THE DIDACTISATION PRACTICES IN PRIMARY SCHOOL MATHEMATICS TEACHERS THROUGH MODELLING
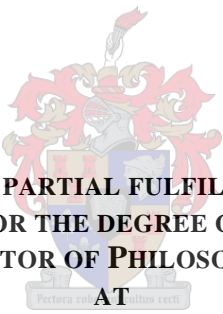
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Declaration

I, the undersigned, hereby declare that the work contained in this dissertation is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

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Abstract

Mathematics teacher development is a source of national and international concern. This study describes how primary school mathematics teachers develop didactisation practices. In considering how teachers could develop, so that student learning is optimised; the concepts of didactisation and the mathematical work of teaching were sourced from existing literature. The concept *didactisation* is explored and defined; and is incorporated with the concept of *mathematical work of teaching*. Nine practices were made explicit through this incorporation: active students, differentiation, mathematisation, vertically aligned lessons, access, probe, connect and assess student thinking, and teacher reflection. These nine practices become the framework for the professional development program and the data generation structure. Five primary school teachers were involved in a professional development program that used model-eliciting activities (MEAs) as a point of departure. A modelling perspective to teacher learning was chosen for the professional development program. The methodology followed the principles of design research and from this, a three phase teaching experiment was designed and implemented. The teachers and researcher met for development sessions and teachers were observed in practice at intervals throughout the program. Their developing didactisation practices were documented through a qualitative analysis of the data. It was established that teachers' didactisation practices did develop during the nine-month program. Furthermore it was found that didactisation practices developed at different rates and consequently, a hierarchy of didactisation practice development is presented. The impact of the program was also gauged through teachers' changing resources, goals and orientations. These three aspects also evolved over time. The program proposed in this study may be a suitable model to develop in-service and pre-service mathematics teachers. The study contributes to understanding teacher action in a classroom and how teachers can change their own thinking and practice.

Key Words

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Abstrak

Die ontwikkeling van wiskundeonderwysers is 'n bron van nasionale en internasionale kommer. Hierdie studie beskryf hoe die didaktiseringspraktyke van laerskool wiskundeonderwysers met die oog op optimalisering van leer ontwikkel het. In die bestudering van die ontwikkeling van onderwysers met die oog op optimalisering van leer, is die begrippe didaktisering en die wiskundige werk van onderrig (mathematical work of teaching) nagespoor uit bestaande literatuur. Die begrip didaktisering is deeglik ondersoek, gedefinieer en saamgevoeg met die begrip wiskundige werk van onderrig. Nege praktyke is eksplisiet gemaak deur hierdie inkorperering: aktiewe studente, differensiasie, matematisering, vertikaalgerigte lesse, toegang, indringende ondersoek, gekonneerdheid en assessering van studente-denke, en onderwyserrefleksie. Hierdie nege praktyke het die raamwerk gevorm vir die professionele ontwikkelingsprogram en die data genereringstruktuur.

Vyf laerskool onderwysers was betrokke in 'n professionele ontwikkelingsprogram waarin model-ontlokkende aktiwiteite (MOA's) as 'n vertrekpunt gebruik is. 'n Modelleringsperspektief is vir onderwyserleer in die ontwikkelingsprogram gekies. Die metodologie volg die beginsels van ontwerpnavorsing waarna 'n drie-fase onderrig-eksperiment ontwerp en in werking gestel is. Die navorser en die onderwysers het byeengekom vir ontwikkelingsessies; die onderwysers is op 'n gereelde basis tydens die program besoek om hul onderwyspraktyk waar te neem. Hul ontwikkelende didaktiseringspraktyke is gedokumenteer en die data is kwalitatief geanaliseer. Onderwysers se didaktiseringspraktyke het wel gedurende die negemaande program ontwikkeling getoon. Hierdie didaktiseringspraktyke het egter teen verskillende tempo's ontwikkel en daarom kon 'n hierargie van die ontwikkeling van didaktiseringspraktyke saamgestel word. Die impak van hierdie program op onderwysers se veranderende hulpbronne, doelstellings en oriëntasies is ook gemeet. Die drie aspekte het in hierdie nege maande verder ontwikkel. Die voorgestelde program in hierdie studie mag moontlik 'n gepaste model wees om indiens en voornemende wiskundeonderwysers te ontwikkel. Die studie lewer 'n bydrae tot 'n beter begrip van onderwyserhandelinge in 'n klaskamer, asook hoe onderwysers hul eie denke en praktyke kan verander.

Sleutelwoorde: Laerskool wiskundeonderwysers, professionele ontwikkeling, didaktisering, wiskundige werk van onderrig, modelleringstake, ontwerpnavorsing

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CHAPTER 1

BACKGROUND AND MOTIVATION

1.1. BACKGROUND

In most parts of the world those involved in mathematics education have set ambitious goals for schools, teachers and students of mathematics (Frid & Sparrow 2009: 36). Since teachers are fundamental to student learning (Ball & Forzani 2009: 497), Smith and Southerland's (2007: 397) description that teachers are the most "critical layer of the school system in terms of efforts to change what happens in schools" provides motivation for this study. The change most anticipated is to shift mathematics learning from traditional binds to more problem-centred, inquiry and modelling-based approaches. To effect a change in what and how students learn mathematics the focus for research should be on what, why and how teachers teach mathematics. However, any "mechanisms for growth and change must ask teachers to act as their own change agents" (Frid & Sparrow 2009: 39). The decision to change (how much, or when) must be made by the teacher if it is to have a meaningful and lasting effect.

All teacher decision making is based on a teacher's current resources, orientation and goals (Schoenfeld 2011: 10). Schoenfeld further states that teacher resources (which include intellectual, material and contextual resources) "fundamentally shapes" teacher decision making. A teacher's intellectual resource is also known as teacher knowledge. Mathematics teacher knowledge has received much attention (Shulman 1986; Ball, Thames & Phelps 2008) in an attempt to improve mathematics education, but the realm of teacher knowledge does not always account for all aspects in the "work of teaching" (Ball & Forzani 2009: 497). Shulman's description of pedagogical content knowledge (PCK) and the more recent classification (Ball *et al*. 2008) of PCK into knowledge of content and students (KCS), knowledge of content and teaching (KCT) and knowledge of content and curriculum (KCC) have made advances in our understanding of the nature of mathematics teaching and what it means to teach mathematics skillfully and competently.

Schoenfeld's (2011) theory of teacher decision making proposes that teacher decisions are influenced by more than just teacher knowledge. He states that teacher orientations and teacher goals also shape a teacher's decisions. Teacher orientations are more widely referred to as teacher beliefs. In following Schoenfeld's (2011: 30) discussion on student orientations, it follows that teacher orientations shape the way a teacher interprets teaching and classroom activities, shape the goals the teacher has and shape the knowledge that a teacher may or may not use in a teaching situation. Teacher beliefs are an area where extensive research has taken place (Liljedahl 2008; Pajares 1992; Guskey 1986) and they are considered instrumental in guiding teacher change.

The complex interrelationship between teacher knowledge, teacher orientations and teacher goals in making decisions should not be underestimated. Schoenfeld proposed that these notions provide a "lens through which teacher activity can be examined" and that research on professional development that documents a teacher's evolution along these ideas is a "rich theoretical arena for future investigation" (2011: 194). This study therefore aims at (see 1.3.2.5) integrating these concepts into a professional development program (see 4.2.2) for primary school mathematics teachers. The professional development program is better suited to a design research approach where iterative cycles can be planned for and reflected on both by teachers and researcher. The professional development program proposed by this study (see 4.2.2), which aims at creating an innovative learning environment in order to develop local instructional theory (Gravemeijer & Cobb 2006: 17), is seated within a design research framework.

The components that make up mathematics teacher expertise are continually evolving as our understanding of the complexity of teaching takes shape. The concept of Mathematics Proficiency for Teaching (MPT) (Kilpatrick, Swafford & Findell 2004; Wilson & Heid 2010) integrates important ideas of teacher knowledge with teacher skills and teacher values into a more holistic term "proficiency". The MPT framework includes three perspectives: mathematical proficiency, mathematical activity and the mathematical work of teaching (Wilson & Heid 2010: 3). The mathematical work of teaching (which is a focus for this study) includes: probing, accessing and assessing the mathematical ideas of learners, knowing and using the curriculum, as well as teacher reflection on practice (Wilson & Heid, 2010: 6). The MPT framework requires knowledge, actions and attitudes from teachers that are informed by

how students learn mathematics. The aim of this study (see 1.3.2.1) is to produce a framework focusing on the mathematical work of teaching, but would include another component – that of making connections (see 3.2.1.3). Accessing, probing, connecting and assessing the mathematical knowledge and ideas of learners are considered to be the fundamental components of the framework for teacher development in this study. What knowledge and understandings do teachers need to access, probe and connect student mathematical ideas? How can teachers better focus on these connections? How can teachers be guided to become aware of the potential wealth in exploring student ideas? These questions, which are central to this study, converge and diverge at the concept of didactisation.

Didactisation is taken from Treffers (1987: 58) and, in this study, remains consistent with his definition that didactisation is the ***essence of didactical action which makes mathematisation possible.*** Treffers further delineates four components of didactisation: active students, differentiation, mathematisation and vertically planned lessons. These components will become the backbone of the professional development program planned in this study. These four components allow a teacher to fulfill the "mathematical work of teaching" at a very high level. Cognitively active students that are engaged in the learning process ensure that accessing and probing their ideas becomes possible. In differentiation, which according to Treffers means that students solve problems using their own (often informal) methods, the teacher will be able to assist students in making connections within mathematical ideas. This is fertile ground for the development of mathematisation, both horizontal (in bridging from real problems to mathematics) and vertical (in connecting to and building of abstract mathematical ideas). The nine didactisation principles that make up the framework for this study become the nine teacher didactisation practices are:

- Active students
- Differentiation
- Mathematisation
- Vertically planned/aligned lessons
- Accessing student thinking
- Probing student ideas
- Connecting student ideas

- Assessing classroom solutions
- Reflecting on practice

The professional development of teachers, advancing their knowledge, understanding and proficiency in didactisation is the focus of this study (see 1.2). Didactisation forms a key construct in the teacher development framework proposed by this study (see 2.4). It will also be shown that improved teacher knowledge of didactisation leads to improved mathematics learning experiences for students. A learning environment to facilitate the development of teacher knowledge, orientations and goals in didactisation is guided by developmental or design research. The design concept of Fosnot, Dolk, Zolkhower, Hersch and Seignoret (2006: 7) is accepted as sound:

> … we engaged in-service teachers in experiences that involved action, reflection, and conversation within the context of learning/teaching. We took the perspective that teachers need to construct new gestalts, new visions of mathematics teaching and learning. To do this they need to be learners in an environment where mathematics is taught as mathematising, where learning is seen as constructing in terms of professional development of teachers.

For new gestalts or new visions of mathematics teaching and learning to develop, "paradigm shifts" (Kuhn 1996: 85) will be necessary. This study intends to elicit this shift through modelling tasks (see 5.2.9.1 and 6.2). Modelling is defined as a problem-solving process whereby complex real-world problems are solved by creating and working a model that describes or explains the problem situation. A modelling perspective on teacher development emphasises that teachers are seen as "evolving experts" (Doerr & Lesh 2003: 127), and this concept is foundational to the design of the learning environment of the teaching experiment phase of this study. Modelling tasks encompass and make visible the didactisation principles set out by this study. By focusing on didactisation when solving modelling tasks, teachers may form a deeper understanding of these principles in more meaningful teaching and learning. From this perspective, the emphasis is not to repair deficiencies in teachers' mathematical knowledge, but rather to use what teachers already know to develop more powerful forms for teaching (Doerr & Lesh, 2003: 130). The thesis for this study is that a modelling based teaching experiment will allow for the development of mathematics teacher didactisation practices (see 5.5.6).

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This study aims at exploring an innovative field in teacher education (see 1.3.1.2). Didactisation as a conceptual avenue to the development of teacher knowledge in the field of mathematics education will bring together a number of current powerful ideas in teacher development. It ties up teacher decision making, the mathematical work of teaching and a modelling perspective through didactisation principles. The study will make an original contribution to the field of teacher knowledge, of developing mathematical teaching proficiency and of teacher development through modelling problems (see 6.3). Adler (2005: 172) summarised research related to teacher education over a ten year period in the SAARMSTE (South African Association for Research in Mathematics, Science and Technology Education) publications. She found that the focus of most papers was INSET and on issues of teacher change. Her opinion is that research on teacher change "makes sense in the South African context". She reminds us that many changes in a variety of spheres have occurred in South Africa over the past 20 years. Teachers have not been immune to these changes. Since 1998 there have been four significant revisions to the curriculum together with many changes to planning, assessment, funding and district support at school level.

South Africa is currently implementing a revised curriculum called CAPS (Curriculum and Assessment Policy Statement), which will "create a need to build models of professional development" (Arbaugh & Brown 2005: 500). The training to prepare teachers for the revised curriculum takes place over 3 days and involves getting to know the new document rather than developing improved teaching strategies. The document for mathematics is highly prescriptive of specific content that must be covered during the year. South African classrooms are also large and many include a wide range of abilities. This study is situated within the current international and South African domains and deals with issues pertaining to both.

1.2 STATEMENT OF THE PROBLEM

How can primary school mathematics teachers be developed to achieve a greater awareness and implementation of didactisation practices?

Sub-questions:

a) What constitutes didactisation practices in mathematics teaching?

b) How do teachers' didactisation practices develop through involvement in a modelling based professional development program?

c) What is the effect of the developmental changes on the teacher's classroom resources, orientations and goals?

1.3 AIMS AND OBJECTIVES

1.3.1 Aims

This study aims to:

1.3.1.1 Formulate a didactisation developmental framework for mathematics teachers.

1.3.1.2 Use this framework to develop mathematics teacher didactisation practices through a modelling-based professional development program.

1.3.2 Objectives

1.3.2.1 Explicate a didactisation teacher development framework incorporating the RME (Realistic Mathematics Education) concept of "didactisation" and the MTP concept of the "mathematical work of teaching".

1.3.2.2 Situate didactisation practices as a mathematics teaching orientation within the development framework.

1.3.2.3 Use modelling tasks as a vehicle in a TDP to develop teachers' didactisation practices.

1.3.2.4 Observe teachers in practice in order to gauge the development of didactisation practices.

1.3.2.5 Consider the changes in teacher knowledge, orientations and goals through the teacher development program.

1.3.2.6 Document the development of didactisation practices of the teachers involved in the TDP.

1.3.2.7 Use a design research framework to create an innovative learning environment to propose local teaching theory about the professional development of mathematics teachers.

1.4 METHODOLOGY

Design research or developmental research is used for this study since it "is directed at innovation and improvement of education" and takes place in "an integrated cyclistic process" (Treffers 1993: 103). This study includes cycles of teacher-researcher activity whereby "invention and revision" (Bakker 2004: 38) form part of the process (see 4.2.2). A learning landscape for teachers will be generated so that a realisation of how teachers develop an understanding of didactisation practices will emerge. The design of such a learning landscape is revised through reflection. Three distinct phases of design research are identified: a preparation and design phase, a teaching experiment and a retrospective analysis (Bakker 2004: 3). The preparation and design phase includes a comprehensive literature study of all the components of the study together with the design of instruments. The teaching experiment is typical of design research in its "characteristic iterative design" (Cobb, Confrey, diSessa, Lehrer & Schauble 2003: 10). The teaching experiment is briefly outlined below: Four to five volunteer teachers will be selected from schools that are in close proximity to the researcher. The researcher is a full-time practicing teacher and therefore a convenient sample is important for the feasibility and success of the study. Time needed to travel to the other schools for observation visits will be kept to a minimum in this way. The focus is on how the teachers' didactisation develops as a result of being exposed to modelling, and how this impacts on their classroom practice. The teachers will be selected on a volunteer basis. The program and all its facets will be explained to them and they will be given the opportunity to ask questions on any feature of the program.

The teachers will meet fortnightly with the researcher for a period of nine months. The researcher will also visit these teachers in their classrooms at intervals throughout the program. The program will take place in various stages or cycles that include collecting data in a number of formats.

1.4.1 The teaching experiment

Activity	Instrument	Timeframe
Selection of interested teachers		July 2012
Questionnaire	Instrument 1	July 2012
Classroom Observation 1	Instrument 3 Instrument 4 Researcher field notes	August (1) 2012 30-60 min
Baseline Structured Questionnaire	Instrument 2	45 min
Teacher Session 1: Information session, aim and methodology of this study. Didactisation principles presented to teachers. Teachers working on modelling task 1	Power-point Presentation. Recording and transcription.	August 2012(3) 2 hours
Take home reflection sheet	Instrument 6	45 minutes
Teacher Session 2: Fishbowl activity task 1 Teacher group discussion on task and different teaching models	Four pupils solve modelling task 1. Powerpoint presentation.	September 2012 (1) 2 hours
Take home reflection sheet	Instrument 7	45 min
Teacher Session 3 Teachers and researcher group discussion	Resource material for teachers. Recording and transcription.	September (3) 2 hours
October 2012 to January 2013: According to a ruling within the Gauteng Department of Education – no research is allowed at schools during this time. The possible impact of this forced break is discussed below.		
Classroom observation 2	Instrument 3 Instrument 4	February 2013 (1) 30-60 min
Informal questionnaire	Instrument 5	February 2013 (1) 45 min
Teacher Session 4: Teachers work on modelling task 2	Recording and transcription.	February 2013 (3) 2 hours
Take home reflection sheet	Instrument 6	45 minutes

Teacher Session 5: Fishbowl modelling task 2	Teachers observe a group of four students solving modelling task 2.	March 2013 (1) 2 hours
Take home reflection activity	Instrument 7	45 minutes
Teacher Session 6: Researcher and teachers' group discussion on planning for lessons.	Recording and transcription. Resources for teachers' classrooms.	March (3) 2 hours
Classroom observation 3	Instrument 3 Instrument 4	April 2013(2)
Informal questionnaire	Instrument 5	April 2013(4) 45 min
Teacher Session 7: Teachers work on modelling task 3	Recording and transcription.	May 2013(2)
Take home reflection sheet	Instrument 6	45 minutes
Teacher Session 8: Fishbowl modelling task 3	Teachers observe a group of four students solving modelling task 3.	May 2013(4) 2 hours
Take home reflection sheet	Instrument 7	45 minutes
Teacher Session 9: Summary	Resources for teachers' classrooms.	June (1) 1 hour
Concluding classroom observation	Instrument 3 Instrument 4	June 2013 (3) 30-40 min
Concluding Questionnaires	Instrument 5 Instrument 2 Instrument 1	June 2013(3) 45 min

(The number in brackets indicates the week during that month).
Table 1.1 Teaching experiment

A new phase in the teaching experiment starts again with a new modelling task.
The developmental process for the first task will influence the design and structuring of the discussions for the following tasks.

The forced break between October and February is a stipulation by the Gauteng Department of Education. Although this could impact negatively on the flow of the professional development it is possible that this break will test the robustness of the professional development program. Although these months comprise a four month break – only two

months of teaching are involved. Exams, end-of-year functions and assemblies, a month vacation and the beginning of year disruptions make up the rest of the time.

Design research contributes to research in three main types of outputs: design principles, curricular products or programs and professional development (McKenney, Nieveen & van den Akker 2006: 73). The formulation of design principles allows others to select their own principles in their own settings. In this study the formulation takes places through the literature review and theory exposition in Chapter 2 and 3. The design principles are then incorporated in the professional development program (see 4.2.2); the curricular products are those activities designed for use in the professional development program that can also be used by the professional community (see 6.3).

1.5 DELINEATION AND LIMITATIONS

The study intends limiting its scope to four or five teachers from three primary schools in South Africa. This means that the results are not generalisable over a greater spectrum. The two grades that will be focused on are grade 5 and grade 6, which further delineate the field of focus.

The study does not attempt to produce a fine-grained analysis of teacher thinking but rather on the impact on teacher didactisation practices in their lessons. The study also does not compare teachers' to each other, but rather evaluates the development in the lessons that are observed. A further limitation of this study relates to researcher and teacher resources. More time to conduct informal interviews or added discussions may prove beneficial to the study. However, this is not always possible because of the researcher's full time employment as a teacher. The participating teachers, being volunteers, had to fit this program into an already full schedule. The program set out above was considered possible and sufficient within the time limitations.

1.6 PROVISIONAL CHAPTERING

Chapter 1: Introduction and background to the study.
Chapter 2: Landscape of factors influencing primary school mathematics teachers. This chapter includes an exposition of the main theoretical constructs in the study.

Realistic Mathematics Education, Theory of didactical situations and Modelling perspective forms the base of the theoretical approach to this study. Furthermore the chapter includes a description of factors in teacher decision making as well as exploring the concept of didactisation.

Chapter 3: The mathematical work of teaching and teacher development. This chapter focuses on various aspects of the mathematical work of teaching, teacher change and professional development.

Chapter 4: Design research methodology. This chapter sets out the aspects of the design research methodology as it relates to this study as well as addressing concerns of validity and reliability.

Chapter 5: Interpretation of data and results of the study. This chapter sets out the chronological collection of the data together with an analysis and discussion of the findings.

Chapter 6: Conclusion and recommendations. This chapter summarises the main findings of the, its theoretical and practical contributions, limitations and recommendations for further research.

1.7 ETHICAL CONSIDERATIONS

The study will work intensively with teachers and pupils at Government Schools. The relevant permission will be obtained from the Department of Education and Principals of each school. Teachers involved in the study will be fully briefed as to the intentions and processes involved in the study. Teachers will sign assent documentation and will partake voluntarily.

A small number of students will be involved in the study. They and their parents will be briefed as to the focus and intentions of this study. Parents will be asked to sign consent documentation.

All names of schools, teachers and students will remain anonymous.

1.8 ADDITIONAL INFORMATION

The use of multiple instruments, together with varied and numerous data sources, increases the reliability and validity of the study. The integration of existing and new instruments also assists in raising the level of reliability in the study. The modelling tasks are taken from

existing literature and therefore meet the design principles for modelling. Transcriptions allow for a more intensive retrospective analysis and discussion with others in the field. The retrospective analysis allows for changes in the design of new cycles and therefore ensures that instruments are continually reviewed and remain relevant.

1.9 ACRONYMS AND ABBREVIATIONS

PCK: pedagogical content knowledge

MWT: mathematical work of teaching

CAPS: Curriculum Assessment and Policy Statement (Revised National Curriculum Statement of South Africa – implementation 2013).

ANAs: Annual National Assessments. This refers to the systemic evaluation in South Africa that takes place during September each year. Grades 1-6 and Grade 9 Language and Mathematics are involved.

1.10 SPELLING PREFERENCE

South African English spelling typically follows the UK spelling rules. Words such as mathematise, summarise or analyse is preferred to mathematize, summarize or analyze. The same applies to the spelling of modelling as opposed to modeling. The researcher is following the most common spelling within the South African context.

CHAPTER 2

LANDSCAPE OF FACTORS INFLUENCING PRIMARY SCHOOL MATHEMATICS TEACHING

2.1 INTRODUCTION

Teaching is a complex and multifaceted activity. Petersen (2005: 5) states that the greatest obstacles to implementing an inquiry model of instruction are the teacher's mathematical proficiency, pedagogical proficiency and administrative policies. This chapter aims to create a framework for understanding the work of teaching mathematics by exploring teacher knowledge and teaching knowledge. It is necessary to formulate a framework that takes into account what it means to be a proficient teacher of mathematics. Doerr (2006: 5) reminds us that teaching means "knowing how to see and interpret the complex and ill structured domain of classroom practice". Specifically what are the resources and knowledge that teachers require to teach mathematics proficiently? How do resources impact on the day-to-day decisions that teachers make in the classroom? What knowledge domains would add to what it means to be a proficient mathematics teacher?

This chapter also highlights certain theoretical approaches to teaching and learning mathematics. This study accepts the suggestion of Bikner-Ahsbahs and Prediger (2010: 483) that called for a "networking of theories" that enables one to exploit the diversity of theoretical approaches in mathematics education as a resource for richness. It is also accepted that "theory guides research practices and are influenced by them" (Bikner-Ahsbahs & Prediger 2010: 488). It is envisaged that by presenting and connecting more than one theoretical approach to this study it will result in "better communication and understanding" and "better collective capitalization of research results"(Bikner-Ahsbahs & Prediger 2010: 490).

This chapter also presents the didactisation framework for this study that rests on the Realistic Mathematics Education (RME) concept of didactisation and the mathematical work of teaching which is part of MPT. These two constructs prepare the underlying principles for the professional development program designed in this study.

2.2 FOUNDATIONAL THEORIES OF MATHEMATICS TEACHING

The central theory guiding this study is of a social constructivist nature. Ernest (1993: 42) sets out that from this viewpoint, mathematics is seen as social construction. He adds that social constructivism takes from quasi empiricism "its fallibilist epistemology, including the view that mathematical knowledge and concepts develop and change" (Ernest 1993: 42). Ernest further adds that social constructivism is a descriptive rather than a prescriptive philosophy of mathematics. He explains that the term social constructivism rests on the following three ideas:

1. The basis of mathematical knowledge is linguistic, and language is a social construction.

2. Interpersonal social processes change an individual's subjective knowledge into objective mathematical knowledge.

3. Objectivity itself will be understood to be social (Ernest 1993: 42).

This study sets out didactisation principles (see 2.3 and 3.2) that are aligned to social constructivist principles. It is envisaged that teachers will include social constructivist ideas in their classrooms such as understanding that students' individual contributions can add to, restructure or reproduce mathematical knowledge and that subjective knowledge is largely internalised, constructed objective knowledge (Ernest 1993: 44).

When mathematics is seen as a social construction, teaching takes on a different nature. It now requires that student understanding, student processes and student representations become more important than student reproductions. A theoretical approach that has as a fundamental pillar, mathematics as a human activity is Realistic Mathematics Education theory. It forms a central tenet of the study in that the didactisation principles (see 2.4) are accepted as they have been defined in RME theory.

2.2.1 Realistic Mathematics Education

Freudenthal, as one of the pillars of Realistic Mathematics Education, proposed many of the tenets used in this study (see 2.4). Freudenthal's didactical principles include guided reinvention (1991: 45). He clarifies that students must be guided to "reinvent mathematising rather than mathematics" (p. 49) and that the actual guiding from the teacher means "striking a delicate balance between the force of teaching and the freedom of learning" (p. 55). It is this pendulum between teaching and learning that is somewhat out of balance in many

mathematics classrooms. Treffers (1987: 251) typified different classroom scenarios according to the balance between teaching and learning as follows: mechanistic, structuralistic, empiristic and realistic. Each style is shaped by a different focus on vertical and horizontal mathematisation.

	Horizontal Mathematisation	Vertical Mathematisation
Mechanistic Instruction	-	-
Empirist Instruction	+	-
Structuralist Instruction	-	+
Realistic Instruction	+	+

Table 2.1 Treffers' instruction styles

It is the horizontal theme-approach and the specific vertical mathematising combination that distinguishes RME from mechanistic, structuralist and empiricist domains (Treffers 1993: 102). Mathematisation, as one of the didactisation principles is at the heart of teacher proficiency (see 1.3.2.1) and therefore central to this study. Horizontal mathematisation is compared to translating a real world word problem into a mathematical problem, while vertical mathematisation involves working within mathematics and creating more advanced mathematical understandings and concepts (see 2.4.4).

Mechanistic or traditional instruction that focuses on memorizing and applying already known procedures does not need students to mathematise either horizontally or vertically. Students do not need to make sense of any real problems nor do they need to structure or build their own mathematical ideas. Mechanistic teaching is definition based, but van Hiele (1959: 18) warns that using definitions when concepts are unknown to students is of no use because it presupposes a field of knowledge which the students do not yet have.

Within empiricist instruction the focus is on solving many problems, and includes a strong horizontal mathematising element, although this is largely where it ends, and students do not build mathematical concepts through vertical mathematisation. It may be in line with what Kilpatrick (1985: 9) termed as problem solving by "osmosis" where students are immersed in an environment of problems with the assumption that techniques will be absorbed.

A structuralist teaching approach focuses on how mathematical concepts are built and not how these structures can be found and described in real situations. This teaching approach teaches mathematics for its own sake. A realistic approach to mathematics focuses on both horizontal and vertical mathematisation since the guiding principle of realistic mathematics education is guided reinvention through progressive mathematisation by a phenomenological exploration (Gravemeijer 1994a: 90). Day-to-day situations are investigated which give rise to horizontal mathematisation. These aspects in the problem that have been mathematised are then the focus of students' "self-developed models" (Gravemeijer 1994a: 91). These models become a source for reflection where vertical mathemazation allows students to build or reinvent more abstract mathematical concepts for themselves.

Freudenthal offers a concise definition of didactics being the "organisation of the teaching/learning process" (1991: 45). It is especially important that teaching and learning are viewed as different sides to the same coin. Freudenthal's organisation of mathematics is through his concept of "guided reinvention" (1991: 46). He explains that the inventions are the steps in the learning process while the learning environment where learning takes place points to the "guided" part. He provides three firm pedagogical arguments for students learning by guided reinvention. Firstly, that learning through activity bonds better and is more readily available. Secondly, that learning by reinvention is motivating to students, and thirdly, that it enables students to understand mathematics as a human activity (1991: 47). So Freudenthal's proposed two principles for teaching mathematics are:

1. Choose learning situations within the learner's reality that are appropriate for horizontal mathematisting.

2. Offer means and tools for vertical mathematising.

He adds further that a full formulation of guided reinvention is given by adding systems specified by Treffers such as:

3. Interactive instruction, which is Treffers's first didactisation principle of activity (see 2.4.2)

4. Learner's own productions, which is Treffers' second didactisation principle of differentiation (see 2.4.3). Gravemeijer (1994a: 85) confirms that in RME a standard procedure or algorithm is taught by "letting it evolve from informal ones".

5. Intertwining learning strands, which means connecting long-term learning processes. This allows the teachers to align lessons to learning and not the other way around. Treffers' fourth

didactisation principle is that lessons are vertically aligned. This means that teachers prepare for vertical mathematisation (see 2.4.4.2) within mathematical learning.

In RME, the role of problems is central to the realisation of its principles. Problems are realistic in that students find them possible in their environments. The problems are not simply there for students to solve, but for students to learn mathematics meaningfully through building informal notions into more abstract mathematical ideas.

RME theory allows one to focus on teaching activities that promote effective mathematical learning. A theory that focuses closely on the interaction between teacher, students and the learning environment is discussed next as it is simply not enough for the teacher to *do* certain things; he or she has to create the environment for these actions to take place meaningfully.

2.2.2 Theories of didactical situations

2.2.2.1 Theory of didactical situations (TDS)

Brousseau (1997: 30) theorises on didactical situations. He explains that students learn by adapting to a *milieu* (an environment or situation) that includes contradictions and difficulties. New knowledge is a result of the student adapting to challenges and obstacles. The teacher should present problems to students that he/she knows the students will accept and solve on their own. This handing over of problems to students, Brousseau terms the *devolution* of a problem. In this sense he explains that it is more than simply giving the students problems to solve, as there must be an accepting of responsibility for the problem by the student while the teacher refrains from communicating the knowledge or information she wants the student to learn. This devolution of a problem to the student, and the student adapting to the situation, constitutes an *adidactical* situation. Clearly the roles of both student and teacher are changed when the teacher hands over the responsibility of solving the problem to the student. However, Herbst and Kilpatrick (1999: 6) warn that adidactical situations do not provide a "naturalistic paradise" for students, but rather offer a landscape that ensures meaningful production of knowledge that emerges as a solution to a problem. Brousseau also mentions that the teacher is not relieved of "all didactical responsibility" in an adidactical situation

since "a milieu without didactical intentions is manifestly insufficient to induce in the students all the cultural knowledge that we wish her to acquire" (1997: 30).

Creating this adidactical situation is part of moving a teacher from reacting to responding. The problems devolved to students must be carefully chosen, and while the teacher is not actually demonstrating the knowledge required he or she is evaluating students' progress and interacting with students. Brousseau comments that a traditional axiomatic approach to mathematics teaching assures the teacher a way of ordering and accumulating the maximum number of knowledge items in the shortest time. To facilitate teaching even further Brousseau explains the didactical transposition common in traditional teaching, whereby certain concepts and properties are isolated and taken away from the activities which provide their origin, meaning, motivation and use. Herbst and Kilpatrick (1999: 6) maintain that adidactical situations are helpful since they point out the epistemological importance of the student's milieu. According to them this means that the student's activity then evolves producing knowledge that "may eventually lead to a valid institutionalization of the target knowledge". The roles of both the teacher and the student are altered in creating a milieu that will encourage meaningful learning. It is specifically the role of the teacher that is in focus in this study (see 1.3.1.1 and 5.1-5.5).

Brousseau (1997: 23) does explicate the roles of both teacher and student in mathematical classrooms where real mathematical learning takes place. The teacher must produce a recontextualisation and repersonalisation of the mathematical knowledge students are required to learn. The students must redecontextualise and redepersonalise their knowledge. According to Brousseau, each item of knowledge must originate from adaption to a specific situation or from solving devolved problems. According to Sriraman and English (2010: 23) the most significant contribution of Brousseau's theory of didactical situations is that it "allows researchers from different theoretical traditions to utilize a uniform grammar to research analyze and describe teaching situations".

In terms of this study, the theory of didactical situations ensures that it is not simply changing teacher knowledge or teacher actions alone that are important, but the integration of these in creating a milieu whereby student and teacher are responding to the situation in order to grow

and develop. Just how does a teacher hand over a suitable problem or create a situation where students willingly accept the work (both action and thinking) allocated to them? This study proposes a framework for teacher development that may answer this concern (see 4.2.1.3).

2.2.2.2 Anthropological theory of didactics (ATD)

Any attempt to understand teaching and learning situations must take into account where the teaching and learning takes place, therefore organisational understanding or institutional understanding is valuable. Claxton and Langer (in Mason 2010: 23) proposed two useful distinctions that are fundamental to understanding teaching. Mason specifically distinguishes *reacting* from *responding* in a teaching situation. Just how can teachers be guided towards more didactically sensitive responses (and not just reactions) in a mathematics classroom is part of the rationale of the study. As this study aims at developing teachers (see 1.3.1) it is vital that the places where teachers do the work of teaching are understood. This holistic nature of the study (see 3.3) in terms of developing teachers' didactisation practices may contribute to the successful development of teachers.

Understanding teaching and learning as a social endeavour and social action also needs to be considered. According to Ligozat and Schubauer-Leoni (2010: 1616) the "basics of ATD are that (1) ways of thinking of individuals are shaped by the collective practices to which they partake and (2) these collective practices are oriented by purposes whose coherence defines the primary goal of an institution as a social organisation bound to achieve a type of task. In the case of educational institutions, the transmission of a socially agreed culture is the core of the activity, relayed by an 'intention to teach' and an 'intention to learn' at the level of the teacher and the students respectively".
These authors add that

> the interpretative process of the collective meanings by individuals are shadowed by the schemes of institutional practices that (over)structures local purposes and psychological processes (2010: 1619 italics in original).

The culture of the institution wherein a teacher practices teaching contributes to what a teacher does and thinks within the classroom. Similar to Brousseau's ideas of contextualisation and personalisation, Chevallard's (1989: 4) notion of didactical transposition process takes an effect in mathematics education.

According to Sriraman and English (2010: 24) this notion was developed to study the changes in mathematical knowledge as it passes from mathematicians to curricula and policies to teachers and eventually to students. Bosch, Chevallard and Gascon (2005: 1257) explain didactical transposition with the following diagram:
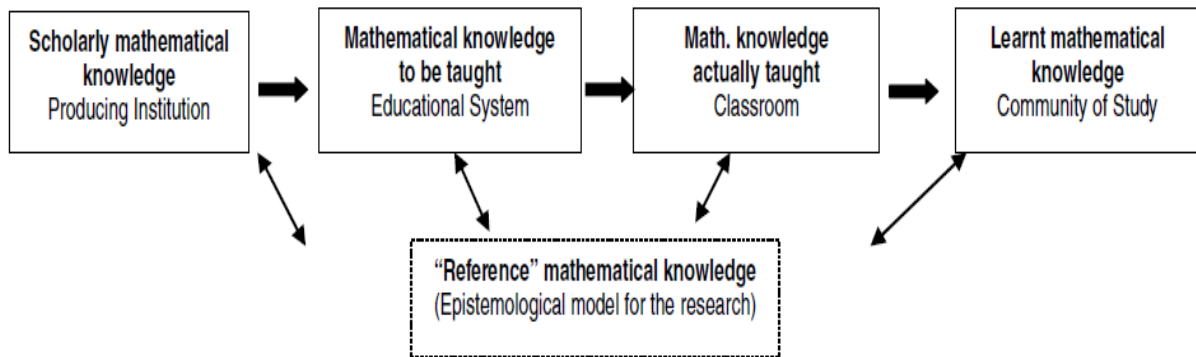
Figure 2.1 The process of didactic transposition

According to these authors, didactics of mathematics studies mathematical cognition in relation to the conditions that make the development of mathematical knowledge possible in social institutions (Bosch *et al*. 2005: 1256). They further explain that in school the mathematical praxeologies are often presented by syllabi and textbooks as disconnected, which hinders the mathematics the teacher teaches in the classroom. They also state that it is "rather impossible for the teacher to 'give meaning' to the mathematical praxeologies to be taught", because the rationale of concepts "cannot be integrated in the mathematical practice that is actually developed at this level" (Bosch *et al*. 2005: 1259).

Chevallard (1989: 6) makes it clear that the need for didactic transposition of knowledge arises from knowledge that should be used becoming knowledge that is to be taught. The distinction Chevallard considered is whether knowing something is the same as knowing how to do something? The knowledge that students are therefore taught is useful in another sphere of life. Chevallard explains that knowledge used in another sphere of life "will not survive the transitions from the specific social practice to the teaching institution" (1989: 8). Therefore, Chevelard tells us, didactic environment will have to be "rebuilt from scratch". Very often, this is not possible, so the school system teaches a version of knowledge that is manageable and can survive the school set up.

For the teacher to do justice to real mathematical thought in the classroom, he or she needs to go against the grain of institutionalised ideas and practices. Breen (1999: 118) says that the

more successful an innovative professional development program is, the more likely the teacher will "disrupt the school because they will be undermining the status quo - and the system will move to restore its equilibrium". This study will be sensitive to this (see 4.2.2 where no specific lessons are prescribed to teachers). Teachers are not islands in the educational system; they are responsible and accountable to many role players (students, parents, school management, administrators and government) in education.

From a modelling perspective where knowledge is transposed from a real world to a mathematical world, Cabassut (2009: 136) explains that "in the whole mathematisation cycle […] mathematical knowledge and techniques and extra-mathematical ones have to be transposed and interfere". The nature of modelling tasks will assist teachers in understanding the transposition involved in mathematical learning that emanates from problem-based learning. It is probable and possible that modelling will allow teachers to build contexts whereby knowledge can be known in the contexts where it is really used and that schools can therefore not simply provide society with a "common unauthorized version of knowledge" (Chevallard 1989: 9) but can be seen as meeting the needs of society.

A carefully considered professional development program should also take cognisance of the didactic transposition diagram to fully understand the more holistic view of mathematics knowledge and its teaching. This study is specifically concerned with the way in which mathematical knowledge is transposed in classrooms (see 2.4.4) and whether it can be offset through improved didactisation practices of teachers involved in the study so that the mathematics students' experience is more meaningful. A theory that embraces some of the features of the above theories, but also tried to account for both teachers and students interpretative activity (Ligozat & Schubauer-Leoni 2010: 1621), follows.

2.2.2.3 Joint-Action theory in didactics (JATD)

This theory uses the transposition process of the Anthropological theory of didactics (ATD) and the didactical contract of the Theory of didactical situations (TDS) as the "starting point of a *hybridizing plot*" (Ligozat & Schubauer-Leoni 2010: 1621, italics in original). These authors propose that the interpretation of classroom events cannot be seen only from a teacher or a student's actions. They proposed the concept of "joint-action" so that the interdependence of classroom actions is considered. These authors suggest acknowledging the classroom role-players' interpretations of the situations. This may lead researchers to reconsider the transposition of knowledge from a bottom-up view (2010: 1622). Sensevy (2010: 1652) describes JATD as focusing on the diffusion process or an *actional turn* of knowledge in a classroom while ATD and TDS focus on the nature of knowledge.

In terms of this study, the concept of joint action is important. This study therefore sets out to propose a teacher development program (see 1.3.2.1) that has mathematical understanding as pivotal and that this understanding should be generated through combined actions of teacher and students. The didactisation principles set out in this study permeate the actions of both teachers and students. The nine principles discussed later (see 2.4 and 3.2.1) comprise of six that relate to the actions of teaches and three relating to the actions of students. The nine principles however are co-ordinated principles and co-exist with each other. Most importantly, teachers should realise that they need to plan their responses and lessons based on students' responses and ideas. This means that teachers must probe and understand those responses. Students will need to be active for responses to be elicited from them in the classroom.

2.2.3 Problem-centered teaching and learning

The problem-centred paradigm referred to in this study is one conceptualised by Murray, Olivier and Human (1999: 35). They describe the approach as one whereby a teacher will pose problems to the students without the need for routine mechanised solving methods. Students have to construct solutions with the tools that they have available to them. Students then share their ideas, they discuss various methods and justify and explain to each other with the teacher facilitating the discussions. The teacher does not demonstrate solution methods,

nor does the teacher indicate a preferred method (Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier & Human 1997: 115).

This approach to teaching and learning is one whereby problem solving is a "vehicle for learning" (Murray, Olivier & Human 1998: 171). Gravemeijer (1994a: 92) explained that when teaching becomes problem-centered the aim *is* the problem and not the mathematical tools. He further explains that the aim is not to fit a problem to a "pre-designed system" but rather students need to find central relations in the problem. These early descriptions of those relations "can be sketchy" and students use "self-invented symbols". The translation and interpretation of the problem are easier according to Gravemeijer because the symbols are meaningful to the student. Gravemeijer further explains that "the description does not automatically answer the question, but simplifies the problem by describing relations and distinguishing matters of major and minor importance" (1994a: 93). Problem solving in this way allows students to develop mathematical concepts and mathematical language that become part of their formal mathematical knowledge (1994a: 94).

A problem-centred approach to teaching and learning is more challenging and demanding for teachers. It requires that teachers are confident to allow for varied thinking in students. Teachers need to organise and facilitate these learning episodes so that student learning is constructive and co-operative. The principles of problem-centred learning can be daunting for teachers to implement in their classrooms on a micro-level. What this study proposes (see 1.3.2.3) is to use modelling tasks which encapsulate a problem-centred approach but which allow teachers to experience problem-centred teaching in a holistic and integrated way (see 3.6) within much less time. One of the features of modelling problems with students is that the typical developmental cycles can be acted out within 60 to 90 min. (Lesh & Harel 2003: 161).

2.2.4 Modelling and a model-eliciting perspective to teaching

Modelling or model-eliciting activities (MEAs) (Lesh & Doerr 2003a: 3) is one of many problem-based learning activities. According to Doerr and Lesh (2011: 253) modelling activities "clearly satisfy even the most strongly specified definitions of problem-based learning". Modelling as an activity has been described as "an advance on existing classroom problem solving" (English & Sriraman 2010: 273) in five significant ways.

1. The quantities and operations needed to mathematise realistic situations go beyond traditional school mathematics.

2. Modelling offers a richer learning experience than word-problems. Students have to elicit their own mathematics to solve the problems.

3. The explicit use of real world problems from several disciplines are used.

4. Students have to develop generalisable models – they have to extend their thinking to the structure of the problem.

5. They are designed for small groups to work in as a "local community of practice" (English & Sriraman 2010: 273-274).

Modelling tasks are set in real-life complex situations. They require that groups of students work collaboratively over at least 45 minutes to solve the problem. In solving a modelling problem, students create a model for the situation that takes into account the complexity and relationships between elements of the problem. The models that students develop involve "sharable, manipulatable, modifiable , and reusable conceptual tools for constructing, describing, explaining, manipulating, predicting or controlling mathematically significant systems" (Lesh & Doerr 2003a: 3). It is specifically the process of modelling that creates an innovative learning environment. Lesh and Doerr (2003a: 5) emphasise that modelling activities can lead to "significant forms of learning". An example of a modelling task is presented that may assist in understanding some of the principles of modelling.

Quilts are often made from pictures that are found. This means that each type of shape must be cut from paper. Often quilters find beautiful pictures of patterns they would like to make and have to figure out the size of the pieces. The members of the Quilt Club often have difficulties converting photographs like these into templates that are exactly the right size and shape so that they too can make the quilts. **You have been asked to write a letter to the Quilt Club to explain to them how make pattern pieces (for the quilt on the next page) that are exactly the right size and shape. Also include in your letter the templates for each shape. (Note: the quilt is for a double bed so should be 200cm x 236cm when finished.) You will have to think about a 5mm edge for each piece so that it can be sewn.**	 Task from Lesh and Carmona (2003: 85). Additional information from: www.engineering.purdue.edu/ENE/Research/SGMN
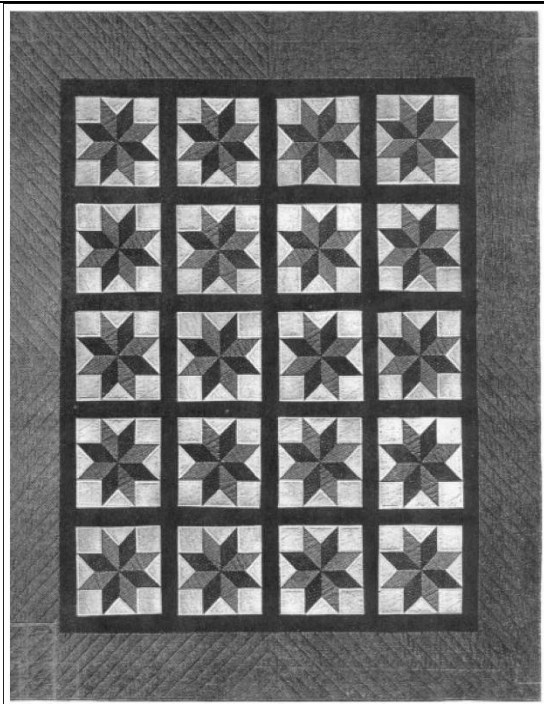
Table 2.2 Example of a modelling task

This modelling task will elicit model construction from students. To achieve this, student thinking will move through the phases of the modelling cycle (Fig 2.2). The following is a normative modelling cycle, which means that when students actually solve modelling tasks they may proceed through the cycle many times or move backwards and forwards through the cycle depending on their progress.
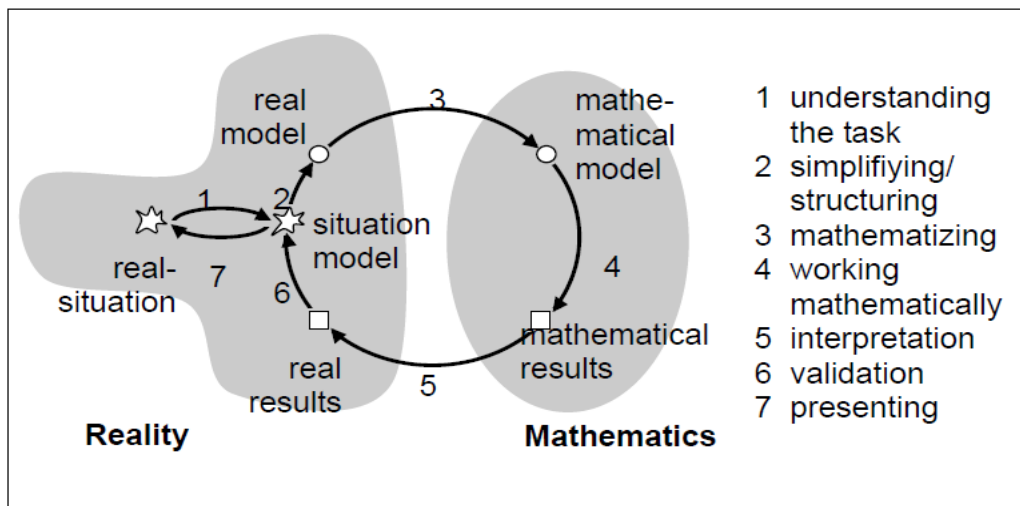
Figure 2.2 Modelling cycle (Blum & Leiss in Borromeo Ferri 2006: 87)

Studies that used the quilting task (Lesh & Carmona 2003: 85; Lesh & Harel 2003: 168; Biccard 2010: 104) explored student reasoning possibilities that were elicited. Lesh and Carmona found that student thinking about parts and wholes developed notably and that the final model was a rich integration of geometry, measurement and estimation. Lesh and Harel (2003: 175) explained that students move through several improved interpretations of their own thinking since modelling tasks allow students to express and test their own thinking and this enabled students to produce sophisticated ways of thinking about units of measurement (Lesh & Harel 2003: 177). Biccard (2010: 134) found that the visual representation, and students' closely connecting this to similar scaling up tasks in social studies and technology, assisted pupils in successfully solving this problem.

Stillman, Galbraith, Brown and Edwards (2007: 689) capture the essence of modelling by comparing the "direction" students work in when solving application problems i.e. (mathematics →reality) with modelling problems i.e. (reality →mathematics) where students work in the opposite "direction" Modelling problems also provide multiple entries into the problem, depending on the students' interpretation of what the solution should entail. Students will turn towards mathematics that they are comfortable using in solving these problems. This does place teachers in a different position to traditional teaching. In traditional teaching a teacher will 'funnel' students into adopting a prescribed method, while with modelling, the teacher needs to assist pupils reach their own solution from a number of possible entry points. Modelling tasks allow students to explore the 'mathematical terrain' in which the problem is embedded more so than with traditional instruction. Galbraith, Stillman and Brown (2006: 239) explain that a modelling or problem-orientated classroom involves variation, unpredictability, blockages and unexpected teacher and student actions. Teachers need to be prepared for this change in their work. This change is neither trivial nor easily assimilated into current traditional practices. The assimilation must be anticipated in terms of teacher knowledge, orientation and beliefs.

The modelling cycle also reflects a larger domain of problem solving than typical word problems. Most word problems end in the mathematical solution or a superficial interpretation back to the real situation. Modelling problems necessitate a deeper interpretation of the mathematical solutions. Niss (2013: 44) suggests the term "de-mathematisatization" for this

process whereby mathematical results are then interpreted against the real problem from where they emanated. Biccard (2010: 72) stated that a reverse mathematisation takes place when students interpret their solutions. Jablokna and Gellert (2007: 5) argue that often mathematical results do not match the problems from which they come from so "modified mathematisations" become necessary. The interpretation phase of the modelling cycle may provide many challenges for teachers and students alike as they are placed in different roles in a problem-centred classroom where interpretation is part of the mathematical terrain.

Modelling tasks meet the criteria elaborated on by Hiebert *et al*., (1997: 18-22) that tasks should "encourage reflection and communication"; "allow students to use tools"; and "leave important residue behind". Modelling tasks assist in producing mathematical understandings that are related and connected. These tasks allow a teacher to do the "mathematical work of teaching" and apply didactisation principles set out in this study to a very high degree. Notably, when students develop a model "many inferences can be made about the nature of their mathematical knowledge and development" (Lesh, Carmona & Post 2002: 1) which means that modelling is a particularly useful activity for teacher development. Schorr and Lesh (2003: 146) explain how interacting with and experiencing modelling tasks allows teachers to develop mathematical content knowledge and understand how students learn mathematics which results in more effective teaching. Schorr and Lesh also found that teachers' interpretations of student work changed enormously and moved in a direction that related to the "overall integrity and quality of their students' work" (2003: 151). In particular teachers became increasingly aware of the importance of students' using their own strategies and defending and justifying their solutions. Another critical change was that teacher "conceptualizations about what constituted mathematical ability changed" (2003: 154) through their modelling task experiences. Doerr and Lesh (2011: 267) remind us that when students are engaged in modelling activities it places new demands on teachers such as listening to students, responding with useful representations, hearing unexpected solutions and making connections to different mathematical ideas. These new demands are not easy to adapt to or anticipate. Teachers will need development opportunities and support to take on this new role and new challenge.

A major intersection in this study is how a modelling perspective supports didactisation principles and how modelling allows teachers to become more proficient in their mathematical work as teachers (see 3.6). This intersection lead out of a need to answer what Blum (2011: 24 italics in original) considered an "interesting open research question in which elements of *teachers' competencies* precisely are necessary and how these elements contribute to successful teaching". The didactisation teacher development program suggested by this study coincides with a possible answer to Blum's question. The power of modelling tasks for teacher development lies in the words of Lesh and Doerr (2003b: 556) in that modelling is needed for systemic change as it provides a "small number of 'leverage points' that significantly impact the most important systemic characteristics of the interacting systems we're trying to understand and influence".

2.2.5 Conclusion

From the above theoretical perspective it appears that the task of teaching in primary school mathematics classrooms is more complex than simply telling or showing or explaining mathematics to students. It requires from teachers that they prepare a mathematical terrain whereby the students construct mathematical ideas in an active and interactive way. This requires that teachers reflect and reorganise their current ways of thinking. It requires that teachers hand over the task of doing mathematics to the students. However this handing over needs to take place in a supported and guided manner so that students learn mathematics in a manner that lays foundations for more abstract concepts. Their mathematical foundations should not be based on rote learning of methods or procedures but rather on true understanding of the concepts. This takes place when students are given real problems, to which they can 'hook' their mathematical ideas. Then, while solving the real problem, they are able to understand how the mathematics inherent in the problem works. In doing so, students may experience difficulties and frustration, but the teacher should be equipped to anticipate and guide students without prescribing a set method. Furthermore the teacher will need to probe student thinking and connect their informal ideas to more formals ones. The teacher will also have to connect different ideas regarding the problem solution to each other. This will enable students to gain a much better picture of the mathematical landscape. It is the aim of this study (see 1.3.1.2) to equip teachers with the knowledge and skills through the

didactisation practices proposed in the professional development program to develop their teaching proficiency (see 5.3.7; 5.4.7; 5.5.8).

2.3 THREE PILLARS OF TEACHER DECISION MAKING

Schoenfeld's (2011:10) theory of teacher decision making is centered on three interdependent concepts of: teacher resources, goals and orientations. He also proposed that these notions provide a "lens through which teacher activity can be examined" and that research on professional development that documents a teacher's evolution along these ideas is a "rich theoretical arena for future investigation" (2011: 194). Torner, Rolka, Rosken and Sriraman (2010: 403) followed Schoenfeld's earlier conceptualisation named the theory of teaching-in-context since these three pillars minimise the variables in teaching processes so that the most significant ones can be identified. Effective teacher development necessitates understanding how and why teachers make micro decisions daily. Schoenfeld's theory provides pillars on which to structure an understanding of changing teachers' decision making.

2.3.1 Teacher resources: the case of mathematics teacher knowledge

Schoenfeld (2011: 25) focuses specifically on knowledge when discussing teacher resources. He defines knowledge as information that the teacher has available that may be involved in problem solving or in pursuing goals. He proposes some essential claims about knowledge (2011: 27). Firstly, knowledge matters in problem solving. Teacher problem solving is solving problems about the challenges involved in teaching. Secondly: knowledge is association and it comes in "packages". Thirdly: memory is associative. Things that belong together are remembered together. Fourthly: knowledge gets activated and accessed in ways that entail related actions associations. Fifthly: knowledge structures are connected, generative and regenerative. Knowledge is neither static nor absolute. In short, Schoenfeld defines knowledge as a "very precious intellectual resource" (2011: 29).

In following Schoenfeld's ideas of teacher decision making (Schoenfeld 2011: 10), knowledge can be seen as a teacher resource that influences teacher decision making. He elaborates on knowledge as being the intellectual, material and contextual resources available to the teacher. Petersen (2005: 2) also proposes that teaching is based on decisions that

consider content, students and professional knowledge. Teacher knowledge is a complex and multifaceted domain. It includes knowledge of teaching gained through formal pre-service education and in-service development programs. It also includes knowledge gained through professional experience. This means that teacher knowledge cannot be revealed by a pen and paper test, nor can it be written down as a list. What Schoenfeld also makes clear is that the knowledge a teacher has is not necessarily "correct", but that the knowledge a teacher *does have* generates a particular decision or response. Therefore, for this study, is it considered vital to develop teacher knowledge so as to shape a change in the teacher's actions and responses in a classroom. As stated by Blum and Krauss (2006: 1) many questions about the content of teacher knowledge, its structure, or on the way in which it influences teaching and learning, are still empirically unanswered. This study aims at (see 1.3.2.5) developing teacher knowledge and developing teacher proficiency.

2.3.1.1 Dimensions of teacher knowledge

What knowledge domains are significant for mathematics teaching? How does the knowledge of more proficient teachers differ? The various facets of teacher knowledge are explored in this section. Doerr and Lesh (2003: 128) make distinctions in the nature of expert knowledge in that it is characterised by the following types of complexities and variabilities.

a) Knowledge is pluralistic: This means that a student may have many ways of thinking about a problem and different students will think about the problem in different way and ways. This means that teachers should have knowledge about the multiplicity of ways in which students will think about a particular problem and how they will build on these ways of thinking.

b) Knowledge is multidimensional: The authors explain multidimensionality of knowledge as meanings and explaining that evolve along dimensions. For example, concrete to abstract; simple to complex; external to internal; sequential to simultaneous; discrete to continuous; particular to general and static to dynamic. A teacher should have the ability to identify different knowledge dimensions and be able to assist the student in developing each knowledge dimension.

c) Knowledge is variable: Students may be thinking about a problem situation along any of the above dimensions outlined above. Students can be at different levels along various dimensions of knowledge when solving a problem. This compounds what the teacher will deal with in a classroom. The teacher should be able to identify the student knowledge dimensions and develop and evolve student thinking.

d) Knowledge is contextual: Doerr and Lesh (2003: 129) remind us that concepts such as proportionality are first mastered in small and restricted types of problem and then extend gradually to larger classes of problems. Teachers should therefore not at the outset aim for students to have general problem solving competencies but should allow competencies to develop out of contextual situations.

e) Knowledge is continual: Doerr and Lesh (2003: 129) also remind us that conceptual development is characterised by a gradual increase in competence, and not by a general, all-purpose cognitive structure. Essentially teachers should have an understanding of conceptual development as knowing and as knowledge. This means that teachers need knowledge of how student knowledge manifests itself and how student knowledge grows.

These distinctions of the characteristics of knowledge illuminate the complexity that is teacher knowledge. It is this variability and flexibility in knowledge that this study aims to develop (see 1.3.2.5) through didactisation principles. It is through teachers understanding of these principles that teacher knowledge, teacher didactisation and teacher proficiency may develop (see 5.2.9; 5.3.8; 5.4.8; 5.5.8).

According to Ball, Thames and Phelps (2008: 396) teachers need to know *more* and *different* mathematics while Schoenfeld and Kilpatrick (2008: 322) confirm that proficient teachers' knowledge is both broad and deep while Hill and Ball (2009: 70) refer to "horizon knowledge". Teacher knowledge needs to be broad in that teachers should conceptualise the teaching content in multiple ways, and represent it in a variety of ways. Teaching knowledge should be deep in that teachers should know the curricular origins and direction of the content. So, mathematical knowledge for teaching (MKT) (Thompson & Thompson 1996: 2) is strongly related to the quality of instruction and cannot be based on a teacher's

mathematical knowledge alone. It may also affect teachers' use of explanation, representation and the teacher's responsiveness to students' mathematical ideas (Hill, Blunk, Charalambous, Lewis, Phelps, Sleep & Ball 2008: 437). In a study proposing a framework for understanding the transformation of MKT, Silverman and Thompson (2008: 502) saw MKT as a transformation of conceptual understandings that had "pedagogical potential" to having "pedagogical power". Furthermore they believed that the answer to this transformation could be answered by applying Piaget's notion of reflecting abstraction (see 2.4.4.2) to teacher learning. When applied to teacher development, it means that the teacher must place her or himself in the place of the student and try to understand the operations and concepts from that perspective and ask how s/he can assist the student in thinking about the concept as s/he does (Silverman & Thompson 2008: 508). This study aims at transforming teachers' pedagogical potential to pedagogical power through developing teachers' didactisation practices (see 1.3.2.4).

From a modelling perspective, teacher knowledge takes yet another dimension in that teachers have to anticipate multiple representations and multiple solutions to the problems presented. In the professional development program set out in this study (see 4.2.2), teachers will first have to experience and develop these multiple representations and solutions. The knowledge gained from these experiences should springboard teachers into acknowledging and co-ordinating this type of experience in their own mathematics classrooms. Modelling tasks therefore allow teachers to develop alternative knowledge domains which are essential to teaching effective problem solving, and to teach students to solve problems. In all modelling tasks, teachers have to deal with the context, the content, multiple solutions and multiple representations. Treilibs, Burkhardt and Low (1980: 60) warn that a "critical element" in teaching modelling is a "willingness to follow the pupil down whatever tracks his thought processes may lead" and that this requires a change in the teacher's role and personal confidence. Although they advocate that this can be built through experience, it is the proposal of this study (see1.3.2.7) to accelerate this by increasing teacher knowledge and experience in didactisation practices through a design research teaching experiment (see 4.2.2).

2.3.1.2 Pedagogical content knowledge (PCK)

Pedagogical content knowledge first defined and described by Shulman allowed researchers to focus on an elusive form of teaching and teacher knowledge. Shulman identified some major categories of teacher knowledge:

- General pedagogical knowledge

- Knowledge of learners

- Knowledge of educational contexts

- Knowledge of educational ends

- Content knowledge

- Curriculum knowledge

- Pedagogical content knowledge (Shulman 1987: 8)

Pedagogical content knowledge is defined as a "particular form of content knowledge that embodies the aspects of content most germane to its teachability" (Shulman 1986: 9). Shulman includes in his definition of PCK the importance of representations. In his words –

> the teacher must have at hand a veritable armamentarium of alternative forms of representation, some of which derive from research whereas others originate in the wisdom of practice.

This prominence given to forms of representations is comparable to a modelling approach. Hill, Ball and Schilling (2008: 377) further demarcate PCK into three domains – knowledge of content and student (KCS); knowledge of content and teaching (KCT); and knowledge of the curriculum, while Rowland, Huckstep and Thwaites (2005: 259) used the term "knowledge quartet" for understanding teacher knowledge. The quartet is made up of foundational knowledge, transformational knowledge, connection and contingency. Manizade and Mason (2011: 187) constructed four components of PCK from an existing literature. The four components based on their definition for PCK are:

- Knowledge of connections among big mathematical ideas.
- Knowledge of learning theories describing students' developmental capabilities.
- Knowledge of students' common misconceptions and subject specific difficulties
- Knowledge of useful representations and appropriate instructional techniques for the teaching of the content.

This study will incorporate these aspects and components of PCK as they become integrated into a framework that describes and tracks teacher didactisation practices (see 5.2 - 5.5) of which PCK and its definitions and dimensions form an integral part. This study accepts the proposal of Doerr and Lesh (2003: 131) that how PCK develops needs to be documented and explained. This study will specifically focus on how knowledge and experience of didactisation principles will affect mathematics teachers' teaching proficiency.

2.3.1.2. (a) Knowledge of content and students (KCS)

Ball *et al*. (2008: 401) describe KCS as the intersection between "knowing students" with "knowing about mathematics". This relies on considerable anticipation on the part of the teacher as to what students may find confusing or motivating. This anticipation is therefore drawn out of reflection from the teacher. KCS as the intersection of knowing content and knowing students means teacher knowledge of how students learn mathematics.

Cognitively Guided Instruction (CGI) as a teacher development program epitomises what it means for teachers to know and use student thinking. The focus of CGI is to build on existing teacher knowledge by offering "opportunities for teachers to build on their existing ideas to create continually evolving organizing framework of children's mathematical thinking" (Carpenter, Fennema, Franke, Levi & Empson 2000: 2). This study incorporates some of the fundamentals of CGI to develop a teacher program (see 3.3) that improves teacher knowledge of didactisation principles in an effort to improve teaching practice. The thesis for this study (see 1.3.2.5) is that teachers' improved knowledge will start a cycle of improved inquiry and decision making that will guide teachers' actions (see 5.2 - 5.5). The authors of CGI did document the levels of teacher engagement with children's mathematical thinking levels through the CGI program. They assigned and described the following levels to show change in teacher actions through the growth of teacher knowledge of student understandings.

Level 1: The teacher does not believe that the students in his or her classroom can solve problems unless they have been taught how. Does not provide opportunities for solving problems. Does not ask the children how they solved problems. Does not use children's mathematical thinking in making instructional decisions.

Level 2: A shift occurs as the teachers begin to view children as bringing mathematical knowledge to learning situations. Believes that children can solve problems without being explicitly taught a strategy. Talks about the value of a variety of solutions and expands the types of problems they use. Is inconsistent in beliefs and practices related to showing children how to solve problems. Issues other than children's thinking drive the selection of problems and activities.

Level 3: The teacher believes it is beneficial for children to solve problems in their own ways because their own ways make more sense to them and the teachers want the children to understand what they are doing. Provides a variety of different problems for children to solve. Provides an opportunity for the children to discuss their solutions. Listens to the children talk about their thinking.

Level 4A: The teacher believes that children's mathematical thinking should determine the evolution of the curriculum and the ways in which the teachers individually interact with the students. Provides opportunities for children to solve problems and elicits their thinking. Describes in detail individual children's mathematical thinking. Uses knowledge of thinking of children as a group to make instructional decisions.

Level 4B: The teacher knows how what an individual child knows fits in with how children's mathematical understanding develops. Creates opportunities to build on children's mathematical thinking. Describes in detail individual children's mathematical thinking. Uses what he or she learns about individual students' mathematical thinking to drive instruction.

Table 2.3 Levels of engagement with children's mathematical thinking (Franke, Carpenter, Levi & Fennema 2001: 662)

The description of these levels allows one to track changes in teacher knowledge, orientations and goals. It is useful as a broad guide that can be used in any classroom context. CGI's starting point was to provide knowledge about the development of student's mathematical thinking and then "letting the teachers decide how to make use of that knowledge in the context of their own teaching practice" (Franke & Kazemi 2001: 102). These authors explain

that in this way CGI is more a philosophy than a recipe. Understanding student thinking is important to successful mathematics lessons teachers need to incorporate these ideas in their into their own thinking schemas.

2.3.1.2. (b) Knowledge of content and teaching (KCT)

This subset of teacher and teaching knowledge involves knowing the mathematical content that needs to be covered from a teaching perspective. Possibly Ball and Bass' (2000: 95) description of "pedagogically functional mathematical knowledge", which they contest is essential to effective teaching, is a relevant term here. They explain that when students' produce multiple approaches to problems, there is a "profound mathematical imperative to inspect, analyze and reconcile them" (2000: 96). It is when teacher inspection, analysis and reconciliation of students approaches takes place that knowledge of content and teaching becomes increasingly complex and demanding. Ball and Bass (2000: 99) also maintain that being able to use mathematical knowledge in teaching means that mathematical knowledge must be used sensibly when considering subtle pedagogical questions. There needs to be a careful consideration of the content to be taught and of the metaphors and representations used, together with knowledge of how students will interpret these (2000: 99).

Ball *et al.* (2008: 402) also state that understanding the impact that certain representations and language have on the development of student understanding is a very important part of KCT. Another important aspect of KCT set out by these authors is teacher decisions about what to do about student difficulties. The didactisation framework set out in this study (see 2.5) endeavours to prepare teachers to anticipate and deal with student difficulties as they arise in teaching and to more closely align knowledge of mathematical content with knowledge of teaching so that the best of both worlds can become relevant in mathematics classrooms. KCS and KCT appear to be knowledge based on experience and practice. The categories in the knowledge quartet contain knowledge aspects that teachers can learn through the type of pre-service and in-service training.

2.3.1.2. (c) Foundational knowledge

This refers to the academic training a teacher has undergone to prepare for teaching in a classroom (Rowland *et al*. 2005: 260). These authors take the view that foundational knowledge has potential to guide teachers' decisions in an important way. Included in this category of knowledge are: knowledge and understanding of mathematics itself and beliefs about the nature of mathematical knowledge, the purpose of mathematics education and how mathematics is learnt (Rowland *et al*. 2005: 261). In this respect the authors explain that this knowledge is made up of what teachers learn in their personal education and in their training. The authors further include procedural and instrumental understanding of mathematical topics as well as the use of mathematical vocabulary as foundational knowledge. Foundational knowledge can therefore be seen as everything the teacher knows about his/her role in the classroom and everything he/she knows about the mathematics that has to be taught. The professional development program developed by this study includes a biographical section (see Appendix 5) to assist in developing a profile of a teacher's foundational knowledge.

2.3.1.2. (d) Transformational knowledge

This is the knowledge needed to make the subject available to students. Rowland *et al*. (2005: 262) use Shulman and Ball's conceptions of transformation to define this category as the teacher's "behaviour that is directed towards a pupil (or a group of pupils)". This is how the teacher decides to present or develop a lesson. The use of representations, teacher explanations, examples and demonstrations form part of a teachers' transformational knowledge (Rowland *et al*. 2005: 265). They confirm that transformational knowledge is informed by foundational knowledge.

2.3.1.2. (e) Connection knowledge

Rowland *et al*. (2005: 262) explain connection as the coherence of the lesson and the sequencing of topics within and between lessons. This co-incides with the didactisation principle of vertically planned or aligned lessons (see 2.4.5). Ensuring that students make connections between mathematical procedures and mathematical concepts require what Rowland *et al*. (2005: 263) have described as knowledge of "structural connections" and an

awareness of the cognitive difficulty involved. According to Stein, Engle, Smith and Hughes (2008: 330) making connections means making a "judgment about the consequences of different approaches". They also maintain that it is important to show students how the same ideas can be "embedded" in different strategies that may look quite dissimilar at first. Treffers (1987: 62) also explained "differentiation" as each student judging his/her solution against those of others. The difference between the considerations of Stein *et al*. and Treffers is that Stein *et al*. feel that the important part is not just that each student understand his own approach, but that all students understand how all the approaches related, while Treffers has each pupil work at his or her own level and a summary is made by the teacher at the end of the lesson. It would appear that each student would make some connection to other methods, but to expect all students to relate all methods may be improbable. Freudenthal adds to this an important distinction that teaching should not be differentiated in advance by the teacher, but rather that the students differentiate themselves on levels that are accessible to them. He called this "spontaneous differentiation" as opposed to "imposed differentiation" (Frendenthal 1991: 117). It therefore appears important that a teacher is prepared for the different levels of thought that may occur during a lesson, as well as be prepared to accommodate other levels that he/she has not necessarily planned for. It is this unplanned mental activity that may prove most difficult for teachers.

Guo and Pang (2011: 4) proposed variation theory that explains that for students to learn a concept they must experience variation. This variation is defined by four patterns: contrast, separation, fusion and generalisation. Contrast means that students experience different aspects of the concept. Separation occurs when students focus on one aspect of the concept. Fusion takes place when students discern several aspects of the concept that vary simultaneously. Generalisation occurs when students fully experiences the concept through many examples. Kant (in Radford 2010: 2 italics in Radford) contended that generalisation is both being able to synthesise *resemblances* between *different* things and also *differences* between *resembling* things. Connections therefore represent a fundamental way of understanding how students can tell different things from similar things.

Variation theory is an important concept as it builds on what it means for teachers to make connections. It allows one to view the task of making connections from a number of vantages.

This means that it will be easier for a teacher to facilitate making connections because it is not presented as a narrow linear concept. Making connections part of teacher knowledge will affect the type of connections that are made within the mathematics classroom. Making connections is considered to be a fundamental aspect to the mathematical work of teaching and is therefore included in the didactisation principles for this study (see 3.2.1.3).

2.3.1.2. (f) Contingency knowledge

Rowland *et al*. (2005: 263) describe this knowledge as the ability to think on one's feet and the ability to take contingent action. This refers to responding to students' ideas as they occur during a lesson. Brown and Wragg (in Rowland *et al.* 2005: 263) made an interesting observation that the ability to listen diminishes with anxiety. A teacher's lack of confidence in his/her own mathematical ability may help us understand why teachers prefer to teach to a script where student responses are controlled to some degree. The more knowledgeable and confident the teacher is, the more likely the teacher will deal with off script responses as opportunities to understand student learning and to promote student learning. It is therefore considered important in this study that teachers experience the same type of tasks and learning principles as students themselves (see 3.3.1.3). This will enable teachers to develop confidence and therefore develop contingency knowledge.

2.3.1.3 Summary

Conceptions of teacher knowledge are vast. Many authors have foreseen that this component of a teacher's resources is significant. Didactisation principles as set out in this study encompass many of these descriptions of teacher knowledge in all its hues and nuances. The nine principles (active students, differentiation, mathematisation, vertically aligned lessons, accessing, probing, connecting, assessing and reflecting) set out as didactisation principles in this study correspond to the various knowledge domains described above. They are however not a repetition of the above knowledge domains. They are in a format that is accessible to both teachers and researchers of mathematics teacher education. The didactisation principles proposed in this study may provide teachers with the bridge from theoretical knowledge conceptions to practical understanding and implementation of these conceptions. The nine principles provide researchers with the vocabulary that may make it possible to discuss

teacher knowledge in a more productive way. In this respect this study is significant for mathematics teacher development programs on a small or large scale. It is predicted that when teachers implement the didactisation principles they will be building their teacher knowledge base substantially (see 5.2.9; 5.3.8; 5.4.8; 5.5.8). Teachers will be able to build PCK through meaningful practical and reflected-on experiences.

2.3.2 Teacher goals

According to Schoenfeld (2011: 21) almost all human behaviour is goal-orientated. A teacher responds and reacts in certain ways in order to achieve certain goals and they will use their resources during goal-orientated activities (2011: 25). Knowledge, goals and orientations are artificially divided in this section so as to focus on each one, but in reality they work together. Torner, Rolka, Rösken and Sriraman (2010: 409) found in the comments of a teacher during an interview that "goals and beliefs can hardly be separated" and that they present a "complex network or dependencies".

Schoenfeld characterised three goal types: overarching goals, major instructional goals and local goals.

Teacher goals are overt – such as wanting to cover a particular unit of work in a single lesson or more covert – such as their overall goals for their students over the year. Here goals may include student achievement, student enjoyment and teacher satisfaction. Teacher goals are inextricably linked to teacher knowledge and teacher pedagogical skills. A teacher may have a specific trajectory in mind that is based on his/her specific knowledge about the mathematical concept. The teacher may have the exposition of a certain pedagogical skill (e.g. groupwork) as a goal if this is part of the teacher's knowledge base. What is important when considering teacher goals is that they impact on a teacher's decision making and that the teacher will take certain actions in pursuit of these goals. Often the goals are not explicitly stated because they belong to the affective realm and are set in a greater network of the teacher's orientations.

It is clear that teacher goals and knowledge are interwoven. So, a change in one may effect a change in the other which will affect teacher decision making. Following this assumption, increasing teacher knowledge through knowledge of modelling and knowledge of didactisation principles is likely to change teacher goals for student activities and for student

learning. Carpenter, Fennema, Franke, Levi and Empson (2000: 1) found that through the CGI program, if teachers understood the development of children's mathematical thinking this could lead to "fundamental changes in teachers' beliefs and practices". This study proposes to adopt some of the CGI principles but to also include clear guidelines as to teacher activity (see 3.2.1) in a classroom to bring about a change in teachers' beliefs about mathematics teaching and learning (see 5.2.9; 5.3.8; 5.4.8; 5.5.8).

2.3.3 Teacher orientations

Schoenfeld (2011: 15) defines orientations "as an inclusive term to encompass beliefs, dispositions, values, tastes and preferences". So a teacher's orientations includes what he/she perceives teaching to be, how mathematics should be taught, what students should do that would construe *doing* mathematics and what he/she would do to be *teaching* mathematics. Orientations are therefore heavily dependent and interdependent on teacher goals and knowledge. Cobb (1986: 4) determined that activities that are carried out to achieve goals are "expressions of beliefs" and that these beliefs can be said to be "assumptions about the nature of reality that underlie goal-orientated activity".

Ernest (1994: 339) explains that the

> didactical consequences of prescriptive philosophies, such as Logicism and Formalism, follow from their identification of mathematics with rigid and logically structured theories…of mathematics as an objective, absolute, incorrigible body of knowledge.

Although teachers may not have overt philosophical ideas of mathematics, if a philosophy is understood as "an attitude that guides ones behaviour" (Concise Oxford English Dictionary) then even underlying philosophical notions of mathematics are pertinent in teacher decision making and inform teacher action. Ernest also warns of another didactic consequence of an absolutist paradigm being a transmission approach to teaching mathematics as is common in more traditional classrooms.

In explaining how student orientations influence their problem solving, Schoenfeld (2011: 30) found that orientations:

- Shaped the way students interpret the task;
- Shaped the goals that were established for the task; and

- Shaped the knowledge they did and did not use in solving the problem.

It must be surmised that since teachers are solving problems and making decisions while teaching, that teachers' orientations also shape the way they interpret the task of teaching, the goals that they establish for teaching and the knowledge that they use and do not use in solving teaching problems or decisions. Van der Sandt (2007: 345) generated a research framework on teacher behaviour. She condensed existing literature on teacher beliefs and set out the following categories.

1. Teacher beliefs about the nature of mathematics

- Problem solving view – teaching is driven by dynamic problems.
- Instrumentalist view – teaching presents mathematics as an unrelated collection of facts, rules etc.
- Platonist view – teaching presents mathematics as a system of interconnecting structures that have to be discovered.

2. Teacher beliefs about mathematics teaching

- Learner focused – focused on the learner's construction of knowledge.
- Content focused (emphasis on conceptual understanding) – teaching is driven by the content itself.
- Content focused (emphasis on performance) – teaching is driven by making sure learners master procedures and algorithms.
- Classroom focused – a well structured and well organised classroom is the focus.

Van der Sandt therefore links teacher beliefs to the type of activities present in a classroom. More so, she presents a link between beliefs about a problem solving paradigm and learner-centred classrooms. She presents the pinnacle of a teaching pyramid to be learner-centred and problem-centred – however these two are congruent.

Focusing on teacher beliefs is an important aspect when considering teacher development. Pajares (1992: 314) reminds us that all teachers hold beliefs about their work, students, and mathematics. As stated by Pajares (1992: 309) on the topic of existential presumptions, "people believe them because, like Mount Everest, they are there". Nespor (in Pajares 1992: 311) argues that belief systems are more inflexible and less dynamic than knowledge systems and that when they change it is neither via argumentation nor reasoning but rather a

"conversion or gestalt shift". Lasley (in Pajares 1992: 316) also maintains that beliefs are of an enduring nature, unless "deliberately challenged". A professional development program should therefore not only focus on teacher beliefs, but make teachers "aware of their own beliefs" (van der Sandt 2007: 349). It is through this awareness, reflection and challenge that change in teacher orientations and beliefs may occur.

Changing teacher beliefs is difficult as explained by Pajares (1992: 317). He maintains that the earlier a belief is incorporated into a belief structure the stronger it is. So, newly acquired beliefs are, according to him, most vulnerable. They need time and use to become "robust". This may explain why professional development programs of longer duration are more successful. It also explains that a teacher's understanding of the "work of teaching" is formed through many years of observing teaching as students themselves.

It would appear that reconstructing the cognitive-affective filter will require tasks and activities to challenge teachers' assumptions and beliefs about their role as teachers, the nature of mathematical knowledge and how mathematics is best learned. It would appear pointless to simply tell teachers about these things since their filter would block building of different belief models. It is the filter itself that needs to be changed.

Pennington in a study of language teachers (1995: 705) described teacher change as behavioural and perceptual which she said made it attitudinal and cognitive. Freeman (in Pennington 1995: 705) explains that "teaching is the integration of thought and action" and that a key ingredient to teacher change is *awareness* or at least the teacher wanting to experiment with available alternatives. It is further explained that the teacher's awareness and knowledge of alternatives is "colored by that teacher's experience and philosophy of teaching, which act as a psychological barrier, frame or selective filtering mechanism" (Pennington 1995:705).

The filter image which is shown in Figure 2.3, shows how input from professional development that does get past the teachers' cognitive-affective filter (which is determined by the teachers cultural and personal values or beliefs) can enter the change cycle and be taken into teacher practice. The teacher then reflects on the intake and it is processed at increasingly

deeper levels. These ideas that have been reflected on then become uptake in the teacher's individual value system and classroom behaviour (Pennington 1995: 722).

This study follows a design research framework (see 4.2) in order to create the type of change cycle described by Pennington. The cycles of experiencing modelling tasks, student observation, discussion sessions and reflection (see 4.2.2) set out in this study will enable a reconstruction of a teacher's cognitive-affective filter and may therefore allow teachers to "take up" (Adler 2005: 173) the didactisation principles in their teaching knowledge framework.
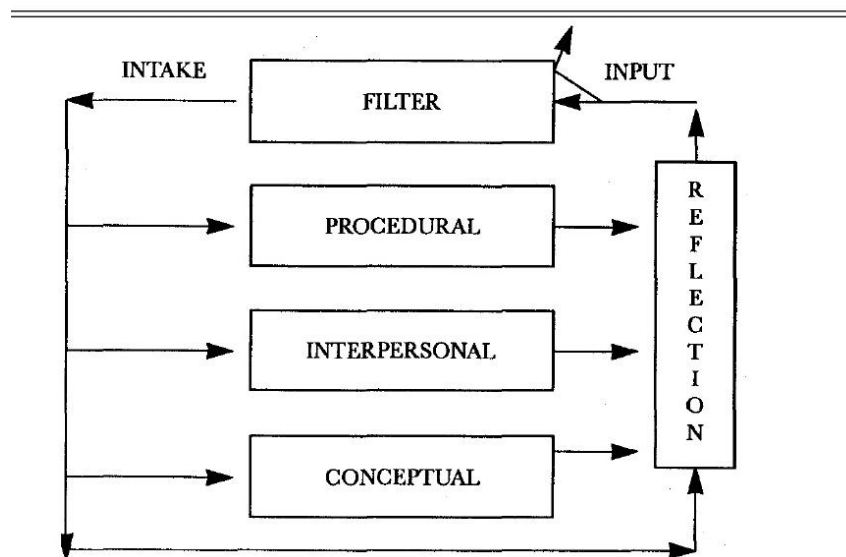
Figure 2.3: Pennington's Teacher Change Cycle (Pennington 1995: 722)

The question that may arise is, if changing a teacher's classrooms actions will bring about a change in his/her beliefs, or, if changing teacher beliefs will bring about a change in his/her classrooms actions? Guskey's model (1986: 7) of teacher change advocates that changes in teachers' classroom practices lead to changes in student learning which ultimately result in a change in teachers' attitudes and beliefs. Perhaps the sentiment of Phillip (in McDonough, Clarkson & Scott 2010: 397) that it is more important to support teachers to change their beliefs and actions jointly than worry about which comes first is helpful. It means that Pennington's beliefs as a filter is a powerful image. The cognitive-affective filter can be reconstructed or opened by reflective thinking (Pennington 1996: 348). Smith and Southerland (2007: 399) explain that teachers assimilate modified messages via filtration

"rather than revise their preexisting beliefs". The filter image is also presented by Cohen and Ball (1990: 238) where they explain how new reform methods are "filtered through an older and much more traditional mathematical and pedagogical structure". Battista (1994: 465) said that teacher beliefs "blocked their understanding and acceptance of the philosophy of the reform movement". Pennington (1996: 348) also suggests that since the filter can inhibit change in teachers, part of a teacher development program should bring "value clashes" to the surface to challenge the filter and coerce it to open up. Smith and Southerland (2007: 399) also maintain that teachers will not change their beliefs unless they are sufficiently challenged.

Another teacher orientation factor that encourages traditional transmission type teaching is a concept of "defensive teaching" defined by McNeil (1986: 157). She found through an extensive study that issues around classroom control were a determining factor in classrooms. Her study was conducted in social science classrooms but the findings are equally relevant to mathematics classrooms. She maintains that teachers have reduced authority and control in what happened in classrooms and this encouraged them to simplify content or present content with little reference to student experiences in an endeavour to create as little resistance as possible in order to "get through the day" (McNeil 1986: 160). Changing the type of students that were admitted to schools, changing the way the school tracked students and their own ability to affect student learning, were examples of what teachers perceived as a loss of control. She described classroom control as a "negotiation of efficiencies" in that teachers weighed up the "smallness of their financial rewards and professional incentive in relation to the potential for classroom disorder, dissent and conflict" (McNeil 1986: 160). This caused teachers not to present their own knowledge of the topic dealt with in the classroom. McNeil did indicate that ironically when "students see minimal teaching, they respond with minimal effort" (McNeil 1986: 161). It therefore appears that issues of teacher beliefs and teacher loss of control are determining factors in teachers remaining within a traditional teaching paradigm.

2.3.4 Teacher decision making – summary

Remillard and Bryans (2004: 353) remind us that if we wish teachers to teach differently, then the teachers themselves need opportunities to learn mathematics in new ways so that they will

consider new ideas about teaching and learning. These authors also reiterate that when teachers were faced with and had to interpret students' work on unfamiliar mathematical tasks – their ideas were challenged and altered. (2004: 355).

At the end of the day, as summarised by Hargreaves (1994: ix) "It is what teachers think, what teachers believe, and what teachers do at the level of the classroom that ultimately shapes the kind of learning that young people get."

Schoenfeld's theory on teacher decision making makes it clear that professional development needs to address three components:

- What teachers believe – since teacher orientations affect teacher goals
- What teachers think – since teacher resources determine teacher goals and will ultimately determine -
- What teachers do – which in due course influences what and how students learn.

A visual representation of Schoenfeld's theory where the relationship between teacher beliefs, knowledge and actions and student beliefs, knowledge and actions is interlinked and can be visualised as a filtering process that may change the cycle of unfavorable beliefs about mathematics.
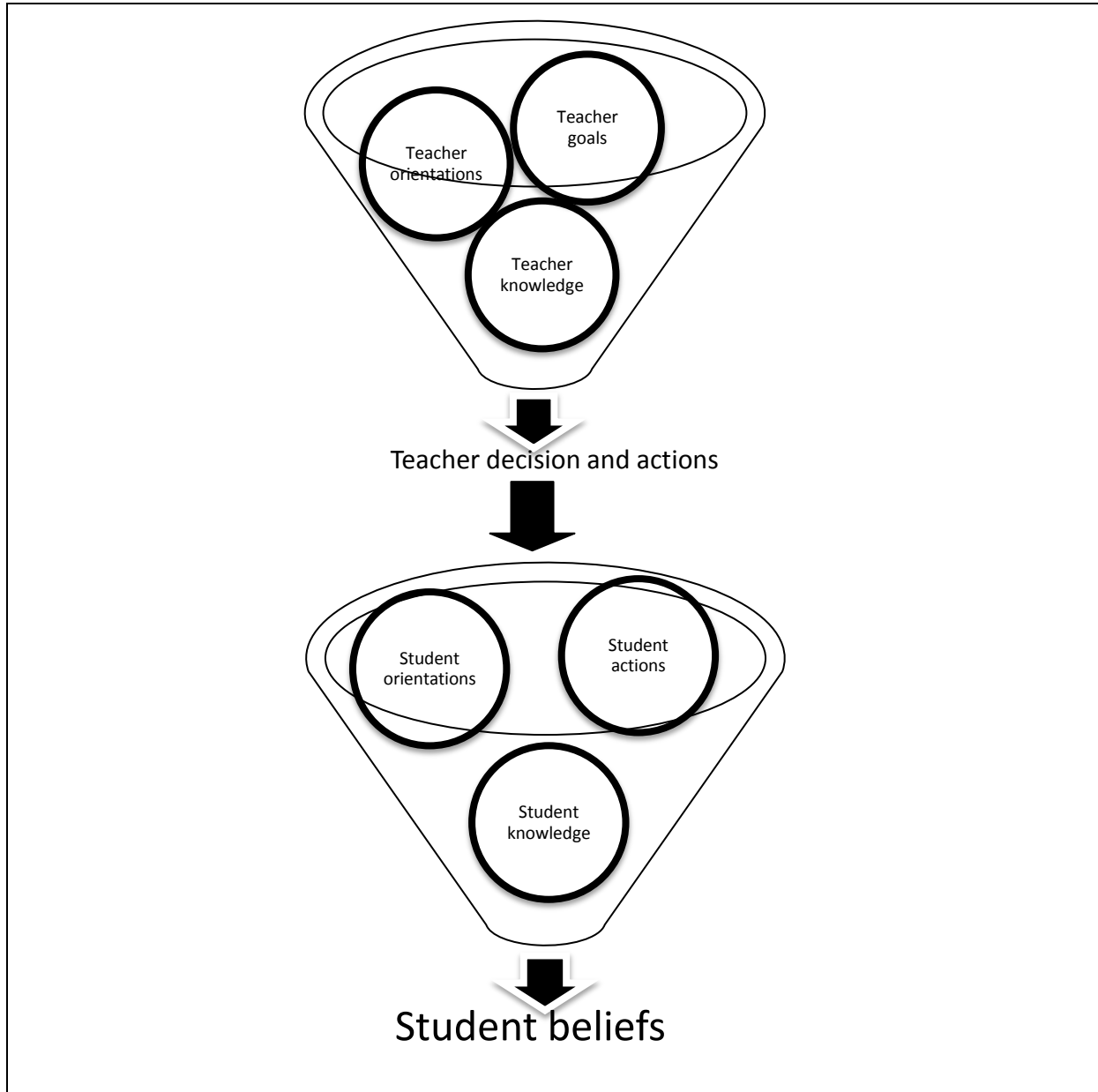
Figure 2.4: Teacher decision making influence on students

In the same vein as Schoenfeld, Leatham (2006: 92) describes teachers' beliefs as a "sensible system". He explains teachers' actions that sometimes appear inconsistent as inherently sensible if one understands what goal or what belief is strongest at the time. He found that when inconsistencies in teachers' beliefs or behaviours have arisen it is because the teacher has made a "sensible" decision based on an underlying belief. He argues that teachers "have reasons for the many decisions they make" (2006: 100). For mathematics education research to weld theory and practice together, an appreciation of the many micro-moments that need to be managed by a teacher is necessary. There are a number of goals (individual understanding,

covering of content, building student abstract thinking, classroom management etc) that a teacher needs to reach within any thirty minute session, and often a trade-off is necessary so that various goals can be attended to. How can teachers be assisted in establishing, prioritizing and reaching these goals? This study aims at developing teachers so that a number of these goals can be explored through didactisation practices (see 1.3.1.2).

2.4 DIDACTISATION

Treffers defines didactisation as "the essence of didactical action which makes mathematisation possible" (1987: 58). He also describes four characteristics that a learning process will have as a result of didactisation. They are: activity; differentiation; mathematisation; and vertically planned lessons. Activity means that students work actively in the lesson. Students are motivated (through teacher devolution) to make the problem their own and to solve it. Differentiation means that students will show different approaches to the problem – which the teacher encourages and supports. Vertically planned lessons indicate that a particular lesson fits into the *whole* of mathematics instruction or has a particular purpose in moving students to higher or more complex mathematics, i.e. vertical mathematisation. Mathematising means that there should be evidence of different aspects of mathematisation in the lesson. The focus on mathematisation within Realistic Mathematics Education is very strong and therefore considered the backbone of the framework proposed by this study. Freudenthal (1991: 87) emphasises that focusing on learning processes is a didactical principle. Didactisation from this point of view sees teaching and learning as opposite sides of the same coin.

In RME work with student teachers, Goffree and Oonk (1999: 209) described Freudenthal's consideration that mathematisation and didactisation play an important part in the learning processes of teachers. They further outline a cyclical process whereby "mathematical problems, mathematisation, reflective problem solving and mastering teaching methods follow naturally from each other, and in each successive domain become more sophisticated" (1999: 210) . This cyclical process informs the design research approach for this study (see 4.2).

Yackel, Stephan, Rasmussen and Underwood (2003: 102) further add to the description of didactising, by saying that it includes (but is not limited to) designing teaching activities, planning notational means that might foster conceptual development and figuring out productive interactional situations. This description allows one to gauge the extent and depth of teacher knowledge and experience that is involved in effective didactisation. Didactical action is not simply presenting knowledge at the appropriate level, but also involves decisions that will lead students to learn mathematics with understanding. According to Winslow (2007: 533), didactical knowledge can be seen as an expansion of mathematical knowledge with the focus being on *teaching* mathematics. He further states that didactics studies the conditions of well-established practices as an "inverse problem": from knowledge to manageable situations of learning. He adds that didactics of mathematics can be seen as a type of "conquest" of mathematics. However, the manner in which mathematics was investigated by mathematicians in history is still unattainable to students in most classrooms. Students memorise ideas and procedures out of a context that would necessitate these ideas and procedures.

In effect, this study aims at exploring the landscape of some South African primary school mathematics classrooms; specifically the landscape of teacher decision making that is influenced by teacher knowledge and teacher beliefs (see 2.3). The principles set out by Treffers are considered to be the *what* of effective mathematics teaching while the mathematical work of teaching (see 3.2) is the *how* of effective didactical action.

2.4.1 Didactisation and the mathematical work of teaching (MWT) framework

Freudenthal (1991: 45) defines didactics as the organisation of the teaching and learning processes relevant to a subject. He adds that "didacticians are organizers: educational developers, textbook authors, teachers of any sort, maybe even students who organize their individual or group learning processes". Learning is seen as a process. Didactisation fits the term "mathematical work of teaching" very well, since organising implies conscious goal directed and purposeful activity on the part of the teacher.

2.4.2 Active students

Freudenthal (1991: 1) describes mathematics as a "mental activity" and, if viewed as an activity is different from mathematics that is printed in textbooks (1991: 14). Polya's (in Kilpatrick 1985: 12) first principle of teaching is what he termed active learning. In the process of the devolution of a problem (see 2.2.2.1) the teacher hands over the responsibility and cognitive activity of the problem to the student. Winslow (2007: 525) outlines the task of the teacher to ensure a movement of students' "personal" knowledge towards "public" knowledge and adds that personal knowledge is contextualised while public knowledge is decontextualised. Brousseau (1997: 22) talks about a teacher's role as having to recontextualise and repersonalise public knowledge so that it can becomes the student's own knowledge. This process cannot happen silently or smoothly. It requires challenge and mental activity on the part of the student. Hmelo-Silver (2004: 236) confirms that problem-based learning helps students become more active because it makes students more "responsible for their learning".

Freudenthal (1991: 90) states that observing a learning process is a didactical principle. This observation of students is made possible if students are actively involved in some non-trivial learning activity. He further elaborates in a discussion on levels in learning processes that "thinking is continued acting" (1991: 96) and in discussing the van Hiele levels in learning processes, he reminds us that a "learner's lower level *activity* becomes the *object of analysis* to him on a higher level" (1991: 98, italics in original). Radford (2010: 3) proposes that objects of knowledge exist only in the form of activity. Volmink (1994: 60 italics in original) states that "as we engage with the problematic through actions, we are forced to *reflect*". Confrey (in Volmink 1994: 60) states that "mathematics is constructed by performing actions on problem situations and then through reflection a structure is created within which the problematic can be understood and explained". Student activity therefore is a matter of significant consequence within a mathematics classroom and it is the quality of mathematical activity that this study focuses on (see Fig 5.57). It is however surprising that many mathematics classrooms are devoid of real student activity. It is still the teacher who is participating in mathematics while the students passively observe.

Freudenthal (1991: 119) reminds us that mathematics is a learner's activity. This is based on his premise that mathematics is a human activity. It is noteworthy that many mathematics lessons see the teacher involved in the mathematical activity and the students mimicking this. In discussing two postulates of assimilation and accommodation processes of learning, Piaget (1978: 7) stated that one must assume activity on the part of the subject for these processes to take place. Treffers, in setting out the four didactisation principles, specified that it is the children who work actively in a mathematics lesson (1987: 58). He reminds us that it is not the activity that is the be all and end all of mathematics learning, but it is the starting point for mathematics education (1987: 61). Gravemeijer (2002: 2) presents the idea that "formal mathematics is seen as something that grows out of the students' activity". Rasmussen, Zandieh, King and Teppo (2005: 53) take the view that mathematical activity is preferable to the term mathematical thinking. They however do state that activity encompasses both thinking and doing and that mathematical activity includes participating in different practices including socially or culturally situated mathematical practices (2005: 52).

Active learning therefore can be seen as the opposite end of a pendulum that swings from a traditional paradigm where student activity is minimal. Student activity in traditional settings is based on memory (Gattegno 1971: 5). Traditional teaching comprises an "accumulation of props, all to sustain the poor weak memory". The props according to Gattegno are cyclical in the form of: exercises, homework, reviewing (repeating) and testing. Gattegno advocated the use of imagery in learning mathematics since this assisted students to remain in contact with their "mental energy" (Gattegno 1971: 26). Gattegno surmises that if the only activity a teacher emphasises is memory work, then it is the teacher that is a barrier to learning (1971: 53). One of the tasks Gattegno places before teachers is to consider the economy of learning. He asks why teachers spend many hours on repetition in the classroom when "to live is to change time into experience" so one should swap time for its "equivalent worth in terms of experience, which when accumulated, becomes growth" (1971: 64). Teaching that relies on memory and repetition may be short changing our students in terms of active learning and meaningful experiences. Students and teachers may be investing too much time in exchange for too little meaningful experience in mathematics classrooms.

Treffers also promotes the use of "interactivity" (Treffers 1987: 249) which means that students work with and/or alongside other students. He maintained that the productions and constructions of other students could stimulate students to either shorten their own path (vertical mathematisation) or to become aware of positive or negative aspects of their own ideas. In describing interactivity, Gravemeijer (1994b: 451) says "explicit negotiation, intervention, discussion, cooperation, and evaluation are essential elements in a constructive learning process in which the student's informal methods are used as a lever to attain the formal ones". This assists one in deciding to what extent students are active in a classroom. Their informal methods, followed by discussion of these methods, are keys to this.

In an exposition of reflective abstraction, as a key to understanding mathematisation (see 2.4.4) Simon, Tzur, Heintz and Kinzel (2004: 320) explain that through reflection, students "abstract a relationship between their *activity* and its effect". Reflective abstraction is therefore only possible if students are mentally active. These authors specify a lesson design process that includes teachers' identifying an activity sequence (Simon *et al*. 2004: 322). This activity sequence that is based on a student's existing concepts should include an activity that will lead to "an abstracted activity-effect relationship corresponding to the pedagogical goal". Dubinsky (1991b: 99) also states that reflective abstraction deals with the interrelationships among actions as opposed to objects.

From a modelling perspective, the goal is to "leave the problem solving to the students" (Zawojewski, Lesh & English 2003: 356) - students have to take part in the cognitive activity required for the task. These authors therefore suggest that during initial modelling tasks teacher "interventions that do not remove the cognitive demand of the task are possible". Modelling and the didactisation principle of cognitively active students are congruent.

This study will encourage teachers to increase the meaningful mathematical activity of students in their classrooms (see Fig 5.57). Once students are involved in *doing* mathematics, their thinking can be promoted by supporting differentiated thinking. Differentiation in this study adheres to the following discussion.

2.4.3 Differentiation

Treffers (1987: 62) explains differentiation as students working on meaningful problems at his/her own level of schematisation. He maintains that it is through the meaningful discussion where the teacher summarises the various methods of solving the problem where the student can judge his solution process against the others. This process relies on a teacher skillfully setting out connections between different approaches. Treffers further elaborated that one can distinguish (qualitatively) the differences of level on two planes: general solving processes and specific calculation procedures.

Differentiation as discussed by Freudenthal (1991: 117) refers to the levels at which students will work during classroom activities. He conceptualised both spontaneous differentiation where students choose for themselves at which level they will work; and imposed differentiation, where a teacher decides in advance at which level the students will work. Differentiation is a natural consequence of mathematics teaching and starts with a problem. Students then have to develop their own methods for solving the contextual problem. Hiebert *et al*. (1997: 24) point out that as a result of students developing their own methods "they develop general approaches for inventing specific procedures or adapting ones they already know to fit new problems". This means that students become independent and freed from being bound by having to remember and apply a correct algorithm taught to them.

2.4.4 Mathematisation

According to Gravemeijer (1994b: 446) teaching activities "capitalize on mathematizing as the main learning principle. Mathematizing enables students to reinvent mathematics". Freudenthal (1993: 72) defined mathematising as "turning a non-mathematical invention into mathematics, or a mathematically underdeveloped matter into more distinct mathematics". Mathematising has always had this double-edged definition. Treffers formalised this into horizontal and vertical mathematisation. Treffers and Goffree (1985: 102) do admit that this can be an artificial split since these two forms of mathematisation develop together. In a lecture by van den Heuvel-Panhuizen (1998: 1) she defined vertical mathematisation as "the process of reorganisation within the mathematical system itself, like, for instance, finding shortcuts and discovering connections between concepts and strategies and then applying

these discoveries". This means that students are able to use mathematical objects more abstractly. Freudenthal also reminds us that "mathematising and reflecting are closely connected to each other" (1991: 101). Nelissen and Tomic (1993: 23) also state that mathematisation is a constructive, interactive and reflective activity. The link between mathematisation, activity and reflection is also forged.

Sfard (2010: 128) defines mathematising as participation in mathematical discourse. Discourse is not a one-sided communication, but requires, in a classroom setting, active participation of the teacher and student. Furthermore, discourse entails an extended discussion (Word Web 2003) so it does not involve short yes/no or question/answer sessions. In a traditional classroom setting, communication between teachers and students is largely teacher led and directed. From the results of the 1999 TIMMS video study (NCES 2003: 109) it was found that "teacher talk" is greater than "student talk" by a ratio of between 16:1 and 8:1 in those countries that took part.
Trelinski (1983: 283) observed problem-solving behaviour that included: students making global descriptions that did not account for details; students not being able to transfer problem solving to an area outside of mathematics and where students disregard the importance and impact of assumptions. He suggested the necessity of including mathematisation in school and teacher training curricula.

Freudenthal (1991: 67) states that mathematising is didactically translated into reinventing. If mathematics is to be taught as mathematising then teacher actions should be those of guided re-invention. Gravemeijer (1994a: 82) explains the concept of level-raising and mathematisation as promoting features that characterise mathematics. He set out four features that clarify mathematisation and level-raising. Generality, the act of generalising- this is done by looking for analogies, classifying and structuring. Certainty – this, according to Gravemeijer, involves reflecting, justifying and proving. Exactness – involves limiting interpretations and validity. Brevity – involves symbolising and schematising. Gravemeijer further explains that mathematising specifically involves generalising and formalising.

From a modelling perspective, mathematising takes centre stage in the modelling cycle (see Fig 2.2). It is the point that students find most demanding (Biccard 2010: 134). Within the

modelling cycle, the two forms of mathematisation can be found in translating the real problem to a mathematical one as well as generating the model. De Villiers (1988: 9) explains the two types of models that arise from students modelling, a descriptive model and an analytical model. The descriptive model parallels horizontal mathematisation while the analytical model parallels vertical mathematisation. The two model view of de Villiers allows us to see that all mathematisation can be seen as modelling.

2.4.4.1 Horizontal mathematisation

Horizontal mathematisation takes place when students reflect on a situated problem. Freudenthal calls this moving from the world of life to the world of symbols (1991: 41). Gravemeijer (1994a: 92) describes this process as describing the problem so as to identify the central relations and to understand the problem better. The description can be "sketchy" and use "self-invented symbols" and "it needs not be presented in commonly accepted mathematical language". Gravemeijer (1994a: 93) further explains that this description of the problem does not automatically answer the question, but simplifies the problem by identifying major and minor aspects of the problem. The term description necessitates language which Tall (2008: 6) refers to as the way we describe and refine important ideas. Van Hiele (1959: 20) also makes the comment that at the first level of thinking students should be allowed to speak freely about their experiences with materials or problems. Horizontal mathematisation infers an understanding of the problem, the language and the intention of the question. It involves an understanding of the context of the problem and an attempt to make mathematical that which appears not be so. Menon (2012: 262) suggests that the horizontal mathematisation process deals with ordering, schematizing and building models of real situations so that they become open to mathematics. Once students have transcribed or described the problem they have to work within the mathematical field.

Menon (2012: 262) further clarifies the real contexts that are used in mathematical problems. Contexts need to be "anchoring contexts" which provide students with a need to engage in the problem and bring forward important mathematical ideas. The context serves as a "haven where you return whenever there is a need" (2012: 263). Treffers and Goffree (1985: 102) specify that the contexts in the horizontal activities need to function as models.

Simply solving the problem with the first mathematical tools that come to mind does not result in level-raising. The process of vertical mathematisation needs to be accessed in order for students to construct more advanced mathematical constructs.

2.4.4.2 Vertical mathematisation

Stein *et al*.(2008: 330) propose that a mathematical discussion should not only consist of separate presentations of different ways of solving problems, but also to have student presentations "build on each other to develop powerful mathematical ideas". It is this building of more powerful mathematics that constitutes vertical mathematisation. It is the discussion around the mathematics in the problem that distinguishes horizontal mathematisation from vertical mathematisation.

Freudenthal (1991: 61) describes vertical mathematisation as "the progression of shortening". When students start shortening their path to a mathematical result and in so doing, use a more sophisticated path, this can be seen as the essence of vertical mathematisation. Freudenthal (1991: 51) gives examples such as when counting sets of eyes in a group a student starts counting in twos instead of ones. Gravemeijer and Terwel (2000: 783) define vertical mathematisation as "most clearly visible if a student explicitly replaces his or her solution method by one on a higher level. This could be a shift to a solution method, or a way of describing that is more sophisticated, better organized, or, in short, more mathematical." Gravemeijer, Cobb, Bowers and Whitenack (2000: 267) discuss critical shifts in young students' view of numbers. When numbers move from "adjectives" (eight beads) to nouns (eight) or when numbers move from "referents" (1/2 a bar) to "entities" (1/2), it is related to the shift in *model of* to *model for* in mathematics (see 2.4.4.7). This is a vertical mathematisation shift.

Generalisations and being able to generalise from specific instances to common structures can also be seen as vertical mathematisation. Dorfler (1991: 63) defines generalizing as a "psychological process within the cognition of an individual" where schemas and constructs are produced. He does add that this individual process is socially mediated and that this process is socially communicated but cannot be directly observed. Dorfler (1991: 71) sets out

an action or a series of actions as the starting point for generalisations. This ties in with the didactisation principle of active students. Furthermore, Dorfler does state that it is the task of didactics and the teacher to formulate "starting situations" that will result in a series of actions that may result in reflections that will lead to generalisations. The problem-centred approach and model-eliciting tasks provide such "starting situations" that elicit both horizontal mathematisation and vertical mathematisation through the formation of generalisations.

According to Dorfler (1991: 83) actions direct, control and lead to relations between the elements of this action where mental or symbolic operations can take place. This leads to symbolizing the relations between the elements of the action. Abstraction takes place when "cognitive construction" is "directed and regulated by the action" (Dorfler 1991: 84). This leads to objectifying of the symbols and the construction of variables. Freudenthal (1991: 21) contends that "something that is finished can be described structurally" while Dubinsky (1991a: 179) defines generalisation as an existing schema that is used and represented in a different situation from a previous one. The generalisation process starts with mathematical objects. Actions on these objects can become interiorised processes and when the student has a "high degree of awareness of a process in its totality, this process can be encapsulated to obtain an object" (Dubinsky 1991a: 181).

Gravemeijer (1994a: 94) in a description of the reinvention process in RME explains that vertical mathematisation takes place when students' informal descriptions of problem situations develop into mathematical language and students can start shaping algorithms that have evolved through their own informal translation and interpretations of problem situations. In this way algorithms become meaningful to students and not, as explained by van Hiele (1959: 22), that when a student has "not learned to develop the algorithm himself, he will have to be taught a new one for every new situation". Gravemeijer (1994a: 109) explains that students should begin their mathematical activity by mathematising from reality and then analysing their own mathematical activity because this analysis contains the vertical component. So, as deduced by Gravemeijer the key principle to vertical mathematisation is reflection. This description assists in developing a general idea of vertical mathematisation that can be used to identify instances of student mathematisation during teaching and learning episodes. Furthermore a general description of vertical mathematisation is important to this

study since it is a central component of the didactisation principles set out in this study. Therefore the ultimate aim of implementing these didactisation principles should be to enable student reflection on their own mathematical activity. This will not be possible unless students are able to think of mathematics within situations.

Douady (1991: 115) describes the change of concept-as-tool to concept-as-object as taking place through the changing of settings. Douady describes a concept as a tool when it is used to solve a problem and its meaning is context dependant. A concept becomes an object when it can be considered independently of any context. Within traditional teaching "the pupils are faced with objects and must use them as tools" while from a problem-centred paradigm "pupils must deal with tools, and must build them (or part of them) as objects" (1991: 119). The shift from tool to object is facilitated through what Douady (1991: 116) calls "changing of settings" which is described as obtaining different formulations of problems to create an "imbalance" so that new tools and techniques are activated. This will create a "network of relations". This process of tools becoming objects is a component in vertical mathematisation and Douady reiterates that it is through problems that students can *do* and *know* mathematics. Douady also explains that each problem mobilises a "conceptual window" (1991: 118) of objects and relations. According to Douady students must learn by solving problems and working with concepts-as-tools as it "meaning-producing". Douady's sentiment that decontentualisation allows one to constitute "objects" corresponds with vertical mathematisation. Traditional teaching approaches have the reverse where mathematics is first encountered in decontextualised settings and then it is anticipated that students will be able to recontextualise the objects as tools. This reversal is consistent with the Freudenthal's (1991: 85) concept of "anti-didactical inversion". This inversion means that classroom mathematics is presented to student as a completed product. The activity of mathematics entails memorizing, copying and applying these products. The activity should rather be that students go through the process themselves of (re) inventing the mathematical product themselves.

In trying to explore vertical mathematisation in all its different forms and guises due to differences in language and terminology, the following figure from Pegg and Tall (2005: 473) shows how researchers have been looking to define levels in concepts construction. Pegg and Tall highlight several theoretical frameworks showing the cognitive progress of actions to concepts.

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The fundamental cycle of conceptual
construction from action to object

Constructing a Concept via Reflective Abstraction on Actions					
SOLO [Structure of Observed Learning Outcome]	Davis	APOS	Gray & Tall	Fundamental Cycle of Concept Construction	
Unistructural	Visually Moderated Sequence as Procedure	Action	Base Object(s)	Known objects	S C H E M A
			Procedure [as Action on Base Object(s)]	*Procedure as Action* on Known Objects	
Multistructural			[Alternative Procedures]	[*Alternative Procedures*]	
Relational	Process	Process	Process	*Process* [*as Effect* of actions]	
Unistructural [new cycle]	Entity	Object	Procept	Entity as *Procept*	
		Schema			

Figure 2.5: Pegg and Tall's fundamental cycle of concept construction

In the above diagram it is evident that the construction of a mathematical object from action has as a critical point a process. If *process* is accepted as "the performance of some cognitive composite activity" [www.definitions.net] then two important elements are highlighted - that of *composite activity*. A vital link in the action-process step is given by Tall and Poynter (2005: 1271). They explain that it is focusing on the effect of action that allows students to understand or develop a process and give the following example (2005: 1272):

> For instance the process of counting is compressed to the concept of number by focusing on the effect of counting in terms of the last number spoken in the counting schema.

The process of developing or building a schema would require a composition of prior knowledge. The authors, in the above diagram, assist in defining and describing vertical mathematisation. Mathematical process cannot be understood if only isolated concepts are known. It entails a series of co-ordinated actions where objects can be used to reach a mathematical goal. This level of thinking is very difficult to attain through traditional rote learning of procedures. The isolated concepts are rarely connected since the action involved is to reproduce set methods.

In their discussion about the underlying local cycle of concept construction, these authors may have provided a very succinct definition of mathematisation, both horizontal and vertical.

They state that "the local cycle of construction in the embodied world occurs through a shift of attention from the doing of the action to an embodiment of the *effect* of the action" (Pegg & Tall 2005: 474 italics in original). More importantly for this study is the summation by these authors that all these cycles begin

> … with a situation that presents complications to the learner, who may focus at first on single aspects, but then sees other aspects and makes links between them to build not just a more complex conception, but also a richer compressed conception that can be operated as a single entity at a higher level. (Pegg & Tall 2005: 474).

Piaget (1985: 29) termed reflecting (reflective as used by some authors) abstraction as an advanced thinking process. Piaget (as quoted in Simon *et al.* 2004: 312) explained that it is through reflective abstraction that advanced concepts can develop out of existing concepts. Brun (in Simon *et al*. 2004: 312) suggested that the goal of teaching should be to promote reflective abstraction.

Piaget's reflecting abstraction involves two activities (Piaget 1985: 29): a projection phase where action at one level becomes objects of reflection at the next, and a conscious reorganisation phase (reflexion). It therefore appears that Piaget's concept of reflecting abstraction, and the RME notion of vertical mathematisation, may hold common ground. Of note is that Piaget (2001: 53) specified that the projection extracts from the lower level certain *connections* that were used implicitly and transforms the connections into objects of thought on a higher level. This supports the initiative of this study to include connections into the didactisation principles (see 3.2.1.3). According to Campbell (2001: 12) the outcome of reflecting abstraction is a new scheme built out of a prior scheme. It also entails, not only new knowledge at a higher level, but knowledge about knowledge. Furthermore, according to Piaget (2001: 102) the realisation that there is a reason for observed outcomes (defined as necessity by Piaget) is the very product of reflecting abstraction. This necessity means that the student can explain the connections he/she observed on the lower level. It is therefore imperative that teachers are equipped to deal with connections in mathematics lessons.

Van Hiele (1986: 5) reminds us that the psychology of Piaget is about development and not learning. Stages of maturation, although relevant in teaching students do not equate to stages of learning or higher levels of thinking that result from teaching and learning. Van Hiele (1986: 67) differentiates between the study of exploration and the study of problem solving as

another essential difference between psychologies of development and psychologies of learning. Dreyfus (1991: 26) however states that mathematical and psychological processes cannot easily be separated. Abstract thinking is therefore both a developmental and teaching process. It is specifically the development of abstract thinking through teaching that is of interest in this study: specifically how teachers can promote student abstraction through didactical action and didactical decision making.

Dreyfus (1991: 38) explains why abstracting is so difficult for students. He asks how we "generate mental structures which are so often linked to visual images, if they should represent relationships that are removed from the concrete objects which they were originally linked to?" thereby linking visualisation and representation to abstraction. He then sets out four stages (1991: 39) in a learning process that links representing and abstracting. These four stages show links to the SOLO levels of concept construction (Biggs 1999: 67):

Dreyfus (1991: 38)	Biggs (1999: 67)
1. Students using a single representation.	Unistructural – identify or use a simple procedure
2. Students using more than one representation in parallel.	Multistructural – describe, combine or do algorithms
3. Students making links between parallel representations.	
4. Students integrating representations and flexibly switching between them.	Relational – compare, contrast, analyse, explain causes, relate and apply

Table 2.4 Learning stages in abstraction

Dubinsky (1991b: 101) described five different kinds of construction in reflective abstraction:

1. Representation

2. Co-ordination

3. Encapsulation

4. Generalization

5. Reversal

The process of reflective abstraction needs to start with the construction of internal processes to make sense out of perceived phenomena (Dubinsky 1991b: 101). Representations are therefore the first step in abstraction and will be explored in the next section. It is noteworthy that Dubinsky does not end his constructions with generalisation, but extends it to reversals. He explains that for a reversal to take place the process must first exist internally before a student can consider the reverse. The reverse according to Dubinsky does not mean an undoing, but rather a new process which consists of reversing the original process (Dubinsky 1991b: 102). So reversals must be the highest form of vertical mathematisation.

Rasmussen *et al.* (2005: 54) explain that vertical mathematising can only be understood in terms of students' horizontal mathematising. This makes it more clear why students who are exposed to structuralistic mathematics experiences do not build advanced mathematical structures, since, even if students are exposed to extensive vertical ideas, the ideas should be built on students' own horizontal activities for the vertical building to make lasting sense. Another significant suggestion of these authors is their illustration of how a student's symbolizing activity is reflective of horizontal and vertical mathematisation.

A different conception that involves integration of horizontal and vertical mathematisation is offered by Niss where he introduces the concept of "*implemented anticipation*" (2013: 55, italics in original). He identifies three key points in mathematisation of a modelling problem that a student has to anticipate and implement the anticipation:

1. The real situation has to be "prepare[d] for mathematisation" in terms of a "first anticipation of [its] potential mathematisation".
2. In doing so, the student has to anticipate mathematical representations that can encapsulate the situation. The resulting mathematisation is an "implementation of this anticipation".
3. The student has to determine how the "mathematical apparatus" and problem solving strategies used in mathematisation can answer the questions posed. Once again, according to Niss, the mathematisation is a consequence of an implemented anticipation of these problem solving strategies. (Niss 2013: 56-57).

Stillman and Brown (2012: 681) call Niss' successful implementation of anticipation a "three step foreshadowing process". The introduction of an anticipation of how and why certain

representations and strategies are more useful than others enables one to acknowledge the difficulties and challenges in mathematisation and to appreciate the need for a blending of the concepts of horizontal and vertical mathematisation.

Since Rasmussen *et al*. (2005: 57) also explain that when students shift from using symbols for recording and communicating to using symbols to further their reasoning – this progression mirrors the shift from horizontal to vertical mathematisation. It is therefore apparent that symbolizing and representations take centre stage in the discussion of mathematisation and advanced mathematical thinking.

2.4.4.3 Representations in mathematisation

When mathematisation involves describing problem situations, and further reflecting on those descriptions as suggested by Gravemeijer (1994a: 109), then students will need to rely on representations to convey their descriptions on a horizontal mathematisation level and their definitions on a vertical mathematisation level. Dreyfus (1991: 32) suggests that for students to be successful in mathematics they should have rich mental representations about concepts. This means that the representation contains "many linked aspects of that concept" while he explains that if the representations are weak, they will have too few elements that will produce inflexibility in students' ability to solve problems. The nature and the role of representations in students' problem solving abilities need to be considered.

Any mathematical concept that is present in the mind must be represented in some way (Davis 1984: 203). According to Goldin (2008: 182) one way to explore a student's understanding of a mathematical concept is to reflect on the variety of distinctive appropriate or inappropriate representations that a student has and to describe and analyse the relationships that have developed. Vergnaud (in Boero *et al*. 2008: 263) on the other hand defined a concept as a system that consists of three components: a referential situation, operational invariants and symbolic representation. Representations are further categorised as internal and external with internal representations being "mental" representations and external representations being things such as language, written symbols, pictures and physical objects used in communicating internal representations to others (Hiebert & Carpenter 1992: 66).

Skemp (1986: 104) compares visual and verbal-algebraic imagery in mathematics, which explains the dominance of verbal-algebraic over visual imagery. This helps to explain why teachers' use of representations may not include the variety that is necessary in primary school classrooms. Teachers may find visual intuitive representations difficult to deal with as they can make the mathematical classroom space messy and difficult to structure. However, this study aims (see 1.3.2.2) to empower teachers to deal with multi-level, multi-representational teaching and learning.

Visual	Verbal-algebraic
- Abstracts spatial properties, such as shape, position - Harder to communicate - May represent more individual thinking - Integrative, showing structure - Simultaneous - Intuitive	- Abstracts properties which are independent of spatial configuration such as number - Easier to communicate - May represent more socialized thinking - Analytical, showing detail - Sequential - Logical

Table 2.5 Skemp's comparison of visual and verbal-analytic imagery

Shulman's definition of PCK includes the strongly worded section that a teacher should "have at hand a veritable armamentarium of alternative forms of representation" (1986: 9). Douady (1991: 109) also asserts that the type of activity in a classroom will "depend significantly" on the teachers' representations in mathematics activities. It is therefore not simply enough to allow students to use their own representations as the teacher should encourage and introduce students to different representations. Teachers should be able to structure representations along a vertical path so as to build an "iceberg" (see Fig 2.9) metaphor of a concept. The critical role of the teacher being able to assist students in making the connections between the different representations is part of the skills teachers need. This study aims (see 1.3.1.2) at developing these skills in teachers (see Table 5.1; 5.9; 5.16; 5.23).

Goldin (2008: 187) explains the processes of abstraction and its complement of contextualisation in the realm of representations. This discussion affects how mathematisation is viewed. Abstraction, in his view, is to learn to generalise, while contextualisation is to "move toward the concrete in a new representational situation, and to take these steps

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spontaneously and flexibly". The process of horizontal mathematisation can be seen as creating representations of a contextual situation, while vertical mathematisation involves what Goldin (2008: 187) described as the process of abstractions: "reaching the autonomous stage in the functioning of a representational system". Visually this may be denoted as:
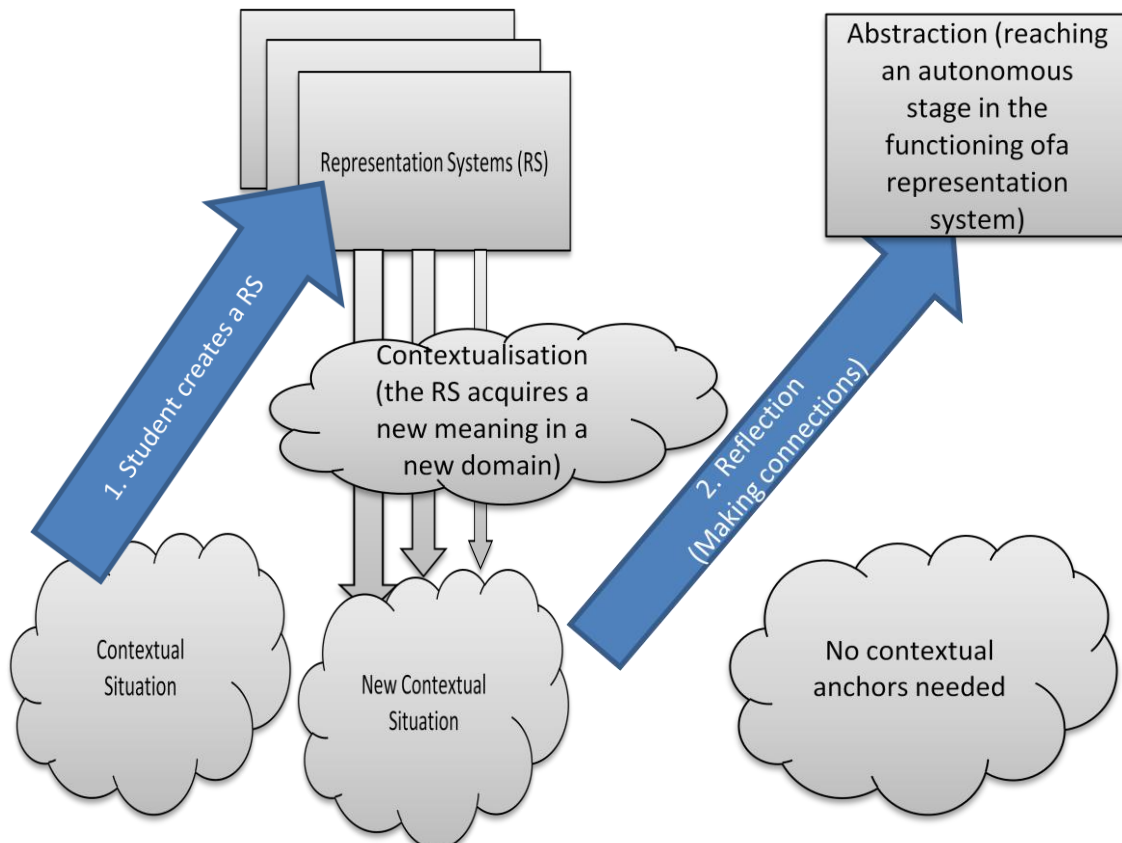
Figure 2.6 Adapted from Goldin's (2008: 187) abstraction description.

The two blue arrows (1 and 2) correspond with horizontal and vertical mathematisation respectively.

Davis (1984: 306) explained the relationship between representations and problem solving. Within his description, both horizontal and vertical mathematisation can be detected. Davis (1984: 306) explains a seven-step representation building process in solving problems. Firstly the problem is examined for cues and clues to guide retrieval from the knowledge base. Thereafter some knowledge representation structures are retrieved from memory. Then there is a gradual building up of a representation for the problem and a building up of the knowledge representation. He then explains two "mappings". Firstly the problem must be mapped to its representation and secondly the problem representation must be mapped to the knowledge representation. The last two steps involve evaluating the constructions and

mappings and cycling back through the process as often as necessary. The two red arrows (first and second mapping) correspond with horizontal and vertical mathematisation.
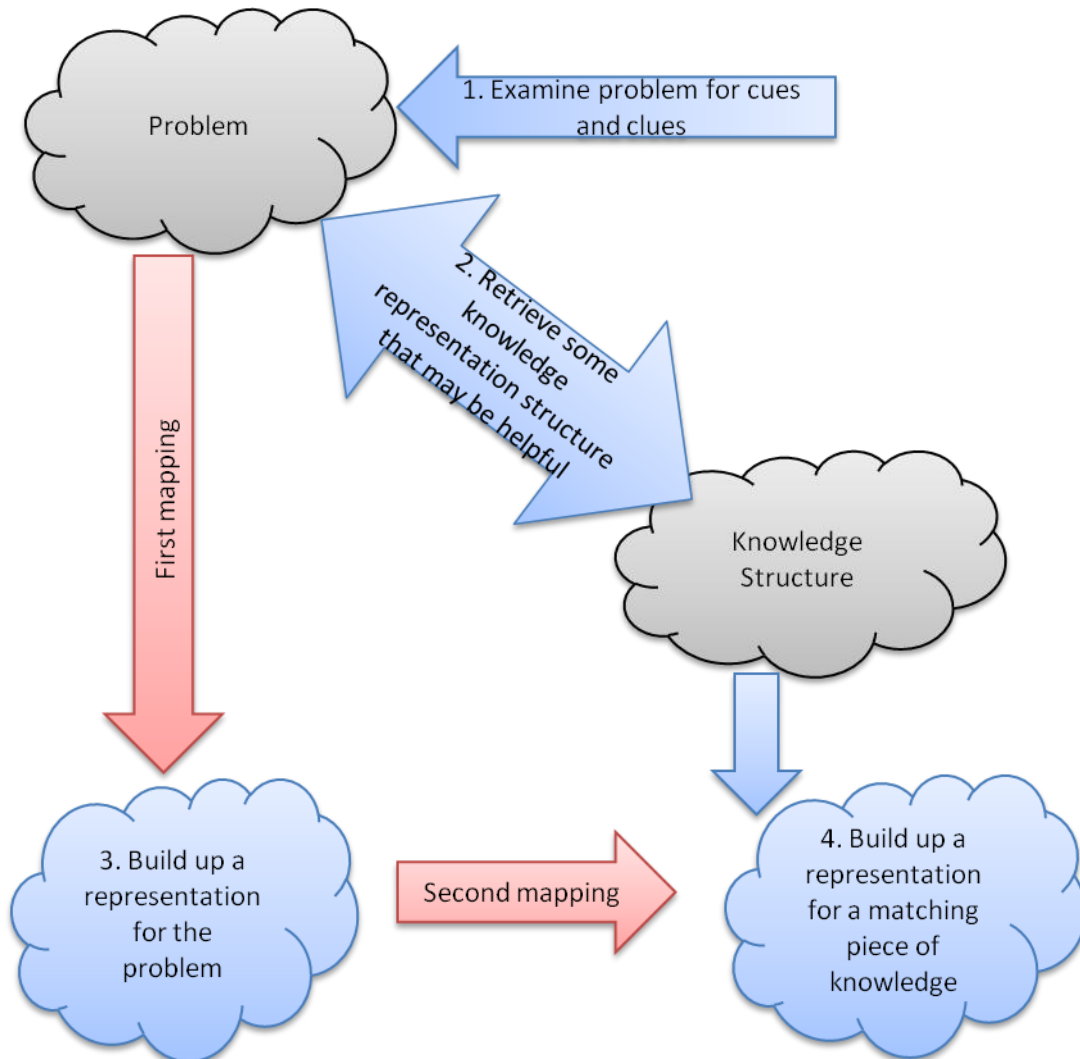
Figure 2.7 Adapted of Davis (1984: 306): how we build representations

The two diagrams show that both vertical and horizontal mathematisation can occur as a result of learning through problems. A problem is a pre-condition for effective mathematisation. The PCA has an advantage over other teaching approaches in that fertile ground for mathematisation is established through *starting* with problems.

From a modelling perspective there is a close relationship between modelling and representations. According to Dreyfus (1991: 34) the situation in modelling is physical and the model is mathematical. In representing, the "object to be represented is the mathematical structure, and the model is the mental structure". As can be seen in the next section, the

iceberg image, as well as the chains of signification, also points to the constructive nature of representations and how students need to build and connect between different levels of representations in order to move along the path of vertical mathematisation. The focus Wessels (2006: 76) gives to representations in mathematical learning is accepted for this study. She states that in modelling "representations play an all-important role".

It is in modelling tasks that Duval's (2006: 107) statement regarding the problem of mathematical comprehension: "what matters is not representations but their transformation" is directly relevant. He further explains why semiotic representation transformation is the essence of mathematical activity. Firstly, there is a cognitive paradox to accessing mathematical objects. He reminds us that we cannot access mathematical objects by perception or instruments but only by using signs and semiotic representations. The conflict arises from students being unable to get access to the mathematical object apart from the semiotic representation, and they cannot distinguish between the object and its representation. Secondly, mathematics is a domain where the largest range of semiotic representation systems is found and often mathematical systems require more than one set of representations.

2.4.4.4 The van Hiele levels in mathematisation

Van Hiele proposed five levels of thinking in mathematics (1986: 53): a visual level; a descriptive level; a theoretical level, a formal level and a logical level. Within the learning process a student progresses through these levels in five stages described by van Hiele (1986: 53-54):

1. An information stage.
2. A guided orientation stage – this is where students are guided by a teacher or themselves through tasks which have different relations of the network that has to be formed.
3. An explication phase – where students become aware of the relations, they try to express them in words. They also use the technical words used within the subject.
4. A free orientation phase – this is how students learn by general tasks to find their own way within the network of relations that make up the concept.
5. An integration phase – where students build an overview of all they have learnt using the newly formed relations at their disposal.

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These phases are based on van Hiele's ideas regarding student development of concepts and represented a new way of thinking about student mathematisation. These five phases are compatible with Freudenthal's (1991:45) notion of "guided reinvention". Van Hiele proposes these beacons along the way of guiding student thinking. Too often traditional classrooms expect students to work at the latter stages without providing the guided scaffolding of the initial phases. Teachers may not be aware of the importance of reflecting on the lower level activities and that this reflection can result in building of concepts.

Of importance to this study is the central role that the teacher and tasks play in student learning and in level-raising. Phases two and three point again to the valuable role that didactisation plays in mathematics teaching and learning. Van Hiele (1986: 56) also maintains that using the levels of thinking to teach is advantageous. Not only for a teacher's own knowledge, but to know that "there is a well-founded base to begin with – the visual-base level". Piaget (2001: 87) also gives prominence to "empirical abstraction" since it leads to "precision" concerning generalisation. Van Hiele (1986: 63) also explains that the teacher needs to give the student guidance during the learning process, but this does not mean that the teacher must tell students how to act in a situation. This will result in students developing networks that are not connected with the first level of thinking. Therefore, for this study, the inclusion of making connections (see 3.2.1.3) in the mathematical work of teaching framework is considered essential.

Gravemeijer (1994a: 22) also explains that Van Hiele analysed the communication between teacher and student and found that the concepts used by teachers and students have different meanings since they come from different levels of thinking. It is therefore important that teachers understand that their frame of reference is not the same as the student's and that the student should be guided through progressively formalizing his/her pre-knowledge. This study proposes that teachers anticipate this formalizing from intuitive beginnings through applying didactisation principles set out by the study.

2.4.4.5 Advanced mathematical thinking as vertical mathematisation

Within the domain of advanced mathematical thinking, Gray and Tall (1991: 72) described three important components of mathematical thinking: process, concept and procept. They considered how a mathematical process and mathematical concept can be amalgamated into the idea of a procept (1991:73). A procept, since it is represented by the same symbolism for process and concept, can represent either a process or a concept. In a student's progress to encapsulation or reification he or she needs to go through process and procept cycles in creating new facts or structures. The procept idea also assists in understanding that horizontal mathematisation and vertical mathematisation should be seen as integrative in the same way as process and concept are integrated into the concept of procept. This also assists in understanding the vital role of vertical mathematisation in elevating students' structural thinking about mathematics. According to Gray and Tall (1991: 78) "the less able child who is fixed in process can only solve a problem at the next level up by coordinating sequential processes….if they are faced with problem two levels up, then the structure will almost certainly be too burdensome for them to support".

Mamona-Downs and Downs (2008:156) describe advanced mathematical thinking as a situation where a set of conditions are given from which one has to adduce a new property. Tall (1991: 20) explains the move from elementary to advanced mathematical thinking involving a move from describing to defining. This ties in with Gravemeijer's (see 2.4.4.1) idea that horizontal mathematising involves descriptions.

Dreyfus (1991: 26) explains that advanced mathematical concepts are likely to be complex, but that this complexity can be managed by two "powerful processes" of abstracting and representing. According to Dreyfus these processes allow one to move from one level of detail to another. Dreyfus (1991: 33) also identifies "translation" as a related process, which is closely connected to switching between representations. From an advanced mathematical thinking point of view, translating takes place when students recognise one formulation of a problem in a related or similar one. In applied problems students need to understand the context of the problem as well as "establish a close and clear correspondence" between seemingly unrelated concepts. Duval (2006: 112) however prefers to use the term "conversions" to denote a transformation of representation that consists of "changing a

register" without changing the objects denoted. Duval further explains the difficulties involved when the reverse direction of the change in register takes place and students are asked to recognise a concept. So according to Duval difficulty arises when students have to recognise the same mathematical object through two representations from different registers.

The role of analogies is also relevant here. Analogy enables students to apply information from one problem to one that is structurally similar (Richland, Holyoak & Stigler 2004: 39). They define analogies as the "construction of similarity between the relational structures of the source and target objects" (2004: 43) and provided five separate components in the process.

1. Representation of the source.

2. Representation of the target.

3. Finding a mapping.

4. Making inferences and adapting them to the target.

5. Providing a solution to the target.

Their study analysed TIMSS video footage and found that most source problems and target problems were de-contextualised mathematics problems and that teachers produced nearly 90% of all mapping statement. (2004: 49, 55). This may indicate that making analogies is difficult for students and as such teachers take over this role.

Tall (2008: 6) explains that students enter a mathematics classroom with three "fundamental set-befores" that shape mathematical thinking and learning:

- recognition of patterns, similarities and differences;
- repetition of sequences of actions until they become automatic; and
- language to describe and refine the way they think.

Students build their mathematical knowledge based on experiences that they have "met before", so Tall (2008: 6) calls these constructs "met-befores". Some ideas students have met before are consistent when they encounter new information, while some ideas are inconsistent with the new information. Building advanced mathematical thinking in students requires a

judicious interplay between the set-befores and the met-befores. The role of connections is again reiterated while Piaget's (1978: 6) notions of accommodation and assimilation are relevant.

2.4.4.6 Language and vertical mathematisation

Dewey (1910: 170) insisted that while language is not thought, it is necessary for thinking as well as for its communication. In understanding how students learn patterns in mathematics, the following sequential order is suggested by Orton and Orton (1999: 117). In student progress towards an algebraic representation they first are able to provide a correct verbal statement that corresponds with the pattern. The next stage is a credible attempt at an algebraic expression before reaching the correct algebraic expression. The importance of student verbalisation must not be underestimated in the mathematisation process. Van Hiele (1986: 86) reported on the difference in student language on different levels of thinking (see 2.4.4.4). At the first level, students speak about visual observations; at the second level, new language structures and explanation are used; while on the third level, the language used by students is more abstract than the descriptive language on the second level. So a teacher who is observant of the nature and changes in student language should be able to adapt the teaching/learning process to support student mathematisation. This requires however that students are talking about what they are doing in a classroom. As stated by van Hiele (1986: viii) discussion is the "most important part of a teaching-learning process".

Vygotsky (1978: 25) concludes that "speech and action are part of one and the same complex psychological function, directed toward the solution of the problem at hand". Traditional teaching approaches largely silence students in that speech is limited to answering teacher questions. Modelling tasks however cannot be solved without extensive verbal and written communication and negotiation. Vygotsky also notes that the more complex a task is, the greater the need for speech. He found that if students were not permitted to speak they often were not able to solve a given task. Purely silent classrooms may be problematic to students' advanced mathematical thinking. Stein *et al.* (2008: 332) maintain that since the move towards student-orientated inquiry methods of teaching, teachers have struggled to orchestrate classroom discussion that move the class towards important mathematical ideas.

Boero *et al*. (2008: 264) outline the functions of language in the theoretical work of mathematics. They see natural language as: a mediator between mental processes and logical organisation in mathematical activities; a flexible tool that helps students develop meta-linguistic awareness; as a mediator between experience, the emergence of mathematical concepts and the development of these concepts into "embryonic theoretical systems"; and a tool in activities involving validation and argumentation. These authors (2008: 271) conclude that the "flexible mastery of ordinary language should be a necessary step to mathematical proficiency" and assign an educational implication to develop natural language in mathematical activities as it is the "key for accessing control of algebraic problem solving processes".

In order to develop mathematisation within students, the parallel growth in language is necessary. Students cannot grow in language or in mathematisation ability without being given opportunities to talk about or write about mathematics. This once again touches on the didactisation principle of active students (see 2.4.2). Here the activity is that of communication in classroom discussions. Boero *et al*. (2008: 284) see communication as a way of developing linguistic representations of knowledge, they reiterate that "communication reflects and influences the development of thought".

Teachers will need to grapple with this new role if they wish to promote mathematising in their classrooms. They will need to develop new strategies of listening to students, since this will help them to understand and structure different ideas in solving problems. Not only do teachers need to listen, they also need to interpret the type of language students are using. How are the explanations students' are giving mirroring their mathematical thinking? Teachers will have to equate students' language use with their mathematical thinking which is a skill that requires both a solid knowledge base and experience in listening to students.

Suurtamm and Vezina (2010: 2) found that when teachers do attend to their students' mathematical thinking the benefits included higher levels of conceptual understanding by students as well as more positive attitudes. They did however confirm that a "listening orientation" is "hard work" because of the variety in student thinking and how this thinking "sound[s] and look[s] different from ordinary mathematics". They further explained that it

takes time for teachers to incorporate a listening approach into their teaching. Their suggestion was to engage teachers in unfamiliar problems to assist this shift to listening for student ideas. The professional development program proposed in this study seeks to assist teachers in developing student mathematisation by helping teachers develop a listening orientation (see 5.5.6) that is focussed on student thinking - through didactisation principles.

2.4.4.7 Emergent modelling as mathematisation

Gravemeijer (2002: 2) distinguished between initial models that students create of a contextual situation and a model that changes and becomes an entity of its own. In the first instance the intuitive model is a *model of* the contextual situation which can later function as a *model for* more formal mathematical reasoning. He explains that the shift coincides with a shift in student thinking, when student thinking moves away from the situation to the mathematical relationships that were elicited from the contextual situation. Gravemeijer (1994a: 101-102) explains the levels of emergent modelling in more general terms. He specifies the realistic education model begins with real situations followed by models *of* the situation which is followed by a model *for* which gives rise to formal knowledge. He calls the four hierarchical levels: a *situational level* where students work within the context of the problem. This is followed by a *referential level* where their models refer to the situation. This is followed by a *general level* where the mathematical focus is on strategies that govern the reference to the context, and finally a *formal level* where students work with conventional procedures and algorithms. This description will assist one in identifying varying vertical mathematisation levels within student work. He further adds that a higher level is a formalisation of a lower one while the lower level is a referential level for the higher one.
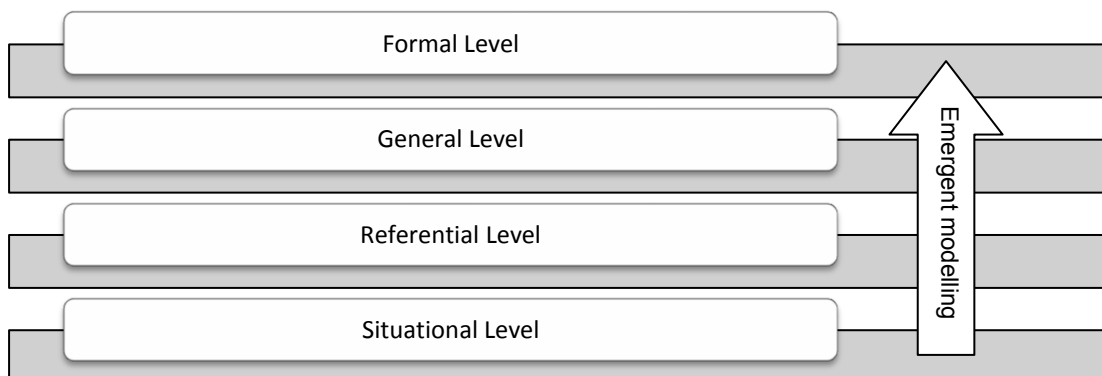
Figure 2.8 Levels in emergent modelling (Gravemeijer 1994a: 102)

Gravemeijer (1994a: 82) further specifies domain strategies within vertical mathematisation. They are strategies for: generality; certainty; exactness; and brevity. He adds that mathematising refers to a "posteriori construction of connections". While emergent modelling described by Gravemeijer and the modelling perspectives described in 2.2.4 are similar in the development of advanced mathematical thinking, there are some differences that can be highlighted. Firstly, emergent modelling is not only used to describe a short activity or sequence of activities but can be used to structure an entire learning sequence or curriculum. Secondly, a modelling perspective is prescriptive about the types of tasks or activities involved in the learning process while emergent modelling can be adapted to a larger variety of mathematical learning activities.

 Yackel *et al.* (2003: 105) explain Gravemeijer's idea that the aim of teaching is to provide students with experiences where they can generate models of a situation which can become models for their thinking on a higher level. They further hypothesise (2003: 108) that a chain of signification may support a model of/model for shift in taken-as-shared reasoning during teaching. Their idea of chains of signification assists one in understanding vertical mathematisation in that the chains grow in progressive abstraction. The visual metaphor of a chain also assists in highlighting the importance of connections and the reliance of complex concepts on simpler informal concepts. An iceberg image is used to describe progressive understanding of a concept while if the chain is rotated it forms a similar "ice-berg" image.

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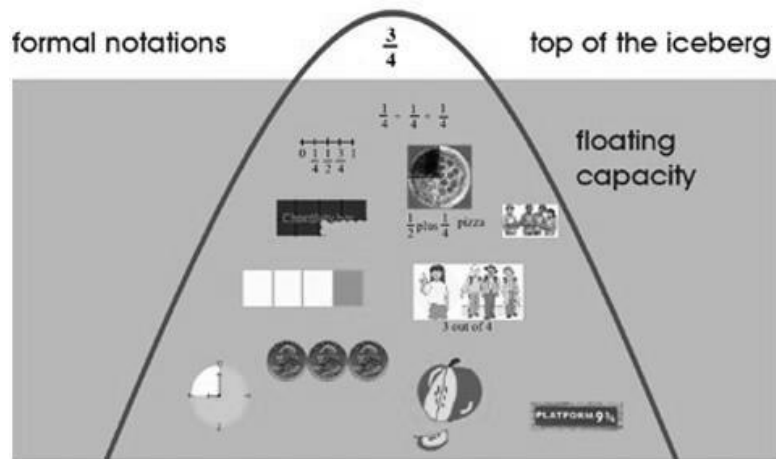
Figure 2.9 Iceberg image [taken from
http://people.usd.edu/~kreins/learningModules/RealisticMathematicsEducation.htm]
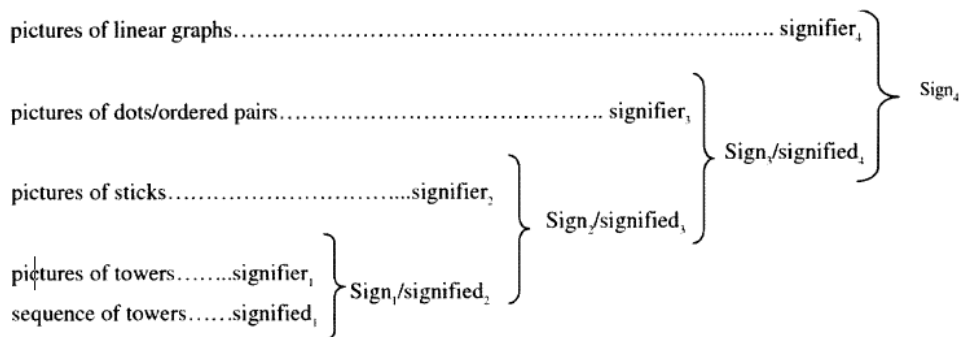
Figure 2.10 Chains of signification. Taken from Yackel *et al*. (2003: 108).
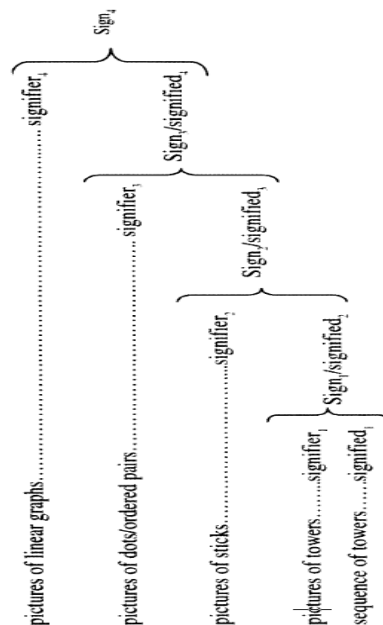
Figure 2.11 Chains of signification rotated to show iceberg property

Gravemeijer *et al*. (2000: 241) explain that teachers can formulate a proposed trajectory or developmental route for the classroom community so that students first model a problem situation in an informal way (model of) and then mathematise their informal activity to produce a model for their reasoning. These authors further specify four levels of activity (Gravemeijer *et al*. 2000: 243) that lead from a model of to a model for. At the first level, students are active within the task setting. The next level requires a referential activity where the students' model is grounded in their experience of the problem situation. At the third level, students are involved in general activity where they lose the dependency on the problem situation and at the fourth level, students can reason using conventional symbolisations.

Rasmussen and Marrongelle (2006: 391) modified the RME design heuristic of emergent modelling for teaching. They state that the model of to model for development is compatible with Sfard's notion of reification. These authors however wanted a day-to-day "transformational record". They explained this as a "weakening" of the emergent modelling heuristic since reification would not take place on a day-to-day basis but they wanted something that could be translated to a micro-level. They called this creating "pedagogical content tools" (2006: 389) so as to adapt RME heuristics for teaching. Emergent modelling as a design principle in RME needs to be *re-understood* as a teaching activity. The didactisation principles set out in this study can also be thought of as "pedagogical content tools" that assist teachers in making decisions that are educationally accountable in the mathematics classroom (see 5.5.8).

2.4.5 Vertically planned lessons

Freudenthal emphasised intertwining of learning strands (1991: 118) in mathematics as a valuable organizing (didactising) tool. According to Treffers (1987: 62), vertical planning is based on the idea that a "lower activity" (e.g. counting) offers the needed basis of experience for a "higher activity" (e.g. combinatory problems). Teachers do use this idea in their planning, but in mathematics it is especially important that the process of vertical building is explicitly planned for. It is more than simply moving from concrete to abstract or moving from one vertical mathematisation process to the next.

Ball and Bass (2000: 100) described this as teachers developing "a sense of the trajectory of the topic over time" specifically to "develop its intellectual core in students' minds and capacities so that they eventually reach mature and compressed understandings and skills". Ball *et al*. (2008: 403) explained the term "horizon knowledge" that refers to knowledge of how a topic develops through the curriculum as well as how teachers see the connections to later mathematical ideas. It is this structural connectivity that Ferrini-Mundy, Burril and Schmidt (2007: 312) define as "curricular coherence".

This also ultimately means that teachers need much greater depth in mathematical curriculum knowledge than what is required in their current teaching grade. Chauvot (2008: 84) describes Shulman's curricular knowledge in its four components. She describes a lateral and vertical component to curricular knowledge. The lateral component includes knowledge of curriculum materials that the teacher is dealing with at the time, while vertical knowledge is a familiarity of topics that will be taught in later years. Teacher vertical knowledge of their curriculum is important since it makes the teacher more sensitive to the development of a concept that will take place. This means that the teacher may not easily resort to tricks or premature shortening of methods since that will hamper true understanding of the concept at a later level.

2.5 DEVELOPMENTAL FRAMEWORK FOR A MODELLING APPROACH TO DIDACTISATION.

The important elements of this chapter now come together to form the framework that explains how a modelling approach can develop didactisation practices in primary school mathematics teachers. Modelling as a mathematical task encompasses what is deemed essential to mathematics teaching and learning. It fits Freudenthal's guided re-invention principle (see 2.2.1) as well as Brousseau's adidactical situation (see 2.2.2.1). Modelling tasks also cover the essential elements of problem-centred learning (see 2.2.3). Furthermore, the framework can only be worked against the background of who teachers are and how they learn and grow. Teacher decision making (see 2.3) plays a paramount role in conceptualizing how a modelling approach can develop didactisation practices. Didactisation principles as set out in RME theory, as well as the "mathematical work of teaching" (see 3.2) as brought into magnification by Wilson and Heid, can come together to serve teacher professional development. The didactisation principles set out in this study provide a means to

"understanding students' mathematical thinking [and] can provide a unifying framework for the development of teachers' knowledge" (Carpenter, Fennema & Franke 1996: 4). The many complex links to a chained network of teacher didactical action need coverage in order to do justice to the real work of teachers in real classrooms.

Although a lengthy discussion on these aspects has taken place in this chapter what may prove to be more effective in conclusion is to describe the interactions and interrelations in visual form.

2.5.1 A visual representation of the framework for this study
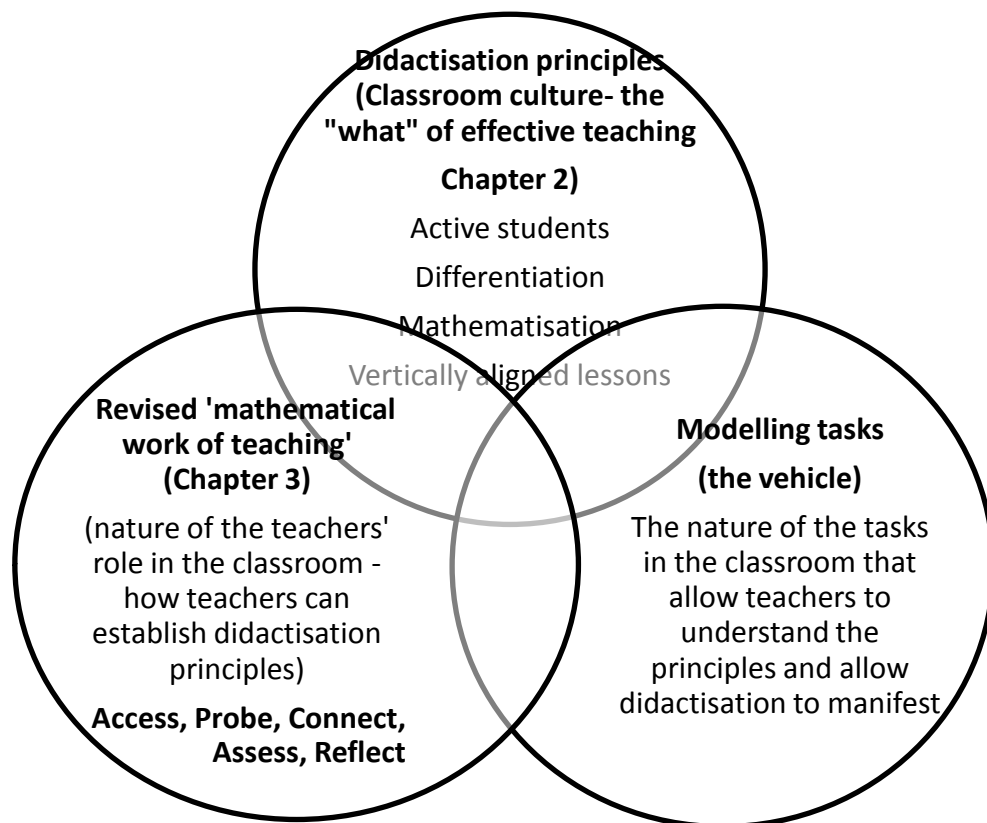
Figure 2.12 Visual representation of the professional development framework for the study

The intersecting area of the three circles represents the professional developmental space of this study. The numerous factors that are taken into consideration in this study for professional development arise from a similar understanding to that of Stigler and Hiebert (1998: 5) that it will be "difficult, if not impossible, to improve teaching by changing

individual elements or features". The problem of teacher change needs to be approached from a more holistic point of view.

2.6 SUMMARY

This study aims to design, develop and study the impact of the formative didactisation framework for in-service teachers' professional development (see 1.3.1.2). By integrating and developing existing significant theories in mathematics education, this study represents design research (see 4.2) in that its purpose is to "develop a class of theories about both the process of learning and the means that are designed to support that learning" (Cobb, Confrey, diSessa, Lehrer & Schauble 2003: 10).

The chapter set out some major theoretical constructs of the study. Realistic Mathematics Education (RME) provides a strong foundation on which to build new ideas in mathematics teaching and learning. The concepts of guided reinvention can be achieved by setting out didactisation practices as beacons. The RME conceptualisation of horizontal and vertical mathematisation is useful in understanding the need for student activity in the horizontal realm so that vertical activity can take place. However, a blended conception of horizontal and vertical mathematisation may be more feasible in understanding mathematisation in a classroom context. The concept of didactisation and its related practices as set out in this chapter permeate the mathematics classroom landscape.

The ideas proposed by the theory of didactical situations (TDS) allow one to focus more clearly on how mathematical work should be devolved to students and how teachers can facilitate this devolution by creating and maintaining a milieu. From this perspective didactisation practice of student activity is a quality feature in mathematics classrooms. The joint action theory of didactics (JATD) builds on the joint action of teachers and students. Mathematics classrooms are better conceptualised with a focus on the interaction between teacher and students. This theory allows one to propose the didactisation practices such as accessing, probing and connecting student thinking as the integration of teacher and student actions.

The problem centred approach allows one to activate didactisation within a real classroom setting. The springboard for didactisation is active students working on meaningful problems whereby they learn mathematics through solving them. The problems allow students to reflect on their actions within the problem situation and to transform their thinking about the solving the immediate problem to thinking about the mathematics that results from these actions. The problem centred approach contributes to didactisation because of it allows teachers to realise didactisation principles such as active students, mathematisation and differentiation. Modelling tasks as reality based problems, by their nature, allow groups of students to share, modify and extend their current thinking. More so, modelling tasks allow teachers insight into students existing ways of thinking and to observe changes to these ways of thinking. Since modelling tasks require facilitative teacher input, teachers can observe the socially shared constructions that arise through modelling.

The explication of mathematisation was also made in this chapter. This lies at the center of mathematics teaching and learning. Mathematisaion is a challenging construct to pin down and many authors have illuminated and clarified the mathematisation process. The extensive literature field is not a negative factor in understanding mathematisation, in fact, the numerous expositions and related terminology allows one to understand mathematisation from a varied and wide perspective.

Schoenfeld's theory of teacher decision making was explored in this chapter as an overarching structure in which to situate teacher development. The three main areas of teacher knowledge, teacher goals and teacher orientations, together with their interrelationships allow one to understand the complex world of teacher decision making. These ideas mean that it is a challenging undertaking to change teacher actions in a classroom through professional development since teacher knowledge, goals and orientations are well set within already practicing teachers.

What is envisaged by mapping the landscape of teaching and teacher decision making is what Jaworski (1999: 181) termed "pedagogical power", which she defined as "the ability to draw on whatever pedagogical knowledge is needed to solve problems". Mathematics teachers solve a diverse array of problems relating to mathematics itself, time constraints, curriculum

requirements and student understandings. The following chapter describes some of the practical constructs in mathematics teaching and teacher development that may allow pedagogical power to be realised.

CHAPTER 3

THE MATHEMATICAL WORK OF TEACHING AND TEACHER DEVELOPMENT

3.1 INTRODUCTION

Elbers (2003: 80) outlined three principles of teaching mathematics. Firstly the problems given to students should play a central role. Secondly a basic element of teaching is to motivate students to mathematise, and thirdly - students should develop good arguments to support their solutions. The previous chapter outlined the didactisation principles for this study. Those comprised what effective mathematics teaching includes. This chapter seeks to understand how these elements can become a reality in mathematics classrooms. The framework of Wilson and Heid is adapted for this purpose. The mathematical work of teaching, and aspects critical to teacher development programs, are the focus of this chapter. How can mathematics teachers change their practice in meaningful incremental ways? Elements of successful teacher development programs will be considered and will inform the decision made for the design of a professional development program for this study. What are the factors that affect teacher development and how can a successful teacher development program be designed that will have maximum benefit for the teachers taking part in the program? These are the questions this chapter seeks to answer.

3.2 MATHEMATICS PROFICIENCY FOR TEACHING (MPT)

Wilson and Heid (2010: 6) proposed a framework to describe and define mathematical proficiency for teaching. They opted for a "proficiency" framework as opposed to a knowledge framework since proficiency is "the observable application of a teachers' knowledge and therefore reveals knowledge held by the teacher" (2010: 19). The framework consists of three strands:

Mathematical proficiency - which consisted of:
Conceptual understanding;
Procedural fluency;
Strategic competence;

Adaptive reasoning;

Productive disposition and

Historical and cultural knowledge.

Mathematical activity - which consisted of:

Mathematical noticing (structure of mathematical systems, symbolic form, form an argument, connect within and outside mathematics).

Mathematical reasoning (justifying/proving, reasoning when conjecturing and generalizing, constraining and extending).

Mathematical creating (representing, defining, modifying/transforming/manipulating)

Integrating strands of mathematical activity.

Mathematical work of teaching (MWT)

Probe mathematical ideas

Access and understand the mathematical thinking of learners

Know and use the curriculum

Assess the mathematical knowledge of learners

Reflect on the mathematics of practice

This study focuses on this third strand (Mathematical work of teaching – MWT) as an avenue to affect teacher change. According to Wilson and Heid (2010: 16), being proficient in the MWT enables teachers to "integrate their knowledge of content and knowledge of processes to increase their students' mathematical understandings".

Hiebert, Morris and Glass (2003: 202) worked on a teacher development model that had two goals. Firstly, teachers becoming mathematically proficient and secondly, to develop teachers' knowledge competencies and dispositions that will assist their students to become mathematically proficient. It is this second aim which is the focus of this study (see 1.3.1.2). The aim of this study is to construct a framework based on didactisation principles and the mathematical work of teaching to guide a professional development program.

3.2.1 The mathematical work of teaching

The suggestion of Medley (in Ball 1991: 3) that it is what teachers *do* rather than what a teacher *is* that results in this study looking further than just teacher knowledge as a factor in effective mathematics teaching (see 1.3.1). The focus while analyzing the data will be on teachers' actions and their decisions regarding these actions (see 5.2.7, 5.3.6, 5.4.6, 5.5.6). Teacher subject knowledge is certainly a component of effective teaching, but it is the work of teaching – a teacher's actions that determine how and what students learn in the classroom. Ball (1991: 4) presents a third factor being that of teacher thought and decision that are part of what constitutes effective teaching. A teacher's knowledge is one of many factors. In this section the concept of the work that teachers perform in a mathematics classroom will be examined. What actions should teachers take to ensure or promote effective learning in their classrooms? Sensevy, Schubauer-Leoni, Mercier, Ligozat and Perrot (2005: 158) explain the teacher's work as a triple dimension made up of mesogenesis, topogenesis and chronogenesis. Mesogenesis is the process whereby a teacher organises the milieu and includes actions whereby the teacher plans to take students from one point to another. Topogenetic action describes the division of the activity between the teacher and the students. Chronogenesis describes the "evolution of the knowledge proposed by the teacher and studied by the students". These authors also talk about "didactic time" which means that the teacher may slow down the exposition of ideas or solutions so as to achieve a didactic goal.

Although the above three processes are valuable in explaining or observing teacher actions, it becomes more difficult to specify these three processes to teachers within a professional development program. The "mathematical work of teaching" as proposed by Wilson and Heid (2010) is used in this study as it provides relevant and real actions that teachers can relate to in their day-to-day actions. The components outlined by Wilson and Heid allow one to cultivate a professional development program where didactisation principles can be made tangible. The components outlined by Wilson and Heid are to: access students thinking and understandings, probe student thinking, assess student thinking and reflect on practice. As discussed in 3.2.1.3, this study suggests a further component to make this framework a coherent entity, the component of *connecting* students' understandings is added. Although Wilson and Heid also include the aspect of "know and use the curriculum", it was decided to include this in the Treffers' principle of vertically aligned lessons. South African public

schools are currently implementing a third revision of the curriculum. It was decided that if a teacher was able to align lessons vertically so that there was clear conceptual development, the teacher's knowledge and use of the curriculum is at a high level.

3.2.1.1 Access and understand the mathematical thinking of learners

Treffers in describing the five instruction principles that lead to progressive mathematisation presents the notion of phenomenological exploration. This assists one in understanding the didactical value of accessing student understandings. He proposes that mathematical activity takes place within a concrete real context so that the concepts can be explored as "multifariously" as possible (Treffers 1987: 248). The aim of this phenomenological exploration is to set up a rich collection of intuitive notions that the students have about the concepts so as to lay the basis of concept formation. Accessing student ideas must provide the basis on which concepts can be probed and connected. Wilson and Heid (2010: 16) also explain that teachers who are proficient in uncovering student ideas are able to "see mathematics from a learner's perspective". They add that it is through quality discourse that student ideas are made evident and that a proficient teacher is able to "interpret imprecise student explanations". Franke and Kazemi (2001: 104) explain that when teachers listen to students thinking, it transforms teachers into learners. This means that listening should form a large part of what teachers do in their classrooms. However it should not stop here which is why the didactistion principles set out in this study assist teachers with taking action on student thinking.

Real world problems to start concept development enable teachers to access student understandings. Modelling problems present students with complex, real world problems that will allow a teacher insight into student understandings and ideas. Modelling tasks are designed to be open multi-level - multi-solution tasks. They allow a rich domain for interaction and discussion. A typical modelling cycle (Fig 2.2) allows one to access student thinking at different nodes. The cycle shows how different modelling competencies are necessary. This relates to different modes of mathematical thinking which are accessed via modelling tasks.

Mousoulides, Christou and Sriraman (2008: 294) indicate an important function of modelling being that modelling "can result in opportunities for students to elicit their own mathematics as they work the problems and to make sense of the realistic situations they need to mathematize". Lesh and Doerr (2003a: 22) also explain that since significant forms of conceptual development occur when students work on modelling tasks, it is "possible to observe the processes that students use to extend, differentiate, integrate, refine or revise" their concepts and constructs. Students, through modelling, are able to provide evidence, via representations, of their thinking. Lesh and Doerr (2003a: 25) confirm that students produce "auditable trails of documentation" when involved in modelling tasks. Once student thinking is made explicit, the next step would be for teachers to probe this thinking. Lesh and Doerr (2003a: 25) also remind us that initial student responses to modelling tasks are barren, distorted and unstable. This is often met with surprise by teachers since they are under the impression, that concepts and ideas that have been taught, are necessarily also understood by students. Treilibs, Burkhardt and Low (1980: 53) believe that students' ability to apply mathematics "lags at least three years behind their first learning of it". This statement is not necessarily a truism since Harel and Lesh (2003: 381) have shown that students go through Piaget-type levels of development during a 90 minute modelling session. The quality of teaching and learning needs to be taken into consideration. Modelling problems provide a suitable avenue to access student understanding of the mathematics they know and the mathematics they can use

3.2.1.2 Probe mathematical ideas

According to Schorr, Warner, Gearhart and Samuels (2007: 431) research has shown that teachers "rarely probe students to determine whether their answers make sense; rarely ask students to explain, justify or share their reasoning". Probing student mathematical ideas is not just an avenue for the teacher to see what ideas can be put on the table, but as Freudenthal (1991: 100) suggests, if the principle of guided reinvention is used for teaching and learning, the "guide should provoke reflective thinking". Freudenthal (1991: 100) emphasises that reflection is a characteristic of mathematical thought. So, in probing student ideas, a teacher is really allowing students to think reflectively. Reflective thought, according to Freudenthal

(1991: 100) is a "forceful motor of mathematical invention". Reflective thought can be stimulated by probing questions from the teacher.

According to the Maryland State Department of Education (in Sahin and Kulm (2008: 224) probing questions "push students to use previous knowledge to explore and develop new concepts and procedures". These authors also add that probing questions have didactical purposes such as to extend knowledge and to encourage explanations. They do however comment that teachers still ask more factual questions than any other, and that this has not changed in nearly 100 years (Sahin & Kulm 2008: 238). In identifying probing questions, Sahin and Kulm used the following criteria:

- Asking students to explain or elaborate
- Asking students to use prior knowledge and apply it to a current idea or problem
- Asking students to justify or prove their ideas

In a study by Henning, McKeny, Foley and Balong (2012: 458-459) that focused on mathematical discussion in classrooms, they proposed that "teacher follow-up moves" (which is considered to be similar to probing) comprised of the following teacher activities in student orientated classrooms: providing hints, elaborating on student responses, reformulating student responses, providing explanations or recapping several student ideas. While in teacher orientated classrooms, the follow up moves by a teachers were to validate or reject student responses. Probing of student thinking from a problem-centred of modelling perspective therefore requires more teacher anticipation and synthesis of divergent ideas than is expected from traditional classrooms. This new role for teachers is challenging and requires a greater sense of confidence from teachers. The study of Henning *et al.* (2012: 472) presents evidence that these "teacher-guided follow-up moves" make up a smaller percentage of classroom discussion in framing and application discourse than discussions around eliciting student responses. Probing seems to play a significant role in the development of lessons and student thinking about ideas during lessons whereas in traditional approaches teachers spend more time eliciting and correcting responses.

Probing mathematical ideas is an extensive part of the modelling process. The ideas presented by students are broad and need to be probed so that students can determine the effect each idea may have on the outcome of the problem. Probing student ideas enables them to project and predict the possible outcome of their thinking. Modelling tasks provide suitable starting points in their problems that lead to many avenues for probing student ideas. They also allow for teachers to explore the underlying structures and concepts in students thinking. Teachers will need to experience modelling problems and experience students solving modelling problems so that they can situate the important role probing student ideas requires.

3.2.1.3 Making connections

According to Eli, Mohr-Schroeder and Lee (2011: 298) "constructing, unpacking and understanding connections are fundamental in carrying out the work of teaching mathematics". Hodson (in Eli *et al*. 2011: 298) points out it is a problem situation that "leads naturally to the establishment and use of connections". Seeking and bridging mathematical ideas and concepts that students have is a vital role that teachers need to develop instead of presenting concepts with connections that are taken for granted. Eli *et al*. (2011: 299) also maintain that connections are a natural outcome of a constructivist theory while Treffers and Goffree (1985: 109) equated "raising the structure level in the corresponding problem field" with vertical mathematisation. Connections enable a teacher to make on the spot decisions about student assessment and student progress. If teachers are sensitive to connecting student knowledge and understandings it opens up the mathematical space in a classroom and allows for differentiation (see 2.4.3) to take place. Making connections and finding corresponding structures allows for vertical mathematisation to manifest in mathematics classrooms.

(a) Connecting the mathematical understanding of students

Noss and Hoyles (in Mitchelmore & White 2007: 2) characterised abstractions as "a process of connection rather than ascension" and proposed the concept of "webbing" as the structure students can draw on for reconstruction and support as they construct mathematical meaning. Hiebert and Carpenter (1992: 67) defined mathematical understanding of a concept as being determined by the "number and strength of its connections". The number and nature of connections made in a mathematics classroom enable one to gauge the extent to which

students are constructing meaning or the extent to which teachers are transmitting readymade connections.

> When connections were lost, teachers most often took over challenging aspects of the problems or shifted the focus to procedures, answers, or superficial or vague treatment of concepts. Regardless of whether or not connections were made, in about half of all implementations, teachers did most of the mathematical work, in about 8% of implementations students did it, and in the remainder, the work was shared more or less equally. (Birky 2007: 2)

In a study comparing USA classrooms and Japanese classrooms, Stigler and Hiebert (1998: 4) found that in Japanese classrooms the meaning in mathematics lessons is found in the connections between different parts of the lesson. Stein *et al.* (2008: 331) maintain that there are many ways that a teacher could assist students in making connections. They propose that a teacher ask how strategies or presentations are similar or different to each other. This does however mean that the teacher has knowledge of various strategies that students could use and why students may present them.

Douek (in Boero *et al*. 2008: 263) considers conceptualisation as a complex process that consists of students constructing the components of the concept; constructing links between different concepts and developing consciousness about these links. Therefore, the mathematical tasks that students engage with should foster conceptual understanding. This, according to Hiebert and Carpenter (1992: 69), means that representations need to be connected in a growing structured and cohesive network. They add that it is the connections which create the networks that form the relationships such as similarities and differences. This is made possible by solving problems in different ways so that the development of mathematical concepts is "supported by shifting between representations, comparing strategies and connecting different concepts and ideas" (Fennema & Romberg 1999 and Silver *et al*. in Leiken & Levay-Waynberg 2007: 350). A discussion on concept connections also leads one to discuss the idea of mental schemas. Marshall (1995: 15 in Eli *et al*. 2011: 299) defines a schema by the presence of connections. Connections emanate from a schema, while a schema consists of a network of connections. The relationship between connections and schemas are consistent.

Skemp (1986: 37) explains a schema as a conceptual structure where concepts are connected. He further adds that schemas act as tools for further learning and make relational understanding possible. New learning takes place by adding to existing schemas where the new knowledge or idea can be placed into the existing schema. Skemp (1986: 41) also mentions that knowledge that does not fit into an existing schema is not learnt at all. He further warns that unsuitable schemas are therefore a hindrance to learning. Students should have schemas in place that have relevant connections between the concepts. If new knowledge does not fit into the schema, the schema must be restructured (Skemp, 1986: 41). Poorly constructed schemas will therefore implode when students are confronted with knowledge or ideas that they cannot assimilate to any part of a schema. Teaching via and for connections is vital to the building of useful schemas for students.

Dubinsky (1991a: 164) defined a schema as a consistent collection of cognitive objects together with the mental processes for manipulating these objects. He added that a schema involves activities and procedures that can be used for solving problems and that may become part of later constructions. Students take action on mathematical objects and these can become processes (Dubinsky 1991a: 181). Dubinsky explains that a schema can be generalised when it can be applied to a new situation. This will require that students can make connections within their mathematical knowledge flexibly and adaptively.

While Andrews (2009:115) reminds us that research shows that flexible or adaptive mathematical knowledge requires an integrated and connected set of concepts and procedures, Silver *et al.* (in Leikin & Levav-Waynberg 2007: 351) state that a limitation in teachers' mathematical knowledge may hinder the use of multiple solutions in a classroom. This in turn will affect how students make connections which will hinder their ability to construct objects, processes and schemas.

Trzcieniecka-Schneider (1993: 258) models a mathematical concept as a core that can be transformed. She further affirms the link between representations and flexible concepts by stating that "the higher the amount of transformation allowed, the more plastic the core becomes". The focus on connections in mathematics teaching and learning is not solely for students to develop a wider network of concepts within a concept, but to develop concepts

that are flexible and can be transformed and used in different situations. This competence is lacking in traditional approaches.

Birky (2007: 80) discussed and coded features of teachers' attempts to make connections as:

- Comparison of mathematics of solution methods
- Connection between representations
- Examining a concept
- Generalization
- Justification
- Problem solving

By examining this list, it is possible to understand how modelling tasks will enable teachers to assist students in making connections. Modelling tasks require that teacher knowledge should include "an understanding of the multiplicity of children's models as those models develop along multiple dimensions" (Lesh & Doerr 2003b: 554).

Lester and Mau (1993: 8) define the teachers' role in teaching via problem solving as bringing possible generalisations to light during discussion with a class while working through solution possibilities for a problem. Being able to generalise is a higher order thinking skill, and from a modelling perspective, being able to generalise a model is an advanced form of thinking about a problem situation. Gravemeijer (1994a: 83) explains that generalizing is to be understood in a reflective sense. He says that it refers to the construction of connections rather than application of general knowledge. Therefore the ultimate aim in teachers' assisting students in making connections is that students may then be able to understand the generalised structure of a mathematical concept. Rasmussen *et al*. (2005: 65) point out that a lack of generality in students' procedures when solving problems is indicative of horizontal mathematising. Once students are able to develop algorithms they are advancing their "mathematical activity" (2005: 65).

Hiebert *et al*. (1997: 4) define mathematical understanding as seeing how something is related or connected to other things that a student knows. They discuss three ways by which a teacher should share information in a classroom. These procedures will assist students in making connections within the classroom community. Firstly, the teacher should share

mathematical conventions for recording and communicating ideas (names, written symbols, special terminology) within the wider mathematical community. Secondly, teachers should share alternative methods and thirdly, teachers should articulate the ideas in students' methods by highlighting the mathematical ideas (Hiebert *et al*. 1997: 36). These three activities by the teacher should assist students in making connections within the ideas and concepts developed through the lesson. These authors propose two important but integrated concepts that assist in developing connections: reflecting and communication. These two activities place students "in the best position to build useful connections in mathematics" (Hiebert *et al*. 1997: 6). It will be shown (2.4) that didactisation principles set out by Treffers allow for students to reflect and communicate in a mathematics classroom.

Freudenthal (1991: 92) spoke about learning discontinuities or jumps, a spontaneous shortening of a mathematical process. It is within the realm of making connections that these jumps or discontinuities may give rise to vertical mathematisation or shortening of mathematical methods and processes by students. Making connections and mathematisation are linked. Horizontal mathematisation (see 2.4.4.1) can be seen as connecting the real world with mathematical ideas and constructs while vertical mathematisation (see 2.4.4.2) can be seen as forging connections and links between more complex and abstract mathematical ideas and constructs. Gravemeijer (1994a: 103) explains that generalizing, as a component of vertical mathematisation, does not mean that students are able to apply a routine procedure, but an "a posteriori construction of connections between various situations". Modelling tasks enable students and teachers to make these meaningful connections in their mathematical knowledge. For teachers, modelling tasks should enable them to consider the thinking paths of their students and assist them in understanding the mathematical learning reality of their students.

(b) Representations in making connections

Goldin (2008: 184) identified five different types of representation and proposed that each of the internal representational systems (verbal, imaginistic, formal notational, meta-cognitive and affective) allow students to produce a vast arrangement of complex and subtle external constructions that other people can interpret meaningfully. Goldin further adds that

representational system develops through three stages: an inventive stage, a period of structural development and an autonomous stage. Barmby, Harris, Higgins and Suggate (2007: 42) constructed a definition of mathematical understanding as making connections between mental representations of the concept and the network of representations that is associated with that mathematical concept. Mathematical understanding is therefore very closely linked to connections and representations. It appears that connections form the bridge between representations and understandings.

Gattegno (1971: 26) described the power of the mind when using imagery in mathematics learning. He states that calling on "mental evocations to advance mathematical understanding" allows one to remain in contact with mental energy and keeps continuity between the initial and final forms of the images. The concept of mental energy ties in with Treffers' didactiation principle of active learning and active students. In a description of mental representations, von Glaserfeld (1991: 52) alludes to the selectiveness within graphic and schematics representations. Since a representation includes "precisely those aspects one wants to or happens to focus on" they are considered "didactic" because they "can help focus the naïve perceiver's attention on the particular operations that are deemed desirable" (von Glaserfeld 1991: 52). In this way we can understand that representations do not display a whole idea, but the essence of the idea. This means that representations are valuable when attempting to assist students in making connections in mathematical concepts.

Davis (1984: 39) categorises representations into active and static structures, and suggests that students either retrieve or construct representations. He adds that representations are "fundamental to mathematical thought", and that problem solving relies on two aspects of representations: how the problem is represented, and how relevant knowledge is learnt in the past, is represented (1984: 78). Davis (1984: 234) also reminds us that knowing one type of representation does not easily allow one to construct another type. This is where a teacher needs to assist students in making connections through various representations. A teacher should not only point out connections between various representations but rather allow students to present and explain various presentations to each other. Dreyfus (1991: 39) in explicating the stages of learning processes explains that in reaching the higher stages students are able to switch between representations which will make them aware of the

underlying concept, and this will "positively influence abstraction". This explains the link between making connections, representations and abstraction.

Rasmussen and Marongelle (2006: 389) introduced the notion of a pedagogical content tool (PCT), which they explained as

> a device, such as a graph, diagram, equation, or verbal statement, that a teacher intentionally uses to connect student thinking while moving the mathematical agenda forward.

Teachers need to understand the nature of connections and representations. They need to value the interdependence that one has on the other and the role both connections and representation play in effective teaching.

From a modelling perspective, Cramer (2003: 450) explains that the development of deep understandings means that students and teachers make connections between and within different modes of representation and that a "translation requires a reinterpretation of an idea from one mode of representation to another". She also adds that "this movement and its associated intellectual activity reflect a dynamic view of instruction and learning". This is the view that is supported by this study and a view that this study hopes to grow in teachers (see Table 5.1; 5.9; 5.15; 5.23). A models and modelling perspective assists in growing this view of teaching and learning as well as developing connections through representations. Students' ability to develop connections through representations and to develop relations between representations will lead them to vertical mathematisation and abstraction.

3.2.1.4 Assess student understandings

Part of the mathematical work of teaching is assessing students. If teachers do apply didactisation principles in the form of active students, differentiation, mathematisation and vertically aligned lessons, students own constructions and productions (Treffers 1987: 249) provide the concrete visible evidence of student progress. The view of Volmink is relevant here. He states that assessment has been used to contrast and compare students. He feels that we need to develop assessment that is "illuminatory instead of discriminatory" (Volmink 1994: 63) and that assessment should not "discriminate, but rather celebrate the value of each person" (1994: 63). Assessment should relate to individual students and across students. This

can be achieved if the teacher understands the mathematical knowledge that is being assessed and has a framework of the various levels that may result around a particular concept.

Goldin (1992: 82) explains that an assessment framework based on a cognitive model should have the following characteristics:

- It should be based on an independent characterization of the understanding we want to assess, so that we can infer cognitive capabilities from behaviors without identifying abilities with behaviors.
- It should be descriptive, capable of informing us what a student can and cannot do, and capable of describing concepts and representations that are partially developed.
- It should be reflective allowing students not only to grapple with mathematical discovery and conceptual constructions but to reflect on these processes.

Van den Heuvel-Panhuizen (1996: 85) explains that within RME, assessment means a "didactical assessment". The principles for assessment set out by RME theory are dynamic and formative. Van den Heuvel-Panhuizen elaborated on RME assessment principles by stating the purpose, content, procedures and tools should all be didactical. This means that assessment should focus on educational evaluation and educational development. The content should reflect a wide variety of goals and should provide insight into a student's mathematisation activities (1996: 86). In keeping with RME principles, van den Heuvel-Panhuizen explains that there will be integration between teaching and assessment which means teachers should make use of more informal forms of assessment such as observation, and oral turns. It is important that assessment show what students do know (de Lange in van den Heuvel-Panhuizen 1996: 87).

De Lange's five principles for assessment within a RME framework are:

1. Students are active participants and should receive feedback on their learning.

2. Assessment should allow students to demonstrate what they know, not what they do not know.

3. The task should not only focus on the product, but on the process that leads to the product.

4. The quality of the assessment is not defined by its ease of objective scoring.

5. Assessment should be able to be carried out in school practice.

(Van den Heuvel-Panhuizen 1996: 86-90)

Assessment should also be considered in more holistic terms in terms of how students are progressing in their mathematical endeavours. For the purposes of this study, assessing is gauged during classroom interaction. It is closer aligned to informal assessment that a teacher makes to gauge the progress of the lesson. Since the focus of this study is on a number of didactisation principles to improve teacher practice, assessment is based on how a teacher responds to student ideas and thinking during the lesson, and not on a specific task given for a teacher to formally assess. This latter aspect falls outside the scope of this study but may be significant in a related study on didactisation practices and assessment.

3.2.1.5 Reflect on the mathematics of practice

Reflection plays a significant role in any learning. Von Glaserfeld (1991: 46) explains that reflecting on an experience is not the same as having the experience. He presents von Humboldt's aphorisms on reflections which state that "in order to reflect, the mind must stand still for a moment in its progressive activity, must grasp as a unit what was just presented, and thus posit it as object against itself" (1991: 47). This leads him to conclude that reflection is the simplest kind of abstraction. Freudenthal (1991: 105) specified a number of modes of reflection or "shifting standpoints". Although he described them in the light of mathematics learning, they are equally powerful for analyzing teacher learning. He called shifting from A to B in order to look back at A as reciprocal shifting, shifting from A to B while considering C as directed shifting and shifting A's environment to B's as parallel shifting. A number of activities and instruments are part of the study in light of the important role reflection plays and these three dimensions will assist in documenting the shifting standpoints teachers have when it comes to the mathematical work of teaching or didactisation. Tall and Poynter (2005: 1265) remind us that within pragmatic cultures teachers "work hard with long hours and relatively little time scheduled for analysis and reflection".

Wilson and Heid (2010: 18) recommend using a "mathematical lens" to reflect on one's practice. These authors remind us that often reflection takes place as teachers make split-second decisions (2010: 19). They feel that it is important for teachers to revisit these quick reflections and decisions in order to learn from one's teaching.

Reflection as a professional development activity is factored into the teacher program of this study (see 4.2.2). Since reflection is dynamic and ongoing, the design research approach is best suited to this since it includes multiple cycles. This means that reflection can be factored into these cycles. If teachers are supported on an ongoing basis in their reflective practices, they will be in a stronger position to factor this into their daily teaching work, which will assist them reach higher levels of teacher proficiency.

3.2.2 Hypothetical learning trajectories (HLT) facilitating the mathematical work of teaching

Simon (1995: 133) and Simon and Tzur (2004: 91) described the necessity for a hypothetical learning trajectory in improving mathematics teaching and learning. Simon found a need for exploring "the ongoing and inherent challenge to integrate the teacher's goals and direction for learning with the trajectory of students' mathematical thinking and learning" (1995: 121). He further defines a hypothetical learning trajectory as a teacher's consideration of the learning goal, the learners' activities and the type of thinking students might engage in (1995: 133). When a teacher formulates a mental or verbal hypothetical learning trajectory for a lesson or lessons - the teacher co-ordinates the mathematical work of teaching in a more meaningful way. Simon (1995: 135) reminds us that the hypothetical learning trajectory formulated by the teacher is a "prediction as to the path by which learning might proceed". Simon (1995: 136) presents the following diagram to explain how hypothetical learning trajectories form part of a teaching cycle. He presents the concept of hypothesis formulation for mathematics teachers.
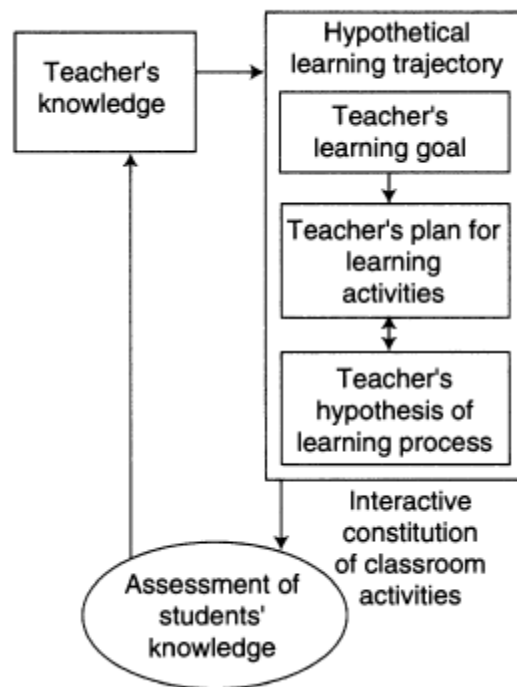
Figure 3.1 Simon's mathematics teaching cycle

In formulating a hypothesis about a lesson or learning activity the teacher will therefore integrate the various forms of teacher knowledge (see 2.3.1.) to construct a learning trajectory. This trajectory will include what the teacher anticipates and how students will respond to the problem or situation he/she presents. In this way, the teacher can therefore plan for the mathematical work of teaching. He or she will plan how to access, probe, connect and assess student thinking based on a prediction that is informed by his/her current knowledge, orientations and goals. The reflection phase of the mathematical work of teaching also becomes more meaningful and urgent. The teacher's hypothesis will need to be accepted or refuted. Based on the result of reviewing the hypothesis for the lesson, the teacher can plan the following lesson or activity. In this way hypothetical learning trajectories assist teachers in vertically aligning lessons (see 2.4.5) In formulating a hypothesis the teacher has a much higher personal stake in the lesson than if he/she was simply teaching to an end goal only. In a study by Zodik and Zaslavsky (2008: 175) on teachers' use of examples in lessons, the authors found that in the category of teachers using examples to attend to student errors – only 21% of the examples were pre-planned by the teachers, while 79% were spontaneous examples. The formulation of a HLT should assist teachers pre-plan for student difficulties

and not only deal with errors in a spontaneous manner. This means that the teacher approaches a lesson with more sensitivity where the components of the mathematical work of teaching integrate in a holistic way.

Simon (1995:138) maintains that a teacher can make changes to the HLT whether it is the goal, the activities, or the hypothetical learning process. Simon and Tzur (2004: 96) elaborate on the HLT and indicate that it is the selection of the activity that will lead students to the intended learning that is a crucial part of a HLT. This study presents problem-centred learning and modelling as suitable areas for activity and task selection (see 2.2.3 and 2.2.4).

There are several strengths to Simon's HLT. Firstly – and of relevance to this study on teacher development, is that student thinking is taken seriously and given a central place and secondly, that changes in teacher knowledge are reflected in changes in the HLT (1995: 141). This study adds that a HLT can assist teachers to implement didactisation principles and to integrate the elements of the MWT into their planning and preparation for lessons. Furthermore, a HLT is not limited to teachers and classrooms. This study, through design research, also uses HLT in guiding the teaching experiment (see 4.2.2).The formulation of a HLT is integrated into the professional development program of this study.

3.3 TEACHER DEVELOPMENT

Calls to reform mathematics teaching have proposed changes to what teachers do in their classrooms and how they conduct lessons and assess their students. These changes proposed a move away from traditional transmission methods towards inquiry based methods. As stated clearly by Guskey and Sparks (1991: 75) student learning is not likely to improve unless there is a change in teacher knowledge, skills, practices and eventually teacher attitudes and beliefs. This ties in with Schoenfeld's three pillars for changing teacher decision making (see 2.3). According to Engle and Conant (2002: 403) there are four norms that teachers should embody in their classrooms to encourage productive engagement of students: problematizing content, giving students authority, holding students accountable to others, and for teachers to provide relevant resources. They use the term authority in that students should be authorised to solve mathematics for themselves and they use the term accountability in that students should account for their ideas. In this way students should be the authors of their thinking. This is in

contrast to traditional classrooms where teachers author everything and are solely accountable for the lesson. Students have to transcribe and memorise. Moving teachers from a traditional approach to a reformed approach will rely on improved knowledge of how students learn.

In a study on professional development effects on teacher efficacy (Ross & Bruce 2007: 58) the following strategies were found to increase the mastery experiences teachers may encounter. Firstly they strengthened the teacher's ability to manage classroom discussion. This was done by providing rich tasks, modelling the use of tasks in simulations, allowing teachers to apply the principles in the TDP in their own classrooms and debriefing the classrooms experiences. The second strategy was to re-define lesson success and the third was to provide opportunities for teachers to interact with their peers about their implementing reform practices. Ross and Bruce (2007: 50) defined "teacher efficacy" as a teacher's self-perception of his or her measure of teaching effectiveness or the "teacher's expectation that he or she will be able to bring about student learning". They proposed that teacher efficacy generated stronger student achievement. Teachers with a higher efficacy scores:

- Were more likely to try new teaching ideas
- Used approaches that stimulated student autonomy
- Attend more closely to the needs of lower ability students
- Modified student perceptions of their own academic abilities
- Were persistent in their attempts for students to be successful.

Gattegno (1971: 51) outlined important aspects that professional development proposers should consider if any suggestions were to find an audience with teachers or if teachers would give the ideas "a chance". According to Gattegno, although teachers will lend their ears to improvement of practice, it is the proposer of the new ideas that has to ensure that teachers are presented with all that is needed to make the change "a possibility". He adds further that "the more a particular change demands from teachers, the more the proposer must work on details and provide special demonstrations to clarify the way this change can occur" (Gattegno 1971: 52). These ideas led to the development of an intensive program for teachers that included many differing activities and instruments (see 4.2.2) in order to make change a possibility.

3.3.1. Elements of successful TDP

It is important to state at this point that this study supports the view of Doerr and Lesh (2003: 127) that teachers should be seen as evolving experts. This means that there is no set benchmark in this study of excellence or weakness in teaching. All teachers therefore are considered as moving towards becoming experts. Where the teachers involved in the study are in terms of their teaching is not as important as whether the study can move them closer towards an expert status. It is the study that carries the critical component, and not the teachers. As stated by Parker (in Hammerman 2001: 21) "a teacher cannot teach differently until they have experienced mathematics differently". In order to experience mathematics differently to the way they were taught and since "students should learn through meaningful problem solving experiences" then "teachers should learn through personally meaningful problem solving experiences." (Schorr *et al.* 2007: 433).

Successful teacher development programs that focused on teachers work on representations, explanations and communication outperformed similar teacher development programs (Hill & Ball 2009: 70). In order to delineate elements that would result in an effective teacher development program, the modelling perspective of Koellner-Clark and Lesh (2003: 161) that "for almost any principle we apply to children there is an analogous situation for teachers" was used to crystallise which elements to include in the professional development program of this study.

Clarke and Hollingsworth (in Dole, Nisbet, Warren and Cooper 1999: 39) identified six perspectives to teacher change.
- Change as training: where teachers are trained in specific skills.
- Change as adaptation: where teachers change as a result of a change in their working environment (new policies, increase in class size).
- Change as personal development: where teachers themselves identify their own needs and work on improving these areas.
- Change as local reform: where teachers work together to change their working environment.
- Change as systemic restructuring: where external bodies impose change.

- Change as growth or learning: where teachers come together as a group to discuss and work through important issues.

It is envisaged that through the development program designed in this study, change as personal development and change as growth are the guiding principles (see 5.2.9; 5.3.8; 5.4.8; 5.5.8). Handling change on these two levels should enable teachers to deal with change that may come about through the other perspectives.

It is important to understand the concept of development. Zawojewski *et al.* (in Brodie & Shalem 2011: 2) argue that professional development should not expect teachers to converge to a particular standard but teachers should grow and improve as a result of taking part in the professional development program. Brodie and Shalem further delineate important principles for teacher development. Two principles are highlighted here – that professional development should design both educational objects and provide opportunities of systematic reflection; and that teacher networks are essential for teacher learning. Teachers need to learn within a community of other teachers.

According to Dole *et al*. (1999: 38) the inference from existing teacher development literature is that, to transform teaching, teachers should experience new teaching ideas and be encouraged to try the new ideas in their own classes. Guskey (1986: 6) also reminds us that teachers attending professional development with a "very pragmatic orientation". He adds that they want "specific, concrete, and practical ideas that directly relate to the day-to-day operation of their classrooms". Kilpatrick *et al.* (2004: 391) reiterate that professional development that focuses on helping teachers understand specific mathematics content domains, and students thinking in that domain, will more often effect a change in the teachers' instructional practice.

Davis (2003: 6) in a study on science teacher change to reform orientated practices highlighted important points to consider for teacher development. Reform programs should:
- enable teachers to reflect upon and make explicit their personal practical knowledge;
- consider teachers' knowledge and practices as the starting point of change;
- provide teachers with experience and training in reform based strategies;

- provide teachers with opportunities to see these approaches modeled and to reflect upon these models;
- enable teachers to design their own practices in a supportive environment where feedback is provided;
- provide teachers with collaborative settings with other educations; and
- provide teachers with access to experienced professionals as mentors and guides.

Although the above points are important for development programs – this study also focuses on mathematisation (see 2.4.4) as both a theoretical and practical construct in how students form new and more abstract mathematical ideas.

Porritt and Earley (2010: 7) identified four approaches that were "at the heart of successful practice" for the professional development of teachers. They were:

- Participants ownership of the project
- Engagement in a variety of activities and opportunities
- Time for reflection and feedback
- Collaborative approaches

This study has strong elements of all these approaches in bringing didactisation principles to teachers in a professional development program (see 4.2.1.3)

The domain of teacher professional development is complex and consists of many interrelated aspects. The following criteria are considered to be fundamental to a professional development program where teacher decision making and improved didactisation principles through modelling are considered. In line with design research, the following section is part of formulating "design principles" (McKenney *et al*. 2006: 73) which are one of the products of design research. These authors explain that design principles are "not intended as recipes for success" but rather to assist others in selecting and applying the most appropriate knowledge for developing tasks "in their own settings".

3.3.1.1 Professional development focusing on teacher beliefs

According to Hammerman (2001: 4) professional development that is designed to change teacher thinking is a key in ongoing learning. He also stated that there is a complex

relationship between a teacher's beliefs and a teacher's practices and that according to Guskey (in Hammerman 2001: 5), a change in practice can effect a change in teacher beliefs. Harkness (2009: 245) divulged that perhaps it is easier for teachers to:

> doubt mathematics that is not procedures or memorized rules to be followed than it is to look at the students' mathematics with a believing lens. Possibly, doubting also caters to teachers' own mathematical understanding or weak understanding.

The focus on teacher development through beliefs is best conceptualised using Pennington's cognitive-affective filter. (see Fig. 2.3) In an attempt to reconstruct the filter, teacher beliefs need to be challenged during the professional development program. In a review essay that critically examined mathematics teacher change (Goos & Geiger 2010: 505) a common thread in modern literature on teacher change was the role of "productive tensions in creating opportunities for mathematics teacher change". This study seeks to create productive tension in the professional development program by using modelling tasks to elicit cognitive conflict (see Table 5.6).

3.3.1.2 Professional development using classroom cases

The use of classrooms cases has also produced success in teacher development. According to the National Research Council (NRC 2001: 394) teachers learned mathematics from professional development programs that did use classroom cases. Those teachers who "gained a greater repertoire of ways to represent mathematical ideas, were able to articulate connections among mathematical ideas, and developed a deeper understanding of mathematical structures."

It is important that teachers feel that professional development is for them and about them and not imposed upon them from some outside organisation. Teachers should be able to apply the ideas from professional development in their classrooms. In fact, Guskey and Sparks (1991: 74) found that the contexts that cultivate support and shared decision making are best for successful development programs. According to Chamberlain, Farmer and Novak (2008: 439), student-centred professional development programs effected classroom changes in that the teachers involved made students thinking a central feature in their lessons, used less drill and practice, actively engaged students in mathematics and demonstrated confidence and beliefs that were more aligned to reform ideals.

In this particular study, the use of a communal *fishbowl* classroom case is used (see 4.2.1.3). This means that the teachers involved in the study have a direct communal point of reference for the activities and discussions of the program. They may also feel less threatened by discussing the fishbowl classroom than defending their own practices in their own classrooms. The communal fishbowl classroom places all teachers within the same milieu and will allow for a shared experience for discussion. Guskey (2002: 383) spells out a crucial point for successful professional development programs. He states that:

> it is not the professional development per se, but the experience of successful implementation that changes teachers' attitudes and beliefs. They believe it works because they have seen it work, and that experience shapes their attitudes and beliefs.

3.3.1.3 Professional development that has teacher as student

Ross and Bruce (2007:52) indicated that participant interaction increased the opportunities for "vicarious experiences". Garet, Porter, Desimone, Birman and Yoon (2001: 925) discussed a core feature of professional development as promoting teachers' active learning. This could take many forms: observing and being observed, planning for classroom implementation, reviewing student work, presenting, reading and writing. Steinberg, Empson and Carpenter (2004: 238) found that research showed that teachers who engaged in practical enquiry were able to change their teaching. This is why a teacher development program that focuses on teacher activity in both thought and action was sought after for this study (see 1.3 2.1). Moreover, a modelling based program where teachers granted the same experiences as their students is considered "absolutely essential" (Schorr & Lesh 2003: 143) and would advance teachers' own knowledge of mathematics, of how students build ideas and of teaching mathematics in such a way that students could develop powerful models (Schorr & Lesh 2003: 143). When teachers themselves complete the activities they will give to their students, this provides them with "experience [of] how initial interpretations of the problem are likely to be primitive and unstable" (Zawojewski *et al.* 2003: 356).

Tzur (2010: 58) does however warn about teacher development that only focuses on the teachers' conceptions and provides a sound alternative that includes professional development that focuses on the student's conceptions too. He called for teachers to have an understanding

of student mathematics as being "qualitatively different from the teachers' understanding and, thus, as the conceptual force that constrains and affords the mathematics students can 'see' in the world" (Tzur, 2010: 51). Or as stated by Hiebert (1984: 507) that teachers must become aware of "how mathematics looks" to children. Tzur (2010: 52) refers to Steffe's (1995: 495) distinction of first order models and second order models. A first order model is one's own model of a mathematical concept and a second order is the teacher's model of the students' conceptions. For this reason, the professional development program designed for this study included three levels of model development for teachers (see 4.2.1.3) A first order model is where teachers solve the task for the first time, a second order model, where they observe and interact with a group of primary school students solving the task and a third order model, or an interactional model, where the teachers adjust their practices. This concept is not unlike multi-tiered modelling teacher development programs (Schorr & Lesh 2003: 149, Koellner-Clark & Lesh 2003: 161).

3.3.1.4 The use of tasks in professional development

In a study on mathematics teacher professional development tasks, Ferrini-Mundy *et al*. (2007: 313) looked at how teachers developed a deeper understanding of the "didactical dimensions" through tasks. This phrase provides an ideal scope to assess tasks for this study (see 4.2.1.2). They also tried to create tasks for teacher development that would raise issues and "upset current assumptions" (Zaslavsky in Ferrini-Mundy *et al*. 2007: 313). This parallels the view of this study in using modelling tasks for teacher professional development (see Table 5.6). Garet *et al.* (2001: 920) examined three core features of professional development activities. The degree to which the activity had content focus; the extent to which the activity offered opportunities for active learning and the degree to which the activity promoted coherence in teacher professional development by including experiences that were aligned with state standards and assessment and by encouraging continuing professional interaction between teachers. They viewed the "degree of content focus as a central dimension of high-quality professional development" (2001: 925). Brown, Smith and Stein (in Hill 2004: 217) found that teachers, who experienced novel curriculum materials in their professional development, used more cognitively complex student tasks in their classrooms.

Liljedahl (2010: 420) described a push-pull rhythm of change that explained a teacher's rapid and profound change. It encompasses a series of up to four phases related to tasks or activities teachers are involved in during professional development. In the first phase (x) called exo-spection, the teachers are involved with an activity that they focus on and solve; here they are *holding* the activity. In the second phase (X), eXo-spection, teachers sense disparity between this type of task and their usual classroom teaching. They then decide that the disparity lies outside of themselves (school, curriculum, external tests etc) – here according to Liljedahl, they are *pushing* the activity away. The third phase (N) called eNdo-spection, finds the teacher changes his/her outlook on the activity and the teacher tries to fit the activity into their own practice and classroom, and this is the *pulling* phase. The final phase (n) is endo-spection where the teacher's attention shifts to individual student(s). The teacher pulls the problem closer to him/her self and focuses on his/her influence. Liljedahl (2010: 421) also says that there is a delicate shift in the teachers' communication from teaching to learning.

3.3.1.5 Reflective thinking in professional development

Llinares and Krainer (2006: 449) identified within PME (Psychology of Mathematics Education) research the acknowledgement that reflection promotes teacher change and development. Pennington (1996: 342) defined reflection as a "way of reorganizing or reconstructing experience to gain new understandings of action situations in classrooms, of the self as teacher and of assumptions previously taken for granted about teaching". Reflection in a teacher development program needs to be continuous, consistent and woven into all aspects of the development program. All teacher thinking and action trajectories proposed in the professional development program should include purposeful, context and content relevant reflective thinking. This is made possible by a design research framework where reflective thinking activities are woven into the professional development cycles.

Breen, Candlin, Dam and Gabrielsen (in Pennington 1995: 706) concluded that lasting change in teachers occurred when teachers were motivated to try new things, to reflect on the consequences of those and then adjust their practice and thinking according to the results. Pennignton (1995: 706) explained two senses of reflection: reflection as "deliberating" on experience and reflection as "mirroring" the experience. She further concluded that critically

reflective teachers should deliberate on an experience and change their actions rather than simply mirror the external content of a professional development experience. She quotes Hunt (Pennington 1995: 707) and advised that the trajectory for teacher is "from outside in to inside out". She explains that reflection leads to change in teacher behaviour, attitude and beliefs (Pennington 1996: 342). Tzur confirms the important role of teachers reflecting on their own practice in stating that "as long as (problematizing) their own teaching cycle is not the source of teacher learning, suggested improvements will, at best, be adopted superficially and bound for quick decay" (Tzur 2010: 51).

Llinares and Krainer (2006: 449) explain the double role that reflections play in teacher development. They increase teachers' understanding and they help researchers interpret teachers' learning. This double role therefore is a valuable tool to use in teacher professional development and will be incorporated in the program proposed by this study (see 4.2.2).

3.3.1.6 The role of time frames in professional development

Longer TDPs show better results than short term ones. Firstly, longer term programs provide an "opportunity for in-depth discussion of content, student conceptions and misconceptions, and pedagogical strategies" and it is "more likely to allow teachers to try out new practices in the classroom" (Garet *et al*. 2001: 922). According to Garet *et al.* (2001: 921) locating professional development within a teacher's normal working day may make connections to teachers' classrooms and may be easier to sustain over time. This study takes this into account in the design of the professional development program that will be implemented (see 4.2.1.3) Their study revealed that time span and contact hours had a "substantial positive influence on opportunities for active learning" (2001: 933) and that active learning is related to enhanced knowledge and skills and that enhanced knowledge and skills have a substantial influence on changing teachers' practice. Goldenberg and Gallimore (1991: 69) suggested saying "goodbye to quick-fix workshops" and rather generating frameworks in teachers' working lives that support and maintain meaningful changes. While Smith, Hofer, Gillespie, Solomon and Rowe (2003: xiii) found that the most important professional development factors for teacher change are time and quality. This study aims at balancing the time needed for the professional

development with the time that teachers can give to the program while still remaining focused and enthusiastic (see Table 4.1).

3.3.1.7 Creating communities in professional development

A basic tenet of this study is stated by Smith (2003: 3) that "learning is, thus, not seen as the acquisition of knowledge by individuals so much as a process of social participation". The format for the teacher development program in this study is envisaged and informed by a number of theoretical ideas. The major focus of the program is on four or five teachers collaborating and working together on the activities and tasks presented (see 4.2.1.3). That means that there is a social learning aspect that needs to be placed within a theoretical context. One of which is Lave and Wenger's community of practice and community of inquiry. A community of practice is the basic building block of a social learning system. Communities of practice grow out of experiences that involve mutual engagement and a means to "negotiate competence" through participation (Wenger 2000: 229). Wenger (2000: 226) also distinguished between three modes of belonging as: engagement, imagination and alignment. Engagement means working on things together, imagination means constructing an image of ourselves in order to reflect, and alignment refers to a mutual co-ordination of perspectives and actions to reach higher goals. It is envisaged that the teachers taking part in this study will benefit from the community of practice, the divergent nature of the tasks and activities, the length of the program and a common, vested interest in learning.

Wenger (1998: 5) outlines a framework to illustrate how the four components of a social theory of learning are connected. The four components are: community (learning as belonging), practice (learning as doing), meaning (learning as experience) and identity (learning as becoming).

Figure 3.2 Wenger's component's of a social theory of learning

Graven (2005: 209) found this framework particularly useful for describing and explaining teacher learning in a professional development program in a South African context. South African teachers spend very little time interacting with other teachers in the same field. At the most they meet once every three month and these meetings are mostly to disperse instruction or information from a district office. Occasionally these meetings include a share and tell aspect. Winslow introduces an important aspect to teacher education, that of "team teaching mathematics" (Winslow 2007: 532) where he states that teacher education should go beyond craftsmanship of individuals but also that it is fundamental to have shared experiences to discuss, construct and manage teaching milieus. Team teaching is relatively uncommon in South African mathematics classrooms.

Hargreaves (1994: 245-247) explained that collaboration was a productive response to unpredictable problems. He further explained that collaboration had extensive and diverse promise since it provides moral support, increased efficiency, encourages risk taking, supports greater diversity in teaching strategies, increases teachers' opportunities to learn, increases capacity for reflection and encourages teachers to see change as an unending process of continuous improvement.

This study seeks to include a collaborative community aspect to the professional development program (see 4.2.2). This will assist in providing a supportive environment for the teachers and allow teachers to find common ground. It will also allow teachers to provide feedback to the researcher so that the cycles within the design research framework can be adapted and modified in order that it provides support, knowledge and skills in didactisation principles that

teachers feel they need. It may also provide teachers with colleagues to whom they turn in future (see 5.5.8).

3.3.1.8 Teacher development as an activity system

Leading from the previous sections, teacher development that is considered as an activity system is conceived by this study to be of merit. Hung and Chen (2002: 247) see learning *about* and learning *to be* as being two essential intertwined concepts in learning within a community of practice. According to them, "practice shapes our dispositions and beliefs system – our identity in a particular profession". Their definition of enculturation as participation in an activity within a community that can change a person's behaviour or change an identity and therefore the person becomes prepared to engage in similar activities. So participating leads to "learning to be" or identity formation (Hung & Chen 2002: 248). This view assists one in understanding that teacher professional development needs to be an active learning environment whereby teachers participate in exactly the type of activities the development program is hoping teachers will adopt in their own classrooms. These authors also remind us that engaging in activity causes "an internalization of the actions, activities, and processes through which an identity is formed congruent with that of the activity system" (2002: 248). Then if professional development is identity forming and identity changing the sentiment of Pimm (1993: 31) needs to be reflected upon by researchers that "it is dangerous to lose sight of how difficult personal change can be – and we should not talk lightly or glibly about it, let alone expect or demand it."

Teacher change can be anticipated, supported and positive if professional development program designers "embark on a journey with the assembled inservice mathematics teachers" where the "path is co-determined, devised from a rich and growing set of possibilities" (Dawson 1999: 160).

3.3.1.9 Conclusion

This study therefore supports Liljedahl's (2008: 3) ideas that pre-service teacher beliefs can be changed using three approaches. Firstly, teachers' beliefs have to be challenged, secondly; they should be involved as learners of mathematics and thirdly they should be involved in mathematical discovery. The third dimension is said by Liljedahl to have a "profound, and

immediate, transformative effect on the beliefs regarding the nature of mathematics, as well as their beliefs regarding the teaching and learning of mathematics". So too was the conclusion of Amit and Hillman (1999: 24) that professional development programs that offer experiences to do, reflect and discuss new teaching approaches are "a step in the right direction for challenging and reshaping teachers' conceptions". This study aims to use these ideas in a teacher development program (see 4.2.2) involving in-service teachers to further generalise these approaches.

Didactisation principles constitute the *what* of effective mathematics classrooms. These classrooms have students who are mentally active and involved in creating and constructing mathematics. Students are encouraged to use their own methods and solutions, there is a strong element of mathematisation and lessons are vertically aligned to advance mathematical thinking. Just how do teachers reach these dimensions of teachers? This study proposes an adapted "mathematical work of teaching" framework with day-to-day teacher activities of: accessing, probing, connecting, assessing and reflecting (see 3.2.1). The study will present modelling tasks to teachers to catalyze cognitive conflict and to develop didactisation practices in a dynamic and formative way. The type of teacher change envisaged through this study is non-trivial (see 5.5.6), so an understanding of how teachers change and what constitutes teacher change is necessary.

3.4 WHAT CONSTITUTES TEACHER CHANGE?

The word change is not unproblematic. According to Adler (2005: 173) in discussing teacher change, "teachers will always be found lacking". Adler and Reed (in Adler 2005: 173) propose a shift in language from "change" to "take up" since they felt that the word change produces a "deficit discourse in relation to teachers". The concept of development is more than that of growth or advancement; it also includes concepts such as the start of a new idea or experience (as defined by the Concise Oxford English Dictionary). In photographic terms it means to make something visible. If change is associated with improvement then it is the improvement that is of concern and not the starting or ending points. Therefore, in this study, current teacher ideas or actions (see 5.2) are seen as essential to the notion of development. The difference between pre and post teacher actions and thoughts is not given any value

judgment by the researcher. A wholesome conception of change given by Graven (2005: 223) as being more about "stimulating and supporting a lifelong process of teacher learning" is accepted for this study. Graven further explains that teacher change is a problematic concept in South Africa. If change is seen in terms of moving from poor to good it can be problematic. Rather, change should be seen as doing things differently where the only person to pass value judgment is the teacher herself about the quality of what she is doing differently.

Freeman (in Richards, Gallo & Renandya 2001: 45) on discussing teacher change, proposes the following conception of change:

- Change can mean a change in awareness
- Change can be an affirmation of current practice
- Change is not necessarily immediate or complete
- Some changes are directly accessible
- Some types of change can come to closure and others are open-ended.

It is foreseen that this study will promote a change in awareness of student learning through a change in teacher action (see 5.5.6), and that the change will not be complete and will remain open-ended. This will mean that the development program will have a longer lasting effect than simply the time that it is active, but rather it is hoped that the change will be evolutionary in the teachers.

Steinberg, Empson and Carpenter (2004: 259) discussed three conditions for teacher change from research literature that outlined the conceptual framework of their study. These were:

- membership in a discourse community;
- processes for reflectively generating, debating and evaluating new knowledge and practices; and.
- ownership of change.

This study integrates these three principles. Teacher sessions are created to support and acknowledge differences in teachers and their environments. The discourse community that will be created is strongly linked to reflection, debating and evaluation of new knowledge. The various instruments to collect data are largely reflective in nature. Throughout the

sessions teachers will be encouraged to take what they can from the sessions and apply those aspects and ideas with which they feel comfortable. In this sense, a greater ownership of change is envisaged.

Smith *et al. (*2003: xiii) asked pertinent questions about teacher change through professional development. They identified four types of change when considering the ways in which teachers.

- No change to minimal change
- Thinking change
- Acting change
- Integrated change.

They then combined thinking change and acting change since they did not feel that one was preferable to another. However, this study anticipates that there will be combined changes to teacher thinking and teacher action (see 5.2.9; 5.3.8; 5.4.8; 5.5.6; 5.5.8), and these two categories are not easily distinguishable.

Nickerson and Masarik (2010: 28) suggested three dimensions along which their development program changed teachers:

- The teacher's role in supporting student learning.
- The teacher's perception of what it means for a student to understand mathematics.
- Differentiation of the teacher's instructional strategies for moving students along a trajectory.

Remillard and Bryans (2004: 362) categorised observed lessons according to the mathematical emphases of the lesson. They looked at four different emphases a lesson could have: technical, steps or skill; working with materials and models; meaning understanding and strategy development and student explanations. This provides a good way of looking at developing didactisation that occurs throughout lessons and will be considered in lesson observation during the professional development program of this study.

Andreasen, Swan and Dixon (2007: 25) set out a framework for identifying four stages of teacher change:

- Resistance of the teacher to consider change.

- Teacher starts to talk about change and what he/she could do differently.

- Mimicking – this happens when the teacher uses the same materials that were used in the development program in his/her classroom.

- Changing practice – the teacher starts to apply the principles of the development program and develop his/her own lessons.

These four stages can be used by the researcher to highlight the major shifts in teachers' didactisation practices. However, a finer grain is needed for data analysis for this study as incremental changes need to be captured and reflected upon so that the subtle changes to teacher development can be reported on accurately. This is where the cyclical nature of design research allows for capturing the changes during the cycles of a teaching experiment (see 5.2 - 5.5).

Zuljan (2007: 32) set out and explained a hierarchical taxonomy of categories that ranged from traditional transmission instruction to constructivist mode of instruction. The details of this taxonomy are clear enough to implement as a classroom observation tool. In terms of categories of the teachers' role the categories are:

- Transmission: the teacher transmits, checks, demands and assesses.

- Encouraging understanding: the teacher takes learners' previous knowledge into consideration but there is no building on learner ideas. The teacher explains well and encourages learners to ask questions.

- Providing direction for learner development: the teacher encourages contextual understanding. Learners' prior knowledge and experience is central to instruction. Learners participate in shaping the learning process.

- Encouraging personal growth: the teacher tries to get students to see things differently, to think in a different way.

It appears that many of the categorisations specific to mathematics teaching in existing literature operate on a four point scale. This is useful for documenting teacher changes; however this study used a three point scale in the instruments (see Appendix 6) to ensure a greater accuracy of documenting any changes. However the day-to-day data analysis of the

study will be more detailed and fine grained to further assist with accurate, reliable and valid interpretations of the data in this study (see 5.2 – 5.5).

Jackson (1992: 64) proposed four ways in which professional development contributed to teacher development.

- Tell teachers how to teach or how to improve their teaching.
- Improve the conditions under which teachers work.
- Relieve psychological discomforts.
- Assist teachers in seeing their practice differently, in becoming more aware of the deeper significance of their work, and create an awareness of their work.

This last point is where this study is situated. It dovetails with the sentiments of Doerr and Lesh (2011: 247) that reveals, from a modelling perspective, that highly competent people not only do things differently, they see things differently. On careful analysis of the above categorisations it is evident that authors either see an improvement in teachers as moving from thinking towards actions. Some authors detail the change towards the focus and quality of teachers' actions by placing students at the centre of teaching. This increased awareness or sensitivity to the learning of students is what this study ultimately hopes to achieve with the teachers involved (see 5.5.6). It is through improved didactisation that teachers will become adept at accessing, probing, connecting, assessing and reflecting on student understandings in their classrooms. Then teacher change in these mathematics classrooms may result.

3.4.1 The theory of conceptual change

Verschaffel and Vosniadou (2004: 445) ascertain that the theory of conceptual change can be a productive component of a theory of mathematics teaching and learning. They define the term conceptual change as a type of learning necessary when new information comes into conflict with existing knowledge. The existing knowledge is usually based on experiences. Liljedahl, Rolka and Rosken (2007: 278) brought the theory of conceptual change into the teacher affective arena.

Liljedahl *et al*. (2007: 281) explain the four criteria for the theory of conceptual change to be relevant.

- It can be applied when misconceptions are formed through lived experiences and where formal instruction is absent.
- There is a need for theory rejection.
- There is a phenomenon of theory replacement.
- There is the possibility to form synthetic models.

As suggested by Liljedahl *et al*. (2007: 281) the theory of conceptual change is achieved through cognitive conflict. This means that before a new idea, theory of belief can be adopted, the current theory needs to be rejected. He adds that cognitive conflict is the vehicle through which current theories are rejected. Liljedahl (2010: 412) found that in some rare occurrences teachers undergo "rapid and profound changes in their beliefs and practices". He ascribed one particular teacher's change from belief rejection to belief replacement as a special form of conceptual change (2010: 414). The theory of conceptual development assists in understanding how the reconstructing of the cognitive-affective filter of teachers involved in the study may occur. In meeting the criteria for conceptual development this study's participants have the relevant lived experience that occurred during their many years as learners, students and teachers. In the proposed professional development program it is envisaged that teachers will experience cognitive and affective conflict due to the nature of modelling tasks (see 2.2.4 and 3.6). Teachers may therefore go through a process of belief rejection and belief replacement and finally construct synthetic models. Cohen and Ball (1990: 237) suspected that teachers "would have much to learn – and *unlearn*" (italics in original) in order to make changes from traditional teaching. They add further that like students, teachers would learn in "partial and halting ways".

Although similar to Piaget's notions of assimilation and accommodation (1978: 6), the theory of conceptual change has the dimension that includes current or existing knowledge and the possibility for changing this. Since teachers' beliefs are part of their existing framework or cognitive-affective filter, then a theory that accounts for existing knowledge, and not only how new knowledge is learnt, is important. It is an aim of this study that by developing teacher didactisation practices through modelling and didactisation principles set out in 2.4 that teachers may be able to examine their current beliefs. In examining their own beliefs, teachers would fall into two categories. There are teachers who in their own classrooms have experienced a discrepancy between their beliefs about mathematics teaching and student

learning and have started a change cycle. The second category (and possibly more common) is those teachers who have not yet perceived an inconsistency between their actions (guided by cognitive-affective filter) and student learning. It is hoped that teachers may experience some challenges in assimilating and accommodating didactical principles and thereby starting the conceptual change cycle. Piaget (1978: 18) also describes disturbances in knowledge structures that assist in regulating assimilation and accommodation processes. He stresses that a gap in knowledge can create the disturbance that may lead to a modification. Woodbury (2000: 39) reminds us that teachers must be thinking about problems within their own teaching for them to be open to reform suggestions. She advocates creating "cognitive dissonance" by explicitly challenging teachers to examine their teaching and beliefs.

Kuhn (1996: 85) who is attributed with the beginnings of conceptual change coined the term "paradigm shift" and said a "switch of gestalt" occurred in a full-scale paradigm shift. He stated that within science advances, crises are an "appropriate prelude to the emergence of new theories". He noted that a crisis "loosens the stereotypes and provides incremental data necessary for a fundamental paradigm shift" (1996: 89). The theory of conceptual change extends this by including this idea in teacher change paradigms. Many of teacher orientations and beliefs can be viewed as stereotypes of mathematics teacher behaviour. Woodland (2000: 46) suggested that research focus on which contextual features will facilitate and support conceptual change in teachers.

This study aims to determine if a teacher development program that includes didactisation principles and creates cognitive dissonance through modelling tasks will effect meaningful teacher conceptual change and a change in classroom practices (see 1.3 and 5.2 – 5.5). Doerr and Lesh (2011: 251) remind us that model-eliciting activities are designed so that significant conceptual changes occur during brief periods of time. Not only will the modelling tasks provide mathematical conceptual changes in the teachers but with the professional development program focus on didactisation principles, it is envisaged that conceptual changes regarding the teacher's teaching practices may evolve.

3.4.2 Changing cultural activities

If teaching is viewed as a cultural activity (Stigler & Hiebert 1999: 86), then, according to these researchers, it is not all learnt through teacher training but much is learnt through "informal participation over long periods of time". They maintain that one learns to teach by growing up in a culture, rather than by studying teaching. Many of the enigmas around teacher change are more easily understood when viewed from a cultural activity point of view. A cultural view of teaching means that the people involved in the teaching learning environment have certain mental pictures or "scripts" (Stigler & Hiebert 1998: 2) of what happens in a classroom. These scripts are learnt through many years of being involved in teaching and learning, mostly as a student. These scripts are based on a small and implicit set of beliefs about the nature of the subject, how students learn and the role of the teacher (Stigler & Hiebert 1998: 2). Nickson (1994: 9) also mentions that teachers "act as agents of a particular culture, and in this role they make judgments and choices about aspects of that culture". She further mentions (1994: 28) that the culture of a mathematics classroom is determined by the knowledge, beliefs and values the teacher and students bring into that classroom and that these affect the social interactions within the classroom.

When teaching is viewed as a cultural activity, the differences between teaching approaches in different counties become more understandable. In Stigler and Hiebert's study they found that Japanese teachers acted as though mathematics was inherently interesting, thought that frustration and confusion were a natural part of learning mathematics, used a chalkboard to generate a cumulative record of a lesson and viewed differences in student's range of abilities and ideas as a resource for discussion. Teachers from the United States on the other hand, had a different cultural view of teaching that deeply affected their teaching beliefs, teaching thoughts and teaching actions. These teachers wanted students to master a set of skills. Presenting the concept, piece by piece, incrementally was therefore common. They wanted to have students' undivided attention, so the use of an overhead projector was widespread. It is therefore possible to understand the difficulties involved in teacher change when viewing teaching as a cultural activity.

Jacobs and Morita (2002: 156) remind us that American teachers hold traditional views on teaching and learning. The same can be said of South African teachers, while these authors

report that Japanese teachers use more problem-solving in their lessons. This difference in teaching approaches is ascribed to these groups of teachers having different beliefs about mathematics teaching and learning. Jacobs and Morita further synthesised research on these two groups of teachers and explained that beliefs differed in American and Japanese teachers in terms of classroom management, needs of individual students and teaching students equitably. They also stated that research showed that American teachers used teacher-directed teaching methods while the Japanese teachers presented more constructivist approaches. They concluded that "within countries, there may be a strong consonance between views about good teaching and common classroom practices" (2002:157) while Whitman and Lai (1990: 80) contend that the concept of "effective teaching" and teacher education differs from culture to culture and should be studied within a cultural context.

Hillman and Ventura (in Breen 1999:118) emphasise how problems involving change "magnify if there is a definite direction in which you are supposed to change". This should sound warning to professional development designers to look at how narrowly they are presenting a hypothetical learning trajectory of teacher change and to what extent this picture of change is presented to the teachers involved in the program. The dilemma now is how can the nature of the change that researchers wish to see be presented to teachers so that it is not a narrowly confined picture. Caution must be taken not to present a "perfect image that no-one will ever be able to reach, which means we have built in an inevitable result of failure" (Breen 1999: 118). This study has endeavoured to view the end point of change as open as the change itself. It will also endeavour to provide beacons of change factors in the form of didactisation principles that teachers may use/adapt/apply to their own teaching mental spaces.

3.4.3 Personal practice theories

Hammerness (2001: 144) reported on how personal visions should serve as foundation for school change. She presents the powerful structures of human personal vision and characterised three dimensions to teachers' personal vision: focus, range and distance. The focus being the center interest of a teacher's vision, the range being the scope of the focus and the distance being how far or how close is the teacher's current action from the vision. Hammerness (2001: 160) explained that exploring and understanding teachers' visions may

make it possible to appreciate resistance to reform or change. In a study of a science teacher, Cornett, Yeotis and Terwilliger (1990: 520) defined a "personal practice theory" as the beliefs a teacher has which guide the teacher. These beliefs come from prior personal and classroom experiences. The teacher in their study had never articulated her beliefs or personal practice theories. By the end of their study the teacher had more "knowledge of the conceptual framework functions or her theories" and her theories were "reflected to a significant degree in her decision making in instructional practice" (Cornett *et al*. 1990: 526). This is what Clark and Peterson (in Cornett *et al*. 1990: 526) called a "maturing professional". Furthermore, Cornett *et al*. (1990: 527) proposed that teachers need knowledge of their own personal practice theories so that they could receive development based on their needs, and not simply receive development without personal conceptual awareness. From this, it is considered important for this study to provide an avenue for teachers to verbalise their personal practice theories. This need is accommodated within the research design by providing a non threatening collaborative environment.

Even in a South African setting, Hobden (in Breen 2005: 238) reported that pre-service teachers and their lecturers did not realise the "strong influence" of personal theories on learning nor on how perceptions are coloured by personal theories. According to Hargreaves (1994:12) what lies at the heart of teacher change for most teachers is the question of whether the proposed changes are "practical". He explains that this sense of practicality is more than just a sense of whether a particular change will work. It is about whether it suits the person and is in tune with their particular "whether it helps or harms their interests". Guskey (2002: 382) echoed this in stating that since teachers are "quite pragmatic", professional development programs had to provide teachers with practical ideas if these programs were to be successful. This study, by its design research methodology, does exactly this. The program is directly linked to the teacher's classroom and grade and curriculum- specific ideas are generated. The design research methodology allows the program to be revised so that it maintains this scope and vision. The teaching ideas presented and explored in the program are hopefully taken up into a teacher's own personal practice theory since these ideas have been aligned to the teacher's own practice. This is further facilitated by the researcher also being a practicing teacher influenced by the same curriculum and challenges.

3.5 PROFESSIONAL DEVELOPMENT IN SOUTH AFRICA

This study takes place in South Africa so the South African context needs to be taken into account. Teachers in South African government schools have a three or four year teaching qualification. Primary school teachers do take mathematics methodology as a course in their degree or diploma; very few take pure mathematics as a subject in their degree. Those who are interested or confident in mathematics turn to secondary school programs. Once they are qualified, they can be expected to teach any of the primary school subjects. Once in a school, they will teach a national curriculum (NCS or CAPS –currently being implemented) and in mathematics their students will write the Annual National Assessments. Classroom sizes vary from 35 to 45 students. At any school a teacher has the following senior members of staff: a departmental head (this person may or may not be a proficient mathematics teacher), a deputy head and a headmaster. South Africa's CAPS curriculum is content orientated and, as stated by Galbraith and Haines (2001: 342), a content-orientated curriculum will have different "priorities" than a proficiency based curriculum. It must therefore be assumed that the teacher will have different priorities in a content orientated curriculum such as the current South African curriculum. This study is set in the context of teachers having to cover certain content and content-based competencies. This backdrop needs to be taken as a contextual factor when evaluating teacher actions and practices.

Teachers in South Africa have been subjected to three types of professional development programs. Those organised and delivered by the education department; those organised and delivered by teacher unions; and those organised and delivered by private, non-governmental individuals or organisations. Research (such as this study) conducted by university masters' and PhD students are included in the third type. Teachers are expected to pay for the transport and/or costs of the professional development but some schools that are financially sound do refund teachers for this. An interesting recommendation made by Smith *et al.* (2003: xiv) is that teachers should be paid to attend professional development programs. Teachers paid by the government are not given a major incentive to further their studies. The financial remuneration is a once-off payment significantly below what business and commerce pay for improved qualifications. Currently CAPS training is underway on a national scale for grade 4 to grade 6 teachers. This is the fourth revision of the curriculum that teachers are implementing. A pertinent point is made by Graven (2002: 26) that although national

department may design new or different roles for teachers, they cannot re-design the local identities of teachers. It is the aim of this study, through developing teachers' didactisation practices (see 1.3.1.2) that they will lead to teachers redefining their personal identities in becoming more proficient mathematics teachers.

The effectiveness of the three types of development is not the focus of this study, but there has not been any consistent mathematics teacher development program for the teachers involved this study. Graven (2005: 212) described South African INSET carried out as a cascade model having "low impact" and being "ineffective". Telese (2008: 7) reminds us that "although professional development is an expensive endeavor, [it] is a critical aspect of teachers' professional life". He also affirms that teachers who take part in a content-specific professional development are more likely to give prominence to conceptual learning (2008: 8). It is anticipated that since the teachers involved in this study volunteered for the study, they hold professional development in high regard.

Adler (2005: 172) summarised research related to teacher education over a ten year period in the SAARMSTE (South African Association for Research in Mathematics, Science and Technology Education) publications. She found that the focus of most papers was INSET and on issues of teacher change. Her opinion is that research on teacher change "makes sense in the South African context". She reminds us that many changes in many spheres have occurred in South Africa over the past 20 years. Graven (2004: 190) echoes this by stating that "long term studies that take into account the wide range of contextual factors which affect teachers is especially important". This study is therefore situated within a current international and South African domain and deals with issues pertaining to both.

3.6 MODELLING AS VEHICLE FOR TEACHER DEVELOPMENT PROGRAMS

Doerr and Lesh (2003: 133) specify the overall goals of model-eliciting activities for teachers. They are:

- To have teachers reveal their current ways of thinking
- To test, revise and refine those ways of thinking
- To share with colleagues for replication

- To reuse their ways of thinking in multiple contexts.

The instructional design principles for modelling are used by Doerr and Lesh (2003: 133) to design similar principles for teacher development programs. The principles in terms of teacher development are:

1. The Reality principle: this means that teachers must interpret student work from their own classrooms or use concepts that they teach in their classrooms as part of the development program.

2. The Multilevel principle: this means that the tasks used for teachers in a development program should address the multiple levels of the teaching and learning environment. Addressing only student thinking is not enough. Content, strategies and psychological aspects must also be considered.

3. The Multiple contexts principles: the variability of settings, students and contexts should be accounted for. This will lead to teacher thinking that is increasingly generalisable.

4. The Sharing principle: ideas about teaching and learning should be shared among many teachers.

5. Self-evaluation principle: teachers should be in a position to judge for themselves that their interpretations or actions are moving in the right direction.

Schorr and Lesh (2003: 146) illustrated in their model for teacher development that a modelling approach to teacher development would impact on teachers' mathematical content knowledge understanding how children learnt mathematics which in turn would result in "effective instructional decision". Nilsson and Ryve (2010: 245) talk about a "focal event" that occurs in rich and explorative learning activities. Modelling tasks will be used in this study as the focal event of the sessions that take place in the teacher development program. Each modelling task is visited twice – once as a task for the teachers and once as a task for students that teachers are observing.

Steffe (1991:181,182, 185,186,187,189, 190) explicates teacher actions that constitute a constructivist approach to mathematics learning:

- using the mathematics of students rather than what is prescribed in the curriculum;
- using interactive communication;

- focusing on students qualitative processes and not focussing only on the product of their activity;

- seeing learning as modifications of schemes;

- understanding that mathematical concepts are constructed through goal directed activity and not through disembodied forms of learning;

- taking responsibility for learning children's mathematical knowledge;

- using students mathematical activity to teach;

- seeing the learning environment as a variable whose contents are specified by the participants ;and

- acknowledging the mathematical power of students and not taking what students can learn as being specified by an a priori curriculum.

These actions should be facilitated when teachers' didactisation practices develop. It is possible to present these practices to teachers by using modelling tasks. Modelling tasks allow a teacher to improve their actions as specified by Steffe.

Arbaugh and Brown (2005: 503) developed a program for teachers that involved critically analyzing tasks in their teaching. They considered this an emotionally safe way to engage teachers in an initial examination of their practices. They found that this type of professional development supported a growth in their pedagogical content knowledge and a change in the type of tasks teachers chose (2005: 527). Smith and Stein (1998: 348) drew up a table of characteristics of mathematical tasks and ranked them from lower to higher order. The table clearly shows how modelling tasks are higher level demand tasks which focus on doing mathematics. It is also evident that higher order thinking tasks are rich in student-making connections as well as involving a high level of student cognitive activity.

Levels of Demands
Lower-level demands (memorization):

 • Involve either reproducing previously learned facts, rules, formulas, or definitions or committing facts, rules, formulas or definitions to memory.
• Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.
• Are not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be re-produced is clearly and directly stated.
• Have no connection to the concepts or meaning that underlie the facts, rules, formulas, or definitions being learned or reproduced.

Lower-level demands (procedures without connections):

• Are algorithmic. Use of the procedure either is specifically called for or is evident from prior instruction, experience, or placement of the task.
• Require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it.
• Have no connection to the concepts or meaning that underlie the procedure being used.
• Are focused on producing correct answers instead of on developing mathematical understanding.
• Require no explanations or explanations that focus solely on describing the procedure that was used.

Higher-level demands (procedures with connections):

• Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.
• Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts
• Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations helps develop meaning.
• Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.

Higher-level demands (doing mathematics):

• Require complex and nonalgorithmic thinking - a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example.
• Require students to explore and understand the nature of mathematical concepts, processes, or relationships.
• Demand self-monitoring or self-regulation of one's own cognitive processes.
• Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task.
• Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions.
• Require considerable cognitive effort and may involve some level of anxiety for the student because of the un- predictable nature of the solution process required.

Table 3.1 Task demand levels (Smith & Stein 1998: 348).

They follow much the same as Polya's (in Kilpatrick 1985: 4) classification of problems where modelling fits the last description.

1. One rule under your nose – this problem is solved by mechanical application of a rule just taught.

2. Application with some choice – this problem can be solved by applying a rule taught earlier so that the solver has to use some judgment.

3. Choice of a combination – this problem requires that the solver combine two or more rules.

4. Approaching research levels – a problem that also requires a novel combination of rules but that has many ramifications and requires a high degree of independence.

Modelling tasks encapsulate the didactisation principles set out in this study. When students are presented with a modelling problem, they are actively working on constructing meaning. The tasks allow different points of entry and therefore differentiation takes place. Mathematisation at both a horizontal and vertical level take place since students develop a model that reflects both horizontal and vertical mathematisation because the task instruction will demand this of them. When students are modelling, it allows the teacher to access, probe, connect and assess their thinking while providing a wide spectrum of reflection possibilities. When students solve modelling problems, the teacher takes a behind-the-scenes approach, this allows the teacher the time to focus on what the students are thinking and doing. It often provides the teacher with the time to formulate questions and guidance that may be necessary. It will assist the teacher in changing the classroom from teacher orientated to student orientated.

3.6.1 A modelling view of mathematics education

Galbraith (2007: 59) formulated significant differences between a modelling culture and an educational culture by considering the "implications for didactics from Zone theory". His comparative table presents these differences from a variety of views (social interaction, technology etc.) which is compatible with Lesh, Yoon and Zawojewski's (2007: 315) difference between "making mathematics practical" and "making practice mathematical":

	Modelling culture	**Education culture**
1	Mathematics involves both 'thinking' and 'hands-on' ability.	Mathematics is done in the head.
2	Mathematics involves written and oral communication.	Mathematics is about calculations and bookwork.
3	Life-related mathematical activity involves both predictable and unpredictable element.	Classrooms mathematical activity occurs in a passive and controlled environment.

4	Some data are external to the classroom.	All needed data are internal to the classroom.
5	Mathematics involves both individual and team activity.	Mathematics is ultimately an individual activity.
6	Mathematics takes place where and whenever the need occurs.	Mathematics occurs only in formal schedules sessions or through structured homework.
7	Success is measured by the solution of the problem.	Success is measured by individual performance on tests.
8	Assessment involves a range of outcomes and criteria for success.	Assessment requires standardised conditions and instruments.
9	The real world is an essential component	The real world is an optional extra.
10	Technology use is chosen to maximise problem-solving success.	Technology use is subject to local policy and availability.
11	Choice of resources is decided by what needs to be addressed.	Choice of resources is decided by curriculum detail and availability.
12	Teamwork is managed by the need to maximise problem-solving capability.	Group work is managed by organisational rather than problem-orientated decision making.

Table 3.1 Galbraith (2007: 59)

Blum and Leiß (2007: 222) define "good mathematics teaching" as providing diverse experiences for students to acquire competencies (especially modelling), and creating multiple connections in and outside of mathematics. For them, it also means stimulating cognitive and metacognitive activities for students and co-ordinating learner-centred classroom management. It would appear that Galbraith's modelling culture adequately meets the definition given by Blum and Leiß. Bonotto (2007: 273) however, encapsulated a modelling approach as wanting to "encourage the children to recognize a wide variety of situations as mathematical situations, or more precisely as 'mathematisable' situations". The strong interrelationship between modelling, model-eliciting activities and mathematisation is well forged.

Lesh (2007: 159) also explained that modelling and developing models involves major forms of concept development and that this development can be achieved by students who are considered average or below average in traditional school assessment. Lesh and Kaput (2007: 298) extend modelling to future professions that students will follow. They state that countries that are performing well at international tests recognise that these basic skills are not

sufficient for the future workplace and economy of their countries. Other competencies need to be fostered in mathematics education such as:

> the development of creativity, and problem solving ability, as well as the ability to produce useful mathematical descriptions of complex systems, the ability to work productively within diverse teams of specialists, and the ability to adapt to rapidly evolving technical tools.

Lesh and Doerr (2003b: 533) distinguish four teaching objectives that are important to student learning and that form part of students' mathematical interpretations of their experiences:

1. Behaviour objectives such as basic facts and skills,

2. Process objectives such as habits of mind that are not connected to any particular mathematical constructs,

3. Affective objectives such as attitudes, beliefs, feelings, and

4. Cognitive objectives such as models and accompanying conceptual systems or constructs.

Traditional school instruction will not produce many of the above competencies. These proficiencies will need to be integrated into resources and orientations so that they become visible in classrooms. Stillman (2010: 309) linked teaching and learning through modelling to increasing the cognitive demand of an activity. A modelling approach to teaching and learning will need to be fostered to raise the level of student abilities to include this description. However, it will be necessary to firstly place this description in the domain of teacher development.

3.7 SUMMARY

In an attempt to provide meaningful professional development for primary school mathematics teachers, the domains of didactisation and its foundational concept of mathematisation are considered important in this study. This area of teacher knowledge is conceived and defined by what Ball *et al.* (2008: 389) called "specialized content knowledge".

The ideas presented in this chapter serve as a guide for the design of the professional development program for this study. The professional development that is proposed by this study is "highly interventionist" and may show "discontinuity between typical forms" (Cobb

et al. 2003: 10) of professional development offered in South Africa. The focus of this study, being integrated development of didactisation practices of primary school mathematics teachers, necessitates its design research nature. These authors add that although design experiments develop theories, they "must place these theories in harm's way" (Cobb *et al*. 2003: 10). The proceeding chapter explains how these theories will be developed and tested in an authentic and naturalistic environment. Brown's (1992: 173) observation that "the question becomes, what are the absolutely essential features that must be in place to cause change under conditions that one can reasonable hope to exist in normal school settings?" is considered in the next chapter where the design research methodology of the study is set out in more detail. The preceding chapters were part of what McMillan and Schumacher (2006: 316 italics in original) refer to as "*describe and explore*" in qualitative research while the next two chapters will add the "*describe and explain*" element of qualitative research. Just how the integration of didactisation principles, modelling and teacher development through design research was realised in this study is set out in the next chapter.

CHAPTER 4

METHODOLOGY

4.1 INTRODUCTION

This study undertakes a qualitative research design. According to McMillan and Schumacher (2006: 315) this means that the following assumptions are present:

- It is based on a constructive philosophy.
- It assumes reality is a multilayered, shared social experience that is interpreted by individuals.
- Reality is a social construction.
- People form constructions to make sense of their world while also reorganizing these as viewpoints, perceptions and beliefs.

These authors further explain that qualitative research goals include understanding social events and incidents from the participants' perspective. This study also aims at (see 1.3.2.7) generating theory or "empowerment" (McMillan & Schumacher 2006: 316) (see 6.3.1).

Design research as a qualitative methodology guides this study. Cobb *et al*. (2003: 9) outlined five cross cutting features of design research studies. Firstly design experiments develop local theories both about learning processes and how to support those learning processes. Secondly, design experiments are "highly interventionist" in nature but design research uses "prior research to both specify a design and justify the differentiation of central and ancillary conditions is central to the methodology" (Cobb *et al*. 2003: 10). The third feature of design experiments is that they are both prospective and reflective. The designs include a hypothetical learning trajectory (HLT) as well as a retrospective analysis. The third feature leads to the fourth – that design experiments have an iterative design. The fifth feature is that the theory developed through design research is relatively humble. Cobb *et al*. (2003:10) explain that the theory is also "accountable to the activity of design". The theory generated by design research is practical on a meso and micro instruction level. This study aims at generating theory that can be used in other professional development programs (see 6.3).

Gravemeijer (1994a: 196) described developmental research as "the kind of research that is needed to bring about educational change in mathematics education". He further (1994a:196) described this type of research as evolutionary (it is gradual, iterative and cumulative); stratified (the theory development takes place at different levels from teaching activities to a subject specific teaching theory); and reflexive (the theory development is fostered by reflective relations between the different levels of theory).

It has further been proposed (Baumgartner *et al.* 2003: 5) that design-based research exhibits the following characteristics: the central goals and developing theories are intertwined; development and research take place through continuous cycles of design and analysis; the designs lead to sharable theories; the research accounts for how designs function in authentic settings; and the development of such accounts can connect "process of enactment to outcomes of interest".

In all things there are certain trade-offs that need to take place when implementing ideas from a variety of views and perspectives. Graven (2005: 211) identified five "dilemmas" in the design of her INSET program. She decided to compromise with a long term intervention that would necessitate a small-scale design, otherwise it would be "highly labour intensive". This study had to consider the same compromise. Therefore a small group of teachers participating over a longer duration will be factored into the design.

This study is set in primary schools teaching and classrooms in South Africa. It is noteworthy that in a study on educational design based-research worldwide by Anderson and Shattuck (2012: 20) no published studies came from South Africa. This study therefore may be part of only a few design based-research studies that focus on the South African context. Design research assists with "a greater understanding of a learning ecology" (Cobb *et al*. 2003: 9). It is envisaged that the program will be indicative of this authentic South African environment and will be equally applicable to similar environments.

4.2 DESIGN RESEARCH AS METHODOLOGY

Design research comprises three phases. A planning and preparation phase, a teaching experiment; and a retrospective analysis. A distinguishing feature of design research is the iterative cyclical nature of the intervention or experiment. The word experiment may be problematic if taken from a quantitative background. In design research it has more the meaning of being "*experimental* but not an experiment" (Kelly 2006: 114 italics in original). Kelly adds that it is more about cultivating and generating hypothesis than testing it, although this study does evaluate its thesis (see 6.2).

Design research needs "robust designs – ones that produce impressive results, not only under ideal conditions, but also under severe but realistic constraints" (Walker 2006: 13). Collins (1999 in Kelly 2006: 112) explained that design researchers:

- Conduct research in a messy setting
- Involve many dependent variables
- Characterize (not control) variables
- Flexibly refine the design rather than follow a set procedure
- Value social interaction over isolated learning
- Generate profiles and do not test hypotheses
- Value participant input to researcher judgment

Goffree and Oonk's (1999: 209) suggested program for student teachers is for a cyclical process in which mathematical problems, mathematisation, reflective problem solving and mastering teaching methods follow naturally. This cycle is consistent with design research and is part of this study.

The teaching experiment has many aspects of the teacher level thought raising program described by Schorr and Lesh (2003: 146) which led to effective instructional decisions. This study however extends their design in that there are more purposeful sessions of reflection built into the program. This may increase participant input in the study.

4.2.1 Planning and preparation phase

The planning and preparation for this study included a thorough evaluation of relevant literature. It has included a strong theoretical element since according to Adler, Ball, Krainer, Lin and Novotna (2005: 372) developing "strong and effective theoretical languages" will result in a distance between the researcher and what is being studied. The literature study in design research serves not only to orientate the researcher in the field and to integrate relevant research, but to "identify central organizing ideas for a domain" (Cobb *et al*. 2003: 11).

Gravemeijer and Cobb (2006: 19) explain that the goals of this phase of design research are to put together a local instruction theory and to clarify the study's theoretical standpoint. This study proposes an instruction theory for teacher development and as stated by McKenney *et al*. (2006: 72) the program is the product of this study. The preceding chapters have allowed these two goals to be met which will allow for a well thought teaching experiment to follow. Gravemeijer and Cobb (2006: 21) also state that in putting together a local instruction theory (i.e. in terms of a teacher development program) during the planning phase of a design research study – conjectures about the learning process and means of supporting that learning process are necessary. They add that this includes creating productive learning activities, the tools necessary and the envisioned classroom culture. Although they warn that available research may provide limited guidance, this is where they use the French term "bricoleur" to describe a designer using and adopting any necessary materials. The varied tools and learning activities in this study are taken from existing literature and new tools and activities have been designed. These are constructed in an original way to produce the professional development program.

After a literature study, the selection of modelling tasks, the design and activities in the teacher development program and the instruments had to take place. These had to be carefully considered in terms of the literature study and in terms of those tasks and activities that would best allow for didactisation principles to become apparent. At this point a questionnaire was sent to teachers in two different schools. The questionnaire dealt with the type of development program teachers thought would be most useful and the type of content they would like to see present in a teacher development program. Teachers returned these questionnaires anonymously.

As suggested by Bakker (2004: 39) the use of a Hypothetical Learning Trajectory (HLT) is meaningful in design research. Gravemeijer (1999: 157 italics in original) prefers the term *local instruction theory* since in design research it encompasses a whole instructional sequence and is more general than lesson-based HLT. In this study, the instructional sequence is one that is designed for teachers. In preparing the teaching experiment (teacher development program) a HLT was part of the thought-experiment of this phase. A HLT was defined by Simon (1995: 136) as a prediction of how learning and understanding will evolve over the learning activities. Bakker (2004: 40) further explains the function of a HLT within a design research approach. During the preparation phase the HLT guides the design of the materials that will be used. During the teaching experiment it guides the researcher on what to focus on during observing, interviewing and teaching. The HLT can change during the teaching experiment depending on the needs of the participating teachers or the researcher. It means that the researcher needs to be very sensitive to the participants and incidents during the teaching experiment. During the retrospective analysis phase of design research the HLT guides the researcher as to what to focus on during the analysis and the "researcher can contrast those anticipations with the observations made during the teaching experiment" (Bakker 2004: 40). Bakker further explains that it is via this type of analysis that instruction theory can be developed and research questions can be answered (see 6.2).

4.2.1.1 Selection of participants

a) Teacher selection

Teacher selection is based on a convenience sampling since the researcher for this study is a full time teacher. Since the intervention requires a number of classroom observation visits, close proximity of the schools to which the researcher will need to travel is an important consideration for the success of this study. Teachers were selected based on their interest and willingness to participate in the study. McMillan and Schumacher (2006: 322) explain that the sample size is determined by the purpose and focus of the study, the data collection strategy and redundancy of data. Since this study relies on a single researcher and a descriptive cross case analysis focusing on a number of didactisation principles, the sampling is appropriate. In

all decisions around research there is a trade-off between what is available and the purpose of the study.

b) Student selection

Student selection is based on students volunteering to take part in the program. Many students at schools have a keen interest in mathematics and are not necessarily involved in the school's extra mural program. Students and their parents were fully briefed as to the intentions of the program and the role of the students. Confidentiality was assured. It was pointed out that students would not be assessed in any way. Parents signed assent letters on behalf of students. Students would take part as a group and would be requested to solve modelling problems. Students would be selected from the researcher's own school for convenience.

4.2.1.2 Modelling task selection

The three tasks for inclusion in the teacher development program were selected from existing modelling tasks in the literature. This assists in increasing the validity and reliability of the study. Tasks are subjected to six criteria before they are considered modelling tasks. These six criteria or six principles were outlined by Lesh, Hoover, Hole, Kelly & Post, 2000: 608 and Lesh, Hoover & Kelly, 1992: 113) and are:

- The Reality principle - requires that the task encourages students to make sense of the situation and use their own experience and intuitions about the situation.
- The Model Construction principle - requires that the problem must ensure that students develop or construct a model.
- The Self-Evaluation principle - requires that the task allows students be able to judge for themselves when their ideas are good enough.
- The Model-Documentation principle - the task must elicit a response to the problem that requires that students reveal their thinking about the situation.
- The Simple Prototype principle - requires that the situation is as simple as possible while still creating the need for a model.
- The Model Generalization principle - does problem require that students are able to construct a generalised model of and for the specific situation and similar situations?

The tasks serve three purposes in this study. Firstly, teachers experience modelling through these tasks; secondly didactisation principles are built into these tasks; and thirdly teachers can observe students solving modelling task – this means that they are able to observe didactisation principles through the execution of the task. The tasks are from different content sections of mathematics and can be linked to the CAPS curriculum. It is an important consideration to develop professional development that relates to what teachers are dealing with in their classrooms.

Task 1 can be linked to:

- "solving problems involving whole numbers and decimal fractions, including measurement contexts" (DBE 2011: 15).

Task 2 can be linked to:

- "draw enlargements and reductions of 2D shapes to compare size and shape of triangles and quadrilaterals" (DBE 2011: 23); and
- "calculations and problem solving involving length" (DBE 2011: 25).

Task 3 can be linked to:

- "solve problems involving whole numbers and decimal fractions including financial contexts" (DBE 2011: 263); and
- "analyse data by answering questions related to: data sources and contexts" (DBE 2011: 268); and
- "solve problems involving whole numbers, including comparing two quantities of different kinds (rate)" (DBE 2011: 285).

These tasks also assist teachers in preparing specifically set mathematics assessment tasks prescribed by CAPS. A project has to be set at least once per year with the following criteria given:

> Projects are used to assess a range of skills and competencies. Through projects, learners are able to demonstrate their understanding of different Mathematics concepts and apply them in real-life situations [...] Good projects contain and display of real data, followed by deductions that can be substantiated (DBE 2011: 295).

Modelling tasks become the vehicle whereby didactisation principles are made visible to teachers. In subsequent classroom visits by the researcher it is not expected that teachers

present a modelling lesson. It is envisaged that teachers will integrate the didactisation principles to a greater extent in their usual classroom activities.

The three tasks chosen are:

1. The Airplane problem (Appendix 3). The problem is described in Lesh & Doerr (2003a: 7) and

https://engineering.purdue.edu/ENE/Research/SGMM/CASESTUDIESKIDSWEB/index.htm This task was selected because it allows several entry points and levels of complexity. It involves decimal number measurement in several domains. It would be suitable for the teachers participating in the study as well as the students that will be observed (the fishbowl activity). It requires that students determine how to award an airplane contest winner based on three distinct but interrelated measurements: the length of time the airplane stayed in the air, the total distance it flew and the distance it was from a pre-set target. The complexity here is that some measurements are linear while others can be used as vector measurements.

2. Tangram Toys (Adapted from Brousseau 1997: 177) (Appendix 4)

This problem is based on proportional reasoning and also requires accurate drawing of common geometrical shapes. The task requires that students/teachers calculate a scaling up factor to produce the correct size pattern piece. Students have to work between multiplication and division to consider the enlargement that is needed. The scale factor of 1.75 adds complexity to the problem.

3. Hire or Fire? This task was adapted from https://engineering.purdue.edu/ENE/Research/SGMM/CASESTUDIESKIDSWEB/index.htm (Appendix 5)

This task requires an understanding and application of unit rates. A qualitative aspect is brought into the problem in that the hours worked are given according to how busy the venue was. The amount of money made by each student is also displayed in this way. Students will

have to decide what to do about this qualitative information and will have to integrate it into their model.

The three tasks were selected from different content areas of mathematics so that teachers could experiences modelling from different content areas. The tasks were also selected as those most closely matching actual curriculum content as set out in the CAPS curriculum. It is hoped that through these tasks the didactisation principles set out in this study will become relevant and meaningful to teachers taking part in the program (see 5.2 – 5.5).

4.2.1.3 Program design

The program is designed in three main cycles. Each cycle starts with the researcher visiting the teachers in their own classroom at a time and lesson of their own choice. The observation is followed by three teacher-researcher sessions that take place after school hours. Each session is approximately 2 hours long. After each session teachers complete a reflection sheet in their own time at home. Three sessions were conducted in a school term (10 weeks). A number of questionnaires at the start and end of the program are also completed.

a) Teachers working on modelling task as a group

Teachers work together on modelling task. They experience the task as the students themselves would. The researcher promotes discussion during the session. The researcher discusses the principles set out in this study. Teachers are made aware of their activity level, differentiation and mathematisation. The researcher also facilitates the discussion so that teachers understand the principles of access, probe, assess and connect mathematical understandings. The session is transcribed by the researcher. Teachers also complete a reflection sheet (Instrument 6).

b) Fishbowl activity

This session involves four students (Grade 6- approximately 12 years old) that solve the same modelling problem the teachers solved and reflected on in the previous session. Teachers observe the group and interact with the student if necessary. A researcher led discussion takes

place after the students have left. Teachers are asked to complete a take-home reflection sheet to complete individually based on this session.

c) Resource sessions

The third sessions explores the didactisation principles further in that teachers are given the opportunity to find and extend these principles within the normal context they are expected to teach. This is where the true strength of this program lies. It seeks to equip teachers with didactisation principles that they can use on a day-to-day basis with day-to-day content. It is not possible to teach the content of an entire curriculum through modelling tasks. As suggested by Lesh, Cramer, Doerr, Post & Zawojewski (2003: 42) it would be foolish to abandon fundamentals. What is needed is a sensible mix of complexity in the form of applied problems and fundamentals. The program seeks to equip teachers to make decisions that will enhance their teaching within any context and any curriculum. In this session many resources are provided to teachers that are directly relevant to their curriculum.

d) Reflection after each session.

Teachers involved in this program would spend much time discussing and collaborating so a means of guiding the structuring the discussions is needed otherwise unfocussed discussions could end up in a number of loose strands.

Allowing teachers to collaborate is important. Winslow (2007: 532) explained that an important idea, not always associated with the theory of didactical situations, is that of team teaching. He said that:

> to go beyond the craftsmanship of individual experience with constructing lessons, it is crucial to have a shared and systematic approach to discuss and construct didactical milieus and to manage the phases of didactical situations.   (Winslow 2007: 532).

The reflection takes place after teachers have worked on a modelling task themselves, after they have observed students solve the problem and after observation lessons.

The various reflection sheets would allow teachers to think critically about didactisation principles and to explore their own thinking and teaching practices. Their responses will allow

the researcher insight into possible changes in their resources, orientations and goals that may be ascribed to the development program.

4.2.1.4 Instrument design

Instruments were designed to capture as much of the data as possible. The use of multi-instruments that are eclectic increase the validity and reliability of the study in that they allow for triangulation of data sources.

Instrument 1: Teacher questionnaire.

This will assist in capturing as much relevant personal and historical information from the teachers as possible. This will create a profile of each teacher. It allows the researcher to gauge to what extent the teacher understands and applies the didactical principles described in this study. The questionnaire includes the coding for analysis.

Instrument 2: Baseline questionnaire

This instrument would be used twice during the program, after the first and last class visit by the researcher. The questions are coded according to the didactisation principles of the study. The questions explore areas of mathematics teaching and will assist in gauging teachers' orientations and goals.

Instrument 3 and 4: Classroom Observation Guides

These instruments are used by the researcher during class visits. These protocols have been used before in previous literature and add to the validity and reliability of the study. Instrument 4 is a combination of existing questions and new questions included by the researcher. These instruments have also been coded according to the didactisation principles of this study. These instruments are based on a three point scale which is consistent with the other instrument and increases the accuracy and therefore the reliability of the instruments. The first classroom observation, together with the baseline interview and teacher questionnaire to form a clear picture of a teacher's current practice is "crucial to evaluating impact" (Porritt & Earley 2010: 9).

Instrument 5: Lesson questionnaire

This instrument was used for the second, third and fourth classroom observation visits. It involves questions on how the teacher did and can address didactisation principles described in the study. It is intended to assist teachers in describing their challenges and difficulties in implementing didactisation principles. This instrument differs from Instrument 2 in that teachers have been through one cycle of the program and can now comment and discuss the didactisation principles set out in the study. It also allowed teacher and researcher to assess how didactisation principles could be supported in the day-to-day content in classrooms. It further assisted the researcher in adjusting the next cycle of the design research program to maximise teacher development.

Instrument 6: Take-home reflection sheet (Post teacher modelling session)

This instrument is for teachers to complete individually after they have taken part as students in solving a modelling task. It is intended to bridge the modelling task they completed with their current thinking about classroom practices. It also addresses didactisation principles inherent in the task. Coding is included on the sheet in terms of didactisation principles.

Instrument 7: Take-home reflection sheet (Post fishbowl activity)

This instrument allows teachers to reflect on the fishbowl activity that they have observed. It allows teachers to focus on the essential features of modelling tasks and to reflect on the didactisation principles inherent in modelling tasks.

Instrument 8: Discussion guidelines

The instrument provided structure and a focus to discussion on the differences between the mathematics question examples provided. The teachers categorise questions into traditional type problems and more problem-based questions. They furthermore discuss the potential of each type of problem in terms of the didactisation principles. This activity is intended to assist teachers find resources and questions for their own classrooms and own content that will assist them in applying didactisation principles.

4.2.2 Teaching experiment

The teaching experiment or professional development program proposed by this study fulfils several roles. It includes the suggestions of Pajares (1992: 327) that a study that includes teacher beliefs should include open-ended interviews, responses to dilemmas and vignettes, and observation of behaviour so that "richer and more accurate inferences" can be made. It also seeks to provide a "combination of experiences" (Putnam & Borko 2000: 7) so that teachers can learn "deeper understandings of mathematical learning and teaching". The learning combination of the teaching experiment includes: teacher-researcher discussions, teacher-group modelling sessions, fishbowl sessions and take-home reflection sessions. Teachers were observed in their classroom by the researcher during the program.

This professional development takes place over an implementation period for a revised curriculum (CAPS). Teachers would therefore have a foot in each curriculum (NCS and CAPS) over the development period. It was therefore felt that to focus on a specific content area of mathematics may limit the program. It is envisaged that the didactisation principles set out in this study would be included in the teachers' day-to-day activity (see 5.3 – 5.5). Another factor is the nature of classroom visits in South African classrooms. Teachers are usually visited by a senior member of the staff for evaluation and scoring purposes. Classroom visits from the researcher in this study will be dealt with very sensitively and with the full co-operation and collaboration of the teacher.

The teaching experiment takes place over two main settings, the teacher's own classroom but mostly in an outside setting where the five teachers meet with the researcher. Putnam and Borko (2000: 6) provide valid reasons for doing so. They state that

> the classroom is a powerful environment for shaping and constraining how practicing teachers think and act. Many of their patterns of thought and action have become automatic –resistant to reflection or change. Engaging in learning experiences away from this setting may be necessary to help teachers 'break set' – to experience things in new ways.

The teaching experiment table from Chapter 1 is repeated here for ease of reference for the reader. More information is added in this table.

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Detailed programme of teaching experiment
1. Contact with interested teachers Teachers who indicate a verbal interest in being part of the program are invited formally to take part in the program. They are issued with a consent document where a number of confidentiality issues are described. Discuss the researcher's intent and aims with the program. Teachers ask questions on the nature/aims/content of the program.
2. Teacher Questionnaire Instrument 1 Teachers answer this questionnaire individually and in their own time. They complete this before meeting the other teachers in the program.
3. Classroom Observation 1 **Start of a cycle 1.** The researcher visits each teacher before the program starts. Teachers invite the researcher on a day that suits both parties. The researcher requests that the teacher prepare a 'typical' lesson for this visit. Instrument 3 Instrument 4
4. Baseline Questionnaire Instrument 2 Teachers answer a questionnaire based on their current standing as mathematics teachers.
5. Teacher Session 1: Researcher discusses traditional teaching and problem based teaching with teachers. Researcher also discusses didactisation principles as they relate to students (active students, differentiation, mathematisation) and teachers (access, probe, connect, assess, reflect, vertically align lessons). Recording and transcription
6. Teachers work on modelling task : task 1 - Airplane problem Discussion with researcher on their thoughts about these types of problems. Take home reflection sheet - Instrument 6
7. Teacher Session 2: Fishbowl activity task 1 Four students are asked to solve task 1. Teachers observed. Five teachers and researcher to discuss this session afterwards. Take home reflection sheet - Instrument 7
8. Teacher Session 3: **Focus: bringing problems forward to promote conceptual development.** Researcher and teachers' group discussion traditional vs. problem-based learning. Resources and ideas for teachers' own classrooms.
Retrospective analysis
9. Classroom observation 2 **Start of cycle 2** Instrument 3 Instrument 4
10. Questionnaire based on the observed lesson Instrument 5

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11. Teacher Session 4: Teachers work on modelling task 2 Take home reflection sheet Recording and transcription Instrument 6
12. Teacher Session 5: Fishbowl modelling task 2 Four pupils solve task 2. Teachers as observers. Discussion with researcher afterwards. Instrument 7
13. Teacher Session 6: **Focus: Teacher planning** Researcher and teachers' group discussion. The focus for this discussion is on teacher lesson planning. Recording and transcription
Retrospective analysis
14. Classroom observation 3 **Start of cycle 3** **Focus: Proportional reasoning through the curriculum** Instrument 3 Instrument 4
15. Questionnaire based on observed lesson Instrument 5
16. Teacher Session 7: Teachers work on modelling task 3 Take home reflection sheet Recording and transcription Instrument 6
17. Teacher Session 8: Fishbowl modelling task 3 Four students solve task 3. Teachers observe. Discussion with researcher afterwards. Instrument 7
18. Teacher Session 9: Resources and ideas for teachers' classrooms.
19. Concluding classroom observation 4 **Final visit** Instrument 3 Instrument 4
20. Questionnaire Instrument 5 Instrument 2
Baseline Questionnaire Instrument 1
Retrospective analysis

Table 4.1: Detailed program of teaching experiment

The repetition of the cycles allows for "progressive refinement" of the design and allows the researcher to make changes based on the actions of the participants (Collins, Joseph & Bielaczyk 2004: 18). The cycles allow for fine tuning and adjusting of the teaching experiment based on new information that comes to light or reflecting on the situation by the researcher (see 5.2.8; 5.3.7; 5.4.7). The cycles can be viewed as follows:

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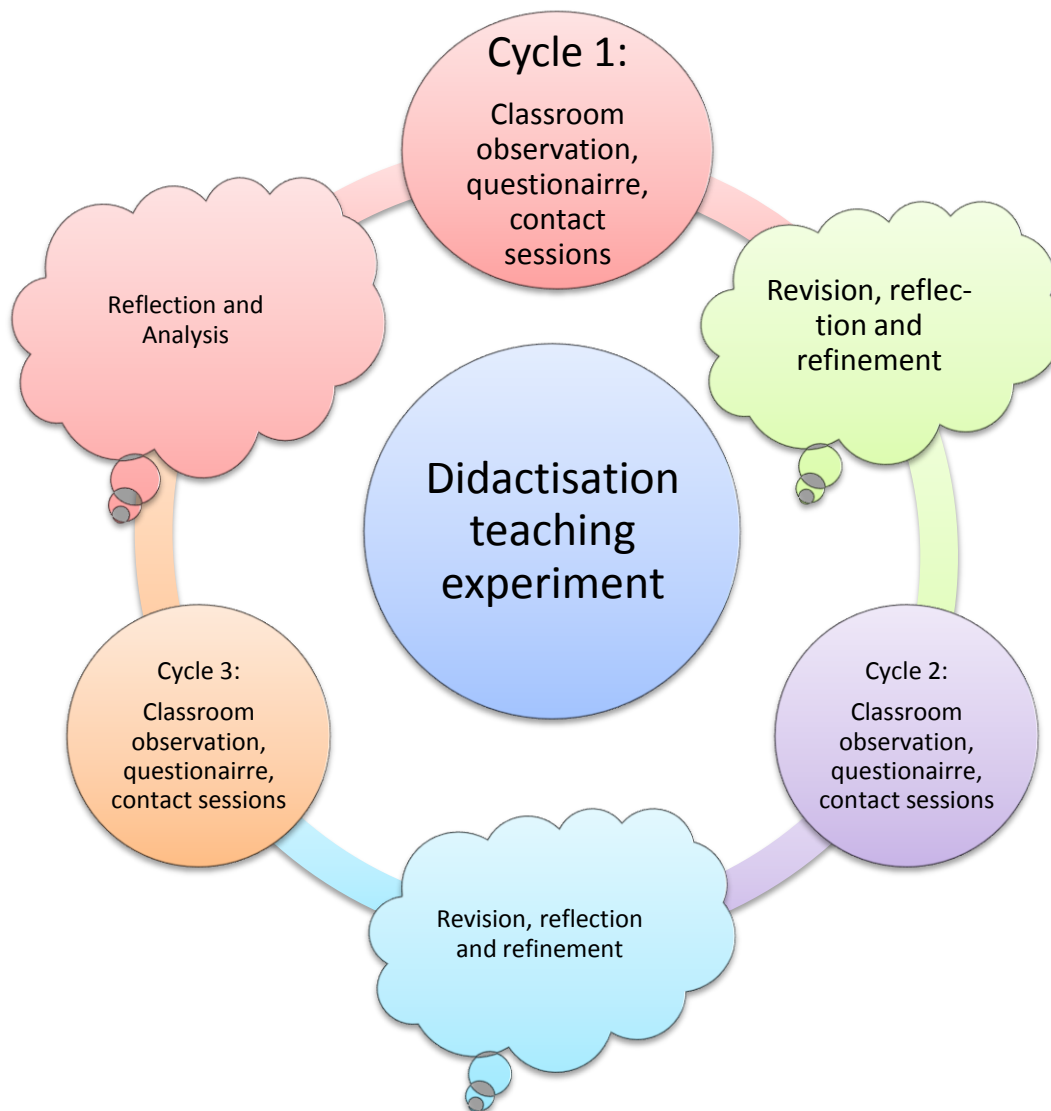
Figure 4.1 Didactisation teaching experiment

4.2.2.1 Collecting information by participant observation

Collection of valid and reliable information during the design experiment is critical to the success of the study. According to Strydom (2005: 274) observation is a typical procedure of a qualitative design. Specifically for this study participant observation is used since the researcher plays a dual role of data-collector and data-interpreter (Coertze in Strydom 2005: 277). In this role a researcher must "become an instrument that absorbs all sources of information (Neuman in Strydom 2005: 277). This role means that reliability and validity are of concern but are not insurmountable since the trade-off of the type of information that will be recovered from such a participant observer role is valuable to understanding didactisation

practices in primary school mathematics teachers. One of the first hurdles to overcome is gaining permission to enter the field. This was done by securing permission from the Gauteng Department of Education (see Appendix 1), permission from school principals, as well as participant consent.

The advantages of participant observation are given by Strydom (2005: 283). The aspects relevant to this research are that participant observation:

- Allows the researcher to stand as in "insider" and not an "outsider" to the phenomenon studied.
- Gives a comprehensive and in-depth perspective on the problem under investigation.
- It is a flexible procedure that allows the problem to be re-defined without detracting from the scientific nature of the study. This ties in with design research where a cyclical design is incorporated.
- It has a special link to practice and prevents results from becoming too theoretical. This ties in with design research where the output is both theory building and practical application.
- It is ideal for gathering non-verbal behavior. This allows the study to build more robust validity and reliability since it adds to the data collected.

4.2.2.2 Collecting information by questionnaires

A number of questionnaires were designed for this study. After careful consideration this was preferred to interviews for some practical reasons. Teachers are not always available after the classroom observation for an interview. On the other hand, questionnaires designed after sessions require some time for reflection before responses can be constructed. Written responses are preferred to verbal ones since the teachers would take care to construct a written response while verbal responses can be off the cuff. Written responses entail more commitment than a verbal response and teachers may therefore only commit to what they are sure of. The other intention with a written questionnaire is that the level of reflection required for a written response is greater than with a verbal response. It also allows teachers the freedom to draw or make necessary diagrams to assist their responses.

4.2.2.3 Collecting information by document study

This study requires that teachers share their lesson planning sheets with the researcher as well as complete a number of written reflection documents throughout the sessions. The lesson planning sheets in particular "are not written with a view to research" (Strydom & Delport 2005: 315), while the instruments designed for this study are written specifically with a view to understanding a phenomenon. These two types of document analysis assist in building a more robust study. Furthermore, the lesson planning sheets will also be indicative of how much systemic pressure is present in teachers' planning and how this may affect their practice in the classroom. Strydom and Delport (2005: 318) cite the non-reactivity of documents as being an advantage of document study. The other instruments that teachers are required to fill in allow the researcher insight into teachers' thoughts and ideas at various intervals during the study. It is also envisaged that a balance between verbal and written interaction of teachers is important in collecting a fuller set of data. This study deemed it necessary for investigating teachers' didactisation principles to use document study to fully understand the phenomenon. The nature of the research question warrants the use of document study.

4.2.3 Retrospective analysis

According to Bakker (2004: 45) the analysis phase leads to the development of instruction theory. The retrospective analysis required the following:

1. Scrutinizing the teacher questionnaires
2. Analyzing the questionnaires
3. Conducting teacher classroom observations
4. Transcribing and exploring teacher sessions
5. Analyzing classroom observation instruments
6. Studying written teacher reflection instruments
7. Exploring researcher field notebook.

After each session with teachers, the didactisation principles that have become apparent will be identified and documented. This will assist in charting a teachers' development per didactisation principle. At the end of the program teachers' development or changes in teacher

knowledge, beliefs or goals will also be documented. To best use the rich data that will be generated by the professional development program a systematic process of analysis is necessary. This will mean that the descriptions and findings reported on are valid, credible and based on that data collected.

Analysis per cycle:

Teacher	Student activity	Student differentiation	Student mathematisation	Teacher Access	Teacher Probe	Teacher connect	Teacher Assess	Teacher Reflect	Vert. aligned lessons
A									
B									
C									
D									
E									

Table 4.2 Didactisation practices analysis

Global analysis:

Teacher	Change in knowledge	Change in orientations	Change in goals

Table 4.3 Global teacher analysis

Analysis of pedagogy, use of context and mathematical content scale. (Instrument 3).

Teacher	Changes in pedagogy scale	Changes in use of context	Changes in mathematical content
A			
B			
C			
D			
E			

Table 4.4 Changes to pedagogy, use of context and use of content scale

These data sources are scrutinised for evidence of improved didactisation practices in participative teachers. This means that the didactisation principles described in Chapter 2 become the coding criteria (see 4.3). The instruments are designed so as to gauge growing didactisation principles. The analysis of the instruments will take place after each session. This will assist the researcher in having fresh perspective on what has taken place so that the sessions that follow can be evaluated in terms growing and developing understandings. As

explained Cobb *et al*. (2003: 12) the research team (or in this case the researcher) "deepens its understanding of the phenomenon under investigation while the experiment is in progress". Furthermore Cobb *et al*. explain that this entails an obligation from the researcher to generate data that will support the analysis of that being studied. Since the study involves one researcher, it is manageable to collect this data for five teachers.

The number of instruments allow for a multiple analysis perspective and the use of multiple theoretical perspectives on the data. This will assist in improving the reliability and validity of the study. Cobb *et al*. (2003: 12) affirm that "multiple sources of data ensure that retrospective analyses conducted when the experiment has been completed will result in rigorous, empirically grounded claims and assertions".

The retrospective analysis needs to be a trustworthy account whereby the events can be seen as potentially reproducible patterns (Cobb *et al*. 2003: 12) while these authors also feel that the "situated nature of the retrospective analysis is the strength of the methodology".

4.3 CODING

Instruments were designed and used so that it would be possible for a single researcher to capture the necessary data that could be used as evidence to substantiate claims. Brown (1992: 163) reminds us that it is a "non-trivial task to capture rich social and intellectual life of a classroom with a level of analysis that would permit one to look at real conceptual change taking place over time". Since the study has set out didactisation principles explicitly, these principles become the basis of coding for the study. The coding is set out in the instruments as follows:

Teachers	Coding	Students	Coding
Accessing student understandings/concepts	TA	Active students – mentally active and challenged	SA
Probe student understanding	TP	Differentiation – provide own methods/ideas	SD
Connecting student understandings	TC	Mathematisation – both vertical and horizontal	SM
Assessing student understanding	TA		
Reflecting on practice	TR		
Vertically aligned lessons/Know and use the curriculum	TVAL		

Table 4.5 Coding for didactisation principles

The nine didactisation principles identified in this study through the literature study were categorised into those that depicted teacher actions and those that represented student actions. Therefore the initial T was given to 6 of the principles while the initial S was given to 3 that related specifically to student actions.

Cobb *et al*. (2003: 12) explain the issues of measurement in design research. They explain that "all measurements (even observations) are indexes to constructs of interest, not the constructs themselves". When extracting evidence from varied forms of data, a coding system was developed to facilitate cross transfer of information.

Instrument number or type	Name of instrument	Coding for direct reference
1	Baseline Questionnaire/Post Questionnaire	BQ, N, date PQ, N, date
2	Lesson Questionnaire	LQ, N, date
3	Lesson Observation Instruments	LI-X, N, date
4	Lesson Observation Instruments	LI-X, N, date
Researcher Notes	Lesson Observation	LON-X, N, date
Researcher Notes	Session Observation	SO-XN, date
Transcriptions	Sessions	ST-X, date
Informal discussions		ID, date

Table 4.6 Coding for data sources

N= Pseudonym of the teacher

X= The number indicates the first, second or cycle of the design experiment

4.4 VALIDITY AND RELIABILITY

The design-based research collective (2003: 7) explain that the "reliability of research findings and measures can be promoted through triangulation from multiple data sources, repetition of analyses across cycles of enactments, and the use (or creation) of standardised instruments". This study promoted validity and reliability of the instruments, analyses and conclusions by various means. This section describes the process of upholding validity and reliability.

4.4.1 Rigor, relevance and collaboration

Mckenney *et al.* (2006: 77) have specified three tenets that should shape design research: rigor, relevance and collaborations. In terms of rigor, the study must be supported in terms of its internal validity, external validity, reliability and utilisation. In terms of relevance, the study sets out the target setting and takes place in a "naturally occurring test bed" (2006: 77). Collaboration must take place *with* and not *for* the participants involved (2006: 77 italics in original). The data research methods are (as suggested by these authors) mutually beneficial. It is envisaged that the professional development program will be meaningful for teachers involved since they are structured to "stimulate dialogue, reflection or engagements among participants" (2006: 77).

McKenney *et al.* (2006: 85) provide guidelines whereby academic rigor in design research studies can be strengthened. These guidelines include:

- Explicating a conceptual framework. This study has undertaken a literature review that has both breadth and depth. A conceptual framework established in the preceding chapters will enable others to make analytic generalisation.

- Congruent study design. This study has strong links with previous research, theory, research questions and research design.

- Triangulation. This study includes a number of data sources, data collection means and instruments. Teachers participating in the study come from different schools and have experience in different contexts and environments.

- Three different types of data analysis. The retrospective analysis of the data will include deductive, interim and inductive analysis which can also be seen as a measure of triangulation.

- Full description. This study aims at providing the context-rich description (see 5.2 – 5.5) of the situation, design decisions, and research results as suggested by McKenney *et al*. (2006: 86).

4.4.2 Validity

a) Internal Validity

Bakker (2004: 46) explains internal validity as the quality of the data and the soundness of the reasoning that is derived from that data. Bakker used source triangulation, looked for counter

examples, multiple theoretical instruments and substantiated claims using transcripts. This study makes use of these strategies in order to strengthen the internal validity. According to Gravemeijer (1994b: 455) this means "seriously searching for counterexamples or by searching for alternative explanations". This study draws from many theoretical perspectives (see 2.2) in its design and implementation and therefore these varying perspectives are drawn upon to make inferences and draw conclusions. This should support the internal validity.

b) External Validity

This refers to what extent the results of this study can be transferred or used in another study in a different setting or context. The teaching experiment is accurately documented in this study which will allow for replication in other settings. Decisions that are made are recorded and explained so that other researchers may be aware of the dynamic nature of the research and make similar (or different) decisions if necessary.

McMillan and Schumacher (2006: 314) included a number of strategies to enhance validity of a qualitative study. Those strategies that relate directly to this study are:

- Prolonged field work – the professional development program designed in this study extends over a number of months, not simply a once-off workshop.

- Multi-method strategies – the study includes numerous activities and instruments.

- Verbatim accounts – in this study, teachers own writing or own words (in vivo) are used to support any conclusions.

- Mechanically recorded data – all written accounts and transcriptions are mechanically recorded by the researcher. This also assists in building the researcher's fluency with the accounts which will heighten sensitivity during the analysis of the data.

- Participant researcher – in this study the researcher is not a silent neutral observer but is the person designing and running the program. This means that the researcher is involved and committed to all aspects of the program and understands how actions and decisions will impact on the program.

- Negative cases – if one or more of the teachers involved does not show development or increased didactisation principles this has to be recorded, explored and explained which will contribute to the validity of the theory proposed by the study.

- Low-inference descriptors – detailed and literal accounts of the participants and the lessons are made in this study so that the reader can make the same inferences as the researcher.

4.4.3 Reliability

4.4.3.1 Internal Reliability

The internal reliability of the study is improved through the explication and discussion of didactisation principles throughout the study and then by using those same principles for coding and analysis. This means that the reader can judge the "the reasonableness and argumentative power of inferences and assertions" (Bakker 2004: 46). Gravemeijer and Cobb (2006: 47) refer to a notion of "trustworthiness", being how reasonable and justifiable the interferences and assertions are. They admit that in design research a number of probable analyses may be made from a set of data depending on the purpose of the research. They affirm that the issue is that the analysis "credible" as a result of being "both systematic and thorough".

4.4.3.2 External Reliability

This refers to the "replicability" (Bakker 2004: 46) and "trackability" (Gravemeijer 1994b: 456) of the study. The study must be documented in such a way that the "experience can be transmitted to others to become like their own experience" (Freudenthal 1991: 161). Bakker suggests that the "failures and successes, procedures followed, the conceptual framework used, and the reasons to make certain choices must all be reported" to increase the external reliability of a design research study. Gravemeijer maintains that other researchers must be able to "retrace the learning process' to enable them to enter into the discussions". The detailed teaching experiment outlined in this study will enable other researchers to implement this design in their own settings and environments.

4.4.4 Triangulation

Triangulation can be informed from various dimensions. Triangulation of data collection and data analysis promotes validity and reliability of a qualitative study. According to McMillan and Schumacher (2006: 325) triangulation refers to multi-method strategies, multiple researchers, and multiple theories to interpret the data, multiple data sources and multiple disciplines. This study makes use of multi-method strategies, multiple data sources and multiple theories to strengthen validity and reliability of the data collected and of the interpretation thereof. By including five teachers, rather than a single case study, the triangulation of data will be strengthened.

4.4.5 Verification strategies

Morse, Barrett, Mayan, Olson and Spiers (2002: 17) define verification as the process of checking and confirming the mechanism used during the process of research to contribute to the reliability and validity of the study. They identified five strategies (2002: 18):

- There needs to be methodological coherence in that the method must match the question. For this study the question as to how teachers' didactisation practices can be promoted is suitably matched with a teacher development program that is sensitive to various components of teacher development and didactisation principles. The meshing of didactisation principles, the mathematical work of teaching and the program form a coherent whole that is informed by theoretical and practical measures.
- The sample must be appropriate. The sample consists of participants who best represent the research topic and will ensure sufficient data to account for all aspects of the study. The sample in this study consists of five teachers at various schools. They are of varying ages and experience levels. It is possible that a negative case or outlying case may exist in the five teachers. It is not envisaged that teacher development is linear, prescriptive or predictive in all ways.
- The collecting and analyzing of data concurrently forms mutual interaction between what is known and what needs to be known. According to these authors this forms the essence of attaining reliability and validity. The data in this study is analysed at regular intervals and informs the next cycle of the teacher development program as is consistent with a design research methodology.

- The investigator should think theoretically. Ideas that emerge from data should be reconfirmed in new data. These authors warn against making cognitive leaps. The investigator should inch forward by constantly checking and rechecking the data. In this study the data is gathered from multiples sources and this means it can be analysed from multiple perspectives.

- Theory development means that theory is developed through two mechanisms. Firstly as an outcome of the research process and secondly as a template for comparing and furthering the development of the theory.

4.4.6 Researcher as designer and evaluator

In this study (as is common in other design research studies) the researcher is the designer and evaluator of the program. This according to McKenney *et al.* (2006: 83) means that the researcher plays conflicting roles of advocate and critic. They agree with Putnam and Borko (2000: 13) view of "rather than pretending to be objective observers, we must be careful to consider our role in influencing and shaping the phenomena we study". McKenny *et al.* see this acknowledgment as the first step toward dealing with the Hawthorne effect and hypothesis guessing. These authors do advise that making a research setting as natural and genuine as possible also reduces these threats. Brown (1992:162) explains how the Bartlett effect influences the data that is selected for validation of the researcher's claims. Morse *et al.* (2002: 17) refer to the "lack of responsiveness of the investigator" as the biggest single threat to validity. Stating these potential problems from the outset will establish that the researcher is aware that these can lead to shortcomings in research and that he/she needs to guard against them constantly. However Kelly (2004: 124) remarks that "researcher bias is inevitable. It is not a moral flaw; it is an expression of human judgment". Anderson and Shattuck (2012: 18) argue that inside knowledge from the researcher "adds as much as it detracts from the research validity". Of importance is that the researcher specifies which aspects of the data he/she is using and why this has been chosen. By keeping the number of participants in the study small it will be possible to analyse all the data generated and not to have "miles of videotape left unwatched and students 'artifacts' unread" (Kelly 2004: 124).

McMillan and Schumacher (2006: 327) suggest the term "reflexivity" as being necessary so that the researcher can scrutinise his/her "personal and theoretical commitments". These authors argue that this will assist in understanding how these commitments guided the researcher in the research questions, generating data, relating to the participants and making interpretations around the data. Once again, one does not deny subjectivity but rather one needs to be aware of its impact on the research. McMillan and Schumacher quote Pillow's (2003) suggestions regarding reflexivity strategies:

- The researcher should have personal awareness.
- The researcher should be aware of the participants – let them speak for themselves.
- The researcher should see his/her role as "truth gathering".
- The researcher should act transcendentally so that the report is represented accurately.

They suggest the following strategies to enhance reflexivity in a researcher; the use of a field log which includes times and dates and activities, a field journal to record decisions made, a written account of ethical considerations, audit-ability – keeping recorded codes and categories for data collection and analysis and conducting formal corroboration interview to substantiate findings. This study has included all of these in the research design. The researcher is aware of, and prepared for, the potential dilemma of acting as both designer and researcher.

Another contributing factor to this research is the relationship of the researcher to the participants. The three of the five participants know the researcher from previous interactions in the field of mathematics education (e.g. attendance at meetings). Two of the three are at the same school as the researcher. This may be cited as a limitation of the study in that replicability under these conditions may be compromised. An outside researcher would have a different relationship with the participants but the program should produce the same results.

In order to facilitate the research proposed in this study at a high quality level, the results of Linder (2011: 48) are taken into consideration. It was found that there are three essential characteristics facilitators of professional development should have. These are:

- A high level of content knowledge;
- The ability to problematize instructional practices; and
- The ability to modify experiences to fit participants' context.

It can be surmised by the design research methodology presented, that these three aspects are present in the researcher of this study. Linder's study further revealed that in terms of being an influential facilitator (2011: 60) the following factors are important:

- Credibility – this is the content and pedagogical knowledge of the facilitator as well as his his/her own classrooms experience. Evidence of effectiveness and professionalism also determine the facilitator's credibility.

- Support- providing assistance to participants and the facilitators' reactions towards participants during the sessions.

- Motivation – content and pedagogical knowledge of the facilitator as well as his/her classroom experience.

- Management – the facilitator's physical actions during the sessions, making session content meaningful and organizing materials for the sessions contribute to how participants perceive the facilitator in terms of management.

- Personality – the positive presentation of self by the facilitator.

The researcher of this study as facilitator of the professional development sessions is cognoscente of these aspects and has ensured that these five aspects will contribute to meaningful sessions and a successful development program as viewed by the participants.

4.5 RESEARCH ETHICS

McMillan and Schumacher (2006: 314) in giving an overview of qualitative research identified some aspects to research ethics. Three aspects have been selected for discussion in the next section as being most relevant to this study. That is: informed consent; confidentiality and anonymity; and privacy and empowerment.

4.5.1 Informed consent

Both teachers and students involved in the study were given a detailed informed consent/ informed assent letter. This included the purpose, procedures, benefits, risks, confidentiality, participation and withdrawal and rights of participants. The teachers and students/parents of students signed this document. They were encouraged to ask questions regarding the nature of

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the research and the activities. It was explained to teachers that they could withdraw at any point of the study.

4.5.2 Confidentiality and anonymity

This was assured to the participants through the informed consent documentation above. Furthermore, the participants would be given pseudonyms. The names of the schools, teachers and students would not be identifiable in print in any dissemination of the research afterwards. Transcriptions would also make use of pseudonyms. All recordings, written artifacts and transcriptions would be kept by the researcher under secure conditions.

4.5.3 Privacy and empowerment

Participants will be able to view the instruments, analysis and research findings before it is published. This will assist in adding validity to the study. Participants may veto any of the information presented in this research in order to protect their privacy. Since the research is not a single case study, this should not be problematic. Data are collected on face value and no deceptions are used, nor is secret information gleaned from the participants.

4.6 OUTPUT OF DESIGN RESEARCH

Design research has two main areas of contribution; a contribution to theory and contribution to practice. As stated by Gravemeijer (1999: 156) a main goal of design research is

> to explicate the local instruction theory that underlies this reconstructed instructional sequence and to justify this theory in terms of theoretical deliberations and empirical assessments.

Since design research makes explicit all its phases, not only the results, it has value to both theoretical contribution and practical contribution. Its fine grained analysis of itself allows for local theory development.

4.6.1 Theory

According to Kelly (2004: 125) we need to move towards identifying what is *necessary* in a learning situation so that the theoretical output of design research can become scientifically more mature. Kelly states that this is where "the real work of theory building occurs". Gravemeijer (1994b: 453) explains that the theory proposed is not put to the test after the development has been concluded; rather "it is the developmental process itself that has to underpin the theory". So with design research, the theory has been tested through its emanation from the design experiment and retrospective analysis. Since design research rests upon cycles of refinement, progressive cycles are only successful if the theory on which they rest is sound. The cycles therefore serve not only to make the theory practically relevant but also to refine and test the theory (see 6.3.1 and 6.3.2).

Barab and Squire (2004: 5) explain that design-based research continually connects design interventions with existing theory, and that design based research generates new theories and does not only test existing theories. It is envisaged that this research will contribute to new theories in mathematics teacher education. By developing didactisation practices in current primary school mathematics teachers, this study meets the challenge of what Barab and Squire call "continually making both types of arguments, arguments that have both experience-near significance and experience-distant relevant" (2004: 6). The real test, according to these authors, is that a study with respect to design based research should demonstrate "consequentiality" of the work (2004: 8). The consequentiality can be seen in the contribution that design research makes to practice.

4.6.2 Practice

According to Gravemeijer (1994b: 454) design research is more concerned with making sense of what is going on than on predictions. This means that design research is never far from practice and that its implications are very relevant for practice.

This research takes place within the messy arena of real school practice. It is designed for teachers, and their real classroom environments are never left out of the design. The design itself is practically useful as other researchers can make use of the practical designs as well as the theory that underpins such designs. The retrospective analysis of design research is

practically useful, not only in its considerations, but also by means of the process of the analysis. Design research is not only practically useful in its final assertions, but in all its phases. The planning phase and teaching experiments have practical applications and transcendence.

4.7 CONCLUSION

Design research has received criticism (Engestrom 2011: 602) in three areas. The first area of concern is that the unit of analysis is left vague. This study, following design research has set out a very clear unit of analysis in defining and clarifying didactisation principles from a strong theoretical point. A second criticism is that the design process is linear and leads to completion or perfection. This study anticipates the development of teachers but no specific end-points have been specified, nor is there any desired concluding skill or attitude expected from the teachers taking part in the study. Growth is a process that cannot have any predetermined path or conclusion. This study takes the view of Schorr and Lesh (2003: 145) that "there can be no final or fixed state of expertise or excellence".

Brown (1992: 172) advocates three stages to design research. This study constitutes what she termed the alpha stage where ideal supportive conditions exist for the study. The next stage (beta stage) involves trying the program out at a few carefully chosen sites, and finally, the gamma stage – the program should be implemented on a widespread basis. The nature of this study allows only the alpha stage to take place. However a successful alpha stage makes for a positive implementation in the beta and gamma stages.

The study has met the conditions for successful design research to take place. It is envisaged that data of sufficient quality can be gathered to answer the research questions and thereby to make a meaningful contribution to mathematics education research.

CHAPTER 5

RETROSPECTIVE ANALYSIS AND RESULTS OF THE STUDY

5.1 INTRODUCTION

Collins *et al*. (2004: 38) propose five elements that are necessary when reporting on design research. These are:

1. Goals and elements of the study need to be reported on in detail;
2. The settings where the study was implemented;
3. The description of each phase;
4. Outcomes found; and
5. The lessons learned.

The goals of this study (see 1.3) were to consider didactisation principles that would form the foundation of a teacher development program and to determine if teacher didactisation practices could be developed through the program. Five teachers took part in the empirical part of the study on a voluntary basis. The empirical study took place over a period of one year – although contact with teachers was maintained for about nine months. For ease of reading and for logical progression the description of each phase, which includes the results and retrospective analysis, are presented in chronological order. Each teacher's lesson is presented. The lessons are presented to the reader in descriptive narrative as observed by the researcher. The description of the lesson is a chronological account of the main aspects of the lesson. Thereafter, each didactisation principle (active students, differentiation, mathematisation, accessing student thinking, probing student thinking, connecting student ideas, assessing classroom solutions, reflecting on lessons and vertically aligned lessons) is presented and discussed across the five lessons. Thirdly, the developmental changes that were made to the program itself are noted since design research lends itself to this iterative process; and finally, changes in teacher resources, orientations and goals are set out. Integration of these elements assists in holistically evaluating the development of didactisation practices.

Two of the didactisation practices are deeply seated within the mental structures of the participants i.e. mathematisation and reflecting on practice. Individual student

mathematisation or student achievement falls outside the scope of the study, therefore, mathematisation is discussed as a potential in each lesson. It is the teachers' alignment of mathematisation moments or activities that forms part of the discussion.

With regard to reflection on lessons, this was gauged because of prompting by the researcher in terms of the questions teachers were asked.

A number of graphical representations are set out through the discussion. They are presented to include a 'pictorial' view of the developing didactisation practices. Although they are 'number' based they are used to show ideas and shifting trends. They however, do not encapsulate the entire essence of the didactisation principle or the progression of the didactisation practice in its entirety i.e. they cannot replace the discussion and analysis that accompanies each section.

5.2 TEACHING EXPERIMENT CYCLE 1

Cycle 1 started with making contact with the teachers and their school principals. It involved securing permission to conduct the study and filling in the preliminary paperwork. Once the initial paperwork was completed, a date was set with each teacher for the baseline classroom visit. Before fully understanding the setting of the three schools involved in the study a few terms need to be explained.

5.2.1 Schools and textbooks in the study

5.2.1.1 Ex-model C schools

This term is used by the researcher to encapsulate a number of aspects regarding these schools. Firstly, it means that these schools were white under the previous government. This means that they were well resourced and maintained by the government. Teacher salaries were also paid by the government. According to Roodt (2011:1) these schools enjoy superior resources and are high-functioning. Hofmeyer (2000: 6) explains that during the 1990s schools were allowed to choose one of three Models that the parents of the school voted on. Model A meant that the school wanted to become totally private with no state subsidy. Model B meant that the status quo would prevail and that the funding from government would be

reduced to 70%. Model C meant that the government would continue to pay for staff salaries, but the school could determine its own admission policy. Hofmeyer states that most schools became Model C schools and opened their doors to all races. Although these tags were done away with in 1994, when a school became either independent (private) or government, schools are still related to in terms of an ex model status. Hofmeyer also explains that there is a high level of integration at English ex-Model C schools because of "an overwhelming desire of black parents to have their children learn English, as the route to jobs and higher education" (2000: 6).

The three schools in this study are Section 21 schools under the South African Schools Act. This means that they charge school fees which are administered by a School Governing Body. They receive a subsidy by the government but the amount received is not as high as schools in the poorer communities. Poorer schools do not charge any school fees and are therefore less resourced. The government pays teacher salaries in a ratio of 1 teacher to 40 pupils in all schools. The governing bodies of ex-model C schools often employ extra teachers to bring down this ratio.

5.2.1.2 Use of textbooks

At the start of the program and for the baseline classroom visits, teachers were following the Revised National Curriculum Statement Grade 4-6 (RNCS). This is the revision of the OBE (Outcomes Based Education) curriculum implemented in the early 2000s. Mathematics and English specifically were revised again and a Foundation for Learning (FFL) program was formulated and very strongly recommended for schools to follow. This program set out specific topics and sections that had to be covered within the term. Within OBE and RNCS training and further cluster meetings, the use of textbooks was never advocated for teachers since they had to make the curriculum relevant to their own schools and communities. Teachers therefore have been producing their own worksheets for pupils on a daily basis. With the further streamlining of the curriculum in the FFL program, some publishers did produce textbooks that matched the program. Only one of the teachers at the three schools in the program was using a textbook although with the fourth revised curriculum that had to be implemented from 2013 called CAPS (Curriculum Assessment and Policy Statement) the use

of a textbook that is accredited by the National Department of Education would be compulsory. This curriculum sets out specific topics and content that must be covered within each week of the year. The textbooks are produced by different authors and publishing houses but need to go through an accreditation process with the National Department of Education. Funds are allocated to schools to purchase these approved textbooks.

5.2.2 Baseline classroom visit: Mrs A

Mrs A teaches Grade 6 mathematics at a suburban school. She has been teaching for three years. She is well qualified with qualifications in psychology and two honours degrees. She has attended some courses on mathematics including the latest CAPS training (3 days) run by the Department of Basic Education. She was very excited at being part of this study. She became a mathematics teacher when her school asked if she would help out with two classes of mathematics. She *loved the subject so jumped at the chance to teach mathematics permanently* (BQ, A, 07-12).

At the school where Mrs A teaches, the buildings and grounds are well looked after. The teaching staff is well qualified both academically and professionally. The school has a well stocked library and computer laboratory. The teachers have access to computers and some teachers (including Mrs A) use interactive smart boards in their classrooms. The pupils at this school are mostly from previously disadvantaged backgrounds. The school's language of teaching and learning is English, but many pupils (between ½ to ¾ - according to Mrs A) are learning in their second language. Mrs A teaches mathematics to four grade 6 classes and the classes are of mixed abilities. She is involved in the school's extramural program about 3 afternoons per week. She also attends staff meetings, grade meetings, subject meetings and teacher appraisal meetings during the afternoons. She is often involved in evening functions held at the school.

Mrs A expressed a belief that a *good* mathematics teacher:

- Has passion for the subject
- In-depth understanding of the subject
- Understands the fears many have of the subject
- Is creative in teaching

- Has an openness to learn

- Has the ability to make learners think differently   (BQ, A, 07-12)

The lesson started with students working on mental multiplication and division followed by correcting homework. After this, she presented pupils with a problem of buying four milkshakes and the bill coming to R50. Mrs A led the pupils carefully from the real problem through the mathematics necessary in solving it. She then asked them for a hidden cost when paying a restaurant bill.

Pupil1: *the thing for the waiter*

Mrs A: *what is that called?*

Pupil 2: *a tip*

Mrs A: *in South Africa, what is the amount we pay for a tip?*

Pupil 3: *10 %*

Mrs A: *so does that mean my bill will go up or down?*

Pupil 4: *up*

Mrs A: *am I adding to the bill or subtracting?*

Pupil 4: *adding to it*

Mrs A then took them through the process of calculating 10 % of R50. She suggested writing 10 percent as $\frac{10}{100}$ and to further reduce this to $\frac{1}{10}$. She used the interactive board to go through the calculations slowly and carefully. She showed them another example where a percentage discount had to be calculated. She showed them the same process carefully again, this time calculating a 25 % discount on a R12 000 motorbike.

She started by writing $\frac{25}{100}$ on the interactive board.

Teacher A: *When we did fractions, I told you to remember that 25 over 100 will always be what?*

Pupil1: *one quarter*

She then wrote $\frac{1}{4} \times 1200000$ (She suggested they convert the rands to cents).

$$\overset{\frown}{} \underset{\div 4}{\underbrace{}}$$

She then asked different pupils to assist her through this calculation. She inserted arrows to show the divide by four, followed by a multiply by 1.

Her students listened attentively and were interested in the lesson. In her reflection of the lesson Mrs A said that her pupils "*need to be part of the lesson constantly, helping with calculations, answering and asking questions.*" (LQ1, A, 08-12).

Mrs A then gave pupils similar problems from their textbook to solve individually. The lesson included a horizontal mathematisation element although this could have been more fully integrated into the latter parts of the lesson. In terms of vertical mathematisation, the procedural links between concepts was re-enforced through student repeating steps shown by the teacher. Students may have experienced challenges if the conceptual links between percentages and fractions were not built in previous lessons. From a modelling perspective on mathematics education (see 3.6.1) this lesson required that students developed behavioural and cognitive objectives. Constructs were developed in the context of word problems. These constructs may be reduced to checklists and students may lose the holistic concept. Process objectives in terms of habit of mind were not necessarily developed through the lesson while students may have developed beliefs that mathematics is about following procedures.

5.2.3 Baseline classroom visit: Mrs B

Mrs B teaches Grade 5 mathematics at a suburban school. She has 15 years experience as a teacher and has a four year qualification. She has attended a number of workshops – none of which she feels has had a major impact on her teaching (BQ, B, 07-12). She has been teaching Mathematics for five years. When she started teaching at this school one class of Grade 5 mathematics was included in her timetable. She now teaches all the three Grade 5 classes and is head of the subject at her school. She says that "*I love teaching Maths –never thought that I would, always struggled with it and avoided it*" (BQ, B, 07-12). She suggested that mathematics teachers should "*teach at the students level and lift the bar constantly*" (BQ, B, 07-12).

During the week that the school was visited there were a number of notice boards that displayed Maths Week challenges. The school also ended the week with a Maths Challenge outside on the field that included many practical activities. The school does not stream classes; all classes have more or less the same mix of student abilities. This practice is in keeping with departmental policy. The school is neat and well maintained. Mrs B teaches

mathematics to about 100 Grade 5 pupils. She also teaches computer literacy and research skills. She is involved in the school's extra mural program four afternoons per week. One afternoon per week is dedicated to teaching of extra mathematics lessons to the pupils who require it.

Mrs B teaches in a shared classroom which also doubles up as a computer laboratory at the school. The laboratory setting means that some desks are in straight rows placed against a wall so students cannot interact with each other. About half the class is able to interact with other pupils if necessary, since the desks are placed in the middle of the room facing each other. There were 36 pupils in the class, many from previously disadvantaged backgrounds. Although the school's language of teaching and learning is English, most pupils in the class would not be learning in their home language. The following figures were provided by the teacher: 15 to 20 percent had good language ability, 15 to 20 percent were "*inbetween*" and about 60 percent had "*marginal*" language ability and could only function on a "*basic*" language level (ST3, 09-12).

The pupils were working in lined notebooks but did not use textbooks (see discussion on why this is a common practice in section 5.2.1.2). On the day of the baseline classroom visit, Mrs B was wrapping up the section on Area and Perimeter. She started by questioning pupils on basic ideas of area and perimeter. She asked them to draw a rectangle in the air to consolidate that rectangles have two lengths and two breadths. She cautioned pupils in learning a formula for perimeter and preferred that they knew what they were doing but adding all the lengths of the shape. She had pre-prepared summary notes on area and perimeter on the classroom board. The pupils not only had to copy the notes for the up-coming test, but also arrange them meaningfully in mind-maps in their books. For the remainder of the lesson, pupils worked quietly on their mind maps. Mrs B then called several pupils (one at a time) to her desk where she checked previous work and re-explained when necessary. Towards the end of the lesson Mrs B reminded the pupils about the upcoming test and reminded them about the correct format for answering an area question (formula, calculation and answer). She also spoke about irregular shapes, that one could calculate the perimeter of an irregular shape. Here she gave the example of a farm that may need to be fenced and mentioned that farms are not perfect rectangles. The pupils managed easily to tell her how to calculate such a perimeter.

Mrs B intended the mind maps to form a type of scaffold – to assist students in remembering how to calculate area or perimeter and to assist students in understanding the difference between concepts of area and perimeter. She added in her reflection of this lesson that she wanted students to have a clear understanding of the difference between the two concepts and that she was surprised that "*some still don't know the meaning of the two concepts and therefore cannot put all the other facts together and see the whole picture. I strive that all learners should always see the whole picture - then tackle the finer detail*" (LQ1, B, 08-12). Mathematisation took place on an individual level during this lesson as there was little whole class or group discussion or interactions. Mrs B did try to link the idea of perimeter to the fence around a farm but the focus on the lesson was on the difference between calculating perimeter and area of given shapes. From the modelling perspective provided in 3.6.1, this lesson encouraged behavioural objectives by focusing on the basic formulas involved in calculating area and perimeter, while students may have developed habits of mind in constructing their mind maps. Their conceptual understanding may have been improved by creating of the mind map while the constructs of area and perimeter were not modeled in a complex problem. By being allowed to design their own mind map students may have developed affective objectives that they have some control over organising their understanding.

5.2.4 Baseline classroom visit: Mrs C

Mrs C has been teaching mathematics for 14 years. She has a three year teaching qualification and has attended a few mathematics workshops including the latest CAPS training by the Department of Basic Education. Her interest in mathematics teaching started at University where she "*enjoyed the way the lecturers transferred knowledge across to us as students. I experienced something different during those classes that I did not experience at high school*" (BQ, C, 08-12).

Mrs C's school is well-resourced and the buildings and grounds are well cared for. The pupils come from a variety of social and cultural backgrounds. About one third of the pupils were possibly second language learners from previously disadvantaged backgrounds. Mrs C teaches mathematics to about 135 Grade 6 pupils. She is involved in the school's extra mural program on two to three afternoons per week.

The classroom is neat and clean with mathematics posters displayed on the walls. The classroom is equipped with a board and overhead projector. The pupils were working in lined notebooks but did not use textbooks. The lesson observed was about conversions between common fractions, decimals fractions and percentages. Mrs C spent a long time in discussion with the students about their understanding of these representations. She questioned them regularly and expected them to explain their thinking. After dealing with grids containing 100 blocks and 10 blocks, she presented the following shape on the overhead:
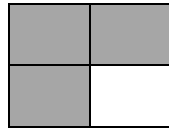
Figure 5.1 Lesson example cycle 1

Mrs C*: What fraction is not shaded here?*

Pupil 1: ¾

Class: *No….!*

Pupil 1: *I mean ¼*

Mrs C: *What fraction is shaded?*

Pupil 2: ¾

Mrs C: *so what percentage is that?*

Pupil 2: *75%.*

Mrs C: *How do you know its 75%, its ¾ - nothing tells me that it is over 100?*

Pupil 2: *I timesed it by 25.*   (LON1, C, 08-12)

The lesson continued in this type of question and answer method. She asked as many different pupils as possible to answer and by the end of the lesson checked to see that everyone had an opportunity of answer a question. In the baseline questionnaire she gave "*question and answer method*" (BQ, C, 08-12) as her way of getting students to take part in a lesson. The lesson continued in this way where Mrs C moved students through converting visual representations to fractions and percentages. She added in the written lesson questionnaire that students learn mathematics best "*visually and practically*" (LQ1, C, 08-2012) and that she was pleased that the students could explain their answers to her. Towards the end of the lesson Mrs C provided pupils with a worksheet where they had to do these conversions individually, in written form. Mrs C carefully built the discussion around these conversions so that she provided scaffolding for vertical mathematisation. She led the discussion from the diagrams

which were models students could use to develop their understanding of these conversions. The lesson encouraged students to model basic facts and skills, while Mrs C guided students' habits of mind through her earlier questioning. There was no sense however of students developing a model for thinking about equality in a more complex system.

5.2.5 Baseline classroom visit: Mrs D

Mrs D is a grade 6 mathematics teacher. She has two years teaching experience. She has a BEd qualification with Mathematics 3 and mathematics methodology courses. She is currently studying further for her BEd (Honours) in Curriculum design. She considers the "*investigative model preferably using the up-down approach*" (BQ, D, 08-2012) to be a good way to introduce pupils to a new concept. Mrs D teaches three classes of Grade 6 mathematics and the school had just undergone a timetable change, so she was also allocated Social Sciences and Physical Education. She is involved in the school's extra mural policy four afternoons per week. She also attends lectures every second Friday afternoon.

Mrs D has her own classroom where mathematics type posters are displayed. The classroom was arranged in groups, from two to six pupils sitting in groups. Mrs D started the lesson with a "sum search". The whole class searched a grid for addition problems. Once the class had found the first ten, pupils had to work individually to see if they could find a total of 30. Mrs D set a time limit after which she asked pupils how many addition sums they could find in the grid. She then asked them to add up the total number of addition sums their group had found. She asked for these totals. She then wanted pupils to find the average for their group. She never used this word, but if there were six in a group she asked the group to divide their total by 6 "*because there are six of you*" (LON1, D, 08-2012). She then wrote each group's average down on the overhead projector. First she wrote down the results as 18rem2, then asked them to write down the common fraction form for this, $18\frac{2}{6}$. The next step was to simplify or reduce the fractions, so in this case $18\frac{1}{3}$. The class worked together through these. Finally she asked them to decide which group had done the best at finding the addition sums. This raised some discussion as pupils were not sure if it was the lowest number or highest number. This part of the lesson moved fairly quickly. Pupils had to listen carefully and were constantly questioned by Mrs D with many pupils involved in answering questions. Mrs D

then presented pupils with cross-number puzzles, but this time negative numbers were included which is an extension on the current grade 6 curriculum. The students continued to work individually for the remainder of the lesson.

Mrs D reinforced a number of mathematical operations during this lesson, so in terms of mathematisation flexibility and speed with mental calculations was developed. Student strategies for mental calculations were not juxtaposed nor compared. Students calculating the average for their group could constitute a horizontal mathematisation element although this was the only occurrence thereof. This lesson displayed students' developing basic facts and skills, habits of mind (in terms of speed and flexibility) and cognitive objectives (see 3.6.1). However students were not expected to provide descriptions or explanations of their thinking as proposed by a modelling perspective.

5.2.6 Baseline classroom visit: Mrs E

Mrs E is a mathematics teacher with sixteen years experience, eight years as a grade 5 mathematics teacher at the school where she currently works. She has a four year teaching diploma. She has attended some mathematics workshops and was a trainer for the Gauteng Department of Education training teachers to implement CAPS in 2013. Her career as a mathematics teacher started when the principal of the school informed her that "*this is what you will teach next year*" (BQ, E, 08-12) when the school needed a grade 5 mathematics teacher. She believes that pupils learn mathematics best by doing something, "*either read and solve a problem, cut and sort shapes or fractions, discuss in pairs and show each other*" (BQ, E, 08-12). Mrs E teaches mathematics to about 135 Grade 5 pupils. She is involved in the school's extra mural program where she could be busy for up to four afternoons per week.

A few days before the baseline observation visit, Mrs E explained that she was concerned about her pupils' ability to multiply two and three digit numbers. She had spent many lessons in class on this concept. The lesson that she had planned for the visit was therefore taking a few steps back and consolidating multiplying by multiples of 10 and 100 since she considered this to be the problematic area. She got the pupils to write out these lists (10 x2, 20x2, 10 x 20, 20 x 20 etc) which she said in her lesson reflections is "*not much fun*" (LQ1, E, 08-12). Mrs E had spent the previous few weeks showing pupils various multiplication methods but

she felt that no progress was being made because these multiplication facts were stumbling blocks. She was particularly keen that the grade 5s used "*the breaking up method*" (as advocated by the CAPS document) and to do this, pupils had to know their multiplication facts. The fact that she had to spent more time on this section was worrying to her, not only because the pupils were having difficulty but also because there was not much extra time allowed if she were to complete the Foundations for Learning program. She did comment that often, as a teacher "*I need to move on in the curriculum before all pupils are ready*" (LQ1, E, 08-12). Mrs E also commented that she allowed herself to adjust her lessons on a day to day basis depending on pupils' understanding. In terms of the scope for mathematisation during this lesson, student mental calculations and memorisation of these calculations was forged on an abstract level. Since Mrs E was pressed for time, she did not present other representations of these calculations or encourage students to build their own methods. In terms of a modelling perspective (see 3.6.1) the focus of the lesson was on learning basic facts and skills while habits of mind may have included a dependency on memorisation which Mrs E was anticipating students would transfer to a conceptual understanding of multiplication. Mrs E wanted student to construct multiplication patterns although they may have needed to sort out existing place value structures first.

5.2.7 Baseline didactisation principles

The five teachers each have various strengths as mathematics teachers. Their high level of care for their pupils and the high level of maintaining the integrity of the subject were noticed. Their professional behaviour was beyond reproach and they form core pillars at their respective schools in fulfilling many other roles (grade tutor, sport, administrative duties etc). It can be said that they add value to their schools in more ways than just mathematics teaching.

The five teachers are mainly teacher-centered and embody "pedagogy of presentation" (Nicol 1999: 47). Each teacher invested much time and energy in ensuring that the concepts were presented as clearly as possible. Each teacher using what appeared to be a preference to a visual and auditory approach to developing understanding. The teachers understood the role

of questioning in a mathematics lesson and used questioning to assess whether students were following their presentation and exposition of the lesson.

A contextual factor is the number of extra-mural hours these teachers are required to put into their schools. This will affect their reading, planning, reflection and assessment time for mathematics teaching. Each teacher is involved in between 4 and 10 hours of extra mural activities per week excluding meetings, marking, preparation or other administrative tasks such as filling support documentation for weak students or compiling statistics. This was also a factor in the time teachers had to interact with the researcher on occasions other than the specified session times or observation lessons.

The following discussion applies to the baseline lessons presented. For each didactisation practice a graph is presented. The graph was generated from the instruments used in classroom observation and the notes taken during these lessons. These visual representations complement the discussion on each didactisation practice and cannot replace the discussion. The vertical axis represents the number of lessons (the maximum being five) and the horizontal axis represents the developing didactisation practice. Each individual teacher's developing practice is not represented graphically since this would present a comparison of the teachers. This would go against the non-judgmental underpinnings and ethical considerations of this study. The overall development of the teachers involved in this study is more significant than the development of each single teacher. Studies where the development of an individual teacher is tracked would complement this study and is recommended for future research (see 6.5).

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5.2.7.1 Student activity: baseline lessons
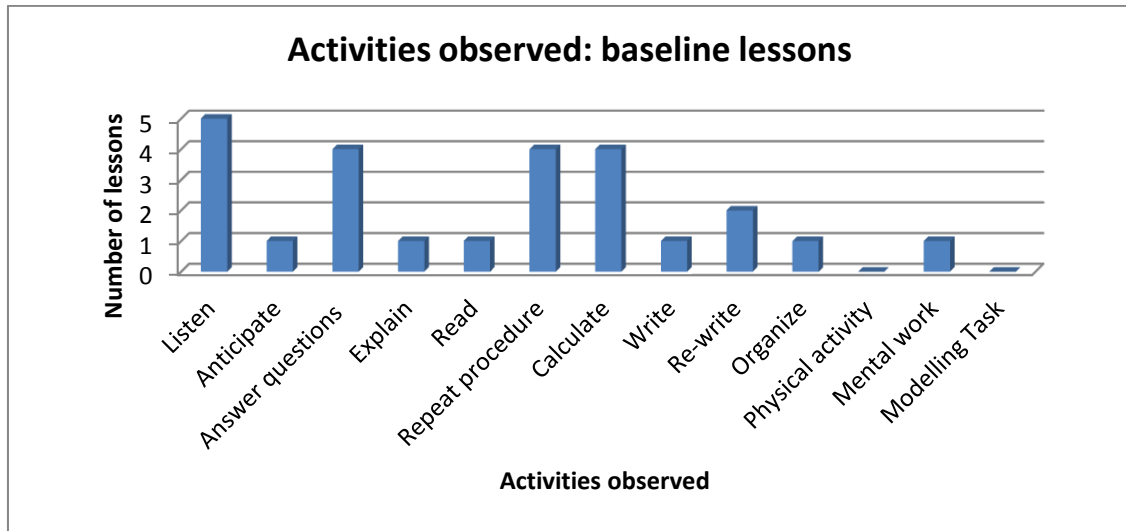
Figure 5.2 Student activity: baseline lessons

Fig 5.2 shows what activities the students were involved in during this first lesson observation. The preferred activity seems to focus on the mathematical task for the day, so student activities such as: listen, answer questions, repeat the procedure and calculate, featured mostly over these lessons. The five lessons were teacher centered and this would have a direct impact on the other didactisation practices, since they all emanate from activity. The type and level of student activity in these lessons will result in student working at an abstraction level that they may not necessarily be ready for. Mathematisation may not be supported sufficiently if student activity is not supported. Low student activity may also make it difficult for teachers to gauge how students are coping with the lesson until the next day or even later.

5.2.7.2 Differentiation: baseline lessons
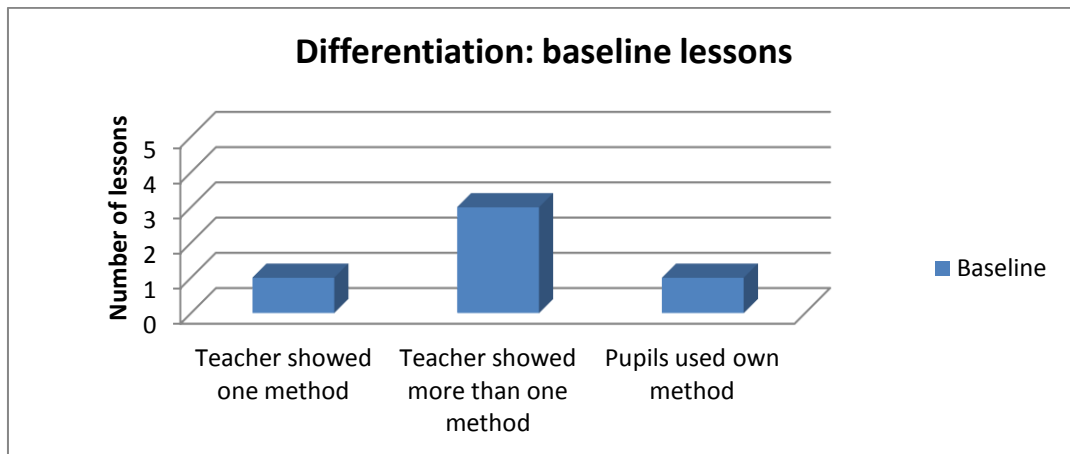
Figure 5.3 Differentiation: baseline lessons

In terms of differentiation (Fig 5.3) the five teachers do have sensitivity towards students'
understanding the mathematics that they are working with or assisting students in performing
the required calculation more easily. The teachers show or explain an easier (shorter) way of
reaching an answer. Mrs E's lesson was based on mechanistic calculation while in four
lessons (Mrs A, B, C and D), teachers all showed another way of arriving at the answer.
During the individual work at the end of the two lessons (Mrs A and Mrs C), pupils were not
specifically prescribed a method. The lessons involved aligning student thinking **to** the
teachers thinking.

5.2.7.3 Mathematisation: baseline lessons
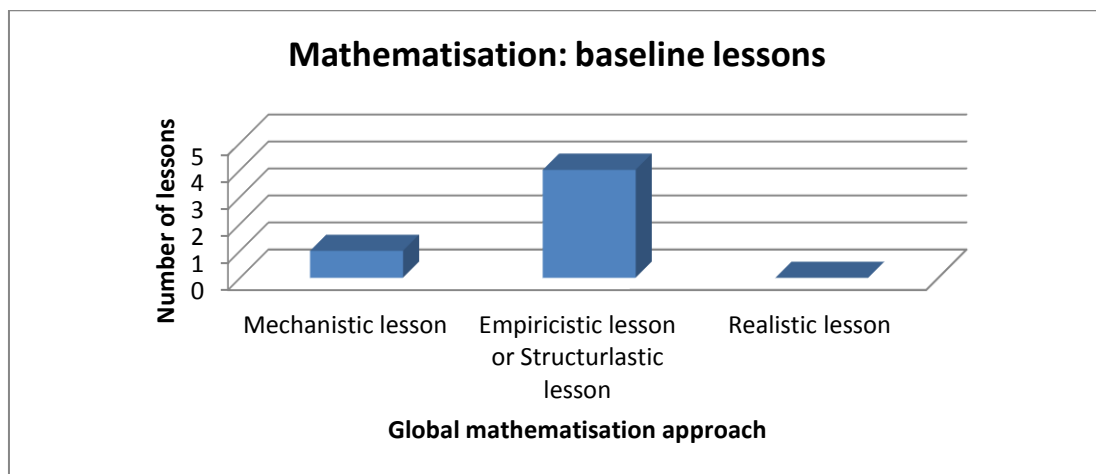
Figure 5.4 Mathematisation: baseline lessons

The lessons observed are discussed according to the RME conception of lessons. Most lessons took on a structuralistic form with a strong emphasis on formal knowledge. Teachers wanted to move students onto a more elegant and shorter version of mathematics and therefore presented suggested models of calculation to students. The teachers had all, in their own way, thought long and hard about how to **present** mathematics so that students would understand it, but students were not involved in making their own connections, shortening or reflecting their own ways of working, which are important components of mathematisation. Since Van De Walle, Karp and Bay-Williams (2010: 27) explain that translations between and within different representations can assist students in developing concepts together and considering the adaptations made to Goldin and Davis (see Fig 2.6 and Fig 2.7), it is important to consider the representations in the five lessons.

Lesson	Representations	Teacher or student	Is representation student constructed or applied?
Mrs A Gr. 6	Verbal Written	Teacher and students Teacher and students	Constructed Applied
Mrs B Gr. 5	Verbal Written	Teacher and students Teacher and students	Constructed Applied
Mrs C Gr. 6	Verbal Written Pictures	Teacher and students Teacher and students Teacher	Constructed Applied -
Mrs D Gr. 6	Verbal Written	Teacher and students Teacher and students	Constructed Applied
Mrs E Gr. 5	Verbal Written	Teacher Teacher and students	Constructed Applied

Table 5.1 Baseline lessons: teacher representations

Table 5.1 shows that teachers use written and verbal representations mostly during their teaching. This may be limiting to the type of connections that students can build and to their ability to reflect on these connections and therefore limiting to their ability to mathematise. The reliance on written or verbal communication confirms the teachers' pedagogy of presentation as opposed to one of construction. Richland *et al.* (2004: 56) also found that most analogies in US classrooms are verbal. Skemp (1996:104) helps us understand the reliance on verbal-algebraic imagery over more visual ones. Table 2.5 (see 2.4.4.3) is produced here again:

Visual	Verbal-algebraic
- Abstracts spatial properties, such as shape, position - Harder to communicate - May represent more individual thinking - Integrative, showing structure - Simultaneous - Intuitive	- Abstracts properties which are independent of spatial configuration such as number - Easier to communicate - May represent more socialized thinking - Analytical, showing detail - Sequential - Logical

Table 2.5 Skemp's comparison of visual and verbal-analytic imagery

In terms of Gravemeijer's (1994a: 101) models of mathematics education (see 2.4.4.7) which include 3 possible lesson models where a bottom-up approach is pursued

Formal knowledge	Formal knowledge	Formal knowledge
Model	Model	Model-for
	Situations	Model-of
		Situations
Structuralist	**Intermediate model**	**Realistic**

Table 5.2 Gravemeijer's models of lessons

and Gravemeijer's (1994a: 82) four strategies (for generality, for certainty, for exactness and for brevity) that promote mathematisation (see 2.4.4), the five lessons can be interpreted as follows.

Mrs A's lesson: can be seen as a structuralist lesson which involves only a teacher presented model (algorithm for calculating area) and formal knowledge while students are involved in exactness (modelling the teachers actions) and brevity (symbolizing and schematizing).

Mrs B's lesson: can be classified as a structuralist lesson since the problems are there only to house the model (method to calculate a percentage) of formal knowledge. Although a problem was presented to start the lesson, it was used as means to practise a specific computation model.

Mrs C's lesson: this lesson is also structuralist since it involves a model (how to convert fractions to percentages) and formal knowledge.

Mrs D's lesson: this lesson is structuralist as it includes models of mental calculating and formal knowledge.

Mrs E's lesson: this is classified in terms of Treffers' mechanistic lesson. There were no composite activities in this lesson. The drill was part of forming connections to the formal algorithm for multiplying two and three digit numbers.

The four strategies that promote mathematisation should ideally not occur in isolated instances nor where brevity is the main focus of the mathematical activity. Douady's conception of tool and object formation (see 2.4.4.2) is also relevant here since students had to use given objects as tools and were not required to build the objects first.

5.2.7.4 Accessing student thinking: baseline lessons
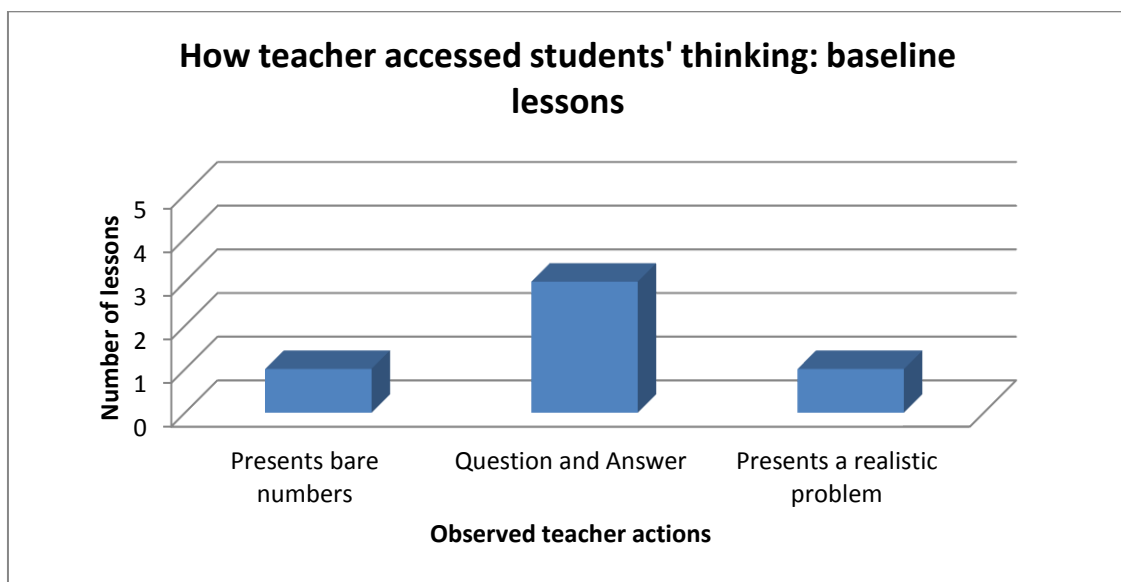
Figure 5.5 Accessing student thinking: baseline lessons

Figure 5.5 displays how teachers started their respective lessons. The horizontal axis presents three different ways teachers could start a lesson: presenting a calculation with bare numbers only, eliciting student thinking about a particular topic or content by asking questions or presenting students with realistic problems. Asking questions was central to the lessons observed. This technique was used in varying degrees, with Mrs C using this most effectively in the lesson. One lesson started with a realistic problem but the problem was then used to show how regular calculations could be performed on similar problems.

5.2.7.5 Teacher probing: baseline lessons

Teacher probing of student thinking: baseline lessons
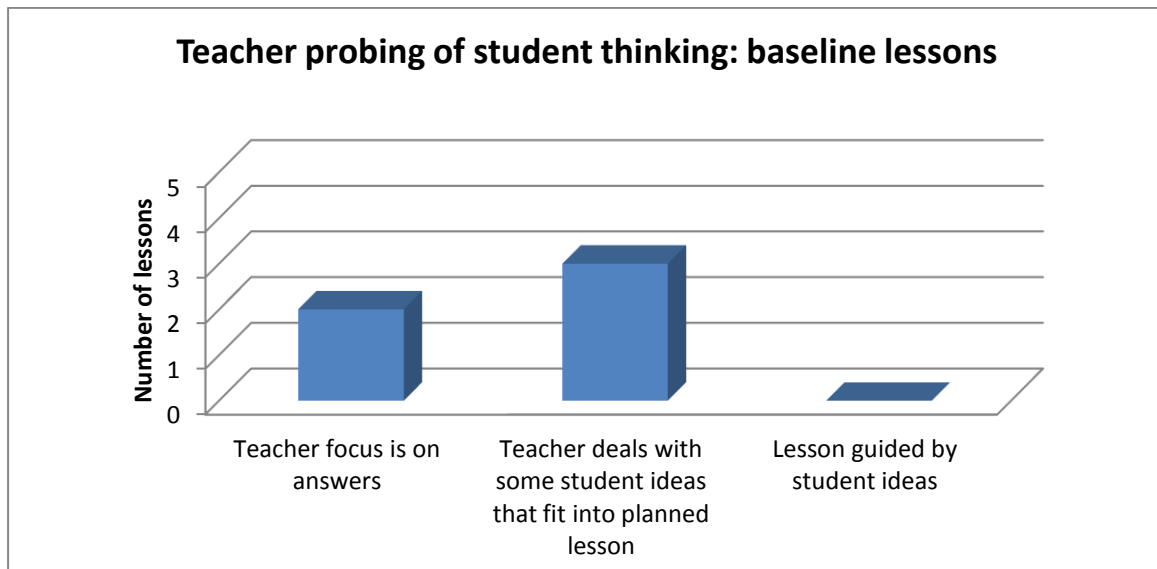
Figure 5.6 Teacher probing of student thinking: baseline lessons

Fig 5.6 shows how teachers probed student thinking during the lesson. The teachers were well prepared and knew the section of work well that they were covering. However, their goal orientation may have contributed to student ideas not being the focus on the lessons. The following were the written goals given by the teachers for their lessons:

Teacher	Goal for baseline lesson
A Gr. 6	*That with a discount you need to do a subtraction, over and above working out what the percentage of the value is.*
B Gr. 5	*Have a clear understanding of perimeter and area and the difference between the two concepts.*
C Gr. 6	*Write a decimal number or common fraction as a percentage.*
D Gr. 6	*That they would not get worried when they get a BODMAS sum.*
E Gr. 5	*A solid knowledge of the basic tables and the extended tables (up to 120 x 900) as these are essential in order to multiply large numbers accurately.*

Table 5.3 Teacher goals for baseline lessons   (LQ1, 07-12)

Goals were largely to convey mathematics to the students, although Mrs B and Mrs D had more student centered goals. Although teachers did ask and answer questions, the discussion was steered back to the goals the teacher had for the lesson. Contextual factors include a very full curriculum to cover; students at varying mathematical ability levels; and multi-lingual

classrooms. The following table summarises the teachers' roles during the part of the lesson where an activity was allocated for students to complete on their own. The categories have been adapted from the classroom observation instrument of the SPIRIT (Timms 2006: 4) project.

Teacher Probing role	Teacher as supervisor of the activity. Answers questions or clarifies if students ask.	Teacher as director or manager. Initiates discussion, controls the topic. Allows or invites input.	Teacher as facilitator. Sets up structure, interacts with students. Students interact with each other and materials.
A	X		
B	X		
C	X		
D	X		
E	X		

Table 5.4 Teacher probing: baseline lessons

5.2.7.6 Connecting student ideas: baseline lessons
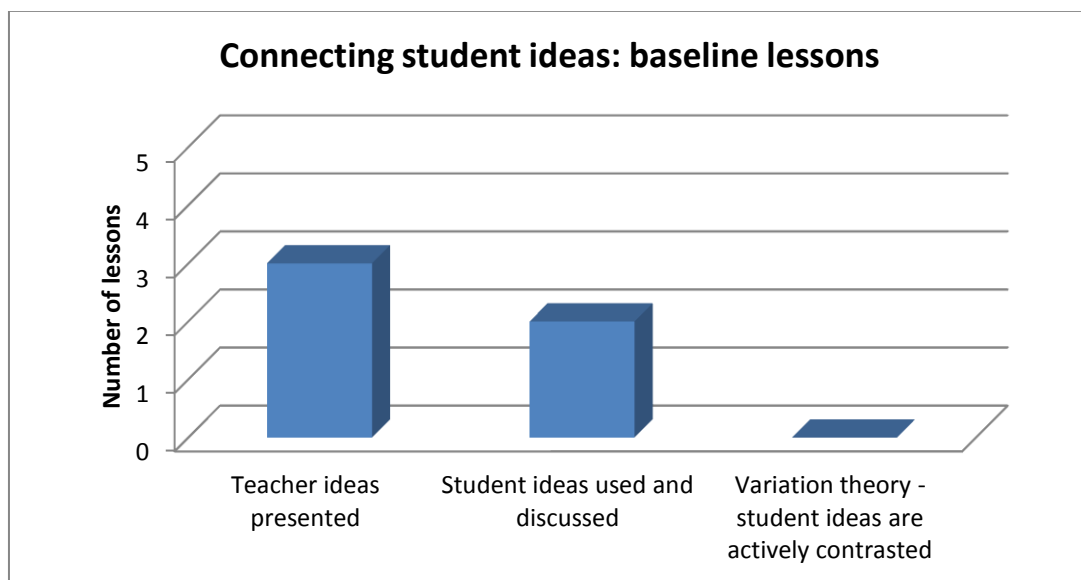
Figure 5.7 Connecting student ideas: baseline lessons

Fig 5.7 shows the results of how teachers facilitated constructions during these lessons. The focus of the ideas in the five lessons was largely the mathematics that had to be taught and covered and not as much about student ideas about these concepts. With larger classes it is easier for a teacher to present the most common understanding of an idea. The aim for these

teachers was to get students to think about the concept in the same way. Teacher goals for their lessons can also be linked to the development of concepts and connections during the lesson. In terms of Hiebert *et al.*'s (1997: 4) three ways (sharing mathematical conventions, suggesting alternative methods and generalizing) that a teacher can encourage connections in a classroom (see 3.2.1.3); all teachers were sharing mathematical conventions with students in these lessons; Mrs A and Mrs C's lessons students did suggest alternative methods to simplify fractions so that the calculation of an amount or conversion to a percentage was easier; while Mrs B's lesson involved students generalizing. Gravemeijer reminds us that generalizing involves the construction of connections (1994a: 83). Teachers did build explicit connections between the verbal and algebraic imagery in their lessons, although weaker students may not have had access to this. Since the students are between the ages of 10 and 12, they may have needed more varied representations in dealing with these mathematics ideas. However, a more suitable approach may have been for students to construct these connections themselves. This would have needed independent student activity first.

5.2.7.7 Assessing classroom solutions: baseline lessons
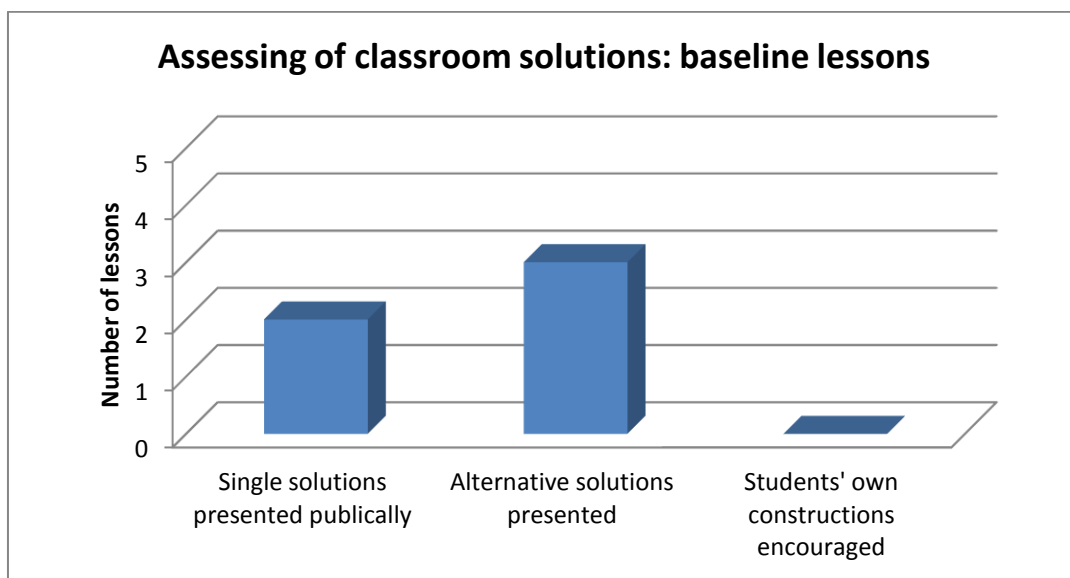
Figure 5.8 Assessing classroom solutions: baseline lessons

Although teachers were presenting alternative solutions or methods, ratifying of solutions in these lessons was a teacher function. Students in all classrooms, however, had, and exercised the freedom to decide and say if a solution was plausible during class discussions. Teachers

did not specifically request that students use or find different solutions or methods. Discussion around alternative solution methods would have a direct impact on the type and quality of connections students could build.

5.2.7.8 Teacher reflection: baseline lessons
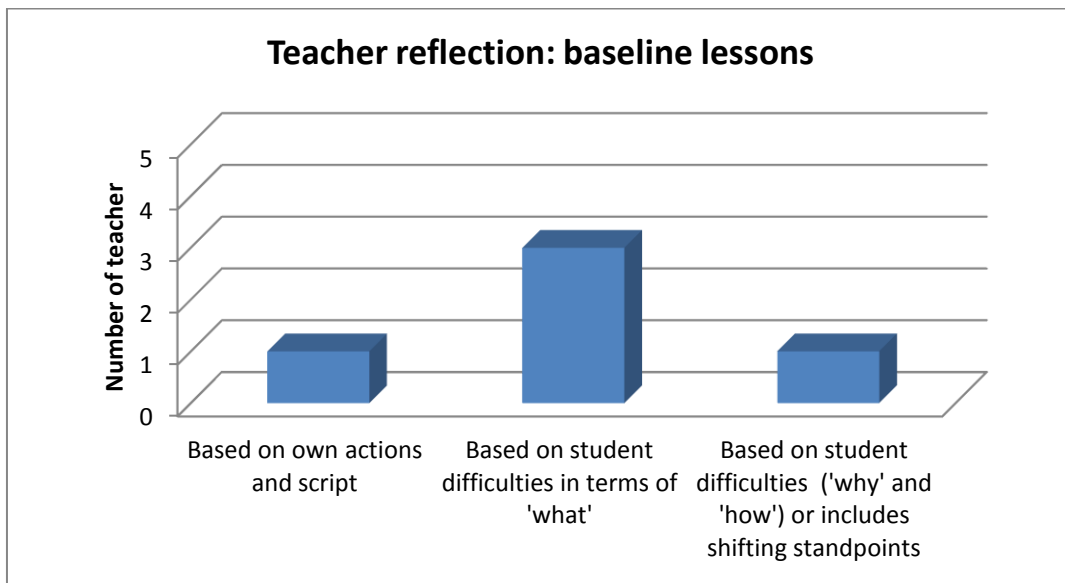
Figure 5.9 Teacher reflection: baseline lessons

Teacher reflection over the five teachers was varied with most teachers focusing on *what* mathematics students found problematic. The teachers have a good rapport with their classes and are largely able to judge if the lesson went well before conducting any formal assessment. All teachers commented on the varying student abilities in their classes. This is due to students not having to be retained in a grade if they had poor mathematics results. This will also be a factor in how difficult it will be to connect student ideas since, the more varied the levels of understanding, the more skillful a teacher needs to be to make the connections. These connections could assist students to progress vertically and the desired mathematical goal to be reached.

When asked to reflect on what they found 'surprising' in their lessons, the following responses were given:

Teacher	Response	Response based on content, students or teacher
A Gr. 6	*The most surprising was that many were aware that there was an additional calculation to do in order*	Content

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	to work out how much they needed to pay for a discount	
B Gr. 5	*Some still don't know the meaning of the two concepts* [area and perimeter] *and therefore cannot put all the other facts together and see the whole picture.*	Content and student
C Gr. 6	*That they could answer my questions. They could explain their answers to me.*	Student
D Gr. 6	*The excitement about working in a group and the fact that they were happy with their averages.*	Content and student
E Gr. 5	*They have some basic understanding of 4x7, but 40 x 70 or 40 x 700 is not understood and this inability to extend the tables came as a surprise. Multiplying by big numbers will help when we do the breaking up method in the next section.*	Content and student

Table 5.5 Teacher reflection: baseline lessons   (LQ1, 07-12)

Table 5.5 shows that teacher reflection on their lessons is based on content and students' responses to the content. The table shows that reflection allows teachers to learn from their teaching (Wilson & Heid 2010: 19, see 3.2.1.5). Reflecting on a lesson from various perspectives is not included in teachers' current planning.

5.2.7.9 Vertically aligned lessons: baseline lessons
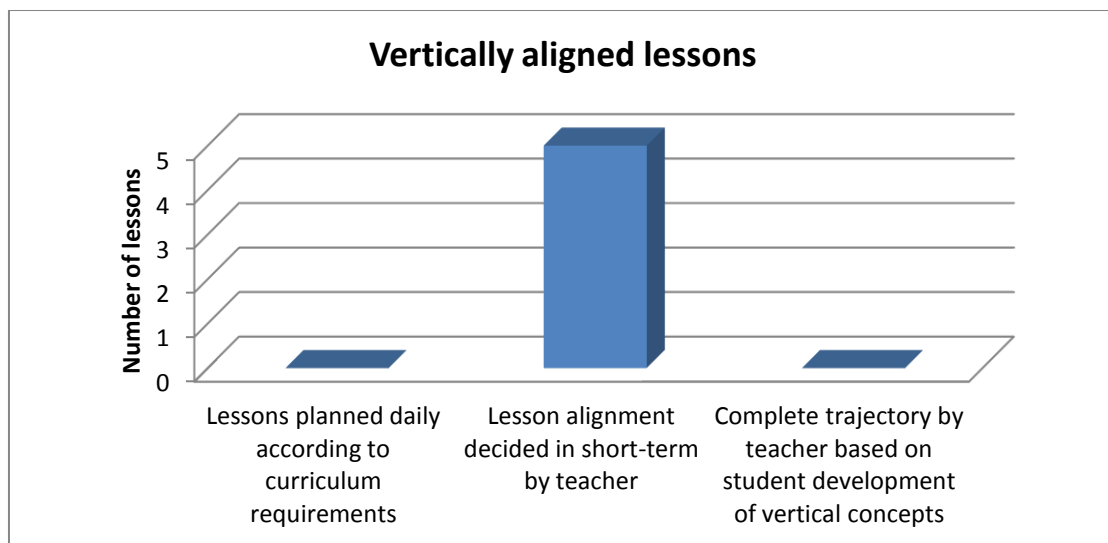
Figure 5.10 Vertically aligned lessons: baseline lessons

Teachers are bound by what was stated in their curriculum documents. The pressure of local district departments and the rigidity of the forthcoming CAPS curriculum have led teachers to follow curriculum order very closely so as to avoid confrontation with local district officials. Teachers' knowing and using the curriculum has evolved through the various curriculum documents set out at a national level. An outcomes-based curriculum was set out in 1998. It allowed the teacher to select relevant content to match the outcomes set out for a phase (a three year period). This curriculum was revised and a revised national curriculum statement (RNCS) was set out for implementation. In this document, mathematics teachers were given assessment standards per grade. These standards had to be met within the year. Teachers were required to 'cluster' different assessment standards and chart the vertical progression of content. A few years later, a program was designed and developed called Foundations for Learning. This document set out already clustered content and vertical progression of the content to be covered per term (3 month period). Teachers were able to arrange the content within the term only. With the introduction of CAPS in 2013, content and vertical progression of concepts are set out in order that they are to be covered and the topics are assigned a time frame. Teachers have to cover specific sections per week. The funneling of the curriculum from allowing teachers a vertical progression over a year to specifying hourly time frames for content is a relevant contextual factor in understanding the didactisation practice of vertically aligned lessons. At this point teachers are following the Foundations for Learning program and can arrange lessons within the term. For this reason, all teachers fall into the middle section of Fig. 5.10.

5.2.7.10 Summary: baseline lessons

The five lessons observed at this stage of the program are scored against the instruments used in the classroom observation and the *average* of those scores is taken to reflect a general guide of where teachers' didactisation practices are currently. The vertical axis is now out of a possible score of 3 since the instruments comprised three categories. The 3 categories were given a score of 0 for traditional teaching, a 1 for signs of change and a 3 for teachers implementing reform-based ideas for that didactisation practice. The graphs are a visual representation of what was observed during lessons and are based on the researcher's interpretations. The graphs will enable a visual comparison at the end of each classroom

cycle visit. This visual representation may be useful to others who would like to focus on the general trend or findings of the study. The resulting graph is as follows:
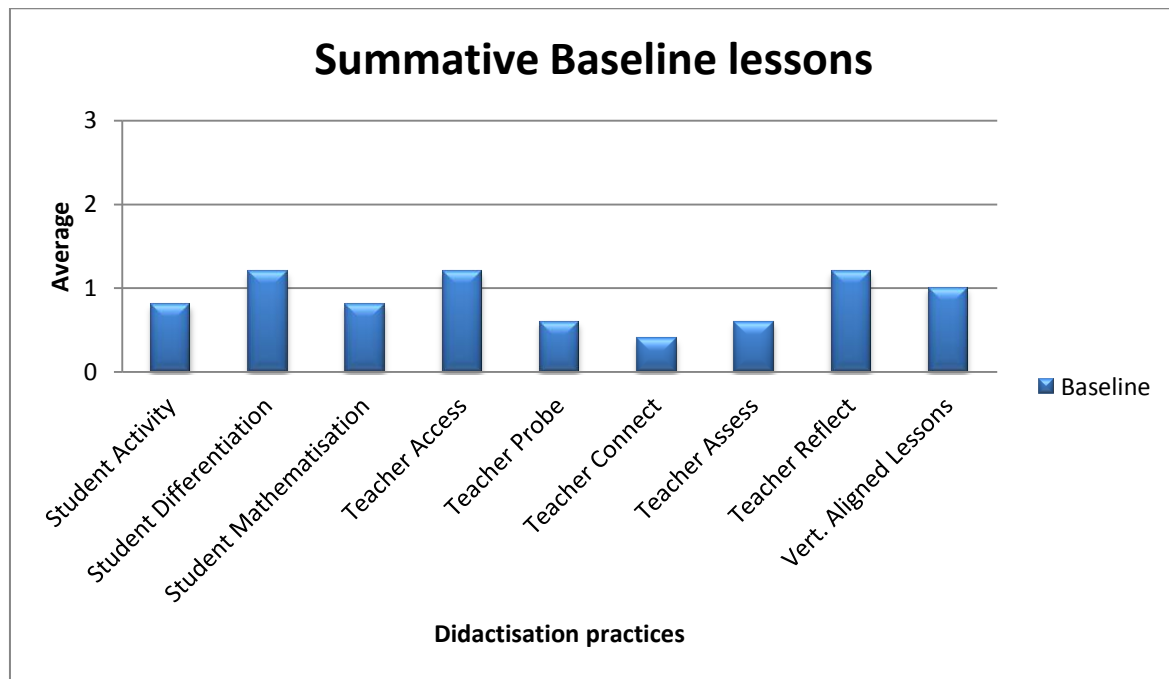
Figure 5.11 Baseline classroom observation

5.2.8 Developmental/Design changes to the teaching experiment in cycle 1

The classroom visits were followed by three researcher/teacher sessions (see Table 4.1). During the first session, teachers solved modelling task 1 as a group and discussed their thoughts about these problems. During session 2, groups of students solved the same problem while teachers observed the session. A follow-up discussion and reflection took place. The third session involved discussion differences between traditional bare number computation and problem solving. The researcher presented different problems to teachers for discussion. The changes made in terms of design research during the first cycle are documented to account for the design research nature of the study. It became necessary to change the baseline interview to a questionnaire. The teachers' schedules did not allow for discussion after the first classroom visit as they all had other classes to attend to. This change did not compromise the data collected in any way, in fact it may have improved the quality of the responses since teachers would have more time to formulate a written response than to simply 'say what come into mind first' as could happen during an interview.

During the first session, one of the teachers shared that she had some knowledge of a Singaporean method of solving word problems by using models. The researcher followed this up by providing each teacher with an article describing this the following week. It was an opportunity to present teachers with research that was topical and that emanated from the group discussion. When describing what would take place during the fishbowl session, teachers requested if they could observe two groups of mixed ability grouping. The teachers were concerned that weaker students would not manage the modelling problem. The teaching experiment was therefore changed to accommodate this without compromising the research quality.

During the third teacher session, the researcher had requested that teachers fill in an A3 piece of paper to summarise their discussion around the two different problem types and their use in improving didactisation practices. It was however found that this instrument hampered the discussion (which was being recorded). When teachers were talking they stopped writing and when they were writing they stopped talking. So instead of producing two sets of much thinner data, the researcher decided to keep the recorded discussion as the primary data source. The headings on the instrument were however used by the researcher to guide the discussion. This was found to produce rich discussions. In the light of how few opportunities South African mathematics teachers have to talk to and with each other, it was decided that the discussion would be valuable for their professional development. It was decided to discontinue the use of this instrument for teachers to produce written notes.

The end of the third teacher session would signify the end of research at schools for the year. This meant that four months would pass before research could continue. It was therefore decided to provide teachers with resources that they could use in their classrooms as well as relevant assessment questions from TIMSS. The reasons were twofold. Firstly to keep teachers motivated and reflective during the four month delay as well as providing material that met the aims of this program and what would be required of them during the implementation of CAPS from 2013. The assessment criteria in CAPS (DBE 2011: 296) are based on four cognitive levels (knowledge 25%, routine procedures 45%, complex procedures 20% and problem solving 10%) that are based on TIMSS questioning. Teachers were given copies of Grade 4 and Grade 6 TIMSS past questions. It was felt that the success of the

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program would be closely linked to the relevance of the program material to teachers' everyday teaching.

5.2.9 Changing resources, orientations and goals during cycle 1

As explained by Schoenfeld a teacher's resources, orientations and goals are interdependent and provide a "lens through which teacher activity can be examined" (2011: 194). These three concepts are used to analyse possible changes in the teachers' thinking. Although the focus of this study is on teacher action in their classrooms i.e. didactisation practices, these are supported by an interconnected and complex network of orientations and goals. This section allows one to start gauging whether the study meets the objective set out in 1.3.2.7.

5.2.9.1 Changes in teacher resources

By exposing teachers to modelling problems and later problem-centred approaches for their own classrooms, the resources available to teachers were expanded by this research (see 1.3.2.5 and 2.3.1). There were many discussions regarding these types of problems and this led to expressions of changes in orientations and changes in goals. These problems as resources created cognitive conflict which through reflection could result in a change in goals or orientations or an acceptance of current goals orientations. Teachers were specifically asked what *surprised* them about observing the group of students solving the modelling problem (airplane task). A summary of their responses shows that modelling may be considered as a resource by these teachers since they saw many benefits to modelling as a teaching and learning activity. It also became evident that this task did make teachers consider their beliefs and orientations in mathematics.

Mrs A

- The enthusiasm and participation of all taking part
- Understanding formulated by the learners
- The input given by all learners was surprising
- The interpretation of the question
- The substantiation of the answer
- Implementation and understanding of 'prior' knowledge in the concepts presented in the question (S2R,A, 09-12)

Mrs B

- The group realised that the smallest value would be the winner for the third column and the other two values the greatest value is the best (S2R, B, 09-12)

MrsC

- They needed very little assistance
- The could actually get on with the problem
- They interacted well
- They were very interested (S2R, C, 09-12)

Mrs D

- The learners worked out that everyone was supposed to share their ideas
- The learners did not read the whole activity thoroughly
- The learners were not in a hurry to complete the task
- Minimal off task talking was observed
- They easily participated - they saw pleasure in solving the problem (S2R, D, 09-12)

Mrs E

- The pupils were engaging with the problem - they took a while to understand what to do, but were soon taking part
- Both groups shared with each other and arrived at different conclusions that they were able to justify (S2R,E,09-12)

Table 5.6 Changing teacher resources: cycle 1

It appears that there is a surprise factor to teachers' first experience of pupils solving modelling problems. This can be seen as meeting the objective set out in 1.3.2.3 and serving as cognitive conflict (see 3.3.1.1 and 3.4.1). Modelling served a very valuable role in the design experiment. It facilitated the discussions in subsequent sessions on problem-centred teaching. In fact, teachers were learning by the same process that problem-centred teaching and a modelling perspective advocates. Mrs B started the observation session by stating that:

This was not going to work

Researcher: *how would you do this?*

Mrs B: *I would spend 80% of the time doing the problem with them.*

Researcher: *would you discuss the context in more detail?*

Mrs B: *I would clarify the aim of the competition more – that it is the airplane closest to the target*.   (S2ON, 09-12).

 Later in her reflection of the session she wrote:

> "*I think one has to experience this first before the penny drops. We underestimate the children at the best of times. Their confidence will also be boosted once they have solved a few problems themselves. I'm putting my main emphasis on implementing this way of teaching in 2013 and look forward to the results.*" (S2R, B, 9-12).

Modelling tasks are a powerful resource for both researcher and teachers.

5.2.9.2 Changes in teacher orientations

This section considers how the study meets objective 1.3.2.5 in terms of teacher orientations (see 2.3.3). During the second teacher session, the focus on the discussion was centered on placing problems first in teaching concepts and not waiting for the end of a section to present problems to students. The researcher presented examples from her own teaching with grade 7 integers. During the third session, Mrs C reported:

Mrs C: *I tried that…bring the problem forward…and it worked…*

Researcher: *what did you do?*

Mrs C: *You know, I have to teach mode and median and they don't understand it. I used the shoe shop. I gave them a little problem –the owner of a shoe store. I asked what information could they get from the shoe sizes he sells. They were able to say to me that from the numbers I gave them…that is the common shoe size…that is what is called a mode...then they got it!*

Teachers discussed how two different types of questions on the same concepts could either alienate students or be more inclusive. In their discussion of the differences between the different types of questions the following is pertinent.

Researcher: *what about differentiation?*

Mrs E: *these are all memory, they must remember what to divide and when.*

Researcher: *and if I don't remember?*

Mrs A: *then it's over.*

Mrs C: *but that* (problem-centred questions) *they can still figure out – its words.*

Mrs A: *they don't have to remember that you have to do that and then that…*

Mrs A: *I think their mathematical understands builds a lot more here* (problem-centred questions) *than in a traditional way, and they think further, more outside the box that they currently are.*

Mrs C*:* (pointing to one of the problems) *that's more realistic. They can identify with it. It's a problem they can understand.*

Researcher: *this problem could possibly exist*

Mrs E: *the problems are like what they could experience in their lives.*

The teachers also brought up the issue of what the students believe mathematics teaching is and how they do get resistance from students when they are presented with problems

Mrs E: *I taught multiplication with a story, and when I taught division, and I taught them the short division method…they were so happy to see me there teaching them, telling them what to do, not giving them something where they had to read and think.*

Mrs C: *but this* (problems) *is less de-motivating for the weaker pupils.*

Mrs A: *you're right.*

Mrs C: *this* (calculation problems) *is right or wrong...all they see is crosses...that* (problems) *allows you...you are marking what they think. Some just cannot do raw manipulation of numbers, but give them problems and they manage.*

At the end of the 2012 school year (about 2 months) after the last session with the teachers the researcher met Mrs D. She told me that she had enrolled for a Masters' program in educational change. She was very excited and added that this program had made her "*think*

differently" and that it was a pity that not all the mathematics teachers at the school were involved. (ID, 12-12)

5.2.9.3 Changes in teacher goals

Expression of teacher goals was guided by the words 'want to', 'have to' or 'would like to'. One of the changes that can be noted at the end of the first cycle is that the type of discussion teachers were taking part in during sessions had changed. At the beginning, when teachers had been given the first modelling problem, the discussion was centered on the problem and obstacles they felt were inherent in the modelling problem (too vague, students needed guidance etc). Much time was spent discussing how they felt the problem should be changed. The researcher had to continually guide the discussion around the didactisation principles inherent in using these problems in a mathematics class. By the end of the first cycle when teachers had to sort the given problems out into two groups (traditional and problem centered) the discussion changed to the strengths and weaknesses of each type of problem. Mrs B is used as an example. Her comment about the airplane problem and the data given:

Mrs B: *I don't think you can really base anything on this unless you have uniformity on what you are going to use.* (ST1, 08-12)

During the third teacher session she focused on what problem centered teaching could offer student thinking:

Mrs B: *I would start here and then they could probably come up with the next method and the next method….I need to get them active. They have to grow in their creative processes…I think…*

Teacher cognitive conflict and teacher reflection (see 3.4.1 and 3.2.1.5) were also noted during the sessions. During the third teacher session, when teachers were grouping and comparing traditional problems to problem-centered questions, they discussed how traditional teaching means that students have to memorise methods and if they do forget a method they cannot continue to solve the problem. They were comparing the following problems:

105 ÷ 15	*Around March each year the Grade 6s and 7s go on camp. Here are the numbers that are going this year.* *Gr 6a: 27 Gr 6b: 31* *Gr 7a: 23 Gr 7b: 24* *Pupils travel in small buses. Each bus can carry 15 pupils. Four teachers are also going on the camp. How many buses must their teacher Mrs Smith organise?*

Table 5.7 Session example

Researcher: *so the second problem is more accessible to them, they will work out that there are 105 children, instead of you telling them. It allows all levels of ability to tackle the problem.*

Mrs E: *you see, I am still telling them…I have to go to where they work it out for themselves…that is where I am struggling.*

Mrs B: *I have to be forced to do a thing…that's why I am liking these sessions and now I am going to take it to the next level…*   (ST3, 09-12)

5.3 TEACHING EXPERIMENT CYCLE 2

Cycle 2 started in February of the following year. A number of factors need to be considered as having impact on this study. All five teachers would be implementing the revised CAPS curriculum for the first time and all five teachers would be working from a compulsory (Nationally approved) textbook.

5.3.1 Cycle 2 classroom observation: Mrs A (Grade 6)

The visit to Mrs A took place during the first week of February. She started her lesson with mental work (division and multiplication tables). This is now a compulsory requirement of a mathematics lesson according to CAPS (DBE 2011: 13). Her exposition of the lesson (which was on equivalent fractions) was based on well considered questions to the class. It was evident that the didactisation principles of accessing and probing student thinking were being explored by Mrs A. Her line of questioning went as follows:

Mrs A: **what** *are equivalent fractions?*

Student: *its different types of fractions that are equal*

Mrs A: **how** *are the fractions equal?*

Student: *like* $\frac{2}{4}$ *it is equal to* $\frac{1}{2}$

Mrs A: *why is it equal?*

Student: *The two quarters are the same size as the half*

Her probing of mathematical ideas was not left to one or two questions. She was investigating real understanding of her students. She continued with this line of questioning when she wanted to find out what students understood by the words numerator and denominator of a fraction. She asked a number of students "*what does the number* (numerator or denominator) *tell me?*" She allowed students to answer freely and did not correct their definitions immediately. She allowed them to speak without interruption. She then used this information to guide her through the work she had prepared on the interactive smart board. She presented each student with a slab of chocolate. In dividing the slab into different fraction quantities she kept asking the student to explain "*why are they equal*" and "*how are they equal*". She spent time getting the class to define the whole before moving onto fractional quantities of the whole. Students had to move between visual, written and verbal representations. The class spent time together dividing the slab (consisting of 16 parts) into quarters and halves. Mrs A continually asked them how many parts in each group? What makes them equal? How many pieces do you have? If half of the chocolate was represented by 8 blocks – she insisted that they say "*8 out of 16*". Mrs A spent a great deal of time paying attention to language detail. She spent longer interacting with the students in this lesson than in the previous lesson. Mrs A then handed out a worksheet which consisted of splitting the chocolate into various fractions and finding equivalent forms. The students explored equivalence with a denominator of 12, 10 and then 8. The students worked in pairs for the remainder of the lesson, packing and sorting their chocolate pieces in exploring equivalent fractions. Towards the end of the worksheet some students were able to complete the equivalent forms without needing to pack out the pieces. This vertical mathematisation would support Mrs A's next lesson which would be to write equivalent forms without packing or counting pieces. Mrs A moved between the groups listening to their discussions, looking at their responses to the worksheets and guiding where necessary. If a group was uncertain she asked them to repack the chocolate into halves and quarters before moving on. All groups remained on task for the duration of the lesson.

Figure 5.12 Lesson observation cycle 2

Figure 5.13 Lesson observation cycle 2

Mrs A's own comment on this lesson was that "*learners embraced the lesson and participated with enthusiasm. Even learners who seldom do work or who are very introverted took part*" (LQ2, A, 02-2013). The researcher commented that very few students called for the teacher's assistance during this activity. Mrs A reflected on this comment as such:

"*Few raised their hands for help, I believe, because they were working with a partner and because of the practicality of the exercise*" (LQ2, A, 02-2013).

Mrs A was focusing on the understanding of the students in her class. According to her she developed this lesson idea so that the students could "*see the concept in practice, which allowed it to make sense*" (LQ2, A, 02-2013). She also felt that her questioning allowed students to "*tell in their own words, what they understood about the concept*" (LQ2, A, 02-2013).

Mrs A comments regarding implementing problem-centred lessons into CAPS:

> "*CAPS expects too much to be completed in a term. An exercise like this one needs two lessons to complete effectively. This infringes on work that needs to be completed*

by the end of the term. The children's workload is unrealistic in comparison to the results they expect to see" (LQ2, A, 02-2013).

In terms of the modelling perspectives outlined in 3.6.1, this lesson included students working with basic facts and skills, while Mrs A's questioning may have resulted assisted students in developing their own meta-cognitive scripts. Mrs A's intention during this lesson was for students to sort out existing conceptual understanding about equivalence.

5.3.2 Cycle 2 classroom observation: Mrs B (Grade 5)

Mrs B had decided to consolidate work on inverse operations on the day of the lesson observation since the students found the idea difficult. She wanted to spend one more period working on this before she continued with the prescribed work as set out by CAPS. She had moved classroom from the previous year and the desks were set out in groups. There were mathematics posters and mathematics information on the walls.

Mrs B started the lesson by drawing a chicken on the board. She asked the class what parts the chicken was made up of. Several students gave answers such as feathers, beak, legs etc. She then said that if she had a machine that put chickens together she would need to feed all the parts into the machine and the machine would produce a chicken. She asked the students what would happen with the reverse process of this machine and they understood that feeding a whole chicken into the machine would results in all its constituent pieces on the other side of the machine. She then related this to inverse operations using input/output flow diagrams. She said "*let's try it with numbers*". (LON2, B, 02-13)

She gave students combinations of numbers that added up to 10. She discussed how if $1 + 9$ equal 10, then $10 - 9 = 1$.

She then wrote an input/output diagram $\boxed{x \ 2 - 1}$ on the board where some of the outputs were written down while the input numbers were left blank. She stressed the OUT in output and the IN in input. She made sure that the students would relate this to the words in and out. She asked the pupils how they would go about working the opposite way (from output to input). Some pupils knew that this required the inverse operation. The pupils used the word *opposite* while Mrs B reminded them that they were working with "*inverse operations*". Having assured herself that she had covered this idea well enough the students set to work on a worksheet on input/output diagrams. She walked around the class checking on how pupils

were doing. When she found an example or idea she wanted to share – she called the class to attention and discussed this example with them. The period ended and students were assigned the rest of the worksheet for homework.

Mrs B commented after the lesson that the students were battling with the idea of inverse operations. She had not intended to re-do the lesson as the curriculum did not allow much time to re-teach concepts but she felt it was important that the students could tackle input/output diagrams. She tried to encourage understanding during the lesson by asking 'why' questions on numerous occasions.

Mrs B: *but why do I minus it? Why don't I divide it?*

S: *to get the answer*

Mrs B: *yes, but why minus?*

S: *because it's the opposite*   (LON2, B, 02-13)

The lesson was very much teacher orientated but Mrs B expressed concern and frustration that students were finding difficulty with inverse operations. In her reflection on this lesson, Mrs B said that through a question and answer technique she '*found some knew how but did not know why'* (LR2, B, 02-2013).

In terms of vertical mathematisation, Mrs B was trying to get pupils to work flexibly with whole number multiplication and division. She wanted them to be able reflect on the nature of the operations themselves and not only on the result of the operations. In terms of Dubinsky's encapsulation during reflective abstraction (see 2.4.4.2) this conversion of a dynamic process into a static object (Dubinsky 1991b: 100) is most important for mathematics but also particularly difficult for students. If students are unable to encapsulate these processes then expecting the reversal of a process will prove very difficult especially if, like Mrs B, the teacher wants the students to do this spontaneously. Teachers may therefore resort to traditional drill methods.

Mrs B's questioning and probing of student ideas was more evident during this lesson than the previous one. She seemed to want to be sure that the students were with her at regular intervals. She also expressed her goal of wanting her students to be able to work with input/output diagrams and this goal set the direction and subsequent actions for the lesson. Her goal was therefore supported by her orientations and resources and this manifested in a teacher-centered lesson. The lesson incorporated basic skills and facts, habits of mind as well

as building conceptual systems. Students may have needed to sort out their existing multiplication and division constructs. This could have been facilitated by bridging with mathematical representations.

5.3.3 Cycle 2 classroom observation: Mrs C (Grade 6)

Mrs C had reached a section of work "fractions of whole numbers" (DBE 2011: 226). She wrote the following on the board

$$\frac{2}{3} \ of \ 30$$

and asked students if they could think of any real life situation where you would need to calculate this. Some of the students answered in terms of money:

Student1: *when you have to divide money between two people*

Student 2: *if I get R100 pocket money, my dad suggests that I save one quarter of it*

Mrs C then said she was going to hand over solving these problems to students. She divided the students into groups and gave each group a number of small wooden cubes. She also gave each group a big sheet of paper.

Mrs C: *you are going to use these things in your own way... at the end one person from each group will come up and explain how to do it.*

Mrs C then asked each group how many blocks they had and she asked them to calculate a fraction of the number of blocks (e.g. one group had 18 blocks so she asked them to calculate $\frac{5}{6} \ of \ 18$. Each group had a non-unitary fraction – no numerators were 1. Mrs C then walked around from group to group listening and questioning them. She specifically asked them "*what are you doing*?" and "*how did you get that*?" in this phase of the lesson. Groups worked for about five minutes before she called the class to order and each group presented their problem and their solution. She asked them to pack out the blocks on the overhead projector and to explain what the problem looked like with the blocks. Some groups wanted to show a written way but she explained that they had to explain using the blocks. These groups had packed out the blocks and verified their working by calculating mathematically.

Before the groups presented their working she said they should imagine that the blocks were sweets and that she was giving each group the fraction of sweets in the problem. They had to tell her how many sweets to give them. The following photographs from this lesson:
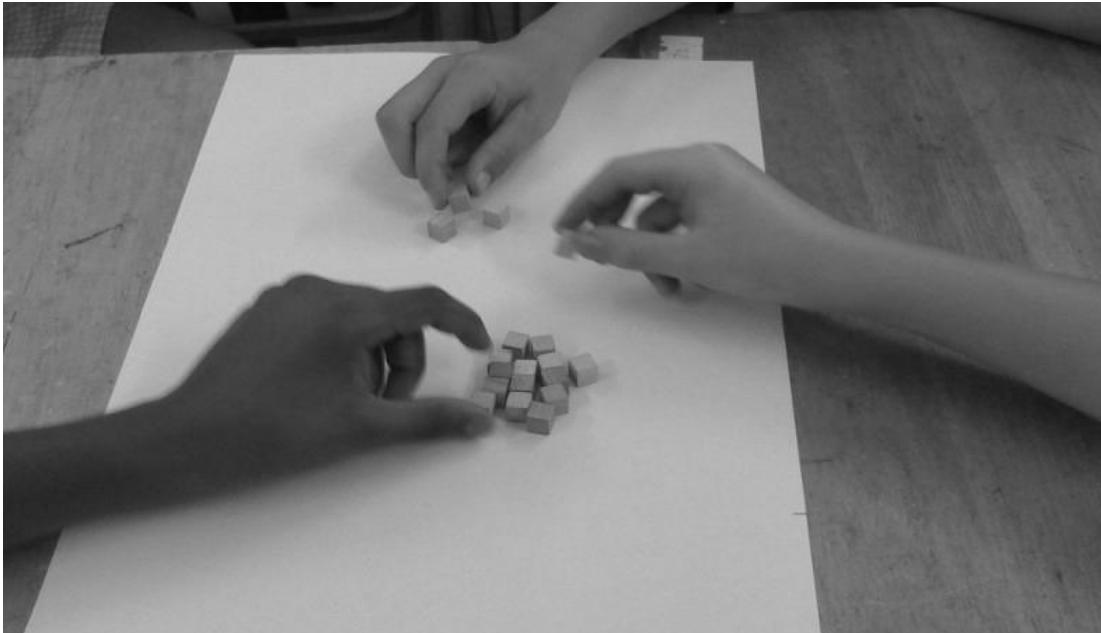
199

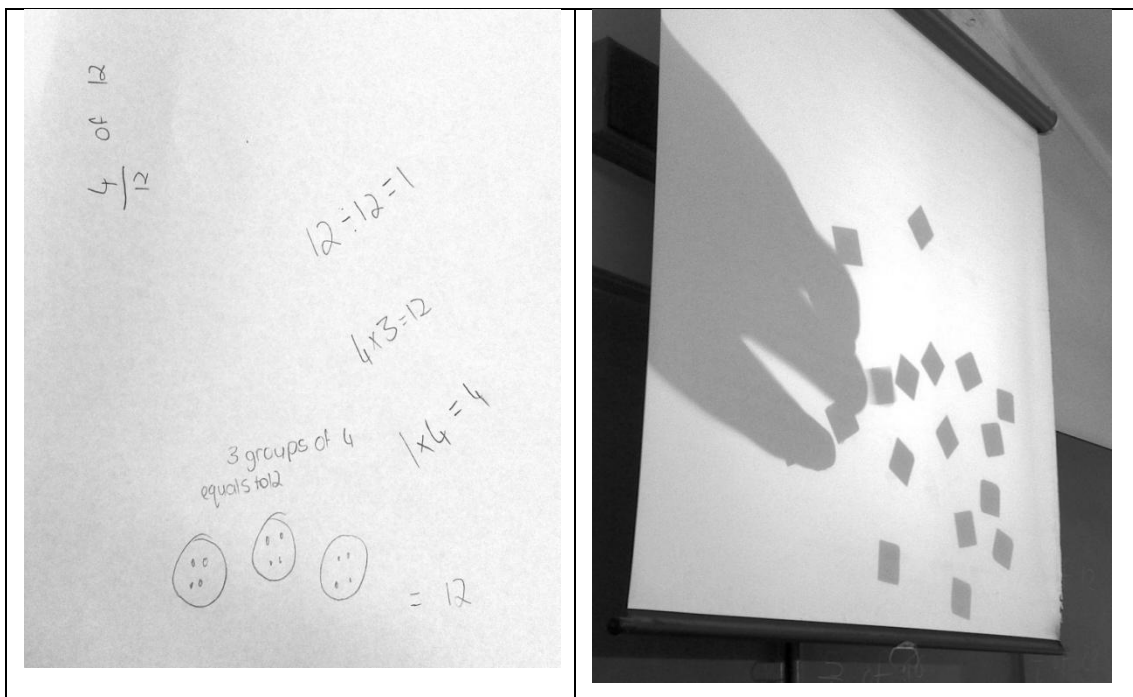Figure 5.14 Students using blocks to calculate

Figure 5.15 Some groups verifying their solution

Figure 5.16 Group presentations to the class

Once each group had presented their solution Mrs C then asked pupils to calculate the number of blocks (sweets) that were remaining and to write that as a fraction of the original number of blocks (sweets).

In terms of mathematisation promoted by this lesson, once students had worked with the physical manipulates some groups verified their solutions mathematically (see Fig. 5.15). They did this without being asked to by the teacher. One group packed out a solution and then produced a different solution when they worked on paper. This allowed the group to try to find connection between the physical manipulative and the formal calculations to find why the solutions were different. The concepts of numerator and denominator could be reflected upon and the student understanding of this projected onto a higher plane. It is important to point out to teachers that students may begin the formalizing and shortening process spontaneously if their mathematising at a lower level is good. This may assist in showing teachers that the extra time taken to allow students to learn with understanding will pay off in the long run. The lesson provided students with an opportunity to sort out existing constructs as well as building new ones as proposed by a modelling perspective.

5.3.4 Cycle 2 classroom observation: Mrs D (Grade 6)

Mrs D was also following the CAPS curriculum supported by a compulsory textbook. The lesson started by pupils looking at the diagram below that was in their textbooks while the teacher questioned them on how many parts the whole had been divided into and how many parts were shaded. Mrs D got the students to read the questions from the textbook as a class. Since many in her class are second language learners this was a good strategy on the part of Mrs D to assist them with listening to the instruction as well as possibly improving their reading. In her first lesson Mrs D did all the reading needed. In the final session of cycle 1 the researcher and the teachers spent some time discussing ideas to assist second language learners in the classroom. Mrs D asked the students which of these numbers was the numerator and which was the denominator of a fraction. She stressed that the denominator consisted of ten *equal* parts.

Figure 5.17 Lesson example cycle 2

The next diagram showed:

Figure 5.18 Lesson example cycle 2

Mrs D took them through identifying how many parts the whole had been divided into and how many parts were shaded. She stressed the phrase "*two **equal** parts out of 10*". She then asked them to simplify this fraction. A number of pupils suggested $\frac{1}{5}$.

Mrs D: *why is it one fifth?*

Student 1: *because half of 2 is one and half of 10 is five.*

She then drew the following on the board:

1	2	3	4	5	6	7	8	9	10

Figure 5.19 Lesson example cycle 2

Mrs D: *how many groups of two are there?*

Student 2: *five*

Mrs D: *and how many of these groups did I shade in?*

Student 3: *one over five…*

A number of pupils react with an *'ohh…'*

Student 4: *so m'am, you can write two tenths equals one fifth?*

Mrs D then moved her class to working with a fraction wall that was printed in the textbook. She took them through finding equivalent fractions using the fraction wall. She then gives them a number of fractions and asks them to find equivalent fractions using the fraction wall. The lesson moved quickly and the pupils are all actively enjoying finding as many equivalent forms as possible. Mrs D asked all pupils to answer questions not only those who had their hands up. Many were correcting each other as they gave answers. Mrs D followed a number of responses with a 'why?'

The lesson moved to looking at fraction blocks out of 100. At one point a pupil gives her a response and indicates that he has his own way of simplifying fractions. The question was to reduce $\frac{20}{100}$. Mrs D was taking the class through division by 2. The pupil offered the answer ($\frac{1}{5}$) immediately.

Mrs D: *how did you get $\frac{1}{5}$?*

Student 5: *Me, I say how many times does 20 go into 100. It's five, that's how **I** do it* (emphatically-pointing to his chest).

A number of times during the rest of the lesson this pupil interrupted the lesson to show his way of calculating. Later he said:

Student 5: *let me show you **my** way.*

Mrs D kept showing the class the various methods that were being suggested by the boy. She presented different and more varied examples to contrast and compare the methods suggested. She did try to get him to see the value in other calculation options.

Initially Mrs D was not very comfortable with this interaction. When asked specifically about this incident Mrs D wrote as follows:

> *Regarding my interaction, I welcomed the learners' efforts as it was encouraging to know that I can lift the cap on how much and how far the classroom extension should go* (ID (email), D, 02-2013).

In understanding the initial hesitation to allow the discussion and then Mrs D's full interaction with the student, one must consider which played a strong role at the time – her beliefs or her goal for the lesson? The interaction did cost her some time in getting through the work allocated for the day but may therefore have fulfilled the above orientation. Mrs D's first lesson was more strongly guided by her goal for the lesson.

Mrs D, in her reflection on her development since the first lesson observed, wrote that *"I have matured. My probing was slower. The textbook also assisted me in channeling the questions"* (LQ2, D, 02-2013).

Although pointed out earlier that the impact of CAPS on this study was uncertain, here Mrs D has reflected a positive result of working from the prescribed textbooks. During a later session, Mrs D did say that she felt her teaching was focused on where the students really were conceptually, rather than where they had to be (as she did in the past). She said that the weaker pupils have *"got me thinking harder and wiser and stronger"* (ST4, 02-2013).

Mrs D was trying to assist students in the vertical progression of fraction equivalence. The visual representations may have assisted. The teacher may have gained more insight into student thinking by asking them to present models of the equivalence.

5.3.5 Cycle 2 classroom observation: Mrs E (Grade 5)

Mrs E had started a section of work on number patterns as specified by the CAPS document (DBE 2011: 136). She started the lesson by writing the following number sequence on the board:

1350; 1340; 1330; _____

Many of the students raised their hands and many excited tones were audible.

Mrs E: what do you notice about this pattern?

Student1: *it's subtracting*

Mrs E: *what's subtracting?*

Student 2*: ten!*

Mrs E: *where are you subtracting 10?*

Student 3: *the next one is 1320*

Mrs E: *what is this pattern doing?*

Student 4: *it is minus-ing ten*

Mrs E: *what is a better word for 'minus-ing ten'?*

Students (chorus): *subtracting ten.*

Mrs E then wrote the following patterns on the board and dealt with this in the same way – asking students to explain the pattern and to predict the next few numbers.

132; 133; 136; 141

Mrs E: *what number could be added next?*

Student: *seven*

Mrs E: *why do you say seven?*

Student: *because it's odd numbers*

Mrs E then explains that every pattern is governed by a 'rule'. She said this was the way you could explain the pattern to a friend. Another student explained the above pattern as '*you add different odd number in the correct order every time'*

Mrs E: *what is the word for that? Do you know what that word is? We call that consecutive. It's a lovely word isn't it? Otherwise you could say you are adding the next odd number.*

Some students copy her in saying 'consecutive'.

She then wrote the following down:

2; 4; 8; 16, ___

Student 7: *you times by 2*

Mrs E: *how do you know that? You could also plus 2.*

Student 7: *um, ya*

Mrs E: *why do you say times 2 and not plus 2?*

Silence. Some discussion amongst students.

Student 7: *because you have to times the 4 by 2 to get 8.*

Mrs E: *you weren't fooled were you? Well done.*

Student smiles.

Mrs E: *so if you say 2 times 2 you get 4, if you say 4 times 2 you get 8*

Student 8: *2+2; 4+4, 8+8...*

Mrs E: *so you can either double the number or you can multiply it by 2.*

Student 8: *I found another pattern there, plus 2, plus 4, plus 8...*

The above discussion highlights how Mrs E accessed and probed student thinking in her classroom. Also noteworthy is that the number patterns she uses for classroom discussion are of a higher level than those set in the textbook.

Mrs E graded the type of examples she presented to the class. She arranged them from easier to more difficult. She specifically chose the last pattern to discuss the difference between adding two consecutively and multiplying by two. Students were able to project their understanding of the process of multiplying by two and compare this with the process of adding two. The vertical step in comparing the product of these processes was noted in this lesson. Not all students were able to encapsulate the process into a product but many were. In Mrs E's reflection on the lesson, she made the following comments about what she thought was different about herself in this lesson compared to the first lesson observed:

> *I continue to think about what I'm doing in the class as concepts/material must be covered, however it is easier to 'do' the work and 'tell' them how to – it's a challenge not to keep falling into this method.* (LQ2, E, 02-2013)

This observation allows one to recognise that Mrs E is more aware of another way of teaching or of different approaches to teaching mathematics. She however makes mention of the volume of content that needs to be covered. This volume appears to her to be in contrast with fully implementing the didactisation practices set out in this study.

5.3.6 Changes in didactisation practices

Teacher didactisation practices at the beginning of cycle 2 are now compared to those observed at the beginning of cycle 1. This will assist in determining the extent to which the aims in 1.3.1.2, 1.3.2.4 and 1.3.2.6 are being achieved. The graphical representation reflects the trend over all five lessons. This study is not so much about the 'score' of individual teachers as it is about their development (see 3.3). Presenting the data as a trend over five lessons enables one to evaluate the effectiveness of the professional development program and not to be judgmental of individual teachers. These comparisons during the teaching experiment are necessary since according to Lerman and Zehetmeir (in Brown & Coles 2010: 378) research that is concerned with effectiveness always has the question of "robustness" nearby. There is a professional sensitivity necessary when dealing with an opinion or judgment of people. According to Fullan (in Brown & Coles 2010: 380), capacity building comes through being non-judgmental. Furthermore they describe the notion of feedback. Feedback can be direct or indirect but needs to remain non-judgmental. Feedback should therefore be "descriptive". This study has attempted to provide a descriptive, non-judgmental feedback system in the presentation and analysis of the data.

Lessons themselves were not discussed with teachers afterwards for a number of reasons. Firstly, indirect feedback would be given during the sessions in terms of teacher practices and actions that do promote didactisation practices. In keeping with a filter metaphor (see Fig. 2.3), it was hoped that teachers would filter what they could include into their own teaching from the program instead of being given a list of do's and don'ts to follow. Since teachers are volunteers in this program, it was assumed that their own goals and orientations were open to change.

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5.3.6.1. Student Activity: cycle 2
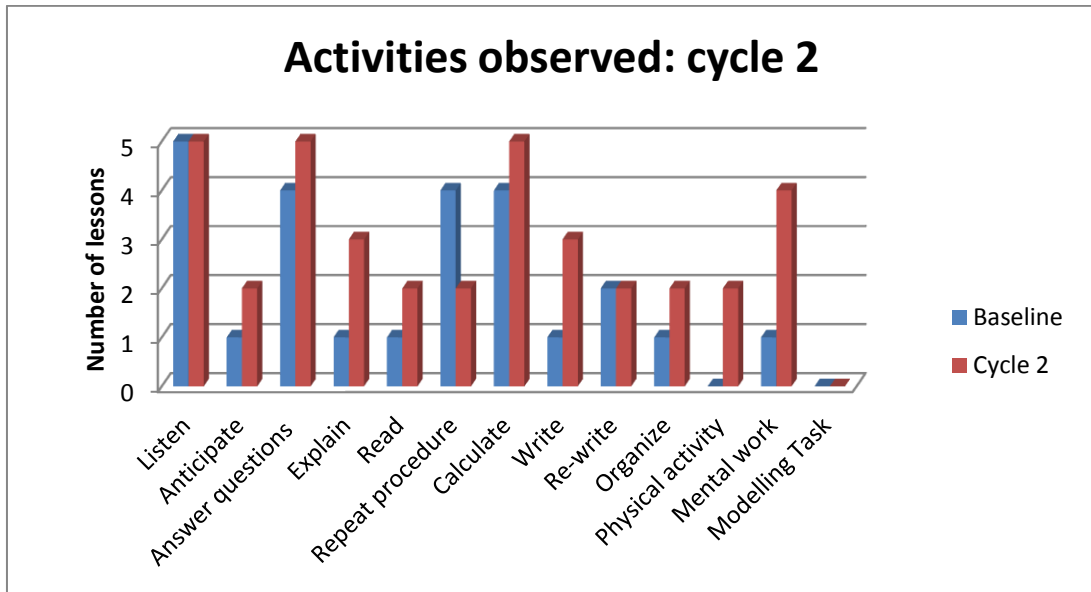
Figure 5.20 Activities observed: cycle 2

The inclusion of daily mental work as a stipulation of CAPS was observed during Cycle 2 classroom visits. Two lessons included students physically manipulating objects which was not observed during the baseline visits. Repeating of procedures had decreased slightly. Improved teacher questioning had an impact on students having to explain why they made statements. The vertical alignment of lessons has a direct link to the impact of CAPS where content to be covered is set out fairly rigidly on a week by week basis. Four of the teachers did add in their own lesson in addition to the textbook activities to explore or develop concepts they thought were important.
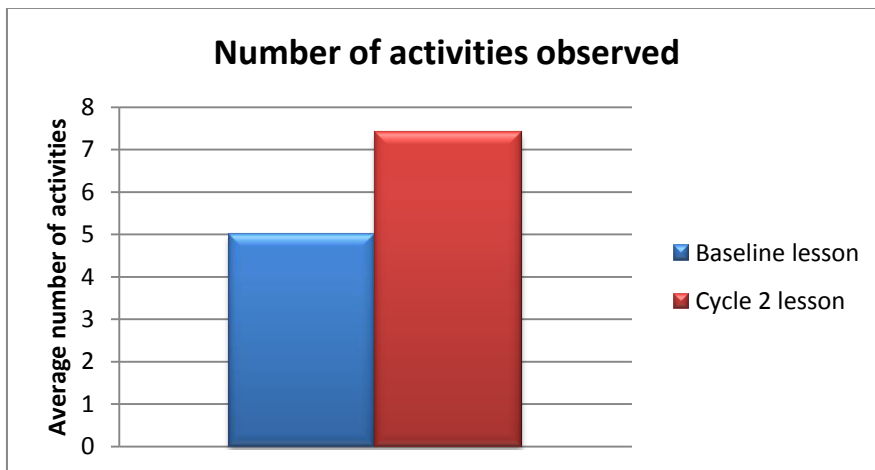
Figure 5.21 Average number of activities observed: cycle 2

The number of activities for each of the five lessons was gauged by their appearance in the lesson and not necessarily on the length of time they took during the lesson nor on their mathematical significance. An average over the five lessons is presented for the purpose of comparison. A comparison of the number of activities over the five lessons also shows that the number of activities that students were involved in has increased. Possible reasons for this are the CAPS policy which specifies ten minutes of mental mathematics activities each day, and the expected work to be covered daily per lesson may have resulted in increased activities in the classroom. Not forgetting that one of the principles of this study is student activity, an increase in student activity is not surprising. The value of each activity is not judged here. The graph (Fig 5.21) shows the average for the five lessons.

5.3.6.2 Differentiation: cycle 2
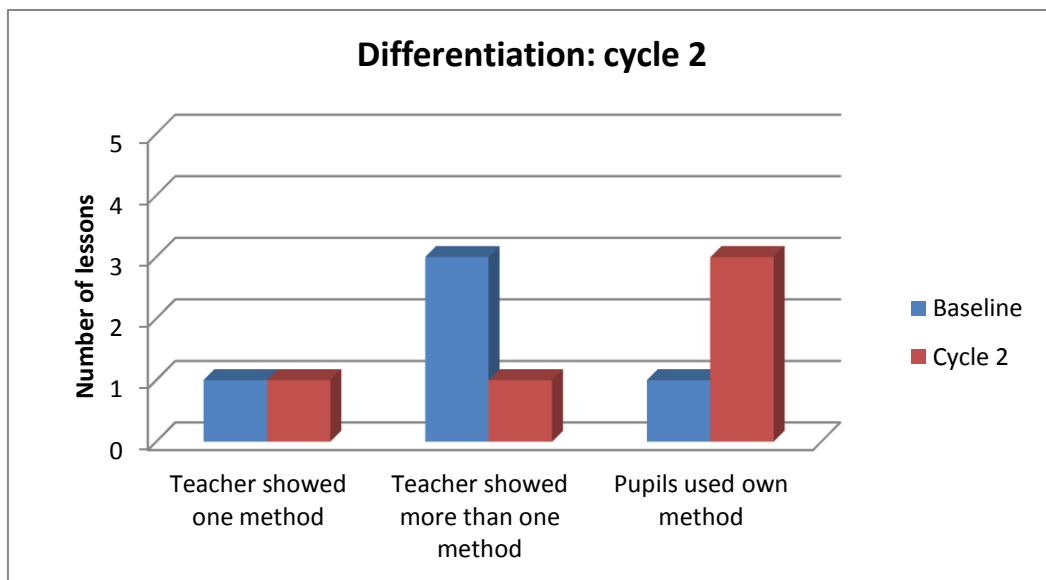
Figure 5.22 Differentiation: cycle 2

The above graph (Fig. 5.22) shows that students were using more of their own solution ideas/methods during cycle 2 lessons, although one teacher (Mrs B) had chosen to show a preferred solution method. The differentiation practices are a direct result of increased student activity in the lessons. It is important for the follow on of mathematisation that differentiation takes place. Students need to analyse and reflect on their own mathematical activity (see 2.4.4.2) so that they can structure or shorten their activity.

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Lesson	Activity that allowed for differentiation
Mrs A Gr. 6	Packing of physical manipulatives, collaborative sharing of ideas, students could move to next question in their own time.
Mrs B Gr. 5	Example of chicken and its parts, teacher questioning.
Mrs C Gr. 6	Packing of physical manipulatives, group collaboration, presenting solution, teacher questioning during presentations.
Mrs D Gr. 6	Teacher questioning on textbook representation of fraction wall.
Mrs E Gr. 5	Teacher questioning, whole class discussion.

Table 5.8 Activity that allowed for differentiation: cycle 2

5.3.6.3 Mathematisation: cycle 2
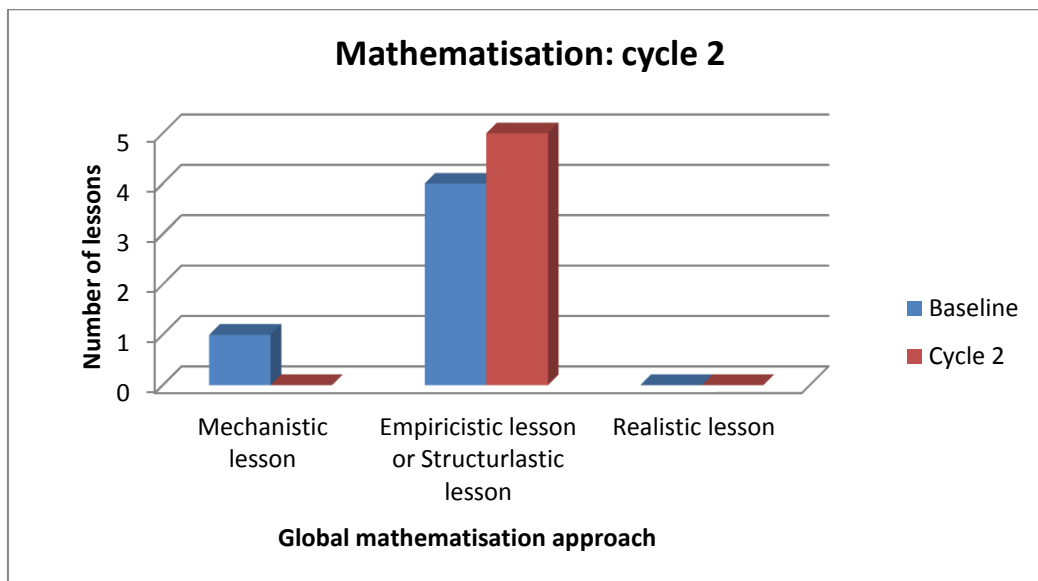
Figure 5.23 Mathematisation: cycle 2

The above graph (Fig 5.23) shows that the overarching mathematisation focus is still on structuralistic lessons, although an important shift is noted and discussed below (see Table. 5.10). None of the teachers presented realistic lessons during the observation visit. It is more probable that teachers will make incremental changes before they display a change from traditional lessons to realistic lessons. The changes to their lesson models are discussed below.

Didactisation principles, such as activity, access, probing and differentiation are the gateway principles to improving other principles such as mathematisation. Mathematisation experiences of students in the classrooms begin when they engage in the lesson activities and are able to structure, schematise and reflect on their actions (see 2.4.4). The mathematical experiences of the learners in the five classrooms are different to the first lessons observed since their involvement in constructing the ideas has increased, which means that the potential for mathematical understanding has increased.

The representations that were part of these lessons are as follows, with an added note that many of the representations that assisted in construction of student ideas were *socially mediated* in these lessons. The students were involved in using and presenting their understanding, especially in Mrs A and Mrs C's lessons where they had to consider how the physical models could convey meaning. Students had to consider and reflect on the connections between the verbal representation, the written and physical models. Increasing the number and types of representations will also allow more students access to mathematics.

Lesson	Representations	Teacher or student use	Is the representation student constructed or applied?
Mrs A Gr. 6	Verbal Written Manipulative models	Teacher and students Students Students	Constructed Constructed Constructed
Mrs B Gr. 5	Verbal Written Pictures	Teacher and students Teacher and students Teacher	Constructed Applied -
Mrs C Gr. 6	Verbal Written Manipulative models	Teacher and students Students Students	Constructed Constructed Constructed
Mrs D Gr. 6	Verbal Written Pictures	Teacher and students Teacher and students Teacher and students	Constructed Constructed Applied
Mrs E Gr. 5	Verbal Written	Teacher and students Teacher and students	Constructed Constructed

Table 5.9 Lesson representations: cycle 2

The analysis of the lessons in terms of horizontal and vertical mathematisation is as follows:

Lesson	Horizontal or vertical mathematisation	Gravemeijer's strategies that promote mathematisation

Mrs A	Horizontal mathematisation: projecting connections from physical manipulatives to symbolic numbers. Vertical mathematisation: task involved moving from sharing and diving chocolate pieces to writing equivalent fractions using the multiplication of numerator and denominator by same number (intuitively using one as the identity element of multiplication which can be seen as re-invention). Using algebraic laws in generalisations.	Generality and brevity
Mrs B	Horizontal mathematisation: chicken metaphor as a horizontal activity. Vertical mathematisation. Task and class discussion involved a deeper realisation of multiplication and division as inverse operations and addition and subtraction as inverse operations. The final level was for the input to be calculated from the output which involves using algebraic laws. At this age, it may be considered a reversal (Dubinsky 1991b: 102).	Generality and exactness
Mrs C	Horizontal mathematisation: projecting from manipulatives to symbolic form. Vertical mathematisation: the task involved the connection between the symbolic numbers and the physical manipulatives, moving towards symbolism. Moving to using mathematical operations to reflect the packing and counting of the blocks. Re-invention of packing groups that leads to short symbolic form.	Generality and certainty
Mrs D	Horizontal mathematisation: projecting from a picture of fraction wall to symbolic form. Vertical mathematisation: Projecting the connections of fraction wall representation to its symbolic form. Thereafter finding equivalent fractions using mathematical operations of multiplication and division (by one).	Brevity
Mrs E	Horizontal mathematisation: using the number pattern to predict the next number. This may be more a horizontal activity or starting situation (Dorfler 1991: 71). Vertical mathematisation: the task and class discussion involved identification of the underlying structure in number patterns and then being able to continue the pattern. Moving from describing the pattern to defining that mathematical operation.	Generality and certainty

Table 5.10 Mathematisation analysis: cycle 2

Table 5.10 shows that mathematisation was a stronger feature of these lessons. There was

more emphasis on students doing mathematics e.g. structuring, symbolizing, describing and

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defining relationships and the language use that is involved. It is surmised that teachers' goals (having to complete a section of work), their orientations (mathematics cannot be developed through real situations) and their resources (little knowledge of reform teaching approaches) may contribute to the focus on vertical mathematisation. Lessons have increased in the number of strategies that focus on mathematisation.

A review of Gravemeijer's conceptions of lesson models is now made. However, in the light of the increase to student activity and student construction of ideas in these lessons, an adaptation to Gravemeijer's lesson models (see 5.2.7) is proposed. A pre-intermediate step is needed to show the shifting responsibility to students, although lessons are not at an intermediate level yet.

Formal knowledge Model	Formal Knowledge Student structured model	Formal knowledge Model Situations	Formal knowledge Model-for Model-of Situations
Structuralist	**Pre-intermediate model**	**Intermediate model**	**Realistic**

Table 5.11 Revised lesson models

Mrs B's lesson is still structuralistic while the other four lessons can be said to reflect a pre-intermediate model since there was a focus on student constructions to develop the formal mathematics.

5.3.6.4 Accessing student thinking: cycle 2
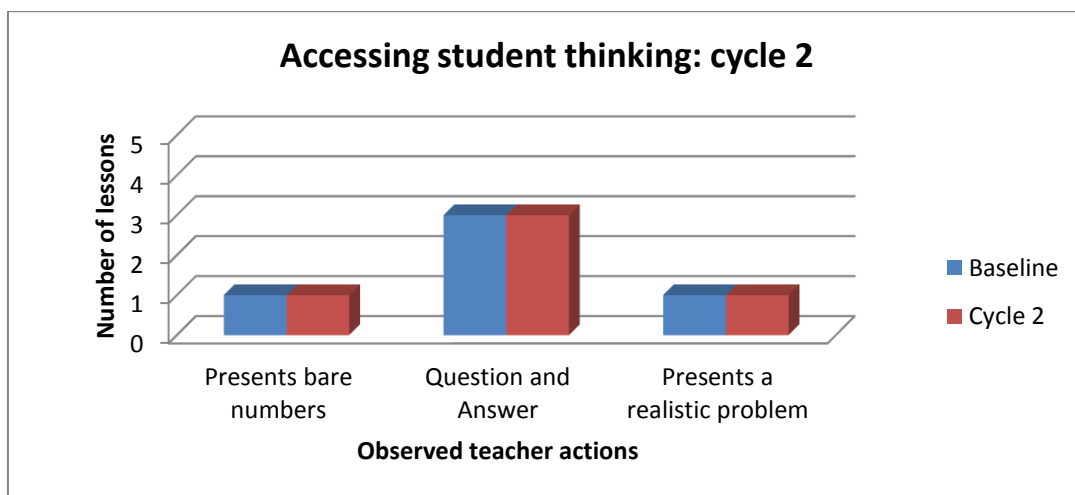
Figure 5.24 Accessing student thinking: cycle 2

Accessing student thinking (Fig. 5.24) at the beginning of a lesson presented the same spread as the baseline observation. However, the teacher who presented a realistic problem during cycle 2 was not the same teacher as in the baseline lessons. So, at this point two teachers had tried a problem-initiated lesson. Teacher questioning and probing of ideas has shown signs of development which is reflected as a separate didactisation practice. Mrs E indicated she "*started with an easy sequence that they could all access*" (LQ2, E, 02-13).

5.3.6.5 Teacher probing: cycle 2
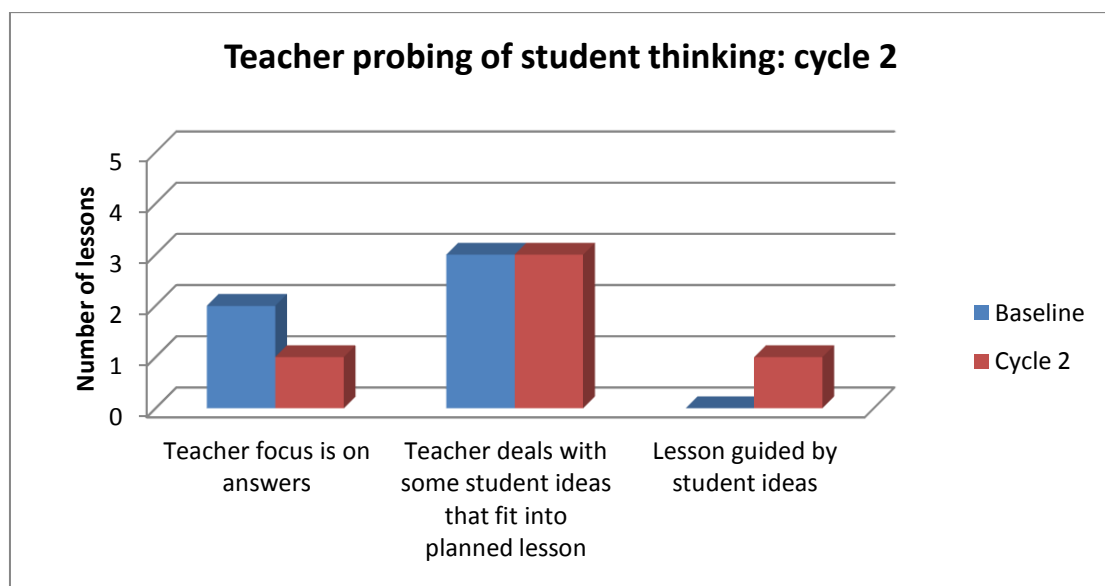
Figure 5.25 Teacher probing of student ideas: cycle 2

What is evident at this point of the study is the improved teacher questioning that was observed during lessons. This had led to a greater focus on student thinking and understanding. The didactisation principle of teacher probing is developing. The time allocated for student interaction (Mrs A and Mrs C) in the lessons, allowed for students to think reflectively about their working with the manipulatives which could assist in projecting these reflections onto a higher plane as suggested by Piaget (see 2.4.4.2).

Although teacher questioning improved during the earlier phases of the lesson, the latter phases of the lessons were also considered in continuing probing student ideas. When an activity was allocated for students to complete, the following changes occurred from the baseline lesson:

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Teacher Probing role	Teacher as supervisor of the activity. Answers questions or clarifies if students ask.	Teacher as director or manager. Initiates discussion, controls the topic. Allows or invites input.	Teacher as facilitator. Sets up structure, interacts with students. Students interact with each other and materials.
A Baseline	X		
B Baseline	X		
C Baseline	X		
D Baseline	X		
E Baseline	X		
A Cycle 2			X
B Cycle 2	X		
C Cycle 2			X
D Cycle 2		X	
E Cycle 2		X	

Table 5.12 Teacher probing: cycle 2

5.3.6.6 Connecting student ideas: cycle 2
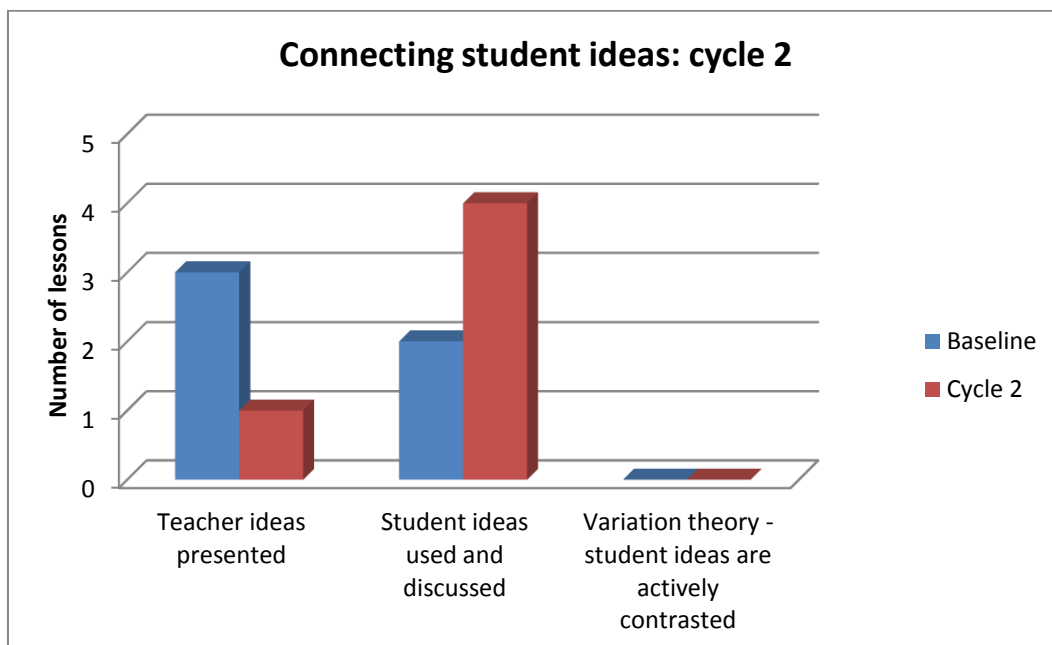
Figure 5.26 Connecting student ideas: cycle 2

In considering how teachers elicited and connected student ideas about concepts in the classroom, the above graph (Fig. 5.26) summarises the development of this principle.

Students were more verbal in these lessons because of improved teacher questioning and their ideas were presented. In Mrs B's lesson the connection between multiplication and division was being forged although it was teacher led. Mrs A allowed the structuring of her worksheet to forge the connections between the manipulatives and the concepts. At no point during this lesson did students *have to* use the chocolate blocks. A few pair groups only used the chocolate blocks for the first questions and proceeded without the packing and counting for the remainder of the worksheet. Mrs C however insisted that students use their blocks in presenting the solution to the class but she did not tell them *how* to use the blocks. One of the groups came to a different solution when packing blocks as to when they tried to calculate the answer so they had to resolve this disparity by looking at which of their actions when packing the blocks matched the mathematical calculation. This was dealt with by the group while Mrs C was attending to another group so it did not become part of the discussion. When asked in the reflection on the lesson how she connected the different understandings in the classroom she responded "*I tried to guide them to the correct answer if they got it wrong*" (LQ2, C, 02-13). Mrs D's lesson focused on the connections between the visual fraction wall and the calculation that would result in finding the equivalent fraction without the wall. The lessons that included social interaction may have produced more understanding though connections as Hiebert and Carpenter (1992: 72) remind us that shared experiences can be very powerful in building connections and understanding. Utilising all aspects of the problem also allows for a fuller understanding of the problem. It is this skilful integration of connections that is a challenging part of teaching.

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5.3.6.7 Assessing classroom solutions: cycle 2
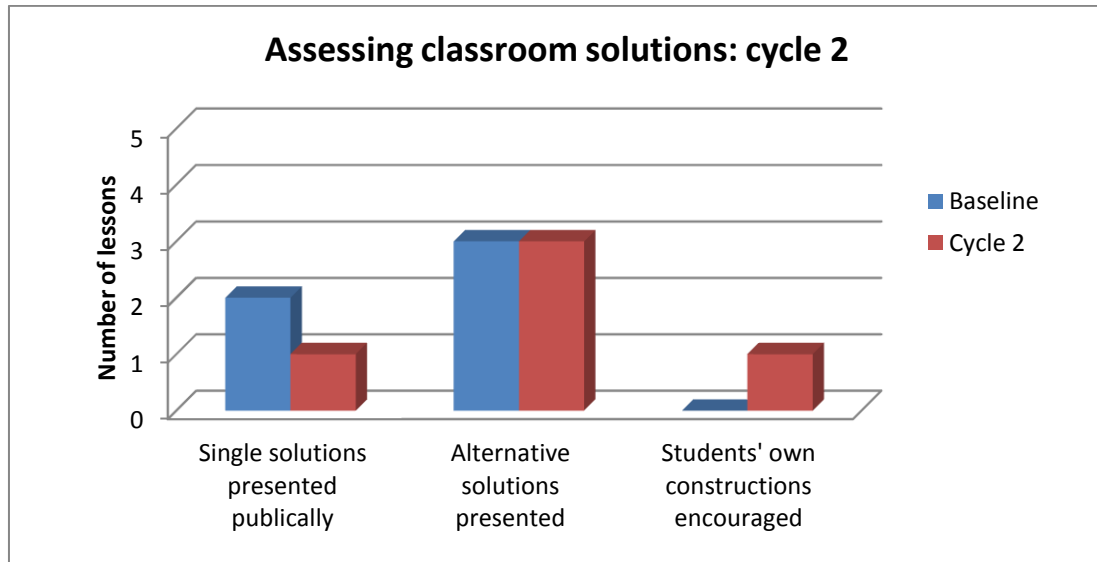
Figure 5.27 Assessing classroom solutions: cycle 2

In terms of assessing classroom solutions, some shifts are evident. In two of the lessons in this cycle pair/peer assessment was used. Since teacher questioning has improved in both scope and depth, it would mean that classroom solutions and the presentation of student ideas are encouraged. Often classroom solutions are used by a teacher to verify that students are following the exposition of the concept. The solution itself is sometimes seen by teachers to carry full understanding of the concept. This is not always the case, so teachers should probe a little to ensure that the student does understand his/her own solution. Teachers do not always stop and check if the solution and its meaning for the student are the same. This will result in fuller discussion that can assist teachers in enabling students to make connections between different methods, representations or responses.

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5.3.6.8 Teacher reflection: cycle 2
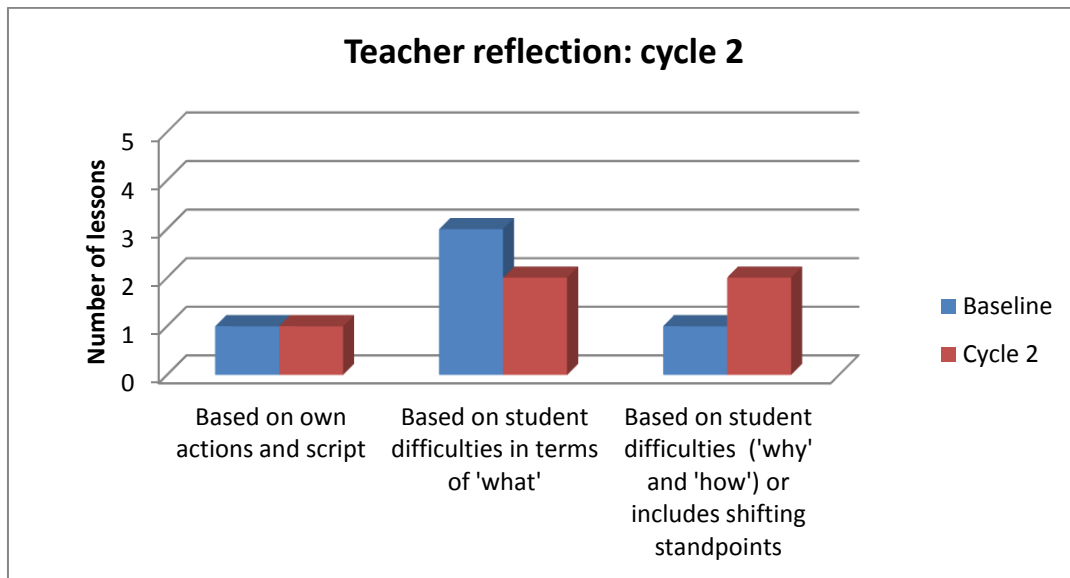
Figure 5.28 Teacher reflection: cycle 2

Teacher reflection is still based on *what* students found problematic and not always on *why* this is so. To answer the *why* question necessitates that teachers' PCK is fairly deep. Teachers are starting to see that there is a difference between their perception of a lesson and the students' perspective. Some of the teachers' written statements regarding their reflection on the lesson include:

Teacher	How do you feel about the lesson; what went well, or not so well; how do you think students feel? (LQ2, 02-13)
Mrs A Gr.6	*Learners loved the hands-on approach. Stronger learners tended to finish quickly, while weaker learners (some for the first time) began to understand what fractions actually mean.*
Mrs B Gr. 5	*I will continue with this concept until they are confident. Lots of practice in other words. Some don't fully understand while others do.*
Mrs C Gr. 6	*I think they did most of the mathematics. I think it went well, it was interesting to see how different groups interpreted the questions and the different ways they derived the answer.*
Mrs D Gr. 6	*The learners had an opportunity to find the answers first before answering.*
Mrs E Gr. 5	*I still talk too much. They did talk a bit – don't give them enough time to process? Students can sometimes find the next number, but were still struggling to explain how they did this as a rule in words*

Table 5.13 Teacher reflection: cycle 2 lesson

5.3.6.9 Vertically aligned lessons: cycle 2
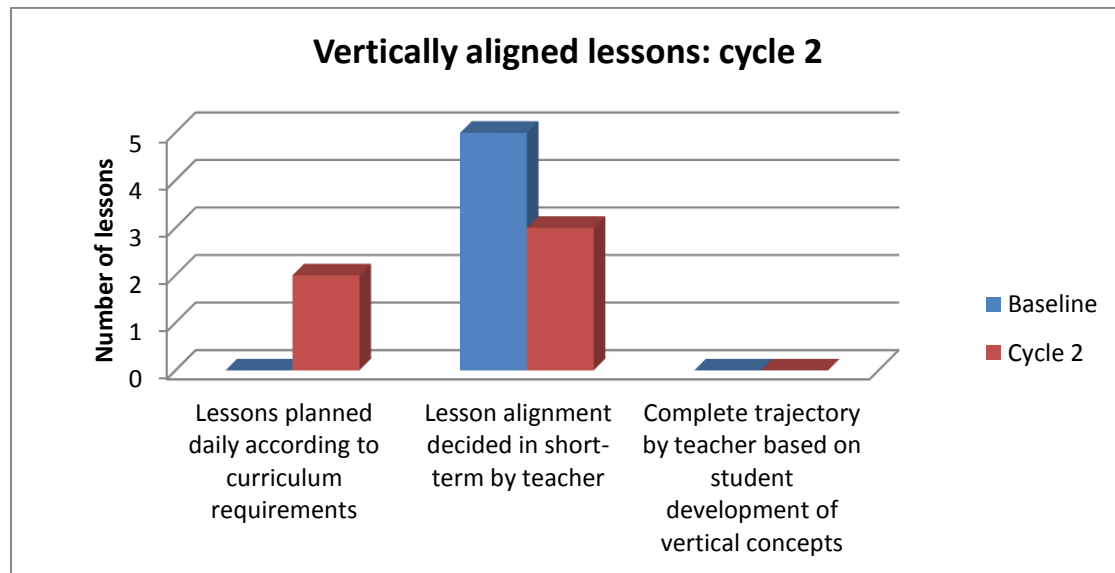
Figure 5.29 Vertically aligned lessons: cycle 2

Vertical alignment of lessons was gauged over the time period that teachers were shaping the prescribed activities. The term *short-term* refers to a section of work with its specified time allocation (e.g. 5 to 9 hours). It is within this timeframe that most of the teachers had made changes or suggested other activities to the learning trajectory. The vertical alignment of lessons is at the pinnacle of didactisation practices. It requires a teacher to be fully in control of the trajectory of a curriculum or even a section of work. The CAPS curriculum already has a vertical alignment built into it and is structured for vertical progression of concepts. However, not all students are able to understand the concepts as quickly or as fully as set out in the curriculum. The understanding at a lower level is not always explored enough or consolidated to provide scaffolding for developing the concept at the lower level. Teachers often complain that students 'forget' concepts taught earlier in the year. This is where van Hiele's (1959: 22) reminder is directly relevant, that if the student him/herself has not developed the algorithm, then he or she will have to be taught a new one for every situation.

5.3.6.10 Summary: cycle 2

In the summative figure (Fig 5.30), each of the didactisation principles was gauged during the classroom observation and compared to the baseline classroom observation. Each teacher's

practices were scored on a 3 point scale as described in 5.2.6.10. These scores were then averaged to show a global trend across the five teachers. This graph is more about the teacher's practices while the previous graphs were also about the lesson and its characteristics. The average score itself is not as important as the comparison it allows to be made across cycles of the teaching experiment. This further allows one to determine how the study meets the aims set out in 1.3.1.2, 1.3.2.4 and 1.3.2.6.
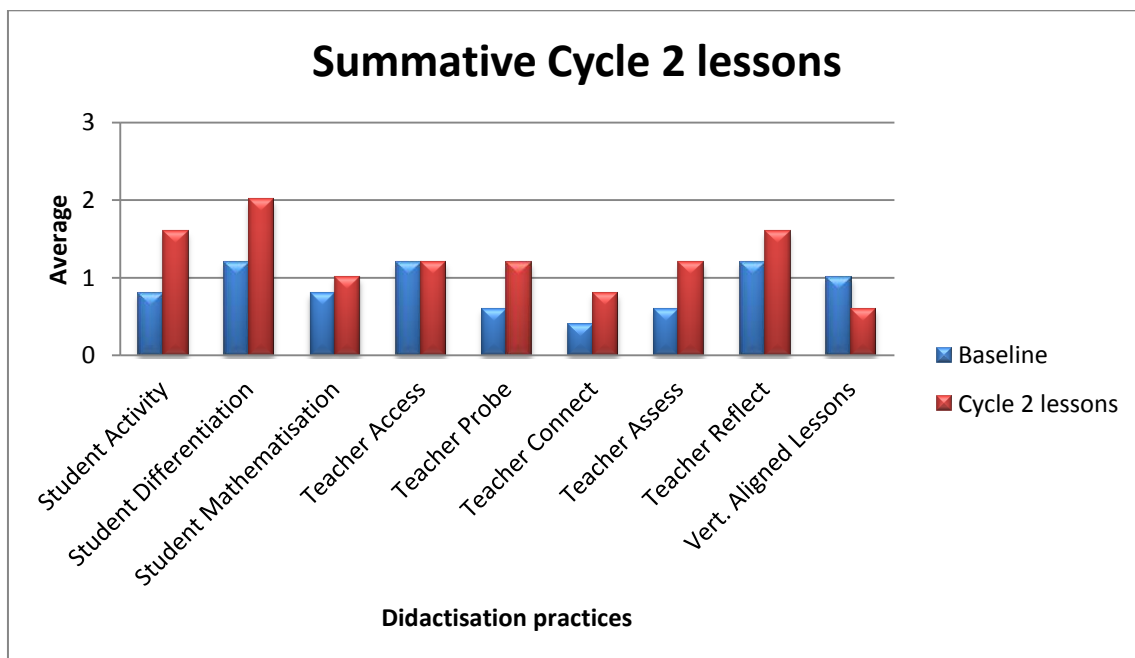
Figure 5.30 Summative classroom observation: cycle 2

5.3.7 Developmental changes during cycle 2

After the classroom visits had been conducted, the fourth teacher-researcher session was held. Teachers worked in groups to complete the Tangram Toys task (see Appendix 4) where proportional reasoning was necessary. At this point the CAPS document included 'inverse operations' as a topic for grade 4 to grade 6, so some discussion on this was also factored into the end of the session. Teachers' responses to the tangram problem included avoiding the problem, additive reasoning and multiplicative reasoning. When teachers started drawing the new tangram, deficiencies in their own additive reasoning became obvious. Two of the teachers did however calculate the scale factor of 1.75 immediately and worked towards explaining this to the other teachers. At the end of the task the researcher shared some of the

solutions used by students in the past. Many teachers were surprised by the variety of student thinking about the problem. The rest of the session was spent discussing inverse operations and possible strategies to teach this concept with understanding. Teachers were asked to compare a modelling approach to teaching proportional reasoning as opposed to traditional approaches.

Session 5 saw the teachers observe two groups of mixed ability students solve the tangram task. Teachers observed how one group was able to calculate the ratio mathematically, while another group enlarged every 4 cm to become 9 cm, they enlarged every 2cm to become 4.5cm. The teachers commented that this was a particluarly difficult task. It did show deficiencies in student thinking about ratios. The teachers felt they understood why students' first approach is additive reasoning.

Later in this session, teachers were discussing (amongst themselves) the structure of CAPS and its impact. They mentioned that there were 'gaps' in the textbooks in terms of conceptual development. This was addressed by the researcher during session 6 when the discussion was centred around improving textbook activities. The session added in how to guide teachers in incorporating the didactisation practices set out in the study within the policy and textbook guidelines. The teachers and their needs are the most important components in a professional development study. This sensitivity towards the needs of the teachers is important since according to Sarason (1971: 78):

> what is not recognised or verbalised cannot be dealt with, and if it is important and not recognized, effort to introduce substantive change, particularly in the classroom, result in the illusion of change.

Session 6 also introduced teachers to planning mathematics lessons with a hypothetical learning trajectory. This was after their current planning documents prescribed by each school were scrutinised. The planning documents required a combination of these from teachers:

- Writing the relevant topic and scope as given by the curriculum document
- The textbook activity/worksheet students will be involved in completing
- Homework activity
- A time frame

- Assessment (if any for the lesson)
- Reflection (only one school included this aspect, it was used for the teacher to make a note of any changes that would be needed for the following year.

Pages from teachers' current textbooks were copied. The researcher and teachers also discussed how to develop textbook lessons by including different planning questions (Instrument 8b). It was anticipated that this discussion would impact on teachers' resources, orientations and goals during the third cycle of the teaching experiment. The discussion for session 6 was structured around some of the ideas by Van De Walle *et al*. (2010: 59) together with hypothetical learning trajectories (Simon 1995: 136). Sarason (1971: 77) explained the difference between curriculum as "suggestion" and curriculum as "requirement". The teachers in this study have come to understand that a national policy is an *instruction*. Session 6 assisted teachers in considering how planning could enhance mathematisation within a lesson. It was suggested that once a mathematical goal was established for the lesson that teachers verbalise how students will reach this goal. This prepared them for the vertical mathematisation of the lesson and also to anticipate difficulties in the lesson. By including these aspects into their thinking about lessons, it was hoped that the mathematical goal of the lesson would be taken into account as well as anticipating difficulties that students may have in the lesson. Since a HLT was only dealt with as part of session 6, it cannot be said that teachers would be proficient in planning lessons this way. The session was used to improve didactisation principles.

When asked how planning through hypothetical learning trajectories is different from the teacher's current planning the following responses are relevant:

"*extra things are taken into account: what needs to be completed by when, according to CAPS. What needs to be covered for the ANAs and common exams. How can it be covered effectively despite the time restrains*" [referring to her own planning] (S6R, A, 03-2013).

"*the planning is different because it is more focussed on what they* [students] *are thinking and reflect on what they are learning*" (S6R,E, 03-2013).

Teachers commented that the following aspects of a hypothetical learning trajectory would be a challenge to their current way of planning:

"*I sometimes feel, oh yes, they can do this easily, this they certainly know and then they don't. Continuing to be in touch with what they are thinking is challenging*" (S6R, E, 03-2013).

"Predicting what learners are going to do, or how they will approach the solution" (S6R, C, 03-2013).

"Planning teacher questions instead of feeling the questions I would find difficult" (S6R, A, 03-2013).

" A learning trajectory. It gives the learners greater opportunity to be involved in the lesson. Expectations from the educator are higher" (S6R, D, 02-2013).

The factors within a hypothetical learning trajectory that teachers felt would improve their lessons were:

"All of these would improve lessons as they focus my thinking on what the children are learning/thinking and trying to anticipate the difficulties beforehand" (S6R, E, 03-2013).

"Forseeing the problem areas and using it for future use to better the lessons. (S6R, C, 03-2013).

"Reflection: not just focusing on time crunches but on the value, for the learners, that the lesson had" (S6R, A, 03-2013).

" Hypothesis formation. It is the more scientific manner of planning and assists with the timeline" (S6R, D, 03-2013).

Currently teachers have been planning the *what* part of lessons. The introduction of other aspects of lesson planning has developed reflection for the teachers. Teachers were left with these planning ideas over the vacation period of about two weeks that separated cycle 2 from cycle 3. The planning ideas would become relevant during the classroom observation at the beginning of cycle 3 where they were asked to complete instrument 5 before the lesson observation.

5.3.8 Changing resources, orientations and goals during cycle 2

Development of teacher resources, orientation and goals (see 2.3) noted during this phase of the teaching experiment need to be continually juxtaposed with the possible effect of CAPS implementation. Although CAPS is often referred to as a *new* curriculum - it is rather a repackaging of the existing NCS (National Curriculum Statement). Teachers who were following the FFL program (Foundations for Learning) would not notice too many changes to

the curriculum. While FFL set out prescribed content and sequence of content per term – CAPS provides the time-frames within which the content should be covered and assessed. FFL was not accompanied by a compulsory textbook while schools had a choice of about six nationally-accredited textbooks. Four of the five teachers had not worked with a prescribed textbook in the previous years. The five teachers are also intensely aware of an Annual National Assessment that their students would have to write at the end of September. During this cycle some schools were involved in the piloting of the Annual National Assessment paper.

Change is not a linear, hierarchical or predictable process, although some positive developments were noted at the end of cycle 1 and during the classroom observations at the beginning of this cycle. There needs to be some *plateau-ing* at times during a program. This may allow effective reflection to take place.

The teachers presented sensitivity towards improved questioning during cycle 2 lesson observation. This was also evident in teachers listening more to students and responding to students rather than only following their own scripts. Student ideas and own constructions were more prevalent during cycle 2 lessons. Two of the five lessons presented showed use of inquiry-based teaching although this cannot be called reality-based problem solving yet. In these lessons student activity was also raised. What may be added at this point of the research is that teachers are mostly working on level 2 of CGI engagement levels (see 2.3.1.2.1).

A development in teacher resources and teacher orientations impacts upon teacher goals. This was evident in the lessons that were presented. The observation of the five lessons allowed the researcher to make important adaptations to the content of the last session. A bridging between CAPS and didactisation principles was necessary. Building teachers' reflective strategies was incorporated into the program during session 6. Hypothetical learning trajectories lesson planning was the focus of the last session. It is envisaged that the lesson planning discussed in the last session of cycle 2 will result in improved didactisation practices. Considering this aspect of an HLT may become a resource for teachers that will assist in developing didactisation practices. It also allows for a raised quality of teacher reflection.

It is relevant to bring in the data collected from Instrument 3 (see Appendix 6). Although not specifically focused on individual didactisation practices, the instrument gauges three significant global areas of mathematics teaching, that of: pedagogy; use of context; and use of mathematical content. The category *pedagogy* contains the didactisation principles of active students, accessing, probing and connecting student understandings as well as teacher reflection. The category *use of context* scale includes the didactisation principles of differentiation as well as accessing, probing and connecting student understandings while teacher reflection is also relevant. The third category in Instrument 3, that of *mathematical content* scales include the didactisation principles of vertically aligned lessons, mathematisation, assessing student understandings and teacher reflection. Visually it can be represented as follows:

Pedagogy	Use of context	Mathematical content
Active students	Differentiation	Mathematisation
Access, probe, and connect student understanding	Access, probe and connect student understanding	Vertically aligned lessons
Teacher reflection	Teacher reflection	Assess student understanding
		Teacher reflection
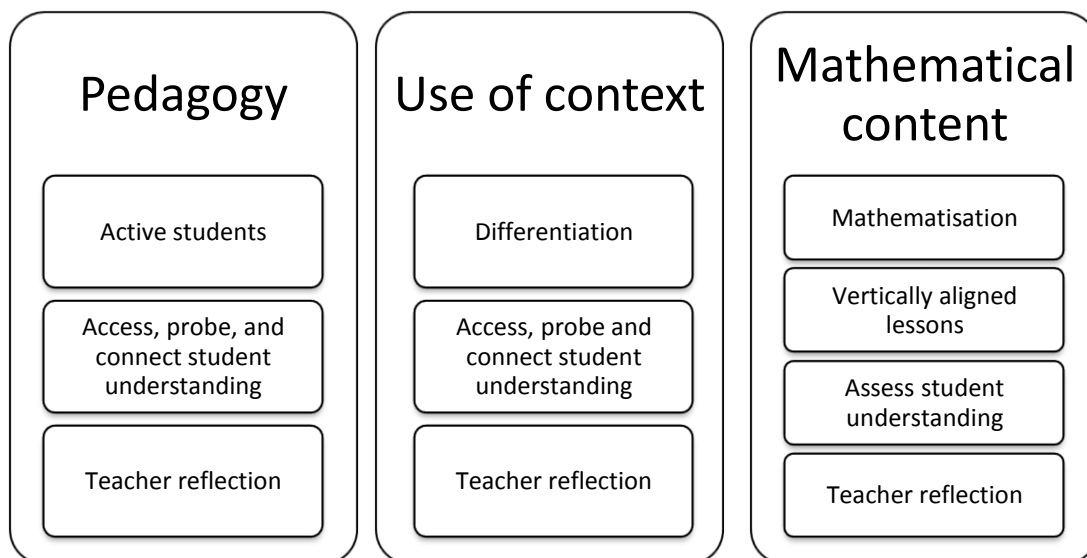
Figure 5.31 Alignment of didactisation principles in Instrument 3

At this stage of the teaching experiment the following graphs were deduced from lesson observations. The instrument included a number of operational behaviours that could be used to define the activities in the classroom as a 1, 2 or 3.

The observations were not based on a minute-by-minute account of the lesson but by the global focus of the lesson. The values on the vertical axis are therefore for comparative purposes only.

Pedagogy Scale: cycle 2
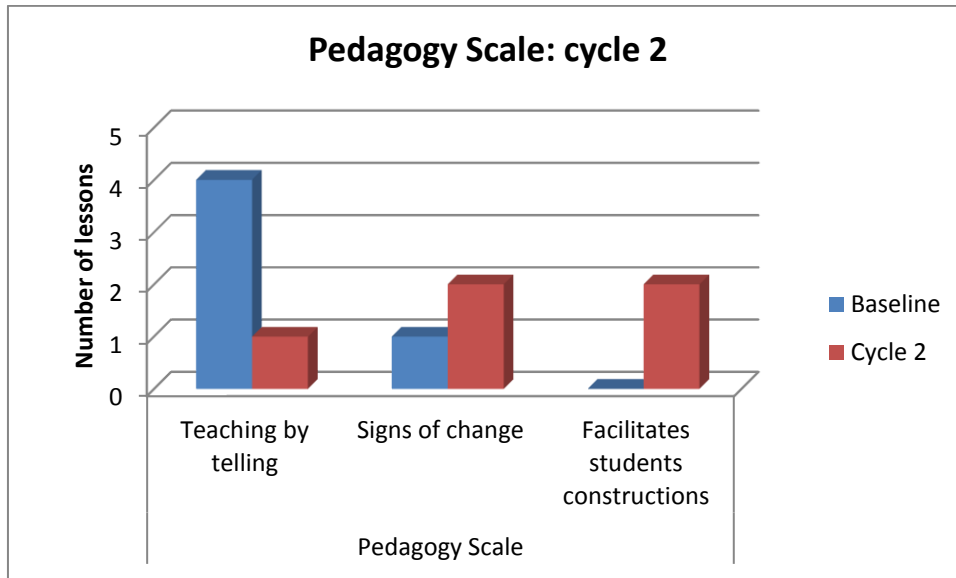
Figure 5.32 Pedagogy scale: cycle 2

The baseline lessons were consistent with 'teaching by telling' with Mrs C's lesson more in line with eliciting ideas from students. Cycle 2 lessons, included students working with partners and activities that invited more puzzlement from students (packing chocolates and blocks). The students were doing more and the teacher was talking less.

Use of context scale: cycle 2
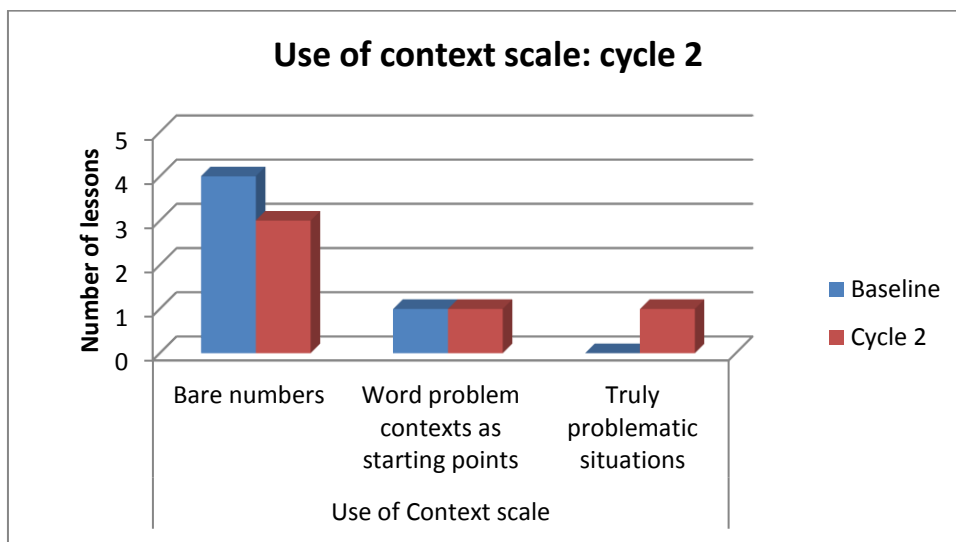
Figure 5.33 Use of context scale: cycle 2

The baseline lessons saw teachers present bare numbers from the outset of the lessons, although Mrs B did start the lesson with a problem. In cycle 2 lessons, Mrs C's lesson can be

considered truly problematic (for the grade and experience with independent group work that these students have).

Mathematical content scale: cycle 2
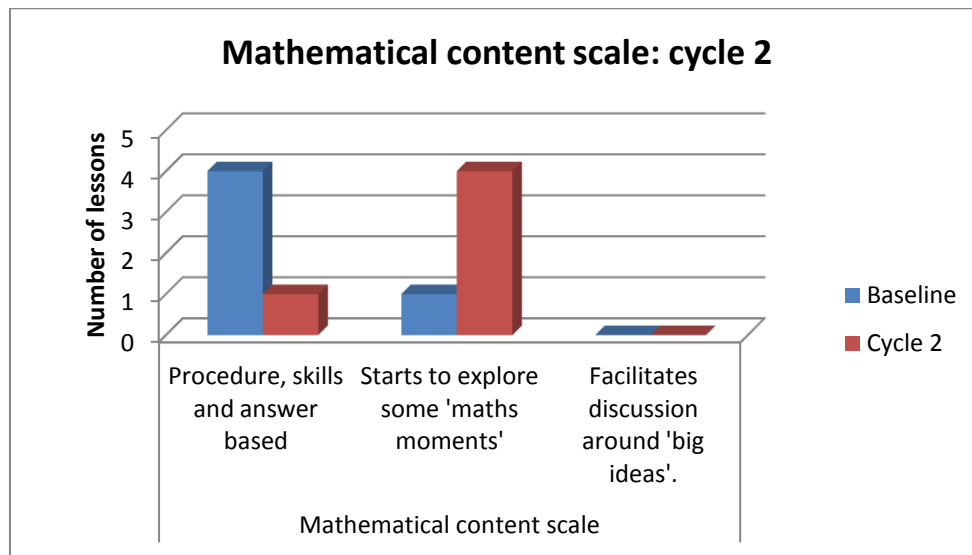
Figure 5.34 Mathematical content scale: cycle 2

The baseline lessons were skill and procedure based, although Mrs C did allow for 'math moments' in her discussion with students. Cycle 2 lessons show a definite shift to dealing with more of student understandings in the lessons. At this point a true 'math congress' on the big ideas in mathematics is not evident. It cannot be said that teachers understand teaching mathematics as mathematising at this stage. However, these adaptations and changes are slow and incremental due to the nature of the change having links to teachers' beliefs and orientations (see 2.3.2 and 2.3.3).

A modelling perspective also assists in gauging teachers' development. The six instructional design principles for modelling (see 4.2.1.2) are now reformulated so that they can be used for considering the development of the lessons from a modelling perspective. This perspective allows for different insights into the lessons so that the analysis of the lessons is varied and integrated within different theoretical perspectives.

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Design principle	Baseline lessons	Cycle 2 lessons
1. **Reality principle**. Real life situations are used and students need to make sense of these situation.	Mrs A used this to start the lesson.	No lessons involved real life situations.
2**. Model construction principle**. Are students involved in constructing, describing, extending or explaining a structurally significant system?	None. Students were involved in the "working mathematically" part of the modelling cycle (see Fig 2.2)	Mrs B, Mrs C and Mrs E's lessons involved describing and explaining systems.
3**. Self evaluation principle**. Students able to or were expected to judge their own responses.	None. Validation was done by the teacher.	Students could judge their responses in all the lessons. However final validation was still the teachers' function.
4. **Documentation principle**. What type of documentation was expected? Did students have to explain their thinking?	Calculations and answers were written down in books.	Mrs C's lesson involved students using a combination of writing and manipulatives in their explanations, while Mrs E's lesson involved writing the structure of the pattern in words
5. **Simple prototype principle**. Will the experiences assist students in making sense of other structurally similar situations?	All lessons had vertical mathematisation elements that meant students would have to memorise a procedure.	All lessons involved students identifying an underlying structure through a strong emphasis on vertical mathematisation which may lead to meaningful memorisation. Mrs A – noticing equivalence; Mrs B – lesson may lead to flexibility with inverse operations; Mrs E – structural understanding of patterns.
6. **Generalisation principle**. What constructions can be modified and applied?	Mrs E's lesson (a model of multiplication of multiples of 10 or 100) involved constructions could be modified and applied.	All lessons involved reflection on the underlying structure (e.g. equivalence, pattern defining and describing) that could assist in generalizing.

Table 5.14 Modelling design principles: cycle 2

5.4 TEACHING EXPERIMENT CYCLE 3

Cycle three of the teaching program started with classroom visits to the five teachers. The visits were conducted at the beginning of the second term (each school term is about 10 weeks long) after a two week school vacation. Teachers would be preparing for a term test during the term as well as an exam near the end of the term (requirement of CAPS). Government schools are also required to send out a report at the end of each term reflecting that term's assessment. By the end of the third term all schools throughout the country would write the Annual National Assessments. Each teacher was asked to complete a questionnaire based on their planning *before* the observed lesson. This was to establish if session 6 would have any impact on their didactisation practices.

5.4.1 Cycle 3 classroom observation: Mrs A (Grade 6)

Mrs A was working through a section on multiplication of whole numbers with her students. This particular lesson was based on solving problems that would require multiplication of large numbers. Methods for multiplying numbers had been done in the preceding lessons. The problems were realistic and included working with money. Mrs A anticipated that students may find multiplication of decimal numbers problematic. However this did not appear to be problematic during the observed lesson. Mrs A used the extension activity in the prescribed textbook.

Example of word problems:

1. Your school's soccer first team would like to go on a tour of Zimbabwe. The 23 members of the team have each collected R2659 so far.

a) How much money have they collected so far?

b) If they need to collect R90 000, how much more money must they collect as a team?

(Tiaden, Farrell & John 2012: 62)

She asked the students to work in pairs to solve the word problems. The only whole class discussion that took place at the beginning of the lesson was to specify the format for answering word problems i.e. the three step compulsory format of:

1. Open number sentence

2. Calculation

3. Concluding sentence

Mrs A checked that all students knew that they had to follow the format. She spent some time questioning them on the concluding sentence. She asked what they understood by this. Not all students knew why they had to end each problem with a concluding sentence.

Student 1: *you need it to explain your answer, your working out.*

Mrs A: *but you have shown your calculation in the previous step.*

Student 2: *you are answering the question that was asked...*   (LON3, 04-13)

Mrs A asked students to work in their books and to show the three steps. She reminded them about the classrooms norm that each member of the group should be able to explain any part of the problem to her if she asked. Mrs A then walked around the class questioning groups about what they were doing and why they were working in this way. She asked some groups why they did not add repeatedly but multiplied instead. They explained that it was faster to multiply. The groups worked enthusiastically on the problems and there was discussion between the pairs as can be seen by their hand movements in these photographs.

Figure 5.35 Student collaboration on word problems

When questioned about her choice of using pairs for this lesson Mrs A said that it was a very important component of the lesson and that individual work or larger groups would not work as well. She said that with the pair work they had to "*explain their understanding of the problem to each other first*" (ID, A, 04-13) and this assisted in working through the problems. This shows the emphasis Mrs A has placed on both horizontal and vertical mathematisation. Mrs A said that she would follow up this lesson by doing worked corrections the next day.

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She would discuss each problem and ask students to explain how they solved the problem. In planning for this lesson Mrs A indicated that she was going to assess the success of the lesson if learners "*can explain what they need to do and can work out the problem to completion*" (ID, A, 04-13). Mrs A added a qualitative aspect to her assessment of this lesson. It was also noted that very few pupils called for the teacher's assistance during the lesson. This could be ascribed to the partner work and the teacher continually moving from group to group. It may be that the students are more confident to work on their own or are becoming accustomed to working without continual teacher support.

5.4.2 Cycle 3 classroom observation: Mrs B (Grade 5)

Mrs B started her lesson by placing a large, colourful, three dimensional prism onto each desk shared by two students. The shapes she handed out included cubes, rectangular prisms, triangular prisms, cones and cylinders. She then asked the students to close their eyes and to feel the shape. She asked them to feel if the shape had a base and if it had a top. She asked them to feel the sides and to feel for corners where sides met and to feel for "*flatness*" and "*curves*".

Students then opened their eyes and could look more carefully at the shapes. She asked students a number of questions that allowed them to focus on the shape. Mrs B was very specific about the language she used and from time to time explained what she was asking. E.g. "*What is the shape of the polygon that makes up the sides of your shape*?" (LON3, B, 04-13). She then stopped to recap what a polygon is.

Mrs B then gave every student a blank sheet of paper and asked them to write down the names of the two dimensional polygons that made up the base, top and sides of the prism. Thereafter she asked them to draw the 3D shape, together with how many of each polygon made up the shape.

Mrs B had asked students to each bring a few toothpicks and a small packet of jelly sweets with them for the lesson. She had also brought these items for any student who did not have them,although most groups placed their items together in a pool for all to use. She now asked students to construct the 3-D shape they had been given out of the toothpicks and sweets.

Figure 5.36 Students constructing 3D shapes

Mrs B then questioned students about the corners, toothpicks and sides which assisted her to introduce the required vocabulary of vertices, edges and faces.

In the next lesson, students were to work from their textbooks as in this example.
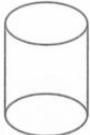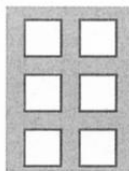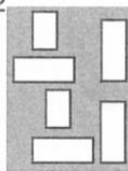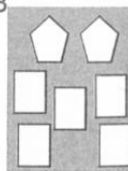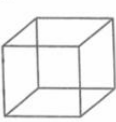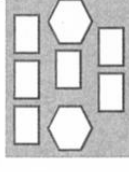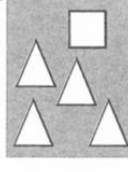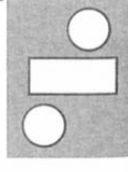
Figure 5.37 Textbook activity (Austin, Jones, Hechter & Marchant 2011: 102)

Mrs B's structuring of the lesson was an indication of evolving vertical mathematising in her planning. The preliminary activities of feeling the shape, then looking at the shape and describing it, followed by making drawings and naming the polygons that make up the shape, assisted students in developing vocabulary in context and enabling them to make their jelly-sweet constructions. It would appear that the planning for this lesson was more focused on the mathematical goal of the lesson as was suggested to teachers during session 6. Focusing on

the mathematical goal of the lesson as opposed to focusing on the next page of the textbook is a significant factor in assisting teachers to structure vertical mathematisation into their lessons.

5.4.3 Cycle 3 classroom observation: Mrs C (Grade 6)

Mrs C and her class were working on factors of numbers and constructing factor trees. She started the lesson by asking the students to tell her what factors, multiples and prime numbers were. When students gave a definition for these she followed this by asking them for an example. Once she had established this, she explained that they were going to construct factor trees. She first asked them if they knew what a family tree was. Most students had seen a family tree and could explain to her what it was. There was some discussion around why the grandparents were at the top and not the bottom of a family tree. Part of the opening discussion:

Mrs C: *but first tell me, what is multiplication? How do we work it out? Do you ask yourself why do we do it?*

Student 1: *because it's a simpler form of addition*.

Mrs C: (repeats what student 1 says slowly) *Does anyone disagree with that?*

Silence

Mrs C: *does anyone have a different way of explaining it?*

Some talking amongst the students.

Mrs C: *what do we mean it's a simpler form of addition?*

Student 2: *it's a shorter sum.*

Mrs C: *can you explain that a little bit more; maybe there is someone who doesn't understand when you say it's a shorter sum?*

Student 3: *instead of saying fifty plus fifty, you can just say fifty times two and it will give you the same answer.*

She followed this with an example of a parent who wanted to give each child at the school a bag of 30 apples. She said that there were about 900 students at the school. She explained that calculating 30 x 900 could be made simpler by breaking down 30 into its factors and then multiplying 10 x 3 x 900. The class agreed that this was easier. She asked students to work in pairs and gave them each a blank sheet of paper. She then wrote the numbers 18, 36, 48 and

60 on the board. She asked the students to find the factor trees of each of these numbers. Mrs C then walked around the class asking students questions about their trees. Some students realised that there could be more than one factor tree for each of the numbers. She reminded students that working in pairs meant that they were each responsible for understanding and calculating the solution. She suggested that they explain to each other what they were doing. The students went to work enthusiastically. Very few pupils called for the teacher to assist them. When most groups had finished this she called the class to attention and asked the groups to share their factor trees. She wrote down all the possible factors trees for each number and pointed out that each factor tree eventually resolved into the same factors. She made connections between the different factor tress for the same number. Some students expressed surprise that this happened. At this point the lesson ended for the day. The following photograph shows a group working on their factor trees. Although they are working together, each member of the group was required to write the factors trees on their own page. Mrs C indicated that she would continue with the other numbers in the next lesson. She was pleased that the students had found it surprising that the prime factorisation for each number was the same.

Figure 5.38 Students working on factor trees

In terms of the mathematisation involved in the lesson, a vertical component is more prominent than a horizontal component. Mrs C did try to show students that multiplying larger numbers could arise from a real situation; the goal of the lesson was for students to see how calculating the factors of a number could assist in laborious calculations. The textbook presented factor trees as one of many *methods* of multiplication although it would become

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significant in factorisation by grade 9. Mrs C was pleased with the students' factor trees in that she felt it would improve their flexibility with inverse operations.

5.4.4 Cycle 3 classroom observation: Mrs D (Grade 6)

Mrs D started her lesson on space and shape by placing three-dimensional prisms on top of each other to form new shapes e.g. she paced a pyramid on top of a cylinder or cube.

Figure 5.39 Complex 3D shapes

She asked students what the complex shape could be and they responded with a variety of answers (hut, rocket etc). She presented a number of combinations and the students enjoyed telling her what they thought it could be in reality. She then took the class for a walk around the school building where she pointed out various structures of the school building and asked students to try to name the three dimensional shapes. They found a variety of cycliners, rectangular prisms and triangular prisms. The students were taken to the carport and were asked to count the number of rectangular prisms in one structure. Mrs D then took them to another part of the school where each group was given an area. Students worked in groups to identify the various 3D prisms in the area and to name the two dimensional shapes that each prism consisted of.

In the next lesson Mrs D indicated that she would ask the groups to report back on their findings and then she would continue with the prescribed textbook work as in the following example:

1. Match each pyramid with the correct set of faces. Write the name and matching letter.
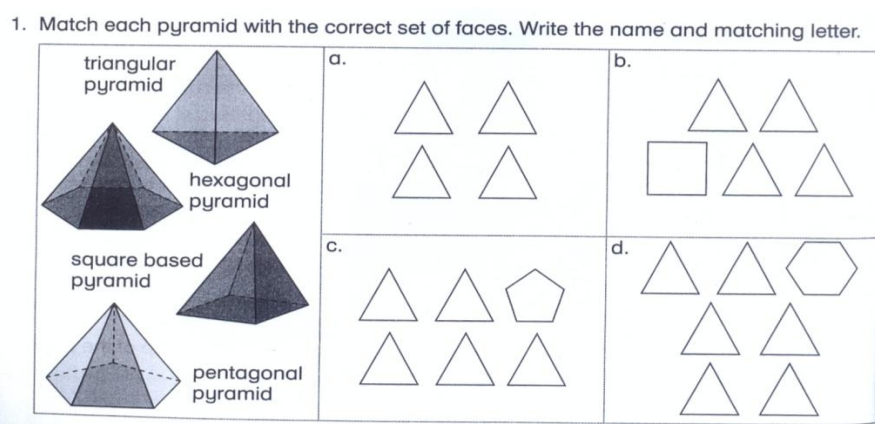
Figure 5.40 Textbook activity (Austin, Jones, Hechter & Marchant 2012: 76)

In the outside activity, students could identify some of the shapes very easily (pillars were cylinders) but when shapes were inverted or there were a number placed together (a metal tower of triangular prisms) the identification became more challenging. Mrs D wrote that her mathematical goal for this lesson was for the students to identify 3D objects in context. She also anticipated that students may find this difficult. Mrs D structured the vertical mathematisation of the lesson. She first presented students with the three dimensional shapes that are traditionally used in mathematics teaching. She then created combinations of these shapes (huts, rockets etc) and finally she took students outside to find these shapes in real contexts before returning to traditional textbook work. The didactisation principles of student activity together with a mathemtical focus on the goal of the lesson seems to have assisted Mrs D in developing this lesson.

5.4.5 Cycle 3 classroom observation: Mrs E (Grade 5)

Mrs E started her lesson reminding the students that she has two children. She then said that often at home the children have to share things. In this case Mrs E took out a chocolate and said that she had to share this chocolate between her two children. Mrs E then showed the class different possibilities of sharing the chocolate by cutting it in two pieces. In each case she explained that she could cut the chocolate in two, but it was not always fair. The students told her that to be fair she had to make sure the two pieces were *equal*. They told her to cut the chocolate in half.

Mrs E: *Where is half?*

Chorus: *in the middle!*   (LON3, E, 04-13)

Mrs E then handed out a page to each student. On the page were 3 pizzas and 4 slabs of chocolate. She gave them a problem. She said that they needed to share the 3 pizzas between 4 children so that the children each received the same amount of pizza. She then told them to cut out the pizzas and to "*show me clearly*" (LON, E, 04-13) that every child gets the same amount. The students got to work, many discussing with their neighbour, and, Mrs E walking around asking what students were doing and why they were doing so. Some students were reluctant to start and thought that the activity was difficult. She told them to think about it first. Some students used their rulers and made some measurements; some students cut each pizza into four and then gave each child ¼ until all the pieces were used up. Within a few minutes all students were enthusiastically working on the problem. Although some students needed reassurance in getting started, very few needed Mrs E to verify that their solutions were correct. They were able to see for themselves that their answers were correct. Mrs E then set them the next task, which was to share the 4 slabs of chocolate equally between 12 children. None of the students asked for her assistance in starting this problem.
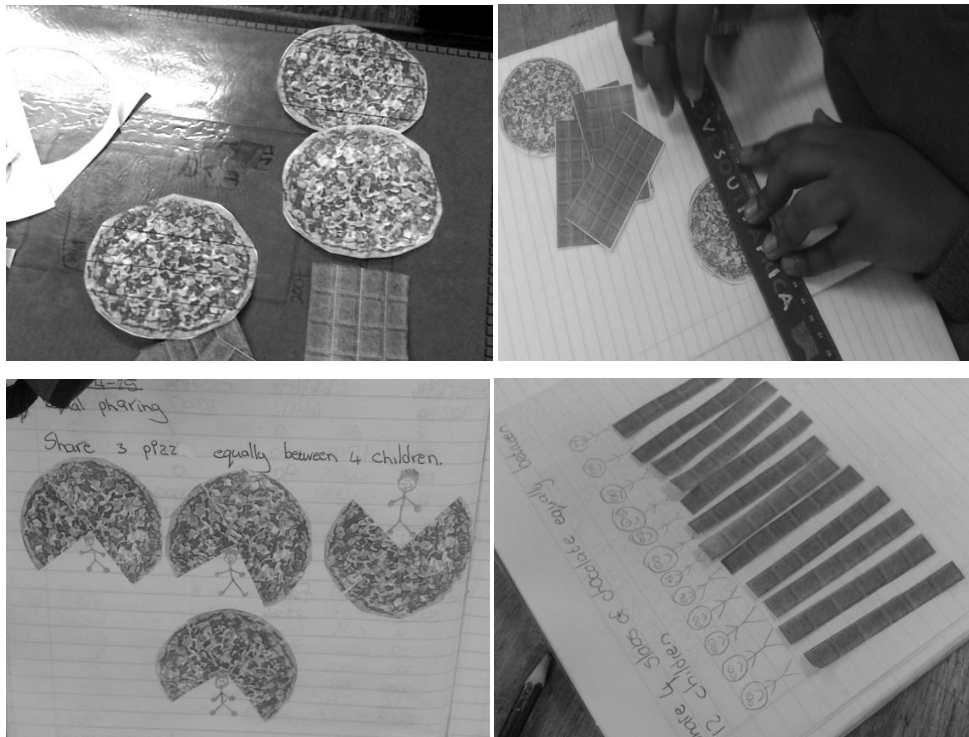
Figure 5. 41 Fraction sharing activity

In terms of the mathematisation involved in the lesson, students built on their concept of sharing (division) and were able to extend it to sharing that involved fractions. The problem they had to solve necessitated that they move from wholes to halves and then quarters. This

concept of equal sharing assisted them to understand the concept of equivalence which is fundamental in understanding fraction addition or subtraction. In the first photograph the student drew four vertical lines to share the pizza. This enabled Mrs E to question the student about sharing the pizzas *equally*. Without any other prompting the student asked for a new sheet of paper and cut the pizzas differently. Students enjoyed this lesson and worked quite quickly through the activity. This lesson appeared to have the quality of what the theory of didactical situations (see 2.2.2.1) aims at – in creating a milieu that the students are responding to. The problem has been devolved and the students willingly accept responsibility for solving it.

5.4.6 Changes in didactisation practices

The changes in didactisation practices are now compared through all three cycles of the program. The final classroom observation visit will take place at the conclusion of the program to gauge the possible effect of the professional development program during cycle 3. The following graphs show each didactisation practice as it is gauged from the start of the professional development program.

5.4.6.1 Student Activity: cycle 3
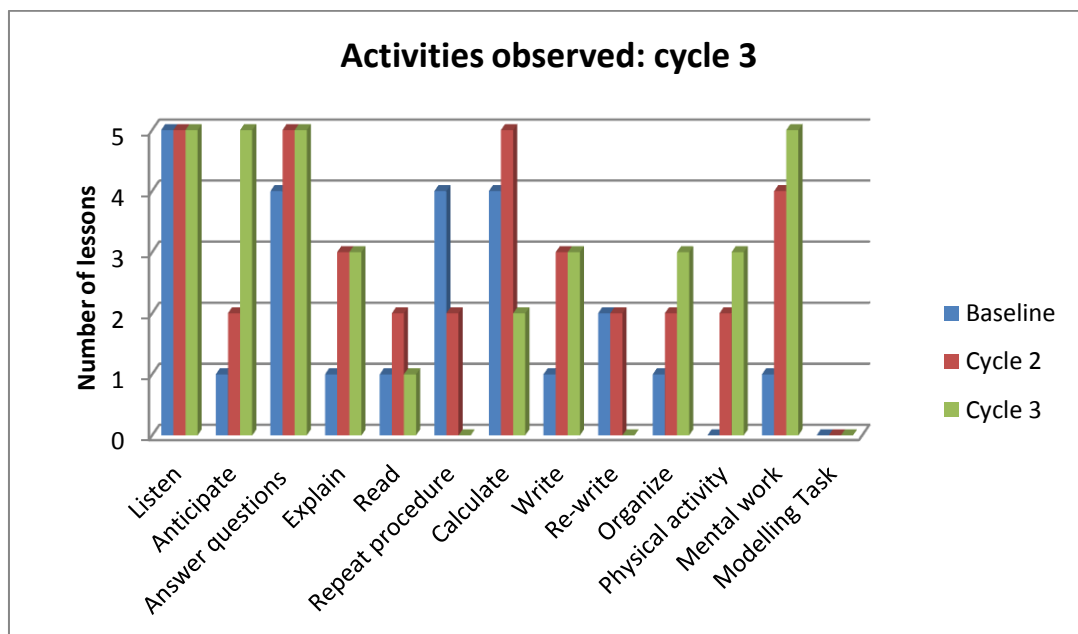
Figure 5.42 Activities observed: cycle 3

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The lessons observed were more varied in terms of student activity, and the trade off is that students are not always involved in calculating during these lessons. This may take place in the succeeding lessons. Students were more active physically and verbally which meant that for the observed lessons the writing and reading was less evident. Mental work is still a requirement of the curriculum and the teachers do mental work daily.
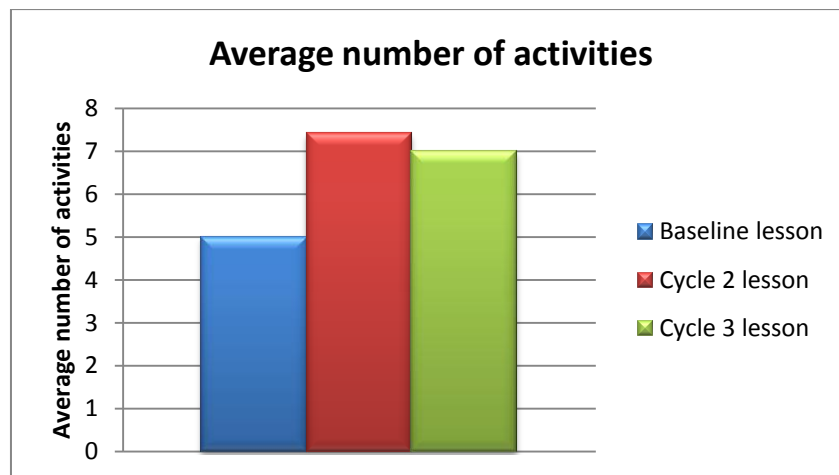
Figure 5.43 Average number of activities: cycle 3

The average number of activities as seen in Fig 5.44 is similar to cycle 2, although it is still higher than the baseline lessons. Activities in the lessons seem to have been selected to build concepts at a lower level first and then for teachers to use these activities as a base on which to build concepts at a higher level. When teacher goals for their lessons are scrutinised, there seems to be a focus on the words "use" and "understand". There may be connection between the goals and the orchestration of the activities.

Lesson	Cycle 3 lesson goal
Mrs A Gr. 6	*Interpret and solve word problems. To **use** knowledge of multiplication.*
Mrs B Gr. 5	*To build a 3D shape, **understand** that it is made up of 2D polygons.*
Mrs C Gr. 6	*Multiplying numbers using factors. Draw a factor tree and **use** this to multiply large numbers.*
Mrs D Gr. 6	*Identifying different shapes in the outside activity. **Apply** 3D objects into contextual problems.*
Mrs E Gr. 5	*To **understand** that a fraction is an equal part of a whole.*

Table 5.15 Teacher goals: cycle 3   (LQ3, 04-13)

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5.4.6.2 Differentiation: cycle 3
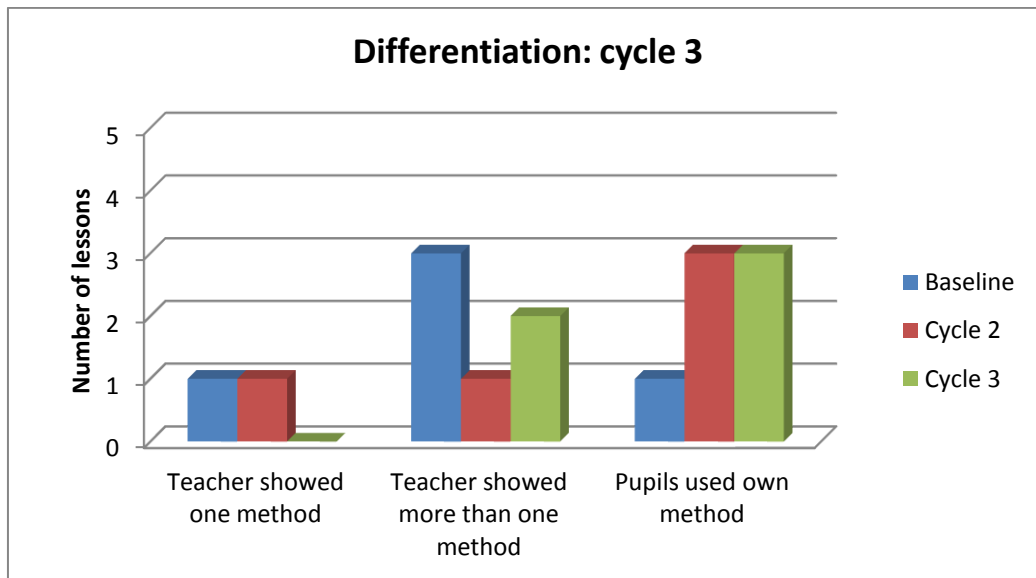
Figure 5.44 Differentiation: cycle 3

The above figure (Fig. 5.44) presents a focus on methods used for the lessons. The lessons do however need to be looked at within the context of the curriculum. Two of the lessons were on space and shape which do not necessarily include 'methods' of calculation. The rich visual representations (3D shapes, drawings, buildings) made it possible for students of varying abilities to have access to the concepts. Three lessons (Mrs B, Mrs D and Mrs E) had a high level of student activity before the formal concepts were to be introduced. Teachers were allowing more for spontaneous differentiation than imposed differentiation (see 2.4.3).

5.4.6.3 Mathematisation: cycle 3
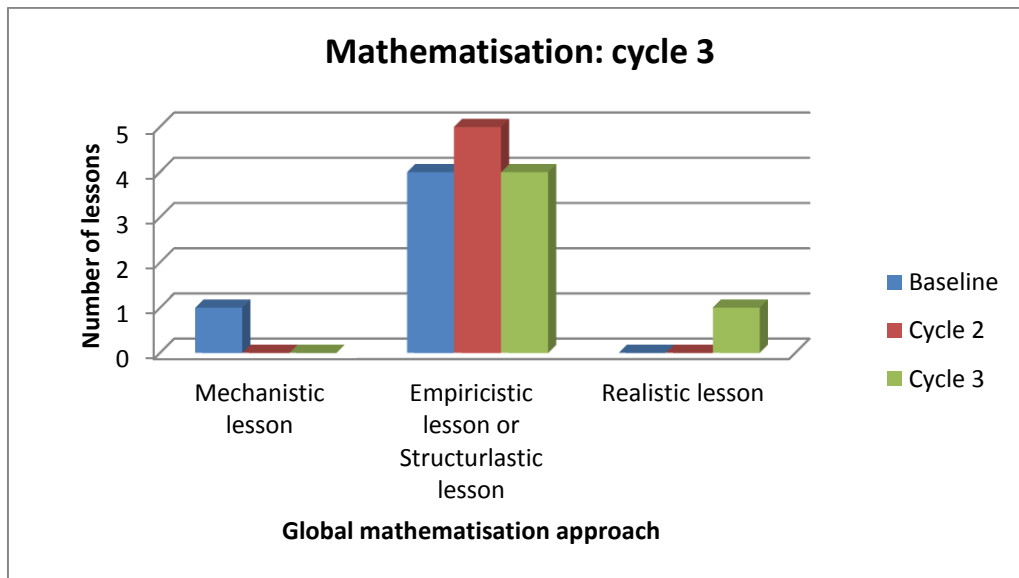
Figure 5.45 Mathematisation: cycle 3

This cycle included the first realistic mathematics lesson while no pure mechanistic lessons were observed (Fig 5.45). When teachers prepare for active students who can access a problem at many levels, it facilitates the scope for mathematisation. If horizontal and vertical mathematisation is coordinated in a lesson the mathematical experience is more meaningful. In this cycle, four teachers (A,B,D,E) focused on horizontal mathematisation and vertical mathematisation while Mrs C's factor trees could be considered a horizontal activity for the vertical development of prime factorisation. All teachers used pair or group working to facilitate discussion between students which would assist in making connections and mathematising. Teacher E's lesson reflects the true sense of Brousseau's devolution of a problem (see 2.2.2.1).

The means of representation in these lessons is as follows:

Lesson	Representations	Teacher or student use	Is student representation constructed or applied?
Mrs A Gr. 6	Verbal Written	Teacher and students Students	Constructed Applied

Mrs B Gr. 5	Verbal	Teacher and students	Constructed
	Written	Teacher and students	Constructed
	Drawings	Students	Constructed
	Manipulative models	Students	Constructed
Mrs C Gr. 6	Verbal	Teacher and students	Constructed
	Written	Teacher and students	Applied
	Drawings	Teacher and students	Applied
Mrs D Gr. 6	Verbal	Teacher and students	Constructed
	Written	Students	Constructed
	Manipulative models	Teacher	-
	Physical models	Teacher and students	Constructed
Mrs E Gr. 5	Verbal	Teacher and students	Constructed
	Written	Students	Constructed
	Pictures	Teacher and students	Constructed
	Physical model	Teacher	-

Table 5.16 Lesson representations: cycle 3

The increase in representations (Table 5.16) is apparent not only for teacher representations but also the representations that students produced. These representations will impact on the access students have to understanding, and if connections are reflected on it, will impact on the scope for vertical mathematisation in the lessons. As stated by Hiebert and Carpenter (1992: 72) the language and the materials are essential for students in understanding mathematics. In four of the lessons there was a building up from the pictures/physical models or manipulatives to organizing, abstracting and formalizing ideas from these. The physical representations assisted students in structuring the mathematical ideas. The field of teacher representations may be linked to teacher knowledge. This professional development program did not to focus on improving teachers' mathematical knowledge. Teachers are confident in using representations within their set curricula. Improving the depth and scope of teacher content knowledge may positively impact this didactisation principle. Teachers also mentioned being under time constraints, so they select the more common or relevant representations to use and present to a class.

When considering the revision of Gravemeijer's lesson models (see Table 5.11), the lessons reflect the following:

Lesson	Model shift
Mrs A (Gr. 6)	Pre-intermediate to Intermediate
Mrs B (Gr. 5)	Structuralistic to Intermediate
Mrs C (Gr. 6)	Pre-intermediate

| Mrs D (Gr. 6) | Pre-intermediate to Intermediate |
| Mrs E (Gr. 5) | Pre-intermediate to Realistic |

Table 5.17: Lesson models: cycle 3

The lesson shift to an intermediate level to made possible by including more realistic contexts within the lessons (word problems, building physical models, observing school buildings and sharing pizzas). Students had to create representations of these contextual situations and further abstract on these representations.

Further analysis of the mathematisation evident in these lessons can be summarised as follows:

Lesson	Horizontal or vertical mathematisation	Gravemeijer's strategies that promote mathematisation
Mrs A Gr. 6	Horizontal mathematisation made possible through inclusion of word problems that included working with money. Vertical mathematisation included shortening of calculation (as socially constructed by pairs), schematizing and symbolizing their understanding of the problem. Since all problems involved multiplication some generalizing may have taken place.	Generalizing, Exactness and Brevity
Mrs B Gr. 5	Horizontal mathematisation through analysing real shape before constructing their own. Vertical mathematisation through the progressive schematizing before building the shape. Identifying the 2D shapes, naming them, considering how many of each is needed and drawing these. Finally, a deeper realisation of the properties of the shape. Re-invention of the properties.	Generalizing, Exactness and Certainty
Mrs C Gr. 6	Starting situation could act as a horizontal activity since at grade 6 it could function as a model (Treffers & Goffree 1985: 102). Vertical mathematisation through reflection on the re-organisation while considering different factor trees for each number. Re-invention of prime factorisation.	Generalizing and Certainty
Mrs D Gr. 6	Horizontal mathematisation through analyzing structures within the school and connecting these to the names of 2D and 3D shapes that are known. Vertical mathematisation through re-organisation and structuring of known shapes to make up composite real shapes so that they can be used abstractly.	Generalizing

Mrs E Gr. 5	Horizontal mathematisation through interpretation of the real situation. Vertical mathematisation through extending on the sharing concept to a division concept that includes organizing fractional sizes. Mapping will also take place when sharing fractional quantities. Use of an area model to build deeper realisation of fractional sharing. Re-invention of division by a fraction.	Generalizing and certainty. The cutting and sharing of pictures acted as a model *of* which when was reflected on becomes the model *for* divison.

Table 5.18 Mathematisation analysis: cycle 3

The classroom environments of interactive student collaboration and student-constructed physical models provide the milieu for Freudenthal's guided re-invention (see 2.4.4). In terms of Freudenthal's delicate balance (see 2.2.1) in guided re-invention, the pendulum is moving towards freedom of learning as opposed to force of teaching. What may however be necessary in future lessons and professional development is to focus on how the teacher could "provoke reflection" (Freudenthal 1991: 100).

5.4.6.4 Accessing student thinking: cycle 3
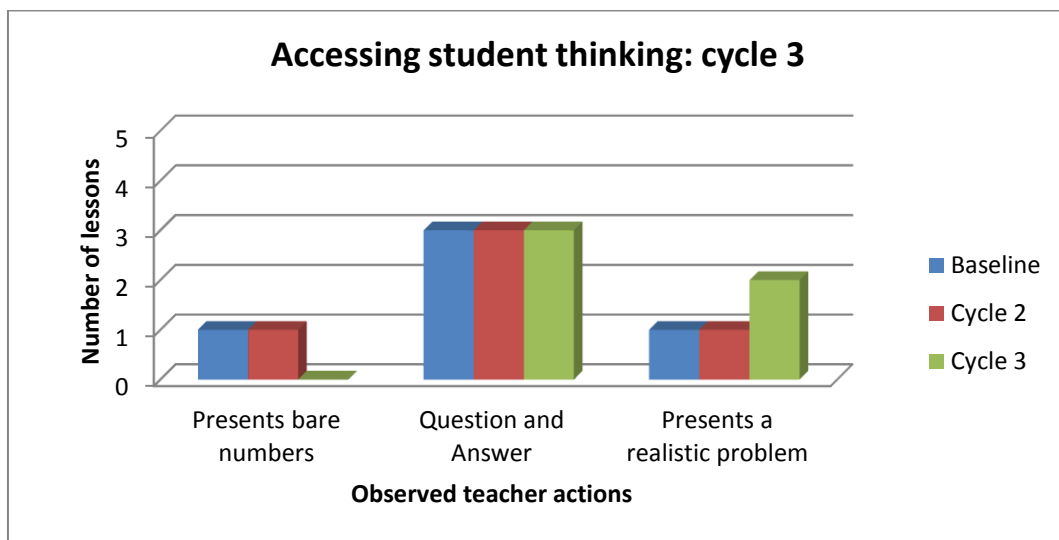
Figure 5.46 Accessing student thinking: cycle 3

The five teachers all displayed good questioning skills and wanted students to take a more active part in answering questions. Two lessons presented realistic problems from the onset, although only one lesson used the problem to structure the rest of the lesson. Realistic problems that become the "focal point" result in a "contextualization" where students solve

problems that they perceive as their "obligation" to solve (Nilsson & Ryve 2010: 245). Students were working more actively in these lessons which increase teachers' ability to access student thinking. Three lessons had students physically working (jelly sweets and toothpicks lesson, finding 3D prisms on school premises and the cutting of pizzas) which increases the horizontal mathematisation aspect and increases the potential for vertical mathematisation. It may be valuable to include discussion about a *focal point activity* in future teacher development programs.

5.4.6.5 Teacher probing: cycle 3
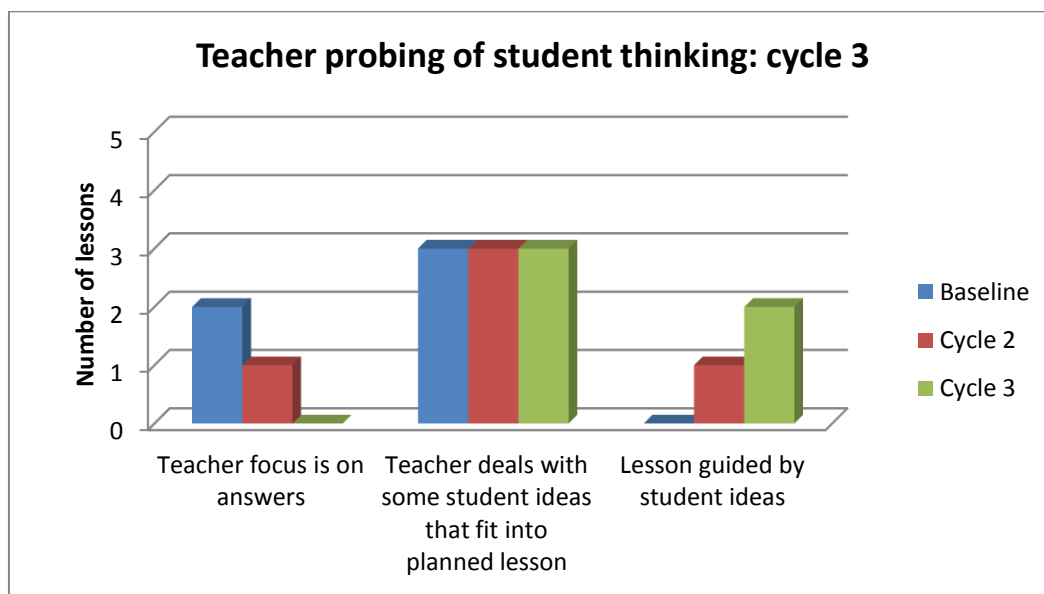
Figure 5.47: Probing student thinking: cycle 3

The extremities of this spectrum of Fig. 5.47 are showing signs of change in the initial phases of the lessons. Teachers are planning for students' ideas and are working towards using students' ideas to teach the content or concept for that lesson. In the latter phases of the lessons, where students are engaged in an activity, the following shows the development of the teachers' roles in probing student ideas:

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Teacher Probing role	Teacher as supervisor of the activity. Answers questions or clarifies if students ask.	Teacher as director or manager. Initiates discussion, controls the topic. Allows or invites input.	Teacher as facilitator. Sets up structure, interacts with students. Students interact with each other and materials.
A Baseline	X		
B Baseline	X		
C Baseline	X		
D Baseline	X		
E Baseline	X		
A Cycle 2			X
B Cycle 2	X		
C Cycle 2			X
D Cycle 2		X	
E Cycle 2		X	
A Cycle 3			X
B Cycle 3		X	
C Cycle 3		X	
D Cycle 3		X	
E Cycle 3			X

Table 5.19 Teacher probing: cycle 3

The skill of fully connecting student ideas to the concepts to be taught is more difficult than providing the right type of activity or questioning within the lesson. The didactisation principles are inter-linked and form a web to improving mathematics teaching, but it may be necessary to arrange them in a sort of hierarchy of development (see 5.5.7). This can be seen in the next didactisation principle (Fig. 5.48) where teachers are not yet effectively contrasting and presenting student ideas to structure a vertical shift in student thinking about the topic. Stein *et al*. (2008: 322) suggest ways to make teachers more proficient in coordinating classroom discussions. According to them, for teachers to help students make connections five other carefully orchestrated practices (anticipating, monitoring, selecting, sequencing and finally connecting) are necessary. Teachers assisting students in making mathematical connections is a complex practice. Although teachers may present a shift in their knowledge, beliefs and goals about connecting student ideas, the skill necessary to do this needs to be

practiced so that teachers can become confident. The quality of teacher-student and student-student interaction has developed through the program.

5.4.6.6 Connecting student ideas: cycle 3
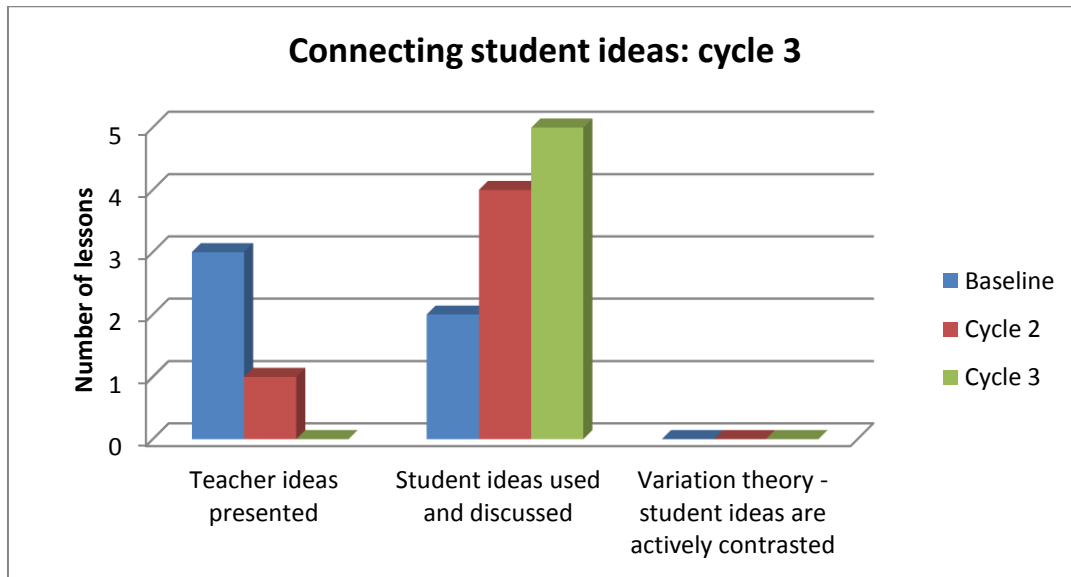
Figure 5.48 Connecting student ideas: cycle 3

The increased student activity, representations and social constructions mean that the possibilities for students to make connections are increased. As already shown, the quality of horizontal and vertical mathematisation has developed in these lessons compared to the baseline lessons. The lessons allowed students to create connections under implicit and explicit conditions. Although Hiebert and Carpenter (1992: 72) explain that students can construct connections under both conditions a student constructed connection is of a more lasting nature since it will require assimilation and accommodation (Piaget 1978: 6).

Mrs A's lesson: connections between word sums and the result of multiplication process. Mrs A explicitly pointed out these connections in a whole class discussion in the follow-up lesson.

Mrs B's lesson: connections between the physical properties, the mathematical language that describes the 2D shapes that makes up the 3D shape and the construction of the shape. The sequencing of the activities led to these connections more than the teacher facilitating this.

Mrs C's lesson: connections between the different factor trees that lead to the same prime factorisation of the number. Students presented different factors trees and this brought the connections to the fore. Mrs C did not specifically plan to engage with this.

Mrs D's lesson: connections between the different structures at school with the 3D shape and the 2D polygons that make up the shape. Connections are also made to the mathematical language (vertices, faces etc.). The sequencing of the activities assisted in making the connections apparent.

Mrs E's lesson: connections between sharing, division and fractional amounts. Connections between area models and numerals that represent these models could be made. Mrs E did not specifically focus on these connections; it was the activity itself that may have led to this. She followed up on this activity in the next lesson, where she reported: *they all got it right* (ID, E, 04-13).

Although lessons have shown development in this didactisation practice, what is ideally sought is for the teacher to take a more active role in developing connections in the lesson. The teachers may need to sum up with a whole class discussion at the end of the lessons to facilitate the construction of connections.

5.4.6.7 Assessing classroom solutions: cycle 3
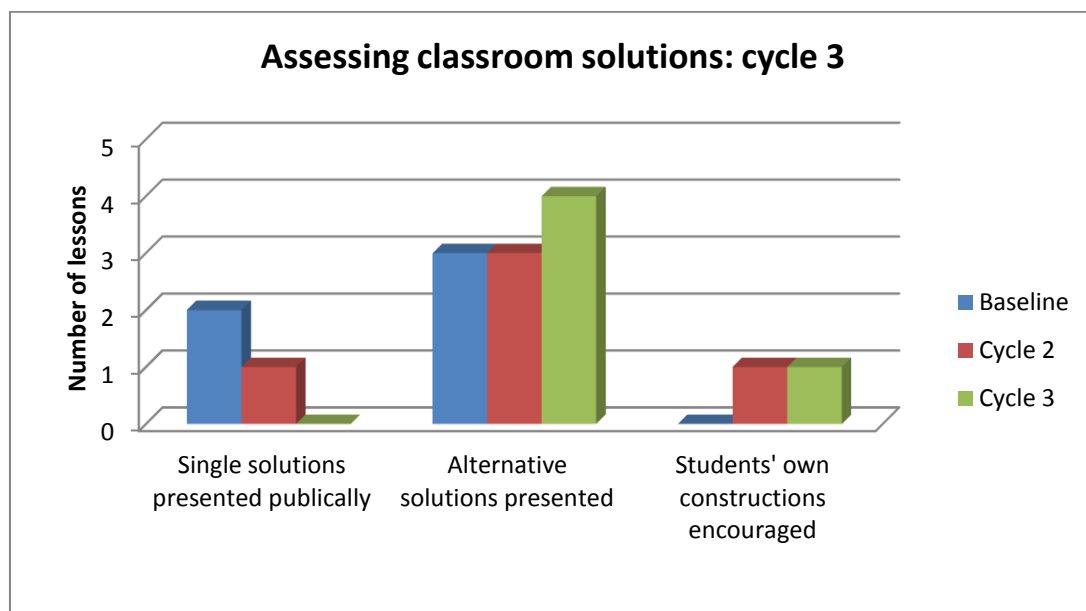
Figure 5.49 Assessing classroom solutions: cycle 3

In the observed lessons, teachers were encouraging students to understand and talk to each other about the concepts. Students' own constructions were more noticeable during lessons

while single solutions presented by the teacher have decreased. All the lessons observed in this cycle were structured in such a way that the students were largely able to assess their own ideas. This was facilitated through pair and group work, looking at physical structures or building their own 3D shapes. In one incident in Mrs B's lesson, a student produced the following 3D structure for a square based pyramid:
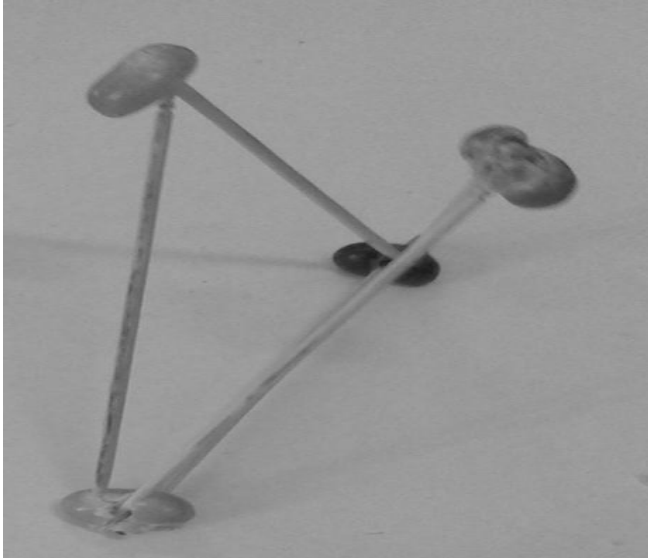
Figure 5.50 Student construction of 3D shape

Mrs B: *what is the problem here?*

Student 1: (looking, but no response)

Student 2: *it doesn't have a base* (she points to part of the structure)

Mrs B: *can you fix it?*

Student 1: (nods).   (LON3, B, 04-13)

5.4.6.8 Teacher reflection: cycle 3
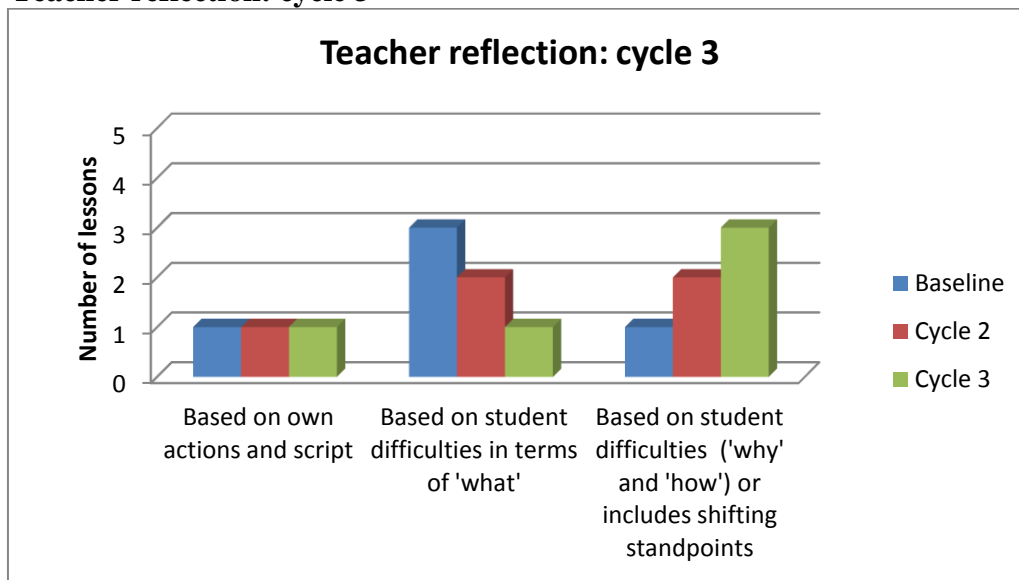
Figure 5.51 Teacher reflection: cycle 3

Teachers' reflection was supported by a focus on teacher planning and questions relating to how teaches think students perceive their lesson. Teacher reflection can develop if teacher knowledge and resources develop. One can only reflect on what one knows. Reflecting on a lesson is enhanced if teachers are equipped with vocabulary to describe what they are seeing. Reflection on a lesson also changes when teachers consider different aspects of the lesson. In this professional development the reflection sheets asked teachers to judge the lesson from the students' perspective which is a new orientation for teachers. It will be important to develop reflective vocabulary in teachers if this didactisation principle is to fully develop. The role of professional development should not only rest on teacher knowledge or teaching skills. Professional development cannot be undertaken as a quick-fix or within a short time span. The network of skills and practices that make up teaching proficiency should be developed together.

Session 6 of the second cycle focused teachers on planning for mathematics lessons. This included making teachers aware that the mathematical goal of the lesson is not to complete a set of exercises or to take part in some mathematical activity. The mathematical goal of the lesson was to be verbalised by teachers. This then allowed teacher to reflect on whether the class had reached this mathematical goal or on what part of reaching the goal would be difficult. It is important for teachers to focus on this in their planning. It affects the way they

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structure the lesson, type of representations and activities they incorporate in the lesson. It also contributes to how teachers vertically plan their lesson.

Teachers were asked what information from the lesson would assist them in assessing the success of the lesson. Their responses show an integration of taking the students' responses and the students' success into account when reflecting on a lesson. This is part of the "shifting standpoints" that Freudenthal proposed for reflection. Teachers are also more focused on a more formative type of reflection.

Mrs A: *Can the learners* **explain** *what they need to do and can they work out the multiplication sum to completion?*

Mrs B: *If they can build the shape and* **see** *the 2D shapes that make up the shape.*

Mrs C: *Feedback from the learners – if they are able to complete the* [factorisation] *process.*

Mrs D: *If students are able to* **apply** *what they already know about shapes to the outside activity.*

Mrs E: *I will see if they are able to divide the shapes equally and* **explain** *what they are doing.*

(LQ3, 04-13)

5.4.6.9 Vertically aligned lessons: cycle 3
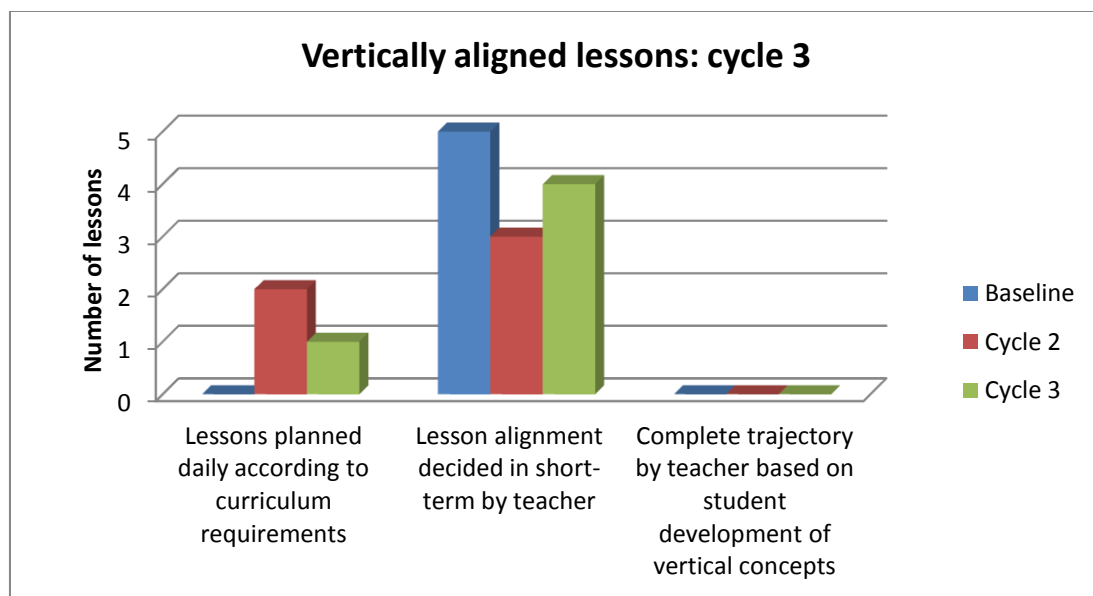
Figure 5.52 Vertically aligned lessons: cycle 3

Teachers were still planning lessons based on the curriculum requirements but were deciding within that framework how to best align specific activities and concept development. Four

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teachers designed their own activity based lesson in order to improve student understanding of the concepts. Teachers' ability to vertically align lessons may be linked to their own connection knowledge within concepts and topics in the curriculum. Their responses to the question "why is today's concept important for future understanding of mathematics?":

Lesson	Why is today's concept important for future understanding of mathematics?
Mrs A Gr. 6	*Learners need to know how to interpret word problems (real life situations, to a certain extent) and implement a solution to get an answer.*
Mrs B Gr. 5	*That they will know how to work with 3D shapes and see how they are made up.*
Mrs C. Gr 6.	*They will need this for factorization in high school.* (Algebra)
Mrs D. Gr. 6	*The will be able to apply their knowledge of 2D and 3D shapes to contextual problems.*
Mrs E Gr. 5	*Fractions are covered in maths a lot. A good understanding of this means that you can build on this concept easily and hopefully with some success.*

Table 5.20 Teacher alignment of concepts   (LQ3, 04-13)

Table 5.20 reveals that although teachers have good knowledge of short-term vertical alignment, their knowledge or application of their knowledge to alignment may need development. Mrs E's testament that fractions require a good understanding may have resulted in her lesson that had a high level of connections and constructions.

5.4.6.10 Summary: cycle 3
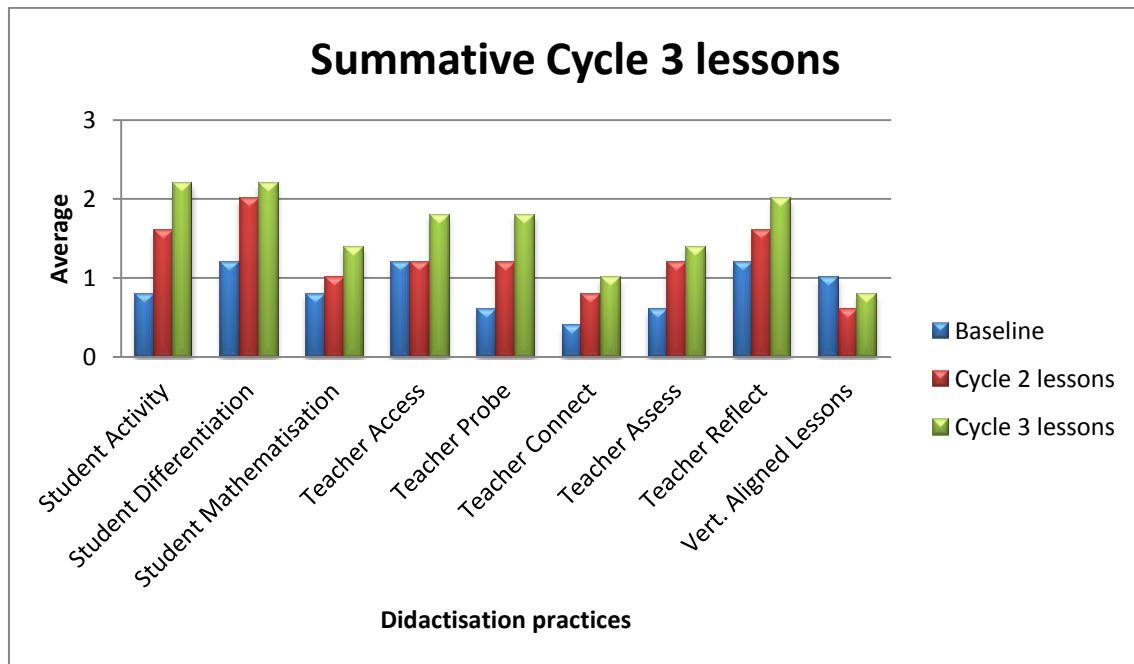
Figure 5.53 Summative classroom observation: cycle 3

From the above visual representation (Fig. 5.53) of the overall didactisation practices a number of changes have taken place. The number and variety of activities students were involved in has increased in all classrooms. This would have a direct impact on the differentiation spectrum for students of varying abilities to be involved in and would therefore increase the potential for vertical mathematisation in the lesson. A number of the didactisation practices have remained at the same level. This should not be viewed negatively since real change is incremental and slow. Teacher capacity to align lessons vertically is one didactisation practice that may need support. Two main factors need to be taken into consideration, firstly, the tightly scheduled curriculum that teachers are compelled to follow and secondly, the nature of their own mathematical content knowledge and orientations. The figure also shows that the aims and objectives in 1.3 (1.3.1.2, 1.3.2.4 and 1.3.2.6) are being met.

Looking more carefully at the graph – although student activity has improved, some of the follow-on didactisation practices are more 'difficult' to develop. Creating the type of activities that lend themselves to better differentiation and increased mathematisation are the first step

in improving didactisation practices. The other practices such as connecting student ideas and reflection from "shifting standpoints" (Freudenthal 1991: 05) are more difficult and require more experience and intervention. The progression of didactisation practices as observed across the five teachers is presented in 5.5.7 as it became evident that the practices were developing in a similar way.

5.4.7 Developmental changes during cycle 3

Teachers met with the researcher for session 7 where a new modelling problem (Hire or Fire) was presented to them (see Appendix 5). Teachers worked on the problem while the researcher led the discussion on how this problem could be used in a classroom. The researcher also presented a short talk on proportional reasoning and why students (and adults) find this so difficult. Teachers worked through some shorter problems (in Van De Walle *et al*. 2010: 352-355) in order to understand the difference between additive and multiplicative reasoning and how to foster multiplicative reasoning in their classrooms. Mrs B specifically asked:

*but **how** do I get them to bridge the gap?* (ST7, 05-13)

By the end of the session, she felt that the scaffolding presented by the vertical alignment of problems in the session could be used in her classroom. The five teachers were pondering how to assist students in the vertical shift from additive reasoning to multiplicative reasoning. The summary of research (Van De Walle *et al*. 2010: 350) was presented to them and a number of other proportional reasoning problems in many different contexts were discussed. The teachers indicated that some of their students would feel overwhelmed by the amount of data in the modelling problem and would not know where to start. Some teachers took down notes when the researcher showed how some problems could be accessed via mathematics that the students know (e.g. fractions or percentages).

During session 8, eight students worked in groups to solve the same modelling problem. At first some of the teachers were skeptical about how successful students would be. At the end of the 45 minutes allocated to the problem both groups had managed to produce a model for their solutions. It must be noted however that both groups had not considered the qualitative aspect of the numbers given in the Hire or Fire problem. The students also successfully solved

some of the shorter problems presented to the teachers in session 7. The researcher facilitated these problems with the groups. Teachers observed this interaction and answered a questionnaire on this session. The last session of the program included providing teachers with a number of problem-based ideas for various topics in the curriculum. Teachers indicated to the researcher that they needed assistance in finding problems that covered the curriculum topics. The researcher also provided some alternative forms of representations that teachers could include in teaching of fractions.

5.4.8 Changing resources, orientations and goals during cycle 3

The clustering of didactisation principles using instrument 3 is presented for the lesson observations. These graphs present a summary of the development of didactisation principles and also provided the researcher with an opportunity to cross-check the data gathered from the other instruments. It was found that the data remained consistent across the instruments. At this point in the program, teachers seem to be moving towards a joint-action approach (see 2.2.2.3). They are considering student thinking as a resource for their lessons and are starting to provide an activity scaffolding to elicit student responses. It may therefore be possible that once teachers see the positive impact that this may have, that they will begin to change their beliefs in a significant way as suggested by Guskey (1986: 7). When evaluating teachers using the CGI scale (see 2.3.1.2.1) these teachers are working confidently on level 2 with some teachers working on level 3.
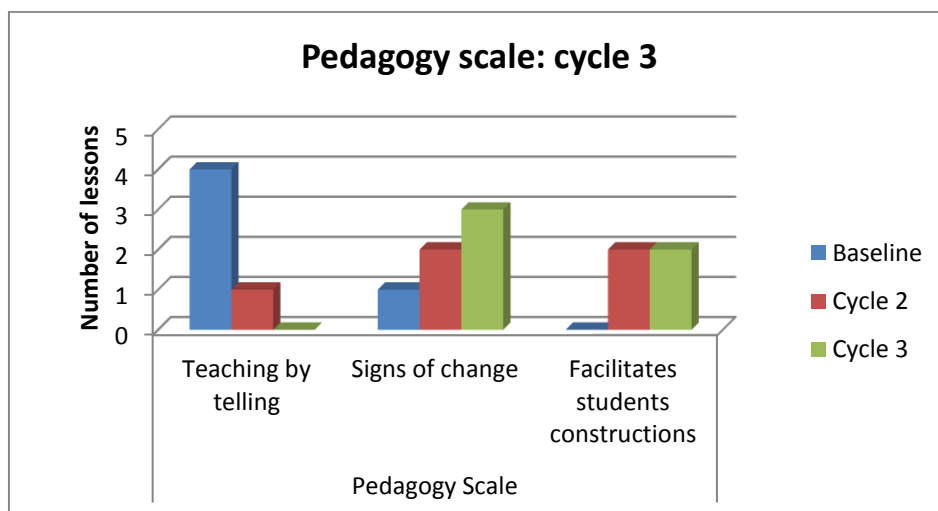
Figure 5.54 Pedagogy scale: cycle 3

The third cycle shows a move away from teachers' teaching by telling and a greater focus on asking students to explain their thinking. Students are spending more time working with each other and verifying their work with each other. Mrs B and Mrs E's lessons include questions that stimulate student thinking rather than seek specific answers.
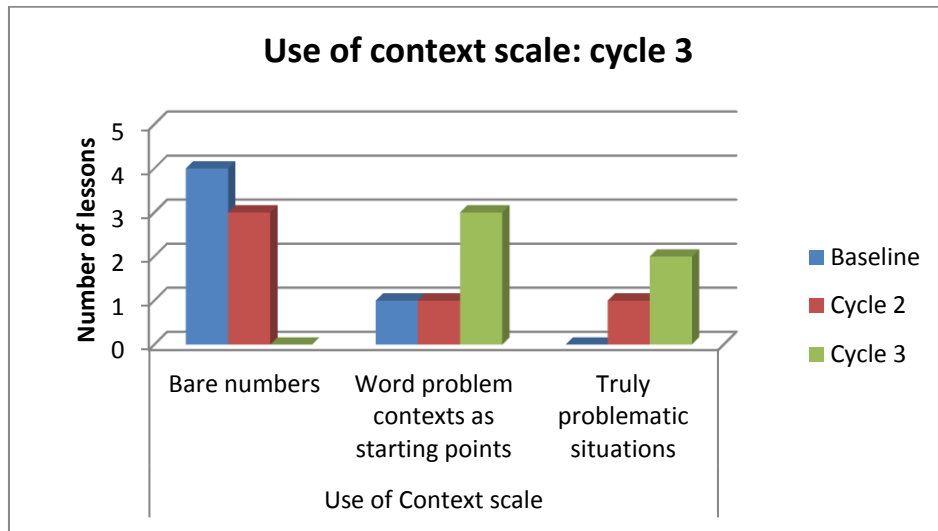
Figure 5.55 Use of context scale: cycle 3

In this cycle, all lessons involved a 'starting situation' although Mrs C and Mrs D's lesson remained within formal mathematics. Since the situation resulted in reflective thinking in students, they were placed in the second set.
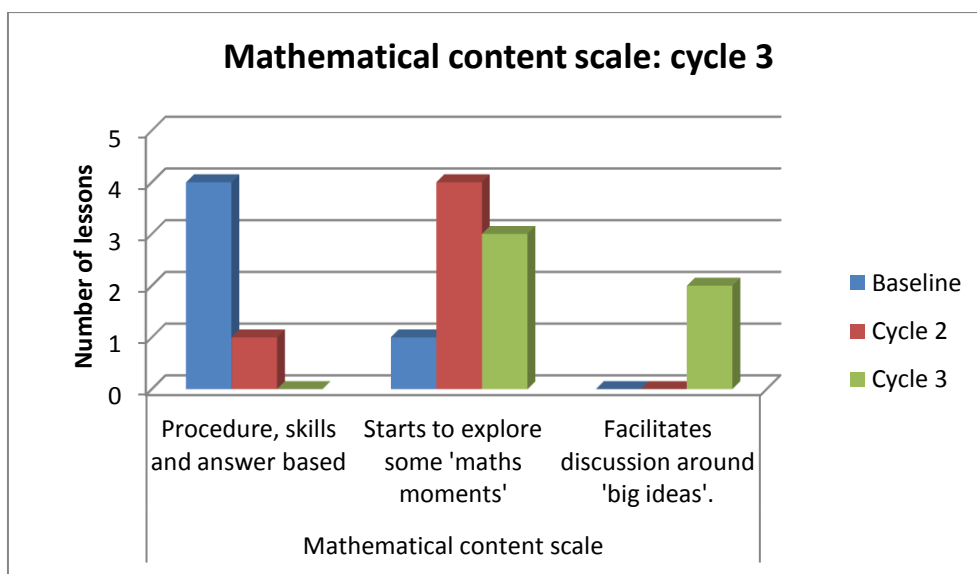
Figure 5.56 Mathematical content scale: cycle 3

Lessons in this cycle reflected many 'math moments' due to the increase in student activity and teacher questioning. Mrs C and Mrs E's lessons involved a development of some of the 'big ideas' from the students' perspective (fraction sharing and prime factorisation). A modelling perspective also assists in gauging teachers' development. A modelling perspective provides an alternative analysis of the lessons. This assists by providing more vocabulary to discuss the lessons and to validate other data collected. This alternative perspective is important in providing a holistic, multi-dimensional analysis of lessons. When these lessons are analysed from a modelling perspective they display the following characteristics:

Design principle	Baseline lessons	Cycle 2 lessons	Cycle 3 lessons
1. **Reality principle**. Real life situations are used and students need to make sense of these situation.	Mrs A used this to start the lesson.	No lessons involved real life situations.	Reality based lessons: Mrs E Contextually richer lessons where contexts acted as models: Mrs A,B,C and D.
2**. Model construction principle**. Are students involved in constructing, describing, extending or explaining a structurally significant system?	None. Students were involved in the "working mathematically" part of the modelling cycle (see Fig 2.2)	Mrs B, Mrs C and Mrs E's lessons involved describing and explaining systems.	Mrs B's lesson involved building a 3D model, Mrs C's lesson included a model of whole number prime factorisation, Mrs E's lesson involved constructing a model for sharing that involved fractions.
3**. Self evaluation principle**. Students able to or were expected to judge their own responses.	None. Validation was done by the teacher.	Students could judge their responses in all the lessons. However final validation was still the teachers' function.	Mrs A, B, C and D's lessons: students were able to judge their own responses since pair work played a role. In Mrs E's lesson students were fully in control of judging their responses.

4. **Documentation principle**. What type of documentation was expected? Did students have to explain their thinking?	Calculations and answers were written down in books.	Mrs C's lesson involved students using a combination of writing and manipulatives in their explanations, while Mrs E's lesson involved writing the structure of the pattern in words.	Writing down of procedures and answers in workbooks (Mrs A and Mrs C). Making drawings on blank sheets of paper (Mrs B and Mrs C). Constructing (Mrs A) Cutting, organising (Mrs E). In four lessons students had to explain their thinking.
5. **Simple prototype principle**. Will the experiences assist students in making sense of other structurally similar situations?	All lessons had vertical mathematisation elements that meant students would have to memorise a procedure.	All lessons involved students identifying an underlying structure through a strong emphasis on vertical mathematisation which may lead to meaningful memorisation. Mrs A – noticing equivalence; Mrs B – lesson may lead to flexibility with inverse operations; Mrs E – structural understanding of patterns.	Mrs A: students applying known mathematical operation. Mrs B: vertical progression allows for connections to similar situations. Mrs C: students' vertical progression to prime factorisation will allow algebraic factorisation. Mrs D: connections between individual and complex shapes. Mrs E: connections through differentiation may lead to studies developing analogies.
6. **Generalisation principle**. What constructions can be modified and applied?	Mrs E's lesson (a model of multiplication of multiples of 10 or 100) involved constructions could be modified and applied.	All lessons involved reflection on the underlying structure (e.g. equivalence, pattern defining and describing) that could assist in generalising.	Mrs A: multiplication procedural fluency Mrs B, C and D: Reflection on connections may promote generalisation.

Table 5.21 Modelling design principles: cycle 3

During session 8, teachers observed the groups of students working through modelling problems. They were asked to complete a questionnaire during this session. Some of the teachers stated the benefits of a problem-centred approach are that students could:

"*Explain if there are uncertainties*" (S8R, C, 05-13)

"*Help each other*" (S8R, E, 05-13)

"*Participate, they get to discuss and negotiate the answer*" (S8R, D, 05-13)

"*They develop their thinking by discussing the problems. Listening to others and building ideas. This is very valuable and children must be exposed to it*" (S8R, B, 05-13).

While Mrs A said:

"*After understanding the idea behind the concept, learners are aware of what to do with the 'raw' numbers*" (S8R, A, 05-13).

These comments reflect a possible change in teacher beliefs and orientations (see 1.3.2.5 and 2.3.3) regarding what students do know about mathematics. Teachers here see that students arrive in their classrooms with their own ideas and thoughts that can be used to further their mathematical understandings. It may also show that their beliefs about being the fountain of knowledge in a classroom may be changing.

Teachers also identified the following challenges that still exist in their schools, classrooms or themselves that make it difficult for them to implement some of the didactisation practices proposed by this study. All five teachers identified smaller class sizes as a school-based challenge (class sizes are determined by the department of education and not by the school) because of the noise levels involved, while three stated that the volume of work set out in the curriculum was still a challenge.

When asked what changes they may still need to make in *themselves,* teachers wrote the following which can be seen as a development of their teaching goals:

"*to have the patience not to give groups who are struggling the answer, but to guide them*" (S8R, C, 05-13)

"*patience with their methods/ideas – not to <u>tell them</u> how to work out the answer. Talk less and listen more*" (S8R, E, 05-13)

"*facilitate more and control less*" (S8R, E, 05-13)

"*adapt my way of thinking and putting it into practice*" (S8R, B, 05-13)

These comments also show that teachers are becoming more reflective (see 3.3.1.5) of themselves in this study and not only reflective on their teaching in terms of outside factors.

Teachers were also asked to compare the problems that the groups worked through during the session with traditional textbook problems (bare numbers without a context). They were asked to comment about accessibility to the problem by all levels of thinking. Their responses show a shift in the value that solving problems may have for weaker students. This may be seen as a shift in their differentiation practices:

"*The weaker learners will have a change to understand*" (S8R, C, 05-13).

"*The focus is more on reasoning and thinking; all learners will be able to give some input*" (S8R, E, 05-13).

"*Weaker learners can also share their ideas and the discussion can help them structure their own thoughts – some guidance will still be required*" (S8R, B, 05-13)

"*Learners formulate methods that make sense to them … learners are met and guided on their levels*" (S8R, A, 05-13).

"*Builds mathematical understanding, in such a way to allow for the understanding to be moulded, developed and built on*" (S8R, A, 05-13).

Teachers are starting to consider the implication that problem-centred learning would have on mathematical understanding. They remain concerned about the practical school implications such as length of each period, size classes, noise levels and covering the required curriculum.

5.5 FINAL CLASSROOM OBSERVATIONS

The final classroom observations took place either just before a mid-year exam or just after the schools' mid-year exams. Teachers were either in a revision cycle or were trying to catch up sections of work before the winter vacation. The final observations allow one to gauge to what extent the program has met the aims set out in 1.3, specifically 1.3.1.2, 1.3.2.4, 1.3.2.5 1.3.2.6 and 1.3.2.6. Figure 5.67 will allow one to determine to what extent didactisation practices have developed, to what extent the researcher has documented this development

through capturing the data in the design experiment as well as considering the changes to teachers' resources, orientations and goals.

5.5.1 Final classroom observation - Mrs A (Grade 6)

Mrs A's class was ordering decimal numbers. Students were working in pairs on an activity where she presented them with decimal numbers which they had to arrange in ascending order. They were working in their books and each student in the pair was responsible for their own work but they were encouraged to discuss what they were doing and why they were doing it. Each pair made up part of a bigger group of about 8 students. Mrs A stated that she had arranged these groups for the exam revision games. The exam was over and she decided to keep the group arrangement. Each group of 8 comprised one strong student, one weaker student with mixed ability students making up the rest. Each pair had to report to the bigger group and one person was selected by the group to show and explain their solutions. Mrs A allowed 15 minutes for the activity. Mrs A moved around the groups asking questions such as:

"Are you sure that this is not going to help you?" (LON4, B, 06-2013)

She then asked each group reporter to present their solution to the class but also to explain where and why they had placed each number in the order. Mrs A was very particular in making sure that their explanations were mathematically correct and clear.

Although the lesson was based on bare numbers, Mrs A created an environment where students had to communicate their understandings. When the pairs reported back to the bigger group, any discrepancy had to be negotiated within the bigger group. The bigger groups also tended to select the better communicators to present the solution to the class.

The lesson may also show that, when under pressure, teachers' goals and orientations may push them back to a more traditional teaching approach. However, the inclusion of collaborative group work and presentations to the class elevate the meaningfulness of the lesson.

5.5.2 Final classroom observation - Mrs B (Grade 5)

Mrs B presented a single realistic problem to her class. The problem was from her textbook. It presented a shopping list together with cost prices for the items and what the items were sold

for. The textbook problem then required students to calculate some of the profit margins of individual items where a simple selling price, minus cost price, would be required. Mrs B gave students the information but she raised the complexity of the problem. She added in that the items were repackaged into smaller quantities and then sold.

Items bought:
- 2kg (peanuts and raisins) @ R16-00.

She repackaged this into 50g packets and sold them at R4-00 each
- 100 x 15g packets sherbet @ R10-00

She sold each packet for R2-00.

Which item would give her the most profit if she sold all the packets?

Table 5.22 Mrs B's partner problem

She asked students to calculate which item out of two on the list would bring in the most profit if they were repackaged in smaller quantities and then sold. She also asked students to work in pairs on the problem and provided them with a sheet of blank paper to do their calculations. This is the first time Mrs B presented partner work for a lesson observation. Mrs B then moved around the class asking questions and assisting groups. One or two pairs were unsure where to start. Mrs B explained the context of the problem to them again. She assisted them to visualise a big packet of peanuts and raisins which would be repackaged into smaller bags. Some groups used repeated subtraction while others used division to calculate how many smaller packets could be made.

At the end of the lesson Mrs B called the class to attention and presented the solution in a step-by-step manner because time had run out.

Mrs B: *ok everyone, look at me, everyone pens down. I am very proud of all of you, you really did well.*

Student1: *yes!*

Mrs B: *Let me give you the solution. It is very important that when we do these problems that you settle your mind and ask "Where am I going to start?" Many of you, when I give you the first little clue, then you can run with it. That's fantastic, but in time to come you will have to see that through for yourself. Let's look at the problem...*

Her presentation of the solution was carefully explained and she spent much time talking about how items are repackaged and how each packet would be sold. She tried to show them how to have an internal dialogue with themselves.

Mrs B*: I wanted you to think about how many 50g packets can I make if I have 2 kilograms? In order for you to have done that you would have said "there are a thousand grams in a kilogram, ok, so I have got 2000grams, so now I have to divide". Remember that when we share, actually what we are doing here is sharing out into 50 gram packets.*

Ideally the lesson could have ended with the students presenting their ideas for solving the multi-step problem. This lesson presented students with both horizontal and vertical mathematisation possibilities. Students collaborated with their partner and were able to work on the problem independently of the teacher.

The weaker groups managed to correctly calculate 3 out of the 7 steps involved before the end of the lesson and were particularly pleased with themselves. This lesson may have needed more time for students to work and for a class discussion at the end.

5.5.3 Final classroom observation - Mrs C (Grade 6)

Mrs C had reached the point in the curriculum where she had to teach decimal numbers. Mrs C presented a modelling task that she adapted so that it could be completed within the 40 minute lesson. The problem involved the correct ordering of decimal times given for 100m and 800m races. She presented students with all the data together with her allocation of the first five places for both races. The task instruction indicated that students were unhappy about the results and were querying her ranking system. She asked them to look at the data, and in groups, answer the following five questions:

1. Were the students correct in querying her placement of top five positions?

2. What mistake (if any) had she made?

3. If necessary – write down who the top five for each race are

4. Who should the overall winner for the track events be?

5. Should Mrs C be used as a placement judge again?

The students got to work very eagerly on this task. She also said to them that:

"I am not happy that my work has being queried, so I don't want to be part of your decisions"

(LON4, C, 06-13) which worked very well in getting the message to the students that they were to work on their own.

There were some heated discussions within the groups as some students realised that the biggest decimal did not mean the student had won the race, but had actually come last. Many of the group responses to question 2 were:

Group 1: *The mistake was putting the lowest time to the highest* (position).

Group 2: *You put the slowest people down first.*

You put the highest five times and not the lowest. (LON4, C, 06-13)

The groups successfully completed the task within the time allocated and were all willing to take part in the wrap-up discussion at the end of the lesson. In a discussion later, Mrs C said: "*My next section in decimals is to order and round off decimals – it should be a breeze*" (ID, C, 06-2013)

Later that same week, Mrs C informed the researcher that in one of the classes she thought she would do the ordering and rounding off textbook activity first and then go onto the modelling problem. She reported that the students found the textbook activity difficult while the other classes who had completed the modelling task first had not experienced any difficulty. She stopped the lesson, gave them the modelling problem and then set them back onto the textbook work. She reported that after doing the modelling problem the students could complete the textbook activity easily.

The mathematisation opportunities in this lesson were enhanced by the modelling problem and by the teacher allowing the students to do the mathematical work. The lesson was rich in both horizontal and vertical mathematisation. The last two questions were truly open questions where groups could give a variety of responses. This is where connecting student ideas is a skillful undertaking. Teachers need to contrast, separate, fuse and generalise (Guo & Pang 2011: 4) student ideas when connecting.

5.5.4 Final classroom observation - Mrs D (Grade 6)

Mrs D was doing revision of the four operations for the examination the following next. Mrs D started the lesson by arranging students in the classroom. She paired students in combinations so that a weaker and stronger student would work together. She then handed out two word problem sheets, a set A and a set B. The problems on the sheets were matched in that the context for each problem was the same but the numbers were different. The word problems included all four operations. She then asked students to work in pairs to solve the

problems on their sheet. This was the first time Mrs D presented partner work for the purposes of observation. The students were very excited about the lesson. They worked diligently on the problems. Mrs D stated that during the following lesson – she would ask students to present their working and solutions. She would alternate a problem from set A and the corresponding one from set B and see if the students could see that the two problems were related and could be solved in the same way. She was hoping that the students would see the connection between the two problems. This lesson allowed for different levels of students to access the problems. Some students used repeated addition while others multiplied. One group suggested that they each work independently and then compare their answers. Mrs D moved around the class where most students were collaborating with their partners.

5.5.5 Final classroom observation - Mrs E (Grade 5)

Mrs E started the section of work on 3D shapes by recapping the names of the 2D shapes that the class had previously dealt with by asking the class to name as many as they could remember. Most students knew the names of 2D shapes required for grade 5. Mrs E then handed out a number of large plastic cubes to the students. She asked them to hold them and look at them for a few minutes. She then asked them what would happen if she cut the edges of the cube open and laid it flat.

Mrs E: *If I take a knife and I cut it here along the top and along the sides and I lie it flat on the paper.*

Some students: *oohh...*

Mrs E*: I want you to image what it would look like flat on your paper. How would you draw that? On your desk you've got a ruler and a pencil. You are going to draw what you think the cube will look like if I cut it open and I put it flat on the paper. Once it's flat on the paper, you must be able to cut out what you have drawn on the paper and build the cube from what you have drawn on the paper.*   (LON4, E, 06-13).

She asked them to imagine what it would look like? She gave them an opportunity to talk to each other about it. Mrs E then walked around watching what students were drawing and asking them questions about their shapes. Some students requested a new sheet of paper since they did not plan their drawings before starting out. Mrs E was able to gauge very quickly from this lesson which students appeared to have spatial and perceptual difficulty with 3D

shapes. Most students had little difficulty (other than the accuracy of their drawings) in creating the net. They had made boxes in the previous year in the Technology class. Mrs E then asked the class if they thought their drawings would result in constructing perfect cubes. They indicated that their drawings were not really good enough. She then handed out a net of a cube printed on a sheet of cardboard and asked them to compare their drawing to the one she gave them:

"Look carefully at what is the same and what is different?" (LON4, E, 06-13). The discussion got louder at this point with pairs excitedly comparing their own nets to the given one.

She then asked them to construct their cube and the cube she had given them. They spent the rest of the period enthusiastically constructing cubes. She noted in her lesson reflection:

"They love doing anything that is not 'bookwork'". I hope they remember the maths and not only the fun" (LQ4, E, 06-13). This comment from Mrs E also reflects what may be contradictions in her beliefs about 'bookwork' and 'fun'. However she was happy that the lesson did reach the desired mathematical objective for her students.

Mrs E indicated that in the following lesson she would turn to the textbook activity on 3D cubes. Although Mrs E anticipated that her students' nets would not produce good cubes, she allowed the activity so that students could make the comparisons between the nets.

5.5.6 Changes in didactisation practices - overview

The program undertaken in this study took place over the period of 9 months. The changes in teacher didactisation practices are now presented from the baseline classroom visit to the final classroom observation. The context of the final classroom observations provides a platform to test the robustness of the development of didactisation practices. All teachers were either preparing students for a midyear examination or had completed the exams and were attempting to complete the required sections of work. They were under more time pressure than usual; this meant that in terms of their beliefs and orientations they could revert to teaching styles that were familiar to them.

5.5.6.1 Student Activity: baseline to final
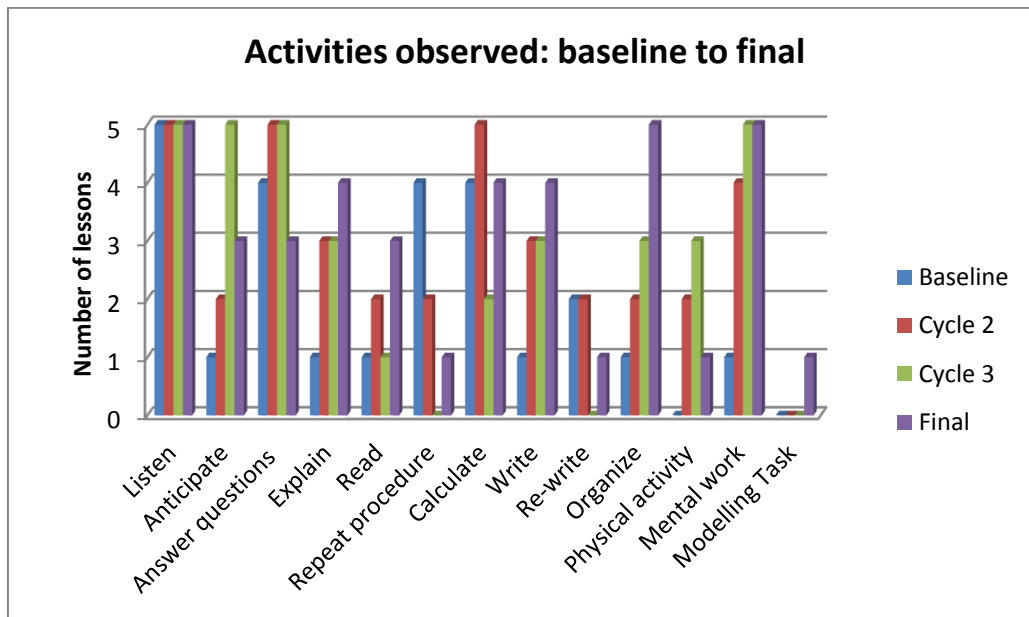
Figure 5.57 Activities observed: baseline to final lessons

Student activity showed the greatest improvement and this was evident from the baseline lessons where students were seated and silent, to the inclusion of problems and modelling tasks in the later lessons. Students were given greater responsibilities in the latter lessons – this is evident in the increase in 'explain' and 'organise' columns. As teachers' questioning and probing developed, so the increase in student anticipation as an activity increased. This as the suggestion in 2.4.4.6 is teachers developing a 'listening orientation'. In only one of the 15 session visits that took place after the baseline visits did students have to simply repeat a given procedure. In this particular lesson students were paired together and were allowed to discuss their working and had to present and explain their answer.
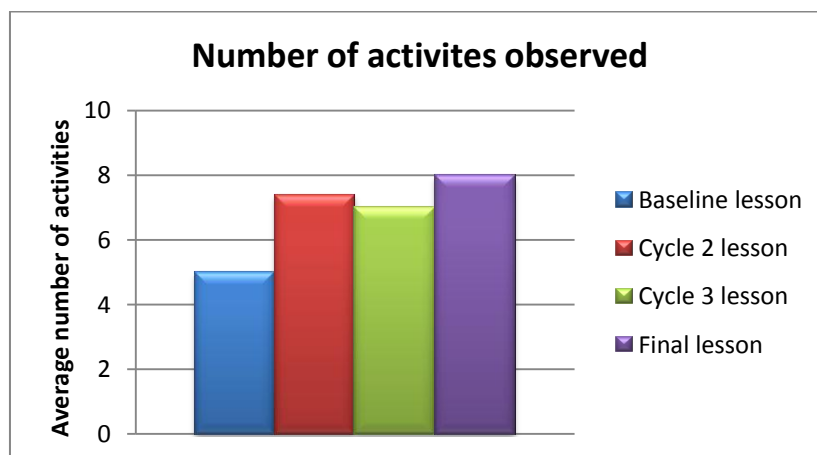
Figure 5.58 Average number of activities: baseline to final lessons

The average number of activities that students are involved in during lessons has remained consistent through the cycle lessons. Care must be taken not to assume that more is better. It is the quality of the activity and its potential for mathematising that is important. Challenging problems such as modelling do include a number of competencies for students (Biccard 2010: 67). In the cycle lessons teachers have included more for students to do. This means that the students are taking on more responsibility for the mathematical work. This is concurrent with Brousseau's concept of 'devolution' of a problem (see 2.2.2.1). From this increased activity, students may be able to create their own understanding as suggested by a constructivist framework (see 2.2).

5.5.6.2 Differentiation: baseline to final
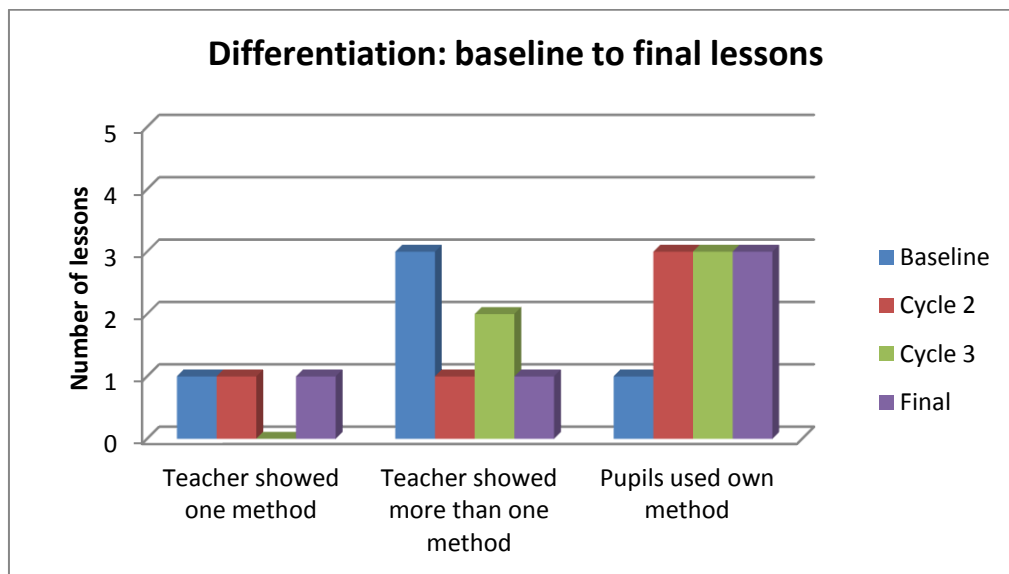
Figure 5.59 Differentiation: baseline to final lessons

All teachers allow different methods to be used by students. The prescribed textbooks also present various algorithms for calculation. Although students were allowed to calculate in their own way – teachers did not always ask students to present their methods or explain what they were doing. There was always a sense of time constraints that impeded this. This may contribute to the lower development of teacher connecting the different ideas in the classroom as can be seen in Figure 5.68.

5.5.6.3 Mathematisation: baseline to final
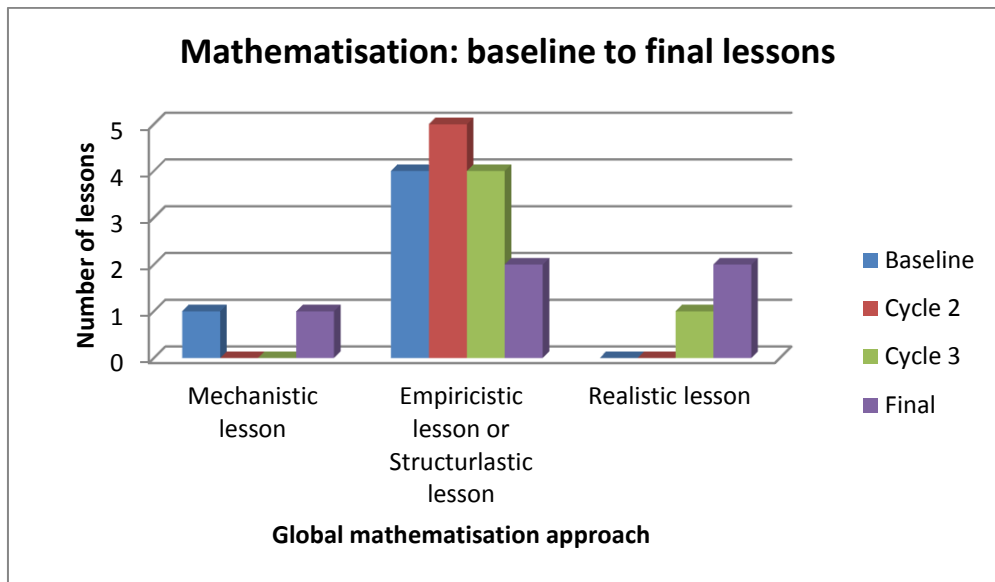
Figure 5.60 Mathematisation: baseline to final lessons

Mathematisation is considered the pinnacle of didactisation if the definition of Treffers is followed that didactisation is teacher action that allows mathematisation to take place. Mathematisation is the reason for mathematics lessons. Although the graph presents the avenues and opportunities for mathematisation it cannot show the moment-by-moment mathematisation that occurs in each lesson. Mathematisation can occur in any of the three types of lessons outlined here. It is the quality, robustness and meaningfulness of the learning that changes. There is a time and place for each of the three types of lessons. The increase in teachers facilitating realistic lessons is encouraging. The development of the other didactisation practices points to increasing mathematisation within the lessons observed. The representations that were evident in this lesson are:

Lesson	Representations	Teacher or student use	Is student representation constructed or applied?
Mrs A Gr. 6	Verbal Written	Teacher and students Teacher and students	Constructed Applied
Mrs B Gr. 5	Verbal Written Pictures	Teacher and students Teacher and students Teacher	Constructed Constructed -

Mrs C Gr. 6	Verbal	Teacher and students	Constructed
	Written	Teacher and students	Constructed
	Tables and lists	Teacher and students	Constructed
Mrs D Gr. 6	Verbal	Teacher and students	Constructed
	Written	Teacher and students	Applied
Mrs E Gr. 5	Verbal	Teacher and students	Constructed
	Written (drawings)	Students	Constructed
	3D manipulatives	Teacher	-

Table 5.23 Lesson representations: final lessons

As suggested earlier by Cramer (see 3.2.1.3 b) there is a connection between deep understandings and they are related to the connections between different representations. The didactisation practice of teacher connections is one that will need further research and development. It may rest on the type of mathematical knowledge that teacher has – not *what* they know but *how* they know it. This falls outside the scope of this study but will become one of the recommendations of this study (see 6.5). However, single representations as is common in traditional classrooms have decreased. What may need to be considered and strengthened in the lessons is Hiebert and Carpenter's (1992: 69) suggestion that representations should be connected and organised so that they facilitate mathematical understanding.

In considering the horizontal and vertical mathematisation within these lessons, the following summary applies:

Lesson	**Horizontal or vertical mathematisation**	**Gravemeijer's strategies that promote mathematisation**
Mrs A Gr. 6	Horizontal mathematisation: familiar context of decimal numbers to project place value understanding. Vertical mathematisation: reflecting on place value structure of decimal notation. Explaining the process of ordering numbers.	Generality, brevity, certainty
Mrs B Gr. 5	Horizontal mathematisation: context of buying in bulk and repackaging to sell smaller. Vertical mathematisation: structuring and symbolizing multiple operations. Reflection on concept of 'most profitable'. Re-invention of profit formula.	Certainty, exactness and brevity. Initial calculations are a model *of* the situation. Students (with teacher assisted reflection) create a model *for* calculating profit from sales.
Mrs C Gr. 6	Horizontal mathematisation: results of races, finding the winners mathematically. Vertical mathematisation: ordering,	Certainty, exactness, generalizing and brevity. Analysis of the contextual

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	structuring and explaining a complex system. Re-invention of place value.	problem is the model *of* the situation while reflecting on this creates a model *for* comparing decimal numbers.
Mrs D. Gr. 6	Horizontal mathematisation: word problem situations. Vertical mathematisation: comparing two similar problems with differing contexts.	Generality and certainty.
Mrs E Gr. 5	Horizontal mathematisation: 3D shape provided a context that acted as a model. Vertical mathematisation: considering the 3D shape abstractly, reflecting on the implicit properties of the shape. Re-invention of properties of cubes.	Certainty and exactness.

Table 5.24 Mathematisation analysis: final lessons

Finally, a comparison of lesson development in terms of Gravemeijer's models of lessons is presented. The numbering of the lessons is as follows; baseline lesson (1), cycle 2 lesson (2), cycle 3 lesson (3) and final lesson (4).

Teacher	Structuralist lesson	Pre-intermediate lesson	Intermediate lesson	Realistic lesson
A	1	2	3 and 4	
B	1 and 2		3	4
C	1	2 and 3		4
D	1	2	3 and 4	
E	1	2	4	3

Table 5.25 Lesson models: baseline to final

Teachers were including a much stronger situational component in their latter lessons. It would be of benefit to teachers if the development program had run for a longer time period in order to gauge if teachers adopt a realistic type lesson for more lessons.

5.5.6.4 Accessing student thinking: baseline to final
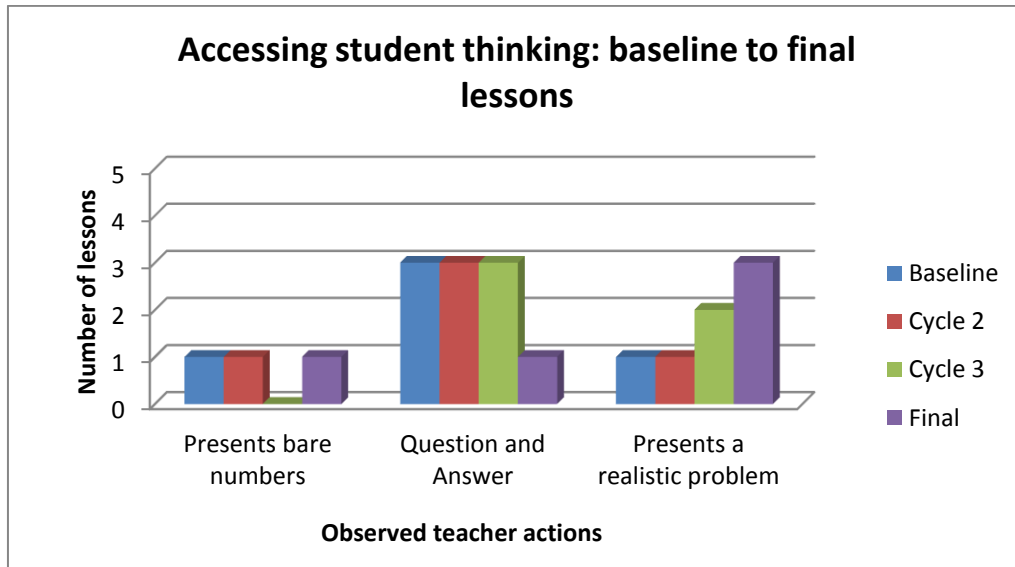
Figure 5.61 Accessing student thinking: baseline to final

Students' being involved in a variety of activities assists in accessing their thinking. While students are busy on a problem, it allows the teacher time to observe what students are doing. All teachers spent less time at the front of the classroom and more time moving around the classroom in latter lessons. Teachers are being guided by the question and answer sessions as well as student responses to problems.

5.5.6.5 Teacher probing: baseline to final
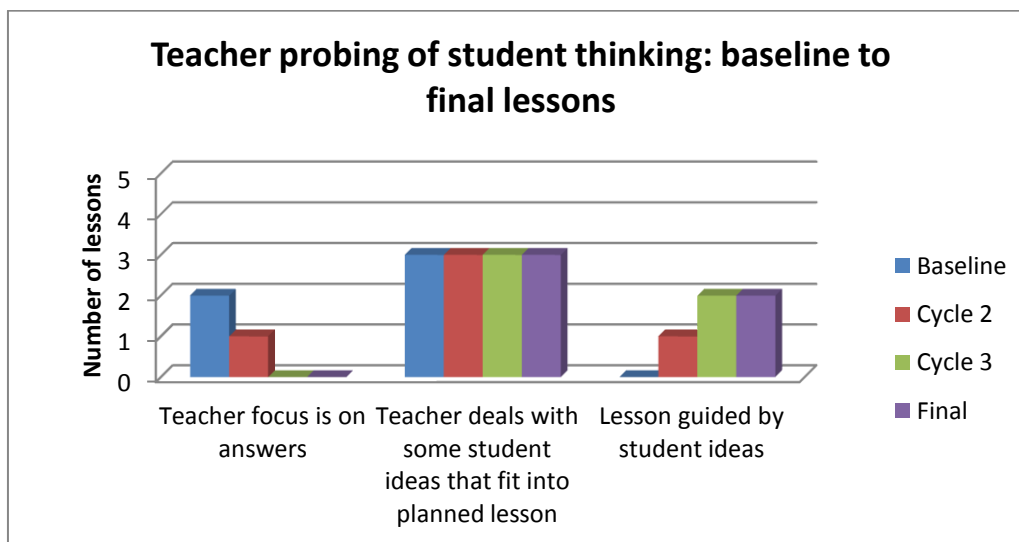
Figure 5.62 Teacher probing of student thinking: baseline to final lessons

This practice has remained constant for the latter lessons after the baseline lesson where teachers' type of questioning has allowed them to engage more with student thinking in their classrooms. During the latter part of the lessons, teacher probing roles were considered and compared across the four lessons.

Teacher Probing role	Teacher as supervisor of the activity. Answers questions or clarifies if students ask.	Teacher as director or manager. Initiates discussion, controls the topic. Allows or invites input.	Teacher as facilitator. Sets up structure, interacts with students. Students interact with each other and materials.
A Baseline	X		
B Baseline	X		
C Baseline	X		
D Baseline	X		
E Baseline	X		
A Cycle 2			X
B Cycle 2	X		
C Cycle 2			X
D Cycle 2		X	
E Cycle 2		X	
A Cycle 3			X
B Cycle 3		X	
C Cycle 3		X	
D Cycle 3		X	
E Cycle 3			X
A Final			X
B Final			X
C Final			X
D Final			X
E Final			X

Table 5.26 Teacher probing: baseline to final lessons

Teachers moved from their initial roles to a combination of roles during the later lesson observations. It is encouraging to note this development in teacher probing. Once students are challenged to think about what they are doing, it increases the scope for group activity and to focus the group. It also generates a wider scope for connecting ideas whether it is a self-

connection on the part of the individual students or the group or for the teacher to consider in the wrap-up discussions.

5.5.6.6 Connecting student ideas: baseline to final
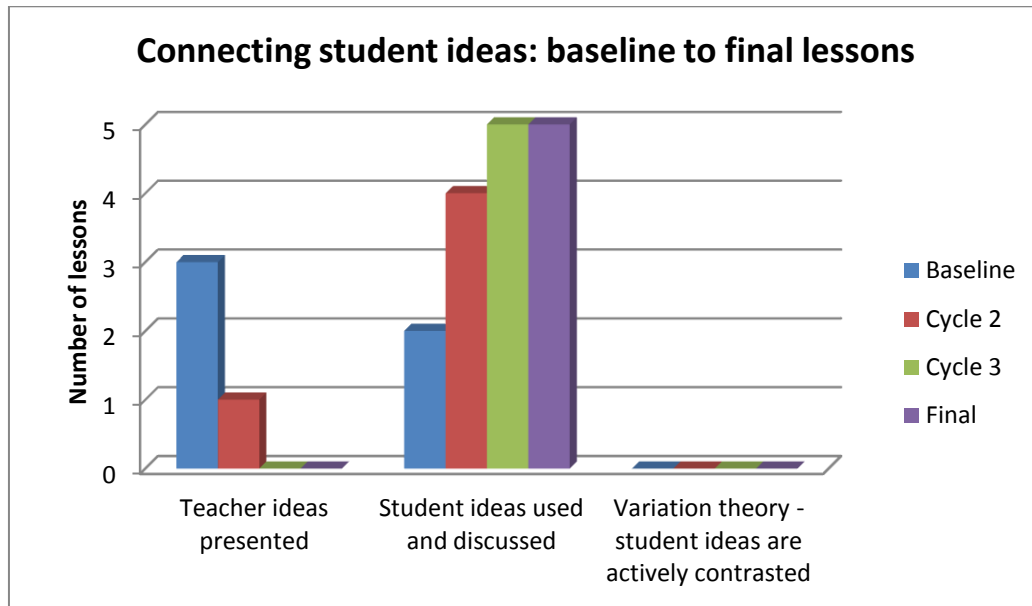
Figure 5.63 Connecting student ideas: baseline to final lessons

This didactisation practice appears to be more difficult to develop or may need longer time or more focus in a professional development program. A number of factors could have contributed to this: a shortcoming in the program, a limitation in how teachers know their mathematical knowledge or not enough experience in how to skillfully engage with student ideas and connect them. This study added this practice to Treffers' didactisation principles as well as Wilson and Heid's mathematical work of teaching (see 3.2.1.3). It appears to be a challenging area for primary school mathematics teachers. Although teachers were including student ideas by asking more pertinent questions in their lessons, the ideas were put on the table and not explored further. Teachers may feel that time constraints, or possibly 'confusing' weaker students, stops them from fully exploring connections. Another factor that needs to be recognised is teachers' own connections in their mathematical knowledge. How did teachers themselves come about their mathematical knowledge? In-service primary school teachers did not necessarily need a mathematics course as part of their degree or diploma. Many become mathematics teachers when the school requires one and they offer or are

assigned the class. The concept of didactical transposition (see 2.2.2.2) assists one in understanding that the 'type' of mathematical knowledge that enters the classroom, whether it be scholarly teacher knowledge or mathematical knowledge to be taught in the education system. However, in primary school mathematics classrooms, it may be necessary to include teachers already learnt mathematical knowledge at the beginning of the cycle of didactic transposition. It may be necessary to run a mathematics course that focuses on connections between the big ideas in mathematics. Hiebert *et al*. (1997: 4) remind us that "we understand something if we see how it is related or connected to other things we know" while Randall (2005: 12) explains that "Big Ideas have connections to many other ideas, understanding Big Ideas develops a deep understanding of mathematics".

From a modelling perspective connections are forged when students shift their thinking from one model to another, so evidence is needed that students are "thinking in terms of: (i) different elements, (ii) different relations, (iii) different operations, or (iv) different patterns and regularities" (Lesh & Carmona 2003: 96). Connections, therefore, also relates to "local conceptual developments" where models students develop are "gradually extended to larger classes of problems" (2003: 96).

5.5.6.7 Assessing classroom solutions: baseline to final
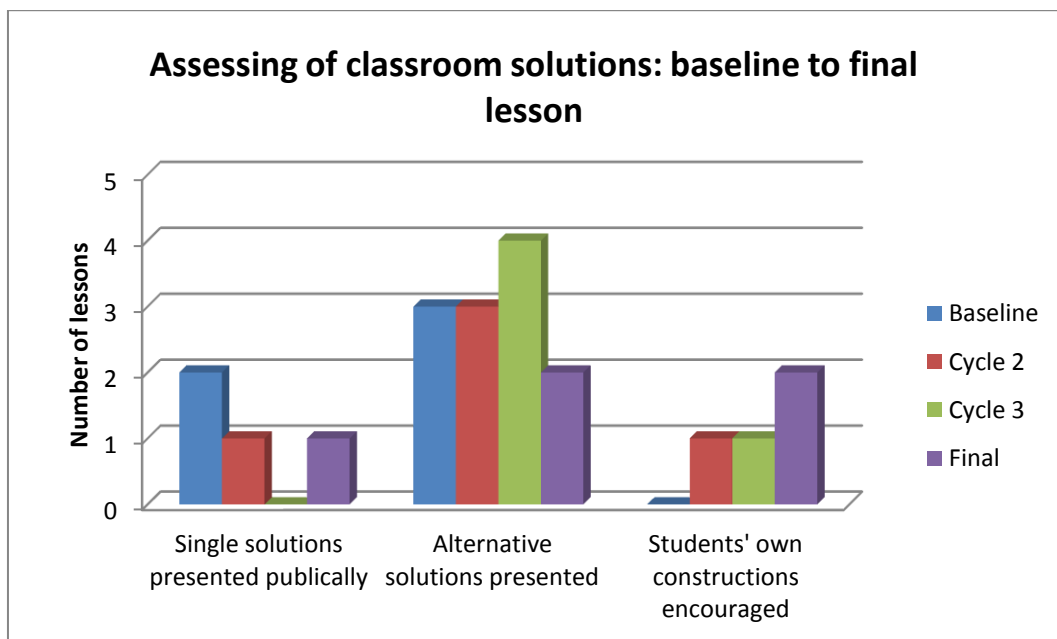
Figure 5.64 Assessing classroom solutions: baseline to final lessons

The increase in partner work and group work in lessons meant that this practice developed. Not only the teacher was involved in deciding if a solution or a path to a solution was correct. When teachers did present solutions there was always some discussion that accompanied it.

5.5.6.8 Teacher reflection: baseline to final lessons
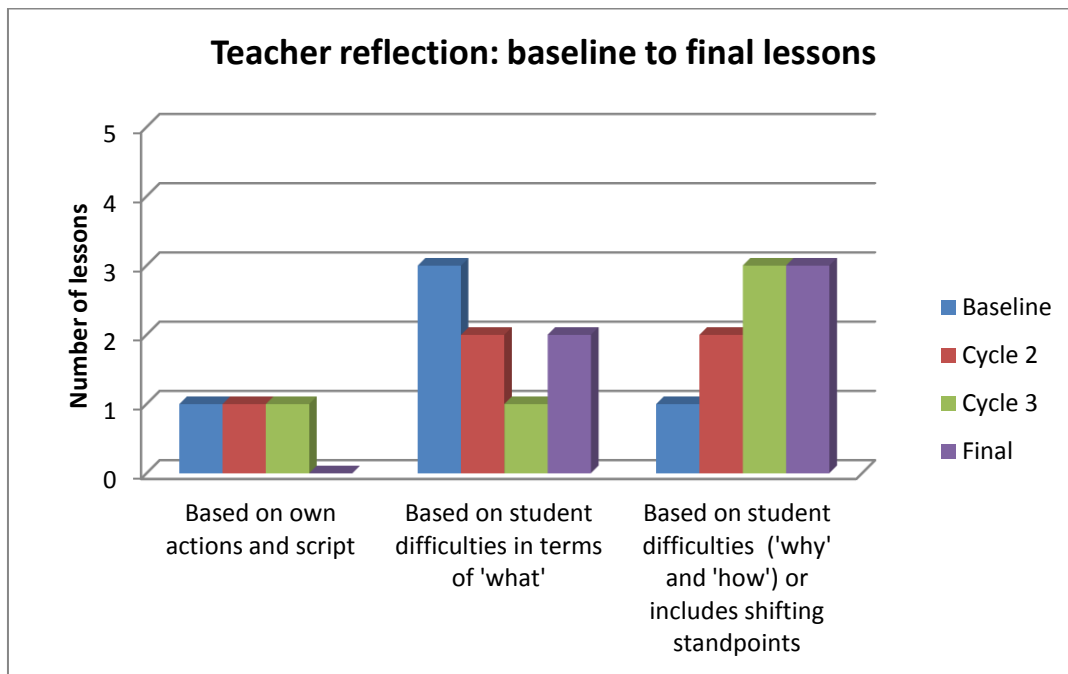
Figure 5.65 Teacher reflection: baseline to final lessons

Teacher reflection is not a process that teachers undertake easily. The teachers mentioned that they found the questions in the instruments 'difficult'. It is important to guide teachers through reflection of their lessons. It means providing teachers with the necessary vocabulary, to have them establish mathematical goals for their lessons and to consider a hypothetical learning trajectory. This allows a specific focus on the reflection. Teachers gave the following responses to "how do you feel about the lesson, what went well, not so well? How do you think students feel?" Their responses can be seen as developing a more holistic reflection on the lesson and not simply considering if they had completed the work set out for the day.

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Teacher	How do you feel about the lesson, what went well, not so well? How do you think students feel?"
A Gr. 6	*It went much better than I expected. They picked up the concept faster than expected. The learners were confident with the concept.*
B Gr. 5	*I enjoyed it and the children felt a great sense of accomplishment.*
C Gr. 6	*I think it went well. Every child had some input in a lesson like this. My weaker learners enjoyed it most because they could do the work.*
D Gr. 6	*I found that they enjoyed the questions. The curve ball question was interesting. I found it nerve-wracking to let the learners work on their own and then comforting when I heard them speak using mathematical terms.*
E Gr. 5	*They love doing anything that is not 'bookwork' so they loved it. I hope they remember the maths and not only the fun.*

Table 5.27 Teacher reflection: final lessons   (LQ4, 06-13)

5.5.6.9 Vertically aligned lessons: baseline to final lessons
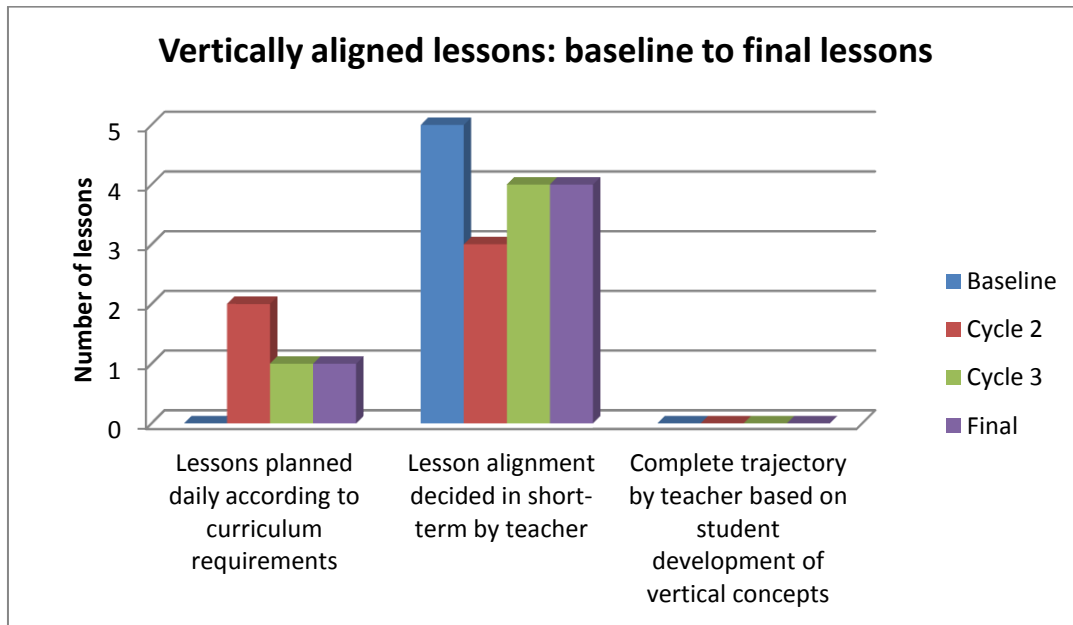
Figure 5.66 Vertically aligned lessons: baseline to final lessons

Although teachers are following the lessons as prescribed by the curriculum, all teachers had arranged lessons to best suit the development of concepts. All teachers had in the session visits included activities that were not included in the textbook. These activities included active collaboration; exploration and modelling were used by teachers to facilitate the

development of concepts. This adjustment to the curriculum was however within smaller sections of the content. This principle is not so much about the teachers' competency but about the impact of an overly structured curriculum. However, teachers are able to plan lessons based on vertical alignment of concepts in smaller sections even within this constraint. Most teachers were working on level 3 of the framework with some level 4A of the CGI framework (see 2.3.2.1.1).

5.5.6.10 Summary: baseline to final lessons

The following graph shows the development of all the didactisation principles as observed through the four lessons. The first lesson took place in July/August 2012 and the final lesson in June 2013.
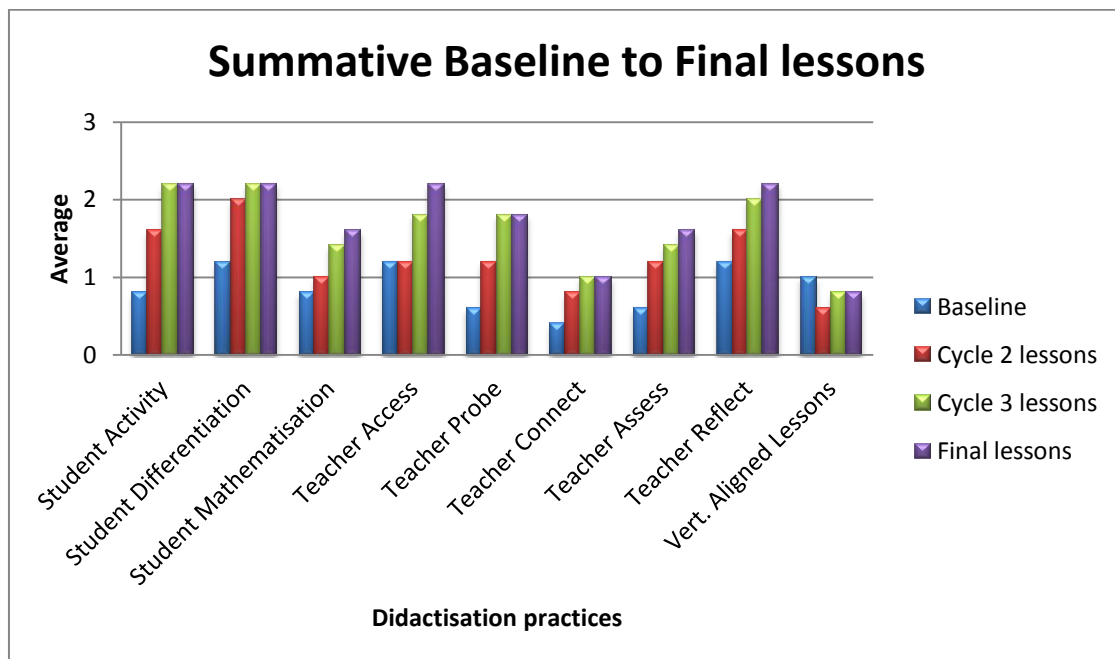
Figure 5.67 Overview of didactisation practices development

The first observation is that all didactisation practices developed over this period of time. This is not surprising since the entire program was focused on this and the researcher was specifically alert to these practices while observing the teachers. The teachers involved in the program developed their resources in terms of their knowledge of modelling, using problem-centred learning and considering different planning questions. The sessions included many discussions on teacher beliefs and orientations. Teachers also understand how the

administrative pressures constrict their teaching freedom but have discussed ideas on how to work within the confines of curriculum and bureaucratic paperwork. The graph shows that student activity increased most while teacher connecting of student ideas showed least gain.

5.5.7 Progression of didactisation practices implementation

Earlier in the chapter it was suggested that didactisation practices are not equally easy or difficult to implement. Didactisation practices could be arranged in the following hierarchy of progression in terms of their development in classroom environments and within curricula such as those described in this study. This hierarchy helps to understand the development of didactisation practices while it was through the development of didactisation practices that the hierarchy could be structured. Increased student involvement was the practice to be developed by all teachers early in the program followed by student differentiation as a result thereof. Providing teachers with suitable tasks and discussing the type of problems that are more conducive to mathematisation made this possible. Teacher questioning was the next didactisation practice to show signs of significant development. Teaching tends to take place as a session of asking questions so teachers are very good at asking questions. Restructuring the type of questions teachers asked also took place. During the discussion sessions with teachers, emphasis was placed on allowing students to do the mathematics and for a teacher to avoid simply telling students what to do. Mathematisation and teacher reflection appear to be equally 'difficult' for teachers. Although lessons became more student-centered, assisting students in making vertical connections in mathematical ideas was not always observed in lessons. Sometimes teachers allowed the activity and discussion but wrapped up the lesson very quickly. Teacher reflection was rated using a questionnaire that prompted teachers into thinking about their lessons from various perspectives. It is not certain to what extent teachers are reflecting in all their lessons in this manner. Although teachers were allowing students to become more involved in doing mathematics and in discussing ideas with each other, teachers did not always use these ideas to develop concepts further as a whole class discussion. Student ideas that were raised during the lessons were linked to the teacher's desired outcome of the lesson and were not always used to understand student thinking. At the end of the spectrum are the ability to align lessons vertically and the skill of connecting student ideas as they surface during a lesson. These practices could be considered more *difficult* didactisation

practices based on the observations in this study. Even though these practices are rated as more difficult it is encouraging that these practices did develop through the program. Earlier (see 3.2.1) it was decided to incorporate Wilson and Heid's "know and use the curriculum" into the idea of Treffers' "vertically aligned lessons". This didactisation practice was placed at the end of the table since teachers were still referring to 'covering the curriculum' as a challenge and not focusing on the 'big ideas' (Randall 2005: 9) within the curriculum. Sections such as capacity, mass and length all revolve around the same decimal principles. Teachers could ensure that students understand these together, and not as separate entities that have to be taught individually. They could be used as contexts for teaching the decimal number system. This is an area that can be further developed with teachers. It could contribute significantly with teachers accepting more student-centered ways of teaching by allowing them to find more 'time' in the curriculum.

Didactisation practices		
Connecting student ideas		Vertically aligned lessons/Know and use the curriculum
Assessing classroom solutions		
Access student ideas	Student mathematisation	Reflect on lesson
Differentiation		Probe student ideas
Student Activity		

Increasing difficulty
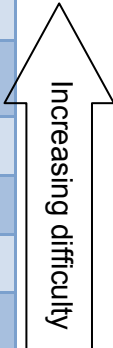
Figure 5.68 Didactisation implementation hierarchies

5.5.8 Changes in teacher resources, orientations and goals

Teacher resources, orientation and goals are gauged again at the end of the program. Together with the set didactisation practices that developed, teachers' resources, orientations and goals also shifted during the program.

Problems and problem solving became resources to teachers as a result of the program. It was evident during the latter visits that all teachers were ensuring that students were active in constructing mathematics. Teachers also developed very good questioning skills and were allowing students more talk time during lessons. Modelling tasks themselves became a resource for teachers' reflective thinking regarding the role of problems in mathematics teaching. These developing resources also impact on teacher orientations and goals. Teachers'

lessons started to include problem solving sessions. Teachers were also using pair and group work during the latter lessons.

Baseline lessons	Individual seat work evident in all five lessons
Session lessons	Pair work, group work, practical work or whole class discussion

Table 5.28 Student involvement comparison

These resources were factored into the goal of the lesson while the teachers' orientations about mathematics teaching may change as a result of successful lessons.

At the end of the last classroom observation teachers were asked to comment on what they thought was different about themselves during the last lesson compared to the baseline lesson.

Mrs A: *Learners interacted more with one another than they did in the first lesson.*

Mrs B: *I facilitated and the children were in control.*

Mrs C: *In the first lesson I did a lot of talking. In the last lesson I only read through the task, the learners had to find the mistake with no help given.*

Mrs D: *I was the centre of teaching, now I try to facilitate learning*

Mrs E: *I try to allow the children to talk and do more in the lesson.*

(LQ4, 06-2013)

Comparison can also be made in the goals set out by teachers for the baseline lesson and the final lesson.

Teacher	Goal for baseline lesson	Goal for final lesson	Changes in goals
A	*With a discount you need to do a subtraction.*	*Ordering decimal numbers, showing their answers while explaining what they were doing.*	More focus on student activity.
B	*Have a clear understanding of perimeter and area and the difference between the two concepts.*	*That they would know where to start in solving the problem, order their working and find the solution.*	A better problem solving orientation.
C	*Write a decimal number or common fraction as a percentage.*	*To order numbers (decimals) by solving the problem.*	Introduced a modelling problem to learn through.
D	*That they would not get worried when they get a BODMAS sum.*	*The mathematical reasoning necessary to solve word problems.*	Content to concept change.
E	*A solid knowledge of the basic tables and the*	*That 3D shapes are made up of 2D shapes -*	Content to concept change.

extended tables (up to 120 x 900) are essential in order to multiply large numbers accurately.	*that cubes are similar to squares.*	

Table 5.29 Teacher goal comparison   (LQ1, 08-12 and LQ4, 06-13)

Further insight into teachers' orientations may be gauged from their changing ideas about what determines a 'good' mathematics teacher.

Teacher	Baseline Questionnaire	Post Questionnaire	Change in orientation
A	*In-depth understanding of the subject.*	*Someone who understands the subject inside and out and can convey that understanding in numerous ways to reach those taught.*	Included the value of varying representations. Also reflects a change in goals – increase knowledge of representations.
B	*You have to have a very sound understanding of the content in order to dissect it for children.*	*Dedication, creating mental pictures for children, routine and passion for the subject.*	More holistic and student-centered.
C	*Knowledge of subject matter, experience, dedication and understanding how children learn.*	*Make the subject interesting – introduce different methods. Build confidence in learners' skills and abilities. Identify when a concept is not working and able to change the lesson.*	More student-centered and more reflective. Also reflects change in goals – varied representations.
D	*Time, effort, experience, subject knowledge and passion.*	*One who is prepared, knowledgeable and has a teachable spirit. One that welcomes his or her mistakes and always considers learning as the most important concept.*	More reflective ideas. Change in goals – a focus on learning.
E	*Positive attitude and willingness and desire to learn.*	*Enjoyment of the subject, willingness to try new ways of doing things – flexibility, understanding of key concepts.*	Focus on teacher able to be flexible. Change in goals – to try new ideas.

Table 5.30 Teacher orientations comparison   (BQ, 08-12 and PQ, 06-13)

Teachers also responded to several other questions in the baseline and post-evaluation questionnaires which point to teachers' changing orientations in terms of what it means to do mathematics and redefining what a successful mathematics lesson would entail. Their comments also show an increased ability for reflective thinking about themselves as mathematics teachers.

How do you know if a mathematics lesson has gone well?

July 2012 *When the assessment is positive* (BQ, B, 07-12)	June 2013 *When children are confident and want to try again* (PQ, B, 06-13)

What do you think is the best way to introduce a new concept in mathematics?

Previously I would have said "take what they already know and build until you reach the new concept". Now I would say, using a problem to understand where they are at and then being able to build on that knowledge as opposed to assuming what level they are on (PQ, A, 06-13)

How do you think students learn more abstract forms of mathematics?

July 2012 *By working through the basics first thoroughly – then being led.* (BQ, E, 07-12)	June 2013 *By being actively involved in what they are doing* (PQ, E, 06-13)

How do you know if students have developed/understood a concept/idea during the lesson?

July 2012 *By assessing their written work (classwork and homework)* (BQ, C, 07-12)	June 2013 *Once they have done an activity, by the questions asked. Most times they will let the teacher know if they have understood.* (PQ, C, 06-13)

How do you deal with students who have different understandings of the concept you are trying to teach or have different ways of working?

July 2012 *I always teach two methods when available and then a third method if they can at least master one* (BQ, D, 07-12)	June 2013 *I welcome it. I discuss their ways and then re-address the same question.* (PQ, D, 06-13)

Table 5.31 Teacher reflection comparison

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Teachers were also asked to reflect on the value of the professional development program itself. The collaborative nature of the program (see 3.3.1.7) and the increase in their own knowledge and awareness of their decision making were noted:

Mrs E: *I have enjoyed the opportunity to share experiences with colleagues and I have enjoyed spending time considering how and why I do what I do in the class.*

(PQ, E, 06-13)

Mrs A: *Makes sense, I see the increased love of maths from learners and learners can feel less overwhelmed when they can have an opinion.* (PQ, A, 06-13)

Mrs B: *It encouraged me to engage the children more and provide opportunities for them to experience self discovery through problem solving.* (PQ, B, 06-13)

When considering teachers' decision making, it can be reiterated that the didactisation practices teachers have been exposed to can become pedagogical content tools (see 2.4.4.7). The professional development program that teachers have been exposed to has contributed to their developing knowledge, orientation and goals thereby becoming a factor in their decision making for their classrooms. The program will not necessarily change all decisions that teachers make, but it will become one of the resources that may affect teachers' decisions.

Once again the graphical representations of instrument 3 are also valuable in gauging this change in a holistic way as it manifests within a classroom lesson. The resulting graphs are congruent with the above discussion on the development of didactisation principles. There are numerous signs that teachers are moving towards facilitating student constructions and including problematic situations. The multiple data sources and instruments have allowed for crystallisation of why and how didactisation practices develop. Multiple instruments have allowed for congruence of the findings in this study.
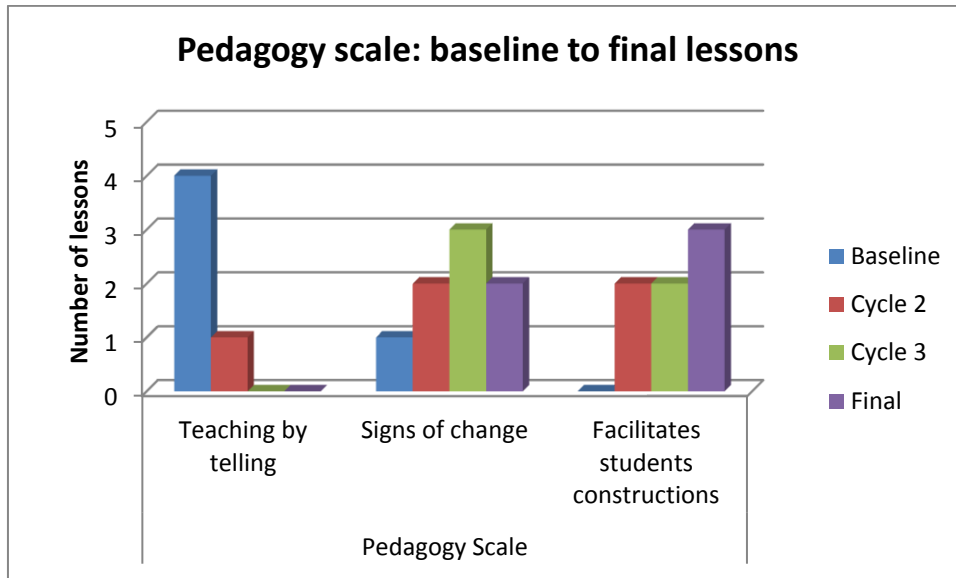
Figure 5.69 Pedagogy scale: baseline to final lessons

In the final lessons, 'teaching by telling' is not a feature of the lessons. The students are involved in most of the mathematical work for the lesson. More focus on student pair or group work is evident. Students are expected to explain their thinking to each other. Mrs B, Mrs C and Mrs E's lesson involved questions that stimulated thinking in students.
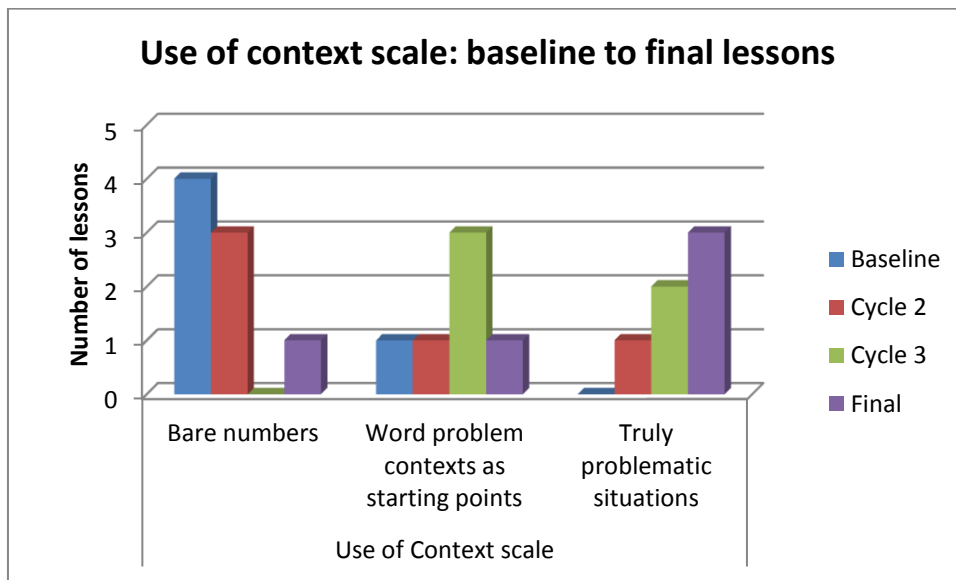
Figure 5.70 Use of context scale: baseline to final lessons

In the final lessons, Mrs B, C and E used contexts to elicit mathematical ideas and to promote discussion. The contexts had inherent potential for both horizontal and vertical mathematisation to take place in an integrated way. These contexts allowed for "conceptual

window[s]" (Doudy 1991: 118) whereby students could explore mathematical ideas, and by projecting these ideas, could come to more abstract realisations.
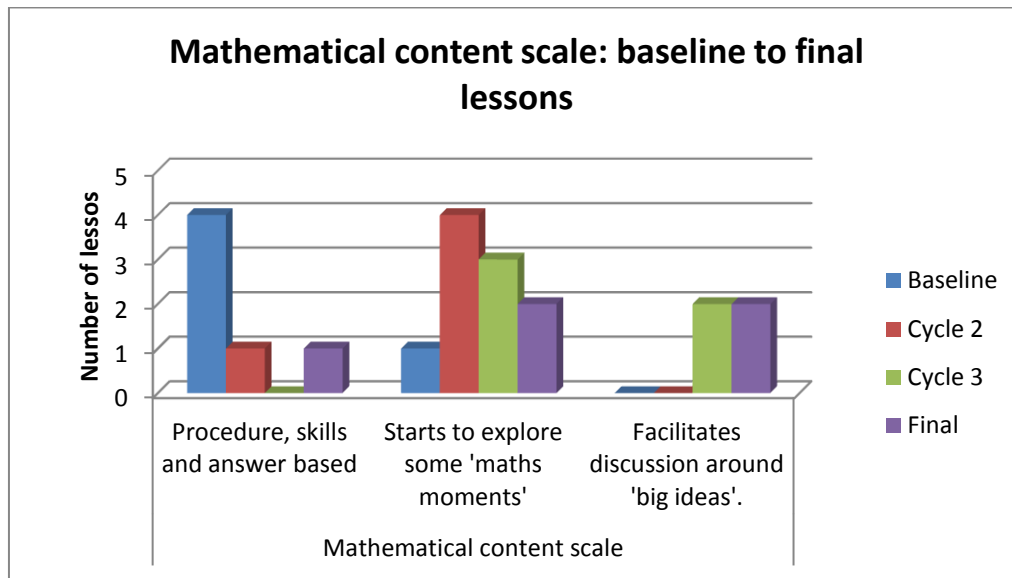
Figure 5.71 Mathematical content scale: baseline to final lessons

The final lessons involved a spread along the mathematical content scale. What needs to be considered is the shift within the cycle lessons away from the baseline lessons as a trend across all five teachers in the fifteen lessons across cycle 2, 3 and final lessons.

The concluding consideration is that from a modelling perspective. The table now reflects the changes from the baseline lesson through to the final lesson. This consideration may be valuable to other researchers. It shows how a modelling perspective can be used to analyse a lesson within any context and content. The table shows that students are exposed to a much fuller mathematical experience in the latter lessons. While the baseline lessons were similar from a modelling perspective, the latter lessons were varied and therefore needed more explanation. A modelling perspective provides researchers with another perspective together with vocabulary to analyse lessons.

Design principle	Baseline lessons	Cycle 2 lessons	Cycle 3 lessons	Final lessons
1. **Reality principle**. Real life situations are used and students need to make sense of these situation.	Mrs A used this to start the lesson.	No lessons involved real life situations.	Reality-based lesson: Mrs E Contextually richer lessons where contexts acted as models: Mrs A,B,C and D.	Two reality-based lessons (Mrs B and C). Mrs E's lesson context was based on 3D objects, Mrs D's lesson word problem based.
2**. Model construction principle**. Are students involved in constructing, describing, extending or explaining a structurally significant system?	None. Students were involved in the "working mathematically" part of the modelling cycle (see Fig 2.2)	Mrs B, Mrs C and Mrs E's lessons involved describing and explaining systems.	Mrs B's lesson involved building a 3D model, Mrs C's lesson included a model of whole number prime factorisation, Mrs E's lesson involved constructing a model for sharing that involved fractions.	Mrs B, C and E's lessons were reality-based model construction lessons. Mrs A and D's lesson involved model application.
3**. Self evaluation principle**. Students able to or were expected to judge their own responses.	None. Validation was done by the teacher.	Students could judge their responses in all the lessons. However final validation was still the teachers' function.	Mrs A, B, C and D's lessons: students were able to judge their own responses since pair work played a role. In Mrs E's lesson students were fully in control of judging their responses.	Mrs C's lesson involved full self evaluation of responses. Mrs E's lesson, students could build their net to verify its quality. Mrs A, B and D lessons involved pairs or groups that could validate own work.
4. **Documentation principle**. What type of documentation was expected? Did students have to explain their thinking?	Calculations and answers were written down in books.	Mrs C's lesson involved students using a combination of writing and manipulatives in their explanations, while Mrs E's lesson involved writing the structure of the pattern in words.	Writing down of procedures and answers in workbooks (Mrs A and Mrs C). Making drawings on blank sheets of paper (Mrs B and Mrs C). Constructing (Mrs A) Cutting, organising (Mrs E). In four lessons students had to explain their thinking.	Mrs A and D's lesson: pair calculations and solutions in book. Mrs B's lesson: pairs worked on blank paper – it allows for more freedom of writing ideas. Mrs C's lesson: blank paper, students created own lists and tables. They had to give reason for their responses.

				Mrs E's lesson: constructing own nets and building nets.
5. **Simple prototype principle**. Will the experiences assist students in making sense of other structurally similar situations?	All lessons had vertical mathematisation elements that meant students would have to memorise a procedure.	All lessons involved students identifying an underlying structure through a strong emphasis on vertical mathematisation which may lead to meaningful memorisation. Mrs A – noticing equivalence; Mrs B – lesson may lead to flexibility with inverse operations; Mrs E – structural understanding of patterns.	Mrs A: students applying known mathematical operation. Mrs B: vertical progression allows for connections to similar situations. Mrs C: students' vertical progression to prime factorisation will allow algebraic factorisation. Mrs D: connections between individual and complex shapes. Mrs E: connections through differentiation may lead to studies developing analogies.	Mrs A and Mrs D's lessons: application of known procedures. Identifying underlying structure of decimal place value and four operations. Mrs B, C and E: building of self-invented methods or applying fully understood mathematical procedures will allow for students to apply what they learnt from this lesson.
6. **Generalisation principle**. What constructions can be modified and applied?	Mrs E's lesson (a model of multiplication of multiples of 10 or 100) involved constructions could be modified and applied.	All lessons involved reflection on the underlying structure (e.g. equivalence, pattern defining and describing) that could assist in generalising.	Mrs A: multiplication procedural fluency Mrs B, C and D: Reflection on connections may promote generalisation.	Mrs A and D: already shortened methods to apply if students understand and remember them. Mrs B, C and E: students' own constructions that they understand.

Table 5.32 Modelling design principles: baseline to final lessons

Teachers are showing signs of change. The changes are incremental and reflect a number of challenges to reform mathematics teachers: their beliefs, their own mathematical knowledge, their environment and the specifications of the curriculum. These factors need to be given attention when deciding to what extent teachers have changed or can change. The nature of the change is also an important consideration. It is only when observing teachers in their own environments that this judgment can be made. Sensitivity and improved mathematical interaction in the classroom creates fertile ground for many meaningful changes.

5.6 CONCLUSION

This chapter set out to present and discuss the data collected through the teaching experiment designed in this study. The teaching experiment designed had been successful in producing the data needed to evaluate the development of didactisation practices in primary school mathematics classrooms. The retrospective analysis as a component of design research allowed a number of conclusions to be reached, including confirming the hypothesis of the study which stated that a modelling teaching experiment would allow for the development of mathematics teacher didactisation practices.

Didactisation practices did not develop simultaneously or at the same rate. It did however appear that practices were implemented in the same order by the teachers. Although teachers are dictated to by curriculum materials, they could make improvements to the materials and lessons. The qualitative nature of the study allowed the development of the didactisation practices to be seen in the context of teachers' day-to-day experiences.   Teachers were all moving towards creating milieus through tasks and questioning that allowed students more activity and more responsibility in the lessons.

The multi data sources and multi theoretical perspectives allowed practices to be viewed from a number of perspectives. Individual didactisation principles were gauged while a holistic view of the teachers' pedagogy, use of context and mathematical content were also determined. A modelling perspective was also included in the lesson evaluations since modelling allows for an integration of the didactisation principles at a very high level. All

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teachers showed signs of change in at least one of these aspects. Changes to teachers' resources, orientations and goals were also tracked. These showed positive incremental development.

CHAPTER 6

SUMMARY, CONTRIBUTIONS AND RECOMMENDATIONS

6.1 SUMMARY

This study aimed at formulating a didactisation framework that could be used as the backbone for a professional development program using modelling tasks (see 1.3). The study also aimed at developing teachers' didactisation practices through the professional development program. Furthermore, development of teachers' didactisation practices was anticipated and documented. In order to achieve these goals the study followed a design research framework to create the environment to run the teaching experiment that lead to the development of didactisation practices in mathematics teachers. In doing so, the study answered the following research questions:

1. What constitutes didactisation practices in mathematics teaching?
2. How can didactisation practices develop through involvement in a modelling-based professional development program?
3. What is the effect of the teacher development program on teachers' resources, orientations and goals?

In answering these questions, the following aims and objectives set out in 1.3 were met:

1.3.1.1: This study did formulate a didactisation developmental framework for mathematics teachers (see 2.5).

1.3.1.2: The framework was used to develop mathematics teachers' didactisation practices through a modelling orientated professional development program (see 3.6, 4.2.2 and 5.6.6).

1.3.2.1: This study clarified the framework by incorporating the RME concept of didactisation and the MTP concept of the mathematical work of teaching. Furthermore, this study included connections as part of the framework (see 2.4, 3.2.1 and 3.6).

1.3.2.2: Didactisation practices were situated as a mathematics teaching orientation within the framework (see 3.6).

1.3.2.3: Modelling tasks were used as a vehicle to develop teachers' didactisation practices by creating a scaffold for teachers to think about their own actions in terms of didactisation practices (see 4.2.2, Table 5.6 and Appendix 3-5).

1.3.2.4: Teachers were observed in their own practice to gauge the development of didactisation practices (see 5.2 – 5.5).

1.3.2.5: Changes in teacher knowledge, goals and orientations were considered and documented (see 2.3, 5.2.9, 5.3.8, 5.4.8 and 5.5.8).

1.3.2.6: The development of teacher didactisation practices was presented and analysed (see chapter 5).

1.3.2.7: A design research framework created an innovative learning environment to propose local theory about professional development of mathematics teachers (see 4.2 and 6.3).

This study presented and incorporated theoretical perspectives (see 2.1) to establish sound practices and data collection methods. This made it possible for the aims and objectives of the study to be realised. The concept of didactisation was explored, and incorporated significant ideas from Treffers (1987: 58) and Wilson and Heid (2010: 3) (Chapter 2 and 3). The formulation of a rich description of didactisation allowed the researcher to gauge the development of didactisation practices in classrooms that included different teaching styles, grades and schools. The study of teacher development and what constitutes successful teacher development programs (Chapter 3) led to the inclusion of elements that were essential to the success of the teaching experiment. These perspectives allowed the researcher to analyse the data and reach theoretically based conclusions.

Design research (Chapter 4) provided the means for several revisions of the teaching experiment, so that the needs of the teachers could be incorporated into the cycles. The teachers were active participants in the program and guided some of the changes that took place during the design revisions. The use of modelling tasks as a vehicle for the teacher development sessions provided a resource for cognitive conflict necessary for teachers to explore socio-constructivist ideas in mathematics education. It furthermore provided teachers with the resources to understand problem-centred teaching and learning.

This chapter will conclude the study by discussing the findings, contributions to theory and practice, the limitations of the study as well as to provide suggestions for further research.

6.2 SUMMARY OF FINDINGS

The main findings that come from this study on the development of didactisation practices of primary school mathematics teachers are:

- The didactisation practices proposed and formulated into a framework for professional development allow one to view teacher work more holistically and in a more integrated way than single focus studies. This was possible through the widespread literature study where this study not only integrated various theoretical perspectives but also advanced them.

- Didactisation practices do develop through professional development programs such as the one followed in this study. This was made possible by theoretically-orientated design research that incorporated various perspectives to teacher development.

- Teachers' resources, orientations and goals shifted during the program. These aspects are more difficult to change and equally difficult to document. By focusing on the didactisation practices in teachers' classrooms these foundational pillars of teacher decision making were accessed.

- Teachers' developed didactisation practices in a similar way. This lead to a hierarchy being developed (see Fig 5.68) where some practices appeared earlier or more frequently than others. The implication is that other professional development programs could factor this into their own frameworks.

- Making students more active is a necessity for the other didactisation practices to evolve. It therefore appears at the pinnacle of Figure 5.68. It is a gateway practice and is essential for teacher development of other didactisation practices. It also supports students in experiencing more meaningful mathematics; making significant connections and enables them to mathematise the consequence of their activity.

- Making connections appears to be the most skillful of the didactisation practices. This is evident in the extensive literature. This study, by adding connections into the mathematical work of teaching (Wilson & Heid 2010) and further validating that this plays a critical role in classrooms, has contributed to the literature. This concept can

be extended to include teachers' own connections within their mathematical knowledge domain. Vertically aligning activities and concepts within lessons and aligning a sequence of lessons that build on each other, is a central aspect of teachers' own connection knowledge.

- Modelling tasks became a resource for teachers. These tasks are able to produce cognitive conflict in teachers. The tasks also enabled teachers to see some of the shortcomings of traditional teaching.

6.3 CONTRIBUTIONS TO THEORY AND PRACTICE

The earlier comment from Sarason (1971: 78):

> what is not recognised or verbalised cannot be dealt with, and if it is important and not recognized, effort to introduce substantive change, particularly in the classroom, result in the illusion of change.

In terms of the contributions of the study, Kelly's earlier (see 4.6.1) statement that design research needs to move towards identifying what is necessary in a learning situation is relevant. This study sought to provide such guidance at a micro and meso level of professional development. The study produced both instructional activities, design principles and a curricular product, as is typical of design research contributions (McKenney *et al*. 2006: 73).

6.3.1 Theorizing and theory building

As suggested by Anderson and Shattuck (2012: 17) design based research does not create de-contextualised principles or grand theories – design based research reflects on the conditions in which the principles operate. The theorizing proposed by this study is from a context orientation. This study, presented a professional development program that was successful because of a number of theoretical conditions that were incorporated: didactisation as a theoretical construct proposed by Treffers; the mathematical work of teaching as a framework to understand teachers' daily activities; and the conditions resourced from literature that signify a successful program (see 3.3.1 and 4.2.2). Furthermore, the inclusion of connections to the mathematical work of teaching framework proved to be an important construct for mathematical teaching proficiency. This aspect should be included in further studies on

teaching proficiency. Teachers were self-driven in the program in that no particular end-point or specific action was prescribed to them. Teachers reflected on the program and integrated the didactisation practices within their current lessons in their own way.

The analysis used in this study continually moves from a fine grain of individual teacher to a broader view of the impact on all teachers. This means that the impact of the study as a whole on a small group of teachers is considered. Therefore the findings can be generalised over a larger domain than a single case study. This study has made a significant contribution to knowing and understanding primary school mathematics teachers in South Africa. Furthermore it has made a preliminary analysis of the impact CAPS may have on teaching in mathematics classrooms.

This study proposed design principles for a professional development program that would lead to improved didactisation practices in teachers. Furthermore the study produced a program that may be "of value to school or a broader education community" (Mc Kenney *et al*. 2006: 72). The third factor, is the contribution of the program to the teachers themselves. Teachers were enriched through exposure to modelling tasks, working with colleagues who have similar experiences, and problem-centred resources.

Gravemeijer and Cobb (2006: 46) elaborate on the theory development in design research in terms of three levels:

- The instructional activities (microtheories) level
- The instructional sequence (local instruction theories) level
- The domain-specific instruction theory level

This study contributed to the micro-theory level by proposing activities such as modelling tasks for teacher professional development and did not necessarily seek its replication in the classroom. The modelling activities became springboards for discussion to elaborate didactisation practices in the classroom, and to create avenues of conceptual change in teachers.

On a local instruction theory level, the components of the development program allowed for a successful teacher development program which aimed at presenting theory to teachers on a

practical level. As stated by Gravemeijer and Cobb (2006: 46) design research has the potential to bridge the gap between theory and practice. The research focus was on the lessons presented by the teachers; it was grounded within teachers' current practice. The didactisation practices and development in resources, goals and orientations were based largely on teacher practice. Furthermore, research that is grounded in teacher practice can make contributions to the practice of mathematics teaching.

From a domain-specific point, the study highlighted the central role of mathematisation in primary school mathematics lessons. The other practices should be evaluated in terms of their contribution towards student mathematisation. Professional development programs should endeavor to facilitate the co-ordination of the other didactisation practices towards reaching meaningful mathematisation for students.

6.3.2 Contribution to practice

 Kilpatrick (in Chauvot 2008: 97) reminds us that:

"No one should expect to draw strong implications for practice from the results of a single research study. The results of a study may be its least important part. Research in mathematics education gains its relevance to practice or to further research by its power to cause us to stop and think. It equips us not with the results we can apply but rather with tools for thinking about our work"

This study has made some important contributions to mathematics education, mathematics teaching and mathematics teacher development practice. The tools that were developed in this study make a significant contribution to practice as they are ready to take into classrooms. The tools developed for use in the sessions as well as tools used for the analysis of the data can contribute to other successful teacher development programs. This study also established nine didactisation practices that could be studied through teacher actions in the classroom. The study used a number of instruments that already exist in literature. Their success in a study that is divergent from where they were sourced adds to establish these instruments as sound. Other instruments were developed through the needs of the study and may prove to be useful in other studies.

The teachers involved in this study are very mindful of the curriculum. They each worked at covering the set content within the set time-frames. The curriculum appears to be a very powerful document in terms of 'what' teachers have to teach, but it does not specify the 'how' to teach. The suggestion of Galbraith (2012: 60) that a curriculum should comprise of two parallel strands is insightful. For a country like South Africa, it may mean that problem-centred learning and modelling may become a reality in classrooms. The 'how' of teaching is what this study focused on, but the 'what' to teach was a relevant factor in the manner in which teachers took up didactisation practices. The teachers mentioned the wide range of content that needed to be covered in the year as a contextual factor in their own didactisation practices development. Teachers would welcome Galbraith's suggestion. Parallel strands in the curriculum may allow teachers the freedom and confidence to try new approaches. It may also allow researchers better access to develop and run professional development programs within real school settings.

This study represents one of relatively few studies that focus on primary school mathematics teachers in South Africa. It explored the landscape of some South African primary school mathematics classroom and looked at teacher decision making against this context. This contributes to its significance, but does not mean that it is not noteworthy in an international community. The didactisation practices outlined, together with the instruments developed, can be used in any classroom. The study does reiterate that professional development should run close to teachers' daily practice and experiences. However, professional development is not factored into the daily or weekly job description of South African teachers. A policy decision needs to be made to factor consistent professional development into teachers' activities. It would mean that research studies could access teachers and achieve more significant goals.

This study set out to increase the meaningful mathematical activity of students in classrooms (see 2.4.2) and this was achieved. At the foundation of all didactisation practices, stimulating activity would be necessary for all other practices to follow. Teachers declared that they would like to increase their repertoire of representations (see Table 5.30). This, as a foundational construct to mathematisation, together with that of making connections, will make a significant difference to their lessons.

6.4 LIMITATIONS

This study focused on five primary school mathematics teachers. This means that the results cannot be generalised over the entire spectrum of mathematics teachers. The teachers were grade 5 or grade 6 teachers, so no generalisation can be extended to other grades. The study is set in the South African context, so may not be representative of other areas in South Africa or other countries; although countries with similar backgrounds will find the research beneficial. The schools in this study were similar in resources and size and therefore different results may be obtained from schools with different resources and sizes.

The study did not give judgmental feedback to the teachers on their individual lessons. A non-judgmental (see 5.3.6) approach was preferred where it was hoped the teachers would filter, from session resources and activities, those elements that they could implement in their classrooms. The researcher did not want to mirror current teacher experiences with class visits that are judgmental in nature. A more judgmental approach where the researcher discussed each teacher's individual lesson with her may have resulted in better or faster development of didactisation practices.

6.5 RECOMMENDATIONS FOR FURTHER RESEARCH

Further research may be necessary in terms of the focal lens of this research. This study incorporated many principles, but few participants. Other research that will be relevant is a case study on a single participant focusing on all the didactisation principles or investigating many participants and only one of the didactisation practices. This will establish different data than the current study and may strengthen theoretical and practical understanding of teacher didactisation practices implementation.

The didactisation practice of making connections appears to be the one that took the longest time to develop. Further studies on how to develop this practice will be valuable. The role of connections should be closely linked to the nature and role of representations. Teacher development should have representations as the vehicle to facilitate teacher understanding of

connections in mathematics. Both the inherent connections in the subject matter as well as building student connections within each topic.

The impact on teachers' developing didactisation practices on students' needs to be evaluated. Both student mathematics results and affective changes in students will make a valuable contribution to mathematics education.

Teacher content knowledge was not evaluated in this study. It may be of value to gauge teacher knowledge before a professional development program, as it may shed light on why or how teachers are developing didactisation practices. It may also be beneficial to profile teachers before a program of this nature and to document the changes in different profiling teachers. This study was not longitudinal in nature; a study that evaluates the development of didactisation practices that includes a longitudinal component will be beneficial.

The teachers themselves make some comments that can be used as a starting point for further research. When asked: Is there anything you want to add about being a primary school mathematics teacher in the South African context? The following responses are relevant.
"*Language is a massive barrier, class sizes are a problem…the admin and expectations of what needs to be done* [in the curriculum] *and the time allocated can be unachievable*"
(PQ, A, 06-13)
"*We are limited to some extent by the Curriculum policy document and the nationally approved textbooks and workbooks that we are required to use. Also the threat of poor performance in the ANAs means that focus is sometimes on covering the curriculum quickly rather than on pupils learning."* (PQ, E, 06-13)
Possible avenues for further research are the impact of class sizes on mathematics learning in the primary schools and the impact on the Annual National Assessment on primary school mathematics teachers in South Africa.

6.6 CONCLUSION

Professional development of primary school mathematics teachers in South Africa should become a priority within academic and governmental organisations. The current curriculum

is seen as a quick fix to the problems of poor mathematics results in the Annual National Assessments (DBE 2012: 33-35). The Grade 5 and Grade 6 mathematics national average were 30, 4% and 26, 7% respectively. The schools involved in this study performed far above these national averages. Curriculum documents alone will not bring about a real change in mathematics classrooms. Consistent, meaningful and sustained teacher development will be the key to change. Teacher development should not stop at making teachers competent at only teaching the curriculum documents. Teacher development should pursue more difficult areas of change such as using representations, making connection, orientations and goals. Teacher development programs should aim at teachers' fostering student activity and developing student thinking in the classroom. Professional development may be costly in terms of time and money, but it needs to be seen as an investment that will produce rewards for students, teachers, schools and education departments.

The National Commission on Teaching and America's future's statement in 1996 (1996: 2) that "the need for excellent teaching grows ever more pressing" is still relevant today. They further add that it is not because teachers do not *want to* implement reform methods in their classrooms, it is because they do not know *how to* make changes to their teaching (1996: 5, emphasis added). This is where research has a responsibility to teachers and students in bridging the theoretical and practical divide.

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8. APPENDICES

Appendix 1: Permission from Gauteng Department of Education

education
Department: Education
GAUTENG PROVINCE

For administrative use:
Reference no. D2013/67

GDE RESEARCH APPROVAL LETTER

Date:	11 June 2012
Validity of Research Approval:	11 June 2012 to 30 September 2012
Name of Researcher:	Biccard P.
Address of Researcher:	652 16th Avenue
	Rietfontein
	Pretoria
	0084
Telephone Number:	082 787 4299
Fax Number:	012 460 3717
Email address:	pbiccard@yahoo.com
Research Topic:	Developing didactisation principles in Primary school Mathematics teachers through modelling
Number and type of schools:	FOUR Primary Schools
District/s/HO	Tshwane South

Re: Approval in Respect of Request to Conduct Research

This letter serves to indicate that approval is hereby granted to the above-mentioned researcher to proceed with research in respect of the study indicated above. The onus rests with the researcher to negotiate appropriate and relevant time schedules with the school/s and/or offices involved to conduct the research. A separate copy of this letter must be presented to both the School (both Principal and SGB) and the District/Head Office Senior Manager confirming that permission has been granted for the research to be conducted.

The following conditions apply to GDE research. The researcher may proceed with the above study subject to the conditions listed below being met. Approval may be withdrawn should any of the conditions listed below be flouted:

1. The District/Head Office Senior Manager/s concerned must be presented with a copy of this letter that would indicate that the said researcher/s has/have been granted permission from the Gauteng Department of Education to conduct the research study.

Making education a societal priority

Office of the Director: Knowledge Management and Research
9th Floor, 111 Commissioner Street, Johannesburg, 2001
P.O. Box 7710, Johannesburg, 2000 Tel: (011) 355 0506
Email: David.Makhado@gauteng.gov.za
Website: www.education.gpg.gov.za

GAUTENG PROVINCE

Department: Education
REPUBLIC OF SOUTH AFRICA

For administrative use:	
Reference no. D2013/238A	

GDE AMENDED RESEARCH APPROVAL LETTER

Date:	**15 November 2012**
Validity of Research Approval:	**4 February 2013 to 27 September 2013**
Previous GDE Research Approval letter reference number	**D2013/67 dated 11 June 2012**
Name of Researcher:	**Biccard P.**
Address of Researcher:	**652 16th Avenue**
	Rietfontein
	Pretoria
	0084
Telephone Number:	**082 787 4299**
Fax Number:	**012 460 3717**
Email address:	**pbiccard@yahoo.com**
Research Topic:	**Developing didactisation principles in Primary school Mathematics teachers through modelling**
Number and type of schools:	**FOUR Primary Schools**
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Making education a societal priority

Office of the Director: Knowledge Management and Research
9th Floor, 111 Commissioner Street, Johannesburg, 2001
P.O. Box 7710, Johannesburg, 2000 Tel: (011) 355 0506
Email: David.Makhado@gauteng.gov.za
Website: www.education.gpg.gov.za

Appendix 2: Ethical clearance from Stellenbosch University

UNIVERSITEIT·STELLENBOSCH·UNIVERSITY
jou kennisvennoot · your knowledge partner

18 October 2012

Tel.: 021 - 808-9003
Enquiries: Mrs S. Oberholzer
Email: oberholzer@sun.ac.za

Reference No. DESC_ Biccard2012

Ms P Biccard
Dept of Curriculum Studies

LETTER OF ETHICS CLEARANCE

With regard to your application, **DESC_Biccard2012** I would like to inform you that the project, *Developing didactisation practises in primary school mathematics teachers through modelling*", was approved on the following proviso's:

The researcher will remain within the procedures and protocols indicated in the proposal, particularly in terms of any undertakings made in terms of the confidentiality of the information gathered.

1. The research will again be submitted for ethical clearance if there is any substantial departure from the existing proposal.
2. The researcher will remain within the parameters of any applicable national legislation, institutional guidelines and scientific standards relevant to the specific field of research.
3. The researcher will consider and implement the foregoing suggestions to lower the ethical risk associated with the research.
4. This ethics clearance is valid for one year from **18 October 2012 – 17 October 2013**

We wish you success with your research activities.

Best regards

MRS S. OBERHOLZER
REC Coordinator: Research Ethics Committee: Human Research (Humaniora)
Registered with the National Health Research Ethics Council (NHREC): REC-050411-032

Afdeling Navorsingsontwikkeling • Division for Research Development

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www.sun.ac.za/research

Appendix 3: Modelling Task 1

Task 1:
The Greatest Airplane Contest

A neighbouring school held their annual Paper Airplane Contest last week. Children took part in teams and each team that took part had to design and make a paper airplane. They then had to throw the plane from a starting point and aim to get it to a finish point marked on the school field. Each team was allowed to have three throws.

Path

Finish ●

Start
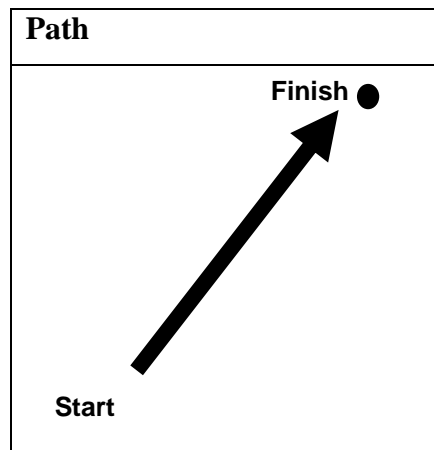
Problem:

In past competitions, the judges have had problems deciding how to select a winner for this competition. They don't know what to consider determining who wins the award. The data and a description of how measurements were made have been included.

You will have to prepare a presentation for the judges of the contest. Explain a method that will assist them in choosing this years' winner. Your explanation must enable them to use your method future competitions also.

Measurements	Explanation
Amount of time in air	Number of seconds from time of throw to landing
Length of throw	Straight-line distance from the start point to the landing point
Distance from target	Straight-line distance from the landing point to the finish point

Table of 2012 Results:

TEAM	Measurements		
	Amount of time in air (seconds)	Length of throw (meters)	Distance from target (meters)
Team 1	3.1	11	1.8
	0.1	1.5	8.7
	2.7	7.6	4.5
Team 2	3.8	10.9	1.7
	4.2	13.1	5.4
	1.7	3.4	8.1
Team 3	4.2	12.6	4.5
	5.1	14.9	6.7
	3.7	11.3	3.9
Team 4	2.3	7.3	3.25
	2.7	9.1	4.9
	0.2	1.6	9.1
Team 5	4.9	7.9	2.8
	2.5	10.8	1.7
	5.1	12.8	5.7
Team 6	0.2	1.8	8.8
	2.4	10.1	4.6
	4.7	10.3	5.4

Appendix 4: Modelling Task 2

Tangram Toys

Many teachers use Tangrams in their teaching. These are often made in plastic or wood. Students enjoy making different pictures with tangram pieces.
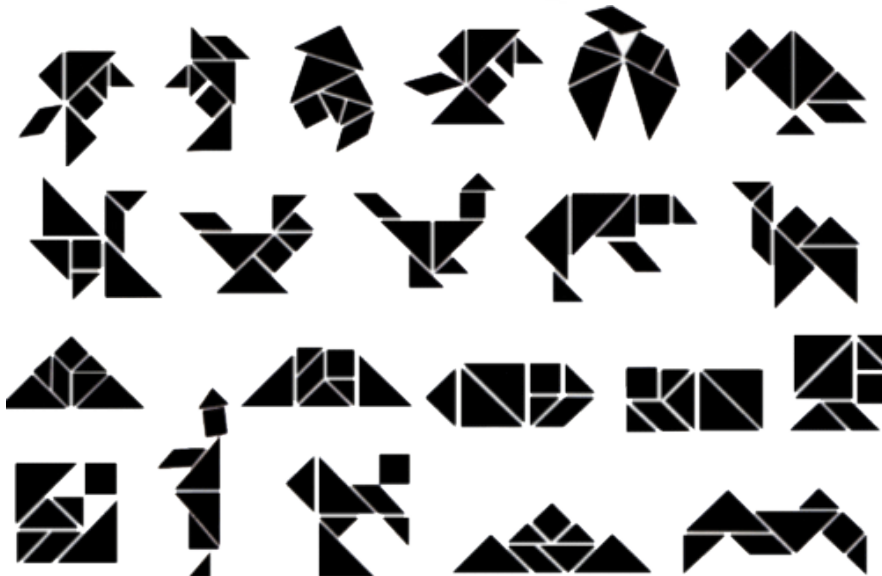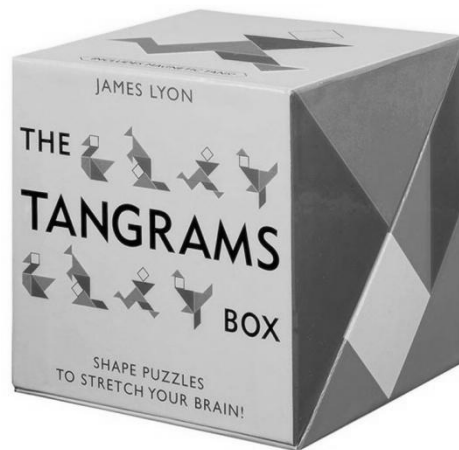
An educational toy factory has received orders for plastic and wooden tangrams like the one below. However, part of the order is for tangrams that are a little bigger than the one shown as they are

going to be used by younger students. The principal of the school who ordered the bigger sets suggested that 'everything that is 4 cm should be 7cm on the bigger set'. Can you assist the toy factory in correctly making the conversions to the shapes? You will know that all the shapes are correctly enlarged when they all fit together in a square again. You will have to explain how you did the conversion because other schools may want even bigger tangrams or one that is smaller than the original tangram. Please cut out a set of the larger tangram pieces to show the factory owner.
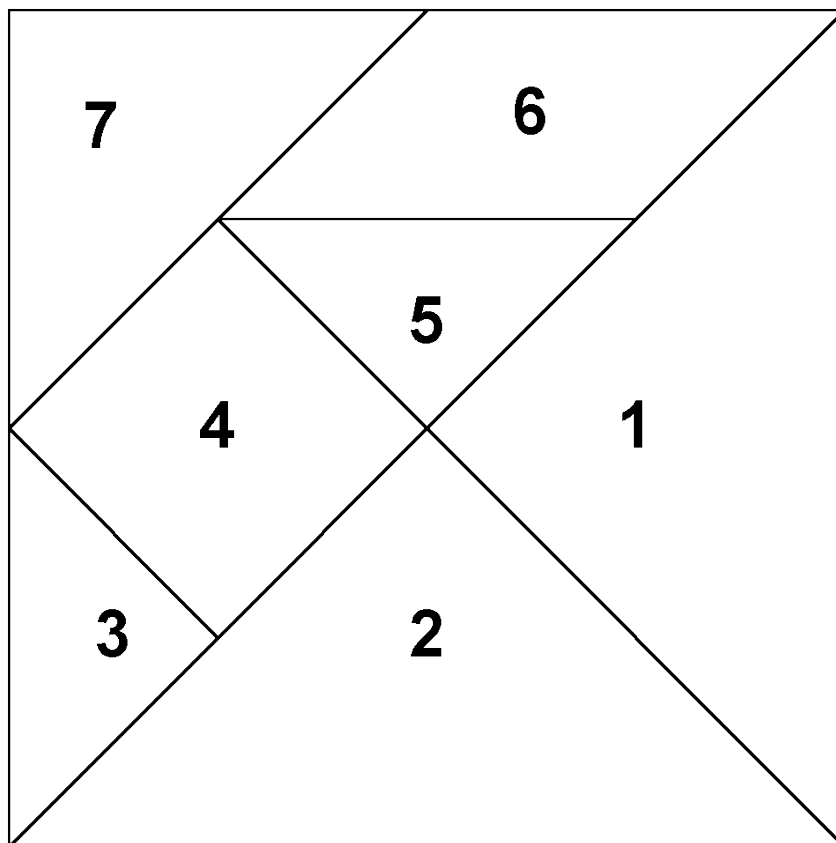
Appendix 5: Modelling Task 3

Hire or Fire?

Mr Ndlovu runs a fast food business. Last year he had nine students who sold food on Saturday mornings at the flea market. This year, he can only afford to pay two students. These are the records he kept from last years' sales for each student. Which two students should he hire this year…and why?

	HOURS WORKED LAST SUMMER		
	Dec-12		
	Busy	Steady	Slow
MARIA	12.5	15	9
KARABO	5.5	22	15.5
TONY	12	17	14.5
JOSE	19.5	30.5	34
CHAD	19.5	26	0
LETHABO	13	4.5	12
ROBIN	26.5	43.5	27
TONY	7.5	16	25
SIYA	0	3	4.5

	MONEY COLLECTED LAST SUMMER		
	Dec-12		
	Busy	Steady	Slow
MARIA	690	780	452
KARABO	474	874	406
TONY	1047	667	284
JOSE	1263	1188	765
CHAD	1264	1172	0
LETHABO	1115	278	574
ROBIN	2253	1702	610
TONY	550	903	928
SIYA	0	125	64

The number are given for times when park attendance was high (busy), medium (steady), and low (slow).

Appendix 6: Instruments

Coding used in instruments

Teachers	Coding	Students	Coding
Accessing student understandings/concepts	TA	Active students – mentally active and challenged	SA
Probe student understandings	TP	Differentiation – provide own methods/ideas	SD
Connecting student understandings	TC	Mathematisation – both vertical and horizontal	SM
Assessing student understandings	TA	Vertically aligned lessons (teacher)	TVAL
Reflecting on practice	TR		

Instrument 1:
Baseline Teacher Questionnaire/Post Evaluation Quesionnaire
(More space was allocated on actual pages used)
Instrument to be used at the beginning and at end of the program.

Name
Age
Number of years teaching Experience:
Qualifications: Professional Development courses/workshops attended. Next to each one, please indicate if/how it changed your knowledge about mathematics or about how to teach mathematics.
Number of years teaching mathematics:
Why or how did you become a mathematics teacher?
What do you think determines a good mathematics teacher?
What do you think is the best way to introduce a new concept/section of work in mathematics? **TA**
To what extent do students take part in your lesson, what do they do? **TP/SA**
How do you deal with students who have different understandings of the concept you are trying to teach or different ways of working? **TC/SD**
How do you assess the students in your class? When, how often, using what? **TAS**
How do you know if a mathematics lesson has gone well? **TR/SM**
How do you know if students have 'understood' the lesson? **TR/SM**
How do you know that students have developed a concept/idea during a lesson? **TR/TC/SM**
What is your experience of problem solving? When do you think problem solving works well?
How do you think students learn more complicated/abstract forms of mathematics? **TC/SM**
How do you plan your lessons over a period of a week/month/year? **TVAL**

May I have a copy of your lesson planning sheet? If you prefer a blank one with the relevant headings would be sufficient.
Date completed: _____

Instrument 2
Baseline Structured Questionnaire/ Post Evaluation Questionnaire
Instrument to be used at the beginning and end of the program.

Teacher: _____

Date of Observation: _____

Contextual factors: _____

1. Tell me how this maths lesson fits into your week, term and year plan? (TVAL)

2. What were you hoping that students' would learn today? (TM)

3. How much should/did students take part in a lesson? (SA)

3. What surprised you about what they did know and what they didn't know? (TA, SM)

4. How will what you taught today affect the student in a years' time? (TC, TVAL)

    5. What information from this lesson will you use to plan the next lesson? (TC, TVAL, TM)

6. What are the things you do well as a maths teacher?

7. What are the things you would like to improve as a maths teacher?

8. How do you think students learn mathematics best?

9. What resources do you think will improve your teaching?

10. How to you prepare or plan (mentally and otherwise) for (TVAL):

The next day's lesson:

Next week's lessons:

Next month's lessons:

Instrument 3

Pedagogy scale, use of context scale and mathematical content scale for classroom observation.

(Fosnot, D.T., Dolk, M., Zolkower, B., Hersch, S. & Seignoret, H. 2006: 10)

The **Pedagogy Scale** depicts teacher development from teaching by telling (level 1), to teaching to facilitate students' mathematical constructions (level 3).

| 1. **Transmission or direct instruction** modality of teaching; teaching by telling (the teacher does all the explaining and showing). Emphasis is placed on reinforcement and practice and on arriving at the answer that the teacher has in mind.

Operational behaviors to look for when coding:
• teacher explanation;
• teacher acknowledgement of correct answers, rather than asking student to convince and explain;
• asking students to practice;
• if questioning is used, teacher's answer is being sought rather than student reasoning. If manipulatives are used, they are used to show and demonstrate concepts and/or procedures. | 2. **Signs of change** away from telling towards facilitating students' constructions, but more rote use of pedagogical strategies rather than in relation to the development of student reasoning about mathematical content.
Operational behaviors to look for when coding:
• allowance of "wait time" and the posing of questions that invite reasoning,
• children working with partners and discussing their work—but these and other pedagogical techniques are not always used at the proper times or places and, thus, tend to function merely as "tricks."
• The teacher's focus seems to be on his/her own behavior—a set of new pedagogical strategies, rather than in relation to student thinking. For example, teacher may provide manipulatives, but more as a "pedagogical" strategy, rather than in relation to student's thinking.
• Math congresses may be in place, but they are more a sharing of children's ideas, rather than a carefully executed discussion around the scaffolding of strategies, or important mathematical ideas. | 3. **Teaching to facilitate construction.**

Operational behaviors to look for when coding:
• the use of wait time,
• genuine questioning,
• the encouragement of classroom dialogue, and the generation of puzzlement are all done in relation to student thinking and mathematical development and, thus,
• work towards facilitating students' mathematical constructions.
• Teacher waits and allows for thinking after asking a question that requires deep thought rather than a quick answer. Teacher questions students' mathematical reasoning and encourages other students to comment on it; and,
• attempts to facilitate puzzlement around big ideas, strategies that could be more efficient, etc.
• Teacher uses, chooses, and thinks about manipulative use in relation to students' thinking |

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Description: The **Use of Context Scale** depicts teacher's development from a mechanical use of context merely as a locus for applying taught procedures, towards the use of (realistic: i.e. realizable, imaginable) contexts and `truly problematic' situations both as starting points for mathematical constructions and as a didactic to facilitate mathematical development.		
1. Lack of context, or mechanical use of context: either the mathematical work is done entirely within the **domain of bare numbers (no context at all),** or contexts—mostly limited to stereotypical word problems—are used for the application of previously learned concepts and procedures. **Operational behaviors to look for when coding:** • Teacher explains to children that they have been working on a topic, i.e. addition, and now they will do some problems where it is used. • No context is used at all, or when used, problems are trivialized "school type" word problems to see if children can apply the operations and procedures they have already been taught.	2. **Word problem types of contexts** are used as a starting point for construction, in contrast to application of previously learned knowledge as depicted in level one. But this serves merely the purpose of motivation or to elicit children's thinking; no attention is paid to the process whereby mathematical ideas and/ or strategies may emerge or originate from suggestions or constraints in rich contexts. **Operational behaviors to look for when coding:** • Teacher embeds "school math" into a word problem. • Children's names may be used in problems to motivate, spark interest, but context is trivial and not likely to generate new strategies or bring big ideas up for discussion or exploration.	Use of realistic contexts and **truly problematic situations as a didactic.** **Operational behaviors to look for when coding:** • contexts are purposely designed so as to bring to develop mathematical big ideas, models, and strategies. • Contexts have implicit potentially realized suggestions of ideas or strategies built-in, or context has potential constraints to learners' strategies built-in. • Teacher adapts and modifies context as she works with different children, in relation to learner's reasoning. For example, teacher may decide to not allow cubes, or may place numerals, rather than dots, on a die, to encourage children to "count on." • Teacher uses models like the open number line, or the open array, as a didactic to bridge from the informal to the formal.

Description: The **Mathematical Content Scale** depicts the development in teacher's ability to see the mathematics in students' work, to becoming able to identify relevant connections across solutions, and to bring students' solutions to a higher level of mathematical sophistication.		
1. The teacher seems **blind to the mathematical potential of students**' work; all "math moments" in relation to the "landscape of learning" are missed. This may happen either because the teacher is unaware of critical big ideas, strategies, and mathematical models due to his/her lack of mathematical/didactic knowledge or because s/he is too intent on obtaining his/her expected answers. **Operational behaviors:** • teacher questioning is centered on procedures, or tool use, such as which manipulative child used, or whether the correct answer has been produced. • Potential discussions around big ideas or powerful strategies are missed. • Focus is on rote procedures, skills, and answers.	2. The teacher **begins to focus on the mathematics in students' work**, exploring some "math moments" with an eye to facilitating the construction of big ideas, models, and strategies; yet, many are still missed. **Operational behaviors:** • Teacher questions around concepts and understanding, but not with an eye towards critical big ideas or the developing strategies. • Emphasis is placed on mathematics as concepts from the discipline to be understood, rather than the cognitive, mathematizing of the learner. • For example, teacher may use open arrays as a way to explore students' conceptual understanding of strategies for multiplication, but may miss the fact that the array itself may be being mathematized differently by different learners…some learners may be struggling with how a square can represent a row and a column simultaneously.	3. Teacher takes advantage of most or all of the "math moments," thus, taking a proactive role in facilitating the development of **students' mathematical constructions.** **Operational behaviors:** • Teacher questions and facilitates discussion around important mathematical big ideas and strategies. Uses mathematical models as a bridge to enable children to move from their initial level of mathematizing towards a more formal one. For example, in mental math mini lessons, teacher scaffolds discussion from less efficient to more efficient strategies and uses the open number line to illustrate the jumps. • Teacher's lesson is both designed and implemented around the development of the big ideas and strategies and models in the "landscape of learning." Mathematics is understood as mathematizing and the potential for the development of it is noticed and acted upon throughout the lesson.

Instrument 4
Lesson observation instrument.
1. Access and understand the mathematical thinking of learners (Code: TA)

a) What introduction to the lesson did teacher use?

1. Number, Procedure or Method	2. Teacher states concept to be discussed or explored.	3. Teacher presents a realistic problem that needs mathematising.

b) How many students were involved in the discussion regarding the introduction of the lesson?

1. None.	2. One or Two	3. At least half the class.

c) Variety of Teacher representation (Schoenfeld & Kilpatrick, 2008).

1. One single representation	2. Two similar representations	3. More than two and connections between them.

2. Probe mathematical ideas (Code: TP)

1. Only teacher initiated ideas are presented in the lesson. Teacher sticks to the 'script' that he/she has prepared in advance for the lesson. Very little interaction between teacher/students. There is focus on 'right' and 'wrong' answers.	2. Moving towards dealing with student ideas that the teacher is comfortable with and disregarding ideas/suggestions/questions that don't fit into the planned lesson. Some interaction between teacher and students but not between students yet. There is focus on 'why' and 'how' questions.	3. Being guided by student ideas during the lesson. Making connections between student ideas and the planned lesson outcomes. Allowing interaction between students. Continually asks leading questions and waits for responses.

3. Connecting student understandings, ideas and concepts.
Code: TC

1. Teacher presents one or two 'methods' for class to practice.	2. Teacher asks students to present their solutions. No discussion on how methods are similar/different/better/faster etc.	3. Students present solution methods. Teacher applies 'variation theory' (Guo & Pang 2011) to contrast, separate, fuse and generalize student ideas.

b) Variety of Teacher representation (Schoenfeld & Kilpatrick, 2008).

1. One single representation	2. Two similar representations	3. More than two and connections between them.

4. Assess the mathematical knowledge of learners. Code (TAS)

a) Specific features of solutions in the classroom (NCES, 2003)

1. Single solutions presented publically.	2. Alternative methods/solutions presented Students had a choice of method.	3. Students own construction and interpretations encouraged. Connections between student constructions.

5. Teacher Reflection on Practice. Code (TR)
The teacher's reflection on the lesson

1. Is based on his/her own actions, understandings of what constitutes a good lesson. Is based on the ease of which students were able to do set work. Is based on how much work was covered. Is defeatist in nature.	2. Is based on student difficulties, but 'what' they were and not necessarily 'why' or 'how they came about. Is focussed on using the information for the next lesson to eliminate difficulties.	3. Is focussed on 'how' and 'why' students had difficulties.

6. Student Activity Code: (SA)

1. Students are largely: Silent Seated Practising methods given by teacher Do same type of work every day,	2. Students are given the opportunity to ask/answer questions. Solve a variety of problems. Individual work/pair work.	3. Students solve interesting, realistic, challenging problems. Collaborate with each other during parts of the lesson.

7. Student Differentiation Code: (SD)

1. Teacher ideas are presented.	2. Student ideas used and discussed.	3. Variation theory – student ideas are actively contrasted.

8. Student Mathematisation Code: (SM)

1. Mechanistic lesson: No horizontal or vertical mathematisation	2. Empiricist Lesson: Horizontal mathematisation only Structuralist Lesson: Vertical mathematisation only	3. Realistic Lesson. Both horizontal and vertical mathematisation, both purposefully used.

9. Teacher Vertically Aligned Lessons Code: (TVAL)

1. Teacher is presenting lessons or concepts as decided by an 'outside' source (textbook etc)	2. Teacher is planning for the next few lessons in terms of vertically mathematisation possibilities.	3. Teacher is fully in control of a trajectory that anticipates and prepares students for developing of vertical concepts.

Instrument 5
To be used for follow-up classroom observation visits.
Teacher: _____Date: _____

This study deals with nine principles we have spoken about. In your lesson which of the principles do you think came out strongly (S) and which would you like to improve (I)? (Use S or I in the second column). In the third column could you tell me why you say so or what challenges you faced in developing these principles in your lesson?

Principle	S/I	Reason?
Active students – what did they have to do (mentally, socially, physically, verbally etc)? Did they do most of the mathematics or did you?		
Student differentiation – how could all levels in the class take part in the lesson?		
Mathematisation- how could students build from their own informal ideas to more abstract ideas? How did you help them build these ideas?		
Building the concept- have you laid the foundation for the next lesson, next section, and next year?		
Accessing what students already know – how did you start the discussion about the concept you wanted them to learn about today?		
Probing what students understand about this concept. How did you find out about what students already know about the concept you wanted them to learn today?		
Connecting the different understanding in the classroom. How did you connect different student understandings and then their understandings with what you wanted them to learn?		
Reflecting on your lesson – seeing the lesson from students' perspective. How do you feel about the lesson? What went well, not so well? How do you think students feel?		
Other things I need to know? (Difficult class, class is behind, test coming up etc)		
General: What would you say was different about you or your lesson in this lesson compared to the first lesson I observed?		

Instrument 5

Teacher: _____Date:_____

Please complete as preparation for the classroom observation.

What is the mathematical goal of the lesson?	What are you anticipating students will find difficult?
What are the students going to **do** in this lesson?	How are the weak/strong pupils going to react to this lesson?
What are the new mathematical ideas students will have to connect to what they already know?	Why is today's concept important for future understanding of mathematics?
What information that you get from today's lesson will assist you in assessing the success of the lesson?	Any other things I need to know?

Instrument 6
Take-home reflection activity
Modelling Problem: ___   Teacher: _____

Didactisation Principle:
TA What did this problem access in terms of mathematical concepts and understandings? How important are these ideas? Could you treat these ideas equally well in a traditional way?
TP What type of questions should/could a teacher ask while students are working on this problem so that the teacher can probe student ideas to make them available to themselves and the rest of the group?
TC How can you deal with the different ways students may use to approach this problem? How can you overcome classroom issues in dealing with this? Can you compare and contrast the different ideas that are elicited in solving the problem? Can you arrange them in order of complexity?
TAS What is the most important aspect of assessing a problem such as this one? How can you overcome classroom issues?
TR How different is this problem/way of working from your normal teaching? Is learning from this type of activity possible or valuable or necessary?
SA What forms of activity are evident from this problem? Why are they important? Could you reach this type of challenge through traditional teaching?
SD In how many different ways could this problem (or part thereof) be approached? Why is this important?
SM How were students able to use mathematics in these problems? Did they develop more complex mathematics as they progressed? Did they have to have a 'different' understanding of mathematics? Does school mathematics prepare them for these types of problems?

Teachers provided written feedback to these questions.

Instrument 7
Take-home reflection (after session 1)

Didactisation principle	You currently do this	You feel you could improve this with the right type of problems	What pressure is on you or your classroom that makes it difficult for you to do this	Any other comment/suggestion you want me to be aware of?
Make pupils more active in your class				
Use a problem that they can all access and then lead to the concept				
Make sure that pupils can apply the mathematics or can build on the concept you have taught them				
Accessing student understanding by using real/word problems				
Probe their thinking				
Connecting the different ways that students in the class understand a concept				
Assessing students using problems				
Reflecting on what has happened in you class - not just what students don't understand but **why** they don't understand it				

Instrument 7. Take-home reflection (after session 2,5,7)
Teacher: _____Date: _____

1. When observing the groups of students solving the modelling task, what stood out/surprised you/impressed you etc?

2. Comment on what appears to be an inversion of mathematics teaching. Problems should be taught before bare manipulation of concepts and numbers. The concepts can be better understood through seeing it in the problem. We normally teach the concept and number manipulation and then ask pupils to apply it in 'word sums'…(if we have time…)

Instrument 7: Take home reflection (after session 6)

1. How is this type of planning different to your current way of thinking about mathematics lessons?

2. Which element will you find most helpful in your planning? Why?

3. Which element will you find difficult? Why?

4. Which element do you think will improve your lessons?

Instrument 8a: Guideline for discussions

	Activity	**Differentiation**	**Mathematisation**	**Vertically aligned lessons**
Traditional presentation				
Problem-based presentation				

Examples of problem for this session:

Look at the following tasks and arrange them into the two groups above. One group being a more traditional approach to lessons and one group showing how didactisation can be improved. Discuss the potential of each problem types according to the cards given to you.

Present your findings on the chart above.

1 min =_____ sec	Around March each year the Grade 6s and 7s go on camp. Here are the numbers that are going this year. Gr 6a: 27 Gr 6b: 31 Gr 7a: 23 Gr 7b: 24 Pupils travel in small buses. Each bus can carry 15 pupils. Four teachers are also going on the camp. How many buses must their teacher Mrs Smith organise?	Of the 180 students in the seventh grade at Hillcrest school, $\frac{2}{3}$ of them are on a sports team, and $\frac{1}{4}$ of those active in sports are on a basketball team. What fraction of all the students are on a basketball team?
300 sec = _____ min	At the camp there are canoes to rent. There are 2, 3, 4 or 5 seater canoes. Mrs Smith prefers to order the same size canoe, how many canoes of what size must she order? (The teachers will not be going in canoes).	$\frac{1}{4}\ of\ 120$
About how old is a person who is a million seconds old?	105 ÷ 15 =	In a recent survey of 300 Kempton Park residents, 216 people said the airport noise is too loud. Mike (a reporter) for *Fraction Times* writes the following

		headline: "216 People Think Airport Noise Too Loud." He realises that this is not a good headline and decides against using it. **1.** Explain why Mike decides not to use the headline. **2.** Write a better headline to summarise the situation.
How long does it take an average pupil to tie shoelaces?	Is 105 divisible by 2,3 4, or 5?	Write as a fraction and simplify: 216 out of 300.

Some problems taken from: Abels, M. & Wijers, M. 2010 and Keijzer, R *et al.* A. 2010.

b) Guideline for discussion (Session 6)
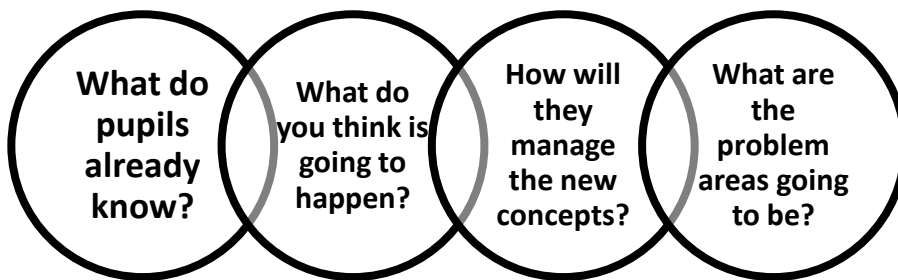
Adapted from Van De Walle *et al.* (2010: 59)