I} A_i \lhd K$.*

Proof. The kernel of A clearly is the same as that of K. We need only show that the second condition of Definition 5.3.4 holds. We first will consider the case with $I = \{1, 2\}$. We need only show that $A_1, A_2 \lhd K$ gives $A_1 \wedge A_2 \lhd A_1$ as Lemma 5.3.6 will imply that $A_1 \wedge A_2 \lhd K$. Assume that B is an exponentially finitely generated extension of $A_1 \wedge A_2$ in A_1. We have that

$$\begin{aligned} \delta(B/A_1 \wedge A_2) &= \delta(B/B \wedge A_2) && (B \text{ is a } \Gamma\text{-subfield of } A_1) \\ &\geqslant \delta(BA_2/A_2) && (\text{submodularity}) \\ &\geqslant 0 && (\text{Lemma 5.3.5}). \end{aligned}$$

By induction, we have, if I is finite, that $\bigwedge_{i\in I} A_i \lhd K$. Now suppose that I is infinite and assume that B is an exponentially finitely generated Γ-field extension of A in K. We have

$$A = B \wedge A = \bigwedge_{i\in I}(A_i \wedge B).$$

Each Γ-field $A_i \wedge B$ is in the lattice of intermediate Γ-fields between A and B. By definition of exponentially finitely generated Γ-fields, it follows that this lattice is isomorphic to the lattice of vector subspaces between $\Gamma(B)$ and $\Gamma(A)$ which are finite dimensional extensions of $\Gamma(A)$. The vector space lattice has no infinite chains, so the Γ-field lattice does not either. Thus, there is a finite subset $I_0 \subseteq I$ such that

$$\begin{aligned} A &= \bigwedge_{i\in I}(A_i \wedge B) \\ &= \bigwedge_{i\in I_0}(A_i \wedge B) \\ &= A_{I_0} \wedge B \end{aligned}$$

where $A_{I_0} = \bigwedge\limits_{i \in I_0} A_i$. Since I_0 is finite, it follows from our previous arguments that $A_{I_0} \lhd K$. It follows from submodularity that

$$\begin{aligned} \delta(B/A) &= \delta(B/A_{I_0} \wedge B) \\ &\geqslant \delta(A_{I_0}B/A_{I_0}) \\ &\geqslant 0 && \text{(Lemma 5.3.5).} \end{aligned}$$

Thus $A \lhd K$. ☐

We can define a closure operator in terms of strong extensions.

Definition 5.3.8 ([6, Definition 4.6, p. 17])**.** *Let K be a Γ-field and let $A \subseteq K$. We define the* hull of A in K *or* strong closure of A in K *as*

$$\lceil A \rceil_K = \bigwedge \{K' \lhd K \mid A \subseteq K'\}.$$

The following property of the strong closure is not difficult to deduce.

Lemma 5.3.9. *Let K be a Γ-field and let L be a Γ-field extension of K. If $K \lhd L$ and A is a Γ-field exponentially finitely generated over K in L, then $\lceil A \rceil_L$ is finitely generated over K.*

Proof. If $A \lhd L$, then $\lceil A \rceil_L = A$ and we are done. Assume that A is not strong in L so that there exists a Γ-field B exponentially finitely generated over A in L such that $\delta(B/A) < 0$. By the addition formula, we have

$$\delta(B/A) = \delta(B/K) - \delta(A/K).$$

Since $K \lhd L$, it follows that $\delta(B/K) \leqslant \delta(A/K)$. Take B such that $\delta(B/K) \leqslant \delta(A/K)$ is minimal. Then, for any Γ-field C exponentially finitely generated over B in L, we must have $\delta(C/B) \geqslant 0$. Assume otherwise so that, by addition formula, we would have

$$\delta(C/B) = \delta(C/A) - \delta(B/A)$$

which implies, by our assumption, that $\delta(C/A) < \delta(B/A)$ contradicting the minimality of our choice of B. Thus $B \lhd L$. By definition of the hull, we have have that $\lceil A \rceil_L \subseteq B$. Since B is exponentially finitely generated over A, it follows, by Definition 5.3.8, that $\lceil A \rceil_L$ must be finitely generated over A. ☐

The strong closure can be characterized in terms of finitely generated Γ-fields. This lemma follows the proof of [6, Definition 4.6, p. 17], but is stated differently.

Lemma 5.3.10. *Let K and L be Γ-fields such that $K \lhd L$ and let A be a Γ-field exponentially finitely generated over K in L. Then $\lceil A \rceil_L$ is equal to*

$$\bigcup \{\lceil A_0 \rceil_L \mid A_0 \textit{ is a finitely generated } \Gamma\textit{-field in } A\}.$$

We say that the hull operator has finite character.

Proof. Let $\mathcal{A}$ be the union in the statement above. If $A_0 \subseteq A$, then we clearly have $\lceil A_0 \rceil_L \subseteq \lceil A \rceil_L$, so that $\mathcal{A} \subseteq \lceil A \rceil_L$. If B, C are Γ-fields finitely generated in A, then BC is also a Γ-field finitely generated in A. It follows that $\lceil BC \rceil_L$ is contained in $\mathcal{A}$. We deduce that $\mathcal{A}$ is a directed union of Γ-subfields of L. Using lattice theory, we deduce that $\mathcal{A}$ is a Γ-subfield of A. Clearly $A \subseteq \mathcal{A}$. We need only show that $\mathcal{A}$ is strong in L which would imply that $\lceil A \rceil_L \subseteq \lceil \mathcal{A} \rceil_L = \mathcal{A}$. Let X be an exponentially finitely generated extension of $\mathcal{A}$ with basis $b \in X_\Gamma^n$ for some $n \in \mathbb{N}$. Then, by finite character of δ, there exists a finitely generated Γ-subfield C of $\mathcal{A}$ such that, for any intermediate Γ-field $C \subseteq B \subseteq \mathcal{A}$, $\delta(b/\mathcal{A}) = \delta(b/B)$. In particular, we may choose B to be of the form $\lceil B_0 \rceil_L$ for some finite set $B_0 \subseteq A$. Since $\lceil B_0 \rceil_L \lhd L$, it follows that $\delta(X/\mathcal{A}) = \delta(b/\mathcal{A}) \geqslant 0$. Since X was arbitrary, we deduce that $\mathcal{A} \lhd L$ so that $\lceil A \rceil_L \subseteq \mathcal{A}$. □

5.4 A finitary closure operator and an analogue to the Bays-Kirby-Wilkie result

We now have all necessary prerequisites to prove the main results of this chapter. We will first begin by defining a key closure operator in the main arguments of this thesis and will use properties of the predimension function δ to give an analogue to the Bays-Kirby-Wilkie result [7, Theorem 1.2, p. 2]. As before, we follow the presentation section 4.3 of [6] up to Proposition 5.4.5.

Recall that E is some elliptic curve defined over $\mathcal{K}_0$ with $\mathcal{O}_E$ defined over $\mathcal{K}_0$ and F is an algebraically closed field which we consider as a Γ-field (with respect to $H(F) = (\mathbb{G}_a \times E)(F))$ by taking F_Γ to be some divisible $\mathcal{O}_E$-submodule of $H(F)$.

The following definition, slightly different from [6, Definition 4.9, p. 18], plays a crucial role in our arguments.

Definition 5.4.1. *A Γ-subfield A of F is Γ-closed in F, written $A \lhd_{\mathrm{cl}} F$, if, for any $A \subseteq B \subseteq F$ with B exponentially finitely generated over A and $\delta(B/A) \leqslant 0$, we have $B = A$ as Γ-fields.*

The following is immediate from definition.

Lemma 5.4.2 ([6, Lemma 4.10, p. 18])**.** *If $A \lhd_{\mathrm{cl}} F$, then $A \lhd F$.*

Definition 5.4.3 ([6, Definition 4.11, p. 18])**.** *Let $A \subseteq F$. The* Γ-closure of A in F *is defined to be the smallest* Γ*-closed* Γ*-subfield containing* A*,*

$$\Gamma\mathrm{cl}^F(A) := \bigwedge\{B \lhd_{\mathrm{cl}} F \mid A \subseteq B\}.$$

This induces a map $\Gamma\mathrm{cl}^F : \mathcal{P}(F) \to \mathcal{P}(F)$*.*

We now work towards proving that Γ-closure is a finitary closure operator by giving a characterisation.

Lemma 5.4.4 ([6, Lemma 4.12, p. 18])**.** *For any* Γ*-subfield* K *of* F*, we have* $\Gamma\mathrm{cl}^F(K) = \bigcup \mathcal{B}$ *where* $\mathcal{B}$ *is the set of all* Γ*-subfields* B *of* F *such that* B *is an exponentially finitely generated* Γ*-field extension of* $\lceil K \rceil_F$ *and* $\delta(B/\lceil K \rceil_F) = 0$*.*

Proof. Combining Lemma 5.4.2 and Lemma 5.3.7, by definition of $\Gamma\mathrm{cl}^F$, we have that $\Gamma\mathrm{cl}^F(K) \lhd F$. Thus $\lceil K \rceil_F \subseteq \Gamma\mathrm{cl}^F(K)$. Since $K \subseteq \lceil K \rceil_F$, it follows that $\Gamma\mathrm{cl}^F(K) = \Gamma\mathrm{cl}^F(\lceil K \rceil_F)$. Thus, we may assume that $K \lhd F$. Let $\mathcal{B}$ be as in the statement of the lemma and set $C = \bigcup \mathcal{B}$. If $A, B \in \mathcal{B}$, we have that

$$\begin{aligned}
\delta(AB/K) - \delta(B/K) &= \delta(AB/B) && \text{(addition formula)} \\
&\leqslant \delta(A/A \wedge B) && \text{(submodularity)} \\
&= \delta(A/K) - \delta(A \wedge B/K) && \text{(addition formula)} \\
&= 0.
\end{aligned}$$

Since $\delta(B/K) = 0$, we deduce that $\delta(AB/K) = 0$. It follows that $\mathcal{B}$ is directed by set inclusion since the compositum of two Γ-fields in $\mathcal{B}$ is an upper bound. Again, using lattice theory. Thus, C is a Γ-subfield of F.

We will now argue that $C = \Gamma\mathrm{cl}^F(K)$ by showing that $C \subseteq \Gamma\mathrm{cl}^F(K)$ and vice versa. Let $b \in F_\Gamma^n$ for some $n \in \mathbb{N}$ such that $\delta(b/C) \leqslant 0$. By finite character of δ, there is a finite $D_0 \subseteq C$ which generates a finitely generated Γ-subfield D such that $\delta(b/C) = \delta(b/D)$. For each $d \in D_0$, there exists a Γ-field B_d exponentially finitely generated over K in C containing d such that $\delta(B_d/K) = 0$. Let B be the compositum of all such B_d's. Then B is exponentially finitely generated over K, $\delta(B/K) = 0$ and $\delta(b/B) = \delta(b/C)$. We then have

$$\begin{aligned}
0 &\geqslant \delta(b/B) \\
&= \delta(b/K) - \delta(B/K) && \text{(Addition formula)} \\
&= \delta(b/K) \\
&\geqslant 0 && \text{(Lemma 5.3.5).}
\end{aligned}$$

Hence, $\delta(b/K) = 0$. It follows that the Γ-field which is generated by b and K is contained in $\mathcal{B}$. This implies that C contains b so that C is Γ-closed. Thus, $\Gamma\mathrm{cl}^F(K) \subseteq C$. We now show that $C \subseteq \Gamma\mathrm{cl}^F(K)$. Suppose that B is an exponentially finitely generated Γ-field extension of K such that $\delta(B/K) = 0$ and let D be a Γ-subfield of F such that $K \subseteq D$ and $D \lhd_{\mathrm{cl}} F$. We then have

$$\begin{aligned} \delta(BD/D) &\leqslant \delta(B/B \wedge D) && \text{(Submodularity)} \\ &= \delta(B/K) - \delta(B \wedge D/K) && \text{(Addition formula)} \\ &\leqslant 0 \end{aligned}$$

where the last line follows by the assumption that $K \lhd F$ and Lemma 5.3.5. Since $D \lhd_{\mathrm{cl}} F$, it follows that $B \subseteq D$. Hence, if $B \in \mathcal{B}$, then $B \subseteq \Gamma\mathrm{cl}^F(K)$. It follows that $C \subseteq \Gamma\mathrm{cl}^F(K)$. □

We now have the necessary tools to prove the following.

Proposition 5.4.5 ([6, Lemma 4.14, p. 19])**.** *$(F, \Gamma\mathrm{cl}^F)$ is a finitary closure operator.*

Proof. Let $A \subseteq B \subseteq F$. Clearly, $A \subseteq \Gamma\mathrm{cl}^F(A)$. Since $\Gamma\mathrm{cl}^F(C)$ is the smallest Γ-closed field that contains some $C \subseteq F$, it also immediately follows that $\Gamma\mathrm{cl}^F(\Gamma\mathrm{cl}^F(A)) = \Gamma\mathrm{cl}^F(A)$ as, using arguments in the proof of Proposition 5.4.4, $\Gamma\mathrm{cl}^F(A)$ is already a Γ-closed field. It is also immediate that $\Gamma\mathrm{cl}^F(A) \subseteq \Gamma\mathrm{cl}^F(B)$ as all Γ-closed fields that contain B will contain A.

We now show that $\Gamma\mathrm{cl}^F$ has finite character. To that end, suppose that $a \in \Gamma\mathrm{cl}^F(A)$. By Lemma 5.4.4, there is a Γ-field K exponentially finitely generated over $\lceil A \rceil_F$ such that $a \in K$ and $\delta(K, \lceil A \rceil_F) = 0$. Let $\beta \subseteq F_\Gamma$ be the finitely many generators for K over $\lceil A \rceil_F$. Then a is in the field generated by β and $\lceil A \rceil_F$. By finite character of δ, there exists a finitely generated Γ-field $A_0 \subseteq \lceil A \rceil_F$ such that for any Γ-field A', with $A_0 \subseteq A' \subseteq \lceil A \rceil_F$, we have $\delta(\beta/A') = 0$. Hence, there is a finite $\alpha \subseteq \lceil A \rceil_F$ such that, $a \in \Gamma\mathrm{cl}^F(A_0 \cup \alpha)$. By Lemma 5.3.10, there is a finite subset $A_{00} \subseteq A$ such that $\lceil A_0 \cup \alpha \rceil_F \subseteq \lceil A_{00} \rceil_F$. We conclude that $a \in \Gamma\mathrm{cl}^F(A_{00})$. □

A key ingredient of Boxall's arguments within [11] was to use o-minimality to describe exponential closure in terms of algebraic closure in certain cases. We work no differently and will now introduce our o-minimal setting that will allow us (in chapter 6) to also describe Γ-closure in terms of algebraic closure.

Let $\mathcal{K}_0 = \overline{\mathbb{Q}}$ so that E is an elliptic curve defined over $\overline{\mathbb{Q}}$. Let $\wp$ be the Weierstrass elliptic function associated to E and let $\mathcal{F}$ be the set consisting of the holomorphic functions $\wp$ and $\frac{1}{\wp}$. Consider the following definition from [23, Definition 2.2, p. 3]:

Definition 5.4.6. *Let $U \subseteq \mathbb{R}^n$ for some $n \in \mathbb{N}$ be open and let $f : U \to \mathbb{R}$ be a function. A* proper restriction *of f is a restriction $f|_D$ where D is an open rectangular box with rational corners such that $\overline{D}$ is contained within U. If $d \in D$, then we say that $f|_D$ is a proper restriction of f around d.*

We will write $\mathbb{R}_{\mathrm{PR}(\mathcal{F})}$ to be the smallest expansion of the real field $\mathbb{R}$ such that all proper restrictions of functions in $\mathcal{F}$ are definable. This structure has the feature of being o-minimal as described at the top of [23, p. 4]. Let $\mathcal{R}$ be a κ-saturated elementary extension of $\mathbb{R}_{\mathrm{PR}(\mathcal{F})}$ for some infinite cardinal $\kappa > 2^{\aleph_0}$. We note that in $\mathbb{R}_{\mathrm{PR}(\mathcal{F})}$, $\wp$ and $\wp'$ are locally definable which means that, for any $f = \{\wp, \wp'\}$, the restriction of f to some neighbourhood of each point of its domain is definable (see [23, p. 1]). Furthermore, if Λ is the lattice associated to E (see [31, Theorem 5.1 and Corollary 5.1.1, p. 173]), the function $\exp : \mathbb{C}\backslash\Lambda \to E(\mathbb{C})$ given by sending $z \in \mathbb{C}\backslash\Lambda$ to $[\wp(z) : \wp(z)' : 1]$ is locally definable. We will identify $F = \mathcal{R}^2$ and consider it a Γ-field by setting F_Γ to be the graph of exp restricted to the infinitesimal elements of F , i.e. the elements of F whose real and imaginary parts are infinitesimal in $\mathcal{R}$. As a convention, we will denote, for any $a = (a_1, \ldots, a_n) \in F^n$ for some $n \in \mathbb{N}$, $(\exp(a_1), \ldots, \exp(a_n))$ as $\exp(a)$ when $\exp(a_i)$ is defined for all $i \in [[1, n]]$.

We now prove an analogue of the Bays-Kirby-Wilkie result for infinitesimals in fields defined as above which, in our setting, follows the presentation of [11, Theorem 3.7, p. 9]. First, we remind the reader that the definition of Γ_1 is given in Definition 5.2.2.

Theorem 5.4.7. *Let $N \in \mathbb{N}$. Let $a = (a_1, \ldots, a_N) \in \Gamma_1(F_\Gamma)^N$ and let $K \subseteq F$ such that* $\mathrm{acl}(\Gamma\mathrm{cl}^F(K)) = K$. *Let $\lambda \in F$ such that $\lambda \notin K$. Suppose $\lambda a = (\lambda a_1, \ldots, \lambda a_N) \in \Gamma_1(F_\Gamma)^N$ and assume that* $\exp(a) = (\exp(a_1), \ldots, \exp(a_n))$ *is additively independent. Then*

$$\mathrm{td}(\exp(a), \exp(\lambda a)/K, \lambda) \geqslant N.$$

Proof. Let $b = (a, \lambda a)$. Since $\lambda \notin K$ and K is Γ-closed, it follows that

$$\mathrm{td}(b, \exp(b)/K) - \mathrm{ldim}_{k_{\mathcal{O}_E}}(b, \exp(b)/K_\Gamma) \geqslant 1$$

which gives that

$$\mathrm{td}(b/K) + \mathrm{td}(\exp(b)/K, b) - \mathrm{ldim}_{k_{\mathcal{O}_E}}(b, \exp(b)/K_\Gamma) \geqslant 1.$$

We have

$$\mathrm{td}(b/K, \lambda) \leqslant \mathrm{ldim}_{k_{\mathcal{O}_E}(\lambda)}(b/K)$$

and

$$\mathrm{td}(\exp(b)/K, b) \leqslant \mathrm{td}(\exp(b)/K, \lambda).$$

It follows that

$$\mathrm{td}(\exp(b)/K, \lambda) + \mathrm{ldim}_{k_{\mathcal{O}_E}(\lambda)}(b/K) - \mathrm{ldim}_{k_{\mathcal{O}_E}}(b, \exp(b)/K_\Gamma) \geqslant 0.$$

We now consider $\ker = \{c \in F \mid \exp(c) = 0_E\}$ and let $\ker_\Gamma = \{(c, \exp(c)) \mid c \in \ker\}$. By finite character of span, there exists $(l, \exp(l)) \in K_\Gamma^n$ for some $n \in \mathbb{N}$ such that

$$\mathrm{ldim}_{k_{\mathcal{O}_E}}(b, \exp(b)/K_\Gamma) = \mathrm{ldim}_{k_{\mathcal{O}_E}}(b, \exp(b)/\langle\{(l, \exp(l))\} \cup \ker_\Gamma\rangle).$$

Let K' be the field generated by $\{l\} \cup \ker$. We have $K' \subseteq K$ so that

$$\mathrm{td}(\exp(b)/K, \lambda) + \mathrm{ldim}_{k_{\mathcal{O}_{EE}}(\lambda)}(b/K') - \mathrm{ldim}_{k_{\mathcal{O}_{EE}}}(b, \exp(b)/K_\Gamma) \geqslant 0.$$

We now have

$$\begin{aligned}
&\mathrm{ldim}_{k_{\mathcal{O}_E}}(b/\ker) - \mathrm{ldim}_{k_{\mathcal{O}_E}}(b, \exp(b)/K_\Gamma)\\
&= \mathrm{ldim}_{k_{\mathcal{O}_E}}(b, \exp(b)/\ker_\Gamma) - \mathrm{ldim}_{k_{\mathcal{O}_E}}(b, \exp(b)/\langle\{(l, \exp(l))\} \cup \ker_\Gamma\rangle)\\
&= (\mathrm{ldim}_{k_{\mathcal{O}_E}}(b, \exp(b), l, \exp(l)/\ker_\Gamma) - \mathrm{ldim}_{k_{\mathcal{O}_E}}(l, \exp(l)/\langle\{b, \exp(b)\} \cup \ker_\Gamma\rangle))\\
&\quad - (\mathrm{ldim}_{k_{\mathcal{O}_E}}(b, \exp(b), l, \exp(l)/\ker_\Gamma) - \mathrm{ldim}_{k_{\mathcal{O}_E}}(l, \exp(l)/\ker_\Gamma))\\
&= \mathrm{ldim}_{k_{\mathcal{O}_E}}(l, \exp(l)/\ker_\Gamma) - \mathrm{ldim}_{k_{\mathcal{O}_E}}(l, \exp(l)/\langle\{(b, \exp(b))\} \cup \ker_\Gamma\rangle)\\
&= \mathrm{ldim}_{k_{\mathcal{O}_E}}(l/\ker) - \mathrm{ldim}_{k_{\mathcal{O}_E}}(l, \exp(l)/\langle\{(b, \exp(b))\} \cup \ker_\Gamma\rangle)\\
&= \mathrm{ldim}_{k_{\mathcal{O}_E}(\lambda)}(l/\ker) - \mathrm{ldim}_{k_{\mathcal{O}_E}}(l, \exp(l)/\langle\{(b, \exp(b))\} \cup \ker_\Gamma\rangle)\\
&= \mathrm{ldim}_{k_{\mathcal{O}_E}(\lambda)}(l/\ker) - \mathrm{ldim}_{k_{\mathcal{O}_E}}(l/\{b\} \cup \ker)\\
&\leqslant \mathrm{ldim}_{k_{\mathcal{O}_E}(\lambda)}(l/\ker) - \mathrm{ldim}_{k_{\mathcal{O}_E}(\lambda)}(l/\{b\} \cup \ker)\\
&= (\mathrm{ldim}_{k_{\mathcal{O}_E}(\lambda)}(l, b/\ker) - \mathrm{ldim}_{k_{\mathcal{O}_E}(\lambda)}(b/l \cup \ker))\\
&\quad - (\mathrm{ldim}_{k_{\mathcal{O}_E}(\lambda)}(l, b/\ker) - \mathrm{ldim}_{k_{\mathcal{O}_E}(\lambda)}(b/\ker))\\
&= \mathrm{ldim}_{k_{\mathcal{O}_E}(\lambda)}(b/\ker) - \mathrm{ldim}_{k_{\mathcal{O}_E}(\lambda)}(b/K')
\end{aligned}$$

where the third and second-last line above follow from the addition formula and the fifth equality follows from [7, Lemma 3.2 (ii) and (iii), p. 3]. We deduce that

$$\mathrm{ldim}_{k_{\mathcal{O}_E}}(b, \exp(b)/K_\Gamma) - \mathrm{ldim}_{k_{\mathcal{O}_E}(\lambda)}(b/K') \geqslant \mathrm{ldim}_{k_{\mathcal{O}_E}}(b/\ker) - \mathrm{ldim}_{k_{\mathcal{O}_E}(\lambda)}(b/\ker)$$

so that

$$\mathrm{td}(\exp(b)/K, \lambda) + \mathrm{ldim}_{k_{\mathcal{O}_E}(\lambda)}(b/\ker) - \mathrm{ldim}_{k_{\mathcal{O}_E}}(b/\ker) \geqslant 0.$$

We need only show that $\mathrm{ldim}_{k_{\mathcal{O}_E}}(b/\ker) - \mathrm{ldim}_{k_{\mathcal{O}_E}(\lambda)}(b/\ker) \geqslant N$. This is done analagously to that at the bottom of [7, p. 4], where we replace $\mathbb{Q}$ with $k_{\mathcal{O}_E}$ as appropriate. We note that our a, b, N will correspond to x, z, n in the relevent sections of [7]. □

Chapter 6

A Finiteness Result

To give the proof of [11, Theorem 1.2, p. 1], Boxall made use of o-minimality and the algebraic nature of the statement of the exponential closure operator given by Kirby in [26]. However, due to the rather complex definition of Γ-closure, there were no obvious analogous properties for Γ-fields to exploit. Looking elsewhere, another similar pregeometry to Γ-closure and exponential closure expressing functional transcendence in the case of the complex holomorphic functions was used by Jones, Kirby and Servi in [23], known as holomorphic closure, which was originally defined by Wilkie in [35]. For both exponential closure and holomorphic closure, respectively, methods of Hrushovski were used similarly to our setting. More importantly, pregeometries using derivations were used to characterise exponential closure and holomorphic closure. It became clear that using derivations would be a promising means of attack, although there was no clear proof to show that Γ-closure would be related to such a pregeometry. However, we eventually discovered work done by Alarcon, a PhD student of Wilkie, in [1] that generalised Wilkie's constructions and results to elementary extensions of an o-minimal expansion of the real ordered field. In particular, Alarcon described a pregeometry using derivations that would apply to our setting. The results in this paper sufficed to allow us to generalise Boxall's auxiliary results in [11, Section 4] so that we may prove the analogue of his main theorem.

6.1 Interlude on certain pregeometries

In this section, we state some results and definitions from [1] that will be of use. Recall that $\mathcal{R}$ is a κ-saturated elementary extension of $\mathbb{R}_{\mathrm{PR}(\mathcal{F})}$ for some infinite cardinal $\kappa > 2^{\aleph_0}$, F is its algebraic closure and that E is some elliptic curve defined over $\overline{\mathbb{Q}}$ and that we identify F with $\mathcal{R}^2$. Henceforth, definable refers to being definable with respect to $\mathcal{R}$. Additionally, in this non-standard setting, and throughout this chapter, a holomorphic function refers to that of

Peterzil and Starchenko defined in [30, Section 2].

The following closure operator can be thought of as the analogue of the exponential closure in [26] or the holomorphic closure in [23].

Definition 6.1.1 ([1, p. 25])**.** *Let* $A \subseteq F$ *and, for any* $n \in \mathbb{N}$*, let* $\mathcal{P}_n(\mathbb{C})$ *denote the ring of definable, with parameters in* $\mathbb{C}$*, holomorphic functions in a definable, with parameters in* $\mathbb{C}$*, neighbourhood of the origin in* F^n*. Define* $\mathrm{Alg/An}(A)$ *as the smallest subset of* F *containing* A *such that*

- *If* $a_1, \ldots, a_n$*, for some* $n \in \mathbb{N}$*, are infinitesimal elements of* $\mathrm{Alg/An}(A)$ *and* $F \in \mathcal{F}_n(\mathbb{C})$*, then* $F(a_1, \ldots, a_n) \in \mathrm{Alg/An}(A)$*, and*
- $\mathrm{Alg/An}(A)$ *is algebraically closed.*

The above definition was only seen to be a finitary closure operator and, as such, Alarcon supplemented the above definition to define a pregeometry of the form below.

Definition 6.1.2 ([1, p. 41])**.** *Let* $A \subseteq F$ *be finite. We define* the completed algebraic-analytic closure*, denoted by* $\overline{\mathrm{Alg/An}}(A)$ *to be the set of all* $a \in F$ *such that there exist an* $n \in \mathbb{N}$ *and a definable open set* $D \subseteq F^n$ *satisfying*

$$a \in \bigcap_{(x_1,\ldots,x_n)\in D} \mathrm{Alg/An}(A \cup \{x_1, \ldots, x_n\}).$$

For general $A \subseteq F$*, we define* $\overline{\mathrm{Alg/An}}(A)$ *as* $\bigcup_{\substack{A_0 \subseteq A \\ \text{with } A_0 \text{ finite}}} \overline{\mathrm{Alg/An}}(A_0)$.

We now introduce derivations that will play a crucial role in giving properties of Γcl.

Definition 6.1.3 ([1, p. 34])**.** *A* derivation on a subfield T of F *is a map* $\delta : T \to T$ *such that, for all* $a, b \in T$*,*

- $\delta(a + b) = \delta(a) + \delta(b)$
- $\delta(ab) = b\delta(a) + a\delta(b)$.

Let $f(z_1, \ldots, z_n)$ *be a holomorphic* F*-valued function,* $a = (a_1, \ldots, a_n) \in F^n$ *and* δ *a derivation on* F*. We say that* δ respects f at a *if*

$$\delta(f(a)) = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(a) \cdot \delta(a_i).$$

We say that δ respects f *if* δ respects f *for all* $a \in \mathrm{dom}(f)$*.*

This next well-known lemma follows easily by using Definition 6.1.3.

Lemma 6.1.4. *For any $q \in \overline{\mathbb{Q}} \subseteq F$ and any derivation δ on F, $\delta(q) = 0$.*

A collection of derivations on F gives rise to the following pregeometry.

Definition 6.1.5 ([1, p. 34])**.** *Given a collection of holomorphic F-valued functions, $\mathcal{G}$, from connected open subsets of F, we denote the set of all derivations that respect every function in $\mathcal{G}$ as $\mathrm{Der}_F(\mathcal{G})$. For any $A \subseteq F$, we define $DD_{\mathcal{G}}(A)$ as the set of all $a \in F$ such that there exists a finite subset $A_0 \subseteq A$ such that every $\delta \in \mathrm{Der}_F(\mathcal{G})$ that vanishes at A_0 will vanish at a.*

To apply the above pregeometry (see Fact 6.1.7), we introduce the following set of functions.

Definition 6.1.6. *We define Δ to be the collection of all holomorphic F-valued functions definable over $\mathbb{C}$ defined in a neighbourhood of the origin.*

To no surprise, as in the case of exponential closure and holomorphic closure, Alarcon characterised $\overline{\mathrm{Alg/An}}$ in terms of derivations.

Theorem 6.1.7 ([1, Theorem 4.4.3, p. 43])**.** $\overline{\mathrm{Alg/An}}$ *and DD_Δ are pregeometries and $DD_\Delta(A) = \overline{\mathrm{Alg/An}}(A)$ for all $A \subseteq F$.*

6.2 Main result

To conclude the thesis, we will prove a few smaller results and one of the main results in the thesis. We now further assume $\mathcal{R}$ to be a strongly κ-homogeneous and κ-saturated extension of $\mathbb{R}_{\mathrm{PR}(\mathcal{F})}$ for some cardinal $\kappa > 2^{\aleph_0}$. Let A denote the infinitesimal elements of $\mathcal{R}$ and let $\tilde{\mathcal{R}}$ be an ultrapower of $\mathcal{R}$ with respect to a non-prinicipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Let $\tilde{\mathbb{N}}$, $\tilde{\mathbb{C}}$ and $\tilde{A}$ denote the canonical extension of $\mathbb{N}$, $\mathbb{C}$, and A, respectively, to $\tilde{\mathcal{R}}$.

In [26], Kirby used work of Ax in [4] to prove a functional transcendence result and he had also done further work in [25] to generalise the work of Ax to semi-abelian varieties. We, too, will make use of Kirby's generalisations.

Definition 6.2.1 ([25, p. 33])**.** *Let $\mathcal{D}$ be a set of derivations on a field K and let $\mathcal{D}_0 = \{\delta_1, \dots, \delta_r\} \subseteq \mathcal{D}$. We define, for any $a = (a_1, \dots, a_n) \in K^n$,*

$$\mathrm{Jac}_{\mathcal{D}_0}(a) := \begin{pmatrix} \delta_1 a_1 & \dots & \delta_1 a_n \\ \vdots & \ddots & \vdots \\ \delta_r a_1 & \dots & \delta_r a_n \end{pmatrix}$$

and define $\mathrm{rkJac}_{\mathcal{D}} := \max\{$*rank of* $\mathrm{Jac}_{\mathcal{D}'}(a) \mid \mathcal{D}' \subseteq \mathcal{D}$ *is finite*$\}$.

Fact 6.2.2 ([25, Theorem 3.8, p. 37])**.** *Let K be a field of characteristic 0, let $\mathcal{D}$ be a collection of derivations on K, and let C be the intersection of their constant fields. Let S be a semi-abelian variety defined over C of dimension n and let $\Omega_S \subseteq LS \times S$ be the solution set of the exponential differential equation of S (that is, the intersection of the solution sets for each $\delta \in \mathcal{D}$) where LS is the tangent space at 0 of S.*

Suppose that $(x, y) \in \Omega_S$ and $\operatorname{td}(x, y/C) - \operatorname{rkJac}(x, y) < n$. Then there is a proper algebraic subgroup G of S and a constant point $\gamma \in TS(C)$ such that (x, y) lies in the coset $\gamma \cdot TG$ where TG, TS denotes the tangent bundle of G and S, respectively.

Using some inspiration from arguments used in the proof of Proposition 10.7 in [6, p. 44], we now have the tools to relate $\Gamma\mathrm{cl}$ to a pregeometry defined using derivations.

Proposition 6.2.3. *Let $C \subseteq F$. Then $\Gamma\mathrm{cl}^F(C) \subseteq \mathrm{DD}_\Delta(C)$.*

Proof. Suppose that C is DD_Δ-closed. We consider C as a Γ-field by taking C_Γ to be $F_\Gamma \cap H(C)$. Let B be a proper exponentially finitely generated Γ-field extension of C in F. Let $b = (b_1, \ldots, b_n) \in B_\Gamma^n$ be a basis for the extension over C for some $n \in \mathbb{N}$ and let $V = \mathrm{Loc}(b/C)$ where $\mathrm{Loc}(b/C)$ denotes the smallest variety containing b defined over C. Let $b_i = (b_{i,1}, \exp(b_{i,1}))$ for some $b_{i,1} \in A$ for all $i \in [[1, n]]$. We need only prove that $\delta(b/C) > 0$ which amounts to showing that $\dim(V) > n$. Arguing as in [23, Proposition 6.8, p. 10], the $\mathcal{O}_E$-linear independence of the b_i implies that b does not lie in any coset $\gamma \cdot TG$ where G is a proper algebraic subgroup of E^n and γ is a point of $TG(C)$. To apply Fact 6.2.2, we will use $\mathrm{Der}_F(\Delta)$ as our set of derivations. We need now only check that b satisfies the exponential differential equation of E for each derivation in $\mathrm{Der}_F(\Delta)$. To that end, for any $\delta \in \mathrm{Der}_F(\Delta)$, we show that

$$\begin{aligned}\delta(\wp(b_{i,1})) &= \frac{\partial \wp(b_{i,1})}{\partial z} \cdot \delta(b_{i,1}) \\ &= \wp'(b_{i,1}) \cdot \delta(b_{i,1})\end{aligned}$$

for all $i \in [[1, n]]$. We note that, as $\wp$ is not holomorphic in a neighbourhood of zero, that $\frac{1}{\wp} \in \Delta$ so that any $\delta \in \mathrm{Der}_F(\Delta)$ respects $\frac{1}{\wp}$. Therefore, for any

$i \in [[1, n]]$, as $\wp(b_{i,1}) \neq 0$, we have that

$$\begin{aligned}
0 &= \delta(1) \\
&= \delta(\wp(b_{i,1}) \cdot \frac{1}{\wp(b_{i,1})}) \\
&= \delta(\wp(b_{i,1})) \cdot \frac{1}{\wp(b_{i,1})} + \wp(b_{i,1}) \cdot \delta(\frac{1}{\wp(b_{i,1})}) \\
&= \delta(\wp(b_{i,1})) \cdot \frac{1}{\wp(b_{i,1})} - \wp(b_{i,1}) \cdot (\frac{1}{\wp(b_{i,1})})^2 \cdot \wp'(b_{i,1}) \cdot \delta(b_{i,1}) \quad (\delta \text{ respects } \frac{1}{\wp}) \\
&= \delta(\wp(b_{i,1})) \cdot \frac{1}{\wp(b_{i,1})} - \frac{\wp'(b_{i,1})}{\wp(b_{i,1})} \cdot \delta(b_{i,1}).
\end{aligned}$$

so that $\delta(\wp(b_{i,1})) = \wp'(b_{i,1}) \cdot \delta(b_{i,1})$. Now, applying Fact 6.2.2, we have that

$$\mathrm{td}(b/C) - \mathrm{rkJac}(b) \geqslant n.$$

Since B is a proper extension of C and C is DD_Δ-closed, it follows that there is a $\delta \in \mathrm{Der}_F(\Delta)$ such that $\delta(b_{i,1}) \neq 0$ for some $i \in [[1, n]]$. Thus, $\mathrm{rkJac}(b) \geqslant 1$ so that

$$\mathrm{td}(b/C) \geqslant n + 1 > n.$$

We deduce that $\dim(V) > n$. □

The following essential result was contributed by G.J. Boxall in discussion of this thesis where dcl refers to the definable closure with respect to the structure $\mathcal{R}$. In particular, this result will allow us to bound $\Gamma\mathrm{cl}$ in a sense.

Lemma 6.2.4. *For any* $C \subseteq F$, $\overline{\mathrm{Alg/An}(C)} \subseteq (\mathrm{dcl}(C \cup \mathbb{C}))^2$.

Proof. Let $C \subseteq F$. By definition of Alg/An, it is clear that $\mathrm{Alg/An}(C) \subseteq (\mathrm{dcl}(C \cup \mathbb{C}))^2$. To show that $\overline{\mathrm{Alg/An}(C)} \subseteq (\mathrm{dcl}(C \cup \mathbb{C}))^2$, let $(a_1, a_2) \in \overline{\mathrm{Alg/An}(C)}$. We want to show that $a_1, a_2 \in \mathrm{dcl}(C \cup \mathbb{C})$. By definition of $\overline{\mathrm{Alg/An}}$, there exists a definable open set D and $(x_1, \ldots, x_n) \in D$, with $x_i = (x_{i,1}, x_{i,2})$ for some $x_{i,1}, x_{i,2} \in F$ for all $i \in [[1, n]]$, such that $(a_1, a_2) \in \mathrm{Alg/An}(C \cup \{x_1, \ldots, x_n\})$. Hence, $(a_1, a_2) \in (\mathrm{dcl}(C \cup \mathbb{C} \cup \{x_1, \ldots, x_n\}))^2$. We will only show that $a_1 \in \mathrm{dcl}(C \cup \mathbb{C})$ as the case for a_2 is analogous. Assume that $a_1 \notin \mathrm{dcl}(C \cup \mathbb{C})$. We may also assume that $\dim_{\mathrm{dcl}}(x_{1,1}, \ldots, x_{n,2}/A \cup \mathbb{C} \cup \{a_1\}) = 2n$. By exchange, we have $x_{i,j} \in \mathrm{dcl}(\{x_{1,1}, \ldots, \hat{x_{i,j}}, \ldots, x_{n,2}, a_1\} \cup C \cup \mathbb{C})$ for some $i \in [[1, n]]$ and $j \in [[1, 2]]$. This contradicts the dcl-dimension of $\{x_1, \ldots, x_n\}$ over $\{a_1\} \cup C \cup \mathbb{C}$. We conclude that $a_1 \in \mathrm{dcl}(C \cup \mathbb{C})$. □

Putting together Theorem 6.1.7, Proposition 6.2.3 and Lemma 6.2.4, we have the following.

Corollary 6.2.5. *For any* $C \subseteq F$, $\Gamma\mathrm{cl}^F(C) \subseteq (\mathrm{dcl}(C \cup \mathbb{C}))^2$.

The following result plays a key role in generalising the arguments of Boxall in [11, Section 4].

Lemma 6.2.6. *Let* $C \subseteq \tilde{\mathbb{C}}$*. Then* $\Gamma\mathrm{cl}^{\tilde{F}}(C) \subseteq \mathrm{acl}(C)$.

Proof. We first show that $\langle C \cup \Gamma\mathrm{cl}^{\tilde{F}}(C)_\Gamma, \Gamma\mathrm{cl}^{\tilde{F}}(C)_\Gamma\rangle_{\tilde{F}} = \Gamma\mathrm{cl}^{\tilde{F}}(C)$ as Γ-fields. Since $C \subseteq \tilde{\mathbb{C}}$, we may consider it as a Γ-subfield of $\tilde{F}$ by setting $C_\Gamma = \{(0, 0_E)\}$. It follows that $\langle C \cup \Gamma\mathrm{cl}^{\tilde{F}}(C)_\Gamma, \Gamma\mathrm{cl}^{\tilde{F}}(C)_\Gamma\rangle_{\tilde{F}}$ is a Γ-subfield of $\tilde{F}$. Denote $\langle C \cup \Gamma\mathrm{cl}^{\tilde{F}}(C)_\Gamma, \Gamma\mathrm{cl}^{\tilde{F}}(C)_\Gamma\rangle_{\tilde{F}}$ by D. By Definition 5.4.3, we need only show that D is Γ-closed. Let B be an exponentially finitely generated Γ-field over D, with basis $b \in B_\Gamma^n$ for some $n \in \mathbb{N}$, such that $\delta(B/D) \leqslant 0$. We then have

$$\mathrm{td}(b/D) - \mathrm{ldim}_{k_{\mathcal{O}_E}}(b, \Gamma\mathrm{cl}^{\tilde{F}}(C)_\Gamma) = \mathrm{td}(b/D) - \mathrm{ldim}_{k_{\mathcal{O}_E}}(b, D_\Gamma) \quad = \delta(b/D) \leqslant 0.$$

This implies that

$$\mathrm{ldim}_{k_{\mathcal{O}_E}}(b, \Gamma\mathrm{cl}^{\tilde{F}}(C)_\Gamma) \geqslant \mathrm{td}(b/D) \geqslant \mathrm{td}(b/\Gamma\mathrm{cl}^{\tilde{F}}(C)).$$

Since $\Gamma\mathrm{cl}^{\tilde{F}}(C)$ is Γ-closed, this implies that $b \in \Gamma\mathrm{cl}^{\tilde{F}}(C)_\Gamma^n$ so that D is Γ-closed. We deduce that $D = \Gamma\mathrm{cl}^{\tilde{F}}(C)$. Since, by Corollary 6.2.5, $\Gamma\mathrm{cl}^{\tilde{F}}(C) \subseteq (\mathrm{dcl}(C \cup \mathbb{C}))^2$, $\langle C \cup \Gamma\mathrm{cl}^{\tilde{F}}(C)_\Gamma, \Gamma\mathrm{cl}^{\tilde{F}}(C)_\Gamma\rangle_{\tilde{F}} = \Gamma\mathrm{cl}^{\tilde{F}}(C)$ and $\tilde{A} \cap \tilde{\mathbb{C}} = \{0\}$, we must have that $C = \Gamma\mathrm{cl}^{\tilde{F}}(C)$. Clearly, as $C \subseteq \mathrm{acl}(C)$, the result follows. ☐

As there are only continuum many definable functions in $\mathbb{R}_{\mathrm{PR}(\mathcal{F})}$, the following is immediate.

Lemma 6.2.7. *Let* $C \subseteq F$*, then* $|\mathrm{dcl}(C \cup \mathbb{C})| \leqslant \max\{|C|, |\mathbb{C}|\}$.

To prove Conjecture 1.0.3, it suffices, by Proposition 4.0.9, to prove the following result. We are thus in the position to prove one of the main aims of the thesis by using the tools proven in this section. We follow the arguments of [11, Theorem 4.4, p. 12].

Theorem 6.2.8. *Let* $C_1, C_2 \subseteq E^3(\mathbb{C})$ *be irreducible closed algebraic curves such that at least one of the curves is not defined over* $\overline{\mathbb{Q}}$*. Suppose that there does not exist a proper algebraic subgroup* $H \subseteq E^3(\mathbb{C})$ *such that* $C_1 \subseteq H$*. Let* $\mathcal{N} = \{n \in \mathbb{N} \mid [n]C_1 \subseteq C_2\}$*. Then* $\bigcup_{n \in \mathbb{N} \setminus \mathcal{N}} \{x \in C_1 \mid nx \in C_2\}$ *is finite.*

Proof. Assume that $\bigcup_{n \in \mathbb{N} \setminus \mathcal{N}} \{x \in C_1 \mid nx \in C_2\}$ is infinite. We will argue for a contradiction. Let $e_1, \ldots, e_k \in \mathbb{C}$ be algebraically independent such that C_1 and C_2 are defined over $K = \mathrm{acl}(e_1, \ldots, e_k)$. Letting $e_i = (e_{i,1}, e_{i,2})$ for some $e_{i,1}, e_{i,2} \in \mathcal{R}$ for all $i \in [[1, k]]$, we may assume that $e_{1,1}, e_{1,2} \ldots, e_{k,1}, e_{k,2}$ are dcl-independent in the sense of the underlying real-closed field. We will break the argument up into various steps.

Step 1: We construct a field automorphism $\phi : F \to F$ where $\phi(e_1), \ldots, \phi(e_k)$ are dcl-independent over $\mathbb{C}$, as a $2k$-tuple in the sense of the full o-minimal structure, and $\phi(e_i) - e_i \in A$ for all $i \in [[1, k]]$.

First, let $p_{1,1}(x)$ be the partial type that expresses x is dcl-independent over $\mathbb{C}$, that $|x - e_{1,1}| < \frac{1}{n}$ for all $n \in \mathbb{N}$ and that x satisfies all formulae $\varphi(x)$ in the underlying real-closed field such that $\mathcal{R} \models \varphi(e_{1,1})$. As $|\mathrm{dcl}(\mathbb{C})| \leqslant 2^{\aleph_0}$ by Lemma 6.2.7 and $\mathcal{R}$ is κ-saturated, we deduce that there exists $c_{1,1} \in \mathcal{R}$ which satisfies the partial type $p_{1,1}(x)$. We then construct the partial type $p_{1,2}(x)$ which expresses that x is dcl-independent over $\mathbb{C} \cup \{c_{1,1}\}$, that $|x - e_{1,2}| < \frac{1}{n}$ for all $n \in \mathbb{N}$ and that x satisfies all formulae $\varphi(c_{1,1}, x)$ in the underlying real-closed field such that $\mathcal{R} \models \varphi(e_{1,1}, e_{1,2})$. Similarly to above, as $|\mathrm{dcl}(\mathbb{C} \cup \{c_{1,1}\})| \leqslant 2^{\aleph_0}$, again, by Lemma 6.2.7 and $\mathcal{R}$ is κ-saturated there exists $c_{1,2} \in \mathcal{R}$ which satisfies the partial type $p_{1,2}(x)$. We proceed in this manner, by induction, so that there exists $c_{1,1}, c_{1,2}, \ldots, c_{k,2} \in \mathcal{R}$ such that $\mathcal{R} \models p_{i,j}(c_{i,j})$ for all $i \in [[1, k]]$ and $j \in [[1, 2]]$. Furthermore, by our construction of $p_{i,j}$, $c_{i,j}$ has the same type as $e_{i,j}$, in the sense of the underlying real-closed field, for all $i \in [[1, k]]$ and $j \in [[1, 2]]$. As $\mathcal{R}$ is strongly κ-homogeneous, it follows that there exists an isomorphism $\psi : \mathcal{R} \to \mathcal{R}$, in the sense of real-closed fields, such that $\psi(e_{i,j}) = c_{i,j}$ for all $i \in [[1, k]]$ and $j \in [[1, 2]]$ and such that ψ fixes $\overline{\mathbb{Q}}$. Defining $\phi : F \to F$ by sending $a = (a_1, a_2) \in F$ to $(\psi(a_1), \psi(a_2))$ gives our wanted isomorphism.

Step 1 allows us to construct curves which are infinitesimal shifts of C_1 and C_2. Let $K' = \phi(K)$, $C_1' = \phi(C_1(F))$, $C_2' = \phi(C_2(F))$ and $e_i' = \phi(e_i)$ for all $i \in [[1, k]]$. As $C_1, C_2 \subseteq E^3(F)$ and ϕ fixes $\overline{\mathbb{Q}}$ and E is assumed to be defined over $\overline{\mathbb{Q}}$, we note that $C_1', C_2' \subseteq E^3(F)$.

Step 2: Given any $n \in \mathbb{N} \backslash \mathcal{N}$, $x = (x_1, x_2, x_3) \in C_1$ and $a = (a_1, a_2, a_3) \in \mathbb{C}^3$ such that $nx \in C_2$ and $\exp(a) = x$, we want to construct some $z = (z_1, z_2, z_3) \in C_1'(F)$ and $b = (b_1, b_2, b_3) \in F^3$ such that $nz \in C_2'(F)$, $a_i - b_i \in A$ and $\exp(b_i) = z_i$ for all $i \in [[1, 3]]$.

As a convention, we set $\exp(0) = 0_E$. There then exists $a = (a_1, a_2, a_3) \in \mathbb{C}$ such that $\exp(a) = x$. If $x_i = 0_E$, then we set $a_i = 0$, $b_i = 0$ and $z_i = 0_E$ for all $i \in [[1, 3]]$. We form the partial type which has the formulas that say that, for each real coordinate of non-zero a_1, a_2, a_3, you can specify all rational numbers that the real coordinate lies between and that $\exp(a) \in C_1$ and $\exp(na) \in C_2$. Since $e_1, \ldots, e_k$ has the same type as $e_1', \ldots, e_k'$, using the partial type above, there exists $z = (z_1, z_2, z_3) \in C_1'$ and $b = (b_1, b_2, b_3) \in F^3$ such that $nz \in C_2'$, $a_i - b_i \in A$ and $\exp(b_i) = z_i$ for all $i \in [[1, 3]]$.

Step 3: Using step 2, we construct elements in $C_1(\tilde{\mathbb{C}})$ and in $C_1'(\tilde{F})$ which will be key in finding a contradiction.

Since $\bigcup_{n \in \mathbb{N} \backslash \mathcal{N}} \{x \in C_1 \mid nx \in C_2\}$ is infinite, by Lemma 4.0.6, there exists an increasing sequence of natural numbers $(n_i)_{i\in\mathbb{N}}$ such that

- there exists $x_i = (x_{i,1}, x_{i,2}, x_{i,3}) \in C_1(\mathbb{C})$ with $n_i x_i \in C_2(\mathbb{C})$ and
- there exists $a_i = (a_{i,1}, a_{i,2}, a_{i,3}) \in \mathbb{C}^3$ with $\exp(a_i) = x_i$ and $\exp(n_i a_i) = n_i x_i$.

Furthermore, by step 2,

- there exists $z_i = (z_{i,1}, z_{i,2}, z_{i,3}) \in C_1'(F)$ with $n_i z_i \in C_2'(F)$ and
- there exists $b_i = (b_{i,1}, b_{i,2}, b_{i,3}) \in F^3$ with $\exp(b_i) = z_i$ and $\exp(n_i b_i) = n_i z_i$ such that $a_i - b_i \in A^3$.

We can now define, in $\tilde{\mathcal{R}}$,

- $\tilde{n} := [(n_i)_{i\in\mathbb{N}}]$,
- $\tilde{x} := [(x_i)_{i\in\mathbb{N}}]$,
- $\tilde{a} := [(a_i)_{i\in\mathbb{N}}]$,
- $\tilde{z} := [(z_i)_{i\in\mathbb{N}}]$ and
- $\tilde{b} := [(a_i)_{i\in\mathbb{N}}]$.

It follows, by properties of the ultrapower, that $\tilde{x} \in C_1(\tilde{\mathbb{C}})$ and $\tilde{z} \in C_1'(\tilde{F})$ such that $\tilde{n}\tilde{x} \in C_2(\tilde{\mathbb{C}})$, $\tilde{n}\tilde{z} \in C_2'(\tilde{F})$, $\exp(\tilde{a}) = \tilde{x}$, $\exp(\tilde{b}) = \tilde{z}$ and $\tilde{a} - \tilde{b} \in \tilde{A}^3$. We note that, by definition, $\wp(\tilde{a})$ is not algebraic over K.

<u>**Step 4**</u>: Use the properties of $\tilde{z}$ and $\tilde{x}$ to obtain a contradiction. We now consider the point $\tilde{z} - \tilde{x}$ which lies on the curve $C_1'(\tilde{F}) - \tilde{x}$ and the point $\tilde{n}\tilde{z} - \tilde{n}\tilde{x}$ which lies on the curve $C_2'(\tilde{F}) - \tilde{n}\tilde{x}$. By Proposition 2.6 (c) of [Silverman - Arithmetic of elliptic curves], it follows that both curves are defined over $K'(\wp(\tilde{a}), \wp'(\tilde{a}), \wp(\tilde{n}\tilde{a}), \wp'(\tilde{n}\tilde{a}))$. We now consider various cases for a contradiction.

<u>Case 1</u>: Suppose that $\tilde{z} - \tilde{x}$ is additively independent. Let u be a maximal subtuple of the tuple $(\wp(\tilde{a}), \wp'(\tilde{a}), \wp(\tilde{n}\tilde{a}), \wp'(\tilde{n}\tilde{a}))$ such that u is acl-independent over $K \cup \{n\}$. As $C_1'(\tilde{F}) - \tilde{x}$ is a curve defined over $K'(\wp(\tilde{a}), \wp'(\tilde{a}), \wp(\tilde{n}\tilde{a}), \wp'(\tilde{n}\tilde{a}))$, it follows that

$$\mathrm{td}(\tilde{z} - \tilde{x}/K'(\wp(\tilde{a}), \wp'(\tilde{a}), \wp(\tilde{n}\tilde{a}), \wp'(\tilde{n}\tilde{a}))) \leqslant 1.$$

Similarly, we have

$$\mathrm{td}(\tilde{n}\tilde{z} - \tilde{n}\tilde{x}/K'(\wp(\tilde{a}), \wp'(\tilde{a}), \wp(\tilde{n}\tilde{a}), \wp'(\tilde{n}\tilde{a}))) \leqslant 1.$$

Since $K'(\wp(\tilde{a}), \wp'(\tilde{a}), \wp(\tilde{n}\tilde{a}), \wp'(\tilde{n}\tilde{a})) \subseteq \operatorname{acl}(\Gamma\operatorname{cl}(K', K, u), \tilde{n})$, it follows that

$$\operatorname{td}(\tilde{z} - \tilde{x}, \tilde{n}\tilde{z} - \tilde{n}\tilde{x}/\operatorname{acl}(\Gamma\operatorname{cl}(K', K, u)), \tilde{n}) \leqslant 2.$$

By the manner in which we defined $\tilde{z}$ and $\tilde{n}\tilde{z}$ in step 3, we have that $\tilde{z} - \tilde{x} = \exp(\tilde{b}-\tilde{a})$ and $\tilde{n}\tilde{z}-\tilde{n}\tilde{x} = \exp(\tilde{n}\tilde{b}-\tilde{n}\tilde{a}) = \exp(\tilde{n}(\tilde{b}-\tilde{a}))$ where $\tilde{b}-\tilde{a}, \tilde{n}(\tilde{b}-\tilde{a}) \in \tilde{A}^3$. Thus, by Theorem 5.4.7, we must have $\tilde{n} \in \operatorname{acl}(\Gamma\operatorname{cl}(K', K, u))$. Since $e_1', \ldots, e_k'$ are dcl-independent over $\tilde{\mathbb{C}}$, they are acl(Γcl)-independent over $\tilde{\mathbb{C}}$ so that, by exchange, we must have $\tilde{n} \in \operatorname{acl}(\Gamma\operatorname{cl}(K, u))$. By Lemma 6.2.6, $\tilde{n} \in \operatorname{acl}(K, u)$. Since $\tilde{n}$ is infinite, it is immediate that $\tilde{n} \notin \operatorname{acl}(K)$. Using the exchange principal once more, we deduce that we contradict the acl-independence of u over $K \cup \{n\}$.

Case 2: Suppose that $\tilde{z} - \tilde{x}$ has additive dimension 2. Then there is an algebraic subgroup $G \subseteq E^3(\tilde{F})$ of dimension 2 such that $\tilde{z} - \tilde{x} \in G$. By Proposition 4.0.1, G is defined using elements of $\mathcal{O}_E$ which implies that $z_i - x_i \in G$ for all $i \in I$ for some $I \subseteq \mathcal{U}$. It follows that $n_i z_i - n_i x_i \in H$ for all $i \in I$, so that $\tilde{n}\tilde{z} - \tilde{n}\tilde{x} \in G$. We deduce that $\tilde{x} \in G + \tilde{z}$ and $\tilde{n}\tilde{x} \in G + \tilde{n}\tilde{z}$.

Subcase 2a: Suppose that both $C_1(\tilde{F}) \subseteq G + \tilde{z}$ and $C_2(\tilde{F}) \subseteq G + \tilde{n}\tilde{z}$. This implies, where $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{O}_E$, that

$$\sum_{j=1}^{3} \lambda_j(\tilde{x}_j) = \tilde{y}$$

for some $\tilde{y} \in E(\tilde{F})$. Since $\bigcup_{m \in \mathbb{N} \setminus \mathcal{N}} \{x \in C_1 \mid mx \in C_2\}$ is infinite, $\tilde{y}$ must be a torsion point. This contradicts the fact that C_1 is not contained in a proper algebraic subgroup of $E^3(\mathbb{C})$.

Subcase 2b: Suppose that $C_2(\tilde{F}) \nsubseteq G + \tilde{n}\tilde{z}$. We must have that $C_2(\tilde{F}) \cap G + \tilde{n}\tilde{z}$ at only finitely many points, otherwise, by previous arguments in Lemma 4.0.6, it would follow that $C_2(\tilde{F}) \subseteq G + \tilde{n}\tilde{z}$. We deduce that $\wp(na) \in \operatorname{acl}(\wp(nb), K, K')$. Now, as $e_1', \ldots, e_k'$ are dcl-independent over $\tilde{\mathbb{C}}$, and hence acl-independent over $\tilde{\mathbb{C}}$, and, by our definition of $\tilde{n}\tilde{a}$, $\wp(\tilde{n}\tilde{a})$ cannot be algebraic over K, it follows that $\wp(\tilde{n}\tilde{a}) \notin \operatorname{acl}(K, K')$. Hence, by exchange, $\wp(\tilde{n}\tilde{b}) \in \operatorname{acl}(\wp(\tilde{n}\tilde{a}), K, K')$. If $\tilde{n}\tilde{x}$ is a torsion point, then there are infinitely many $x_i \in C_1(F)$ that are torsion points. By Corollary 4.0.4, this implies that C_1 is contained in a proper algebraic subgroup of $E^3(\mathbb{C})$ which is a contradiction. So, $\tilde{n}\tilde{x}$ is non-torsion. We deduce, as $\wp(\tilde{n}\tilde{a})$ cannot be algebraic over K, that $\wp(\tilde{n}\tilde{a}) \notin \operatorname{acl}(K)$, so that, by exchange, $\wp(\tilde{n}\tilde{b}) \in \operatorname{acl}(\wp(\tilde{n}\tilde{a}), K, K')$. It follows that

$$\operatorname{td}(\tilde{n}\tilde{z} - \tilde{n}\tilde{x}/K, K', \tilde{n}\tilde{x}) = 0.$$

As in Case 1, let u be a acl-basis for $(\wp(\tilde{a}), \wp'(\tilde{a}), \wp(\tilde{n}\tilde{a}), \wp'(\tilde{n}\tilde{a}))$ over $K \cup \{\tilde{n}\}$. By Theorem 5.4.7 and that

$$\operatorname{td}(\tilde{z} - \tilde{x}, \tilde{n}\tilde{z} - \tilde{n}\tilde{x}/\operatorname{acl}(\Gamma\operatorname{cl}(K, K', u)), \tilde{n}) \leqslant 1,$$

we have that $\tilde{n} \in \operatorname{acl}(\Gamma\operatorname{cl}(K, K', u))$. As in Case 1, we obtain a contradiction.

Subcase 2c: The subcase where $C_1(\tilde{F}) \nsubseteq G + \tilde{z}$ is analogous to Subcase 2b where we instead have $\wp(\tilde{a}) \notin \operatorname{acl}(K, K')$.

Case 3: Suppose that $\tilde{z}-\tilde{x}$ has additive dimension 1. Then there is an algerbaic subgroup $G \subseteq E^3(\tilde{F})$ of dimension 1 such that $\tilde{x} \in G + \tilde{z}$ and $\tilde{n}\tilde{x} \in G + \tilde{n}\tilde{z}$. By Proposition 4.0.7, $C_1(\tilde{F}) \nsubseteq G + \tilde{z}$ and $C_2(\tilde{F}) \nsubseteq G + \tilde{n}\tilde{z}$. So $\wp(\tilde{a}) \in \operatorname{acl}(K, \wp(\tilde{b}))$ and $\wp(\tilde{n}\tilde{a}) \in \operatorname{acl}(K, \wp(\tilde{n}\tilde{b}))$. Using similar reasoning to Subcase 2b, we have that $\wp(\tilde{b}) \in \operatorname{acl}(K, K', \wp(\tilde{a}))$ and $\wp(\tilde{n}\tilde{b}) \in \operatorname{acl}(K, K'\wp(\tilde{n}\tilde{a}))$. Let u be a acl-basis for $(\wp(\tilde{a}), \wp'(\tilde{a}), \wp(\tilde{n}\tilde{a}), \wp'(\tilde{n}\tilde{a}))$ over $K \cup \{\tilde{n}\}$. By Theorem 5.4.7 and that

$$\operatorname{td}(\tilde{z} - \tilde{x}, \tilde{n}\tilde{z} - \tilde{n}\tilde{x}/\operatorname{acl}(\Gamma\operatorname{cl}(K, K', u)), \tilde{n}) = 0,$$

we have that $\tilde{n} \in \operatorname{acl}(\Gamma\operatorname{cl}(K, K', u))$. We again obtain a contradiction as in the previous cases.

Case 4: Suppose that $\tilde{z}-\tilde{x}$ has additive dimension 0. Then $\tilde{z}-\tilde{x} = (0_E, 0_E, 0_E)$, so that $z_i = x_i$ for infinitely many $i \in \mathbb{N}$. Thus, the curves $C_1(F)$ and $C_1'(F)$ have infinitely many points in common. It follows that by similar arguments to that in Lemma 4.0.6 that $C_1(F) = C_1'(F)$. Similarly, $C_2(F) = C_2'(F)$. Then $C_1(F)$ and $C_2(F)$ are both defined over $K \cap K' = \overline{\mathbb{Q}}$ which is a contradiction. □

Appendix A

Appendices

A.1 Appendix A: Topology

This chapter follows various sections of [16].

Definition A.1.1. *A* topological space *is a pair* (X, τ_X) *where* X *is a set and* τ_X *is a family of subsets of* X *such that*

1. *Both* $\varnothing \in \tau_X$ *and* $X \in \tau_X$.
2. *For any* $U, V \in \tau_X$, $U \cap V \in \tau_X$.
3. *For any family of sets* $\{U_i\}_{i \in I}$ *in* τ_X, $\bigcup\limits_{i \in I} U_i \in \tau_X$.

We call τ_X *a* topology of X *and elements of* τ_X open sets*. Complements of open sets are called* closed sets*. We will usually denote the pair* (X, τ_X) *by* X *whilst noting that it is a topological space with an associated topology* τ_X.

We may analogously define a topological space (X, τ_X) by specifying a topology with closed sets instead of open sets where we will instead need to check that the closed sets are stable under finite union and infinite intersection. In this case, τ_X known as a *closed set topology* where Definition A.1.1 τ_X gives an *open set topology.*

Definition A.1.2. *Let* X *be a topological space,* $\{U_i\}_{i \in I}$ *be a family of open sets in* X *and* V *an open subset of* X*. We say that* V *has an* open covering (in X) *if there exists a family* $\{U_i\}_{i \in I}$ *of open subsets in* X *such that* $V = \bigcup\limits_{i \in I} U_i$.

Definition A.1.3. *Let* X *be a topological space. A* basis, $\mathcal{B}$, *for the topology* X *is a collection of open sets of* X *such that any open set in* X *can be written as the union of elements in* $\mathcal{B}$.

Lemma A.1.4. *Let* X *be a topological space with base* $\mathcal{B}$*. If* $U, V \in \mathcal{B}$ *and* $x \in U \cap V$*, then there exists* $W \in \mathcal{B}$ *such that* $x \in W$ *and* $W \subseteq U \cap V$.

Proof. Since $U \cap V$ is open, $U \cap V$ is a union of elements in $\mathcal{B}$ by definition of $\mathcal{B}$. It follows that, by definition of arbitrary unions, that there exists some $W \in \mathcal{B}$ such that $x \in W$, thus giving the result. ☐

Definition A.1.5. *A topological space is* quasicompact *if every open cover has a finite subcover.*

Ordinarily in topology, one would call such a topological space "compact", however, we reserve the term "compact" for topological spaces that are both Hausdorff and quasicompact.

Definition A.1.6. *Let X be a topological space and $Y \subseteq X$ be non-empty. We say that Y is* irreducible *if $Y = Y_1 \cup Y_2$, for Y_1 and Y_2 closed in X, implies that $Y_1 = Y$ or $Y_2 = Y$. If a subset of X is not irreducible, then we call it* reducible. *By convention, we assume the empty set to be reducible.*

Definition A.1.7 ([20, p. 5])**.** *Let X be a topological space. We define the dimension of X (if it exists), denoted as* $\dim(X)$*, to be the supremum of all $n \in \mathbb{N}$ such that there exists a chain*

$$X_0 \subseteq X_1 \subseteq ... \subseteq X_n$$

of distinct irreducible closed subsets of X.

Definition A.1.8 ([20, p. 5])**.** *A topological space X is* Noetherian *if it satisfies the following property known as the* descending chain condition*: For any sequence*

$$X_1 \supseteq X_2 \supseteq ...$$

of closed subsets in X, there exists an $n \in \mathbb{N}$ such that $X_n = X_m$ for all $m \in \mathbb{N}$ where $n \leqslant m$.

Proposition A.1.9 ([20, p. 5])**.** *Let X be a Noetherian topological space. Then every non-empty closed subset Y of X can be expressed, for some $n \in \mathbb{N}$, as a finite union*

$$Y = Y_1 \cup ... \cup Y_n$$

where Y_i is an irreducible closed subset of Y for all $i = 1, ..., n$. If $X_i \nsubseteq X_j$ for all $i, j \in \{1, ..., n\}$ such that $i \neq j$, then the Y_i are unique (up to permutation). We call such irreducible sets the irreducible components of Y.

Proof. We first show existence of such a decomposition any closed subset of X. Let Y be such a closed set and Y cannot be written as a finite union of irreducible closed subsets. Thus, Y is reducible so that there exists non-empty closed subsets $Y_1, Y_1' \subseteq X$ such that $Y = Y_1 \cup Y_1'$. We must have that Y_1' is reducible, otherwise Y is irreducible. Therefore, there exists closed $Y_2, Y_2' \subseteq X$ such that $Y_1' = Y_2 \cup Y_2'$. It follows that $Y = Y_1 \cup Y_2 \cup Y_2'$. Similarly, Y_2' must be reducible. Continuing in this manner, we can construct a sequence of sets as follows: Let $Z_1 = Y$. We then inductively construct $Z_{i+1} = Z_i \backslash Y_i$ for all $i \in \mathbb{N}$. We thus have the sequence of inclusions

$$Z_1 \supset Z_2 \supset ...$$

Since this sequence does not stabilize, we have a contradiction since X is Noetherian. Thus Y can be written as a finite union of irreducible closed subsets.

We now show that the irreducible are unique (up to permutation). Let Y be a closed subset of X and let there be two decompositions $Y = Y_1 \cup ... \cup Y_n = Y_1' \cup ... \cup Y_m'$ for some $n, m \in \mathbb{N}$. Set $i \in \{1, ..., n\}$. We have $Y_1 \subseteq \bigcup_{j=1}^{m} Y_j'$ so that $Y_1 = \bigcup_{j=1}^{m} Y_i \cap Y_j'$. Since Y_i is irreducible, we must have $Y_i = Y_i \cap Y_j'$ which implies $Y_i \subseteq Y_j'$ for some fixed $j \in \{1, ..., m\}$. Similarly shown, we have $Y_j' \subseteq Y_k$ for some fixed $k \in \{1, ..., n\}$. Therefore, $X_i \subseteq X_j' \subseteq X_k$ which, by assumption, implies $i = k$. Hence $X_i = X_j'$. By applying the same argument to the X_j' for all $j \in \{1, ..., m\}$, we conclude that $m = n$ and that each decomposition is a permutation of the other. □

Proposition A.1.10 ([18, Exercise 2.19, p. 16]). *Let X be a topological space. If there exists an finite open covering $\{X_i\}_{i \in I}$ of X where $I = \{1, ..., n\}$ for some $n \in \mathbb{N}$ and X_i is Noetherian for all $i \in I$, then X is Noetherian.*

Proof. Assume that there exists a sequence of closed subsets in X such that

$$Y_1 \supseteq Y_2 \supseteq ...$$

Fix some $i \in \{1, ..., n\}$. We have that the

$$X_i \cap Y_1 \supseteq X_i \cap Y_2 \supseteq ...$$

stabilizes from (and including) some index $N_i \in \mathbb{N}$. Since $Y_i = \bigcup_{j=1}^{n} X_i \cap Y_i$ for all $i \in \mathbb{N}$, it follows that $(Y_i)_i$ stabilizes from (and including) $\max\{N_i \mid i \in \{1, ..., n\}\}$ so that X is Noetherian. □

Lemma A.1.11. *Let X be an irreducible topological space. Every open set in X is dense.*

Proof. Let $U \subseteq X$ be open. Then $U \cup (X \backslash U) = X$. It follows that $\overline{U} \cap (X \backslash U) = X$. Since X is irreducible, we must have either have $\overline{U} = X$ or $(X \backslash U) = X$, but clearly the latter cannot hold. □

Lemma A.1.12. *Let X be an irreducible topological space. Every open set is irreducible in X.*

Proof. We prove the result by contradiction. Let $U \subseteq X$ be open and assume that U is reducible so that there exists some closed $V, W \subseteq X$ such that $U = V \cup W$. By Lemma A.1.11, U is dense in X which implies that

$$V \cup W = \overline{V \cup W} = \overline{U} = X.$$

This contradicts that X is irreducible. □

Lemma A.1.13. *Let X, Y be topological spaces and let $f : X \to Y$ is a continuous map. If $Z \subseteq X$ is irreducible, then $f(Z)$ is irreducible.*

Proof. Assume otherwise, so that Z is irreducible, but $f(Z)$ is not irreducible. Then there exists closed sets Y_1 and Y_2 of $f(Z)$ such that $f(Z) = Y_1 \cup Y_2$ with $Y_1, Y_2 \neq f(Z)$. It follows that $Z = (Z \cap f^{-1}(Y_1)) \cup (Z \cap f^{-1}(Y_2))$. Since f is continuous, $f^{-1}(Y_1)$ and $f^{-1}(Y_2)$ are both closed. Furthermore, since $Y_1, Y_2 \neq f(Z)$, it follows that $(Z \cap f^{-1}(Y_1)), (Z \cap f^{-1}(Y_2)) \neq Z$ so that Z is reducible. A contradiction. □

Lemma A.1.14. *Let X be a topological space. If $Z \subseteq X$ is irreducible, then $\overline{Z}$ is irreducible.*

Proof. Let $Z \subseteq X$ be such that Z is not irreducible. We prove that $\overline{Z}$ is irreducible. Suppose that $\overline{Z} = Z_1 \cup Z_2$ for some closed sets $Z_1, Z_2 \subseteq \overline{Z}$. We want to prove that $Z_1 = \overline{Z}$ or $Z_2 = \overline{Z}$. By the definition of closed sets of Z, we have that $Z_1 \cap Z$ and $Z_2 \cap Z$ are closed in Z. It follows that $Z = (Z_1 \cap Z) \cup (Z_2 \cap Z)$. Since Z is irreducible, then $Z = Z_1 \cap Z$ or $Z = Z_2 \cap Z$. If $Z_1 \cap Z = Z$, then $\overline{Z_1 \cap Z} = \overline{Z}$ and $\overline{Z_1 \cap Z} \subseteq \overline{Z_1} = Z_1 \subseteq \overline{Z}$, since Z_1 closed in $\overline{Z}$. Therefore, $Z_1 = \overline{Z}$. Similarly, if $Z_2 \cap Z = Z$, then $\overline{Z}_2 = Z$. This implies that Z is irreducible, a contradiction. □

A.2 Appendix B: Commutative algebra

We will assume that rings are both commutative and have unity henceforth. The entirety of this appendix is referenced in various places of and can found in [17], unless explicitly stated.

A.2.1 Modules

Definition A.2.1. *Let R be a ring. An* R-module *(or module if R is understood) is a set M together with two binary operations $+ : M \times M \to M$ and $\cdot : R \times M \to M$ such that*

1. $(M, +)$ *is an Abelian group,*
2. $(a + b) \cdot m = a \cdot m + b \cdot m$ *and* $a \cdot (m + n) = a \cdot m + a \cdot n$*,*
3. $(a \cdot b) \cdot m = a \cdot (b \cdot m)$*,*
4. $1 \cdot m = m$

for all $m, n \in M$ *and* $a, b \in R$*.*

For any ring R, a typical example of a module is the n-fold product for any $n \in \mathbb{N}$ equipped with component-wise addition and scalar multiplication defined in the usual way.

Definition A.2.2. *Let* R *be a ring and let* M *be an* R*-module. A non-empty* $N \subseteq M$ *is called a* submodule *of* M *if* $m + n \in N$ *and* $a \cdot m \in N$ *for all* $m, n \in N$ *and* $a \in R$*.*

One easily obtains the following.

Lemma A.2.3. *Let* R *be a ring and let* M *be a* R*-module with submodule* N*. Then*

$$M/N := \{x + N \mid x \in M\}$$

is a R*-module where we define* $(a+N)+(b+N) := (a+b)+N$ *and* $r\cdot(a+N) := (r \cdot a) + N$ *for all* $a, b \in M$ *and* $r \in R$*.*

Definition A.2.4. *Let* R *be a ring and* M *an* R*-module. For any subset* $S \subseteq M$*, the set*

$$\langle S \rangle := \{a_1 s_1 + ... + a_n s_n \mid n \in \mathbb{N}, a_1, ..., a_n \in R \text{ and } s_1, ..., s_n \in S\} \subseteq M$$

is called the submodule generated by S*. If* $S = \{s_1, ..., s_n\}$ *for some* $n \in \mathbb{N}$*, then we write* $\langle S \rangle$ *as* $\langle s_1, ..., s_n \rangle$*. The module* M *is* finitely generated *if* $M = \langle S \rangle$ *for some finite* $S \subseteq M$*.*

Definition A.2.5. *Let* R *be a ring and let* M *be a* R*-module with submodule* N*. We call* M/N *the quotient module of* M *modulo* N*.*

Definition A.2.6. *Let* R *be a ring. A* morphism (or R-linear map) *of* R*-modules from* M *to* N *is a map* $\phi : M \to N$ *such that* $\phi(m+n) = \phi(m) + \phi(n)$ *and* $\phi(am) = a\phi(m)$ *for all* $m, n \in M$ *and* $a \in R$ *(if* R *is understood, we also call* ϕ *a* linear map*). A morphism of* R*-modules is an* isomorphism *if it is bijective.*

Definition A.2.7. *Let R be a ring and let M, N, S be R-modules. A map $\phi : M \times N \to S$ is called a R-bilinear map (or bilinear map if R is understood) if, for all $m \in M$, the map $\phi_m : N \to S$, which sends $n \in N$ to $\phi(m, n)$, is a linear map and if, for all $n \in N$, the map $\phi_n : M \to S$, which sends $m \in M$ to $\phi(m, n)$, is a linear map.*

Definition A.2.8. *Let R be a ring and let M be a finitely generated R-module. We say that a family of elements $\{m_1, ..., m_n\}$, for some fixed $n \in N$ and $m_1, ..., m_n \in M$, is a basis of M if the R-module morphism $\phi : R^n \to M$, which sends $(a_1, ..., a_n)$ to $a_1m_1 + ... + a_nm_n$ is an isomorphism. If M has a basis, then we call M a* free *R-module.*

One immediately observes that, by definition, the cardinality of a basis of a finitely generated module is unique.

A.2.2 Tensor product

Definition A.2.9. *Let R be a ring and let M, N, S be R-modules. The* tensor product *of M and N over R is an R-module T together with a bilinear map $\mu : M \times N \to T$ such that the following universal property holds: For every bilinear map $\alpha : M \times N \to S$, there exists a unique linear map $\phi : T \to S$ such that the following diagram commutes:*

$$\begin{tikzcd}
M \times N \arrow[r, "\alpha"] \arrow[d, "\mu"] & S \\
T \arrow[ur, dashed, "\phi"'] &
\end{tikzcd}$$

The elements of a tensor product are called tensors. *We write T as $M \otimes_R N$ (or as $M \otimes N$ if R is understood). Moreover, we write $\mu(m, n)$ as $m \otimes n$ and call $m \otimes n$ the tensor product of m and n for all $m \in M$ and $n \in N$.*

Proposition A.2.10. *Let R be a ring and let M, N be R-modules. Then the tensor product of M and N over R exists and is unique (up to isomorphism).*

Proof. We begin by proving uniqueness. Assume that T_1 and T_2 are tensor products of M and N with associated bilinear maps $\mu_1 : M \times N \to T_1$ and $\mu_2 : M \times N \to T_2$, respectively. Since T_1 is a tensor product, there exists $\phi_1 : T_1 \to T_2$ such that $\mu_2 = \phi_1 \circ \mu_1$. Similarly, there exists $\phi_2 : T_2 \to T_1$ such that $\mu_1 = \phi_2 \circ \mu_2$. We thus have the following diagram:

$$\begin{tikzcd}
M \times N \arrow[r, "\mu_1"] \arrow[d, "\mu_1"] & T_1 \\
T_1 \arrow[ur, dashed, "\phi_2 \circ \phi_1"'] &
\end{tikzcd} \tag{A.1}$$

Since, the identity map $id_{T_1} : T_1 \to T_1$ is another map which makes (A.1) commute, it follows by the universal property of the tensor product that $\phi_2 \circ \phi_1 = id_{T_1}$. Similarly, one shows that $\phi_1 \circ \phi_2 = id_{T_2}$ so that T_1 and T_2 are isomorphic.

We now show existence. Let F denote the set of functions from $M \times N$ to R that vanish for all but finitely many elements in $M \times N$. This is a R-module with $+ : F \times F \to F$ and $\cdot : R \times F \to F$ defined such that $(f+g)(x) := f(x) + g(x)$ and $(r \cdot g)(x) := r(g(x))$ for all $f, g \in F$, $x \in M \times N$ and $r \in R$. Let $\mu : M \times N \to F$ be the map which sends $(m, n) \in M \times N$ to the function $f : M \times N \to R$ which vanishes for all $M \times N$ except at (m, n) where it takes value 1. It is routine to check that this is indeed a linear map. We will denote $\mu(m, n)$ as $\overline{(m, n)}$ for all $(m, n) \in M \times N$. Let G be the submodule of F generated by the elements

1. $\overline{(m_1 + m_2, n)} - \overline{(m_1, n)} - \overline{(m_2, n)}$,
2. $\overline{(m, n_1 + n_2)} - \overline{(m, n_1)} - \overline{(m, n_2)}$,
3. $\overline{(am, n)} - a\overline{(m, n)}$,
4. $\overline{(m, an)} - a\overline{(m, n)}$

for all $a \in R$, $m, m_1, m_2 \in M$ and $n, n_1, n_2 \in N$ and set $T = F/G$. The map $\mu : M \times N \to T$, which sends $(m, n) \in M \times N$ to $\overline{(m, n)} \in T$ is R-bilinear by definition. We now show that T is indeed the wanted tensor product. Let $\alpha : M \times N \to S$ be a bilinear map for some R-module S. We define $\phi_\alpha : T \to S$ as follows: Let $\overline{(m, n)} \in T$ so that there exists a lifting $(m, n) \in M \times N$. Set $\phi_\alpha(\overline{(m, n)}) := \alpha(m, n)$ which we may extend by linearity so that, for some $k \in \mathbb{N}$,

$$\phi_\alpha(a_1\overline{(m_1, n_1)} + ... + a_k\overline{(m_k, n_k)}) = a_1 \cdot \alpha(m_1, n_1) + ... + a_k \cdot \alpha(m_k, n_k)$$

where $a_i \in R$ and $(m_i, n_i) \in M \times N$ is a lifting for $\overline{(m_i, n_i)} \in T$ for all $i = \{1, ..., k\}$. Since α is bilinear, it follows that ϕ_α is well-defined. By definition of ϕ_α, we have $\phi_\alpha \circ \mu = \alpha$ so that the universal property is satisfied. Hence, the tensor product of M and N over R exists. $\square$

Definition A.2.11. *Let R be some ring. For any R-modules M, N, tensors in $M \otimes N$ of the form $m \otimes n$ for some $(m, n) \in M \times N$ are called* pure.

It is not difficult to see, by definition, that these pure tensors, in fact, generate $M \otimes N$, however, they may not necessarily be unique.

Lemma A.2.12. *Let R be a ring and let M, N, S be R-modules, then the following hold:*

1. $M \otimes N \cong N \otimes M$,
2. $M \otimes R \cong M$,
3. $(M \otimes N) \otimes S \cong M \otimes (N \otimes S)$,
4. $(M \oplus N) \otimes S \cong (M \otimes S) \oplus (N \otimes S)$.

Proof. We begin by showing 1. Define $\alpha : M \times N \to N \otimes M$ which sends (m, n) to $n \otimes m$. We claim that this map is bilinear. To prove the claim, fix some $n \in N$ and let $m_1, m_2 \in M$ and $a \in R$. We want to show that the map $\alpha_M : M \to N \otimes M$, defined by sending $m \in M$ to $\alpha(m, n)$, is linear. We have

$$\alpha_M(m_1 + m_2) = n \otimes (m_1 + m_2) = n \otimes m_1 + n \otimes m_2 = \alpha_M(m_1)\alpha_M(m_2)$$

by definition of the tensor product. Furthermore, we have

$$\alpha_M(am_1) = n \otimes (am_1) = a(n \otimes m_1) = a \cdot \alpha_M(m_1)$$

again by definition of the tensor product so that α_M is a linear map. One similarly shows that α is linear in the other coordinate so that α is bilinear. Thus α induces a unique linear map $\phi : M \otimes N \to N \otimes M$ where $\phi(m \otimes n) = n \otimes m$ for all $m \in M$ and $n \in N$ by the universal property of $M \otimes N$. Analogously, we have $\psi : N \otimes M \to M \otimes N$ where $\psi(n \otimes m) = m \otimes n$ for all $m \in M$ and $n \in N$. It follows that $(\psi \circ \phi)(m \otimes n) = m \otimes n$ for all $m \in M$ and $n \in N$. As pure tensors generate $M \otimes N$ and $\psi \circ \phi$ is bilinear, it follows by linearity that $\psi \circ \phi = id_{M \otimes N}$. Similarly, we have $\phi \circ \psi = id_{N \otimes M}$ so that $M \otimes N$ is isomorphic to $N \otimes M$.

We next show 2. The map $\alpha : M \times R \to M$, which sends $(m, a) \in M \times R$ to $am \in M$ is clearly bilinear using arguments similar to the proof of 1. From α, we obtain $\phi : M \otimes R \to M$, which sends $m \otimes a$ to $a \cdot m$ for all $m \in M$ and $a \in R$, by the universal property of $M \otimes N$. We then define the map $\psi : M \to M \otimes R$ which sends $m \in M$ to $m \otimes 1$. This is clearly a linear map and we claim that it is the inverse to ϕ. Indeed, we have $(\phi \circ \psi)(m) = m$ and $(\psi \circ \phi)(m \otimes a) = am \otimes 1 = m \otimes a$ for all $m \in M$ and $a \in R$ by definition of the tensor product. Again, since pure tensors generate a tensor product, using extension by linearity, we conclude that M and $M \otimes R$ are isomorphic.

To prove 3, we require several repeated steps of using the universal property of the tensor product. Define the bilinear map $\alpha_m : N \times S \to (M \otimes N) \otimes S$ where $(n, s) \in N \times S$ is sent to $(m \otimes n) \otimes s$ for all $m \in M$. Setting $m \in M$, it follows that there exists a linear map $\phi_m : N \otimes S \to (M \otimes N) \otimes S$ where $n \otimes s \mapsto (m \otimes n) \otimes s$ for all $n \in N$ and $s \in S$ by the universal property of $N \otimes S$. We next define $\alpha : M \times (N \otimes S) \to (M \otimes N) \otimes S$ where $(m, n \otimes s) \mapsto (m \otimes n) \otimes s$

for all $m \in M$, $n \in N$ and $s \in S$. We claim that α is a bilinear map. Indeed, we have, for all $m \in M$, $n, n_1, n_2 \in N$, $s, s_1, s_2 \in S$ and $a \in R$,

$$\begin{aligned} \alpha(m, n_1 \otimes s_1 + n_2 \otimes s_2) &= \alpha_m(n_1 \otimes s_1 + n_2 \otimes s_2) \\ &= \alpha_m(n_1 \otimes s_1) + \alpha_m(n_2 \otimes s_2) \qquad (\alpha_m \text{ is bilinear}) \\ &= \alpha(m, n_1 \otimes s_1) + \alpha(m, n_2 \otimes s_2) \end{aligned}$$

and

$$\begin{aligned} \alpha(m, a(n \otimes s)) &= \alpha_m(a(n \otimes s)) \\ &= a \cdot \alpha_m(n \otimes s) \qquad (\alpha_m \text{ is bilinear}) \\ &= a \cdot \alpha(m, n \otimes s). \end{aligned}$$

Linearity in the first coordinate follows similarly. Hence, α is a bilinear map as claimed. It follows that there exists a linear map $\phi : M\otimes(N\otimes S) \to (M\otimes N)\otimes S$ where $m \otimes (n \otimes s) \mapsto (m \otimes n) \otimes s$ for all $m \in M$, $n \in N$ and $s \in S$ by the universal property of $M \otimes (N \otimes S)$. Analagously, there exists a linear map $\psi : (M \otimes N) \otimes S \to M \otimes (N \otimes S)$ where $(m \otimes n) \otimes s \mapsto m \otimes (n \otimes s)$ for all $m \in M$, $n \in N$. Having ϕ and ψ inverse to one another on pure tensors, extending both by linearity, we may deduce that ϕ and ψ are inverse to one another. Hence $M \otimes (N \otimes S)$ and $(M \otimes N) \otimes S$ are isomorphic.

We conclude the proof by showing 4. We define the map $\alpha : (M \oplus N) \times S \to (M \otimes S) \oplus (N \otimes S)$ which sends $((m, n), s) \in (M \oplus N) \times S$ to $(m \otimes s, n \otimes s)$. We now show that α is bilinear. Let $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, $s \in S$ and $a \in R$. We have

$$\begin{aligned} \alpha((m_1, n_1) + (m_2, n_2), s) &= ((m_1 + m_2) \otimes s, (n_1 + n_2) \otimes s) \\ &= (m_1 \otimes s + m_2 \otimes s, n_1 \otimes s + n_2 \otimes s) \\ &= \alpha((m_1, n_1), s) + \alpha((m_2, n_2), s) \end{aligned}$$

and

$$\begin{aligned} \alpha(a(m, n), s)) &= (am \otimes s, an \otimes s) \\ &= (a(m \otimes s), a(n \otimes s)) \\ &= a(m \otimes s, n \otimes s) \\ &= a \cdot \alpha((m, n), s) \end{aligned}$$

where the second step of each calculation follows by definition of the tensor product. One similarly shows linearity in the second coordinate so that α is bilinear. Thus, by the universal property of $(M \oplus N) \otimes S$, there exists a linear map $\phi : (M \oplus N) \otimes S \to (M \otimes S) \oplus (N \otimes S)$ where $(m, n) \otimes s \mapsto (m \otimes s, n \otimes s)$ for all $m \in M$, $n \in N$ and $s \in S$. Similarly shown, we have the linear maps $\psi_1 : M \otimes S \to (M \oplus N) \otimes S$, where $m \otimes s \mapsto (m, 0) \otimes s$, and $\psi_2 : N \otimes S \to (M \oplus N) \otimes S$, where $n \otimes s \mapsto (0, n) \otimes s$, for all $m \in M$, $n \in N$ and $s \in S$. We obtain, by the universal property of the coproduct $(M \otimes S) \oplus (N \otimes S)$, the linear map $\psi : (M \otimes S) \oplus (N \otimes S) \to (M \oplus N) \otimes S$ where $(m \otimes s, n \otimes s) \mapsto (m, 0) \otimes s + (0, n) \otimes s = (m, n) \otimes s$ for all $m \in M$, $n \in N$ and $s \in S$. It is obvious that ψ is inverse to ϕ so that $(M \otimes S) \oplus (N \otimes S)$ is isomorphic to $(M \oplus N) \otimes S$. $\square$

A.2.3 Rings and algebras

Definition A.2.13. *Let R be a ring. An R-*algebra *is a pair (A, ϕ_A) consisting of a ring A with a ring homomorphism $\phi_A : R \to A$. The ring homomorphism ϕ_A is referred as the structure homomorphism of A. We will often denote a R-algebra or algebra (when R is understood) by the underlying ring A and denote its structural homomorphism when necessary.*

Remarks A.2.14. Let A be a R-algebra.

1. The structure homomorphism of A, ϕ_A, gives to A a structure of an R-module defined as follows: $r \cdot a = \phi_A(r) \cdot a$, for all $a \in A$ and $r \in R$.
2. When R is a field. The structure homomorphism ϕ_A is injective. Indeed, a homomorphism with domain a field is always injective. This permits us to identify R as a subring of A. In other word, ϕ_A is an embedding of R into A.

Definition A.2.15. *Let R be a ring and let A be an R-algebra and let ϕ_A be the structural homomorphism of A. An R-*subalgebra *is a subring B of A such that $\phi_A(R) \subseteq B$.*

Definition A.2.16. *Let R be a ring and let A, B be R-algebras with structure homomorphisms ϕ_A and ϕ_B, respectively. A* homomorphism from A to B *is a ring homomorphism $\psi : A \to B$ such that the following diagram commutes*

$$\begin{tikzcd} & R \arrow[dl, "\phi_A"'] \arrow[dr, "\phi_B"] & \\ A \arrow[rr, "\psi"] & & B. \end{tikzcd}$$

Two R-algebras A and B are isomorphic *if and only if there exists a R-algebra homomorphism ψ from A to B such that ψ is a bijection. We define the category of R-algebras denoted by* **R-*Alg*** *whose objects are R-algebra and morphisms are homomorphisms of R-algebra.*

Definition A.2.17. *Let* $((A_i)_{i\in I}, (f_{j,i})_{\substack{i,j\in I\\ i\leqslant j}})$ *be an inverse system in the category of Rings (resp. R-algebras). Denote* $\pi_i : \prod_{i\in I} A_i \to A_i$ *as the canonical projection, for all* $i \in I$*. We define the* inverse limit of the the inverse system $((A_i)_{i\in I}, (f_{j,i})_{\substack{i,j\in I\\ i\leqslant j}})$ *as*

$$A = \{a \in \prod_{i\in I} A_i \mid f_{j,i}(\pi_j(a)) = \pi_i(a),\ \forall i,j \in I,\ i \leqslant j\}.$$

We denote A *by* $\varprojlim((A_i)_{i\in I}, (f_{j,i})_{\substack{i,j\in I\\ i\leqslant j}})$*. When the inverse system is clear from the context we write simply* $\varprojlim A_i$*.*

Lemma A.2.18 ([10, p. 191])**.** *Let* $((A_i)_{i\in I}, (f_{j,i})_{\substack{i,j\in I\\ i\leqslant j}})$ *be an inverse system in the category of Rings (resp. R-algebras). Then* $\varprojlim A_i$ *is a ring (resp. R-algebra).*

Proof. Let $A = \varprojlim A_i$. The addition and a multiplication on A are those operations defined $\prod_{i\in I} A_i$ induced by those of A_i component wise.

Since $(0)_{i\in I} \in \varprojlim A_i$ and hence $\varprojlim A_i$ is non-empty, we need only show that A is closed under the addition and multiplication so that it is a subring of $\prod_{i\in I} A_i$. Let $(a_i)_{i\in I}, (b_i)_{i\in I} \in A$. We have that $(a_i)_{i\in I} + (b_i)_{i\in I} = (a_i + b_i)_{i\in I}$ with the additional property that $f_{j,i}(a_j + b_j) = f_{j,i}(a_j) + f_{j,i}(b_j) = a_i + b_i$ for all $i,j \in I$ such that $i \leqslant j$. Hence $(a_i + b_i)_{i\in I} \in A$. A similar result follows for multiplication. Thus A is a subring of $\prod_{i\in I} A_i$.

Moreover, we want to prove that the inverse limit for R-algebras is an R-algebra for some ring R. Assume A_i is a R-algebra for all $i \in I$ and assume $f_{i,j}$ is a R-algebra homomorphism for all $i,j \in I$ such that $i \leqslant j$. It suffices to show that $\phi_{\prod_{k\in I} A_k}(R) \subseteq A$. Let $i,j \in I$ such that $i \leqslant j$. It follows, for all $r \in R$, that

$$\begin{aligned} f_{j,i}(\pi_j(\phi_{\prod_{k\in I} A_k}(r))) &= f_{j,i}(\phi_{A_j}(r)) \\ &= \phi_{A_i}(r) \\ &= \pi_i(\phi_{\prod_{k\in I} A_k}(r)) \\ &= \pi_i(\phi_{\prod_{k\in I} A_k}(r))) \end{aligned}$$

since π_i, π_j and $f_{i,j}$ are R-algebra homomorphisms. Thus $\phi_{\prod_{k\in I} A_k}(R) \subseteq A$ so that A is a sub-algebra of $\prod_{k\in I} A_k$. □

Proposition A.2.19 ([10, p. 191]). *Let $((A_i)_{i\in I}, (f_{j,i})_{\substack{i,j\in I\\ i\leqslant j}})$ be an inverse system of rings (resp. R-algebras) and let A denote its inverse limit. Then A satisfies the following universal property: For any ring (resp. R-algebra) C such that there exists homomorphisms $\psi_i : C \to A_i$ for $i \in I$ where $f_{i,j}\circ\psi_i = \psi_j$ for all $i, j \in I$ such that $i \leqslant j$, there exists a unique homomorphism $u : C \to A$ such that*

$$\begin{tikzcd}
 & C \arrow[ddl, "\psi_j"'] \arrow[d, dotted, "u"] \arrow[ddr, "\psi_i"] & \\
 & A \arrow[dl, "\pi_j"] \arrow[dr, "\pi_i"'] & \\
A_j \arrow[rr, "f_{j,i}"] & & A_i
\end{tikzcd} \tag{A.2}$$

commutes for all $i, j \in I$ such that $i \leqslant j$. Moreover, the inverse limit is the unique ring (resp. R-algebra) up to isomorphism that satisfies the universal property above.

Proof. Let C and, for all $i \in I$, ψ_i be as in (A.2). We observe, if $u : C \to A$ is assumed to exist and makes (A.2) commute, that we have $\pi_i(u(c)) = \psi_i(c)$ for all $i \in I$ and $c \in C$. This implies that the i-th index of $u(c)$ must be $\psi_i(c)$ by definition of π_i for all $i \in I$. We deduce that $u : C \to A$ must be defined by $c \mapsto (\psi_i(c))_{i\in I}$ for all $c \in C$. Therefore, let $u : C \to A$ be given by $c \mapsto (\psi_i(c))_{i\in I}$ for all $c \in C$. We now show that by u is a well-defined ring homomorphism. To do so, we prove that $(\psi_i(c))_{i\in I} \in A$. This is a consequence of the commutativity of the outer triangle in (A.2), i.e. $f_{j,i} \circ \pi_j \circ u = \pi_i \circ u$, for all $i, j \in I$.

We have that u is a ring (resp. R-algebra) homomorphism which follows from the fact that ψ_i is a ring (resp. R-algebra) homomorphism for all $i \in I$. Moreover, assume that there is some other ring (resp. R-algebra) C that satisfies the universal property. Then, by the universal property of A and C, there exists unique homomorphisms $u : C \to A$ and $v : A \to C$ making (A.2) commute. By composing them, we have $u\circ v = \mathrm{id}_A$ which follows from the fact that there is a unique homomorphism from A to A making (A.2) commute by the universal property of A. Similarly, we have $v \circ u = \mathrm{id}_C$ so that u and v are mutually inverse and hence A is isomorphic to C. ☐

Definition A.2.20 ([10, p. 203]). *Let $((B_i)_{i\in I}, (g_{i,j})_{\substack{i,j\in I\\ i\leqslant j}})$ be a direct system in the category of Rings (resp. R-algebras). We define an equivalence relation on $\bigsqcup_{i\in I} B_i$, denoted by $\sim$, as follows: Let $a, b \in \bigsqcup_{i\in I} B_i$. Then there is a unique $i, j \in I$ such that $a \in B_i$ and $b \in B_j$. We set $a \sim b$ if and only if there exists $k \in I$ such that $i \leqslant k$ and $j \leqslant k$ where $g_{i,k}(a) = g_{j,k}(b)$. It follows that the relation*

$\sim$ *defines an equivalence relation. We call* $B = \bigsqcup_{i \in I} B_i \,/ \sim$ *the* direct limit of the direct system $((B_i)_{i \in I}, (g_{i,j})_{\substack{i,j \in I \\ i \leqslant j}})$. *We denote* B *by* $\varinjlim((B_i)_{i \in I}, (g_{i,j})_{\substack{i,j \in I \\ i \leqslant j}})$. *When the direct system is clear from the context, we write simply* $\varinjlim B_i$. *By definition of* B*, each element of* B *is an equivalence class of some* $b_i \in B$ *where* $i \in I$. *We will denote such an equivalence class by* $[b_i]$.

Lemma A.2.21. *Let* $((B_i)_{i \in I}, (g_{i,j})_{\substack{i,j \in I \\ i \leqslant j}})$ *be a direct system in the category of Rings (resp. R-algebras). Then* $\varinjlim B_i$ *is a ring (resp. R-algebra).*

Proof. Let $\sim$ be as in the definition of B. We define the operations $+, \cdot : B \times B \to B$ as follows: Let $[b_i], [b_j] \in B$. Then there exists $b_i \in B_i$ and $b_j \in B_j$ for some $i, j \in I$. Since I is a directed set, there exists $k \in I$ such that $i, j \leqslant k$. By definition of B, it follows that $[b_i] = [g_{i,k}(b_i)]$ and $[b_j] = [g_{j,k}(b_j)]$. We define $+$ by $([b_i], [b_j]) \mapsto [g_{i,k}(b_i) + g_{j,k}(b_j)]$ and similarly define $\cdot$ by $([b_i], [b_j]) \mapsto [g_{i,k}(b_i) \cdot g_{j,k}(b_j)]$. One easily checks that both operations satisfy the conditions for a ring where we note that the multiplicative identity of B is $[1_i]$ for any $1_i \in B_i$ where $i \in I$. Moreover, if $((B_i)_{i \in I}, (g_{i,j})_{\substack{i,j \in I \\ i \leqslant j}})$ is a direct system of R-algebras for some ring R, then B is easily seen as a R-algebra with structural homomorphism $\phi_B := \iota_i \circ \phi_{B_i}$ for any choice of $i \in I$. $\square$

Proposition A.2.22 ([10, p. 203])**.** *Let* $((B_i)_{i \in I}, (g_{i,j})_{\substack{i,j \in I \\ i \leqslant j}})$ *be a direct system of rings,* B *denote its direct limit and let* $\iota_i : B_i \to B$ *denote the natural inclusion for all* $i \in I$. *Then* B *has the following universal property: For any ring* C *such that there exists maps* ψ_i *for* $i \in I$ *where* $\psi_i = \psi_j \circ g_{i,j}$ *for all* $i, j \in I$ *such that* $i \leqslant j$*, there exists a unique homomorphism* $u : C \to B$ *such that*

$$\begin{tikzcd}
B_i \arrow[rr, "g_{i,j}"] \arrow[dr, "\iota_i"] \arrow[ddr, "\psi_i"'] & & B_j \arrow[dl, "\iota_j"'] \arrow[ddl, "\psi_j"] \\
& B \arrow[d, "u"] & \\
& C &
\end{tikzcd} \tag{A.3}$$

commutes for all $i, j \in I$ *such that* $i \leqslant j$. *Moreover,* B *is the unique ring up to isomorphism with the universal property above.*

Proof. Let $((B_i)_{i \in I}, (g_{i,j})_{\substack{i,j \in I \\ i \leqslant j}})$ be an direct system with direct limit B and let C be as in (A.3). We now show the existence and uniqueness of $u : B \to C$. If we assume that u in (A.3) exists, then, for the diagram to commute, u must map $[b_i] \in B$ to $\iota_i(b_i)$ for all liftings $b_i \in B_i$ with $i \in I$. Thus u is entirely determined by ι_i for all $i \in I$ and hence there is only one possible choice of u. We need only show that this definition of u is indeed a well-defined ring homomorphism. Let $u : B \to C$ be defined by $[b_i] \mapsto \iota_i(b_i)$ where $b_i \in B_i$ is a lifting of $[b_i]$ for some $i \in I$. We need to check that u is indeed well-defined. By definition of $[b_i] \in B$, we must have, for any two liftings $b_i \in B_i$ and $b_j \in B_j$ of $[b_i]$ for some $i, j \in I$, that $g_{i,j}(b_i) = b_j$ if $i \leqslant j$ or $g_{j,i}(b_j) = b_i$ if $j \leqslant i$. However, the outer triangle of (A.3) commutes so that $\psi_i(b_i) = \psi_j(b_j)$ for all $i, j \in I$ such that $i \leqslant j$. Thus our choice of lifting does not affect u and hence u is well defined. We have that u is also ring homomorphism: Let $[b_i], [b_j] \in B$ with respective liftings $b_i \in B_i$ and $b_j \in B_j$ for some $i, j \in I$. Because I is a directed set, there exists a $k \in I$ such that $i \leqslant k$ and $j \leqslant k$, so that $u([b_i] + [b_j]) = u([g_{i,k}(b_i) + g_{j,k}(b_j)])$. Therefore,

$$\begin{aligned} u([b_i] + [b_j]) &= u([g_{i,k}(b_i) + g_{j,k}(b_j)]) \\ &= \psi_k(g_{i,k}(b_i) + g_{j,k}(b_j)) \\ &= \psi_k(g_{i,k}(b_i)) + \psi_k(g_{j,k}(b_j)) \\ &= u([b_i]) + u([b_j]) \end{aligned}$$

and similarly $u([b_i] \cdot [b_j]) = u([b_i]) \cdot u([b_j])$. It follows from definition of u that the diagram commutes. If we further assume that $((B_i)_{i \in I}, (g_{i,j})_{\substack{i,j \in I \\ i \leqslant j}})$ is a direct system of R-algebras for some ring R and that C is a R-algebra with R-algebra homomorphisms ψ_i for all $i \in I$, then we once again need only show that u is a R-algebra homomorphism. We thus have

$$\begin{aligned} u \circ \phi_Y &= u \circ \iota_i \circ \phi_{Y_i} \\ &= \psi_i \circ \phi_{Y_i} \\ &= \phi_C \end{aligned}$$

for some $i \in I$. Thus u is a R-algebra morphism which satisfies the wanted properties. Moreover, the universal property is shown similarly to that of Proposition A.2.19. □

Lemma A.2.23 ([13, p. 254])**.** *Let R be a ring. Every proper ideal in R is contained in a maximal ideal.*

Proof. Let I be an ideal and let $\mathcal{I}$ be the set of all ideals containing I. This set is partially ordered by set inclusion. By making use of Zorn's lemma, we want to prove that $\mathcal{I}$ contains one maximal element being our wanted maximal ideal. We need only show that any chain in $\mathcal{I}$ has an upper bound. Let $\mathcal{J} \subseteq \mathcal{I}$ be a chain of ideals. Define $J := \bigcup \mathcal{J}$. If we show that this is an ideal of R containing I then we are done. Clearly, J contains I and hence is non-empty. Let $a, b \in J$ and $r \in R$. We have that $a \in I'$ and $b \in I''$ for some ideals I' and I''

in $\mathcal{J}$. By definition of a chain, $\mathcal{J}$ is totally ordered so that $I' \subseteq I''$ or $I'' \subseteq I'$. it follows that $a + b$ is in either I' or I'' and hence $a + b \in J$. Since $a \in I' \subseteq J$, it follows that $a \cdot r \in I' \subseteq J$ as well. We conclude that J is an ideal. □

Lemma A.2.24. *Let R be a ring and I, J ideals in R. Then $IJ \subseteq I \cap J$.*

Proof. Let $x \in IJ$, then $x = ij$ for some $i \in I$ and $j \in J$. By definition of an ideal, $ij \in I$ and $ij \in J$ so that $x \in I \cap J$. □

Proposition A.2.25 ([15, p. 21])**.** *Let R be a ring. The following statements are equivalent:*

1. *The set of non-units in R form an ideal.*
2. *There is a unique maximal ideal, $\mathfrak{m}$, in R.*

Proof. Let I be the set of non-units in R. We have that every proper ideal is contained in I, otherwise some proper ideal would contain a unit which implies that $I = R$.

<u>*1* $\Rightarrow$ *2*</u>: Suppose I is an ideal. Consider the ideal I' where $I \subseteq I' \subseteq R$. We would like to show that I' is either I or R. Assume $I \neq I'$ so that $I \subset I'$. It then follows that $\exists a \in I'$ such that a is a unit, which implies that $I' = R$. Hence $I' = I$ or $I' = R$ so that I is maximal. Since all proper ideals are contained in I, we conclude that I is the unique maximal ideal in R.

<u>*2* $\Rightarrow$ *1*</u>: Assume that $\mathfrak{m}$ is the unique maximal ideal in R. We now show that I is indeed just $\mathfrak{m}$. By the uniqueness of m and proposition A.2.23, we have $(n) \subseteq \mathfrak{m}$ for all $n \in I$. It follows that $I \subseteq \mathfrak{m}$. However, as we showed above, all proper ideals are contained in I. Hence $\mathfrak{m} \subseteq I$ so that $I = \mathfrak{m}$. Therefore I is an ideal. □

Proposition A.2.26. *Let R be a ring and let $(I_a)_{a \in A}$ be a family of ideals in R where A is some set. Then the set $\sum\limits_{a \in A} I_a$ defined by*

$$\{f_{a_1} + ... + f_{a_n} \mid f_{a_i} \in I_{a_i} \textit{ for some } a_i \in A \textit{ where } n \in \mathbb{N} \textit{ and } i \in \{1, 2, ..., n\}\}$$

is an ideal (in R) and is the smallest ideal containing $\bigcup\limits_{a \in A} I_a$.

Proof. We begin by showing that $\sum\limits_{a \in A} I_a$ satisfies the basic axioms of an ideal. Let $x, y \in \sum\limits_{a \in A} I_a$. Then there are $B, C \subseteq A$ with B and C finite and $f_b \in I_b$ for all $b \in B$ and $g_c \in I_c$ for all $c \in C$ so that $x = \sum\limits_{b \in B} f_b$ and $y = \sum\limits_{c \in C} g_c$. Define $f_b = 0$ for $b \in C \backslash B$ and $g_c = 0$ for $c \in B \backslash C$. It follows that $x = \sum\limits_{b \in B \cup C} f_b$

and $y = \sum_{c \in B \cup C} g_c$. Therefore $x + y = \sum_{d \in B \cup C} f_d + g_d$. Since $f_d + g_d \in I_d$ for all $d \in B \cup C$, we deduce that $x + y \in \sum_{a \in A} I_a$. Let $r \in R$. Since I_d is an ideal for all $d \in R$, it is obvious that $xr = rx \in \sum_{a \in A} I_a$. We deduce that $\sum_{a \in A} I_a$ is an ideal.

The fact that $\sum_{a \in A} I_a$ is the smallest ideal containing $\bigcup_{a \in A} I_a$ follows easily from definition. Let I' be an ideal containing $\bigcup_{a \in A} I_a$ which is already an ideal. Then I' must contain all finite sums of elements in $\bigcup_{a \in A} I_a$. This is precisely the form of elements in $\sum_{a \in A} I_a$. Therefore all ideals that contain $\bigcup_{a \in A} I_a$ must contain $\sum_{a \in A} I_a$ which gives the wanted result. □

Definition A.2.27. *A ring R is called a* local ring *if it satisfies either of the statements above in Proposition A.2.25.*

Proposition A.2.28. *Let R, R' be local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$, respectively, and let $f : R \to R'$ be a ring homomorphism. Then the following statements are equivalent:*

1. $f(\mathfrak{m}) \subseteq \mathfrak{m}'$.
2. $\mathfrak{m} \subseteq f^{-1}(\mathfrak{m}')$.
3. $\mathfrak{m} = f^{-1}(\mathfrak{m}')$.

Proof. We clearly have (1) $\Leftrightarrow$ (2) by the properties of image and inverse image and it is also clear that (3) $\Rightarrow$ (2) by set-inclusion.

We need only show that (2) $\Rightarrow$ (3). Let R and R' be local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$, respectively and let $f : R \to R'$ be a ring homomorphism. Assume that $\mathfrak{m} \subseteq f^{-1}(\mathfrak{m}')$. We know by basic commutative algebra, that the inverse image of a ring homomorphism of a proper ideal is also a proper ideal Hence, $f^{-1}(\mathfrak{m}')$ is a proper ideal. Since $\mathfrak{m}$ is a maximal ideal, the inclusion of ideals $\mathfrak{m} \subseteq f^{-1}(\mathfrak{m}') \subseteq R$ implies that $\mathfrak{m} = f^{-1}(\mathfrak{m}')$. Thus giving the wanted result. □

Definition A.2.29. *If a ring homomorphism $f : R \to R'$ between local rings R, R' satisfies either of the conditions in Proposition A.2.28, then we call f a* local homomorphism.

Definition A.2.30. *Let S be a subset in a ring R containing the multiplicative identity of R and is closed under multiplication. We call S a* multiplicative set *(in R). We define the* localization $S^{-1}R$ *to be the set of all equivalence classes of pairs (r, s) where $r \in R$, $s \in S$ such that $(r, s) \sim (r', s')$ for some*

$r' \in R$ and $s' \in S$ if and only if there exists $h \in S$ such that $h(rs' - r's) = 0$. By convention, we will denote the equivalence class of (r, s) as $\frac{r}{s}^{S}$ or simply $\frac{r}{s}$ where there is no confusion of what S may be.

Definition A.2.31. *Standard examples of the above definition which we will make use of are as follows:*

1. *Let R be a ring and let $f \in R$. Take $S = \{f^n \mid n \in \mathbb{N}\}$ so that S is a multiplicative set. We define the* localization of R at f *as $R_f := S^{-1}R$.*

2. *Another regular historically used construction is if we consider $S = R \setminus \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of R. Clearly, $1 \in S$ and by taking the contrapositive of the definition of $\mathfrak{p}$, we see that $a \notin \mathfrak{p}$ and $b \notin \mathfrak{p} \Rightarrow ab \notin \mathfrak{p}$ which shows that S is multiplicatively closed. We define the* localization of R at $\mathfrak{p}$ *as $R_{\mathfrak{p}} := S^{-1}R$.*

3. *If R is an integral domain and we take S as $R\setminus\{0\}$, then we call $S^{-1}R$ the* field of fractions of R.

Lemma A.2.32. *Let R be a ring and let $\mathfrak{p}$ be a prime ideal of R. Then the localization $R_{\mathfrak{p}}$ is a local ring.*

Proof. By Proposition A.2.25, it is enough to show that the non-units in $R_{\mathfrak{p}}$ form an ideal.

Claim: For some $a \in R$ and $c \in R \setminus \mathfrak{p}$, $\frac{a}{c} \in R_{\mathfrak{p}}$ is a unit in $R_{\mathfrak{p}}$ if and only if $a \in R \setminus \mathfrak{p}$.

Indeed, the backwards is easily seen since the inverse for $\frac{a}{c}$ would be $\frac{c}{a} \in R_{\mathfrak{p}}$. The forward inclusion is slightly trickier. Let $a \in R$, $c \in R \setminus \mathfrak{p}$ and assume that $\frac{a}{c}$ is invertible in $R_{\mathfrak{p}}$ with inverse $\frac{b}{d}$ for some $b \in R$ and $d \in R \setminus \mathfrak{p}$. Then $\frac{ab}{cd} = 1$ which implies that $hab = hcd \in R \setminus \mathfrak{p}$ for some $h \in R\setminus\mathfrak{p}$. If $a \in \mathfrak{p}$, then $hab \in \mathfrak{p}$ since $\mathfrak{p}$ is an ideal. Thus $a \in R \setminus \mathfrak{p}$, giving the wanted result.

Let I be the set of non-units in $R_{\mathfrak{p}}$. We can now show that I is an ideal. Let $\frac{a}{c}, \frac{b}{d} \in I$ for some $a, b \in R$ and $c, d \in R \setminus \mathfrak{p}$. We have $\frac{a}{c} + \frac{b}{d} = \frac{ac+bd}{cd}$. Since $a, b \in \mathfrak{p}$ by the claim above, it follows that $ac + bd \in \mathfrak{p}$ so that $\frac{ac+bd}{cd} \in I$. Let $\frac{e}{f} \in R_{\mathfrak{p}}$ for some $e \in R$ and $f \in R \setminus \mathfrak{p}$, then $\frac{a}{c} \cdot \frac{e}{f} = \frac{ae}{cf} \in I$ by the claim since $ae \in \mathfrak{p}$. We conclude that $R_{\mathfrak{p}}$ is a local ring. □

Lemma A.2.33 ([3, Proposition 3.1, p. 37])**.** *Let R and R' be a ring, let $S \subseteq R$ be multiplicatively closed and let $\psi : R \to R'$ be a ring homomorphism. Then there exists a unique ring homomorphism $h_{\psi,S}$ such that the following diagram* *commutes:*

$$\begin{tikzcd}
R \arrow[r, "\psi"] \arrow[d, "\iota_{R,S}"] & R' \arrow[d, "\iota_{R',\psi(S)}"] \\
S^{-1}R \arrow[r, "h_{\psi,S}"] & \psi(S)^{-1}R'
\end{tikzcd} \tag{A.4}$$

where $\iota_{R,S}$ *and* $\iota_{R',\psi(S)}$ *are the usual embedding homomorphisms.*

Proof. In order to make the diagram commute we see that $h_{\psi,S^{-1}}$ must map ${\frac{r}{1}}^{S}$ to ${\frac{\psi(r)}{1}}^{\psi(S)}$ for all $r \in R$. Thus, to specify $h_{\psi,S^{-1}}$, we need only consider where $h_{\psi,S^{-1}}$ maps ${\frac{1}{s}}^{S}$ as ${\frac{1}{s}}^{S} \cdot {\frac{r}{1}}^{S} = {\frac{r}{s}}^{S}$ for all $r \in R$ and $s \in S$. Since $S \subseteq R$, it follows that $h_{\psi,S^{-1}}({\frac{s}{1}}^{S}) = {\frac{\psi(s)}{1}}^{\psi(S)}$ for all $s \in S$. For ψ to be a homomorphism, we must map $\psi(a^{-1})$ to $\psi(a)^{-1}$ for all $a \in S^{-1}R$. It follows that $h_{\psi,S^{-1}}$ must map ${\frac{r}{s}}^{S}$ to ${\frac{\psi(r)}{\psi(s)}}^{\psi(S)}$ for all $r \in R$ and $s \in S$. So $h_{\psi,S^{-1}}$ is entirely determined by ψ and A.4. We need only show that this map is indeed a ring homomorphism. Let $r, r' \in R$ and $s, s' \in S$. We have

$$\begin{aligned}
h_{\psi,S}({\frac{r}{s}}^{S} + {\frac{r'}{s'}}^{S}) &= {\frac{\psi(rs' + r's)}{\psi(ss')}}^{S} \\
&= {\frac{\psi(r)\psi(s') + \psi(r')\psi(s)}{\psi(s)\psi(s')}}^{S} \\
&= {\frac{\psi(r)}{\psi(s)}}^{S} + {\frac{\psi(r')}{\psi(s')}}^{S} \\
&= h_{\psi,S}({\frac{r}{s}}^{S}) + h_{\psi,S}({\frac{r'}{s'}}^{S}).
\end{aligned}$$

Similarly, we have $h_{\psi,S^{-1}}({\frac{r}{s}}^{S} \cdot {\frac{r'}{s'}}^{S}) = h_{\psi,S}({\frac{r}{s}}^{S}) \cdot h_{\psi,S}({\frac{r'}{s'}}^{S})$ so that $h_{\psi,S}$ is a ring homomorphism. ☐

Proposition A.2.34. *Let R be a ring and let I be an ideal of R. Then $\sqrt{I}$ is the intersection of all prime ideals containing I.*

Proof. We first show that $\sqrt{I} \subseteq \bigcap_{\mathfrak{p} \in \mathrm{Spec}(I)} \mathfrak{p}$. Let $a \in \sqrt{I}$. Then there exists $n \in \mathbb{N}$ such that $a^n \in I$. Thus $a^n \in \mathfrak{p}$ for any prime ideal containing I. By definition of a prime ideal, we must therefore have that $a \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(I)$. Thus $a \in \bigcap_{\mathfrak{p} \in \mathrm{Spec}(I)}$.

We now show the opposite inclusion. Let $a \in R$ such that $a \notin \sqrt{I}$. Hence, there is no $n \in \mathbb{N}$ such that $a^n \in I$. Now, we form the set $\mathcal{S}$ of all ideals J in R such that $I \subseteq J$ and there exists no $n \in \mathbb{N}$ such that $a^n \in J$. Clearly, $\mathcal{S}$ is partially ordered by set inclusion and is non-empty as it contains I. To make use of Zorn's lemma, we need to check that any chain has an upper bound. Let $\mathcal{T}$ be a chain in $\mathcal{S}$. Analogously to the proof of Lemma A.2.23, the set $\bigcup \mathcal{T}$

is an ideal of R which is an upper bound for $\mathcal{T}$. Therefore, by Zorn's lemma, there exists a maximal element $M \in \mathcal{S}$. If we show that this ideal is a prime ideal, then we are done. Assume that $b, c \in R$ with $bc \in M$ and $b, c \notin M$. Then $I_b = \langle M, b \rangle$ and $I_c = \langle P, c \rangle$ are strictly larger ideals than M so $I_b, I_c \notin \mathcal{S}$. This implies that there exists $n, m \in \mathbb{N}$ such that $a^n \in I_b$ and $a^m \in I_c$. It follows that $a^{nm} \in I_a \cdot I_b \subseteq I_{bc} = \langle M, b, c \rangle$. But $I_{bc} = M$ by our assumptions. Thus M must be prime. ☐

Lemma A.2.35. *Let R, S be rings, let $I \subseteq S$ be an ideal and let $\psi : R \to S$ be a homomorphism. Then $\psi^{-1}(I)$ is an ideal of R.*

Proof. Let $x, y \in \psi^{-1}(I)$ and $r \in R$. We need only show that $x+y, x \cdot y \in f^{-1}(I)$ and $rx \in \psi^{-1}(I)$. Since $\psi(x) + \psi(y) = \psi(x+y)$, $\psi(x) \cdot \psi(y) = \psi(x \cdot y)$, $\psi(r) \cdot \psi(x) = \psi(rx)$ are all in I, we are done. ☐

Lemma A.2.36. *Let R and S be rings and let $I \subseteq R$ be an ideal. Let $\psi : R \to S$ be a surjective homomorphism. Then $\psi(J)$ is an ideal of S.*

Proof. Let $x, y \in \psi(I)$ and $s \in S$. To show that $x + y, x \cdot y, s \cdot x \in \pi(I)$, let $x', y' \in R$ be any lifting such that $\psi(x') = x$ and $\psi(y') = y$. Clearly, $x + y = \pi(x') + \pi(y') = \pi(x' + y')$ and $x \cdot y = \pi(x') \cdot \pi(y') = \pi(x' \cdot y')$ are in $\pi(I)$. We are left to show that $s \cdot x \in \pi(I)$. Since ψ is surjective, there exists some $s' \in R$ such that $\psi(s') = s$ so that $s \cdot x = \psi(s' \cdot x') = \psi(s') \cdot \pi(x')$. Hence $s \cdot x \in \psi(I)$. ☐

Corollary A.2.37. *Let R and S be rings, let $\psi : R \to S$ be a surjective homomorphism and let $I = \ker(\psi)$. There is a bijection between the ideals of R containing $\ker(\psi)$ and the ideals of S given by sending the ideal $J \subseteq S$ to $\psi^{-1}(J)$.*

A.3 Appendix C: Category theory

This chapter states various definitions found in [22].

Definition A.3.1. *A* category, $\mathbf{C} = (\mathbf{C_0}, \mathbf{C_1}, \mathbf{d_C}, \mathbf{c_C}, \mathbf{m_C})$, *is a system containing the data:*

1. *A class $\mathbf{C}_0$ whose elements we call objects.*
2. *A class $\mathbf{C}_1$ whose elements we call morphisms.*
3. *Two functions $\mathbf{d_C}, \mathbf{c_C} : \mathbf{C}_1 \to \mathbf{C}_0$ called* domain *and* codomain, *respectively. If $f \in \mathbf{C_1}$ and $\mathbf{d_C}(f) = A$ and $\mathbf{c_C}(f) = B$, we write $f : A \to B$. We denote the class of all morphisms from A to B as* $\hom_{\mathbf{C}}(A, B)$.

4. *A binary operation,* $\mathbf{m_C}$*, called composition which sends elements of* $\{(f,g) \in \mathbf{C}_1 \times \mathbf{C}_1 \mid \mathbf{d_C}(f) = \mathbf{c_C}(g)\}$ *to* $\mathbf{C}_1$ *where we denote* $\mathbf{m_C}(f,g) = f \circ_{\mathbf{C}} g$ *for* $f, g \in \mathbf{C}_1$*. If* $\mathbf{C}$ *is understood, then we will usually drop the subscript on the notation for* $\circ$*.*

Such that the following hold:

- *For* $A, B, C \in \mathbf{C}_0$ *and* $f, g, h \in \mathbf{C}_1$ *such that* $f : A \to B$*,* $g : B \to C$ *and* $h : C \to D$*, we have* $\mathbf{d_C}(g \circ f) = A$*,* $\mathbf{c_C}(g \circ f) = C$ *and* $h \circ (g \circ f) = (h \circ g) \circ f$*.*
- *Let* $X, Y \in \mathbf{C}_0$*. For each* $A \in \mathbf{C}_0$*, there exists* $\mathrm{id}_A : A \to A$ *in* $\mathbf{C}_1$ *such that, for all* $f : A \to X$ *and* $g : Y \to A$*, we have* $\mathrm{id}_A \circ f = f$ *and* $g \circ \mathrm{id}_A = g$*.*

Definition A.3.2. *Let* $\mathbf{C} = (\mathbf{C}_0, \mathbf{C}_1, \mathbf{d_C}, \mathbf{c_C}, \mathbf{m_C})$ *be a category. We define the* opposite category of $\mathbf{C}$ *as the category* $\mathbf{C}^{\mathrm{op}} = (\mathbf{C}_0, \mathbf{C}_1, \mathbf{c_C}, \mathbf{d_C}, \mathbf{m}'_{\mathbf{C}})$ *where* $m'_{\mathbf{C}}(g,f) = m_{\mathbf{C}}(f,g)$ *with* $(f,g) \in \{(f,g) \in \mathbf{C}_1 \times \mathbf{C}_1 \mid \mathbf{d_C}(f) = \mathbf{c_C}(g)\}$*.*

Definition A.3.3. *Let* $\mathbf{C}$ *and* $\mathbf{C}'$ *be categories. A* functor *from* $\mathbf{C}$ *to* $\mathbf{C}'$ *is a pair of maps* $(\mathcal{F}_0 : \mathbf{C}_0 \to \mathbf{C}'_0, \mathcal{F}_1 : \mathbf{C}_1 \to \mathbf{C}'_1)$ *such that* $\mathcal{F}_1(\mathrm{id}_C) = \mathrm{id}_{\mathcal{F}_0(C)}$ *and* $\mathcal{F}_1(g \circ h) = \mathcal{F}_1(g) \circ \mathcal{F}_1(h)$ *for all* $C \in \mathbf{C}_0$ *and all morphisms* $g, h \in \mathbf{C}_1$*. As an abuse of notation, we will denote a functor* $(\mathcal{F}_0, \mathcal{F}_1)$ *by a single symbol* $\mathcal{F}$ *where* $\mathcal{F}(A) = \mathcal{F}_0(A)$ *for all* $A \in \mathbf{C}_0$ *and* $\mathcal{F}(f) = \mathcal{F}_1(f)$ *for all* $f \in \mathbf{C}_1$*.*

Definition A.3.4. *Let* $\mathbf{C}$ *and* $\mathbf{C}'$ *be categories and let* $\mathcal{F}, \mathcal{G} : \mathbf{C} \to \mathbf{C}'$ *be functors. A* natural transformation *is a family of morphisms* $(\alpha_A : \mathcal{F}_0(A) \to \mathcal{G}_0(A))_{A \in \mathbf{C}_0}$ *such that the following diagram commutes*

$$\begin{tikzcd}
\mathcal{F}_0(A) \arrow[r, "\alpha_A"] \arrow[d, "\mathcal{F}_1(f)"] & \mathcal{G}_0(A) \arrow[d, "\mathcal{G}_1(f)"] \\
\mathcal{F}_0(B) \arrow[r, "\alpha_B"] & \mathcal{G}_0(B)
\end{tikzcd}$$

for all morphisms $f : A \to B$ *in* $\mathbf{C}_1$ *where* $A, B \in \mathbf{C}_0$*.*

Definition A.3.5 ([22])**.** *Let* $\mathbf{C}$ *be a category, let* $A, B, C \in \mathbf{C}$ *and let* $f : A \to C$ *and* $g : B \to C$ *be morphisms of* $\mathbf{C}$*. The* pullback of f and g *is a triple* (D, p, q)*, where* $D \in \mathbf{C}$ *and* $p : D \to A$ *and* $q : D \to B$ *are morphisms in* $\mathbf{C}$*, such that* $f \circ p = g \circ q$ *with the following universal property: If* (E, α, β) *is a triple where* $E \in \mathbf{C}$ *and* $\alpha : E \to A$ *and* $\beta : E \to B$ *are morphisms in* $\mathbf{C}$ *such that* $f \circ \alpha = g \circ \beta$*, then there exists a unique morphism* $\mu : E \to D$ *such that the following diagram commutes:*

$$\begin{tikzcd}
E \arrow[rrd, bend left, "\beta"] \arrow[rd, dotted, "\mu"] \arrow[rdd, bend right, "\alpha"'] & & \\
& D \arrow[r, "q"] \arrow[d, "p"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"] & C
\end{tikzcd}$$

Definition A.3.6 ([10]). *Let I be a preordered set and $\mathbf{C}$ be a category. Let $(A_i)_{i\in I}$ be a family of objects in a category $\mathbf{C}$ such that for all $i \leqslant j$ where $i, j \in I$, there are $f_{j,i} : A_j \to A_i$ morphisms in $\mathbf{C}_1$ such that the following hold:*

1. $f_{i,i} = \mathrm{id}_{A_i}$ *for all* $i \in I$.
2. $f_{k,i} = f_{j,i} \circ f_{k,j}$ *for all* $i \leqslant j \leqslant k$ *in* I.

We call the pair $((A_i)_{i\in I}, (f_{j,i})_{\substack{i,j\in I\\ i\leqslant j}})$ *an* inverse system.

Definition A.3.7 ([10]). *Let I be a directed preordered set and let $\mathbf{C}$ be a category. Let $(B_i)_{i\in I}$ be a family of objects in $\mathbf{C}$ and, for all $i \leqslant j$ where $i, j \in I$, let $g_{i,j} : B_i \to B_j$ be morphisms in $\mathbf{C}$ such that the following hold:*

1. $g_{i,i} = \mathrm{id}_{B_i}$ *for all* $i \in I$
2. $g_{i,k} = g_{j,k} \circ g_{i,j}$ *for all* $i \leqslant j \leqslant k$ *in* I.

We call the pair $((B_i)_{i\in I}, (g_{i,j})_{\substack{i,j\in I\\ i\leqslant j}})$ *a* direct system.

The following is well-known.

Proposition A.3.8 ([29, p. 63]). *Let R be a ring and let A, B be R-algebras. Then $(A \otimes_R B, \iota_A, \iota_B)$ is the coproduct of A and B, where $\iota_A : A \to A \otimes_R B$ is the linear map sending $a \in A$ to $a \otimes 1$ and $\iota_A : A \to A \otimes_R B$ is the linear map that sends $b \in B$ to $1 \otimes b$.*

Proof. We need only show that $A \otimes_R B$ has the universal property of a coproduct. Let C be an R-algebra and let $f : A \to C$ and $g : B \to C$ be R-algebra homomorphisms. We want to show that there exists a unique homomorphism $\mu : A \otimes_R B \to C$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "\iota_A"] \arrow[rd, "f"'] & A \otimes_R B \arrow[d, dotted, "\mu"] & B \arrow[l, "\iota_B"'] \arrow[ld, "g"] \\
& C &
\end{tikzcd} \tag{A.5}$$

commutes. Assume that μ exists. Then it is forced to satisfy, for all $a \in A$, $b \in B$,

$$\begin{aligned}\mu(a \otimes_R b) &= g((a \otimes_R 1)(1 \otimes_R b))\\ &= \mu(a \otimes_R 1)\mu(1 \otimes_R b)\\ &= f(a)g(b).\end{aligned}$$

Thus, by extending μ by linearity, μ would be completely determined. It is easy to see that $A \times B \to C$ defined by sending $(a, b) \in A \times B$ to $f(a)g(b)$ is an R-algebra homomorphism, so that by the universal property of the tensor product, there exists a unique map $A \otimes_R B \to C$ that sends $a \otimes_R b$ to $f(a)g(b)$ for all $a \in A$ and $b \in B$ which is precisely μ. □

A.4 Appendix D: Spectra of rings

This chapter follows [28, Section 2.1].

Definition A.4.1. *Let R be a ring. We define the* spectrum of R, *denoted by Spec(R), as the set of all prime ideals in R. For any ideal I in R, define*

$$\mathbb{V}_{\text{sch}}(I) := \{\mathfrak{p} \in \mathit{Spec}(R) \mid I \subseteq \mathfrak{p}\}.$$

To ease notation, if $S \subseteq R$ is not an ideal, then $\mathbb{V}_{\text{sch}}(S)$ will denote $\mathbb{V}_{\text{sch}}((S))$. Similarly, if $f \in R$, then we will denote $\mathbb{V}_{\text{sch}}((f))$ as $\mathbb{V}_{\text{sch}}(f)$.

Proposition A.4.2. *Let R be a ring. Then the following properties hold:*

1. *Let I, J be ideals in R such that $I \subseteq J$, then $\mathbb{V}_{\text{sch}}(J) \subseteq \mathbb{V}_{\text{sch}}(I)$.*
2. $\mathbb{V}_{\text{sch}}(R) = \varnothing$ *and* $\mathbb{V}_{\text{sch}}(0) = \text{Spec}(R)$.
3. *For ideals I and J in R, we have $\mathbb{V}_{\text{sch}}(I) \cup \mathbb{V}_{\text{sch}}(J) = \mathbb{V}_{\text{sch}}(IJ) = \mathbb{V}_{\text{sch}}(I \cap J)$.*
4. *Let $(I_a)_{a\in A}$ be a family of ideals in R with some index set A, then $\bigcap\limits_{a\in A} \mathbb{V}_{\text{sch}}(I_a) = \mathbb{V}_{\text{sch}}(\sum\limits_{a\in A} I_a)$.*
5. *Let I, J be ideals in R. Then $\mathbb{V}_{\text{sch}}(I) \subseteq \mathbb{V}_{\text{sch}}(J)$ if and only if $\sqrt{J} \subseteq \sqrt{I}$.*

Proof. 1. and 2. follow immediately by definition.

We now prove 3. by means of set inclusion. Let I, J be ideals in R. Since $I \cap J \subseteq I$ and $I \cap J \subseteq J$, it follows from 1. that $\mathbb{V}_{\text{sch}}(I) \cup \mathbb{V}_{\text{sch}}(J) \subseteq \mathbb{V}_{\text{sch}}(I \cap J)$. By Lemma A.2.24 and by 1., we have $\mathbb{V}_{\text{sch}}(I \cap J) \subseteq \mathbb{V}_{\text{sch}}(IJ)$, so that the inclusion $\mathbb{V}_{\text{sch}}(I) \cup \mathbb{V}_{\text{sch}}(J) \subseteq \mathbb{V}_{\text{sch}}(I \cap J) \subseteq \mathbb{V}_{\text{sch}}(IJ)$ holds. We now need only show that $\mathbb{V}_{\text{sch}}(IJ) \subseteq \mathbb{V}_{\text{sch}}(I) \cup \mathbb{V}_{\text{sch}}(J)$. Let $\mathfrak{p} \in \mathbb{V}_{\text{sch}}(IJ)$. To prove this inequality, we argue by contradiction. Assume that $I \nsubseteq \mathfrak{p}$ and $J \nsubseteq \mathfrak{p}$. Then there exists some $i \in I$ and $j \in J$ such that $i, j \notin \mathfrak{p}$, but we have $ij \in \mathfrak{p}$ which contradicts $\mathfrak{p}$ being a prime ideal. Hence $\mathbb{V}_{\text{sch}}(IJ) \subseteq \mathbb{V}_{\text{sch}}(I) \cup \mathbb{V}_{\text{sch}}(J)$.

Next, we show that 4. holds in a similar fashion to 3.. Let $(I_a)_{a\in A}$ be a family of ideals where A is some set. Since $I_b \subseteq \sum\limits_{a\in A} I_a$ and hence, by 1., $\mathbb{V}_{\text{sch}}(\sum\limits_{a\in A} I_a) \subseteq \mathbb{V}_{\text{sch}}(I_b)$ for all $b \in A$, it follows that $\mathbb{V}_{\text{sch}}(\sum\limits_{a\in A} I_a) \subseteq \bigcap\limits_{a\in A} \mathbb{V}_{\text{sch}}(I_a)$. To show $\bigcap\limits_{a\in A} \mathbb{V}_{\text{sch}}(I_a) \subseteq \mathbb{V}_{\text{sch}}(\sum\limits_{a\in A} I_a)$, consider a prime ideal $\mathfrak{p} \in \bigcap\limits_{a\in A} \mathbb{V}_{\text{sch}}(I_a)$. By definition we have $I_a \subseteq \mathfrak{p}$ for all $a \in A$ so that $\bigcup\limits_{a\in A} I_a \subseteq \mathfrak{p}$. It follows from Proposition A.2.26 that $\mathfrak{p}$ contains $\sum\limits_{a\in A} I_a$ so that $\mathfrak{p} \in \mathbb{V}_{\text{sch}}(\sum\limits_{a\in A} I_a)$.

Finally, 5. is easily seen to hold since $\sqrt{I}$, by definition, is the intersection of all prime ideals containing I. ☐

Lemma A.4.3. *If $I \subseteq R$ is an ideal, then $\mathbb{V}_{\text{sch}}(I) = \mathbb{V}_{\text{sch}}(\sqrt{I})$.*

Proof. Since $I \subseteq \sqrt{I}$, it follows from Proposition A.4.2 that $\mathbb{V}_{\text{sch}}(\sqrt{I}) \subseteq \mathbb{V}_{\text{sch}}(I)$. We need only show that $\mathbb{V}_{\text{sch}}(I) \subseteq \mathbb{V}_{\text{sch}}(\sqrt{I})$. Let $\mathfrak{p} \in \mathbb{V}_{\text{sch}}(I)$ so that $I \subseteq \mathfrak{p}$. It follows from Lemma A.2.34 that $\sqrt{I} \subseteq \mathfrak{p}$ so that $\mathfrak{p} \in \mathbb{V}_{\text{sch}}(\sqrt{I})$. ☐

The statements 2., 3. and 4. of Proposition 4 indicate that the $\mathbb{V}_{\text{sch}}(I)$, for all ideals I in R, define a closed set topology. We call this topology the *Zariski topology*. Thus the open sets of the Zariski topology are of the form $\text{Spec}(R) \setminus \mathbb{V}_{\text{sch}}(S)$ for some ideal generated by $S \subseteq R$.

The Zariski topology has a natural construction of a base which we will now consider. Let R be a ring. Denote, for $f \in R$,

$$\mathbb{D}_{\text{sch}}(f) := \text{Spec}(R) \setminus \mathbb{V}_{\text{sch}}(f)$$

which we call the *distinguished open subset of X associated to f*. If U is open in R under the Zariski topology, then $U = \text{Spec}(R) \setminus \mathbb{V}_{\text{sch}}(S)$ for some $S \subseteq R$. Thus

$$U = \text{Spec}(R) \setminus \mathbb{V}_{\text{sch}}(\sum_{f\in S}(f)) = \text{Spec}(R) \setminus \bigcap_{f\in S}\mathbb{V}_{\text{sch}}(f) = \bigcup_{f\in S}\text{Spec}(R) \setminus \mathbb{V}_{\text{sch}}(f)$$

where the first equality follows by the definition of $\sum\limits_{f\in S}(f)$ and the second equality follows by Proposition A.4.2. Hence, the distinguished open subsets of R constitute a basis for the Zariski topology of R. A useful property is that these particular sets are also closed under finite intersection:

$$\begin{aligned}\bigcap_{i=1,\ldots,n} Spec(R) \setminus \mathbb{V}_{\text{sch}}(f_i) &= \operatorname{Spec}(R) \setminus \bigcup_{i=1}^{n} \mathbb{V}_{\text{sch}}(f_i) \\ &= \operatorname{Spec}(R) \setminus \mathbb{V}_{\text{sch}}(\prod_{i=1}^{n} f_i) \quad \text{(Proposition A.4.2)} \\ &= Spec(R) \setminus \mathbb{V}_{\text{sch}}(f)\end{aligned}$$

for some $n \in \mathbb{N}$ and $f = \prod_{i=1,\ldots,n} f_i$ such that $f_i \in R$ for $i = 1, \ldots, n$.

A.5 Appendix E: Sheaves

This chapter follows [28, Section 2.2.1]. Let X be a topological space. Consider the category $\mathbf{Ops}_X$ of open sets in X with morphisms the usual inclusion functions for every $U \subseteq V$ where U, V are open sets in X. We can now give our first definition:

Definition A.5.1 ([14, 11]). *Let X be a topological space and consider $\mathbf{Ops}_X$ as above and* **Ring***, the category of rings. We define a* presheaf of rings *(with respect to X) to be a functor $\mathcal{F} : \mathbf{Ops}_X^{\text{op}} \to$* **Ring** *such that $\mathcal{F}(\varnothing) = 0$.*

If $V \subseteq U$ are open sets in a topological space X and $\mathcal{F}$ is a presheaf with respect to X, then we will denote the image of the inclusion map $i_{U,V} : U \to V$ (which is a morphism in the category $\mathbf{Ops}_X^{\text{op}}$) over $\mathcal{F}$ as $\rho_{U,V}^{\mathcal{F}} := \mathcal{F}_1(i_{U,V}) : \mathcal{F}_1(U) \to \mathcal{F}_1(V)$ which we refer to as restriction map from U to V. Furthermore, if $f \in \mathcal{F}(U)$, then we will also denote the restriction of f to U as $f|_U$, i.e. $f|_U := \rho_{V,U}^{\mathcal{F}}(f)$. Note that we can replace the target category of rings with any other category of an algebraic construction to get other desirable presheafs (such as abelian groups and algebras). An element $f \in \mathcal{F}(U)$ is called a *section of $\mathcal{F}$ over U* where U is some open set in X.

Definition A.5.2. *Let X be a topological space. We call a presheaf $\mathcal{F}$ (with respect to X) a* sheaf *if it satisfies the two following conditions: Let U be an open subset of X and let $\{U_i\}_{i \in I}$ an open covering of U.*

1. *(Uniqueness condition) Let $s \in \mathcal{F}(U)$. If $s|_{U_i} = 0$ for all $i \in I$, then $s = 0$.*

2. *(Gluing condition) Let $s_i \in \mathcal{F}(U_i)$ for all $i \in I$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$. Then there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.*

The uniqueness condition of Definition A.5.2 is not merely a strange naming convention as it implies that the section that exists by the gluing condition is indeed unique: Let X be a topological space, $\mathcal{F}$ be a sheaf (with respect to X), U an open set of X with open covering $\{U_i\}_{i\in I}$ and let $s, t \in \mathcal{F}(U)$ be such that $s|_{U_i} = t|_{U_i} = s_i \in \mathcal{F}(U_i)$ for all $i \in I$. Then $(s-t)|_{U_i} = s|_{U_i} - t|_{U_i} = 0$ for all $i \in I$ so that $s - t = 0$ and hence $s = t$. Therefore showing that s is unique in this context.

To better understand the local properties of a sheaf, we will construct a direct system which will prove to be quite useful for our understanding. Consider any given point x in a topological space X. Let I be the set of neighbourhoods of x, this will be the index set for our direct system. We equip I with the partial order $\leqslant$ such that $U \leqslant V$ if and only if V is contained in U for all $V, U \in I$. In particular, $\leqslant$ is a directed partial order since for any $U, V \in I$, we have $U \cap V \in I$ and $U, V \leqslant U \cap V$. This gives us the family $(\mathcal{F}(U))_{U\in I}$. Furthermore, we have the inclusion maps $\rho^{\mathcal{F}}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ for all $U, V \in I$ such that $U \leqslant V$. Among these maps, we also have $\rho^{\mathcal{F}}_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$ for $U \in I$ and that $\rho^{\mathcal{F}}_{U,W} = \rho^{\mathcal{F}}_{V,W} \circ \rho^{\mathcal{F}}_{U,V}$ for $W, V, U \in I$ such that $U \leqslant V \subseteq' W$. Hence $((\mathcal{F}(U))_{U\in I}, (\rho^{\mathcal{F}}_{U,V})_{\substack{U,V\in I\\ U\leqslant V}})$ forms a direct system. We refer to this direct system as the direct system given by $\mathcal{F}$.

Definition A.5.3. *Let X be a topological space, let $x \in X$, let $\mathcal{F}$ be a presheaf with respect to X and let I be the partially ordered set of open sets containing x as discussed above. We define the* stalk of $\mathcal{F}$ at x *to be the direct limit given by $\mathcal{F}$ denoted by*

$$\mathcal{F}_x := \varinjlim((\mathcal{F}(U))_{U\in I}, (\rho^{\mathcal{F}}_{U,V})_{\substack{U,V\in I\\ U\leqslant V}}).$$

Note that by Lemma A.2.21, it follows that $\mathcal{F}_x$ is also a ring.

Let $s \in \mathcal{F}(U)$ for some open set U in X. For some $x \in U$, denote s_x as the image of s in $\mathcal{F}_x$ which we call the germ of s at x ([28, 35]).

Lemma A.5.4. *Let $\mathcal{F}$ be a sheaf with respect to X. If $s, t \in \mathcal{F}(X)$ such that $s_x = t_x$ for all $x \in X$, then $s = t$.*

Proof. Let X be a topological space and let $\mathcal{F}$ be a sheaf with respect to X and let ι_X denote the natural inclusion of $\mathcal{F}(X)$ in $\varinjlim_{U\ni x} \mathcal{F}(U)$.Let $s, t \in \mathcal{F}(X)$ such that $s_x = t_x$ for all $x \in X$. We have

$$0 = s_x - t_x = \iota_X(s) - \iota_X(t) = \iota_X(s - t).$$

Let $r = s - t \in \mathcal{F}(X)$ so that $r_x = 0$ for all $x \in X$. By definition of $\mathcal{F}_x$, there exists an open neighbourhood U_x for all $x \in X$ such that $r|_{U_x} = 0$. Since $X = \bigcup_{x\in X} U_x$, it follows that $r = 0$ by the uniqueness axiom of sheaves. Thus $s = t$. □

Definition A.5.5. *Let X, Y be topological spaces, $f : X \to Y$ be a continuous map and $\mathcal{O}_X$ be a sheaf with respect to X. We define the* pullback of $\mathcal{F}$ by f, *denoted by $f_*\mathcal{F}$, to be the sheaf with respect to Y given by by sending objects $U \in \mathbf{Ops}^{\mathrm{op}}_{X,0}$ to $\mathcal{F}(f^{-1}(U))$ and sending morphisms $i_{U,V} \in \mathbf{Ops}^{\mathrm{op}}_{X,1}$, for some $U, V \in \mathbf{Ops}^{\mathrm{op}}_{X,0}$, to $\rho^{\mathcal{F}}_{f^{-1}(U),f^{-1}(V)}$.*

We need to show that this is indeed a sheaf.

Proposition A.5.6. *Let X, Y be topological spaces, $f : X \to Y$ be a continuous map and $\mathcal{O}_X$ be a sheaf with respect to X. Then the* pullback of $\mathcal{F}$ by f, *denoted by $f_*\mathcal{F}$, is a sheaf with respect to Y given*

Proof. We begin by showing that $f_*\mathcal{F}$ is a presheaf. Clearly, $f_*\mathcal{F}$ assigns each open $U \subseteq Y$ to a ring $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ since $f^{-1}(U)$ is open in X by the continuity of f. We now need only to show that the restriction maps commute. Let $W \subseteq V \subseteq U$ with W, V, U open in Y. Then

$$\begin{aligned} \rho^{f_*\mathcal{F}}_{V,W} \circ \rho^{f_*\mathcal{F}}_{U,V} &= \rho^{\mathcal{F}}_{f^{-1}(V),f^{-1}(W)} \circ \rho^{\mathcal{F}}_{f^{-1}(U),f^{-1}(V)} \\ &= \rho^{\mathcal{F}}_{f^{-1}(U),f^{-1}(W)} && (\mathcal{F} \text{ is a sheaf}) \\ &= \rho^{f_*\mathcal{F}}_{U,W}. \end{aligned}$$

It is also clear that $\rho^{f_*\mathcal{F}}_{U,U} = \rho^{\mathcal{F}}_{f^{-1}(U),f^{-1}(U)} = \mathrm{id}^{\mathcal{F}}_{\mathcal{F}(f^{-1}(U))} = \mathrm{id}_{f_*\mathcal{F}(U)}$ which shows that $f_*\mathcal{F}$ satisfies the conditions for a presheaf.

We conclude the proof by showing that $f_*\mathcal{F}$ is a sheaf. Let $U \subseteq Y$ be open. Let $\{U_i\}_{i\in I}$ be an open covering of U and let $s_i \in f_*\mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$. Let $V = f^{-1}(U)$ and let $V_i = f^{-1}(U_i)$ for all $i \in I$, then V and the V_i are open sets in X by the continuity of f and $V = \bigcup\limits_{i\in I} f^{-1}(U_i) = \bigcup\limits_{i\in I} V_i$ so that $\{V_i\}_{i\in I}$ is an open covering of V. It follows that $s_i \in f_*\mathcal{F}(U_i) = \mathcal{F}(V_i)$ for all $i \in I$, so that there exists a unique $s \in \mathcal{F}(V) = f_*\mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$ by the sheaf properties of $\mathcal{F}$ applied to the open covering $\{V_i\}_{i\in I}$ of V. Hence $f_*\mathcal{F}$ is a sheaf. ☐

Definition A.5.7. *Let X be a topological space with topology τ_X and let $\mathcal{F}$ and $\mathcal{G}$ be presheaves (resp. sheaves) with respect to X. A* morphism between presheaves (resp. sheaves) $\mathcal{F}$ *and* $\mathcal{G}$ *is a natural transformation $\alpha = (\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U))_{U\in\tau_X}$. That is, the diagram*

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\alpha_U"] \arrow[d, "\mathcal{F}(i_{U,V})"] & \mathcal{G}(U) \arrow[d, "\mathcal{G}(i_{U,V})"] \\ \mathcal{F}(V) \arrow[r, "\alpha_V"] & \mathcal{G}(V) \end{tikzcd}$$

commutes, for all $V \subseteq U$ such that U, V open in X.

Definition A.5.8. *Let X be a topological space and let $\alpha = (\alpha_U)_{U \in \tau_X}$ be a morphism of sheaves. We say that α is injective (resp. an isomorphism) if α_U is injective (resp. an isomorphism) for all open U in X.*

A morphism of sheaves induce ring homomorphisms between stalks. Indeed, if $\mathcal{F}$ and $\mathcal{G}$ are sheaves with respect to a topology X and $(a_U)_{U \in \tau_X}$ is a morphism between $\mathcal{F}$ and $\mathcal{G}$, then we can, for all $x \in X$, canonically define $a_x : \mathcal{F}_x \to \mathcal{G}_x$ such that $s_x \mapsto a_x(s_x) := (a_U(s))_x$ for any choice of open $U \subseteq X$ and $s \in \mathcal{F}(U)$ such that $x \in U$.

The following is not found in [28], however, it is not difficult to prove.

Definition A.5.9. *Let X be a topological space and let $\alpha = (\alpha_U)_{U \in \tau_X}$ be a morphism of sheaves. We say that α is surjective if α_x is surjective for all $x \in X$.*

Proposition A.5.10. *Let X be a topological space and let $\alpha = (\alpha_U)_{U \in \tau_X}$ be a morphism of sheaves. Then α is an isomorphism if and only if α_x is an isomorphism for all $x \in X$.*

Proof. The forward implication is easy to see: Assume that α is an isomorphism. Let $x \in X$. We define $\alpha_x : \mathcal{F}_x \to$ by sending $s_x \in \mathcal{F}_x$ to $(\alpha_U(s))_x$ for any choice of open $U \subseteq X$ and $s \in \mathcal{F}(U)$ such that $x \in U$. The inverse for α_x is exactly $\alpha_x^{-1} : \mathcal{G}_x \to \mathcal{F}_x$ which sends $t_x \in \mathcal{G}_x$ to $(\alpha_U^{-1}(t))_x$ for any choice of open $U \subseteq X$ and $t \in \mathcal{G}(U)$ such that $x \in U$. Since x was arbitrary, we get the wanted result.

We now show the backwards implication. Assume that α_x is an isomorphism for all $x \in X$. Let $U \subseteq X$ be open and let $s \in \mathcal{F}(U)$. Assuming that $\alpha_U(s) = 0$, we have, for every $x \in U$, that $\alpha_x(s_x) = (\alpha_U(s))_x = 0$. Since α_x is a ring homomorphism, this implies that $s_x = 0$ for all $x \in U$. Thus, by Lemma A.5.4, $s = 0$. We chose U to be arbitrary, so that we may deduce that α is injective. We now want to show that α_U is surjective for every open $U \subseteq X$. Let $U \subseteq X$ be open, let $\{U_i\}_{i \in I}$ be an open covering of U and let $t \in \mathcal{G}(U)$. Let $t_i = t|_{U_i}$ for all $i \in I$. Fix some $i \in I$. We have that there exists an $s_{i,x} \in \mathcal{F}(V_{i,x})$ such that $V_{i,x} \subseteq X$ is open, contains x and $\alpha_x((s_{i,x})_x) = (t_i)_x$ for every $x \in U_i$. We may assume that $V_{x,i} \subseteq U_i$ as we may just take $V_{x,i}$ to be $V_{x,i} \cap U_i$. Since α_x is injective for all $x \in U_i$, it follows that $s_{i,x}|_{V_{i,x} \cap V_{i,x'}} = s_{i,x'}|_{V_{i,x} \cap V_{i,x'}}$ for all $x, x' \in U_i$. Thus by gluing using the open covering $\{V_{i,x}\}_{x \in U_i}$, we obtain a $s_i \in U_i$ such that $\alpha_{U_i}(s_i) = t_i$. Now, since α is injective, it follows that $s_j|_{U_j \cap U_k} = s_k|_{U_j \cap U_k}$ for all $j, k \in I$. Therefore, we may use glueing with the open covering $\{U_i\}_{i \in I}$ and $\{s_i\}_{i \in I}$ to obtain a $s \in \mathcal{F}(U)$. We need only show that $\alpha_U(s) = t$. However, since

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "a_U"] \arrow[d, "\mathcal{F}(i_{U,U_i})"] & \mathcal{G}(U) \arrow[d, "\mathcal{G}(i_{U,U_i})"] \\
\mathcal{F}(U_i) \arrow[r, "a_{U_i}"] & \mathcal{G}(U_i)
\end{tikzcd} \tag{A.6}$$

is commutative for all $i \in I$, it follows that $\alpha_U(s) = t$. □

Proposition A.5.11. *Let X be a topology and let $\mathcal{B}$ be a base for X. Furthermore, let $\mathcal{F}$ and $\mathcal{G}$ be sheaves with respect to X and assume additionally that $\mathcal{B}$ is stable under finite intersections. If there exists homomorphisms (resp. isomorphisms) $a_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for all $U \in \mathcal{B}$ such that the following diagram commutes*

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "a_U"] \arrow[d, "\mathcal{F}(i_{U,V})"] & \mathcal{G}(U) \arrow[d, "\mathcal{G}(i_{U,V})"] \\
\mathcal{F}(V) \arrow[r, "a_V"] & \mathcal{G}(V)
\end{tikzcd} \tag{A.7}$$

for all $U, V \in \mathcal{B}$ such that $V \subseteq U$, then there is a unique morphism (resp. isomorphism) of sheaves $\alpha : \mathcal{F} \to \mathcal{G}$ such that $\alpha_U = a_U$ for all $U \in \mathcal{B}$.

Proof. We now construct a morphism of sheaves α by describing α_U for all open $U \subseteq X$. Let U be an open set of X and set $\alpha_V = a_V$ for all $V \in \mathcal{B}$. We define $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ as follows: Let $s \in \mathcal{F}(U)$. Since $\mathcal{B}$ is a base of X, there exists an open covering $\{U_i\}_{i\in I}$ of U such that $U_i \in \mathcal{B}$ for all $i \in I$. We choose $\{U_i\}_{i\in I}$ to include all subsets V of U such that $V \in \mathcal{B}$. We decompose s into a collection of its restrictions on each U_i so that we have $\{s_i\}_{i\in I}$ where $s_i = s|_{U_i}$ for all $i \in I$. To ease notation let $t_i = \alpha_{U_i}(s|_{U_i})$ for all $i \in I$. Thus $\{t_i\}_{i\in I}$ is a collection of elements such that $t_i \in \mathcal{G}(U_i)$ for all $i \in I$ and we claim that $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$ for all $i, j \in I$. Set some $i, j \in I$. To see the claim, we note that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ since restrictions commute (explicitly given by $s|_{U_i \cap U_j} = s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$). By assumption $\mathcal{B}$ is stable under finite intersection, so that $U_i \cap U_j \in \mathcal{B}$. Therefore, by A.6, we have $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$. Thus, by the gluing axiom of sheaves, there exists a unique $t \in \mathcal{G}(U)$ such that $t|_{U_i} = t_i$ for all $i \in I$. We define $\alpha_U(s) = t$. By our choice of open covering of U, there is no other possible option for $\alpha_U(s)$ so that this map is well-defined. We have that α_U is indeed a homomorphism: Let $s, s' \in \mathcal{F}(U)$. Then $\alpha_U(s + s')$ is the unique element $t'' \in \mathcal{G}(U)$ such that $t"|_V = \alpha_{U_i}((s + s')|_{U_i}) = \alpha_{U_i}(s|_{U_i}) + \alpha_{U_i}(s'|_{U_i})$ for all $i \in I$. In particular, $\alpha_U(s) + \alpha_U(s') = \alpha_{U_i}(s|_{U_i}) + \alpha_{U_i}(s'|_{U_i})$ for all $i \in I$. We conclude, by the uniqueness axiom of sheaves, that $\alpha_U(s + s') = \alpha_U(s) + \alpha_U(s')$. Similarly, the result holds for $\alpha_U(s \cdot s') = \alpha_U(s) \cdot \alpha_U(s')$. To confirm that our construction α

is indeed a morphism of sheaves, we need to check that the diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\alpha_U"] \arrow[d, "\mathcal{F}(i_{U,V})"] & \mathcal{G}(U) \arrow[d, "\mathcal{G}(i_{U,V})"] \\
\mathcal{F}(V) \arrow[r, "\alpha_V"] & \mathcal{G}(V)
\end{tikzcd} \tag{A.8}$$

commutes for all open $U, V \subseteq X$ such that $V \subseteq U$. Let $s \in \mathcal{F}(U)$. Since $s|_W = (s|_V)|_W$, it follows that $\alpha_U(s)|_W = \alpha_V(s|_V)|_W$ for all $W \in \mathcal{B}$ such that $W \subseteq V$. Thus, since all $W \subseteq V$, such that $W \in \mathcal{B}$, are subsets of U as well, we conclude that $\alpha_U(s)|_V = \alpha_V(s|_V)$ so that A.8 commutes. Furthermore, we check that α is unique. Assume that there is a morphism of sheaves $b = (b_U : \mathcal{F}(U) \to \mathcal{G}(U))_{U \in \tau_X}$ such that b_U is a ring homomorphism for all $U \in \tau_X$ and that $b_V = a_V$ for all $V \in \mathcal{B}$. Let U be open in X and let $s \in \mathcal{F}(U)$. We must have, by definition of a morphism of a sheaf, that $b_U(s)|_V = b_V(s|_V) = a_V(s_V)$ for all $V \in \mathcal{B}$ such that $V \subseteq U$, but then by the uniqueness axiom sheaves, we must have that $b_U(s) = \alpha_U(s)$. Thus α is unique.

In addition, assume that α_U is an isomorphism for all $U \in \mathcal{B}$. To show that α is an isomorphism of sheaves, we need only check that α_U is an isomorphism for all open U in X. We construct α_U^{-1} as follows: Let U be open in X, let $t \in \mathcal{G}(U)$ and let $\{U_i\}_{i \in I}$ be an open covering of U containing all $V \in \mathcal{B}$ such that $V \subseteq U$. Similarly shown as in the case with α_U above, we have $s_i = \alpha_{U_i}^{-1}(t|_{U_i}) \in \mathcal{F}(U_i)$ for all $i \in I$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$. Thus there exists a unique s such that $s|_{U_i} = s_i$ for all $i \in I$. We define α_U^{-1} which, by construction, clearly gives an inverse to α_U. □

Proposition A.5.12 ([33, Section 00AK]). *Let X be a topological space, let $\{U_i\}_{i \in I}$ be an open covering of X and let $\mathcal{F}_i$ be a sheaf with respect to U_i for all $i \in I$. If, for each $i, j \in I$, $\phi_{i,j} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ is an isomorphism of sheaves such that, for all $i, j, k \in I$, the diagram*

$$\begin{tikzcd}
\mathcal{F}_i|_{U_i \cap U_j \cap U_k} \arrow[rr, "\phi_{i,k}"] \arrow[dr, "\phi_{i,j}"'] & & \mathcal{F}_k|_{U_i \cap U_j \cap U_k} \\
& \mathcal{F}_j|_{U_i \cap U_j \cap U_k} \arrow[ur, "\phi_{j,k}"'] &
\end{tikzcd} \tag{A.9}$$

commutes, then there exists a sheaf $\mathcal{F}$ on X with isomorphisms $\phi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ for all $i \in I$ such that the diagram

$$\begin{tikzcd}
\mathcal{F}|_{U_i \cap U_j} \arrow[rr, "\phi_i"] \arrow[dr, "\phi_j"'] & & \mathcal{F}_j|_{U_i \cap U_j} \\
& \mathcal{F}_i|_{U_i \cap U_j} \arrow[ur, "\phi_{i,j}"'] &
\end{tikzcd} \tag{A.10}$$

is commutative for all $i, j \in I$.

Proof. We begin by constructing the sheaf $\mathcal{F}$ and showing that it has the desired properties. Define $\mathcal{F}(V)$ as

$$\{(s_i)_{i\in I} \mid s_i \in \mathcal{F}_i(V \cap U_i), \phi_{i,j,V\cap U_i\cap U_j}(s_i|_{V\cap U_i\cap U_j}) = s_j|_{V\cap U_i\cap U_j} \text{ for all } i, j \in I\}$$

for all open $V \subseteq X$. We first show that $\mathcal{F}$ is a presheaf. We clearly have $\mathcal{F}(\varnothing) = \{(0)_{i\in I}\}$ which is the desired trivial ring. We now define, for open $W \subseteq V$, $\rho^{\mathcal{F}}_{V,W} : \mathcal{F}(V) \to \mathcal{F}(W)$ such that $(s_i)_{i\in I} \in \mathcal{F}(V)$ is sent to $(s_i|_{W\cap U_i})_{i\in I}$. By definition, we see that $\mathcal{F}(\mathrm{id}_V) = \rho^{\mathcal{F}}_{V,V}$ for all open $V \subseteq X$. We need only show commutativity of the restriction maps. Let $W, V, U \subseteq X$ be open such that $W \subseteq V \subseteq U$. We have, for some $(s_i)_{i\in I} \in \mathcal{F}(U)$, that

$$\begin{aligned}
(\rho^{\mathcal{F}}_{V,W} \circ \rho^{\mathcal{F}}_{U,V})((s_i)_{i\in I}) &= \rho^{\mathcal{F}}_{V,W}((s_i|_{V\cap U_i})_{i\in I}) \\
&= ((s_i|_{V\cap U_i})|_{W\cap U_i})_{i\in I} \\
&= (s_i|_{V\cap U_i\cap W\cap U_i})_{i\in I} && (\mathcal{F}_i \text{ is a sheaf}) \\
&= (s_i|_{W\cap U_i})_{i\in I} \\
&= \rho^{\mathcal{F}}_{U,W}((s_i)_{i\in I}).
\end{aligned}$$

Thus $\mathcal{F}$ is a presheaf.

We now show that $\mathcal{F}$ is a sheaf. Let $V \subseteq X$ be open and let $\{V_j\}_{j\in J}$ be an open covering of V. To show uniqueness, let $s = (s_i)_{i\in I} \in \mathcal{F}(V)$ such that $s|_{V_j} = 0$ for all $j \in J$. We have $s|_{V_j} = (s_i|_{V_j\cap U_i})_{i\in I} = (0)_{i\in I}$ so that, in particular, $s_i|_{V_j\cap U_i} = 0$ for all $i \in I$ and $j \in J$. Since $s_i|_{V_j} = 0$ for all $j \in J$, we have by the uniqueness condition of $\mathcal{F}_i$ that $s_i|_V = 0$ for all $i \in I$. Thus $s = (s_i)_{i\in I} = (0)_{i\in I}$. To show the glueing condition, let $s_j = (s_{j,i})_{i\in I} \in \mathcal{F}(V_j)$ for all $j \in J$ such that $s_j|_{V_j\cap V_k} = s_k|_{V_j\cap V_k}$ for all $j, k \in J$. We have $s_{j,i} \in \mathcal{F}_i(V_j \cap U_i)$ for all $i \in I$ and $j \in I$ so that, by the glueing condition of $\mathcal{F}_i$, there exists $s_i \in \mathcal{F}_i(V \cap U_i)$ such that $s_i|_{V_j} = s_{j,i}$ for all $j \in J$ and all $i \in I$. To confirm that $s := (s_i)_{i\in I}$ is in $\mathcal{F}(V)$, we need to check that $\phi_{i,k,V\cap U_i\cap U_k}(s_i|_{V\cap U_i\cap U_k}) = s_k|_{V\cap U_i\cap U_k}$ for all $i, k \in I$. However, since $\phi_{i,k}$ is a morphism of sheaves, we have the commutative diagram

$$\begin{tikzcd}
\mathcal{F}_i(V \cap U_i \cap U_k) \arrow[r, "\phi_{i,k,V\cap U_i\cap U_k}"] \arrow[d, "\mathcal{F}_i(i_{V\cap U_i\cap U_k, V_j\cap U_i\cap U_k})"] & \mathcal{F}_k(V \cap U_i \cap U_k) \arrow[d, "\mathcal{F}_j(i_{V\cap U_i\cap U_k, V_j\cap U_i\cap U_k})"] \\
\mathcal{F}_i(V_j \cap U_i \cap U_k) \arrow[r, "\phi_{i,k,V_j\cap U_i\cap U_k}"] & \mathcal{F}_j(V_j \cap U_i \cap U_k)
\end{tikzcd} \tag{A.11}$$

for all $j \in J$ and, furthermore, we have

$$\phi_{i,k,V_j\cap U_i\cap U_k}(s_i|_{V_j\cap U_i\cap U_k}) = s_k|_{V_j\cap U_i\cap U_k}$$

for all $j \in J$. By the uniqueness axiom of the $\mathcal{F}_i$, we deduce that

$$\phi_{i,k,V\cap U_i\cap U_k}(s_i|_{V\cap U_i\cap U_k}) = s_k|_{V\cap U_i\cap U_k}$$

must hold for (A.11) to commute. Thus $\mathcal{F}$ is a sheaf.

We now show that there exists an isomorphism $\psi_i : \mathcal{F}_i \to \mathcal{F}|_{U_i}$, which will be inverse to our wanted $\phi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$, for all $i \in I$. Set $k \in I$ and let $V \subseteq U_k$ be open. We define $\psi_{k,V} : \mathcal{F}_k(V) \to \mathcal{F}|_{U_k}(V)$ such that $s \in \mathcal{F}_i(V)$ is mapped to $(\phi_{k,i,V\cap U_i}(s|_{V\cap U_i}))_{i\in I}$. Let $V_i = V \cap U_i$ for all $i \in I$ and set $i, j \in I$. To see that this map is well-defined, we need to verify that

$$\phi_{i,j,V_i\cap U_j}(\phi_{k,i,V_i}(s|_{V_i})|_{V_i\cap U_j}) = \phi_{k,j,V_j}(s|_{V_j})|_{V_j\cap U_i}.$$

Since $\phi_{k,i}$ and $\phi_{k,j}$ are morphisms of sheaves, the above simplifies to

$$\phi_{i,j,V_i\cap U_j}(\phi_{k,i,V_i}(s|_{V_i\cap U_j})) = \phi_{k,j,V_i\cap U_j}(s|_{V_i\cap U_j})$$

which follows immediately by (A.9). We now check that $\psi_{i,V}$ is an isomorphism. To show it is one-to-one, assume that $\psi_{i,V}(s) = \psi_{k,V}(s')$ for some $s, s' \in \mathcal{F}_k(V)$. We therefore have $(\phi_{k,i,V}(s))_{i\in I} = (\phi_{k,i,V}(s'))_{i\in I}$. Hence

$$\phi_{k,i,V\cap U_i}(s|_{V\cap U_i}) = \phi_{k,i,V\cap U_i}(s'|_{V\cap U_i})$$

for all $i \in I$. Since $\phi_{k,i}$ is an isomorphism, it follows that $s|_{V\cap U_i} = s'|_{V\cap U_i}$ for all $i \in I$. Because $\{V \cap U_i\}_{i\in I}$ is an open covering of V, we deduce that $s = s'$ by the sheaf axioms of $\mathcal{F}_i$. Thus $\psi_{k,V}$ is injective. To show that $\psi_{i,V}$ is onto, let $t = (t_i)_{i\in I} \in \mathcal{F}|_{U_k}(V)$. Since $\phi_{k,i,V\cap U_i}$ is an isomorphism, there exists $s_i \in \mathcal{F}_k(V \cap U_i)$ such that $\phi_{k,i,V\cap U_i}(s_i) = t_i$. Because $\{V \cap U_i\}_{i\in I}$ covers V, if we can prove

$$s_i|_{V\cap U_i\cap U_j} = s_j|_{V\cap U_i\cap U_j} \tag{A.12}$$

for all $i, j \in I$, by the sheaf axioms, there then exists $s \in \mathcal{F}_k(V)$ such that $\psi_{k,V}(s) = (\phi_{k,i,V\cap U_i}(s|_{V\cap U_i}))_{i\in I} = (t_i)_{i\in I}$. Indeed, to show (A.12), we have

$$\begin{aligned}
s_i|_{V\cap U_i\cap U_j} &= \phi^{-1}_{k,i,V}(t_i)|_{V\cap U_i\cap U_j} \\
&= \phi^{-1}_{k,i,V\cap U_i\cap U_j}(t_i|_{V\cap U_i\cap U_j}) \\
&= \phi^{-1}_{k,i,V\cap U_i\cap U_j}(\phi^{-1}_{i,j,V\cap U_i\cap U_j}(t_j|_{V\cap U_i\cap U_j})) \\
&= \phi^{-1}_{k,j,V\cap U_i\cap U_j}(t_j|_{V\cap U_i\cap U_j}) \\
&= s_j|_{V\cap U_i\cap U_j}.
\end{aligned}$$

Thus $\psi_{k,V}$ is surjective and hence an isomorphism. The commutativity of (A.10) is a direct consequence of $\phi_k = \psi_k^{-1}$ being an isomorphism of sheaves.

□

Definition A.5.13. *Let X, Y be topological spaces, let $\mathcal{F}$ and $\mathcal{G}$ be a sheaves with respect to Y, let $f : X \to Y$ be a continuous map and let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. We define $f_*\alpha : f_*\mathcal{F} \to f_*\mathcal{G}$ to be the morphism of sheaves given by $(\alpha_{f^{-1}(U)})_{U\in\tau_X}$.*

A.6 Appendix F: Locally ringed spaces

This chapter uses various sources for the results and definitions stated.

Definition A.6.1 ([20, p. 72])**.** *A* ringed space *is a pair* $(X, \mathcal{O}_X)$ *where* X *is a topological space and* $\mathcal{O}_X$ *is a sheaf of rings on* X*. A* locally ringed space *is a ringed space* $(X, \mathcal{O}_X)$ *such that* $\mathcal{O}_{X,x}$ *is a local ring for every* $x \in X$*. We will denote* $(X, \mathcal{O}_X)$ *as* X *when* $\mathcal{O}_X$ *is understood.*

Let $(X, \mathcal{O}_X)$ be a (locally) ringed space and let $U \subseteq X$ be open so that we may endow U with a ringed space structure induced by X, i.e. $(U, \mathcal{O}_U)$ is a ringed space where $\mathcal{O}_U = \mathcal{O}_X|_U$. In the case of locally ringed spaces, this is well-defined as we have $\mathcal{O}_{X,x} \cong \mathcal{O}_{U,x}$ by definition of their respective stalks at x.

Definition A.6.2 ([20, p. 72])**.** *Let* X *and* Y *be ringed spaces with respective sheaves* $\mathcal{O}_X$ *and* $\mathcal{O}_Y$*. A* morphism of ringed spaces *from* X *to* Y *consists of a pair* $(f, f^\#)$ *with* $f : X \to Y$ *a continuous between topological spaces and* $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ *a map of sheaves on* Y*. Moreover, if* X *and* Y *are locally ringed spaces, a* morphism of locally ringed spaces $(f, f^\#)$ *from* $(X, \mathcal{O}_X)$ *to* $(Y, \mathcal{O}_Y)$ *is a morphism of ringed spaces such that the induced map between stalks* $f^\#_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ *is a local homomorphism. In each case,* $(f, f^\#)$ *is an* isomorphism *if it has an inverse.*

Lemma A.6.3 ([33, Section 01HD])**.** *Let* $(X, \mathcal{O}_X)$ *be a (resp. locally) ringed space and let* $U \subseteq X$ *be open. If* $\iota : U \to X$ *is the usual inclusion map and, for all open* $V \subseteq X$*,* $\iota^\#_V : \mathcal{O}_X(V) \to \iota_*\mathcal{O}_U(V)$ *is the map from* V *into* $V \cap U$ *induced by the restriction map from* $\mathcal{O}_X(V)$ *to* $\mathcal{O}_X(\iota^{-1}(V))$*, then* $(\iota, \iota^\#)$ *is a morphism of (resp. locally) ringed spaces where* $\iota^\# = (\iota^\#_V)_{V \in \tau_X}$*.*

Proof. Since we know that ι is a continuous map, we need only check that $\iota^\#$ is a morphism of ringed spaces. Let $W, V \subseteq X$ be open such that $W \subseteq V$. Considering the diagram

$$\begin{tikzcd}
\mathcal{O}_X(V) \arrow[r, "\iota^\#_V"] \arrow[d, "\rho^{\mathcal{O}_X}_{V,W}"] & \iota_*\mathcal{O}_U(V) \arrow[r, "\cong"] \arrow[d, "\rho^{\iota_*\mathcal{O}_U}_{V,W}"] & \mathcal{O}_X(\iota^{-1}(V)) \arrow[d, "\rho^{\mathcal{O}_X}_{\iota^{-1}(V),\iota^{-1}(W)}"] \\
\mathcal{O}_X(W) \arrow[r, "\iota^\#_W"] & \iota_*\mathcal{O}_U(W) \arrow[r, "\cong"] & \mathcal{O}_X(\iota^{-1}(W)),
\end{tikzcd}$$

one observes that both inner rectangles commute as restriction maps of $\mathcal{O}_X$ commute. It follows that $(\iota, \iota^\#)$ is a morphism of ringed spaces. Moreover, if we assume that $(X, \mathcal{O}_X)$ is a locally ringed space, we have by definition of $\mathcal{O}_U$ that $\mathcal{O}_{X,f(x)} \cong \mathcal{O}_{U,x}$ for all $x \in U \subseteq X$ so that $(\iota, \iota^\#)$ is a morphism of locally ringed spaces.

□

Definition A.6.4 ([33, Section 0090])**.** *Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ and $(Z, \mathcal{O}_Z)$ be (resp. locally) ringed spaces and let $(f : X \to Y, f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X)$ and $(g : Y \to Z, g^\# : \mathcal{O}_Z \to g_*\mathcal{O}_Y)$ be morphisms of (resp. locally) ringed spaces. For all open $U \subseteq Z$, we define $(g \circ f)^\#_U := g_* f^\#_U \circ g^\#_U$ illustrated by the diagram*

$$\begin{tikzcd}
\mathcal{O}_Z(U) \arrow[rr, "(g\circ f)^\#"] \arrow[dr, "g^\#_U"'] & & g_*(f_*\mathcal{O}_X(U)) \\
& g_*\mathcal{O}_Y(U) \arrow[ur, "g_*f^\#_U"'] &
\end{tikzcd}$$

Finally, we define the composition $(g, g^\#) \circ (f, f^\#)$ as $(g \circ f, (g \circ f)^\#)$.

Definition A.6.5. *Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be (resp. locally) ringed spaces and let $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of (resp. locally) ringed spaces. Let $U \subseteq X$ is open, $(U, \mathcal{O}_U)$ be the (resp. locally) ringed space induced by $(X, \mathcal{O}_X)$ and let $(\iota, \iota^\#) : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ be the inclusion morphism described in Lemma A.6.3. We define the restriction of $(f, f^\#)$ to $(U, \mathcal{O}_U)$, denoted by $(f, f^\#)|_U$, as $(f, f^\#) \circ (\iota, \iota^\#)$.*

Proposition A.6.6 ([28, Exercise 2.11, p. 40])**.** *Let X and Y be (resp. locally) ringed spaces, let $\{U_i\}_{i\in I}$ be an open covering of X and let $(f_i : U_i \to Y, f_i^\# : \mathcal{O}_Y \to f_{i*}\mathcal{O}_X)$ be a morphism of (locally) ringed spaces for all $i \in I$ such that $(f_i, f_i^\#)|_{U_i \cap U_j} = (f_j, f_j^\#)|_{U_i\cap U_j}$ for all $i, j \in I$. Then there exists a unique morphism of (resp. locally) ringed spaces $(f : X \to Y, f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X)$ such that $(f, f^\#)|_{U_i} = (f_i, f_i^\#)$ for all $i \in I$.*

Proof. We begin by showing existence of such a morphism. Define $f : X \to Y$ such that $x \mapsto f_i(x)$ if $x \in X_i$ for some $i \in I$. Since $\{U_i\}_{i\in I}$ is an open covering of X and by our assumptions the f_i for all $i \in I$, this map is well-defined. We now check that f is continuous. Let $V \subseteq Y$ be open. We have

$$f^{-1}(V) = \bigcup_{i\in I} U_i \cap f^{-1}(V) = \bigcup_{i\in I} f^{-1}|_{U_i}(V).$$

Since $f|_{U_i}$ is continuous, it follows that $f^{-1}|_{U_i}(V)$ is open for all $i \in I$ so that $f^{-1}(V)$ is open. Hence f is continuous.

We now define the needed morphism of sheaves $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$. Let $U \subseteq Y$ be open. We construct $f^\#_U : \mathcal{O}_Y(U) \to f_*\mathcal{O}_X(U)$ as follows. Let $s \in \mathcal{O}_Y(U)$ and let $t_i := f^\#_{i,U}(s) \in \mathcal{O}_X(f_i^{-1}(U))$ for all $i \in I$. Since $\{f_i^{-1}(U)\}_{i\in I}$ is an open covering of $f^{-1}(U)$, to use the gluing condition of sheaves to give a section in $\mathcal{O}_X(f^{-1}(U))$, we need to check that $t_i|_{f_i^{-1}(U)\cap f_j^{-1}(U)} = t_j|_{f_i^{-1}(U)\cap f_j^{-1}(U)}$ for all $i, j \in I$. Set $i, j \in I$, let $U_{i,j} \subseteq U_i$ and let $U_{j,i} \in U_j$. Let $(\iota_{i,j}, \iota^\#_{i,j})$, $(\iota_{j,i}, \iota^\#_{j,i})$

denote the restriction morphisms of ringed spaces, respectively from $U_{i,j}$ to U_i and from $U_{j,i}$ to U_j, given as in Lemma A.6.3. We have

$$\begin{aligned} t_i|_{f_i^{-1}(U)\cap U_{i,j}} &= \iota^{\#}_{i,j,f_i^{-1}(U)}(t_i) \\ &= f_{i*}\iota^{\#}_{i,j,U}(t_i) \\ &= (f_{i*}\iota^{\#}_{i,j,U} \circ f^{\#}_{i,U})(s) \\ &= (f_i \circ \iota_{i,j})^{\#}_U(s) \\ &= (f_j \circ \iota_{j,i})^{\#}_U(s) && \text{(By our assumptions on } f_i \text{ and } f_j\text{)} \\ &= t_j|_{f_j^{-1}(U)\cap U_{j,i}} && \text{(Symmetry of the above equalities).} \end{aligned} \tag{A.13}$$

Substituting $U_{i,j} = U_{j,i} = f_i^{-1}(U) \cap f_j^{-1}(U)$, we achieve $t_i|_{f_i^{-1}(U)\cap f_j^{-1}(U)} = t_j|_{f_i^{-1}(U)\cap f_j^{-1}(U)}$. It follows that there is a unique section $t \in \mathcal{O}_X(f^{-1}(U))$ such that $t|_{f_i^{-1}(U)} = t_i$ for all $i \in I$. We define $f^{\#}_U : \mathcal{O}_Y(U) \to f_*\mathcal{O}_X(U)$ such that $s \mapsto t$. If $s, s' \in \mathcal{O}_Y(U)$, $t = f^{\#}_U(s)$ and $t' = f^{\#}_U(s')$, it is routine to check that $f^{\#}_U(s + s') = t + t'$ and $f^{\#}_U(s \cdot s') = t \cdot t'$ so that $f^{\#}_U$ is a well-defined homomorphism. We need to check, given open $U, V \subseteq X$ such that $V \subseteq U$, that the diagram

$$\begin{tikzcd} \mathcal{O}_Y(U) \arrow[r, "f^{\#}_U"] \arrow[d, "\rho^{\mathcal{O}_X}_{U,V}"] & f_*\mathcal{O}_X(U) \arrow[d, "\rho^{\mathcal{O}_X}_{f^{-1}(U),f^{-1}(V)}"] \\ \mathcal{O}_Y(V) \arrow[r, "f^{\#}_V"] & f_*\mathcal{O}_X(V) \end{tikzcd} \tag{A.14}$$

commutes. To do so, consider the commutative diagram

$$\begin{tikzcd} \mathcal{O}_Y(U) \arrow[r, "f^{\#}_U"] \arrow[d, "\rho^{\mathcal{O}_Y}_{U,V}"] & f_*\mathcal{O}_X(U) \arrow[d, "\rho^{\mathcal{O}_X}_{f^{-1}(U),f^{-1}(V)}"] \\ \mathcal{O}_Y(V) \arrow[r, "f^{\#}_V"] \arrow[d, "\mathrm{id}"] & f_*\mathcal{O}_X(V) \arrow[d, "\rho^{\mathcal{O}_X}_{f^{-1}(V),f_i^{-1}(V)}"] \\ \mathcal{O}_Y(V) \arrow[r, "f^{\#}_{i,V}"] & f_{i*}\mathcal{O}_X(V) \end{tikzcd} \tag{A.15}$$

for all $i \in I$. We would like to use the sheaf properties of $f^{\#}_V$ to show that the upper rectangle of (A.15) commutes. We claim that the outer rectangle of (A.15) commutes. To show this, we first note that

$$f^{\#}_{i,V} \circ \rho^{\mathcal{O}_Y}_{U,V} = \rho^{\mathcal{O}_X}_{f_i^{-1}(U),f_i^{-1}(V)} \circ f^{\#}_{i,U}$$

since $f_i^{\#}$ is a morphism of sheaves. Furthermore, by definition of $f^{\#}_U$, we have that

$$f^{\#}_{i,U} = \rho^{\mathcal{O}_X}_{f^{-1}(U),f_i^{-1}(U)} \circ f^{\#}_U.$$

It follows that

$$\begin{aligned} f^{\#}_{i,V} \circ \rho^{\mathcal{O}_Y}_{U,V} &= \rho^{\mathcal{O}_X}_{f_i^{-1}(U),f_i^{-1}(V)} \circ \rho^{\mathcal{O}_X}_{f^{-1}(U),f_i^{-1}(U)} \circ f^{\#}_U \\ &= \rho^{\mathcal{O}_X}_{f^{-1}(U),f_i^{-1}(V)} \circ f^{\#}_U \\ &= \rho^{\mathcal{O}_X}_{f^{-1}(V),f_i^{-1}(V)} \circ \rho^{\mathcal{O}_X}_{f^{-1}(U),f^{-1}(V)} \circ f^{\#}_U . \end{aligned}$$

Let $s \in \mathcal{O}_Y(U)$. Since $(f^{\#}_U(s)|_{f^{-1}(V)})|_{f_i^{-1}(V)} = f^{\#}_{i,V}(s|_V)$ for all $i \in I$, it follows, by uniqueness of $f^{\#}_V(s|_V)$, that $f^{\#}_U(s)|_{f^{-1}(V)} = f^{\#}_V(s|_V)$. We deduce that (A.14) commutes so that $f^{\#} = (f^{\#}_U)_{U \in \mathcal{T}_Y}$ is a morphism of sheaves.

We conclude the proof by confirming that $(f, f^{\#})$ is indeed unique. Assume that $(g, g^{\#})$ is another morphism of ringed spaces such that $g|_{U_i} = f_i$. Since f and g are functions and agree on an open covering of their domain, it follows that $f = g$, so that we need only check that $f^{\#} = g^{\#}$. Let $V \subseteq Y$ be open and let $s \in \mathcal{O}_Y(V)$. We have, for all $i \in I$,

$$\begin{aligned} f^{\#}_V(s)|_{f^{-1}(V) \cap U_i} &= f^{\#}_V(s)|_{f_i^{-1}(V)} \\ &= f^{\#}_{i,V}(s) \\ &= g^{\#}_V(s)|_{f_i^{-1}(V)} \\ &= g^{\#}_V(s)|_{f^{-1}(V) \cap U_i} . \end{aligned}$$

Hence $f^{\#} = g^{\#}$, so that $(f, f^{\#})$ is unique (up to isomorphism).

If we, moreover, assume that X and Y are locally ringed spaces, we would like to show that f is additionally a morphism of locally ringed spaces. Let $x \in X$ and let $i \in I$ be such that $x \in X_i$. We have, by the universal property of $\mathcal{O}_{Y,f(x)}$, for open $V, W \subseteq Y$ such that $W \subseteq V$, the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[rr, "\rho^{\mathcal{O}_Y}_{V,W}"] \arrow[d, "f^{\#}_V"'] \arrow[dr, "\phi^{\mathcal{O}_{Y,f(x)}}_V"] & & \mathcal{O}_Y(W) \arrow[d, "f^{\#}_W"] \arrow[dl, "\phi^{\mathcal{O}_{Y,f(x)}}_W"'] \\
\mathcal{O}_X(f^{-1}(V)) \arrow[dr, bend right, "\phi^{\mathcal{O}_{X,x}}_{f^{-1}(V)}"] \arrow[dd, bend right, "\rho^{\mathcal{O}_X}_{f^{-1}(V),f_i^{-1}(V)}"'] & \mathcal{O}_{Y,f(x)} \arrow[d, dotted, "f^{\#}_x"] & \mathcal{O}_X(f^{-1}(W)) \arrow[dl, bend left, "\phi^{\mathcal{O}_{X,x}}_{f^{-1}(W)}"'] \arrow[dd, bend left, "\rho^{\mathcal{O}_X}_{f^{-1}(W),f_i^{-1}(W)}"] \\
 & \mathcal{O}_{X,x} & \\
\mathcal{O}_X(f_i^{-1}(V)) \arrow[ur, "\phi^{\mathcal{O}_{X,x}}_{f_i^{-1}(V)}"'] & & \mathcal{O}_X(f_i^{-1}(W)). \arrow[ul, "\phi^{\mathcal{O}_{X,x}}_{f_i^{-1}(W)}"]
\end{tikzcd}$$

Since $\rho^{\mathcal{O}_X}_{f^{-1}(U),f_i^{-1}(V)} \circ f^{\#}_V = f^{\#}_{i,U}$ for any open $U \subseteq Y$, it follows that the diagram

$$\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[rr, "\rho^{\mathcal{O}_Y}_{V,W}"] \arrow[d, "f^{\#}_V"'] \arrow[dr, "\phi^{\mathcal{O}_{Y,f(x)}}_V"] & & \mathcal{O}_Y(W) \arrow[d, "f^{\#}_W"] \arrow[dl, "\phi^{\mathcal{O}_{Y,f(x)}}_W"'] \\
\mathcal{O}_X(f^{-1}(V)) \arrow[dr, bend right, "\phi^{\mathcal{O}_{X,x}}_{f^{-1}(V)}"'] & \mathcal{O}_{Y,f(x)} \arrow[d, "f^{\#}_x"] & \mathcal{O}_X(f^{-1}(W)) \arrow[dl, bend left, "\phi^{\mathcal{O}_{X,x}}_{f^{-1}(W)}"] \\
& \mathcal{O}_{X,x} &
\end{tikzcd}$$

commutes. Since $f(x) = f_i(x)$, it follows that $\mathcal{O}_{Y,f(x)} = \mathcal{O}_{Y,f_i(x)}$. We deduce that $f^{\#}_x = f^{\#}_{i,x}$ as $f^{\#}_{i,x}$ is the unique morphism that makes the above the diagram commute by the universal property of $\mathcal{O}_{Y,f_i(x)}$. Since $f^{\#}_{i,x}$ is a local homomorphism, we conclude that $f^{\#}_x$ is a local morphism so that $(f, f^{\#})$ is a morphism of locally ringed spaces.

□

Lemma A.6.7. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally ringed spaces. If $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an isomorphism of ringed spaces, then it is an isomorphism of locally ringed spaces.*

Proof. We need only show that $f^{\#}_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism for all $x \in X$. Let $x \in X$. By definition of the canonical homomorphism between stalks, we have $f^{\#}_x(s_{f(x)}) := (f_U(s))_{f(x)}$ where $s_{f(x)} \in \mathcal{O}_{Y,f(x)}$ and $s \in \mathcal{O}_Y(U)$ is a lifting of $s_{f(x)}$ for some open $U \subseteq Y$ such that $f(x) \in U$. We now claim that $f^{\#}_x$ is an isomorphism: We first show that $f^{\#}_x$ is surjective. Let $t_{f(x)} \in f_*\mathcal{O}_{X,f(x)}$ so that $t_{f(x)}$ has a lifting $t \in f_*\mathcal{O}_{X,f(x)}(U)$ for some open $U \subseteq Y$ such that $f(x) \in U$. Set the notation that $f^{\#} = (f_U)_{U \in \tau_Y}$. Since f_U is an isomorphism, we have $f_U^{-1}(t) \in \mathcal{O}_Y(U)$. It follows that $f^{\#}_x((f_U^{-1}(t))_{f(x)}) = t_{f(x)}$. We next show that $f^{\#}_x$ is injective. Assume that $f^{\#}_x(s_{f(x)}) = 0$ for some $s_{f(x)} \in \mathcal{O}_{Y,f(x)}$. Let $s \in \mathcal{O}_Y(U)$ denote the lifting of $s_{f(x)}$ for some open $U \subseteq Y$. Then we have $(f_U(s))_{f(x)} = 0$. It follows by the definition of stalks that there exists an open $V \subseteq Y$ such that $f(x) \in V$ and that $f_U(s)|_V = 0$. Since $f^{\#}$ is a morphism of sheaves, it follows that $f_V(s|_V) = 0$. Furthermore, f_V is an isomorphism, and hence injective, so that $s|_V = 0$. Thus, by definition of $\mathcal{O}_{Y,f(x)}$, we have $s_{f(x)} = 0$ so that $f^{\#}_x$ is injective. Following immediately from the fact that $f^{\#}_x$ is an isomorphism, we have that $f^{\#}_x$ is a local homomorphism since $\mathcal{O}_{Y,f(x)}$ and $\mathcal{O}_{X,x}$ are local rings. □

Proposition A.6.8 ([33, Section 00AK])**.** *Let I be some index set, $(X_i, \mathcal{O}_{X_i})$ be a locally ringed space for all $i \in I$ and, for each $i, j \in I$, let $U_{i,j} \subseteq X_i$ be open with an induced locally ringed space structure. Assume that there exists* *an isomorphism* $\phi_{i,j} : U_{i,j} \to U_{j,i}$ *for all* $i, j \in I$ *such that, for all* $i, j, k \in I$, $\phi_{i,j}^{-1} = \phi_{j,i}$ *and that*

$$\phi_{i,j}^{-1}(U_{j,i} \cap U_{j,k}) = U_{i,j} \cap U_{i,k} \tag{A.16}$$

and such that the diagram

$$\begin{tikzcd} U_{i,j} \cap U_{i,k} \arrow[rr, "\phi_{i,k}"] \arrow[dr, "\phi_{i,j}"] & & U_{k,i} \cap U_{k,j} \\ & U_{j,i} \cap U_{j,k} \arrow[ur, "\phi_{j,k}"] & \end{tikzcd} \tag{A.17}$$

commutes. Further assume that $U_{i,i} = X_i$ *and* $\phi_{i,i} = \mathrm{id}_{X_i}$. *Then there exists a locally ringed space* $(X, \mathcal{O}_X)$ *and, for all* $i \in I$, *an open* $U_i \subseteq X$ *with an induced locally ringed space structure and an injective continuous map* $\phi_i : X_i \to X$ *such that*

1. $\phi_i(X_i) = U_i$
2. $X = \bigcup_{i\in I} U_i$
3. $\phi_i(U_{i,j}) = U_i \cap U_j$
4. $\phi_{i,j} = \phi_j^{-1}|_{U_i \cap U_j} \circ \phi_i|_{U_{i,j}}$.

Proof. We will construct X in several steps. We would like to define a topology X. First, we consider an equivalence relation on $\bigsqcup_{i\in I} X_i$. Let $x_i, x_j \in \bigsqcup_{i\in I} X_i$ with lifings $x_i' \in X_i$ and $x_j' \in X_j$, respectively, for some $i, j \in I$. We define $\sim$ on $\bigsqcup_{i\in I} X_i$ such that $x_i \sim x_j$ if and only if $x_i' \in U_{i,j}$, x_j' $\phi_{i,j}(x_i') = x_j'$. To show that $\sim$ is an equivalence relation, we need only check transitivity as reflexivity and symmetry follow easily from our assumption that $U_{i,i} = X_i$ and $\phi_{i,i} = \mathrm{id}_{X_i}$. Assume that $x_i, x_j, x_k \in \bigsqcup_{i\in I} X_i$ with respective liftings $x_i' \in X_i$, $x_j' \in X_j$ and $x_k' \in X_k$ for some fixed $i, j, k \in I$ such that $x_i \sim x_j$ and $x_j \sim x_k$. We have that $x_j' \in U_{i,j} \cap U_{j,k}$. It follows that $x_i' \in U_{i,j} \cap U_{i,k}$ and $x_k' \in U_{k,i} \cap U_{k,j}$ by (A.16) and that $\phi_{i,k}(x_i') = x_k'$ by the commutativity of (A.17). We define $X := \left(\bigsqcup_{i\in I} X_i\right) / \sim$ equipped with the induced topology given by the coproduct topology of $\{X_i\}_{i\in I}$ and quotient topology of $\sim$.

Having defined X, we now denote $\phi_i : X_i \to X$ as the natural inclusion map for all $i \in I$ and set $U_i = \phi_i(X_i) \subseteq X$. Thus properties 1. and 2. hold. We would like to show that ϕ_i is a homeomorphism. It follows by definition of ϕ_i that ϕ_i is both injective and continuous for all $i \in I$. Furthermore, ϕ is a bijection to it's image. Note, by definition of $\sim$, that $\phi_j^{-1}(U_i) = U_{j,i}$ so that 3.

holds. We now show that the image of these maps of open sets are open. Fix some $i \in I$ and let $W_i \subseteq X_i$ be open. We have, by 3., that

$$\phi_j^{-1}(\phi_i(W_i)) = \phi_{j,i}^{-1}(W_i \cap U_{i,j}) \tag{A.18}$$

is open. Combining (A.18) with the fact that ϕ_j is continuous for any $j \in J$, it follows that $\phi_i(W_i)$ is open. Thus, for any choice of $i \in I$, ϕ_i is a homeomorphism to its image. To deduce 4., one simply convenes (A.18) with the fact that ϕ_i, ϕ_j and $\phi_{j,i}$ are isomorphisms, such that $\phi_{j,i}^{-1} = \phi_{i,j}$, for all $i, j \in I$.

We now construct the sheaf $\mathcal{O}_X$. Define $\mathcal{O}_{U_i} := \phi_{i*}\mathcal{O}_{X_i}$ for all $i \in I$. Fix some $i, j, k \in I$. It is clear, by definition of $\mathcal{O}_{U_i}$, that there is an isomorphism $\phi_{i,j} : X_i \to U_j$. Moreover, we have, by 3., the commutative diagram

$$\begin{tikzcd}
U_{i,j} \arrow[rr, "\phi_{i,j}"] \arrow[dr, "\phi_i|_{U_{i,j}}"'] & & U_{j,i} \arrow[dl, "\phi_j|_{U_{j,i}}"] \\
& U_i \cap U_j &
\end{tikzcd} \tag{A.19}$$

It follows that $\phi_{i,j}$ induces an isomorphism $(\psi_{i,j}, \psi_{i,j}^{\#}) : (U_i \cap U_j, \mathcal{O}_{U_i}|_{U_i \cap U_j}) \to (U_i \cap U_j, \mathcal{O}_j|_{U_i \cap U_j})$. We deduce from (A.17) and (A.19) that the diagram

$$\begin{tikzcd}
\mathcal{O}_{U_i}|_{U_i \cap U_j \cap U_k} \arrow[rr, "\psi_{i,k}"] \arrow[dr, "\psi_{i,j}"'] & & \mathcal{O}_{U_k}|_{U_i \cap U_j \cap U_k} \\
& \mathcal{O}_{U_j}|_{U_i \cap U_j \cap U_k} \arrow[ur, "\psi_{j,k}"'] &
\end{tikzcd} \tag{A.20}$$

commutes. Thus, applying Proposition A.5.12, we obtain a sheaf $\mathcal{O}_X$ with isomorphisms $\psi_i : \mathcal{O}_X|_{U_i} \to \mathcal{O}_X|_{U_i}$ that is compatible with $\psi_{i,j}$ for all $i, j \in I$. Therefore, $(X, \mathcal{O}_X)$ is a ringed space. To show that $(X, \mathcal{O}_X)$ is a locally ringed space, we need to show that $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$. This follows easily. Let $x \in X$ and let $U \subseteq X$ be open such that $x \in U$. Since $\{U_i\}_{i \in I}$ is an open covering of X, we have that $x \in U \cap U_i \subseteq U_i$ for some $i \in I$. We deduce, by definition of $\mathcal{O}_{X,x}$, that $\mathcal{O}_{X,x} = \mathcal{O}_{U_i,x}$. Since U_i is isomorphic to X_i as a locally ringed space, it follows that $\mathcal{O}_{U_i,x}$ is a local ring so that $(X, \mathcal{O}_X)$ is a locally ringed space. □

A.7 Appendix G: Schemes

A.7.1 First definitions and examples

This section follows [20, Chapter 2, Section 2].

Definition A.7.1. *Let R be a ring. We define a sheaf with respect to* $\text{Spec}(R)$*,* $\mathcal{O}_{\text{Spec}(R)} : \mathbf{Ops}^{op}_{\text{Spec}(R)} \to \mathbf{Ring}$*, which*

- *sends an open set U of* $\text{Spec}(R)$ *to the ring of functions* $s : U \to \bigsqcup_{\mathfrak{p} \in U} R_{\mathfrak{p}}$ *such that $s(\mathfrak{p}) \in R_{\mathfrak{p}}$ for all $\mathfrak{p} \in U$ and such that, for all $\mathfrak{p} \in U$, there exists an open subset V of U containing $\mathfrak{p}$ such that $s(\mathfrak{q}) = \frac{a}{f} \in R_{\mathfrak{q}}$ where $a \in R$ and $f \in R \setminus \mathfrak{q}$ for all $\mathfrak{q} \in V$. Note that our ring here is equipped with the usual point-wise addition and multiplication operation of functions.*

- *given two open sets $V \subseteq U$ of* $\text{Spec}(R)$*, sends the morphism $i_{U,V} \in$* $\mathbf{Ops}^{op}_{\text{Spec}(R)}$ *to the map $\rho^{\mathcal{O}_{\text{Spec}(R)}}_{U,V} : \mathcal{O}_{\text{Spec}(R)}(U) \to \mathcal{O}_{\text{Spec}(R)}(V)$ which restricts the domain of any element of $\mathcal{O}_{\text{Spec}(R)}$ to V for all open $U, V \subseteq X$ such that $V \subseteq U$.*

One still needs to check that this definition does indeed define a sheaf.

Proposition A.7.2. *Let R be a ring, then $\mathcal{O}_{\text{Spec}(R)}$ is a sheaf.*

Proof. In order to show that $\mathcal{O}_{\text{Spec}(R)}$ is a sheaf, we first need to show that it is indeed a presheaf that satisfies the sheaf conditions. One easily sees that, for all open U of $\text{Spec}(R)$, $\mathcal{O}_{\text{Spec}(R)}(U)$ is a commutative ring with unity $\text{id} : U \to \bigsqcup_{\mathfrak{p} \in U} R_{\mathfrak{p}}$ which sends $\mathfrak{p} \in U$ to $1_{R_{\mathfrak{p}}}$. Let $W, V, U \subseteq \text{Spec}(R)$ be open such that $W \subseteq V \subseteq U$ and let $i_{U,V}, i_{U,W}, i_{V,W}$ be the corresponding inclusion maps in $\mathbf{Ops}^{op}_{\text{Spec}(R)}$. Since $\rho^{\mathcal{O}_{\text{Spec}(R)}}_{U,V}, \rho^{\mathcal{O}_{\text{Spec}(R)}}_{U,W}$ and $\rho^{\mathcal{O}_{\text{Spec}(R)}}_{V,W}$ are function restrictions as we usually understand them, it is obvious that $\rho^{\mathcal{O}_{\text{Spec}(R)}}_{U,W} = \rho^{\mathcal{O}_{\text{Spec}(R)}}_{V,W} \circ \rho^{\mathcal{O}_{\text{Spec}(R)}}_{U,V}$ holds. It follows that $\mathbf{Ops}^{op}_{\text{Spec}(R)}(i_{U,W}) = \rho^{\mathcal{O}_{\text{Spec}(R)}}_{U,W} = \rho^{\mathcal{O}_{\text{Spec}(R)}}_{V,W} \circ \rho^{\mathcal{O}_{\text{Spec}(R)}}_{U,V} = \mathbf{Ops}^{op}_{\text{Spec}(R)}(i_{V,W}) \circ \mathbf{Ops}^{op}_{\text{Spec}(R)}(i_{U,V})$. Furthermore, $\mathcal{F}(\varnothing) = 0$ by definition, so that $\mathcal{O}_{\text{Spec}(R)}$ is a presheaf. Finally, we prove that the sheaf axioms hold for $\mathcal{O}_{\text{Spec}(R)}$. Let U be an open set in $\text{Spec}(R)$, let $\{U_i\}_{i \in I}$ be an open covering of U and let $s_i \in \mathcal{O}_{\text{Spec}(R)}(U_i)$ for all $i \in I$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$. We would like to construct a section $s \in \mathcal{O}_{\text{Spec}(R)}$ by defining $s(\mathfrak{p}) := s_i(\mathfrak{p})$ if $p \in U_i$ for some $i \in I$. However, we first need to show that it is indeed well defined. If $p \in U_i$ for multiple $i \in I$, then by our restriction condition on the s_i, we must have that $s_i(\mathfrak{p}) = s_j(\mathfrak{p})$ for all $i, j \in I$ such that $\mathfrak{p} \in U_i$ and $\mathfrak{p} \in U_j$. Thus, our construction is well-defined and hence we have shown the gluing condition for a sheaf. We conclude the proof by showing that the uniqueness condition holds. Let $t \in \mathcal{O}_{\text{Spec}(R)}(U)$ such that $t|_{U_i} = 0$ for all $i \in I$. Consider the graphs of $t|_{U_i} : U_i \to \bigsqcup_{\mathfrak{p} \in U_i} R_{\mathfrak{p}} \subseteq \bigsqcup_{\mathfrak{p} \in U} R_{\mathfrak{p}}$ for all $i \in I$. We have that $t = \bigcup_{i \in I} t|_{U_i}$ which is to say that $t(\mathfrak{p}) = 0 \in R_{\mathfrak{p}}$ for all $\mathfrak{p} \in U$ so that $t = 0$. Therefore $\mathcal{O}_{\text{Spec}(R)}$ is a sheaf. ☐

Proposition A.7.3 ([20, p.71])**.** *Let R be a ring. The stalk $\mathcal{O}_{\text{Spec}(R),\mathfrak{p}}$ is isomorphic to $R_{\mathfrak{p}}$ for all $\mathfrak{p} \in$* $\text{Spec}(R)$*.*

Proof. To give an isomorphism, we first need to construct a homomorphism between each ring. We define, for all $\mathfrak{p} \in \operatorname{Spec}(R)$, the map $ev_{\mathfrak{p}} : \mathcal{O}_{\operatorname{Spec}(R),\mathfrak{p}} \to R_{\mathfrak{p}}$ which sends a local section $s_{\mathfrak{p}}$ in $\mathcal{O}_{\operatorname{Spec}(R),\mathfrak{p}}$ to the evaluation at $\mathfrak{p}$ of a lifting of $s_{\mathfrak{p}}$, say $s \in \mathcal{F}(U)$ for some open $U \ni \mathfrak{p}$ in Spec(R), i.e. $s_{\mathfrak{p}} \mapsto s(\mathfrak{p})$. By definition of s, $s(\mathfrak{p}) \in R_{\mathfrak{p}}$, so we may deduce that $ev_{\mathfrak{p}}$ is a well-defined ring homomorphism for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Set some $\mathfrak{p} \in \operatorname{Spec}(R)$. We now show that $ev_{\mathfrak{p}}$ is an isomorphism, beginning with surjectivity. Let $\frac{a}{f}_{\mathfrak{p}} \in R_{\mathfrak{p}}$ where $a \in R$ and $f \in R \setminus \mathfrak{p}$. It follows that $\mathbb{D}_{\text{sch}}(f)$ is an open neighbourhood of $\mathfrak{p}$ so that the section $s \in \mathcal{O}_{\operatorname{Spec}(R)}(\mathbb{D}_{\text{sch}}(f))$, defined by

$$s(\mathfrak{q}) = \begin{cases} \frac{a}{f}_{\mathfrak{p}} & \text{if } \mathfrak{q} = \mathfrak{p} \\ 0 & \text{otherwise,} \end{cases}$$

, for all $\mathfrak{q} \in \mathbb{D}_{\text{sch}}(f)$, maps to $\frac{a}{f}_{\mathfrak{p}}$ under $ev_{\mathfrak{p}}$ giving surjectivity. To show that $ev_{\mathfrak{p}}$ is injective, assume that $ev_{\mathfrak{p}}(s_{\mathfrak{p}}) = ev_{\mathfrak{p}}(t_{\mathfrak{p}})$ for some $s_{\mathfrak{p}}, t_{\mathfrak{p}} \in \mathcal{O}_{\operatorname{Spec}(R),\mathfrak{p}}$. Let $s \in \mathcal{O}_{\operatorname{Spec}(R)}(U)$ be a representative of $s_{\mathfrak{p}}$ for some open $U \subseteq \operatorname{Spec}(R)$ and let be $t \in \mathcal{O}_{\operatorname{Spec}(R)}(V)$ be a representative of $t_{\mathfrak{p}}$ for some open $V \subseteq \operatorname{Spec}(R)$. By definition of elements in $\mathcal{O}_{\operatorname{Spec}(R)}(W)$ for any open $W \subseteq \operatorname{Spec}(R)$,we may assume $s(\mathfrak{p}) = \frac{a}{f}_{\mathfrak{p}}$ and $t(\mathfrak{p}) = \frac{b}{g}_{\mathfrak{p}}$ for some $a, b \in R$ and $f, g \in R \setminus \mathfrak{p}$. Since $\frac{a}{f}_{\mathfrak{p}} = \frac{b}{g}_{\mathfrak{p}}$ in $R_{\mathfrak{p}}$, it follows by the definition of a localization that there exists an $h \in R \setminus \mathfrak{p}$ such that $h(ag - bf) = 0$. Thus $\frac{a}{f}_{\mathfrak{q}} = \frac{b}{g}_{\mathfrak{q}}$ in all $R_{\mathfrak{q}}$ where $\mathfrak{q} \in \operatorname{Spec}(R)$ and $f, g, h \notin \mathfrak{q}$. In other words, $\frac{a}{f}_{\mathfrak{q}} = \frac{b}{g}_{\mathfrak{q}}$ for all $\mathfrak{q} \in \mathbb{D}_{\text{sch}}(f) \cap \mathbb{D}_{\text{sch}}(g) \cap \mathbb{D}_{\text{sch}}(h)$. Since $\mathfrak{p} \in \mathbb{D}_{\text{sch}}(f) \cap \mathbb{D}_{\text{sch}}(g) \cap \mathbb{D}_{\text{sch}}(h)$ as well, we deduce that s and t agre in a neighbourhood of $\mathfrak{p}$ which implies $s_{\mathfrak{p}} = t_{\mathfrak{p}}$ in $\mathcal{O}_{\operatorname{Spec}(R),\mathfrak{p}}$. ☐

We deduce the following corollary from the fact that $R_{\mathfrak{p}}$ is a local ring (see Lemma A.2.25).

Corollary A.7.4. *Let R be a ring, then* $(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$ *is a locally ringed space.*

Proposition A.7.5. *Let R be a ring. For any $f \in R$, $\mathcal{O}_{\operatorname{Spec}(R)}(\mathbb{D}_{\text{sch}}(f))$ is isomorphic to R_f.*

Proof. To show that the two rings are isomorphic, we need to give an isomorphism. Let $f \in R$. We define the map $\psi : R_f \to \mathcal{O}_{\operatorname{Spec}(R)}(\mathbb{D}_{\text{sch}}(f))$ where $\frac{a}{f^n}_f \mapsto (s : \mathbb{D}_{\text{sch}}(f) \to \prod_{\mathfrak{p} \in \mathbb{D}_{\text{sch}}(f)} R_{\mathfrak{p}})$ such that $s(\mathfrak{p}) = \frac{a}{f^n}_{\mathfrak{p}}$ for all $n \in \mathbb{N}$ and all $a \in R$. To check that this map is well-defined, let $\frac{a}{f^n}_f = \frac{b}{f^m}_f$ for some $b \in R$ and $m \in \mathbb{N}$. We have $h(af^m - bf^n) = 0$ for some $h \in (f)$. Since $\mathfrak{p} \in \mathbb{D}_{\text{sch}}(f)$, it follows that $h \in R \setminus \mathfrak{p}$ so that $\frac{a}{f^n}_{\mathfrak{p}} = \frac{b}{f^m}_{\mathfrak{p}}$. Hence, $\psi(\frac{a}{f^n}_f) = \psi(\frac{b}{f^m}_f)$ so that ψ is well-defined. We additionally have that ψ is also a homomorphism which follows from the fact that the image of $\frac{a}{f^n}_f + \frac{b}{f^m}_f$ in $\mathfrak{p}$ is $\frac{a}{f^n}_{\mathfrak{p}} + \frac{b}{f^m}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathbb{D}_{\text{sch}}(f)$

and all $m, n \in \mathbb{N}$.

We now show that this homomorphism is injective. Let ${\frac{a}{f^n}}_f, {\frac{b}{f^m}}_f \in R_f$ for some $m, n \in \mathbb{N}$ and assume that $\psi({\frac{a}{f^n}}_f) = \psi({\frac{b}{f^m}}_f)$. This implies that ${\frac{a}{f^n}}_{\mathfrak{p}} = {\frac{b}{f^m}}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathbb{D}_{\text{sch}}(f)$. Therefore, there exists $h_{\mathfrak{p}} \in R \backslash \mathfrak{p}$ such that $h_{\mathfrak{p}}(f^m a - f^n b) = 0$, for all $\mathfrak{p} \in \mathbb{D}_{\text{sch}}(f)$. Let $\mathfrak{a} \subseteq R$ be the set of all elements $\alpha \in \mathfrak{a}$ such that $\alpha(f^m a - f^n b) = 0$. It is easy to check that such a set is an ideal in R. It follows that $h_{\mathfrak{p}} \in \mathfrak{a}$ so that $\mathfrak{a} \nsubseteq \mathfrak{p}$ for all $\mathfrak{p} \in \mathbb{D}_{\text{sch}}(f)$. Hence $\mathbb{V}_{\text{sch}}(\mathfrak{a}) \cap \mathbb{D}_{\text{sch}}(f) = \varnothing$ which implies that $\mathbb{V}_{\text{sch}}(\mathfrak{a}) \subseteq \mathbb{V}_{\text{sch}}(f)$. It follows from item 5. of Proposition A.4.2 that $f \in \sqrt{\mathfrak{a}}$ so that $f^l \in \mathfrak{a}$ for some $l \in \mathbb{N}$. Therefore $f^l(f^m a - f^n b) = 0$. We conclude that ${\frac{a}{f^n}}_f = {\frac{b}{f^m}}_f$ so that ψ is injective.

We now show that ψ is surjective. Let $s \in \mathcal{O}_{\text{Spec}(R)}(\mathbb{D}_{\text{sch}}(f))$. We want to find an element of R_f which equals $s(\mathfrak{p})$ for all $\mathfrak{p} \in \mathbb{D}_{\text{sch}}(f)$. We proceed as follows: By definition of $\mathcal{O}_{\text{Spec}(R)}(\mathbb{D}_{\text{sch}}(f))$, there exists an open covering $\{V_i\}_{i \in I}$ of $\mathbb{D}_{\text{sch}}(f)$ such that $s(\mathfrak{p}) = {\frac{a_i}{g_i}}_{\mathfrak{p}}$ with $g_i \notin \mathfrak{p}$ and $\mathfrak{p} \in V_i$ for all $i \in I$. In particular, $V_i \subseteq \mathbb{D}_{\text{sch}}(g_i)$ for all $i \in I$. Since each open set V_i can be covered by distinguished open subsets such that $s(\mathfrak{p}) = {\frac{a_i}{g_i}}_{\mathfrak{p}}$ for all $\mathfrak{p}$ in any such distinguished subset, we may assume that $V_i = \mathbb{D}_{\text{sch}}(h_i)$ with $h_i \in R$ for all $i \in I$. We have $\mathbb{D}_{\text{sch}}(h_i) \subseteq \mathbb{D}_{\text{sch}}(g_i)$, so that $\mathbb{V}_{\text{sch}}(g_i) \subseteq \mathbb{V}_{\text{sch}}(h_i)$ for all $i \in I$. It follows by 5. of Proposition A.4.2 that $\sqrt{h_i} \subseteq \sqrt{g_i}$ and hence $h_i^n \in (g_i)$ with $n \in \mathbb{N}$ for all $i \in I$. Hence $h_i^n = c g_i$ for some $c \in R$ which implies ${\frac{a_i}{g_i}}_{\mathfrak{p}} = {\frac{c a_i}{h_i^n}}_{\mathfrak{p}}$ with $\mathfrak{p} \in \mathbb{D}_{\text{sch}}(h_i)$ for all $i \in I$. Since $\mathbb{D}_{\text{sch}}(h_i^n) = \mathbb{D}_{\text{sch}}(h_i)$, we may replace h_i^m with h_i and $c a_i$ with a_i for all $i \in I$.

We have representatives of s on possibly infinitely many distinguished open subsets covering $\mathbb{D}_{\text{sch}}(f)$, however, we can consider a sufficiently smaller covering. Observe that $\mathbb{D}_{\text{sch}}(f) = \bigcup\limits_{i \in I} \mathbb{D}_{\text{sch}}(h_i)$ if and only if $\mathbb{V}_{\text{sch}}(f) \supseteq \bigcap\limits_{i \in I} \mathbb{V}_{\text{sch}}(h_i) = \mathbb{V}_{\text{sch}}(\sum\limits_{i \in I} h_i)$. By 5. of Proposition A.4.2, $f^m \in \sum\limits_{i \in I} h_i$ for some $m \in \mathbb{N}$ so that $f^m = \sum\limits_{i \in I} b_i h_i$ for finitely many $b_i \in A$ with $b_i \neq 0$ for all $i \in I$. Let $J \subseteq I$ be the set of indices $i \in I$ such that $b_i \neq 0$. Since $\mathbb{D}_{\text{sch}}(f^m) = \mathbb{D}_{\text{sch}}(f)$, it follows that $\mathbb{D}_{\text{sch}}(f) = \bigcup\limits_{i \in J} \mathbb{D}_{\text{sch}}(h_i)$. This finite open covering is, in fact, all we need since if $\mathbb{D}_{\text{sch}}(h_i)$ and $\mathbb{D}_{\text{sch}}(h_j)$ intersect for some $i, j \in I$ such that $i \neq j$, we have that ${\frac{a_i}{h_i}}_{\mathfrak{p}} = {\frac{a_j}{h_j}}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathbb{D}_{\text{sch}}(h_i) \cap \mathbb{D}_{\text{sch}}(h_j) = \mathbb{D}_{\text{sch}}(h_i h_j)$. Now consider $i, j \in J$ such that $i \neq j$. It follows similarly to arguments of the proof of injectivity of ψ that ${\frac{a_i}{h_i}}_{h_i h_j} = {\frac{a_j}{h_j}}_{h_i h_j}$. Hence

$$(h_i h_j)^n (h_j a_i - h_i a_j) = 0 \tag{A.21}$$

for some $n \in \mathbb{N}$. Since there are only finitely many $j \in J$ and hence only finitely many intersections between $\mathbb{D}_{\text{sch}}(h_i)$ and $\mathbb{D}_{\text{sch}}(h_j)$ for all $i, j \in J$, we may choose n large enough to work simultaneously for all $i, j \in J$. We can rewrite (A.21) as

$$h_j^{n+1}(h_i^n a_i) - h_i^{n+1}(h_j^n a_j) = 0. \tag{A.22}$$

Replacing h_i with h_i^{n+1} and a_i with $h_i^n a_i$, we still have $s(\mathfrak{p}) = \frac{a_i}{h_i}^{\mathfrak{p}}$ where $\mathfrak{p} \in \mathbb{D}_{\text{sch}}(h_i)$ for all $i \in I$ and, furthermore, we have $h_j a_i = h_i a_j$ for all $i, j \in J$ by rewriting (A.22).

Recalling that $f^n = \sum\limits_{i \in I} b_i h_i$, we let $a = \sum\limits_{i \in I} b_i a_i$. For all $j \in J$, we have

$$h_j a = \sum_{i \in J} b_i a_i h_j = \sum_{i \in J} b_i h_i a_j = f^n a_j.$$

This implies that $\frac{a}{f^n}^{\mathfrak{p}} = \frac{a_j}{h_j}^{\mathfrak{p}}$ where $\mathfrak{p} \in \mathbb{D}_{\text{sch}}(h_j)$ for all $j \in I$. We conclude that $\psi(\frac{a}{f^n}^{f}) = s$ so that ψ is surjective. ☐

By taking $f = 1$ in the above proposition, we have the following immediate corollary.

Corollary A.7.6. *Let R be a ring, then $\mathcal{O}_{\text{Spec}(R)}(\text{Spec}(R))$ is isomorphic to R.*

Definition A.7.7. *An* affine scheme *is a locally ringed space that is isomorphic to $(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$ for some ring R.*

Definition A.7.8. *A scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that there is an open covering $\{X_i\}_{i \in I}$ of X with $(X_i, \mathcal{O}_X|_{X_i})$ an affine scheme for all $i \in I$.*

Proposition A.7.9. *Let R be a ring and let $g \in R$. Then the induced locally ringed space of the open set $\mathbb{D}_{\text{sch}}(g) \subseteq \text{Spec}(R)$ is isomorphic (as a locally ringed space) to the affine scheme $\text{Spec}(R_g)$.*

Proof. To give an isormorphism between locally ringed spaces, we need show that there exists a two-sided inverse with an induced local homomorphism between stalks. We begin by noting that $\mathbb{D}_{\text{sch}}(g)$ and $\text{Spec}(R_g)$ have the same underlying set $\{\mathfrak{p} \in \text{Spec}(R) \mid (g) \nsubseteq \mathfrak{p}\}$ and that they also have the same topology given by the closed sets $\mathbb{V}_{\text{sch}}(I) = \{\mathfrak{p} \in \text{Spec}(R) \mid (g) \nsubseteq \mathfrak{p} \;\&\; I \subseteq \mathfrak{p}\}$ for all ideals $I \subseteq \text{Spec}(R)$ such that $(g) \nsubseteq I$. Thus, the map $f : \mathbb{D}_{\text{sch}}(g) \to \text{Spec}(R_g)$, given by sending $\mathfrak{p} \in \mathbb{D}_{\text{sch}}(g)$ to the same underlying $\mathfrak{p} \in \text{Spec}(R_g)$, is continuous.

We now give an invertible morphism between the sheaves $\mathcal{O}_{\text{Spec}(R_g)}$ and $f_*\mathcal{O}_{\mathbb{D}_{\text{sch}}(g)}$. Let U be an open subset of $\text{Spec}(R_g)$ such that $U = \mathbb{D}_{\text{sch}}(h)$ for some $h \in R$. Then by Proposition A.7.5, we have that $\mathcal{O}_{\text{Spec}(R_g)}(U) \cong (R_g)_h$.

Conversely, we have $\mathbb{D}_{\text{sch}}(h) \subseteq \mathbb{D}_{\text{sch}}(g)$ so that $\mathbb{D}_{\text{sch}}(h) = \mathbb{D}_{\text{sch}}(g) \cap \mathbb{D}_{\text{sch}}(h) = \mathbb{D}_{\text{sch}}(gh)$. Noting that $\mathcal{O}_{\mathbb{D}_{\text{sch}}(g)}(U) = \mathcal{O}_{\text{Spec}(R)}(U)$ by definition of the restriction of $\text{Spec}(R)$ to $\mathbb{D}_{\text{sch}}(g)$ as ringed spaces, we can once again apply Proposition A.7.5 so that $\mathcal{O}_{\mathbb{D}_{\text{sch}}(g)}(U) = \mathcal{O}_{\text{Spec}(R)}(U) \cong R_{gh}$. We conclude that $\mathcal{O}_{\mathbb{D}_{\text{sch}}(g)}(U) \cong \mathcal{O}_{\text{Spec}(R_g)}(U)$. Since these isomorphisms respect restrictions between distinguished open subsets, it follows by Propisition A.5.11 that the isomorphism holds for arbitrary open sets as well. Therefore, we have isomorphisms $f_U : \mathcal{O}_{\text{Spec}(R_g)}(U) \to \mathcal{O}_{\mathbb{D}_{\text{sch}}(g)}(f^{-1}(U))$ for all open $U \subseteq \text{Spec}(R_g)$ which gives the morphism of sheaves $f^{\#} = (f_U)_{U \in \tau_{\text{Spec}(R_g)}}$ from $\mathcal{O}_{\text{Spec}(R_g)}$ to $f_*\mathcal{O}_{\mathbb{D}_{\text{sch}}(g)}$. Since f is a homeomorphism, the open sets of $\mathbb{D}_{\text{sch}}(g)$ and $\text{Spec}(R_g)$ are in bijection by mapping $V \in \mathbb{D}_{\text{sch}}(g)$ to $f(V) \in \text{Spec}(R_g)$ thus implying that $(f^{\#})^{-1} = (f^{-1}_{f(V)})_{V \in \tau_{\mathbb{D}_{\text{sch}}(g)}}$ is an inverse to $f^{\#}$ and is a morphism of sheaves.

Since $(f, f^{\#})$ is a isomorphism of ringed spaces, we conclude from Lemma A.6.7 that $f^{\#}_{\mathfrak{p}}$ is local homomorphism so that $\mathbb{D}_{\text{sch}}(f)$ and $\text{Spec}(R_g)$ are isomorphic as locally ringed spaces. □

Proposition A.7.10. *Let X be a scheme and let U be an open subset of X. Then $(U, \mathcal{O}_X|_U)$ is a scheme.*

Proof. Since X is a scheme, there exists an open covering of X by affine schemes $\{U_i\}_{i\in I}$. It follows that $U \cap U_i$ is open so that there exists an open covering $\{V_{i,j}\}_{j\in J_i}$ of $U \cap U_i$ with distinguished open subsets in U_i where each distinguished open subset is an affine scheme by Proposition A.7.9. Thus U has an open covering $\{V_{i,j}\}_{i\in I, j\in J_i}$ by affine schemes and is hence a scheme. □

Definition A.7.11 ([28, p. 44])**.** *Let X be a scheme and let U be an open subset of X. We say that the scheme $(U, \mathcal{O}_X|_U)$ is an* open subscheme *of X.*

Proposition A.7.12 ([18, p. 97])**.** *Let R and S be rings. There is a bijection between* $\text{Hom}_{\textbf{Sch}}(\text{Spec}(R), \text{Spec}(S))$ *and* $\text{Hom}_{\textbf{Ring}}(S, R)$.

Proof. We construct a bijection as follows: Assume that we are given a locally ringed space morphism $(f, f^{\#}) : (\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)}) \to (\text{Spec}(S), \mathcal{O}_{\text{Spec}(S)})$ such that $f^{\#} = (f_U)_{U \in \tau_{\text{Spec}(S)}}$. We want to find a ring homomorphism from S to R. We naturally obtain the ring homomorphism $f_{\text{Spec}(S)} : \mathcal{O}_{\text{Spec}(S)}(\text{Spec}(S)) \to \mathcal{O}_{\text{Spec}(R)}(\text{Spec}(R))$. By Corollary A.7.6, we have an induced homomorphism $\psi : S \to R$ given by composing the isomorphism between S and $\mathcal{O}_{\text{Spec}(S)}(\text{Spec}(S))$ with $f_{\text{Spec}(S)}$ and with the isomorphism between $\mathcal{O}_{\text{Spec}(R)}(\text{Spec}(R))$ and R.

Now, assume we are given a ring homomorphism $\psi : S \to R$. We want to construct a locally ringed space morphism $(f_\psi, f^{\#}_\psi) : (\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)}) \to (\text{Spec}(S), \mathcal{O}_{\text{Spec}(S)})$ such that $f^{\#}_\psi(\text{Spec}(S))$ induces ψ as above. We begin by defining a continuous map $f_\psi : \text{Spec}(R) \to \text{Spec}(S)$ given by mapping

$\mathfrak{p} \in \operatorname{Spec}(R)$ to $\psi^{-1}(\mathfrak{p})$. Since the inverse image of a ring homomorphism of a prime ideal is again a prime ideal, this map is well-defined. All closed subsets of $\operatorname{Spec}(S)$ are of the form $\mathbb{V}_{\text{sch}}(I)$ for some ideal $I \subseteq S$. Hence, for some ideal $I \subseteq S$

$$\begin{aligned} f_\psi(\mathbb{V}_{\text{sch}}(I)) &= \{\mathfrak{p} \in \operatorname{Spec(R)} \mid I \subseteq \psi^{-1}(\mathfrak{p})\} \\ &= \{\mathfrak{p} \in \operatorname{Spec(R)} \mid \psi(I) \subseteq \mathfrak{p}\} \\ &= \mathbb{V}_{\text{sch}}(\psi(I)) \subseteq \operatorname{Spec}(R). \end{aligned}$$

Thus, the inverse image of f_ψ of any closed set is also closed and hence f_ψ is continuous. In particular, we have $f_\psi^{-1}(\mathbb{D}_{\text{sch}}(g)) = \mathbb{D}_{\text{sch}}(\psi(g))$ for every $g \in S$. We now define $f_\psi^{\#}$ by considering distinguished open subsets. Let $\mathbb{D}_{\text{sch}}(g) \subseteq \operatorname{Spec}(S)$ for some $g \in S$. We have, by Lemma A.2.33, that there exists a unique homomorphism $h_{\psi,g}$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\psi"] \arrow[d, "\iota_{S,g}"] & R \arrow[d, "\iota_{R,\psi(g)}"] \\ S_g \arrow[r, "h_{\psi,g}"] & R_{\psi(g)} \end{tikzcd} \tag{A.23}$$

where ι_S, g and $\iota_{R,\psi(g)}$ denote the usual embedding homomorphisms. By Proposition A.7.5, we have the isomorphisms $\mu_{S,g} : S_g \to \mathcal{O}_{\operatorname{Spec}(S)}(\mathbb{D}_{\text{sch}}(g))$ and $\mu_{R,\psi(g)} : R_{\psi(g)} \to f_*\mathcal{O}_{\operatorname{Spec}(R)}(\mathbb{D}_{\text{sch}}(g))$ so that we can extend (A.23) to the commutative diagram

$$\begin{tikzcd} S \arrow[r, "\psi"] \arrow[d, "\iota_{S,g}"] & R \arrow[d, "\iota_{R,\psi(g)}"] \\ S_g \arrow[r, "h_{\psi,g}"] \arrow[d, "\mu_{S,g}"] & R_{\psi(g)} \arrow[d, "\mu_{R,\psi(g)}"] \\ \mathcal{O}_{\operatorname{Spec}(S)}(\mathbb{D}_{\text{sch}}(g)) \arrow[r, "f^{\#}_{\psi,\mathbb{D}_{\text{sch}}(g)}"] & f_*\mathcal{O}_{\operatorname{Spec}(R)}(\mathbb{D}_{\text{sch}}(g)) \end{tikzcd} \tag{A.24}$$

where we define $f^{\#}_{\psi,\mathbb{D}_{\text{sch}}(g)} = \mu_{R,\psi(g)} \circ h_{\psi,g} \circ \mu_{S,g}^{-1}$. Note that this homomorphism is unique in the sense that it makes (A.24) commute. We now show that these homomorphisms respect restrictions between distinguished open subsets. Let $\mathbb{D}_{\text{sch}}(h) \subseteq \mathbb{D}_{\text{sch}}(g)$, so that $\mathbb{D}_{\text{sch}}(h) = \mathbb{D}_{\text{sch}}(h) \cap \mathbb{D}_{\text{sch}}(g) = \mathbb{D}_{\text{sch}}(gh)$ for some $h \in S$. Because of the correspondence between $h_{\psi,g}$ and $f^{\#}_{\psi,\mathbb{D}_{\text{sch}}(g)}$ for all $g \in S$ described in (A.24), we need only check that the following diagram commutes:

$$\begin{tikzcd} S_g \arrow[r, "h_{\psi,g}"] \arrow[d, "\iota_{S_g,h}"] & R_{\psi(g)} \arrow[d, "\iota_{R_{\psi(g)},\psi(h)}"] \\ S_{gh} \arrow[r, "h_{\psi,gh}"] & R_{\psi(gh)} \end{tikzcd} \tag{A.25}$$

where $\iota_{S_g,h}$ and $\iota_{R_{\psi(g)},\psi(h)}$ are again the usual embedding homomorphisms. To show the commutivity, we will trace an element along the diagram. Let $\frac{a}{g^n}^{g} \in S_g$. Along the bottom-left of the rectangle, we have

$$h_{\psi,gh}(\iota_{S_g,h}(\frac{a}{g^n}^{g})) = h_{\psi,gh}(\frac{a}{g^n}^{gh})\frac{\psi(a)}{\psi(g)^n}^{\psi(gh)}.$$

Similarly, along the top right, we have

$$\iota_{R_{\psi(g)},\psi(h)}(h_{\psi,g}(\frac{a}{g^n}^{g})) = \iota_{R_{\psi(g)},\psi(h)}(\frac{\psi(a)}{\psi(g)^n}^{\psi(g)}) = \frac{\psi(a)}{\psi(g)^n}^{\psi(gh)}.$$

Hence (A.25) commutes. Thus, we can apply Proposition A.5.11 so that we have a morphism of sheaves $f_\psi^\# = (f_{\psi,U}^\#)_{U\in\tau_{\mathrm{Spec}(S)}}$.

In order to show that our ringed space morphism $f_\psi^\#$ has the necessary local properties, we need to check that the canonical homomorphisms

$$(f_\psi^\#)_\mathfrak{p} : \mathcal{O}_{\mathrm{Spec}(S),f_\psi(\mathfrak{p})} \to \mathcal{O}_{\mathrm{Spec}(R),\mathfrak{p}}$$

are local homomorphisms for all $\mathfrak{p} \in \mathrm{Spec}(R)$. We now consider another diagram for some $\mathfrak{p} \in \mathrm{Spec}(R)$:

$$\begin{tikzcd}
S \arrow[r, "\psi"] \arrow[d, "\mu_S"] & R \arrow[d, "\mu_R"] \\
\mathcal{O}_{\mathrm{Spec}(S)}(\mathrm{Spec}(S)) \arrow[r, "f^\#_{\psi,\mathrm{Spec}(S)}"] \arrow[d, "\sigma_{S,\psi^{-1}(\mathfrak{p})}"] & \mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R)) \arrow[d, "\sigma_{R,\mathfrak{p}}"] \\
\mathcal{O}_{\mathrm{Spec}(S),\psi^{-1}(\mathfrak{p})} \arrow[r, "(f^\#_\psi)_\mathfrak{p}"] \arrow[d, "\eta_{S,\psi^{-1}(\mathfrak{p})}"] & \mathcal{O}_{\mathrm{Spec}(R),\mathfrak{p}} \arrow[d, "\eta_{R,\mathfrak{p}}"] \\
S_{\psi^{-1}(\mathfrak{p})} \arrow[r, "h_{\psi,\psi^{-1}(\mathfrak{p})}"] & R_\mathfrak{p}
\end{tikzcd} \tag{A.26}$$

where μ_S and μ_R are the isomorphisms given by Proposition A.7.5, $\eta_{S,\psi^{-1}(\mathfrak{p})}$ and $\eta_{R,\mathfrak{p}}$ are the isomorphisms given by Proposition A.7.3, $\sigma_{R,\mathfrak{p}}$ and $\sigma_{S,\psi^{-1}(\mathfrak{p})}$ are the usual embeddings for each stalk and we define $h_{\psi,\psi^{-1}(\mathfrak{p})} = \sigma_{R,\mathfrak{p}} \circ (f_\psi^\#)_\mathfrak{p} \circ \sigma^{-1}_{S,\psi^{-1}(\mathfrak{p})}$. We want to show that $(f_\psi^\#)_\mathfrak{p}$ is a local homomorphism. We observe that, by definition, $\eta_{R,\mathfrak{p}} \circ \sigma_{R,\mathfrak{p}} \circ \mu_R$ is equivalent to the usual restriction from R to $R_\mathfrak{p}$ and the result holds similarly for $\eta_{S,\psi^{-1}(\mathfrak{p})} \circ \sigma_{S,\psi^{-1}(\mathfrak{p})} \circ \mu_S$. It follows that $h_{\psi,\psi^{-1}(\mathfrak{p})}$ is the unique map described in Lemma A.2.33. Hence, it suffices to show that $h_{\psi,\psi^{-1}(\mathfrak{p})}$ is a local homomorphism. Since $h_{\psi,\psi^{-1}(\mathfrak{p})}$ is a homomorphism, we must have $h_{\psi,\psi^{-1}(\mathfrak{p})}(s^{-1}) = h_{\psi,\psi^{-1}(\mathfrak{p})}(s)^{-1}$ for all $s \in S_{\psi^{-1}(\mathfrak{p})}$ which implies that $h_{\psi,\psi^{-1}(\mathfrak{p})}(\psi^{-1}(\mathfrak{p})) \subseteq \mathfrak{p}$ so that $h_{\psi,\psi^{-1}(\mathfrak{p})}$ is a local homomorphism. It follows that $(f_\psi, f_\psi^\#)$ is a locally ringed space morphism.

We conclude the proof by checking that these constructions are indeed inverse to one another. It is clear that given $\psi : S \to R$, we can recover ψ by using $f^\#_{\psi,\mathrm{Spec}(S)}$ from $(f_\psi, f_\psi^\#)$. We need to check, if given $(f, f^\#)$ and hence given $\psi : S \to R$ induced by $f^\#_{\mathrm{Spec}(S)}$, that $(f, f^\#)$ and $(f_\psi, f_\psi^\#)$ are equal. To that end, for any $\mathfrak{p} \in \mathrm{Spec}(R)$, we have the commutative diagram:

$$\begin{tikzcd}
S \arrow[r, "\psi"] \arrow[d, "\iota_{S,f(\mathfrak{p})}"] & R \arrow[d, "\iota_{R,\mathfrak{p}}"] \\
S_{f(\mathfrak{p})} \arrow[r, "f_\mathfrak{p}^\#"] & R_\mathfrak{p}
\end{tikzcd} \tag{A.27}$$

with $\iota_{R,\mathfrak{p}}$ and $\iota_{S,f(\mathfrak{p})}$ being the usual localization homomorphisms. Consider the image of $\mathfrak{p}$ under $\iota_{R,\mathfrak{p}}$ which we will denote as $\frac{\mathfrak{p}_\mathfrak{p}}{1}$. Since both vertical maps are injective and $f_\mathfrak{p}^\#$ is a local homomorphism, it follows that $(\iota^{-1}_{S,f(\mathfrak{p})} \circ (f_\mathfrak{p}^\#)^{-1} \circ \iota^{-1}_{R,\mathfrak{p}})(\frac{\mathfrak{p}_\mathfrak{p}}{1}) = f(\mathfrak{p})$. By the commutivity of (A.27), the left hand side is precisely $\psi^{-1}(\mathfrak{p})$. Thus f and f_ψ agree set-theoretically. Furthermore, by Lemma A.2.33, $f_\mathfrak{p}^\#$ is the unique homomorphism that makes (A.27) commute so that $f_\mathfrak{p}^\# = (f_\psi^\#)_\mathfrak{p}$. Since $\mathfrak{p}$ was arbitrary and hence $f_\mathfrak{p}^\# = (f_\psi^\#)_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$, it follows by Lemma A.5.4 that $f^\# = f_\psi^\#$. ☐

Definition A.7.13. *Let S be a scheme. A* scheme over S, *or simply a S-*scheme, *is a scheme X along with a morphism $f : X \to S$. We call f the* structural morphism *and S the* base scheme *of X. If S is an affine scheme, i.e. $S = \mathrm{Spec}(R)$ for some ring R, then we alternatively denote X as a R-scheme. Let Y be an S-scheme with structural morphism g. A* morphism of S-schemes *between X and Y is a morphism of schemes $h : X \to Y$ such that $f = g \circ h$.*

A.7.2 Properties of schemes

This section follows [28, Section 2.3 and 2.4].

Definition A.7.14 ([20, p. 86])**.** *Let X be a scheme. We define the* dimension *of X (if it exists), denoted by* $\dim(X)$, *as the dimension of the underlying topological space of X.*

Definition A.7.15. *A scheme is irreducible if its underlying topological space is irreducible.*

Definition A.7.16. *A scheme X is* reduced *if the ring $\mathcal{O}_{X,x}$ is reduced for all $x \in X$.*

Proposition A.7.17. *Let X be a scheme. The following are equivalent:*

1. *X is reduced.*
2. *For all open $U \subseteq X$, $\mathcal{O}_X(U)$ is reduced.*
3. *There exists an affine open covering $\{X_i\}_{i\in I}$ of X such that $\mathcal{O}_X(X_i)$ is reduced for all $i \in I$.*

Proof. We first show that 1. implies 2.. Assume that X is reduced. Let $U \subseteq X$ be open and further assume that $s \in \mathcal{O}_X(U)$ such that $s^n = 0$ for some $n \in \mathbb{N}$. For any $x \in X$ such that $x \in U$, denote the inclusion map from $\mathcal{O}_X(U)$ into $\mathcal{O}_{X,x}$ as $\iota_{U,x}$. Since, for all $x \in X$, $\mathcal{O}_{X,x}$ is reduced as X is reduced, we have that $\iota_{U,x}(s) = 0$ since s is nilpotent. However, this holds for all $x \in U$, so that $s = 0$ by Lemma A.5.4.

Clearly, 2. implies 3. We now show that 3 implies 1. Assume that there exists an open affine covering $\{X_i\}_{i\in I}$ of X where we may assume that $X_i = \text{Spec}(A_i)$ for some reduced ring A_i for all $i \in I$. By Proposition A.7.3, we have that $\mathcal{O}_{X,x}$ is isomorphic to $(A_i)_x$ for all $x \in X$. Since A_i is reduced, it follows that $(A_i)_x$ is reduced, so that $\mathcal{O}_{X,x}$ is reduced for all $x \in X$. Hence X is reduced. □

Definition A.7.18. *A scheme X is called* integral *if for every open $U \subseteq X$, $\mathcal{O}_X(U)$ is an integral domain.*

Definition A.7.19. *Let Y be a scheme and let X be a Y-scheme so that there exists a morphism of schemes $f : X \to Y$. We say that X is* of finite type over *Y if there exists a finite opening covering of Y by affine schemes $\{V_i\}_{i\in I}$ such that $f^{-1}(V_i)$ has a finite open covering by affine schemes $(V_{i,j})_{j\in J}$ where each $V_{i,j}$ is isomorphic to* $\text{Spec}(A_{i,j})$ *such that $A_{i,j}$ is a finitely generated R_i-algebra.*

Definition A.7.20. *A scheme X is* locally Noetherian *if it has an open covering by $\{U_i\}_{i\in I}$ such that U_i is isomorphic to* $\text{Spec}(R_i)$ *for some Noetherian ring R_i for all $i \in I$. We say that X is* Noetherian *if it is locally Noetherian and quasi-compact.*

The following result uses the ideas of [20, Exercise 2.15, p. 81].

Lemma A.7.21. *Let X, Y be schemes of finite type over k and let $f : X \to Y$ be a morphism of schemes over k. Then f maps closed points of X to closed points of Y.*

Proof. Assume $x \in X$ is a closed point. Let $f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ denote the induced local homomorphism from $(f, f^\#)$. Furthermore, denote ϕ as the composition of $f_x^\#$ with the projection of $\mathcal{O}_{X,x}$ to $\mathcal{O}_{X,x}/\mathfrak{m}_x$. As $f_x^\#$ is a local homomorphism, we have $f_x^{\#,-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$ with $\mathfrak{m}_x$ and $\mathfrak{m}_{f(x)}$ the maximal ideals of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,f(x)}$, respectively. It follows that $\mathfrak{m}_{f(x)}$ is the kernel of ϕ so that, by the first isomorphism theorem, $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is an extension of the field $\mathcal{O}_{Y,f(x)}/\mathfrak{m}_{f(x)}$. As, by [19, Proposition 3.33, p. 79], $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is a finite extension of k since x is a closed point and X is of finite type over k, we deduce that $\mathcal{O}_{Y,f(x)}/\mathfrak{m}_{f(x)}$ is a finite extension of k. It follows, again, from [19, Proposition 3.33, p. 79]that $f(x)$ is a closed point in Y. ☐

Lemma A.7.22. *If X is a scheme of finite type over k, then $X \otimes_k M$ is of finite type over M.*

Proof. We first cover the case when X is affine. Assume that there exists an isomorphism $f : X \to \mathrm{Spec}(R)$ for some finitely generated k-algebra R. Since $X \otimes_k M$ is a product for X and $\mathrm{Spec}(M)$ over k, it follows that, by the universal property of products that, there exists an isomorphism $f_{k,M} : X \otimes_k M \to \mathrm{Spec}(R) \times_k M$. By arguments in Proposition A.7.26, we have that $\mathrm{Spec}(R) \times_k M \simeq \mathrm{Spec}(R \otimes_k M)$. By [28, Corollary 1.13, p. 5], we have that $(R \otimes_k M)$ is a finitely generated M-algebra. Hence $X \otimes_k M$ is of finite type over M. Now, suppose that there exists an open covering $\{U_i\}_{i \in I}$ of X such that there exists isomorphisms $f_i : U_i \to \mathrm{Spec}(R_i)$ where R_i is a finitely generated k-algebra for all $i \in I$. Let $\pi_X : X \times_k M \to X$ be the projection. By Lemma A.7.24, $\pi_X(U_i)$ is a fibered product of U_i and $\mathrm{Spec}(M)$ over k for all $i \in I$. It follows by our preceeding arguments in the affine case that $\pi_X(U_i)$ is of finite type over M for all $i \in I$. Since $\{\pi_X(U_i)\}_{i \in I}$ is an open covering of $X \times_k M$, we deduce that $X \times_k M$ is of finite type over M. ☐

A.7.3 Fibred product and base change

This section can be found in [20, p. 87-88].

Definition A.7.23. *Let S be a scheme and let X, Y be two S-schemes with respective structural morphisms f and g. We define the fibred product of X and Y over S, denoted by $X \times_S Y$, as the pullback along f and g.*

Lemma A.7.24. *Let S be a scheme and let X and Y be S-schemes. If $U \subseteq X$ is open and $X \times_S Y$ exists, with projections $\pi_Y : X \times_S Y \to Y$ and $\pi_X : X \times_S Y \to X$, then $\pi_X^{-1}(U)$ is a fibred product for U and Y over S*

Proof. Let $\phi_X : X \to S$ and $\phi_Y : X \to S$ denote the structural morphisms of X and Y, respectively and let $\iota_{U,X}$ denote the continuous inclusion map. We will denote $\phi_X^{-1}(U)$ by $U \times_S Y$. We want to show that the triple $(U \times_S Y, \pi_X|_{U \times_S Y}, \pi_Y|_{U \times_S Y})$ is a pullback. Given any scheme Z and morphisms $f : Z \to U$ and $g : Z \to Y$, we have the commutative diagram

$$\begin{tikzcd}
Z \arrow[rd, dotted, "\mu"] \arrow[rrd, bend left, "g"] \arrow[rdd, bend right, "\iota_{U,X}\circ f"'] & & \\
& X \times_S Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] \arrow[rd, phantom, "(1)"] & Y \arrow[d, "\phi_Y"] \\
& X \arrow[r, "\phi_X"] & S
\end{tikzcd}$$

where $\mu : Z \to X \times_S Y$ is the morphism given by the universal property of $X \times_S Y$. Since $f(Z) \subseteq U$, it follows that $\mu(Z) \subseteq \pi_X^{-1}(U)$. Thus, μ can be considered as a morphism from Z to $\pi_X^{-1}(U)$. Furthermore, since the rectangle (1) commutes, it follows that

$$\begin{tikzcd}
U \times_S Y \arrow[r, "\pi_Y|_{U \times_S Y}"] \arrow[d, "\pi_X|_{U \times_S Y}"] & Y \arrow[d, "\phi_Y"] \\
U \arrow[r, "\phi_X"] & S
\end{tikzcd}$$

commutes. We deduce that μ is unique in the sense that it makes the diagram

$$\begin{tikzcd}
Z \arrow[rd, dotted, "\mu"] \arrow[rrd, bend left, "g"] \arrow[rdd, bend right, "f"'] & & \\
& U \times_S Y \arrow[r, "\pi_Y|_{U \times_S Y}"] \arrow[d, "\pi_X|_{U \times_S Y}"] & Y \arrow[d, "\phi_Y"] \\
& U \arrow[r, "\phi_U"] & S
\end{tikzcd}$$

commutes. We conclude that $U \times_S Y$ is the fibred product as wanted. ☐

Lemma A.7.25. *Let S be a scheme and let X and Y be S-schemes and let $\{X_i\}_{i\in I}$ is an open covering of X. If $X_i \times_S Y$ exists for all $i \in I$, then $X \times_S Y$ exists.*

Proof. Let $\pi_{X_i} : X_i \times_S Y \to X_i$ and $\pi_Y : X_i \times_S Y \to Y$ be the projections for all $i \in I$. Set $U_{i,j} := \pi_i^{-1}(X_{i,j}) \subseteq X_i \times_S Y$ where $X_{i,j} = X_i \cap X_j$. It follows from Lemma A.7.24 that $U_{i,j}$ is a fibred product for $X_{i,j}$ and Y over S. Thus, by uniqueness of the fibred product, there exists isomorphisms $\phi_{i,j} : U_{i,j} \to U_{j,i}$ compatible with the projections for all $i, j \in I$ such that the diagram

$$\begin{tikzcd}
U_{i,j} \cap U_{i,k} \arrow[rr, "\phi_{i,k}"] \arrow[dr, "\phi_{i,j}"] & & U_{k,i} \cap U_{k,j} \\
& U_{j,i} \cap U_{j,k} \arrow[ur, "\phi_{j,k}"] &
\end{tikzcd}$$

commutes for all $i, j, k \in I$. Thus, by Proposition A.6.8, we may glue the schemes $X_i \times_S Y$ for all $i \in I$ to obtain a scheme $X \times_S Y$. Furthermore, denoting π_{i,X_i} and $\pi_{i,Y}$ as the projections from $X_i \times_S Y$ to X_i and Y, respectively, for all $i \in I$, by Proposition A.6.6, we may glue these projections to obtain the projections $\pi_X : X \times_S Y \to X$ and $\pi_Y : X \times_S Y \to Y$. We claim that $X \times_S Y$, along with the projections π_X and π_Y, is the fibred product for X and Y over S. Indeed, the commutativity of the diagram

$$\begin{tikzcd}
X \times_S Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "phi_Y"] \\
X \arrow[r, "\phi_X"] & S
\end{tikzcd}$$

follows by restricting $X \times_S Y$ to the relevant scheme $X_i \times_S Y$ for some $i \in I$. Now, consider a given S-scheme Z and morphisms $f : Z \to X$ and $g : Z \to Y$ such that the diagram

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "\phi_Y"] \\
X \arrow[r, "\phi_X"] & S
\end{tikzcd} \tag{A.28}$$

commutes. We would like to show that there exists a unique map $\mu : Z \to X \times_S Y$ such that the diagram

$$\begin{tikzcd}
Z \arrow[drr, bend left, "g"] \arrow[dr, dotted, "\mu"] \arrow[ddr, bend right, "f"'] & & \\
& X \times_S Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "\phi_Y"] \\
& X \arrow[r, "\phi_X"] & S
\end{tikzcd} \tag{A.29}$$

commutes. For all $i \in I$, let $Z_i = f^{-1}(X_i)$ given with $f|_{Z_i} : Z_i \to X_i$ and $g|_{Z_i} : Z_i \to Y$ so that we obtain the map $\mu_i : Z_i \to X_i \times_S Y$ by the universal property of the fibred product of X_i and Y over S. We would like to glue these newly obtained maps to construct our wanted μ. Fixing $i, j \in I$, we first need to check that $\mu_i|_{Z_i \cap Z_j} = \mu_j|_{Z_i \cap Z_j}$. Since $X_i \cap X_j \subseteq X_i$ is open, it follows, from Lemma A.7.24, that $(X_i \cap X_j) \times_S Y$ with the projections $\pi_{i,X_i}|_{(X_i \cap X_j) \times_S Y}$ and $\pi_{i,Y}|_{(X_i \cap X_j) \times_S Y}$ is a fibred product of $X_i \cap X_j$ and Y over S. Similarly, $X_i \cap X_j \subseteq X_j$ is open so that $X_i \cap X_j \times_S Y$ with the projections $\pi_{j,X_j}|_{(X_i \cap X_j) \times_S Y}$ and $\pi_{j,Y}|_{(X_i \cap X_j) \times_S Y}$ is a fibred product of $X_i \cap X_j$ and Y over S. By uniqueness of the fibred product, we deduce that $\pi_{i,X_i}|_{(X_i \cap X_j) \times_S Y} = \pi_{j,X_j}|_{(X_i \cap X_j) \times_S Y}$ and $\pi_{i,Y}|_{(X_i \cap X_j) \times_S Y} = \pi_{j,Y}|_{(X_i \cap X_j) \times_S Y}$. Furthermore, we have the diagram

$$\begin{tikzcd}
Z_i \cap Z_j \arrow[drr, bend left, "g|_{Z_i \cap Z_j}"] \arrow[dr, dotted, "\mu_{i,j}"] \arrow[ddr, bend right, "f|_{Z_i \cap Z_j}"'] & & \\
& (X_i \cap X_j) \times_S Y \arrow[r, "\pi_{i,Y}|_{(X_i \cap X_j) \times_S Y}"] \arrow[d, "\pi_{i,X_i}|_{(X_i \cap X_j) \times_S Y}"] & Y \arrow[d, "\phi_Y"] \\
& X \arrow[r, "\phi_X"] & S.
\end{tikzcd}$$

We deduce that $\mu_i|_{Z_i \cap Z_j} = \mu_{i,j} = \mu_j|_{Z_i \cap Z_j}$. We may compose μ_i with the inclusion $X_i \times_S Y \subseteq X \times_S Y$ to obtain the map $\mu_i : Z_i \to X \times_S Y$ for all $i \in I$. Thus, by glueing, we obtain a map $\mu : Z \to X \times_S Y$. As before, the commutativity of (A.29) can be checked by considering the relevant restrictions. To check that μ is unique, let $\theta : Z \to X \times_S Y$ be another map which would make (A.29) commute. It follows by the universal properties of $X_i \times_S Y$ that $\theta|_{X_i \times_S Y} = \mu_i$ for all $i \in I$. Therefore, by Proposition A.6.6, $\theta = \mu$. ☐

Proposition A.7.26. *Let S be a scheme and let X, Y be two S-schemes. The fibred product of X and Y over S exists.*

Proof. We first consider the affine case and then build on this by considering affine open coverings. Assume that $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$ and $S = \operatorname{Spec}(R)$. Let Z be an S-scheme and let $f : Z \to X$ and $g : Z \to X$ be morphisms such that the diagram

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\phi_Y"] \\
X \arrow[r, "\phi_X"] & S
\end{tikzcd} \tag{A.30}$$

commutes. We claim that $\operatorname{Spec}(A \otimes_R B)$ is the fibred for X and Y over S. Indeed, we have, from Proposition A.3.8, the commutative diagram

$$\begin{tikzcd}
 & S \arrow[dl] \arrow[d] \arrow[dr] & \\
A \arrow[r, "\iota_A"'] & A \otimes_R B & B. \arrow[l, "\iota_B"]
\end{tikzcd}$$

By [28, Lemma 3.23, p. 48] and A.30, it follows, by the universal property of the coproduct, that we have a unique morphism $\mu : A \otimes_R B \to \mathcal{O}_Z(Z)$ such that the diagram

$$\begin{tikzcd}
 & S \arrow[dl] \arrow[d] \arrow[dr] & \\
A \arrow[r, "\iota_A"'] \arrow[dr] & A \otimes_R B \arrow[d, dotted, "\mu"] & B \arrow[l, "\iota_B"] \arrow[dl] \\
 & \mathcal{O}_Z(Z) &
\end{tikzcd}$$

Applying [28, Lemma 3.23, p. 48] once more, we obtain a unique morphism $`\mu : Z \to \operatorname{Spec}(A \otimes_R B)$ such that the diagram

$$\begin{tikzcd}
Z \arrow[dr, dotted, "`\mu"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
 & \operatorname{Spec}(A \otimes_R B) \arrow[r] \arrow[d] & Y \arrow[d, "\phi_Y"] \\
 & X \arrow[r, "\phi_X"] & S
\end{tikzcd}$$

commutes. It follows that $\operatorname{Spec}(A \otimes_R B)$ is the wanted fibred product.

We now consider the non-affine case. Assume that X is an arbitrary S-scheme and that Y and S are affine. Thus X has an open covering $\{X_i\}_{i \in I}$ of affine schemes. It follows that $X_i \times_S Y$ exists for all $i \in I$. Thus, by Lemma A.7.25, $X \times_S Y$ exists. By symmetry of Lemma A.7.25, it follows that $X \times_S Y$ exists if X, Y are arbitrary S-schemes and S is affine. Now, assume that X, Y are arbitrary S-schemes and S is an arbitrary scheme. Denote $\phi_X : X \to S$ and $\phi_Y : Y \to S$ as the structural morphisms of X and Y, respectively, and let $\{S_i\}_{i \in I}$ be an affine open covering of S. Set $i \in I$ and let $X_i = \phi_X^{-1}(S_i)$ and $Y_i = \phi_Y^{-1}(S_i)$. Then $X_i \times_{S_i} Y_i$ exists since S_i is affine. We claim that $X_i \times_{S_i} Y_i$ is the fibred product for X_i and Y over S. Indeed, for all $i \in I$, if we are given an S-scheme Z and morphisms $f : Z \to X_i$ and $g : Z \to Y$ such that the diagram

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "\phi_Y"] \\
X_i \arrow[r, "\phi_{X_i}"] & S
\end{tikzcd}$$

commutes, we must have that the image of g is in Y_i. We deduce that $X_i \times_S Y$ exists for each $i \in I$. Applying Lemma A.7.25, we conclude that $X \times_S Y$ exists. ☐

Definition A.7.27 ([28, p. 81])**.** *Let S be a scheme and let X be an S-scheme. For any other S-scheme S', the projection $\pi'_S : X \times_S S' \to S'$ endows $X \times_S S'$ with the structure of a S'-scheme. This process is called the* base change by *$S \to S'$. We will denote the S'-scheme $X \times_S S'$ by $X_{S'}$. If $S' = \mathrm{Spec}(R)$ for some ring R, we will also denote $X_{S'}$ by X_R.*

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