Interpolatory Bivariate Refinable Functions and Subdivision

by

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Declaration

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Summary

In this thesis, we introduce bivariate refinable functions which are functions that are expressible as linear combinations of the shifts of their own dilation by a factor of a dilation matrix. For the corresponding refinement masks, we define the mask symbols as the Laurent polynomials whose coefficients are the elements of the refinement masks. Of particular interest are interpolatory refinable functions, that is, refinable functions which vanish at all integers except the origin at which they take the value 1. We present simple characterization of the corresponding interpolatory masks in terms of both the delta sequence and the determinant of the dilation matrix. The corresponding interpolatory mask symbols are characterized by some polynomial identities.

An important tool for our work is the Euclidean algorithm, which, in association with the Bezout theorem, helps us to provide an explicit computational algorithm to find the general solution for some polynomial identities. Using the algorithm thus presented, we introduce the general form of an interpolatory mask symbol associated with the dilation matrix $2I$, and the result thus obtained is applied to the mask symbols corresponding to the box splines.

The concepts of interpolatory subdivision schemes and cascade algorithms are also investigated. Subdivision schemes, as usually used to generate curves and surfaces, are interpolatory when the initial data points are preserved at all the steps of the subdivision process. We show that interpolatory subdivision schemes and the cascade algorithm are strongly linked to each other. For a well-chosen dilation matrix and interpolatory refinement mask, we find that the associated cascade algorithm preserves certain properties of the initial functions, allowing us to prove that cascade algorithm convergence implies the existence of a corresponding interpolatory refinable function, which in turn implies subdivision scheme convergence.

Specializing only to the case where the dilation matrix is $M = 2I$, we present some workable methods applied for both non-negative interpolatory masks and interpolatory masks obtained by tensor products in order to investigate the existence of corresponding interpolatory refinable functions. For interpolatory masks constructed to satisfy the sum rules, we provide numerical proofs towards investigating the existence of corresponding interpolatory refinable functions by using the cascade algorithm with an appropriate initial function. Numerical illustrations by means of subdivision graphs are also provided.

Opsomming

In hierdie tesis beskou ons tweeveranderlike verfynbare funksies, oftewel funksies wat uitdrukbaar is as lineêre kombinasies van die skuiwe van hulle eie dilasie deur die faktor van die dilasiematriks. Vir die ooreenkomstige verfyningsmaskers definieer ons die maskersimbole as Laurent polinome waarvan die koëffisiënte die elemente van die verfyningsmaskers is. Van besondere belang is interpolerende verfynbare funksies, dit wil sê verfynbare funksies wat gelyk aan nul is by alle heelgetalle behalwe die oorsprong waar hulle die waarde 1 aanneem. Ons gee 'n eenvoudige karakterisering van die ooreenstemmende interpolerende maskers, beide in terme van die delta ry en die determinant van die dilasiematriks. Die ooreenstemmende interpolerende maskersimbole word gekarakteriseer deur sekere polinoom identiteite.

'n Belangrike stuk gereedskap vir ons werk is die Euklidiese algoritme, wat, tesame met die Bezout stelling, ons help om 'n eksplisiete algoritme te bepaal vir die algemene oplossing van sekere polinoom identiteite. Met behulp van hierdie algoritme stel ons dan bekend die algemene vorm van 'n interpolerende maskersimbool wat ooreenstem met die dilasiematriks $2I$, en die resultaat wat sodanig verkry is word dan toegepas op die maskersimbole wat ooreenstem met 'n sekere klas tweeveranderlike latfunksies ("box splines").

Die konsepte van interpolerende subdivisie skemas en kaskade algoritmes word ook ondersoek. Subdivisieskemas, soos gewoonlik gebruik om krommes en oppervlakke te genereer, is interpolerend indien die begin-datapunte gepreserveer word by elke stap van

die subdivisie proses. Ons toon aan dat interpolerende skemas en die kaskade algoritme sterk aanmekaar verbind is. Vir 'n goedgekose dilasiematriks en interpolerende verfyningsmasker vind ons dat die ooreenstemmende kaskade algoritme sekere eienskappe van die beginfunksie preserveer, met behulp waarvan ons dan kan bewys dat kaskade algoritme konvergensie die bestaan van 'n ooreenstemmende interpolerende verfynbare funksie impliseer, en wat op die beurt dan die konvergensie van die subdivisieskema impliseer.

Deur te spesialiseer na die geval waar die dilasiematriks $M = 2I$, verskaf ons werkbare metodes vir toepassing op beide nie-negatiewe interpolerende maskers en interpolerende maskers soos verkry met behulp van tensor produkte met die doel om die bestaan van ooreenstemmende interpolerende verfynbare funksies te ondersoek. Vir interpolerende maskers wat die somreëls bevredig, gee ons numeriese bewyse ten opsigte van die ondersoek na die bestaan van ooreenstemmende verfynbare funksies, deur die kaskade algoritme met 'n gepaste beginfunksie te gebruik. Numeriese illustrasies deur middel van subdivisie grafieke word ook verskaf.

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Contents

List of symbols

Symbol	Definition
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$, $\mathbb{Z}_+$	the sets of integers and non-negative integers
$\mathbb{Z}^2$, $\mathbb{Z}^2_+$	the sets of integer pairs and non-negative integer pairs
$\mathbb{Q}$, $\mathbb{Q}^2$	the sets of rational numbers and rational pairs
$\mathbb{R}$, $\mathbb{R}^2$	the sets of real numbers and real pairs
$\mathbb{C}$, $\mathbb{C}^2$	the sets of complex numbers and complex pairs
$M(\mathbb{Z})$	the linear space of bi-infinite real-valued sequences in $\mathbb{Z}$, i.e. $c \in M(\mathbb{Z}) \Longleftrightarrow c = \{c_j : j \in \mathbb{Z}\} \subset \mathbb{R}$
$M(\mathbb{Z}^2)$	the linear space of bi-infinite real-valued sequences in $\mathbb{Z}^2$, i.e. $c \in M(\mathbb{Z}^2) \Longleftrightarrow c = \{c_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^2\} \subset \mathbb{R}^2$
$M(\mathbb{R})$	the linear space of real-valued functions in $\mathbb{R}$, i.e. the set $\{f : \mathbb{R} \to \mathbb{R}\}$
$M(\mathbb{R}^2)$	the linear space of real-valued functions in $\mathbb{R}^2$, i.e. the set $\{f : \mathbb{R}^2 \to \mathbb{R}\}$
$M_0(\mathbb{Z})$	the subset of finitely supported sequences in $M(\mathbb{Z})$
$M_0(\mathbb{Z}^2)$	the subset of finitely supported sequences in $M(\mathbb{Z}^2)$
$M_0(\mathbb{R})$	the subset of finitely supported functions in $M(\mathbb{R})$
$M_0(\mathbb{R}^2)$	the subset of finitely supported functions in $M(\mathbb{R}^2)$

$\mathrm{supp}(c)$	the support of the sequence $c \in M_0(\mathbb{Z}^2)$, i.e. the set $\{\mathbf{j} \in \mathbb{Z}^2 : c_{\mathbf{j}} \neq \mathbf{0}\}$
$\mathrm{supp}(f)$	the support of the function $f \in M_0(\mathbb{R}^2)$, i.e. the smallest closed set containing $\{\mathbf{x} \in \mathbb{R}^2 : f(\mathbf{x}) \neq 0\}$
$C(\mathbb{R})$	the subset of continuous functions in $M(\mathbb{R})$
$C(\mathbb{R}^2)$	the subset of continuous functions in $M(\mathbb{R}^2)$
$C_0(\mathbb{R})$	the subset of finitely supported functions in $C(\mathbb{R})$
$C_0(\mathbb{R}^2)$	the subset of finitely supported functions in $C(\mathbb{R}^2)$
$C^\alpha(\mathbb{R})$	the subset of α-times continuously differentiable functions in $C(\mathbb{R})$
$C^\alpha(\mathbb{R}^2)$	the subset of α-times continuously differentiable functions in $C(\mathbb{R}^2)$
$C_0^\alpha(\mathbb{R})$	the subset of finitely supported functions in $C_0(\mathbb{R})$
$C_0^\alpha(\mathbb{R}^2)$	the subset of finitely supported functions in $C_0(\mathbb{R}^2)$
$\sum_j$ and $\sum_{\mathbf{j}}$	the summations $\sum_{j \in \mathbb{Z}}$ and $\sum_{\mathbf{j} \in \mathbb{Z}^2}$
$\sum_{i,j}$	the summation $\sum_{(i,j) \in \mathbb{Z}^2}$
$\sup_{\mathbf{j}}$ and $\sup_{\mathbf{x}}$	the suprema over all $\mathbf{j} \in \mathbb{Z}^2$ and over all $\mathbf{x} \in \mathbb{R}^2$
I	the 2×2 identity matrix
M	dilation matrix, i.e. a 2×2 invertible matrix with integer entries
a	refinement mask in $M_0(\mathbb{Z}^2)$
Π	the space of all polynomials with complex variables
Π_k	the subspace of Π consisting of polynomials of degree at most $k \in \mathbb{Z}_+$
A	mask symbol associated with the refinement mask $a \in M_0(\mathbb{Z}^2)$, i.e. the Laurent polynomial $\sum_{i,j} a_{i,j} z_1^i z_2^j$
ϕ	refinable function, i.e. a function satisfying the refinement equation $\phi = \sum_{\mathbf{j}} a_{\mathbf{j}} \phi(M \cdot - \mathbf{j})$
δ	the delta sequence defined by $\delta_{\mathbf{0}} = 1$ and $\delta_{\mathbf{j}} = 0$ for $\mathbf{j} \neq \mathbf{0}$
$\mathbf{j}^T$	the transpose of $\mathbf{j} \in \mathbb{Z}^2$, i.e. $\mathbf{j}^T = \binom{i}{j}$ for $\mathbf{j} = (i, j)$

S_a	the subdivision operator mapping $c \in M(\mathbb{Z}^2)$ to $S_a c \in M(\mathbb{Z}^2)$, with $(S_a c)_{\mathbf{j}} = \sum_{\mathbf{k}} a_{\mathbf{j}-M\mathbf{k}^T} c_{\mathbf{k}}$, $\mathbf{j} \in \mathbb{Z}^2$
S_a^r	the subdivision operator S_a applied r-times, with the convention that S_a^0 is the identity operator
$c^{(r)}$	the sequence $S_a^r c$, where $c \in M(\mathbb{Z}^2)$
$\lVert \cdot \rVert_\infty$	the uniform norm in $M(\mathbb{Z}^2)$ and in $M(\mathbb{R}^2)$, i.e. $\lVert c \rVert_\infty = \sup_{\mathbf{j}} \lvert c_{\mathbf{j}} \rvert$ for $c \in M(\mathbb{Z}^2)$, and $\lVert f \rVert_\infty = \sup_{\mathbf{x}} \lvert f(\mathbf{x}) \rvert$ for $f \in M(\mathbb{R}^2)$
$S_a^\infty c$	the limit function of a convergent subdivision scheme S_a with initial sequence $c \in M(\mathbb{Z}^2)$
T_a	the cascade operator mapping $f \in M(\mathbb{R}^2)$ to $T_a f \in M(\mathbb{R}^2)$, with $T_a f = \sum_{\mathbf{j}} a_{\mathbf{j}} f(M \cdot - \mathbf{j})$
T_a^r	the cascade operator T_a applied r-times, with the convention that T_a^0 is the identity operator
g	an initial function in $M(\mathbb{R}^2)$ for the cascade algorithm
f_r	the function $T_a^r g$, $r \in \mathbb{Z}_+$
$T_a^\infty g$	the limit function of a convergent cascade algorithm T_a with initial function $g \in C_0(\mathbb{R}^2)$
$\mathcal{D}$	the dyadic set $\left\{ M^{-r} \mathbf{j}^T : \mathbf{j} \in \mathbb{Z}^2, r \in \mathbb{Z}_+ \right\}$ which is dense in $\mathbb{R}^2$
$\tilde{\phi} \cdot \tilde{\psi}$	the tensor product of the univariate functions $\tilde{\phi}$ and $\tilde{\psi}$, i.e. the bivariate function $(x, y) \mapsto \tilde{\phi}(x)\tilde{\psi}(y)$, $(x, y) \in \mathbb{R}^2$

Introduction

A *refinable function*, or a function expressible as a linear combination of the shifts of its own dilations by a factor of a *dilation matrix*, i.e. an invertible matrix with integer entries, is always linked to a certain sequence called the *refinement mask*. The refinement mask corresponds to a Laurent polynomial called the *mask symbol*, the coefficients of which are the elements of the refinement mask. The cardinal B-spline functions presented in [dV07] are among the first examples of univariate refinable functions which have enormous applications in wavelet analysis and approximation theory.

In general, it is hard to investigate whether a given function is refinable, since both the associated refinement mask, as well as the corresponding the dilation matrix have to be found. It is thus better to start with a given dilation matrix and a finitely supported sequence, and investigate the existence of a corresponding refinable function.

Based on a given dilation matrix and a finitely supported sequence, the associated *subdivision scheme* is defined as an operator which recursively produces denser and denser data points by means of linear combinations of the previous ones. The corresponding *cascade algorithm* is also defined as a functional operator which iteratively produces a sequence of functions by means of linear combinations of the previous ones.

Subdivision methods, as initialy introduced by de Rham (1956) and later by Chaikin (1974), play important roles in computer aided geometric design (CAGD) by generating curves and surfaces in computer graphics (see e.g. [Dyn92]). Cascade algorithms, on the other hand, are useful in the sense that cascade algorithm convergence implies the refinability of the limit function.

Specializing only to the case where the dilation matrix is $M = 2I$, our goal in this thesis is to give a purely algebraic method for the study of both bivariate refinable functions and their associated subdivision schemes, in contrast to methods based on Fourier transforms as mostly encountered in the literature. A fundamental theme in this thesis is that of *interpolatory* bivariate refinable functions, that is, refinable functions that take the value 1 at the origin and 0 at all other integers. We proceed to introduce in Chapter 1 a brief overview of interpolatory refinable functions. The corresponding refinement masks, called *interpolatory masks*, and the associated *interpolatory mask symbols* are respectively characterized by (1.8) and (1.10). We refer to the Dubuc-Deslauriers interpolatory refinable function, as investigated in [VGH03] (see also [Hun05, Goo00]) for the univariate setting, and to the interpolatory refinable functions constructed in [RS97] (see also [Jia00]) for the multivariate case.

Several studies of refinement masks have been developed by using the associated mask symbols, which often help to prove the convergence of the subdivision schemes to which they are associated (e.g. [DL02, pages 37-70], [CDM91]). Motivated by this perspective, we take a special interest in interpolatory mask symbols for the special case where the dilation matrix is $2I$. In Chapter 2, an alternative criterion to interpolatory mask symbols which is easier to use than (1.10) is given. In Theorem 2.2.3, we deduce the general form of an interpolatory mask symbol by using some polynomial identities and the *Euclidean algorithm*. The results thus obtained are then applied to the mask symbols corresponding to the well-known box splines.

An interpolatory refinement mask generates an *interpolatory subdivision scheme*, that is, a subdivision scheme for which the initial data points are preserved at all the steps of the recursive process (see [Dyn92]). This is extremely relevant in certain application areas in CAGD, where the initial data are required to be preserved while applying the subdivision process. In Chapter 3, we discuss the convergence of interpolatory subdivision schemes, and we investigate in Section 3.3 the issue of property preservation with respect to the cascade algorithm.

Though remarkable progress by mathematicians in the area have been made, computationally inefficient conditions are still often applied to refinement masks in order to ensure the convergence of the associated subdivision schemes. For instance, the characterization by using the joint spectral radius for subdivision schemes investigated in [HJ98a] can take impractically long to test computationally, whereas the alternative method based on contractivity conditions, as introduced in [DL02] (see also [Dyn02]), can also be a formidable computational task to perform. Under certain restrictions, we therefore develop in Chapter 4 three feasible methods to examine the existence of interpolatory refinable functions from a practical point of view. The presented methods are applied on interpolatory mask symbols, and are based on the results of Micchelli in [Mic96] and on tensor products.

Unfortunately, for the general setting, the existing methods investigating the existence of interpolatory refinable functions are not always feasible to implement. By using the above-mentioned general form of an interpolatory mask symbol, an interesting continuation of this thesis thus include finding easily checkable sufficient conditions on interpolatory mask symbols for them to comply with the conditions of the existing methods.

Chapter 1

Interpolatory bivariate refinable functions

We first give in this chapter a brief introduction to interpolatory bivariate refinable functions and the corresponding interpolatory masks. Then, we elaborate a simple criterion in (1.8) and in (1.10) to recognize simultaneously an interpolatory mask and the associated interpolatory mask symbol. We end the chapter by presenting the box splines as examples of interpolatory bivariate refinable functions.

1.1 Notation and general concepts

We shall denote the set of natural numbers by $\mathbb{N}$, the set of integers and non-negative integers respectively by $\mathbb{Z}$ and $\mathbb{Z}_+$, the set of real numbers by $\mathbb{R}$ and the set of complex numbers by $\mathbb{C}$. Similarly, the symbols $\mathbb{Z}^2$, $\mathbb{R}^2$ and $\mathbb{C}^2$ denote the set of ordered pairs with respectively integer, real number and complex number entries.

For the linear space $M(\mathbb{Z}^2)$ of all real-valued sequences $c = \{c_{\mathbf{j}} \in \mathbb{R} : \mathbf{j} \in \mathbb{Z}^2\}$ which support is denoted by $\operatorname{supp}(c) := \{\mathbf{j} \in \mathbb{Z}^2 : c_{\mathbf{j}} \neq \mathbf{0}\}$, the subspace of finitely supported sequences, i.e. whose supports are finite, constitute a linear subspace denoted by $M_0(\mathbb{Z}^2)$. In the same way, for the linear space $M(\mathbb{R}^2)$ of all real-valued bivariate functions f on $\mathbb{R}^2$ which support $\text{supp}(f)$ is the smallest closed set containing $\{\mathbf{x} \in \mathbb{R}^2 : f(\mathbf{x}) \neq 0\}$, the set of finitely supported functions constitute a linear subspace denoted by $M_0(\mathbb{R}^2)$. Moreover, the subspaces of continuous functions respectively in $M(\mathbb{R}^2)$ and in $M_0(\mathbb{R}^2)$ are denoted by $C(\mathbb{R}^2)$ and $C_0(\mathbb{R}^2)$.

For a given 2×2 invertible matrix M with integer entries, a function $\phi \in M_0(\mathbb{R}^2)$ is termed M-refinable if there exists a sequence $a = \{a_\mathbf{j} : \mathbf{j} \in \mathbb{Z}^2\} \in M_0(\mathbb{Z}^2)$ such that

$$\phi = \sum_{\mathbf{j}} a_\mathbf{j} \phi(M \cdot -\mathbf{j}). \tag{1.1}$$

We shall refer to M as the *dilation matrix*, whereas the sequence a is called the *refinement mask* (or simply the mask), and the equation (1.1) is referred to as the *refinement equation.*

Note that an M-refinable function is therefore expressible as a linear combinations of the shifts of its own dilations with the factor of the dilation matrix M, as specified by the refinement mask a. For convenience, we shall often simplify "M-refinable" to "refinable".

The problem of existence of refinable functions by using refinement masks is fundamental, but most importantly in this thesis, is that our study is focussed on *interpolatory* refinable functions, that is, refinable functions that satisfy

$$\phi(\mathbf{j}) = \delta_\mathbf{j}, \quad \mathbf{j} \in \mathbb{Z}^2, \tag{1.2}$$

where the delta function δ (also called the delta sequence) is defined by

$$\delta_\mathbf{j} = \begin{cases} 1, & \mathbf{j} = \mathbf{0}, \\ 0, & \mathbf{j} \neq \mathbf{0}, \end{cases} , \mathbf{j} \in \mathbb{Z}^2. \tag{1.3}$$

In other words, a refinable function is interpolatory if it vanishes at all integers except at the origin $\mathbf{0} \in \mathbb{Z}^2$ where it takes the value 1. We proceed to characterize the so-called *interpolatory refinement masks* associated with interpolatory refinable functions.

1.2 Interpolatory refinement masks

We present in this section a characterization theory for refinement masks associated with interpolatory refinable functions. Thereafter we introduce the concept of refinement mask symbols and then specialize to the case $M = 2I$, with some examples of bivariate interpolatory refinable functions.

By using the symbol $\mathbf{j}^T$ for the transpose of the integer pair $\mathbf{j} \in \mathbb{Z}^2$, we come first to the following result.

Proposition 1.2.1. *For a given dilation matrix M and a mask $a \in M_0(\mathbb{Z}^2)$, suppose the refinement equation (1.1) holds for a refinable function ϕ. If ϕ is interpolatory, then a satisfies*

$$a_{M\boldsymbol{j}^T} = \delta_{\boldsymbol{j}}, \quad \boldsymbol{j} \in \mathbb{Z}^2. \tag{1.4}$$

Proof. From (1.2) and (1.1), we have that, for $\mathbf{j} \in \mathbb{Z}^2$,

$$\delta_{\mathbf{j}} = \phi(\mathbf{j}) = \sum_{\mathbf{k}} a_{\mathbf{k}} \phi(M\mathbf{j}^T - \mathbf{k}) = \sum_{\mathbf{k}} a_{\mathbf{k}} \delta_{M\mathbf{j}^T - \mathbf{k}} = a_{M\mathbf{j}^T}.$$

□

Our next result was proved for the case $M = 2I$ in [CDM91]. Our general proof is based on a suggestion in [HJ98a].

Proposition 1.2.2. *For a given dilation matrix M and a mask $a \in M_0(\mathbb{Z}^2)$, suppose the refinement equation (1.1) holds for a refinable function ϕ. If ϕ is finitely supported and* *integrable with non-zero integral over* $\mathbb{R}^2$*, then* a *satisfies*

$$\sum_{\boldsymbol{j}} a_{\boldsymbol{j}} = |det(M)|. \tag{1.5}$$

Proof. Suppose that the dilation matrix has the form

$$M = \begin{pmatrix} c & d \\ e & f \end{pmatrix}.$$

Writing $a_{i,j} = a_{\mathbf{j}}$, we can now integrate the refinement equation (1.1) to obtain

$$\int\int_{\mathbb{R}^2} \phi(x,y)\mathrm{d}x\mathrm{d}y = \sum_{i,j} a_{i,j} \int\int_{\mathbb{R}^2} \phi(M(x,y)^T - (i,j))\mathrm{d}x\mathrm{d}y. \tag{1.6}$$

Since the variable transformation $(X,Y)^T = M(x,y)^T$ has Jacobian

$$J(x,y) = \begin{vmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{vmatrix} = \begin{vmatrix} c & d \\ e & f \end{vmatrix} = \det(M),$$

it follows from standard multivariate integration theorems in analysis that

$$\begin{aligned} \int\int_{\mathbb{R}^2} \phi(M(x,y)^T - (i,j))|\det(M)|\mathrm{d}x\mathrm{d}y &= \int\int_{\mathbb{R}^2} \phi((X,Y) - (i,j))\mathrm{d}X\mathrm{d}Y \\ &= \int\int_{\mathbb{R}^2} \phi(X,Y)\mathrm{d}X\mathrm{d}Y. \end{aligned} \tag{1.7}$$

We then deduce from (1.6) and (1.7) that

$$\int\int_{\mathbb{R}^2} \phi(x,y)\mathrm{d}x\mathrm{d}y \;\; = \sum_{i,j} a_{i,j} \frac{1}{|\det(M)|} \int\int_{\mathbb{R}^2} \phi(x,y)\mathrm{d}x\mathrm{d}y.$$

Moreover, since we assume the integral of ϕ to be non-zero over $\mathbb{R}^2$, we obtain

$$\sum_{i,j} a_{i,j} \frac{1}{|\det(M)|} = 1,$$

from which the result (1.5) follows. □

Therefore, given a dilation matrix M, the existence of a compactly supported interpolatory refinable function ϕ with non-zero integral over $\mathbb{R}^2$ requires for a given refinement mask a to satisfy the conditions

$$\begin{cases} a_{M\mathbf{j}^T} = & \delta_{\mathbf{j}}, \quad \mathbf{j} \in \mathbb{Z}^2, \\ \sum_{\mathbf{j}} a_{\mathbf{j}} = & |\det(M)|. \end{cases} \tag{1.8}$$

Now, considering a refinement mask $a = \{a_{\mathbf{j}}\} = \{a_{i,j}\}$, we define the corresponding *refinement mask symbol*, or simply the mask symbol, as the bivariate Laurent polynomial A given by

$$A(z_1, z_2) = \sum_{i,j} a_{i,j} z_1^i z_2^j, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}. \tag{1.9}$$

Also, we say that a refinement mask a is *interpolatory* if it satisfies (1.8). In that case, for brevity, we call a an interpolatory mask. Moreover, its symbol A is called an *interpolatory mask symbol.*

Since, according to (1.9), refinement masks and their symbols are bijectively linked, the restrictions (1.8) on a mask a can equivalently be expressed in terms of the mask symbol A as follows:

$$\begin{cases} \text{The constant term in } A(z_1, z_2) \text{ is } 1, \text{ and } A \text{ has no term in } z_1^{\alpha_1} z_2^{\alpha_2} \\ \text{such that } (\alpha_1, \alpha_2) = M(i,j)^T \neq (0,0) \text{ for some } (i,j) \in \mathbb{Z}^2; \text{ also,} \\ A(1,1) = \sum_{i,j} a_{i,j} = |\det(M)|. \end{cases} \tag{1.10}$$

It is often convenient to use refinement mask symbols instead of their corresponding refinement masks. Indeed, as presented in [CDM91, Mic96, Der99], some properties of masks symbols lead to the existence of compactly supported refinable functions.

The following section presents some examples of interpolatory refinable functions with dilation matrix $M = 2I$.

1.3 Box splines

In this section, we fix the dilation matrix $M = 2I$. The conditions (1.8) on an interpolatory mask a can then be re-written as

$$\begin{cases} a_{2i,2j} = & \delta_{(i,j)}, \quad (i,j) \in \mathbb{Z}^2, \\ \sum\limits_{i,j} a_{i,j} = & 4, \end{cases} \tag{1.11}$$

whereas the conditions (1.10) on an interpolatory mask symbol A become

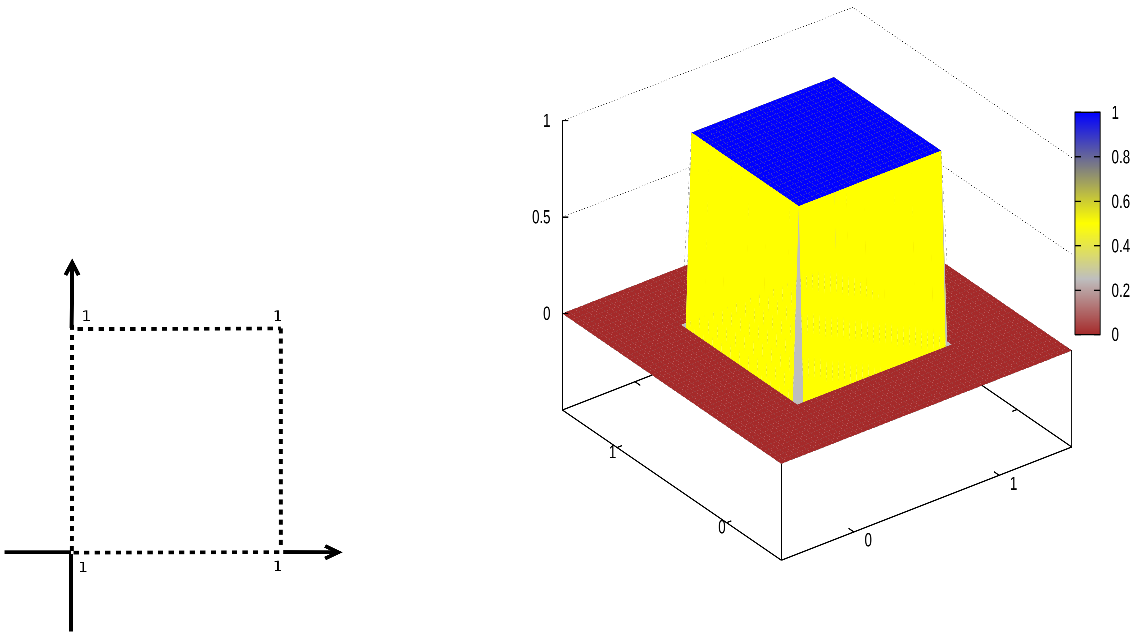
$$\begin{cases} \text{The constant term in } A(z_1, z_2) \text{ is } 1, \text{ and } A \text{ has} \\ \text{no term in } z_1^{2\alpha_1} z_2^{2\alpha_2}, \text{ for any } (\alpha_1, \alpha_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}; \quad \text{also,} \\ A(1,1) = \sum\limits_{i,j} a_{i,j} = 4. \end{cases} \tag{1.12}$$

The box spline N_1

The box spline function N_1 is defined by

$$N_1(x,y) = \begin{cases} 1, & (x,y) \in [0,1)^2, \\ 0, & (x,y) \notin [0,1)^2. \end{cases} \tag{1.13}$$

The graph of N_1 is shown in Figure 1.1 (b), from which we see that N_1 is finitely



(a) Support of $a^{(1)}$ (b) Graph of N_1

Figure 1.1: The box spline N_1

supported, and though it is not continuous, we claim that N_1 is an interpolatory refinable function with respect to the interpolatory mask $a^{(1)}$ which support is delimitated by the dotted lines in Figure 1.1 (a), as given by

$$a^{(1)}_{0,0} = a^{(1)}_{0,1} = a^{(1)}_{1,0} = a^{(1)}_{1,1} = 1; \quad a^{(1)}_{i,j} = 0 \text{ otherwise.} \tag{1.14}$$

To prove this, observe first that, for $x, y \in \mathbb{R}$,

$$N_1(2x, 2y) = \begin{cases} 1, & (x,y) \in [0, \frac{1}{2})^2, \\ 0, & (x,y) \notin [0,1)^2; \end{cases}$$
$$N_1(2x-1, 2y) = \begin{cases} 1, & (x,y) \in [\frac{1}{2}, 1) \times [0, \frac{1}{2}), \\ 0, & (x,y) \notin [0,1)^2; \end{cases}$$
$$N_1(2x, 2y-1) = \begin{cases} 1, & (x,y) \in [0, \frac{1}{2}) \times [\frac{1}{2}, 1), \\ 0, & (x,y) \notin [0,1)^2; \end{cases}$$

$$N_1(2x-1,2y-1) = \begin{cases} 1, & (x,y) \in [\frac{1}{2},1)^2, \\ 0, & (x,y) \notin [0,1)^2. \end{cases}$$

Then, since the squares $[0,\frac{1}{2})^2, [\frac{1}{2},1) \times [0,\frac{1}{2}), [0,\frac{1}{2}) \times [\frac{1}{2},1)$ and $[\frac{1}{2},1)^2$ form a partition of the unit square $[0,1)^2$, we obtain, for $(x,y) \in \mathbb{R}^2$,

$$N_1(x,y) = N_1(2x,2y) + N_1(2x-1,2y) + N_1(2x,2y-1) + N_1(2x-1,2y-1), \quad (1.15)$$

thereby proving that N_1 is refinable with corresponding mask $a^{(1)}$ given in (1.14). Hence, according to (1.14) and (1.9), the corresponding mask symbol A_1 is given by

$$A_1(z_1,z_2) = 1 + z_1 + z_2 + z_1 z_2 = (1+z_1)(1+z_2), \quad z_1, z_2 \in \mathbb{C}. \quad (1.16)$$

Note that the conditions (1.11) and (1.12) are respectively fulfilled by the refinement mask $a^{(1)}$ and its symbol A_1. Moreover, (1.13) shows that $N_1(\mathbf{j}) = \delta_{\mathbf{j}}$, $\mathbf{j} \in \mathbb{Z}^2$, which means that N_1 is an interpolatory refinable function.

The box spline N_2

Using the box spline N_1 given in (1.13), the box spline function N_2 is defined by

$$N_2(x,y) = \int_0^1 N_1(x-t, y-t)\mathrm{d}t, \quad x, y \in \mathbb{R}. \quad (1.17)$$

Let us first prove that N_2 is a continuous function by finding its explicit formula. To this end, observe that, for $t \in (0,1)$ and $x, y \in \mathbb{R}$,

$$\begin{aligned} N_1(x-t, y-t) \neq 0 &\Longleftrightarrow x-t \in [0,1) \text{ and } y-t \in [0,1) \\ &\Longrightarrow 0 < x < 2 \text{ and } 0 < y < 2. \end{aligned} \quad (1.18)$$

Hence, from (1.18) and (1.17), we deduce that $N_2(\mathbf{x}) = 0$, $\mathbf{x} \notin [0,2]^2$.

Moreover, for $x, y \in [0,2)$, we have

$$0 \leq x - t < 1 \Longleftrightarrow x - 1 < t \leq x \text{ and } 0 \leq y - t < 1 \Longleftrightarrow y - 1 < t \leq y,$$

which, together with (1.17), yields

$$N_1(x-t, y-t) \neq 0 \Longleftrightarrow t \in (0,1) \cap (x-1, x] \cap (y-1, y], \quad x, y \in [0,2). \tag{1.19}$$

We then have the following result.

Proposition 1.3.1. *The box spline N_2, as defined in (1.17), is explicitly given by*

$$N_2(x,y) = \begin{cases} \min\{x,y\}, & \text{if } (x,y) \in [0,1)^2, \\ 2 - \max\{x,y\}, & \text{if } (x,y) \in [1,2)^2, \\ 1 + \min\{x,y\} - \max\{x,y\}, & \text{if } (x,y) \in \Delta, \\ 0 & \text{otherwise,} \end{cases} \tag{1.20}$$

where Δ is the set defined by

$$\Delta = \{(x,y) : \min\{x,y\} \in [0,1); \max\{x,y\} \in [1,2); 1 + \min\{x,y\} \geq \max\{x,y\}\}, \tag{1.21}$$

i.e.,

$$\Delta = B \cup E,$$

with B and E as in Figure 1.2. Consequently, the support of N_2 is the polygon $A \cup B \cup C \cup D \cup E \cup F = [0,1]^2 \cup \Delta \cup [1,2]^2$ in Figure 1.2.

Proof. Observe from Figure 1.2 that $[0,1)^2 = A \cup F$, $[1,2)^2 = C \cup D$ and $\Delta = B \cup E$.

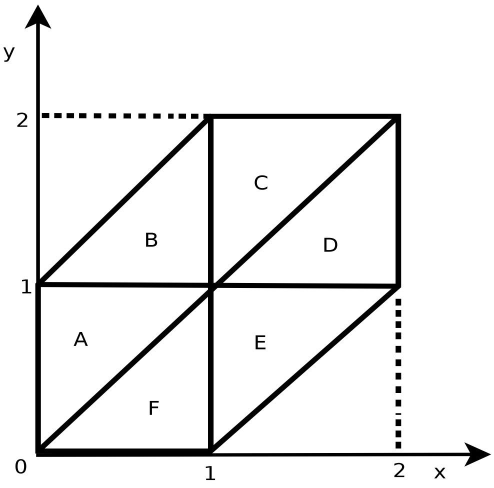


Figure 1.2: Support of the box spline N_2.

Therefore, from (1.19), we have that, for $x \in [0,1)$:

- If $y \in [0,1)$ is such that $y \leq x$ (resp. $y \geq x$), then $t \in [0,y]$ (resp. $t \in [0,x]$), so that $N_2(x,y) = \int_0^y \mathrm{d}t = y \quad \left(\text{resp. } N_2(x,y) = \int_0^x \mathrm{d}t = x\right)$.
- If $y \in [1,2)$, two cases occur:
    - If $y - 1 > x$, then $t \in \emptyset$ and $N_2(x,y) = 0$;
    - If $y - 1 \leq x$, then $t \in (y-1, x]$ and therefore $N_2(x,y) = \int_{y-1}^x \mathrm{d}t = 1 + x - y$.

Similarly, from (1.19), we have that, for $x \in [1,2)$:

- If $y \in [0,1)$, two cases occur:
    - If $y > x - 1$, then $t \in (x-1, y]$ and therefore $N_2(x,y) = \int_{x-1}^y \mathrm{d}t = 1 + y - x$;
    - If $y \leq x - 1$, then $t \in \emptyset$ and $N_2(x,y) = 0$.
- If $y \in [1,2)$ is such that $y \leq x$ (resp. $y \geq x$), then $t \in (x-1, 1]$ (resp. $t \in (y-1, 1]$), so that $N_2(x,y) = \int_{x-1}^1 \mathrm{d}t = 2 - x \quad \left(\text{resp. } N_2(x,y) = \int_{y-1}^1 \mathrm{d}t = 2 - y\right)$.

By taking the appropriate combination of the four cases above, we obtain the desired result (1.20). $\square$

Next, by using Proposition 1.3.1 and Figure 1.2, we deduce that the restrictions of N_2 to the respective regions A, B, C, D, E and F are given as follows:

- ⋄ In the region A: $x, y \in [0, 1)$, with $y \geq x$, we have $N_2|_A(x, y) = x$;
- ⋄ In the region F: $x, y \in [0, 1)$, with $y \leq x$, we have $N_2|_F(x, y) = y$;
- ⋄ In the region B: $x \in [0, 1)$ and $y \in [1, 2)$, with $x \geq y - 1$, we have $N_2|_B(x, y) = 1 + x - y$;
- ⋄ In the region E: $x \in [1, 2)$ and $y \in [0, 1)$, with $y \geq x - 1$, we have $N_2|_E(x, y) = 1 + y - x$;
- ⋄ In the region C: $x, y \in [1, 2)$, with $y \geq x$, we have $N_2|_C(x, y) = 2 - y$;
- ⋄ In the region D: $x, y \in [1, 2)$, with $x \geq y$, we have $N_2|_D(x, y) = 2 - x$.

Hence, N_2 defines a different plane in each of the respective regions A, B, C, D, E and F. It will therefore suffice to prove the continuity of N_2 at the edges of these regions, i.e along the lines $x = 0$, $x = 1$, $x = 2$, the lines $y = 0$, $y = 1$, $y = 2$, as well as the lines $y = x$, $y = x + 1$ and $y = x - 1$.

To this end, observe first that, for the region A (resp. F), when $x \to 0$ (resp. $y \to 0$), we have that $N_2(x, y) \to 0$. Similarly, for the region D (resp. C), when $x \to 2$ (resp. $y \to 2$), we also have that $N_2(x, y) \to 0$.

Next, observe that, when $x \to 1$ (resp. $y \to 1$), we have $N_2|_F(x, y) \to y$ and $N_2|_E(x, y) \to y$ (resp. $N_2|_A(x, y) \to x$ and $N_2|_B(x, y) \to x$), so that N_2 is continuous in the region $F \cup E$ (resp. $A \cup B$). Similarly, when $x \to 1$ (resp. $y \to 1$), we have that $N_2|_B(x, y) \to 2 - y$ and $N_2|_C(x, y) \to 2 - y$ (resp. $N_2|_E(x, y) \to 2 - x$ and $N_2|_D(x, y) \to 2 - x$), so that N_2 is also continuous in the region $B \cup C$ (resp. $E \cup D$).

Finally, along the line $y = x$, we have that $N_2|_A(x,y) = N_2|_F(x,y)$ and $N_2|_C(x,y) = N_2|_D(x,y)$, so that N_2 is continuous in the regions $A \cup F$ and $C \cup D$. Along the line $y = x+1$ (resp. $y = x-1$), we have that $N_2|_B(x,y) = 0$ (resp. $N_2|_E(x,y) = 0$). Thus, we conclude that N_2 is continuous on $\mathbb{R}^2$.

We proceed now to prove that N_2 is refinable. From the refinement equation (1.15), we have that, for $x, y \in \mathbb{R}$,

$$\begin{aligned} N_2(x,y) =& \int_0^1 N_1(x-t, y-t)\mathrm{d}t \\ =& \int_0^1 [N_1(2x-2t, 2y-2t) + N_1(2x-2t-1, 2y-2t) \\ &+ N_1(2x-2t, 2y-2t-1) + N_1(2x-2t-1, 2y-2t-1)]\,\mathrm{d}t. \end{aligned} \tag{1.22}$$

Using the transformations $\tilde{t} = 2t$ for $t \in [0, \frac{1}{2}]$ and $\tilde{t} = 2t-1$ for $t \in [\frac{1}{2}, 1]$, the first integral in (1.22) can be re-written, for $x, y \in \mathbb{R}$, as

$$\begin{aligned} \int_0^1 N_1(2x-2t, 2y-2t)\mathrm{d}t =& \int_0^{\frac{1}{2}} N_1(2x-2t, 2y-2t)\mathrm{d}t + \int_{\frac{1}{2}}^1 N_1(2x-2t, 2y-2t)\mathrm{d}t \\ =& \frac{1}{2}\int_0^1 N_1(2x-t, 2y-t)\mathrm{d}t + \frac{1}{2}\int_0^1 N_1(2x-t-1, 2y-t-1)\mathrm{d}t \\ =& \frac{1}{2}N_2(2x, 2y) + \frac{1}{2}N_2(2x-1, 2y-1), \end{aligned} \tag{1.23}$$

by virtue of the definition of N_2 in (1.17). Similarly, we get, for $x, y \in \mathbb{R}$,

$$\int_0^1 N_1(2x-2t-1, 2y-2t)\mathrm{d}t = \frac{1}{2}N_2(2x-1, 2y) + \frac{1}{2}N_2(2x-2, 2y-1), \tag{1.24}$$

$$\int_0^1 N_1(2x-2t, 2y-2t-1)\mathrm{d}t = \frac{1}{2}N_2(2x, 2y-1) + \frac{1}{2}N_2(2x-1, 2y-2), \tag{1.25}$$

$$\int_0^1 N_1(2x-2t-1, 2y-2t-1)\mathrm{d}t = \frac{1}{2}N_2(2x-1, 2y-1) + \frac{1}{2}N_2(2x-2, 2y-2). \tag{1.26}$$

Consequently, from (1.22), (1.23), (1.24), (1.25) and (1.26), we obtain

$$\begin{aligned} N_2(x,y) =& \frac{1}{2}\left\{N_2(2x,2y) + N_2(2x-1,2y) + N_2(2x,2y-1) + 2N_2(2x-1,2y-1)\right. \\ &\left. + N_2(2x-1,2y-2) + N_2(2x-2,2y-1) + N_2(2x-2,2y-2)\right\}, \end{aligned} \tag{1.27}$$

which shows that N_2 is refinable with corresponding mask $a^{(2)}$ given by

$$\begin{cases} a^{(2)}_{1,1} = 1, \quad a^{(2)}_{0,0} = a^{(2)}_{0,1} = a^{(2)}_{1,0} = a^{(2)}_{2,1} = a^{(2)}_{1,2} = a^{(2)}_{2,2} = \frac{1}{2}, \\ a^{(2)}_{i,j} = 0, \quad (i,j) \notin \{(0,0),(0,1),(1,0),(1,1),(1,2),(2,1),(2,2)\}, \end{cases} \tag{1.28}$$

according to which the corresponding mask symbol A_2 is given by

$$A_2(z_1,z_2) = (1+z_1)(1+z_2)\left(\frac{1+z_1z_2}{2}\right), \quad z_1,z_2 \in \mathbb{C}. \tag{1.29}$$

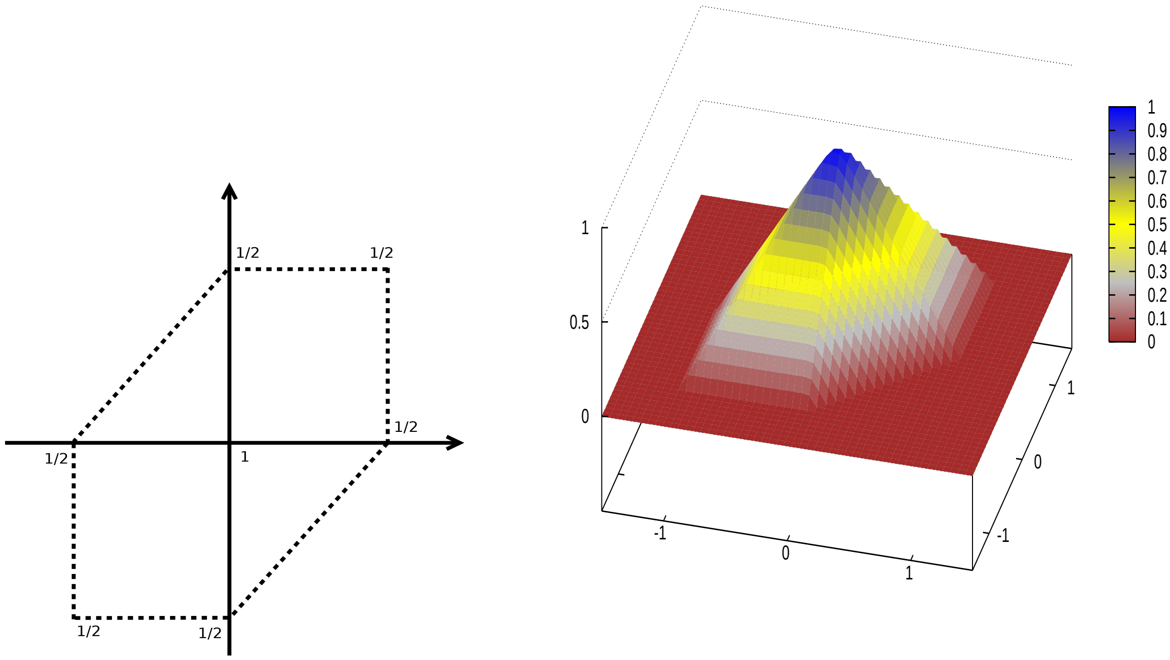
However, observe from (1.28) that $a^{(2)}_{0,0} \neq 1$ and $a^{(2)}_{2,2} \neq 0$ (or, equivalently, the constant term in $A_2(z_1,z_2)$ is not 1 and it has a term in $z_1^2z_2^2$), that is, N_2 is not interpolatory.

The shifted box spline $\tilde{N}_2$

Using the box spline N_2 defined in (1.17), we define the shifted box spline function $\tilde{N}_2$ by

$$\tilde{N}_2(x,y) = N_2(x+1,y+1), \quad x,y \in \mathbb{R}. \tag{1.30}$$

We claim that the function $\tilde{N}_2$, as drawn in Figure 1.3 (b), is an interpolatory refinable function associated with the interpolatory mask $\tilde{a}^{(2)}$ which support is delimitated by the



(a) Support of $\tilde{a}^{(2)}$ (b) Graph of $\tilde{N}_2$

Figure 1.3: The shifted box spline $\tilde{N}_2$

dotted lines in Figure 1.3 (a), as given by

$$\begin{cases} \tilde{a}^{(2)}_{0,0} = 1, \quad \tilde{a}^{(2)}_{1,1} = \tilde{a}^{(2)}_{0,1} = \tilde{a}^{(2)}_{1,0} = \tilde{a}^{(2)}_{-1,0} = \tilde{a}^{(2)}_{0,-1} = \tilde{a}^{(2)}_{-1,-1} = \frac{1}{2}, \\ \tilde{a}^{(2)}_{i,j} = 0, \quad (i,j) \notin \{(0,0),(0,1),(1,0),(-1,0),(0,-1),(1,1),(-1,-1)\}, \end{cases} \tag{1.31}$$

with corresponding mask symbol $\tilde{A}_2$ given by

$$\tilde{A}_2(z_1,z_2) = (1+z_1)(1+z_2)\left(\frac{1+z_1z_2}{2}\right) z_1^{-1}z_2^{-1}, \quad z_1,z_2 \in \mathbb{C}\setminus\{0\}. \tag{1.32}$$

To prove this, we use (1.30) and (1.27) to deduce that, for $x, y \in \mathbb{R}$,

$$\begin{aligned} \tilde{N}_2(x,y) =& N_2(x+1,y+1) \\ =& \frac{1}{2}\{N_2(2x+2,2y+2) + N_2(2x+1,2y+2) + N_2(2x+2,2y+1) \\ &+ 2N_2(2x+1,2y+1) + N_2(2x+1,2y) + N_2(2x,2y+1) + N_2(2x,2y)\} \end{aligned}$$

$$=\frac{1}{2}\Big\{\tilde{N}_2(2x+1,2y+1)+\tilde{N}_2(2x,2y+1)+\tilde{N}_2(2x+1,2y)$$
$$+2\tilde{N}_2(2x,2y)+\tilde{N}_2(2x,2y-1)+\tilde{N}_2(2x-1,2y)+\tilde{N}_2(2x-1,2y-1)\Big\}, \tag{1.33}$$

which implies that $\tilde{N}_2$ is a refinable function with refinement mask $\tilde{a}^{(2)}$ given by (1.31). Moreover, by using (1.31) and (1.9), we find that the corresponding mask symbol $\tilde{A}_2$ is given by (1.32). It can now be verified from (1.31) and (1.32) that $\tilde{a}^{(2)}$ and $\tilde{A}_2$ satisfy respectively the interpolatory conditions (1.11) and (1.12).

To prove that $\tilde{N}_2$ is interpolatory, we use (1.30) and (1.17) to obtain, for $x, y \in \mathbb{R}$,

$$\tilde{N}_2(x,y) = N_2(x+1,y+1) = \int_0^1 N_1(x+1-t,y+1-t)\mathrm{d}t. \tag{1.34}$$

Taking into account the definition of the box spline N_1 in (1.13), we deduce that

$$\tilde{N}_2(0,0) = \int_0^1 N_1(1-t,1-t)\mathrm{d}t = \int_0^1 1\mathrm{d}t = 1, \tag{1.35}$$

whereas, for $(i,j) \neq (0,0)$, we have that

$$\tilde{N}_2(i,j) = \int_0^1 N_1(i+1-t,j+1-t)\mathrm{d}t = 0, \tag{1.36}$$

for if $i \neq 0$ (resp. $j \neq 0$) then $i+1-t \notin [0,1)$ (resp. $j+1-t \notin [0,1)$), for any $t \in (0,1)$. It follows from (1.35) and (1.36) that the interpolatory condition (1.2) is satisfied, thereby showing that the shifted box spline $\tilde{N}_2$ is an interpolatory refinable function.

Note in particular from Figure 1.3 (b) that $\tilde{N}_2$ belongs to $C_0(\mathbb{R}^2) \setminus C_0^1(\mathbb{R}^2)$.

Chapter 2

The interpolatory mask symbols for $M = 2I$

We fix the dilation matrix $M = 2I$ in this chapter. In Section 2.1 below, we produce the alternative criterion (2.9) for interpolatory mask symbols. In Section 2.2, after solving some polynomial identities by means of the well-known Bezout identity and the Euclidean algorithm, we provide in Theorem 2.2.3 a useful characterization result for interpolatory mask symbols. In Section 2.3, we specialise to the case of box splines interpolatory mask symbols.

2.1 Simple characterization

We proceed to establish an alternative characterization to interpolatory mask symbols which is simpler to use than (1.12), and which will be used in Section 2.2. Recall from Chapter 1 that the class of interpolatory mask symbols consists of all Laurent polynomials

A satisfying the conditions (1.12), i.e.

$$\begin{cases} \text{The constant term in } A(z_1, z_2) \text{ is } 1, \text{ and } A \text{ has} \\ \text{ no term in } z_1^{2\alpha_1} z_2^{2\alpha_2}, \text{ for any } (\alpha_1, \alpha_2) \in \mathbb{Z}^2 \setminus (0,0); \text{ also,} \\ A(1,1) = 4, \end{cases} \tag{2.1}$$

where a is the corresponding interpolatory mask, as defined by (1.9), and satisfying the conditions (1.11), i.e.

$$\begin{cases} a_{2i,2j} & = \delta_{(i,j)}, \quad (i,j) \in \mathbb{Z}^2, \\ \sum\limits_{i,j} a_{i,j} & = 4. \end{cases} \tag{2.2}$$

Let us denote by $F \sqcup G$ the union of two sets F and G for which the intersection $F \cap G$ is empty, whereas EE, EO, OE and OO stand for the sets of integer pairs with respectively *even-even*, *even-odd*, *odd-even* and *odd-odd* entries. Observe that the set of integers $\mathbb{Z}^2$ consists of the union of the four disjoint subsets EE, EO, OE and OO, i.e.

$$\mathbb{Z}^2 = EE \sqcup EO \sqcup OE \sqcup OO. \tag{2.3}$$

Given a mask symbol A with corresponding mask $a \in M_0(\mathbb{Z}^2)$, we obtain from (2.3) and (1.9) that, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned} A(z_1, z_2) = & \sum_{i,j} a_{2i,2j} z_1^{2i} z_2^{2j} + \sum_{i,j} a_{2i,2j+1} z_1^{2i} z_2^{2j+1} + \sum_{i,j} a_{2i+1,2j} z_1^{2i+1} z_2^{2j} \\ & + \sum_{i,j} a_{2i+1,2j+1} z_1^{2i+1} z_2^{2j+1}, \end{aligned} \tag{2.4}$$

whereas also, by replacing z_1 by $-z_1$ in (2.4), we have, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$A(-z_1, z_2) = \sum_{i,j} a_{2i,2j} z_1^{2i} z_2^{2j} + \sum_{i,j} a_{2i,2j+1} z_1^{2j} z_2^{2j+1} - \sum_{i,j} a_{2i+1,2j} z_1^{2i+1} z_2^{2j}$$
$$- \sum_{i,j} a_{2i+1,2j+1} z_1^{2i+1} z_2^{2j+1}. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$A(z_1, z_2) + A(-z_1, z_2) = 2 \sum_{i,j} a_{2i,2j} z_1^{2i} z_2^{2j} + 2 \sum_{i,j} a_{2i,2j+1} z_1^{2i} z_2^{2j+1}. \tag{2.6}$$

Now replace z_1 by $-z_1$ and z_2 by $-z_2$ in (2.6) to deduce that, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$A(-z_1, -z_2) + A(z_1, -z_2) = 2 \sum_{i,j} a_{2i,2j} z_1^{2i} z_2^{2j} - 2 \sum_{i,j} a_{2i,2j+1} z_1^{2i} z_2^{2j+1}. \tag{2.7}$$

By adding (2.6) and (2.7), we obtain the identity

$$A(z_1, z_2) + A(-z_1, z_2) + A(z_1, -z_2) + A(-z_1, -z_2) = 4 \sum_{i,j} a_{2i,2j} z_1^{2i} z_2^{2j}, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{2.8}$$

which we can now use to prove the following characterization result.

Theorem 2.1.1. *Suppose that a is a refinement mask such that $\sum_{\boldsymbol{j}} a_{\boldsymbol{j}} = 4$. Then a is interpolatory if and only if the corresponding mask symbol A, as defined by (1.9), satisfies the identity*

$$A(z_1, z_2) + A(-z_1, z_2) + A(z_1, -z_2) + A(-z_1, -z_2) = 4, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}. \tag{2.9}$$

Proof. Suppose first that a is interpolatory. From (2.2), since $a_{2i,2j} = \delta_{i,j}$, we have that

$$\sum_{i,j} a_{2i,2j} z_1^{2i} z_2^{2j} = 1,$$

which, together with (2.8), implies that (2.9) holds.

Conversely, if (2.9) holds, we obtain from (2.8) that

$$\sum_{i,j} a_{2i,2j} z_1^{2i} z_2^{2j} = 1,$$

which proves that $a_{2i,2j} = \delta_{i,j}$. Therefore, (2.2) holds and a is interpolatory. $\square$

Note that, for a given refinement mask a, the condition in the second line of (2.2) is achieved if the corresponding mask symbol A satisfies the identity (2.9), and if there exist positive integers k_1, k_2 and a Laurent polynomial B such that

$$A(z_1, z_2) = (1 + z_1)^{k_1}(1 + z_2)^{k_2} B(z_1, z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{2.10}$$

since then $A(-1, z_2) = A(z_1, -1) = 0$ for any $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, so that (2.9) yields $A(1, 1) = 4$ and thus the mask symbol A is interpolatory. Hence the following result.

Corollary 2.1.2. *For a Laurent polynomial A satisfying the identity (2.9), if there exists a Laurent polynomial B such that (2.10) holds, then A is an interpolatory mask symbol.*

Note that the converse of Corollary 2.1.2 is not necessarily true, for if A is an interpolatory mask symbol that satifies the identity (2.9), then since $A(1, 1) = 4$, we only get that $A(-1, 1) + A(1, -1) + A(-1, -1) = 0$, which does not necessarily imply that A is of the factorized form (2.10).

Motivated by the result of Corollary 2.1.2, we proceed to characterize in Section 2.2 below the interpolatory mask symbols which are in the factorized form (2.10).

2.2 General form

We proceed to give the general form of interpolatory mask symbols that are factorizable in the sense of (2.10). More precisely, we start by solving for the Laurent polynomial A in the identity (2.9) with the help of the Bezout theorem, to finally establish a general formulation of interpolatory mask symbols.

To facilitate our investigation, we henceforth assume that the mask symbol A has the factorized form

$$A(z_1, z_2) = 2^{2-k_1-k_2}(1 + z_1)^{k_1}(1 + z_2)^{k_2}B(z_1, z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{2.11}$$

for some integers $k_1, k_2 \in \mathbb{N}$ and some Laurent polynomial B such that $B(1, 1) = 1$, $B(-1, z_2) \neq 0$ and $B(z_1, -1) \neq 0$ for all $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, so that, from (2.11), it holds that $A(1, 1) = 4$. Also, we shall assume that A satisfies the identity (2.9), in which case, according to Corollary 2.1.2, A is an interpolatory mask symbol.

Polynomial identities

To characterize the mask symbol A, we first prove the following result.

Lemma 2.2.1. *Let $k_1, k_2 \in \mathbb{N}$ and suppose α_1, α_2 are two odd integers in $\mathbb{N}$. Then:*

(a) if $\alpha_1 < 2k_1$, there exists a polynomial S_1 which is odd in z_2, with degree α_2 in z_2, and degree less than k_1 in z_1, such that the general Laurent polynomial solution K_1 of the identity

$$(1 + z_1)^{k_1}K_1(z_1, z_2) - (1 - z_1)^{k_1}K_1(-z_1, z_2) = z_1^{\alpha_1} z_2^{\alpha_2}, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{2.12}$$

is the Laurent polynomial given by

$$K_1(z_1, z_2) = S_1(z_1, z_2) + T_1(z_1, z_2)(1 - z_1)^{k_1}, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{2.13}$$

with T_1 denoting an arbitrary even Laurent polynomial in z_1; also, K_1 is odd in z_2 if and only if T_1 is odd in z_2.

(b) if $\alpha_2 < 2k_2$, there exists a polynomial S_2 which is odd in z_1, with degree α_1 in z_1, and degree less than k_2 in z_2, such that the general Laurent polynomial solution K_2 of the identity

$$(1 + z_2)^{k_2} K_2(z_1, z_2) - (1 - z_2)^{k_2} K_2(z_1, -z_2) = z_1^{\alpha_1} z_2^{\alpha_2}, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{2.14}$$

is the Laurent polynomial given by

$$K_2(z_1, z_2) = S_2(z_1, z_2) + T_2(z_1, z_2)(1 - z_2)^{k_2}, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{2.15}$$

with T_2 denoting an arbitrary even Laurent polynomial in z_2; also, K_2 is odd in z_1 if and only if T_2 is odd in z_1.

Proof. (a) Since the two univariate polynomials $(1+z_1)^{k_1}$ and $(1-z_1)^{k_1}$ have no common factor, there exist by the Bezout theorem two univariate polynomials U_1 and V_1 such that

$$(1 + z_1)^{k_1} U_1(z_1) + (1 - z_1)^{k_1} V_1(z_1) = 1, \quad z_1 \in \mathbb{C}. \tag{2.16}$$

Multiplying both sides of (2.16) by $z_1^{\alpha_1} z_2^{\alpha_2}$ yields, for $z_1, z_2 \in \mathbb{C}$,

$$(1 + z_1)^{k_1} \left[z_1^{\alpha_1} z_2^{\alpha_2} U_1(z_1)\right] + (1 - z_1)^{k_1} \left[z_1^{\alpha_1} z_2^{\alpha_2} V_1(z_1)\right] = z_1^{\alpha_1} z_2^{\alpha_2}, \quad z_1, z_2 \in \mathbb{C}. \tag{2.17}$$

Using the polynomial division theorem, we deduce the existence of two polynomials Q_1 and R_1 satisfying

$$z_1^{\alpha_1} V_1(z_1) = Q_1(z_1)(1+z_1)^{k_1} + R_1(z_1), \quad z_1 \in \mathbb{C}, \tag{2.18}$$

such that the degree of R_1 is less than k_1, and where Q_1 and R_1 are uniquely determined by α_1 and V_1. It then follows from (2.17) that

$$(1+z_1)^{k_1} S_1(z_1,z_2) + (1-z_1)^{k_1} \tilde{R}_1(z_1,z_2) = z_1^{\alpha_1} z_2^{\alpha_2}, \quad z_1, z_2 \in \mathbb{C}, \tag{2.19}$$

where S_1 is the polynomial defined by $S_1(z_1,z_2) = z_1^{\alpha_1} z_2^{\alpha_2} U_1(z_1) + (1-z_1)^{k_1} z_2^{\alpha_2} Q_1(z_1)$, and $\tilde{R}_1$ is the polynomial given by $\tilde{R}_1(z_1,z_2) = z_2^{\alpha_2} R_1(z_1)$, for all $z_1, z_2 \in \mathbb{C}$. We claim that the degree in z_1 of S_1 is less than k_1. To prove this, we first note from (2.19) that

$$(1+z_1)^{k_1} S_1(z_1,z_2) = z_1^{\alpha_1} z_2^{\alpha_2} - (1-z_1)^{k_1} \tilde{R}_1(z_1,z_2), \quad z_1, z_2 \in \mathbb{C},$$

according to which, since the degree of $\tilde{R}_1$ in z_1 is less than k_1, and since $\alpha_1 < 2k_1$, we neccessarily have that the degree in z_1 of S_1 is less than k_1.

Replacing z_1 by $-z_1$ in (2.19), and using the fact that α_1 is odd, we deduce that

$$(1-z_1)^{k_1} \left[-S_1(-z_1,z_2)\right] + (1+z_1)^{k_1} \left[-\tilde{R}_1(-z_1,z_2)\right] = z_1^{\alpha_1} z_2^{\alpha_2}, \quad z_1, z_2 \in \mathbb{C}. \tag{2.20}$$

Substracting the identities (2.19) and (2.20) now yields

$$(1+z_1)^{k_1}[S_1(z_1,z_2) + \tilde{R}_1(-z_1,z_2)] = -(1-z_1)^{k_1}[S_1(-z_1,z_2) + \tilde{R}_1(z_1,z_2)], \quad z_1, z_2 \in \mathbb{C},$$

and thus

$$S_1(z_1,z_2) + \tilde{R}_1(-z_1,z_2) = M_1(z_1,z_2)(1-z_1)^{k_1}, \quad z_1, z_2 \in \mathbb{C}, \tag{2.21}$$

for some polynomial M_1. Since the degree in z_1 of the polynomial in the left-hand-side of (2.21) is less than k_1, we neccessarily have $M_1 = 0$ in (2.21), or, equivalently,

$$S_1(z_1, z_2) = -\tilde{R}_1(-z_1, z_2), \quad z_1, z_2 \in \mathbb{C}, \tag{2.22}$$

$$\tilde{R}_1(z_1, z_2) = -S_1(-z_1, z_2), \quad z_1, z_2 \in \mathbb{C}. \tag{2.23}$$

Using (2.19), (2.22) and (2.23), we find that the polynomial S_1 satisfies

$$(1+z_1)^{k_1} S_1(z_1, z_2) - (1-z_1)^{k_1} S_1(-z_1, z_2) = z_1^{\alpha_1} z_2^{\alpha_2}, \quad z_1, z_2 \in \mathbb{C}, \tag{2.24}$$

which means that S_1 is a particular polynomial solution of the identity (2.12) with a degree in z_1 less than k_1. Moreover, from (2.22), we see that $S_1(z_1, z_2) = -z_2^{\alpha_2} R_1(-z_1)$. Since α_2 is odd, we conclude that S_1 is odd in z_2, and that its degree in z_2 is α_2.

Now, let K_1 denote the general Laurent polynomial solution of (2.12). Substracting (2.12) from (2.24), we obtain, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$(1+z_1)^{k_1} \left[K_1(z_1, z_2) - S_1(z_1, z_2)\right] = (1-z_1)^{k_1} \left[K_1(-z_1, z_2) - S_1(-z_1, z_2)\right]. \tag{2.25}$$

Since $(1+z_1)^{k_1}$ and $(1-z_1)^{k_1}$ have no common factor, it follows from (2.25) that there exists a Laurent polynomial T_1 satisfying

$$K_1(z_1, z_2) - S_1(z_1, z_2) = T_1(z_1, z_2)(1-z_1)^{k_1}, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}. \tag{2.26}$$

Substituting (2.26) into (2.25) yields that $T_1(z_1, z_2) = T_1(-z_1, z_2)$ for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, i.e T_1 is even in z_1. Thus, we deduce from (2.26) that K_1 is given by

$$K_1(z_1, z_2) = S_1(z_1, z_2) + T_1(z_1, z_2)(1-z_1)^{k_1}, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{2.27}$$

where T_1 is an arbitrary even Laurent polynomial in z_1.

Also, since S_1 is odd in z_2, we get from (2.27) that, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned} K_1(z_1, -z_2) = &\quad S_1(z_1, -z_2) + T_1(z_1, -z_2)(1 - z_1)^{k_1} \\ = &\quad -S_1(z_1, z_2) + T_1(z_1, -z_2)(1 - z_1)^{k_1}, \end{aligned} \tag{2.28}$$

whereas also, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$-K_1(z_1, z_2) = \quad -S_1(z_1, z_2) - T_1(z_1, z_2)(1 - z_1)^{k_1}. \tag{2.29}$$

Substracting the identities (2.28) and (2.29) gives, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$K_1(z_1, -z_2) + K_1(z_1, z_2) = (1 - z_1)^{k_1}[T_1(z_1, -z_2) + T_1(z_1, z_2)],$$

from which it then immediately follows that K_1 is odd in z_2 if and only if T_1 is odd in z_2.

(b) The proof is similar to (a). $\square$

The Euclidean algorithm

We present here a detailed method to compute the polynomials S_1 and S_2 in Lemma 2.2.1 by using the Euclidean algorithm.

Under the conditions of Lemma 2.2.1, with $k_1, k_2 \in \mathbb{N}$, and where $\alpha_1, \alpha_2 \in \mathbb{N}$ are odd integers such that also $\alpha_1 < 2k_1$, we first proceed to find the univariate polynomials U_1 and V_1 such that (2.16) holds.

From the polynomial division theorem, there exist univariate polynomials q_0, q_1 and

r_1, r_2 such that, for $z_1 \in \mathbb{C}$,

$$(1+z_1)^{k_1} = q_0(z_1)(1-z_1)^{k_1} + r_1(z_1), \quad \deg(r_1) < k_1, \tag{2.30}$$

$$(1-z_1)^{k_1} = q_1(z_1)r_1(z_1) + r_2(z_1), \quad \deg(r_2) < \deg(r_1). \tag{2.31}$$

Repeated applications of polynomial division then yield the existence of $n \in \mathbb{N}$ and univariate polynomials q_j, $j = 2, \ldots, n+1$ and r_j, $j = 3, \ldots, n+2$, such that, for $z_1 \in \mathbb{C}$,

$$\left.\begin{aligned} r_1(z_1) = {} & q_2(z_1)r_2(z_1) + r_3(z_1), \quad \deg(r_3) < \deg(r_2), \\ & \vdots \\ r_{n-1}(z_1) = {} & q_n(z_1)r_n(z_1) + r_{n+1}(z_1), \quad \deg(r_{n+1}) \geq 1, \\ r_n(z_1) = {} & q_{n+1}(z_1)r_{n+1}(z_1) + r_{n+2}(z_1), \quad r_{n+2}(z_1) = c, \text{ a constant,} \end{aligned}\right\} \tag{2.32}$$

so that, by back substitution, it holds that, for $z_1 \in \mathbb{C}$,

$$r_{j+1}(z_1) = r_{j-1}(z_1) - q_j(z_1)r_j(z_1), \quad j = 0, \ldots, n+1, \tag{2.33}$$

with $r_{-1}(z_1) = (1+z_1)^{k_1}$ and $r_0(z_1) = (1-z_1)^{k_1}$, $z_1 \in \mathbb{C}$. Observe that $c \neq 0$, otherwise, by back substitution and by using (2.33), $(1+z_1)^{k_1}$ and $(1-z_1)^{k_1}$ would have $r_{n+1}(z_1)$ as a common factor, which is impossible since $\deg(r_{n+1}) \geq 1$.

Now define the polynomial sequence $\{T_{i,j}(z_1) : i = 0, 1, 2, 3; \quad j = -1, 0, \ldots, n+2\}$ by

$$\left.\begin{aligned} T_{i,j+1}(z_1) = {} & T_{i,j-1}(z_1) - q_j(z_1)T_{i,j}(z_1), \quad \text{for } i = 0, 1, 2 \text{ and } j = 1, \ldots, n+1, \\ T_{3,j}(z_1) = {} & q_j(z_1), \quad \text{for } j = 0, \ldots, n+1 \\ T_{3,-1}(z_1) = {} & T_{3,n+2}(z_1) = 0, \end{aligned}\right\} \tag{2.34}$$

with also

$$\left.\begin{aligned} T_{0,-1}(z_1) &= (1+z_1)^{k_1}, \\ T_{1,-1}(z_1) &= 0, \\ T_{2,-1}(z_1) &= 1, \end{aligned}\right\} \tag{2.35}$$

$$\left.\begin{aligned} T_{0,0}(z_1) &= (1-z_1)^{k_1}, \\ T_{1,0}(z_1) &= 1, \\ T_{2,0}(z_1) &= 0. \end{aligned}\right\} \tag{2.36}$$

Observe from (2.34), (2.33) and the first lines of (2.35) and (2.36) that then

$$T_{0,j}(z_1) = r_j(z_1), \quad j = 1, 2, \ldots, n+2. \tag{2.37}$$

It follows that the matrix T consisting of the polynomials $[\ T_{i,j}(z_1)\ ]$, for $0 \leq i \leq 3$ and $-1 \leq j \leq n+2$, is given by

$$T = \begin{bmatrix} (1+z_1)^{k_1} & (1-z_1)^{k_1} & r_1(z_1) & r_2(z_1) & \ldots & r_{n+1}(z_1) & r_{n+2}(z_1) \\ 0 & 1 & -q_0(z_1) & 1+q_1(z_1)q_0(z_1) & \ldots & T_{1,n+1}(z_1) & T_{1,n+2}(z_1) \\ 1 & 0 & 1 & -q_1(z_1) & \ldots & T_{2,n+1}(z_1) & T_{2,n+2}(z_1) \\ 0 & q_0(z_1) & q_1(z_1) & q_2(z_1) & \ldots & q_{n+1}(z_1) & 0 \end{bmatrix}.$$

We claim that, for $j = 1, \ldots, n+2$,

$$(1+z_1)^{k_1} T_{2,j}(z_1) + (1-z_1)^{k_1} T_{1,j}(z_1) = r_j(z_1), \quad z_1 \in \mathbb{C}. \tag{2.38}$$

We prove this by induction on j. Observe first from (2.34) (2.30) that (2.38) holds for

$j = 1$. Also, from (2.31), (2.30) and (2.34), we obtain, for $z_1 \in \mathbb{C}$,

$$\begin{aligned}
r_2(z_1) =& (1-z_1)^{k_1} - q_1(z_1)r_1(z_1) \\
=& (1-z_1)^{k_1} - q_1(z_1)[(1+z_1)^{k_1} - q_0(z_1)(1-z_1)^{k_1}] \\
=& [-q_1(z_1)](1+z_1)^{k_1} + [1 + q_1(z_1)q_0(z_1)](1-z_1)^{k_1} \\
=& T_{2,2}(z_1)(1+z_1)^{k_1} + T_{1,2}(z_1)(1-z_1)^{k_1},
\end{aligned}$$

thereby proving that (2.38) holds for $j = 2$.

Suppose now that (2.38) is true for $j-1$ and j with $j \in \{2, \ldots, n+1\}$. Multiplying both sides of (2.38) by $-q_j(z_1)$ yields

$$(1+z_1)^{k_1}[-q_j(z_1)T_{2,j}(z_1)] + (1-z_1)^{k_1}[-q_j(z_1)T_{1,j}(z_1)] = -q_j(z_1)r_j(z_1), \quad z_1 \in \mathbb{C}. \tag{2.39}$$

From the inductive assumption, recall that

$$(1+z_1)^{k_1}T_{2,j-1}(z_1) + (1-z_1)^{k_1}T_{1,j-1}(z_1) = r_{j-1}(z_1), \quad z_1 \in \mathbb{C}. \tag{2.40}$$

Addition of equations (2.39) and (2.40), and using also (2.34) and (2.33), then yield

$$(1+z_1)^{k_1}T_{2,j+1}(z_1) + (1-z_1)^{k_1}T_{1,j+1}(z_1) = r_{j+1}(z_1), \quad z_1 \in \mathbb{C},$$

thereby completing our inductive proof of (2.38).

In particular, by choosing $j = n+2$ in (2.38), and since $r_{n+2}(z_1) = c \neq 0$, we deduce that

$$(1+z_1)^{k_1}U_1(z_1) + (1-z_1)^{k_1}V_1(z_1) = 1, \quad z_1 \in \mathbb{C}, \tag{2.41}$$

where the polynomials U_1 and V_1 are given by

$$U_1(z_1) = \frac{T_{2,n+2}(z_1)}{c} \quad \text{and} \quad V_1(z_1) = \frac{T_{1,n+2}(z_1)}{c}, \quad z_1 \in \mathbb{C}. \tag{2.42}$$

Next, from the polynomial division theorem, there exist univariate polynomials Q_1 and R_1 such that (2.18) holds, that is, for $z_1 \in \mathbb{C}$,

$$z_1^{\alpha_1} V_1(z_1) = Q_1(z_1)(1+z_1)^{k_1} + R_1(z_1), \quad \text{with } \deg(R_1) < k_1, \tag{2.43}$$

so that, from the proof of Lemma 2.2.1 (a), by choosing the polynomial S_1 as

$$S_1(z_1, z_2) = -z_2^{\alpha_2} R_1(-z_1), \quad z_1, z_2 \in \mathbb{C}, \tag{2.44}$$

it follows that (2.24) holds. In other words, we have the identity

$$(1+z_1)^{k_1} S_1(z_1, z_2) - (1-z_1)^{k_1} S_1(-z_1, z_2) = z_1^{\alpha_1} z_2^{\alpha_2}, \quad z_1, z_2 \in \mathbb{C}. \tag{2.45}$$

Moreover, we know from Lemma 2.2.1 (a) that S_1 is odd in z_2, that its degree in z_2 is α_2, and that its degree in z_1 is less than k_1.

We have now proved the following algorithm for the explicit computation of the polynomial S_1 of Lemma 2.2.1 (a)

Algorithm for the computation of S_1:

1. Use polynomial division to obtain the polynomials $\{q_j(z_1) : j = 0, \ldots, n+1\}$ and $\{r_j(z_1) : j = 1, \ldots, n+2\}$, with $r_{n+2}(z_1) = c \neq 0$ as in (2.32).

2. Define the polynomial sequence $\{T_{i,j}(z_1) : i = 0, 1, 2; j = -1, \ldots, n+2\}$ recursively by means of (2.34), (2.35) and (2.36).

3. Define the polynomials U_1 and V_1 by (2.42);

4. Use the polynomial division theorem to find Q_1 and R_1 such that (2.43) holds;

5. The polynomial S_1 is then given by (2.44).

The construction of the polynomial S_2, under the constraint $\alpha_2 < 2k_2$, is analogous to that of S_1.

We proceed to give an example by finding the polynomial S_1 for $k_1 = 2$. The case $k_1 = 1$ will be presented in Section 2.3, and will be used to characterize the mask symbols of the box spline functions from Chapter 1. Under the conditions of Lemma 2.2.1 and the above algorithm, let $k_1 = 2$, $\alpha_1 \in \{1, 3\}$, and let $\alpha_2 \in \mathbb{N}$ be any odd integer. Observe that, for $z_1 \in \mathbb{C}$,

$$
\begin{aligned}
(1 + z_1)^2 =& q_0(z_1)(1 - z_1)^2 + r_1(z_1), \quad \text{with} \quad q_0(z_1) = 1 \quad \text{and} \quad r_1(z_1) = 4z_1, \\
(1 - z_1)^2 =& q_1(z_1)r_1(z_1) + r_2(z_1), \quad \text{with} \quad q_1(z_1) = \frac{1}{4}z_1 - \frac{1}{2} \quad \text{and} \quad r_2(z_1) = 1.
\end{aligned}
$$

It follows that the matrix T is given by

$$
T = \begin{bmatrix}
(1 + z_1)^2 & (1 - z_1)^2 & 4z_1 & 1 \\
0 & 1 & -1 & \frac{1}{4}z_1 + \frac{1}{2} \\
1 & 0 & 1 & -\frac{1}{4}z_1 + \frac{1}{2} \\
0 & 1 & \frac{1}{4}z_1 - \frac{1}{2} & 0
\end{bmatrix},
$$

which, together with (2.42), yields that the polynomials U_1 and V_1 are given by

$$
U_1(z_1) = -\frac{1}{4}z_1, \quad V_1(z_1) = \frac{1}{4}z_1 + \frac{1}{2}, \quad z_1 \in \mathbb{C}. \tag{2.46}
$$

Two cases occur:

- if $\alpha_1 = 1$: we deduce from (2.43) that, for $z_1 \in \mathbb{C}$,

$$z_1 V_1(z_1) = \frac{1}{4}z_1^2 + \frac{1}{2}z_1 = Q_1(z_1)(1+z_1)^2 + R_1(z_1),$$

with $Q_1(z_1) = \frac{1}{4}$ and $R_1(z_1) = -\frac{1}{4}$, and it follows from (2.44) that the polynomial S_1 is given by

$$S_1(z_1, z_2) = \frac{1}{4}z_2^{\alpha_2}, \quad z_1, z_2 \in \mathbb{C}. \tag{2.47}$$

- if $\alpha_1 = 3$: we deduce from (2.43) that, for $z_1 \in \mathbb{C}$,

$$z_1^3 V_1(z_1) = \frac{1}{4}z_1^4 + \frac{1}{2}z_1^3 = Q_1(z_1)(1+z_1)^2 + R_1(z_1),$$

with $Q_1(z_1) = \frac{1}{4}z_1^2 - \frac{1}{4}$ and $R_1(z_1) = \frac{1}{2}z_1 + \frac{1}{4}$, and it follows from (2.44) that the polynomial S_1 is given by

$$S_1(z_1, z_2) = \frac{1}{4}(2z_1 - 1)z_2^{\alpha_2}, \quad z_1, z_2 \in \mathbb{C}. \tag{2.48}$$

Observe in particular from (2.47) and (2.48) that the degree of S_1 in z_1 is less than $k_1 = 2$, and that S_1 is odd in z_2 with degree α_2 in z_2.

First factorization of mask symbols

With the help of Lemma 2.2.1, we can prove the following formula.

Lemma 2.2.2. *For an interpolatory mask symbol A, suppose there exist integers $k_1, k_2 \in \mathbb{N}$ and a Laurent polynomial B such that (2.11) holds, and let α_1 and α_2 be any pair of odd integers such that $\alpha_1 < 2k_1$ and $\alpha_2 < 2k_2$. Then both the following results hold:*

(a) There exist Laurent polynomials K_1, K_2 and T_3 such that the Laurent polynomial B has, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, the form

$$B(z_1, z_2) = 2^{k_1+k_2} z_1^{-2\alpha_1} z_2^{-2\alpha_2} [K_1(z_1, z_2)K_2(z_1, z_2) + T_3(z_1, z_2)(1 - z_2)^{k_2}], \tag{2.49}$$

where the Laurent polynomial T_3 is odd in z_2, and with K_1, K_2 satisfying the respective identities

$$\left\{\begin{array}{lcl} (1+z_1)^{k_1} K_1(z_1, z_2) - (1-z_1)^{k_1} K_1(-z_1, z_2) & = & z_1^{\alpha_1} z_2^{\alpha_2}, \\ (1+z_2)^{k_2} K_2(z_1, z_2) - (1-z_2)^{k_2} K_2(z_1, -z_2) & = & z_1^{\alpha_1} z_2^{\alpha_2}, \end{array}\right. \quad , \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}. \tag{2.50}$$

Moreover, K_1 and K_2 are formulated explicitly by the expressions (2.13), (2.15), with S_1, T_1, S_2 and T_2 as described in Lemma 2.2.1, and where both K_1 and T_1 are odd in z_2.

(b) There exist Laurent polynomials L_1, L_2 and $\tilde{T}_3$ such that the Laurent polynomial B has, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, the form

$$B(z_1, z_2) = 2^{k_1+k_2} z_1^{-2\alpha_1} z_2^{-2\alpha_2} [L_1(z_1, z_2)L_2(z_1, z_2) + \tilde{T}_3(z_1, z_2)(1 - z_1)^{k_1}], \tag{2.51}$$

where the Laurent polynomial $\tilde{T}_3$ is odd in z_1, and with L_1, L_2 satisfying respective identities

$$\left\{\begin{array}{l} (1+z_1)^{k_1} L_1(z_1, z_2) - (1-z_1)^{k_1} L_1(-z_1, z_2) = z_1^{\alpha_1} z_2^{\alpha_2}, \\ (1+z_2)^{k_2} L_2(z_1, z_2) - (1-z_2)^{k_2} L_2(z_1, -z_2) = z_1^{\alpha_1} z_2^{\alpha_2}, \end{array}\right. \quad , \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}. \tag{2.52}$$

Moreover, L_1 and L_2 are formulated explicitly by the expressions (2.13), (2.15), with S_1, T_1, S_2 and T_2 as described in Lemma 2.2.1, and where both L_2 and T_2 are odd in z_1.

Proof. (a) By defining the Laurent polynomial H as

$$H(z_1, z_2) = A(z_1, z_2) + A(z_1, -z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{2.53}$$

we observe that the identity (2.9) is equivalent to

$$H(z_1, z_2) + H(-z_1, z_2) = 4, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}. \tag{2.54}$$

Also, by using (2.11) and (2.53), we have that

$$H(z_1, z_2) = 2^{2-k_1-k_2}(1 + z_1)^{k_1} G(z_1, z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{2.55}$$

where the Laurent polynomial G is defined by

$$G(z_1, z_2) = (1 + z_2)^{k_2} B(z_1, z_2) + (1 - z_2)^{k_2} B(z_1, -z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{2.56}$$

with B denoting the Laurent polynomial for which (2.11) is satisfied.

It then follows from (2.54) and (2.55) that G satisfies the identity

$$2^{-k_1-k_2}(1 + z_1)^{k_1} G(z_1, z_2) + 2^{-k_1-k_2}(1 - z_1)^{k_1} G(-z_1, z_2) = 1, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}. \tag{2.57}$$

Now, choose any pair of odd integers $\alpha_1, \alpha_2 \in \mathbb{N}$ such that $\alpha_1 < 2k_1$ and $\alpha_2 < 2k_2$. Then, for the Laurent polynomial G given by (2.56), we define the Laurent polynomial K_1 by

$$G(z_1, z_2) = 2^{k_1+k_2} z_1^{-\alpha_1} z_2^{-\alpha_2} K_1(z_1, z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}. \tag{2.58}$$

It follows from (2.58) and (2.57) that K_1 satisfies the identity

$$(1 + z_1)^{k_1} z_1^{-\alpha_1} z_2^{-\alpha_2} K_1(z_1, z_2) - (1 - z_1)^{k_1} z_1^{-\alpha_1} z_2^{-\alpha_2} K_1(-z_1, z_2) = 1, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\},$$

or, equivalently,

$$(1+z_1)^{k_1}K_1(z_1,z_2) - (1-z_1)^{k_1}K_1(-z_1,z_2) = z_1^{\alpha_1}z_2^{\alpha_2}, \quad z_1,z_2 \in \mathbb{C}\setminus\{0\}. \tag{2.59}$$

Hence, according to Lemma 2.2.1 (a), there exist a polynomial S_1 and a Laurent polynomial T_1 such that

$$K_1(z_1,z_2) = S_1(z_1,z_2) + (1-z_1)^{k_1}T_1(z_1,z_2), \quad z_1,z_2 \in \mathbb{C}\setminus\{0\},$$

with the polynomial S_1 and the Laurent polynomial T_1 satisfying the properties as stated in Lemma 2.2.1 (a).

Besides, (2.55) and (2.58) yield

$$H(z_1,z_2) = 4(1+z_1)^{k_1}z_1^{-\alpha_1}z_2^{-\alpha_2}K_1(z_1,z_2), \quad z_1,z_2 \in \mathbb{C}\setminus\{0\},$$

according to which, since the Laurent polynomial H defined by (2.53) is even in z_2, we deduce that K_1 is odd in z_2, and hence also, from Lemma 2.2.1 (a), T_1 is also odd in z_2.

Next, we define the Laurent polynomial $\tilde{B}$ by

$$B(z_1,z_2) = 2^{k_1+k_2}z_1^{-2\alpha_1}z_2^{-2\alpha_2}\tilde{B}(z_1,z_2), \quad z_1,z_2 \in \mathbb{C}\setminus\{0\}. \tag{2.60}$$

From (2.58) and (2.56) we then obtain

$$(1+z_2)^{k_2}B(z_1,z_2) + (1-z_2)^{k_2}B(z_1,-z_2) = 2^{k_1+k_2}z_1^{-\alpha_1}z_2^{-\alpha_2}K_1(z_1,z_2), \quad z_1,z_2 \in \mathbb{C}\setminus\{0\}, \tag{2.61}$$

which, together with (2.60), shows that $\tilde{B}$ satisfies the identity

$$(1+z_2)^{k_2}\tilde{B}(z_1,z_2) + (1-z_2)^{k_2}\tilde{B}(z_1,-z_2) = z_1^{\alpha_1}z_2^{\alpha_2}K_1(z_1,z_2), \quad z_1,z_2 \in \mathbb{C}\setminus\{0\}. \tag{2.62}$$

It now remains to find $\tilde{B}$. To this end, we first obtain a particular solution of (2.62) by considering the Laurent polynomial B_1 defined by

$$B_1(z_1, z_2) = K_1(z_1, z_2)K_2(z_1, z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{2.63}$$

for some arbitrary appropriate Laurent polynomial K_2 such that B_1 satisfies (2.62), i.e.

$$(1 + z_2)^{k_2} B_1(z_1, z_2) + (1 - z_2)^{k_2} B_1(z_1, -z_2) = z_1^{\alpha_1} z_2^{\alpha_2} K_1(z_1, z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}. \tag{2.64}$$

Since K_1 is odd in z_2, we have from (2.63) that, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$B_1(z_1, -z_2) = K_1(z_1, -z_2)K_2(z_1, -z_2) = -K_1(z_1, z_2)K_2(z_1, -z_2),$$

so that, from (2.64) and (2.63), and after dividing by $K_1(z_1, z_2)$, we deduce that, if the Laurent polynomial K_2 is chosen to satisfy the identity

$$(1 + z_2)^{k_2} K_2(z_1, z_2) - (1 - z_2)^{k_2} K_2(z_1, -z_2) = z_1^{\alpha_1} z_2^{\alpha_2}, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{2.65}$$

then the Laurent polynomial B_1 defined by (2.63) satisfies the identity (2.64). But according to Lemma 2.2.1 (b), the general Laurent polynomial solution K_2 of the identity (2.65) is given by

$$K_2(z_1, z_2) = S_2(z_1, z_2) + (1 - z_2)^{k_2} T_2(z_1, z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\},$$

with the polynomial S_2 and the Laurent polynomial T_2 satisfying the properties as stated in Lemma 2.2.1 (b).

Substracting the equations (2.62) and (2.64) now yields, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$(1 + z_2)^{k_2} [\tilde{B}(z_1, z_2) - B_1(z_1, z_2)] = -(1 - z_2)^{k_2} [\tilde{B}(z_1, -z_2) - B_1(z_1, -z_2)], \tag{2.66}$$

and, since the univariate polynomials $(1+z_2)^{k_2}$ and $(1-z_2)^{k_2}$ have no common factor, there exists a Laurent polynomial T_3 such that, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$\tilde{B}(z_1, z_2) - B_1(z_1, z_2) = (1 - z_2)^{k_2} T_3(z_1, z_2). \tag{2.67}$$

Substituting the expressions in (2.67) into (2.66), we obtain, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$(1+z_2)^{k_2}(1-z_2)^{k_2} T_3(z_1, z_2) = -(1-z_2)^{k_2}(1+z_2)^{k_2} T_3(z_1, -z_2),$$

from which we deduce that T_3 is odd in z_2.

Also, we deduce from (2.67) that

$$\tilde{B}(z_1, z_2) = B_1(z_1, z_2) + T_3(z_1, z_2)(1 - z_2)^{k_2}, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\},$$

which, together with (2.60) and (2.63), shows that B is indeed given by (2.49).

(b) By defining the Laurent polynomial J as

$$J(z_1, z_2) = A(z_1, z_2) + A(-z_1, z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{2.68}$$

observe that the identity (2.9) is equivalent to $J(z_1, z_2) + J(z_1, -z_2) = 4$, $z_1, z_2 \in \mathbb{C} \setminus \{0\}$. The rest of proof then uses a similar argument as in (a). $\square$

The characterization result

Note that (2.49) and (2.51) yield two different formulae for the Laurent polynomial B in Lemma 2.2.2. We proceed here to give an alternative expression for B which verifies simultaneously (2.49) and (2.51).

Using Lemmas 2.2.1 and 2.2.2, we prove the following result which yields an important characterization for interpolatory mask symbols.

Theorem 2.2.3. *For a Laurent polynomial A, suppose that there exist integers $k_1, k_2 \in \mathbb{N}$ and a Laurent polynomial B such that (2.11) holds. Then A defines an interpolatory mask symbol if and only if for any pair of odd integers α_1 and α_2 such that $\alpha_1 < 2k_1$ and $\alpha_2 < 2k_2$, the Laurent polynomial B has, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, the form*

$$\begin{aligned} B(z_1, z_2) =& 2^{k_1+k_2} z_1^{-2\alpha_1} z_2^{-2\alpha_2} \left[T(z_1, z_2)(1 - z_1)^{k_1}(1 - z_2)^{k_2} \right. \\ &\left. + \left\{ S_1(z_1, z_2) + T_1(z_1, z_2)(1 - z_1)^{k_1} \right\} \left\{ S_2(z_1, z_2) + T_2(z_1, z_2)(1 - z_2)^{k_2} \right\} \right], \end{aligned} \tag{2.69}$$

where the polynomials S_1 and S_2 are as in Lemma 2.2.1, i.e. S_1 and S_2 are respectively odd in z_2 and odd in z_1, they satisfy the respective identities

$$\begin{cases} (1 + z_1)^{k_1} S_1(z_1, z_2) - (1 - z_1)^{k_1} S_1(-z_1, z_2) = z_1^{\alpha_1} z_2^{\alpha_2}, \\ (1 + z_2)^{k_2} S_2(z_1, z_2) - (1 - z_2)^{k_2} S_2(z_1, -z_2) = z_1^{\alpha_1} z_2^{\alpha_2}, \end{cases} \quad z_1, z_2 \in \mathbb{C}, \tag{2.70}$$

where also S_1 has a degree less than k_1 in z_1, and S_2 has a degree less than k_2 in z_2. Besides, the Laurent polynomials T_1, T_2 and T are respectively even in z_1 but odd in z_2, even in z_2 but odd in z_1, and odd in both z_1 and z_2.

Proof. We show that the proof in the necessary direction can be obtained either by starting with the formula given by (2.49) with an appropriate choice for the polynomial L_1, or by starting with the formula given by (2.51) with an appropriate choice for the polynomial K_2. We then prove the theorem in the sufficient direction by using Theorem 2.1.1.

To prove the theorem in the necessary direction, we suppose that A defines an interpolatory mask symbol and consider any pair of odd integers $\alpha_1, \alpha_2 \in \mathbb{N}$ such that $\alpha_1 < 2k_1$ and $\alpha_2 < 2k_2$.

According to Lemma 2.2.2, the Laurent polynomial B for which (2.11) is satisfied, has the forms given by (2.49) and (2.51), where the Laurent polynomials K_2 in (2.49) and L_1 in (2.51) are to be chosen as specified in Lemma 2.2.2.

We see from Lemma 2.2.1 and 2.2.2 that we may choose $L_1 = K_1$, according to which it then holds that both K_1 and L_1 are even in z_1 and odd in z_2. It follows that, from (2.11) and (2.51), it holds that

$$\begin{aligned} A(z_1, -z_2) =& 4(1+z_1)^{k_1}(1-z_2)^{k_2} z_1^{-2\alpha_1} z_2^{-2\alpha_2} \\ & [-L_1(z_1, z_2)L_2(z_1, -z_2) + \tilde{T}_3(z_1, -z_2)(1-z_1)^{k_1}], \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \end{aligned}$$

which, together with (2.11), (2.51) and the second line of (2.52), shows that, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned} A(z_1, z_2) + A(z_1, -z_2) =& 4(1+z_1)^{k_1} z_1^{-2\alpha_1} z_2^{-2\alpha_2} [z_1^{\alpha_1} z_2^{\alpha_2} L_1(z_1, z_2) \\ & + (1-z_1)^{k_1}\{(1+z_2)^{k_2}\tilde{T}_3(z_1, z_2) + (1-z_2)^{k_2}\tilde{T}_3(z_1, -z_2)\}]. \end{aligned} \tag{2.71}$$

Next, we note that, since the Laurent polynomials T_3 and K_1 in (2.49) are, according to Lemma 2.2.2, odd in z_2, we have from (2.11) and (2.49) that

$$\begin{aligned} A(z_1, -z_2) =& 4(1+z_1)^{k_1}(1-z_2)^{k_2} z_1^{-2\alpha_1} z_2^{-2\alpha_2} \\ & [-K_1(z_1, z_2)K_2(z_1, -z_2) - T_3(z_1, z_2)(1+z_2)^{k_2}], \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \end{aligned}$$

which, together with (2.11), (2.49) and the first line of (2.50), shows that, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$A(z_1, z_2) + A(z_1, -z_2) = 4(1+z_1)^{k_1} z_1^{-2\alpha_1} z_2^{-2\alpha_2} [z_1^{\alpha_1} z_2^{\alpha_2} K_1(z_1, z_2)]. \tag{2.72}$$

It then follows from (2.71) and (2.72) that, since also we have chosen $L_1 = K_1$, the Laurent polynomial $\tilde{T}_3$ satisfies

$$(1+z_2)^{k_2}\tilde{T}_3(z_1,z_2) + (1-z_2)^{k_2}\tilde{T}_3(z_1,-z_2) = 0, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\},$$

or, equivalently,

$$(1+z_2)^{k_2}\tilde{T}_3(z_1,z_2) = -(1-z_2)^{k_2}\tilde{T}_3(z_1,-z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}. \tag{2.73}$$

Since the univariate polynomials $(1+z_2)^{k_2}$ and $(1-z_2)^{k_2}$ have no common factor, we deduce from (2.73) the existence of a Laurent polynomial $\tilde{T}_4$ satisfying

$$\tilde{T}_3(z_1,z_2) = \tilde{T}_4(z_1,z_2)(1-z_2)^{k_2}, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{2.74}$$

so that, since $\tilde{T}_3$ is odd in z_1, we find that $\tilde{T}_4$ is odd in z_1. Also, by substituting the expression in (2.74) of $\tilde{T}_3$ into (2.73), we obtain

$$(1+z_2)^{k_2}(1-z_2)^{k_2}\tilde{T}_4(z_1,z_2) = -(1-z_2)^{k_2}(1+z_2)^{k_2}\tilde{T}_4(z_1,-z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\},$$

showing that $\tilde{T}_4$ is also odd in z_2. Combining (2.51) with (2.74), we deduce that, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, the Laurent polynomial B is of the form

$$B(z_1,z_2) = 2^{k_1+k_2} z_1^{-2\alpha_1} z_2^{-2\alpha_2}[L_1(z_1,z_2)L_2(z_1,z_2) + T(z_1,z_2)(1-z_1)^{k_1}(1-z_2)^{k_2}], \tag{2.75}$$

where $T = \tilde{T}_4$ is a Laurent polynomial which is odd in both z_1 and z_2.

Our proof in the necessary direction is now completed by appealing to Lemma 2.2.1 and 2.2.2, and using (2.75), with specifically the Laurent polynomial T_2 in Lemma 2.2.1 (b) chosen to also be odd in z_1.

Note from Lemmas 2.2.1 and 2.2.2 that the result (2.69) can similarly be achieved by means of the choice $K_2 = L_2$ in (2.49).

Next, we prove the theorem in the sufficient direction. To this end, suppose that, for any pair of odd integers α_1 and α_2 such that $\alpha_1 < 2k_1$ and $\alpha_2 < 2k_2$, the Laurent polynomial B has the form given by (2.69). To show that the Laurent polynomial A is an interpolatory mask symbol, it will suffice to prove that A satisfies the identity (2.9) in Theorem 2.1.1.

To this end, since by assumption S_2, T_2 and T are odd in z_1, observe from (2.11) and (2.69) that, for $z_1, z_2 \in C \setminus \{0\}$,

$$\begin{aligned}
A(z_1, z_2)+&A(-z_1, z_2)\\
&=4z_1^{-2\alpha_1} z_2^{-2\alpha_2}(1+z_1)^{k_1}(1+z_2)^{k_2}\left[T(z_1, z_2)(1-z_1)^{k_1}(1-z_2)^{k_2}\right.\\
&\quad+\left\{S_1(z_1, z_2)+T_1(z_1, z_2)(1-z_1)^{k_1}\right\}\left.\left\{S_2(z_1, z_2)+T_2(z_1, z_2)(1-z_2)^{k_2}\right\}\right]\\
&+4z_1^{-2\alpha_1} z_2^{-2\alpha_2}(1-z_1)^{k_1}(1+z_2)^{k_2}\left[-T(z_1, z_2)(1+z_1)^{k_1}(1-z_2)^{k_2}\right.\\
&\quad+\left\{S_1(-z_1, z_2)+T_1(z_1, z_2)(1+z_1)^{k_1}\right\}\left.\left\{-S_2(z_1, z_2)-T_2(z_1, z_2)(1-z_2)^{k_2}\right\}\right],
\end{aligned}$$

which, together with (2.70), yields, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned}
A(z_1, z_2)+&A(-z_1, z_2)\\
&=4z_1^{-2\alpha_1} z_2^{-2\alpha_2}(1+z_2)^{k_2}\left[z_1^{\alpha_1} z_2^{\alpha_2}\left\{S_2(z_1, z_2)+T_2(z_1, z_2)(1-z_2)^{k_2}\right\}\right].
\end{aligned} \tag{2.76}$$

Replacing z_2 by $-z_2$ in (2.76), and using the fact that T_2 is even in z_2, we obtain, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned}
A(z_1, -z_2)+&A(-z_1, -z_2)\\
&=4z_1^{-2\alpha_1} z_2^{-2\alpha_2}(1-z_2)^{k_2}\left[-z_1^{\alpha_1} z_2^{\alpha_2}\left(S_2(z_1, -z_2)+T_2(z_1, z_2)(1+z_2)^{k_2}\right)\right].
\end{aligned} \tag{2.77}$$

Since S_2 satisfies (2.70), adding (2.76) with (2.77) yields, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$A(z_1, z_2) + A(-z_1, z_2) + A(z_1, -z_2) + A(-z_1, -z_2) = 4z_1^{-2\alpha_1} z_2^{-2\alpha_2} [z_1^{\alpha_1} z_2^{\alpha_2} (z_1^{\alpha_1} z_2^{\alpha_2})] = 4,$$

thereby showing that the Laurent polynomial A satisfies the identity (2.9), which concludes our proof. $\square$

2.3 Application to box splines interpolatory mask symbols

Consider the mask symbols A_1 and $\tilde{A}_2$ corresponding respectively to the box spline N_1 given by (1.16) and to the shifted box spline $\tilde{N}_2$ given by (1.32). Then, we have

$$A_1(z_1, z_2) = (1 + z_1)(1 + z_2) B_1(z_1, z_2), \quad z_1, z_2 \in \mathbb{C}, \tag{2.78}$$

$$\tilde{A}_2(z_1, z_2) = (1 + z_1)(1 + z_2) \tilde{B}_2(z_1, z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{2.79}$$

where the polynomial B_1 and the Laurent polynomial $\tilde{B}_2$ are given by

$$B_1(z_1, z_2) = 1, \quad z_1, z_2 \in \mathbb{C}, \tag{2.80}$$

$$\tilde{B}_2(z_1, z_2) = \left(\frac{1 + z_1 z_2}{2} \right) z_1^{-1} z_2^{-1}, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}. \tag{2.81}$$

Recall from Chapter 1 that both A_1 and $\tilde{A}_2$ are interpolatory, so that, according to Theorem 2.2.3, with $k_1 = k_2 = 1$ and $\alpha_1 = \alpha_2 = 1$, B_1 and $\tilde{B}_2$ are of the form (2.69) for some Laurent polynomials T_1, T_2 and T respectively even in z_1 but odd in z_2, even in z_2

but odd in z_1, and odd in both z_1 and z_2, and for polynomials S_1 and S_2 satisfying

$$\begin{cases} (1+z_1)S_1(z_1,z_2) - (1-z_1)S_1(-z_1,z_2) = z_1z_2, \\ (1+z_2)S_2(z_1,z_2) - (1-z_2)S_2(z_1,-z_2) = z_1z_2, \end{cases}, \quad z_1, z_2 \in \mathbb{C}, \tag{2.82}$$

such that S_1 and S_2 are, respectively, odd in z_2 with degree less than k_1 in z_1 and odd in z_1 with degree less than k_2 in z_2.

We now proceed to find the polynomials S_1 and S_2 satisfying (2.82). By using the Euclidean algorithm presented in Section 2.2, we find that the univariate polynomials U_1 and V_1 satisfying

$$(1+z_1)U_1(z_1) + (1-z_1)V_1(z_1) = 1, \quad z_1 \in \mathbb{C},$$

are given by $U_1(z_1) = V_1(z_1) = \frac{1}{2}$, $z_1 \in \mathbb{C}$. Also, by using the polynomial division theorem, we obtain $z_1V(z_1) = z_1\frac{1}{2} = \frac{1}{2}(1+z_1) - \frac{1}{2}$, $z_1 \in \mathbb{C}$, from which it follows that R_1 is given by $R_1(z_1) = -\frac{1}{2}$, and consequently, S_1 is given by

$$S_1(z_1,z_2) = -z_2R_1(-z_1) = \frac{1}{2}z_2, \quad z_1, z_2 \in \mathbb{C}. \tag{2.83}$$

Using a similar argument, we show that S_2 is given by

$$S_2(z_1,z_2) = \frac{1}{2}z_1, \quad z_1, z_2 \in \mathbb{C}. \tag{2.84}$$

Observe in particular that S_1 and S_2 are, respectively, odd in z_2 and odd in z_1.

The box spline mask symbol A_1

Consider the polynomials T_1, T_2 and T defined respectively by

$$T_1(z_1,z_2) = -\frac{1}{2}z_2, \quad T_2(z_1,z_2) = -\frac{1}{2}z_1, \quad T(z_1,z_2) = 0, \quad z_1, z_2 \in \mathbb{C}, \tag{2.85}$$

according to which T_1 is even in z_1 but odd in z_2, T_2 is even in z_2 but odd in z_1, and T is odd both in z_1 and in z_2. Using (2.83), (2.84) and (2.85), we obtain, for $z_1, z_2 \in \mathbb{C}$,

$$\begin{aligned}
&4z_1^{-2}z_2^{-2}\left[T(z_1,z_2)(1-z_1)(1-z_2)\right.\\
&\qquad\left.+(S_1(z_1,z_2)+T_1(z_1,z_2)(1-z_1))(S_2(z_1,z_2)+T_2(z_1,z_2)(1-z_2))\right]\\
&=4z_1^{-2}z_2^{-2}\left[\left(\frac{1}{2}z_2-\frac{1}{2}z_2(1-z_1)\right)\left(\frac{1}{2}z_1-\frac{1}{2}z_1(1-z_2)\right)\right]\\
&=4z_1^{-2}z_2^{-2}\left[\left(\frac{1}{2}z_2z_1\right)\left(\frac{1}{2}z_1z_2\right)\right]\\
&=1\\
&=B_1(z_1,z_2),
\end{aligned}$$

by virtue of (2.78) and (2.80). Hence $B = B_1$ is of the form (2.69), where the polynomials S_1, S_2 are given by (2.83) and (2.84), and the polynomials T_1, T_2 and T given by (2.85).

The shifted box spline mask symbol $\tilde{A}_2$

Similarly, consider the polynomials T_1, T_2 and T defined respectively by

$$T_1(z_1,z_2) = -\frac{1}{4}z_2,\quad T_2(z_1,z_2) = -\frac{1}{4}z_1,\quad T(z_1,z_2) = \frac{1}{16}z_1z_2,\quad z_1,z_2\in\mathbb{C}, \tag{2.86}$$

so that T_1 is even in z_1 but odd in z_2, T_2 is even in z_2 but odd in z_1, and T is odd both in z_1 and in z_2. Using (2.83), (2.84) and (2.86), we obtain, for $z_1, z_2 \in \mathbb{C}$,

$$\begin{aligned}
S_1(z_1,z_2)+T_1(z_1,z_2)(1-z_1) &= \frac{1}{2}z_2-\frac{1}{4}z_2(1-z_1) = \frac{1}{4}z_2+\frac{1}{4}z_1z_2,\\
S_2(z_1,z_2)+T_2(z_1,z_2)(1-z_2) &= \frac{1}{2}z_1-\frac{1}{4}z_1(1-z_2) = \frac{1}{4}z_1+\frac{1}{4}z_1z_2,\\
T(z_1,z_2)(1-z_1)(1-z_2) &= \frac{1}{16}(z_1z_2-z_1^2z_2-z_1z_2^2+z_1^2z_2^2),
\end{aligned}$$

so that, for $z_1, z_2 \in \mathbb{C}$,

$$
\begin{aligned}
T(z_1,z_2)(1-z_1)(1-z_2) + (S_1(z_1,z_2) + T_1(z_1,z_2)(1-z_1))(S_2(z_1,z_2) + T_2(z_1,z_2)(1-z_2))& \\
=\frac{1}{16}(z_1z_2 + z_1^2z_2 + z_1z_2^2 + z_1^2z_2^2) + \frac{1}{16}(z_1z_2 - z_1^2z_2 - z_1z_2^2 + z_1^2z_2^2)& \\
=\frac{1}{8}z_1z_2(1+z_1z_2).& \qquad (2.87)
\end{aligned}
$$

Multiplying both sides of (2.87) by $4z_1^{-2}z_2^{-2}$ yields, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$4z_1^{-2}z_2^{-2}\frac{1}{8}z_1z_2(1+z_1z_2) = z_1^{-1}z_2^{-1}\frac{1}{2}(1+z_1z_2) = \tilde{B}_2(z_1,z_2),$$

by virtue of (2.79) and (2.81). Hence $B = \tilde{B}_2$ is of the form (2.69), where the polynomials S_1, S_2 are given by (2.83) and (2.84), and the polynomials T_1, T_2 and T given by (2.86).

Chapter 3

Interpolatory subdivision schemes

The main theme in this chapter are the concepts of interpolatory bivariate subdivision schemes and the cascade algorithm. In Section 3.2, we discuss the convergence of interpolatory subdivision schemes, whereas, in Section 3.3, we prove that certain properties of the initial function are preserved by the iterates of the cascade algorithm if the interpolatory mask and the dilation matrix are chosen to satisfy the conditions (3.18) and (3.19) below.

3.1 Preliminaries

For a given sequence $a \in M_0(\mathbb{Z}^2)$ and a dilation matrix M, the *subdivision operator* $S_a : M(\mathbb{Z}^2) \to M(\mathbb{Z}^2)$ is defined for any sequence $c \in M(\mathbb{Z}^2)$ by

$$(S_a c)_{\mathbf{j}} = \sum_{\mathbf{k}} a_{\mathbf{j}-M\mathbf{k}^T} c_{\mathbf{k}}, \quad \mathbf{j} \in \mathbb{Z}^2. \tag{3.1}$$

The resulting *subdivision scheme* S_a then generates, for a given sequence $c \in M(\mathbb{Z}^2)$, the sequence $\{c^{(r)} : r \in \mathbb{Z}_+\} \subset M(\mathbb{Z}^2)$ by means of the recursive formulation

$$c^{(0)} = c, \qquad c^{(r+1)} = S_a(c^{(r)}), \quad r \in \mathbb{Z}_+, \tag{3.2}$$

or, equivalently, $c^{(r)} = S_a^r c$, $r \in \mathbb{Z}_+$, where

$$S_a^0 c = c, \qquad S_a^{r+1} c = S_a(S_a^r c), \quad r \in \mathbb{Z}_+. \tag{3.3}$$

The sequence a is called the *subdivision mask*, also referred to as the mask, and if a satisfies the interpolatory conditions in the sense of (1.8), then in (3.1) we have

$$(S_a c)_{M\mathbf{j}^T} = c_{\mathbf{j}}, \quad \mathbf{j} \in \mathbb{Z}^2. \tag{3.4}$$

In that case, by induction on $r \in \mathbb{Z}_+$, we also have in (3.2) that

$$c^{(r+1)}_{M\mathbf{j}^T} = c^{(r)}_{\mathbf{j}}, \quad \mathbf{j} \in \mathbb{Z}^2, \tag{3.5}$$

which means that, at each level of iteration, the subdivision scheme process preserves all the points obtained in the previous subdivision steps. Such a subdivision scheme is then called *interpolatory.*

For a set $\mathcal{M} \subset M(\mathbb{Z}^2)$, we say that the subdivision scheme S_a is *convergent* on $\mathcal{M}$ if, for any sequence $c \in \mathcal{M}$, there exists a function $f \in C(\mathbb{R}^2)$ depending on c, such that

$$\lim_{r\to\infty} \|S_a^r c - f(M^{-r}\cdot)\|_\infty = 0, \tag{3.6}$$

where, for $r \in \mathbb{Z}_+$, $f(M^{-r}\cdot)$ denotes the sequence $\{f(M^{-r}\mathbf{j}^T) : \mathbf{j} \in \mathbb{Z}^2\}$. The limit function f will often be denoted by $S_a^\infty c$.

Similarly, for a given dilation matrix M and a sequence $a \in M_0(\mathbb{Z}^2)$, we define the *cascade operator* $T_a : M(\mathbb{R}^2) \to M(\mathbb{R}^2)$ by

$$T_a f = \sum_{\mathbf{j}} a_{\mathbf{j}} f(M \cdot -\mathbf{j}), \quad f \in M(\mathbb{R}^2). \tag{3.7}$$

The resulting *cascade algorithm* T_a then generates, for a given initial function $g \in M(\mathbb{R}^2)$, the sequence $\{f_r : r \in \mathbb{Z}_+\}$ by means of the recursive formula

$$f_0 = g, \qquad f_{r+1} = T_a f_r, \quad r \in \mathbb{Z}_+, \tag{3.8}$$

or, equivalently, $f_r = T_a^r g$, $r \in \mathbb{Z}_+$, where

$$T_a^0 f = f, \qquad T_a^{r+1} f = T_a(T_a^r f), \quad r \in \mathbb{Z}_+. \tag{3.9}$$

The cascade algorithm T_a is said to be *convergent* on a set $\mathcal{M} \subset C_0(\mathbb{R}^2)$ if, for any initial function $g \in \mathcal{M}$, there exists a function $f \in C(\mathbb{R}^2)$ such that

$$\lim_{r \to \infty} \|T_a^r g - f\|_\infty = 0. \tag{3.10}$$

The limit function f will often be denoted by $T_a^\infty g$.

For convenience, we shall simply say, for a subdivision schemes, “ convergent ” for “ convergent on $M(\mathbb{Z}^2)$ ”, and, for the cascade algorithm, “ convergent ” for “ convergent on $C_0(\mathbb{R}^2)$ ”.

Our following result presents an important relationship between subdivision schemes and cascade algorithms. Our proof uses a similar argument as in [Dyn92] where only the case $M = 2I$ is discussed.

Proposition 3.1.1. *Suppose that M is a dilation matrix and a an interpolatory mask.*

Then, for any sequence $c \in M(\mathbb{Z}^2)$ and for any function $f \in M(\mathbb{R}^2)$,

$$\sum_{\boldsymbol{j}} (S_a^r c)_{\boldsymbol{j}} f(M^r \cdot - \boldsymbol{j}) = \sum_{\boldsymbol{j}} c_{\boldsymbol{j}} (T_a^r f)(\cdot - \boldsymbol{j}), \quad r \in \mathbb{Z}_+. \tag{3.11}$$

In particular, choosing the sequence c in (3.11) as the delta sequence δ defined in (1.3), yields, for any function $f \in M(\mathbb{R}^2)$,

$$T_a^r f = \sum_{\boldsymbol{j}} (S_a^r \delta)_{\boldsymbol{j}} f(M^r \cdot - \boldsymbol{j}), \quad r \in \mathbb{Z}_+. \tag{3.12}$$

Proof. Let $f \in M(\mathbb{R}^2)$ and $c \in M(\mathbb{Z}^2)$. First, note from (3.3) and (3.9) that (3.11) trivially holds for $r = 0$. Next, we use (3.3), together with (3.1) and (3.7), to obtain

$$\begin{aligned}
\sum_{\mathbf{j}} (S_a^r c)_{\mathbf{j}} f(M^r \cdot - \mathbf{j}) =& \sum_{\mathbf{j}} \sum_{\mathbf{k}} a_{\mathbf{j} - M\mathbf{k}^T} (S_a^{r-1} c)_{\mathbf{k}} f(M^r \cdot - \mathbf{j}) \\
=& \sum_{\mathbf{k}} (S_a^{r-1} c)_{\mathbf{k}} \sum_{\mathbf{j}} a_{\mathbf{j} - M\mathbf{k}^T} f(M^r \cdot - \mathbf{j}) \\
=& \sum_{\mathbf{k}} (S_a^{r-1} c)_{\mathbf{k}} \sum_{\mathbf{j}} a_{\mathbf{j}} f(M^r \cdot - M\mathbf{k}^T - \mathbf{j}) \\
=& \sum_{\mathbf{k}} (S_a^{r-1} c)_{\mathbf{k}} \sum_{\mathbf{j}} a_{\mathbf{j}} f(M(M^{r-1} \cdot - \mathbf{k}) - \mathbf{j}) \\
=& \sum_{\mathbf{k}} (S_a^{r-1} c)_{\mathbf{k}} (T_a f)(M^{r-1} \cdot - \mathbf{k}) \\
&\vdots \\
=& \sum_{\mathbf{k}} (S_a^0 c)_{\mathbf{k}} (T_a^r f)(\cdot - \mathbf{k}) \\
=& \sum_{\mathbf{k}} c_{\mathbf{k}} (T_a^r f)(\cdot - \mathbf{k}),
\end{aligned}$$

by virtue of (3.3), thereby showing that (3.11) holds.

In particular, choosing $c = \delta$ in (3.11) yields

$$\sum_{\mathbf{j}} (S_a^r \delta)_{\mathbf{j}} f(M^r \cdot - \mathbf{j}) = \sum_{\mathbf{i}} \delta_{\mathbf{i}} (T_a^r f)(\cdot - \mathbf{i}) = T_a^r f, \quad r \in \mathbb{Z}_+, \quad f \in M(\mathbb{R}^2).$$

$\square$

3.2 Subdivision schemes convergence

Assuming that the interpolatory refinable function exists, we proceed to analyse the convergence of the associated interpolatory subdivision scheme.

Observe first that a dilation matrix M defines a bijective linear application from the set of rational pairs $\mathbb{Q}^2$ into itself, so that the dyadic set $\mathcal{D}$ given by

$$\mathcal{D} = \left\{ M^{-r} \mathbf{j}^T : \mathbf{j} \in \mathbb{Z}^2, r \in \mathbb{Z}_+ \right\}, \tag{3.13}$$

is dense in $\mathbb{R}^2$. We prove the following result.

Theorem 3.2.1. *Suppose that ϕ is an interpolatory refinable function associated with the interpolatory mask $a \in M_0(\mathbb{Z}^2)$ and with the dilation matrix M. Then, for any initial sequence $c \in M(\mathbb{Z}^2)$, the function Φ defined by*

$$\Phi = \sum_{\boldsymbol{j}} c_{\boldsymbol{j}} \phi(\cdot - \boldsymbol{j}), \tag{3.14}$$

satisfies

(i) $\Phi(\boldsymbol{m}) = c_{\boldsymbol{m}}, \quad \boldsymbol{m} \in \mathbb{Z}^2$;

(ii) $\Phi(M^{-r}\boldsymbol{m}) = (S_a^r c)_{\boldsymbol{m}}, \quad r \in \mathbb{Z}_+, \quad \boldsymbol{m} \in \mathbb{Z}^2.$

Consequently, for a sequence $c \in M(\mathbb{Z}^2)$, the subdivision scheme S_a, as defined by (3.1), converges to the function Φ given by (3.14), so that

$$S_a^\infty c = \Phi \quad and \quad S_a^\infty \delta = \phi, \tag{3.15}$$

where δ denotes the delta sequence defined by (1.3).

Proof. Consider a sequence $c \in M(\mathbb{Z}^2)$. Then:

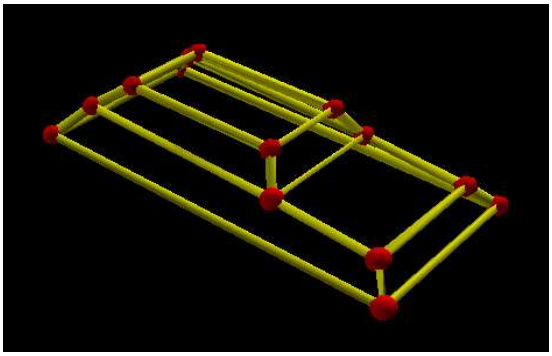
(i) Since ϕ is interpolatory, it follows from (3.14) that

$$\Phi(\mathbf{m}) = \sum_{\mathbf{j}} c_{\mathbf{j}} \phi(\mathbf{m} - \mathbf{j}) = c_{\mathbf{m}}, \quad \mathbf{m} \in \mathbb{Z}^2.$$

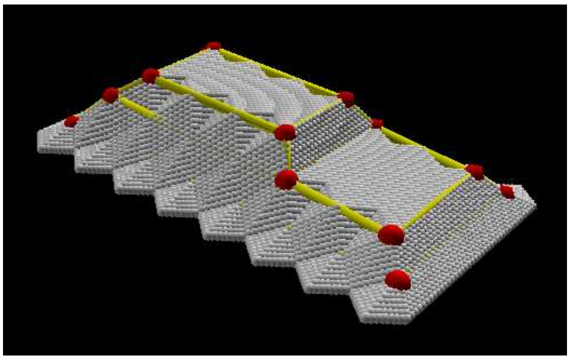
(ii) Since ϕ is refinable, it follows from (3.14), (3.1) and (3.3) that, for $r \in \mathbb{Z}_+$, $\mathbf{m} \in \mathbb{Z}^2$,

$$\begin{aligned}
\Phi\left(M^{-r}\mathbf{m}^T\right) =& \sum_{\mathbf{j}} c_{\mathbf{j}} \phi\left(M^{-r}\mathbf{m}^T - \mathbf{j}\right) \\
=& \sum_{\mathbf{j}} c_{\mathbf{j}} \sum_{\mathbf{k}} a_{\mathbf{k}} \phi\left(M^{-r+1}\mathbf{m}^T - M\mathbf{j}^T - \mathbf{k}\right) \\
=& \sum_{\mathbf{j}} c_{\mathbf{j}} \sum_{\mathbf{k}} a_{\mathbf{k}-M\mathbf{j}^T} \phi\left(M^{-r+1}\mathbf{m}^T - \mathbf{k}\right) \\
=& \sum_{\mathbf{k}} \left[\sum_{\mathbf{j}} a_{\mathbf{k}-M\mathbf{j}^T} c_{\mathbf{j}}\right] \phi(M^{-r+1}\mathbf{m}^T - \mathbf{k}) \\
=& \sum_{\mathbf{k}} (S_a c)_{\mathbf{k}} \phi\left(M^{-r+1}\mathbf{m}^T - \mathbf{k}\right) \\
\vdots \\
=& \sum_{\mathbf{k}} (S_a^r c)_{\mathbf{k}} \phi(\mathbf{m} - \mathbf{k}) \\
=& (S_a^r c)_{\mathbf{m}},
\end{aligned} \tag{3.16}$$

by virtue of the interpolatory property of ϕ.



(a) Initial sequence c



(b) Graph of Φ and c

Figure 3.1: Subdivision $S_{\tilde{a}^{(2)}}$ applied to c

Given the fact that the set $\mathcal{D}$ defined by (3.13) is dense in $\mathbb{R}^2$, we deduce from (3.16) that $\|S_a^r c - \Phi(M^{-r}\cdot)\|_\infty = 0$, $r \in \mathbb{Z}_+$, and therefore (3.6) holds. Hence, for any sequence $c \in M(\mathbb{Z}^2)$, the subdivision scheme S_a converges to the function Φ given by (3.14), i.e. $S_a^\infty c = \Phi$. In particular, choosing $c = \delta$ in (3.14) yields $S_a^\infty \delta = \phi$. $\square$

As an example, consider the shifted box spline $\tilde{N}_2$ from Chapter 1, and the associated interpolatory mask $\tilde{a}^{(2)}$ given by (1.31), i.e.

$$\begin{cases} \tilde{a}^{(2)}_{0,0} = 1, \quad \tilde{a}^{(2)}_{1,1} = \tilde{a}^{(2)}_{0,1} = \tilde{a}^{(2)}_{1,0} = \tilde{a}^{(2)}_{-1,0} = \tilde{a}^{(2)}_{0,-1} = \tilde{a}^{(2)}_{-1,-1} = \frac{1}{2}, \\ \tilde{a}^{(2)}_{i,j} = 0, \quad (i,j) \notin \{(0,0),(0,1),(1,0),(-1,0),(0,-1),(1,1),(-1,-1)\}. \end{cases} \tag{3.17}$$

According to Theorem 3.2.1, the subdivision scheme $S_{\tilde{a}^{(2)}}$ is convergent. Therefore, for any initial sequence $c \in M(\mathbb{Z}^2)$, the limit function $\Phi = S^\infty_{\tilde{a}^{(2)}} c$ is guaranteed to exist.

Choosing the initial sequence c as the red points in Figure 3.1 (a), the graph of the limit function Φ is illustrated in Figure 3.1 (b), showing that the initial points are preserved by means of the subdivision process. Observe, however, that $\Phi \in C(\mathbb{R}^2) \setminus C^1(\mathbb{R}^2)$, i.e. Φ defines a non-smooth surface.

3.3 Property preservation in the cascade algorithm

In this section, we show that certain properties of the initial functions are preserved by the iterates $\{f_r : r \in \mathbb{Z}_+\}$ of the cascade algorithm. More precisely, for an appropriate sequence $a \in M_0(\mathbb{Z}^2)$, we show that the initial function g and its image $T_a g$ share certain properties. By induction on $r \in \mathbb{Z}_+$, we then show that g and $T_a^r g$ have common properties, so that, in the case where the cascade algorithm is convergent, by considering the limit $r \to \infty$, we shall show that the limit function $T_a^\infty g$ also preserves these properties of the initial function g.

For this purpose, we first state (without proof) the following result [HJ98a] (see also [KLY07]), which presents a necessary condition on the interpolatory mask a for the convergence of the corresponding subdivision scheme.

Proposition 3.3.1. *Suppose that the subdivision scheme S_a associated with an interpolatory mask $a \in M_0(\mathbb{Z}^2)$ and a dilation matrix M is convergent. Then a satisfies the condition*

$$\sum_{\boldsymbol{j}} a_{\boldsymbol{k}-M\boldsymbol{j}^T} = 1, \quad \boldsymbol{k} \in \mathbb{Z}^2. \tag{3.18}$$

It should be pointed here that the converse of Proposition 3.3.1 does not hold, that is, the condition (3.18) is not sufficient for the subdivision scheme S_a to converge.

Next we prove the following result on the preservation of properties with respect to the cascade operator.

Theorem 3.3.2. *Suppose that M is a dilation matrix and $a \in M_0(\mathbb{Z}^2)$ an interpolatory mask supported on some finite square $[N_1, N_2]^2$, and such that the sequence a satisfies the* *condition (3.18). Suppose, in addition, that M satisfies the condition*

$$[2\alpha, 2\beta]^2 \subseteq M[\alpha, \beta]^2, \quad \alpha, \beta \in \mathbb{Z}. \tag{3.19}$$

Then, given an initial function $g \in M(\mathbb{R}^2)$, the functions $\{\phi_r = T_a^r g : r \in \mathbb{Z}_+\}$ as generated recursively by means of (3.8), satisfy the following:

(i) If supp$(g) \subseteq [N_1, N_2]^2$, then supp$(\phi_r) \subseteq [N_1, N_2]^2$;

(ii) If $g \in C(\mathbb{R}^2)$, then $\phi_r \in C(\mathbb{R}^2)$;

(iii) If g satisfies the condition

$$g(\boldsymbol{j}) = \delta_{\boldsymbol{j}}, \quad \boldsymbol{j} \in \mathbb{Z}^2, \tag{3.20}$$

then ϕ_r satisfies the condition

$$\phi_r(\boldsymbol{j}) = \delta_{\boldsymbol{j}}, \quad \boldsymbol{j} \in \mathbb{Z}^2; \tag{3.21}$$

(iv) If g satisfies the partition of unity property, i.e.

$$\sum_{\boldsymbol{j}} g(\boldsymbol{x} - \boldsymbol{j}) = 1, \quad \boldsymbol{x} \in \mathbb{R}^2, \tag{3.22}$$

then ϕ_r satisfies the partition of unity, i.e.

$$\sum_{\boldsymbol{j}} \phi_r(\boldsymbol{x} - \boldsymbol{j}) = 1, \quad \boldsymbol{x} \in \mathbb{R}^2. \tag{3.23}$$

Proof. We proceed by induction on r. Recall first from the recursive formula (3.8), together with (3.7), that

$$\phi_{r+1} = T_a \phi_r = \sum_{\mathbf{j}} a_{\mathbf{j}} \phi_r(M \cdot - \mathbf{j}), \quad r \in \mathbb{Z}_+. \tag{3.24}$$

Next, for $r = 0$, suppose that, in (i), (ii), (iii) and (iv) respectively, $\phi_0 = g$ is supported on $[N_1, N_2]^2$, continuous, interpolatory as in (3.20) and satisfying the partition of unity property (3.22).

Let us now fix $r \in \mathbb{Z}_+$. The following holds:

(i) If $\operatorname{supp}(\phi_r) \subseteq [N_1, N_2]^2$, it holds that, for $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{j} \in [N_1, N_2]^2$,

$$\begin{aligned} M\mathbf{x}^T - \mathbf{j} \in [N_1, N_2]^2 &\Longrightarrow M\mathbf{x}^T \in \mathbf{j} + [N_1, N_2]^2 \subseteq [2N_1, 2N_2]^2 \\ &\Longrightarrow \mathbf{x} \in M^{-1}\left(\mathbf{j} + [N_1, N_2]^2\right) \subseteq M^{-1}\,[2N_1, 2N_2]^2 . \end{aligned} \tag{3.25}$$

Since a is supported on $[N_1, N_2]^2$, and since there is only a finite number of integers $\mathbf{j}$ in $[N_1, N_2]^2$, we deduce from (3.25), (3.24) and (3.19) that the support of ϕ_{r+1} satisfies

$$\operatorname{supp}(\phi_{r+1}) \subseteq \bigcup_{\mathbf{j} \in [N_1, N_2]^2} M^{-1}\left(\mathbf{j} + [N_1, N_2]^2\right) \subseteq \bigcup_{\mathbf{j} \in [N_1, N_2]^2} M^{-1}\,[2N_1, 2N_2]^2 \subseteq [N_1, N_2]^2,$$

by virtue of (3.19).

(ii) If ϕ_r is continuous, then the shifts with respects to $\mathbb{Z}^2$ of its dilations are continuous, so that, from (3.24), we deduce that ϕ_{r+1} is also continuous.

(iii) If ϕ_r is interpolatory as in (3.21), we obtain from (3.24) and (1.8) that, for $\mathbf{j} \in \mathbb{Z}^2$,

$$\phi_{r+1}(\mathbf{j}) = \sum_{\mathbf{k}} a_{\mathbf{k}} \phi_r(M\mathbf{j}^T - \mathbf{k}) = a_{M\mathbf{j}^T} = \delta_{\mathbf{j}}.$$

(iv) If ϕ_r satisfies the partition of unity property, then we have for $\mathbf{x} \in \mathbb{R}^2$ that

$$\sum_{\mathbf{k}} \phi_r(M\mathbf{x} - \mathbf{k}) = 1, \tag{3.26}$$

which, together with (3.24) and (3.18), yields, for $\mathbf{x} \in \mathbb{R}^2$,

$$\begin{aligned}\sum_{\mathbf{j}} \phi_{r+1}(\mathbf{x}-\mathbf{j}) =& \sum_{\mathbf{j}} \sum_{\mathbf{k}} a_{\mathbf{k}} \phi_r(M\mathbf{x} - M\mathbf{j}^T - \mathbf{k}) \\ =& \sum_{\mathbf{j}} \sum_{\mathbf{k}} a_{\mathbf{k}-M\mathbf{j}^T} \phi_r(M\mathbf{x}-\mathbf{k}) \\ =& \sum_{\mathbf{k}} \left[\sum_{\mathbf{j}} a_{\mathbf{k}-M\mathbf{j}^T} \right] \phi_r(M\mathbf{x}-\mathbf{k}) \\ =& \sum_{\mathbf{k}} \phi_r(M\mathbf{x}-\mathbf{k}) \\ =& 1,\end{aligned}$$

which then completes our inductive proof. □

In the case where the cascade algorithm is convergent, we show in the result below that the limit function preserves certain properties of the initial function.

Theorem 3.3.3. *Under the conditions of Theorem 3.3.2, with specifically g satisfying the conditions in (i) to (iv) of that theorem, if also $g \in C_0(\mathbb{R}^2)$ and the sequence a is such that the cascade algorithm (3.8) is convergent with limit function ϕ, then the following holds:*

(i) $\phi \in C_0(\mathbb{R}^2)$*;*

(ii) If $supp(g) \subseteq [N_1, N_2]^2$*, then* $supp(\phi) \subseteq [N_1, N_2]^2$*;*

(iii) ϕ *is an interpolatory refinable function with respect to the refinement sequence* a *and the dilation matrix* M*, satisfying also the partition of unity property*

$$\sum_{\boldsymbol{j}} \phi(\boldsymbol{x}-\boldsymbol{j}) = 1, \quad \boldsymbol{x} \in \mathbb{R}^2. \tag{3.27}$$

Proof. (*i*) Since $g \in C_0(\mathbb{R}^2)$, it follows from Theorem 3.3.3 (i) and (ii) that $\phi_r = T_a^r g \in C_0(\mathbb{R}^2)$, $r \in \mathbb{Z}_+$, so that the uniform convergence result $\|\phi - \phi_r\|_\infty \to 0$, $r \to \infty$, then yields $\phi \in C_0(\mathbb{R}^2)$.

(ii) Suppose that $\mathrm{supp}(g) \subseteq [N_1, N_2]^2$, and let $\mathbf{x} \notin [N_1, N_2]^2$, so that Theorem 3.3.3 (i) yields $\phi_r(\mathbf{x}) = 0$, $r \in \mathbb{Z}_+$. Hence,

$$|\phi(\mathbf{x})| = |\phi(\mathbf{x}) - \phi_r(\mathbf{x})| \leq \|\phi - \phi_r\|_\infty \to 0, \quad r \to \infty,$$

and it follows that $\phi(\mathbf{x}) = 0$, i.e. $\mathrm{supp}(\phi) \subseteq [N_1, N_2]^2$.

(iii) According to Theorem 3.3.2 (iii), ϕ_r is interpolatory for every $r \in \mathbb{Z}_+$, so that, for $\mathbf{j} \in \mathbb{Z}^2$,

$$|\phi(\mathbf{j}) - \delta_{\mathbf{j}}| = |\phi(\mathbf{j}) - \phi_r(\mathbf{j})| \leq \|\phi - \phi_r\|_\infty \to 0, \quad r \to \infty,$$

and it follows that ϕ is interpolatory as in (1.2).

To prove that ϕ satisfies the refinement equation (1.1), we use (3.8) and (3.7) to obtain

$$\begin{aligned}\|\phi - T_a\phi\|_\infty \leq & \|\phi - \phi_{r+1}\|_\infty + \|T_a(\phi_r - \phi)\|_\infty \\ \leq & \|\phi - \phi_{r+1}\|_\infty + \left[\sum_{\mathbf{j}} |a_{\mathbf{j}}|\right] \|\phi_r - \phi\|_\infty \to 0, \quad r \to \infty,\end{aligned}$$

i.e. $\phi = T_a\phi$, which is equivalent to (1.1).

Finally, since ϕ is interpolatory and refinable, we deduce from (3.18) that, for $\mathbf{i} \in \mathbb{Z}^2$ and $r \in \mathbb{Z}_+$,

$$\begin{aligned}\sum_{\mathbf{j}} \phi\left(M^{-r}\mathbf{i}^T - \mathbf{j}\right) = & \sum_{\mathbf{j}}\sum_{\mathbf{k}} a_{\mathbf{k}} \phi\left(M^{-r+1}\mathbf{i}^T - M\mathbf{j}^T - \mathbf{k}\right) \\ = & \sum_{\mathbf{j}}\sum_{\mathbf{k}} a_{\mathbf{k}-M\mathbf{j}^T} \phi\left(M^{-r+1}\mathbf{i}^T - \mathbf{k}\right) \\ = & \sum_{\mathbf{k}} \left[\sum_{\mathbf{j}} a_{\mathbf{k}-M\mathbf{j}^T}\right] \phi\left(M^{-r+1}\mathbf{i}^T - \mathbf{k}\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mathbf{k}} \phi\left(M^{-r+1}\mathbf{i}^T - \mathbf{k}\right) \\
&\vdots \\
&= \sum_{\mathbf{k}} \phi\left(\mathbf{i} - \mathbf{k}\right) \\
&= \sum_{\mathbf{k}} \phi(\mathbf{k}) \\
&= 1,
\end{aligned}
$$

from which we conclude, by recalling also the fact that the dyadic set $\mathcal{D}$ in (3.13) is dense in $\mathbb{R}^2$, that ϕ satisfies the partition of unity condition (3.27). $\square$

In conclusion, the important results of this section are that cascade algorithm convergence implies interpolatory refinable function existence, which in turn implies subdivision convergence. Graphical illustrations are provided in Chapter 4.

Chapter 4

Existence of interpolatory refinable functions

For the dilation matrix $M = 2I$, we present in this chapter three methods to prove, for a given refinement mask, the existence of a corresponding interpolatory refinable function. The first method is based on a result by Micchelli [Mic96] for interpolatory mask symbols which are factorizable and which are non-negative on the torus T. The second method, as described in Section 4.2, consists of using tensor products in order to generate bivariate refinable functions from univariate ones. Finally, the third method presented in Section 4.3 is based on deductions from numerical results, as generally applied to interpolatory masks satisfying higher order sum rules.

An important concept is this section is that of symmetry which we proceed to define as follows. For a refinement mask in $a \in M_0(\mathbb{Z}^2)$, consider the following properties:

$$a(-i,j) = a(i,-j) = a(i,j), \quad (i,j) \in \mathbb{Z}^2, \tag{4.1}$$

$$a(-i,-j) = a(i,j), \quad (i,j) \in \mathbb{Z}^2, \tag{4.2}$$

$$a(j,i) = a(i,j), \quad (i,j) \in \mathbb{Z}^2. \tag{4.3}$$

We say that a is *symmetric about the two axes* if a satisfies the property (4.1), *symmetric about the origin* if a satisfies the property (4.2), and *symmetric about the line* $y = x$ if a satisfies the property (4.3).

4.1 For non-negative masks

Consider the torus T and its subset $\tilde{T}$ defined respectively by

$$T = \{(\mathrm{e}^{ix_1}, \mathrm{e}^{ix_2}) : x_1, x_2 \in \mathbb{R}\} \quad \text{and} \quad \tilde{T} = \{(\mathrm{e}^{ix_1}, \mathrm{e}^{ix_2}) : x_1, x_2 \in \mathbb{R}, \quad |x_1|, |x_2| \leq \pi/2\}.$$

A mask $a \in M_0(\mathbb{Z}^2)$ is termed *non-negative* if the corresponding mask symbol A, as defined by (1.9), is non-negative on the torus T, i.e.

$$A(\mathrm{e}^{ix_1}, \mathrm{e}^{ix_2}) \geq 0, \quad x_1, x_2 \in \mathbb{R}. \tag{4.4}$$

The result below presents a sufficient condition on the interpolatory mask for the existence of the corresponding interpolatory refinable function. We refer to [Mic96] for the proof.

Theorem 4.1.1. *Consider the dilation matrix $M = 2I$, and suppose that $a \in M_0(\mathbb{Z}^2)$ is a non-negative interpolatory mask. Suppose, in addition, that there exist integers $k_1, k_2 \in \mathbb{N}$ and a Laurent polynomial B, such that the corresponding mask symbol A is of the form*

$$A(z_1, z_2) = 2^{2-k_1-k_2}(1 + z_1)^{k_1}(1 + z_2)^{k_2} B(z_1, z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{4.5}$$

with $B(1, 1) = 1$ and $B(z_1, z_2) \neq 0$ for $(z_1, z_2) \in \tilde{T}$.

Then the corresponding interpolatory refinable function $\phi_a \in C_0(\mathbb{R}^2)$ exists.

Example 1

Consider the mask symbol G_1 defined by

$$G_1(z_1, z_2) = \frac{1}{4}(1+z_1)^2(1+z_2)^2 z_1^{-1} z_2^{-1}, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}. \tag{4.6}$$

We verify that G_1 satisfies (2.1), i.e. G_1 is interpolatory. Moreover, G_1 is of the form (4.5), with $k_1 = k_2 = 2$ and $B(z_1, z_2) = z_1^{-1} z_2^{-1}$, $z_1, z_2 \in \mathbb{C} \setminus \{0\}$.

Using the expression of G_1 in (4.6), we obtain, for $x_1, x_2 \in \mathbb{R}$,

$$\begin{aligned} G_1(\mathrm{e}^{ix_1}, \mathrm{e}^{ix_2}) =& 1 + \frac{1}{2}\left(\mathrm{e}^{ix_1} + \mathrm{e}^{-ix_1} + \mathrm{e}^{ix_2} + \mathrm{e}^{-ix_2}\right) \\ &+ \frac{1}{4}\left(\mathrm{e}^{i(x_1+x_2)} + \mathrm{e}^{-i(x_1+x_2)} + \mathrm{e}^{i(x_1-x_2)} + \mathrm{e}^{-i(x_1-x_2)}\right) \\ =& 1 + \cos x_1 + \cos x_2 + \frac{1}{2}[\cos(x_1+x_2) + \cos(x_1-x_2)] \\ =& 1 + \cos x_1 + \cos x_2 + \cos x_1 \cos x_2 \\ =& (1+\cos x_1)(1+\cos x_2) \geq 0, \end{aligned}$$

that is, G_1 is non-negative on the torus T. Moreover, since $B(z_1, z_2) = z_1^{-1} z_2^{-1}$, $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, we clearly have $B(1,1) = 1$ and $B(z_1, z_2) \neq 0$, $z_1, z_2 \in \tilde{T}$. Hence, according to Theorem 4.1.1, the corresponding interpolatory refinable function $\phi \in C_0(\mathbb{R}^2)$ exists.

Example 2

Consider next the mask symbol $\tilde{A}_2$, as given by (1.32), i.e.

$$\tilde{A}_2(z_1, z_2) = (1+z_1)(1+z_2)\left(\frac{1+z_1 z_2}{2}\right) z_1^{-1} z_2^{-1}, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \tag{4.7}$$

according to which, $\tilde{A}_2$ is of the form (4.5), with $k_1 = k_2 = 1$ and

$B(z_1, z_2) = \left(\frac{1 + z_1 z_2}{2}\right) z_1^{-1} z_2^{-1}$, $z_1, z_2 \in \mathbb{C} \setminus \{0\}$.

Recall from Chapter 1 that $\tilde{A}_2$ is interpolatory, and that the corresponding interpolatory refinable function is the box spline $\tilde{N}_2 \in C_0(\mathbb{R}^2)$ given by (1.30).

However, the mask symbol $\tilde{A}_2$ is not non-negative on the torus T. As a matter of fact, by using the expression of $\tilde{A}_2$ in (4.7), we obtain, for $x_1, x_2 \in \mathbb{R}$,

$$\begin{aligned}\tilde{A}_2(\mathrm{e}^{ix_1}, \mathrm{e}^{ix_2}) &= 1 + \frac{1}{2}\left(\mathrm{e}^{ix_1} + \mathrm{e}^{-ix_1} + \mathrm{e}^{ix_2} + \mathrm{e}^{-ix_2} + \mathrm{e}^{i(x_1+x_2)} + \mathrm{e}^{-i(x_1+x_2)}\right)\\ &= 1 + \cos x_1 + \cos x_2 + \cos(x_1 + x_2).\end{aligned}$$

Since $\tilde{A}_2(\mathrm{e}^{i2\pi/3}, \mathrm{e}^{i2\pi/3}) = -\frac{1}{2} < 0$, we deduce that $\tilde{A}_2$ is not non-negative on the torus T.

Therefore, observe that there are mask symbols which are not non-negative on the complex unit circle, but for which corresponding interpolatory refinable functions exist. Hence, the conditions for interpolatory refinable function existence in Theorem 4.1.1 are sufficient but not neccessary.

4.2 Tensor products

Tensor products, as briefly discussed in [DL02] (see also [Dyn92]), yield the simplest method to generate bivariate refinable functions. More precisely, given two univariate functions $\tilde{\phi}$ and $\tilde{\psi}$, the bivariate function ϕ, obtained by the tensor product of $\tilde{\phi}$ and $\tilde{\psi}$, inherits some of the properties of the two constituent functions $\tilde{\phi}$ and $\tilde{\psi}$. In particular, if $\tilde{\phi}$ and $\tilde{\psi}$ are interpolatory and refinable, then ϕ is interpolatory and refinable.

Given two functions $\tilde{\phi} \in C^{\alpha_1}(\mathbb{R})$ and $\tilde{\psi} \in C^{\alpha_2}(\mathbb{R})$, $\alpha_1, \alpha_2 \in \mathbb{Z}_+$, we define the *tensor* *product* $\phi = \tilde{\phi} \cdot \tilde{\psi}$ as the function given by

$$\phi(x, y) = \tilde{\phi}(x)\tilde{\psi}(y), \quad (x, y) \in \mathbb{R}^2, \tag{4.8}$$

so that $\phi \in C^{\alpha}(\mathbb{R}^2)$, where $\alpha = \min\{\alpha_1, \alpha_2\}$.

Let $\varphi \in M_0(\mathbb{R})$. We say that φ is *interpolatory* if $\varphi(j) = \delta_j$, $j \in \mathbb{Z}$, that φ satisfies the partition of unity condition if $\sum_j \varphi(x - j) = 1$, $x \in \mathbb{R}$, and that φ is *refinable* if there exists a sequence $a \in M_0(\mathbb{Z})$, called the *refinement mask*, such that $\varphi = \sum_j a_j \varphi(2 \cdot -j)$.

We are now able to present the following result.

Theorem 4.2.1. *Suppose that $\tilde{\phi} \in C_0^{\alpha_1}(\mathbb{R})$ and $\tilde{\psi} \in C_0^{\alpha_2}(\mathbb{R})$, $\alpha_1, \alpha_2 \in \mathbb{Z}_+$, are refinable functions with corresponding masks $\tilde{a}$ and $\tilde{b}$ respectively. Then, the tensor product ϕ defined by (4.8) is a refinable function associated with the dilation matrix $M = 2I$ and the refinement mask a given by*

$$a_{j,k} = \tilde{a}_j \tilde{b}_k, \quad (j, k) \in \mathbb{Z}^2. \tag{4.9}$$

Moreover, if $\tilde{\phi}$ and $\tilde{\psi}$ are both interpolatory refinable functions, then ϕ is an interpolatory refinable function. Also, if $\tilde{\phi}$ and $\tilde{\psi}$ both satisfy the partition of unity condition, then ϕ satisfies the partition of unity condition (3.27).

Proof. Since $\tilde{\phi}$ and $\tilde{\psi}$ are refinable, we deduce from (4.8) that, for $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned}
\phi(x, y) = \tilde{\phi}(x)\tilde{\psi}(y) &= \sum_j \tilde{a}_j \tilde{\phi}(2x - j) \sum_k \tilde{b}_k \tilde{\psi}(2y - k) \\
&= \sum_j \sum_k \tilde{a}_j \tilde{b}_k \tilde{\phi}(2x - j)\tilde{\psi}(2y - k) \\
&= \sum_{j,k} a_{j,k} \phi(2x - j, 2y - k),
\end{aligned}$$

according to which, ϕ is refinable with associated dilation matrix $M = 2I$ and mask a given by (4.9).

If $\tilde{\phi}$ and $\tilde{\psi}$ are both interpolatory, then, for $\mathbf{j} = (i, j) \in \mathbb{Z}^2$,

$$\phi(\mathbf{j}) = \phi(i, j) = \tilde{\phi}(i)\tilde{\psi}(j) = \delta_i \delta_j = \delta_{\mathbf{j}},$$

proving that ϕ is interpolatory as in (1.2).

If $\tilde{\phi}$ and $\tilde{\psi}$ both satisfy the partition of unity, then we have, for $\mathbf{x} = (x, y) \in \mathbb{R}^2$,

$$\sum_{\mathbf{j}} \phi(\mathbf{x} - \mathbf{j}) = \sum_{i,j} \phi(x - i, y - j) = \left[\sum_i \tilde{\phi}(x - i)\right]\left[\sum_j \tilde{\psi}(y - j)\right] = 1,$$

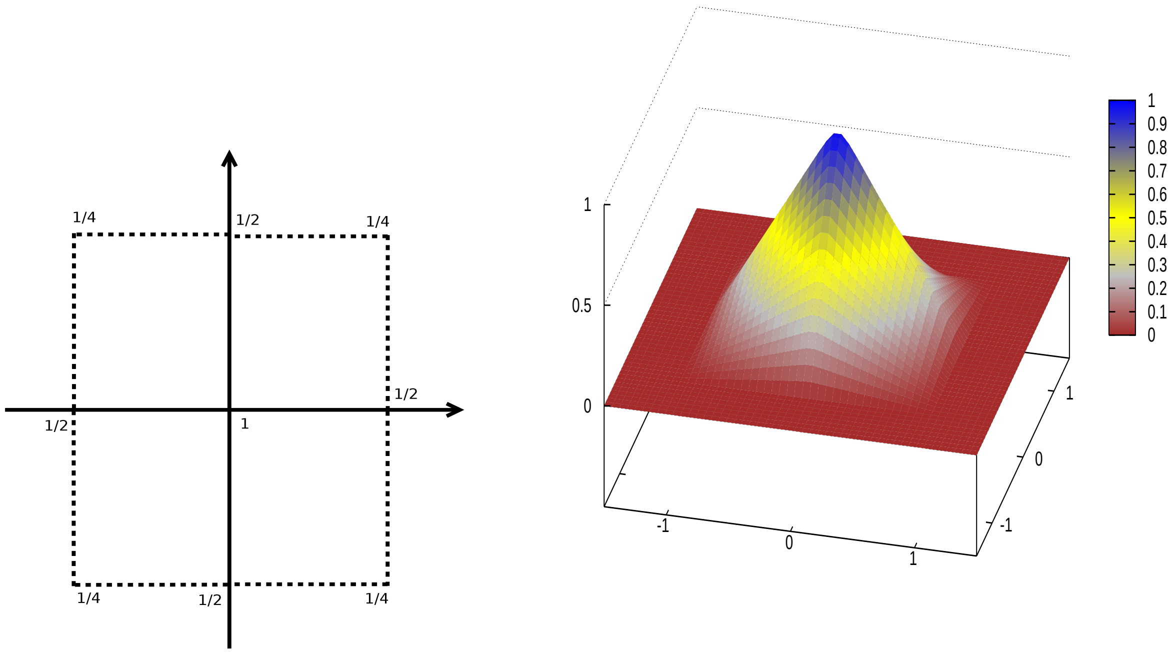
which shows that ϕ satisfies the partition of unity condition (3.27). $\square$

Denoting respectively by $\tilde{A}$, $\tilde{B}$ and A the mask symbols corresponding to the masks $\tilde{a}$, $\tilde{b}$ and a in Theorem 4.2.1, it follows from (4.9) that, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$A(z_1, z_2) = \sum_{j,k} a_{j,k} z_1^j z_2^k = \left(\sum_j \tilde{a}_j z_1^j\right)\left(\sum_k \tilde{b}_k z_2^k\right) = \tilde{A}(z_1)\tilde{B}(z_2). \tag{4.10}$$

The result below is then a direct consequence of Theorem 4.2.1.

Corollary 4.2.2. *Given a mask symbol A, suppose that there exist mask symbols $\tilde{A}$ and $\tilde{B}$ such that (4.10) holds. If there exist interpolatory refinable functions $\tilde{\phi} \in C_0^{\alpha_1}(\mathbb{R})$ and $\tilde{\psi} \in C_0^{\alpha_2}(\mathbb{R})$, $\alpha_1, \alpha_2 \in \mathbb{Z}_+$, corresponding to $\tilde{A}$ and $\tilde{B}$, then the tensor product $\phi = \tilde{\phi} \cdot \tilde{\psi} \in C_0^{\alpha}(\mathbb{R}^2)$, where $\alpha = \min\{\alpha_1, \alpha_2\}$, is an interpolatory refinable function with associated dilation matrix $2I$ and refinement mask symbol A.*



(a) Support of the mask associated with $\phi = \tilde{h} \cdot \tilde{h}$

(b) Graph of $\phi = \tilde{h} \cdot \tilde{h}$

Figure 4.1: The tensor product of the hat function $\tilde{h}$

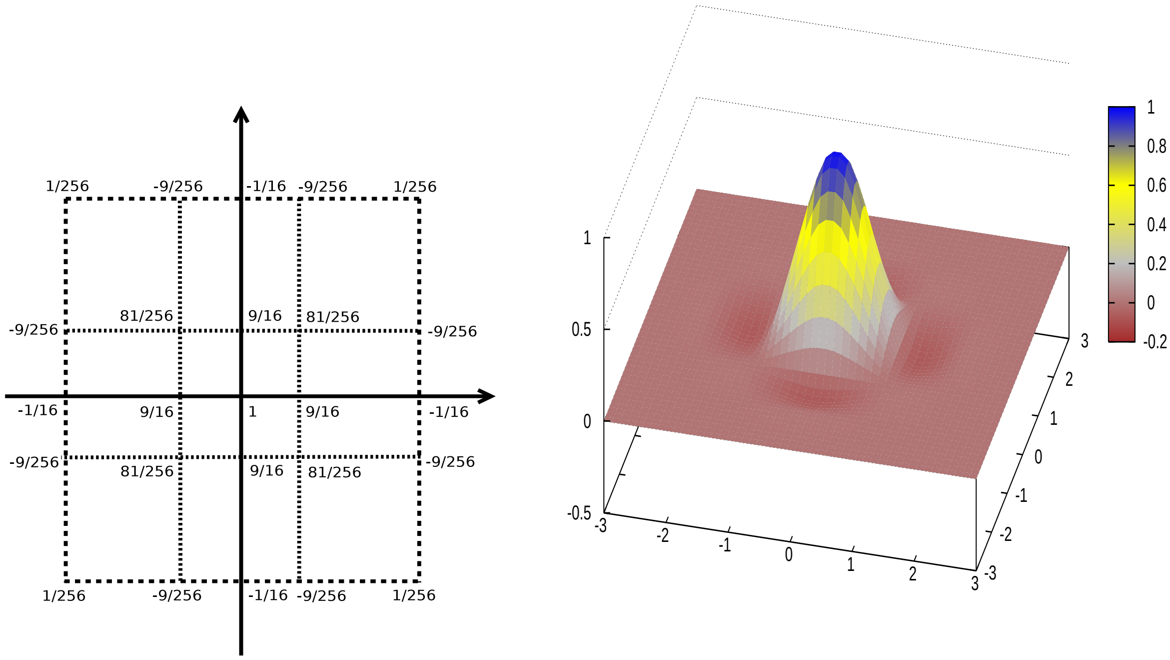
As an example, consider the shifted *hat* function $\tilde{h} \in C_0(\mathbb{R})$, as defined by

$$\tilde{h}(x) = \begin{cases} x+1, & x \in [-1,0), \\ 1-x, & x \in [0,1), \\ 0, & x \in \mathbb{R} \setminus [-1,1), \end{cases} \tag{4.11}$$

which is interpolatory, refinable and supported on the interval $[-1,1]$, and which associated mask symbol $\tilde{\mathcal{A}}_{\tilde{h}}$ is given by

$$\tilde{\mathcal{A}}_{\tilde{h}}(z) = \frac{1}{2}(1+z)^2 z^{-1} = 1 + \frac{1}{2}(z+z^{-1}), \quad z \in \mathbb{C} \setminus \{0\}. \tag{4.12}$$

It follows from Theorem 4.2.1 that $\phi = \tilde{h} \cdot \tilde{h} \in C_0(\mathbb{R}^2)$ is an interpolatory refinable function supported on the square $[-1,1]^2$. The graph of ϕ is given in Figure 4.1 (b), and the support of the corresponding interpolatory mask is delimitated by the dotted lines in Figure 4.1 (a).



(a) Support of A^D (b) Graph of $\phi^D = \tilde{\phi}^D \cdot \tilde{\phi}^D$

Figure 4.2: The tensor product of the Dubuc-Deslauriers $\tilde{\phi}^D$

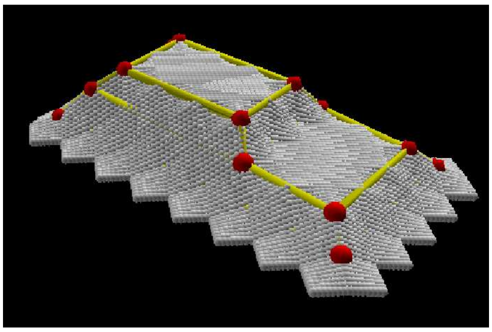
Moreover, we deduce from (4.12) and (4.10) that the associated interpolatory mask symbol $\tilde{\mathcal{A}}$ is given by

$$\tilde{\mathcal{A}}(z_1, z_2) = \tilde{\mathcal{A}}_{\tilde{h}}(z_1) \cdot \tilde{\mathcal{A}}_{\tilde{h}}(z_2) = \frac{1}{4}(1+z_1)^2(1+z_2)^2 z_1^{-1} z_2^{-1}, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}. \tag{4.13}$$

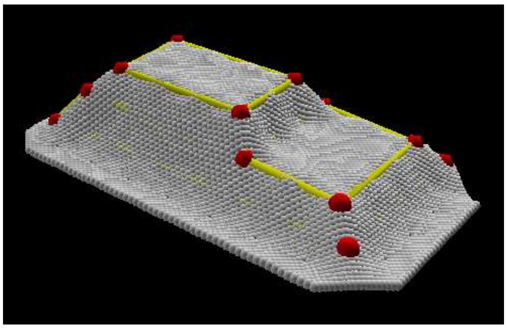
Observe that the mask symbol $\tilde{\mathcal{A}}$ given by (4.13) and the mask symbol G_1 given by (4.6) are the same, which means that they correspond to the same refinable function ϕ which existence is guaranteed by both Theorem 4.2.1 and Theorem 4.1.1.

Next, consider the *Dubuc-Delauriers* function $\tilde{\phi}^D$ [Hun05] (see also [VGH03]) which is interpolatory, refinable and supported on the interval $[-3, 3]$, and which associated mask symbol $\tilde{\mathcal{A}}^D$ is given by

$$\begin{aligned} \tilde{\mathcal{A}}^D(z) =& 1 + \frac{9}{16}(z + z^{-1}) - \frac{1}{16}(z^3 + z^{-3}) \\ =& \frac{1}{16} z^{-2}(1+z)^4(4 - z - z^{-1}), \quad z \in \mathbb{C} \setminus \{0\}. \end{aligned} \tag{4.14}$$



(a) Graph of $\Phi_{\tilde{\mathcal{A}}}$ and c



(b) Graph of Φ_{A^D} and c

Figure 4.3: Subdivisions $S_{\tilde{\mathcal{A}}}$ and S_{A^D} applied to c

Since also $\tilde{\phi}^D \in C_0^1(\mathbb{R})$, it follows from Theorem 4.2.1 that $\phi^D = \tilde{\phi}^D \cdot \tilde{\phi}^D \in C_0^1(\mathbb{R}^2)$ is an interpolatory refinable function supported on the square $[-3,3]^2$. Besides, we deduce from (4.14) and (4.10) that the associated mask symbol A^D is given by

$$A^D(z_1,z_2) = \frac{1}{256}(1+z_1)^4(1+z_2)^4 z_1^{-2} z_2^{-2}(4-z_1-z_1^{-1})(4-z_2-z_2^{-1}), \ \ z_1,z_2 \in \mathbb{C}\backslash\{0\}. \quad (4.15)$$

Observe that the graph of ϕ^D, as shown in Figure 4.2 (b), is indeed a smooth surface as implied by Theorem 4.2.1. The support of the corresponding interpolatory mask symbol A^D is delimitated by the dotted lines in Figure 4.2 (a).

Let us now use the control point c illustrated in Figure 3.1 (a), and denote by $S_{\tilde{\mathcal{A}}}$ and S_{A^D} the subdivision schemes corresponding to the interpolatory mask symbols $\tilde{\mathcal{A}}$ and A^D, as respectively given by (4.13) and by (4.15). We show in Figures 4.3 (a) and (b) the graphs of the limit functions $\Phi_{\tilde{\mathcal{A}}}$ and Φ_{A^D} corresponding respectively to the subdivision schemes $S_{\tilde{\mathcal{A}}}$ and S_{A^D}, with respect to the initial sequence c.

Observe that $\Phi_{A^D} \in C^1(\mathbb{R}^2)$, i.e. Φ_{A^D} defines a smooth surface, whereas both $\Phi_{\tilde{\mathcal{A}}}$ in Figure 4.3 (a) and Φ in Figure 3.1 (b) define non-smooth surfaces. In general, smoother refinable functions can be obtained by tensor products, yet they present the disadvantage of having large supports.

4.3 Mask construction based on sum rules

In this section, we deduce from numerical results the existence of refinable functions associated with interpolatory masks constructed from sum rules.

Borrowing the definition in [HJ00], given a dilation matrix M, we say that a sequence $a \in M(\mathbb{Z}^2)$ satisfies the sum rules of order $k \in \mathbb{N}$ if

$$\sum_{\beta \in M\mathbb{Z}^2} a_{\varepsilon+\beta} p(\varepsilon+\beta) = \sum_{\beta \in M\mathbb{Z}^2} a_\beta p(\beta), \quad \varepsilon \in \mathbb{Z}^2, \quad p \in \Pi_{k-1}, \tag{4.16}$$

where Π_{k-1} denotes the set of bivariate polynomials of total degree (at most) $k-1$. Since Π_{k-1} is generated by the monomial ideal $\langle z_1^{\mu_1} z_2^{\mu_2} : (\mu_1, \mu_2) \in \mathbb{Z}_+^2, \quad \mu_1 + \mu_2 \leq k-1 \rangle$, we observe from (1.8) that, for an interpolatory mask $a \in M_0(\mathbb{Z}^2)$, the property (4.16) is equivalent to

$$\sum_{(\beta_1,\beta_2) \in M\mathbb{Z}^2} a_{\varepsilon_1+\beta_1, \varepsilon_2+\beta_2} (\varepsilon_1+\beta_1)^{\mu_1} (\varepsilon_2+\beta_2)^{\mu_2} = \delta_{(\mu_1,\mu_2)}, \quad \mu_1 + \mu_2 \leq k-1, \tag{4.17}$$

for $(\mu_1, \mu_2) \in \mathbb{Z}_+^2$ and $(\varepsilon_1, \varepsilon_2) \in \mathbb{Z}^2$, where δ denotes the delta sequence defined by (1.3).

Using then a similar argument as in [HJ98b], we claim that, for an interpolatory mask $a \in M_0(\mathbb{Z}^2)$ symmetric about the two coordinates, the sum rules (4.17) holds whenever μ_1 or μ_2 is an odd number.

To prove this, consider an interpolatory mask $a \in M_0(\mathbb{Z}^2)$ and suppose that a is symmetric about the two coordinates. If μ_1 is odd, we have, for $\mu_2 \in \mathbb{Z}_+$ and $(\varepsilon_1, \varepsilon_2) \in \mathbb{Z}^2$,

$$\begin{aligned}
&\sum_{(\beta_1,\beta_2)\in M\mathbb{Z}^2} a_{\varepsilon_1+\beta_1,\varepsilon_2+\beta_2}(\varepsilon_1+\beta_1)^{\mu_1}(\varepsilon_2+\beta_2)^{\mu_2} \\
&\qquad = \sum_{(\beta_1,\beta_2)\in M\mathbb{Z}^2} a_{-\varepsilon_1-\beta_1,\varepsilon_2+\beta_2}(\varepsilon_1+\beta_1)^{\mu_1}(\varepsilon_2+\beta_2)^{\mu_2} \\
&\qquad = -\sum_{(\beta_1,\beta_2)\in M\mathbb{Z}^2} a_{\varepsilon_1+\beta_1,\varepsilon_2+\beta_2}(\varepsilon_1+\beta_1)^{\mu_1}(\varepsilon_2+\beta_2)^{\mu_2},
\end{aligned}$$

and thus

$$\sum_{(\beta_1,\beta_2)\in M\mathbb{Z}^2} a_{\varepsilon_1+\beta_1,\varepsilon_2+\beta_2}(\varepsilon_1+\beta_1)^{\mu_1}(\varepsilon_2+\beta_2)^{\mu_2} = 0 = \delta_{(\mu_1,\mu_2)}.$$

We apply a similar argument for the case where μ_2 is odd.

According to [HJ98b] (see also [HJ00]), given a dilation matrix M and an interpolatory refinable function $\phi \in C_0(\mathbb{R}^2)$, the *shift invariant space* $S(\phi)$ generated by ϕ, as defined by

$$S(\phi) = \left\{ \sum_{\mathbf{j}} c_{\mathbf{j}} \phi(\cdot - \mathbf{j}), \quad c \in M(\mathbb{Z}^2) \right\}, \tag{4.18}$$

contains Π_{k-1} if and only if the interpolatory mask $a \in M_0(\mathbb{Z}^2)$ associated with ϕ satisfies the sum rules of order $k \in \mathbb{N}$.

From this perspective, it seems sensible to have an interpolatory mask that satisfies the sum rules of as high an order as possible. In [HJ98b], some finitely supported interpolatory masks are constructed by solving for the sequence a from the non-linear equations (4.17). However, the existence of the associated interpolatory refinable functions are not investigated.

This motivates us to investigate numerically whether for some of the interpolatory masks constructed in [HJ98b], the corresponding interpolatory refinable functions seem to exist.

Given a dilation matrix M and an interpolatory mask $a \in M_0(\mathbb{Z}^2)$, we use the delta sequence δ defined in (1.3), as well as the dyadic set $\mathcal{D}$ defined in (3.13), to deduce from (3.12) that, for $f \in M(\mathbb{R}^2)$,

$$T_a^r f(M^{-r}\mathbf{k}^T) = \sum_{\mathbf{j}} (S_a^r \delta)_{\mathbf{j}} f(\mathbf{k}-\mathbf{j}), \quad \mathbf{k} \in \mathbb{Z}^2, \quad r \in \mathbb{Z}_+,$$

according to which, if the function f satisfies $f(\mathbf{j}) = \delta_{\mathbf{j}}$, $\mathbf{j} \in \mathbb{Z}^2$, then it holds that

$$T_a^r f(M^{-r}\mathbf{k}^T) = (S_a^r \delta)_{\mathbf{k}}, \quad \mathbf{k} \in \mathbb{Z}^2, \quad r \in \mathbb{Z}_+. \tag{4.19}$$

Considering then an initial function $g \in C_0(\mathbb{R}^2)$ chosen to be interpolatory and refinable, we shall use the cascade algorithm T_a, as defined in (3.7), to draw the graphs of $\phi_0 = g$, $\phi_1 = T_a g$ and $\phi_2 = T_a^2 g$ by means of the formula (3.9). Since evaluating $\phi_r = T_a^r g$ is computationally intense for large values of $r \in \mathbb{Z}_+$, we shall rather use (4.19) in order to represent the graph of ϕ_r. More precisely, for $r \geq 3$, we plot the sequence of points $(M^{-r}\mathbf{j}^T, (S_a^r\delta)_{\mathbf{j}})$, $\mathbf{j} \in \mathbb{Z}^2$, as generated recursively by means of the subdivision scheme S_a^r defined in (3.3).

The interpolatory masks g_2 and h_2

Let the dilation matrix $M = 2I$ be fixed, and let $a \in M_0(\mathbb{Z}^2)$ be an interpolatory mask. From now on, we shall use the shifted box spline $\tilde{N}_2 \in C_0(\mathbb{R}^2)$ defined by (1.30) as the initial interpolatory refinable function for the cascade algorithm T_a^r, $r \in \mathbb{Z}_+$, as given by (3.9).

According to (4.17), the mask $a \in M_0(\mathbb{Z}^2)$ satisfies the sum rules of order $k \in \mathbb{Z}_+$ if

$$\sum_{\beta_1,\beta_2} a(\varepsilon_1 + 2\beta_1, \varepsilon_2 + 2\beta_2)(\varepsilon_1 + 2\beta_1)^{\mu_1}(\varepsilon_2 + 2\beta_2)^{\mu_2} = \delta_{(\mu_1,\mu_2)}, \quad \mu_1 + \mu_2 \leq k-1, \tag{4.20}$$

where $(\mu_1, \mu_2) \in \mathbb{Z}_+^2$ and $(\varepsilon_1, \varepsilon_2) \in \{(0,0), (1,0), (0,1), (1,1)\}$.

The interpolatory mask $a = g_2$ [HJ98b] is contructed in such a way to satisfy the sum rules of order 4, and to be supported on the set $\{(\alpha_1, \alpha_2) : |\alpha_1| + |\alpha_2| \leq 4\}$. It is obtained by solving the linear system (4.20) for $k = 4$, after setting also $a(i,j) = 0$, $|i| + |j| \geq 5$, yielding the values $a(i,j) = g_2(i,j)$ given by

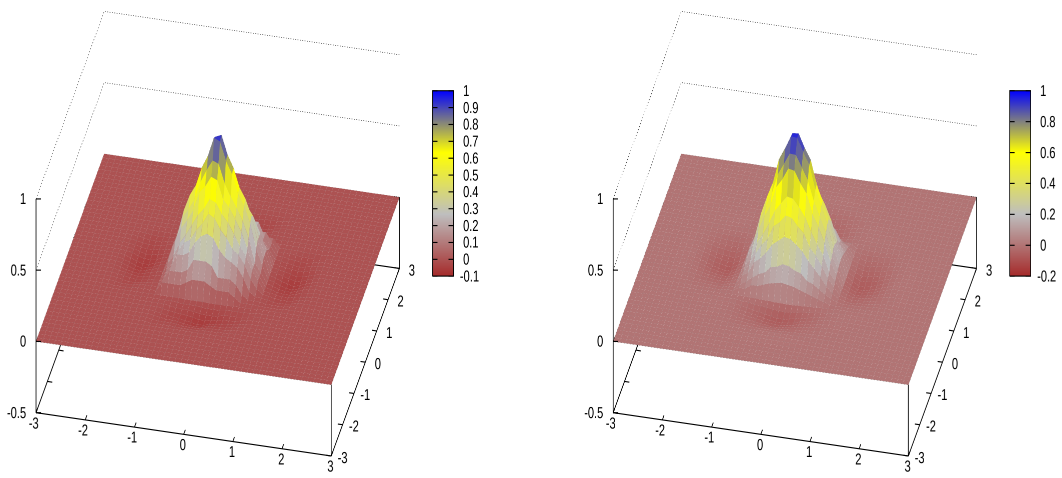
$$\begin{aligned}
g_2(0,0) =& 1, \\
g_2(3,0) =& g_2(-3,0) = g_2(0,3) = g_2(0,-3) = \frac{-1}{16}, \\
g_2(1,0) =& g_2(-1,0) = g_2(0,1) = g_2(0,-1) = \frac{9}{16}, \\
g_2(1,1) =& g_2(-1,1) = g_2(1,-1) = g_2(-1,-1) = \frac{5}{16}, \\
g_2(3,1) =& g_2(-3,1) = g_2(3,-1) = g_2(-3,-1) = \frac{-1}{32}, \\
g_2(1,3) =& g_2(-1,3) = g_2(1,-3) = g_2(-1,-3) = \frac{-1}{32}.
\end{aligned}$$

The mask symbol G_2 associated with g_2 is given by

$$\begin{aligned}
G_2(z_1, z_2) = 1 &- \frac{1}{16}(z_1^{-3} + z_1^3 + z_2^{-3} + z_2^3) + \frac{9}{16}(z_1^{-1} + z_1 + z_2^{-1} + z_2) \\
&+ \frac{5}{16}(z_1 z_2 + z_1^{-1} z_2 + z_1 z_2^{-1} + z_1^{-1} z_2^{-1}) - \frac{1}{32} b(z_1, z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \quad (4.21)
\end{aligned}$$

where $b(z_1, z_2) = z_1^3 z_2 + z_1^{-3} z_2^{-1} + z_1 z_2^3 + z_1^{-1} z_2^{-3} + z_1 z_2^{-3} + z_1^{-1} z_2^3 + z_1^{-3} z_2 + z_1^3 z_2^{-1}$, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$. Note that G_2 can be re-written as

$$\begin{aligned}
G_2(z_1, z_2) = \frac{1}{16}(1 + z_1)^2 (1 + z_2)^2 z_1^{-2} z_2^{-2} \Bigg[& z_1 z_2^2 + z_1^2 z_2 - \frac{1}{2}(z_1 z_2^3 + z_1^3 z_2) \\
& - \frac{1}{2}(z_1 z_2^{-1} + z_1^{-1} z_2) + z_1 + z_2 + 2 z_1 z_2 \Bigg], \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}.
\end{aligned}$$



(a) Graph of $T_{g_2}\tilde{N}_2$ (b) Graph of $T_{g_2}^2\tilde{N}_2$

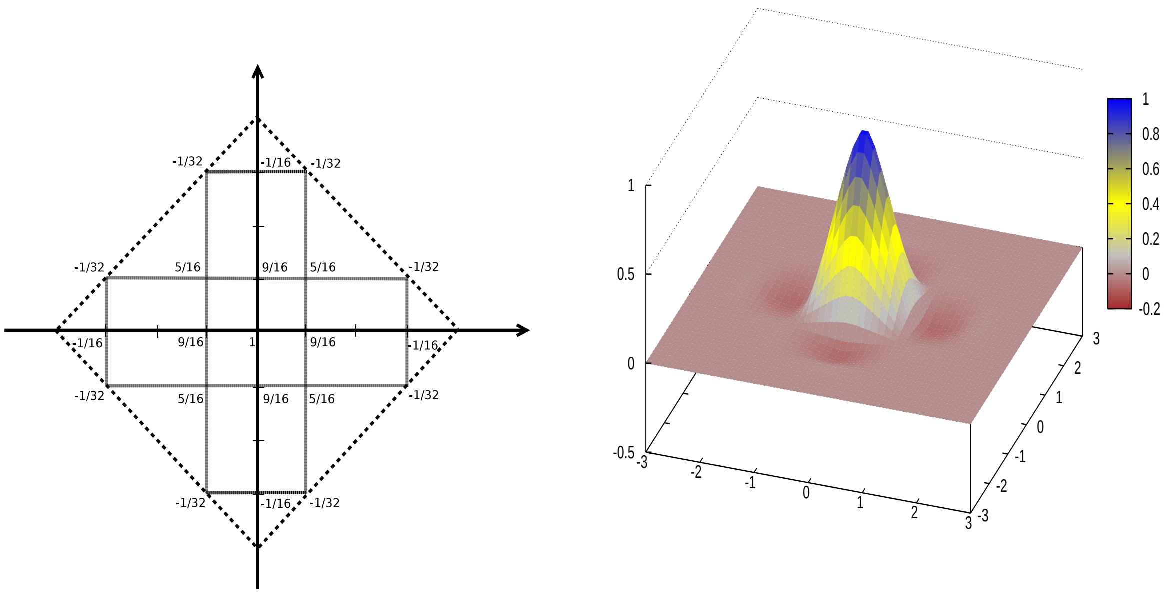
Figure 4.4: Cascade algorithm for the mask g_2

Observe now from (4.21) that, for $x_1, x_2 \in \mathbb{R}$,

$$\begin{aligned} G_2(\mathrm{e}^{ix_1}, \mathrm{e}^{ix_2}) =& 1 - \frac{1}{8}[\cos(3x_1) + \cos(3x_2)] + \frac{9}{8}[\cos\ x_1 + \cos x_2] + \frac{5}{8}\cos(x_1 + x_2) \\ &+ \frac{5}{8}\cos(x_1 - x_2) - \frac{1}{16}[\cos(3x_1 + x_2) + \cos(x_1 + 3x_2)] \\ &- \frac{1}{16}[\cos(x_1 - 3x_2) + \cos(3x_1 - x_2)], \\ =& 1 - \frac{1}{8}[\cos(3x_1) + \cos(3x_2)] + \frac{9}{8}[\cos x_1 + \cos x_2] \\ &+ \frac{5}{4}\cos\ x_1 \cos x_2 - \frac{1}{8}[\cos(3x_1)\cos x_2 + \cos x_1 \cos(3x_2)). \end{aligned}$$

Noting that $G_2(\mathrm{e}^{i7\pi/6}, \mathrm{e}^{i7\pi/6}) = -1.044 \times 10^{-3} < 0$, we deduce that g_2 is not non-negative, so that we can not appeal to Theorem 4.1.1 for the existence of a corresponding refinable function ϕ_{g_2}.

Nevertheless, we observe from Figures 4.4 (a) and (b) that the cascade algorithm T_{g_2} seems to be convergent. Hence, we numerically deduce that the corresponding interpolatory refinable function ϕ_{g_2} exists, as illustrated in Figure 4.5 (b) which also shows that ϕ_{g_2} seems to be of class C^1, i.e. $\phi_{g_2} \in C_0^1(\mathbb{R}^2)$. The support of g_2 is delimitated by the dotted lines in Figure 4.5 (a) according to which g_2 is symmetric about the two axes and



(a) Support of g_2 (b) Graph of ϕ_{g_2}

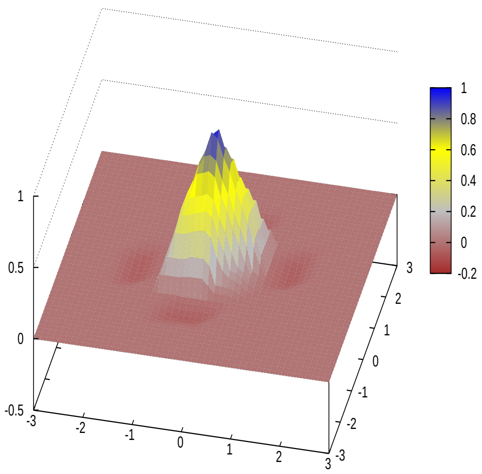
Figure 4.5: Refinable function corresponding to g_2

about the line $y = x$.

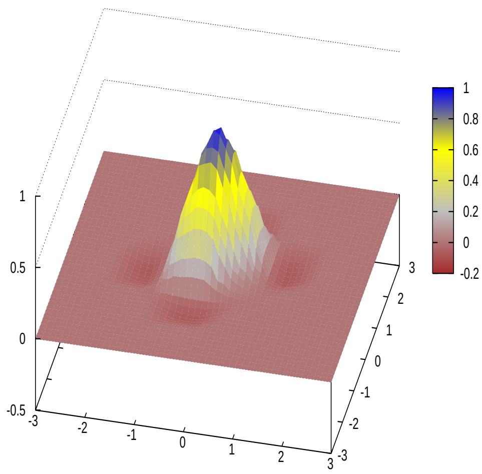
Similarly, the interpolatory mask $a = h_2$ [HJ98b] is constructed in such a way to satisfy the sum rules of order 4, and to be supported on the set $\{(\alpha_1, \alpha_2) : |\alpha_1 + \alpha_2| \leq 4, |\alpha_1 - \alpha_2| \leq 3\}$. It is obtained by solving the linear system (4.20) for $k = 4$, after setting also $a(i, j) = 0$, $|i + j| \geq 5$ or $|i - j| \geq 4$, yielding the values $a(i, j) = h_2(i, j)$ given by

$$
\begin{aligned}
h_2(0,0) =& 1,\\
h_2(3,0) =& h_2(-3,0) = h_2(0,3) = h_2(0,-3) = \frac{-1}{16},\\
h_2(1,0) =& h_2(-1,0) = h_2(0,1) = h_2(0,-1) = \frac{9}{16},\\
h_2(1,1) =& h_2(-1,-1) = \frac{1}{2},\\
h_2(1,-1) =& h_2(-1,1) = \frac{1}{8},\\
h_2(3,1) =& h_2(-3,-1) = h_2(1,3) = h_2(-1,-3) = \frac{-1}{16}.
\end{aligned}
$$

Note that h_2 has a smaller support than g_2, and that the associated mask symbol H_2



(a) Graph of $T_{h_2}\tilde{N}_2$



(b) Graph of $T_{h_2}^2\tilde{N}_2$

Figure 4.6: Cascade algorithm for the mask h_2

is given by

$$\begin{aligned} H_2(z_1,z_2) &= 1 - \frac{1}{16}(z_1^{-3}+z_1^3+z_2^{-3}+z_2^3) + \frac{9}{16}(z_1^{-1}+z_1+z_2^{-1}+z_2) \\ &\quad + \frac{1}{2}(z_1z_2+z_1^{-1}z_2^{-1}) + \frac{1}{8}(z_1^{-1}z_2+z_1z_2^{-1}) \\ &\quad - \frac{1}{16}(z_1^3z_2+z_1^{-3}z_2^{-1}+z_1z_2^3+z_1^{-1}z_2^{-3}), \quad z_1,z_2\in\mathbb{C}\setminus\{0\}, \end{aligned} \tag{4.22}$$

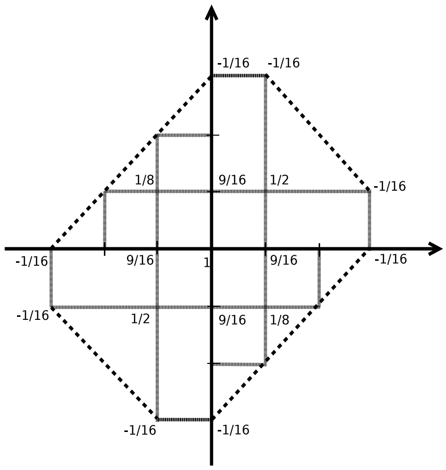
which can be re-written as

$$\begin{aligned} H_2(z_1,z_2) = \frac{1}{16}(1+z_1)(1+z_2)\,&\big[6+z_1+z_2+2(z_1^{-1}+z_2^{-1})-z_1^2-z_2^2 \\ &+ z_1^{-2}z_2^{-1}+z_1^{-1}z_2^{-2}-z_1^{-3}z_2^{-1}-z_1^{-1}z_2^{-3}+6z_1^{-1}z_2^{-1}\big], \quad z_1,z_2\in\mathbb{C}\setminus\{0\}. \end{aligned}$$

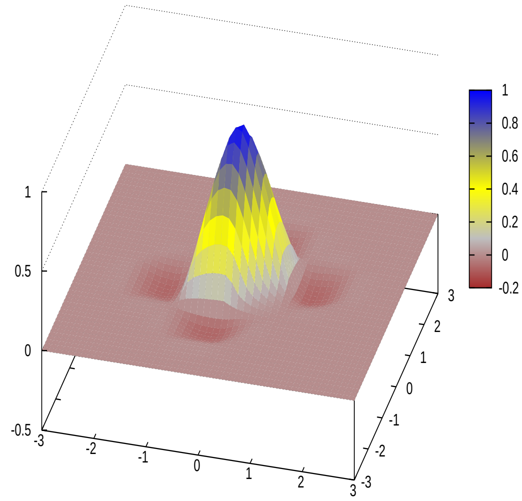
Next, we deduce from (4.22) that, for $x_1,x_2\in\mathbb{R}$,

$$\begin{aligned} H_2(\mathrm{e}^{ix_1},\mathrm{e}^{ix_2}) &= 1 - \frac{1}{8}[\cos(3x_1)+\cos(3x_2)] + \frac{9}{8}[\cos\ x_1+\cos x_2] \\ &\quad - \frac{1}{8}[\cos(3x_1+x_2)+\cos(x_1+3x_2)] + \cos(x_1+x_2) + \frac{1}{4}\cos(x_1-x_2). \end{aligned}$$

Noting that $H_2(\mathrm{e}^{i2\pi/3},\mathrm{e}^{i2\pi/3}) = -\dfrac{1}{2} < 0$, we deduce that h_2 is not non-negative, which



(a) Support of h_2



(b) Graph of ϕ_{h_2}

Figure 4.7: Refinable function corresponding to h_2

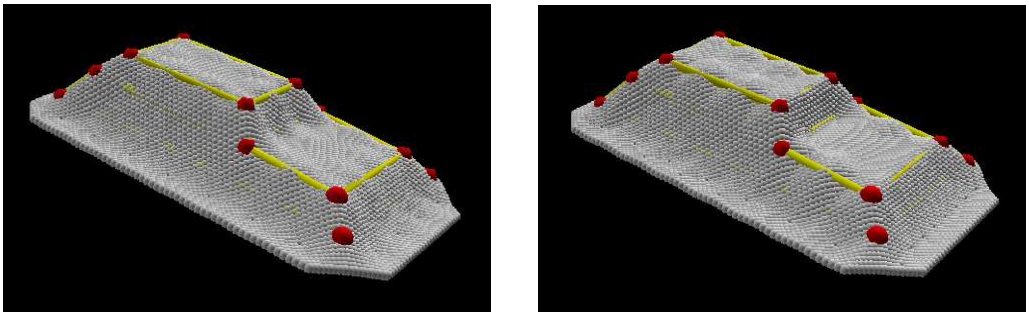
means that we can not appeal to Theorem 4.1.1 for the existence of a corresponding refinable function ϕ_{h_2}.

However, we observe from Figures 4.6 (a) and (b) that the cascade algorithm T_{h_2} seems to be convergent. We then numerically deduce that the corresponding interpolatory refinable function ϕ_{h_2} exists, as illustrated in Figure 4.7 (b) which also shows that ϕ_{h_2} seems to be of class C^1, i.e. $\phi_{h_2} \in C_0^1(\mathbb{R}^2)$. The support of h_2 is delimitated by the dotted lines in Figure 4.7 (a) according to which h_2 is symmetric about both the origin and the line $y = x$.

Note that, given an interpolatory mask a, if the corresponding interpolatory refinable function ϕ exists, then, from (1.1),

$$\phi(\mathbf{j}/2) = \sum_{\mathbf{k}} a_{\mathbf{k}} \phi(\mathbf{j} - \mathbf{k}) = a_{\mathbf{j}}, \quad \mathbf{j} \in \mathbb{Z}^2, \tag{4.23}$$

by virtue of the refinement equation (1.1). It follows from (4.23) that the surface defined by ϕ passes through the points $(\mathbf{j}, a_{\mathbf{j}})$ for all $\mathbf{j} \in \mathbb{Z}^2$.



(a) Graph of Φ_{g_2} and c (b) Graph of Φ_{h2} and c

Figure 4.8: Subdivisions S_{g_2} and S_{h_2} applied to c

For the interpolatory masks g_2 and h_2, observe from Figure 4.5 (b) and Figure 4.7 (b) that the graphs of ϕ_{g_2} and ϕ_{h_2} are consistent with the property (4.23).

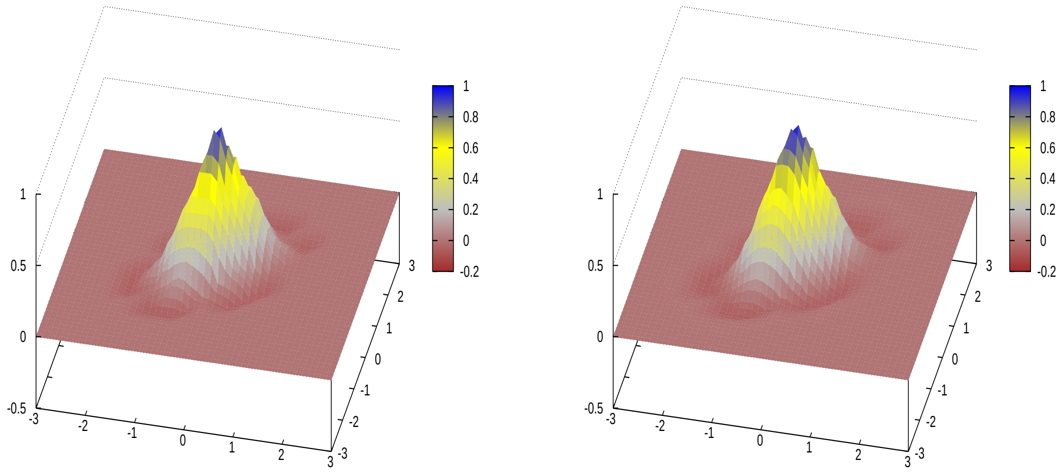
Moreover, using the control point c illustrated in Figure 3.1 (a), we observe from Figures 4.8 (a) and (b) that the corresponding subdivision schemes S_{g_2} and S_{h_2}, with respect to the initial sequence c, yield the limit functions Φ_{g_2} and Φ_{h2} which both define smooth surfaces, which is consistent with the result in [HJ98b] stating that g_2 and h_2 induce C^1 interpolatory subdivision schemes, i.e. for any sequence $c \in M(\mathbb{Z}^2)$, the limit function $S_{g_2}^{\infty}c$ and S_{h_2} belong to $C^1(\mathbb{R}^2)$.

The butterfly interpolatory mask

Let the dilation matrix $M = 2I$ be fixed. We now introduce the well-known butterfly mask developed in [DLG90] and [DL02] (see also [Dyn92]).

For $w \in \mathbb{R}$, the *butterfly* mask symbol $\mathcal{B}_w$ is the Laurent polynomial defined by

$$\mathcal{B}_w(z_1, z_2) = \frac{1}{2}(1 + z_1)(1 + z_2)(1 + z_1^{-1} z_2^{-1})(1 - wC(z_1, z_2)), \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}, \qquad (4.24)$$



(a) Graph of $T_{\mathcal{B}_w}\tilde{N}_2$, $w = 1/16$ (b) Graph of $T^2_{\mathcal{B}_w}\tilde{N}_2$, $w = 1/16$

Figure 4.9: Cascade algorithm for the butterfly mask $\mathcal{B}_w$, $w = 1/16$

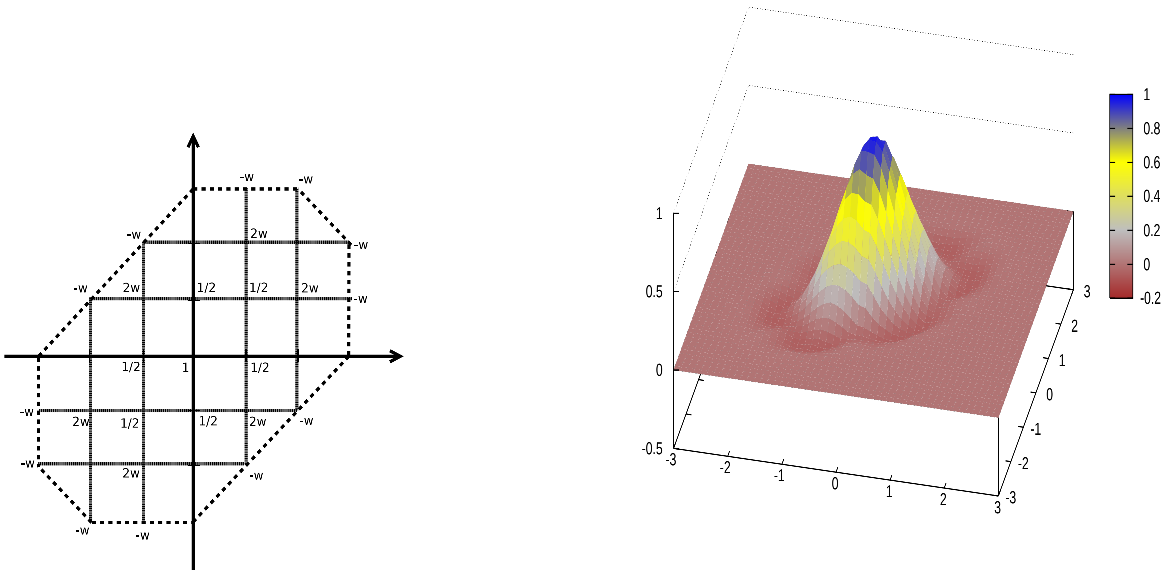
where the Laurent polynomial C is given by

$$C(z_1, z_2) = 2z_1^{-2}z_2^{-1} + 2z_1^{-1}z_2^{-2} - 4z_1^{-1}z_2^{-1} - 4z_1^{-1} - 4z_2^{-1}$$
$$+ 2z_1^{-1}z_2 + 2z_1z_2^{-1} + 12 - 4z_1 - 4z_2 - 4z_1z_2 + 2z_1^2z_2 + 2z_1z_2^2, \quad z_1, z_2 \in \mathbb{C} \setminus \{0\}.$$

Note from (4.24) that, for $w \in \mathbb{R}$, the butterfly mask $\mathcal{B}_w$ is an interpolatory mask symbol supported on the square $[-3, 3]^2$. In particular, we have $\mathcal{B}_0 = \tilde{A}_2$, where $\tilde{A}_2$ denotes the interpolatory mask symbol given by (1.32).

With the choice $w = 1/16$, we observe from Figures 4.9 (a) and (b) that the cascade algorithm $T_{\mathcal{B}_w}$ seems to be convergent. Therefore, we numerically deduce that the corresponding interpolatory refinable function $\phi_{\mathcal{B}_w}$ exists, as illustrated in Figure 4.7 (b) which also shows that $\phi_{\mathcal{B}_w}$ seems to be of class C^1, i.e. $\phi_{\mathcal{B}_w} \in C_0^1(\mathbb{R}^2)$. The support of $\mathcal{B}_w$ is delimitated by the dotted lines in Figure 4.7 (a) according to which $\mathcal{B}_w$ is symmetric about both the origin and the line $y = x$.

Using the control point c illustrated in Figure 3.1 (a) and with $w = 1/16$, we show in Figure 4.11 that the limit function $\Phi_{\mathcal{B}_w}$ resulting from the Butterfly subdivision defines a



(a) Support of $\mathcal{B}_w$ (b) Graph of $\phi_{\mathcal{B}_w}$, with $w = 1/16$

Figure 4.10: Refinable function corresponding to $\mathcal{B}_w$

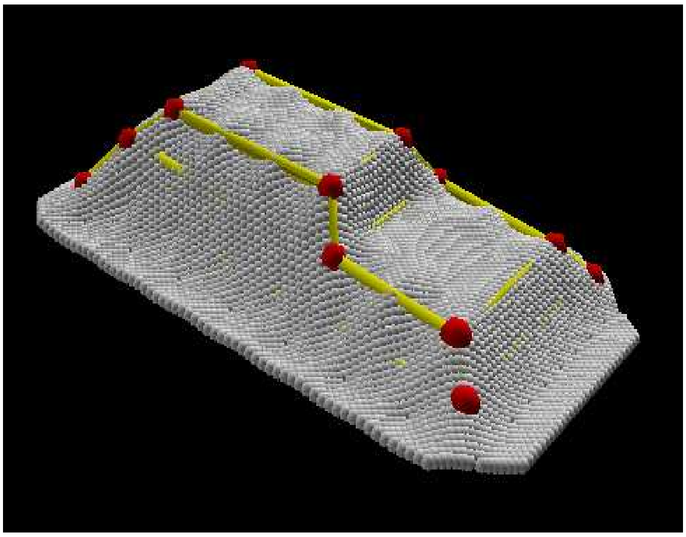


Figure 4.11: Graph of $\Phi_{\mathcal{B}_w}$, $w = 1/16$, showing the Butterfly subdivision applied to c

smooth surface, which is consistent with the result in [DLG90] and in [DL02] stating that, for a sufficiently small $w > 0$, the butterfly scheme $S_{\mathcal{B}_w}$ is a C^1 interpolatory subdivision scheme, that is, for any sequence $c \in M(\mathbb{Z}^2)$, the limit function $S^\infty_{\mathcal{B}_w} c$ belongs to $C^1(\mathbb{R}^2)$.

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