Teachers identifying and responding to learner errors and misconceptions in Numbers, Operations and Relationships in the Intermediate Phase

By

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Declaration

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Abstract

The purpose of this study was to investigate the ways in which Intermediate Phase (Grades 4, 5 and 6) teachers identify and respond to learners' mathematical errors and misconceptions in the content area of Numbers, Operations and Relationships (NOR) as prescribed by the Curriculum and Assessment Policy Statement CAPS (Department of Basic Education [DBE], 2011).

It is evident that errors and misconceptions arise when learners are learning specific aspects of mathematics. The ability to distinguish between, and address, errors and misconceptions is key to learning and improving learners' mathematical achievements within the mainstream class context. The identification of these errors and misconceptions is important for teachers, as this offers them the opportunity to adapt their teaching style and lessons to help eradicate learners' errors and misconceptions.

This study focuses specifically on the ways teachers identify the challenges learners face in understanding numbers, operations and relationships and explores suitable instructional methods for the remediation of those errors and misconceptions. These methods include learner revision strategies, in which learners engage in cognitive conflict by seeking out and addressing the mathematical and/or cognitive nature of their errors.

A qualitative approach was adopted by using three cycles of participatory action research (PAR). The empirical data collected during this study included a variety of data collection instruments. The instruments include interviews, class observations, focus groups and field notes. Data from these instruments were analysed and the findings suggest that, when teachers attend to learner errors and their reasoning behind their errors, they were able to identify and address misconceptions.

Opsomming

Die doel van hierdie studie was om ondersoek in te stel na wyses waarop onderwysers in die Intermediêre Fase (Graad 4, 5 en 6) leerders se wiskundige foute in en wanopvattings oor die inhoudsarea van getalle, bewerkings en verwantskappe, soos voorgeskryf deur die Kurrikulum- en assesseringsbeleidsverklaring (KABV, 2011), identifiseer en wat die onderwysers se reaksie op hierdie kwessies is.

Dit is duidelik dat foute en wanopvattings by leerders ontstaan tydens die onderrig van wiskunde. Die vermoë om te onderskei tussen foute en wanopvattings en die aanspreek van hierdie kwessies, is belangrik vir leer en vir die verbetering van leerders se wiskundeprestasies binne die konteks van ’n hoofstroomklas. Die identifisering van hierdie foute en wanopvattings is beduidend belangrik vir onderwysers, aangesien dit hulle die geleentheid bied om hul onderwysstyl en lesse sodoende aan te pas dat die geïdentifiseerde dilemmas uitgeskakel kan word..

Hierdie studie fokus spesifiek op die wyses waarop onderwysers die uitdagings identifiseer wat leerders ondervind rondom getalbegrip, bewerkings en verwantskappe. Tydens die ondersoek is navorsing gedoen en ondersoek is ingestel na geskikte metodes vir die remediëring van daardie foute en wanopvattings. Hierdie metodes sluit hersieningstrategieë in waar leerders kognitiewe konflik gebruik om die wiskundige en/of kognitiewe aard van die foute uit te lig en aan te spreek.

’n Kwalitatiewe benadering is oor drie siklusse van Deelnemende Aksienavorsing (DAN) gevolg. Die empiriese data is ingewin deur die aanwending van ’n verskeidenheid van dataversamelingstegnieke. Dit sluit in persoonlike onderhoude, klaswaarnemings, fokusgroepe en veldnotas. Data oor hierdietegnieke is ontleed en daar is bevind dat wanneer onderwysers aandag skenk aan leerderfoute en hul redenasie oor die foute, hulle wanopvattings kon identifiseer en dit gevolglik kon aanspreek.

Key words

Intermediate Phase teachers

Learners' mathematical errors and misconceptions

Number, Operations, Relationships

Procedural and conceptual knowledge

Mathematics Assessment

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List of Abbreviations and Acronyms

ANA	Annual National Assessments
BODMAS	Bracket, Of, Division, Multiplication, Addition, Subtraction
BQ	Brain Quest
CAPS	Curriculum and Assessment Policy Statement
DA	Developmental Appraisal
DBE	Department of Basic Education
DIPIP	Data Informed Practice Improvement Project
IP	Intermediate Phase
IQMS	Integrated Quality Management System
LR	Learner Response
MCO	Maths Curriculum Online
NOR	Numbers, Operations and Relationships
PAR	Participatory Action Research
SBA	School-based Assessment
TIMSS	Trends in International Mathematics and Science Study
QR	Qualitative Research

Chapter 1

Motivation, Rationale and Research Questions

1.1 Motivation

My motivation for exploring the topic, 'Ways teachers identify and respond to learner errors and misconceptions in Numbers, Operations and Relationships in the Intermediate Phase', started with my questioning of the impact on teachers of having to cope with large number of learners attending mathematics interventions within a mainstream class. Intermediate Phase teachers are concerned about the effectiveness of their current teaching practice and are interested in exploring different remedial strategies, within a mainstream class, to identify and respond to learners' mathematical errors and misconceptions in the Learning Area of Numbers, Operations and Relationships (Department of Basic Education [DBE], 2011). These concerns were shared during formal and informal discussions with colleagues and teachers in our cluster and district. My interest in this research field is to improve the Mathematics results of learners within the South African context. Even though South Africa's score in the Trends in International Mathematics and Science Study (TIMSS) improved by 20 points from 2011 to 2015, South Africa's overall performance is very close to the bottom of the surveyed countries (Reddy *et al*., 2016).

For the past nine years I have been teaching at Grade 5 level and, on reflection on various statistics [including school-based assessments, past Annual National Assessments' (ANAs) results and systemic results], I picked up a noticeable decline in the outcome of learners' Mathematics achievements. As an Intermediate Phase teacher working daily with diverse learners who are at different Mathematics levels, I feel that, instead of assigning blame for learners' low achieving performance, teachers need to find a more constructive way of identifying and responding to learners' mathematical errors and misconceptions.

Through the IQMS (Integrated Quality Management System) for school-based educators, I was able to identify areas for my personal professional development. Developmental appraisal (DA) is one of the IQMS programmes and is aimed at enhancing and monitoring the performance of the education system (Employment of Educators Act, No.76 of 1998). The purpose of DA is to appraise individual educators in a transparent manner with a view to determining areas of strength and weakness, and to draw up programmes for individual development. Teachers are appraised in classroom management to identify areas in which they need to be developed in teaching and learning. DA is aimed at a general level, i.e. all school subjects, and is not subject-specific, e.g. Mathematics.

As stated earlier, as a Mathematics teacher I am currently observing an annual increase in the number of learners attending Mathematics intervention within a mainstream class. The current minimum pass requirement in the CAPS (Curriculum and Assessment Policy Statement; DBE, 2011) is 40% for the Intermediate Phase and, in our school situation, learners who perform in a low achievement range in Mathematics (below 45%) receive intervention/learning support. These are two weekly sessions of an hour each. As there are no specific periods allocated for learning support on the class timetable, these sessions take place during contact time. Some classes have up to 15 learners attending these sessions out of a class size of 33 learners.

I believe that if teachers are able to identify learners' errors and misconceptions in the content focus area of Numbers, Operations and Relationships and apply remedial strategies, this will lead to a decrease in the number of learners attending intervention sessions and thus allow more contact time in a mainstream class. The prescribed instructional time per week for Mathematics is six hours for Grades 4 to 6. If 40% of the Grade 5 learners attend Mathematics intervention sessions, that accounts for 33% of the instructional time per week.

1.2 Problem Statement

The Learning Area of Numbers, Operations and Relationships in Mathematics in the Intermediate Phase plays a vital role in the acquisition of specific skills (including conceptual and procedural knowledge), and in preparing learners to apply these skills and understanding in the Senior Phase.

Support in Mathematics through intervention sessions is viewed as necessary and important in facilitating an improvement in learner achievements on a national and international level. For these concepts of Numbers, Operations and Relationships to be strengthened and perhaps remediated, teachers within the Intermediate Phase should be able to access learners' prior knowledge within this specific mathematical domain and identify the nature of learners' mathematical errors and misconceptions.

1.3 Rationale

The level of learners' mathematics achievements in Number concepts, especially, has gained much interest on an international level. Many countries who scored in the bottom range, including Morocco, Kuwait and South Africa are concerned about poor learner performance in Mathematics, as revealed by international studies such as the Trends in International Mathematics and Science Study (TIMSS, 2015). One of the goals of TIMSS is to help countries make informed decisions about how to improve teaching and learning in Mathematics and Science. It can be viewed as a valuable tool for countries to use to evaluate the achievement of goals and standards and to monitor trends in learners' achievements in an international context. South Africa has participated in five TIMSS projects (1995, 1999, 2003, 2011 and 2015) and has scored in the bottom range of the surveyed countries each year.

One of the general aims of the South African National Curriculum Statement (DBE, 2011) is that inclusivity should become a central part of organising, planning and teaching at each school. This can only happen if all teachers have a sound understanding of how to recognise and address barriers to learning, and how to plan for diversity, i.e. dealing with learners who are at different levels of understanding.

According to Education White Paper 6: Building an inclusive education and training system (2001), changes within education and training needed to be made so that learners with special needs, including those within the mainstream who have educational needs, can be accommodated adequately. An inclusive education and training system is organised so that it can provide various levels and kinds of support to learners and educators. It is within this framework that I believe an investigation of how Intermediate Phase teachers identify and respond to learners' mathematical errors and misconceptions in Numbers, Operations and Relationships could lead to supportive instruction strategies.

1.4 Research Questions

My focus for this research study is Numbers, Operations and Relationships (NOR), as this area forms the basis of understanding mathematical operations and related concepts and procedures. According to the Curriculum and Assessment Policy Statement (DBE, 2011), the weighting of this area has been increased to 50% across the Intermediate Phase (Grades 4, 5 and 6) in an attempt to ensure that learners are sufficiently numerate when they enter the Senior Phase. This makes it an important area in the intended curriculum and hence the focus of my study.

My research study was guided by the following main question and sub-questions:

Main research question:

How do Intermediate Phase teachers identify and respond to learners' mathematical errors and misconceptions in the content area of Numbers, Operations and Relationships?

The sub-questions:

1. In what ways do teachers recognise and distinguish learners' low and high performances as mathematical errors or misconceptions?

2. How do teachers access learners' prior mathematical knowledge in the focus area of Numbers, Operations and Relationships?

3. How do teachers analyse *procedural and conceptual knowledge* and use their findings to remediate errors and misconceptions?

1.5 Research Aims

This research study intended to achieve the following aims:

1. Investigate how Intermediate Phase teachers interpret learners' reasoning about their own errors and misconceptions in Numbers, Operations and Relationships;

2. Analyse and assess the relevance of establishing learners' prior mathematical knowledge which influences their understanding of new concepts; and

3. Explore the relationship between conceptual and procedural knowledge and examine how the application of these notions contribute to the remediation of errors and misconceptions in *Numbers, Operations and Relationships*.

1.6 Structure of the Study

This study is organised into five chapters. Chapter 1 provides the motivation and rationale for this specific research focus. In it the main research question and sub-questions are outlined, and the research aims are specified. The literature review in Chapter 2 includes explanations of key concepts relevant to this study. The key concepts discussed are learner errors and misconceptions and conceptual and procedural knowledge. A review of teachers' responses to learner errors concludes this chapter. Chapter 3 outlines the research design and methodology employed in this study, namely participatory action research (PAR). It explains the rationale for choosing this method, as well as the data collection tools, i.e. semi-structured interviews, observations and document analysis. The analysis of my findings, which responds to the main research question, is presented in Chapter 4. Chapter 5 concludes the study with recommendations emerging from the study and recommendations for possible further studies.

Chapter 2

Literature Review

2.1 Introduction

The main research question – 'How do Intermediate Phase teachers identify and respond to learners' mathematical errors and misconceptions in the content area of Numbers, Operations and Relationships?' – required a review of the literature:

- on middle grades (intermediate phase) learners' mathematical errors and misconceptions in general
- on the area of Numbers, Operations and Relationships
- on primary school teachers' responses to and understanding of their learners' mathematical errors and misconceptions

Central to the theoretical framework necessary to answer the main research question is "Intermediate Phase (IP) teachers' identification and responses" as the unit of analysis.

The introduction to this chapter motivates the relevance and importance of supporting teachers in finding ways to identify and respond to learners' mathematical errors and misconceptions. Firstly, it presents the specific Mathematics content focus for the content area of Numbers, Operations and Relationships in the Intermediate Phase according to the Curriculum and Assessment Policy Statement (CAPS; DBE, 2011). Secondly, I identified key concepts that provide a framework within which to conduct the literature review. Here a description is provided of the concepts of learner errors and misconceptions, and conceptual and procedural knowledge. This chapter concludes with a review of primary school teachers' responses to learner errors.

The rationale that informs this study is based on my view that teachers in the Intermediate Phase should be exposed to various strategies that could assist them to identify, distinguish and address learners' errors and misconceptions; access learners' prior mathematical knowledge; and analyse procedural and conceptual knowledge in the content area of Numbers, Operations and Relationships. The notion of inclusivity can be accommodated within a mainstream class and possibly lead to a decrease in the number of learners attending learning intervention sessions.

As mentioned in my motivation for this study, I believe that, instead of assigning blame for learners' poor achievements in Mathematics, or viewing learners as having poor mathematical ability, teachers need support in learning how to find constructive ways of identifying and responding to learners' mathematical errors and misconceptions. As teachers we are only able to assist our learners when we are able to work at the level of specific detail and get to know the specific roots of mistakes (Olivier, 1992).

Mathematics in the Intermediate Phase covers five content areas:

- Numbers, Operations and Relationships
- Patterns, Functions and Algebra
- Space and Shape (Geometry)
- Measurement; and
- Data Handling

Each content area contributes towards the acquisition of specific skills. This study will focus on the specific content of Numbers, Operations and Relationships.

Table 2.1: Content knowledge for Numbers, Operations and Relationships in the Intermediate Phase

MATHEMATICS CONTENT KNOWLEDGE		
Content area	**General content focus**	**Intermediate Phase-specific content focus**
Numbers, Operations and Relationships	Development of number sense that includes: • The meaning of different kinds of numbers • Relationship between different kinds of numbers • The relative size of different numbers • Representation of numbers in various ways	• The range of numbers developed by the end of the Intermediate Phase is extended to at least 9-digit whole numbers, decimal fractions to at least 2 decimal places, common fractions and fractions written in percentage form. • In this phase, the learner is expected to move from counting reliably to calculating fluently in all four operations. The learner should be encouraged to memorise with understanding, multiply fluently, and sharpen mental calculation skills.

MATHEMATICS CONTENT KNOWLEDGE		
Content area	**General content focus**	**Intermediate Phase-specific content focus**
	• The effect of operating with numbers • The ability to estimate and check solutions	• Attention needs to be focused on understanding the concept of place value so that the learner develops a sense of large numbers and decimal fractions. • The learner should recognise and describe properties of numbers and operations, including identify properties, factors, multiples, and commutative, associative and distributive properties.

Source: DBE (2011:10)

Smith, Disessa and Roschelle (1993) agree that, where previous research only differentiated between correct and incorrect responses, research about the identification of misconceptions is needed to explain learners' frequent errors. I am of the view that learners' mathematics performance may improve through the implementation of various practices and that their number sense would be sufficiently developed when entering the Senior Phase.

2.2 Learner errors and misconceptions in general and in the domain of Numbers, Operations and Relationships

Some researchers conceptualise the concepts 'errors' and 'misconceptions' as different though interrelated. When Olivier (1992) distinguishes between errors and misconceptions, he classifies errors as 'wrong answers due to planning; they are systematic in that they are applied regularly in the same circumstances'. For example, errors are the symptoms of the underlying conceptual structures that are the cause of errors. It is these underlying beliefs and principles in the cognitive structure that are the cause of systematic conceptual errors called 'misconceptions'. Olivier suggests that, because learners overgeneralise numbers, errors may be predicted for problems where numerical values are critical. The overgeneralisation of number and number properties may be an important underlying cause of learners' misconceptions (Olivier, 1992:6).

Olivier distinguishes between two learning theories that explain different approaches to handling learners' misconceptions – behaviourism and constructivism. The distinction between these learning theories is relevant to this study to classify primary school teachers' responses and understanding of their learners' mathematical errors and misconceptions.

Behaviourism is explained through the view of learners as passive recipients of knowledge, whereby their existing knowledge is irrelevant to learning. Behaviourists view errors and misconceptions as not being important because they do not consider learners' current concepts as relevant to learning. Constructivism, however, views the learner as an active role player in the construction of his/her own knowledge, in terms of which new concepts are interpreted and understood in the light of the learners' own current knowledge – which is gained through his/her previous experiences. From a constructivist perspective, misconceptions are crucially important to learning and teaching, because misconceptions form part of a learners' conceptual structure that will interact with new concepts and influence new learning, mostly in a negative way, because misconceptions generate errors. Olivier concludes that misconceptions cannot be avoided and that making errors is an important part of the learning process.

It therefore is suggested that teachers advocate for classrooms that are tolerant of errors and misconceptions and use them as opportunities to enhance teaching and learning. Teachers are encouraged to help learners make connections between new knowledge and previous learning. Errors can be used by teachers to provide learners with epistemological access to Mathematics and contribute to developing learners' conceptual understanding. Borasi (1987:2) shares the opinion that errors can be a powerful tool to diagnose learning difficulties and consequently direct remediation. She explores errors as having educational potential and views errors as "springboards for inquiry". Borasi suggests that errors can be used as a motivational device and as a starting point for creative mathematical exploration.

According to Hansen's (2011:1) study in primary schools, "errors can be the result of carelessness, misinterpretation of symbols or text, lack of relevant experience or knowledge related to that mathematical topic/learning objective/concept; a lack of awareness or inability to check the answer given; or the result of a misconception".

Hansen (2011:12) explains that misconceptions "could be the misapplication of a rule, an over- or under-generalization of the situation". She uses the three-digit example: *a number with three digits is "bigger" than a number with two digits* works in some situations (e.g. 328 is bigger than 35), but not necessarily in others situations, where decimals are involved (e.g. 3.28 is *not* bigger than 3.5). In this example, knowledge of whole numbers has been overgeneralised when working with decimal numbers. Even though it is often assumed that misconceptions occur with learners who need learning support, Hansen states that misconceptions are not limited to children who need additional support. Children who cope well also make incorrect generalisations.

Earlier research by Rubenstein and Thompson (2002) distinguish between 11 categories of difficulties associated with learning the language of Mathematics. Relevant to this study is the category referred to as "mathematical meanings that are more precise" – e.g. *product* as the solution to a multiplication problem vs. the *product* of a company.

According to Riccomini, Hughes and Fries (2015), the understanding of mathematical vocabulary grants access to concepts. They emphasise that teaching and learning the language of Mathematics is vital for the development of mathematical proficiency (Riccomini *et al*., 2015:236).

Schifter, Monk, Russell and Bastable (2008) examined elementary grade to middle grade learners' understanding of the properties of numbers and found that learners overgeneralise numbers. For example, when learners were questioned about whether the commutative and associative properties were true for addition, subtraction, multiplication and division, some learners incorrectly identified the associative property as true for subtraction and division.

Smith *et al*. (1993) also quote examples from Nesher (1987), showing that misconceptions arise from prior instruction when learners incorrectly generalise prior knowledge of whole numbers or common fractions to order decimal fractions.

In a study of middle-school learners conducted by Booth, Barbieri, Eyer and Paré-Blagoev (2014), they examined six categories of conceptual errors that included fraction errors. Their research found that fraction errors did not represent learners' misunderstanding of the values of fractions themselves, but was a misunderstanding between numerators and denominators.

Ashlock (2010) found that many students did not understand the concept of regrouping to solve addition and subtraction problems, such as 46 + 17 or 46 - 17. Students made systematic errors that suggested they did not have a good understanding of place value.

Watson, Lopes, Oliveira and Judge (2018) conducted a study with elementary learners to investigate the difficulties they have in mastering addition and subtraction calculation tasks. Their focus was on the Number and Operations content strand, which includes understanding numbers, knowing the meaning of operations and computing fluently. The results of their study included common errors among participants in both addition and subtraction tasks, where learners had problems with the conceptual knowledge of decimals, the base–10 system and place value. Learners experienced challenges with conceptual knowledge, for example addition in calculation tasks in which the sum of the given numbers did not result in a larger number than the given digits. Also, with subtraction, learners subtracted a larger number from a smaller one, which also reveals misunderstanding of the concept of subtraction. Identifying learners' misconceptions and/or lack of knowledge of a mathematical concept can thus assist teachers to make informed decisions about effective intervention strategies.

2.2.1 Conceptual and procedural knowledge

As a Mathematics teacher, I was interested in investigating how teachers' understanding of conceptual and procedural knowledge contributes to enhancing Mathematics competency in the area of Numbers, Operations and Relationships. Sidney and Alibali (2015) state that people learn new information in the context of their own prior knowledge, which can be procedural and/or conceptual. When learning new mathematical concepts, students draw on their existing knowledge of related mathematical concepts and procedures. They emphasise that understanding how learners build on prior knowledge is crucial to understanding how cognitive development occurs. Understanding how best to build on what learners already know is at the heart of effective instruction. Mathematical competence rests on developing both conceptual and procedural knowledge (Rittle-Johnson, Schneider & Star, 2015).

Different accounts of mathematical knowledge distinguish between conceptual and procedural knowledge. Procedural knowledge is commonly defined as knowledge of sets of actions that can be used to solve a particular type of problem. In contrast, conceptual knowledge is defined as knowledge of principles that apply within a domain, and knowledge about relationships among elements within a domain, including knowledge of the meanings of structures and processes used in the domain (Rittle-Johnson, Siegler & Alibali, 2001).

Kilpatrick, Swafford and Findell (2001) provide an analysis of 'mathematical proficiency' by identifying five strands that are interwoven and interdependent – conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. They state that mathematical proficiency cannot be achieved by emphasising just one or two of these strands.

Kilpatrick *et al*. (2001) define 'conceptual understanding' as an integrated and functional grasp of mathematical ideas, and procedural fluency as knowledge of when and how to use them appropriately. In their report, they specifically refer to the domain of number, in which procedural fluency plays an important role in supporting conceptual understanding with place value and the meanings of rational numbers. How learners represent and connect pieces of knowledge is a key factor in whether they will understand it deeply and can use it in problem-solving. Learning with understanding is more powerful than simply memorising, because the organisation improves retention, promotes fluency, and facilitates learning related material.

Conceptual understanding helps learners avoid many critical errors in solving problems – they can see the deeper similarities between superficially unrelated situations. An example Kilpatrick *et al*. (2001) refer to is that, if learners understand, for example, that addition is commutative – 3 + 5 = 5 + 3 – their learning of basic addition combination is reduced by almost half. Because learners learn doubling in their elementary years, they can use this understanding to produce closely related sums, e.g. 6 + 7 is just one more than 6 + 6. These relations make it easier for learners to learn new addition combinations because they are generating new knowledge rather than relying on rote memorisation.

The importance of procedural fluency is explained especially in the domain of number, where it supports conceptual understanding of place value and the meanings of rational numbers. Procedural fluency extends to the understanding of similarities and differences between methodologies, which include mental calculations, written procedures, calculating differences, products and quotients, or when using concrete objects for counting. When learners are taught procedures without understanding, they are limited to only applying the learned procedures. Teaching with the intention to help learners learn with understanding will empower them to adapt procedures for easier calculations, e.g. in an addition sum 598 + 647, learners with understanding would recognise that 598 is only 2 less than 600, so they might add 600 + 647 and then subtract 2. In other words, if learners build up a bank of computational tools, they should be able to select the appropriate tool to complete specific calculation tasks. In this way there is a likelihood that concepts and related procedures are integrated.

Emphasising the connectedness between the strands of conceptual understanding and procedural fluency, Kilpatrick *et al*. (2001:122) state that "understanding makes learning skills easier, less susceptible to common errors and less prone to forgetting, and that a certain level of skill is required to learn many mathematical concepts with understanding, and using procedures can help strengthen and develop that understanding".

Earlier research viewed conceptual and procedural knowledge concepts as separate entities, while recent researchers are focused more on the relationships between the two kinds of knowledge. Although conceptual and procedural knowledge are often discussed as distinct entities, they do not develop independently in Mathematics and, in fact, lie on a continuum, which often makes them hard to distinguish (Star, 2005).

Rittle-Johnson and Schneider (2014) review previous research on the relationships between conceptual and procedural knowledge, including the *concept-first view* (where learners acquire concepts first and build procedural knowledge from it), and the *procedures-first view* (where learners first learn procedures and then establish concepts).

Hiebert and Lefevre (1986) provide a distinction between the two kinds of knowledge with the intention to shed light on the teaching and learning process and assist in the understanding of learners' mathematical successes and where misconceptions may occur. They compare conceptual knowledge to procedural knowledge by stating that conceptual knowledge is a connectedness between pieces of knowledge, whereas procedural knowledge has a sequential aspect for completing tasks. "Procedural knowledge is characterized as step-by-step procedures executed in a specific sequence; conceptual knowledge involves a rich network of relationships between pieces of information" (Hiebert & Lefevre, 1986:113).

In a study conducted with prospective teachers, Bartell, Webel, Bowen and Dyson (2013) examined the ability of prospective teachers to recognise evidence of learners' conceptual understanding. They also designed an intervention programme that included examples in which learners used correct procedures that could be mistaken for evidence of understanding. Although the participants in their study were only prospective teachers, it is my belief that the focus of using learner samples that highlight the differences between evidence of procedural knowledge and evidence of conceptual understanding is relevant to both prospective and practising teachers. Their instrument to recognise conceptual understanding is relevant to this current study and it will therefore not focus on the content knowledge of prospective teachers used their study. The Bartell *et al*. study only refers to examples of developing mathematical understanding of key number and operation topics in the content of whole numbers and decimals. Three sample learner responses were presented to the prospective teachers to examine what each learner knew about the addition of decimal quantities. The aim was to assist the teachers in distinguishing between learners' procedural responses and demonstrating conceptual understanding.

Table 2.2: Three learners' responses to a decimal problem: 63.7 + 49.8

Student A	Student B	Student C
I solved the problem by changing it, taking 2-tenths away from 63.7, making it 63.5, and adding 2-tenths to the 49.8 making it 50. Then, 63.5 + 50 is 6 tens plus 5 tens (60 + 50) or 11 tens, which is the same as 1 hundred and 1 ten. Then I need to add the three ones and 5-tenths, making the sum 1 hundred, 1 ten, 3 ones and 5-tenths, or 113.5	$\begin{array}{r} {\scriptstyle 111} \\ 63.7 \\ \underline{+\ 49.8} \\ 113.5 \end{array}$ *First, 7 plus 8 is 15, so I wrote down the 5 and grouped the one with the one's column. Then I plus 3 plus 9 is 13, so I wrote down the 3 and grouped the one with the ten's column. Then I plus 6 plus 4 is 11, and wrote that down, giving me 113.5*	$\begin{array}{r} {\scriptstyle 111} \\ 63.7 \\ \underline{+\ 49.8} \\ 114.5 \end{array}$

Through a discussion, the teachers agreed to the following explanations of each sample response.

Table 2.3: Teacher distinctions between learner's procedural and conceptual responses

Student A	Student B	Student C
Conceptual, with the correct answer.	Procedural, with the correct answer; explanation may seem more conceptual.	Procedural, incorrect answer, not good evidence of understanding or misconception

Other examples included in their instrument were comparing fractions, multiplication of fractions and subtracting fractions. In their responses, learners who showed evidence of conceptual understanding explained their answers using conceptual features like diagrams and prior knowledge of fractions to group numbers. Other solutions by learners described procedural methods and revealed little about conceptual understanding. In their findings, Bartell *et al*. (2013:20) conclude that the ability to distinguish between evidence of conceptual understanding and procedural knowledge is a positive step in Mathematics intervention, but a persistent challenge.

Long (2011) and Rittle-Johnson and Schneider (2014) emphasise procedural and conceptual knowledge, and extend support through an iterative view that accommodates gradual improvements in each type of knowledge over time. Long (2011) made this discovery during a general Mathematics teaching practice course for prospective teachers. Rittle-Johnson and Schneider (2014) reviewed earlier research of a study of elementary school children's knowledge of fractions, with a focus on the relations between conceptual and procedural knowledge in Mathematics. Rittle-Johnson, Schneider & Star (2015:594) state that the "relations between conceptual and procedural knowledge are bi-directional, with increases in conceptual knowledge leading to subsequent increases in procedural knowledge and vice versa".

I was drawn to Hiebert and Lefevre's (1986:22) conclusion that an understanding of the relationship between conceptual and procedural knowledge "is important because it seems to hold the key to many learning processes and problems". One can argue that procedures underpin or inform concepts and vice versa, i.e. concepts underpin procedures. The Mathematics education literature does not offer a standard language agreement on differences between procedures and concepts.

2.2.2 Teachers' responses to learner errors

Although knowledge of content is important for Mathematics teachers, Ashlock (2010) states that teachers must also understand the nature of the errors learners make to provide corrective feedback to learners to eliminate those errors. It is advantageous that teachers identify what affects their learners' progress as early as possible and provide explicit instruction that addresses their individual needs. Reflection on learners' erroneous Mathematics concepts and/or procedures can provide effective instruction or strategies that addresses the diverse needs of learners (Watson *et al.*, 2018). If error patterns or misconceptions are not corrected early, these may persist and affect learners' acquisition of higher mathematical skills such as algebra (Ashlock, 2010; Khan & Chishti, 2011).

In a more recent study of teachers, Jong, Thomas, Fisher, Schack, Davis and Bickett (2017:14) explored the theory of *professional noticing* of learners' mathematical thinking as a set of interrelated skills, including

1. attending to learners' strategies
2. interpreting learners' understanding, and
3. deciding how to respond on the basis of learners' understandings

These authors view the implementation of these three strands as useful strategies when identifying and addressing these misconceptions.

Through their investigation of learners' misconceptions with decimals, Jong *et al*. (2017) proved that the patterns and rules of addition, subtraction, multiplication and division of whole numbers were generalised to decimals.

Earlier research, conducted by Jacobs, Lamb and Philipp (2010), used *professional noticing* with the three interrelated skills to explore *how*, and to what extent, teachers notice learners' mathematical thinking instead of focusing only on what teachers notice.

For their study, Jacobs *et al*. (2010) investigated the professional noticing of learners' mathematical thinking by teacher participants with different teaching experience. The participant group included prospective teachers (who had no formal teaching experience), initial teachers (who had teaching experience but no professional development), advanced teachers (who had teaching experience and two years of professional development) and lead teachers (who had the most teaching experience and four or more years of professional development). Their investigation was therefore a cross-sectional study aimed at assessing the different groups of participants' expertise in attending to, interpreting and deciding how to respond on the basis of the learner's understanding. The teacher participants were presented with evidence of learners' work and had to apply the interrelated skills of attending, interpreting and deciding how to respond to each example.

An example from their study (Jacobs *et al*., 2010:178) is provided below.

"*Todd has 6 bags of M&Ms. Each bag has 43 M&Ms. How many M&Ms does Todd have?*"

Evidence of learners' work showed different representations of calculating the answer. One learner drew six bags with 43 tallies in each bag and added the grouped tallies, and then counted the remaining 3 in each bag to get a final answer. Another learner used repeated addition of 43 and added two groups of tens and two groups of units each time (40 + 40 = 80 and 3 + 3 = 6). Counting in multiples of 40 for six groups and then adding on the multiples of three for six groups was another strategy used by a learner.

The results of each participant group, applying the three interrelated skills, were categorised into three levels of evidence – robust evidence, limited evidence and lack of evidence. Below is a summary of the participant groups' responses:

2.2.1 Attending to learners' strategies

Teachers were asked to describe mathematically significant details of how the learner counted, used diagrams or tools to represent quantities, or decomposed numbers to make them easier to manipulate. Responses by the teachers varied, from tracking the entire strategy of the learner in substantial detail, to a very general description of the learner's strategy, missing all reference to place value and decomposition.

2.2.2 Interpreting learners' understanding

The focus of this skill was on the extent to which the participants' reasoning was consistent with the details of the specific learner strategy. Responses by the teachers included details of the learners' strategy and how those details reflected what the learner understood and what strategies and understandings the learner did not demonstrate, e.g. the learners' ability to group numbers. Other responses from teachers did not provide any evidence of interpretation of learner understanding, even after they had been prompted to explain what they had learned about the learner's understanding.

2.2.3 Responding on the basis of learners' understanding

The focus of this skill was not to expect teachers to find a specific 'next step', but rather to evaluate the extent to which teachers based their decisions on what they had learned about the learners' understandings. Responses by the teachers included explicit considerations of the learners' existing strategy and anticipating a possible next strategy to further the learners' understandings, e.g. providing examples in which the learner can group numbers. Other responses were vague and had little or no reference to building on the learners' understandings or anticipating future strategies for the proposed problem.

The study by Jacobs *et al*. (2010:191) concludes that professional noticing of learners' mathematical thinking "merits attention from teachers" and that conducting a cross-sectional study proves that expertise in attending, interpreting and deciding how to respond can be learned.

An investigative approach based on a constructivist perspective in which the process of teaching and learning involves the adaptation of prior knowledge to accommodate new ideas could provide a lens through which possible intervention strategies may develop. The role of prior knowledge is crucial in constructivism, and learners' interpretations of tasks and instructional activities involving new concepts is filtered in terms of their prior knowledge. Bray (2011) conducted a study which investigated teachers' responses to learner errors by having learners analyse and revise flawed solutions during whole-group discussions. Three dimensions of error-handling practices during class discussion of Mathematics tasks emerged:

1) Intentional focus on flawed solutions in a whole-class discussion;

2) Promotion of conceptual understanding through discussion of errors; and

3) Mobilisation of a community of learners to address errors.

Bray (2011) concludes that current reforms in Mathematics education advocate instruction that emphasises classroom discourse that builds on learners' thinking, promotes conceptual understanding, and mobilises learners as a community of learners and suggests that teachers would benefit from greater awareness of common learner errors and how these errors are related to key Mathematics concepts.

Yorulmaz and Önal (2017) conducted a study aimed at identifying the errors primary school learners make in four operations, based on the views of class teachers. Errors in addition, subtraction, multiplication and division were investigated. Table 2.4 below provides a short summary of the most common error categories.

Table 2.4: Common error categories with common error sources

The four operations	Common error categories	Common error sources
Addition	Carrying errors	Forgetting to add the digits
	Place value errors	Not being able to write the digits one under another
	Counting errors	Difficulty in rhythmic counting
Subtraction	Decomposition errors	Unable to subtract tens Forgetting to subtract ten from the tens digit Not being able to subtract from a number whose two or three digits are "0"
	Operational errors	Subtracting the minuend from the subtrahend when the subtrahend is smaller
	Counting errors	Difficulty in backward rhythmic counting
	Symbolic errors	Confusing the terms of subtraction
Multiplication	Place value errors	Not scrolling digits in two-digit multiplication
	Operational errors	Forgetting the digits in multiplication Failure to transfer the addition to multiplication
	"0" digit errors	Errors in the multiplication by 0
Division	"0" digit errors	Failure to add "0" to the quotient

The four operations	Common error categories	Common error sources
	Place-value errors	Starting to subtract from the ones digit, not from the number on the left while dividing

Adapted from Yorulmaz and Önal (2017:1888)

One of the objectives of Yorulmaz and Önal's (2017) study was to determine the views of primary school teachers for the causes of errors in the four operations. According to the opinions of the teachers, the errors that learners make in the four operations are caused by:

- The learners – due to carelessness, failure to fully understand the four operations and not revising the concepts at home,
- The teacher – failure to make operations concrete, rote learning and insufficient training in rhythmical counting,
- The programme – limited time and inadequate class activities, large class sizes, and
- The learner's family and environment.

Another objective of their study was to find solutions offered by teachers regarding the errors learners make in the four operations. The following possible solutions were suggested by the teachers:

- The content – more examples and revision to be done; to make the subject concrete; more time spent on rhythmic counting activities
- The teacher – examples from daily life to be used; lessons made relevant and interesting; attention-increasing activities
- The learner's family – interest of families should be increase.

The causes of and solutions for learner errors stipulated in the views of the teachers in this study include identifying common learner errors and misconceptions, but do not include the analysis of conceptual and procedural knowledge and the relevance of establishing learner prior knowledge in order to address learners' common errors.

In a study done with the Data Informed Practice Improvement Project (DIPIP) over three years, Brodie (2014) identified three shifts that teachers made in their learning about learner errors. The project worked with senior phase Mathematics teachers to create and sustain professional learning communities with a focus on learner errors. The first shift was from identifying to interpreting errors; the second shift from interpreting to engaging with errors; and the third shift from focusing on learner errors to focusing on their own knowledge.

In the first shift – from identifying to interpreting errors – the teacher participants discovered that there are differences between the errors learners make and the reasoning behind the errors. To gain clarity in an effort to articulate the inner logic of learners' errors, specific questions were posed:

"*What could the learner be thinking in order to make the error? How can we see the error from the learners' perspectives? How might the error make sense to the learner, even if not to the teacher?*"

The second shift – from interpreting to engaging with errors – required teachers to re-think how they would address the errors. Considering that errors and misconceptions arise in the interconnections of different ideas, simply re-teaching concepts was not an effective strategy. Teachers reflected on their teaching practices to discover what might explain the systematic errors of learners.

Through enquiry into learners' knowledge, teachers started to reflect on and enquire into their own knowledge. This shift, from learners' knowledge to teachers' knowledge, is the third shift.

Although these three shifts may not necessarily occur in chronological order because of the nature of shifting practices, Brodie (2014) argues that they suggest a deepening of teachers' thinking in relation to learner errors. When asked to interpret learners' errors, teachers' responses were to re-teach the concept. Brodie (2014:232) explains misconceptions as "taking what they've learned before and applying it in a way that makes it wrong". She uses the example of learners understanding that "multiplication makes numbers bigger", but when you multiply fractions, they incorrectly apply the same rule. Brodie suggests that teachers do not have to re-teach concepts; rather, they have to connect with the misconception at the point at which they are making the misconception. Brodie challenges the perceptions of teachers that the best way to remedy errors is to re-teach, creating the assumption that errors arise from incorrect prior teaching or learning.

Chapter 3

Research Methodology

3.1 Introduction

In an attempt to answer the main research question – 'How do Intermediate Phase teachers identify and respond to learners' mathematical errors and misconceptions in the content area of Numbers, Operations and Relationships?' – a research design was required to guide this study.

The research design may be viewed as the master plan of a research study that sheds light on how the study is conducted. Crotty (1998:3) defines methodology as the "strategy or plan of action which lies behind the choice and use of particular methods – the why, what, from where, when and how data are collected and analysed".

Qualitative researchers are interested in understanding how people interpret their experiences, how they construct their worlds, and what meaning they attribute to their experiences (Merriam, 2009). A qualitative methodological approach shares its philosophical foundation with the interpretive paradigm – many truths and multiple realities (Merriam, 2009:8). There is fairly general consensus that qualitative research is a naturalistic, interpretative approach concerned with understanding the meanings that people attach to phenomena (actions, beliefs, decisions, values, etc.)

Considering the above criteria, this study followed a qualitative approach.

3.2 Participatory action research

I believe that participating in and using participatory action research (PAR) will give insight into how Intermediate Phase teachers identify learners' errors and misconceptions in the content area of Numbers, Operations and Relationships and respond by collectively working towards developing effective practices to remediate those errors and misconceptions through observation and reflection, which will lead to planning and action.

PAR is a research strategy that has been used in schools and could be carried out by individual teachers, small groups of teachers, teams or a single department.

Selener (1997) defines PAR as the process through which research participants can identify a problem, collect and analyse relevant information, and act upon it in order to develop solutions.

MacDonald (2012) considers PAR as a strand of ‘action research’, which is the systematic collection and analysis of data for the purpose of taking action and making change by generating practical knowledge. PAR is a cyclical process that works through multiple iterations of planning, taking action, observing and reflecting (Walter, 2009). Each new cycle is informed by the previous one and continues until a collaborative outcome is achieved. Crane and O’Regan (2010) define the cyclical nature as a key characteristic of PAR, and they emphasise that one PAR project should include multiple cycles. PAR provides opportunities to investigate current practice and develop a deeper understanding of the teaching–learning process. As researcher and Intermediate Phase teacher, I would view myself as a participant in the study, as this approach would offer me the opportunity to develop professionally by investigating current teaching practice.

The diagram in Figure 3.1 below summarises the iterative PAR cycle intended for this study.

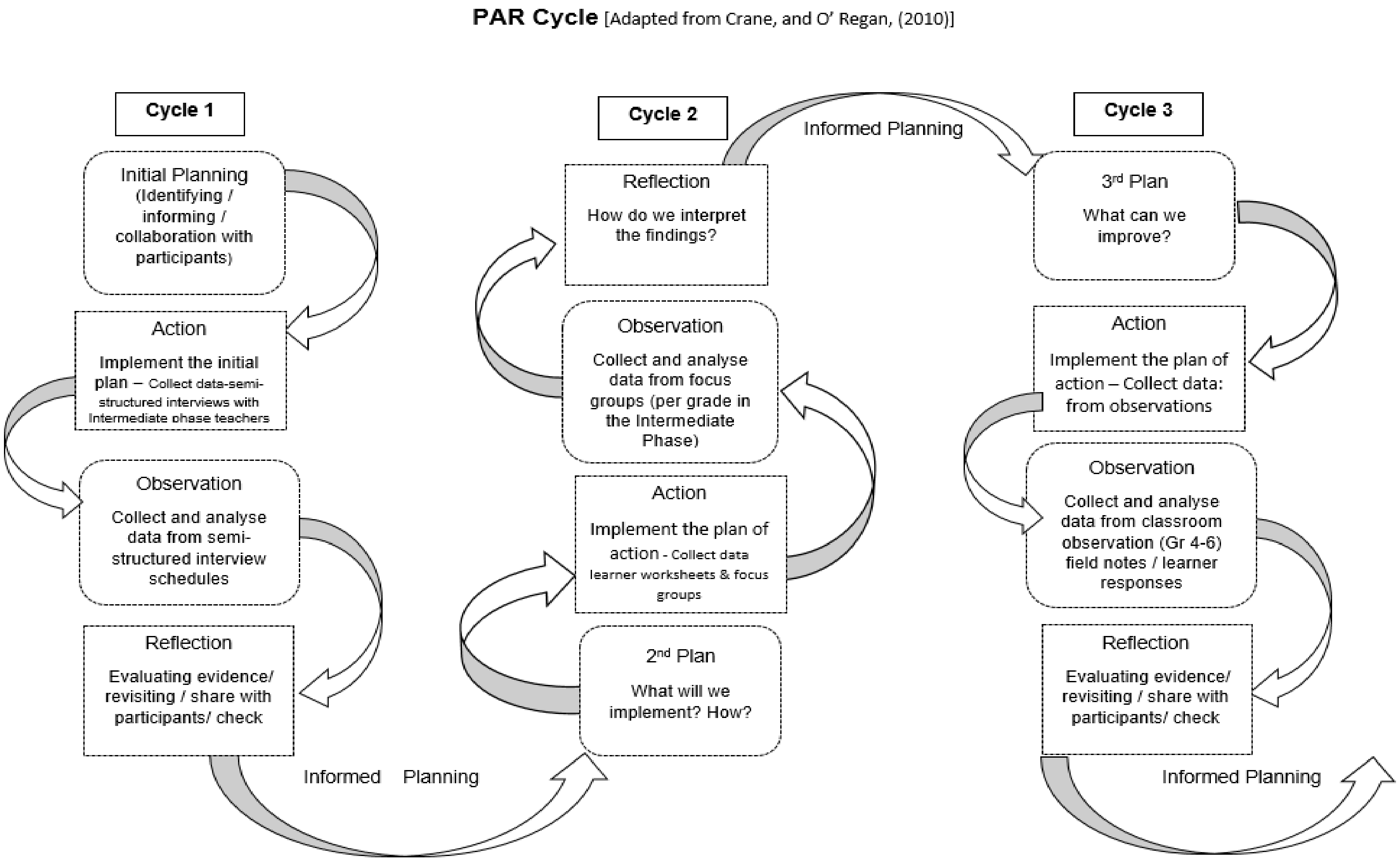


Figure 3.1: The PAR cycle

3.2.1 Sampling

To answer the main research question – 'How do Intermediate Phase teachers identify and respond to learners' mathematical errors and misconceptions in the content area of Numbers, Operations and Relationships?' – an appropriate sampling strategy was required.

Purposeful sampling was applied for participant and site selection for this study. Purposeful sampling assumes that the investigator wants to discover, understand and gain insight and therefore must select a sample from which the most can be learned (Merriam, 2009). The selected site was a middle school in the Western Cape that accommodates learners in Grade 4 to Grade 7. The target population for this study included 12 Intermediate Phase Mathematics teachers who were knowledgeable and experienced informants on this topic. The teachers in the Intermediate Phase implement class teaching for most subjects, including Mathematics. Although there is no predetermined sample size, Patton (2005) argues that the logic and power of purposeful sampling lies in selecting information-rich participants. As members of the Maths department, our subject meetings offered a platform to plan together and track content coverage across the Intermediate Phase. As teacher-researcher I had a working relationship with the Intermediate Phase teachers at the school, and I believed a deeper understanding of the purpose of this inquiry could develop through PAR.

Table 3.1 below illustrates the sample population of teachers with years of teaching experience.

Table 3.1: Teaching experience of participants in years

Male/ female	Teacher/ participant code	Years in Intermediate Phase	Years in Foundation Phase	Years in Senior Phase	Special needs	Total years
F	T1	18				18
F	T2	3			27	30
F	T3	37				37
F	T4	19			1	20
F	T5	22				22
F	T6	17				17
F	T7	20	5	3		28
F	T8	13	17			30
F	T9	29				29
M	T10	5				5
F	T11	11	16			27
M	T12	0				0

Figure 3.2 below summarises the teaching years in the Intermediate Phase of the participating teachers.

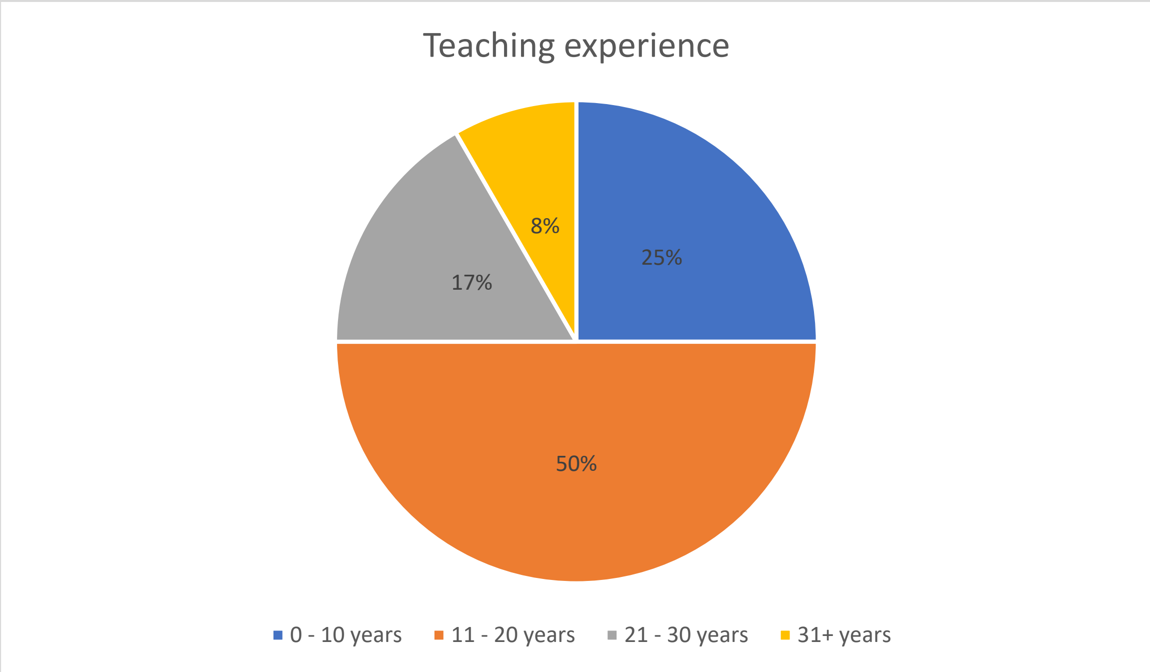


Figure 3.2: Teaching experience in Intermediate Phase

3.2.2 The research site

The selected site is a primary school in the southern (middle-class) suburbs of Cape Town in the Western Cape. The rationale for choosing *Peace Primary school* (a pseudonym) was, firstly, because I am an Intermediate Phase teacher at the school and have an established working relationship with the teachers. Secondly, the school is known as a middle school because it accommodates Grades 4 to 7. The school has one principal, one deputy principal and two heads of departments (one for the Intermediate Phase (Grades 4 to 6) and one for the Senior Phase (Grade 7). There are twelve Intermediate Phase teachers, one Resource Unit teacher and one Learning Support teacher. The school is classified as a full-service school, which means that it offers support via a resource unit class in a mainstream school. The role of the resource teachers is to meet the special needs of learners who experience learning difficulty, disability or other challenges. The school was founded in 1918 and officially opened the doors to the new and current building in 1957. *Peace Primary School* is a well-resourced school in which each teacher is provided with a laptop and internet access and each classroom is fitted with a data projector. The school also has a computer laboratory, which is fitted with a workstation for each learner per class. Intermediate Phase teachers and learners have access to the MCO (Maths Curriculum Online) programme, which is used as a teaching tool for two hourly Mathematics lessons, as well as two termly assessments.

3.2.3 Data collection

Considering a participatory action research approach, three data collection methods were utilised that allowed me to investigate the research questions of this study: observations, semi-structured interviews and focus group interviews. The instruments used to gather and record the data include semi-structured interview schedules, field notes and worksheets based on the teachers' work in the domain of Numbers. Operations and Relationships.

The collection of qualitative data was aimed at producing credible knowledge of interpretations and therefore required criteria that extend beyond reliability and validity. Kara (2016) reviews the debate about quality in qualitative research methods. Lincoln and Guba (1985) consider confirmability, dependability, credibility and transferability as quality criteria for qualitative research.

Tracy (2010) suggests eight key markers, which are designed to be comprehensible, flexible, universal and supportive of dialogue and learning. To evaluate the research integrity of utilising qualitative data, the following quality criteria were assessed:

- a worthy/relevant topic
- rich rigour
- sincerity (considering that full transparency will compromise participant anonymity)
- credibility
- resonance
- significant contribution
- ethics
- meaningful coherence

The aim of the interviews, class observations, document analysis and focus groups was to provide a detailed account of how Intermediate Phase teachers engage with learner errors, specifically in Numbers, Operations and Relationships.

The purpose of interviewing individual participants was to gain insight into their experiences. The format of semi-structured interviews enabled the researcher to pose more flexible questions because it is less structured (Merriam, 2009:90). This method allows access to information through some specific questions, as well as affords the participants the opportunity to reflect on and explore their own experiences.

Observation makes it possible to record behaviour as it is happening (Merriam, 2009). Through observations, I was able to informally observe the teaching and learning process within the classroom context and become familiar with the teaching strategies employed by the participants. Taking the stance of participant as observer allowed me to participate in activities as desired, yet focus mainly on collecting data, through field notes, which assisted in answering the research questions. Worksheets were also used as a tool to establish the learners' competencies in different concepts within the domain of Numbers, Operations and Relationships. Three worksheets (one for each grade in the Intermediate Phase) were used (see Appendixes A to C). The questions were intentionally selected to include the four cognitive levels as prescribed by CAPS: Knowledge, Routine procedures, Complex procedures and Problem solving (DBE, 2011).

Focus groups allowed the participants to share information or experiences that may not have been communicated during individual interviews. Smithson (2008) states that focus groups enable the participants to develop ideas collectively, bringing forward their own priorities and perspectives. The focus groups include three teacher participants per grade in the Intermediate Phase.

3.2.4 Data analysis

The aim of analysing data is to become familiar with the data collected from the data sources used in the study and to interpret the meaning of the data. The data sources analysed for this study were the individual teacher interviews, the Maths Curriculum Online results (https://gsed.co.za/), learners' worksheets, focus group interviews with learners and teachers (per grade in the Intermediate Phase) and field notes.

Data analysis is the process of making sense of the data, thereby answering the main research question. The aim of data analysis is to discover and interpret emerging patterns, themes, concepts and meaning. Qualitative data analysis is defined by Bogdan and Biklen (1982:145) as "working with the data, organising them, breaking them into manageable units, coding them, synthesising them, and searching for patterns". Hsieh and Shannon (2005:1286) view qualitative content analysis as "a research method for the subjective interpretation of the content of text data through the systematic classification process of coding and identifying themes or patterns". They distinguish between three approaches to qualitative content analysis (Hsieh & Shannon, 2005:1286):

- conventional content analysis (coding categories are derived from text data)
- direct content analysis (aimed at validating/extending a theory conceptually)
- summative content analysis (involves counting and comparisons).

An advantage of the conventional approach is that the researcher can gather information and knowledge directly, based on the participants' unique perspectives.

Merriam (2009) emphasises that data collection and data analysis should be a simultaneous process in qualitative research to avoid omitting valuable data during the research study.

To ensure clarity of the volume of data, I analysed the data simultaneously with data collection. Data analysis is not a linear process, and some steps may need to be revisited as the researcher makes meaning of the data. Continuous comments on observations, transcribed interviews and field notes were coded and categorised into themes guided by the sub-questions of this study. Generating codes and categorising data assist the researcher to organise and describe the data and easily retrieve specific segments of the data, leading to a response to the main research question: 'How do Intermediate Phase teachers identify and respond to learners' mathematical errors and misconceptions in the content area of Numbers, Operations and Relationships?'

3.2.5 Ethical considerations

Permission to embark on this research study was obtained from the Research Ethics Committee of Stellenbosch University (see Appendix H), the Western Cape Education Department (see Appendix I), as well as from the School Governing Body and principal (see Appendix G). Consent letters were issued to the teacher participants as well as to obtain consent from the parents to use evidence of learners' work (see Appendixes B and C).

The researcher aimed to be transparent about the research process by informing the participants of the aims of the research study. Research participants were informed that their participation in the research was voluntary and that they may withdraw as a participant at any stage of the research.

The notion of researcher positionality (Milner, 2007) is acknowledged in this study as I was part of the data collection instrument. The concept of self as research instrument reflects the likelihood that the researcher's own subjectivity will come to bear on the research project and any subsequent reporting of findings (Bourke, 2014). In my dual role of teacher-researcher, I bring my own assumptions and beliefs to the study and should therefore be aware of any disconfirming evidence.

The confidentiality and anonymity of the school, the teacher participants and all documents generated by the data were maintained.

Chapter 4

Data Collection, Analysis and Findings

4.1 Introduction

In the previous chapter, the research methodology of PAR was described. This chapter provides a discussion of the results of 'How Intermediate Phase teachers identify and respond to learners' mathematical errors and misconceptions in the content area of Numbers, Operations and Relationships'. The analysis and findings are presented through three cycles of the PAR process, which include the phases of planning, taking action, observation and reflection.

The data that were analysed comprised individual teacher interviews, Maths Curriculum Online results (https://gsed.co.za/), learners' worksheets, focus group interviews with teachers (per grade in the Intermediate Phase) and field notes.

4.2 Cycle 1 – Planning

After obtaining permission from the governing body of the school and the Western Cape Education Department, I arranged a meeting with the Intermediate Phase teachers. I included the Intermediate Phase Learning Support teacher, as she supports learners with barriers in Mathematics. Many of the teachers were unfamiliar with PAR and I therefore explained the motivation for this research project, as well as the process of participatory action research. Thirteen teachers were invited to participate in this study and twelve teachers were willing to participate. Evidence of learners' classwork, worksheets and assessments form an important resource for teachers to identify and respond to errors and misconceptions. The Maths Curriculum Online (MCO) was also used as a tool to review learners' results in Numbers, Operations and Relationships. The MCO provides a curriculum mapped to CAPS for Grades 3 to 9. The implementation of MCO is explained further later in this chapter. Sixty-eight Intermediate Phase learners were identified as possible participants and consent forms were obtained from sixty learners and parents. Reflecting on Hansen's (2011) view that errors and misconceptions are not limited to children who need additional support in Mathematics, the selected learners were from a range of mathematical abilities, based on their achievements. Dates for individual semi-structured interviews were scheduled with the twelve teachers.

4.3 Cycle 1 – Action

Conducting semi-structured interviews with individual teachers allowed opportunities for them to share their experiences in their own words. The semi-structured interviews consisted of a range of questions based on the teachers' experiences of learner errors in Numbers, Operations and Relationships. The aim of the interviews was to examine the current ways teachers notice learners' mathematical errors and/or misconceptions and how they interpret the errors. All the interviews were recorded and then transcribed. After transcribing the interviews, the teacher-researcher verified with the participants that what was transcribed was in fact what they had said. The interviews provided opportunities for the teachers to reflect, engage and plan towards improving the current situation.

4.4 Cycle 1 – Observation

Data analysis was done by reading and re-reading the transcribed interviews and finding emerging themes. The following observations were made through the interview sessions. For the purpose of this study, it was important to first determine *when* teachers notice learner errors in the teaching and learning process, as this would lead to the strategies, methods and tools they apply to identify specific errors.

When teachers were asked *when* they started noticing learner errors in general, and specifically in Numbers, Operations and Relationships, the following key concepts emerged: teaching time, tests and assessments, and Maths Curriculum Online. The next section provides an explanation of each key concept.

4.4.1 Teaching time

CAPS (DBE, 2011:32) stipulates that six hours per week is allocated for teaching time in Mathematics. Between three and six hours are allocated for revision per term and, in addition, six hours are allocated for assessment. The distribution of time per topic takes into account the weighting for the content area. In other words, the weighting of content areas represents teaching hours. Relevant to this study is the 50% weighting for Numbers, Operations and Relationships across the Intermediate Phase.

The teachers indicated that they became aware of learner errors during their class teaching. When they start teaching a particular concept, they can immediately pick up that their learners face challenges. A few responses from teachers:

T2 - "When I start teaching my Maths lesson, I immediately notice that learners don't grasp the concept at all."

T7 - *"When I start teaching a new concept, I go back to my learners' previous knowledge to start with what is familiar to them, and then I notice that they didn't grasp some concepts which they should know before I can build on that concept."*

T9 - *"I notice throughout teaching, when I've taught a concept, only a small percentage of my learners will get the concept immediately, then others you have to reteach."*

4.4.2 Assessment in Mathematics

According to CAPS (DBE, 2011:293), assessment is

> a continuous planned process of identifying, gathering and interpreting information regarding the performance of learners, using various forms of assessment. It involves steps: generating and collecting evidence of achievement, evaluating this evidence, recording the findings and using this information to understand and thereby assist the learners' development in order to improve the process of learning and teaching. Assessment should be both informal and formal. In both cases regular feedback should be provided to learners to enhance the learning experience. This will assist the learner to achieve the minimum performance level of 40% (for Intermediate Phase) required in Mathematics for promotion purpose.

The policy distinguishes between different types of assessment, namely baseline assessment, diagnostic assessment, formative assessment and summative assessment. For the purpose of this research study, a distinction between formative and summative assessment sheds light on how teachers use the two types of assessment. Formative assessment is used to aid the teaching and learning process – assessment *for* learning. This type of assessment is commonly used during teaching time, like class activities or verbal questioning during a lesson. In other words, it can inform teaching methods. Summative assessments take place after specific Mathematics topics have been covered – assessment *of* learning. The results of summative assessments are recorded and used for promotion purposes.

Interview transcripts and field notes revealed that most teachers used tests/formal assessments to specifically identify what their learners know and what they do not know.

T1 - *"I notice learner errors in formal assessments."*

T8 - *"When we review our tests and I give learners feedback, I pick up that they make certain errors in Numbers, Operations and Relationships."*

T9 - *"After Test results I usually review to address specific errors."*

4.4.3 Maths Curriculum Online (MCO)

MCO provides a curriculum mapped to CAPS for grades 3 to 9. Each learner has an individual log-in and password. Having a unique username enables teachers to track each learner's progress, and allows learners to track their own progress. Brain Quests (BQ) are quiz-based activities aligned to the curriculum for the entire school year. Learners complete two brain quest activities per week, which is included in the six hours teaching time. Each brain quest is marked automatically and the learners receive immediate feedback on their answers. The number of questions per BQ are varied, depending on the concept it covers. Teachers receive real-time summary feedback per learner and per question for brain quests. Below is a screenshot of a brain quest example to show the marked BQ.

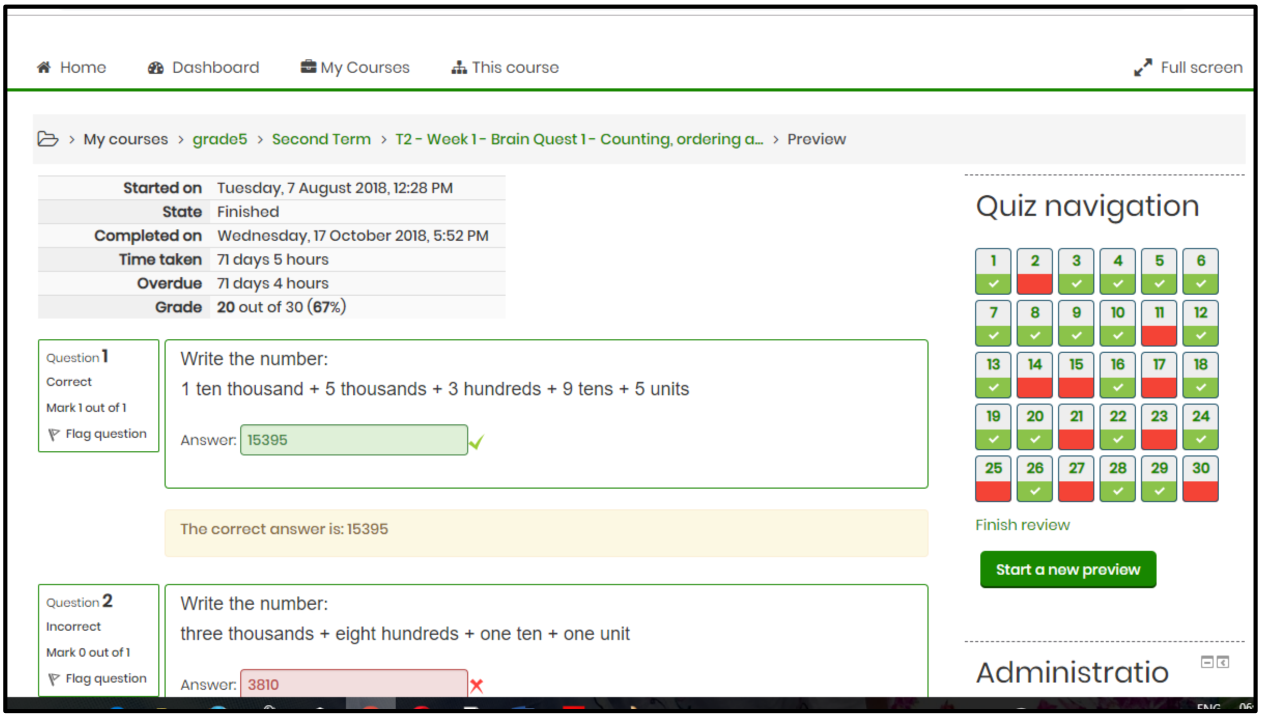


Figure 4.1: Example of an auto-marked Brain Quest

The MCO includes online school-based assessments (SBAs), which are standardised, benchmarked assessments. Two school-based assessments are administered twice termly. Learners only have access to the assessments on a specific date and require a password that is provided by the teacher on commencement of the assessment. Learners and teachers receive immediate feedback and the data is available for teachers via the Green Shoots dashboard (Green Shoots Suite, n.d.). The dashboard allows teachers to view specific content.

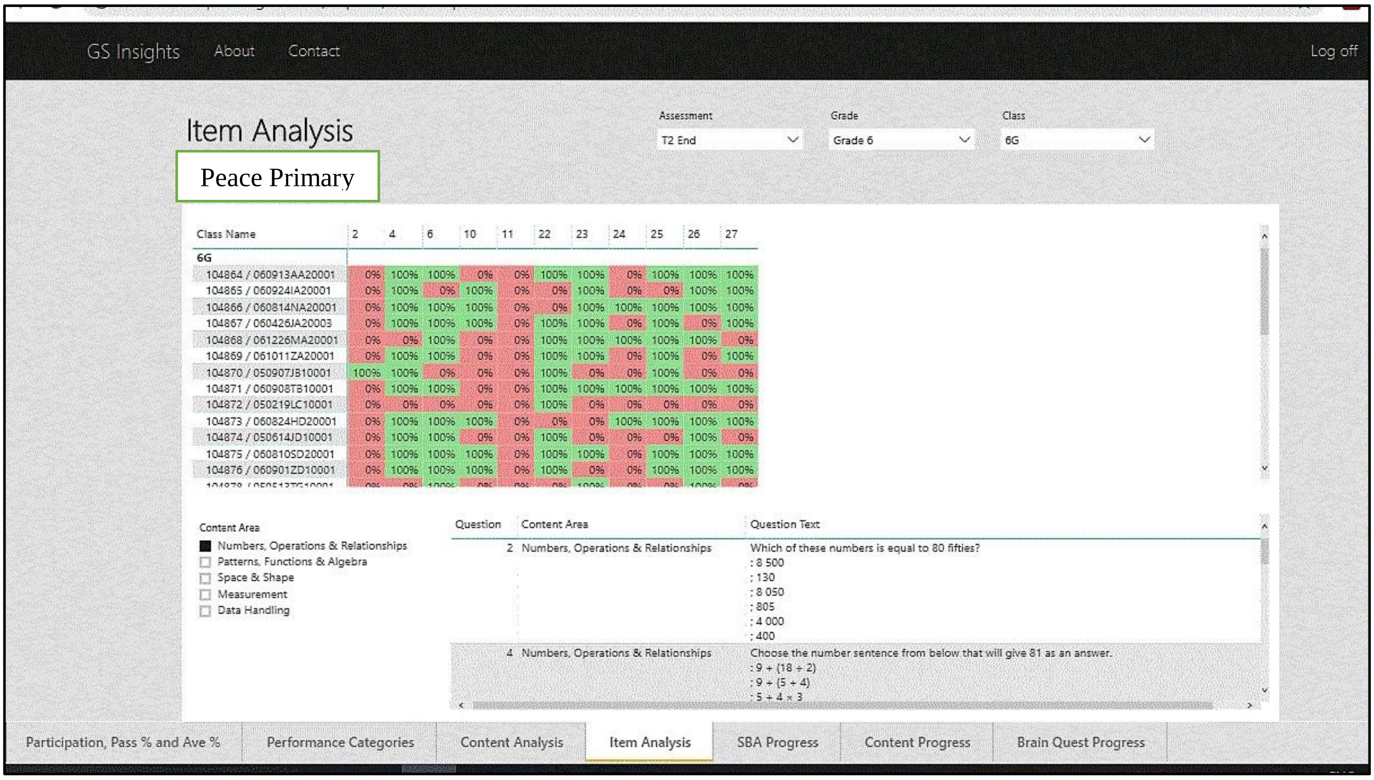


Figure 4.2: Dashboard item analysis for Numbers, Operations and Relationships

The Maths Curriculum Online provides evidence for teachers to view learners' progress, as the programme allows immediate feedback to both teacher and learners. The Item Analysis tab displays specific questions per content area and reflects which questions learners answered correctly and incorrectly. Figure 4.2 shows an example of how teachers could view specific questions in the content area of Numbers, Operations and Relationships. Teachers who engaged with the MCO responded as follows:

T2 - *"When learners are in the computer lab using MCO, they struggle to understand instructions, reading with understanding."*

T7 - *"With MCO I can check the progress of my learners. It gives you immediate feedback and I can identify the areas they are struggling with."*

T8 - *"As class teacher I can analyse MCO data and also as a grade we can see common error areas we need to focus on or spend more time on certain concepts."*

A summary of *when* teachers identify learner errors in the content area of Numbers, Operations and Relationships (NOR) revealed that all 12 participants noticed during their teaching time, seven teachers noticed errors during tests/assessments, and five teachers noticed errors when using the MCO.

4.4.4 Error noticing

Most of the participating teachers with more than five years' teaching experience could list a variety of learner errors because they had noticed a pattern of common errors over the years. Table 4.1 represents the teachers' classification of learner errors in NOR.

The first column, *Common errors in NOR*, includes a summary of eight (8) common errors teachers identified in NOR. The items listed here are specific to the topics, concepts and skills within the domain of NOR. The error items 1, 2, 3, 4, 5, 6 and 8 are specific to whole numbers and error item 7 is related to common fractions. There is an overlap of errors in the different items, viz. borrowing and carrying errors and calculation procedures in basic operations.

The second column, *Possible causes of errors described by teachers*, is a summary of the teachers' interpretations of what they consider to be possible causes for learner errors. The summary in this column is matched to the items in the first column. The possible causes of errors described by the teacher's overlap with the different error items, e.g. learners not understanding place value and mathematical terminology, confusion with calculation procedures, gaps in prior knowledge and carelessness.

Table 4.1: Teachers' classification of learner errors in NOR before error review

No	Common errors in NOR	Possible causes of errors described by teachers
1	Incorrect calculation procedures (setting out) of basic operations	Curriculum includes too many different calculation methods with too many steps, which cause confusion among learners
2	Addition: borrowing and carrying errors; incorrect grouping	Not understanding the place and number value of digits Not knowing number bonds and tables Different calculation methods Gaps in prior knowledge of number concepts Carelessness
3	Subtraction: borrowing and carrying errors; number swop (subtracting a "bigger" digit from a "smaller" digit)	Not understanding the different calculation procedures/ methods Place value and number value Gaps in prior knowledge of number concepts Subtracting from zero (0) Carelessness
4	Multiplication: borrowing and carrying	Confusion with different calculation procedures/methods Not knowing tables Incorrect calculation Carelessness
5	Division: Incorrect operational procedure	Confusion with the steps of the clue board method/ procedures Carelessness
6	Incorrect problem-solving operation	Learners lack comprehension skills and understanding of mathematical terminology
7	Fractions: Equivalent fractions; fractions of a whole number	Gaps in prior knowledge of fractions Not understanding the concept
8	Rounding off: to the nearest 5; 10; 100; 1 000; 10 000	Misinterpretation of the place value of digits

Many teachers shared that, even though errors were highlighted after tests and assessments, their focus was to record the results for progression purposes, and that errors were not always used to inform their teaching.

An interesting finding was that, although teachers noticed common errors during their teaching time and assessments and while using the MCO, they only gave general descriptions of what they thought the cause of learner errors were. For teachers to address learner errors, they should have a clear understanding of why errors are made and what learners' reasoning is about their own errors.

It was also interesting to note that, when teachers were asked about what they thought learners' errors were due to, no reference was made to the cognitive levels or the learners' ability to demonstrate the skills at each level. Most of the participants identified the different methodologies in addition, subtraction, multiplication and division as possible causes of learner errors in NOR. Teachers felt that teaching different ways of solving a mathematical problem confuses the learners.

Where error patterns were noticed, teachers applied different methods to address errors.

Table 4.2 illustrates the teaching tools and strategies currently used by the 12 Intermediate Phase teachers.

Table 4.2: Teacher's teaching tools and strategies in relation to learner errors

Teaching tools	**Teaching strategies**
• Using concrete objects (counters/a fraction wall) • MCO brain quest activities • Extra revision activities/workbooks • Using examples with a smaller number range	• Re-teaching concepts and methodology • Peer teaching • "One-on-one" assistance • Drilling mathematical terminology • Teacher feedback after assessments • Asking a colleague to teach a concept

The above list indicates that the Intermediate Phase teachers were implementing different strategies to address learner errors. Reflecting on the success of the above-mentioned strategies, teachers responded:

T2 - "Some improvement only with concrete apparatus."

T3 - "When using manipulatives some learners show improvement, but others continue to struggle."

T4 - "With my stronger learners I notice improvement, but my weaker group will need continual support to consolidate concepts."

T6 - "Sometimes some will get it while others will struggle and need more time or a different way of explaining."

4.5 Reflection on Cycle 1

What became evident was that although teachers were noticing learner errors, the implementation of their current strategies to address errors, showed minimal improvement. Teachers expressed frustration about the lack of improvement shown, especially after they have re-taught certain concepts.

The teacher-researcher and participants agreed that a "closer look" at specific learner errors in the area of NOR may enable them to gain a better understanding of *why* learners make errors in NOR.

4.6 Cycle 2 – Planning

A collective decision was made by the teacher-researcher and participants to use evidence of learners' work to obtain clarity on specific learner errors in NOR. Due to easy access of the MCO data, it was agreed that reviewing the recent MCO school-based assessment results would reveal the specific content and type of questions learners answered incorrectly. It is important to note that there would be limitations to using the MCO school-based assessments. Firstly, only certain concepts within the domain of NOR were covered in teaching time. Secondly, the MCO results would only reveal the content questions learners answered unsuccessfully and not show the process of how they arrived at their answers. It was agreed that presenting learners with worksheets (see Appendix B), based on those questions, would assist teachers in gaining a better understanding of why errors are made and what learners' reasoning is about their own errors.

According to CAPS (DBE, 2011:295), formal assessments should cater for a range of cognitive levels and abilities of learners. The four cognitive levels to guide assessment tasks are based on those suggested in the TIMSS study of 1999. It therefore was important to ensure that questions of various levels of difficulty were included in the worksheets, as the sample population of learners was selected for this study, based on their different ability levels.

The table below illustrates the four different cognitive levels with the weighted percentage for each level at which assessments must be conducted. Each level has a description of the skills to be demonstrated by learners to show the progression in mathematical thinking and understanding (DBE, 2011:296).

Table 4.3: Curriculum cognitive levels in relation to the skills to be demonstrated

Cognitive levels	Description of skills to be demonstrated
Knowledge - 25%	○ Estimation and appropriate rounding off of numbers ○ Identification and direct use of the correct formula ○ Use of mathematical facts ○ Appropriate use of mathematical vocabulary
Routine procedures - 45%	○ Perform well-known procedures ○ Simple applications and calculations, which might involve many steps ○ Derivation from the given information may be involved ○ Identification and use (after changing the subject) of the correct formula, generally similar to those encountered in class
Complex procedures - 20%	○ Problems involving complex calculations and/or higher reasoning ○ Investigations to describe rules and relationships – there is often not an obvious route to the solution ○ Problems not based on a real-world context – could involve making significant connections between different representations ○ Conceptual understanding
Problem-solving - 10%	○ Unseen, non-routine problems (which are not necessarily difficult) ○ Higher order understanding and processes are often involved ○ Might require the ability to break the problem down into its constituent parts

4.7 Cycle 2 - Action

During the action cycle, the Intermediate Phase teachers reviewed the recent school-based assessment results on MCO to establish which questions learners found most challenging. The results were first reviewed in each grade and then collated to find common error patterns within the phase. This was done because the specification of content shows progression of concepts and skills from Grade 4 to Grade 6. The main progression in NOR in the Intermediate Phase happens in three ways:

- the number range increases
- different kinds of numbers are introduced
- the calculation technique changes

Learners were then presented with worksheets based on the school-based assessment (SBA) questions. The pencil-paper worksheets are intended to support the learners in checking their computation errors after they have received the auto-marked feedback online. Relevant to the scope of this study, the worksheets only included NOR questions from the SBAs.

Teachers in each grade participated in a small focus group to review and discuss learners' responses. This provided opportunities for the teachers in each grade to analyse specific content challenges and the way learners interpreted and responded to the questions. A focus on the different types of questioning in MCO for NOR concepts would guide the teachers to identify the way learners interpreted the questions. The focus group interviews were recorded and the teacher-researcher kept field notes during the discussions.

4.8 Cycle 2 - Observation

4.8.1 - SBA results

Figure 4.3 below illustrates the school-based assessment results in NOR of Grade 4, Grade 5 and Grade 6.

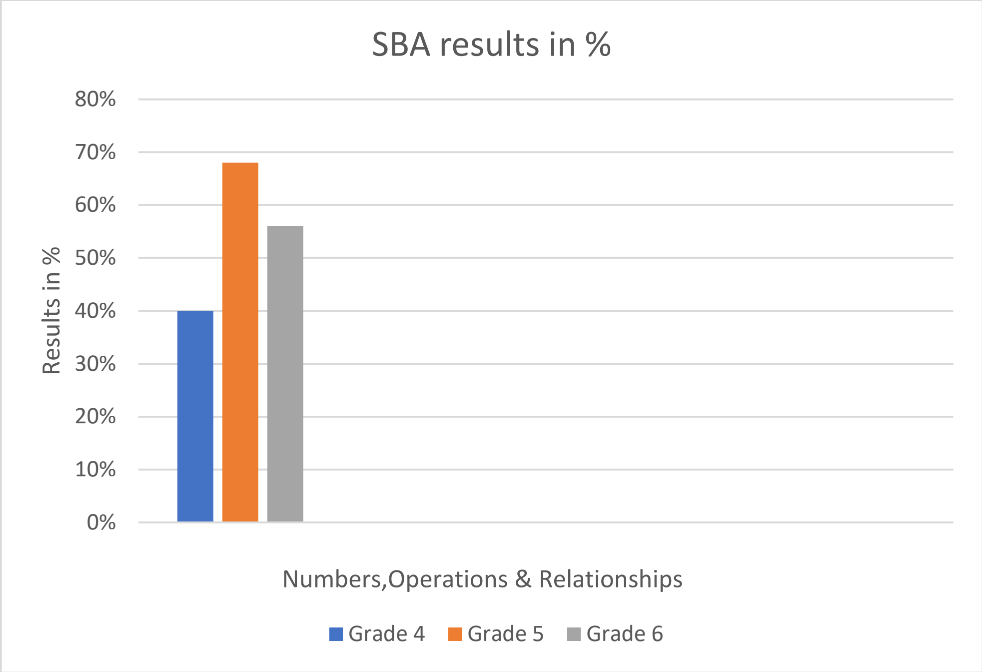


Figure 4.3: SBA results of intermediate grades

It was noticed that the percentages achieved by the Grade 4 was 40%, the Grade 5 learners achieved 68% and the Grade 6 learners achieved 56%. An interesting observation was the difference of 28% between Grade 4 and Grade 5 performance, and then the decline in the results in Grade 6. The teachers in each grade scrutinised the questions that covered NOR to identify the specific concepts learners answered incorrectly. Where the result displays a 100%, the question was answered successfully, and 0% shows questions answered unsuccessfully. Below are screenshots of each grade's result.

The screenshots below show the SBA results from Grade 4 to Grade 6 and the relevant questions relating to the content area of NOR. The different tabs on the dashboard allow teachers to view the learner participation, pass percentages and average percentages per class. The performance categories tab displays the class results in four different achievement categories, viz. 0% to 29%, 30% to 49%, 50% to 79%, and higher than 80%. Although the SBA covers all five content areas, questions are not structured in the order of each content area. The content analysis tab groups the SBA questions in the five content areas of Numbers, Operations and Relationships; Patterns; Functions and Algebra; Space and Shape; and Measurement and Data Handling. Grouping the questions per content area allows teachers to see exactly which questions were covered in each content area. Looking at the overview of the item analysis allows teachers to identify which questions learners answered successfully (100%) and which questions they answered incorrectly (0%). The SBA results can be viewed per grade as well as per class. For the purpose of this study, the Intermediate Phase teachers reviewed the SBA results per grade to identify error trends per grade.

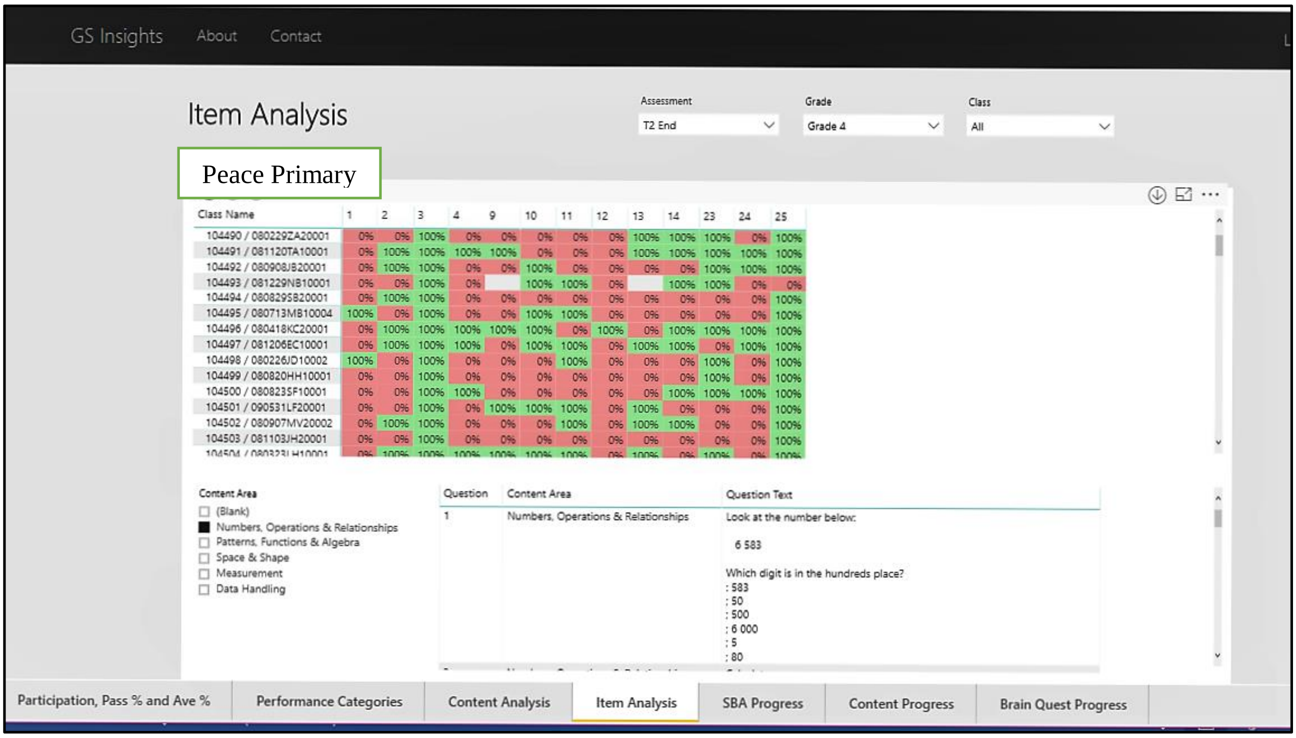


Figure 4.4: Screenshot of Grade 4 NOR item analysis

The Grade 4 item analysis shows that Questions 3, 23 and 25 had the most successful responses, while error trends are shown in Questions 1, 4, 9, 11, 12, 14 and 24, with the most unsuccessful responses. Questions 2, 10 and 13 were not conclusive as either successful or unsuccessful.

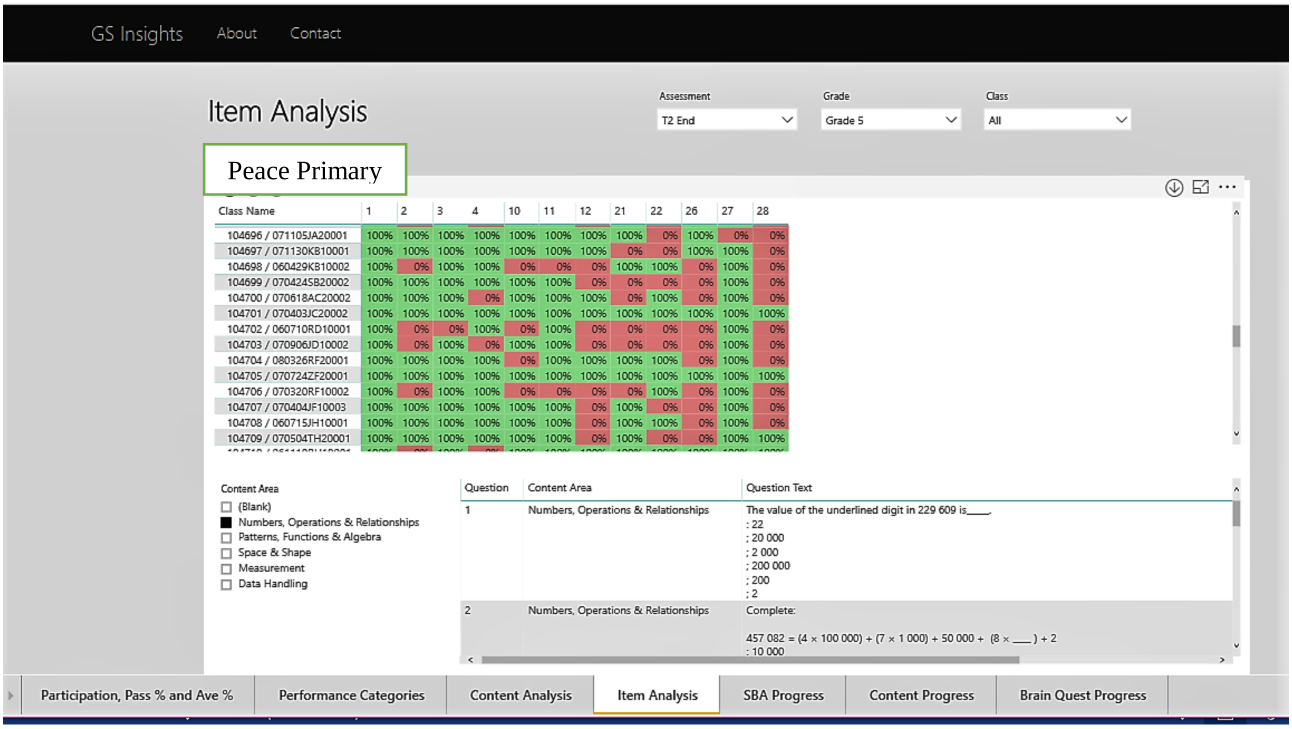


Figure 4.5: Screenshot of Grade 5 NOR item analysis

The Grade 5 item analysis shows that Questions 1, 3, 4, 11 and 27 had the most successful responses and that the error trends are shown in Questions 12, 22, 26 and 28, which had the most unsuccessful responses. Questions 2, 10 and 21 were not conclusive as either successful or unsuccessful.

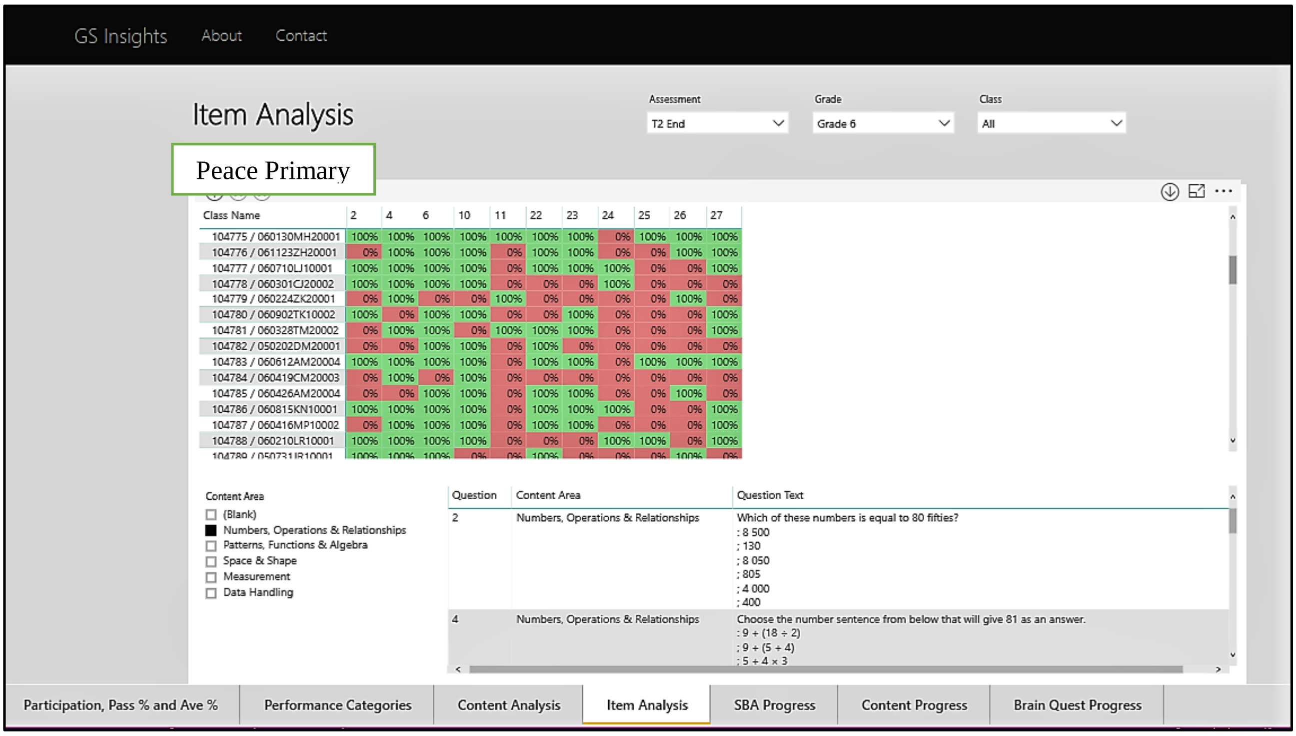


Figure 4.6: Screenshot of Grade 6 NOR item analysis

The Grade 6 item analysis shows that Questions 4, 6, 10 and 22 had the most successful responses and that error trends are shown in Questions 11, 24, 25 and 26, which had the most unsuccessful responses. Questions 2, 23 and 27 were not conclusive as either successful or unsuccessful.

Reviewing the learner worksheets in a focus group was an unfamiliar exercise for most of the participants, and some participants shared that they felt anxious that learners' incorrect responses would reflect on their teaching ability. The teacher-researcher assured the teachers that, through PAR, collaboration with peers and ongoing reflection, new knowledge could be generated, which may assist them to make informed decisions.

To maintain the anonymity of the learner participants, the answers to each question were grouped, so that trends could be investigated across the group of learners. Representing the responses of learners in this way may assist teachers in identifying any error trends in each question.

An interesting observation in all three focus groups was that teachers were initially focused on whether the responses were correct or incorrect. Professional noticing requires teachers to attend to learners' strategies. It therefore was not enough to merely distinguish between errors as correct or incorrect, but rather to engage with the errors with the aim of interpreting the learners' reasoning about their own errors.

4.8.2 - NOR error patterns

The table below provides a summary of learner error categories from Grade 4 to Grade 6, as identified by the Intermediate Phase teachers after reviewing the learner worksheets.

The first column, *Common errors in NOR*, includes the eight (8) common errors teachers identified in NOR before their review of the learner worksheets as in Table 4.1 of Cycle 1. The same content items (whole numbers and common fractions) within the domain of NOR were scrutinised. The intention of using the same items was to determine whether teachers were able to provide more detail about their interpretation of learner errors after reviewing the responses on the worksheets.

The second column, *Causes of errors described by teachers*, is a more detailed summary of the teachers' interpretations of the causes of learner errors compared to the summary presented in Table 4.1 in Cycle 1. The summary in the second column is matched to the items in the first column. In Cycle 1, the following items were generalised as possible causes for learner errors:

- Carelessness
- Gaps in prior knowledge
- Calculation procedures, and
- Not understanding the concept

It was interesting to note that the above-mentioned items were no longer identified as general causes for learner errors, as previously assumed in Cycle 1. Teachers explained basic operation errors more specifically to learners not understanding the inverse relationship between addition and subtraction and between multiplication and division. Misconceptions related specifically to the properties of zero (0) were identified for addition, subtraction and multiplication. From the generalisation of *gaps in prior knowledge* and *not understanding the concept* in Cycle 1, teachers could now identify that learner errors related to fractions were their misconceptions of the relationships between different representations of equivalent fractions and learners' ability to calculate a fraction of a whole number. The teachers' review of errors related to the concept of rounding off numbers highlighted learner challenges when the number range increased and when learners were presented with reversal questions.

Table 4.4: Teachers' classification of learner errors in NOR after error review

No	Common errors in Numbers, Operations and Relationships	Causes of errors as described/identified by the teacher participants
1	Calculation procedures in basic operations	o Not understanding the relationship between basic operations, i.e. the relationship between addition and subtraction, and multiplication and division as inverse operations o Confusion with different methodologies (procedures)
2	Addition	o Incorrect place value grouping o Difficulty in "carrying over" related to place value (hundreds, tens and units). o Not understanding the value of zero ("0") as a placeholder
3	Subtraction	o Subtracting a "bigger" number from a "smaller" number o Swopping numbers around when subtracting from "0" o Place value errors o Carrying and borrowing errors
4	Multiplication	o Not understanding that multiplication is repeated addition o Not understanding the multiplicative property of zero ("0") and one ("1") o Borrowing and carrying errors
5	Division	o Incorrect procedure for setting out calculation o Inability to determine and use multiples as "clues" for clue board method
6	Problem-solving	o Inability to interpret the problem o Poor higher-order reasoning skills
7	Fractions	o Not understanding the relationship between different representations of fractions as equivalent o Inability to calculate a fraction of a whole number o Inability to use the procedural formula of (w ÷ d x n), i.e. the whole number is divided by the denominator of the fraction and then multiplied by the numerator
8	Rounding off	o Confusion based on place value o Difficulty when number range increases o Confusion with interpreting reversal questions

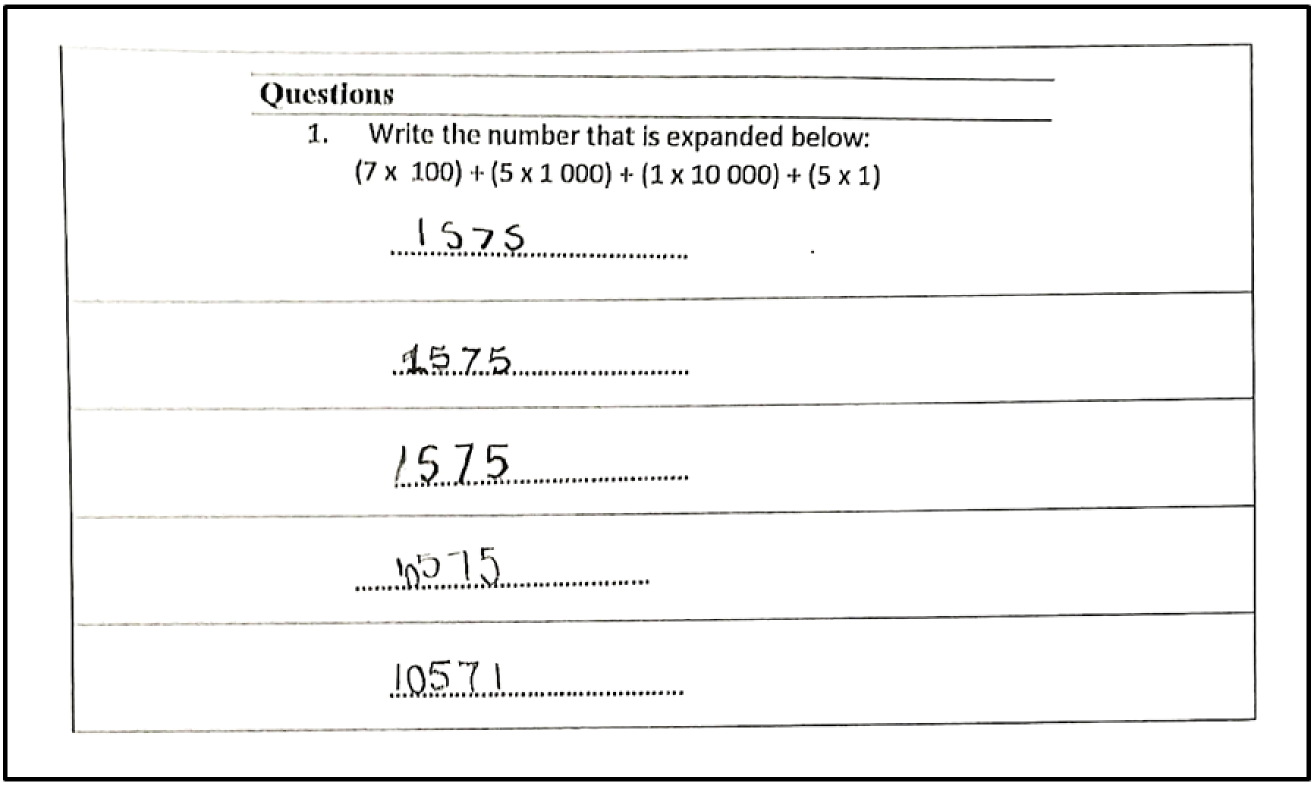
From the above findings, it became apparent that, when teachers attended to learner errors, they were able to provide more detail of their interpretation of the causes of errors, compared to the initial general descriptions provided during the interviews.

The following section deals with specific error trends noticed by the Intermediate Phase teachers. The worksheets for each grade covered a variety of concepts within the domain of NOR. The aim of reviewing the worksheets was to determine common errors or misconceptions. It therefore is important to note that only questions/concepts that highlighted a common trend are discussed.

4.8.2.1 Error patterns in place value

The first concept in NOR in which the teachers discovered an error pattern, was place value. Two examples of questions related to place value are discussed below.

The assumption was that learners would be more confident in solving problems that required the skill of applying basic knowledge and performing routine procedures. Even though errors were prominent in questions related to complex procedures and problem-solving, it became clear that learners had misconceptions related to basic number sense. The first example from the Grade 6 worksheet is illustrated in Figure 4.7 below.



Questions

1. Write the number that is expanded below:
   (7 x 100) + (5 x 1 000) + (1 x 10 000) + (5 x 1)

1575

1575

1575

10575

10571

Figure 4.7: A scan of learner errors related to place value

Learners were asked to write the five-digit number that was represented in expanded notation. In the examples above, learners omitted zero (0) as a placeholder for tens. During the discussion, teachers commented that they would have assumed it was a careless mistake, but now their attention was drawn to the error. They found it concerning that it was only a five-digit number, and that at Grade 6 level learners work with nine-digit numbers. Using the digit zero (0) as placeholder is a concept taught in the early grades (Foundation Phase), when learners build numbers. Teachers also suggested that because the place values were non-sequential in the question, learners became confused with ordering the digits in their different place values.

T9 - *"We usually give learners the expanded notation in place value order, and then ask them to write the number"*

T10 - *"How will this learner understand decimals if there is confusion with whole numbers?"*

The second example of error patterns related to place value is illustrated in the following Grade 4 example. The ability to recognise zero (0) when adding and subtracting the same number was a skill learners lacked when their calculation execution showed them adding the number to get a total, and then subtracting the same number. This further alludes to rhythmic counting and grouping errors.

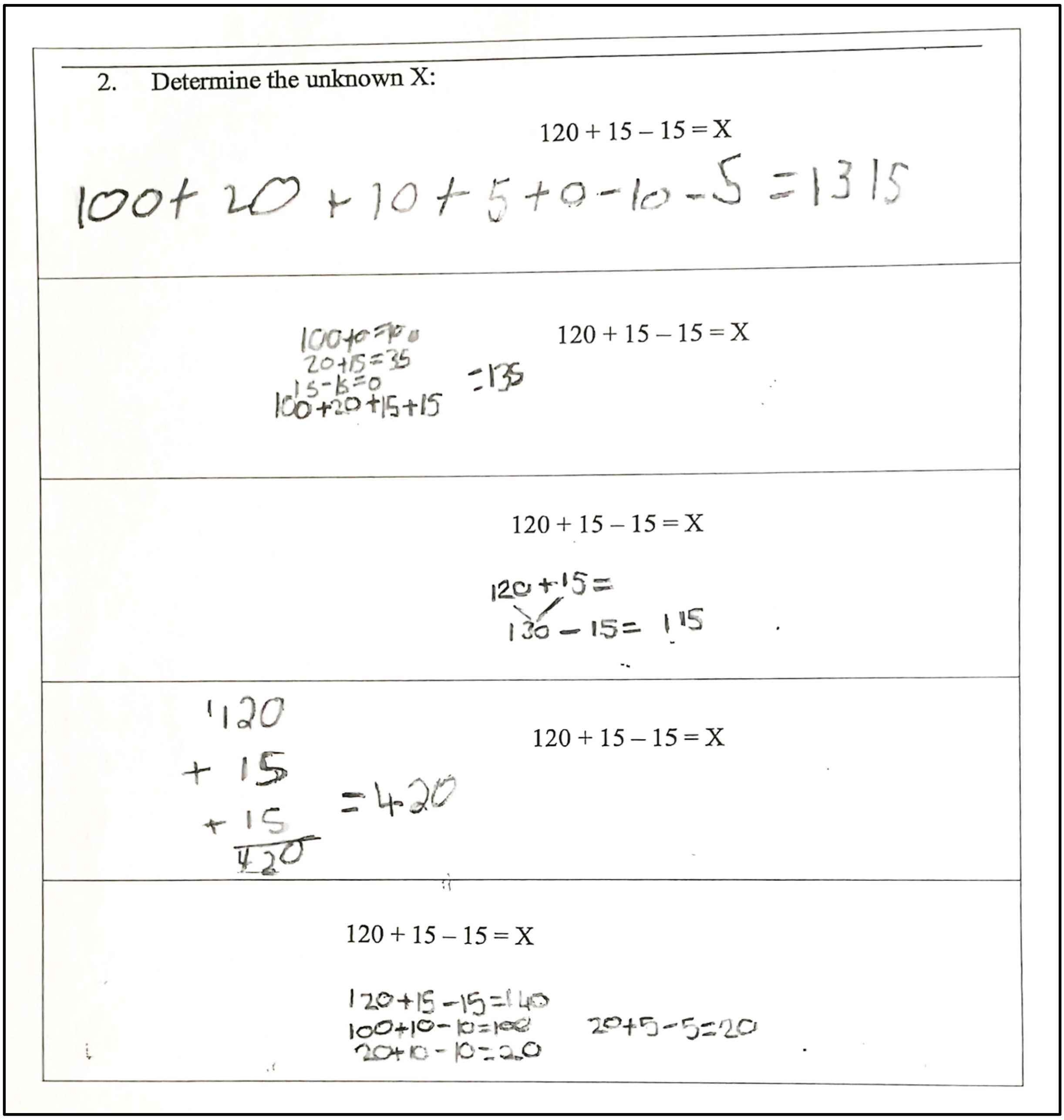


Figure 4.8: A scan of learner errors related to the properties of zero (0)

From the examples in Figure 4.8, teachers realised that learners could not identify the additive property of zero (0) by adding and subtracting the number 15. The number zero plays a central role in Mathematics as the identity element of integers, real numbers and many other algebraic structures. In a study with elementary school students, Hiebert and Lefevre (1986:61) found that “many mistakes in procedure appear to be due not to a lack of procedural knowledge, but to a lack of conceptualization of the symbol system and its principals”.

4.8.2.2 Operational symbols and mathematical vocabulary

During the individual interviews with teachers in Cycle 1, "carelessness" was a common response for possible causes of learner errors in Numbers, Operations and Relationships (NOR). When teachers were analysing learner responses on the worksheets, it became evident that there were underlying factors that influenced errors. An example of this scenario is illustrated in Figure 4.9 below.

Questions

1. Three children wrote the following number sentences for the following instruction:
"Calculate the product of 20 and 5"

d) 20 + 5 = 25
e) 20 x 5 = 100
f) 20 + 20 + 20 + 20 + 20 = 100

Which number sentence do you agree with? Explain your answer:
Becausu 20 × 5 is 25 When we say 20 + 5 is answer still 25

d) 20 + 5 = 25
e) 20 x 5 = 100
f) 20 + 20 + 20 + 20 + 20 = 100

Which number sentence do you agree with? Explain your answer:
because is 20×5 = 25

d) 20 + 5 = 25
e) 20 x 5 = 100
f) 20 + 20 + 20 + 20 + 20 = 100

Which number sentence do you agree with? Explain your answer:
Because they said calculate the product of 20and5.

d) 20 + 5 = 25
e) 20 x 5 = 100
f) 20 + 20 + 20 + 20 + 20 = 100

Which number sentence do you agree with? Explain your answer:
Becece like it said in the sentece calculate the predact of 20 and 5

Figure 4.9: A scan of learner reasoning connecting operational symbols to mathematical vocabulary

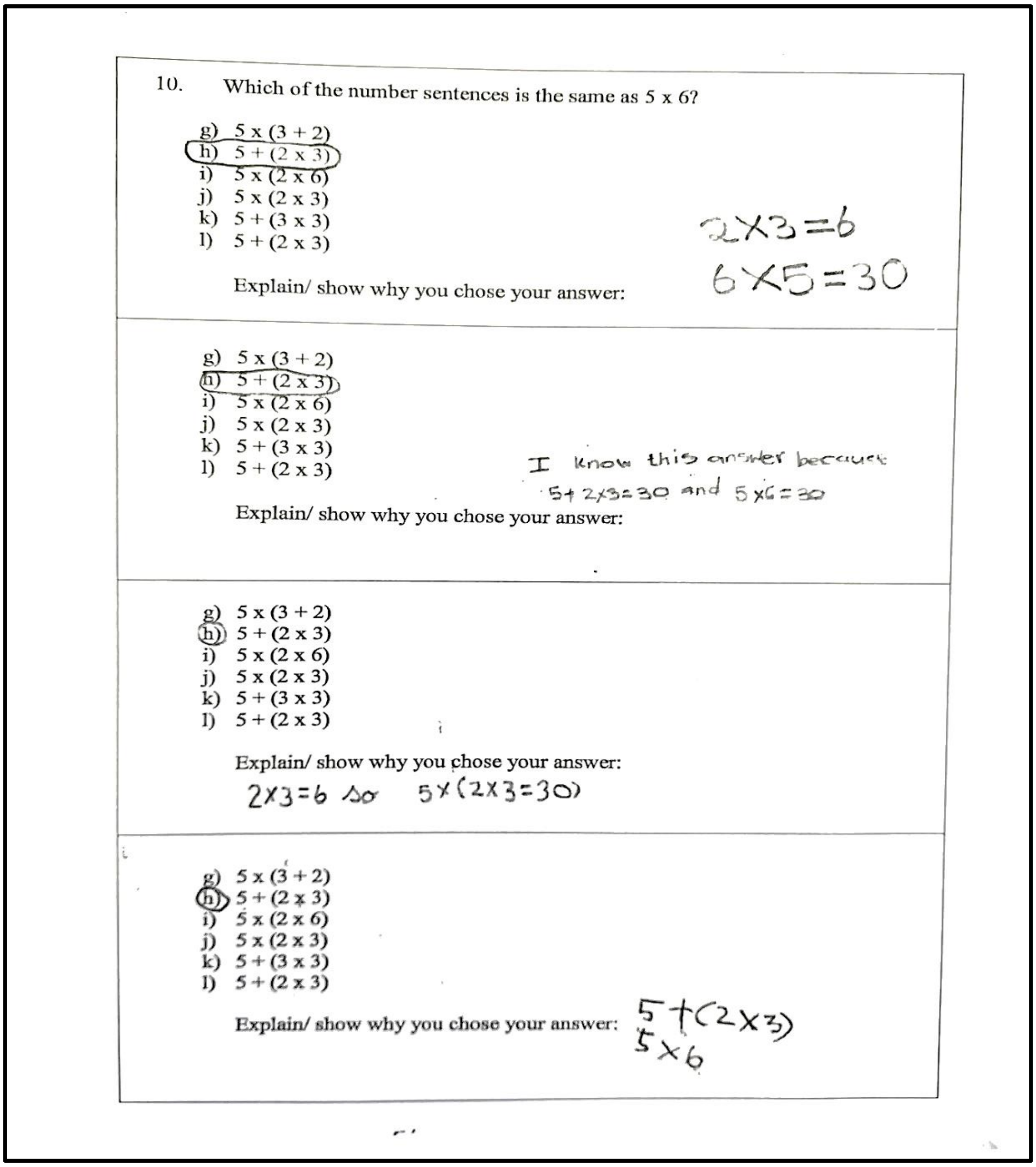
Initially, the teachers assumed that the error was due to the fact that the learners did not understand the mathematical vocabulary of "calculating the product". Concern about learners' lack of understanding mathematical vocabulary was voiced by teachers across the Intermediate Phase. This is supported by the findings of Riccomini *et al*. (2015) and Rubenstein and Thompson (2002). Even though the use of mathematical vocabulary is categorised as a cognitive level 1 skill, many incorrect responses were recorded where learners had to interpret mathematical terminology to execute calculations.

Upon reflection, the Intermediate Phase teachers established that their current strategy of "drilling" vocabulary in isolated lessons was not effective, and agreed that regular introduction of mathematical vocabulary should be implemented consistently with all number concepts. Teaching mathematical vocabulary throughout the year will maximise and facilitate an improved understanding of essential mathematical vocabulary (Riccomini *et al*., 2015). This strategy may help learners realise the relationship between mathematical vocabulary and number concepts so that they are not regarded as separate concepts.

Upon further scrutiny, the teachers discovered that the learners also misinterpreted the operational symbols of multiplication (×) and addition (+).

This misconception of operational symbols (multiplication (×) symbol/procedure being taken as the addition (+) symbol/procedure) is the cause of learners' repetitive errors, especially in basic operation calculations. Learners are introduced to operational symbols in earlier grades (Foundation Phase) and, if the mathematical meaning of the different symbols is not understood in the early stages, it is inevitable that calculation errors will occur.

This misconception of interpreting the addition sign as the multiplication sign is confirmed in the example below. During the action phase of Cycle 1, teachers indicated that learners not knowing their tables was the cause of many basic operation errors. In the following example, it is shown that learners were able to solve $5 \times 6 = 30$, but could not distinguish between the addition (+) and multiplication (×) sign.


10. Which of the number sentences is the same as 5 x 6?

g) 5 x (3 + 2)
h) 5 + (2 x 3)
i) 5 x (2 x 6)
j) 5 x (2 x 3)
k) 5 + (3 x 3)
l) 5 + (2 x 3)

Explain/ show why you chose your answer: 2X3=6 6X5=30

g) 5 x (3 + 2)
h) 5 + (2 x 3)
i) 5 x (2 x 6)
j) 5 x (2 x 3)
k) 5 + (3 x 3)
l) 5 + (2 x 3)

I know this answer because 5+2x3=30 and 5x6=30

Explain/ show why you chose your answer:

g) 5 x (3 + 2)
h) 5 + (2 x 3)
i) 5 x (2 x 6)
j) 5 x (2 x 3)
k) 5 + (3 x 3)
l) 5 + (2 x 3)

Explain/ show why you chose your answer:
2x3=6 so 5x(2x3=30)

g) 5 x (3 + 2)
h) 5 + (2 x 3)
i) 5 x (2 x 6)
j) 5 x (2 x 3)
k) 5 + (3 x 3)
l) 5 + (2 x 3)

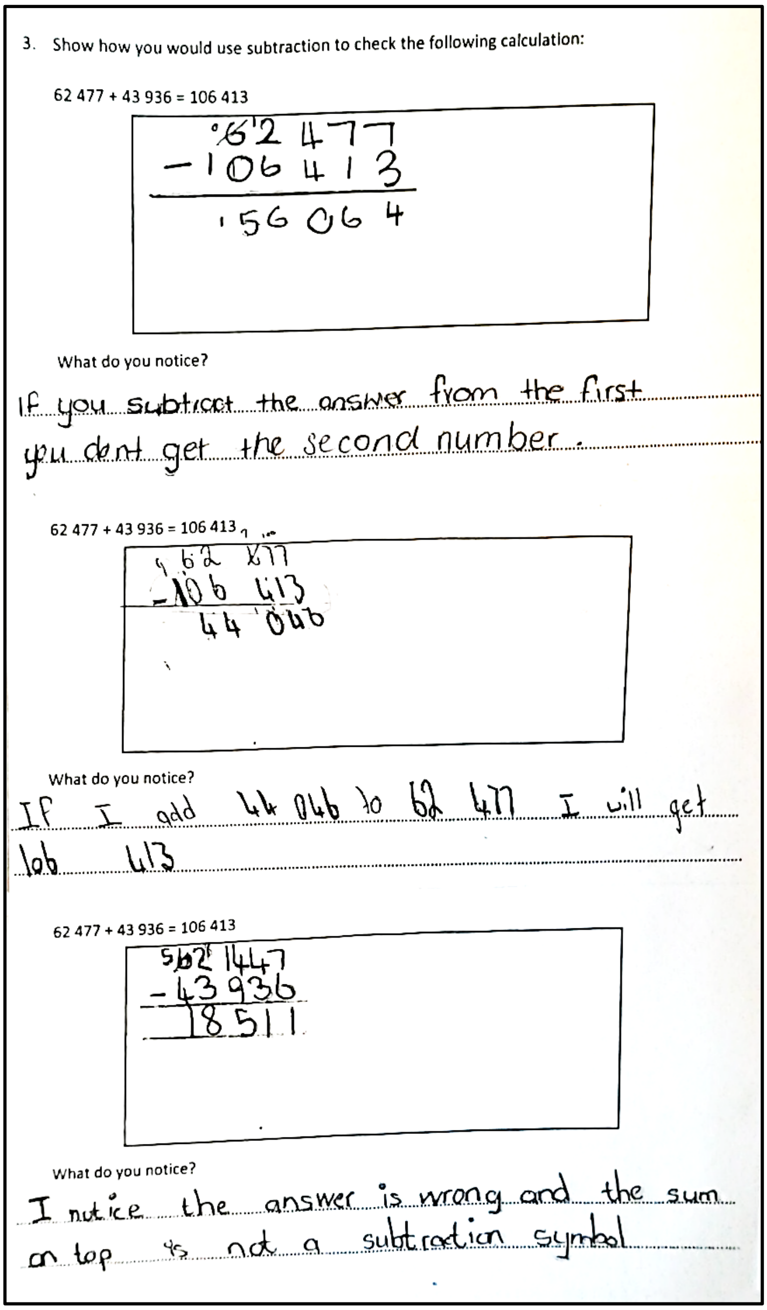
Explain/ show why you chose your answer: 5+(2x3) 5x6

Figure 4.10: A scan of learner explanations and errors with respect to the multiplication operation of numbers using brackets

It was also evident that teachers were baffled by some of the learners' responses. With some calculations, teachers followed the learner's execution step by step but struggled to explain what they thought the learner's reasoning was.

4.8.2.3 Inverse operations

In the example below (Figure 4.11), Grade 6 learners were asked to show how they would check the answer of an addition calculation by using subtraction. In order for learners to successfully check the answer, they should understand the inverse relation of addition and subtraction.



3. Show how you would use subtraction to check the following calculation:

62 477 + 43 936 = 106 413

62 477
− 106 413
156 064

What do you notice?

If you subtract the answer from the first you dont get the second number.

62 477 + 43 936 = 106 413

62 477
− 106 413
44 066

What do you notice?

If I add 44 066 to 62 477 I will get 106 413

62 477 + 43 936 = 106 413

562 1447
− 43 936
18 511

What do you notice?

I notice the answer is wrong and the sum on top is not a subtraction symbol

Figure 4.11: A scan of learner errors and reasoning with respect to the inverse relation of addition and subtraction.

In the first two examples, it is noted that the learners attempted to subtract a larger number value from a smaller number value. Teachers discussed the possible cause(s) of the learner errors and identified basic misconceptions related to place value, basic inverse operations, carrying and borrowing errors.

T12 - *"I would expect my learners to pick it up immediately that the bigger number goes on top."*

T10 - *"But they even got the answer wrong and could not see that it was different to the given numbers."*

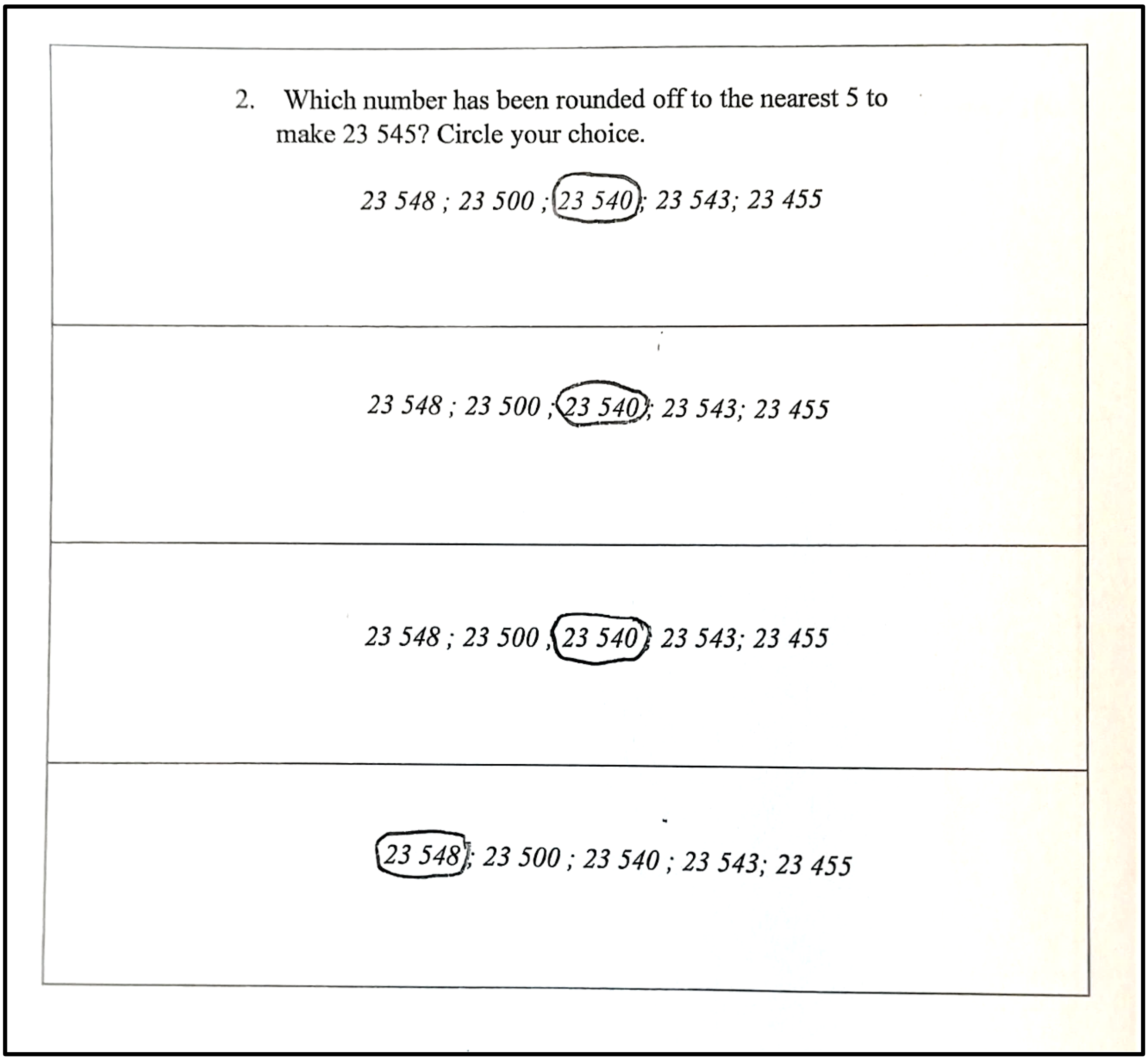
T9 - *"I think they know that subtraction is the inverse of addition, but we (teachers) have not given them enough practice to check their answers with inverse operations."*

In their study of elementary students' multi-digit whole number calculations, Kilpatrick *et al*. (2001:204) found that "subtraction algorithms require more time and support than addition algorithms". They explored three multi-digit subtraction procedures in their study of elementary school students. The procedural steps in all three procedures involved regrouping or borrowing to get 10 or more in the top position. The procedure of alternating between steps creates opportunities for students to make common errors of subtracting a smaller number value from a larger bottom number value. To avoid this error, students are encouraged to ask the regrouping (borrowing) questions, *"Can I subtract in this column? Is the top digit as big as or bigger than the bottom digit?"*

One teacher (T9) responded, "*we have not given them enough practise*", and Kilpatrick *et al*. (2001) state that the "focus of instruction should be on students' understanding and explaining, and not just on routine use". They suggest that "comparing the different calculation methods through classroom discussion is a means of facilitating reflection by students on the conceptual and notational features of arithmetic algorithms" (Kilpatrick *et al*., 2001:213). I believe that teachers should employ strategies focused on key concepts like place value and number properties in order to improve students' understanding of the relationship between multi-digit addition and subtraction calculations.

4.8.2.4 Rounding number errors

In the review below (Figure 4.12 and Figure 4.13) of identifying numbers rounded off to the nearest 5 and 100 (Grade 4 and 5 respectively), teachers discovered a pattern of errors related to the concept of rounding four and five digits of whole numbers. An important aspect of developing number sense is recognising that some numbers are approximate and that some numbers are exact.



2. Which number has been rounded off to the nearest 5 to make 23 545? Circle your choice.

23 548 ; 23 500 ; 23 540 ; 23 543; 23 455

23 548 ; 23 500 ; 23 540 ; 23 543; 23 455

23 548 ; 23 500 , 23 540 ; 23 543; 23 455

23 548 ; 23 500 ; 23 540 ; 23 543; 23 455

Figure 4.12: A scan of Grade 4 learner errors related to rounding off numbers

6. Choose and circle the number that has been rounded off to the nearest 100 to become 6 100.

6 150; 6 048; 6 211; 6 087; 6 178

6 150; 6 048; 6 211; 6 087; 6 178

6 150; 6 048; 6 211; 6 087; 6 178

6 150; 6 048; 6 211; 6 087; 6 178

Figure 4.13: A scan of Grade 5 learner errors related to rounding off numbers

The Grade 4 and 5 teachers explained the rules they teach learners when teaching rounding off.

T5 - *"I don't know why they got this question wrong. I teach them that when we round off to the nearest 100 you look at the tens … and if it is five or more you round off to the next hundred."*

T6 - *"I noticed that the kids always struggle with rounding off to the nearest five. They seem to better understand rounding off to the nearest ten, hundred and thousand."*

T8 - *"I think they didn't understand the question. We don't teach rounding off like that. We always give them a number and ask them to round off to the nearest five, ten, hundred or thousand."*

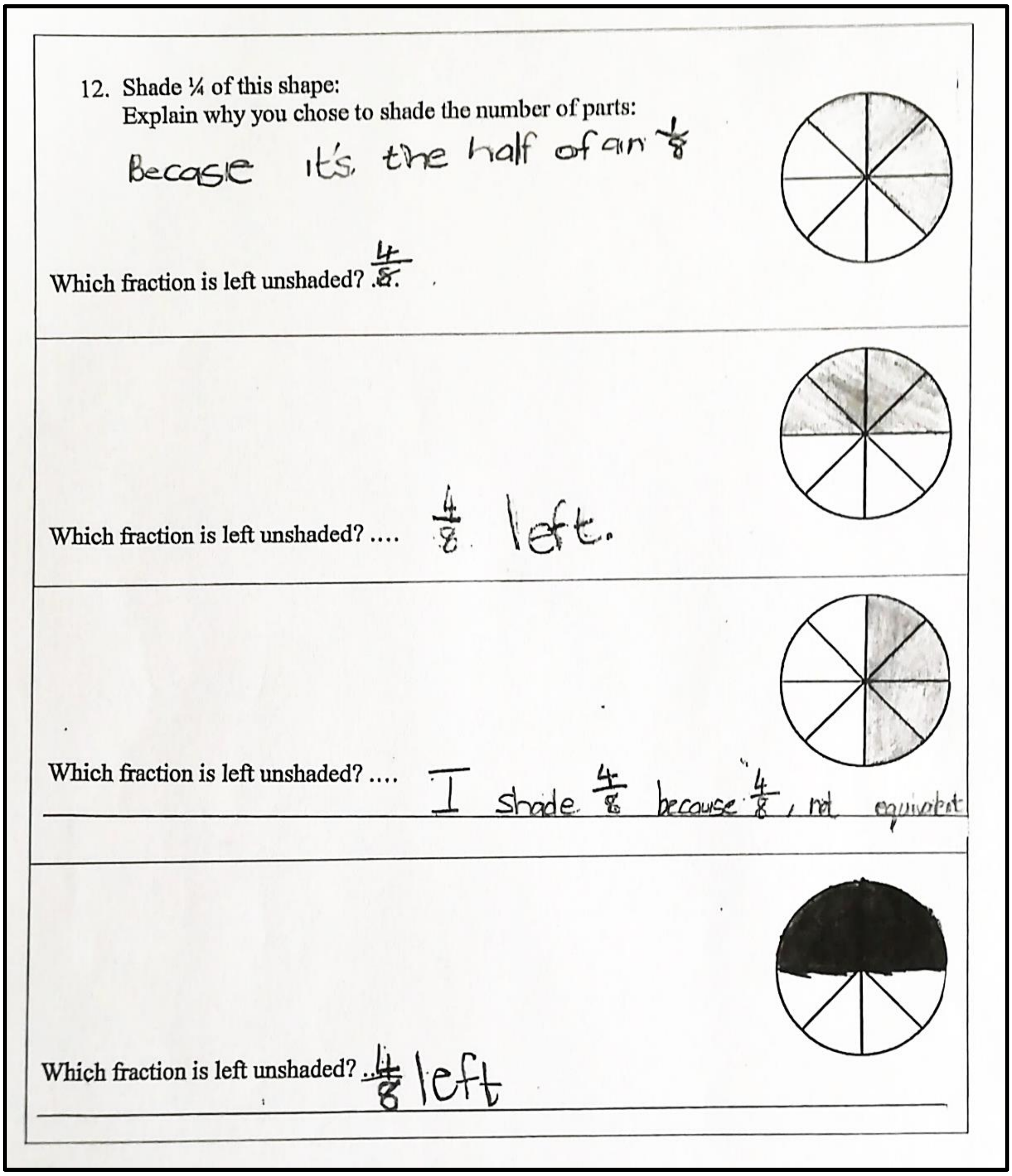
The teachers agreed that the way in which the question was posed was confusing to the learners. Their current teaching strategy exposed learners to only one method for rounding numbers. When learners were faced with a reversed question, their answers revealed a lack of conceptual understanding of rounding numbers.

For learners to identify the numbers that were rounded off, they should have a conceptual understanding of why rounding off is relevant to estimation in everyday situations, as well as a procedural understanding, by knowing the rules of rounding off. “Real world situations encourage learners to think about both the advantages and consequences of rounding numbers” (Kilpatrick *et al*., 2001:165). An example of a real-world situation would be for learners to understand that, if they intend buying something that costs R18.95, they should round up to R20.00 to make sure they have enough money for their purchase. Learners should view rounding numbers as a strategy that not only makes numbers easier to handle, but also makes sense.

4.8.2.5 Fraction errors

Describing and ordering common fractions is one of the skills learners practise at the Grade 4 level within the domain of NOR. Teaching guidelines in the curriculum encourage the aid of diagrams for recognising and using equivalent forms of common fractions.

In the example below (see Figure 4.14), learners were presented with a diagram divided into eight (8) equal parts and asked to shade one quarter of the shape. To calculate the equivalent fraction, learners should have prior knowledge and understanding of the relationship between the different parts of the whole and between eighths and quarters.



12. Shade ¼ of this shape:
Explain why you chose to shade the number of parts:
Becasie it's the half of an $\frac{1}{8}$

Which fraction is left unshaded? $\frac{4}{8}$

Which fraction is left unshaded? .... $\frac{4}{8}$ left.

Which fraction is left unshaded? .... I shade $\frac{4}{8}$ because $\frac{4}{8}$, not equivalent

Which fraction is left unshaded? $\frac{4}{8}$ left

Figure 4.14: A scan of learner errors related to interpreting equivalent fractions

In all of the above responses, learners shaded four of the eight parts, resulting in a half and not a quarter, as instructed in the question. Learners' responses to '*Which fraction is left unshaded*?' therefore were also a half (four eighths of the whole).

When teachers were asked why they thought learners shaded a half and not a quarter of the whole, a few responses of the Grade 4 teachers to the fraction error were:

T1 - *"I've only used a fraction wall to teach fractions so this shape would confuse my learners."*

T5 - *"This is difficult. My kids always struggle with equivalent fractions. But they do know how to write a fraction because they wrote four eights in their answer."*

T4 - *"I think it's reading and interpretation of the question. They saw they had to shade one quarter and shaded the four parts of the denominator, not realising the denominator represents the whole. Also not understanding the rule of equivalent fractions. It's too much for the learners to grasp. The concept hasn't been consolidated."*

In the above fraction example (Figure 4.14), the incorrect responses of learners are indicative of a possible misconception of the values of fractions and of what the numerator and denominator of a fraction represents.

The teachers discussed the importance of using concrete objects to show and help learners understand the equal relations between fractions. Some suggestions were to ask learners to shade one quarter on a fraction wall and then compare the size of the shaded fraction to other fractions to determine their equivalence. Teachers also thought that introducing real-world examples of fractions may help learners to see the relevance and relation of different fractions. In general, teachers still felt that the use of concrete objects should be implemented in the early grades. At the Grade 4 level, they teach the rule of equivalent fractions by telling learners to simplify the numerator and denominator by dividing both by a common factor.

The Grade 4 example below (Figure 4.15) was a grouping problem that could be solved by multiplication. To calculate the total number of legs of eight chickens and seven dogs, learners had to apply reasoning skills.

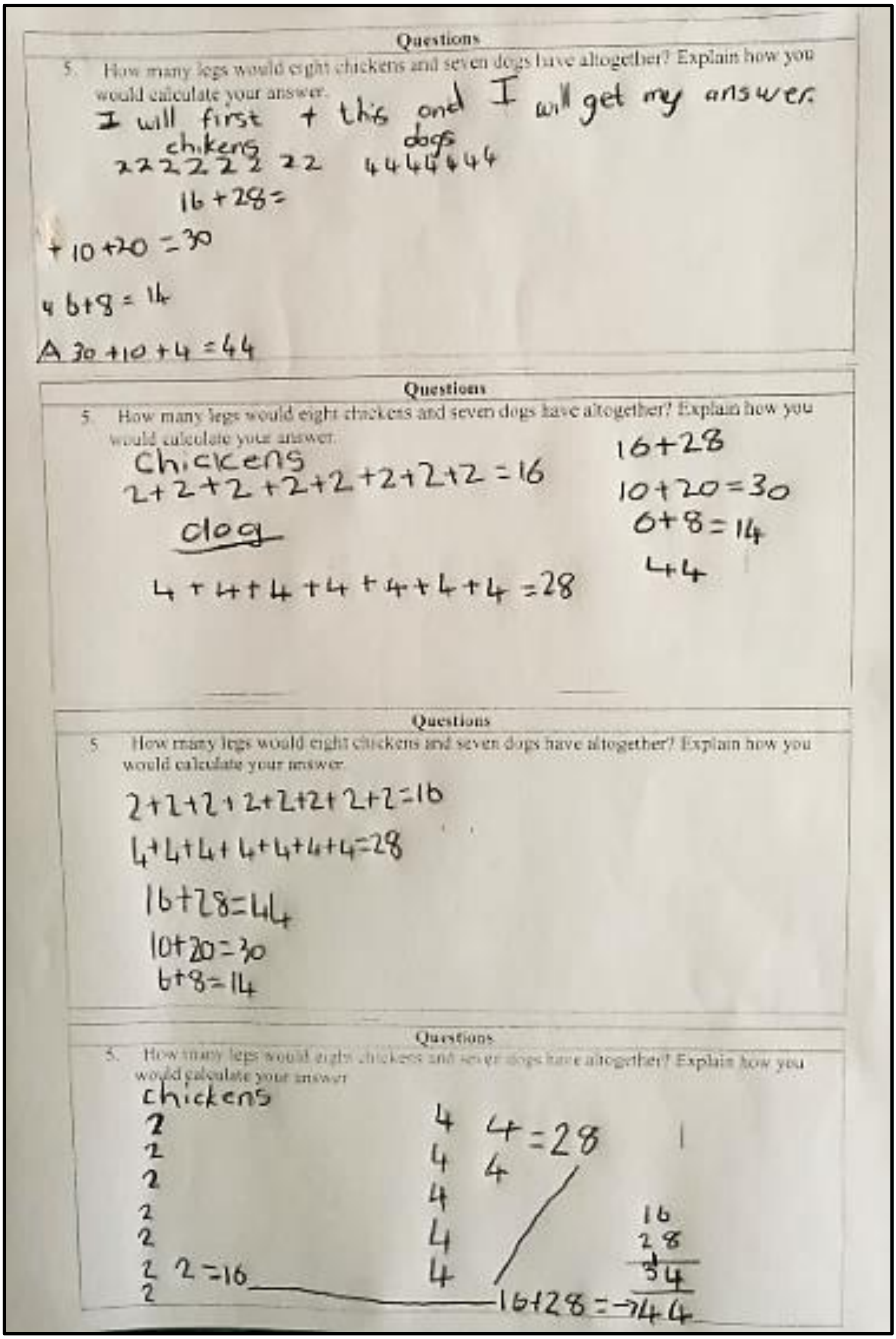


Figure 4.15: A scan of learner errors/explanations with respect to the relationship between repeated addition and multiplication

All four responses show that the learners arrived at the correct answer of 44 legs, which indicates that they had conceptual understanding of the problem. Their procedural application of calculating the answer shows that there was a lack of understanding of the relationship between repeated addition and multiplication. Learners used repeated addition of 2+2+2+2+2+2+2+2 =16 and 4+4+4+4+4+4+4 = 28 instead of recognising the pattern as multiplying 2 × 8 = 16 and 4 × 7 = 28. In this case, the evidence of the learners' work shows that they did not make the connection between previous knowledge of addition to multiplication.

During the review of this problem, teachers debated whether it was more important that learners got the answer correct, or whether their procedural steps used to find the answer were more important.

The teachers agreed that having a conceptual understanding was important, but that learners should have a mathematical understanding of multiplication as repeated addition. If learners were presented with a problem in which larger number values were used, they would find it very challenging and timeous to continue using repeated addition to calculate an answer.

The second part of the above question asked learners to explain how they would calculate their answer. The intention of this question was to determine if learners could apply their reasoning skills and show their understanding of the relationship between repeated addition and multiplication. Some learners responded that they would 'plus' the groupings to get an answer.

4.9 Reflection on Cycle 2

From the above NOR error examples (Figure 4.7 to Figure 4.15), it became apparent that there was more to merely distinguishing between correct and incorrect answers. Reviewing the different errors by their learners afforded the Intermediate Phase teachers the opportunity to analyse learner errors in NOR. Although teachers gained some insight into the errors their learners made, further clarity was needed on what specific misconceptions existed in their learners' mathematical thinking in the domain of NOR.

4.10 Cycle 3 – Planning

To help teachers obtain clarity on learners' errors, they needed to attend to their learners' strategies in NOR. Teachers agreed that they would have to hear from their learners how they apply their mathematical knowledge and skills to solve problems.

In Cycle 3, the teachers planned to present learners with incorrect responses and engage learners in cognitive conflict. Through cognitive conflict, teachers aimed to support and guide learners to discover for themselves where they went wrong. Posing questions to learners about the errors can give direction to learners' thinking processes and guide them to organise their ideas. At this point, it was important for teachers to be aware that the purpose of the questions was to guide learners and not to give them the solution or simply reveal the answers. The urge to help learners may result in lowering or removing the cognitive demand (Henningsen & Stein, 1997). Applying cognitive conflict as a strategy in the classroom would eliminate the process of re-teaching a concept and highlight the point of misconception within a concept.

The teachers also decided that they would incorporate examples of different questioning levels and different question types into concepts in NOR. A different approach to the same task may provide a foundation that builds connections to the mathematical meanings of NOR concepts. Exposing learners to different levels and types of questioning may strengthen their conceptual understanding of NOR concepts.

Based on the evidence of learner errors in Cycle 2, the following NOR concepts would be addressed in Mathematics lessons:

- Place value and rounding off
- Operational symbols and mathematical vocabulary
- Inverse operations, and
- Common fractions

The Intermediate Phase teachers agreed that one teacher in each grade would focus on one of the concepts. This was due to time constraints and, in the light that the progression of the four concepts, would be covered across the phase.

4.11 Cycle 3 – Action

Classroom observations allowed the teacher-researcher opportunities to observe:

- the way in which the teacher introduced the concept
- how teachers accessed learners' prior knowledge of the concept
- what questions teachers posed to learners to make sense of their understanding
- how teachers analysed the conceptual and procedural knowledge of the learners
- how teachers identified misconceptions, and what strategy they employed to address the misconceptions,

thereby answering the sub-questions, "*How do teachers access learners' prior knowledge mathematical knowledge?"* and *"How do teachers analyse procedural and conceptual knowledge and use their findings to remediate errors and misconceptions?"*

4.11.1 Place value and rounding off misconceptions

As teacher-researcher, I observed three mathematics lessons during which the teachers' objectives were to establish at what point misconceptions of place value and rounding off were evident. As researcher I made field notes during the lesson observations. Each lesson is discussed individually below to gain insight into the different teaching and remediating strategies implemented by the Intermediate Phase teachers.

4.11.1.1 Lesson 1 – Grade 4

As an introduction to the lesson, the teacher first wrote one digit on the chalkboard and asked the learners to read the number. Each time a digit was added, learners were asked to read the number aloud until it was a three-digit number. It was evident that learners were confident reading the numbers as they increased in value up to three digits. The learners were asked to describe the different values of each digit within the three-digit number, e.g. 246 as two hundred, four tens and six units, and to show how the number could be written in expanded notation. The teacher used this strategy as a means to determine what the learners knew about the value of digits in numbers. It was noted that the learners only presented the expanded notation in place value order of the given digits:

e.g. $200 + 40 + 6$; $(2 \times 100) + (4 \times 10) + (6 \times 1)$ and 2H + 4T + 6U

No other representation of expanded notation,

e.g. $6 + 40 + 200$ or $40 + 200 + 6$ or $(6 \times 1) + (4 \times 10) + (2 \times 100)$ etc.,

was offered by the learners, which would still result in building the number 246. This was an indication that, when the concept of number and place value is taught in only one notational method, learners may not make the connection between the place and number value of digits.

Next, the question, "*What can we do with a number like 246 in Maths?"* was posed to the learners. The responses from learners included *round off the number*, *write the number in words*, *halve and double the number*, and *move the digits around to form a new number*.

Relevant to the outcomes of Cycle 3 in this study is that the teacher focused on rounding off during the lesson. Rounding off to the nearest ten (10) and nearest hundred (100) were used as examples. Learners were asked to explain how they would round off to the nearest 10 and to the nearest 100. They responded with the correct answers of 246 rounded off to the nearest 10 is 250, and rounded off to the nearest 100 is 200. The explanations learners provided for rounding off to tens were that they *look at the units*, and for rounding the number off to the hundreds they *look at the tens*. The teacher-researcher noted that no further explanation of this concept was expected of the learners by the teacher. Further questioning may allow the teacher to determine whether the learners understand and whether they could explain the rules of rounding off, e.g. if the number you are rounding is followed by 5, 6, 7, 8 or 9, round the number up, and if the number is followed by 0, 1, 2, 3 or 4, round the number down. It was discovered that current examples of classroom activities are limited to only asking learners to round off a given number, as illustrated in Table 4.5 below.

Table 4.5: An example of a rounding-off classroom activity

Round off each number to the nearest 10, 100 and 1 000			
Number	**Nearest 10**	**Nearest 100**	**Nearest 1 000**
834			
799			

If learners have a conceptual understanding of rounding numbers, then executing the rule of rounding numbers would be applying their procedural knowledge. This conclusion is substantiated in the next example. Learners were given the following numbers, *255*; *245*; *262*; *256*, and asked to identify the number that, if rounded off to the nearest 10, would result in 250. It was clear that this form of questioning was an unfamiliar exercise for the learners, as they found it challenging to select the correct answer. The teacher guided them by suggesting they eliminate the options by applying the rule of rounding off to the nearest 10. While a few learners could follow the strategy of using the rule to eliminate the numbers and selected 245 as the correct answer, most of the learners incorrectly answered 260. Learners were asked why they answered 260, which was not one of the given options. A few of the explanations included: *"If I round off 255; 262 and 256 to the nearest 10 it will be 260." "I think there is a mistake, 245 don't belong there." "I can round off 255 to the 260."* The above responses shed light on the following:

- learners interpreted the question incorrectly

- learners had procedural knowledge of rounding off, but could not apply conceptual understanding to solve the problem

4.11.1.2 Lesson 2 – Resource Unit (Grades 4 and 5)

The second lesson was observed in the Resource Unit with learners with learning barriers. Although the lesson was initially planned to address both place value and rounding off, only place value was covered in the allocated time of the lesson.

The introduction to the second lesson observed showed a similar strategy to Lesson 1, of building a number starting from one digit to three digits, e.g. 4…84…184. Learners were asked to read the number each time. The difference in this lesson was that the teacher used columns of place value referred to as ‘houses’ the digits ‘lived in’. The teacher motivated that she uses non-mathematical vocabulary, i.e. ‘houses’, to make Mathematics relatable to the learners. When asked to show how they would represent the number in place value, learners drew circles in each ‘house’ (see example in Table 4.6 below).

H	T	U
0	0000 0000	0000

Table 4.6: An example of learner representation of place value

It was noted that learners did not write digits in the place value column and that drawing circles was a strategy introduced in earlier grades (Foundation Phase) to help learners make connections with counting and number value. At the Intermediate Phase numeracy level, writing digits should be a familiar exercise, but the teacher explained that the learners in the Resource Unit often use manipulatives to express their numeric understanding.

Responding to the teacher’s request to explain the value of each of the ‘houses’, the learners provided the following descriptions: *“Place value is where the number lives, like 80 (eighty) is in the middle column.” “The 4 circles mean only 4 (four), the 8 circles mean 8 tens and the one circle means there is nothing in the hundreds.”* Listening to the responses of each learner, the teacher prompted the learner to name the *‘middle column’* as ‘tens’ and further asked if the *‘middle column’* would always be ‘tens’. When the response from the learner was “*Yes*”, it alluded to the fact that this misunderstanding was derived from the column and ‘houses’ representation used as a teaching strategy, which only includes hundreds, tens and units as examples of place value. In guiding the learner to correct their response, the teacher extended the number to four digits, viz. 3 184, to include thousands as the fourth place value column. Learners were then able to see that the tens were no longer the 'middle column'. Using non-mathematical vocabulary of 'houses' instead of 'place value' was found to be a possible reason for learners' confusion between number value and place value.

Another response from a learner, namely "*the one circle means there is nothing in the hundreds*", highlights the confusion created by using circles as counters for the values of the digits. In this instance, the learner initially viewed the circles as counters when describing the units and tens values, and then viewed the single circle as zero (0) in the hundreds. The teacher suggested that perhaps using a different symbol, e.g. tallies, may have eliminated this confusion.

As part of the lesson's consolidation, the learners were given the number 2 076 and asked how they would go about to form a new number. The response was to *"move the digits around"*. As learners wrote their new four-digit numbers, some answers included 7 062, 6 270, etc. The teacher focused on the answers of 0 726 and 0 267, where learners used zero (0) as a placeholder for the thousands. To determine their understanding of using zero (0) as a placeholder, the teacher asked them to read their numbers out loud. Learners read 0 726 as *"nought thousands seven hundred and twenty-six"* and 0 267 as *"nought thousands two hundred and sixty-seven"*. The teacher drew their attention to reading *"nought thousands"* as having no value and asked how the number would be read if there was no thousands value. Guiding them in this way elicited correct responses of *"seven hundred and twenty-six"* and *"two hundred and sixty-seven"*.

4.11.1.3 Lesson 3 – Grade 6

The third lesson focusing on place value was introduced by the teacher placing chairs labelled with different place values, ranging from units to millions, in order. Each chair was representative of the different place values. Initially the teacher used one digit and asked the learners to identify the value of the digit as it was moved from one chair to the next. This exercise was confidently executed by the learners as they identified the value, e.g. 700 (seven hundred), when the digit was placed on the chair labelled hundreds (H). The next step progressed from one digit to working with four single digits – 2, 4, 6 and 9. Learners were asked to sit on four of the chairs, namely the 2 on units, the 9 on ten thousands, the 4 on millions and the 6 on hundreds, resulting in 4 090 602. Due to the fact that the labels ranged from units to millions, some chairs were empty, in other words, some place values were omitted. Learners were asked: *"What is the value of the vacant chairs?"* They could confidently answer that it was zero (0). When further asked, *"Why do we write zero (0)?"* learners struggled to explain or motivate their answer. One attempted response of a learner, *"It's zero because the chair is empty and there is no number"*, suggests a possible lack of understanding of the importance of zero as placeholder. To guide learners in distinguishing between the number value *with* the zeros as placeholders and *without* the zeros as placeholders, the teacher instructed half of the group to write the number with zeros and the other half of the group without zeros. This resulted in two different numbers, namely 4 090 602 and 4 962. Reading each number aloud revealed that learners found it challenging to read the seven-digit number, compared to confidently reading the four-digit number. In small groups, learners were asked to first discuss and then answer the same question, *"Why do we write zero (0)?"* This time the responses from the small groups included, *"The zero holds the place of a digit where there is no value"*; *"Using the zeros in the number changes the value of the number"*.

4.11.2 Summary of lessons 1 to 3

This summary relates the findings from the observations in the three lessons focusing on place value and rounding off.

The introduction to all three lessons started at the basic levels, with which the learners were confident. The basic level refers to a number range from single digits to three-digit numbers. Teachers wanted to determine what the learners knew, i.e. their prior knowledge, and extend that with gradual progression. The progression of the number ranges and levels of difficulty varied in each class.

- During the lessons, more opportunities were created for learners to explain or motivate their responses.
- Listening to and interpreting the learners' explanations aids teachers in gaining insight into their learners' understanding of their own errors.
- Encouraging learners to find solutions through group discussions showed improved understanding.
- Types of mathematical representations/manipulatives used in class should be selected carefully, as these could add to learners' confusion.

4.11.3 Operational symbols and mathematical vocabulary

The following three lessons (lessons 4, 5 and 6) observed teaching strategies implemented to determine learner misconceptions related to operational symbols and mathematical vocabulary in NOR. The researcher recorded the findings as field notes. Each lesson is discussed individually below to gain insight into the different teaching and remediating strategies implemented by the Intermediate Phase teachers.

4.11.3.1 Lesson 4 – Grade 5

The teacher introduced the lesson dealing with mathematical vocabulary by asking learners to work in pairs and discuss the meaning of, and find synonyms for, the word “times” in mathematics. Learners responded that the term “times” means *to multiply, or when you say 2+2+2+2+2 you keep adding the same number.* Although learners gave an example of repeated addition, it may have been relevant for the teacher at this point to draw the learners’ attention to, and elaborate on, the relationship between addition and multiplication as repeated addition. Learners also incorrectly responded that *finding the quotient also means to multiply*. The teacher rectified this by pointing out that finding the quotient relates to division, which is the inverse operation of multiplication. The response the teacher was hoping to elicit from the learners was that the mathematical term “product” also refers to multiplication. When learners were not forthcoming with the expected response, the teacher provided an explanation of applying the basic operation of multiplication when the term “product” appears in a mathematical problem. To confirm their understanding of the term “product”, the teacher asked learners to work in pairs to investigate responses to a problem, which include two mathematical terms, i.e. *“difference”* and *“product”*. The following scan (Figure 4.16) shows two incorrect responses to the problem: *“Determine the difference between 5 000 and the product of 5 and 1 000.”*

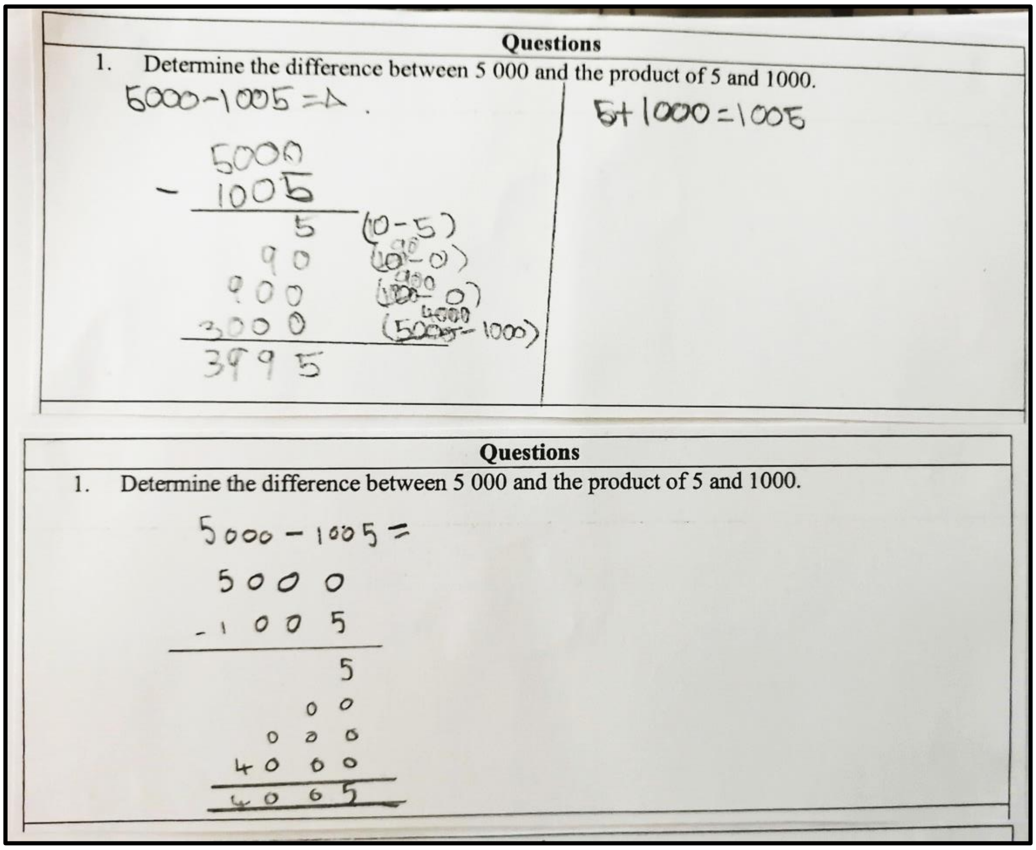


Figure 4.16: A scan of incorrect responses related to mathematical vocabulary:

The instruction to the learners was to review and find the errors in the two examples. The aim of this teaching strategy of engaging learners in cognitive conflict was

- to ascertain their understanding of mathematical vocabulary,
- to discuss and apply reasoning skills, and
- to share solutions

After discussing the errors in pairs, the learners' feedback included the following responses:

LR 1: "Both sums are wrong because the child did a minus sum"

LR 2: "Both children said 0-5=5. They didn't borrow"

LR 3: "In the first example the child didn't times, he added 5 to 1 000 and that's why he got 1 500"

During the introduction of the lesson, the teacher only explored the mathematical meaning of "times" and, at this stage, it became apparent that learners also were not clear about the meaning of the term "difference". The teacher indicated that she assumed the learners understood the meaning of "finding the difference" and that they would be able to apply their previous knowledge to solve the problem. Learners needed further support in reviewing the errors in the two examples. It was clear that learners found it challenging to identify and explain the errors in the two examples. A possible reason for this could be that the learners themselves were unsure of how to solve the problem and that they had their own misconceptions about the mathematical vocabulary of finding the *difference* and the *product* and how to solve the problem.

To guide learners in finding the errors, the teacher posed the following questions:

- *How many operations should be executed to determine the answer?*
- *Which operations should be executed to determine the answer?*
- *Which words will guide you to identify the operations?*

With the assistance of the guided questions, they were able identify the two operations in the problem, i.e. difference meaning subtraction and product meaning multiplication. The next step was to draw learners' attention to the incorrect interpretation of the term "product" as addition in both examples. Both examples show 5+1 000=1 500, instead of 5×1 000=5 000. At this point, the learners were able to identify the misconception of the term *product* as addition in both examples.

Although the focus of this lesson was mathematical vocabulary, the two examples also alluded to another misconception present in both examples, with the incorrect subtraction operation of 0-5=5. When the teacher posed the question, *"Why do they have different answers for the same sum?"*, the learners responded that:

LR: *"Their calculation is wrong because they didn't borrow from the tens"*

LR: *"Their calculation method is wrong"*

It was interesting to note that the learners did not attempt to do their own calculation to determine the correct answer. Doing their own calculation may have provided learners with an opportunity to explain their own reasoning about the errors in the examples.

4.11.3.2 Lesson 5 – Grade 6

The aim of this lesson was to eradicate the misinterpretation of the operational symbols of multiplication (×) and addition (+). The introduction to the lesson started off with the teacher asking the question, *"What is multiplication?"* with LR, *"When you times a number by another number, it's like groups of a number"*. No further explanation of this response was elaborated on to determine if learners:

- could identify the operation symbol for multiplication,
- understood what is meant by “groups of a number”, or
- understood the relationship between repeated addition and multiplication

The teacher wrote a three-digit by one-digit problem (693×4) on the chalkboard and told the learners that they would use the vertical short column method to solve the problem. Although this lesson was aimed at addressing learners’ misinterpretation of the operational signs of multiplication (×) and addition (+), emphasis was placed on the learners’ procedural knowledge of executing a multiplication problem using a specific method.

The learners provided the step-by-step explanation of the vertical short-column method to guide the teacher. Figure 4.17 shows the steps followed to solve the problem.

$$\begin{array}{r} {}^{3}6\ {}^{1}9\ 3 \\ \times \qquad 4 \\ \hline 2\,7\,7\,2 \\ \hline \end{array}$$

Steps as explained by learners:

4x3=12 [carry the 10]
4x90=360+10 = 370 [carry the 300]
4x600=2 400+300=2 700

Figure 4.17 Vertical short-column method as explained by the learners

With every carry-over step, the learners did not name the actual value of the digit, but instead referred to it as a digit with a single value, e.g. 4×3=12, write 2 and *carry 1*, instead of carry 10; 4×9=36 plus 1 equals 37, write 7 and *carry 3*, instead of carry 300. Incorrectly naming the carry-over digit creates the possibility of place value misconceptions. The teacher drew their attention to the value of each digit being carried over and reminded the class that even though they referred to the digit as a single-unit digit, they should remember the actual value of the digit being carried over. The learners’ incorrect response to the last step of the vertical short-column method was, *“You just add the 4 and the 6 and then 300”*. This response was an indication that there was possible confusion with either the learners’ understanding of the procedure of this method of multiplication, or confusion with the mathematical operation signs of multiplication (×) and addition (+). Another possibility could be that the learner generalised what he had learned about the addition step of the vertical short-column method and applied the addition of the carry-over to the multiplication of the digits. No further questioning to clarify the learners’ misconception was done. The teacher reminded the learners that they were working on a multiplication problem and not an addition calculation. With no further explanation, this reminder could possibly lead to confusion in the learners' procedural knowledge of the vertical short-column method, which includes multiplication and addition calculations to solve the problem. A fellow classmate was asked to assist his peer in correcting the last step of the multiplication problem. The response given was *"4×6=24 plus 3=27 and then I just add the 27 to the answer next to the other 7 because there are no more numbers to multiply".* It was assumed by the teacher and class that what the learner meant was to write the digits 27 as 2 000 and 700 place values, although to mathematically "*add the 27*" to the answer would result in a different operation and answer. The language used by learners to explain the steps of their calculation showed a lack of using specific mathematical vocabulary. The response explaining *"no more numbers to multiply"* provided an opportunity to clarify the place value of "no thousands value in the number".

Before presenting learners with another multiplication problem, the teacher posed the question to the learners: *"Why do you think we used the vertical short-column method to calculate 693×4?"*

The learners responded with the following:

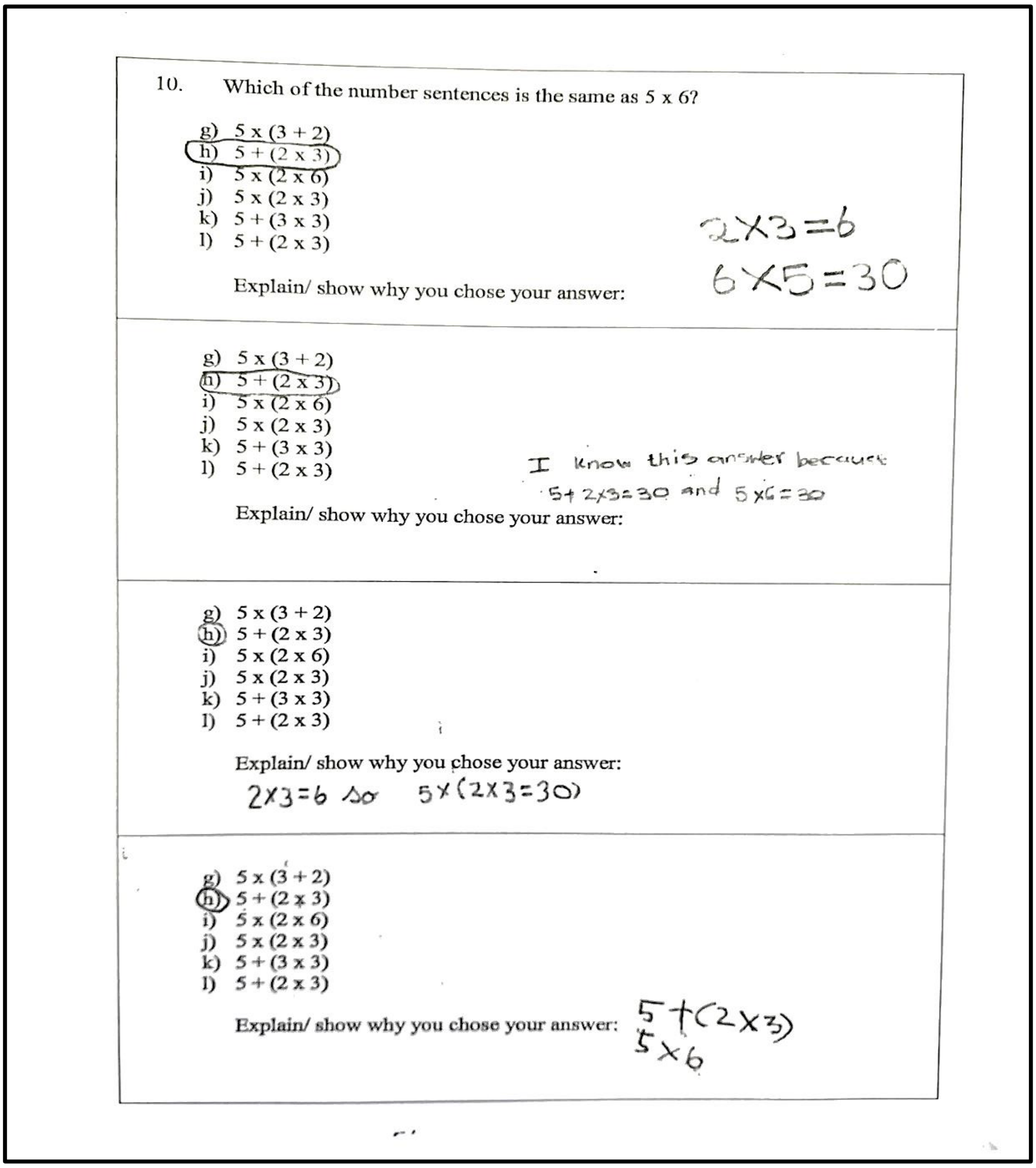
LR1: *"Because it's easy with only one digit"*

LR2: *"We can get to the answer quicker"*

The teacher then asked the class to explain what 693×4 means and how they would represent it. The lack of responses from learners was an indication that they struggled to explain multiplication as adding a number to itself a specified number of times. The teacher then started to write 4+4+4+4+4+4…… on the chalkboard and asked the class if it was easier to keep the repetitive pattern of adding 4 each time. Learners then confidently responded by saying it was easier to multiply 4 by 693 and could realise the relationship between repeated addition and multiplication.

4.11.3.3 Lesson 6 – Grade 4

As an introduction to the lesson, the teacher informed the class that they would be presented with an answer sheet that included responses to a number sentence problem. This strategy was aimed at engaging learners in cognitive conflict by reviewing incorrect responses on the answer sheet. As mentioned in the planning stage of Cycle 3, posing questions to learners about errors can give direction to learners' thinking processes and guide them to organise their ideas. The class was split into four groups with six members each. In their groups, the learners had to review the various responses as presented in Figure 4.10 in Cycle 2:


10. Which of the number sentences is the same as 5 x 6?

g) 5 x (3 + 2)
h) 5 + (2 x 3)
i) 5 x (2 x 6)
j) 5 x (2 x 3)
k) 5 + (3 x 3)
l) 5 + (2 x 3)

Explain/ show why you chose your answer: 2X3=6 6X5=30

g) 5 x (3 + 2)
h) 5 + (2 x 3)
i) 5 x (2 x 6)
j) 5 x (2 x 3)
k) 5 + (3 x 3)
l) 5 + (2 x 3)

I know this answer because 5+ 2x3=30 and 5 x6=30

Explain/ show why you chose your answer:

g) 5 x (3 + 2)
h) 5 + (2 x 3)
i) 5 x (2 x 6)
j) 5 x (2 x 3)
k) 5 + (3 x 3)
l) 5 + (2 x 3)

Explain/ show why you chose your answer:
2X3=6 so 5X(2X3=30)

g) 5 x (3 + 2)
h) 5 + (2 x 3)
i) 5 x (2 x 6)
j) 5 x (2 x 3)
k) 5 + (3 x 3)
l) 5 + (2 x 3)

Explain/ show why you chose your answer: 5+(2x3) 5x6

The problem on the answer sheet required learners to identify the number sentence that was the same as 5×6, i.e. *Which number sentences would result in the same answer, 30?* The answer sheet provided learners with six options to choose from, from *g* to *k*. All the responses on the answer sheet had the same incorrect selection, namely *h, as 5+(2×3),* which resulted in an answer of 11.

The first instruction to the groups was to read the question and say whether they agreed or disagreed with the selections on the answer sheet. The second instruction to the groups was that they had to provide an explanation to motivate the reasoning behind their answers. After the groups were given a few minutes to review and discuss the responses on the answer sheet, their feedback included the following responses:

Group 1: *"We agree that the answers are all right, because 5×6=30 and that number sentence also equals 30."*

Group 2: *"It's all wrong, because the number sentence by 'h' equals 11 and not 30, because 5×6 does not equal 11."*

Group 3: *"We think it's right, but there is a typing mistake. The plus must be times by 'h'."*

Group 4: *"The correct answer should've been 'j', because when you say 2×3=6 and times it by 5 then you get 30. The same as 5×6=30."*

The responses of the groups show that Groups 1 and 3 were unsuccessful in identifying the errors and Groups 2 and 4 could correctly identify that the responses on the answer sheet were incorrect.

After the groups gave their feedback, the teacher asked Groups 1 and 3 to show the class how they calculated their answers. The speaker of Group 1 explained their calculation steps in the following way:

Group 1 speaker: *"We said that 5×6=30, and then we did the brackets like 2×3=6 and then we times the 6 with the 5 to get 30."*

Teacher: *"Why did you multiply the 6 with the 5?"*

Group 1 speaker: *"Because of the brackets."*

Teacher: *"What about the plus (+) sign? Should you not add the 5 to the bracket answer of 6?"*

Group 1 speaker: *"Oh, I don't know. We thought with brackets you must multiply."*

The feedback of Group 1 indicates confusion on the application of the BODMAS rules. The rules of BODMAS are known as the ordering of mathematical operations to solve a mathematical equation – bracket, of, division, multiplication, addition and then subtraction. A possible reason for this misconception of applying the rules of BODMAS is that learners overgeneralised the rules, i.e. applying multiplication for the brackets and ignoring the addition operational sign before the brackets.

At this time, the teacher drew the learners' attention to the rules of BODMAS by asking them to explain the acronym. The learners could confidently identify the order of mathematical operations as bracket, of, division, multiplication, addition and subtraction. The teacher continued to explain that multiplication is only used when there is no operation sign in front of the bracket. The learners in Group 1 were then asked if they should use multiplication if there is another operation sign in front of the bracket. Although some of the group members could answer this correctly by saying "No", it was also clear that some of the group members were unsure.

The speaker of Group 3 explained their answer in the following way:

Group 3 speaker: *"Oh, we thought there is a mistake with the plus sign. Shouldn't it be 5×(2×3)?"*

Teacher: *"No, there is no typing error. So does your group still agree that 'h' is the correct answer?"*

Group 3 speaker: [Brief group discussion] *"We found a times (×) by 'j' 5×(2×3) and that is 30.*

Teacher: *"That's right!"*

When the teacher posed questions to the members of Group 3 about their error, their responses allowed the teacher to gain insight into their thinking process and, through their review group discussion, the learners were able to identify the correct answer on the answer sheet.

Although the speaker of Group 2 was correct in saying that *"h"* was the incorrect choice, they neglected to identify the correct answer as *"j"*. When Group 4 was asked to share with the class how they arrived at their answer, they responded:

Group 4 speaker: *"We first did the sum of 'h' and saw it was wrong. Then we tried the others and 'j' was right answer."*

The responses from Groups 1, 2 and 3 also highlight the fact that when the groups were asked to agree or disagree with the responses on the answer sheet, they did not attempt calculations to eliminate from the list of options. Applying the process of elimination through calculations in a multiple-choice option, may possibly be a skill not yet explored by the learners.

4.11.4 Summary of lessons 4 to 6

This summary relates to the findings observed in the three lessons focusing on operational symbols and mathematical vocabulary.

The structure of the three lessons varied from learners working in pairs, working as a whole class and working in smaller groups. The introductions to Lessons 4 and 6 involved learners engaged in cognitive conflict, where they were presented with incorrect responses and had to identify the errors. Lesson 5 was aimed at eradicating the misconception between the operational symbols of addition (+) and multiplication (×) by working through a specific method of multiplication – the vertical short-column method. This strategy of focusing on procedures proved that learners were limited to only applying the learned process and, within that, the risk of overgeneralising the rules of the method. Although teachers initially planned to attend to learners' strategies and gain insight into their understanding, the lesson observations proved that thorough planning regarding guided questions for learners could be more beneficial.

- Learner responses indicated that there was confusion in their understanding of mathematical vocabulary
- Teachers assumed that learners were confident in their understanding of mathematical vocabulary and in their understanding of the relationship between repeated addition and multiplication as prior knowledge
- Emphasis on procedural knowledge highlighted misconceptions in learners' conceptual understanding

During lessons 4 and 5, the incorrect responses of the learners were rectified by the teacher or other classmates, whereas this opportunity for further questioning could benefit teachers to achieve greater awareness of their errors or misconceptions, as seen in lesson 6. In lesson 6, learners were asked to explain the reasoning behind their answers, which created opportunities for the teacher to follow their thinking processes and, in this way, identify the point of misconception.

- Posing guided questions can help learners to review their reasoning and organise their own ideas to solve problems
- Accessing learners' prior knowledge (the rules of BODMAS) supported the teacher to determine the point of misconception (overgeneralising the rule of multiplication for brackets)

4.11.5 Inverse operations

The following three lessons (lessons 7, 8 and 9) observed teaching strategies implemented to determine learner misconceptions related to inverse operation in NOR. The researcher recorded the findings as field notes. Each lesson is discussed individually below to gain insight into the different teaching and remediating strategies implemented by the Intermediate Phase teachers.

4.11.5.1 Lesson 7 – Grade 4

As an introduction to the lesson, the teacher informed the class that they would be working with "inverse operations" and asked the learners to explain their understanding of inverse operations. The learners were told to work in pairs and decide on an explanation of inverse operations, as well as an example. The following are a few responses:

Pair 1: *"We said that the inverse means the opposite. Like 10+6 =16 and when you say 6+10=16, you get the same answer."*

Pair 2: *"If you say 2×3=6, then you can also say 3×2=6."*

Pair 3: *"We also say it's the opposite, like if you add then you must subtract, or when you times then you must divide."*

Pair 4: *"Yes, it's the opposite sum. If you say 10+2=12 then you do the opposite, 12-10=2."*

The varied responses from the pairs of learners suggested different understandings of inverse operations as operations that reverse the effect of another operation.

Although the first and second pairs could correctly state that inverse operations mean the opposite operation, their examples suggest a misconception of inverse operations in relation to the commutative laws of addition and multiplication. The responses of pairs 1 and 2 suggests that the learners had linked their understanding of inverse or opposite to merely swopping numbers around and getting the same result, which is true for the commutative properties of addition and multiplication. The misinterpretation is that the learners assumed that inverse or opposite is related to swopping digits or numbers around, and not understanding that it is related to the opposite operation.

The responses from pairs 3 and 4 indicate an understanding of what inverse operation means.

To address the incorrect responses of the first and second pair, the teacher posed the following questions:

Teacher [to pair 1]: *"What operation is opposite to addition?"*

Pair 1: *"It is subtraction."*

Teacher: *"So let's use your example of 10+6=16. That is an addition sum. Now how would you change it to subtraction?"*

Pair 1: *"10-6?"*

Teacher: *"Let me ask you. Why would I use the opposite – subtraction?"*

Pair 1: *"To make it less?"*

At this point it became clear that Pair 1 was confused with their interpretation of inverse operations and failed to realise that 10+6=16 also tells them that 16-10 will result in 6. The teacher then posed the same question to whole class:

Teacher: *"Class, why do you think we use inverse operations?"*

Pair 5: *To swop the numbers around but you still get the same answer."*

The teacher then explained that inverse operations are used to check your answer. The number sentence was written on the chalkboard: 10+6=16, and the teacher demonstrated how they would write the inverse number sentence, i.e. 16-10=6. After showing the inverse operation calculation on the chalkboard, the teacher asked Pair 1 to collect counters from the Maths kit and to pack out their example of 10+6, as one group of 10 counters and one group of six 6 counters. The pair was then asked to add the two groups of counters together, which resulted in 16 counters. Learners were then asked to decrease the total of 16 counters by 6 and to count the remaining counters, which resulted in 10. The pair was also asked to write a number sentence for their findings, viz. 16-6=10. The practical activity using manipulatives was aimed at consolidating the learners' conceptual understanding of why subtraction is the inverse operation of addition. It was interesting to note that learners were not asked to find another inverse solution for subtraction by manipulating the numbers, e.g. 16-10=6.

The example of Pair 2 was also written on the chalkboard: 2×3=6, and the learners were asked which inverse operation they should use to check their answer. They correctly identified division as the inverse operation, but responded with an incorrect number sentence of 2÷3, which alludes to another misconception of dividing a larger number into a smaller one.

Another pair responded with the correct number sentences of 6÷2=3 and 6÷3=2.

The examples used to demonstrate inverse operations in this lesson focused only on recognising subtraction as the inverse of addition and division as the inverse of multiplication. It would also be relevant and important at this stage to ensure that learners clearly understood that the same rule would apply for addition as the inverse of subtraction, and for multiplication as the inverse of division, to counter any further misconceptions of under-generalising the rule. The ability to use inverse operations as a means to check answers is a skill teachers assumed learners had been taught previously and that they could access their prior knowledge for double-checking answers.

4.11.5.2 Lesson 8 – Grade 5

This lesson was introduced by dividing the class into smaller groups and handing each group an answer sheet with samples of incorrect responses with the aim of engaging learners in cognitive conflict. The intended outcome of the lesson was twofold:

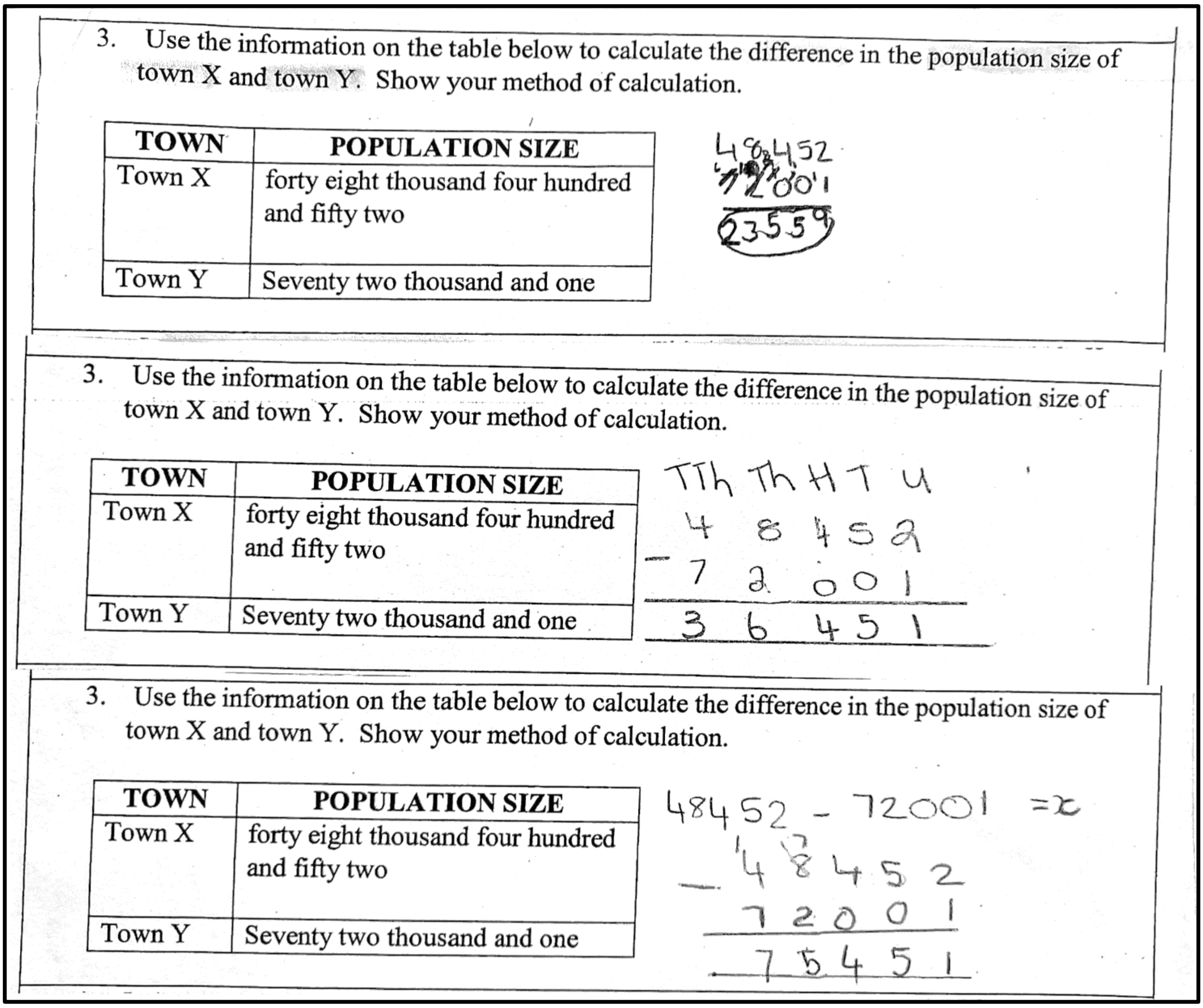
- Firstly, the teacher wanted learners to identify the errors on the answer sheet samples, and
- Secondly, to provide learners with an opportunity to use inverse operations to check their answers

In their groups, learners were asked to identify:

- *what the error was*
- *why they thought the error was made, and*
- *how the error can be rectified*

The problem required learners to use the information in a given table to calculate the difference, and to show their calculation. The teacher indicated that this strategy was used to determine the learners' understanding of mathematical vocabulary, namely finding the difference, as well as converting numbers in words into digits, thereby looking at their understanding of place value.

Below is a scan of three incorrect responses on the answer sheet.



3. Use the information on the table below to calculate the difference in the population size of town X and town Y. Show your method of calculation.

TOWN	POPULATION SIZE
Town X	forty eight thousand four hundred and fifty two
Town Y	Seventy two thousand and one

48452
72001
23559

3. Use the information on the table below to calculate the difference in the population size of town X and town Y. Show your method of calculation.

TOWN	POPULATION SIZE
Town X	forty eight thousand four hundred and fifty two
Town Y	Seventy two thousand and one

TTh Th H T U
4 8 4 5 2
− 7 2 0 0 1
3 6 4 5 1

3. Use the information on the table below to calculate the difference in the population size of town X and town Y. Show your method of calculation.

TOWN	POPULATION SIZE
Town X	forty eight thousand four hundred and fifty two
Town Y	Seventy two thousand and one

48452 − 72001 = x
− 4 8 4 5 2
7 2 0 0 1
7 5 4 5 1

Figure 4.18: Answer sheet with three incorrect responses

Table 4.6 lists the correct and incorrect procedures followed in the three responses to the answer sheet in Figure 4.18.

Table 4.6: Correct and incorrect procedures of subtraction calculations in Figure 4.18

Correct	Incorrect
Identified *calculating the difference* as subtraction calculation	Subtracting a larger number from a smaller number
Transcribed the numbers from words	Borrowing and carrying over from the minuend (smaller number) incorrectly placed as subtrahend (larger number)

In their small groups, the learners reviewed and discussed the responses on the answer sheet. During the feedback, most of the groups could identify the error, with one group stating that *"the child assumed that the numbers in words written in the table must be subtracted in that order"*. The table on the answer sheet first lists *Town X* with a population size of *forty-eight thousand four hundred and fifty-two* and below that *Town Y* with a population size of *seventy-two thousand and one.* In all three responses, the learners used the vertical short-column method to show their calculation by transcribing the numbers from words in the order as they appeared in the table. This means that the *"child ignored the rule that we subtract a smaller number from a bigger number"*, as explained by one group.

Although the class was accurate in identifying the above errors, no mention was made of the incorrect borrowing and carrying over procedure in the first calculation on the answer sheet. This calculation shows that the learner borrowed and carried over from the minuend (larger number), which was incorrectly placed below the subtrahend (smaller number).

One of the groups responded that *"every sum has a different answer, but the first one is close the right answer"*. The teacher asked the group how they found that answer to be closest to the correct answer. The response of this group was that they performed the procedural steps of calculating the answer. It was interesting to note that only one of the class groups attempted their own calculation and then compared their answer to the answers on the answer sheet. Upon instruction by the teacher, the other groups were asked to perform their own calculations and they agreed that the correct answer to the problem was *72 001-48 452=23 549.*

As a follow-up activity, the teacher asked the groups what they would do if two (2) more towns with different population size totals were added to the table and they had to follow the same instruction of *calculating the difference between the largest population size and the smallest population size?* This question was posed to the groups to lead them towards reasoning how the incorrect responses on the answer sheet could be rectified. Below are the responses from two groups.

Group 4: *"We will find the biggest and the smallest number and minus them from each other."*

Group 6: *"You must order the number, from biggest to smallest or from smallest to biggest. Then you take the smallest and biggest number and subtract."*

The teacher was satisfied that learners could identify the errors on the answers sheet, apply their own reasoning about why the error was made, and provide possible steps for how the error could be rectified. The next step of the lesson was to provide learners with an opportunity to use inverse operations to check their answers. Each group was asked to review their calculation of *72 0001-48 452=23 549* and explain how they would go about checking their answers. The following responses are from three groups.

Group 2: *"We used another method, like expanded notation to check our answer."*

Group 3: *"We double-checked our answer so we did the sum again."*

Group 5: *"We took the answer and then we said 72 001-23 549 and then we got 48 452."*

The feedback from Groups 2 and 3 suggests that, although the learners could find ways to check their answers, they were not familiar with using inverse operations to check their answers. Although Group 5 did not apply addition as the inverse of subtraction for their calculation, they were able to recognise the result when swopping the answer and the subtrahend around. The teacher posed the following questions to the groups:

Teacher: *"What operation is the opposite of subtraction?"*

Class response: *"Addition."*

Teacher: *"Now opposite is also known as the inverse operation. For addition it is subtraction and for multiplication it is?"*

Class response: *"Division."*

Teacher: *"In your groups I want you to write a number sentence to show how you would use the inverse of subtraction to check your answer."*

As the groups discussed the problem, the teacher moved between the groups to ensure that they were able to apply addition to check their answer.

Although most groups were successful, with a result of 48 452+23 549=72 001 and 23 549+48 452= 72 001, one group had an incorrect result of 72 001+23 549=48 452. This incorrect number sentence alludes to the inability of the learners to recognise that adding the first two numbers, viz. 72 001 and

23 549, cannot result in a number with less value. Overgeneralising the rule of swopping numbers around when applying inverse operations could be the probable cause of this misconception, resulting in learners not taking cognisance of the value of the numbers.

4.11.5.3 Lesson 9 – Grade 6

This lesson was introduced with a revision activity of the basic operation symbols (+; -; ×; ÷) in which learners are asked to name the four basic Math operations. The lesson starts with single-digit equations and progressed to six-digit numbers at the end of the lesson. The class was shown a slide that displays five number sentences, namely:

a) 2+3=5
b) 2×3=6
c) 2(3)=6
d) 10–2=8
e) 10÷2=5

The learners could confidently identify each operation applicable to each of the number sentences. When the correct response of *"multiplication"* was provided for *"c"*, the teacher asked the learners how they know that they should multiply. The response was that *"the bracket means you should multiply when there is no operation symbol"*.

The second slide displayed the number sentence *2 + 3 = 5* and the teacher asked the class, *"Is there another way of using these numbers to write a different equation?"* The learners responded to the question in the following ways.

LR1: *"We can say 3 + 2 = 5"*

LR2: *"5 - 3 = 2"*

LR3: *"5 – 2 = 3"*

Teacher [to L2 and L3]: *"Right. Now why can we use minus?"*

LR3: *"Because it's the opposite of plus."*

Teacher: *"So what it the word we use instead of opposite?"*

Since there was no response from the class, the teacher instructed them to find the meaning of *"inverse"* in the dictionary.

LR4: *"Inverse means the reverse or the opposite."*

Teacher: *"Now tell me why or when would we use inverse operation in Mathematics?"*

When the class struggled to respond to the question, the teacher led them back to the response of L2 and L3, who used subtraction as the operation in their number sentence, and repeated the question.

LR5: *"To check my answer."*

Teacher: *"Yes. Can anyone think of another reason?* [no response] *No? Well, you can also use inverse to solve number puzzles. But checking answers is the most important reason why we use inverse operation."*

One learner posed the question, *"Plus is opposite of minus, now why can't times be the opposite of plus?"* The teacher asked the class, *"If inverse means opposite, can I reverse a times sum by adding?"*

LR6: *"No, you can't use times and plus as inverse because they both add up to a bigger answer."*

Teacher: *"Remember that an addition undoes a subtraction sum and a division sum undoes a multiplication sum, and the same the other way around."*

The lesson continued with the next activity, in which the learners were asked to provide an answer to an equation, *200+350=?* and then write an answer sentence using the inverse operation.

LR7: *"The answer is 550."*

LR8: *"Then you can say 550–350=200."*

LR9: *"Or 550–200=350."*

The three earners who volunteered answers were accurate with their answers, but when the teacher selected specific learners in the class group to answer the next problem, there was evidence of confusion when using the operation. The next example was *840–310=?*

LR10 [after calculating]: *"The answer for 840–310=530."*

LR11: *"530–840=310."*

The answer of LR11 suggests that there could be a misconception with

- place value and value of numbers when attempting to subtract a larger number from a smaller number, or
- understanding the concept of inverse operations

Teacher: *"Our sum is a subtraction, 840–310. Why would I use subtraction again as my inverse operation? Remember, inverse means the opposite or reverse operation. So what would my inverse operation be for subtraction?"*

LR11: *"Oh yes, it is 310+530."*

LR12: *"Or you can say 530+310=840."*

In the following example, the learners were instructed to complete the activity while the teacher supported individual learners. *Write a number sentence using the inverse operation:*

62 477+43 936=106 413

LR13 writes: *"62 477–106 413=43 936"*

The learner correctly identifies subtraction as the inverse operation, but incorrectly attempts to subtract a larger number from a smaller number. A peer was asked to look at what error LR13 had made and pointed out that *"you can't minus a big number from a small one"*. Once again, it was evident that there were misconceptions around place and value.

It was interesting to note that the learners assumed that the answers in the given number sentences were correct, as they did not attempt their own calculations. A possible way of ensuring that learners perform their calculations might be to give an incorrect answer in the number sentence and allow learners to use operation calculations to check their answers, as this was emphasised as the main function of using inverse operations.

4.11.6 Summary of lessons 7 to 9

This summary relates to the findings from the three lessons focusing on inverse operations.

In the introduction of Lesson 7, learners are asked to work in pairs and explain their understanding of inverse operations with an example. The responses given by the learners gave the teacher an indication that some learners' understanding of the concept was that inverse operations meant switching numbers around. Guided questions posed by the teacher led learners to make the connection between inverse as being the operation that changes to the opposite, and not the numbers. Although learners had the opportunity to work with counters as manipulatives for conceptual understanding, they were not asked to explain their understanding of inverse, as the teacher provided the explanation of inverse operation. It was also noted that multiple solutions for writing the inverse operation number sentence were not explored, which could limit the learners' conceptual understanding of the concept. When multiplication and division were introduced as inverse operations, another misconception of dividing a larger number into a smaller number was highlighted, but not addressed during the lesson.

Lesson 8 provided an opportunity for learners to engage in cognitive conflict by reviewing incorrect responses on an answer sheet. The teacher informed the learners that the answers on the answer sheet were incorrect, which was a missed opportunity for learners to discover this for themselves. The instructions to the learners were to determine what the error was, why the error was made, and how the error could be rectified. The general feedback from the groups showed that most learners could identify the errors on the answer sheet. The action of one group who executed the calculation themselves to check the answer prompted the teacher to encourage the other groups to also perform their own calculations. This lesson also integrated another concept – ordering – by comparing and sorting numbers as progression towards inverse operations. Listening to the responses of the learners informed the teacher of their understanding or misunderstanding of the concept, e.g. overgeneralising the rule. This provided an opportunity for the teacher to pose guided questions to lead learners towards organising their thinking processes, correcting their own errors and explaining the concept.

The introduction to Lesson 9 accessed learners' prior knowledge of the four basic operations and started with single-digit number sentences to introduce inverse operations. The lack of response when learners were asked to explain when they could use inverse operations in Mathematics showed a gap in their conceptual understanding of the relationship between addition and subtraction, and between multiplication and division. The explanation of using inverse operations to check answers was provided by the teacher. As the number range increased to using three-digit and five-digit numbers in number sentences, misconceptions of place value and number value were found to be present when learners attempted to subtract a larger number from a smaller number. The misconception of merely swopping numbers around and still using the same operation symbol to form an inverse operation once again proved that learners related inverse operation to swopping numbers around and not using the opposite operation. It was also noted that, when learners were given a number sentence, they assumed the answer of the number sentence to be correct as did not see the need to check the answer by performing their own calculations.

4.11.7 Common fraction errors

The following three lessons (lessons 7, 8 and 9) observed teaching strategies implemented to determine learner misconceptions related to fractions in NOR. The researcher recorded the findings as field notes. Each lesson is discussed individually below to gain insight into the different teaching and remediating strategies implemented by the Intermediate Phase teachers.

4.11.7.1 Lesson 10 – Grade 4

The aim of this lesson was to determine whether learners understood the concept of common fractions. This lesson was introduced by the teacher asking the learners to name a few common fractions. Among the responses from the learners were $\frac{3}{6}$, $\frac{1}{2}$ and $\frac{1}{4}$. To determine their prior knowledge of fractions, the teacher asked the learners to explain what they understand a fraction to be. The learners shared the following responses:

LR1: *"A fraction is a part of a whole."*

LR2: *"I would say* ***equal*** *parts of a whole."*

Teacher: *"Why would you say equal parts? What is the difference?"*

At this point the learners struggled to answer the question. The teacher drew a pie circle on the chalkboard and poses the following question to the class:

Teacher: *"Into how many parts should I divide the pie if I want quarters?"*

Class response: *"Into four parts."*

The teacher divided the pie into four unequal parts and asked the learners if it was divided into quarters.

LR3: *"No, the parts are different in size. They are not equal."*

Teacher: *"That's right. I have four parts but not four equal parts. Fractions are specifically equal parts of the whole."*

Teacher (pointing to one unequal part of the pie): *"Can I say that this is a quarter?"*

Class response: *"No ma'am."*

The teacher writes the fraction $\frac{1}{4}$ on the chalkboard and asks the learners to label the parts of the common fraction.

LR4: *"The one on top is called the numerator and the four at the bottom is called the denominator."*

Teacher: *"Now do we know what the numerator and denominator actually mean?"*

The learners' responses indicated confusion about explaining the function of the numerator and denominator as represented in a fraction. To guide learners in understanding the function of both the numerator and denominator, the teacher used a cardboard pie circle divided into four equal parts and handed one quarter each to four learners.

Teacher: *"Did anyone get more or less than the others?"*

Class response: *"No!"*

LR5: *"They all got an equal part."*

LR6: *"So the numerator tells us each one got one part and the denominator at the bottom are so many parts."*

Teacher: *"Not just parts, but…."*

LR6: *"Equal parts!"*

Teacher: *"That's right! The whole is divided into four equal parts."*

The teacher then used manipulatives to show the learners different pies and strips divided into different equal parts and the learners were asked to identify the different fractions, e.g. $\frac{1}{5}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{12}$.

Emphasis was also placed on how to name the fractions, i.e. fifths, eighths, tenths and twelfths. To distinguish between the size of the different fractions, learners were asked to explain what they noticed about the denominator and the size of the fraction.

LR7: *"The bigger the denominator, the smaller the size of the fraction."*

The teacher asked the learners to compare the values of the digit 3 and the digit 4 in the fraction $\frac{3}{4}$ to the whole numbers 3 and 4, and to explain how their values differed.

LR8: *"You can turn the $\frac{3}{4}$ into a whole number by adding another one quarter to it."*

Teacher: *"Then what will I get?"*

LR8: *"You will then get $\frac{4}{4}$ and that is equal to one whole."*

To demonstrate this and to ensure conceptual understanding, the learners were asked to use the cardboard pie divided into four parts and show how they would add $\frac{3}{4}+\frac{1}{4}$ to form the whole pie, $\frac{4}{4}$. In smaller groups, the class was instructed to use different pies and strips and break them into different equal parts (fractions) and then build up the whole. The groups were encouraged to use variations when building up the whole e.g. $\frac{6}{10}+\frac{4}{10}=\frac{10}{10}=1\ whole,$ or $\frac{2}{10}+\frac{3}{10}+\frac{5}{10}=\frac{10}{10}=1\ whole,$ etc.

4.11.7.2 Lesson 11 – Grade 5

This lesson was introduced as a continuation of a previous lesson dealing with fractions. The aim of this lesson was for learners to use a fraction wall to compare fractions and find equivalent fractions. To start the lesson, the learners were asked to provide a definition of what their understanding of fractions was.

LR1: *"A fraction is an equal part of a whole."*

To consolidate this definition, the teacher used a sheet of paper and cuts it into four unequal parts.

Teacher: *"Are these four pieces' fractions of the whole?"*

Class response: *"No."*

Teacher: *"Why not?"*

LR2: *"Because they are not equal."*

After showing the class a diagram of a sheet cut into four equal parts, a learner was asked to write a fraction on the chalkboard, viz. $\frac{2}{4}$. The teacher questioned the learners about *"the top number"* and *"the bottom number"*. The learners could confidently label the fraction with numerator and denominator. Further questioning proved that learners were unsure of the functions of the numerator and denominator in a fraction.

Teacher: *"What does the denominator tell us?"*

LR3: *"It tells us that it is 4."*

Teacher: *"What is 4?"*

LR3: *"4 of a shape?"*

Teacher [demonstrates with diagram cut into four equal parts]: *"How many equal parts make up the whole?"*

LR3: *"Oh, the 4 equal parts of the whole."*

Teacher: *"Yes, the denominator tells us how many equal parts the whole has been cut up into."*

Teacher: *"What does the numerator tell you?"*

Although learners had difficulty defining the function of the numerator, they were able to answer the guided questions as the teacher used an example of a pizza cut into slices.

Teacher: *"If I told you that I ate* $\frac{3}{5}$ *of my pizza, it means that I ate …?"*

Class response: *"3 slices."*

Teacher: *"Yes, 3 slices of a pizza that was cut into …?"*

Class response: *"5 slices."*

The teacher continued to access prior knowledge of fractions with a practical activity. The learners worked in pairs to identify the different denominators from sheets of newspaper cut into different equal parts on the floor. Once the pair had identified the denominator, e.g. tenths, they were asked to build a fraction, e.g. two tenths, incorporating the numerator, $\frac{2}{10}$.

The next step of the lesson was aimed at comparing fractions by using a fraction wall, e.g. $\frac{2}{3}$ and $\frac{1}{4}$. A fraction wall is a visual representation that can help learners understand the basics of fractions. Learners were provided with a template and, in small groups, were asked to build the fraction wall (see Figure 4.19).

Figure 4.19: Learners building a fraction wall

The teacher indicated that the learners had previously been introduced to a fraction wall and should be familiar with the different levels of the fraction bricks. After the fraction wall was completed, the groups were asked how they would use the fraction wall to compare two fractions, e.g. $\frac{2}{3}$ and $\frac{1}{4}$?

Teacher: *"Where would you start when comparing fractions?"*

LR1: *"From the top fractions?"*

Teacher: *"No, from which side would you start?"*

LR2: *"You start from the left of the fraction wall."*

Teacher: *"Yes, because you cannot pick a fraction from the right side and hope to compare them like that."*

The above comment made by the teacher, could be indicative of either a misconception held by the teacher or, of a narrow procedural approach in teaching. The fraction wall representation of fractions is symmetric, and can be compared from any side.

Teacher: *"Now let's compare* $\frac{2}{3}$ and $\frac{1}{4}$. *Which level will I look at first?"*

LR3: *"You must look at the thirds."*

Teacher: *"Right, so I must choose the level where the whole has been cut up into three equal parts."*

Teacher: *"How many of those parts am I going to look at?"*

LR4: *"Three!"*

Teacher: *"What is the fraction written on the board?"*

LR5: *"Two thirds!"*

Teacher: *"So how many pieces am I going to be looking at?"*

Class response: *"Oh, two thirds."*

Teacher: *"Yes, I am looking at two parts of the three. Now what am I comparing it to?"*

The teacher shades two equal parts of the third level on the fraction wall.

LR6: *"Now you must look at the fourth row – quarters."*

Teacher: *"How many equal parts must I shade of the four parts?"*

Class response: *"One part!"*

Teacher [shades one equal part]: *"Yes, one quarter."*

Teacher: *"What do we notice about the size of the two thirds compared to the size of one quarter?"*

Class response: *"It is bigger!"*

Teacher: *"What symbol will I use to show bigger than?"*

One learner is asked to write the number sentence on the chalkboard, $\frac{2}{3} > \frac{1}{4}$, and the class reads aloud, *"Two thirds is greater than one quarter"*.

To consolidate using the fraction wall to compare fractions, learners were engaged in an activity of using a deck of cards to build common fractions, as shown below in Figure 4.20.

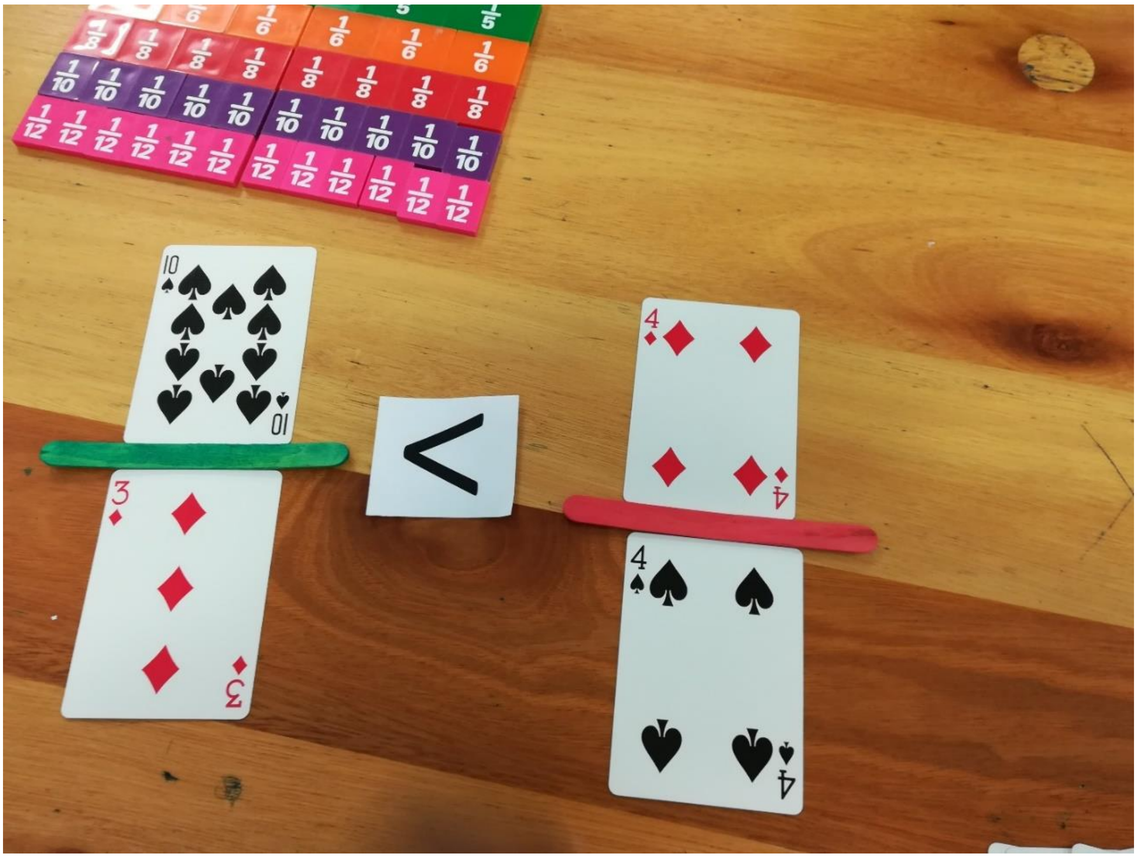


Figure 4.20: Fraction cards to compare fractions

During this activity, learners were supposed to form common fractions with playing cards and use the fraction wall to compare the fractions. The teacher explained that before she taught the rules of equivalent fractions, she wanted the learners to have concrete experiences of developing their own conceptual understanding of fractions. As seen in Figure 4.20, $\frac{10}{3}$ is an improper fraction, meaning the numerator is larger than the denominator. Converting the improper fraction $\frac{10}{3}$ will result in the mixed number $3\frac{1}{3}$. The fraction $\frac{4}{4}$ has the same numerator and denominator and therefore the fraction is equal to one whole. The relationship sign comparing the two fractions incorrectly shows the value of $\frac{4}{4}$ greater than (<) $3\frac{1}{3}$. The fraction wall limits the learner to see only $\frac{3}{3} = 1$, which could mean that the learner restricted her understanding to thinking that all other fractions are less than 1. This suggests that the learner had overgeneralised previous knowledge of fractions with the same numerator and denominator and thought that one whole will always be greater than other fractions.

4.11.7.3 Lesson 12 – Grade 6

This lesson was aimed at revising the concept of calculating a fraction of a whole number. Learners were asked to show how they would calculate $\frac{3}{4}$ of 36 bottle tops. Below (Figures 4.21 and 4.22) are two examples that shows the steps learners used to solve the problem.

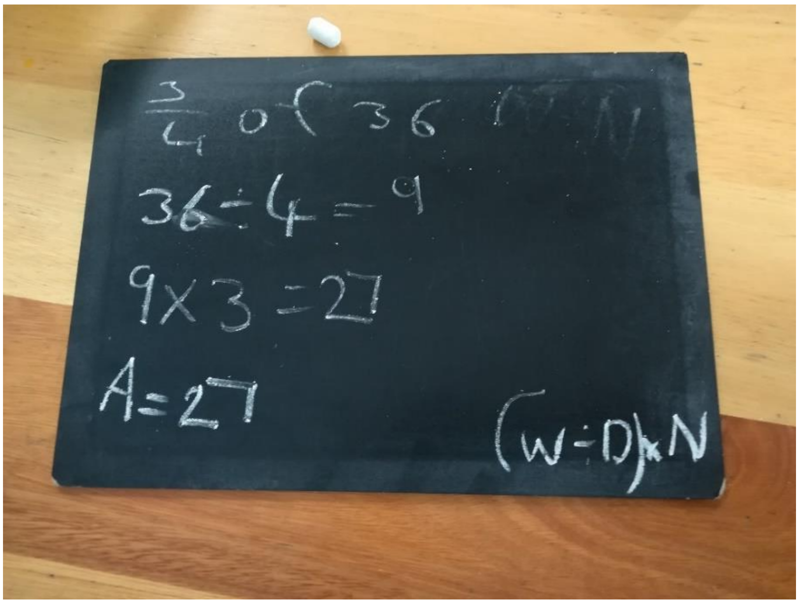


Figure 4.21: Learner calculation of fraction of a whole number

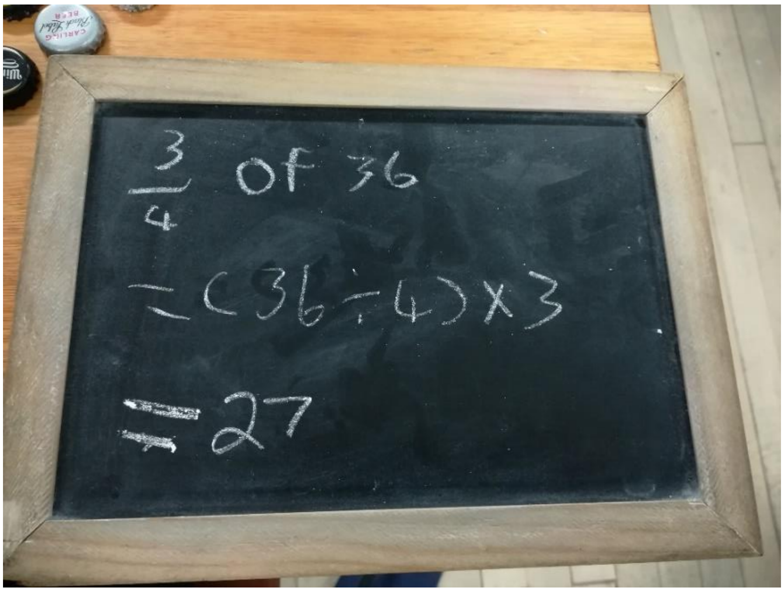


Figure 4.22: Learner calculation of fraction of a whole number

In both examples, we see learners' procedural knowledge being applied to calculate a fraction of a whole number. In the first example in Figure 4.21, the learner wrote the formula for the calculation as $w \div d \times n$, i.e. whole number divided by the denominator and the answer multiplied by the numerator. In the second example (Figure 4.22), the learner used the same formula as seen in the calculation steps. In both examples, the calculation steps resulted in the correct answer of 27. This confirmed that learners were able to apply the rule for calculating a fraction of a whole number.

After solving the problem, learners were asked to explain how they would go about calculating $\frac{3}{4}$ of 36 bottle tops by using actual bottle tops. The teacher provided no guidance to the learners about how they should go about using the bottle tops. It was interesting to note that those learners who correctly counted out 36 bottle tops (the whole number in the equation) had difficulty grouping the bottle tops to represent a fraction of $\frac{3}{4}$. This means that the learners did not recognise the denominator of 4 as four equal parts of the whole and realise that they should divide the bottle tops into four equal parts, as illustrated in the example in Figure 4.23 below.

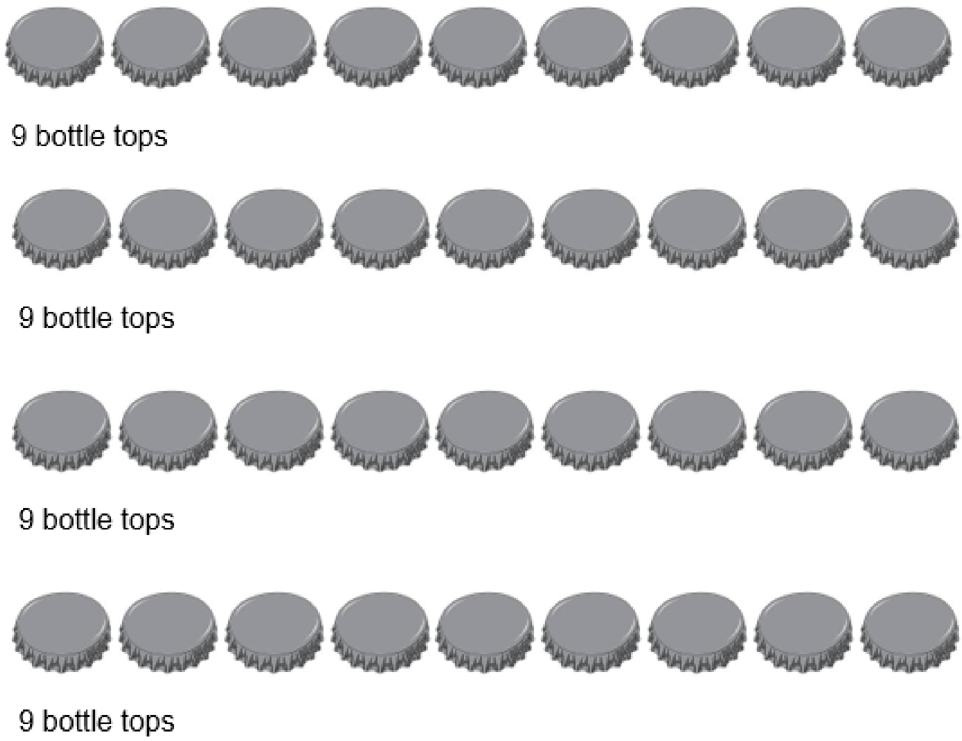


Figure 4.23: Example of dividing the whole into four equal parts

In Figure 4.23, the whole number 36 is divided into four equal groups consisting of nine bottle tops each. Three parts of the whole will consist of three (3) groups of nine (9), resulting in a total of 27, therefore $\frac{3}{4}$ of 36 bottle tops $= 27$ bottle tops.

The learners showed a variety of groupings, as shown in Figure 4.24 below, where the learner made three groups of 12 bottle tops each.

Figure 4.24: Learners' incorrect representation of $\frac{3}{4}$ of 36 bottle tops

The above grouping indicates that the learner could not make the connection of the fraction, $\frac{3}{4}$, as being three equal parts of the whole.

Another learner incorrectly counted out 12 bottle tops and made three groups of four bottle tops each. This suggests a misconception of understanding $\frac{3}{4}$ as a fraction and 36 as a whole number, where the learner treated the numerator and denominator of the fraction as separate whole numbers, as one would do in multiplication.

To help learners make the connection between their procedural steps using the formula $w \div d \times n$ and the practical activity using bottle tops, the teacher posed the following guided questions:

Teacher: *"What is the whole number in the equation* $\frac{3}{4}$ of 36 bottle tops*?"*

LR1: *"36 is the whole number."*

Teacher: *"Right! Now how many bottle tops should I count out?"*

LR3: *"Oh! 36?"*

Teacher: *"Yes! So let's make sure we all count out 36 bottle tops."*

Teacher: *"What is the fraction?*

LR4: *"The fraction is* $\frac{3}{4}$*."*

Teacher: *"So how many parts make up the whole?"*

LR5: *"36?"*

Teacher: *"No, the whole is made up of four (4) parts. But how many parts must we find?"*

LR6: *"$\frac{3}{4}$."*

Teacher: *"Not $\frac{3}{4}$, but three parts of the four. You must look at the numerator."*

Teacher: *"Now let's divide the 36 into four equal parts. How many groups will you have?"*

The learners took quite a while to divide the 36 bottle tops into four groups of nine each. Some learners counted the individual bottle tops into groups. while others used $36 \div 4 = 9$ to arrive at four groups with nine bottle tops each.

Teacher: *"How many groups do you have?"*

LR7: *"Four groups."*

Teacher: *"And how many bottle tops in each group?"*

LR7: *"Nine bottle tops."*

Teacher: *"So how many groups must you count to find ¾ ?"*

LR8 [counting]: *"Three groups. Then it's 9, 18, 27!"*

LR8: *"That's the same answer I got when I did my sum."*

This practical activity clearly showed that, although the learners could apply the formula ($w \div d \times n$) to execute the procedural steps to calculate a fraction of a whole number, there was a lack of conceptual understanding.

4.11.8 Summary of lessons 10 to 12

This summary relates the findings from the three lessons focusing on fractions. All three lessons were done as revision or as continuation of the previous lesson dealing with fractions.

The introduction to lesson 10 explored the learners' prior knowledge and understanding of fractions as equal parts of a whole. Although the learners knew the definition of fractions and had the ability to identify the numerator and denominator, they struggled to explain why fractions are equal parts of a whole and the functions of the numerator and denominator. The learners found it easier to provide explanations, through visual and practical activities, as their conceptual understanding seemed to improve.

To access the learners' prior knowledge, the lesson was introduced with a quick recap of a previous lesson before continuing with a fraction wall to find equivalent fractions. Once again, the learners' understanding of the functions of the numerator and denominator was lacking. The misconception of generalising the value of the numerator and denominator as equal to whole numbers was evident. Using manipulatives like a fraction wall could aid learners in comparing fractions, but a narrow procedural approach in teaching, or a misconception held by the teacher, limits the benefit of using a fraction wall. This is seen in lesson 11 when the teacher states that "*you cannot pick a fraction from the right side and hope to compare them like that*". Showing learners, the symmetric feature of the fraction wall could further support their understanding of comparing fractions. The use of a deck of cards to make fractions further suggests that this presentation is evidence of the teachers' misconception of what a fraction is, and how the notation relates to the concept.

The introduction to lesson 12 was to determine how learners would calculate a fraction of a whole number. It was clear that the learners could apply the procedural rule of $\text{whole number} \div \text{denominator} \times \text{numerator}$ to calculate the answer. The practical activity of using bottle tops as manipulatives was executed to determine the learners' conceptual understanding of finding a fraction of a whole number. In this lesson, the progression of the concept of finding a fraction of a whole number moved from procedural knowledge to conceptual knowledge. It was clear that, although the learners could apply procedural knowledge, they struggled to apply this to showing conceptual understanding during the practical activity. This is evident in the manner they grouped the bottle tops to form equal parts of the whole. The guided questions posed by the teacher assisted learners in realising how they should represent the equal parts of the whole.

4.12 Observation

During the initial planning stage of Cycle 3, the teachers agreed that they would consciously listen to how their learners applied mathematical knowledge and skills during their lessons to solve problems.

Engaging learners in cognitive conflict was one strategy teachers would implement to avoid having to re-teach the entire concept, and to address errors at the point of misconception. The second strategy was to pose guided questions to lead learners to self-discovery of their errors or to solve the problem. The third strategy was to expose learners to different levels of questioning and question types to strengthen their conceptual understanding of NOR concepts.

Although cognitive conflict was identified as a strategy, only three of the lessons presented learners with opportunities to apply reasoning skills to evaluate incorrect responses. The other nine lessons were introduced at an instruction point at which teachers assumed the learners had misconceptions of the concept. These nine lessons were focused on determining learners' prior knowledge of the concept and building the lesson on what learners understood. Accordingly, this was contradictory to addressing errors at the point of misconception compared to re-teaching the concept. If learners were presented with opportunities to engage in cognitive conflict, teachers would gain insight into how they reviewed their reasoning and organised their own ideas by listening to their explanations in their feedback. It is my belief that teachers could gain a deeper understanding of learners' reasoning by not informing them that the responses on the answer sheets are incorrect, but rather attending to the learners' strategies of discovering the incorrect responses.

Although evidence of posing guided questions was recorded during all twelve lessons, careful consideration was needed of which questions would lead learners to self-discovery and improved conceptual understanding. This was evident when teachers were focused on the methodology learners employed to solve problems, e.g. instructing learners to use the short-column method to solve a multiplication calculation and finding a fraction of a whole number. During two of the lessons, I observed teachers providing the correct explanation or answer when learners responded incorrectly, thereby not allowing learners the opportunity to reason and discover their own errors.

Exposing learners to different levels of question and question types was limited in all the lessons. This alluded to the fact that learners had acquired a particular learnt process and found it challenging to solve problems when they were presented in a different way.

Learner misconceptions were identified within the content topics in each lesson and teachers employed strategies to address these misconceptions. Table 4.7 below provides a summary of the misconceptions identified, with the teaching strategies employed to address the misconceptions.

Table 4.7: Misconceptions identified in NOR topics during lesson observations

Lessons	**Content topic of NOR**	**Misconceptions identified during lesson observations**
Lessons 1 to 3	Place value and rounding off	○ Expanded notation limited to one notational method ○ Misapplies learned procedural rules for rounding off ○ Under-generalising the value of zero (0) as place holder
Lessons 4 to 6	Operational symbols and mathematical vocabulary	○ Interpreting "product" as an addition calculation ○ Thinking subtraction is commutative ○ Overgeneralising procedures learned for addition in multiplication ○ Overgeneralising the rules for BODMAS – as ordering of mathematical operations
Lessons 7 to 9	Inverse operations	○ Interpreting inverse as commutative property ○ Overgeneralising previously learned procedures by subtracting a larger number from a smaller number ○ Viewing addition and subtraction as separate operations and not related as inverse operations ○ Overgeneralising addition and subtraction as inverse operations by interpreting subtraction as commutative
Lessons 10 to 12	Common fractions	○ Knowing only limited models for interpreting fractions ○ Not understanding what the numerator and denominator represent in a fraction ○ Restricted interpretation that different fractions giving the same amount are equivalent ○ Treating the numerator and denominator as whole numbers when grouping for finding a fraction of a whole

I also observed the ways in which teachers addressed the abovementioned misconceptions during the lessons. The findings are listed below:

- Using a smaller number range as an example to expand on learners' prior knowledge
- Using visual representations to show how zero as place holder changes a number value

- Building mathematical vocabulary to make meaning in problem-solving contexts
- Engaging learners in practical activities using concrete manipulatives
- Elimination of multiple-choice options by executing calculations
- Using previous knowledge of number sentences to show inverse relations

4.13 Reflection on Cycle 3

The NOR lesson topics covered in Cycle 3 were identified by the Intermediate Phase teachers based on learner errors in Cycle 2, i.e.

- Place value and rounding off
- Operational symbols and mathematical vocabulary
- Inverse operations, and
- Common fractions

The misconceptions identified in Cycle 3 provided more detail of learners' thinking processes when making errors compared to the causes of errors described by the participants in Cycle 2.

The responses by learners during the lesson observations in Cycle 3 confirmed that they were at different levels of understanding the concepts. These different levels of understanding were influenced by:

- how learners link their previous knowledge to the NOR concepts
- whether learners were taught procedures without understanding
- whether learners had an understanding of when and how to use procedural knowledge

Considering that only three lessons allowed learners to engage with incorrect answers, it is clear that this is a strategy teachers need to practise more in their teaching methodologies. Where opportunities allowed for teachers to pose questions to learners about errors, they were able to discover what learners' reasoning was about the errors.

Chapter 5

Conclusion

5.1 Introduction

This chapter provides a summary of the research study and a description of the findings. Secondly, it lists the limitations of this study and makes recommendations for further research.

5.2 Summary of the study

The primary aim of this research project was to investigate how Intermediate Phase (IP) teachers identify and address learner errors and misconceptions in the content area of NOR, thereby answering the main research question: 'How do Intermediate Phase teachers identify and respond to learners' mathematical errors and misconceptions in the content area of Numbers, Operations and Relationships?'

This focus was motivated by an awareness of the patterns of underachievement in Mathematics in the South African context and the belief that, if teachers have a sound understanding of how to recognise and address learner errors and misconceptions, it would contribute to enhancing mathematical competency. The chosen methodology was to engage twelve Intermediate Phase teachers in three participatory action research (PAR) cycles and work collectively towards a collaborative outcome. Data collection methods in the three PAR cycles were observations, semi-structured interviews and focus group interviews.

The results of Cycle 1 revealed the initial premise of this research study – that although teachers were aware of learner errors during teaching time and after formal assessments, these error findings were not used to inform teaching and learning, as suggested by Borasi (1987) and Olivier (1992). The possible causes of learners' NOR errors were initially identified by the IP teachers as being related directly to the learners' lack of procedural knowledge, gaps in prior knowledge and carelessness. These causes of errors are supported in arguments presented by Hansen (2011), Rittle-Johnson *et al*. (2015) and Sidney and Alibali (2015). An interesting finding was that the underlying cause(s) of the errors has not been previously investigated by the IP teachers. Hence, the interrelationship between conceptual and procedural knowledge was not consciously developed in their teaching practice.

Teaching strategies implemented at the time, which included re-teaching and drilling to address learner errors, saw no noticeable improvements in results. The findings of Cycle 1 informed the way forward for Cycle 2.

A finding in Cycle 2 proved that the theory of professional noticing enabled teachers to gain more insight into learners' errors than merely distinguishing between correct and incorrect answers. For instance, when teachers reviewed the worksheet responses, they were able to look beyond incorrect answers and started to attend to learners' strategies and interpret learners' understanding (Jong *et al.*, 2017). This finding, however, proved that teachers needed more clarity about learners' errors and how they applied their mathematical knowledge and skills to solve problems. Furthermore, teachers could describe learner errors and causes of errors in more detail compared to their initial descriptions in Cycle 1.

A significant discovery in Cycle 3 was the success of using cognitive conflict to listen to learners' reasoning about their own errors. Although this strategy was only used in three lessons, it proved to be a successful strategy for the teachers. This was evident when learners were asked to explain the reasoning behind their answers and teachers could follow their thinking processes and, in this way, identify the point(s) of misconception. A further discovery suggests that the potential of using cognitive conflict could be increased if learners are not told that they are reviewing incorrect answers, thus distracting them from discovering flawed responses for themselves. This discovery is in agreement with Henningsen and Stein (1997), who guard against lowering or removing the cognitive demand. The fact that only three teachers used this strategy confirmed that the IP teachers were not familiar with or confident in engaging learners in cognitive conflict.

An interesting finding was the value of posing questions to learners to determine what prior knowledge they had in order to build on their existing knowledge. Researchers such as Bray (2011) and Sidney and Alibali (2015) agree that the role of prior knowledge is crucial for cognitive development. Although I agree with their views, I discovered that careful consideration of which questions to ask to determine prior knowledge is important so as to avoid assumptions of what learners know and miss opportunities to identify the misconception(s).

Another useful finding was the IP teachers' realisation of the importance of developing both the conceptual and procedural knowledge of learners. During some of the lesson observations, calculation procedures were emphasised. As a result, teachers found that learners had misconceptions of the concept due to prior learning experiences focused only on procedural understanding. This discovery of developing both conceptual and procedural knowledge is in line with Hiebert and Lefevre (1986), Kilpatrick *et al*. (2001), Long (2011) and Rittle-Johnson and Schneider (2014).

5.3 Limitations of this study

The first limitation of this study was the sample size and site. Only twelve IP teachers participated in the study. However, purposeful sampling secured participants whose knowledge and experiences were relevant to the scope of this study. Also, all the teachers were from one primary school in the Western Cape. A second limitation was that this research study focused only on one domain of Mathematics, viz. Numbers, Operations and Relationships. Furthermore, only three PAR cycles could be completed due to time constraints.

5.4 Recommendations

This study focused on IP teachers as the unit of analysis. Future research could focus on learners' improvement after teachers have implemented remedial strategies to address misconceptions in NOR. Further research could include opportunities for teachers to deepen their mathematical thinking on concepts and procedures applicable to NOR. This recommendation is based on the learner errors and misconceptions that recurred frequently during the PAR cycle. Research could also be conducted in the other four domains of Mathematics to determine if different strategies will help to identify learner errors and misconceptions.

References

Ashlock, R. B. 2010. *Error patterns in computation: Using error patterns to help each student learn*. 10th edition. Boston: Allyn & Bacon.

Bartell, T. G., Webel, C., Bowen, B. & Dyson, N. 2013. Prospective teacher learning: Recognizing evidence of conceptual understanding. *Journal of Mathematics Teacher Education*, 16(1): 57-79.

Bogdan, R. & Biklen, S. K. 1982. *Qualitative research for education: An introduction to theories and methods*. Boston: Pearson A & B.

Booth, J. L., Barbieri, C., Eyer, F. & Paré-Blagoev, E. J. 2014. Persistent and pernicious errors in algebraic problem solving. *Journal of problem solving*, 7(1): 10-23.

Borasi, R. 1987. Exploring mathematics through the analysis of errors. *For the Learning of Mathematics*, 7(3): 2-8.

Bourke, B. 2014. Positionality: Reflecting on the research process. *The Qualitative Report*, 19(18): 1-9

Bray, W. S. 2011. A collective case study of the influence of teachers' beliefs and knowledge on error-handling practices during class discussion of mathematics. *Journal for Research in Mathematics Education*, 42(1): 2-38.

Brodie, K. 2014. Learning about learner errors in professional learning communities. *Educational Studies in Mathematics*, 85(2): 221-239.

Crane, P. & O'Regan, M. 2010. *On PAR: Using participatory action research to improve early intervention*. Department of Families, Housing, Community Services and Indigenous Affairs, Australian Government.

Crotty, M. 1998. Introduction: The research process. *The foundations of social research: Meaning and perspective in the research process*. London: Sage. 1-17.

Department of Basic Education. 2011. *Curriculum and Assessment Policy Statement: Grades 4 - 6.* Pretoria: Department of Basic Education.

Department of Education. 2001. *Education White Paper 6: Building an inclusive education and training system.* [Online]. Available: https://www.education.gov.za/Resources/Legislation/WhitePapers.aspx [2018, June 22]

Green Shoots Suite. n.d. [Online]. Available: www.gsed.co.za [2018, October 17].

Hansen, A. 2011 *Children's errors in mathematics - Understanding common misconceptions in primary schools*. 2nd edition. London: SAGE Publications.

Henningsen, M. & Stein, M. K. 1997. Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education,* 28(5): 524-549.

Hiebert, J. C. & Lefevre, P. 1986. Conceptual and procedural knowledge in mathematics: An introductory analysis, in Conceptual *and procedural knowledge: The case of mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.1-27.

Hsieh, H.-F. & Shannon, S. E. 2005. Three approaches to qualitative content analysis. *Qualitative Health Research*, 15(9), 1277-1288.

Jacobs, V. R., Lamb, L. L. C. & Philipp, R. A. 2010. Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, 41(2): 169-202.

Jong, C., Thomas, J., Fisher, M., Schack, E., Davis, M. & Bickett, M. 2017. Decimal dilemmas: Interpreting and addressing misconceptions. *Ohio Journal of School Mathematics*, 75(1): 13-21.

Kara, H. 2016. *Creative research methods in the social sciences: A practical guide*. Bristol, UK: Policy Press.

Khan, S. B., & Chishti, S. U. H. 2011. Learners' errors: Supporting learners for participating in mathematics classroom. *International Journal of Academic Research*, 3(1): 656-659.

Kilpatrick, J., Swafford, J. & Findell, B. (eds.). 2001. *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Lincoln, Y. S. & Guba, E. G. 1985. Establishing trustworthiness, in Naturalistic *inquiry*. Thousand Oaks, CA: Sage Publications. 289-331.

Long, C. 2011. Maths concepts in teaching: Procedural and conceptual knowledge. *Pythagoras*, 62: 59-65.

MacDonald, C. 2012. Understanding participatory action research: A qualitative research methodology option. *The Canadian Journal of Action Research*, 13(2): 34-50.

Merriam, S. B. 2009. *Qualitative research: A guide to design and implementation*. 3rd edition. San Francisco, CA: Jossey-Bass.

Milner, H. R. 2007. Race, culture, and researcher positionality: Working through dangers seen, unseen, and unforeseen. *Educational Researcher*, 41(4): 343-375.

Nesher, P. 1987. Towards an instructional theory: The role of student's misconceptions. *For the Learning of Mathematics*, 7: 33-40.

Olivier, A. 1992. Handling pupils' misconceptions. *Pythagoras*, 21: 10-19.

Patton, M. Q. 2005. Qualitative research. *Encyclopedia of statistics in behavioral science*. Chichester, UK: John Wiley & Sons. 1633-1636.

Reddy, V., Visser, M., Winnar, L., Arends, F., Prinsloo, C. & Isdale, K. 2016. *TIMSS 2015: Highlights of Mathematics and Science Achievement of Grade 9 South African Learners*. Pretoria: Human Sciences Research Council.

Republic of South Africa. 1998. Employment of Educators Act 76 of 1998. Pretoria. Government Printer.

Riccomini, P. J., Smith, G. W., Hughes, E. M. & Fries, K. M. 2015. The language of mathematics: The importance of teaching and learning mathematical vocabulary. *Reading and Writing Quarterly*, 31(3): 235-253.

Rittle-Johnson, B. & Schneider, M. 2014. Developing conceptual and procedural knowledge of mathematics, in Oxford *handbook of numerical cognition*. Oxford: Oxford University Press. 1118-1134.

Rittle-Johnson, B., Schneider, M. & Star, J. R. 2015. Not a one-way street: Bidirectional relations between procedural and conceptual knowledge of mathematics. *Educational Psychology Review*, 27(4): 587-597.

Rittle-Johnson, B., Siegler, R. & Alibali, M. W. 2001. Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, 93: 346-362.

Rubenstein, R. & Thompson, D. 2002. Understanding and supporting children's mathematical vocabulary development. *Teaching Children Mathematics*, 9(2): 107-112.

Schifter, D., Monk, S., Russell, S. J. & Bastable, V. 2008. Early algebra: What does understanding the laws of arithmetic mean in the elementary grades? in J. J. Kaput, D. W. Carraher & M. L. Blanton (eds). *Algebra in the early grades*. New York: Routledge. 413-447.

Selener, D. 1997. Participatory action research and social change, in The *Cornell participatory action research network*. 2nd edition. New York: Cornell University Press.

Sidney, P. G. & Alibali, M. W. 2015. Making connections in math: Activating a prior knowledge analogue matters for learning. *Journal of Cognition and Development*, 16(1): 160-185.

Smith, J. P., Disessa, A. A. & Roschelle, J. 1993. Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The Journal of the Learning Sciences*, 3(32): 115-163.

Smithson, J. 2008. Focus groups, in N. Denzin & Y. Lincoln (eds). *The SAGE handbook of qualitative research.* 2nd edition. Thousand Oaks: Sage Publications. 357-370.

Star, J. R. 2005. Re-"conceptualizing" procedural knowledge in Mathematics. *Journal for Research in Mathematics Education*, 36: 404-411.

TIMSS & PIRLS International study centre. 2016. About TIMSS 2015. [Online]. Available: http://timss2015.org/timss-2015/about-timss-2015/ [2018, June 15]

Tracy, S. J. 2010. Qualitative quality: Eight "big-tent" criteria for excellent qualitative research. *Qualitative Inquiry*, 6(10): 837-851.

Walter, M. 2009. Participatory action research, in A. Bryman (ed). *Social research methods*. The Falmer Press. 151-158.

Watson, S. R., Lopes, J., Oliveira, C. & Judge, S. 2018. Error patterns in Portuguese students' addition and subtraction calculation tasks: Implications for teaching. *Journal for Multicultural Education*, 12(1): 67-82.

Yorulmaz, A. & Önal, H. 2017. Examination of the views of class teachers regarding the errors primary school students make in four operations. *Universal Journal of Educational Research*, 5(11): 1885-1895.

Appendix A (Interview schedules)

Interview Schedule: Semi-structured interviews with Intermediate Phase teachers

(Interview protocol to be audiotaped)

Teacher A:

How long have you been teaching Mathematics?

How long have you been teaching Mathematics in the Intermediate Phase?

When did you start noticing learner errors/mistakes in general?

When did you start noticing learner errors in the content area of Numbers, Operations and Relationships?

What specific learner errors/mistakes do you notice in the content area of Numbers, Operations and Relationships?

Have you been addressing learner errors?

What are some of the strategies you have implemented to address learner errors?

How do you know whether these errors are mathematical in nature or not?

What do you think these errors are due to? Please elaborate.

Are there particular revision strategies you have been using? Please provide examples of these.

How did you decide on using these particular strategies?

Have you noticed any improvement with the implementation of these strategies? Please explain.

Focus Group Schedule:

[Three focus group sessions including the teacher participants per grade in the Intermediate Phase]

What kind of learner errors were common in the content area of Numbers, Operations and Relationships?

What strategies were implemented to address learner errors?

Which strategies have shown improvement, if any?

Why do you think these strategies have shown improvement? Please elaborate.

Were learners able to address any of their errors individually or perhaps in group discussions?

Did any of the strategies require learners to use their prior knowledge?

Do you think learners' prior knowledge was relevant in understanding new concepts? Why?/ Why not?

Do you think any other strategies/activities would assist learners in identifying their errors? Please elaborate.

Appendix B (Learner worksheets)

Worksheet based on Numbers, Operations and Relationships

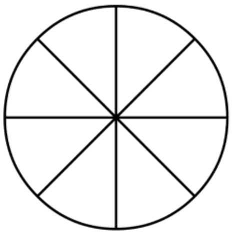
Grade 4

Questions	Rationale
1. Three children wrote the following number sentences for the following instruction: *"Calculate the product of 20 and 5"* a) 20 + 5 = 25 b) 20 x 5 = 100 c) 20 + 20 + 20 + 20 + 20 = 100 Which number sentence do you agree with? Explain your answer: …………………………………………………………….. ……………………………………………………………..	Cognitive level 1 – Knowledge (appropriate use of mathematical vocabulary), and Cognitive level 2 – Routine procedures (perform well-known procedures) To determine whether learners understand the terminology "product" and to assess how they calculate their answer.
2. Determine the unknown X: 120 + 15 – 15 = X	Cognitive level 2 – Routine procedures (simple applications and calculations and derivation from given information) Can learners recognise that adding and subtracting the same number will result in zero?

3. During a recycling project, the Grade 4 learners collected 8 249 cans. They collected 2 578 cans less than the Grade 5s. How many cans did the Grade 5s collect?	Cognitive level 3 – Complex procedures (Conceptual understanding and problem involving higher order reasoning) To assess whether learners understand comparison by difference and show their calculation method.
4. A farmer has 6 000 trees growing in his orchard. After a fire, he counts his trees and finds out that there are only 3 353 trees left. A week later the farmer plants another 675 trees. How many trees altogether are now in the orchard?	Cognitive level 4 – Problem solving (Reasoning – the ability to break the problem down) Are learners able to interpret and successfully identify the different operations required to solve the problem?
5. How many legs would eight chickens and seven dogs have altogether? Explain how you would calculate your answer.	Cognitive level 2: Routine procedures (Derivation from given information) Cognitive level 3: Complex procedures (simple applications and calculations, which may involve more than one step) Grouping problems which are solved with multiplication and/or repeated addition

	Do learners recognise digits in words and apply reasoning skills to solve the problem?
6. Choose the number that has been rounded off to the nearest 100 to become 6 100. *6 150; 6 048; 6 211; 6 087; 6 178*	Cognitive level 2: Routine procedures (Derivation from given information) Cognitive level 1: Knowledge (Appropriate rounding off of numbers) Do learners see place value in the case of tens and apply the rules for rounding off?
7. Father buys 198 potatoes. He cooks 9 potatoes a day. How many days will the potatoes last?	Cognitive level 2: Routine procedures (Derivation from given information – sharing problems solved by division/repeated subtraction) Can learners interpret the problem as a division calculation and show their calculation method?

8. Mr Brown's water tank has a capacity of 6 860 litres. How much water (in litres) is in the tank if it half filled?	Cognitive level 2: Simple applications and calculations. Are learners able to halve the number by halving each place value and then add up to get answer?/or solve by division?
9. Carmen visits the mall and spends two thirds of her birthday money. If she collected R327 for her birthday, how much did she spend?	Cognitive levels 3 & 4: Complex procedures (conceptual understanding) & Problem solving (Reasoning) – Investigations to describe relationships. Can learners determine the relationship meaning of finding a fraction of a whole number?
10. Which of the number sentences is the same as 5 x 6? a) 5 x (3 + 2) b) 5 + (2 x 3) c) 5 x (2 x 6) d) 5 x (2 x 3) e) 5 + (3 x 3) f) 5 + (2 x 3) Explain/show why you chose your answer.	Cognitive level 1: Knowledge (Use of mathematical facts) Cognitive level 2: Routine procedures (Identification and use of correct mathematical rules) Do learners recognise and apply the rule of calculating the brackets first?

11. Arrange the numbers 8 945; 9 854; 8 495; 8 594 in: a) Ascending order: ……………………………..……………………….. b) Descending order: …………………………………………………………	Cognitive level 1: Knowledge (Straight recall) Can learners identify the different place value of the same digits that are rearranged in different place values?
12. Shade ¼ of this shape: Explain why you chose to shade the number of parts:  Which fraction is left unshaded? ….	Cognitive level 3: Complex procedures (Investigations to describe rules and relationships). Are learners able to use their understanding and knowledge of fractions and connect the equivalent fraction?

Worksheet based on Numbers, Operations and Relationships

Grade 5

Questions	Rationale
1. Determine the difference between 5 000 and the product of 5 and 1000.	Cognitive level 1: Knowledge (Appropriate use of mathematical vocabulary) Do learners understand the terminology of finding the "difference" and "product" Cognitive level 2: Routine procedures (Simple applications and calculations, which may involve more than one step)
2. Which number has been rounded off to the nearest 5 to make 23 545? Circle your choice. *23 548; 23 500; 23 540; 23 543; 23 455*	Cognitive level 1: Knowledge (Appropriate rounding off of numbers) Do learners see place value in the case of units and apply the rules for rounding off?

<table>
<tr>
<td>
3. Use the information in the table below to calculate the difference in the population size of town X and town Y. Show your method of calculation.
<table>
<tr><th>TOWN</th><th>POPULATION SIZE</th></tr>
<tr><td>Town X</td><td>forty-eight thousand four hundred and fifty-two</td></tr>
<tr><td>Town Y</td><td>Seventy-two thousand and one</td></tr>
</table>
</td>
<td>
Cognitive level 1: Knowledge (Appropriate use of mathematical vocabulary)

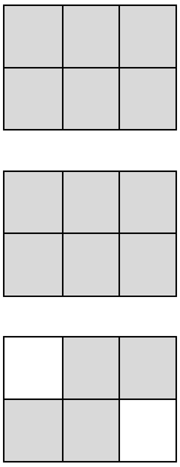
Cognitive level 2: Routine procedures (Perform well-known procedures and derivation from given information)

Can learners write five digit numbers and execute subtraction calculation?
</td>
</tr>
<tr>
<td>4. In 2017, 23 670 guests visited an art museum. This is 9 109 more guests than in 2018. How many guests visited the museum in 2018?</td>
<td>
Cognitive level 3: Complex procedures (Conceptual understanding and reasoning)

To assess whether learners understand comparison by difference and show their calculation method.
</td>
</tr>
<tr>
<td>5. Reuben uses $\frac{2}{3}$ of a cup of oil to bake one plate of biscuits. How many cups did he use if he baked 15 plates?</td>
<td>
Cognitive level 4: Problem – solving (Non-routine problem)

How do learners identify and calculate the relationship meaning of a fraction to a whole number?
</td>
</tr>
</table>

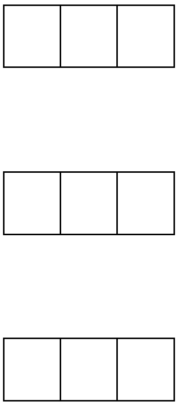
6. Write the mixed number represented by the diagram:



a) Explain your answer:

…………………………………………………………..
…………………………………………………………..
…………………………………………………………..

b) Now shade the diagram below to represent an equivalent mixed number as above:



Cognitive level 2: Routine procedures (Derivation from given information with simple applications)

Can learners interpret the representation of whole numbers and parts of a whole as a fraction?

Cognitive level 3: Complex procedures (Investigation to describe rules and relationships and making connections between different representations)

Are learners able to identify the relationship between four-sixths and two thirds as equivalent?

7. In a test, a grade 5 learner arranged the following numbers in descending order: *22 639; 22 369; 22 693; 22 396* a) Explain the mistake the learner made. …………………………………………………………… …………………………………………………………… b) How would you arrange the numbers in descending order? ………………………………………………………………..	Cognitive level 1: Knowledge (Straight recall and use of mathematical facts) Compare and order numbers Can learners identify the different place value of the same digits that are rearranged in different place values and realise that changing the place value changes the value of the digit?
8. Complete the following by filling in the missing numbers: a) 54 x 35 = 54 x (30 + _____) = 54 x ______ + (54 x _____) = 1 620 + __________ = __________	Cognitive level 2: Routine procedures (Derivation from given information) To determine if learners can identify and apply the distributive property
9. On the day of athletics, the school needed to transport 883 children to the stadium. If each bus can seat 60 children, how many buses must they book for transportation? Explain your final answer: ………………………………………………………………….. …………………………………………………..……………	Cognitive level 2 & 3: Routine and complex procedures Perform simple calculations and display conceptual understanding with higher-order reasoning. Are learners able to interpret and apply the basic operation of division and reason their answer with a remainder to rounding off to the next whole number?

10. There were 63 choir members on stage. When the band joined them for a performance, there were 145 performers. Which number sentence would be best to solve the problem? a) 145 = 63 – m b) 145 + 63 = m c) 145 + m = 63 d) 145 = 63 + m Explain how you made your choice: …………………………………………………………………… …………………………………………………………………..….. ……………………………………………………………………	Cognitive level 4: Problem solving (Non-routine problem which may require reasoning) To determine if learners can interpret the process of finding the unknown and motivate their choice.
11. The following number sentences are incorrect. How would you explain the mistake to the child? a) 8 663 x 0 = 8 663 Rule: …………………………………………………….…. b) 2 188 + 0 = 0 Rule: ……………………………………………………… c) 1 x 6 123 = 1 Rule: ………………………………...…………………..…	Cognitive level 1: Knowledge (Use of mathematical rules) Cognitive level 3: Ability to explain and apply the properties of zero and one Are learners able to identify the additive and multiplicative properties of zero and the multiplicative property of one?

Worksheet based on Numbers, Operations and Relationships

Grade 6

Questions	Rationale
1. Write the number that is expanded below: (7 x 100) + (5 x 1 000) + (1 x 10 000) + (5 x 1) ……………………………………	Cognitive level 1: Knowledge (Straight recall of number and place values) Cognitive level 2: Routine procedures (simple applications) Do determine if learners can order numbers in place value and use zero (0) as a place holder
2. The following question was answered by a grade 6 child: *"Estimate the answer by rounding off both numbers to the nearest 100." 2 532 x 178* Explain what mistake(s) this learner made: $\boxed{2\ 600 \times 100 = 260\ 000}$ …………………………………………………………………… ………………………………………………………………….... …………………………………………………………………… …………………………………………………………………....	Cognitive level 3: Complex procedures (Investigations to describe rules and relationships) Can learners estimate by applying the rule for rounding off to 100 and show ability to check solutions?

3. Show how you would use subtraction to check the following calculation: 62 477 + 43 936 = 106 413 $\boxed{}$ What do you notice? ……………………………………..…………………………………… ………………………………………………………………..………… ……………………………..………………………………………… ………………………………………………………………...…...….	Cognitive level 2: Routine procedures (Derivation from given information) Cognitive level 3: Complex procedures (Investigations to describe rules and relationships) Are learners able to recognise subtraction as the inverse operation for addition and use the inverse operation to check solution?
4. Choose the correct statement to make this number sentence true: 22 + 18 x 9 + 1 = 202 e) 22 + (18 x 9) + 1 f) 22 + (18 x 9 + 1) g) 22 + 18 x (9 + 1) h) (22 x 18) x (9 + 1)	Cognitive level 2: Routine procedures (Derivation from given information and identification and use of correct formula) Are learners able to apply the rules of multiplication and addition?

5. If 2 kg of cheese costs R136, how much (in R) would 250 g cost? Show/explain how you calculated your answer.	Cognitive level 3: Complex procedures (Problem involving calculations and higher order reasoning) Are learners able to use knowledge of measurement in context to practise skills acquired in Numbers, Operations and Relationships and problem solve converting between units of measurement?
6. All the articles in a shop are marked down by 25%. What will Thulani pay (in R) for a shirt that was priced at R240 before the discount?	Cognitive level 3: Complex procedures (Problems involving complex calculations and higher order reasoning) Can learners interpret the problem, represent 25% as a percentage and calculate percentage of whole numbers?

<table>
<tr>
<td>
7. The school's Governing Body raised R14 000 to purchase a new laptop and data projector for the school's media centre. The cost of the equipment is shown in the table below:
<table>
<tr><td>EQUIPMENT</td><td>COST</td></tr>
<tr><td>Laptop</td><td>R7 460</td></tr>
<tr><td>Data projector</td><td>R5 190</td></tr>
</table>
a) How much will it cost to purchase the laptop and the data projector? You may use any method to calculate your answer.

b) Write an open number sentence that will show how much money is left after the equipment is purchased

……………………………………………………………………
</td>
<td>
Cognitive level 2: Routine procedures

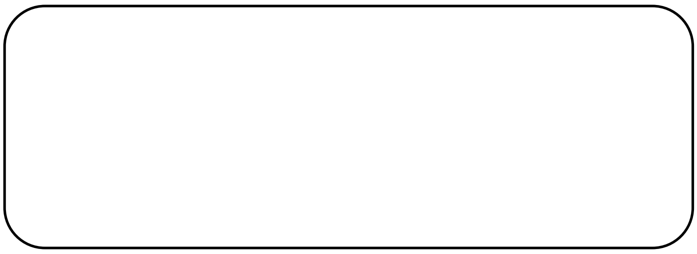
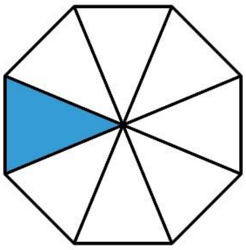
(Simple calculations and derivation of given information)

Cognitive level 3: Complex procedures

(Conceptual understanding)

Can learners identify and represent the unknown?
</td>
</tr>
<tr>
<td>
8. There are 280 children at Sunshine Primary School. Three sevenths are boys.
a) How many boys are there?
</td>
<td>
Cognitive level 3: Complex procedures

(Conceptual understanding)
</td>
</tr>
</table>

b) What fraction are girls?.................. c) How many girls are there? Show your calculation. 	Fraction of a whole: Are learners able to discern part of a whole (a well-known interpretation of fractions)? Can learners identify the missing fraction which makes the whole (seven sevenths)? Linked to previous question – Calculating part of a whole
9. A learner was asked to shade ¼ of the shape below. Explain what error the learner the made.  ……………………………………………………………………… ……………………………………………………………………… ……………………………………………………………………… ………………………………………………………………………	Cognitive level 3: Complex procedures (Making significant connections between different representations) Are learners able to connect numerical and visual representation? Are learners able to identify equivalent fractions?

10. Look at the clues used for clue board division. Can you find a quicker way? 377 ÷ 25 4 x 25 = 100 4 x 25 = 100 4 x 25 = 100 1 x 25 = 25 1 x 25 = 25 1 x 25 = 25	Cognitive level 2: Routine procedures (Simple calculations and derivation of given information) Are learners able to group the given clues (e.g.) 12 x 25 = 300 and 3 x 25 = 75?

Appendix C (Consent to participate in the research study, SU)

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jou kennisvennoot • your knowledge partner

STELLENBOSCH UNIVERSITY

CONSENT TO PARTICIPATE IN RESEARCH

You are invited to take part in a study conducted by Mrs Beryl Elizabeth Bowers, Med Degree, from the Curriculum Studies Department of the Education Faculty at Stellenbosch University. You were approached as a possible participant because the focus of this research study revolves around learner errors in Numbers, Operations and Relationships in the Intermediate Phase. As a Mathematics teacher in the Intermediate Phase, you are eligible to be a participant in this study.

1. PURPOSE OF THE STUDY

This research study titled, *'Teachers Identifying learner errors and misconceptions in Numbers, Operations and Relationships in the Intermediate Phase'*, is aimed at improving learners' achievements in Mathematics, through working with teachers and finding constructive ways of identifying and responding to learners' mathematical errors and misconceptions. The content area of *Numbers, Operations and Relationships* carries a weighting of 50% in the Intermediate Phase in an attempt to ensure that learners are sufficiently numerate when they enter the Senior Phase. This study will focus on the ways teachers identify the challenges learners face in understanding *Numbers, Operations and Relationships* and hope to explore and suggest suitable strategies for the remediation of those errors and misconceptions.

2. WHAT WILL BE ASKED OF ME?

If you agree to take part in this study, you will be asked to participate in an individual semi-structured interview, classroom observations and a focus group interview. These three activities will take place at your school over Term 2 and Term 3. The estimated duration of the individual interview session should not be longer than an hour which will be scheduled during after-school hours. A semi-structured interview allows for more flexible questions and is aimed at gaining insight into your experiences as a Math teacher. Classroom observations will take place during Math lessons and some worksheets may be reviewed to share possible teaching strategies. The focus group interview will be done with other colleagues in your grade as an opportunity to share and develop ideas collectively.

3. POSSIBLE RISKS AND DISCOMFORTS

There are no foreseeable risks as a participant in this research study. The researcher acknowledge that interview sessions will be scheduled after school hours which may inconvenience the participants and the researcher will make every effort to accommodate the participant at a time which is most suitable.

4. POSSIBLE BENEFITS TO PARTICIPANTS AND/OR TO THE SOCIETY

Benefits of being a participant in this research study include opportunities to reflect on current teaching practice, collaborate with colleagues within your grade and within the Intermediate Phase and explore different remedial strategies. Implementation of successful strategies in a mainstream class may lead to a decrease in the number of learners attending learning support and improved learner Mathematics achievements.

5. PAYMENT FOR PARTICIPATION

There will be no payment for participation in this research study. Participation in this study is voluntary.

6. PROTECTION OF YOUR INFORMATION, CONFIDENTIALITY AND IDENTITY

The ethical integrity of this study will be maintained by conducting this study under the auspices of the ethics committee of the University of Stellenbosch.

Any information you share with me during this study and that could possibly identify you as a participant will be protected. Your privacy, confidentiality of information and anonymity will be maintained by replacing your real identity with a pseudonym. As researcher I will code all information from your interviews and classroom observations with a pseudonym. The pseudonym will be used in my research report. You will thus remain anonymous. You will have the opportunity to review and edit audio-recorded interviews before the data is analysed. All biographical information, transcribed interviews, observation sheets and field notes will be kept safe on my personal computer which is password protected and which only I have to access to. Audio recordings will be erased after the data has been analysed and recorded.

7. PARTICIPATION AND WITHDRAWAL

You can choose whether to be in this study or not. If you agree to take part in this study, you may withdraw at any time without any consequence. You may also refuse to answer any questions you don't want to answer and still remain in the study. The researcher may withdraw you from this study if circumstances arise which warrant doing so.

8. RESEARCHERS' CONTACT INFORMATION

If you have any questions or concerns about this study, please feel free to contact:

Principal investigator: Mrs Beryl Elizabeth Bowers

Contact Details: 083 249 1952

Email: beryl.ebowers@gmail.com

Address: 67 Canal Road

Wetton

7780

and/or

Supervisor: Dr Faaiz Gierdien

Contact Details: (021) 808 2289

Email: faaiz@sun.ac.za

Address: Department of Curriculum Studies

Faculty of Education

Stellenbosch University

Private Bag X1

Matieland

7602

RIGHTS OF RESEARCH PARTICIPANTS

You may withdraw your consent at any time and discontinue participation without penalty. You are not waiving any legal claims, rights or remedies because of your participation in this research study. If you have questions regarding your rights as a research participant, contact Ms Maléne Fouché [mfouche@sun.ac.za; 021 808 4622] at the Division for Research Development.

DECLARATION OF CONSENT BY THE PARTICIPANT

As the participant I confirm that:

- I have read the above information and it is written in a language that I am comfortable with.
- I have had a chance to ask questions and all my questions have been answered.
- All issues related to privacy, and the confidentiality and use of the information I provide, have been explained.

By signing below, I ________________________________ *(name of participant)* agree to take part in this research study, as conducted by Mrs Beryl Elizabeth Bowers.

____ ______________________

Signature of Participant **Date**

DECLARATION BY THE PRINCIPAL INVESTIGATOR

As the **principal investigator,** I hereby declare that the information contained in this document has been thoroughly explained to the participant. I also declare that the participant has been encouraged (and has been given ample time) to ask any questions. In addition, I would like to select the following option:

	The conversation with the participant was conducted in a language in which the participant is fluent.
	The conversation with the participant was conducted with the assistance of a translator (who has signed a non-disclosure agreement), and this "Consent Form" is available to the participant in a language in which the participant is fluent.

____ ______________________

Signature of Principal Investigator **Date**

Appendix D (Parent consent to participate in the research study, SU)

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STELLENBOSCH UNIVERSITY

PARENT/LEGAL GUARDIAN CONSENT FOR CHILD TO PARTICIPATE IN RESEARCH

I would like to invite your child to take part in a study conducted by myself, Mrs Beryl Elizabeth Bowers, from the Education Faculty at Stellenbosch University. Your child will be invited as a possible participant because my research study revolves around Mathematics in the Intermediate Phase.

1. PURPOSE OF THE STUDY

This research study titled, *'Teachers Identifying learner errors and misconceptions in Numbers, Operations and Relationships in the Intermediate Phase'*, is aimed at improving learners' achievements in Mathematics, through working with teachers and evidence of learners' work to find constructive ways of identifying and responding to learners' mathematical errors and misconceptions. The content area of *Numbers, Operations and Relationships* carries a weighting of 50% in the Intermediate Phase in an attempt to ensure that learners are sufficiently numerate when they enter the Senior Phase. This study will focus on the ways teachers identify the challenges learners face in understanding *Numbers, Operations and Relationships* and hope to explore and suggest suitable strategies for the remediation of those errors and misconceptions.

2. WHAT WILL BE ASKED OF MY CHILD?

If you consent to your child taking part in this study, the researcher will then approach the child for their assent to take part in the study. If your child agrees to take part in the study, he/she will be asked to allow the researcher and teacher to use evidence of their work to review and discuss possible strategies to address any learner errors in the content area of *Numbers, Operations and Relationships*. There will be classroom observations where the researcher will make notes about how learners respond to their errors. Your child may be asked to participate in a focus group interview with a group of learners. These classroom observations will take place at their school during Math lessons over Term 2 and Term 3.

3. POSSIBLE RISKS AND DISCOMFORTS

There are no foreseeable risks or discomforts to your child at this stage, as the focus of this research study will focus on teachers and will only require evidence of your child's work and participation in a focus group interview. The focus group interview is an interview with a group of learners and not with individuals. There would be no disruption during classroom observations, as the observer will be observing the teaching strategies of the teacher. Your child may be aware of the observer during the classroom observations.

4. POSSIBLE BENEFITS TO THE CHILD OR TO THE SOCIETY

This study is aimed at improving learner Mathematical achievements in the Intermediate Phase. By allowing evidence of your child's work to be used in this study, constructive strategies may emerge which could lead to improved teaching practice.

5. PAYMENT FOR PARTICIPATION

There will be no payment for participation in this research study. Participation in this study is voluntary.

6. PROTECTION OF YOUR AND YOUR CHILD'S INFORMATION, CONFIDENTIALITY AND IDENTITY

The ethical integrity of this study will be maintained by conducting this study under the auspices of the ethics committee of the University of Stellenbosch.

Any evidence of work your child will share during this study and that could possibly identify him/her will be protected. This will be done by instructing learners not to write their names on any of the worksheets which may be used for discussions. Your child will thus remain anonymous during any discussions.

For the focus group interview your child's name will be replaced with a pseudonym.

7. PARTICIPATION AND WITHDRAWAL

You can choose whether your son/daughter should participate in this study or not. If you consent to your child taking part in the study, please note that your child may choose to withdraw or decline participation at any time without any consequence. Your child may also refuse to answer any questions they don't want to answer and still remain in the study. The researcher may withdraw your child from this study if circumstances arise which warrant doing so.

8. RESEARCHERS' CONTACT INFORMATION

If you have any questions or concerns about this study, please feel free to contact:

Principal investigator: Mrs Beryl Elizabeth Bowers

Contact Details: 083 249 1952

Email: beryl.ebowers@gmail.com

Address: 67 Canal Road

Wetton

7780

and/or

Supervisor: Dr Faaiz Gierdien

Contact Details: (021) 808 2289

Email: faaiz@sun.ac.za

Address: Department of Curriculum Studies

Faculty of Education

Stellenbosch University

Private Bag X1

Matieland

7602

9. RIGHTS OF RESEARCH PARTICIPANTS

Your child may withdraw their consent at any time and discontinue participation without penalty. Neither you nor your child are waiving any legal claims, rights or remedies because of your participation in this research study. If you have questions regarding your or your child's rights as a research participant, contact Ms Maléne Fouché [mfouche@sun.ac.za; 021 808 4622] at the Division for Research Development.

DECLARATION OF CONSENT BY THE PARENT/ LEGAL GUARDIAN OF THE CHILD-PARTICIPANT

As the parent/legal guardian of the child I confirm that:

- I have read the above information and it is written in a language that I am comfortable with.
- I have had a chance to ask questions and all my questions have been answered.
- All issues related to privacy, and the confidentiality and use of the information have been explained.

By signing below, I ________________________________ *(name of parent)* agree that the researcher may approach my child to take part in this research study, as conducted by Mrs Beryl Elizabeth Bowers.

____ ________________________

Signature of Parent/Legal Guardian **Date**

DECLARATION BY THE PRINCIPAL INVESTIGATOR

As the **principal investigator,** I hereby declare that the information contained in this document has been thoroughly explained to the parent/legal guardian. I also declare that the parent/legal guardian was encouraged and given ample time to ask any questions.

____ ________________________

Signature of Principal Investigator **Date**

Appendix E (Assent Form)

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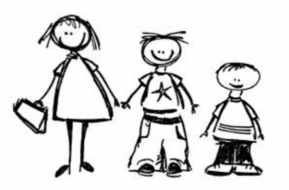
ASSENT FORM FOR MINORS



TITLE OF THE RESEARCH PROJECT: How can teachers help children with mistakes in Numbers, Operations and Relationships in Mathematics?

RESEARCHERS' NAME(S): Mrs BE Bowers

RESEARCHER'S CONTACT NUMBER: (021) 797 4243

What is RESEARCH?

Research is something we do find **NEW KNOWLEDGE** about the way things (and people) work. We use research projects or studies to help us find out more about children and teenagers and the things that affect their lives, their schools, their families and their health. We do this to try and make the world a better place!

What is this research project all about?

This research is to find ways for teachers to help children in Grade 4, 5 and 6 to improve their understanding of numbers in mathematics. Teachers will look at exercises to see what kind of mistakes children make when working with numbers. Teachers may ask the children questions to see what they don't understand. They will try to explain and use different methods to help them to see their mistakes.

Why have I been invited to take part in this research project?

You are invited to take part because Mathematics is one of your subjects and you are in one of the Intermediate Phase grades (Grade 4, 5 or 6).

Who is doing the research?

I am a Grade 5 teacher at your school and I teach Mathematics.

What will happen to me in this study?

Your teacher will ask you to complete some math activities and maybe ask you to explain how you calculated your sums. You may be asked to be part of an interview with other learners (a Focus Group interview) which is an opportunity for you to share your thoughts and ideas in a group.

Can anything bad happen to me?

Nothing bad can happen to you if you participate in this study. You will complete activities in your classroom and lessons will continue as usual. You will be with a group of other learners if you are invited to be part of the Focus Group interview.

Can anything good happen to me?

We don't know what the results of this study will be yet, but we hope that teachers will find different ways to help children with their mistakes with Numbers.

Will anyone know I am in the study?

You will be anonymous in the study. This means that no one will know your name. Your teacher will tell you not to write your name on your worksheets that will be used for this study. If you participate in the Group interview your real name will be replaced with a pseudonym (another name).

Who can I talk to about the study?

If you have any questions about this study you can contact:

Mrs BE Bowers or The Principal at school on (021) 7974243

What if I do not want to do this?

You can choose not to take part in this study. Even if you decide to participate, you may choose to stop being in the study at any time without getting in trouble.

Do you understand this research study and are you willing to take part in it?

YES NO

Has the researcher answered all your questions?

YES NO

Do you understand that you can STOP being in the study at any time?

YES NO

__________________________ _____________________

Signature of Child Date

Appendix F (Research permission letter to SGB)

67 Canal Road

Wetton

7780

29 January 2018

The Principal and SGB

John Graham Primary School

Milford Road

Plumstead

Request for permission to do research

Dear Mrs Johnson and SGB members

I am currently completing my Master's Degree in Education at the University of Stellenbosch. Part of my studies requires me to do research at school with regards to the following:

- **Interviews with Intermediate Phase teachers**
- **Classroom Observations in Grade 4 – 6**
- **Focus group Interviews with teachers in each grade 4 – 6**
- **Evidence of learners' work (on an anonymous basis)**

Implementation of this research study is aimed at starting on 10th April 2018 and ending 28 September 2018.

Attached, please find the following documentation:

- **Approved Research Proposal**
- **Consent Form for teacher participants**
- **Consent Form for parents**
- **Assent Form for learners**

I am happy to meet with you should you require me to elaborate on my research.

In light of the above, I would like to request your permission to do my research at John Graham Primary School.

Hoping my request will be considered favourably.

Yours in education
Beryl Bowers (Mrs)

Appendix G (Research approval letter from the SGB, John Graham Primary)

JOHN GRAHAM PRIMARY SCHOOL

johngrahamps@absamail.co.za
Website: www.johngrahamprimary.co.za
Tel: 021-797 4243
Fax: 021- 797 1174

Milford Road
Plumstead
7800

30 January 2018

Dear Mrs Bowers

Thank you for submitting the application to do research at John Graham Primary School.

On behalf of the Governing Body I wish to inform you that your application has been approved.

We wish you well with your research and your studies.

Yours faithfully

MRS B. JOHNSON
PRINCIPAL

WESTERN CAPE
JOHN GRAHAM
PRIMARY SCHOOL
MILFORD ROAD
PLUMSTEAD, 7800
TEL: 021 797 4243/4
EDUCATION DEPARTMENT

Appendix H (Research approval from the Research Ethics Committee, SU)

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NOTICE OF APPROVAL

REC Humanities New Application Form

28 March 2018

Project number: 6193

Project Title: Teachers Identifying learner errors and misconceptions in Numbers, Operations and Relationships in the Intermediate Phase

Dear Mrs Beryl Bowers

Your REC Humanities New Application Form submitted on 19 February 2018 was reviewed and approved by the REC: Humanities.

Please note the following for your approved submission:

Ethics approval period:

Protocol approval date (Humanities)	Protocol expiration date (Humanities)
28 March 2018	27 March 2021

GENERAL COMMENTS:

Please take note of the General Investigator Responsibilities attached to this letter. You may commence with your research after complying fully with these guidelines.

If the researcher deviates in any way from the proposal approved by the REC: Humanities, the researcher must notify the REC of these changes.

Please use your SU project number (6193) on any documents or correspondence with the REC concerning your project.

Please note that the REC has the prerogative and authority to ask further questions, seek additional information, require further modifications, or monitor the conduct of your research and the consent process.

FOR CONTINUATION OF PROJECTS AFTER REC APPROVAL PERIOD

Please note that a progress report should be submitted to the Research Ethics Committee: Humanities before the approval period has expired if a continuation of ethics approval is required. The Committee will then consider the continuation of the project for a further year (if necessary)

Included Documents:

Document Type	File Name	Date	Version
Research Protocol/Proposal	Approved_Research_Proposal_Bowers_Beryl_15577295	30/01/2018	
Data collection tool	Interview schedule and Focus Group guide	30/01/2018	
Data collection tool	Classroom observation	30/01/2018	
Proof of permission	School Approval Letter	30/01/2018	
Informed Consent Form	Consent Form_Participant	30/01/2018	
Proof of permission	WCED Application Form	30/01/2018	
Proof of permission	WCED_Research approval letter	05/02/2018	
Parental consent form	Consent Form_ParentGuardian	11/02/2018	
Assent form	Assent Form_Minors	11/02/2018	
Default	Rationale with worksheets	11/02/2018	
Default	Worksheets 1 - 4	11/02/2018	

If you have any questions or need further help, please contact the REC office at cgraham@sun.ac.za.

Sincerely,

Clarissa Graham

REC Coordinator: Research Ethics Committee: Human Research (Humanities)

National Health Research Ethics Committee (NHREC) registration number: REC-050411-032.

The Research Ethics Committee: Humanities complies with the SA National Health Act No.61 2003 as it pertains to health research. In addition, this committee abides by the ethical norms and principles for research established by the Declaration of Helsinki (2013) and the Department of Health Guidelines for Ethical Research:

Principles Structures and Processes (2^{nd} Ed.) 2015. *Annually a number of projects may be selected randomly for an external audit.*

Appendix I (Research approval letter from WCED)

Audrey.wyngaard@westerncape.gov.za
Tel: +27 021 467 9272
Fax: 0865902282
Private Bag x9114, Cape Town, 8000
wced.wcape.gov.za

REFERENCE: 20180131–8801

ENQUIRIES: Dr A T Wyngaard

Mrs Beryl Bowers

67 Canal Road

Wetton

7780

Dear Mrs Beryl Bowers

RESEARCH PROPOSAL: TEACHERS IDENTIFYING LEARNER ERRORS AND MISCONCEPTIONS IN NUMBERS, OPERATIONS AND RELATIONSHIPS IN THE INTERMEDIATE PHASE

Your application to conduct the above-mentioned research in schools in the Western Cape has been approved subject to the following conditions:

1. Principals, educators and learners are under no obligation to assist you in your investigation.
2. Principals, educators, learners and schools should not be identifiable in any way from the results of the investigation.
3. You make all the arrangements concerning your investigation.
4. Educators' programmes are not to be interrupted.
5. The Study is to be conducted from **10 April 2018 till 28 September 2018**
6. No research can be conducted during the fourth term as schools are preparing and finalizing syllabi for examinations (October to December).

7. Should you wish to extend the period of your survey, please contact Dr A.T Wyngaard at the contact numbers above quoting the reference number?

8. A photocopy of this letter is submitted to the principal where the intended research is to be conducted.

9. Your research will be limited to the list of schools as forwarded to the Western Cape Education Department.

10. A brief summary of the content, findings and recommendations is provided to the Director: Research Services.

11. The Department receives a copy of the completed report/dissertation/thesis addressed to:

The Director: Research Services

Western Cape Education Department

Private Bag X9114

CAPE TOWN

8000

We wish you success in your research.

Kind regards.

Signed: Dr Audrey T Wyngaard

Directorate: Research

DATE: 01 February 2018