On the definable generalized Bohr compactification of $SL(2, \mathbb{Q}_p)$

by

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Thesis presented in partial fulfilment of the requirements for the degree of Master of Science in Mathematics in the Faculty of Science at Stellenbosch University

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December 2018

Declaration

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Abstract

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This paper provides an overview of existing knowledge regarding the so-called definable generalized Bohr compactification of the group $SL(2, \mathbb{Q}_p)$ of 2×2 matrices with determinant 1 and entries in $\mathbb{Q}_p$. The (open) question of whether this definable generalized Bohr compactification coincides with the Ellis group of the action of $SL(2, \mathbb{Q}_p)$ on its type space is also studied in detail. This includes a discussion on the topologies associated with the space of complete types over $\mathbb{Q}_p$ concentrating on $SL(2, \mathbb{Q}_p)$, as well as an investigation of the possibility of first-countability of this type space.

Uittreksel

Op die gedefinieerbare veralgemene Bohr kompaktifisering van $SL(2, \mathbb{Q}_p)$

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Die artikel gee 'n oorsig van die bestaande kennis in verband met die sogenaamde gedefinieerbare veralgemene Bohr kompaktifisering van die groep $SL(2, \mathbb{Q}_p)$ van 2×2 matrikse met determinant 1 en inskrywings in $\mathbb{Q}_p$. Die (oop) vraag of die gedefineerbare veralgemene Bohr kompaktifisering ooreen stem met die Ellis groep van die aksie van $SL(2, \mathbb{Q}_p)$ op sy tipe spasie word ook deeglik bestudeer. Dit sluit in 'n bespreking oor die topologieë wat geassosieer word met die ruimte van volledige tipes oor $\mathbb{Q}_p$ wat gekonsentreer is op $SL(2, \mathbb{Q}_p)$, sowel as die ondersoek van die eersteaftelbaarheid van hierdie tipe ruimte.

Acknowledgements

I must express immense gratitude to my supervisor, Dr Gareth Boxall, for his role not only in this project, but also in my development as a model theorist. This thesis would not have been possible without his guidance and willingness to assist at all times.

I also thank the National Research Foundation for their willingness to fund this project.

Finally, I extend my thanks to Dr Karin Howell, whose assistance in translating the abstract of this paper into Afrikaans is appreciated.

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Chapter 1

Introduction

Topological dynamics is an area of mathematics concerned with the study of actions of topological groups on topological spaces. Model theory, a field of study within mathematical logic, focuses on the classification of mathematical structures using formal languages. Although the rationale behind the development of each of these fields differs, there do exist mathematical problems of common interest to both.

One prominent application of topological dynamics in the context of model theory is the description of types using group actions. The complete type of a group element, perhaps in some elementary extension, consists of those formulas in the language with parameters in the base structure that are true of that element, and one can consider the action of a definable group G on the space of all complete types containing the formula $x \in G$. This construct is of interest to those who study stability theory, and attempts have been made to use this group action in contexts sans stability to see which favourable properties of stability may be maintained under other conditions. A notable example of such interest is Newelski's investigation into the relationship between the so-called definable Bohr compactification, a group compactification possessing a certain universal property, and the Ellis group associated with the action.

In the case of $SL(2, \mathbb{R})$ it has already been determined [4] that these two constructs do not coincide, since the definable Bohr compactification of this group is trivial whereas its Ellis group is not. However, numerous other cases have yet to be investigated. Another group that has been discussed in this context is $SL(2, \mathbb{Q}_p)$. Progress has been made in the description of the Ellis group associated with the action of $SL(2, \mathbb{Q}_p)$ on the space of complete types over $\mathbb{Q}_p$ concentrating on $SL(2, \mathbb{Q}_p)$, and it has also been shown [12] that the definable Bohr compactification of $SL(2, \mathbb{Q}_p)$ is trivial while its Ellis group is infinitely large. However, one can also consider the definable generalized Bohr compactification of $SL(2, \mathbb{Q}_p)$, a variant of the Bohr compactification, in lieu of the definable Bohr compactification. It is not yet clear whether the Ellis group of the action of $SL(2, \mathbb{Q}_p)$ on its type space coincides with the definable generalized Bohr compactification.

This paper aims to review the findings in this research area thus far, with particular emphasis on $SL(2, \mathbb{Q}_p)$, and also contribute towards the understanding of the type spaces in a topological sense. This, it is hoped, will aid in determining the relationship between the Ellis group and definable generalized Bohr compactification.

The presentation of information in this paper shall be ordered as follows:

The second chapter consists of numerous definitions in model theory that will be used throughout subsequent chapters of the paper. This is done to ensure that accessibility of the material is not limited to scholars of model theory, although some mathematical background will still be required to understand the results in later chapters.

The third chapter provides a thorough introduction to the p-adic number system. This includes an explanation of the p-adic expansion, as well as the p-adic metric and the topology it induces. In addition, some of the critical model theory involving this number system is discussed, including Macintyre's quantifier elimination result and a description of the complete 1-types over the model $M = (\mathbb{Q}_p, +, \times)$.

The fourth chapter focuses on topological dynamics and the study of the Stone and τ-topologies, both of which are integral to subsequent results. Basic concepts from topological dynamics are explained in a manner that distinguishes between the general setting, and that in which maps and groups are definable. The significance of the topological condition of first-countability in the context of the type space is also discussed, along with its implications in the τ-topology.

The fifth chapter provides an abridged, yet sufficiently-detailed account of previous publications on the focal topic of this paper. Particular emphasis is placed on studies of $SL(2, \mathbb{R})$[4] and $SL(2, \mathbb{Q}_p)$ [12].

The final chapter consists of analysis of the definable generalized Bohr compactification of $SL(2, \mathbb{Q}_p)$ and the action of $SL(2, \mathbb{Q}_p)$ on its type space. Progress is also made in the investigation of first-countability of the space of complete types over $\mathbb{R}$ concentrating on $SL(2, \mathbb{R})$, and the space of complete types over $\mathbb{Q}_p$ concentrating on $SL(2, \mathbb{Q}_p)$.

Chapter 2

Model-theoretic Fundamentals

For the sake of the reader, some definitions and explanations of rudimentary concepts in model theory are provided below. However, it shall be assumed that readers of this document have some degree of familiarity with most of these ideas, so exposition is kept to a minimum for the duration of this chapter.

2.1 Basic Model Theory

The following notions are frequently encountered in model theory, so it behooves even recreational readers of model theory to understand these fully. Note that the theory T to which these definitions refer shall always be a complete first-order theory in a language L.
The notion of *types* is critical not only to this paper, but all of model theory. Types are collections of formulas which describe the behaviour of an element, or elements, in a given structure. Types may be classified further using numerous properties, such as completeness or genericity.

Definition 2.1.1 (Types)**.** [7] An *n-type* of a theory T, in a language L, is a set p of formulas $\alpha(\bar{x})$ with free variables in the n-tuple $\bar{x}$, such that for some model M of T and n-tuple $\bar{m} \in M$, $M \models \alpha(\bar{m})$ for each formula $\alpha \in p$. In this case $\bar{m}$ is said to *realize* the type p in M. If no such tuple exists in M, M *omits* the type p.
An n-type over M is a set p of formulas $\alpha(\bar{x})$ with parameters in M such that, for some elementary extension ${}^*M$ of M and n-tuple $\bar{m} \in {}^*M$, ${}^*M \models \alpha(\bar{m})$ for each formula $\alpha \in p$.

A type p is said to be *complete* if, for each formula $\phi \in L$, it is the case that either ϕ or $\neg\phi$ lie in p. The space of all complete n-types over M, for given n, is denoted $S_n(M)$.
A *partial type* is a type that is not complete.

Definition 2.1.2 (Definability of groups)**.** A *definable group* in a model M of a theory T in a language L is a group whose underlying set is a definable set in M, and the graph of whose binary operation is also a definable set.

Quantifier Elimination is a powerful property in logic that greatly simplifies the task of describing definable sets in theories in which it is present. One of the most important results in model theory regarding the p-adic numbers is the development of a language with respect to which the theory of $\mathbb{Q}_p$ has quantifier elimination.

Definition 2.1.3 (Quantifier Elimination)**.** A theory T in a language L admits *quantifier elimination* if every formula ϕ in L is equivalent (mod T) to some other formula ϕ_{QE} that does not contain any quantifiers (the existential quantifier $\exists$ or the universal quantifier $\forall$) i.e. ϕ and ϕ_{QE} define the same set in any model of T.

Saturation is a critical idea in model theory. Its importance is such that model theoretic convention frequently refers to a *sufficiently saturated* model of a theory so that one can assume that the relevant complete types are realized. In addition, it is common practice when working with small structures to move to saturated *elementary extensions* for the sake of realizing types.

Definition 2.1.4 (Saturation)**.** [7] An L-structure M is said to be *κ-saturated* (for a cardinal κ) if, for any subset A of M, if $|A| < \kappa$ then every complete 1-type over A is realized in M.
In the event that $\kappa = |M|$, M is simply said to be *saturated*.

Elementary extensions are often used for scenarios in which types cannot be realized in a particular model. The best-known such example is that of the hyperreal numbers extending $\mathbb{R}$, in which there exist elements x that realize $0 < x < \frac{1}{n} \; \forall n \in \mathbb{N}$.

Definition 2.1.5 (Elementary extension)**.** [7] Consider a language L, L-structures A and B, and a map $f : A \to B$. The map f is an *elementary embedding* if it preserves all first-order formulas.
B is said to be an *elementary extension* of A if f is an elementary embedding. In this case one writes $A \preceq B$.
A and B are *elementarily equivalent* if, for any sentence $\alpha \in L$, $A \models \alpha \Leftrightarrow B \models \alpha$.

The *Tarski-Vaught criterion* is a useful means of identifying elementary substructures. It will prove particularly useful in the final chapter of this paper.

Definition 2.1.6 (Tarski-Vaught criterion)**.** [7] Given a language L and L-structures $A \subseteq B$, the following are equivalent:

(i) $A \preceq B$.

(ii) For any L-formula $\alpha(\bar{x}, y)$ and tuple $\bar{a}$ of A, if $B \models \exists y(\alpha(\bar{a}, y))$ then $B \models \alpha(\bar{a}, c)$ for some $c \in A$.

Definition 2.1.7 (Connected components)**.** [8] Consider a group G definable in a model M. Let ${}^*M$ denote a sufficiently saturated elementary extension of M, and ${}^*G$ the interpretation of G in this extension. Let $A \subseteq {}^*M$ be a set of parameters of size less than the degree of saturation of ${}^*M$.
The *connected components* of ${}^*G$ with respect to A are then defined as follows:

- ${}^*G^0_A$ is the intersection of all A-definable subgroups of ${}^*G$ of finite index,
- ${}^*G^{00}_A$ is the smallest A-type-definable subgroup of ${}^*G$ of bounded index, and
- ${}^*G^{000}_A$ is the smallest A-invariant subgroup of ${}^*G$ of bounded index.

Here *bounded index* means that the index is smaller than the degree of saturation. An A-type-definable subgroup is a set of realizations of some type over A, and is an intersection of A-definable sets [8]. An A-invariant subgroup is invariant with respect to automorphisms of ${}^*M$ that fix A [8].
It is also the case that ${}^*G^{000}_A \leq {}^*G^{00}_A \leq {}^*G^0_A \leq {}^*G$ [8].
In the absence of the *Independence Property*, which will be described later, the parameter set A is inconsequential and so is omitted. [8]

The *Independence Property* is a characteristic possessed by certain complete theories. Theories without this property, known as *NIP* theories, form a field of study in their own right within model theory. Note also that $Th(\mathbb{Q}_p)$, the theory of $\mathbb{Q}_p$, does not possess the Independence Property [12].

Definition 2.1.8 (Independence Property and NIP)**.** [7] Consider a formula $\alpha(\bar{x}, \bar{y})$ and a complete theory T. The formula α is said to possess the *Independence Property* if, in every model M of T, for each $N < \omega$ there is some family of tuples $\bar{b_0}, ..., \bar{b}_{N-1}$ such that for every subset X of N there is a tuple $\bar{a} \in M$ for which $M \models \alpha(\bar{a}, \bar{b}_i)$ iff $i \in X$.
A theory possesses the Independence Property if at least one of its formulas does.
A theory is called *NIP* if it does not possess the Independence Property.

Definition 2.1.9 (Definable amenability)**.** [8] A definable group G in a model of an NIP theory is said to be *definably amenable* if there exists a function f on definable subsets of G with range $[0, 1]$ as follows:

(i) For all definable $X \subseteq G$, $f(X) \geq 0$,

(ii) $f(G) = 1$,

(iii) $f(\bigcup_{i=1}^{n} X_i) = \Sigma_{i=1}^{n} X_i$, for a finite collection of disjoint $X_i \subseteq G$, and

(iv) f is left-invariant.

The first three axioms are those of a *probability measure*. The third axiom, known as *finite additivity*, is a weaker version of *countable additivity*, a property more commonly associated with probability measures, where $f(\bigcup_{i=1} X_i) = \Sigma_{i=1} X_i$ for a countable collection of disjoint $X_i \subseteq G$.
Hence, one could say that definably amenable groups G are equipped with a left-invariant, finitely-additive probability measure.

With an understanding of saturation and elementary extensions, it is now possible to learn about further variants of types that will be encountered in this project.

Definition 2.1.10 (More types)**.** [12]
Let ${}^*M$ be a highly saturated elementary extension of M. Assume that $Th(M)$ is NIP.

A *global type* is a type $p(x) \in S_G(^*M)$. Here $S_G(^*M)$ denotes the set of complete types over $^*M$ containing the formula $x \in G$. One may also refer to this as the space of complete types over $^*M$ concentrating on G. There is a continuous action of G on this type space that will be defined in a later chapter.

A global type is called *f-generic* if its stabilizer is $G(^*M)^{00}$, the smallest type-definable subgroup of $G(^*M)$ of bounded index. In the case of definably amenable G, f-genericity of a global type is equivalent to that type being G^{00}-invariant (note that the action of G^{00} on the type space is the same as that of G).

A type p is *strongly f-generic* if every left G-translate of p is invariant with respect to the group of automorphisms on $^*M$ fixing M.

A formula is said to be *generic* if finitely many translates of that formula cover the entire group, and a type is *generic* if every formula within that type is generic.

A group G is said to be an *fsg* group (and have *finitely satisfiable generics*) if f-generic, strongly f-generic, and generic types all coincide.

The existence of a strongly f-generic type is equivalent to definable amenability of G [12].

Heirs and coheirs are frequently encountered throughout model theory. Poizat ([13]) notes that much of the study of stability is concerned with searching for extensions of particular types. In this project they are of particular relevance in trying to understand the complete types over $\mathbb{Q}_p$, each of which turns out to have both a unique heir and a unique coheir over the space of complete types over $^*\mathbb{Q}_p$, due to the definability each complete type over $\mathbb{Q}_p$.

Definition 2.1.11 (Heir and coheir)**.** [13] Consider $M \preceq {^*M}$, a complete 1-type p over M, and an extension q of p over $^*M$ (so p is the set of formulas in q which only have parameters from M). It is said that q is an *heir* of p if for every formula $\alpha(x, \bar{y}, \bar{z})$, every $\bar{a} \in M$, and every $\bar{b} \in {^*M}$, if $\alpha(x, \bar{a}, \bar{b}) \in q$ then there exists $\bar{b}' \in M$ such that $\alpha(x, \bar{a}, \bar{b}') \in p$.

A type q over an elementary extension $^*M$ of M is said to be a *coheir* of its restriction p to M if it is finitely satisfiable in M (if $\alpha(x, \bar{a}) \in q, \bar{a} \in {^*M}$, one can find $b \in M$ such that $^*M \models \alpha(b, \bar{a})$.

Definable Skolem functions are possessed by certain theories, including that of $\mathbb{Q}_p$. The property of possessing such functions will prove useful in demonstrating that a particular structure is an elementary substructure of an elementary extension $^*\mathbb{Q}_p$ of $\mathbb{Q}_p$ later in this paper. There is a related notion without an assumption of definability, but it is not relevant to this paper's interests.

Definition 2.1.12 (Definable Skolem functions)**.** Consider a theory T and model M of T. T has *definable Skolem functions* if, for every formula $\alpha(x, y)$ with no parameters, there exists some $\emptyset$-definable function f such that if $b \in M$ and $\{m \in M : M \models \alpha(m, b)\}$ is nonempty, then $f(b) \in \{m \in M : M \models \alpha(m, b)\}$.

Filters and *ultrafilters* on a given set are collections of subsets of that set satisfying particular axioms. The Stone topological space, which will be introduced and studied in detail later, consists of ultrafilters.

Definition 2.1.13 (Filter)**.** A *filter* on a partially-ordered set G is a subset F of G satisfying the following axioms:

(i) $F \neq \emptyset$,

(ii) $F \neq G$,

(iii) For all $x, y \in F$ there exists some $z \in F$ such that $z \leq x$ and $z \leq y$, and

(iv) For each $x \in F, y \in G$, $x \leq y \Rightarrow y \in F$.

An *ultrafilter* is a maximal filter.

Semigroup ideals of certain structures will be used in proofs in later chapters. Note that this is a different notion from that typically used for rings. In particular, left ideals of a certain structure will have a recurring role in the study of the Ellis group.

Definition 2.1.14. Semigroup ideal[5] Given a semigroup G and $A \subseteq G$:

- A is a *right ideal* of G if $\{as | a \in A, s \in G\}$ is a subset of A;
- A is a *left ideal* of G if $\{sa | s \in G, a \in A\}$ is a subset of A; and
- A is an ideal of G if both of the previous statements are true.

Chapter 3

The p-adic numbers

3.1 Introduction to the p-adic numbers

The p-adic number system (with p prime) extends the rational numbers via the introduction of a valuation from which is defined an absolute value operator that describes numbers in terms of their divisibility by powers of p. The field $\mathbb{Q}_p$ of p-adic numbers is the completion of the set of rational numbers with respect to the so-called p-adic absolute value. Hensel is credited with first describing them in 1897. The p-adic number system has numerous applications in number theory and remains a topic of general mathematical interest.

Definition 3.1.1 (P-adic valuation)**.** [6] For prime p, the *p-adic valuation* of an integer n is a map $v_p : \mathbb{Z} - \{0\} \to \mathbb{R}$, such that $v_p(n)$ is the unique positive integer satisfying $n = p^{v_p(n)}n'$ where p does not divide n'.
By convention, $v_p(0) = \infty$, which is motivated by the fact that one can divide 0 by p indefinitely with 0 as the answer, since by definition the p-adic valuation is a measure of a number's divisibility by p. This valuation may also be extended to $\mathbb{Q}$:
If $\frac{a}{b}$ is a rational number in its simplest form ($gcd(a, b) = 1$) then

$$v_p(\frac{a}{b}) = \begin{cases} v_p(a) & \text{if p divides a} \\ -v_p(b) & \text{if p divides b} \\ 0 & \text{if p divides neither a nor b} \end{cases}$$

Definition 3.1.2 (p-adic absolute value)**.** [6] The *p-adic absolute value* of $x \in \mathbb{Q}$ is defined as follows:

$$|x|_p = \begin{cases} p^{-v_p(x)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

It is thought that Hensel's interest in the p-adic numbers was born of his observation of similarities between the ring of integers, $\mathbb{Z}$, with field of fractions $\mathbb{Q}$, and the ring of polynomials $\mathbb{C}[x]$ with complex coefficients, whose field of fractions consists of rational functions over $\mathbb{C}$ [6]. In $\mathbb{Z}$, one can write any element (integer) as a product of primes (multiplied by -1 in the case of negative integers), and there is an analogous factorization of any polynomial $f(x) = a(x - a_1)...(x - a_n)$ for $f(x) \in \mathbb{C}[x]$. This gives a correspondence between prime numbers and monomials $(x - a) \in C[x]$.
Using Taylor series [6], for $a \in \mathbb{C}$ one can express a polynomial as a sum $\Sigma_{i=0}^{n} a_i (x - a)^i$. Similarly, a positive integer may be written in base p for prime $p : q = \Sigma_{i=0}^{n} a_i p^i$ with $0 \leq a_i \leq p - 1$. For instance, the number 37 may be written as $2 \times 7^0 + 5 \times 7^1 = 52_7$ (in 7-ary). Of course, the best-known example of base-p arithmetic is the binary system, which plays a significant role in computer science.
Naturally, one would also want to obtain similar expansions for rational numbers, for which one should look to the rational functions over $\mathbb{C}$. Here the Laurent expansion, for $a \in \mathbb{C}$, is used [6]: $h(x) = \frac{f(x)}{g(x)} = \Sigma_{i \geq n_0} a_i (x - a)^i$. Here the starting point of the expansion may be a negative integer. For positive rational numbers, the corresponding process [6] involves the use of long division to obtain an expression of the desired form. For instance, in base 3, $\frac{28}{13} = a + 3b, 0 \leq a \leq 2$ (where a should be selected such that the remainder is divisible by 3). Here $a + 3b = 1 + 3(\frac{5}{13})$, so the first term in the expansion is 1. Next, $\frac{5}{13} = 2 + 3(\frac{-7}{13})$ so the second term is 2, and the subsequent terms are $2, 1, 1, 2, 1, 1, ...$ (so the expression becomes periodic). Using this algorithm one obtains an expression of the form $\Sigma_{i \geq n} a_i p^i$, which in this case is $1 + 2p + 2p^2 + p^3 + p^4 + 2p^5 + p^6 + p^7 + 2p^8 + ...$ If one wishes to verify that this expression is correct, simply multiply this expansion by the expansion of the denominator and ensure that the outcome is the expansion of the numerator [6]. In this example, the expansion of 13 is

$1 + p + p^2$, so one would compute $(1 + p + p^2)(1 + 2p + 2p^2 + p^3 + p^4 + 2p^5 + p^6 + p^7 + 2p^8 + ...)$. The desired outcome is for only $1 + p^3$, the 3-adic expansion of 28, to remain after simplification. This is accomplished by noting that all other powers of p will *vanish*, in the sense that they will indefinitely rise to higher powers of p as they are multiplied and so will approach 0 (which can be seen using the fact that the p-adic valuations of these terms grow extremely large since they are high powers of p, and their p-adic absolute values decrease correspondingly). For instance, if one obtains a term $6p^4$, this would be written $2 \times 3p^4 = 2p^5$, and once this is added to other p^5 terms, the coefficient of the p^5 term would again be divisible by p and thus would rise further. In this manner, only the expansion of the numerator 28 remains static.
Note that all rational numbers have periodic (or eventually periodic) p-adic expansions [6]. The matter of expansions for negative rational numbers is resolved by using an expansion for -1 and multiplying the power series of the two terms [6].

One can equip $\mathbb{Q}$ with a metric $d : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ such that $d(x, y) = |x - y|_p$. This metric induces on $\mathbb{Q}$ the so-called *p-adic topology* which will be used frequently in later chapters. The map $d(x, y)$ satisfies all three of the usual axioms for metrics: $d(x, y) = 0$ iff $x = y$, $d(x, y) = d(y, x)$, and a stronger version of the usual triangle inequality, $d(x, z) \leq max(d(x, y), d(y, z))$ (known as the *strong triangle inequality* [6]). This has some interesting geometric consequences, such as the fact that all triangles in such a space (called an *ultrametric* space) are isosceles [6].

With all this in mind, $\mathbb{Q}_p$ can finally be viewed in its entirety. For each prime p, each $\mathbb{Q}_p$ is a distinct field (but most results on p-adic fields hold true for arbitrary p). Each $\mathbb{Q}_p$ is formally defined as the *completion* of $\mathbb{Q}$ with respect to the p-adic metric (recall that a field is *complete* with respect to a metric if every Cauchy sequence in that field has a limit)[6]. By the definition of completion [6], $\mathbb{Q}$ is a dense subset of each $\mathbb{Q}_p$, and each $\mathbb{Q}_p$ is equipped with an absolute value $||_p$ which induces upon $\mathbb{Q}$ the p-adic absolute value defined earlier. One may also characterize $\mathbb{Q}_p$ as the set of numbers with unique expansions $\Sigma_{i=m}^{\infty} a_i p^i$, where $m \in \mathbb{Z}$ may be negative. Elements of $\mathbb{Q}$ have expansions that are either periodic or eventually periodic, whereas the expansions of elements of $\mathbb{Q}_p - \mathbb{Q}$ are not periodic and so cannot be expressed in a convenient manner [6].

The topology on $\mathbb{Q}_p$, as with other metric spaces, is characterized by open and closed balls [6]. Open balls are sets of the form $B(a,r) = \{x \in \mathbb{Q}_p | d(x,a) < r\}$ and closed balls are sets $\bar{B}(a,r) = \{x \in \mathbb{Q}_p | d(x,a) \leq r\}$ [6]. The open balls are both open and closed, on account of the p-adic absolute value being *non-archimedian*, meaning that it satisfies the strong triangle inequality mentioned earlier. One may also characterize an *archimedian* absolute value as one such that for any x and y with $x \neq 0$, there exists some $n \in \mathbb{Z}_+$ such that $|nx| > |y|$, whereas a *non-archimedian* absolute value is one for which $sup\{|n|\ n \in \mathbb{Z}\} = 1$ [6].

Lemma 3.1.3. [6] The open balls in $\mathbb{Q}_p$ (with respect to the topology induced by the metric) are both open and closed.

Proof. [6] Consider an open ball $B(a,r)$. The fact that $B(a,r)$ is open is trivial, so it remains to show that this is a closed set.
Consider a boundary point x of $B(a,r)$, and the ball $B(x,r_1)$ with $0 < r_1 \leq r$. By the definition of a boundary point, it follows that $B(a,r)$ and $B(x,r_1)$ have nonempty intersection (at least one point b lies in both balls).
Hence $|b-a|_p < r$ and $|b-x|_p < r_1$. Next, one applies the non-archimedian property: $|x-a|_p \leq max\{|x-b|_p, |b-a|_p\}$
(recall that $d(x,y) \leq max\{d(x,z), d(z,y)\}$ for any $x,y,z \in \mathbb{Q}_p$ since the absolute value is non-archimedian[6])
$< max\{r, r_1\}$
$= r$ (since $r \geq r_1$ by assumption).
This proves that x lies within $B(a,r)$. But x was an arbitrary boundary point of $B(a,r)$, so $B(a,r)$ contains all its boundary points and thus is closed. □

Among other properties, the ultrametric grants the ability to regard any element of a ball (open or closed) in the space as its centre [14], a great boon for subsequent topological investigations in this paper.

The valuation ring of the p-adic valuation $\{x \in \mathbb{Q}_p || x|_p \leq 1\}$ is known as the ring of *p-adic integers*, $\mathbb{Z}_p$ [6]. This ring may also be defined as the completion of $\mathbb{Z}$ with respect to the p-adic absolute value [6]. In terms of the p-adic expansion $\Sigma_{i=m}^{\infty} a_i p^i$, $a_i = 0$ for all $i < 0$ in the case of a p-adic integer (its expansion contains no negative powers of p)[6]. Topologically, it is the closed ball of radius 1 centered at 0. Elements within $\mathbb{Z}_p$ all have non-negative p-adic valuations, and it can also be observed that $\mathbb{Q}_p = \mathbb{Z}_p(\frac{1}{p})$ [6] (for any $x \in \mathbb{Q}_p$, one can find $n \geq 0$ such that $p^n x \in \mathbb{Z}_p$).

Hensel's Lemma describes a key property of the p-adic numbers. This result allows one to identify roots of polynomials that lie in $\mathbb{Z}_p$, by introducing a condition on the polynomial's formal derivative and using this in conjunction with an approximate root. Many incarnations of the lemma exist, but the following version has been selected on the basis of simplicity.

Theorem 3.1.4 (Hensel's Lemma). *[3] Suppose $f(X) \in \mathbb{Z}_p[X]$, and there exists $a \in \mathbb{Z}_p$ such that $f(a) \equiv 0 \ (mod p)$ and $f'(a) \not\equiv 0 \ (mod p)$.*
Then there exists a unique $b \in \mathbb{Z}_p$ such that $b \equiv a \ (mod p)$ and $f(b) = 0$.
(Here $x \equiv y \ (mod p)$ iff $|x - y|_p < 1$).

The applications of Hensel's Lemma are numerous [6]. These include the ability to identify roots of unity in $\mathbb{Q}_p$ (using the polynomial $f(x) = x^n - 1$) or to determine the squares of $\mathbb{Q}_p$ (it turns out that any square $x \in \mathbb{Q}_p$ is of the form $x = p^{2n}y^2$ with $n \in \mathbb{Z}$ and y an invertible element of $\mathbb{Z}_p$ [6]).

3.2 Model-theoretic Insights

Much effort has been invested in the study of the p-adic numbers from a model-theoretic perspective. In particular, quantifier elimination for a p-adically closed field K may be achieved via the introduction of additional predicates. This is necessary since the pure language of valued fields (the language $L_{VF} = \{+, -, \times, ^{-1}, 0, 1, \mathcal{O}\}$ where $\mathcal{O}$ is a unary predicate for the valuation ring) does not admit quantifier elimination, as was demonstrated in [9]. Recall that the valuation ring consists of elements with valuation ≥ 0. In the case of $\mathbb{Q}_p$, the valuation ring is $\mathbb{Z}_p$. The valuation map is given by $v(x) = -log_p(|x|_p)$, using the earlier definition of the p-adic absolute value.

Macintyre's quantifier elimination is intended for p-adically closed fields, a class of valued fields to which $\mathbb{Q}_p$ belongs. The axioms defining this class of field are not relevant to the following discussions, but it is worth noting that any model of the theory of p-adically closed fields is elementarily equivalent to $\mathbb{Q}_p$.

In order to achieve quantifier elimination for p-adically closed fields, predicates for the n-th powers are added to the language of valued fields. This gives rise to a language $L_{QE} = L_{VF} \bigcup \{P_n : n = 2, 3, ...\}$ where each P_n is a predicate for n-th powers.

Theorem 3.2.1 (Quantifier elimination for $\mathbb{Q}_p$). *[9] The theory $T = th(\mathbb{Q}_p)$ admits quantifier elimination in the language $L_{QE} = \{+, -, \times, ^{-1}, 0, 1, \mathcal{O}, P_n(n = 2, 3, ...)\}$.*

It is worth noting that the inclusion of the predicate $\mathcal{O}$ is not necessary with regards to $\mathbb{Q}_p$ once predicates for n-th powers have been added (particularly the predicate P_2), as one may use Hensel's Lemma to define $\mathbb{Z}_p$ thus [2] :

$\mathbb{Z}_p = \{y \in \mathbb{Q}_p | \exists t \in \mathbb{Q}_p, t^2 = 1 + p^3 y^4\}$.

An important consequence of quantifier elimination is the classification of definable sets in the structure.

Theorem 3.2.2 (Definability in p-adically closed fields). *[9] Suppose $M \models th(\mathbb{Q}_p)$ and that α is an L_{QE}-formula with free variables $v_1, ..., v_n$.*

Then, by quantifier elimination, α is equivalent to a boolean combination of formulas defining sets of the following forms:

(i) $\{(m_1, ..., m_n) \in M^n | g(m_1, ..., m_n) \neq 0\}, g \in M[x_1, ..., x_n]$
(where $M[x_1, ..., x_n]$ denotes the ring of polynomials in $x_1, ..., x_n$ with coefficients in M).

(ii) $\{(m_1, ..., m_n) \in M^n | M \models \mathcal{O}(h(m_1, ..., m_n)) \wedge g_2(m_1, ..., m_n) \neq 0\}, h = \frac{g1}{g2}, g_i \in M[x_1, ..., x_n]$.

(iii) $\{(m_1, ..., m_n) \in M^n | M \models P_k(h(m_1, ..., m_n)) \wedge g_2(m_1, ..., m_n) \neq 0\}, h = \frac{g1}{g2}, g_i \in M[x_1, ..., x_n]$.

Proof. To see why this classification holds true, one can consider formulas that may be formed in the language $L_{QE} = \{+, -, \times, ^{-1}, 0, 1, \mathcal{O}, P_n(n = 2, 3, ...)\}$. Also apply the convention that, in the event of a zero denominator, the value of a rational function is regarded as zero. Atomic formulas in this language would be of the forms $h(x_1, ..., x_n) = 0, P_n(h(x_1, ..., x_n))$, (stating that h is an n-th power), or $\mathcal{O}(h(x_1, ..., x_n))$, for rational functions h (note that considering polynomials does not suffice on account of the $^{-1}$ symbol in the language - it is necessary to consider rational functions).
The first kind of atomic formula resembles the negation of a formula defining a type-i set in 3.2.2, with an obvious difference in the fact that the atomic formula defines a zero set of a rational function while type-i sets are complements of zero sets of standard polynomials. The set $\{(m_1, ..., m_n) \in M^n | h(m_1, ..., m_n) = 0\}$ (with $h = \frac{g_1}{g_2}$) given by such an atomic formula may also be expressed as the union of the complements of two type-i sets, $\{(m_1, ..., m_n) \in M^n | g_1(m_1, ..., m_n) = 0\} \bigcup \{(m_1, ..., m_n) \in M^n | g_2(m_1, ..., m_n) = 0\}$. This is defined by a boolean combination of type-i formulas as required.
The second and third kinds of atomic formula are almost the same as those formulas defining type-ii and type-iii sets respectively. In 3.2.2, there is an added condition that the function g_2 in the denominator be nonzero, to avoid a zero denominator in keeping with mathematical convention.
Hereafter one need only consider negation, conjunction and disjunction of these formulas due to the absence of quantifiers, and so it is clear any formula is equivalent to a boolean combination of formulas as described in the theorem.

□

In [9] further descriptions of the sets defined by these formulas are provided:
Type-*i* formulas define open subsets of M^n, and their complements are zero-sets of polynomials.
When discussing the sets defined by type-*ii* formulas, and their complements, one should bear in mind that the valuation ring is both open and closed. A type-*ii* formula defines the intersection of the sets $\{(m_1, ..., m_n) \in M^n | M \models \mathcal{O}(h(m_1, ..., m_n))\}$ and $\{(m_1, ..., m_n) \in M^n | g_2(m_1, ..., m_n) \neq 0\}$. The first set is not necessarily open since h is not continuous when the denominator g_2 is zero (due to the convention that $(0)^{-1} = 0$), but by intersecting this set with the set of points such that g_2 is nonzero, one obtains an open set.
The complement of a type-*ii* set is the union of the complement of the set from the valuation ring, which is open, and the complement of the set $\{(m_1, ..., m_n) \in M^n | g_2(m_1, ..., m_n) \neq 0\}$, which is a (closed) polynomial zero-set.
According to [9], a set defined by a type-*iii* formulas is the union of an open set and a closed set. The open set is given by
$\{(m_1, ..., m_n) \in M^n | M \models P_k(h(m_1, ..., m_n))$
$\wedge g_1(m_1, ..., m_n) \neq 0 \wedge g_2(m_1, ..., m_n) \neq 0\}, h = \frac{g1}{g2}, g_i \in M[x_1, ..., x_n]$, and the closed set is
$\{(m_1, ..., m_n) \in M^n | M \models g_1(m_1, ..., m_n) = 0\}$.
The open set accounts for nonzero k-th powers, while the closed set is included since 0 is trivially a k-th power for all k.
However, it may be more correct to describe the second set in this union as $\{(m_1, ..., m_n) \in M^n | M \models g_1(m_1, ..., m_n) = 0 \wedge g_2(m_1, ..., m_n) \neq 0\}$, since a tuple $\bar{m}$ such that $g_2(\bar{m}) = 0$ and $g_1(\bar{m}) = 0$ would lie in the second set of the earlier description, but would not be contained in the type-*iii* set. It is thus necessary to include the condition that $g_2(m_1, ..., m_n) \neq 0$.
In [9] the complement of a set defined by a type-*iii* formula is described as the union of $\{(m_1, ..., m_n) \in M^n | M \models g_1(m_1, ..., m_n) \neq 0\}$, the closed set $\{(m_1, ..., m_n) \in M^n | M \models g_1(m_1, ..., m_n) = 0 \vee M \models g_2(m_1, ..., m_n) = 0\}$, and the open set $\{(m_1, ..., m_n) \in M^n | M \models g_1(m_1, ..., m_n) \neq 0 \wedge M \models g_2(m_1, ..., m_n) \neq 0 \wedge P_k(h(m_1, ..., m_n))\}$.
However, a direct approach yields a somewhat different solution and so there may have been a typographical error in the original source. The

type-*iii* set is a union of two sets, so the application of DeMorgan's Law would be appropriate in finding its complement: $(\bigcup A_i)^c = \bigcap(A_i^c)$ (the complement of a union of sets is given by the intersection of the individual complements of those sets). The complement of a type-*iii* set would be the intersection of the complement of the set $\{(m_1, ..., m_n) \in M^n | M \models P_k(h(m_1, ..., m_n)) \wedge g_1(m_1, ..., m_n) \neq 0 \wedge g_2(m_1, ..., m_n) \neq 0\}, h = \frac{g1}{g2}, g_i \in M[x_1, ..., x_n]$, and the complement of the set
$\{(m_1, ..., m_n) \in M^n | M \models g_1(m_1, ..., m_n) = 0 \wedge g_2(m_1, ..., m_n) \neq 0\}$.
The complement of the second set is $\{(m_1, ..., m_n) \in M^n | M \models g_1(m_1, ..., m_n) \neq 0\} \bigcup \{(m_1, ..., m_n) \in M^n | M \models g_2(m_1, ..., m_n) = 0\}$.
The complement of the first set, which describes nontrivial k-th powers, would be the union of those sets in which at least one of the conditions fails, so $g_1(m_1, ..., m_n) = 0, g_2(m_1, ..., m_n) = 0$, or $\neg P_k(h(m_1, ..., m_n))$. One can view this as the union $\{(m_1, .., m_n) \in M^n | g_1(m_1, ..., m_n) = 0 \vee g_2(m_1, ..., m_n) = 0\} \bigcup \{(m_1, ..., m_n) \in M^n | M \models g_1(m_1, ..., m_n) \neq 0 \wedge M \models g_2(m_1, ..., m_n) \neq 0 \wedge \neg P_k(h(m_1, ..., m_n))\}$. The first set in this union is included in Macintyre's description, but the second differs since here it is specified that h is not a k-th power.
Hence the complement of a set described by a type-*iii* formula would be formally described as
$(\{(m_1, ..., m_n) \in M^n | M \models g_1(m_1, ..., m_n) \neq 0\} \bigcup \{(m_1, ..., m_n) \in M^n | M \models g_2(m_1, ..., m_n) = 0\}) \bigcap (\{(m_1, ...m_n) \in M^n | g_1(m_1, ..., m_n) = 0 \vee g_2(m_1, ..., m_n) = 0\} \bigcup \{(m_1, ..., m_n) \in M^n | M \models g_1(m_1, ..., m_n) \neq 0 \wedge M \models g_2(m_1, ..., m_n) \neq 0 \wedge \neg P_k(h(m_1, ..., m_n))\})$.
Hensel's Lemma, in conjunction with Macintyre's quantifier elimination result, has been used to classify the complete 1-types over $\mathbb{Q}_p$ as a structure in the language $\{+, \times\}$ (call this M) [12].

Lemma 3.2.3 (Complete 1-types over $\mathbb{Q}_p$)**.** [12] Let M denote $\mathbb{Q}_p$ as a structure in the language $\{+, \times\}$. The complete 1-types over M are as follows:

(i) The type of each $a \in \mathbb{Q}_p$ over M (these are obviously realized types),

(ii) the types $p_{a,C}$ for each coset C of $(^*M)^\times)^0$ (the connected component of the multiplicative group of an elementary extension of M) and each $a \in \mathbb{Q}_p$, stating that x is infinitesimally close to a (so $v(x - a) > n$ $\forall n \in \mathbb{N}$) and $(x - a) \in C$, and

(iii) the types $p_{\infty,C}$ stating that $x \in C$ and $v(x) < n \; \forall n \in \mathbb{Z}$.

Here the valuation v is the map defined in ${}^*M$ by the same formula as that defining the valuation map v_p in M.

It is also worth noting that, in practical terms, there is actually little difference between the languages $\{+, \times\}$, $\{+, -, \times, 0, 1\}$, L_{VF}, and L_{QE}. The types over each of these languages are the same, since the symbols added to create each successive language are actually definable in the preceding simpler language, albeit in some cases quantifiers would have to be used, hence the creation of L_{QE}. This observation will be of value in later chapters when studying types over $\mathbb{Q}_p$ and elementary extensions thereof.

The space of complete n-types over $\mathbb{Q}_p$ is a topic of great interest since, for a group G definable in M (such as $SL(2, \mathbb{Q}_p)$), the space of complete types over M concentrating on G (i.e. containing the formula $x \in G$) may be viewed as a *definable analogue* of the Stone space βG [12]. In particular, the possibility of first-countability of this space is worth investigating in the interest of resolving some topological nuances that will be discussed in later chapters.

Chapter 4

A topological perspective

4.1 General Topology

The *net* is a topological construct analogous to the sequence. Sequences are functions with domain $\mathbb{N}$ and codomain a topological space. Note the following properties for a map f between topological spaces:

(i) $f : X \to Y$ is *topologically continuous* if for any open $V \subseteq Y$, $f^{-1}(V) = \{x \in X | f(x) \in V\}$ is an open subset of X.

(ii) $f : X \to Y$ is *sequentially continuous* if, for any $x \in X$ and sequence $(x_i) \to x$ in X, $f(x_i)$ converges to $f(x)$.

Topological continuity of a map automatically grants sequential continuity, but the reverse implication is not true in general since not all topological spaces are *first-countable*. The significance of this condition shall be explained using an example later in this section.

Definition 4.1.1 (First-countable)**.** [16] A space X is first-countable if each element x in that space has a *countable neighbourhood base*. This means that x has a countable collection (U_i) of neighbourhoods in X such that, given an arbitrary neighbourhood N of x, at least one $U_i \subseteq N$.

It is noted [16] that the notion of a sequence is effectively an ordering of certain elements of the topological space X using the positive integers. Hence, in order to retain equivalence of the two properties above without an assumption of first-countability, perhaps one should still order collections of elements in X using an ordered set as a domain. This warrants the use of *directed sets*.

Definition 4.1.2. Directed set [16] A *directed set* D is a set with a relation $\leq$ satisfying the following axioms:

(i) $d \leq d \; \forall d \in D$;

(ii) $d_1 \leq d_2$ and $d_2 \leq d_3 \Rightarrow d_1 \leq d_3 \; \forall d_1, d_2, d_3 \in D$; and

(iii) for any $d_1, d_2 \in D$ there is some $d_3 \in D$ such that $d_1 \leq d_3$ and $d_2 \leq d_3$.

The concept of the *net* allows for the equivalence of topological and sequential continuity in a broader topological context by replacing sequences, defined over countable linearly-ordered sets, with a similar construct defined over directed sets.

Definition 4.1.3 (Net)**.** [16] A *net* in X is a function $P : D \to X$, where D is a directed set.

Definition 4.1.4 (Limits of Nets)**.** [16] If (x_α) is a net from a directed set A into X, and $Y \subseteq X$, (x_α) is *eventually in* Y if there exists some $\gamma \in A$ such that for every $\beta \in A$ with $\beta \geq \gamma$, the point x_β lies in Y.
If (x_α) is a net in a topological space X and $x \in X$, the net has *limit* x (or $lim(x_\alpha) = x$ iff (x_α) is eventually in U for every neighbourhood U of x).

To develop a concrete understanding of the importance of first-countability, consider the possibility of a map $f : X \to Y$ with a domain X that is not first-countable. Then at least one $x \in X$ lacks a countable neighbourhood base, which poses an indexing problem in the case of a sequence converging to x.

Suppose one assumed the sequential continuity of a map f and wished to demonstrate topological continuity via contradiction, so began by assuming the existence of at least one neighbourhood U of $f(x)$ in Y whose preimage $T = f^{-1}(U)$ is not a neighbourhood of x in X (and so $f(N) \nsubseteq U$ for any neighbourhood N of x). If first-countability of X was assumed, there would be a countable neighbourhood base $V_1 \supseteq V_2 \supseteq ...$ of x such that no $f(V_i)$ is contained in U. One could then select a sequence $(x_i)_{i \in \mathbb{N}}$ with $x_i \in V_i$ for each i such that $f(x_i) \notin U$, so $(x_i) \to x$ but $f(x_i) \not\to f(x)$,

contradicting the assumption of sequential continuity. However, this reasoning fails in the absence of first-countability since there is no longer a countable neighbourhood base, and so the construction of the sequence (x_i) fails.
In the absence of first-countability, one can prove that an analogue of sequential continuity, using nets in place of sequences, is equivalent to topological continuity. One uses nets by treating the set of open neighbourhoods of x as a directed set with respect to reverse containment (recall that a net must have a directed set as its domain, so instead of indexing by the components of the neighbourhood base as in the first-countable case, an alternative indexing set has been constructed). Thereafter the proof is conducted in a very similar manner to that of the other case - since no open neighbourhood N of x is contained in T, one can extract an element x_α from each open neighbourhood N_α such that each x_α misses T. Then $f(x_\alpha) \notin U$, as before, with the eventual conclusion that $f(x_\alpha) \not\to f(x)$.

The equivalence of the two notions of continuity, while interesting, is not of particular concern to the aims of this paper. The following consequence of first-countability is of greater relevance to this paper's setting, and so will see application in later sections of this paper. The proof provided below is based on that of a similar result in [15].

Lemma 4.1.5. If X is first-countable and $A \subseteq X$, the following sets are equal:

(i) $\{x \in X | \exists \text{ sequence } (x_n) \in A \text{ such that } (x_n) \to x\}$

(ii) $\{x \in X | \exists \text{ net } (x_\alpha) \in A \text{ such that } (x_\alpha) \to x\}$

Proof.

- $(i) \subseteq (ii)$
  Consider $x \in X$ such that one can find a sequence (x_n) in A converging to x. The sequence is a specialized case of the net, so this sequence is also a net in A converging to x.

- $(ii) \subseteq (i)$
  Suppose there exists some $x \in X$ such that one can find a net $(x_\alpha) \in A$ converging to x. Using the definition of limits of nets, this means that (x_α) is eventually in U for every neighbourhood U of X, so for each neighbourhood U there exists some γ such that $x_\beta \in U$ for every

$\beta > \gamma$.
X is first-countable so one can find a countable neighbourhood base $V_1 \supseteq V_2 \supseteq ...$ of x in X.
For each n, there exists some α_n such that $x_{\alpha_n} \in V_n$. From this one obtains a sequence $(x_{\alpha_n})_{n\in\mathbb{N}}$.
Each neighbourhood N of x contains V_m for some $m \in \mathbb{N}$. Since the neighbourhoods $(V_n)_{n\in\mathbb{N}}$ are nested, this means that the neighbourhood N would contain all V_i for $i \geq m$. In turn, for $n \geq m$, each x_{α_n} also lies within N. This is the case for every neighbourhood N of x (with different values m for different neighbourhoods), and so the sequence converges to x.

□

4.2 The Ellis group and the definable generalized Bohr compactification

Before it is possible to continue to topics such as the Ellis group, some basic concepts from topological dynamics should first be observed. Ellis is credited by Glasner [5] as a progenitor of the study of topological dynamics. The former developed the algebraic theory of flows, which in turn led to the development of group compactifications from this perspective. The latter is a construct of particular interest to the aims of this paper, and will be studied in detail later, but it is first useful to understand more basic structures from flow theory.

4.2.1 The general setting

Definition 4.2.2 (Flow)**.** [12] A *flow* (G, X) consists of a Hausdorff (but not necessarily compact) topological group G that acts continuously on a Hausdorff topological space X - there is a continuous map $f : G \times X \to X$ such that $f(id_G, x) = x$ for $x \in X$, and $f(gh, x) = f(g, f(h, x))$ for $g, h \in G$ and $x \in X$.[5]

One can regard a group G as a topological group by equipping it with the discrete topology [12].

Points $x, y \in X$ are said to be *proximal* with respect to the flow (G, X) if there exists some net $(g_\alpha) \in G$ and $z \in X$ such that both $(g_\alpha x)$ and $(g_\alpha y)$ converge to z. The flow itself is proximal if every pair of elements of X is proximal.

A *subflow* (G, Y) of (G, X) consists of the action group G together with a closed, G-invariant, non-empty subspace Y of X (a G-invariant subspace Y is a subspace Y of X such that $G \cdot Y = Y$, where $\cdot$ denotes the group action). [5]

A flow is *minimal* if it has no proper subflows.

The flows of the *enveloping semigroup*, an important construct which will be introduced later in this chapter, have an interesting property that is worth noting. It is demonstrated in [12] that minimal subflows of the enveloping semigroup coincide with minimal left ideals of the enveloping semigroup (which will be defined later).

Definition 4.2.3 (Homomorphism of flows)**.** [5] Consider two flows (G, X) and (G, Y). A continuous map $f : X \to Y$ is a *flow homomorphism* if, for every $g \in G$, it is the case that $f(g \cdot x) = g \cdot f(x)$ for each $x \in X$.
A *flow automorphism* is a flow homomorphism $f : X \to X$ which is invertible.

The *group compactification* is the basic notion from which the Bohr compactification and its variants are derived.

Definition 4.2.4 (Group compactification)**.** [5] A *group compactification* of a topological group G consists of a compact Hausdorff group C along with a homomorphism from G into C with dense image.

Definition 4.2.5 (Group extension)**.** [8] Consider a homomorphism of minimal flows, $f : (G, Y) \to (G, X)$. The map f is a *group extension* if there exists some compact Hausdorff group K satisfying the following:

(i) K acts faithfully on Y on the right (id_K is the only $k \in K$ such that $yk = y \; \forall y \in Y$) ;

(ii) K acts continuously on the right on Y (the map sending (y, k) to $y \cdot k$ is continuous);

(iii) $f^{-1}(f(y)) = yK \; \forall y \in Y$; and

(iv) $(g \cdot y)k = g \cdot (yk) \; \forall y \in Y, k \in K$, and $g \in G$.

Each $k \in K$ corresponds to an automorphism of the flow (G, Y). Although the formal definition of group extension refers to the homomorphism f, one may also refer to (G, Y, K) as the group extension of (G, X).
Each flow has a unique *universal* group extension, but this universal property will not be used directly and so is not described here. [5]

Related to the concept of a group extension is that of a *compactification flow*, which provides a correspondence of sorts between proximal flows and arbitrary flows.

Definition 4.2.6 (Compactification flow)**.** [8] A flow (G, X) is a *compactification flow* of G if the group of all automorphisms of (G, X) is a compact

Hausdorff topological group (with respect to the so-called *topology of pointwise convergence*, in which convergence of a sequence of elements is equivalent to the pointwise convergence of those elements when construed as functions). In this case, the group of automorphisms is referred to as a *generalized compactification* of G.
A flow (G, X) is a compactification flow iff (G, X, K), where K denotes the aforementioned group of automorphisms of (G, X) is a group extension of some proximal flow. [8]

The *Bohr compactification* is a universal group compactification of an arbitrary topological group. The study of the Bohr compactification and its definable analogue will constitute a nontrivial part of this paper.

Definition 4.2.7 (Bohr Compactification)**.** [5] A *Bohr compactification* of a topological group G consists of a compact Hausdorff topological group C and homomorphism $f : G \to C$ with the following universal property: given another compactification $h : G \to D$, one can find a unique continuous surjective homomorphism $g : C \to D$ such that $h = gf$.

A further variant of the Bohr compactification, known as the *generalized Bohr compactification*, will form the basis of much of the study later in this paper. The following is the formal definition of this construct, but is seldom used and is only included here for interest's sake. Most of the study of this group makes use of the quotient formalization, which can only be introduced after some additional topological discussion. However, it should be emphasized that the version used in later chapters is a *characterization*, whereas Glasner's definition as below is the canonical version.

Definition 4.2.8 (Generalized Bohr Compactification)**.** [5] Let (G, X) denote a minimal proximal flow and (G, Y, K) denote the universal group extension of (G, X). Then (G, Y) is the *universal compactification flow* of G and K is the *generalized Bohr compactification* of G.

Note that the Bohr compactification and generalized Bohr compactification of G coincide if G possesses certain properties [8].

It is possible to construe the generalized Bohr compactification as a quotient of the Ellis group. This result is related to the focal question of this paper (which is concerned specifically with the definable context). Before this characterization is presented, it is of course necessary to define the Ellis group, which in turn necessitates knowledge of the *enveloping semigroup* of a flow.

Definition 4.2.9 (Enveloping semigroup)**.** [12] Given a flow (G, X), the *enveloping semigroup* $E(X)$ is the closure in the space X^X (with the product topology) of the set of maps $\pi_g : X \to X, \pi_g(x) = gx$, equipped with composition.
Proximality of (G, X) is equivalent to the statement $\forall x, y \in X \exists f \in E(X)$ such that $f(x) = f(y)$[12].
$E(X)$ is a compact Hausdroff space, and there is an action $g \cdot f = \pi_g \circ f$ of G on $E(X)$ by homeomorphisms.[12]

With regards to the flow $(G, E(X))$, Ellis investigated the correspondence between minimal subflows and ideals of $E(X)$ (note that these are *semigroup ideals*). It was found that minimal closed left ideals I of $E(X)$ coincide with minimal subflows[12]. Although the result is stated in [12] without proof, a full proof is provided below for the interested reader.

Theorem 4.2.10. *Minimal closed left ideals of $E(X)$ coincide with minimal subflows of the flow $(G, E(X))$.*

Proof. $\subseteq$: Suppose $(G, E_1(X))$ is a minimal subflow of $(G, E(X))$. Then $E_1(X)$ is G-invariant. The aim is to show that $E(X)E_1(X) = \{a \circ b | a \in E(X), b \in E_1(X)\} \subseteq E_1(X)$.
This gives rise to two cases:
1 : If $a = \pi_{g_i}$ for some $g_i \in G$, then $a \circ b = g_i \cdot b$ for each $b \in E_1(X)$. By G-invariance, $g_i \cdot E_1(X) \subseteq E_1(X)$ since $G \cdot E_1(X) = E_1(X)$.
2 : If a is the limit point of some net (π_{g_i}), then $a \circ E_1(X) = lim(\pi_{g_i}) \circ E_1(X) = \{lim(\pi_{g_i} \circ b) | b \in E_1(X)\}$. This is a limit of a net that lies in $E_1(X)$, on account of the fact that $G \cdot E_1(X) = E_1(X)$ by G-invariance, so each $\pi_{g_i} \circ b$ lies within $E_1(X)$. But $E_1(X)$ is closed (by definition) and Hausdorff (since it is a subspace of a Hausdorff space), so this limit necessarily lies in $E_1(X)$. Consequently $a \circ E_1(X) \subseteq E_1(X)$.
It follows that $E(X)E_1(X) \subseteq E_1(X)$.

$\supseteq$: Suppose $E_1(X)$ is a closed left ideal of $E(X)$, so $E(X)E_1(X) \subseteq E_1(X)$.

Consider the subset $E_G(X) = \{\pi_g | g \in G\}$ of $E(X)$.
$E_G(X)E_1(X) = \{\pi_g | g \in G\} \circ E_1(X) = G \cdot E_1(X) \subseteq E_1(X)$. Conversely, $E_1(X) = \pi_{id_G} \circ E_1(X) \subseteq G \cdot E_1(X)$. Thus $G \cdot E_1(X) = E_1(X)$ so $E_1(X)$ is G-invariant.

Hereafter one can observe that the condition of minimality is trivial - any minimal closed left ideal will be minimal as a flow, and vice-versa. This concludes the proof. □

Recall that an idempotent element u is such that $u \cdot u = u$. Denoting the set of idempotents of $E(X)$ by J, and given a minimal closed left ideal I, $I \bigcap J \neq \emptyset$ [12] (every minimal left ideal I contains at least one idempotent of $E(X)$).

Definition 4.2.11 (Ellis Group)**.** [12] Let I denote a minimal closed left ideal of the enveloping semigroup $E(X)$, and let J denote the set of idempotents of $E(X)$. For $u \in I \bigcap J, (u \circ I, \circ)$ is called an *Ellis group*. Here $\circ$ denotes composition of maps.
Based on the definition, one can observe that multiple Ellis groups exist. However, Ellis groups (for various u and I) are isomorphic, so the isomorphism class is referred to as the Ellis group attached to the flow (G, X).

The quotient form of the generalized Bohr compactification makes use of the *Stone topology* and *τ-topology*, which will be discussed in the following sections.

4.2.12 The definable setting

The concepts presented in this section thus far have been for use in a general setting, in which definability has not been assumed. However, the results discussed in this paper are intended for a specific setting in which groups and maps are definable in the model. Much of the theory for this definable setting has, in fact, been developed with *external definability*, as is seen in [8]. However, types over $\mathbb{Q}_p$ are known to be definable, so external definability need not be used here.
It is worth stating the definable versions of notions previously presented so that one can see how, if at all, definability affects them.

Definition 4.2.13 (Definability of maps)**.** [8] Consider a complete theory T in a language L, and a structure M such that $M \models T$. Let G be a group definable with parameters from M, and let C be a compact group.
A map $f : G \to C$ is said to be *definable* if, for any two disjoint closed $C_1, C_2 \subseteq C$, there exists a definable set $G_0 \subseteq G$ such that $f^{-1}(C_1)$ and $f^{-1}(C_2)$ are separated by G_0 (this means that G_0 contains one of these sets and completely omits the other).

Definition 4.2.14 (Definable compactification)**.** [8] A *definable compactification* of a group G is a group compactification of G with a definable homomorphism.

There is also a notion of definability for group actions that should be remembered.

Definition 4.2.15 (Definable action)**.** [8] A *definable action* of a definable group G on a compact space X is an action of G on X by homeomorphisms such that for each $x \in X$ the map $f_x(g) : g \mapsto gx$ is definable.

Definition 4.2.16 (Definable flow)**.** [8] A *definable flow* (G, X) is a flow in which, for each $x \in X$, the map $f_x(g) : g \mapsto gx$ is definable. This is the same as stating that the group action is definable.

Definition 4.2.17 (Definable group extension)**.** [8] A definable group extension is a group extension as previously defined, with the added condition that the flows (G, X) and (G, Y) are definable.

A definable analogue to the Bohr compactification, unsurprisingly known as the *definable Bohr compactification*, is also used.

Definition 4.2.18 (Definable Bohr Compactification)**.** The *definable Bohr compactification* of a group G is a group compactification of G with a definable map f with dense image such that, given another group compactification consisting of a compact Hausdorff group D and definable map $g : G \to D$ with dense image, there exists a unique continuous surjection $h : C \to D$ such that $g = hf$.

4.2.19 The Stone topology

The *Stone-Čech compactification* of a topological group is a well-known object in the context of general topology, and is also of relevance to the study of type spaces in this project.

Definition 4.2.20 (Stone-Čech compactification)**.** A *Stone-Čech compactification* βG of a topological group G is a compact Hausdorff space of which G is a dense subset, and with a universal property - every map from G into a compact Hausdorff space X can be extended to a unique map from βG to X.
The points of this space are ultrafilters on G. There is a strong similarity between these ultrafilters and complete types, as will be discussed in greater detail.
A base for the topology on this space can be described as follows [11]:
For $A \subseteq G$, the Stone set $\hat{A} \subseteq \beta G$ of A is given by
$\hat{A} = \{p \in \beta G | A \in p\}$ and a basis for open sets in βG is given by $B = \{\hat{A} | A \subseteq G\}$.

Definition 4.2.21 (Space of complete n-types)**.** [10] The space of complete n-types over a structure M is denoted $S_n(M)$. This space is equipped with a topology in which the following sets are open:
$\{p(x) \in S_n(M) : \phi(x) \in p(x)\}$ for each formula ϕ with parameters in M.
Since any complete type p contains exactly one of ϕ and $\neg\phi$, these sets are also closed since one can simply substitute any formula ϕ with its negation [10].
With this topology, the space is now a *boolean space* (a totally disconnected, compact topological space). It is sometimes known as the *Stone space of n-types over M*, but this paper will refrain from using this terminology to prevent confusion between βG and the space of complete types which, although related, are not the same space.

One may view the space $S_G(M)$ of complete types over M concentrating on G as a *definable analogue* to the space βG [12]. When $M = \mathbb{Q}_p$ and $G = SL(2, \mathbb{Q}_p)$, the space $S_G(M)$ is a subspace of $S_4(M)$.
The n-types over M may also be characterized using an equivalence relation. Consider the following relation on formulas ϕ and γ with parameters in M and free variables $x_1, ..., x_n$:

$\phi \equiv \gamma \Leftrightarrow M \models \forall x_1, ..., x_n(\phi(x_1, ..., x_n) \leftrightarrow \gamma(x_1, ..., x_n))$.
The set of equivalence classes of formulas under this relation forms a boolean algebra whose ultrafilters correspond to complete n-types over M.

4.2.22 τ-topology

The most obvious application of nets in the context of this project may be seen in the definition of the τ-topology on the Ellis group [5]. This topology is induced by the *τ-closure operator*.

One can now study the flow $(G(M), S_G(M))$, where $S_G(M)$ is equipped with a semigroup operation $*$ defined for types p and q as follows: $p * q = tp(ab/M)$ where a is a realization of p and b realizes the unique heir of q over (M, a). One can also view the action of G in terms of this semigroup action - for $g \in G$ and $p \in S_G(M)$, $g \cdot p = tp(g/M) * p$.

The definition provided for the τ-topology here shall apply to the definable context. Fix $M = \mathbb{Q}_p$ and $G = SL(2, \mathbb{Q}_p)$. Also note that, on account of the definability of types over $\mathbb{Q}_p$, $E(S_G(M)) = S_G(M)$ [12], so the Ellis group is simply $(u * \mathfrak{M}, *)$ where $\mathfrak{M}$ denotes a minimal closed left ideal of $S_G(M)$ and u an idempotent therein, and the $*$-operation is as above. Hereafter, $u * \mathfrak{M}$ shall be denoted $u\mathfrak{M}$.

For $A \subseteq u\mathfrak{M}$, $cl_\tau(A) = (u \circ A) \bigcap (u\mathfrak{M})$ with $u \circ A = \{x \in S_G(M) | \exists \text{nets } (x_i) \in A, (t_i) \in G$ such that $lim(t_i) = u$ and $lim(t_i x_i) = x\}$.

Here $t_i x_i$ is computed as $tp(t_i/M) * x_i$. Also note that the limits are taken with respect to the Stone topology.

Lemma 4.2.23. The operator cl_τ is a closure operator.

Proof. [5] Let $A, B \subseteq u\mathfrak{M}$.

(i) $A \subseteq cl_\tau(A)$
$A = uA$ by idempotence of u, and $uA \subseteq u \circ A = (u \circ A) \bigcap u\mathfrak{M} \subseteq cl_\tau(A)$.

(ii) $A \subseteq B \Rightarrow cl_\tau(A) \subseteq cl_\tau(B)$
This is an immediate consequence of the definition: $cl_\tau(A) = (u \circ A) \bigcap u\mathfrak{M} \subseteq (u \circ B) \bigcap u\mathfrak{M} = cl_\tau(B)$.

(iii) $cl_\tau(cl_\tau(A)) = cl_\tau(A)$
$u \circ ((u \circ A) \bigcap u\mathfrak{M}) \bigcap u\mathfrak{M} \subseteq u \circ (u \circ A) \bigcap u\mathfrak{M} = ((u \circ u) \circ A) \bigcap u\mathfrak{M} = (u \circ A) \bigcap u\mathfrak{M} = cl_\tau(A)$, so $cl_\tau(cl_\tau(A)) \subseteq cl_\tau(A)$.
Thereafter $cl_\tau(cl_\tau(A)) \supseteq cl_\tau(A)$ by (i), so it follows that $cl_\tau(cl_\tau(A)) = cl_\tau(A)$.

$\square$

With this newfound understanding of the Stone and τ-topologies, it is possible to comprehend and appreciate the expression of the definable generalized Bohr compactification as a quotient of the Ellis group. This characterization is preferable to the formal definition provided earlier due to analysis of the Ellis group in [12] which makes it more likely that a simplification may be achieved using this version. For the interested reader, it is demonstrated in [8] that the following characterization is in fact equivalent to the formal definition.

Definition 4.2.24 (definable generalized Bohr compactification as quotient)**.** [8] Consider the Ellis group $u\mathfrak{M}$ and let $H = H(u\mathfrak{M}) = \bigcap\{cl_\tau V | V \in N\}$ where N the collection of all neighbourhoods (with respect to the τ-topology) of u in $u\mathfrak{M}$, known as the *neighbourhood filter*.
Then H is a normal subgroup of $u\mathfrak{M}$, and the *definable generalized Bohr compactification* of G is simply the quotient $u\mathfrak{M}/H$.

4.3 A topological review of $SL(2, \mathbb{Q}_p)$

The possibility of replacing nets with sequences in the context of the τ-topology and the $\circ$ operation in its definition is definitely worthy of investigation. In doing so, one would have to demonstrate first-countability of the space of complete types. To investigate this further, one should temporarily depart the realm of type spaces and return to the study of $SL(2, \mathbb{Q}_p)$ for a closer look with the newly-defined topologies in mind. The following criterion for first-countability should prove to be of value.

Lemma 4.3.1. (i) A metrizable space is first-countable.

(ii) A first-countable space need not be metrizable. A counterexample is the Sorgenfrey line (the topology on $\mathbb{R}$ generated by the basis of half-open intervals with real endpoints) which is first-countable but not metrizable.

Note that one can consider $SL(2, \mathbb{Q}_p)$ as a subset of $\mathbb{Q}_p^4$, so a countable neighbourhood base for the latter would suffice for the interests of this paper. Of course, $\mathbb{Q}_p$ is equipped with the p-adic metric, so it is first-countable. This passes to products, so both $\mathbb{Q}_p$ and $\mathbb{Q}_p^n$ have countable neighbourhood bases with respect to the topologies induced by their p-adic metrics. It is worth studying the process used to determine the existence of a countable neighbourhood base for $\mathbb{Q}_p$, since some of the techniques and observations may prove useful in later studies of the type spaces. The proof provided here is essentially the same as that found in [14], with the addition of a few minor details.

Theorem 4.3.2. *[14] Each element of $\mathbb{Q}_p$ has a countable neighbourhood base.*

Proof. Consider an arbitrary open ball $B(a, r) = \{x \in \mathbb{Q}_p | \; |a - x|_p < r\}$. Note that $r = p^{-s}$ for some $s \in \mathbb{Z}$, since $|a - x|_p = p^{-v_p(a-x)}$ and $-v_p(a - x)$ is an integer by definition unless $a = x$.
Since $a \in \mathbb{Q}_p$, there exists $m \in \mathbb{Z}$ such that $a_m \neq 0$ and $a = \sum_{n=m}^{\infty} a_n p^n$. This follows from the definition of the p-adic expansion.
Let $a_0 = \sum_{n=m}^{s} a_n p^n$. One can see that $a_0 \in \mathbb{Q}$ since it is a finite sum of rational numbers.
Since $|a - a_0|_p < p^{-s}$ it follows that $a_0 \in B(a, p^{-s})$.
It is evident that $B(a, r) = B(a_0, p^{-s})$ using the ultrametric property, since

$r = p^{-s}$.
Thus the set of radii of balls in $\mathbb{Q}_p$ is countably large, and although uncountably many centres are possible, one can resolve this by using the fact that $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, so any ball with a centre in $\mathbb{Q}_p$ is equal to some ball with a centre in $\mathbb{Q}$, which is countable. Thus the set of open balls in $\mathbb{Q}_p$ is countable. $\square$

Each element of $\mathbb{R}^n$ also has a countable neighbourhood base since any metric space with a dense countable subset (in this case, $\mathbb{Q}^n$) will have a countable base. Of course, one could also have used this reasoning to deduce first-countability of $\mathbb{Q}_p$ immediately, but a full proof of the latter result was included in aid of further investigations.

Chapter 5

Revision of established knowledge

This project is by no means a pioneering effort. Numerous studies have been done both on general p-adic model theory and the study of group compactifications. The content of the following chapter is primarily inspired by [12], but also draws heavily from [5], as well as sporadically making use of information from numerous other sources. This section will focus on the progress made in [4] and [12] with regards to studies of $SL(2, \mathbb{R})$ and $SL(2, \mathbb{Q}_p)$ respectively.

5.1 The $SL(2, \mathbb{R})$ case

The action of the group $G = SL(2, \mathbb{R})$ (the group of 2×2 matrices with real entries and determinant 1) on its type space has already been studied ([4]), with numerous facts regarding that structure's Ellis group having been discovered. In particular, Newelski's question of whether, in the case of G being definable in an NIP theory, $G(^*M)/G(^*M)^{00}$ would coincide with the group $(u\mathfrak{M}, *)$ for a minimal, closed, G-invariant subset $\mathfrak{M}$ of the space $S_G(M)$, with $u \in \mathfrak{M}$ idempotent, was refuted in the case where $M = \mathbb{R}$ and $G = SL(2, \mathbb{R})$ by the finding that, in this particular setting, $(u\mathfrak{M}, *)$ is nontrivial (in fact, it consists of exactly 2 elements).

An earlier finding of interest by Pillay is that Newelski's claim holds true for fsg groups (i.e. groups with finitely satisfiable generics) in the case of

NIP theories. In the case of a saturated real closed field K, $G(K) = SL(2, K)$ is a simple group (except for a finite centre) and so $G(K) = G(K)^{00}$. The following material may all be found in [4].

$G = SL(2, \mathbb{R})$ is studied via the semigroup $(S_G(M), *)$, where the operation $*$ is defined for types $p, q \in S_G(M)$ by $p * q = tp(ab/M)$, where $b, a \in G(^*M)$ realize the type q and the unique coheir of p over (M, b) respectively. This is the same semigroup structure that was defined in the previous chapter. Structurally, the group centre of G consists only of I and $-I$, (where I denotes the identity 2×2 matrix). $SL(2, \mathbb{R})$ is semialgebraic (a subset of a real closed field defined by a finite boolean combination of polynomial equations and inequalities), and so many of its structural properties hold for $SL(2, K)$ with abitrary real closed fields K.
Several spaces associated with $SL(2, \mathbb{R})$ are worth noting. The first of these is $H(\mathbb{R})$, the subgroup of $SL(2, \mathbb{R})$ consisting of upper triangular 2×2 matrices $\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}$
with a and a^{-1} positive real numbers, b a real number, and determinant 1. The torus $T(\mathbb{R}) = \{ \begin{bmatrix} x & y^{-1} \\ y & x \end{bmatrix}$ with $x^2 + y^2 = 1\}$ is such that only the 2×2 identity matrix I lies in the intersection of $H(\mathbb{R})$ and $T(\mathbb{R})$. In addition, any element of $SL(2, \mathbb{R})$ has a unique factorization ht where $h \in H(\mathbb{R})$ and $t \in T(\mathbb{R})$.
Before the nontriviality of the Ellis group $u\mathfrak{M}$ can be demonstrated, it is of course necessary to describe $u\mathfrak{M}$. This necessitates the construction of an appropriate ideal $\mathfrak{M}$ as well as an idempotent u therein. One defines the type u as $tp(th/\mathbb{R})$ where $h \in H$ realizes the type
$p_0 = tp(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} /\mathbb{R})$ (with a infinite ($a > \mathbb{R}$) and b infinite over a, i.e. $b > dcl(\{a\})$) and $t \in T$ realizes the unique coheir of the type $q_0 = tp(\begin{bmatrix} x & y^{-1} \\ y & x \end{bmatrix} /\mathbb{R})$ (with y a positive infinitesimal and x the positive root of the equation $1 - y^2$) over $\mathbb{R}, h$. The type q_0 describes an element of T that is positive and infinitesimally close to the identity.
The choice for $\mathfrak{M}$ is $cl(G(\mathbb{R})u)$, where the closure is taken with respect to the Stone topology. One can show that this is a minimal $G(\mathbb{R})$-subflow,

and that u is an idempotent in $cl(G(\mathbb{R})u)$. Thus an explicit description of the Ellis group of the action of $SL(2,\mathbb{R})$ on $S_G(M)$ has been obtained.
Finally, it can be demonstrated that this Ellis group consists of only two elements by making use of bijections between $\mathfrak{M}$ and a certain type space over $\mathbb{R}$, $S_{V,na}(\mathbb{R})$. This is the space of complete nonalgebraic types over $\mathbb{R}$ concentrating on the quotient $V = G(\mathbb{R})/H(\mathbb{R})$. Using these bijections it suffices to show that $uS_{V,na}(\mathbb{R})$ consists of only two elements, and thus so does $u\mathfrak{M}$.

5.2 The $SL(2,\mathbb{Q}_p)$ case

A substantial amount of material on the study of $G = SL(2,\mathbb{Q}_p)$ from the perspective of topological dynamics can be found in ([12]). As with other similar ventures, the intention was to create analogues of notions from stable group theory in an unstable setting. The flow $(G(M), S_G(M))$, where $M = (\mathbb{Q}_p, +, \times)$, is the primary object of interest. Critical findings include the description of the Ellis group of the action of $SL(2,\mathbb{Q}_p)$ on its type space as a semidirect product of $B(\mathbb{Z}_p)$ (the Borel group of upper-triangular 2×2 matrices with determinant 1 and entries in $\mathbb{Z}_p$) and $\hat{\mathbb{Z}}$ (the inverse limit of quotients of $\mathbb{Z}$ by subgroups of $\mathbb{Z}$ of finite index, known as the *profinite completion* of $\mathbb{Z}$), as well as the identification of $cl(\mathcal{I} * \mathcal{J})$ as a minimal subflow of the flow $(G(M), S_G(M))$ (where $\mathcal{I}$ is the unique minimal subflow of $SL(2,\mathbb{Z}_p)$ on its type space and $\mathcal{J}$ is a minimal subflow of the action of $B(\mathbb{Q}_p)$ on its type space). The following content may be found in [12].

The description of the flow $(G(M), S_G(M))$ as a semidirect product involves the use of the *Iwasawa decomposition*, a method of expressing matrices as a product of an orthogonal matrix and an upper-triangular matrix. Such a decomposition proves useful here since the components of the decomposition of $SL(2,\mathbb{Q}_p)$ are definably amenable, whereas $SL(2,\mathbb{Q}_p)$ itself is not definably amenable in M. Since the groups in the Iwasawa decomposition are definably amenable, the existence of a strongly f-generic type in each is guaranteed [12], and every left translate of each of these types is invariant with respect to the group of automorphisms of ${}^*M$ which fix M. This property will be used in later computations involving these types.
The Iwasawa decomposition of $SL(2,\mathbb{Q}_p)$ is a product of the maximal compact subgroup $K = SL(2,\mathbb{Z}_p)$ and the Borel subgroup $B(\mathbb{Q}_p)$. In turn, the Borel group is in fact the semidirect product of the multiplicative and additive groups of $\mathbb{Q}_p$.

$B({}^*M)$, the Borel subgroup of ${}^*M$, is a semidirect product of $({}^*M, +)$ and $({}^*M^\times, \times)$. A particular global f-generic type of $B({}^*M)$ is then described as follows: Letting $C_0 = ({}^*M^\times)^0$, the types p_{0,C_0} and p_{∞,C_0} (as defined in the earlier chapter on the p-adic numbers) are global f-generics of $({}^*M^\times, \times)$ and $({}^*M, +)$ respectively. Given realizations a of p_{0,c_0} and b of p_{∞,C_0} such that $tp(a/M, b)$ is finitely satisfiable in M, one then considers the type $\bar{p}_0 = tp(ab/^*M)$ and its restriction p_0 to M. The latter will later prove useful in describing the Ellis group, particularly in the construction of an idempotent element of a minimal subflow of $(G(M), S_G(M))$.

The $B(^*M)$-orbit $\bar{\mathcal{J}}$ of the type $\bar{p}_0$ is a minimal $B(^*M)$-subflow of $S_B(^*M)$. Also, its restriction $\mathcal{J}$, the $B(M)$-orbit of p_0, is a minimal subflow of $S_B(M)$. In fact, it is the Ellis group of $(B(M), S_B(M))$, and is also one of the components of the minimal subflow of $(G(M), S_G(M))$ that will soon be described.
The unique minimal subflow of the flow $(G(M), S_G(M))$ can be written as $cl(\mathcal{I} * \mathcal{J})$ (with the closure taken with respect to the Stone topology on the type space), where $\mathcal{I}$ is the set of generic types in $S_K(M)$ and is also the unique minimal subflow of $S_K(M)$ where $K = SL(2, \mathbb{Z}_p)$. Fixing a generic type $q_0 \in S_K(M)$ concentrating on K^0 , the element $q_0 * p_0$ is also an idempotent in $cl(\mathcal{I} * \mathcal{J})$.
Having constructed a minimal ideal $cl(\mathcal{I} * \mathcal{J})$ and idempotent $q_0 * p_0$, one can simply define the Ellis group $u\mathfrak{M}$ using $u = q_0 * p_0$ and $\mathfrak{M} = cl(\mathcal{I} * \mathcal{J})$. Furthermore, one can express this Ellis group as the semidirect product $B(\mathbb{Z}_p) \ltimes \mathcal{J}$, though the motivation behind this alternative characterization is far from trivial and not sufficiently relevant to this paper's aims to warrant inclusion.
Finally, it is noted in [12] that it is still unclear whether this Ellis group coincides with the definable generalized Bohr compactification of $SL(2, \mathbb{Q}_p)$. It is hoped that some of the results presented in the following chapter will prove to be of use in this endeavour.

Chapter 6

Investigations

In addition to providing an account of existing knowledge regarding the Ellis group and the p-adic numbers, attempts have also been made in this paper to further the knowledge of these areas from a model-theoretic perspective.

6.1 The Ellis group

In trying to resolve the question of whether the definable generalized Bohr compactification of $SL(2, \mathbb{Q}_p)$ is trivial, the following result was proposed and proven. However, unbeknownst to the writer at the time, Glasner [5] had in fact already demonstrated a proof of a more general result. Nonetheless the proof is included for the sake of the interested reader.

Theorem 6.1.1. *Given a group G definable in a structure M, where $M = \mathbb{Q}_p$ or $M = \mathbb{R}$, and a minimal ideal $\mathfrak{M}$ of the space of complete types over M concentrating on G with idempotent u, the subgroup $H(u\mathfrak{M})$ (as defined earlier) is trivial iff the τ-topology on $u\mathfrak{M}$ is Hausdorff.*

Proof.

- $\Rightarrow$
  - Suppose $H(u\mathfrak{M})$ is known to be trivial i.e. it consists only of the element u. Note that the τ-topology is known to be $T1$ [8] (so any two distinct elements can be separated from one another).
  - Assume that the space is not Hausdorff, so there exists at least one pair of distinct elements $a, b \in u\mathfrak{M}$ such that any neighbourhoods of a and b have a nonempty intersection.

- Claim: $b^{-1}a$ and u are such that, $\forall$ neighbourhoods N_1, N_2 of $b^{-1}a$ and u respectively, $N_1 \bigcap N_2 \neq \emptyset$.
  Proof of claim: By [5], multiplication by an element of $u\mathfrak{M}$ (on the right or left) is a homeomorphism. Due to this invariance, $b^{-1}a$ and u (obtained via multiplication by b^{-1} of a and b) will also have the property that any neighbourhoods of these elements will have a nonempty intersection (the product of b^{-1} and the element(s) in the original intersection will be in the intersection of $b^{-1}a$ and u).
- Claim: $b^{-1}a \in cl_\tau(N) \forall$ neighbourhoods N of u
  Proof of claim: Recall that $a \in cl_\tau(X)$ if every open set in the space containing a also contains at least one $x \in X$. Neighbourhoods of u and $b^{-1}a$ always have a nonempty intersection, so one may consider an open set containing $b^{-1}a$. By definition, this is a neighbourhood of $b^{-1}a$ since it contains an open set that contains $b^{-1}a$ (itself). Thus this open set must have a nonempty intersection with any neighbourhood N of u and so $b^{-1}a \in cl_\tau(N) \forall$ neighbourhoods N of u.
- However, this contradicts the assumption that H is trivial since H must now contain at least one element other than u.
- Thus triviality of H implies that the τ-topology on $u\mathfrak{M}$ is Hausdorff. Note that $b^{-1}a \neq u$ because otherwise $a = b$.

• $\Leftarrow$

- Suppose that the τ-topology is Hausdorff, so that given any two elements $a, b \in u\mathfrak{M}$, one may find disjoint neighbourhoods N_a of a and N_b of b in $u\mathfrak{M}$.
- Recall the definition of $H(u\mathfrak{M})$:
  $H(u\mathfrak{M}) = \bigcap\{cl_\tau(V)|V$ is a τ-neighbourhood of u in $u\mathfrak{M}\}$.
  Suppose (in search of a contradiction) that H contains at least one element besides u (call it v). By the definition of H, $u, v \in cl_\tau(V) \forall$ neighbourhoods V of u in $u\mathfrak{M}$ (so any open sets containing v will also contain at least one element of V).
- As per the Hausdorff assumption, one may find disjoint neighbourhoods N_u and N_v of these elements in $u\mathfrak{M}$. Consider an

open subset S_v of N_v containing v. S_v is also a neighbourhood of v.

– Thus S_v must contain at least one $k \in N_u$, which would contradict the Hausdorff assumption since $N_v \bigcap N_u = \emptyset$ and $S_v \subseteq N_v$.

– Thus H must be trivial.

$\square$

6.2 The type space of $\mathbb{Q}_p$

Another matter of interest is the possibility of first-countability of the space of complete types over $\mathbb{Q}_p$ concentrating on $SL(2, \mathbb{Q}_p)$. Should the type space possess this property, one would be able to replace nets with sequences in several results, as well as in the definition of the τ-topology provided earlier, which could potentially lead to a better understanding of this topology.
The similarities between $\mathbb{R}$ and $\mathbb{Q}_p$ have already been noted in earlier discussions. In particular, the density of $\mathbb{Q}$ in $\mathbb{R}$ has interesting topological consequences in $\mathbb{R}$, and since $\mathbb{Q}$ is also a dense subset of $\mathbb{Q}_p$ it is perhaps worth investigating the type space of $\mathbb{R}$ in the hopes of observations that would lead to new discoveries on the type space of $\mathbb{Q}_p$. However, it is first necessary to familiarize oneself with the *algebraic closure*, which will be necessary in any attempts to demonstrate first-countability.

The *algebraic closure* will be frequently encountered in the discussions that follow. It is important to note that two different notions of algebraic closure exist, one each from field theory and model theory, and they do not necessarily coincide.
In model theory, given a structure M, subset S and $a \in M$, $a \in acl(S)$ if a is an element of some finite S-definable set. The set S is *algebraically closed* if $S = acl(S)$. Note that the intersection of all finite S-definable sets containing a is the unique minimal finite S-definable set containing a. This set consists precisely of all the realizations of $tp(a/S)$. Its elements are called *conjugates* of a over S.
In field theory, the algebraic closure of a field K is the set consisting of all elements that are algebraic over K in the sense that they are solutions to nonzero polynomials with coefficients in K.
It is known that these algebraic closures coincide in $\mathbb{R}$, and the following result indicates the relationship between the two definitions in the p-adic setting.

Theorem 6.2.1. *[9] [1] Suppose $M \models Th(\mathbb{Q}_p)$ and $X \subseteq M$. Then the model-theoretic algebraic closure of X in M is simply the field-theoretic algebraic closure of $\mathbb{Q}(X)$ in M.*

It is best to begin with 1-types and revisit the classification of complete 1-

types over $M = \mathbb{Q}_p$ in the language $\{+, \times\}$ described in the earlier chapter on the p-adic numbers, since it turns out that the space of these types is first-countable.

First-countability of 1*-types over* $\mathbb{Q}_p$. First, note that $({}^*M^\times)^0 = \bigcap P_n({}^*M^\times)$, and P_n denotes $P_n({}^*M^\times)$ [12].
The types $tp(a/M)$ for each $a \in \mathbb{Q}_p$ are realized in $\mathbb{Q}_p$. Each realized type has a singleton neighbourhood base, which is countably large.
The types $p_{a,C}$ state that $(x - a) \in C$ and $v(x - a) > n$ $\forall n \in \mathbb{N}$, where C is a coset of $\bigcap P_n$. There are countably many formulas $v(x - a) > n$, and one can consider the formulas $(x - a) \in C_n$ for cosets C_n of P_n to determine the correct coset of $\bigcap P_n$.
Finally, the types $p_{\infty,C}$ state that x lies within a coset of $\bigcap P_n$ and $v(x) < n$ $\forall n \in \mathbb{Z}$. Here, there are countably many formulas $v(x) > n$, and once again one can identify in which coset of P_n the element lies and thereafter determine the corresponding coset of $\bigcap P_n$. $\square$

In trying to demonstrate first-countability of $S_n(\mathbb{Q}_p)$, one can consider a type $p \in S_n(\mathbb{Q}_p)$ and try to show that this type is fully implied by formulas that define open sets. If this can be accomplished, one could perhaps then prove that only countably many open sets are needed, using arguments similar to those employed in demonstrating that $\mathbb{Q}_p$ has only countably many open balls. The outcome would be a countable neighbourhood base for any element of the type space.

A possible method of simplifying this investigation is to consider n-types as 1-types. For example, since $SL(2, \mathbb{Q}_p)$ is of particular interest, one could construe a 4-type p as a succession of 1-types over different structures: $tp((a, b, c, d)/M) = tp(a/M) \bigcup tp(b/M, a) \bigcup tp(c/M, a, b) \bigcup tp(d/M, a, b, c)$.
Of course, for such an argument to work, one would require dense subsets of field extensions of $\mathbb{Q}_p$. To this end, it is worth examining the case of $\mathbb{Q}_p(a_1, ..., a_n)$, the field generated by finitely many (a_n) in an elementary extension ${}^*\mathbb{Q}_p$ of $\mathbb{Q}_p$.
First, an attempt can be made to apply this logic to $\mathbb{R}$, assuming that $\mathbb{Q}(a_1, ..., a_n)$ is dense in $\mathbb{R}(a_1, ..., a_n)$ (where $a_1, ..., a_n$ may lie in an elementary extension of $\mathbb{R}$).

A method to show first-countability of $S_G(\mathbb{R})$ with an assumption of the density of $\mathbb{Q}(a_1, ..., a_n)$ in $\mathbb{R}(a_1, ..., a_n)$

Proof Attempt 6.2.2. Let q be a complete type over $\mathbb{R}$ concentrating on $SL(2,\mathbb{R})$ that is realized by some (a,b,c,d)(it is easiest to regard this realization as a 4-tuple). This realization may lie in an elementary extension of $\mathbb{R}$.

Case 1: The transcendence degree of (a,b,c,d) over $\mathbb{R}$ is 4.

Then a, b, c and d satisfy no polynomial equations with real coefficients. The type q is implied by the cut realized by a over $\mathbb{R}$, together with the cut realized by b over $\mathbb{R}(a)$, the cut realized by c over $\mathbb{R}(a,b)$, and the cut realized by d over $\mathbb{R}(a,b,c)$ (it is necessary to add each element as a generator to preserve the relationship between them. For instance, given a transcendental α over $\mathbb{R}$, $tp(\alpha, \alpha+1/\mathbb{R})$ would not encode the same information as $tp(\alpha/\mathbb{R})$ and $tp(\alpha+1/\mathbb{R})$ as the latter two types would each have a range of realizations in an elementary extension of $\mathbb{R}$ and selecting an arbitrary realization for each may no longer preserve the difference of 1 between the terms).

The cut realized by a over $\mathbb{R}$ is fully determined by the collection of open intervals in $^*\mathbb{R}$ which are definable over $\mathbb{R}$ and contain a. This cut can be described using formulas $r_1 < x < r_2$ with $r_1, r_2 \in \mathbb{R}$ (these are intervals in $\mathbb{R}$) which are realized when $x = a$. Using the density of $\mathbb{Q}$ in $\mathbb{R}$, it should suffice to only consider intervals with rational endpoints, which could be described using only countably many formulas. In the event that a is infinitesimally close to an irrational number, the introduction of an additional parameter may be necessary since there may not be any rational numbers between a and this irrational number.

Thereafter one would consider the cut realized by b over $\mathbb{R}(a)$, which is described using formulas $r'_1 < y < r'_2$ with $r'_1, r'_2 \in \mathbb{R}(a)$. Now one can simply replace these endpoints with endpoints $q'_1, q'_2 \in \mathbb{Q}(a)$, which is dense in $\mathbb{R}(a)$. On account of the previous step, the number of possible endpoints, even with a possible additional parameter, is still countable.

Repeating this for the cuts realized by c over $\mathbb{R}(a,b)$ and d over $\mathbb{R}(a,b,c)$ yields the same result - these cuts can be described using only countably many formulas, and so first-countability is satisfied.

Case 2: The transcendence degree of (a,b,c) over $\mathbb{R}$ is 3 and $d \in acl(\mathbb{R} \cup \{a,b,c\})$.

Note that since $\mathbb{R}$ has an ordering, its definable closure and algebraic closure coincide. There exists some function f definable over $\mathbb{R}$ such that $f(a,b,c) = d$. Then q is implied by the cut realized by a over $\mathbb{R}$, together with the cut realized by b over $\mathbb{R}(a)$, the cut realized by c over $\mathbb{R}(a,b)$, and $d = f(a,b,c)$.

The cases for the cuts realized by a, b and c receive the same treatment as in case 1, and the function f is of little consequence since it is definable in $\mathbb{R}$ and so only uses finitely many parameters (in a first-order logic setting).

Case 3: The transcendence degree of (a,b) over $\mathbb{R}$ is 2 and $c, d \in acl(\mathbb{R} \cup \{a,b\})$.

There exist definable functions f and g such that $f(a,b) = d$ and $g(a,b) = c$.The type q is implied by the cut realized by a over $\mathbb{R}$, together with the cut realized by b over $\mathbb{R}(a)$, $c = f(a,b)$, and $d = g(a,b)$.

Once again, a and b are handled in the same manner as before, and both of the functions are definable in $\mathbb{R}$ using only finitely many parameters, so only countably many formulas are needed.

Case 4: The transcendence degree of a over $\mathbb{R}$ is 1 and $b, c, d \in acl(\mathbb{R} \cup \{a\})$.

There exist definable functions f,g, and h such that $f(a) = b$, $g(a) = c$, and $h(a) = d$. Then q is implied by the cut realized by a in $\mathbb{R}$, together with $b = f(a), c = g(a)$, and $d = h(a)$.

This case is handled in the same way as the prior cases.

Case 5: $a, b, c, d \in acl(\mathbb{R})$.

This case is trivial since the type is realized in $\mathbb{R}$.

Note also that the ordering of elements in this argument is not of concern. For example, if a, b and d had transcendence degree 3 over $\mathbb{R}$ and $c \in acl(\mathbb{R} \bigcup \{a,b,d\})$, one would consider $c = f(a,b,d)$ along with the cuts realized by a over $\mathbb{R}$, b over $(\mathbb{R}, a)$ and d over $(\mathbb{R}, a, b)$ respectively.

Unfortunately, $\mathbb{Q}(a_1, ..., a_n)$ is not dense in $\mathbb{R}(a_1, ..., a_n)$ since there is at least one positive infinitesimal element $\epsilon \in \mathbb{R}(a_1, ..., a_n)$ and so there exists an uncountably large set of pairwise disjoint open intervals in $\mathbb{R}(a_1, ..., a_n)$. Consequently it is impossible for $\mathbb{R}(a_1, ..., a_n)$ to contain a countable dense subset. Since $\mathbb{Q}(a_1, ..., a_n)$ is countable it fails as a candidate for a dense subset of $\mathbb{R}(a_1, ..., a_n)$. This strategy thus fails as an attempt to demonstrate first-countability of $S_n(M)$ for $M = \mathbb{R}$ and $n > 1$.

Investigations into the case when $M = \mathbb{Q}_p$ present complications of an even greater magnitude.
Using Macintyre's classification for definable sets in $\mathbb{Q}_p$, it should be possible to determine whether a 1-type over a field extension of $\mathbb{Q}_p$ is implied by a collection of formulas that define open sets in $\mathbb{Q}_p$. Consider the introduction of an additional parameter $a_{n+1} \in {}^*\mathbb{Q}_p$ and a formula $\alpha(a_1, ..., a_n, x_{n+1})$ over $\mathbb{Q}_p(a_1, ..., a_n)$.
By Macintyre's quantifier elimination ([9]), formulas over $\mathbb{Q}_p$ are combinations of formulas of 3 different forms. Formulas of the first form are complements of zero-sets of polynomials, and since a_{n+1} is transcendental over $\mathbb{Q}_p(a_1, ..., a_n)$ by assumption, it follows that every formula of this form lie in its type. All such formulas define open subsets of $\mathbb{Q}_p$. Of course, if a_{n+1} is non-transcendental, then its type is isolated and so trivially has a countable base.
Formulas of the second form define open sets, so these do not present a problem either.
The final kind of formula defines a union of two sets. The first set is open, and the second is an intersection of an open set and a closed set (the zero-set of a polynomial). The intersection is not of relevance since a_{n+1} cannot be the solution of any polynomial due to the assumption of transcendentality. Thus only the open set remains.
Next, one should consider the complements of sets defined by type-(i), (ii) and (iii) formulas in [9].
The complement of a type-i set is the zero-set of a polynomial. This can be disregarded since the element a_{n+1} is transcendental over $\mathbb{Q}_p(a_1, ..., a_n)$.
The complement of a type-ii set can be expressed as the union of an open set and a (closed) polynomial zero-set. Of course, the polynomial zero-set is resolved for the same reasons as the previous case, so only an open set remains.
Finally, the complement of a type-iii set is described as $(\{(m_1, ..., m_n) \in M^n | M \models g_1(m_1, ..., m_n) \neq 0\} \bigcup \{(m_1, ..., m_n) \in M^n | M \models g_2(m_1, ..., m_n) = 0\}) \bigcap (\{(m_1, ...m_n) \in M^n | g_1(m_1, ..., m_n) = 0 \vee g_2(m_1, ..., m_n) = 0\}$
$\bigcup \{(m_1, ..., m_n) \in M^n | M \models g_1(m_1, ..., m_n) \neq 0 \wedge M \models g_2(m_1, ..., m_n) \neq 0 \wedge \neg P_k(h(m_1, ..., m_n))\})$.
The first set is the union of a type-i set and the zero-set of a polynomial. The element a_{n+1} cannot lie in the latter due to transcendentality, so only

the type-i set needs to be considered. The second set is a union of polynomial zero-sets, but both of these can be ignored since a_{n+1} is transcendental. The third set is an intersection of two type-i sets, which are open, and the complement of a closed set (which is defined by the predicate P_k). The intersection of the type-i sets and the complement of the set given by P_k is thus open (since it is a finite intersection of open sets), as is the intersection of this set with the first type-i set.
It follows that $tp(a_{n+1}/\mathbb{Q}_p(a_1, ..., a_n))$ is implied by formulas which define open sets.

Of course, this is far from sufficient, since although any open set in the topology may be expressed as a union of open balls, one needs to know in which open balls over $\mathbb{Q}_p(a_1, ..., a_n)$ the element a_{n+1} lies as well as certain information regarding the predicates P_n. By [12] it would suffice to know in which open balls over $\mathbb{Q}_p(a_1, ..., a_n)$ the element a_{n+1} lies, as well as, for each $k \in \mathbb{N}$ and $a \in \mathbb{Q}_p(a_1, ..., a_n)$, in which coset of P_k the element $a_{n+1} - a$ lies. However, the fact that $\mathbb{Q}_p(a_1, ..., a_n)$ is uncountable poses a problem in terms of expressing this using only countably many formulas.
In addition to the problem encountered in $\mathbb{R}$ (the lack of density of $\mathbb{Q}(a_1, ..., a_n)$), the p-adic numbers pose the additional problem of the necessity of uncountably many formulas to handle the P_n predicates, with no clear solution.
One may also pose the question of whether the type space possesses *second-countability* (a *second-countable* space T is one in which there exists a countable collection $(U_i) \subseteq T$ of open sets such that any open set $S \subseteq T$ may be written as a union of sets from the collection (U_i)).

Lemma 6.2.3. The set of complete 1-types over $\mathbb{Q}_p$ is not second-countable.

This can be seen immediately since there are uncountably many realized 1-types. Analysis of the 1-types in [12] indicates that the non-realized 1-types may be divided into two categories, and it is worth checking whether or not the non-realized types satisfy second-countability. However, this undertaking would not be of particular relevance to the interests of this paper, and so shall be omitted.

6.3 A return to charted territory

It is worth returning to the τ-topology and the τ-closure operation to identify potential benefits of first-countability of the type-space. The nets described in the definition of the $\circ$ operator used in defining the τ-topology could, of course, be replaced by sequences. In understanding the closure operator, it is worth noting the extremal possibilities.

Lemma 6.3.1. (i) $cl_\tau(\{u\}) = \{u\}$

(ii) $cl_\tau(u\mathfrak{M}) = u\mathfrak{M}$

Proof. (i) $cl_\tau(\{u\}) = \{x \in S_G(M) | \exists$ nets $(x_i) \in \{u\}$ and $(t_i) \in SL(2, \mathbb{Q}_p)$ such that $lim(t_i) = u$ and $lim(t_i x_i) = x\}$. The only possible net in $\{u\}$ is the constant net consisting of u itself, and by continuity of the multiplication of t_i with u, $lim(t_i u) = lim(t_i)u = uu = u$ by idempotence of u.

(ii) $cl_\tau(u\mathfrak{M}) = \{x \in S_G(M) | \exists$ nets $(x_i) \in u\mathfrak{M}$ and $(t_i) \in SL(2, \mathbb{Q}_p)$ such that $lim(t_i) = u$ and $lim(t_i x_i) = x\}$. Noting that any element in $u\mathfrak{M}$ is of the form um for some $m \in \mathfrak{M}$, it follows that $lim(um) = um$. Thus $cl_\tau(u\mathfrak{M}) = u\mathfrak{M}$.

□

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