Economic capital allocation to market and survival risk for pure endowment products

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Abstract

Economic capital allocation to interest and longevity rate risks is a topic of interest for life insurers. This study aims to provide approaches to allocate the overall economic capital amount into market and survival risk components for a pure endowment product. The calculation of the economic capital figure can be done using either analytical or simulation-based methods. An allocation approach found in literature is then applied to the simulation-based capital quantification. An allocation approach for the analytical method is proposed. A pure endowment contract which faces risks that have been calibrated to a regulatory shock environment over one year backed by a six-month fixed interest risk free asset was used as case study for these allocation approaches. Both methods deliver comparable results, and both conclude that interest rate risk is much more important than the longevity component. The importance of interest rate risks depending on method range between 97% and 99.97% of economic capital allocated to this risk. Sensitivity analysis proved particularly insightful in this study. We found that the analytical approach is more sensitive to the methodology choice in the decomposition step. Both methods provide sensible behaviour across different parameter values. The main advantage of the simulation-based approach is flexibility. The analytical approach delivers a closed form solution for capital allocation which reduces computing time and ease of implementation. This research demonstrates capital allocation provides a valuable tool for understanding the behaviour of capital relative to various risks. This enables insurers to better manage their risk exposure as they can start to see the drivers of risk.

Key words:

Economic Capital, Capital Allocation, Insurance Modelling, Interest Rate Risk, Survival Rate Risk, Variance Decomposition, Euler Method.

Opsomming

Ekonomiese kapitaaltoewysing aan rente- en langlewendheidskoersrisiko's is 'n onderwerp van belang vir lewensversekeraars. Hierdie studie het ten doel om benaderings te verskaf om die algehele ekonomiese kapitaalbedrag in mark- en oorlewingsrisikokomponente vir 'n suiwer uitkeerproduk toe te deel. Die berekening van die ekonomiese kapitaalsyfer kan met óf analitiese óf simulasie-gebaseerde metodes gedoen word. 'n Toedelingsbenadering wat in literatuur gevind word, word dan toegepas op die simulasie-gebaseerde kapitaalkwantifisering. 'n Toekenningsbenadering vir die analitiese metode word voorgestel. As gevallestudie is 'n suiwer uitkeerkontrak met ‘n termyn van een jaar oorweeg. Dit is gekalibreer vir die omgewing deur regulasies geïmpliseer en die versekeraar belê in 'n ses maande vaste rente risikovrye bate. Albei metodes lewer vergelykbare resultate, en albei kom tot die gevolgtrekking dat rentekoersrisiko baie belangriker is as die langlewendheidskomponent. Die belangrikheid van rentekoersrisiko wissel tussen 97% en 99.97% van ekonomiese kapitaal wat aan hierdie risiko toegewys is afhangende van metode. Sensitiwiteitsanalise het in hierdie studie besonder insiggewend bewys. Ons het gevind dat die analitiese benadering meer sensitief is vir die metodologiekeuse in die ontbindingstap. Beide metodes verskaf sinvolle gedrag oor verskillende parameterwaardes. Die grootste voordeel van die simulasie-gebaseerde benadering is dat dit in ‘n verskeidenheid situasies toegepas kan word. Die analitiese benadering lewer 'n geslote vorm oplossing vir kapitaaltoewysing wat rekenaartyd verminder makliker implementeer kan word. Hierdie navorsing demonstreer kapitaaltoedeling bied 'n waardevolle hulpmiddel om die gedrag van kapitaal relatief tot verskeie risiko's te verstaan. Dit stel versekeraars in staat om hul risikoblootstelling beter te bestuur aangesien hulle die drywers van risiko kan begin sien.

Kernwoorde:

Ekonomiese Kapitaal, Kapitaaltoedeling, Versekeringmodelering, Rentekoersrisiko, Langlewendheidskoersrisiko, Variansie ontbinding, Euler Metode.

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CHAPTER 1
INTRODUCTION

In this chapter we introduce the topic along with some background. The research problem is then identified along with secondary research questions that we will consider. The research objectives are then discussed, and the study outline provided.

1.1 TOPIC AND BACKGROUND

We want to determine the amount of economic capital that we can allocate to survival and interest rate risks. Specifically, we wish to determine this allocation for a pure endowment product.

Determining the overall economic capital required by an insurer, as well as capital allocation, form part of an insurer's Enterprise Risk Management. These practices enhance a corporate's ability to identify, measure, price and control its risks (Dheane *et al*., 2012:1). When we allocate capital specifically to risk sources, we further our understanding of the potential financial impact of the risks an insurer accepts. Such an allocation can also be used as part of an insurer's capital management process.

A firm understanding of economic capital is needed to understand the relevance of economic capital allocation, and we therefore introduce this here. Economic capital is the excess amount of assets (relative to the value of the liabilities) that an insurer deems it necessary to hold to remain solvent in a range of adverse scenarios with consideration of their risk appetite. A requirement of economic capital is that the values attached to the assets and liabilities are determined in a way that they are in accordance with economic principles.

1.2 RESEARCH PROBLEM

Capital allocation studies have focused on cases where the risk sources are additive. These would include an allocation to product lines or business units. Allocating risk capital to risk sources has received comparatively little attention in literature (Menzietti & Pirra, 2017:246).

The central research problem this study aims to address is how much economic capital we would need to hold for the interest and survival rate risk of a pure endowment product?

Additionally, we have the following secondary research questions that will be addressed:

- What methodology would be best suited for determining economic capital?
- How would we apply this methodology to the economic capital calculation for a pure endowment product?

- How would we determine the economic capital amount specifically for a pure endowment product?
- How would we decompose the overall risk captured in the economic capital amount to risk sources?
- What are the best allocation methods and how would we apply these to a pure endowment product?
- What amount of economic capital would we allocate to each risk source for a specific pure endowment product?

1.3 RESEARCH OBJECTIVES

To address the research problems, we have the following aims for the study:

- Set out appropriate methodology and assumptions (considering others) for an economic capital determination.
- Apply this methodology to a single period pure endowment product.
- Decompose the risk underlying the economic capital amount to survival and interest rate.
- Consider different allocation methods and apply these to a single period pure endowment product.
- Determine the economic capital amount for a case study.
- Investigate the sensitivity of this amount to various parameters used in modelling the pure endowment product.
- Allocate capital and investigate the findings for a case study.

1.4 RESEARCH METHODOLOGY

We first undertake a literature review to understand the theory that has been developed around the allocation of economic capital. The most important findings of this as they relate to the current study are included. We would need to explore the literature around economic capital modelling in general as well.

We use stochastic modelling to determine the economic capital amount. This model then in turn forms the basis for the allocation exercise that we will undertake. This stochastic model can be used to determine and allocate capital using either a simulation or an analytical approach. These approaches are compared.

We will apply these to a case study that we parameterise considering the current South African regulatory capital modelling framework. We then aim to interpret these results and draw conclusions regarding the survival and interest rate risk an insurer faces from a pure endowment product.

1.5 OUTLINE OF THE STUDY

We have structured the paper towards these aims. After introducing the study in chapter 1 we discuss economic capital in chapter 2. Applying this to a pure endowment product is then discussed in chapter 3. We then need to decompose the risk of the economic capital to survival and interest rate risk as is done in chapter 4. Capital allocation and how this relates to a single period pure endowment product with our modelling setup is then discussed in chapter 5. Chapter 6 contains a case study where we apply chapters 2 to 5 for a pure endowment product with a term of one year that the insurer has matched with 6-month bonds. Chapter 7 concludes. We can illustrate the flow of chapters graphically as is done in Figure 1.

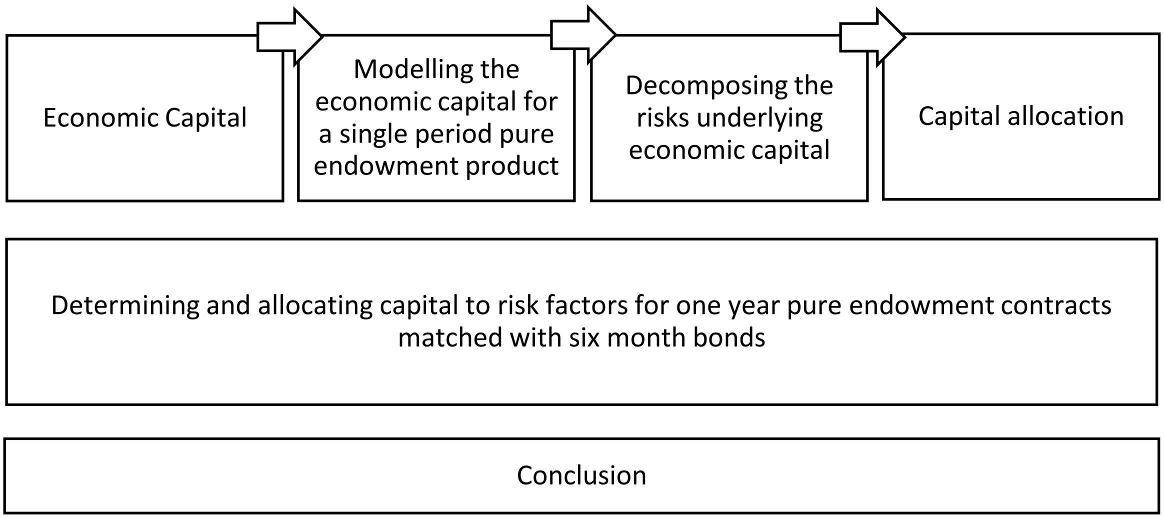


Figure 1

CHAPTER 2
ECONOMIC CAPITAL

In this chapter we consider the theory of economic capital in more depth. We start by defining economic capital before we move our attention to risk measures which form an integral component of defining a firm's economic capital. We then set out the concept of coherence which is determined by certain economic axiomatic requirements. We then look at the advantages and disadvantages of different risk measures and motivate the choice of risk measure for the remainder of the report.

2.1 DEFINITION

As pointed out by Finkelstein, Hoshino, Ino and Morgan (2006:4) there are many possible definitions for economic capital. Importantly this report provides a relatively standard definition for both available and required economic capital. Each of these can be seen as economic capital although they measure different things.

The available economic capital is defined as:"…the excess of the value of the company's assets over the value of its liabilities on a realistic or market consistent basis." (Finkelstein *et al*., 2006:4) This definition is concerned with understanding the solvency position of the firm from an economic point of view. Required economic capital on the other hand focusses on the inherent uncertainty of the insurer's business and what capital is required to support this business. When referring to economic capital in the remainder of the report we are referring to required economic capital rather than available economic capital.

Required economic capital can be viewed as a figure linking probabilistic future profit or loss with a desired confidence level for solvency. Therefore, required economic capital forms the link between the uncertainty of future realisation for the insurer's business and the solvency of the firm. Finkelstein *et al*. (2006:4) defines this simply as the amount of "capital required to support a business with a certain probability of" solvency.

Kapel, Antioch and Tsui (2013:3) define this concept as:

"…an internal calculation of capital required, based on the company's view of risk, with calculations based on "economic principles"."

In this definition it becomes clear that the axiomatic principles for coherence referred to for risk measures above as well as those that will be outlined for capital allocations are important. This is since these axioms have been developed to ensure consistency with economic principles.

We can identify, as was done in Kapel *et al*. (2013:6), several key issues in defining economic capital. In this context we are looking at required economic capital. These are:

- The basis of the balance sheet
- The period of assessment
- Measure of risk
- Confidence level
- Risks to include
- Quantification methodology
- Aggregation

Kapel *et al*. (2013:5) make the distinction between the first four factors corresponding to "methodological decisions that define the conceptual basis", and the last three which reference more practical considerations. Such a distinction although helpful for understanding the involved process of capital management should not lose view that there are practical as well as conceptual considerations in all the issues listed. We can take the example of choosing a one-year time horizon for our calculation which is key to the concept we are evaluating but we may choose to stay away from greater periods of time due to practical issues as well as the greater uncertainty linked to longer maturities. Similarly, the choice of quantification methodology may be driven by practical considerations (such as the availability and reliability of past data) but form a key part of the conceptual basis with which we are dealing. It is therefore important with all economic capital calculations to appreciate the company's view of risks and how this underpins these calculations.

The economic capital can be determined by applying a risk measure to a random variable signifying the gain or loss of the company given by X. We denote a risk measure as ρ such that the economic capital is given by $\rho(X)$. The risk measure links the outcome which is stated in probabilistic terms to an economic capital figure.

We can also use risk measures by the same mechanism to calculate economic capital at different levels of detail, i.e. on a product or risk level and not for the company as a whole as eluded to above. Aggregation then involves how we combine these economic capital amounts to arrive at an overall amount for the company. Kapel *et al*. (2013:20) stress the importance of taking account of the dependencies of different risks in applying an aggregation. A classical example of this is to account for some diversification whereby the total economic capital being less than the sum of the individual amounts. This is since we assume that all risk will not occur simultaneously to the extent implied by the individual economic capital calculations.

The following section will look at different measures of risk that can be used to link the capital amount to outcomes and their associated probabilities.

2.2 CHOICE OF RISK MEASURE

A risk measure is defined as a number that gives us information on the risk taken on. It is natural that risk measures form an integral part of the capital determination and management of an insurer. In the context of capital modelling, the risk measure links the outcome and the capital required to achieve such an outcome to each other. The outcome would be defined in terms of a required confidence level and a time horizon.

As pointed out by Karabey (2012:8-9) a risk measure can be thought of as a mapping from a set of risks denoted χ of which the elements are treated as random variables to real numbers. This can be given by:

$$\rho: \chi \rightarrow \mathbb{R}.$$

The form of the risks in the set will influence the relationship between the risk measure and the economic capital. Let the elements of χ be given by the random variable X which represents the future gain or loss at a specified future time. For the purposes of this section negative values of X represent a loss whereas positive values constitute a gain.

Numerous risk measures have been proposed but we will consider the following three: Value at Risk, Expected Shortfall, and Standard Deviation.

Value at Risk was first introduced to a broader audience in a technical report by JP Morgan in 1995. Morgan introduced this measure as a risk management tool (cited in Karabey 2012:10). It can be thought of as a loss of which the probability of any larger loss is set to a certain level which we will denote by α. The confidence level would then be given by $1-\alpha$ and the corresponding risk measure by:

$$VaR_\alpha(X) = -\inf\{x \in \mathbb{R}: \mathbb{P}(X \leq x) > \alpha\} .$$

The infimum statement can also be interpreted as the greatest lower bound. As the infimum statement is a negative infimum this can be thought of as most negative real value of x that satisfies the probability condition specified. We note that the probability of the random variable being more negative than the $VaR_\alpha(X)$ would be α.

If the Value at Risk is held in cash to absorb possible losses, the greatest loss that can be absorbed will be the greatest loss such that the probability of a greater loss is no more than α. This interpretation illustrates how Value at Risk can be interpreted and used in capital calculations. For a continuous distribution we would have that the Value at Risk would be given by:

$$VaR_\alpha(X) = -F^{-1}(\alpha).$$

We also have the Expected Shortfall which can be thought of as the average loss given the loss is equal to or exceeds a specified quantile of a distribution. For a continuous distribution of X the expected shortfall for quantile α is given by

$$ES_{\alpha}(X) = \frac{1}{1-\alpha}\int_{\alpha}^{1} VaR_u(X)du.$$

The Value at Risk represents the quantiles as shown above. The integral weights the outcomes according to probabilities while dividing the integral by $1 - \alpha$ averages the losses in the tail.

For continuous distributions we can equivalently state the Expected Shortfall as

$$ES_{\alpha}(X) = -E[X|X \leq -VaR_{\alpha}(X)]$$

The negative symbol in front of the expression on the right-hand side ensures that the resulting risk measure is a positive quantity. This reinforces the interpretation of Expected Shortfall as the conditional probability weighted outcome.

It should be noted that if we choose the same α then the Expected Shortfall will be larger than Value at Risk. This is since we consider the entire tail more extreme than a certain quantile of the distribution in for the Expected Shortfall whereas VaR only corresponds to this quantile. This can be seen in **Figure 2**.

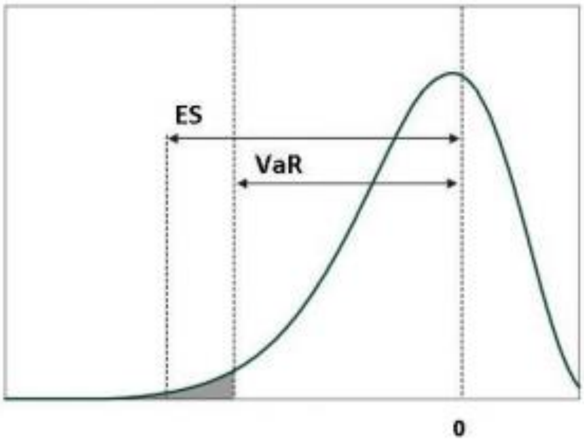


Figure 2 **Source: Karabey (2012:17)**

Standard deviation can also be used as a measure of risk for setting the amount of capital required. This makes sense as standard deviation seeks to capture the variability of the outcomes, it is this same variability we wish be able to absorb by holding capital. We are however interested in the most extreme events so we would need to scale the standard deviation by a factor c such that

$$\rho(X) = c * \sigma(X),$$

,where

$$\sigma(X) = \sqrt{Var(X)} = \sqrt{E[(X-\mu)^2]}.$$

and,

$$\sigma_c(X) = c * \sqrt{Var(X)} = c * \sqrt{E[(X-\mu)^2]} = \rho(X)$$

where c is chosen to reflect a certain confidence level of the eventual result.

We can use Chebyshev - inequality to ensure that irrespective of the distribution, the losses will not exceed the capital held with a predetermined level of confidence. This approach is also shown in Tasche (2008: 8).

$$P[X \leq E[X] - \sigma_c(X)] \leq \frac{1}{1+c^2}$$

where we simply solve for $\frac{1}{1+c^2} = 1 - \alpha$

2.3 COHERENCE OF RISK MEASURES

The theory of coherent risk measures outlines a series of desirable properties that align with finance theory and economic intuition. Evaluating the coherence of the risk measures we can choose from can inform an appropriate choice of risk measure for economic capital determination. These axioms were developed by Artzner, Delbaen, Eber and Heath, (1999:209-210). An overview of the properties follows. Whether or not a risk measure fullfills these properties is given in the next section along with other advantages and disadvantages of the risk measures set out in the previous section.

Monotonicity

$$X_1 \leq X_2 \rightarrow \rho(X_2) \leq \rho(X_1)$$

Which simply means that a position with a larger loss should be reflected by a larger risk measure. As the values for X_1 are smaller than those for X_2, X_1 will have larger losses and therefore require a greater amount of money to be held to offset the risks.

Positive homogeneity

The size of the risk depends linearly on the size of the position. For any $\lambda \geq 0$ it should hold that

$$\rho(\lambda X) = \lambda\rho(X)$$

Translation invariance

Adding a constant amount to a position reduces the risk by the same amount i.e.

$$\rho(X + a) = \rho(X) - a.$$

This can be understood to mean that the risk of the portfolio consisting of the original position and the constant is equal to the risk of that original position less the constant that represents a cash injection. Which would be sensible as we would expect adding cash to any position to reduce the riskiness of said position. This is a necessary condition as we would want the risk to no longer be present if we add the amount of the risk measure in risk free assets to the position. This can be given in formula form as

$$\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0.$$

Subadditivity

The risk of the total position should not exceed the sum of the risk of the individual positions

$$\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2).$$

This property is associated with diversification meaning that if risk measure of the combined portfolio is less than that of the parts, the difference can be explained as a diversification benefit. If we assume a non-negative diversification benefit the result can be rewritten as

$$\rho(X_1 + X_2) = \rho(X_1) + \rho(X_2) - diversification\ benefit.$$

2.4 ADVANTAGES AND DISADVANTAGES OF RISK MEASURES FOR ECONOMIC CAPITAL DETERMINATION

Table 1 gives some advantages and disadvantages of the different possible choices of risk measure.

Table 1

Risk measure	**Advantages**	**Disadvantages**
Value at Risk	Readily used and understood in the market and regulation.	Not sub-additive. Requires additional approximations and calculations to be used for Euler allocation in a simulation setting (Tasche 2008:10-11).
Expected shortfall	Satisfies all axioms of a desirable risk measure. (Acerbi & Tasche, 2002:385). Can easily be used with Euler allocation. (See section 5.2)	Data for entire tail needed.
Standard deviation	Well understood measure of risk. Can be calibrated to accommodate considerable uncertainty for the underlying factor risk model.	Not translation invariant (Karabey, 2012:19).

Weighing up stated advantages and disadvantages of these main risk measures, expected shortfall is identified as the most appropriate for this situation. We would want a risk measure that is coherent and easy to use with the allocation method we show to be the most appropriate for this situation in section 5.3. Expected shortfall has these properties and this consistency with economic principles is weighed more heavily than the fact that this is not currently used in South African solvency rules. Additionally, we should note that the disadvantage would not limit us as we use stochastic modelling. We will use expected shortfall as risk measure for the remainder of the study. The following chapter will look at how we would model the pure endowment product and use this to determine the economic capital required.

CHAPTER 3

MODELLING THE ECONOMIC CAPITAL REQUIRED FOR A SINGLE PERIOD PURE ENDOWMENT PRODUCT

To calculate the economic capital we would need to define a random variable of the present values of a pure endowment contract. This is since the risk measure would need a range of outcomes and associated probabilities to deliver an economic capital amount. This chapter will derive this random variable. In this chapter we will consider the modelling setup, which consists of the approach, distribution and dependence specification and motivation.

We will then show two approaches to calculate the economic capital, namely a simulation approach and an analytical approach.

3.1 MODELLING SETUP

3.1.1 Modelling approach

To orient ourselves in the context of capital modelling we specify the key methodology decisions outlined under section 2.1 in the previous chapter in Table 2.

Table 2

Issue	Option chosen
Basis of balance sheet	Economic
Period of assessment	One year
Measure of risk	Expected Shortfall
Confidence level	99.5%
Risks to include	Mortality risk Interest rate risk
Quantification methodology	Stochastic modelling
Aggregation	Stochastic modelling

We will need to specify a liability distribution that is a combination of risk sources. These risk sources that are combined to arrive at the overall liability distribution will be specified as stochastic random variables.

This means that the risk from the different risk sources is included by modelling these as stochastic variables. Combining the stochastic variables as part of the modelling step mean that the risk from different sources is already aggregated. This is as the economic capital is determined on an aggregate level. Additionally, this stochastic modelling setup allows the dependence between these sources to be accounted for in the modelling step.

The liability distribution is used in conjunction with the risk measure to use possible future outcomes with associated probabilities to determine an economic capital amount. We can summarise how this is determined for our modelling approach as the amount by which the expected shortfall exceeds the best estimate liability, this is explained in more detail in the next section.

We can contrast this with the other typical approach of modelling the capital amount of each source separately and then aggregating these thereafter. This would then require us to choose an aggregation method and make assumptions of the dependence of risk sources in our aggregation step.

It is tempting to think that modelling the risk sources separately will make capital allocation simpler. However, this is not the case as the total requirement will typically be less than the sum of the individual capital amounts. This results from a diversification effect that would need to be allocated to the different risk sources which would again require an allocation method.

We therefore opt to use stochastic modelling as this reduces the amounts of steps we need to take to arrive at a capital allocation and the case is well suited to this approach. The assumptions we make with regards to modelling the present value, the distribution of the risk sources and the relationship between them is discussed in the following two sub-sections.

3.1.2 Model specification

A pure endowment contract is one in which the insurer promises to make a payment upon the insured surviving to the end of the year. In insurance the present value of claims that are to be paid will typically be expressed as the probability weighted discounted cashflow due under the contract. This is given in standard actuarial notation for a pure endowment assurance product over one year in deterministic terms as:

$$PV \quad = Benefit \cdot (1 - q_x) \cdot v$$

$$= Benefit \cdot \frac{1-q_x}{1+i_{(1)}}$$

where q_x is the probability of death for the individual with age x to which the policy pertains. The $1+i_{(1)}$ by which we divide simply puts the benefit payable at the end of the year in present value terms to reflect the time value of money. The rate $1+i_{(1)}$ in the above formula represents the rate at which the assets accumulate over the course of one year, meaning that $i_{(1)}$ is the annualized return.

It is possible to simplify our modelling by setting the benefit equal to one and scaling this later to reflect the benefit that is specified for a specific contract. We should take care when we apply this scaling especially when we evaluate the square of the expected liability values as in a variance calculation. This is since the square of one is simply one but this would not be the case for larger benefit values.

We get the stochastic equivalent of the discounted cashflows by assuming the mortality rate and interest rate vary stochastically for the period under observation. In this setting we would let T be the eventual liability for the portfolio and (X, Y) be random variables representing the mortality rate and interest rate respectively. We can model the denominator and numerator of the liability present value as X and Y respectively. We can denote this stochastic present value term as T.

This would mean that we have chosen to model $1-q_x$ with X and $\left(1+i_{(1)}\right)$ as Y. We define function f as the function that captures this dynamic and we can express the present value as:

$$T = f(X,Y)$$

$$= \frac{X}{Y}.$$

3.1.3 Selecting appropriate underlying distributions

We require distributions for the variables X and Y. Normal distributions for both variables X and Y have been assumed. We motivate this decision in the remainder of this section.

The law of large numbers can be used to motivate that given the company writes a large amount of business our experience would approach a normal distribution for the mortality experience.

Secondly, it has become typical to model the increment at which the interest rate changes using a normal distribution. This is in essence what happens when we use Brownian motion to capture the change over a period as is done when using stochastic calculus for interest rate modelling. It should be noted that we evaluate a larger increment than is typical for stochastic calculus. Even though Y does not represent the increment this argument is still valid as we can restate Y as:

$$Y = \mu_Y + \varepsilon$$

$$Y = 1 + E[i_{(1)}] + \varepsilon.$$

Here we would model the increment by $\varepsilon \sim N(0, \sigma_Y^2)$. We should note that there are some shortcomings in using this assumption for the interest rates. Empirical studies have shown the following factors which could serve to discredit the distribution assumption (BWIN 613 Reader: subject CM2 2020 study guide, 2020:160).

- Momentum effects
- Mean reversion
- Fat tailed changes
- Non-symmetrical changes in interest rates.

Lastly using a normal distribution enables the analysis to be more straightforward as an analytical approximation can be arrived at as well as allowing us to calibrate our distributions easily.

We have chosen to use the normal distribution as a first step in developing the methods for allocating capital to risk sources for pure endowment products. It is natural to develop these methods for other distributions as well, but the normal assumption is thought satisfactory for this study.

3.1.4 Dependence of risk factors

We need to consider whether we can observe and justify an underlying dependence structure between the mortality and interest rates. There are however bound to be constraints and difficulty in allowing adequately for such dependence. The weigh up can then be made between the additional complexity and the value this would add.

The standardised formula Solvency Capital Requirement in South Africa assumes a constant correlation coefficient which is not equal to zero. This would imply that the assumption is made that there is some degree of linear dependence between interest rates and mortality rates. This assumption however is made in that context to ensure that an appropriate diversification benefit is considered. It does not necessarily mean that regulation implies that there is a relationship between these variables. The aim is rather to allow for the diversification benefit by assuming non-perfect positive dependence.

There are sources that have justified a dependence between interest rates and mortality experience. These have sought to model this dependence and the curious reader is referred to Dacorogna and Apicella (2016).

Some arguments for dependence can be given by considering the following scenarios:

- A large mortality shock could influence the productivity of the economy as a whole and this may then correspond to interest rates rising or falling, depending on policy taken and the net effect on the economy.
- A change in mortality and interest rates can both occur due to a factor such as a pandemic and the resulting economic actions taken. CoVid-19 can serve as an example of this.

As can be seen from these examples, which are by no means meant to be exhaustive, the dependence between mortality and interest rates can be complex. We would need to consider a variety of factors and take account of likely decisions by policymakers. Additionally, the behaviour of these in the tails may remain highly uncertain even if we wish to model these. This will be the result of data scarcity in the tails of the observed distributions. Using methods such as copula's to capture the dependence structure between interest and survival rates although possibly more accurate would require considerable additional modelling and justification.

As we have chosen to use normal distributions to model the stochastic variables denoting the cashflow probability (survival rate) as well as the accumulation rate of assets we can assume a bivariate normal distribution and incorporate dependence with a correlation term. The next section will evaluate how we can determine the economic capital amount for this model, distribution and dependence setup using both a simulation and analytical approach.

3.2 DETERMINING THE ECONOMIC CAPITAL AMOUNT

Having specified the distribution of X and Y, we need to be able to translate this to an economic capital amount required to back the risk posed by interest and survival rates jointly. This can be done using a simulation approach as discussed in Section 3.2.1 where a subset of the simulated outcomes is used to calculate the expected shortfall. Alternatively, we can seek to approximate the distribution of $T = \frac{X}{Y}$ and use this distribution to calculate the expected shortfall in a closed form equation if one exists for the distribution of T. We approximate the distribution of T with a lognormal distribution which has a closed form solution and therefore an analytically approximated result is obtained. This is done in Section 3.2.2. The best estimate liability (being the expected present value) would then be deducted from this expected shortfall to arrive at the economic capital amount.

3.2.1 Simulation based approach

We randomly simulate the realisations for X and Y from a bivariate normal distribution n times. We then have a vector of realisations for X and Y say $\boldsymbol{X}$ and $\boldsymbol{Y}$. We then calculate the ratio $\frac{X}{Y}$ for each of the values in this vector to arrive at a vector of outcomes that we will denote $\boldsymbol{T}$. The

Expected Shortfall is then given by taking the average of the most extreme values of $\boldsymbol{T}$. The values we consider are those that are larger than the Value at Risk with the confidence level specified. This is most easily expressed by ordering the values of T such that $T_{(1)}$ is the smallest value and $T_{(n)}$ is the largest. We can then express the expected shortfall at confidence level $(1-\alpha)$ as:

$$ES_\alpha(T) = \frac{1}{\alpha \cdot n} \cdot \sum_{i=(1-\alpha)\cdot n}^{n} T_{(i)}$$

The notation used for the expected shortfall is similar to that in Section 2.2. However, we are modelling the present value of the future cashflows and not the profit or loss as was contemplated in that section. We therefore consider the realisations that are larger than the best estimate amount and consider the most extreme of these to determine the risk measure.

The economic capital (EC) amount is then expressed as the difference between this amount and the best estimate liability. We define the best estimate liability, denoted by BEL, as the expected present value in a risk-free setting. This would mean that the survival and interest rates correspond to their best estimate values.

This can be expressed as:

$$BEL = E[T|X = \mu_X, Y = \mu_Y]$$

$$= \frac{\mu_X}{\mu_Y}$$

We then have the following formula for the economic capital amount.

$$EC = ES_\alpha(T) - BEL.$$

3.2.2 Analytical approximation approach

We will derive an analytical approximation for the distribution of the present values and use this approximation to calculate the economic capital requirement.

We have as before:

$$T = \frac{X}{Y}$$

Where $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$. We also define $X^* \sim N(0, \sigma_X^2)$ and $Y^* \sim N(0, \sigma_Y^2)$. This transformation is done to improve the accuracy of the approximation of T. The above expression can then be restated as:

$$T = \frac{\mu_X + X^*}{\mu_Y + Y^*}$$

$$= \frac{\mu_X}{\mu_Y} \cdot \frac{1 + \frac{X^*}{\mu_X}}{1 + \frac{Y^*}{\mu_Y}}.$$

It becomes possible to approximate this function if we take the log of T.

$$\ln\ T = \ln\left(\frac{\mu_X}{\mu_Y}\right) + \ln\left(1 + \frac{X^*}{\mu_X}\right) - \ln\left(1 + \frac{Y^*}{\mu_Y}\right).$$

This can be approximated using a Taylor series expansion on the functions $g\left(\frac{X^*}{\mu_X}\right) = \ln\left(1 + \frac{X^*}{\mu_X}\right)$ and $h\left(\frac{Y^*}{\mu_Y}\right) = \ln\left(1 + \frac{Y^*}{\mu_Y}\right)$ about zero. A Maclaurin expansion for a function $k(\lambda) = \ln(1+\lambda)$ yields the following equation:

$$\ln(1+\lambda) = \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{3} + \cdots$$

We take the first order approximation for g and h and arrive at the following expression:

$$\ln(T) \approx \ln\left(\frac{\mu_X}{\mu_Y}\right) + \frac{X^*}{\mu_X} - \frac{Y^*}{\mu_Y}.$$

It follows that $\ln(T)$ is approximately normally distributed since both X^* and Y^* are distributed normally.

We define a variable T^* as the first order Taylor approximation for T such that:

$$\ln(T^*) = \ln\left(\frac{\mu_X}{\mu_Y}\right) + \frac{X^*}{\mu_X} - \frac{Y^*}{\mu_Y}.$$

$\ln(T^*)$ is therefore distributed normally and T^* is log-normally as:

$$T^* \sim lognormal(\mu_{T^*}, \sigma_{T^*})$$

$$\mu_{T^*} = E[\ln(T^*)] = \ln\left(\frac{\mu_X}{\mu_Y}\right)$$

$$\sigma_{T^*}^2 = Var[\ln(T^*)] = \frac{\sigma_X^2}{\mu_X^2} + \frac{\sigma_Y^2}{\mu_Y^2} - \frac{2 \cdot Cov(X,Y)}{\mu_X \cdot \mu_Y}.$$

We can restate the variance of T^* in terms of the coefficient of variation of X and Y. This alternate formulation will be used in analysing the sensitivity of the economic capital amount to input parameters in Section 6.4.2. The coefficient of variation of a variable is the standard deviation as a proportion of the mean. We can express this coefficient of variation (given by δ) in the following manner for X and Y:

$$\delta_X = \frac{\sigma_X}{\mu_X}$$

$$\delta_Y = \frac{\sigma_Y}{\mu_Y}.$$

The variance of T would then become:

$$\sigma_{T^*}^2 = \delta_X^2 + \delta_Y^2 - 2 \cdot \rho \cdot \delta_X \cdot \delta_Y$$

Where the ρ is the correlation between X and Y.

We have then for both formulations approximated the distribution of outcomes with specifying parameters which are stated in terms of the risk factor distribution parameters. This approximation allows us to determine the economic capital amount as a function of the input parameters and a confidence level.

We can do this evaluating the expected shortfall of a log-normal distribution. The derivation to arrive at the equation for this quantity as set out below is given in **Appendix A**. The assets required are given by the expected shortfall as:

$$ES_\alpha(T^*) = \exp\left(\mu_{T^*} + \frac{\sigma_{T^*}^2}{2}\right) \cdot \frac{\Phi\left(\sigma_{T^*} - \Phi^{-1}(1-\alpha)\right)}{\alpha}$$

The economic capital is then again calculated as the difference between the expected shortfall and the best estimate liability.

We can confirm that the best estimate liability can equivalently be calculated as the expected value of T^* given the variance is equal to zero. We have that the expected value of the log-normal distribution depends in part on the variation of the interest and survival rates. This is as:

$$E[T^*] = \exp\left(\mu_{T^*} + \frac{\sigma_{T^*}^2}{2}\right)$$

If we then set the standard deviation of T^*equal to zero we have:

$$\begin{aligned} E[T^* | \sigma_{T^*} = 0] &= \exp(\mu_{T^*}) \\ &= \exp\left(ln\left(\frac{\mu_X}{\mu_Y}\right)\right) \\ &= \frac{\mu_X}{\mu_Y} \end{aligned}$$

This is also equal to the value calculated under the simulation approach as well as the median of the distribution of T^*. The fact that the best estimate liability is equal to the median is a desirable result as the probability of the value being too great or too small is equal. The resulting economic capital under the analytical approach is given by:

$$EC = ES_\alpha(T^*) - BEL.$$

We now have developed a simulation and analytical approach to evaluate the economic capital for a single period pure endowment product where we have modelled the present value stochastically with the use of normal distributions. We will explore how we can decompose the risk underlying this amount to survival and interest rate risk in the next section.

CHAPTER 4

DECOMPOSING THE RISKS UNDERLYING ECONOMIC CAPITAL

In this chapter we will look at how the risk causing the need for economic capital arises due to survival rate and interest rate uncertainties. We then discuss decomposition methods that have been proposed in literature. We apply a variance decomposition to the simulation and analytical approach we have developed in the previous chapter.

4.1 RISKS

The risks relating to future dated life contingent cashflows arise from the assets used to back such cashflows and the contingencies upon which such cashflows depend. The economic capital would represent the amount of assets the insurer needs to hold such that the assets are sufficient to cover the cashflow required under a sufficient range of scenarios.

The amount of assets required at the present time to match the expected liability cashflows in the future would be determined by discounting contingent cashflows. Such a present value is determined as the probability weighted contingent cashflow discounted to the current time. The rate at which the probability weighted cashflows are discounted would be the rate that can be attained on the assets the insurer holds with respect to this liability.

For a pure endowment contract the survival risk would be from more people surviving than we expected which would mean that the best estimate liability proves to be insufficient.

Interest rate risk represents the risk arising from changes to the nominal and real yield curves. In this report we are only evaluating liabilities and assets linked to the nominal yield curve, however this can easily be extended for products linked to inflation rates that are therefore real yield dependent. Therefore, we only consider the risk of a change in the nominal yield curve.

If the insurer is invested in assets that match the cashflows exactly, the insurer would not face any interest rate risk. This is as the yields of matching assets are locked in when they are purchased and any changes to these rates over the year would not influence the insurer's ability to meet claims. The insurer will however face an interest rate risk when the assets are not matched to the liabilities by term.

If the assets are shorter dated than the liabilities, the insurer would be at risk of the rates at which it can reinvest being less favourable than they were at outset. This will be the case when liabilities are particularly long dated, and assets of appropriate term are not available. An annuity would be an example of such a scenario. The interest rate risk relates to the interest rates being less than we expected at the time we need to reinvest the assets. The value of the cashflows in the future would be the same but the returns on the assets over the remainder of the term would be less than initially anticipated. The insurer needs to hold more assets (represented by a portion of the economic capital) to cover this risk if it invested in assets of a shorter term.

If the assets are longer dated, they would need to be sold at the time when claims become due at the then current price. The interest rate risk therefore arises when the prices of such securities are lower than anticipated at the time the insurer would need to sell these due to interest rate changes. As a result of the inverse relationship between the quoted interest rates on securities and their prices, the risk of security prices being lower than anticipated corresponds to interest rates associated with these assets being higher than anticipated.

Even though the interest rate risk arises from different moves in market interest rates over the term depending on the duration of the assets the same modelling setup can be used for both. This is the approach set out considers interest earned over the period, looking at the purchase and sale prices. From this perspective the risk is that the earned interest over the period is lower than anticipated.

This report focusses on the risks as economic capital is held for the insurer to be able to absorb these risks. It is however worth noting that the same uncertainty that can lead to losses for the insurer also has the possibility to cause survival and interest rate surplus.

We have now discussed how the risks arise and the following section looks at how we can decompose the overall risk into these components.

4.2 DECOMPOSITION METHODS

The traditional capital allocation methods require the overall risk that the firm faces to be a linear combination of the risk sources (a summation of these sources). This would be the case when we are considering investments or separate business units. In life insurance discounting of cashflows and guarantees both violate this requirement. In seeking to apply capital allocation techniques to these risks it has become typical to resort to a linearisation of the total loss with respect to the constituent factors (Karabey, Kleinow & Cairns, 2014:35).

For a decomposition we denote the risk factors by $(Z_1, Z_2, \ldots, Z_k)$ that influence the present value of future cashflows denoted by T. Schilling, Bauer, Christiansen and Kling (2020:5742-5744) demonstrate how we apply linearisation methods to decompose the deviation between the exposure and the expected exposure. This deviation is representative of the risk we wish to decompose. In this case this means that we subtract the expected value from the random present value. We can express the exposure given by T and the risk given by R as:

$$T = f(Z_1, Z_2, \dots, Z_k)$$

$$R = f(Z_1, Z_2, \dots, Z_k) - E[f(Z_1, Z_2, \dots, Z_k)]$$

$$= T - E[T]$$

For the modelling setup we have for a single period pure endowment product we can express this risk in terms of the random variables representing the accumulation and survival rate. We note that the function f gives the relationship between X and Y as input variables and T as response.

$$T = f(X, Y) = \frac{X}{Y}$$

We wish to find a decomposition of these to a linear function of ideally k terms, such that

$$R = \sum_{i=1}^{k} R_i$$

,where R_i is the resulting decomposition with respect to each factor Z_i.

It should also be noted that the above decomposition is the ideal and that terms reflecting joint effects of different variables can also be included. A Hoeffding decomposition results in such a joint effect term as can be seen in Table 3.

In applying such a decomposition, we have reduced the problem to one which can be solved within the general framework for capital allocation as described in Chapter 5.

Various decompositions for two variables as well as how to apply these to more variables is shown by Schilling *et al*. (2020:5742-5744) and a summary of the most common methods is given in Table 3. Here we show how we would apply these methods to the variable R with the setup the same as above. We let (x_0, y_0) be any chosen point from (X, Y) and denote the expected value of X as μ_X and that of Y as μ_Y.

Method		$\boldsymbol{R_X}$	$\boldsymbol{R_Y}$	$\boldsymbol{R_{X,Y}}$
Variance Decomposition	Conditioning on $X = \mu_X$ first	$T - E[T \mid X = \mu_X]$	$E[T \mid X = \mu_X] - E[T]$	
	Conditioning on $Y = \mu_Y$ first	$E[T \mid Y = \mu_Y] - E[T]$	$T - E[T \mid Y = \mu_Y]$	
Hoeffding Decomposition		$E[T \mid Y = \mu_Y] - E[T]$	$E[T \mid X = \mu_X] - E[T]$	$T - E[T \mid Y = \mu_Y] - E[T \mid X = \mu_X] + E[T]$

Taylor Expansion	$\frac{\partial f}{\partial x_0}(x_0, y_0) * (X - x_0) = \frac{X - x_0}{y_0}$	$\frac{\partial f}{\partial y_0}(x_0, y_0) * (Y - y_0) = \frac{x_0 \cdot (X - x_0)}{y_0^2}$	
One at a Time	$f(X, Y = y_0) - E[T]$	$f(X = x_0, Y) - E[T]$	

Table 3

Schilling *et al*. (2020:5741-5745) discuss advantages and disadvantages of these various methods. A very brief overview of the most important of these is given in the remainder of this section. An advantage of the first two methods in Table 3 is that these deliver variables that sum to the total risk of the portfolio. The shortcoming of the Hoeffding decomposition is that the decomposition does not result in the same amount of risk factors as risk drivers. Whereas the shortcoming with the variance decomposition is that the order in which the calculation is done influences the results we obtain.

The point chosen in the Taylor expansion and one at a time decomposition is not straightforward (Schilling *et al*., 2020:5744). The results obtained from the last two methods will vary largely on the points chosen. If the point (x_0, y_0) is chosen as (μ_X, μ_Y) the risk factors R_X and R_Y will be the same for the one at a time and Hoeffding decomposition. However, the Hoeffding decomposition will have a joint effect given by $R_{X,Y}$ which will let this decomposition sum up to the total amount.

In the next section we will consider how we can apply the variance decomposition to a single period pure endowment product. This method has been chosen as it decomposes into two risk factors and sums to the total risk both of which are desirable properties. The shortcoming that the order in which we condition the results influences these results is investigated. We show all formulae in section 4.3 and results in section 6.3 (which is our case study) for both the orders in which we can condition the decomposition.

4.3 APPLYING THE VARIANCE DECOMPOSITION METHOD TO SINGLE PERIOD PURE ENDOWMENT

We apply the variance decomposition for each approach separately. We find that this yields a capital allocation when applied to our analytical approach whereas this is only a step in determining the allocation under the simulation approach. In applying the variance decomposition method, we model the expected value in Table 3 as the best estimate liability ($BEL = \frac{\mu_X}{\mu_Y}$) as defined in Chapter 3.

4.3.1 Simulation approach

We can simulate values for X and Y and apply the variance decomposition to the vector that corresponds to the present values of the future contingent cashflows.

There are two variance decomposition answers as the order in which the decomposition is done is important. We therefore have the following two decompositions depending on the order of conditioning. If we first condition that the value of Y equal to the expected value thereof we arrive at the following decomposition:

$$R_Y^1 = \frac{X}{Y} - \frac{X}{E[Y]}$$

$$R_X^1 = \frac{X}{E[Y]} - \frac{\mu_X}{\mu_Y}$$

Alternatively, we have the following decomposition if we first condition that the value of X is equal to its expected value:

$$R_X^2 = \frac{X}{Y} - \frac{E[X]}{Y}$$

$$R_Y^2 = \frac{E[X]}{Y} - \frac{\mu_X}{\mu_Y}$$

4.3.2 Analytical approach

For the analytical approach we use the interpretation that setting the value of a variable equal to its expected value is the same as conditioning the approximated present value (T^*) distribution on the variance of that variable being equal to zero. This is as a stochastic variable with no variance is equal to its expected value. Similar to above we have capital amounts (instead of vectors R_X and R_Y) that we will denote RC_X and RC_Y. Again the superscripts are used to distinguish the order of decomposition. We would need to apply an allocation approach to the vectors for R_X and R_Y to arrive at allocated capital amounts, whereas this is not necessary with the variance decomposition of the analytical approach.

If we first condition on variance of Y being equal to zero we have the following capital decomposition:

$$RC_Y^1 = ES_\alpha(T^*) - ES_\alpha(T|\sigma_Y = 0)$$

$$RC_X^1 = ES_\alpha(T|\sigma_Y = 0) - ES_\alpha(T^*|\sigma_Y = 0, \sigma_X = 0)$$

If we reverse the order of conditioning, we have:

$$RC_X^2 = ES_\alpha(T^*) - ES_\alpha(T^*|\sigma_X = 0)$$

$$RC_Y^2 = ES_\alpha(T^*|\sigma_X = 0) - ES_\alpha(T^*|\sigma_Y = 0, \sigma_X = 0)$$

We use the formula for Expected Shortfall in the above for all expressions except where we set the variance of Y equal to zero. This is since when we set this variance equal to zero we have that $T = \frac{X}{\mu_Y}$ which we would not need to approximate the distribution for as we know it follows a normal distribution since X follows a normal distribution. For this term we use the expected shortfall expression for a normal distribution.

We take note that $ES_\alpha(T^*|\sigma_Y = 0, \sigma_X = 0) = \frac{\mu_X}{\mu_Y} = BEL$ as the Expected Shortfall would be equal to the best estimate liability as the whole distribution collapses to a single point if there is no variance.

This decomposition would lead to a full allocation irrespective of the order in which this decomposition is done. We will show this for the case where we first condition on the variance of Y being equal to zero. If we sum the risk capital components we see that this is equal to the economic capital amount. This is shown below:

$$\begin{aligned} RC_X^1 + RC_Y^1 &= ES_\alpha(T^*) - ES_\alpha(T|\sigma_Y = 0) + ES_\alpha(T|\sigma_Y = 0) - ES_\alpha(T^*|\sigma_Y = 0, \sigma_X = 0) \\ &= ES_\alpha(T^*) - BEL \\ &= EC \end{aligned}$$

We have therefore already derived a capital allocation for the analytical approach by applying the variance decomposition, whereas a decomposition of some sort is only a necessary step to allocate capital if we use the simulation approach.

CHAPTER 5
CAPITAL ALLOCATION

We still need to apply a capital allocation method to the random vectors produced by applying the variance decomposition to the simulation approach. In this chapter we review literature relating to capital allocation. We also have a look at how we can evaluate different allocation methods to choose the most appropriate method for our study.

5.1 DEFINITION

Capital allocation is determining the amount of economic capital that should be assigned to the various sources of risk. The sources of risk this report focusses on are distinct risks. However, the theory in this chapter can also be applied to business units or product lines as these can also be viewed as distinct sources of risk. Bauer and Zanjani (2013:865) identify pricing and performance measurement as the main practical needs that drive the use of capital allocation methods. These needs necessitate the allocation of capital to business units, product lines or even individual insurance contracts. This makes sense as we would be interested in the return earned on the capital employed for some subsection of the insurer's larger business. This would then in turn give a measurement of performance that is in accordance with the greater capital framework. We can also take cost of capital into account when pricing life insurance products to enable us to ensure that the required returns are made.

Furthermore when capital allocation is integrated effectively throughout the insurer, it may be used in the management and risk control functions of the insurer by aiding risk monitoring and control (Kapel *et al.*, 2013:30). Additionally, capital allocation can be used as a tool to steer and aim an insurer by means of informing risk-based decision making and assisting in business and strategic planning.

A capital allocation to risk sources can constitute a link between risk and capital management in a firm. This is especially in line with the wider shift towards capital being determined on a market consistent basis. In linking these, the company can better understand its exposure, informing better risk-based management decisions.

Menzietti and Pirra (2017:246-247) justify capital allocation to different distinct risk sources by pointing out the following.

- The risk sources interact in such a way that different risk sources have an influence on the potential losses from a specified sub-portfolio of the insurer's books. When the portfolio is complex and it is not sensible to allocate capital to each instrument or product, an allocation to risk sources may be more useful.

- Understanding the risks that have an influence on the liabilities of the life insurer
- Regulation requires the insurer to offer a quantification of the risk sources that it is faced by. There is the option for an insurer to use their own internal model in which capital allocation can be employed to inform this quantification.

In the context of capital allocation, it is typical to view either the total gain or loss for the insurer as the sum of random variables denoting the gain or loss from each of n risk sources (typically business units) given by $R_1, R_2, \dots, R_n$. It then follows that

$$R = \sum_{i=1}^{n} R_i .$$

It is more common to use the notation X to denote this possible gain or loss, but we have changed it to R for consistency with the notation used throughout the study. We introduce weight variables to the above function that can be expressed as a vector $(u_1, u_2, \dots, u_k)$. This can be thought of as the size of the position relative to each component (be it sub-portfolio, business line or risk). The size of the position refers to the exposure the company has to a specified source.

The need for this may at first not be intuitive. However, it is necessary for defining any allocation that relies on the change in total riskiness with respect to a change in the exposure to a source of risk. This also allows the possibility of using differing proportions to determine an optimal portfolio to hold. This procedure involves using risk adjusted capital and can only be employed if the method used is compatible with a Return on Risk Adjusted Capital measure (Tasche, 2008:3-4). This would lead to the above formula when $\boldsymbol{u}$ is equal to $\mathbf{1}$ and in general is given by

$$R(\boldsymbol{u}) = \sum_{i=1}^{n} u_i R_i \ .$$

We will denote the capital allocated to the i^{th} source of risk as a_i and define the set $N = \{1,2, \dots, n\}$. Capital allocation can be defined by making use of a functional as is done by Karabey (2012). He lets D be the set of risk capital allocation problems, with pairs (N, ρ) being composed by a set of n lines and a coherent risk measure ρ. Then an allocation can be defined as a functional $\Pi: D \rightarrow \mathbb{R}^n$ mapping each allocation problem to a unique allocation a_i, such that

$$\begin{bmatrix} \Pi_1(N,\rho) \\ \Pi_2(N,\rho) \\ \dots \\ \Pi_n(N,\rho) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix} .$$

A capital allocation can then be seen as a way to arrive at a capital amount to be held for each of the n sources of risk.

This means that there is no capital held for certain risk sources jointly. This does not mean that the insurer cannot eventually use the capital allocated to a certain source to cover losses from another source. This is a decision as to how the capital is to be managed with knowledge of the allocated amounts. It could however be possible to allocate a capital amount that can be used jointly by the factors.

5.2 TRADITIONAL ALLOCATION METHODS

Many allocation rules have been proposed. In selecting which allocation methods will be contrasted with our proposed method, we can reference those that are most widely used and have straightforward applications. We have also chosen to focus on methods and risk measures that lend themselves more naturally to a simulation-based approach being followed. A summary of the allocation methods described is given in Table 4.

A natural allocation method would involve allocating the risk in proportion to the capital required when we consider the risk source in isolation. This is known as the proportional method.

The variance-covariance approach takes account of the extent to which the risk variables vary with each other. A drawback of this approach is that this is done over the entirety of the distribution of both the overall loss and the risk source to which we are allocating the capital. This approach was proposed by Overbeck (2000: 15-30).

The Euler method evaluates the change in risk measure from the addition of a source of risk using a gradient of the risk function relative to the source. The allocated capital can be thought of as an instantaneous change in capital resulting from the addition of the risk source to which we wish to allocate capital.

Table 4

Allocation method	$\boldsymbol{a_i}$
Proportional	$\frac{\rho(u_i R_i)}{\sum_{j\in N}\rho\left(u_j R_j\right)}\rho(R(\boldsymbol{u}))$
Variance – Covariance	$\frac{Cov(u_i R_i, R(\boldsymbol{u}))}{Var(R(\boldsymbol{u}))}\rho(R(\boldsymbol{u}))$
Euler	$u_i \frac{\partial\rho(R(\boldsymbol{u}))}{\partial u_i}$ in the point $\boldsymbol{u} = \mathbf{1}$

The resulting formulae for the Euler contributions under various risk measures can be found in Tasche (2008:8-10) and a summary of this is given in Table 5.

Table 5

Risk measure	**Euler contribution**
Value at Risk	$-E[R_i \mid R = -VaR_\alpha(R)]$
Expected Shortfall	$-E[R_i \mid R \leq -VaR_\alpha(R)]$
Standard Deviation based measure	$c * \frac{Cov(R_i, R)}{\sqrt{Var(R)}}$

Where the confidence level is given by α.

When we use a simulation-based approach, the expectations are given by the mean values of the simulated values which correspond to the condition holding for the Expected Shortfall case. To determine a meaningful result for the contribution using the Value at Risk measure we need to make use of Kernel Estimators derived from the simulated data (Tasche, 2008:10-11). This adds additional complexity and further motivates the use of expected shortfall as it is easier to determine an Euler allocation in a simulation setting for this risk measure.

It is intuitive to state that using simulations in this context provides many unnecessary simulations realisations as we are only interested in the tails of the distribution. The use of importance sampling or other techniques can be investigated to address this but run times where not found to be excessively long for the case study we considered so we have not employed these methods.

5.3 EVALUATING THE CONSISTENCY OF AN ALLOCATION METHOD WITH ECONOMIC THEORY

The method that is to be used can be based on a set of desirable properties or axioms of coherence that a method should adhere to. These axioms aim to ensure consistency of the allocation method to economic theory. As we are allocating economic capital such coherence is desirable in the method we choose to apply. In the next section we consider such a set of axioms.

5.3.1 Axioms

The axioms we discuss in this section have been adapted from Denault (2001:5-6).

Full allocation principle

This principle ensures that the allocated capital towards each source of risk will sum to the total capital the company wishes to allocate.

$$\rho(R) = \sum_{\forall i} a_i$$

No undercut

This simply means that any subset of risks cannot have a greater amount allocated towards it than the capital held with respect to said subset.

$$\forall M \subseteq N \, , \sum_{\forall i \subset M} a_i \leq \rho\left(\sum_{\forall i \subset M} R_i \right)$$

In the current setting this corresponds to a degree of diversification being present in every subset. This would however imply that the full allocation principle would need to be met for every possible subset of risk sources. Therefore, a technique which satisfies the full allocation principle when we only have two risk sources naturally meets this requirement.

Symmetry

If a subset of risks is composed of risk sources which both make the same contribution to the overall risk, these will have equal amounts of capital allocated to each.

Therefore, say that risk sources i and j make the same contribution to the subset of which they are part then in including these in the portfolio the capital held with respect to each should be equal.

Risk less allocation

This principle is sensible in the case of assessing the usefulness of an allocation method for allocating capital to different assets in a portfolio. In the context of the current allocation problems we are not considering assets and there would not be any element that can be seen as completely risk free. It is however included here for completeness.

If we consider a risk-free asset corresponding to the i^{th} risk source that has a positive return equal to r_f. Then we have $X_i = \alpha r_f$ with corresponding allocation equal to

$$a_i = \rho(\alpha r_f) = -\alpha$$

This then means that the addition of a risk-free asset decreases the amount of capital required. This makes sense as the best asset to hold with respect to capital should be a risk-free asset.

The idea of cost of capital sprouts from this principle, as we are forfeiting returns in excess of the risk-free rates by holding the capital in safe investments.

5.3.2 Choice of allocation method

Buch and Dorfleitner (2008) study the coherence of the Euler allocation when various risk measures are used. They find that the Euler allocation when expected shortfall is used satisfies the axioms that are economically significant. They argue that symmetry might be too strict a requirement for an allocation method.

We have identified the Euler allocation principle as the one best suited to the study. We apply the Euler allocation for the simulation approach to arrive at the capital amounts.

We have shown that the variance decomposition in the analytical approach yields a full allocation and therefore it would also satisfy the no undercut principle. It intuitively satisfies the risk less allocation since the economic capital would reduce if we subtracted a larger amount than the expected value from the expected shortfall amount. This would be the same as adding cash to our position as we would increase the amount that we can cover without the use of economic capital from the best estimate liability to that amount plus a cash amount. Therefore, the allocation developed under the analytical approach also satisfies all the economically significant axioms.

The next chapter uses the theory in the preceding chapters and determines capital requirement and allocation for a specific case.

CHAPTER 6

DETERMINING AND ALLOCATING CAPITAL TO RISK FACTORS FOR ONE YEAR PURE ENDOWMENT CONTRACTS MATCHED WITH SIX MONTH BONDS

We apply the economic capital determination and allocation methodology for a single period pure endowment product for a specific case in this chapter. We will be considering a one-year pure endowment product with premiums paid at the start of the year. We assume that the policy relates to a 65 year old male. The insurer will be liable for a payment of one if he survives to the end of the year. The insurer will invest both the best estimate liability as well as the economic capital in six-month government bonds. We deliberately choose an asset that does not match the liability by term to introduce an interest rate risk.

The procedure that we will apply for the simulation and analytical approach is given in Figure 3.

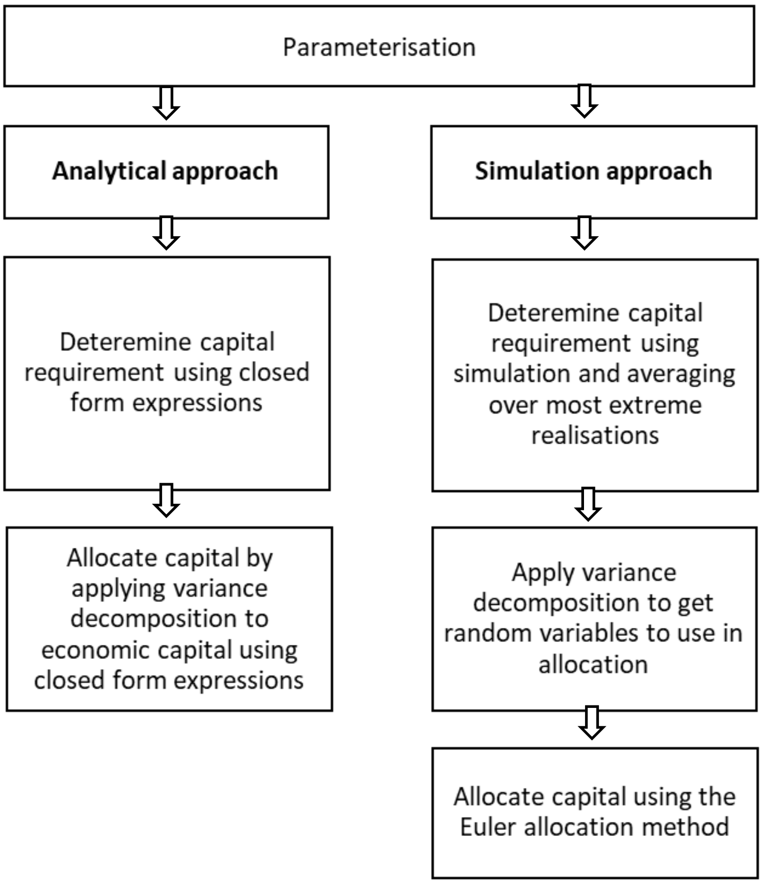


Figure 3

We accordingly discuss our parameterisation next. Thereafter, we determine the capital amounts and allocation. We compare our results for the analytical and simulation approach throughout. A study of the sensitivity of our results to the parameters is also included in this chapter.

6.1 PARAMETERISATION

We need to parameterise the normal distributions we will use for the accumulation and survival rate to determine the capital requirement and allocation. This means that we would need to specify the mean and standard deviation of both these variables as well as the correlation and confidence level.

The specification of the means of these factors would correspond to the best estimates of these variables. For the interest rates we would need to determine both the return we would expect to earn on the government bond over the following six-month period as well as the return for the six months thereafter. The return for the second period would be determined by the rate at which we can reinvest in another six-month government bond.

Annualised rates for three-month government bonds are freely available and can be used as proxy for the annualised return of the initial six-month bond as well as the best estimate for the interest rate at which we can reinvest in six months. We use this rate as of 14 June 2022. We denote this rate as $i_{(1)}$. We then have that $\mu_Y = 1 + i_{(1)}$, with $i_{(1)} = 6.25\%$ (South Africa 3 Months Bond, n.d.).

The choice of best estimate for the cashflow amount was based on q_{65} from the South African Assured Lives Mortality Table 1985-1990 (Actuarial Society of South Africa, n.d.). This would result in $\mu_X = 1 - 0.02440 = 0.9756$.

To parameterise the variation of these variables we could base this on historical data or we can deduce an estimate based on the regulatory amount. We use the fact that the regulatory shocks provide us with a quantile for the cashflow amount and the accumulation rate to calibrate the standard deviation. We therefore assume that this quantile resulted from a normal distribution. Therefore, we estimate the standard deviation of X and Y by setting the shocks that correspond to a 1-in-200 year event (as imagined by regulation) to the 99.5% quantile of the respective distributions. We derive this regulatory shock from the methodology for risk capital requirement as set out in the Prudential Standard Financial Standards for Insurers (Republic of South Africa, 2018). This regulation will be referred to as FSI throughout the rest of the study. It should be noted that only the shock and not the whole distribution is specified in regulation.

We can use the standard normal distribution given by $Z \sim N(0,1)$ to derive an expression that makes it easier to evaluate 99.5% quantile. This enables us to restate the distribution of X and Y in terms of Z. This is done by scaling the variability of the standard normal distribution and adding the mean outcome to this.

This is given as:

$$X = \mu_X + \sigma_X \cdot Z$$

$$Y = \mu_Y + \sigma_Y \cdot Z$$

We have opted to base the notation of VaR on the risk that a source poses. We can use the fact that Z is standard normally distributed to get an expression for $VaR_{99.5\%}(X)$ as:

$$VaR_{99.5\%}(X) = \mu_X + \sigma_X \cdot \Phi(0.995)$$

The risk we have for the accumulation factor is that the return we earned is less than we anticipated. The risk measure would therefore need to correspond to the 0.5% of the distribution for Y. We can use the negative of the 99.5% quantile of the standard normal distribution as this would be equivalent to the 0.5% quantile of this distribution. This can be expressed as:

$$\Phi(0.005) = -\Phi(0.995)$$

We can then express $VaR_{99.5\%}(Y)$ by:

$$VaR_{99.5\%}(Y) = \mu_Y - \sigma_Y \cdot \Phi(0.995)$$

The corresponding shocks specified for longevity risk is a decrease of 10% for the mortality rate (Republic of South Africa, 2018). This means that

$$\begin{aligned} VaR_{99.5\%}(X) &= 1 - (q_{65} - 0.1 \cdot q_{65}) \\ &= 1 - 0.9 \cdot q_{65}\ . \end{aligned}$$

If we equate the two expressions we have for $VaR_{99.5\%}(X)$ and use the fact that $\Phi(0.995) = 2.58$ we can determine the standard deviation of X. This is shown below.

$$\begin{aligned} \mu_X + \sigma_X \cdot \Phi(0.995) &= 1 - 0.9 \cdot q_{65} \\ 1 - q_{65} + 2.58 \cdot \sigma_X &= 1 - 0.9 \cdot q_{65} \\ 2.58 \cdot \sigma_X &= 0.1 \cdot q_{65} \\ \sigma_X &= \frac{0.1}{2.58} \cdot q_{65} \end{aligned}$$

$$\sigma_X = 0.000946$$

The interest rate risk shocks for different terms are specified in Attachment 3 of FSI 4.1 and is a relative decrease in the interest rate for this case (Republic of South Africa, 2018). We have selected the decrease which corresponds to 6 months. We apply this shock to the interest rate we expect to achieve upon reinvestment in a six-month bond which we have assumed to be the same interest as the interest rate we are locked into for the first six months.

We would therefore need to multiply the nominal interest rate we expect to attain upon reinvestment $i_{(1)}$ with a term $(1 + s_{nom}^{down}(0.5))$ to get the shocked interest rate used to calibrate the variance of the accumulation rates. $s_{nom}^{down}(0.5)$ is specified as -0.4806.

We note that the rates are given in annualised terms for both the initial investment as well as the reinvestment. This means that we would need to raise the accumulation rate given for each interest rate to 0.5. This is done as the interest will only be earned for half a year.

In our scenario we can therefore express the equation for the value at risk of Y at 99.5% confidence level as:

$$VaR_{99.5\%}(Y) = \left(1 + i_{(1)}\right)^{0.5} \cdot \left(1 + i_{(1)} \cdot \left(1 + s_{nom}^{down}(0.5)\right)\right)^{0.5}.$$

If we now set the expressions we have for $VaR_{99.5\%}(Y)$ equal to each we can again determine the standard deviation of Y:

$$\mu_Y - \sigma_Y \cdot \Phi(0.995) = \left(1 + i_{(1)}\right)^{0.5} \cdot \left(1 + i_{(1)} \cdot \left(1 + s_{nom}^{down}(0.5)\right)\right)^{0.5}$$

$$1 + i_{(1)} - 2.58 \cdot \sigma_Y = \sqrt{\left(1 + i_{(1)}\right) \cdot \left(1 + i_{(1)} \cdot (1 - 0.4806)\right)}$$

$$\sigma_Y = \frac{\sqrt{\left(1+i_{(1)}\right)\cdot\left(1+i_{(1)}\cdot(0.5194)\right)} - (1+i_{(1)})}{-2.58}$$

If we then substitute for $i_{(1)} = 6.25\%$ we have:

$$\sigma_Y = 0.00586.$$

We assume that there is no correlation between these risk sources and conduct a million simulations when we apply the simulation approach.

Table 6 contains the parameter values used to model the present value of the future cashflows.

Table 6

Parameter	**Symbol**	**Value**
Confidence level	$1-\alpha$	99.5%
Mean probability of survival	μ_x	0.9756
Standard deviation of probability of survival	σ_x	0.000946
Mean accumulation rate	μ_y	1.0625
Standard deviation of accumulation rate	σ_y	0.00586
Correlation between interest and mortality rates	$\rho_{x,y}$	0
Simulation number	n	1000000

The following section uses these parameters to calculate the economic capital amount whereafter we will determine the allocation and conduct a sensitivity analysis.

6.2 ECONOMIC CAPITAL AMOUNT

In this section we determine the economic capital amount under both approaches. We now first show how we calculate the best estimate liability as it aids understanding and is required to calculate the capital under either approach. The best estimate liability is calculated as defined in Section 3.2 as:

$$
\begin{aligned}
BEL &= \frac{\mu_X}{\mu_Y} \\
&= \frac{0.9756}{1.0625} \\
&= 0.9182
\end{aligned}
$$

If we employ the simulation approach, we simulate values for X and Y in RStudio with the built-in random number generator. The random number generator returns values for these variables that correspond to random realisations of the variables. These simulation outputs are then stored in vectors X and Y. We set the seed of our random number generation equal to 123 to ensure the results can be replicated. We conduct a million simulations.

If we draw a scatterplot of these realisations as in Figure 4 we observe an elliptical shape to the realisations. This makes sense as we have used an elliptical distribution to model these variables. Specifically, we have used an independent bivariate normal distribution to model these variables. The realisations seem to be quite closely grouped with some more extreme values being observed from time to time.

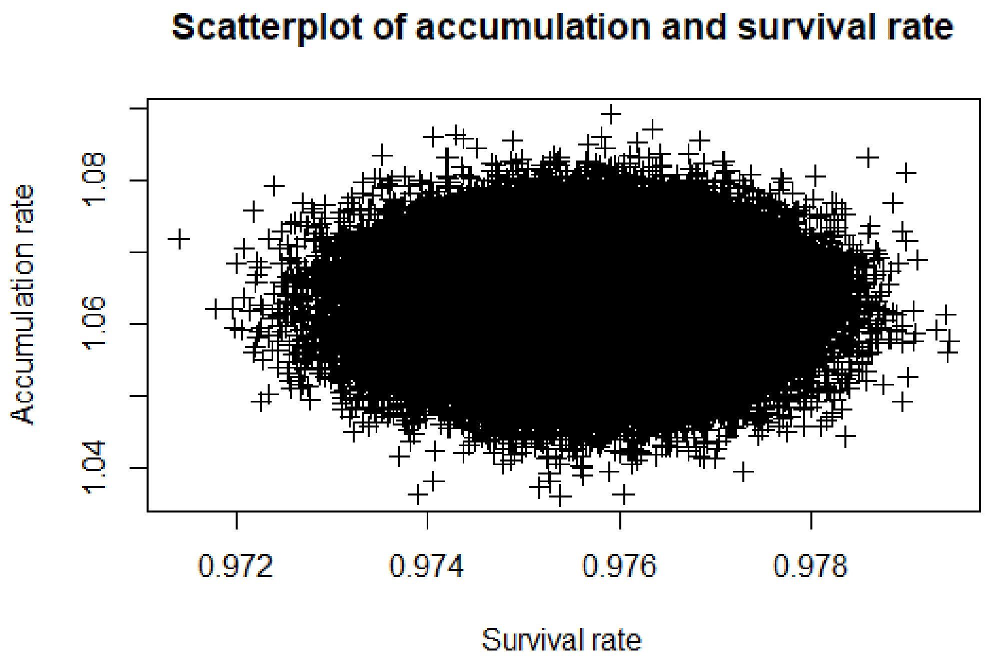


Figure 4

We then calculate the present value of the future cashflows for each pair of realisations and record these in a vector $\boldsymbol{T}$. We can visualise the results by making use of a histogram as given in Figure 5.

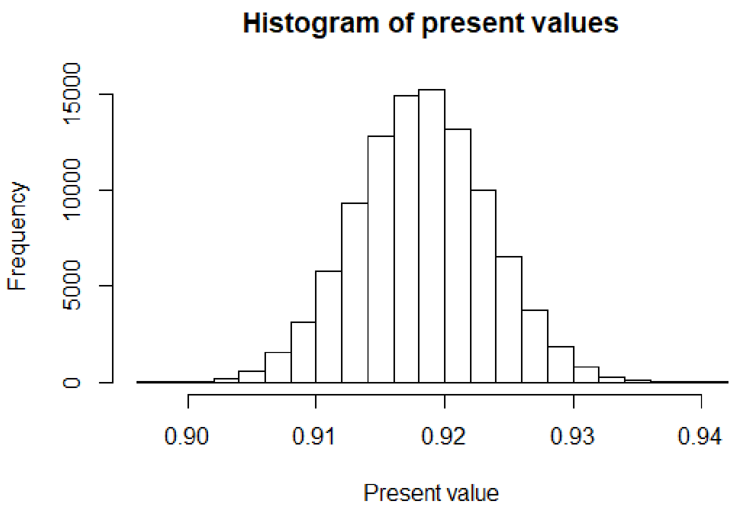
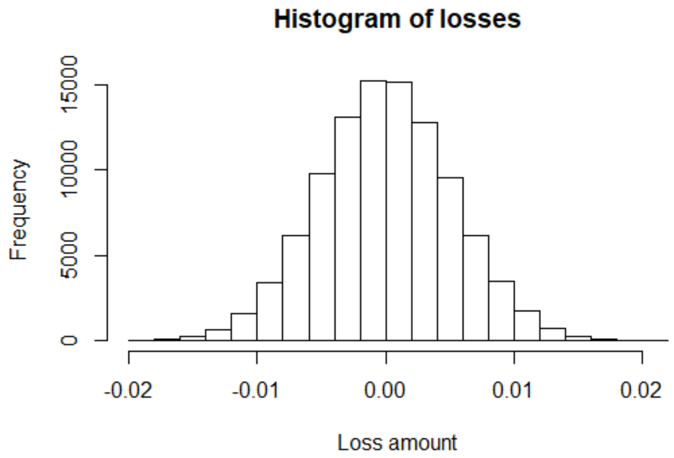


Figure 5

In Figure 5 we see that the majority of present value's fall around the best estimate liability. The distribution of T looks about symmetrical as well.

We are however concerned with the present value of all losses we can incur if we only have assets available to cover the best estimate liability. We can visualise these realisations again with



use of a histogram as in

Figure 6. We will observe the same distribution only shifted to reflect that we have the assets available to cover the best estimate liability. The losses seem to be centred around zero. Negative losses are meant to imply profits where we have more assets than what would be required.

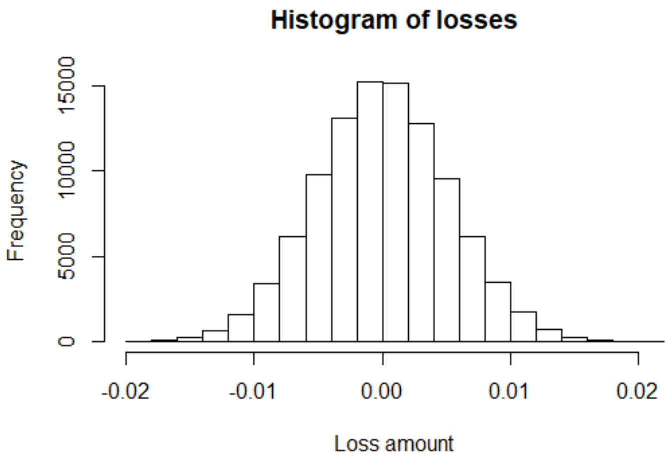


Figure 6

We choose to hold economic capital such that assets equalling economic capital and best estimate liability would be sufficient to avoid loss with a sufficient level of confidence. If this is the only product the insurer sells, we can rephrase this in terms of the insurer's solvency. If the economic capital and best estimate liability amount is held in assets, the company would not become insolvent over the next year with the specified level of confidence.

We apply the approach outlined in Section 3.2.1 to determine the economic capital required using the simulation approach. We need to use the average function over the most extreme realisations of T to determine the economic capital using the simulation-based approach. The risk in

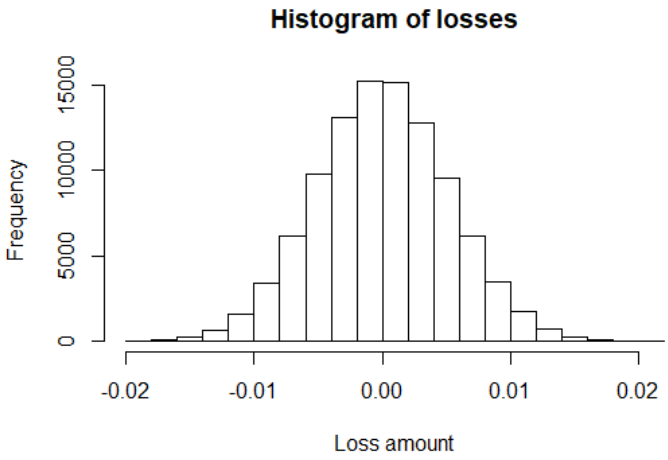


Figure 6 is at the right-hand side and represents the realisations being larger than that for which the insurer is prepared. The capital amount is set to correspond to the expected shortfall at a 99.5% confidence level. If we take the average of the 0.5% of realisations for T on the right-hand side of the above graph we arrive at the simulation based economic capital.

We can plot the set of values of accumulation rate and future cashflow (or survival rate) that correspond to these extreme values of T that are used to determine the expected shortfall and corresponding economic capital, as in Figure 7. We can see that the realisations for accumulation rates are considerably lower than the mean of 1.0625%, whilst the future cashflows are both smaller and larger than the mean of 0.9756. We see that for lower accumulation rates a wider range of future cashflows correspond to the subset of most extreme present value realisations we consider. Conversely, this can be stated as a higher interest rate allows a more extreme future cashflows to be absorbed in the area that is not shown in the below graph.

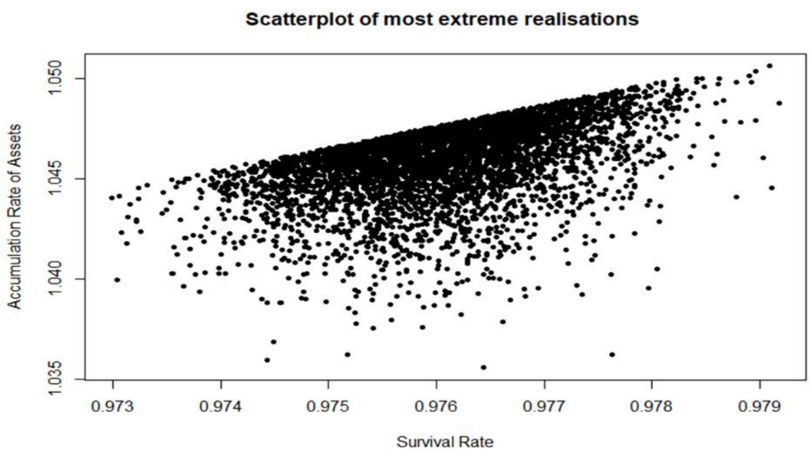


Figure 7

We also apply the analytical approach as outlined in Section 3.2.2. This returns the economic capital figure if we substitute the parameter values derived in the previous section (given in Table 6) into the formula.

The results are given in Table 7. The economic capital figures are very comparable, with the simulation approach yielding a larger value that the analytical method. It should be noted that the analytical approach has an approximation for the distribution and the exact distribution for T is not used.

Table 7

Approach	Economic capital
Simulation	0.01510735
Analytical	0.01499224

This economic capital amount reflects the capital amount required for both risk factors. The joint behaviour of these risk factors is therefore already reflected in the amount determined. We consider how to allocate this capital amount to each of the risk factors in the next section.

6.3 VARIANCE DECOMPOSITION AND CAPITAL ALLOCATION

We apply the variance decomposition under both approaches. Under the analytical approach this yields the allocated capital amounts. We would additionally apply the Euler allocation to the output from such a decomposition in the simulation approach.

6.3.1 Simulation approach

We apply the variance decomposition linearisation approach given in Section 4.3.1 to express the entire risk such that an existing allocation approach can be applied. We then have two decompositions of the total risk R. These are given by $R = R_X^1 + R_Y^1$ and $R = R_X^2 + R_Y^2$. These variables consist of a million realisations as this is done for each realisation before we apply the Euler allocation.

We then apply the Euler allocation method to determine the economic capital amount allocated to each risk source as in Table 5 in Section 5.2. We let RC be the economic capital amount and apply the Euler allocation in the following manner:

$$RC_X^i = E\left[R_X^i \middle| R \geq VaR_\alpha(R)\right],$$

$$RC_Y^i = E\left[R_Y^i \middle| R \geq VaR_\alpha(R)\right], \qquad i = 1,2\ .$$

The results of this allocation for both orders of conditioning are given in Table 8. We observe that the majority (close to 97%) of the economic capital is allocated to interest rate risk. This can be interpreted to mean that more of the variability of the outcomes is due to variability of the interest rates than that of the survival rates. We note that we have a full allocation, meaning that the sum of the risk capital amounts equals the overall economic capital requirement. This is since the decomposition and allocation methods we have chosen both satisfy this property. The capital allocation is also almost completely insensitive to the order of conditioning in the variance decomposition step.

Table 8

Order of conditioning	Measurement	Capital allocated to interest rate risk	Capital allocated to survival rate risk	Total capital requirement
Interest rate first	Amount	0.01466903	0.00043832	0.01510735
	Percentage	97.098625%	2.901375 %	100%
Survival rate first	Amount	0.01466282	0.00044452	0.01510735
	Percentage	97.057580%	2.942420 %	100%

6.3.2 Analytical approach

The amount and percentage of economic capital allocated to each risk for both orders of conditioning are given in Table 9. We have calculated the allocated amounts by applying the variance decomposition to the economic capital amount as set out in Section 4.3.2. The majority of the capital is still allocated to interest rate risk, slightly more than under the simulation approach. The order of conditioning is slightly more important in when we apply the variance decomposition to our approximated analytical distribution. When we condition on interest rate risk first, we allocate almost all the economic capital to interest rate risk.

Table 9

Order of conditioning	Measurement	Capital allocated to interest rate risk	Capital allocated to survival rate risk	Total capital requirement
Interest rate first	Amount	0.01498777	0.00000447	0.01499224
	Percentage	99.970165%	0.029834%	100%
Survival rate first	Amount	0.01476407	0.00022817	0.01499224
	Percentage	98.478047%	1.521953%	100%

6.3.3 Comparison

Both approaches indicate that interest rate risk is much more important in this scenario and would require a greater capital amount allocated to this risk. We observe that the order of conditioning is more important when we employ the analytical approach, whereas the effects thereof are negligible under the simulation approach. The analytical approach allocates more capital to interest rate risk than the simulation approach. It is interesting to note that under the analytical approach we allocate more capital to interest rate risk when we condition on this risk first.

6.4 SENSITIVITIES

6.4.1 Confidence level

We would expect a greater economic capital requirement for higher levels confidence levels. This is since we would require more assets to avoid insolvency in more adverse and less likely scenarios. This would correspond to a decreasing economic capital amount as α increases. This is since the confidence level is specified by $(1-\alpha)$ and less economic capital is required to withstand future events at a lower confidence level.

Analytical approach

When we plot the economic capital requirement that the analytical approximation yields against α we find an inverse relationship as expected. This plot is provided as Figure 8. The relationship is not linear with a much larger economic capital requirement as we move to smaller and smaller levels of α. This relationship looks nearly exponential.

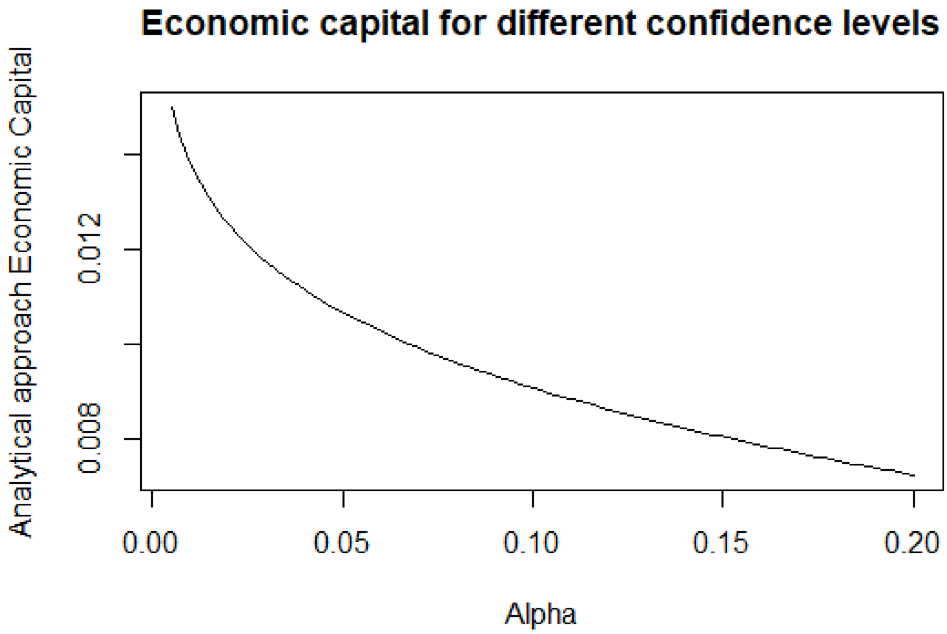


Figure 8

Simulation approach

We have calculated the capital requirement for different values of α by repeating the procedure outlined in Section 3.2.1 for each of these values. This leads to a number of economic capital amounts we can plot against the values of α to understand how these behave. This is given in Figure 9.

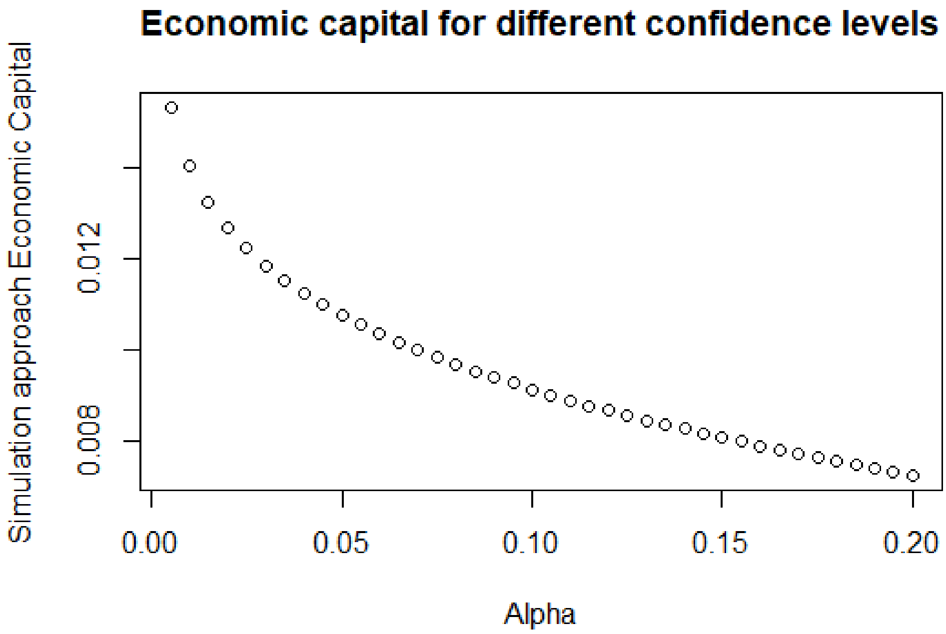


Figure 9

We observe the same relationship and behaviour under the simulation approach. This means that the analytical approach behaves reasonably across α and in line with results from the simulation approach.

Conclusion

The results in the previous section correspond to an α of 0.5% or 0.005 in the above graph (as α is expressed as a number and not a percentage in Figure 9). This parameter influences the point on the outcome distribution in which we are interested whereas the following two sections look at parameters that influence this outcome distribution.

6.4.2 Coefficient of variation

Recall that the coefficient of variation is given by

$$\delta_X = \frac{\sigma_X}{\mu_X}$$

$$\delta_Y = \frac{\sigma_Y}{\mu_Y}.$$

This measure therefore gives us an indication of the severity of the variability of each variable compared to its mean. This can also be thought of as a proportional variation of each variable. We consider the impact on the economic capital amount of changing these coefficients of variation. The range from half to one and a half times the current coefficient of variation is considered in this section for both the accumulation and survival rate.

An increase in the proportional variation of each factor would lead to an increase in the variation of the outcomes given the other parameters stay the same. This increased variation would then in turn act to increase the economic capital requirement.

We can explore whether we observe this intuitive relationship for our analytical and simulation approach. Assessing relative sensitivity of the economic capital on the coefficient of variation of each risk amount provides us with insight as to the importance of each risk. This links quite closely to the central topic of the research as an allocation is in essence a question of importance. We can assess this sensitivity for the analytical as well as the simulation approach.

Analytical approach

We can plot the economic capital amount against the coefficient of variation for each of the variables. This is easy to do as we have a formula that delivers the capital requirement as a function of amongst other things the coefficient of variation of each risk source.

The economic capital shows an exponential relationship to the proportional variation of survival rates as can be seen in Figure 10. Meaning that an increase in the coefficient of variation at a higher level would lead to a larger increase in capital than the same increase at a lower level.

We note that the change of the economic capital amount across the range of coefficient of variations we have considered is in the area of just under 0.0005 which would seem to be a relatively small change compared to the overall capital amount.

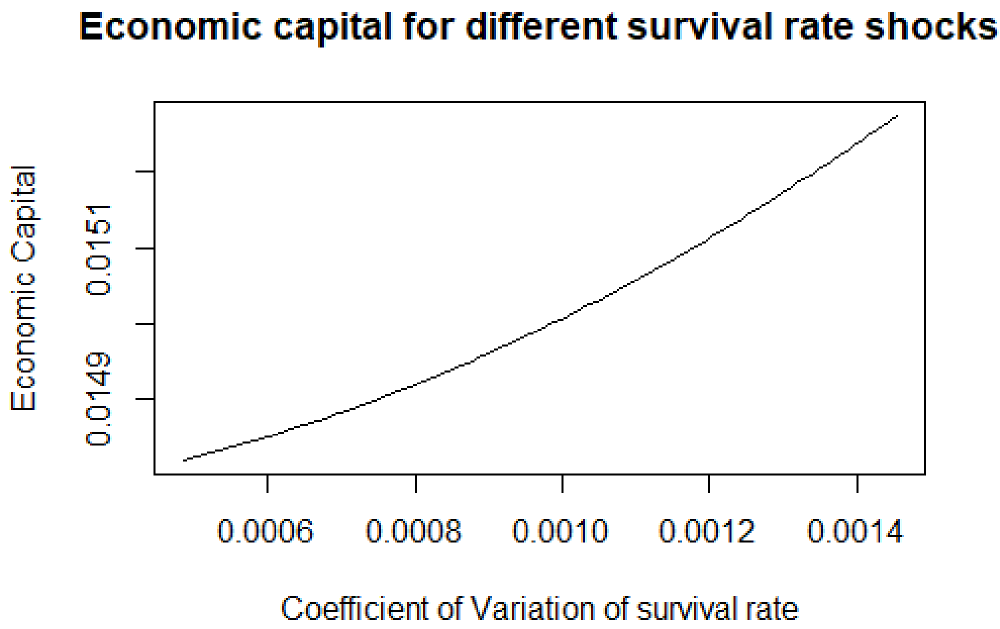


Figure 10

We observe a more linear relationship between the economic capital and the coefficient of variation for the accumulation rate as can be seen in Figure 11. We see that the change in the economic capital amount across the range of coefficients of variation considered is substantially larger in the area of just over 0.01, which is much higher than we observed for survival risk. This greater sensitivity to the coefficient of variation of accumulation rates supports our findings that interest rate risk is a more important component of the economic capital amount. Therefore, it is sensible to allocate more capital to interest rate risk than survival rate risk.

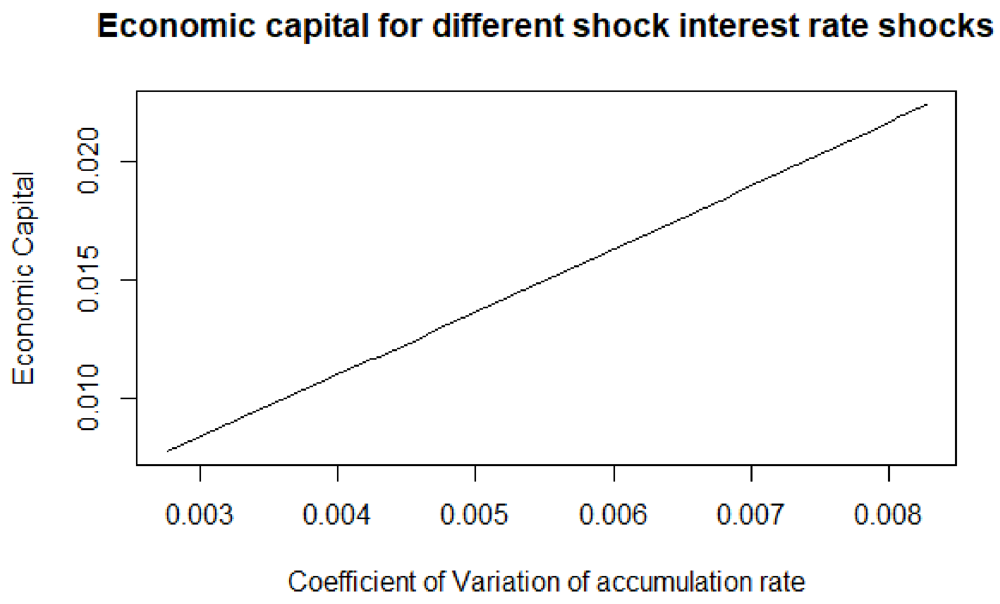


Figure 11

Simulation approach

We can visualise the impact that this has on the economic capital amount by plotting the realisations of both accumulation and survival rate that correspond to the tail of the losses as is done in Figure 12 on the next page. The extent of the tail that we consider is consistent with our confidence level. This is all realisations that are so extreme that they would only occur in the most unlikely 0.05% of cases. The base case is the middle plot in

Figure 12

As we move up the coefficient of variation of the accumulation rate increases as can be seen by the increased spread along the y axis. As we move from left to right the coefficient of variation for the future cashflows increase. This can be seen as the figure increases in width.

We can also see that the coefficient of variation has an impact on how extreme the values of accumulation and interest rates are that correspond to the largest realisations of the present value variable. We can see that a greater coefficient of coefficient of variation for interest rates would lead to more extreme interest rate values (lower accumulation rates) being associated with the subset we are considering. We can see how the subset moves to lower accumulation rates as we move to the graphs that correspond to greater proportional variation for accumulation rates.

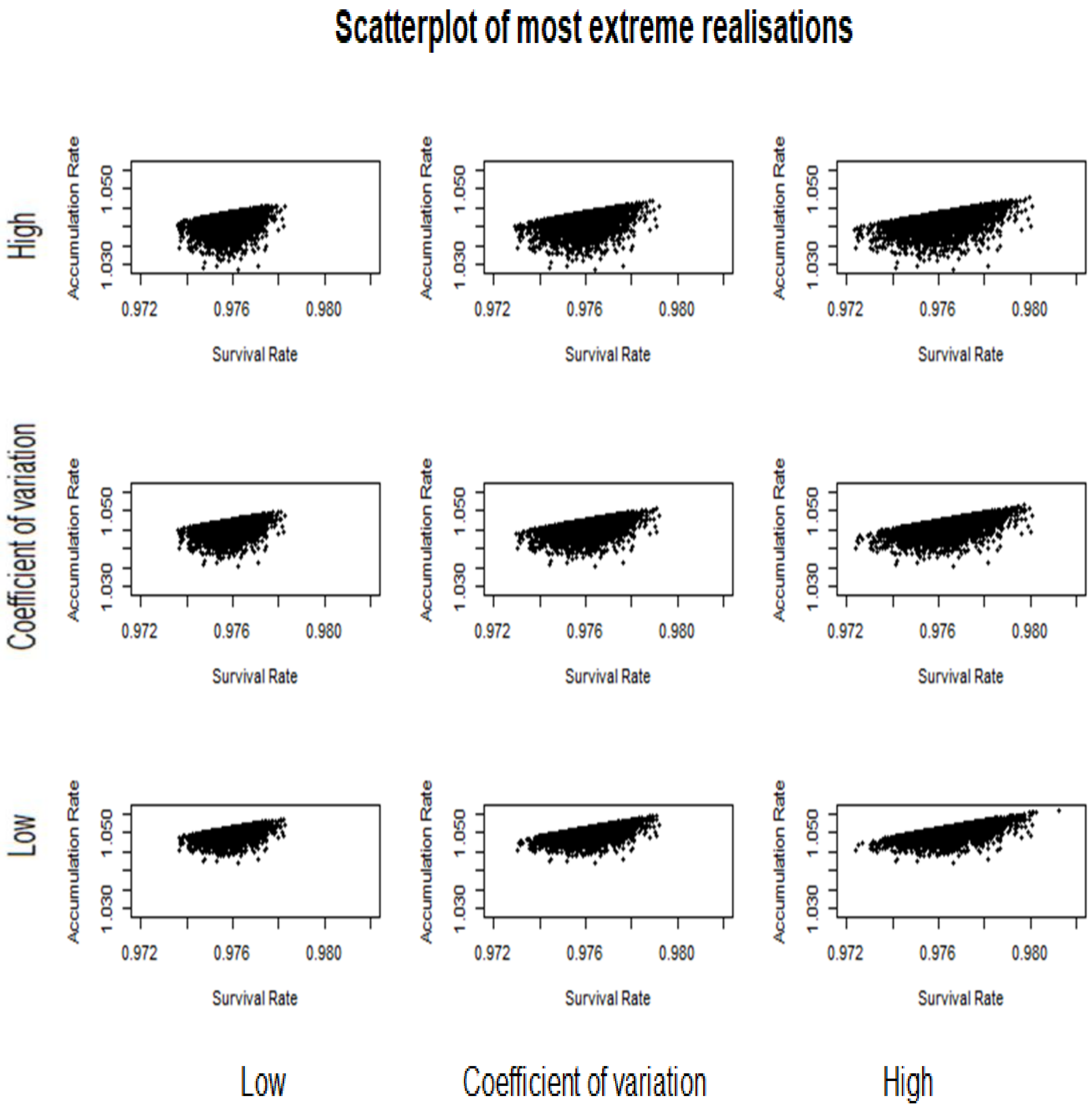

Scatterplot of most extreme realisations
High
Coefficient of variation
Low
Accumulation Rate
1.030
1.050
0.972
0.976
0.980
Survival Rate
Low
Coefficient of variation
High

Figure 12

6.4.3 Correlation

We have assumed zero correlation in the preceding calculations and representations, in this section we consider the impact of non-zero correlation on the amount of economic capital required.

Analytical approach

When we plot the economic capital requirement for different correlation coefficients, we observe that there is an inverse relationship between these factors. This can be seen by the negative gradient in Figure 13. We note that higher interest rates lead to greater accumulation rates whereas higher survival rates lead to greater values for X. Therefore if X and Y are positively correlated we will have more situations with higher accumulation rates (and therefore greater discounting effects) correspond to situations where we would need to pay a greater benefit as the survival rate was greater than anticipated. The opposite is then true for a low interest, low survival realisations. The realisations with greater benefit payments would therefore be discounted more if we have a positive correlation leading to a lower economic capital requirement. Negative correlation would have the opposite effect and lead to a higher economic capital requirement.

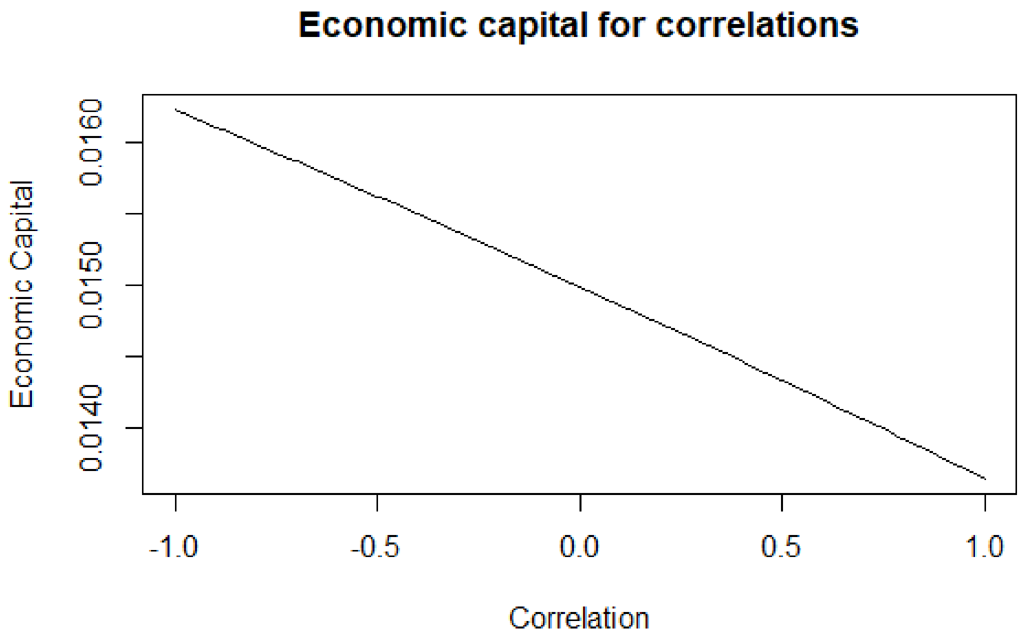


Figure 13

Simulation approach

Again, we can visualise the effects of changing the correlation using a plot of the realisations that correspond to the 0.05% tail of the present values. The plot in the middle of Figure 14 is the base case which is the same as Figure 7 and the middle plot in Figure 12. The plot to the left has a negative correlation of -0.7 that would make the elliptical distribution tilt backwards, giving a negative slope to the whole elliptical. The interest rates where the plot is cut off does not change much as the correlation changes. For the negative correlation case the tail would consist of more realisations with high future cashflows leading to higher economic capital requirements. We observe the converse for the case where we have a positive correlation as in the right plot. The correlation here is equal to 0.7. The elliptical has a more positive slope leading to lower future cashflows and a corresponding lower economic capital figure.

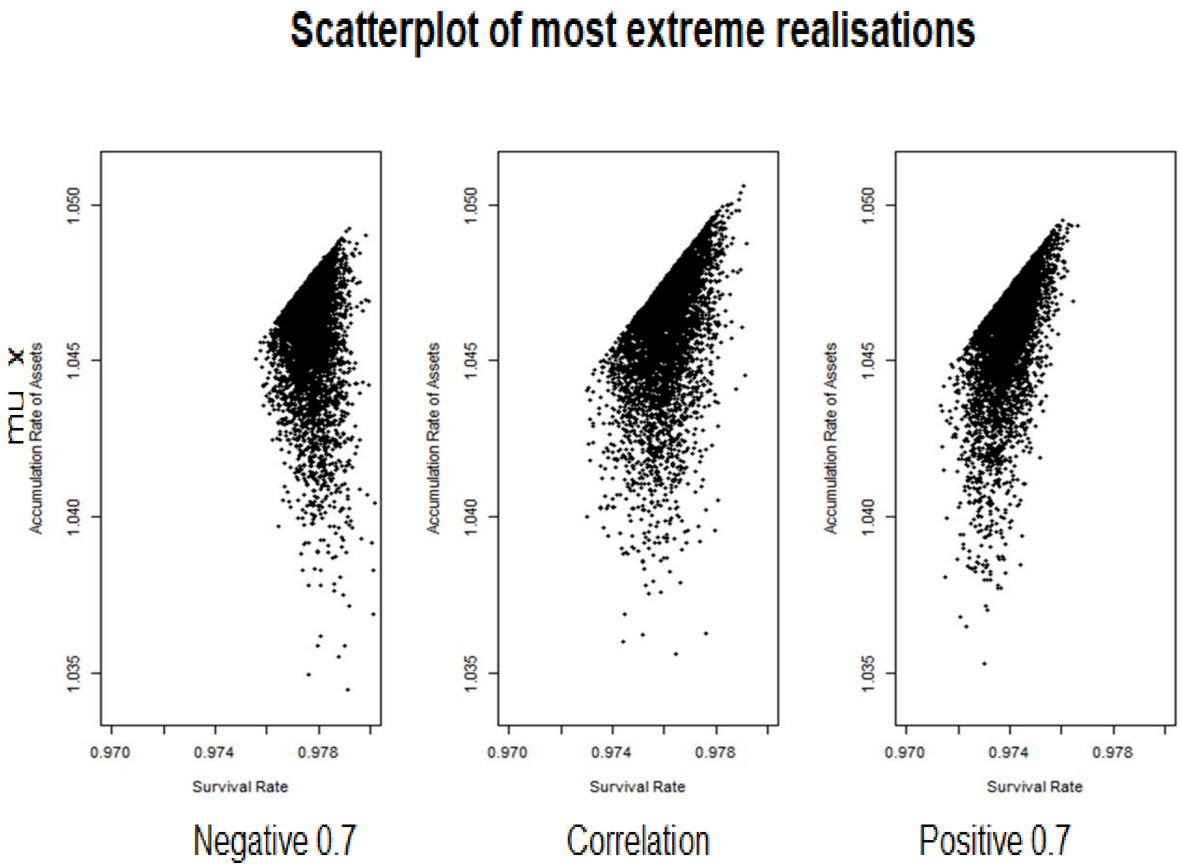


Figure 14

We conclude in the next chapter.

CHAPTER 7
CONCLUSION

It is important to be able to allocate economic capital to different risk sources to understand the importance of each. We evaluate market and survival risk for a simple one-year pure endowment contract.

The methodology and assumptions underlying the economic capital amount need to be specified to phrase the allocation in a relevant manner. We have opted to use expected shortfall as risk measure, since it is coherent with economic theory.

Stochastic modelling was used to model the market and survival risk of the single period pure endowment. We stipulate the distribution of these risk components and model the present values of future liabilities as a function of the mortality and interest rate experience.

A simulation as well as an analytical approach is used to model this function and then determine the economic capital required. The risk is then decomposed to market and survival risk using a variance decomposition for both approaches. In the case of the analytical approach this yields a capital allocation. However, applying such a decomposition to the simulated present values produces random vectors that correspond to each risk source. These random vectors can then be used to apply allocation methods proposed in literature.

The proposed analytical approach therefore requires one less methodology decision (and corresponding calculation) to allocate the economic capital to risk sources. For the simulation approach we then apply the Euler method to allocate the economic capital amount.

We evaluate these quantification and allocation methods by use of a case study. We use a one-year pure endowment contract that the insurer has decided to match with a six-month government bond. We parameterise our model to be in accordance with the regulatory prescribed shocks for longevity and interest rate risk. We find that the analytical method we propose gives a comparable economic capital requirement to the commonly employed simulation approach.

Interest rate risk regardless of approach is identified to be much more important than survival risk. Dependent on the decomposition assumptions and allocation method between 97% and 99.97% of economic capital is allocated to interest rate risk.

We compare the sensitivity of the allocation to the assumptions made in the decomposition of the combined risk. We find that the analytical approach is more dependent on the order in which we apply the variance decomposition.

The sensitivity of the economic capital requirement to parameter values is then evaluated. We find that an increasing confidence level leads to an exponential increase in capital requirement regardless of our approach.

The analytical approach enables us to easily consider the sensitivity of capital amount to the coefficient of variation of risk sources and the correlation between sources. We find that the requirement is more sensitive to the interest rate variation which supports our allocation findings. If interest rates are positively correlated to survival rates, we find that the overall requirement decreases. This is as higher interest rates result in greater discounting and lower present values.

We can also visualise the sensitivity of economic capital to the coefficient of variation and correlation using scatterplots under the simulation approach. This is done by plotting the values of the accumulation and survival rate that correspond to the tail of the present value distribution. The findings from these plots support what we found under the analytical approach sensitivity analysis.

The analytical approach we propose for capital determination and allocation is useful and can further our understanding of the importance of risk sources to the overall risk an insurer faces. As advantages it requires less methodology decisions and calculations as well as enabling easy analysis of parameter importance. However, the slightly increased sensitivity of the allocation to decomposition assumptions can make deciding on a final allocation more difficult.

The structured way used in this study of understanding the importance of survival and interest rate risk for a one-year pure endowment product can be extended to different products, longer periods and alternative modelling distributions. We can also consider different dependence structures between risk sources. The use of copula's could be particularly promising in this area. The situations where the capital and liability amount are invested in different assets can also be considered.

The situation where an insurer needs to hold economic capital for future dated life contingent cashflows that cannot be matched exactly is one that insurers will continue to face.

We found that interest rate risk is more important than survival risk for a one-year pure endowment product with shorter dated backing assets. Economic capital allocation enables an insurer to be managed more effectively as it considers the actual operation of the insurers' liabilities and assets. This allocation provides a measure of the relative importance of investment decisions, asset availability and longevity uncertainty on the solvency of the insurer.

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APPENDIX A:
LOGNORMAL EXPECTED SHORTFALL

Given $X \sim logN(\mu, \sigma^2)$ we have the following conditional expectation.

$$ES_\alpha(X) = E[X|X \geq k] = \exp\left(\mu + \frac{\sigma^2}{2}\right) \cdot \frac{\Phi\left[\frac{\mu+\sigma^2-\ln k}{\sigma}\right]}{1-\Phi\left[\frac{\ln k-\mu}{\sigma}\right]}$$

Where k is such that this corresponds to the confidence level required from the Expected Shortfall statement. This is achieved by letting

$$\alpha = 1 - \Phi\left[\frac{\ln k - \mu}{\sigma}\right]$$

Then we have

$$\Phi^{-1}[1-\alpha] = \Phi^{-1}\left[1 - 1 + \Phi\left[\frac{\ln k-\mu}{\sigma}\right]\right]$$

$$= \Phi^{-1}\left[\Phi\left[\frac{\ln k-\mu}{\sigma}\right]\right]$$

$$= \frac{\ln k-\mu}{\sigma}$$

Then

$$\frac{\mu+\sigma^2 - \ln k}{\sigma} = \sigma - \frac{\ln k-\mu}{\sigma}$$

$$= \sigma - \Phi^{-1}[1-\alpha]$$

Then the Expected Shortfall becomes:

$$ES_\alpha(X) = \exp\left(\mu + \frac{\sigma^2}{2}\right) \cdot \frac{\Phi[\sigma - \Phi^{-1}(1-\alpha)]}{\alpha}$$

This form of expressing the Expected Shortfall simplifies the equation to a function of the confidence level we require and the parameters of the distribution of X.