An associated result of the Van Aubel configuration and its generalization

This note presents some novel generalizations to similar quadrilaterals, similar parallelograms, and similar triangles of a result associated with Van Aubel's theorem about squares constructed on the sides of a quadrilateral. These results provide opportunities for interesting, challenging explorations for talented students using dynamic geometry at high school or for university students.

Keywords: Van Aubel's theorem; similarity; parallelogram; kite; triangles; cyclic quadrilaterals; proof; dynamic geometry; transformation geometry

Subject classification codes: 97G50; 97D40; 97D50

Introduction

Van Aubel's celebrated quadrilateral theorem states that if squares are constructed on the sides of any quadrilateral, then the segments connecting the centres of the squares on opposite sides are equal and perpendicular (Van Aubel, 1878). Apparently not as well-known is the interesting associated result that the respective midpoints *F* and *G*, of the diagonals *AC* and *BD* of *ABCD*, and the respective midpoints *I* and *H* of the segments connecting the centres of the squares on opposite sides form a square *GHFI* (see Figure 1).

[PLACE FIGURE 1 MORE OR LESS HERE]

The historical origin of this result is not known to the author who first saw the result mentioned in Tabov (1996) where it was proved in two different ways, first with complex numbers, and then also with transformation geometry. More recently, this result was mentioned & proved in Silvester (2006, p. 10) as well as at Bogomolny (Undated).

In this classroom note, this associated result of the Van Aubel configuration is generalized by using the 'what if not' problem posing strategy proposed by Brown & Walter (1993). By starting with the familiar Van Aubel arrangement of squares, the result is generalized to other similar quadrilaterals all placed exterior or interior on the sides of a quadrilateral in two different ways. A special case of similar parallelograms on the sides is also explored before looking at further generalisations just involving similar triangles on the sides. The first case of similar quadrilaterals follows.

[PLACE FIGURE 2 MORE OR LESS HERE]

Similar quadrilaterals on the sides

Theorem 1: Given four points *A*, *B*, *C*, *D*, and four directly similar quadrilaterals AP_1P_2B, BQ_1Q_2C, CR_1R_2D, DS_1S_2A with respective centroids[1] *P*, *Q*, *R*, *S.* Further let *F*, *G*, *H, I* be the midpoints of the segments *AC*, *BD*, *QS, PR* respectively (see Figure 2). Then *GHFI* is a parallelogram.

Dynamic geometry sketches illustrating Theorem 1 (and Theorems 3, 4, 5, 6 and 7 further on) are available online for the reader to explore at: http://dynamicmathematicslearning.com/van-aubel-associated-similar-quads.html

The result is quite easy and straightforward to prove by using the following very useful special case of a general similarity theorem proved in DeTemple & Harold (1996, p. 21), and also used in De Villiers (1998, p. 408). More recently, Abel (2007) and Fried (2021) have also provided easily accessible proofs of Theorem 2 below.

[PLACE FIGURE 3 MORE OR LESS HERE]

Theorem 2: If the corresponding vertices of two directly similar figures are connected, then the midpoints of those ‘connecting segments’ form another figure, similar to the other two.

Proof (Theorem 1): Consider Figure 3 which just shows the quadrilateral *ABCD* with the respective centroids *P* and *R* on the opposite sides *AB* and *CD*, and the two respective midpoints *G* and *F* of diagonals *BD* and *AC*. Since △*BPA* is similar to △*DRC* by construction, it immediately follows from Theorem 2 that △*GIF* is similar to both △*BPA* and △*DRC*.

[1] With the centroid of a quadrilateral is meant here, its ‘centre of mass’ or ‘balancing point’, determined by the placement of equal point masses at the four vertices of the quadrilateral. Geometrically, this ‘point mass’ centroid is located at the centre of the Varignon parallelogram formed by the midpoints of the sides of the quadrilateral (see Hanna & Jahnke, 2002).

With reference to Figure 2, it now follows in exactly the same way that △*FHG* is similar to △*ASD* and △*CQB*. But since △*ASD* is similar to △*BPA*, it follows that △*GIF* is similar to △*FHG*. However, since triangles *GIF* and *FHG* share the same side *GF* and correspondingly have two angles congruent, it follows that the two triangles are congruent; hence *GHFI* is a parallelogram. Q.E.D.

From the similarity between △*GIF* and △*BPA* it also follows that the ratio of their corresponding sides is in the same ratio; ie. *FI*/*IG* = *AP*/*PB*.

Obviously when ∠*BPA* = 90°, the parallelogram *GHFI* will become a rectangle, since from similarity, *GHFI* now has a right angle. Also when *PB* = *PA*, the parallelogram *GHFI* will become a rhombus, since from similarity, *GHFI* now has a pair of adjacent sides equal.

In the further special case[2] where ∠*BPA* = 90° and *PB* = *PA*, the parallelogram *GHFI* obviously becomes a square, since from similarity, *GHFI* now has a right angle and a pair of equal adjacent sides. Since this case is equivalent to the original Van Aubel arrangement with squares on the sides of *ABCD*, we will have *PR* = *QS* and *PR* ⊥ *QS*.

[PLACE FIGURE 4 MORE OR LESS HERE]

Another different arrangement of the similar quadrilaterals on the sides of *ABCD* is possible as defined in the theorem below.

Theorem 3: Given four points *A*, *B*, *C*, *D*, and four directly similar quadrilaterals AP_1P_2B, CQ_1Q_2B, CR_1R_2D, AS_1S_2D with respective centroids *P*, *Q*, *R*, *S*. Further let *F*, *G*, *H, I* be the midpoints of the segments *AC*, *BD*, *QS, PR* respectively (see Figure 4). Then *GHFI* is a kite.

From theorem 2, it follows immediately that *GHFI* is a kite symmetrical around *GF*. Q.E.D.

[2] Note that the similar quadrilaterals need not be squares for this to occur. This can be easily checked by the reader in the provided dynamic sketch by dragging points P_1 and P_2. This is because ∠*BPA* = 90° and *PB* = *PA* are not sufficient conditions to ensure that quadrilaterals AP_1P_2B, etc. are squares.

As in theorem 1, from the similarity between △*GIF* and △*BPA* it follows that the ratio of their corresponding sides is in the same ratio; ie. *FI/IG* = *AP/PB*.

In the special case[3] when ∠*BPA* = 90°, ∠*GIF* also becomes 90° and the kite *GHFI* becomes cyclic, and is called a 'right kite' (Wikipedia). Also note that the placement of the similar triangles *BPA*, *DSA*, *DRC and BQC* in this special case is equivalent to the arrangement of the corresponding triangles for similar rhombi on the sides as described in De Villiers (1998) and Silvester (2006). Hence, from the results obtained in these two papers, it follows that if ∠*BPA* = 90°, then *PR* = *QS,* and the angle between *PR* and *QS,* ∠*RXS* = 2 x ∠*PAB*.

From the aforementioned, it further follows that if ∠*BPA* = 90° and ∠*PAB* = 45°, then *PR* = *QS* and *PR* ⊥ *QS.* Since this condition implies that △*BPA* becomes an isosceles right triangle[4], it immediately follows from the similarity of △*BPA* to △*GIF*, that the kite *GHFI* becomes a square.

Note that the arrangement in Theorem 3 of the directly similar quadrilaterals AP_1P_2B, CQ_1Q_2B, CR_1R_2D, AS_1S_2D is different from those of the generalizations of similar rectangles and similar parallelograms discussed in De Villiers (1998) and Silvester (2006). However, if similar parallelograms (and similar rectangles) are arranged according to the arrangement in these two papers, we obtain the following theorem in relation to the quadrilateral *GHFI*.

[PLACE FIGURE 5 ABOUT HERE]

Theorem 4: Given four points *A*, *B*, *C*, *D*, and four directly similar parallelograms with respective centroids *P*, *Q*, *R*, *S,* arranged as shown in Figure 5. Further let *F*, *G*, *H, I* be the midpoints of the segments *AC*, *BD*, *QS, PR* respectively. Then *GHFI* is a cyclic quadrilateral.

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[3] As before, note that the similar quadrilaterals need not be rhombi for this to occur, and the reader is encouraged to check this using the link to the provided dynamic sketch.

[4] As before, note that the similar quadrilaterals need not be squares for this to occur, and the reader is encouraged to check this using the link to the provided dynamic sketch.

Proof: As before, applying Theorem 2 to the pair of similar triangles on opposite sides *AB* and *CD*, it follows that $\triangle GIF$ is similar to $\triangle BPA$, and therefore $\angle BPA = \angle GIF$. Applying Theorem 2 to the different pair of similar triangles on opposite sides *BC* and *AD*, it follows that $\triangle GHF$ is similar to $\triangle BQC$, and therefore $\angle BQC = \angle GHF$. But $\angle BPA + \angle BQC = 180°$ since these are two adjacent angles formed by the diagonals of the similar parallelograms. Hence, $\angle GIF + \angle GHF = 180°$, which implies that *GHFI* is a cyclic. Q.E.D.

From the given similarities we can derive even more. For example, for the two pairs of similar triangles $\triangle GIF$ and $\triangle BPA$, and $\triangle GHF$ and $\triangle BQC$, we correspondingly have $\angle PAB = y = \angle IFG$ and $\angle QCB = z = \angle HFG$. But $\angle PAB + \angle QCB = \angle PAB + \angle PAP_1 = y + z = \angle P_1AB$. However, using complex numbers as in Silvester (2006), vectors, or transformation geometry as shown in the Appendix, the angle between *PR* and *QS* can be shown to be equal to $\angle P_1AB$.

Hence, $\angle IVH = y + z = \angle IFH$, but since these two angles are subtended by the same chord chord *IH*, it follows that the Van Aubel point *V* is concyclic with the other four points *G*, *H*, *F* and *I*. This result, also presented in Pellegrinetti & De Villiers (In press) with a different synthetic proof, is a nice generalization of the Pellegrinetti circle (Pellegrinetti, 2019) for the case when the similar quadrilaterals are squares.

Similar triangles on the sides

As can clearly be seen from the earlier examples above, the original theorem of Van Aubel, as well as its various generalizations (including those in De Villiers (1998) and Silvester (2006)), are really about similar triangles (and their apex vertices) and not really about similar quadrilaterals (and their centres/centroids[5]) on the sides of a quadrilateral. For example, Theorems 1, 3 and 4 would hold as long as the particular sets of triangles on the sides of *ABCD* are similar (and related to each other as required).

[PLACE FIGURE 6 ABOUT HERE]

[5] In fact, the same results would hold if the points *P*, *Q*, *R* and *S* are chosen to lie in the same 'relative position' in respect of each quadrilateral, where the same relative position means that the same similarity transformation that maps the one quadrilateral on to the other, also maps the corresponding points *P*, *Q*, *R* and *S* on to each other.

Working with similar triangles on the sides instead of quadrilaterals, it is now easy to see that Theorems 1 and 3 further generalize as follows to Theorem 5.

Theorem 5: If a pair of directly similar triangles *BPA* and *DRC* are constructed on opposite sides *AB* and *CD* of quadrilateral *ABCD*, and another pair of directly similar triangles *ASD* and *CQB* are constructed on opposite sides *AD* and *CB* so that $\angle ASD = \angle BPA$, and if *F*, *G*, *H, I* are the midpoints of the segments *AC*, *BD*, *QS, PR* respectively, then *GHFI* is a quadrilateral with a pair of equal opposite angles at vertices *H* and *I* (see Figure 6).

Proof: As before, the result follows immediately from the application of Theorem 2.

In addition, since $\angle GIF = \angle GHF$, it follows that the circumcircles of $\triangle GIF$ and $\triangle GHF$ lie symmetrically around *GF*.

[PLACE FIGURE 7 ABOUT HERE]

Another interesting generalisation of the associated Van Aubel result for squares on the sides is the following.

Theorem 6: If a pair of directly similar isosceles triangles *BPA* and *DRC* are constructed on opposite sides *AB* and *CD* of quadrilateral *ABCD*, and another pair of directly similar isosceles triangles *ASD* and *CQB* are constructed on opposite sides *AD* and *CB*, and if *F*, *G*, *H, I* are the midpoints of the segments *AC*, *BD*, *QS, PR* respectively, then *GHFI* is a kite with *HI* its axis of symmetry (see Figure 7).

Proof: As before, from Theorem 2, it follows that $\triangle GIF$ and $\triangle GHF$ are both isosceles triangles (with *GF* as their common base). Hence, *GHFI* is a kite (with *HI* its axis of symmetry). Q.E.D.

More-over, from the similarity between triangles *PAB* and *IFG*, and triangles *SAD* and *HFG*, we have $\angle IFH = \angle PAB + \angle SAD$. Hence, it follows that if $\angle PAB + \angle SAD = 90°$, then *GHFI* becomes a 'right kite' (Wikipedia) and is cyclic with right angles at *G* and *F*.

[PLACE FIGURE 8 MORE OR LESS HERE]

We can also further generalize Theorem 4 as follows.

Theorem 7: If a pair of directly similar triangles *BPA* and *DRC* are constructed on opposite sides *AB* and *CD* of quadrilateral *ABCD*, and another pair of directly similar triangles *ASD* and *CQB* are constructed on opposite sides *AD* and *CB* so that $\angle ASD =$

180° - ∠*BPA*, and if *F*, *G*, *H, I* are the midpoints of the segments *AC*, *BD*, *QS, PR* respectively, then *GHFI* is a cyclic quadrilateral (see Figure 8).

Proof: As before, the result follows immediately from the application of Theorem 2.

As with the previous theorem*,* it follows from the similarity of the triangles that if ∠*PAB* + ∠*SAD* = 90°, then right angles are formed at *G* and *F*, and *HI* is the diameter of the cyclic quadrilateral. Also note that unlike the special case in Theorem 4, as illustrated in Figure 8, the Van Aubel point *V* is not necessarily concyclic with *G*, *H*, *I* and *F*.

Conclusion

The Van Aubel associated generalizations given in Theorems, 1, 3, 4, 5, 6 and 7 provide nice, accessible challenges to mathematically talented high school learners or for undergraduate university students to first explore with dynamic geometry, and then to prove (and explain) using Theorem 2. While providing learners and students with ready-made dynamic sketches is valuable in terms of saving time, challenging learners and students to make their own dynamic constructions of the results in this paper, can also be a very fruitful learning experience for them.

Disclosure statement

No potential conflict of interest was reported by the author.

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APPENDIX

[PLACE FIGURE 9 MORE OR LESS HERE]

The transformation proof below is adapted from Pellegrinetti & De Villiers (In press).

Theorem 8: If directly similar parallelograms AP_1P_2B, CBQ_1Q_2, R_2DCR_1 and S_1S_2AD with respective centres *P*, *Q*, *R* and *S* are erected on the sides of *ABCD*, then the angle between *PR* and *QS* is equal to $\angle P_1AB$, and $QS/PR = AB/P_2B$ (see Figure 9).

Proof: Since the parallelograms are similar, it follows that △'s ABQ_1 and P_2BC are similar since $\angle ABQ_1 = \angle P_2BC$ and the corresponding sides surrounding these angles are in the same ratio (i.e. $AB/BQ_1 = P_2B/BC = \delta$). Hence, a counter-clockwise rotation around *B* of $\triangle ABQ_1$ by $\angle ABP_2$ followed by a dilation from *B* (of P_2B/AB) maps it onto $\triangle P_2BC$. Therefore, the corresponding sides AQ_1 and P_2C of these two similar triangles are inclined towards each other by $\angle ABP_2$, or equivalently, inclined towards each other by its supplementary angle = $180° - \angle ABP_2 = \angle P_1AB$. Moreover, from the similarity of the two triangles, we have $AQ_1/P_2C = \delta$.

Since *P*, *F* and *Q* are the respective midpoints of sides AP_2, *AC* and Q_1C, it follows by applying the triangle mid segment theorem to △'s AP_2C and CAQ_1, that $\angle PFQ = \angle P_1AB$, and $FQ/FP = \delta$.

Similarly it can be shown that $\angle RFS = \angle P_1AB$, and $FS/FR = \delta$. Hence, △'s *PFR* and *QFS* are similar since $\angle PFR = \angle QFS$ and the corresponding sides surrounding these angles are in the same ratio (i.e. $FR/PF = FS/QF = \delta$).

A rotation around *F* of $\triangle PFR$ by $\angle PFQ = \angle P_1AB$ and a dilation from *F* (of QF/PF) maps it onto $\triangle QFS$. Therefore, the corresponding sides *PR* and *QS* of these two similar triangles are inclined towards each other by $\angle P_1AB$ (which is equivalent to its supplement $180° - \angle P_1AB$) and $QS/PR = \delta = AB/P_2B$. This completes the proof.

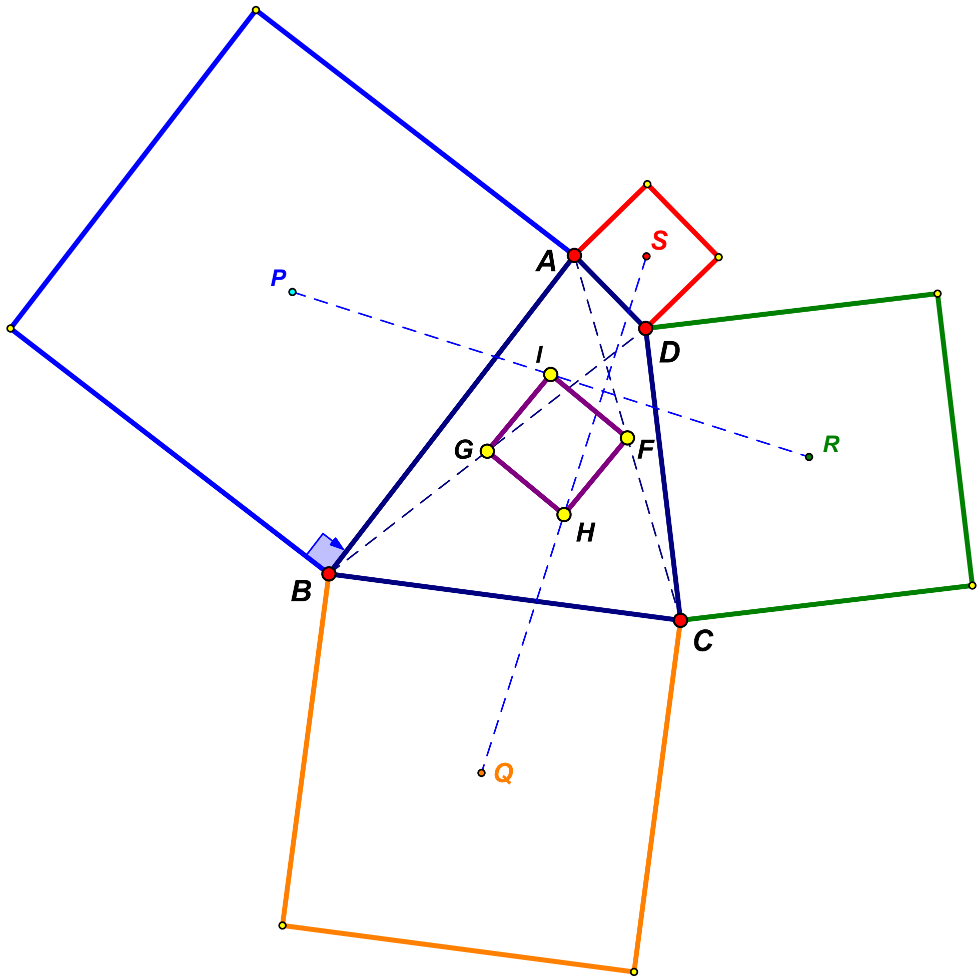

FIGURE 1. Van Aubel's configuration with formed square GHFI

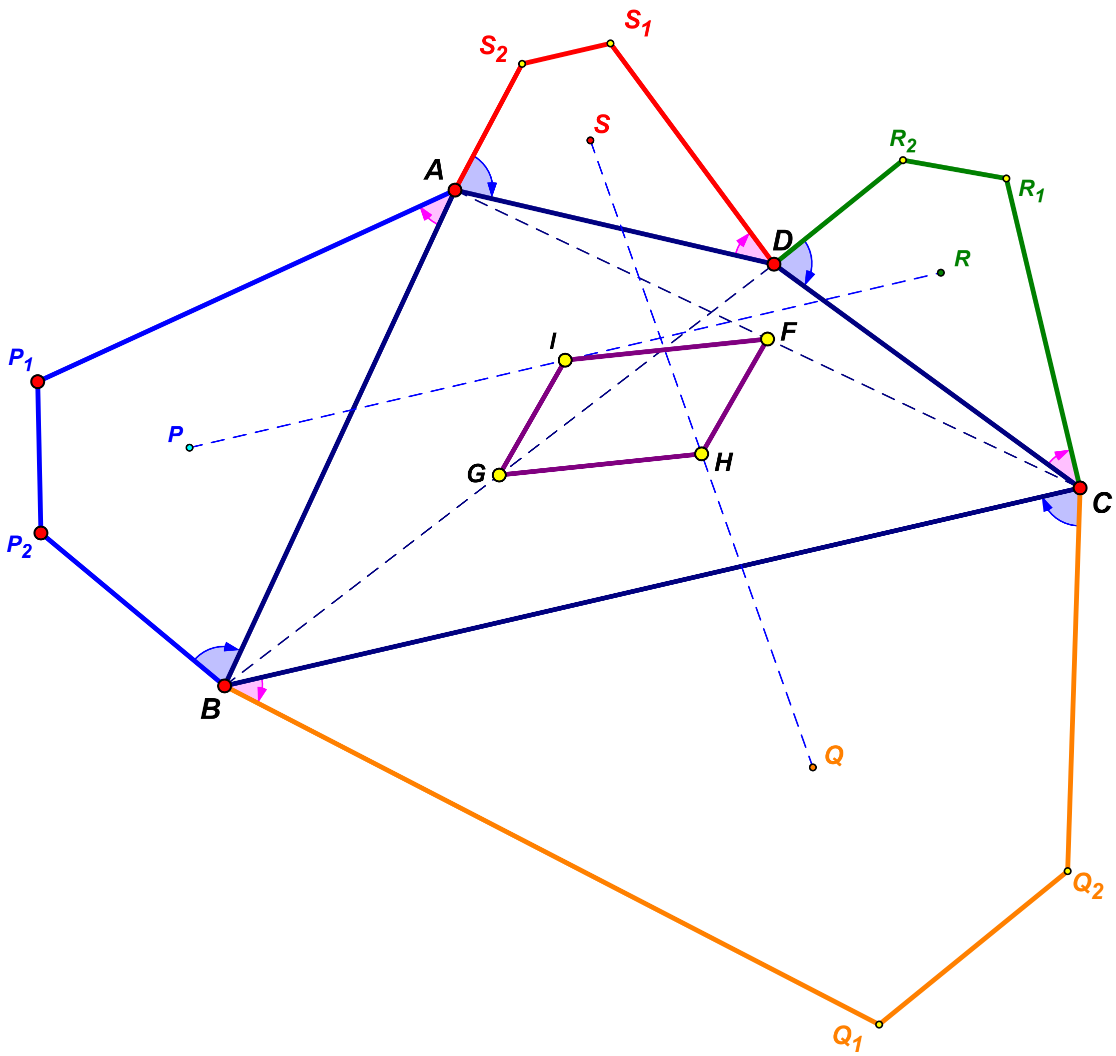

FIGURE 2. Parallelogram *GHFI* of Theorem 1

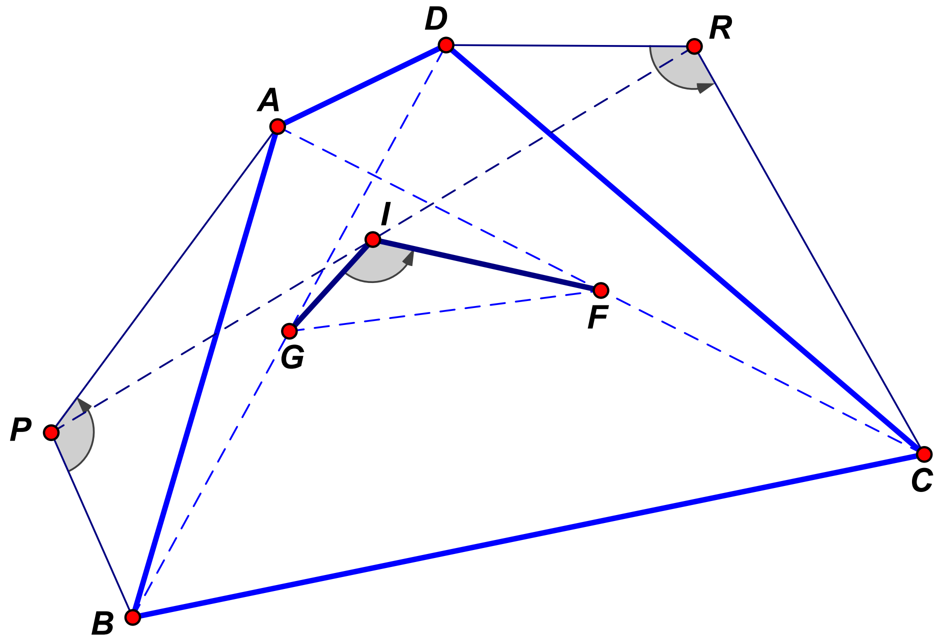

FIGURE 3. Applying Theorem 2

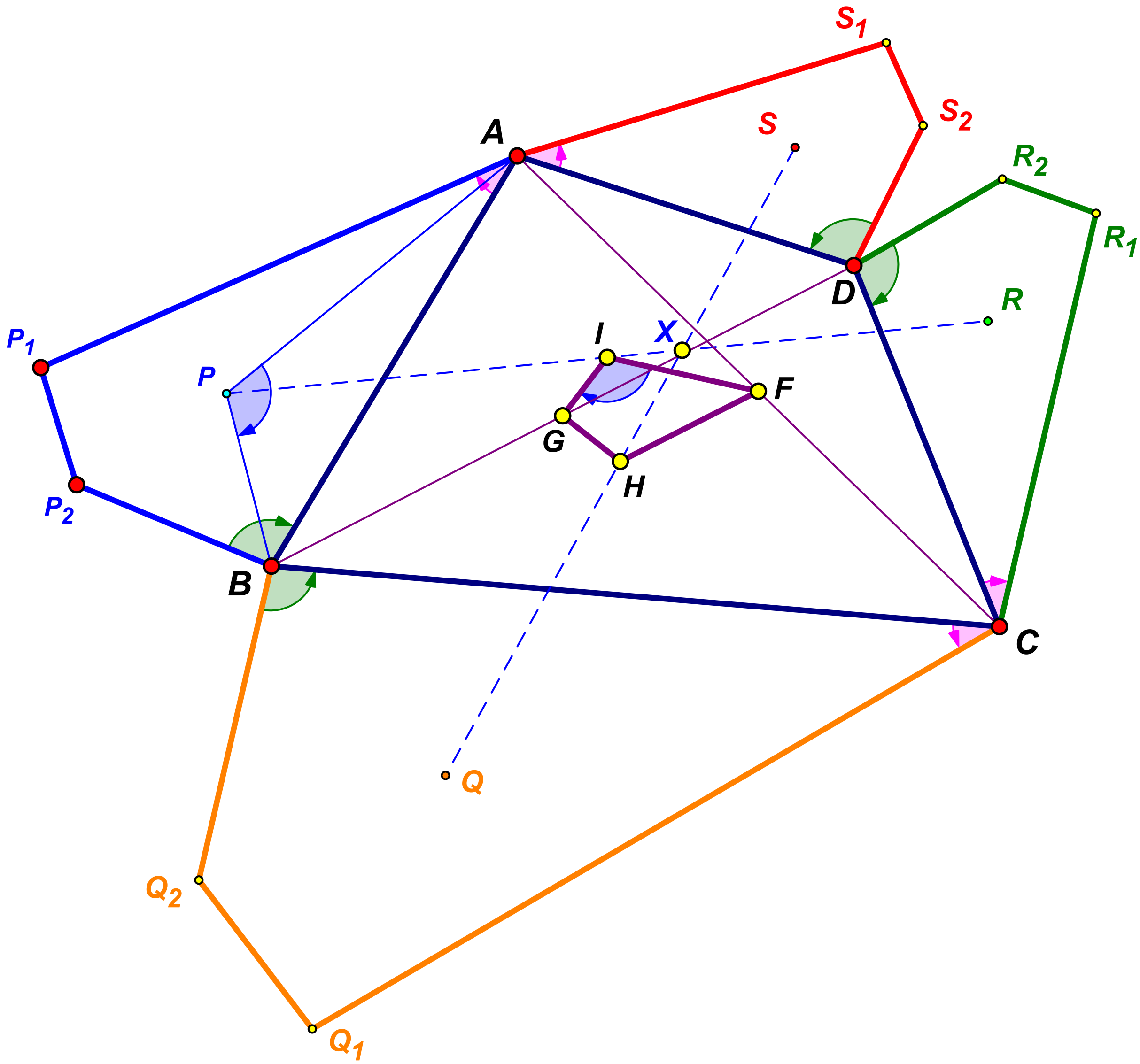

FIGURE 4. Kite *GHFI* of Theorem 3

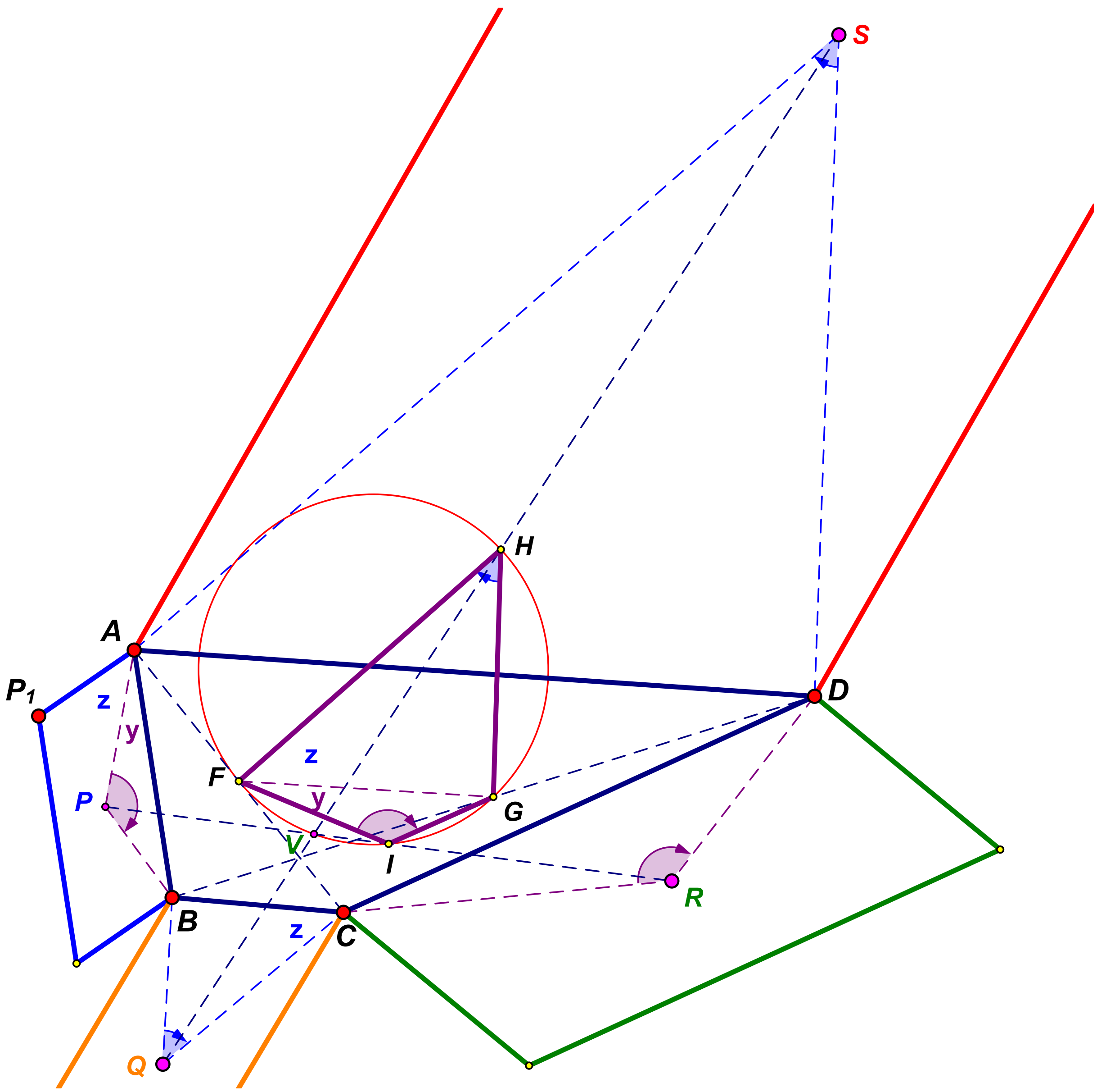

FIGURE 5. Cyclic quadrilateral *GHFI* of Theorem 4

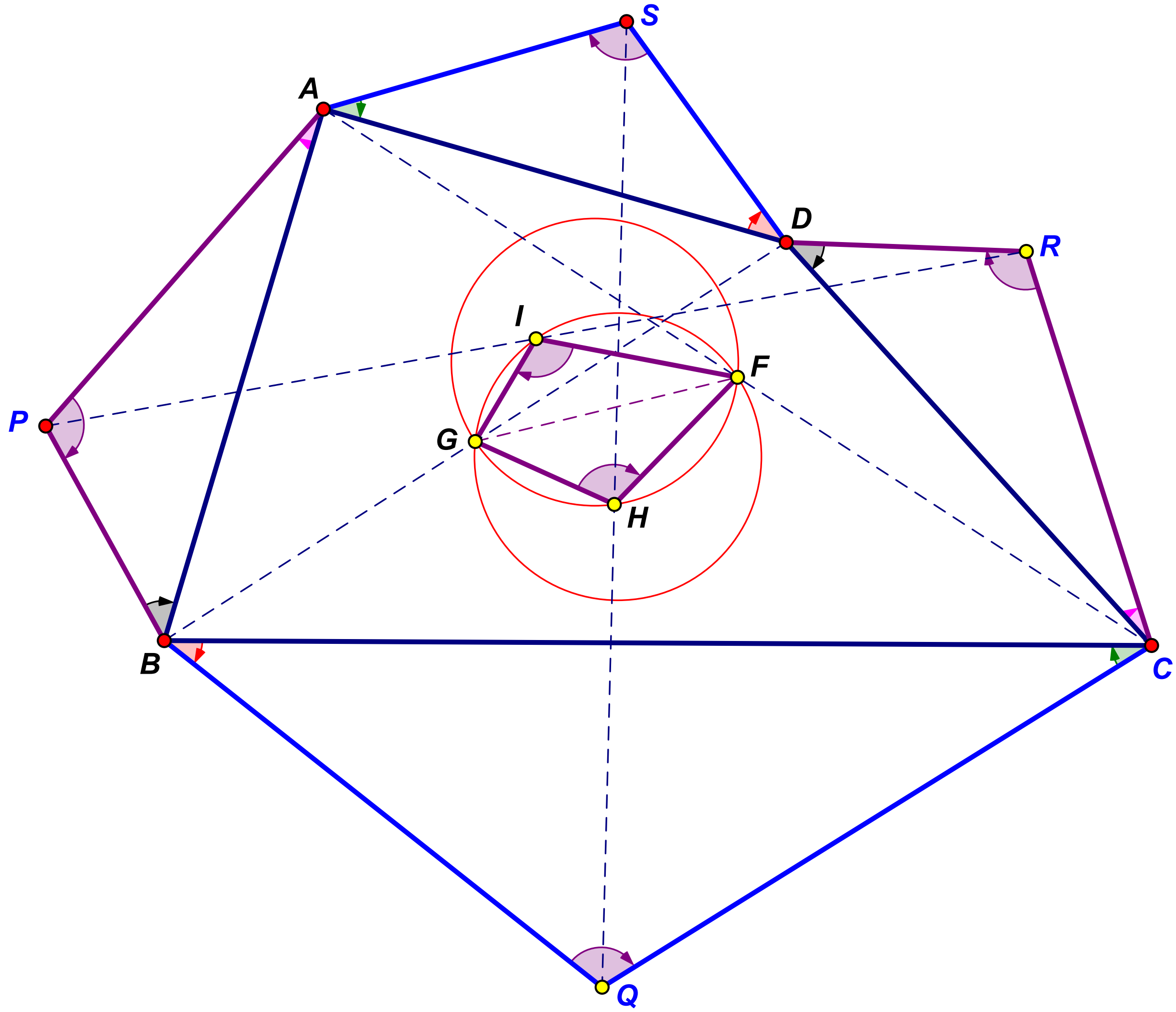

FIGURE 6. Generalization of Theorems 1 and 3

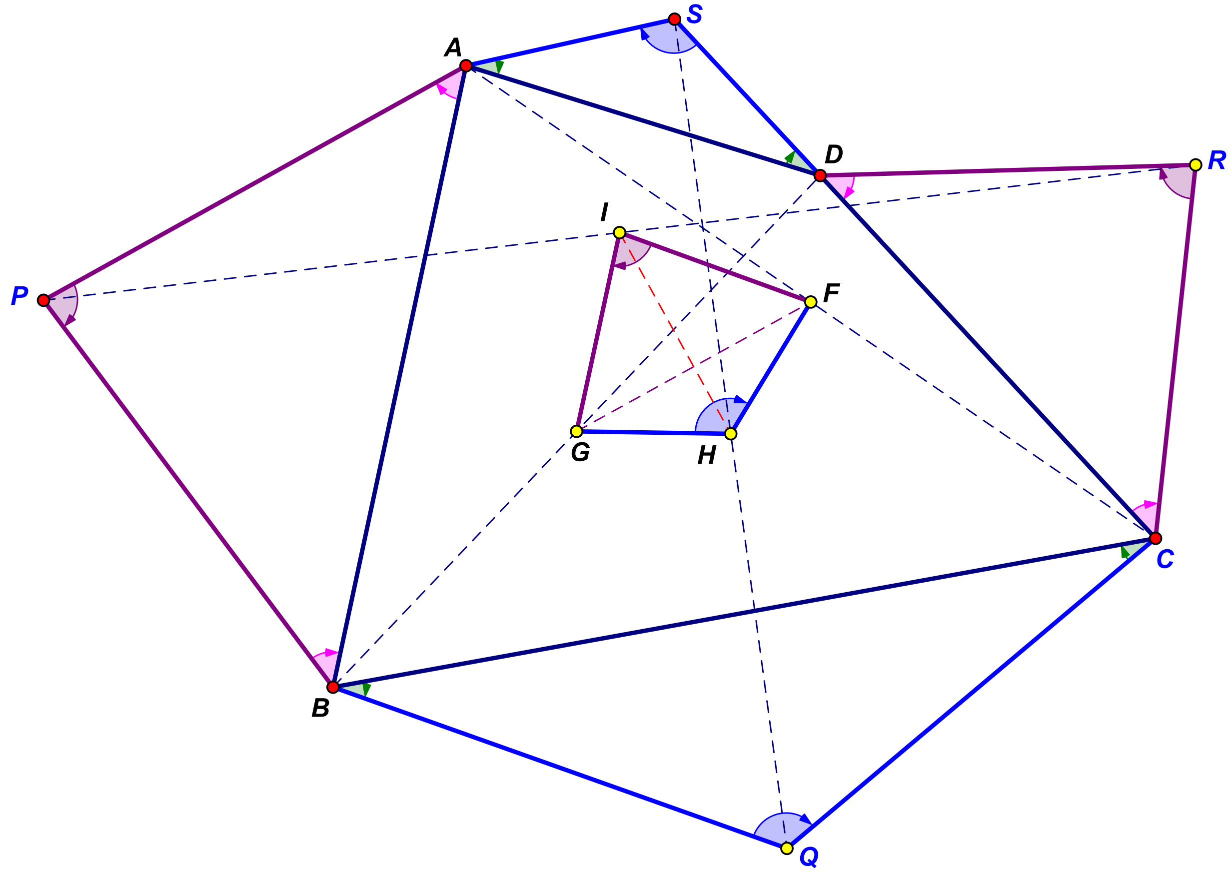

FIGURE 7. Similar isosceles triangles on sides

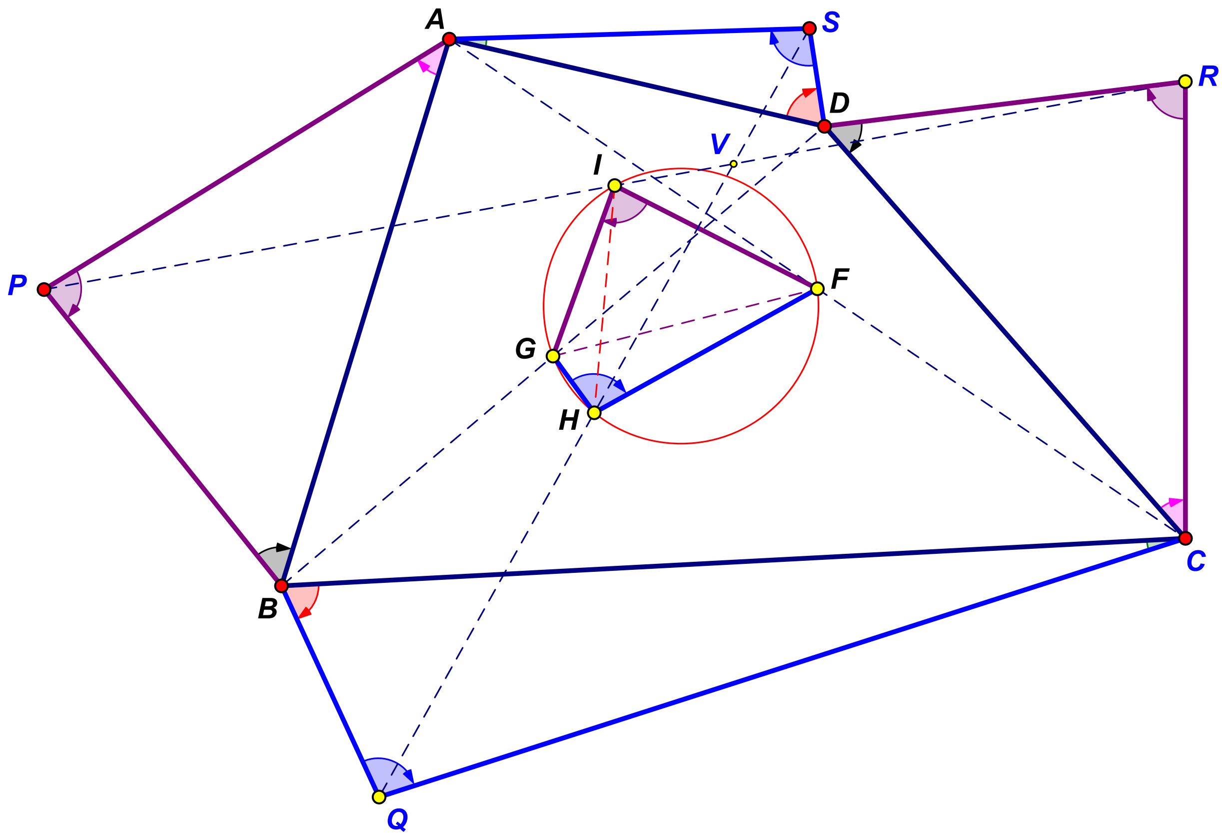

FIGURE 8. Generalization of Theorem 4

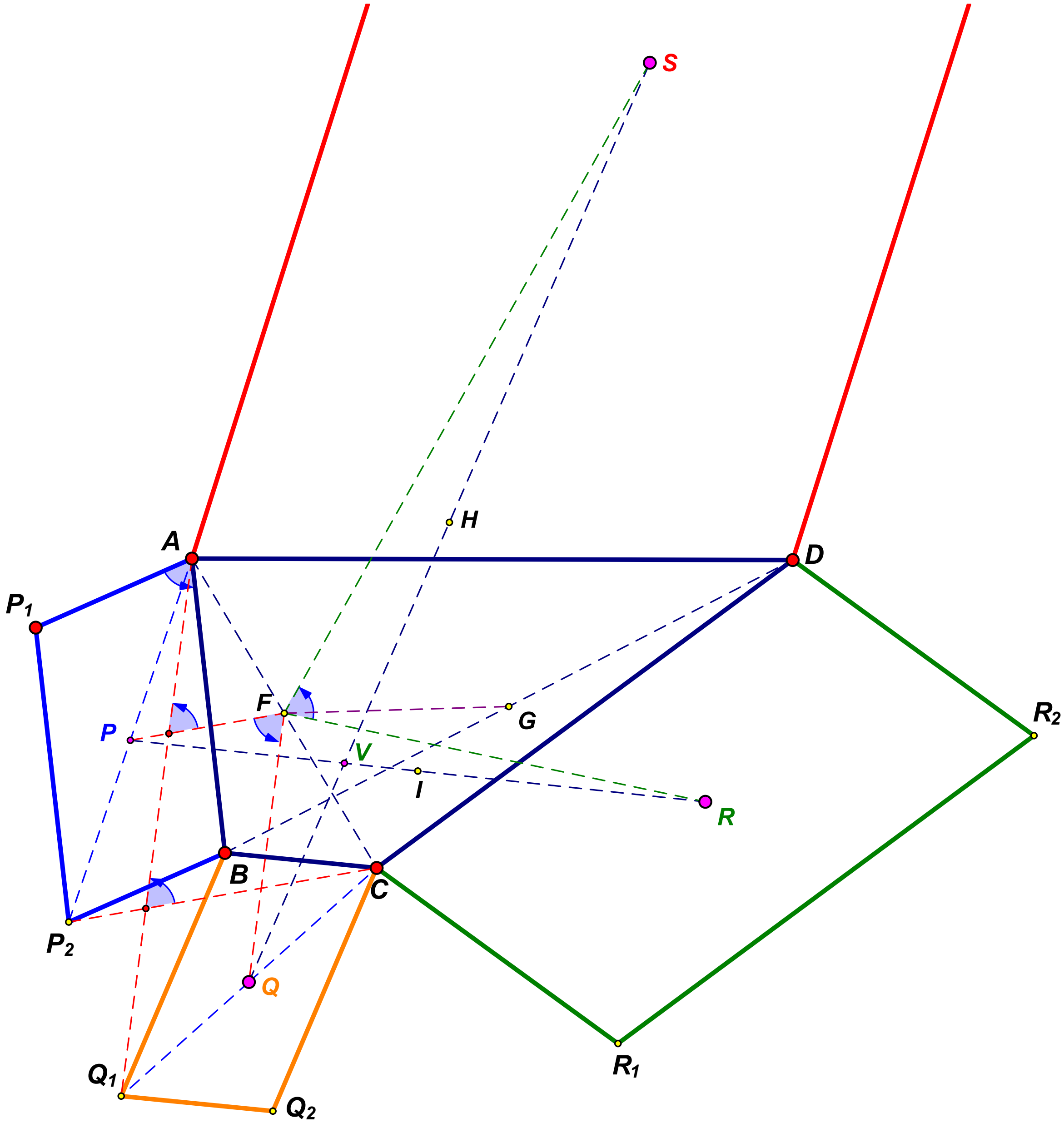

FIGURE 9. Segment properties of Van Aubel generalization to similar parallelograms