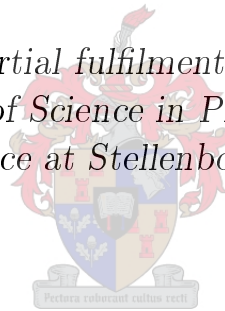


Relativistic Quantum Mechanics on Non-commutative Spaces

by

Paul Henry Williams

*Thesis presented in partial fulfilment of the requirements for
the degree of Master of Science in Physics in the Faculty of
Natural Science at Stellenbosch University*



Department of Physics,
University of Stellenbosch,
Private Bag X1, Matieland 7602, South Africa.

Supervisor: Prof. F.G. Scholtz

December 2015

Declaration

By submitting this thesis electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the sole author thereof (save to the extent explicitly otherwise stated), that reproduction and publication thereof by Stellenbosch University will not infringe any third party rights and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

Date:

Copyright © 2015 Stellenbosch University
All rights reserved.

Abstract

Relativistic Quantum Mechanics on Non-commutative Spaces

P.H. Williams

*Department of Physics,
University of Stellenbosch,
Private Bag X1, Matieland 7602, South Africa.*

Thesis: MSc in Physics

December 2015

A brief review of non-relativistic and relativistic quantum mechanics is carried out, with particular attention to concepts that are modified in the non-commutative setting. The Hilbert-Schmidt operator formulation of non-relativistic, non-commutative quantum mechanics is generalized to the relativistic setting of 4-dimensional non-commutative space-time. The generators of Lorentz transformations are derived in this formalism and compared with the commutative case. It is shown that the non-interacting, non-commutative Dirac equation is Lorentz invariant. Lorentz invariance can be maintained when electromagnetic interactions are included provided that the gauge and Dirac fields are composed through an appropriate star product. The appropriate gauge transformations for the non-commutative Dirac equation are found. The non-commutative C, P, T symmetries of the interacting Dirac equation are investigated. The free Dirac equation and the interacting Dirac equation for the case of a constant background magnetic field are studied on the operator level. Systems confined to a specific space-time volume, and the associated boundary conditions, are studied using appropriate projection operators. As a specific example the Dirac equation in an infinitely long cylinder is considered. An operator valued action, which yields the interacting Dirac equation as the equation of motion, is derived and evaluated in a coherent state basis. The constant background magnetic field is revisited in the coherent state basis. This establishes the link to the standard star product formulation of non-commutative quantum field theories. Useful properties of the star product are derived in an appendix.

Uittreksel

Relatiewistiese Kwantummeganika op Non-commutatiewe Ruimtes

(“Relativistic Quantum Mechanics on Non-commutative Spaces”)

P.H. Williams

*Departement Fisika,
Universiteit van Stellenbosch,
Privaatsak X1, Matieland 7602, Suid Afrika.*

Tesis: MSc in Fisika

Desember 2015

'n Kort hersiening van nie-relatiewistiese kwantummeganika op nie-kommutatiewe ruimtes word gedoen, met die fokus op konsepte wat verander in die nie-kommutatiewe raamwerk. Die Hilbert-Schmidt operator formulering van nie-relatiewistiese, nie-kommutatiewe kwantummeganika word veralgemeen na die relatiewistiese raamwerk van 4-dimensionele nie-kommutatiewe ruimte-tyd. Die generatore van Lorentz transformasies word herlei in hierdie formulering en word vergelyk met die kommutatiewe geval. Dit word aangetoon dat die vrye nie-kommutatiewe Dirac vergelyking Lorentz invariant is. Lorentz invariansie kan behou word, in die teenwoordigheid van elektromagnetiese wisselwerkings, as die yk en Dirac velde met die gepaste ster produk saamgestel word. Die regte yk transformasies van die nie-kommutatiewe Dirac vergelyking word gevind. Die nie-kommutatiewe C, P, T simmetrieë van die Dirac vergelyking met wisselwerkings word ondersoek. Die vrye Dirac vergelyking en die Dirac vergelyking 'n konstante agtergrond magneetveld word opgelos op die operator vlak. Stelsels beperk tot 'n spesifieke ruimte-tyd volume, en die geassosieerde randvoorwaardes, word bestudeer met gepaste projeksie operatore. Die Dirac vergelyking in 'n oneindig lang silinder word as 'n eksplisiete toepassing aangebied. 'n Operator aksie wat die Dirac vergelyking met wisselwerkings as bewegingvergelyking lewer, is gevind en in 'n koherentetoestand basis geëvalueer. Die geval van 'n konstante agtergrond magneetveld is in die koherentetoestand basis herondersoek. Dit vestig 'n verband met die standaard ster produk formulering van nie-kommutatiewe kwantumveldeteorieë. Nuttige eienskappe van die ster produk word in 'n bylaag herlei.

Acknowledgements

I would like to express my sincere gratitude to the following people and organisations:

- My supervisor F.G. Scholtz.
- My parents and family.
- The NRF for funding my MSc.

Contents

Declaration	i
Abstract	ii
Uittreksel	iii
Acknowledgements	iv
Contents	v
1 Introduction	1
1.1 Motivation and Aim	1
1.2 Commutative Quantum Mechanics	2
1.3 Non-commutative Non-relativistic Quantum Mechanics	5
2 Relativistic Non-commutative Quantum Mechanics	9
2.1 Formulation	9
2.2 Lorentz Symmetry	12
2.3 The Free Non-commutative Dirac Equation	14
3 The Non-commutative Dirac Equation With Electromagnetic Interactions	15
3.1 Twisted Lorentz Symmetry	15
3.2 Electromagnetic Interactions	17
3.3 Gauge Symmetry	18
3.4 Discrete Symmetries	19
3.5 Constant Background Magnetic Field	22
4 The Confined Non-commutative Dirac Equation	29
4.1 Boundary Conditions	29
4.2 Solutions	32
5 The Non-Commutative Action and The Coherent State Basis	36
5.1 The Coherent State Basis	37
5.2 Constant Background Magnetic Field Revisited	40

<i>CONTENTS</i>	vi
5.3 The Symbol Map	43
6 Conclusion and Outlook	47
Appendices	49
A Commutator of Generators of on Non-commutative Space Lorentz Transformations	50
B Lorentz Generator Acting on the Modified Interaction	52
C Properties of the $\hat{*}$ Product	54
D Uncertainty for the Constant Magnetic Field	56
E Boundary Conditions	58
F Symbols of Derivatives	61
Bibliography	62

Chapter 1

Introduction

In this thesis we formalize and investigate relativistic quantum mechanics on non-commutative space. Before we begin we must justify this inquiry. Also it is necessary to review commutative relativistic quantum mechanics in order to understand the parallels between this and non-commutative theories. We also review non-relativistic non-commutative quantum mechanics, which guides us in our formulation.

1.1 Motivation and Aim

It is now generally accepted that a consistent theory of quantum gravity will require a modification of the notion of space-time at the Planck scale [1]. The notion of space-time as a flat Minkowski space should break down [2] at these length scales, as any attempt to localize events at a scale shorter than the Planck length will lead to gravitation instability. One such modification, to build in this minimum length scale, is that of non-commutative space-time. Non-commutativity is not a new idea [3], and has also emerged from some string theory constructions [4]. Even though there have been many investigations into the possible physical implications of non-commutativity in quantum mechanics and quantum many body systems [5], [6], [7], [8], [9], [10] quantum electrodynamics [11], [12], [13], the standard model [14] and cosmology [15], [16], a comprehensive understanding of the physical implications of non-commutativity is still lacking, mainly due to a lack of a systematic framework for non-commutative quantum mechanics. Recently, this shortcoming was addressed in two [17] and three dimensional systems [18], which has made headway into a systematic understanding of non-commutative physics, but a relativistic framework is still lacking. Nevertheless there have been studies of non-commutative field theories [19], [20], [21]. But to gain a deeper understanding of the physics of non-commutative field theories a systematic consistent framework is required. In this thesis we aim to begin this process, by formalizing relativistic quantum mechanics on non-commutative spaces.

In particular we would like to study the Dirac equation on such spaces and specifically we would like to:

- Generalize the formulation of non-relativistic quantum mechanics on non-commutative spaces to the relativistic case;
- Study the Lorentz symmetry of the Dirac equation and address the issue of the restoration of Lorentz invariance, with interactions included;
- Solve the Dirac equation and study the dispersion relation, for the free case and the case of a simple interaction;
- Study the Dirac equation confined to a region of non-commutative space-time;
- Derive an abstract action for the non-commutative Dirac equation on the operator level.

1.2 Commutative Quantum Mechanics

In this section we review commutative quantum mechanics to prepare for the modified formulation on non-commutative space. We use natural units and set $c = \hbar = 1$. A good reference is [22] for commutative quantum mechanics.

To start we need a Hilbert space, a complete inner product vector space, \mathcal{H} . On this Hilbert space we have an inner product $(\cdot, \cdot) : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathbb{C}$. Linear functionals F are defined as a linear map $F : \mathcal{H} \rightarrow \mathbb{C}$. The space of linear functionals denoted by \mathcal{H}^* is also a Hilbert space, called the dual space of \mathcal{H} . Riesz's theorem says there is one to one correspondence between linear functionals and vectors in the Hilbert space such that $(\phi, \psi) = F_\phi(\psi)$. We can then use the Dirac bra-ket notation where we denote vectors in \mathcal{H} by $\psi \equiv |\psi\rangle$ and linear functionals by $F_\phi \equiv \langle\phi|$ and then $\langle\phi|\psi\rangle \equiv (\phi, \psi)$. Also we can define operators on the Hilbert space $\hat{A} : \mathcal{H} \rightarrow \mathcal{H}$ and since we have an inner product, the adjoint is also defined $(\phi, \hat{A}\psi) = (\hat{A}^\dagger\phi, \psi)$. In Dirac notation $\hat{A}|\psi\rangle \in \mathcal{H}$ and $\langle\phi|\hat{A} = \langle\hat{A}^\dagger\phi|$. The object $|\psi\rangle\langle\phi|$, called the outer product, can easily be seen to be an operator, $(|\psi\rangle\langle\phi|)|\xi\rangle = (\phi, \xi)|\psi\rangle$. Hermitian (self-adjoint) operators have the property $A = A^\dagger$, and eigenstates $A|a\rangle = a|a\rangle$ which are a complete basis for the Hilbert space. From the spectral theorem we know $A = \int da a |a\rangle\langle a|$.

The vectors in $\phi \in \mathcal{H}$ represent the physical states of the system and we therefore refer to them as states. The correspondence principle says that to every physical observable there corresponds a linear hermitian operator. Thus the observable position x has a hermitian operator \hat{x} and momentum p has \hat{p} . The uncertainty principle says $\Delta x \Delta p > \frac{1}{2}$, and using the uncertainty relation (the general form can be found in [23]) this condition becomes $[x^i, p^j] = i\delta^{ij}$. Since we have no restriction on position or momentum $[x^i, x^j] = [p^i, p^j] = 0$.

In other words \hat{x}^i, \hat{p}^i satisfy the Heisenberg algebra, which is their defining property. A key element, therefore, in the quantization of a classical system is to find a unitary representation of the Heisenberg algebra on the Hilbert space \mathcal{H} .

1.2.1 Non-relativistic Quantum Mechanics

The description above is not yet complete without an equation describing dynamics. Non-relativistically this is the Schrödinger equation:

$$i \frac{\partial}{\partial t} |\psi, t\rangle = \hat{H} |\psi, t\rangle. \quad (1.2.1)$$

In the case of a particle interacting with a static electromagnetic field the Hamiltonian, from minimal substitution, is $\hat{H} = \frac{(\hat{p} - q\vec{A}(\hat{x}))^2}{2m} + qV(\hat{x})$. This yields the time dependent interacting Schrödinger equation:

$$i \frac{\partial}{\partial t} |\psi, t\rangle = \frac{(\hat{p} - q\vec{A}(\hat{x}))^2}{2m} |\psi, t\rangle + qV(\hat{x}) |\psi, t\rangle. \quad (1.2.2)$$

Multiply from the left by $\langle \vec{x} |$ to express the time dependent Schrödinger equation in the position basis:

$$i \frac{\partial}{\partial t} \psi(\vec{x}, t) = \left(\frac{(\vec{\partial}_x - q\vec{A}(\vec{x}))^2}{2m} + qV(\vec{x}) \right) \psi(\vec{x}, t). \quad (1.2.3)$$

If the Hamiltonian is not explicitly time dependent we can separate variables by writing $\psi(x, t) = e^{-iEt} \psi(x)$ to obtain the time independent Schrödinger equation:

$$E\psi(\vec{x}) = \left(\frac{(\vec{\partial}_x - q\vec{A}(\vec{x}))^2}{2m} + qV(\vec{x}) \right) \psi(\vec{x}). \quad (1.2.4)$$

This is an eigenvalue equation for functions $\psi(\vec{x})$. So for a particular V, A we solve for the allowed $\psi(\vec{x})$, which in turn gives the physical information i.e allowed energies and momenta. The inner product is $\int d^3x \psi^*(\vec{x}) \psi(\vec{x})$ and the Hilbert space is now L^2 , the square integrable functions, as required for a sensible probabilistic interpretation. $\psi(\vec{x})$ is a function $\psi(\vec{x}) : \mathbb{R}^3 \rightarrow \mathbb{C}$. \mathbb{R}^3 is called the configuration space, which is also a Hilbert space, the space that the physical system occupies. There exist states which are not in this Hilbert space but are limit points of this space, e.g. free particle solutions. These are known as scattering states and extra care must be taken when making a probabilistic interpretation, however they pose no real problem.

We have a unitary representation of the Heisenberg algebra $[x, -i\partial_x] = i$ on the space L^2 , and from the Stone-von Neumann theorem we know this is a unique representation upto unitary transformations.

How do we extract the physical information from the states of a system. Consider the case when the system is in a mixed state, i.e. the system has the probability p_i of being in a state ψ_i . We can then define the density matrix:

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle \langle \psi_i|. \quad (1.2.5)$$

If the system is known to be in a state ψ_n (a pure state) then the density matrix is simply:

$$\hat{\rho} = |\psi_n\rangle \langle \psi_n|. \quad (1.2.6)$$

The allowed values of a measurement of an observable A , represented by a hermitian operator \hat{A} are the eigenvalues $a_n : \hat{A}|a_n\rangle = a_n|a_n\rangle$. Then the probability $P(a_n)$ of obtaining the value a_n when measuring the observable A for a system with a density matrix $\hat{\rho}$ is:

$$P(a_n) = \langle a_n | \hat{\rho} | a_n \rangle. \quad (1.2.7)$$

The density matrix contains all the probability distributions $P(\cdot)$ of the possible observables. The expectation values are obtained by:

$$\langle A \rangle = \text{Tr}(\hat{\rho}\hat{A}). \quad (1.2.8)$$

1.2.2 Relativistic Quantum Mechanics

In the relativistic case the Schrödinger equation is replaced by a different dynamical equation. In the case of spinless particles the corresponding equation is the Klein-Gordon equation, which for the free particle is given by:

$$(\partial^\mu \partial_\mu - m^2)\psi(t, \vec{x}) = 0. \quad (1.2.9)$$

When electromagnetic interactions are included the corresponding equation, obtained from minimal substitution, is given by:

$$((\partial - qA(x^\nu))^\mu (\partial - qA(x^\nu))_\mu - m^2)\psi(t, \vec{x}) = 0. \quad (1.2.10)$$

A is now the real four potential. In this thesis we don't focus on this equation, since we rather want to investigate matter particles that have spin, i.e. fermions.

The Klein-Gordon equation is second order in time. To obtain a first order equation in time and simultaneously preserve Lorentz covariance, Dirac factorized [24] the Klein-Gordon equation, using gamma matrices, to obtain the Dirac equation which reads for particles with spin:

$$(i\cancel{\partial} - m)\psi(t, \vec{x}) = 0. \quad (1.2.11)$$

When electromagnetic interactions are included the corresponding equation, again obtained through minimal substitution, reads:

$$(i\cancel{\partial} - q\cancel{A} - m)\psi(t, \vec{x}) = 0. \quad (1.2.12)$$

These equations describe the dynamics of spin $\frac{1}{2}$ fermions, and is the main focus of the thesis.

Due to problematic features, such as negative norms for the solutions of the Klein-Gordon equation and negative energies in both the Klein-Gordon and Dirac theories, the single particle probabilistic interpretation of ψ fails. However the charge is given by $Q = e \int d^3x \psi^\dagger \psi$. A proper interpretation, as fields, can be obtained when we consider many-particle systems i.e. quantum field theories.

1.3 Non-commutative Non-relativistic Quantum Mechanics

In this section we review non-commutative non-relativistic quantum mechanics specifically focusing on the two dimensional case, since the two dimensional commutation relations are easily extended to four. In three dimensions it is necessary to modify the commutation relations, to restore rotational symmetry but we don't deal with this in this thesis as this can be found in [18].

As motivated earlier a minimum length should be introduced into quantum mechanics. One way to do this, and probably the only consistent way as compellingly argued in [2], is to adopt an uncertainty relation for coordinates. The simplest possibility is:

$$\Delta x^i \Delta x^j > \frac{|\epsilon^{ij}\theta|}{2}, \quad (1.3.1)$$

where θ is related to the minimum length scale. The way to implement these uncertainty relations is to adopt non-commutative coordinates, which in the simplest case considered here reads:

$$[x^i, x^j] = i\epsilon^{ij}\theta. \quad (1.3.2)$$

Once this has been done, coordinates become operators and one needs to carefully consider how to formulate quantum mechanics on such spaces. In [17] this problem was considered and it was shown that the standard quantum mechanical formalism and interpretation carries over unaltered to these spaces. The only modification required is that one must consider more general Hilbert spaces in which to represent the states of the system and find the unitary representations of the non-commutative Heisenberg algebra. Additionally, since the coordinates do not commute and one cannot measure them simultaneously

due to the uncertainty relations, one has to consider weak, rather than strong, measurements when position is measured. We now proceed to review this formalism briefly as it will form the basis of later developments.

The first step in this formalism is to find a unitary representation of the non-commutative coordinate algebra (1.3.2) on a Hilbert space \mathcal{H}_c , c for configuration space or classical Hilbert space to distinguish it from the Hilbert space of quantum states or quantum Hilbert space. We use the angular ket for states in this space $|n\rangle \in \mathcal{H}_c$. For the two-dimensional non-commutative space (1.3.2) one can identify the most natural configuration space and representation of (1.3.2) by introducing a pair of boson creation and annihilation operators,

$$\hat{b} = \frac{1}{\sqrt{2\theta}}(\hat{x} + i\hat{y}), \hat{b}^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x} - i\hat{y}), \quad (1.3.3)$$

such that $[\hat{b}, \hat{b}^\dagger] = \mathbb{I}_c$. The configuration space is then simply the boson Fock space:

$$\mathcal{H}_c = \text{span}\{|n!\rangle = \frac{1}{\sqrt{n!}}(\hat{b}^\dagger)^n |0\rangle\}_{n=0}^{n=\infty}. \quad (1.3.4)$$

Now our states (wave-functions) are functions of the non-commutative coordinate operators $\psi(\hat{x}^1, \hat{x}^2) \equiv \hat{\psi}$. With this in mind, the most natural choice for the quantum Hilbert space, in which the states of the system are to be represented, is the space of Hilbert-Schmidt operators acting on the configuration space \mathcal{H}_c , i.e.:

$$\mathcal{H}_q = \{\hat{\psi} : \text{Tr}_c(\hat{\psi}^\dagger \hat{\psi}) < \infty\}. \quad (1.3.5)$$

We use a subscript q for quantum to distinguish this from the configuration space. The finite trace condition is a generalisation of square integrability. However we can still consider scattering theory [25], in the same way as commutatively by considering functions outside this Hilbert space (free particle solutions), these are limits points of \mathcal{H}_q . We use the rounded ket to represent these states $\hat{\psi} \equiv |\psi\rangle$ and reserve the symbol \dagger specifically for the adjoint on \mathcal{H}_c . The natural inner product on this space is:

$$(\hat{\phi}, \hat{\psi}) = \text{Tr}_c(\hat{\phi}^\dagger \hat{\psi}). \quad (1.3.6)$$

Once again we have linear functionals and a dual space which are defined in the same way as commutatively, but now using the trace inner product.

The next step is to introduce a unitary representation of the non-commutative Heisenberg algebra (1.3.2) on the quantum Hilbert space. For this purpose we introduce the operators \hat{X}^i, \hat{P}^i and use capital letters to distinguish them from (classical) operators acting on configuration space. These are often called super operators since they operate on operators. We use the symbol \ddagger for the adjoint on \mathcal{H}_q . We can find a unitary representation of the non-commutative

Heisenberg algebra:

$$\begin{aligned} [\hat{X}^i, \hat{X}^j] &= i\epsilon^{ij}\theta, \\ [\hat{X}^i, \hat{P}^j] &= i\delta^{ij}, \\ [\hat{P}^i, \hat{P}^j] &= 0, \end{aligned} \quad (1.3.7)$$

in the following way [17]:

$$\begin{aligned} \hat{X}^i\hat{\psi} &= \hat{x}^i\hat{\psi}, \\ \hat{P}^i\hat{\psi} &= \frac{1}{\theta}\epsilon^{ij}[\hat{x}^j, \hat{\psi}]. \end{aligned} \quad (1.3.8)$$

The non-commutative time independent Schrödinger equation is given then, for a free particle:

$$E\hat{\psi} = \left(\frac{(\hat{P}^i)^2}{2m} \right) \hat{\psi}. \quad (1.3.9)$$

Electromagnetic interactions can be introduced as in the commutative case by minimal substitution:

$$E\hat{\psi} = \left(\frac{(\hat{P}^i - q\vec{A}(\hat{X}^i))^2}{2m} + qV(\hat{X}^i) \right) \hat{\psi}. \quad (1.3.10)$$

The formalism outlined above is still a general operator construction. It can be reformulated in terms of functions in a way similar to the Wigner phase space construction of quantum mechanics using the Moyal product [26]. The essence of this construction is to introduce an invertible map (similar to the Wigner-Weyl transform) from operators to functions, i.e. $\Phi : f(\hat{x}^i) \rightarrow f(x^i)$, e.g. the symbol that appears in chapter five or the Seiberg-Witten map [4], but with a modified product rule, denoted by $*$, such that Φ is a homomorphism i.e.,

$$\Phi(f(\hat{x}^i)) * \Phi(g(\hat{x}^i)) = \Phi(f(\hat{x}^i)g(\hat{x}^i)). \quad (1.3.11)$$

A star product that implements this homomorphism in the case of the simplest non-commutative coordinate algebra 1.3.2 is the Moyal product given by:

$$* = e^{\frac{i}{2}\theta\epsilon^{ij}\overleftarrow{\partial}^i\overrightarrow{\partial}^j}. \quad (1.3.12)$$

The homomorphism is established by noting:

$$x^i * x^j - x^j * x^i = i\epsilon^{ij}\theta. \quad (1.3.13)$$

Then it is possible to write only objects f which depend on the real numbers x^i , but all regular products become star products. However these commuting coordinates cannot be interpreted as true physical coordinates and, indeed they have no clear physical meaning. In addition it should be noted that the star

product that implements the homomorphism is not unique, but that there are infinitely many. Another star product commonly used is the Voros product:

$$* = e^{\frac{i}{2}\theta\epsilon^{ij}\overleftarrow{\partial^i}\overrightarrow{\partial^j} + \frac{1}{2}\theta\overleftarrow{\partial^i}\overrightarrow{\partial^i}}. \quad (1.3.14)$$

Mathematically these formulations are of course equivalent, but there are subtle issues when it comes to the physical interpretation. A discussion of this can be found in [27], in particular it is noted that the allowed functions that can be composed with these products represent different subspaces of L^2 . Additionally the coordinates in the Moyal functions are combinations of position and momentum observables, so are points in phase space, thus cannot be interpreted as coordinates in position space. These issues become particularly relevant when trying to write an action or other physical quantities in some basis, in such a way to resemble commutative physics.

The formalism of non-commutative quantum mechanics follows in the same way as commutatively the only difference being the choice of Hilbert spaces and, of course, the Heisenberg algebra is modified.

Chapter 2

Relativistic Non-commutative Quantum Mechanics

In this chapter we generalize to the relativistic case of non-commutative quantum mechanics. We follow closely the 2 dimensional formulation of non-relativistic non-commutative quantum mechanics. Once the appropriate Hilbert space in which the physical states of the system are to be represented has been identified, we study the representation of the Lorentz group on this space by computing the generators. Using these, the transformation properties of various dynamical quantities, such as position and momenta, are computed and the conditions under which the Dirac equation is covariant are verified.

2.1 Formulation

First we consider a four dimensional space-time, so we now have four coordinate operators \hat{x}^μ and our states(wave-functions) depend on all four $\psi(\hat{x}^\mu) \equiv \hat{\psi}$, which satisfy:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}. \quad (2.1.1)$$

We now have a time coordinate operator $\hat{x}^0 = \hat{t}$. This is necessary since we consider Lorentz transformations, which mix both space and time coordinates. This is not unusual since when we consider commutative relativistic field theories we are forced to either elevate time to an operator or demote position to a parameter, these are equivalent [28]. There have been claims that time non-commutativity produces non-unitary field theories [29]. However, it has been shown that this was due to improper definitions of non-commutative field theories [30]. In this thesis we leave the form of the θ matrix general, its only property being the obvious antisymmetry. It is likely however that $\theta^{12} = \theta^{13} = \theta^{23}$, i.e. all the spatial dimensions have the same minimum length scale, and also $\theta^{01} = \theta^{02} = \theta^{03}$ i.e. the time-space uncertainty is the same in every direction as there is no reason to pick a preferential direction for the intrinsic non-commutativity of space-time. However, if the non-commutativity

arises as an effective theory, e.g. quantum hall systems [38], this may not be the case.

The configuration space is the space which carries a unitary representation of the coordinate operators. To find an explicit construction we simplify by performing a common coordinate transformation (also found in [31] for three dimensions). Consider a transformation $\hat{x}^\mu \rightarrow \hat{\tilde{x}}^\mu$ such that:

$$[\hat{\tilde{x}}^\mu, \hat{\tilde{x}}^\nu] = \tilde{\theta}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \tilde{\theta}^{03} \\ 0 & 0 & \tilde{\theta}^{12} & 0 \\ 0 & -\tilde{\theta}^{12} & 0 & 0 \\ -\tilde{\theta}^{03} & 0 & 0 & 0 \end{pmatrix}, (\tilde{\theta}^{-1})^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\tilde{\theta}^{03}} \\ 0 & 0 & -\frac{1}{\tilde{\theta}^{12}} & 0 \\ 0 & \frac{1}{\tilde{\theta}^{12}} & 0 & 0 \\ \frac{1}{\tilde{\theta}^{03}} & 0 & 0 & 0 \end{pmatrix}. \quad (2.1.2)$$

Now there are 2 sets of non-commuting operators $\hat{\tilde{t}}, \hat{\tilde{z}}$ and $\hat{\tilde{x}}, \hat{\tilde{y}}$ but these two sets commute with each other. It is now possible to define a creation and an annihilation operator:

$$\hat{b} = \frac{1}{\sqrt{2\theta^{12}}}(\hat{\tilde{x}} + i\hat{\tilde{y}}), \hat{b}^\dagger = \frac{1}{\sqrt{2\theta^{12}}}(\hat{\tilde{x}} - i\hat{\tilde{y}}), \quad (2.1.3)$$

we can also define a second set of creation and annihilation operators:

$$\hat{a} = \frac{1}{\sqrt{2\theta^{03}}}(\hat{\tilde{t}} + i\hat{\tilde{z}}), \hat{a}^\dagger = \frac{1}{\sqrt{2\theta^{03}}}(\hat{\tilde{t}} - i\hat{\tilde{z}}). \quad (2.1.4)$$

The configuration space is then simply the two boson Fock space:

$$\mathcal{H}_c = \text{span}\{|n, m\rangle = \frac{1}{\sqrt{n!m!}}(a^\dagger)^n(b^\dagger)^m|0, 0\rangle\}_{n,m=0}^{n,m=\infty}. \quad (2.1.5)$$

Now that the configuration space has been defined, we can introduce, as before, the notion of functions on this space. In general functions on the configuration space are simply the operators generated by the coordinate operators and, since the configuration space carries a representation of the coordinate algebra, these functions (operators) act, by default, on configuration space.

In the formulation of non-relativistic non-commutative quantum mechanics, we have to restrict the class of functions to the class of Hilbert-Schmidt operators in order for the standard probability interpretation of quantum mechanics to hold.

In relativistic quantum mechanics the standard probability interpretation does not apply on the first quantized level. Even in non-relativistic quantum mechanics, one does not require square integrability with respect to integration over time. In what follows we shall however impose the condition that the functions we consider are Hilbert-Schmidt. Although this is not a physically motivated requirement in the relativistic case, this condition is imposed for technical reasons as it ensures that the mathematical manipulations we carry

out are well defined. We therefore only consider the space of functions on configuration space that are Hilbert-Schmidt i.e.:

$$\mathcal{H}_q = \{\hat{\psi} : \text{Tr}_c(\hat{\psi}^\dagger \hat{\psi}) < \infty\}. \quad (2.1.6)$$

This space can again be endowed with the standard inner product to turn it into a Hilbert space. We use the same notation, \mathcal{H}_q , as before to denote this space, although it physically has a different status. We note that the solutions of the Dirac equation will be a spinor of Hilbert-Schmidt operators which act on states in the configuration space as follows:

$$\hat{\psi} |z\rangle = \begin{pmatrix} \hat{\psi}_1 |z\rangle \\ \hat{\psi}_2 |z\rangle \\ \hat{\psi}_3 |z\rangle \\ \hat{\psi}_4 |z\rangle \end{pmatrix}, \quad (2.1.7)$$

this is the same as for commutative spinors:

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}. \quad (2.1.8)$$

In relativistic field theory we do not consider the Heisenberg algebra of coordinates and momenta as coordinates merely have the status of labels rather than physical observables. Instead, one requires equal time (anti-)commutation relations of the fields and their canonically conjugates.

However, to generalize the commutative Klein-Gordon and Dirac equations to the non-commutative case, we need the notion of a derivative of a function, which now has to be defined in a purely algebraic fashion.

Algebraically the operation with all the properties of a derivative is commutation. We therefore define the derivative of a functions as follows:

$$\hat{\partial}_\mu \hat{\psi} = -i(\theta^{-1})_{\mu\nu} [\hat{x}^\nu, \hat{\psi}]. \quad (2.1.9)$$

The formal correspondence of this definition with a derivative also follows by noting from the well known identity $[A, f(B)] = cf'(B)$ if $[A, B] = cI$, that (2.1.9) formally amounts to taking the operator derivative $\frac{\partial}{\partial \hat{x}^\mu}$.

From the Jacobi identity it follows that the derivatives commute as usual. In addition the following properties are readily verified:

$$\hat{\partial}^\mu \hat{x}_\rho = -i(\theta^{-1})^{\mu\nu} i\theta_{\nu\rho} = \delta_\rho^\mu, \quad (2.1.10)$$

from this we can see $[\hat{x}^\mu, \hat{\partial}^\nu] = -\delta_\nu^\mu$ so we know from the Stone-von Neumann theorem this representation of the derivative is unique upto unitary transformations,

$$\hat{\partial}_\mu(\hat{\psi}\hat{\phi}) = -i(\theta^{-1})^{\mu\nu}(\hat{\psi} [x_\nu, \hat{\phi}] + [x_\nu, \hat{\psi}] \hat{\phi}) = \hat{\psi}(\hat{\partial}_\mu \hat{\phi}) + (\hat{\partial}_\mu \hat{\psi})\hat{\phi}. \quad (2.1.11)$$

With these notations in place, the free commutative Klein-Gordon equation,

$$(\partial^\rho \partial_\rho + m^2)\hat{\psi} = 0, \quad (2.1.12)$$

becomes:

$$(\theta^{-1})^{\mu\rho}(\theta^{-1})_{\rho\nu} \left[\hat{x}^\nu, \left[\hat{x}_\mu, \hat{\psi} \right] \right] + m^2 \hat{\psi} = 0. \quad (2.1.13)$$

Similarly the free commutative Dirac equation,

$$(i\hat{\not{\partial}} - m)\hat{\psi} = 0, \quad (2.1.14)$$

becomes:

$$\gamma^\mu (\theta^{-1})_{\mu\nu} \left[\hat{x}^\nu, \hat{\psi} \right] - m\hat{\psi} = 0. \quad (2.1.15)$$

Another useful property follows if we consider the trace of a derivative of a Hilbert Schmidt operator. Then using (2.1.6):

$$\begin{aligned} \text{Tr}(\hat{\partial}_\mu \hat{\psi}) &= \text{Tr}(-i(\theta^{-1})_{\mu\nu} \left[\hat{x}^\nu, \hat{\psi} \right]) \\ &= -i(\theta^{-1})_{\mu\nu} \left(\text{Tr}(\hat{x}^\nu \hat{\psi}) - \text{Tr}(\hat{\psi} \hat{x}^\nu) \right) \\ &= -i(\theta^{-1})_{\mu\nu} \left(\text{Tr}(\hat{x}^\nu \hat{\psi}) - \text{Tr}(\hat{x}^\nu \hat{\psi}) \right) \\ &= 0. \end{aligned} \quad (2.1.16)$$

The commutative analogue of this is to integrate a total derivative. If the function vanishes at infinity, this integral also vanishes. This also clarifies the role of the condition (2.1.6) we imposed on the functions, namely it corresponds to the requirement that the functions we consider vanish at infinity. With this result in place we can also derive the non-commutative analogue of partial integration:

$$\text{Tr} \left((\hat{\partial}_\mu \hat{\psi}^\dagger) \hat{\phi} \right) = - \text{Tr} \left(\hat{\psi}^\dagger (\hat{\partial}_\mu \hat{\phi}) \right). \quad (2.1.17)$$

Note again the absence of boundary terms due to the condition (2.1.6). It follows easily from this that $i\hat{\partial}_\mu$ is hermitian.

2.2 Lorentz Symmetry

Now that we have identified the space of operator functions and defined derivatives on it, we can investigate the representations of the Lorentz group carried by this space. Consider an infinitesimal spatial Lorentz transformation parametrized by $\omega^{\mu\nu}$. Adopting the same transformation properties for the non-commutative coordinates as for the commutative ones, we have:

$$\hat{x}^\mu \rightarrow \hat{x}^\mu + \omega_\nu^\mu \hat{x}^\nu. \quad (2.2.1)$$

Noting $\omega^{\mu\alpha}g_{\alpha\nu} = \omega_\nu^\mu$, where $\omega^{\mu\alpha}$ is an antisymmetric tensor, in other words the superscript is always the first index of the tensor and the subscript always the second index of the tensor. Now consider a Lorentz scalar function transforming:

$$\psi(\hat{x}^\mu) \rightarrow \psi(\hat{x}^\mu + \omega_\nu^\mu \hat{x}^\nu). \quad (2.2.2)$$

To write this as a differential operator acting on $\hat{\psi}$, consider the Weyl representation [19]:

$$\hat{\psi} = \int d^4k \phi(k) e^{ik^\mu \hat{x}_\mu} : k \in \mathbb{R}. \quad (2.2.3)$$

Then:

$$\begin{aligned} \psi(\hat{x}^\mu + \omega_\nu^\mu \hat{x}^\nu) &= \int d^4k \phi(k) e^{ik_\mu(\hat{x}^\mu + \omega_\nu^\mu \hat{x}^\nu)} \\ &= \int d^4k \phi(k) e^{ik_\mu \omega_\nu^\mu \hat{x}^\nu} e^{ik_\mu \hat{x}^\mu} e^{\frac{i}{2} k_\mu k_\lambda \omega_\nu^\lambda [\hat{x}^\nu, \hat{x}^\mu]} \\ &= \int d^4k \phi(k) (1 + ik_\mu \omega_\nu^\mu \hat{x}^\nu) (1 + \frac{i}{2} k_\mu k_\lambda \omega_\nu^\lambda \theta^{\nu\mu}) e^{ik_\mu \hat{x}^\mu} + O(\omega^2) \\ &= \int d^4k \phi(k) (1 + ik_\mu \omega_\nu^\mu \hat{x}^\nu + \frac{i}{2} k_\mu k_\lambda \omega_\nu^\lambda \theta^{\nu\mu}) e^{ik_\mu \hat{x}^\mu} + O(\omega^2) \\ &= \int d^4k \phi(k) (1 + \omega_\nu^\mu \hat{x}^\nu \hat{\partial}_\mu - \frac{i}{2} \omega_\nu^\lambda \theta^{\nu\mu} \hat{\partial}_\mu \hat{\partial}_\lambda) e^{ik_\mu \hat{x}^\mu}. \end{aligned} \quad (2.2.4)$$

The generators of the non-commutative Lorentz transformations are therefore given by [32]:

$$\hat{J}_\omega = \hat{x}^\nu \omega_\nu^\mu \hat{\partial}_\mu - \frac{i}{2} \omega_\nu^\lambda \theta^{\nu\mu} \hat{\partial}_\mu \hat{\partial}_\lambda. \quad (2.2.5)$$

Since:

$$[\hat{J}_\omega, \hat{J}_{\omega'}] = \hat{J}_{\omega \times \omega'}, \quad (2.2.6)$$

where $\omega \times \omega' = \omega_\nu^\rho \omega'^\mu - \omega'^\rho \omega_\nu^\mu$, this satisfies the standard commutation relations of the Lorentz algebra and therefore constitute a valid representation, but in this case on a different Hilbert space. The commutation relations (2.2.6) are explicitly verified in appendix (A). The coordinates transform in the expected way, like covariant 4 vectors in commutative space:

$$\hat{J}_\omega \hat{x}^\mu = \omega_\nu^\mu \hat{x}^\nu, \quad (2.2.7)$$

also $\hat{\partial}^\mu$ transform like covariant 4 vectors since:

$$[\hat{J}_\omega, \hat{\partial}^\mu] = \omega_\nu^\mu \hat{\partial}^\nu. \quad (2.2.8)$$

2.3 The Free Non-commutative Dirac Equation

The free non-commutative Dirac equation is now known to be Lorentz covariant since from (2.2.8) we know this has the same transformation property as the free commutative equation, the spin transformation will be discussed in the next chapter, however we note this remains unmodified in non-commutative space. The form of the free equation:

$$(i\hat{\mathcal{D}} - m)\hat{\psi} = 0. \quad (2.3.1)$$

Inserting the operator valued versions of the commutative solutions in the non-commutative Dirac equation, for positive energy, where the χ are now eigenvectors of σ^3 , with eigenvalues $s = \pm 1$:

$$\begin{aligned} (i\hat{\mathcal{D}} - m)e^{-ik^\mu \hat{x}_\mu} \frac{\not{k} + m}{\sqrt{2m(m + k^0)}} \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} &= e^{-ik^\mu \hat{x}_\mu} \frac{(\not{k} - m)(\not{k} + m)}{\sqrt{2m(m + k^0)}} \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} \\ &= 0. \end{aligned} \quad (2.3.2)$$

This gives the same condition as in the commutative case $(\not{k} - m)(\not{k} + m) = 0$, i.e.,

$$(k^0)^2 = E^2 = \vec{k}^2 + m^2. \quad (2.3.3)$$

This is the desired dispersion relation. This outcome has already been obtained for many non-commutative theories, that is the free solutions retain the same form as commutatively and the spectrum is the same, the 2-D Schrödinger case can be found in [17]. A different, but also valid solution has the exponential $e^{-ik^0 \hat{t}} e^{-ik^1 \hat{x}} e^{-ik^2 \hat{y}} e^{-ik^3 \hat{z}}$, and its various orderings are also solutions of (2.3.2) only differing by a global phase. The anti-particle solutions can also easily be constructed, in particular via the use of the *CPT* transformation which will be discussed in the next chapter.

We now have the relativistic formulation for the free particle, and it remains to included interactions. Also we can see that for free equations there is no physical difference between commutative space and non-commutative space.

Chapter 3

The Non-commutative Dirac Equation With Electromagnetic Interactions

In this chapter we include electromagnetic interactions in the non-commutative Dirac equation. However it turns out this is not simply a case of using the same approach as commutatively as this breaks Lorentz symmetry. We show that this can be rectified by the introduction of a modified composition rule for potentials and states. Then non-commutative analogues for C, P, T operations can be found and applied to the interacting equation to study the non-commutative transformation properties. As a specific application we also solve the Dirac equation in a constant background magnetic field, and study it in an approximately restricted volume.

3.1 Twisted Lorentz Symmetry

In the previous chapter we found the non-commutative generators of Lorentz transformations. The transformation properties of the free non-commutative Dirac equation was the same as commutatively. However, products of functions do not transform in the same way as commutatively since:

$$[\hat{J}_\omega, \hat{x}^\rho] = \omega_\nu^\rho \hat{x}^\nu - \frac{i}{2}(\theta^{\nu\rho} \omega_\nu^\mu - \theta^{\nu\mu} \omega_\nu^\rho) \hat{\partial}_\mu. \quad (3.1.1)$$

When we introduce interactions this becomes relevant as we multiply states with potentials. Furthermore,

$$\hat{J}_\omega \hat{\psi} \hat{\phi} = (\hat{J}_\omega \hat{\psi}) \hat{\phi} + \hat{\psi} (\hat{J}_\omega \hat{\phi}) - \frac{i}{2}(\theta^{\nu\rho} \omega_\nu^\mu - \theta^{\nu\mu} \omega_\nu^\rho) (\hat{\partial}_\mu \hat{\psi}) (\hat{\partial}_\rho \hat{\phi}). \quad (3.1.2)$$

While commutatively,

$$\hat{J}_\omega \psi \phi = (\hat{J}_\omega \psi) \phi + \psi (\hat{J}_\omega \phi). \quad (3.1.3)$$

Thus the Leibniz rule does not hold for the standard product of operator functions. So things which were Lorentz covariant in the commutative case, will not be in the non-commutative case if we simply multiply two functions together. It is necessary to combine functions in a different way to restore Lorentz covariance. Consider products of the form:

$$\hat{\psi} \hat{*} \hat{\phi} \quad \text{where} \quad \hat{*} = e^{-\frac{i}{2} \theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu}. \quad (3.1.4)$$

This is similar to the $*$ star product used to combine ordinary functions to induce non-commutativity, as discussed previously in two dimensions. However this differs by a sign and includes operator derivatives. Also this product is introduced for a different reason, not to induce non-commutativity, but to preserve Lorentz symmetry, it is as if one is undoing the regular star product. Combining functions in this way preserves Lorentz covariance as we verify in (B).

$$\hat{J}_\omega(\hat{\psi} \hat{*} \hat{\phi}) = (\hat{J}_\omega \hat{\psi}) \hat{*} \hat{\phi} + \hat{\psi} \hat{*} (\hat{J}_\omega \hat{\phi}). \quad (3.1.5)$$

Objects combined with the $\hat{*}$ do indeed transform in the correct way, as the \hat{J}_ω obey a Leibniz's rule, like in commutative space. We can now write a Lorentz covariant non-commutative version of the Dirac equation using $\hat{*}$.

3.1.1 Tensor Method

A slightly different approach to think about how to fix the Lorentz invariance of the products of operators $\hat{\phi}, \hat{\psi}$ is to look at the multiplication map m which usually is:

$$m(\hat{\phi} \otimes \hat{\psi}) |n\rangle = \hat{\phi}(\hat{\psi} |n\rangle), \quad (3.1.6)$$

i.e. the usual operator product, where $|n\rangle$ is in the configuration space. Then the action of \hat{J}_ω is given by, looking at equation (3.1.2) :

$$\begin{aligned} \hat{J}_\omega m(\hat{\phi} \otimes \hat{\psi}) &= m((\hat{J}_\omega \hat{\phi}) \otimes \hat{\psi}) + m(\hat{\phi} \otimes (\hat{J}_\omega \hat{\psi})) + \\ &\quad \frac{i}{2} (\theta^{\nu\rho} \omega_\nu^\mu - \theta^{\nu\mu} \omega_\nu^\rho) m((\hat{\partial}_\mu \hat{\psi}) \otimes (\hat{\partial}_\rho \hat{\phi})) \\ &= m((\hat{J}_\omega \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{J}_\omega + \frac{i}{2} (\theta^{\nu\rho} \omega_\nu^\mu - \theta^{\nu\mu} \omega_\nu^\rho) (\hat{\partial}_\mu \otimes \hat{\partial}_\rho)) (\hat{\phi} \otimes \hat{\psi})). \end{aligned} \quad (3.1.7)$$

We call this action on the tensor products the co-product, also found in [32] and a three dimensional discussion [31]:

$$\Delta_\theta(\hat{J}_\omega) = \hat{J}_\omega \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{J}_\omega + \frac{i}{2} (\theta^{\nu\rho} \omega_\nu^\mu - \theta^{\nu\mu} \omega_\nu^\rho) (\hat{\partial}_\mu \otimes \hat{\partial}_\rho). \quad (3.1.8)$$

This co-product is called deformed, since commutatively the co-product is:

$$\Delta_0(\hat{J}_\omega) = \hat{J}_\omega \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{J}_\omega. \quad (3.1.9)$$

Using this we can write:

$$\Delta_\theta(\hat{J}_\omega) = F^\dagger \Delta_0(\hat{J}_\omega) F, \quad (3.1.10)$$

where:

$$F = e^{\frac{i}{2}\theta^{\mu\nu}\hat{\partial}_\mu\otimes\hat{\partial}_\nu}. \quad (3.1.11)$$

F is clearly unitary. Now in order to preserve Lorentz invariance in light of this deformed co-product we must also deform the multiplication map $m \rightarrow m_\theta$, try:

$$\begin{aligned} m_\theta(\hat{\phi} \otimes \hat{\psi}) &= m(e^{-\frac{i}{2}\theta^{\mu\nu}\hat{\partial}_\mu\otimes\hat{\partial}_\nu}(\hat{\phi} \otimes \hat{\psi})) \\ &= m(F^\dagger(\hat{\phi} \otimes \hat{\psi})). \end{aligned} \quad (3.1.12)$$

Then applying Δ_θ :

$$\begin{aligned} \hat{J}_\omega m_\theta(\hat{\psi} \otimes \hat{\phi}) &= m(\Delta_\theta(\hat{J}_\omega)F^\dagger(\hat{\phi} \otimes \hat{\psi})) \\ &= m(F^\dagger\Delta_0(\hat{J}_\omega)(\hat{\phi} \otimes \hat{\psi})) \\ &= m_\theta(\Delta_0(\hat{J}_\omega)(\hat{\phi} \otimes \hat{\psi})). \end{aligned} \quad (3.1.13)$$

So operators combined using m_θ transform via the undeformed co-product. However it is clearly equivalent to combine objects via m_θ or $\hat{\star}$.

3.2 Electromagnetic Interactions

The general form of the non-commutative Dirac equation coupled to a Lorentz covariant electro-magnetic field is given by:

$$(i\hat{\mathcal{D}} - q\hat{A}\hat{\star} - m)\hat{\psi} = 0. \quad (3.2.1)$$

Here q is the charge, and \hat{A} is the real four potential i.e. $\hat{A} = \hat{A}^\dagger$, the same choice that is made commutatively. This equation differs from the commutative case since it is assumed that the \hat{A} couples via the $\hat{\star}$ product otherwise this equation would not transform covariantly under Lorentz transformations. This makes it technically more difficult to solve this equation, in particular some potentials will become highly non-local like $\frac{1}{r}$, where the $\hat{\star}$ generates derivatives of all orders, fortunately the higher order terms become increasingly small like θ^n . The full generators for Lorentz transformations for spinors are given by:

$$\hat{J}_{\omega\frac{1}{2}} = \hat{S}_\omega + \hat{J}_\omega. \quad (3.2.2)$$

Where $\hat{J}_\omega = \hat{x}^\nu\omega_\nu^\mu\hat{\partial}_\mu - \frac{i}{2}\omega_\nu^\lambda\theta^{\nu\mu}\hat{\partial}_\mu\hat{\partial}_\lambda$ this transforms the coordinates as previously calculated. $\hat{S}_\omega = \frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}$ gives the spinorial transformation properties of $\hat{\psi}$, which we take to be the same as commutatively since the spinor structure

is not changed by non-commutativity. Here $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. A commutative discussion can be found in [33]. To calculate the action of \hat{S}_ω , the following is needed:

$$\begin{aligned} [\omega_{\mu\nu}\sigma^{\mu\nu}, \gamma^\lambda] &= \omega_{\mu\nu}2i(g_\mu^\lambda\gamma_\nu - g_\nu^\lambda\gamma_\mu) \\ &= 4i(\omega^{\lambda\nu}\gamma_\nu). \end{aligned} \quad (3.2.3)$$

Transforming the Dirac equation, first by \hat{S}_ω :

$$\hat{S}_\omega(i\hat{\not{D}} - q\hat{A}\hat{*} - m)\hat{\psi} = (-i\omega_\nu^\mu\gamma^\nu\hat{\partial}_\mu + q\omega_\nu^\mu\gamma^\nu\hat{A}_\mu\hat{*})\hat{\psi} + (i\hat{\not{D}} - q\hat{A}\hat{*} - m)\hat{S}_\omega\hat{\psi}, \quad (3.2.4)$$

and then by \hat{J}_ω :

$$\begin{aligned} \hat{J}_\omega(i\hat{\not{D}} - q\hat{A}\hat{*} - m)\hat{\psi} &= (i\gamma^\mu\omega_\mu^\nu\hat{\partial}_\nu - q\gamma^\mu(\hat{J}_\omega\hat{A}_\mu)\hat{*})\hat{\psi} + (i\hat{\not{D}} - q\hat{A}\hat{*} - m)\hat{J}_\omega\hat{\psi} \\ &= (i\gamma^\mu\omega_\mu^\nu\hat{\partial}_\nu - q\gamma^\mu\omega_\mu^\nu\hat{A}_\nu\hat{*})\hat{\psi} + (i\hat{\not{D}} - q\hat{A}\hat{*} - m)\hat{J}_\omega\hat{\psi}, \end{aligned} \quad (3.2.5)$$

since \hat{A} transforms, by definition, like a 4-vector under coordinate transformations. Putting this together:

$$(\hat{S}_\omega + \hat{J}_\omega)(i\hat{\not{D}} - q\hat{A}\hat{*} - m)\hat{\psi} = (i\hat{\not{D}} - q\hat{A}\hat{*} - m)(\hat{S}_\omega + \hat{J}_\omega)\hat{\psi}. \quad (3.2.6)$$

The Dirac equation is therefore covariant as it transforms in the same way as $\hat{\psi}$. Note however the Dirac operator $(i\hat{\not{D}} - q\hat{A}\hat{*} - m)$ is Lorentz invariant. This implies that a Lorentz transformation transforms the solutions of the Dirac equation into each other. This provides an additional motivation for the $\hat{*}$ coupling, since otherwise the action of \hat{J}_ω would have produced an additional term as seen previously.

Some properties of $\hat{*}$ are listed in appendix (C). We use these often during our calculations, in particular we note $\hat{A}\hat{*}\hat{\psi} = \hat{\psi}\hat{*}\hat{A}$, so the star product effectively makes the four potential commutative, however, as we will see later this does not mean there are no more non-commutative effects to be observed.

3.3 Gauge Symmetry

Commutatively multiplication by a unitary operator (phase) doesn't change the inner product $\langle\psi|\psi\rangle$ and thus produces the same physics. Then the equations of motion must stay the same, which is one way to introduce the covariant derivative into the free commutative Dirac equation. What transformation produces the same effect non-commutatively, multiplication by unitary operators will not since these transformations break Lorentz symmetry. The transformation must be of the form:

$$\hat{\psi} \rightarrow \hat{U}\hat{*}\hat{\psi}. \quad (3.3.1)$$

What property must \hat{U} have to leave the inner product invariant, try:

$$\hat{U}^\dagger \hat{\star} \hat{U} = \mathbb{I}. \quad (3.3.2)$$

Then looking at the inner product, using (C):

$$\begin{aligned} \text{Tr}((\hat{\psi}^\dagger \hat{\star} \hat{U}^\dagger)(\hat{U} \hat{\star} \psi)) &= \text{Tr}((\hat{\psi}^\dagger \hat{\star} \hat{U}^\dagger) \hat{\star} (\hat{U} \hat{\star} \psi)) \\ &= \text{Tr}(\hat{\psi}^\dagger \hat{\star} ((\hat{U}^\dagger \hat{\star} \hat{U}) \hat{\star} \psi)) \\ &= \text{Tr}(\hat{\psi}^\dagger \hat{\star} (\mathbb{I}_q \hat{\star} \psi)) \\ &= \text{Tr}(\hat{\psi}^\dagger \hat{\star} \psi) \\ &= \text{Tr}(\hat{\psi}^\dagger \psi). \end{aligned} \quad (3.3.3)$$

This indeed leaves the inner product unchanged but \hat{U} has a modified unitarity condition, which involves $\hat{\star}$ instead of regular operator multiplication. Commutatively the A^μ also transforms to leave the Dirac equation invariant. Now what gauge transformation must \hat{A}^μ undergo to leave the non-commutative Dirac equation invariant. Try:

$$\hat{A} \rightarrow \hat{U} \hat{\star} \hat{A} \hat{\star} \hat{U}^\dagger + \frac{1}{q} i(\hat{\partial} \hat{U}) \hat{\star} \hat{U}^\dagger. \quad (3.3.4)$$

Now simultaneously transforming $\hat{\psi}$ and \hat{A} the Dirac equation becomes:

$$\begin{aligned} (i\hat{\partial} - q(\hat{U} \hat{\star} \hat{A} \hat{\star} \hat{U}^\dagger + \frac{1}{q} i(\hat{\partial} \hat{U}) \hat{\star} \hat{U}^\dagger) \hat{\star} - m)\hat{U} \hat{\star} \hat{\psi} &= 0 \\ \implies (i\hat{\partial} \hat{U}) \hat{\star} \hat{\psi} + \hat{U} \hat{\star} (i\hat{\partial} \hat{\psi}) - q\hat{U} \hat{\star} \hat{A} \hat{\star} \hat{U}^\dagger \hat{\star} \hat{U} \hat{\star} \hat{\psi} - i(\hat{\partial} \hat{U}) \hat{\star} \hat{U}^\dagger \hat{\star} \hat{U} \hat{\star} \hat{\psi} - m\hat{U} \hat{\star} \hat{\psi} &= 0 \\ \implies \hat{U} \hat{\star} ((i\hat{\partial} - q\hat{A} \hat{\star} - m)\hat{\psi}) &= 0 \\ \implies (i\hat{\partial} - q\hat{A} \hat{\star} - m)\hat{\psi} &= 0. \end{aligned} \quad (3.3.5)$$

This is a proper gauge transformation, however, not with unitary matrices but matrices that satisfy the modified condition (3.3.2).

3.4 Discrete Symmetries

We now consider the discrete Symmetries of the non-commutative Dirac equation, comparing to the commutative case. A commutative discussion can be found in [24]. A non-commutative discussion in the phase-space like formulation can be found in [34].

3.4.1 Parity

Commutatively the spatial parity transformation $\hat{P}^{(0)}$, is $x^i \rightarrow -x^i$ and $x^0 \rightarrow x^0$. The full Parity transformation for spinors is $\hat{P} = \gamma^0 \hat{P}^{(0)}$. We use the same

operator in the non-commutative case. For the free equation,

$$\hat{P}(i\hat{\partial}\psi(\hat{x}, \hat{t})) = \hat{P}(-i\gamma^\mu(\theta^{-1})_{\mu\nu} [\hat{x}^\nu, \psi(\hat{x}, \hat{t})]). \quad (3.4.1)$$

We can readily check what happens to $i\theta^{\mu\nu} = [\hat{x}^\mu, \hat{x}^\nu]$ under the $\hat{P}^{(0)}$ transformation,

$$\begin{pmatrix} 0 & \theta^{01} & \theta^{02} & \theta^{03} \\ -\theta^{01} & 0 & \theta^{12} & \theta^{13} \\ -\theta^{02} & -\theta^{12} & 0 & \theta^{23} \\ -\theta^{03} & -\theta^{13} & -\theta^{23} & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -\theta^{01} & -\theta^{02} & -\theta^{03} \\ \theta^{01} & 0 & \theta^{12} & \theta^{13} \\ \theta^{02} & -\theta^{12} & 0 & \theta^{23} \\ \theta^{03} & -\theta^{13} & -\theta^{23} & 0 \end{pmatrix}, \quad (3.4.2)$$

then for the inverse of theta $(\theta^{-1})^{\mu\nu}$,

$$c \begin{pmatrix} 0 & -\theta^{23} & \theta^{13} & -\theta^{12} \\ \theta^{23} & 0 & -\theta^{03} & \theta^{02} \\ -\theta^{13} & \theta^{03} & 0 & -\theta^{01} \\ \theta^{12} & -\theta^{02} & \theta^{01} & 0 \end{pmatrix} \rightarrow c \begin{pmatrix} 0 & \theta^{23} & -\theta^{13} & \theta^{12} \\ -\theta^{23} & 0 & -\theta^{03} & \theta^{02} \\ \theta^{13} & \theta^{03} & 0 & -\theta^{01} \\ -\theta^{12} & -\theta^{02} & \theta^{01} & 0 \end{pmatrix}, \quad (3.4.3)$$

where $c = \frac{1}{\theta^{01}\theta^{23} - \theta^{02}\theta^{13} + \theta^{03}\theta^{12}}$.

$$\begin{aligned} \hat{P}(i\hat{\partial}\psi(\hat{x}, \hat{t})) &= \gamma^0 \left(-i(i\gamma^0)(-\theta^{-1})_{0i} [-\hat{x}^i, \psi(-\hat{x}, \hat{t})] + \right. \\ & \left. i(i\gamma^i)(\theta^{-1})_{ij} [-\hat{x}^j, \psi(-\hat{x}, \hat{t})] + i(i\gamma^i)(-\theta^{-1})_{i0} [\hat{x}^0, \psi(-\hat{x}, \hat{t})] \right) \\ &= \gamma^0(-i(i\gamma^0)(\theta^{-1})_{0\nu} [\hat{x}^\nu, \psi(-\hat{x}, \hat{t})] + i(i\gamma^i)(\theta^{-1})_{i\nu} [\hat{x}^\nu, \psi(-\hat{x}, \hat{t})]) \\ &= (\gamma^0 i\gamma^0 \hat{\partial}_0 - \gamma^0 i\gamma^i \hat{\partial}_i) \psi(-\hat{x}, \hat{t}) \\ &= i\hat{\partial}\gamma^0 \psi(-\hat{x}, \hat{t}) \\ &= i\hat{\partial}\hat{P}\psi. \end{aligned} \quad (3.4.4)$$

We have that $[\hat{\partial}, \hat{P}] = 0$, parity is conserved in the free case. So we can label free states by their parity eigenvalues, even or odd, just like in commutative space. For interactions,

$$\begin{aligned} \hat{P}(-e\hat{A}(\hat{x}, \hat{t}) \hat{\star} \psi(\hat{x}, \hat{t})) &= -\gamma^0 e\hat{A}(-\hat{x}, \hat{t}) (\hat{P}^{(0)} \hat{\star}) \psi(-\hat{x}, \hat{t}) \\ &= \gamma^0 (-e\hat{A}(-\hat{x}, \hat{t}) e^{-\frac{i}{2} \left((-\theta^{0i}) \hat{\partial}_0 (-\hat{\partial}_i) + (-\theta^{i0}) (-\hat{\partial}_i) \hat{\partial}_0 + \theta^{ij} (-\hat{\partial}_i) (-\hat{\partial}_j) \right)}) \psi(-\hat{x}, \hat{t}) \\ &= -e(\gamma^0 A_0(-\hat{x}, \hat{t}) - \gamma^i A_i(-\hat{x}, \hat{t})) \hat{\star} \gamma^0 \psi(-\hat{x}, \hat{t}) \end{aligned} \quad (3.4.5)$$

this is exactly the same transformation as in the commutative case. So if we map any parity invariant gauge potential in commutative space A to \hat{A} in non-commutative space this will be parity invariant.

3.4.2 Charge Conjugation

Charge conjugation takes us from $q \rightarrow -q$. Commutatively the spatial charge conjugation transform was produced by complex conjugation $\hat{K}^{(0)}\psi = \psi^*$, the transformation for spinors is given by $\hat{C} = i\gamma^2\hat{K}^{(0)}$. The non-commutative analogue of complex conjugation, is the adjoint \dagger on the configuration space. However, we cannot simply take \dagger since we do not want to take the adjoint of the spinors, but only the adjoint of each component. Let $\hat{K}^0 = \hat{\Theta}\hat{\mathbb{L}}_4$, where

$$\hat{\Theta}\hat{\phi} = \hat{\phi}^\dagger, \text{ note } \hat{\Theta}\hat{\partial}^\mu\hat{\Theta}^{-1} = \hat{\partial}^\mu. \text{ Then } \hat{\Theta}\hat{\mathbb{L}}_4 \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \hat{\chi}_1 \\ \hat{\chi}_2 \end{pmatrix} = \begin{pmatrix} \hat{\phi}_1^\dagger \\ \hat{\phi}_2^\dagger \\ \hat{\chi}_1^\dagger \\ \hat{\chi}_2^\dagger \end{pmatrix} \text{ for now we will}$$

denote this by $\hat{\psi}^\dagger(x, y)$, and $\hat{\Theta}^2 = \hat{\mathbb{I}}_c$.

$$\begin{aligned} \hat{C}(i\hat{\not{\partial}}\psi(\hat{x}, \hat{t})) &= i\gamma^2(-i(\gamma^\mu)^*\hat{\partial}_\mu\psi^\dagger(\hat{x}, \hat{t})) \\ &= (i\hat{\not{\partial}}(i\gamma^2)\psi^\dagger(\hat{x}, \hat{t})) \\ &= i\hat{\not{\partial}}\hat{C}\psi. \end{aligned} \tag{3.4.6}$$

We have that $[\hat{\not{\partial}}, \hat{C}] = 0$, this is same transformation as in commutative space. Like in commutative space, non-commutative free particles and antiparticles evolve in the same way. For interactions:

$$\begin{aligned} \hat{C}(-e\hat{A}(\hat{x}, \hat{t})\hat{\star}\psi(\hat{x}, \hat{t})) &= +e\gamma^\rho(i\gamma^2)(A_\rho(\hat{x}, \hat{t})\hat{\star}\psi(\hat{x}, \hat{t}))^\dagger \\ &= +e\gamma^\rho(i\gamma^2)\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\theta^{\mu\nu}\right)^n((\hat{\partial}_\nu)^n\hat{\psi})^\dagger((\hat{\partial}_\mu)^n\hat{A}_\rho)^\dagger \\ &= +e\gamma^\rho(i\gamma^2)\psi^\dagger(\hat{x}, \hat{t})\hat{\star}\hat{A}_\rho^\dagger(\hat{x}, \hat{t}) \\ &= +e\gamma^\rho\hat{A}_\rho^\dagger(\hat{x}, \hat{t})\hat{\star}(i\gamma^2)\psi^\dagger(\hat{x}, \hat{t}), \end{aligned} \tag{3.4.7}$$

this is the same transformation as in the commutative case but only because of the inclusion of $\hat{\star}$. Just like in the commutative case \hat{C} gives the solutions that satisfy the Dirac equation with a different sign for charge but with \hat{A}^\dagger instead of A^* .

3.4.3 Time Reversal

Commutatively the time reflection transformation is $\hat{T}^{(0)}\psi(x, t) = \psi(x, -t)$, where temporal time reversal is given by, $\hat{K}^{(0)}\hat{T}^{(0)}\psi(x, t) = \psi^*(x, -t)$. The full transformation for spinors is $\hat{T} = i\gamma^1\gamma^3\hat{K}^{(0)}\hat{T}^{(0)}$. Non-commutatively we take $\hat{T} = i\gamma^1\gamma^3\hat{\Theta}\hat{\mathbb{L}}_4\hat{T}^{(0)}$. $i\theta^{\mu\nu} = [\hat{x}^\mu, \hat{x}^\nu]$ undergoes the same transformation as in

the case of $\hat{P}^{(0)}$ since $[\hat{x}^0, -\hat{x}^i] = [-\hat{x}^0, \hat{x}^i]$.

$$\begin{aligned} \hat{T}(i\hat{\not{\partial}}\psi(\hat{x}, \hat{t})) &= i\gamma^1\gamma^3 \left(-i(i\gamma^0)^*(-(\theta^{-1})_{0i}) [\hat{x}^i, \psi^\dagger(\hat{x}, -\hat{t})] + \right. \\ & \left. i(i\gamma^i)^*(\theta^{-1})_{ij} [\hat{x}^j, \psi^\dagger(-\hat{x}, \hat{t})] + i(i\gamma^i)^*(-(\theta^{-1})_{i0}) [-\hat{x}^0, \psi^\dagger(\hat{x}, -\hat{t})] \right) \\ &= i\gamma^1\gamma^3(i\gamma^0\hat{\partial}_0 - i(\gamma^i)^*\hat{\partial}_i)\psi^\dagger(\hat{x}, -\hat{t}) \\ &= i\hat{\not{\partial}}(i\gamma^1\gamma^3)\psi^\dagger(\hat{x}, -\hat{t}) \end{aligned} \quad (3.4.8)$$

$$= i\hat{\not{\partial}}\hat{T}\hat{\psi} \quad (3.4.9)$$

We have that $[\hat{\not{\partial}}, \hat{T}] = 0$, in the free case. For interactions,

$$\hat{T}(-e\hat{A}(\hat{x}, \hat{t})\hat{\star}\psi(\hat{x}, \hat{t})) = -e(\gamma^0 A_0^\dagger(\hat{x}, -\hat{t}) - \gamma^i A_i^\dagger(\hat{x}, -\hat{t}))\hat{\star}(i\gamma^1\gamma^3)\psi^\dagger(\hat{x}, -\hat{t}). \quad (3.4.10)$$

This is again of the same form as commutatively, so if we map a time reversal invariant A to \hat{A} this will also be invariant.

3.4.4 CPT

The non-commutative CPT transformation is given by $\widehat{CPT} = i\gamma^5\hat{P}^{(0)}\hat{T}^{(0)} = -\gamma^2\gamma^0\gamma^1\gamma^3\hat{P}^{(0)}\hat{T}^{(0)}$. From the above results we can see \widehat{CPT} symmetry is conserved in the non-commutative setting, if we consider non-commutative potentials \hat{A} that are mapped from CPT invariant commutative potentials A . We note that \widehat{CPT} symmetry is also conserved for potentials which are not multiplied with $\hat{\star}$, these are in general not Lorentz invariant, since \widehat{CPT} doesn't involve a $\hat{\Theta}$, however these types of potentials will break C, T individually, but of course, the combined CT is still conserved.

3.5 Constant Background Magnetic Field

Now that we have the interacting non-commutative equation, let us solve it for the simple case of a constant background magnetic field.

3.5.1 Solutions

For the case of a constant background non-commutative magnetic field in $+z$ direction we choose the gauge potential

$$\hat{A}^\mu = \begin{pmatrix} 0 \\ -\hat{y}B \\ 0 \\ 0 \end{pmatrix}. \quad (3.5.1)$$

This is inspired by the form of the commutative potential since in the limit when θ goes to zero we should recover this. Putting this into (3.2.1) and calculating the $\hat{\star}$ contribution gives:

$$(i\hat{\partial} - \gamma^1 Bq(\hat{x}^2 - \frac{i}{2}\theta^{2\mu}\hat{\partial}_\mu) - m)\hat{\psi} = 0. \quad (3.5.2)$$

We can see one additional θ term is generated, because the potential was only linear. Setting $\hat{\psi} = \begin{pmatrix} \hat{\phi} \\ \hat{\chi} \end{pmatrix}$ gives two equations for the electron and positron components,

$$(-i\sigma_j\hat{\partial}_j + \sigma_1(Bq\hat{y} - Bq\frac{i}{2}\theta^{2\mu}\hat{\partial}_\mu))\hat{\chi} = (i\hat{\partial}_0 - m)\hat{\phi}, \quad (3.5.3)$$

$$(-i\sigma_j\hat{\partial}_j + \sigma_1(Bq\hat{y} - Bq\frac{i}{2}\theta^{2\mu}\hat{\partial}_\mu))\hat{\phi} = (i\hat{\partial}_0 + m)\hat{\chi}. \quad (3.5.4)$$

Solving for one of the spinors:

$$(-i\sigma_j\hat{\partial}_j + \sigma_1(Bq\hat{y} - Bq\frac{i}{2}\theta^{2\mu}\hat{\partial}_\mu))^2\hat{\phi} = (-\hat{\partial}_0^2 - m^2)\hat{\phi}. \quad (3.5.5)$$

Multiplying out the square:

$$\begin{aligned} &(-\hat{\partial}_j^2 + (Bq\frac{i}{2}\theta^{2\mu}\hat{\partial}_\mu)^2 + (Bq\hat{y})^2 - \{\sigma_1, \sigma_j\}(iBq\hat{y}\hat{\partial}_j + Bq\frac{1}{2}\theta^{2\mu}\hat{\partial}_\mu\hat{\partial}_j) \\ &- [\hat{y}, \hat{\partial}_j]iBq\sigma_j\sigma_1 - (Bq)^2\hat{y}i\theta^{2\mu}\hat{\partial}_\mu)\hat{\phi} = (-\hat{\partial}_0^2 - m^2)\hat{\phi}. \end{aligned} \quad (3.5.6)$$

Making use of the anti commutation relations of the σ matrices:

$$\begin{aligned} &(-\hat{\partial}_j^2 - (Bq\frac{1}{2}\theta^{2\mu}\hat{\partial}_\mu)^2 + (Bq\hat{y})^2 - 2(iBq\hat{y}\hat{\partial}_1 + Bq\frac{1}{2}\theta^{2\mu}\hat{\partial}_\mu\hat{\partial}_1) \\ &- Bq\sigma_3 - (Bq)^2\hat{y}i\theta^{2\mu}\hat{\partial}_\mu)\hat{\phi} = (-\hat{\partial}_0^2 - m^2)\hat{\phi}. \end{aligned} \quad (3.5.7)$$

Setting $\hat{\phi} = \hat{\phi}e^{-ik^0\hat{t}+ik^1\hat{x}^1+ik^3\hat{x}^3}$, i.e. the plane wave solution in the $x - z$ plane, and noting $\theta^{22} = 0$ gives:

$$(-\hat{\partial}_2^2 + (k^3)^2 - Bq\sigma_3 + (Bq\hat{y} + k^1 - Bq\frac{1}{2}\theta^{2\mu}k_\mu)^2)\hat{\phi} = ((k^0)^2 - m^2)\hat{\phi}. \quad (3.5.8)$$

We could have also chosen the ordering $\hat{\phi} = e^{-ik^0\hat{t}+ik^1\hat{x}^1+ik^3\hat{x}^3}\hat{\phi}$, then moving the exponentials all the way to left produces a shift $\hat{y} \rightarrow \hat{y} + \theta^{\mu 2}k_\mu$. So the equation for this $\hat{\phi}$ is the same but the sign of the θ terms changes. These solutions are both valid but with the same energy. Now pick $\hat{\phi}$ to be eigenvectors of σ_3 ,

$$\hat{\phi}_+ = \begin{pmatrix} F_+(\hat{y}) \\ 0 \end{pmatrix}, \hat{\phi}_- = \begin{pmatrix} 0 \\ F_-(\hat{y}) \end{pmatrix}. \quad (3.5.9)$$

The equation then becomes:

$$(-\hat{\partial}_2^2 + (k^3)^2 \mp Bq + (Bqy + k^1 - Bq\frac{1}{2}\theta^{2\mu}k_\mu)^2)F_\pm(\hat{y}) = ((k^0)^2 + m^2)F_\pm(\hat{y}). \quad (3.5.10)$$

The magnetic length is defined as,

$$l_B = \frac{1}{\sqrt{B|q|}}, \quad (3.5.11)$$

this introduces a length scale which depends on the magnetic field. Choosing a new variable:

$$\hat{\xi} = \frac{1}{l_B} \left(\hat{y} + \text{sgn}(q)l_B^2k^1 - \frac{1}{2}\theta^{2\mu}k_\mu \right), \quad (3.5.12)$$

and setting,

$$a_\pm = l_B^2 \left((k^0)^2 - m^2 - (k^3)^2 \pm \text{sgn}(q)\frac{1}{l_B^2} \right), \quad (3.5.13)$$

also noting that since,

$$[\hat{x}^i, \hat{\xi}] = i\frac{1}{l_B}\theta^{i2}, \quad (3.5.14)$$

we can write,

$$\frac{\partial}{\partial \hat{y}} \hat{\psi} = \frac{1}{l_B} \frac{\partial}{\partial \hat{\xi}} \hat{\psi}, \quad (3.5.15)$$

the following equation is obtained,

$$\left(\frac{d^2}{d\hat{\xi}^2} - \hat{\xi}^2 + a_\pm \right) F_\pm(\hat{\xi}) = 0. \quad (3.5.16)$$

The commutative version of this can be found in [33] and [35]. These solutions match the form of the commutative case except that we have a modified $\hat{\xi}$ parameter and these are operator valued functions, in particular we can see that in our case the effect of non-commutativity was to shift k^1 . Solutions of this differential equation are operator valued Hermite functions,

$$F_{n_\pm}(\hat{\xi}) = N_{n_\pm} e^{-\frac{\hat{\xi}^2}{2}} H_{n_\pm}(\hat{\xi}). \quad (3.5.17)$$

where H is a hermite polynomial and $a_\pm = 2n_\pm + 1, n_\pm \in \mathbb{N}$. This gives for the energies:

$$E_{n_\pm}^2 = \frac{1}{l_B^2} (2n_\pm + 1 \mp \text{sgn}(q)) + m^2 + (k^3)^2. \quad (3.5.18)$$

We see that exactly the same spectrum as in the commutative case is obtained, which is expected since the energy doesn't depend on k^1 . There is a discrete 2 fold degeneracy, e.g. when $q = -e$ then $E_{n_+ - 1} = E_{n_-}$ so in this case we can label the energies by $n_- = n$ then $E_n = \frac{2n}{l_B^2} + m^2 + (k^3)^2$, then also only the

ground state is non degenerate $E_{n_-=0}$. There is also a continuous degeneracy in k^1 . From this point we use the transformed (2.1.2) $\theta \rightarrow \tilde{\theta}$ coordinate system for the rest of the thesis, so $\hat{\xi} = \frac{1}{l_B} (\hat{y} + k^1(\text{sgn}(q)l_B^2 - \frac{1}{2}\theta^{12}))$. Since:

$$\theta^{2\mu} k_\mu \rightarrow \theta^{21} k_1. \quad (3.5.19)$$

Using the notation:

$$\hat{\phi}_{+,n} = \begin{pmatrix} F_{n_+}(\hat{\xi}) \\ 0 \end{pmatrix}, \hat{\phi}_{-,n} = \begin{pmatrix} 0 \\ F_{n_-}(\hat{\xi}) \end{pmatrix}, \quad (3.5.20)$$

we can find $\chi_{\pm,n}$, put $\phi_{\pm,n}$ into equation (3.5.3),

$$\begin{aligned} \hat{\chi}_{\pm,n} &= \frac{1}{E+m} \begin{pmatrix} k^3 & -\partial_2 + Bqy + k^1 - \frac{1}{2}Bq\theta^{21}k_1 \\ \partial_2 + Bqy + k^1 - \frac{1}{2}Bq\theta^{21}k_1 & -k^3 \end{pmatrix} \phi_{\pm,n}, \\ &= \frac{1}{E+m} \begin{pmatrix} k^3 & \frac{1}{l_B}(-\frac{d}{d\xi} + \text{sgn}(q)\hat{\xi}) \\ \frac{1}{l_B}(\frac{d}{d\xi} + \text{sgn}(q)\hat{\xi}) & -k^3 \end{pmatrix} \phi_{\pm,n}. \end{aligned} \quad (3.5.21)$$

Using the following recursions relations for hermite functions f_n found in [36],

$$f'_n(x) = \sqrt{\frac{n}{2}} f_{n-1}(x) - \sqrt{\frac{n+1}{2}} f_{n+1}(x), \quad (3.5.22)$$

$$x f_n(x) = \sqrt{\frac{n}{2}} f_{n-1}(x) + \sqrt{\frac{n+1}{2}} f_{n+1}(x). \quad (3.5.23)$$

Equation (3.5.21) gives, if we take for the normalization:

$$N_n = \sqrt{\frac{\theta^{12}}{2^n n!}} \sqrt{\frac{1}{\pi l_B^2}}, \quad (3.5.24)$$

which we see is correct when we compute the Tr_c (D).

$$\begin{aligned} \hat{\chi}_{+,n} &= \frac{1}{E+m} \begin{pmatrix} k^3 F_{n_+}(\hat{\xi}) \\ \frac{\text{sgn}(q)}{l_B} \sqrt{2n_+ + (1 - \text{sgn}(q))} F_{(n_+ - \text{sgn}(q))}(\hat{\xi}) \end{pmatrix} \\ \hat{\chi}_{-,n} &= \frac{1}{E+m} \begin{pmatrix} \frac{\text{sgn}(q)}{l_B} \sqrt{2n_- + (1 + \text{sgn}(q))} F_{(n_- + \text{sgn}(q))}(\hat{\xi}) \\ -k^3 F_{n_-}(\hat{\xi}) \end{pmatrix}. \end{aligned} \quad (3.5.25)$$

These are, however, not the most general solutions. Since there is a degeneracy in k^1 , we can integrate with some function $f(k^1)$ i.e. $\int dk^1 f(k^1) \psi$, and still obtain the same energy eigenvalue. We can choose $f(k^1) = e^{-\alpha^2(k^1)^2}$, instead of the $\delta(k^1 - (k^1)')$ for the plane-wave solution, this will make computation easier and introduces a localization in \hat{x} determined by the length scale α^2 . The normalization of these states is given by $N_n = \sqrt{\sqrt{\frac{2\alpha^2}{\pi}} \sqrt{\frac{\theta^{12}}{2^n n!}} \sqrt{\frac{1}{\pi l_B^2}}}$.

After verifying these are states in \mathcal{H}_q we also calculate the uncertainty relation (D). So no matter what choice of α or B we make, we will never obtain a localization below the length scale θ^{12} unlike commutative physics. We can also see that there are two length scales involved in this problem l_B^2, θ^{12} . When $l_B^2 \gg \theta^{12}$, i.e. a weak magnetic field, we see the theta terms become irrelevant, and l_B is the dominate length scale which determines the uncertainty, since $\frac{\theta^{12}}{2l_B^2} \approx 0$. This is to be expected since for weak magnetic fields, the states are not very localized and thus the minimum length scale becomes unimportant. If $B \rightarrow 0$ then the states are de-localized and everything behaves as commutatively as previously seen for the free Dirac equation. If $\frac{\theta^{12}}{2l_B^2} = 1$ depending on $\text{sgn}(q)$ we can obtain the minimum uncertainty case. When $l_B^2 \ll \theta^{12}$, i.e. a strong magnetic field, we see that that the uncertainty in the y -direction saturates at θ^{12} , while the uncertainty in the x -direction diverges. Thus the states are maximally localized in the y -direction and de-localized in the x -direction.

3.5.2 Magnetic Flux

Now consider these equations in some volume $V = L_x L_y L_z$. From equation (3.5.17) we can see that the solutions for $\psi_{n,\pm}$ are proportional to a Gaussian in \hat{y} "localized" at $-k^1(\text{sgn}(q)l_B^2 - \frac{\theta^{12}}{2})$. At this point it is convenient to introduce the symbol of $\hat{\psi}$, which is the expectation value in a coherent state basis, i.e. $\psi(z, z^*, w, w^*) = \langle z, w | \hat{\psi} | w, z \rangle$. Since it is a minimum uncertainty basis the real and imaginary parts of z and w are the closest to our conventional notion of space-time coordinates as obtained from a measurement. A detailed discussion of symbols and coherent states is given in chapter five. For our present purposes it is sufficient to note that, to leading order in $\theta^{\mu\nu}$ the symbol can be obtained by $\hat{x}^\mu \rightarrow x^\mu$, where the x^μ are commutative real coordinates. Now choose no boundary conditions for the z direction, $L_z \rightarrow \infty$, and choose periodic boundary conditions for the x direction, we have charged particles moving in cyclotron orbits either clockwise or anticlockwise depending in the sign of k^1 , then $k^1 = \frac{2\pi l}{L_x}, l \in \mathbb{Z}$ since we have a constant magnetic field in the z -direction. Then the spacing between two peaks of the Gaussian for a fixed k^3 and E_n is given by:

$$\Delta y = \left| -\frac{2((l+1) - l)\pi(\text{sgn}(q)l_B^2 - \frac{\theta^{12}}{2})}{L_x} \right| = \left| -\frac{2\pi(\text{sgn}(q)l_B^2 - \frac{\theta^{12}}{2})}{L_x} \right|. \quad (3.5.26)$$

Now in the y direction consider a boundary condition, where no states are localized outside $[0, L_y]$. This is a rough approximation for confinement where we are ignoring the issues that occur at the boundary[37]. Then the maximum number of states we can fit in the $x - y$ plane, is given by, comparing the

dimensionless lengths $\frac{\Delta y}{l_B}, \frac{L_y}{l_B}$:

$$\begin{aligned} N_{x,y} &= \frac{L_y}{\Delta y} = \frac{A}{2\pi(\text{sgn}(q)l_B^2 - \frac{\theta^{12}}{2})} \\ &= \frac{BqA}{2\pi(1 - \frac{Bq\theta^{12}}{2})}. \end{aligned} \quad (3.5.27)$$

Where $A = L_x L_y$ is the area in the $x - y$ plane. So the degeneracy of the landau level is given $\frac{AB_\theta}{\Phi_0}$, where Φ_0 is the fundamental flux unit $\Phi_0 = |\frac{q}{2\pi}|$. This is the same result as in the commutative case, except for the magnetic field B_θ and flux Φ_θ that are modified by non-commutative corrections:

$$B_\theta = \frac{B}{1 - \frac{Bq\theta^{12}}{2}}, \quad (3.5.28)$$

and

$$\Phi_\theta = AB_\theta. \quad (3.5.29)$$

We can see an explicit physical effect caused by the non-commutativity. If we consider the limit $B \rightarrow \infty$, then $B_\theta = -\frac{2}{q\theta^{12}}$ and the number of states becomes $N_{x,y} = \frac{A}{\pi q\theta^{12}}$. So even with an infinite magnetic field the number of states, or flux quanta accommodated, is still limited since the non-commutativity restricts the maximum localization to a length scale θ . This result also implies that the number of particles that can be accommodated in a Landau level is modified by non-commutativity and correspondingly that incompressibility will occur at modified filling factors. This is reminiscent of the fractional quantum Hall effect and offers the possibility of viewing interacting electrons moving in a constant magnetic field as an effective non-commutative theory of non-interacting electrons moving in the same magnetic field. This point of view was already discussed in [38]. We also see a critical point $Bq\theta^{12} = 2$ where the effective magnetic field diverges, a similar non-relativistic result is obtained in [10].

The density of states $\rho(E)$ in V can also be computed, e.g. for electrons $q = -e$, and is given by:

$$\begin{aligned} &\sum_{\lambda=+,-} \sum_n \int dk_z \frac{AB_\theta}{\Phi_0} \delta\left(E - \sqrt{2Ben + m^2 + (k^3)^2}\right) \Theta(E^2 - m^2 - 2Ben) \\ &= 2 \frac{AB_\theta}{\Phi_0} \sum_{\lambda=+,-} \sum_n \sqrt{\frac{E^2}{E^2 - m^2 - 2Ben}} \Theta(E^2 - m^2 - 2Ben). \end{aligned} \quad (3.5.30)$$

We include a step function Θ since from (3.5.18) if $2neB > E^2 - m^2$, k^3 will be imaginary. Again we see the same density but with a modified flux.

Summarizing chapter three, we have found a Lorentz invariant Dirac equation (3.2.1). It was shown that it behaves the same way as the commutative

*CHAPTER 3. THE NON-COMMUTATIVE DIRAC EQUATION WITH
ELECTROMAGNETIC INTERACTIONS***28**

case under $\hat{C}, \hat{P}, \hat{T}, \widehat{CPT}$ transformations. Spectrum and solutions in constant background magnetic field were found (3.5.18) and a modified magnetic field and flux observed (3.5.29).

Chapter 4

The Confined Non-commutative Dirac Equation

One of the main motivations for the topic of this thesis is to study the thermodynamics of relativistic gases, confined to a finite volume, as one expects that non-commutativity will manifest itself in the high temperature and high density limits. Particularly the latter limit can be realized in very dense astrophysical objects, but since the Fermi energy in these systems is very high, the energies of the particles are ultra-relativistic, which makes a relativistic treatment essential. The purpose of this chapter is to lay the foundation for such a study by investigating the non-commutative Dirac equation in the presence of boundaries and confining potentials.

4.1 Boundary Conditions

To confine the commutative Schrödinger equation in $1 - D$ position basis, a constant piece wise potential is added:

$$V(x) = \begin{cases} v & x \leq 0 \\ V & 0 \leq x \leq L . \\ v & L \leq x \end{cases} \quad (4.1.1)$$

Taking the limit $v \rightarrow \infty$ would confine the solutions to the interval $[0, L]$. On the operator level this takes the form of a projection operator $V \int_0^L dx |x\rangle \langle x| + v(\int_{-\infty}^0 dx |x\rangle \langle x| + \int_L^{\infty} dx |x\rangle \langle x|)$. However this procedure does not work with the Dirac equation when sending $v \rightarrow \infty$ the problem of the Klein paradox occurs. Instead, commutatively, a position dependent mass is introduced,

$$m(x) = \begin{cases} M & x \leq 0 \\ m & 0 \leq x \leq L . \\ M & L \leq x \end{cases} \quad (4.1.2)$$

CHAPTER 4. THE CONFINED NON-COMMUTATIVE DIRAC EQUATION 30

For $|E| < M$ we have bound states and for $|E| > M$ scattering states. The free equation goes like e^{ikx} when $x > L$ a particle propagating in the positive x-direction, if $x < 0$ it goes like e^{-ikx} , a particle propagating in the negative x-direction. In the limit $M \rightarrow \infty$, $k \rightarrow \pm i\infty$ since $k = \pm\sqrt{E^2 - m^2}$. So the solutions go to zero outside the boundary. Non-commutatively a similar approach is used. In terms of projection operators that separate regions of non-commutative space, this can be expressed as follows:

$$\hat{m} = M\hat{R} + m\hat{Q}. \quad (4.1.3)$$

Here \hat{Q} and \hat{R} are projection operators on configuration space i.e ,

$$\begin{aligned} \hat{R}^2 &= R, \\ \hat{R} + \hat{Q} &= \mathbb{I}_c, \\ \hat{Q}\hat{R} &= 0. \end{aligned} \quad (4.1.4)$$

A non-relativistic non-commutative discussion of this construction can be found in [39]. So we want solutions that satisfy:

$$(i\hat{\partial} - (m\hat{R} + M\hat{Q}))\hat{\psi} = 0. \quad (4.1.5)$$

These equations are not Lorentz invariant, but this is the usual case for confining potentials. In the regions demarcate by the projection operators R , Q the Dirac equations are respectively:

$$\begin{aligned} (i\hat{\partial} - m)\hat{\psi}_{in} &= 0, \\ (i\hat{\partial} - M)\hat{\psi}_{out} &= 0. \end{aligned} \quad (4.1.6)$$

From this ψ is given by:

$$\psi = \hat{R}\hat{\psi}_{in} + \hat{Q}\hat{\psi}_{out}. \quad (4.1.7)$$

Also multiplying (4.1.6) by \hat{R} and \hat{Q} we obtain,

$$\begin{aligned} (\hat{R}i\hat{\partial} - m\hat{R})\hat{\psi}_{in} &= 0, \\ (\hat{Q}i\hat{\partial} - M\hat{Q})\hat{\psi}_{out} &= 0. \end{aligned} \quad (4.1.8)$$

Putting together (4.1.5) and (4.1.7) and then inserting equations (4.1.6) we obtain the following matching condition, using (4.1.4):

$$\begin{aligned} [i\hat{\partial}, \hat{R}]\hat{\psi}_{in} + [i\hat{\partial}, \hat{Q}]\hat{\psi}_{out} &= 0 \\ \implies [i\hat{\partial}, \hat{R}]\hat{\psi}_{in} &= [i\hat{\partial}, \hat{R}]\hat{\psi}_{out}. \end{aligned} \quad (4.1.9)$$

This provides boundary conditions for operators $\hat{\psi}_{in}$ and $\hat{\psi}_{out}$. The same method can be used to derive the boundary conditions for the commutative

CHAPTER 4. THE CONFINED NON-COMMUTATIVE DIRAC EQUATION **31**

Schrödinger equation in which the necessary commutator is $[\hat{p}^2, \int_0^L dx |x\rangle \langle x|] |\psi_{in}\rangle = [\hat{p}^2, \int_0^L dx |x\rangle \langle x|] |\psi_{out}\rangle$. This produces the usual continuity condition at the boundary.

Calculating the commutator yields:

$$[\hat{\phi}, \hat{R}] = \begin{pmatrix} \mathbb{I}_2[\hat{\partial}_0, \hat{R}] & \sigma_i[\hat{\partial}_i, \hat{R}] \\ -\sigma_i[\hat{\partial}_i, \hat{R}] & -\mathbb{I}_2[\hat{\partial}_0, \hat{R}] \end{pmatrix}. \quad (4.1.10)$$

Setting $\hat{\psi} = \begin{pmatrix} \hat{\phi} \\ \hat{\chi} \end{pmatrix}$ gives two equations, for the electron and positron components:

$$\begin{aligned} [\hat{\partial}_0, \hat{R}]\hat{\phi}_{in} + \sigma_i[\hat{\partial}_i, \hat{R}]\hat{\chi}_{in} &= [\hat{\partial}_0, \hat{R}]\hat{\phi}_{out} + \sigma_i[\hat{\partial}_i, \hat{R}]\hat{\chi}_{out}, \\ [\hat{\partial}_0, \hat{R}]\hat{\chi}_{in} + \sigma_i[\hat{\partial}_i, \hat{R}]\hat{\phi}_{in} &= [\hat{\partial}_0, \hat{R}]\hat{\chi}_{out} + \sigma_i[\hat{\partial}_i, \hat{R}]\hat{\phi}_{out}. \end{aligned} \quad (4.1.11)$$

It is convenient to introduce two operators which simplify our problem:

$$\hat{\rho} = \hat{\partial}_1 + i\hat{\partial}_2, \hat{\rho}^\dagger = \hat{\partial}_1 - i\hat{\partial}_2. \quad (4.1.12)$$

Then (4.1.11) becomes:

$$\begin{aligned} & \begin{pmatrix} [\hat{\partial}_0, \hat{R}] & 0 \\ 0 & [\hat{\partial}_0, \hat{R}] \end{pmatrix} \hat{\phi}_{in} + \begin{pmatrix} [\hat{\partial}_3, \hat{R}] & [\hat{\rho}^\dagger, \hat{R}] \\ [\hat{\rho}, \hat{R}] & [\hat{\partial}_3, \hat{R}] \end{pmatrix} \hat{\chi}_{in} \\ &= \begin{pmatrix} [\hat{\partial}_0, \hat{R}] & 0 \\ 0 & [\hat{\partial}_0, \hat{R}] \end{pmatrix} \hat{\phi}_{out} + \begin{pmatrix} [\hat{\partial}_3, \hat{R}] & [\hat{\rho}^\dagger, \hat{R}] \\ [\hat{\rho}, \hat{R}] & [\hat{\partial}_3, \hat{R}] \end{pmatrix} \hat{\chi}_{out}, \\ & \begin{pmatrix} [\hat{\partial}_0, \hat{R}] & 0 \\ 0 & [\hat{\partial}_0, \hat{R}] \end{pmatrix} \hat{\chi}_{in} + \begin{pmatrix} [\hat{\partial}_3, \hat{R}] & [\hat{\rho}^\dagger, \hat{R}] \\ [\hat{\rho}, \hat{R}] & [\hat{\partial}_3, \hat{R}] \end{pmatrix} \hat{\phi}_{in} \\ &= \begin{pmatrix} [\hat{\partial}_0, \hat{R}] & 0 \\ 0 & [\hat{\partial}_0, \hat{R}] \end{pmatrix} \hat{\chi}_{out} + \begin{pmatrix} [\hat{\partial}_3, \hat{R}] & [\hat{\rho}^\dagger, \hat{R}] \\ [\hat{\rho}, \hat{R}] & [\hat{\partial}_3, \hat{R}] \end{pmatrix} \hat{\phi}_{out}. \end{aligned} \quad (4.1.13)$$

Now in order to continue further we must choose a specific \hat{R} . The action of (4.1.12) is:

$$\hat{\rho}\hat{\psi} = \sqrt{\frac{2}{\theta_{21}}}[\hat{b}, \hat{\psi}], \hat{\rho}^\dagger\hat{\psi} = -\sqrt{\frac{2}{\theta_{21}}}[\hat{b}^\dagger, \hat{\psi}]. \quad (4.1.14)$$

Now consider:

$$\begin{aligned} \hat{b}^\dagger\hat{b} &= \frac{1}{2\theta^{12}}(\hat{x} - i\hat{y})(\hat{x} + i\hat{y}) \\ &= \frac{1}{2\theta^{12}}(\hat{x}^2 + \hat{y}^2 - \theta^{12}). \end{aligned} \quad (4.1.15)$$

The radius operator $\hat{r}^2 = \hat{x}^2 + \hat{y}^2$ is given by,

$$\hat{r}^2 = \theta^{12}(2\hat{b}^\dagger\hat{b} + \mathbb{I}_c). \quad (4.1.16)$$

CHAPTER 4. THE CONFINED NON-COMMUTATIVE DIRAC EQUATION **32**

So the operator which gives a 3D cylinder along the z axis on non-commutative space is given by $\hat{V}_{cylinder} = 2\pi\hat{r}^2\hat{z}$. Since $[\hat{b}^\dagger\hat{b}, \hat{z}] = 0$ we can find simultaneous eigenstates $\hat{V}_{cylinder} |z, n\rangle = z\theta^{12}(2n+1)|z, n\rangle$ where $z \in \mathbb{R}, n \in \mathbb{N}$. Consider a projection operator $\hat{R} = \int_0^L dz \sum_{n=0}^M |z, n\rangle \langle z, n|$, this will restrict to the interior of a cylinder with volume $2\pi L\theta^{12}(2M+1)$. Now it is possible to explicitly calculate the condition (4.1.13) in (E).

The 4 conditions at $\xi = 0$ are:

$$\begin{aligned} \langle 0, l | \hat{\chi}_{in}^{(i)} | \xi', l' \rangle &= \langle 0, l | \hat{\chi}_{out}^{(i)} | \xi', l' \rangle, \\ \langle 0, l | \hat{\phi}_{in}^{(i)} | \xi', l' \rangle &= \langle 0, l | \hat{\phi}_{out}^{(i)} | \xi', l' \rangle, \end{aligned} \quad (4.1.17)$$

and the 4 conditions at $\xi = L$ are:

$$\begin{aligned} \langle L, l | \hat{\chi}_{in}^{(i)} | \xi', l' \rangle &= \langle L, l | \hat{\chi}_{out}^{(i)} | \xi', l' \rangle, \\ \langle L, l | \hat{\phi}_{in}^{(i)} | \xi', l' \rangle &= \langle L, l | \hat{\phi}_{out}^{(i)} | \xi', l' \rangle. \end{aligned} \quad (4.1.18)$$

Finally the 2 conditions at $c = \xi \in (0, L)$ and $l = M$ are:

$$\begin{aligned} \langle c, M+1 | \hat{\phi}_{in}^{(1)} | \xi', l' \rangle &= \langle \xi, M+1 | \hat{\phi}_{out}^{(1)} | \xi', l' \rangle, \\ \langle c, M+1 | \hat{\chi}_{in}^{(1)} | \xi', l' \rangle &= \langle \xi, M+1 | \hat{\chi}_{out}^{(1)} | \xi', l' \rangle, \end{aligned} \quad (4.1.19)$$

and the 2 conditions at $c = \xi \in (0, L)$ and $l = M+1$ are:

$$\begin{aligned} \langle c, M | \hat{\phi}_{in}^{(2)} | \xi', l' \rangle &= \langle \xi, M | \hat{\phi}_{out}^{(2)} | \xi', l' \rangle, \\ \langle c, M | \hat{\chi}_{in}^{(2)} | \xi', l' \rangle &= \langle \xi, M | \hat{\chi}_{out}^{(2)} | \xi', l' \rangle. \end{aligned} \quad (4.1.20)$$

4.2 Solutions

In the two regions we have a free particle with different masses so that the general solutions for ψ , for positive energy, are given by:

$$\begin{aligned} \hat{\psi}_{in} &= \int d^3k f(k_z, k_x, k_y) \delta(E^2 - \vec{k}^2 + m^2) e^{-ik^\mu \hat{x}_\mu} \frac{\not{k} + m}{\sqrt{2m(m+k^0)}} \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} \\ \hat{\psi}_{out} &= \int d^3K f(K_z, K_x, K_y) \delta(E^2 - \vec{K}^2 + M^2) e^{-iK^\mu \hat{x}_\mu} \frac{\not{K} + m}{\sqrt{2m(m+K^0)}} \begin{pmatrix} \chi_s \\ 0 \end{pmatrix}. \end{aligned} \quad (4.2.1)$$

Here $k^\mu k_\mu = m^2$, $K^\mu K_\mu = M^2$ and $k^0 = K^0 = E$. Only the exponentials are operator valued so we begin by calculating their matrix elements. It is convenient to introduce two dimensionless numbers:

$$\kappa = \sqrt{\frac{\theta^{12}}{2}}(k^1 + ik^2), \kappa^* = \sqrt{\frac{\theta^{12}}{2}}(k^1 - ik^2). \quad (4.2.2)$$

CHAPTER 4. THE CONFINED NON-COMMUTATIVE DIRAC EQUATION 33

$$\begin{aligned}
 e^{-ik^\mu \hat{x}_\mu} &\sim e^{-ik^0 \hat{t} + ik^1 \hat{x} + ik^2 \hat{y} + ik^3 \hat{z}} \\
 &\sim e^{-ik^0 \hat{t} + ik^3 \hat{z} + i\kappa^* \hat{b} + i\kappa \hat{b}^\dagger} \\
 &\sim e^{-ik^0 \hat{t}} e^{ik^3 \hat{z}} e^{i\kappa \hat{b}^\dagger} e^{i\kappa^* \hat{b}}.
 \end{aligned} \tag{4.2.3}$$

The exponentials are equal up to a global phase so to simplify this problem we only use the final form.

For the matrix element we find:

$$\begin{aligned}
 \langle \xi, l | e^{-ik^0 t} e^{ik^3 z} e^{i\kappa \hat{b}^\dagger} e^{i\kappa^* \hat{b}} | \xi', l' \rangle &= \langle \xi, l | e^{k^0 \overleftarrow{\partial}_\xi} e^{ik^3 z} e^{i\kappa \hat{b}^\dagger} e^{i\kappa^* \hat{b}} | \xi', l' \rangle \\
 &= e^{-ik^3(\xi + k^0 \theta^{03})} \langle \xi + k^0 \theta^{03}, l | e^{i\kappa \hat{b}^\dagger} e^{i\kappa^* \hat{b}} | \xi', l' \rangle \\
 &= e^{-ik^3(\xi + k^0 \theta^{03})} \langle \xi + E\theta^{03}, l | \sum_n \frac{1}{n!} (i\kappa \hat{b}^\dagger)^n \sum_m \frac{1}{m!} (i\kappa^* \hat{b})^m | \xi', l' \rangle \\
 &= e^{-ik^3(\xi + k^0 \theta^{03})} \sum_n \sum_m \frac{1}{n! m!} (i\kappa)^n (i\kappa^*)^m \\
 &\quad \left(\sqrt{\frac{l! l'}{(l-n)!(l'-m)!}} \delta(\xi + E\theta^{03} - \xi') \delta_{l-n, l'-m} \right).
 \end{aligned} \tag{4.2.4}$$

Now consider two cases $l > l'$ and $l \leq l'$ and sum these equations differently, by either doing the n or m sum first. If $l > l'$, $n = l - l' + m$:

$$\begin{aligned}
 e^{-ik^3(\xi + k^0 \theta^{03})} \delta(\xi + E\theta^{03} - \xi') \sqrt{l! l'} \kappa^{l-l'} \sum_m \frac{i^{2m+(l-l')} (|\kappa|^2)^m}{m! ((l'-m)! (m+(l-l'))!)} \\
 = e^{-ik^3(\xi + k^0 \theta^{03})} \delta(\xi + E\theta^{03} - \xi') \kappa^{l-l'} \sqrt{\frac{l!}{l'}} L_{l'}^{l-l'} (|\kappa|^2),
 \end{aligned} \tag{4.2.5}$$

where L is an associate Laguerre polynomial, similarly if $l < l'$, $m = l' - l + m$:

$$e^{-ik^3(\xi + k^0 \theta^{03})} \delta(\xi + E\theta^{03} - \xi') (\kappa^*)^{l'-l} \sqrt{\frac{l!}{l'}} L_l^{l'-l} (|\kappa|^2). \tag{4.2.6}$$

We can integrate over a small interval around ξ' on both sides of the boundary conditions and thus drop the δ function. Note that we can factor $f(k_z, k_y, k_x) = f(k_x) f(\kappa, \kappa^*)$. We pick $f(\kappa, \kappa^*) = e^{-im\theta}$ where $\kappa = |\kappa| e^{i\theta}$ this projects on to angular momentum states. The condition of fixed energy means the integral over k_z becomes a sum over $\pm k_z$ and we set $f(k_z) = A$, $f(-k_z) = B$. The integrals over k_x, k_z become an integral over θ . Now we evaluate the boundary conditions starting with, (4.1.17) at $\xi = 0$, when $l > l'$:

$$\sum_{k_z, -k_z} \int_{-\pi}^{\pi} d\theta e^{i\theta} e^{-ik^3 k^0 \theta^{03}} (|\kappa| e^{i\theta})^{l-l'} \sqrt{\frac{l!}{l'}} L_{l'}^{l-l'} (|\kappa|^2) \frac{1}{\sqrt{2m(m+k^0)}} \begin{pmatrix} (k^0 + m) \chi_s \\ (\vec{k} \cdot \vec{\sigma}) \chi_s \end{pmatrix}. \tag{4.2.7}$$

CHAPTER 4. THE CONFINED NON-COMMUTATIVE DIRAC EQUATION 34

Now we have to consider the two possibilities of s . First we consider $s = +1$, noting the identity $\delta_{a,b} = \frac{1}{2N} \int_{-N}^N dx e^{ix(a-b)}$:

$$\begin{aligned} & \int_{-\pi}^{\pi} d\theta e^{-im\theta} (|\kappa| e^{i\theta})^{l-l'} \frac{\sqrt{l'!} L_{l'}^{l-l'}(|\kappa|^2)}{\sqrt{l! 2m(m+k^0)}} \begin{pmatrix} (k^0 + m)(Ae^{-ik^3 k^0 \theta^{03}} + Be^{ik^3 k^0 \theta^{03}}) \\ 0 \\ k_z (Ae^{-ik^3 k^0 \theta^{03}} - Be^{ik^3 k^0 \theta^{03}}) \\ \sqrt{\frac{2}{\theta^{12}}} \kappa (Ae^{-ik^3 k^0 \theta^{03}} + Be^{ik^3 k^0 \theta^{03}}) \end{pmatrix} \\ &= \frac{\sqrt{(l-m)! (|\kappa|)^m}}{\sqrt{l! 2m(m+k^0)}} \begin{pmatrix} (k^0 + m) L_{l-m}^m(|\kappa|^2) (Ae^{-ik^3 k^0 \theta^{03}} + Be^{ik^3 k^0 \theta^{03}}) \\ 0 \\ k_z L_{l-m}^m(|\kappa|^2) (Ae^{-ik^3 k^0 \theta^{03}} - Be^{ik^3 k^0 \theta^{03}}) \\ \sqrt{\frac{2(l-(m-1))}{\theta^{12}}} |\kappa| L_{l-(m-1)}^{m-1}(|\kappa|^2) (Ae^{-ik^3 k^0 \theta^{03}} + Be^{ik^3 k^0 \theta^{03}}) \end{pmatrix}. \end{aligned} \quad (4.2.8)$$

Now restrict to the case of $M \rightarrow \infty$, $\hat{\psi}_{out} = 0$. Since these equations have to hold for all l , there is not a solution for $|k|$ that will always make L zero. So we have two conditions:

$$Ae^{-ik^3 k^0 \theta^{03}} + Be^{ik^3 k^0 \theta^{03}} = 0, \quad (4.2.9)$$

$$Ae^{-ik^3 k^0 \theta^{03}} - Be^{ik^3 k^0 \theta^{03}} = 0. \quad (4.2.10)$$

Unfortunately these conditions cannot be satisfied simultaneously. The phases are never zero so these conditions give $Ae^{-ik^3 k^0 \theta^{03}} = Be^{ik^3 k^0 \theta^{03}} = 0$. So the whole wave-function vanishes. In summary we cannot satisfy (4.1.17), (4.1.18) simultaneously for either $s = \pm 1$.

We now look at conditions (4.1.19) and (4.1.20). Dropping the ξ dependence, since $\xi \in [0, L]$ is arbitrary, for $l > l'$ and $s = +1$:

$$\begin{aligned} & \int d\theta \frac{e^{-im\theta}}{\sqrt{2m(m+k^0)}} \begin{pmatrix} (k^0 + m)(|\kappa| e^{i\theta})^{(M+1)-l'} \sqrt{\frac{l'!}{(M+1)!}} L_{l'}^{(M+1)-l'}(|\kappa|^2) \\ 0 \\ k_z (|\kappa| e^{i\theta})^{(M+1)-l'} \sqrt{\frac{l'!}{(M+1)!}} L_{l'}^{(M+1)-l'}(|\kappa|^2) \\ \sqrt{\frac{2}{\theta^{12}}} \kappa (|\kappa| e^{i\theta})^{M-l'} \sqrt{\frac{l'!}{M!}} L_{l'}^{M-l'}(|\kappa|^2) \end{pmatrix} \\ &= (|\kappa|)^m \sqrt{\frac{(M+1-m)!}{(M+1)!}} \frac{1}{\sqrt{2m(m+k^0)}} \begin{pmatrix} (k^0 + m) L_{M+1-m}^m(|\kappa|^2) \\ 0 \\ k_z L_{M+1-m}^m(|\kappa|^2) \\ \sqrt{\frac{2(M+1)}{\theta^{12}}} |\kappa| L_{M+1-m}^{m-1}(|\kappa|^2) \end{pmatrix}. \end{aligned} \quad (4.2.11)$$

Unfortunately we cannot find κ 's that simultaneously satisfy $L_{M+1-m}^m(|\kappa|^2) = L_{M+1-m}^{m-1}(|\kappa|^2) = 0$, so (4.1.19) and (4.1.20) cannot be satisfied simultaneously for either $s = \pm 1$ or any m . The previous equation should not be interpreted

as a four spinor which is a solution to the Dirac equation. For convenience it is a list of boundary conditions.

It turns out that none of our boundary conditions can be satisfied if we want the wave-function to vanish everywhere outside the boundary $\hat{\psi}_{out} = 0$. This is not a problem of non-commutative quantum mechanics, this is exactly the same problem as commutatively when one tries to impose vanishing boundaries on the Dirac equation. So this is problem of relativistic quantum mechanics itself perhaps because it is not a complete theory and one might need to consider an interacting field theory approach to confinement. There are lesser conditions that can be satisfied instead of a vanishing wave-function, we could ask that only the current J_μ vanish outside the boundary but this is not sufficient to quantize the momenta [40]. There is also the MIT bag model [41] where a linear boundary condition, which implies current conservation, is used to confine quarks. This has been applied to the case of the Dirac equation [42], where it did quantize the momenta. It is, however, unclear what the non-commutative analogue of the bag model should be as its boundary condition are introduced in an ad-hoc way. Another approach is to ask that only 2 of the components of the spinor vanish. This is interesting to us since if we only look at (4.1.19), this condition matches the condition obtained in [39] for the two dimensional non-relativistic spherical well. At the very least we have shown that our formulation is consistent with non-relativistic calculations.

Chapter 5

The Non-Commutative Action and The Coherent State Basis

In this chapter an action, which through variation gives the non-commutative Dirac equation, is defined. It is done in an abstract way without reference to a particular basis or coordinate system. This is useful since this general form can later be used to write an action for any basis, and it is possible to study the general properties of non-commutative theories in a basis independent way. This can also serve as the starting point for developing non-commutative field theories in a basis independent way and to develop abstract Feynman rules. We then consider the action in the coherent state basis and also our previous calculation for the magnetic field in the coherent state basis. First consider the Dirac equation and a possible action written as an inner product:

$$S[\hat{\psi}] = (\gamma^0 \hat{\psi}, \hat{\mathcal{D}} \hat{\psi}) = \text{Tr}_c(\hat{\psi} \hat{\mathcal{D}} \hat{\psi}). \quad (5.0.1)$$

Under a variation $\epsilon \hat{\eta}$,

$$\begin{aligned} S[\hat{\psi} + \epsilon \hat{\eta}] - S[\hat{\psi}] &= (\gamma^0 \hat{\psi} + \epsilon \hat{\eta}, \hat{\mathcal{D}}(\hat{\psi} + \epsilon \hat{\eta})) - (\gamma^0 \hat{\psi}, \hat{\mathcal{D}} \hat{\psi}) \\ &= \epsilon^*(\hat{\eta}, \hat{\mathcal{D}} \hat{\psi}) + \epsilon(\gamma^0 \hat{\psi}, \hat{\mathcal{D}} \hat{\eta}) + O(\epsilon^2). \end{aligned} \quad (5.0.2)$$

Since ϵ^*, ϵ are independent, this can only be zero if both terms are zero. Since η is arbitrary two conditions follow:

$$\hat{\mathcal{D}} \hat{\psi} = 0, \quad (5.0.3)$$

The first is just the Dirac equation. To clarify the second condition consider:

$$\hat{\mathcal{D}}^\dagger \gamma^0 \hat{\psi} = 0, \quad (5.0.4)$$

for the Dirac equation with interactions, $\hat{\mathcal{D}} = i\hat{\partial} - q\hat{A}\hat{*} - m$. We recall that $\hat{A}, i\hat{\partial}$ are hermitian, and that $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$, and also that $(\hat{A}\hat{*}\hat{\psi})^\dagger = \hat{\psi}^\dagger \hat{*} \hat{A}^\dagger =$

$\hat{A}^\dagger \hat{*} \hat{\psi}^\dagger$ (3.4.7). So all we have to calculate is $(\hat{A} \hat{*})^\ddagger$, we do this directly by looking at the inner product:

$$\begin{aligned}
 (\hat{\phi}, \hat{A} \hat{*} \hat{\psi}) &= \text{Tr}_c(\hat{\phi}^\dagger (\hat{A} \hat{*} \hat{\psi})) \\
 &= \text{Tr}_c((\hat{\phi}^\dagger \hat{*} \hat{A}) \hat{\psi}) \\
 &= \text{Tr}_c((\hat{A}^\dagger \hat{*} \hat{\phi})^\dagger \hat{\psi}) \\
 &= (\gamma^0 \hat{A} \gamma^0 \hat{*} \hat{\phi}, \hat{\psi}).
 \end{aligned} \tag{5.0.5}$$

Using this we can easily see $\hat{\mathcal{D}}^\ddagger = \gamma^0 \hat{\mathcal{D}} \gamma^0$, the second condition then becomes $\hat{\mathcal{D}}^\ddagger \gamma^0 \hat{\psi} = \gamma^0 \hat{\mathcal{D}} \hat{\psi} = 0$. This condition is equivalent to the first and also reproduces the Dirac equation. We have now shown that our abstract definition of the action does indeed produce the right equation of motion.

5.1 The Coherent State Basis

Now we consider the action in a particular basis, as an example, but also because the coherent state basis is of interest since it is a minimum uncertainty basis. It is possible to define a four dimensional coherent state in configuration space [43]:

$$|z, w\rangle = e^{-\frac{|z|^2}{2}} e^{-\frac{|w|^2}{2}} e^{z\hat{a}^\dagger} e^{w\hat{b}^\dagger} |0, 0\rangle. \tag{5.1.1}$$

In the quantum Hilbert space we define a state, which is shown to have the interpretation as a minimum uncertainty state in [17]:

$$|z, w\rangle = |z, w\rangle \langle w, z|. \tag{5.1.2}$$

where:

$$z = \frac{1}{\sqrt{2\theta^{03}}}(x^0 + ix^3), w = \frac{1}{\sqrt{2\theta^{12}}}(x^1 + ix^2), \tag{5.1.3}$$

and z, w are dimensionless numbers, these states form an over complete basis for the configuration space and it is possible to write the identity as:

$$\mathbb{I}_c = \frac{1}{\pi^2} \int dz dz^* dw dw^* |z, w\rangle \langle z, w|. \tag{5.1.4}$$

As these are minimum uncertainty states, the real and imaginary parts of z and w are the closest to our conventional notion of space-time coordinates as obtained from a measurement.

Writing $dzdz^*dw dw^*$ as d^4x , noting $-|a|^2 - |b|^2 = -|a - b|^2 - a^*b - ab^*$:

$$\begin{aligned}
 S &= \text{Tr}_c(\hat{\psi}\hat{D}\hat{\psi}) = \frac{1}{\pi^2} \int d^4x \langle z, w | \hat{\psi}\hat{D}\hat{\psi} | z, w \rangle \\
 &= \frac{1}{\pi^2} \int d^4x \langle z, w | \hat{\psi} \left(\frac{1}{\pi^2} \int d^4x' |z', w'\rangle \langle z', w'| \right) \hat{D}\hat{\psi} | z, w \rangle \\
 &= \frac{1}{\pi^4} \int d^4x d^4x' e^{-|z-z'|^2 - (z^*z') - (z^*z')^* - |w-w'|^2 - (w^*w') - (w^*w')^*} \times \\
 &\langle 0, 0 | e^{z^*\hat{a}} e^{w^*\hat{b}} \hat{\psi} e^{w'\hat{b}^\dagger} e^{z'\hat{a}^\dagger} | 0, 0 \rangle \langle 0, 0 | e^{(z^*)'\hat{a}} e^{(w^*)'\hat{b}} \hat{D}\hat{\psi} e^{w\hat{b}^\dagger} e^{z\hat{a}^\dagger} | 0, 0 \rangle. \quad (5.1.5)
 \end{aligned}$$

Set $w' = w + u$, $z' = z + v$ and define $\psi(z, w, z^*, w^*) \equiv \langle z, w | \hat{\psi} | z, w \rangle$ commonly referred to as the symbol of the operator $\hat{\psi}$ as mentioned in chapter three. Using the formula for translations of functions $e^{a\frac{d}{dx}}f(x) = f(x+a)$, we find:

$$\begin{aligned}
 S &= \frac{1}{\pi^4} \int d^4x du du^* dv dv^* e^{-|u|^2 - |v|^2} \\
 \bar{\psi}(z, w, z^*, w^*) e^{v\overleftarrow{\partial}_z + u\overleftarrow{\partial}_w} e^{v^*\overrightarrow{\partial}_{z^*} + u^*\overrightarrow{\partial}_{w^*}} \langle z, w | \hat{D}\hat{\psi} | z, w \rangle. \quad (5.1.6)
 \end{aligned}$$

Consider the integral over v, v^* first:

$$\int dv dv^* e^{-|v|^2} e^{v\overleftarrow{\partial}_z} e^{v^*\overrightarrow{\partial}_{z^*}}. \quad (5.1.7)$$

Changing to polar coordinates yields:

$$\begin{aligned}
 &\int_0^\infty r dr \int_{-\pi}^\pi d\theta e^{-r^2} e^{re^{i\theta}\overleftarrow{\partial}_z} e^{re^{-i\theta}\overrightarrow{\partial}_{z^*}} \\
 &= \int_0^\infty r dr \int_{-\pi}^\pi d\theta e^{-r^2} \sum_{n,m} \frac{1}{n!m!} (r\overleftarrow{\partial}_z)^n (r\overrightarrow{\partial}_{z^*})^m e^{i\theta(n-m)} \\
 &= 2\pi \sum_n \frac{1}{n!n!} (\overleftarrow{\partial}_z\overrightarrow{\partial}_{z^*})^n \int_0^\infty dr e^{-r^2} r^{2n+1} \\
 &= 2\pi \sum_n \frac{1}{n!n!} (\overleftarrow{\partial}_z\overrightarrow{\partial}_{z^*})^n \frac{n!}{2} \\
 &= \pi e^{\overleftarrow{\partial}_z\overrightarrow{\partial}_{z^*}}. \quad (5.1.8)
 \end{aligned}$$

Similarly the integral over u gives:

$$\int du du^* e^{-|u|^2} e^{u\overleftarrow{\partial}_w} e^{u^*\overrightarrow{\partial}_{w^*}} = \pi e^{\overleftarrow{\partial}_w\overrightarrow{\partial}_{w^*}}. \quad (5.1.9)$$

So the action becomes:

$$S = \frac{1}{\pi^2} \int d^4x \bar{\psi}(z, w, z^*, w^*) e^{\overleftarrow{\partial}_z\overrightarrow{\partial}_{z^*}} e^{\overleftarrow{\partial}_w\overrightarrow{\partial}_{w^*}} \langle z, w | \hat{D}\hat{\psi} | z, w \rangle. \quad (5.1.10)$$

Using (5.1.3):

$$\partial_z = \sqrt{\frac{\theta^{03}}{2}} \left(\frac{\partial}{\partial x^0} - i \frac{\partial}{\partial x^3} \right), \partial_w = \sqrt{\frac{\theta^{12}}{2}} \left(\frac{\partial}{\partial x^1} - i \frac{\partial}{\partial x^2} \right), \quad (5.1.11)$$

yields,

$$e^{\overleftarrow{\partial}_z \overleftarrow{\partial}_{z^*}} e^{\overleftarrow{\partial}_w \overleftarrow{\partial}_{w^*}} = e^{\frac{i}{2} \theta_V^{\mu\nu} \overleftarrow{\partial}_\mu \overleftarrow{\partial}_\nu}, \quad (5.1.12)$$

known as the Vorus product. Here:

$$\theta_V^{\mu\nu} = \begin{pmatrix} -i\theta^{03} & 0 & 0 & \theta^{03} \\ 0 & -i\theta^{12} & \theta^{12} & 0 \\ 0 & -\theta^{12} & -i\theta^{12} & 0 \\ -\theta^{03} & 0 & 0 & -i\theta^{03} \end{pmatrix}. \quad (5.1.13)$$

This matrix differs from $\theta^{\mu\nu} = -i[x^\mu, x^\nu]$ since it has on-diagonal terms and is not antisymmetric. To distinguish between the two, it is convenient to denote the latter by $\theta_M^{\mu\nu}$, since this corresponds to the matrix used in the Moyal product.

The operator derivative just becomes the usual derivative in the coherent state basis we verify this is (F).

$$\hat{\partial}_0 \rightarrow \frac{\partial}{\partial x^0}, \hat{\partial}_1 \rightarrow \frac{\partial}{\partial x^1}, \hat{\partial}_2 \rightarrow \frac{\partial}{\partial x^2}, \hat{\partial}_3 \rightarrow \frac{\partial}{\partial x^3}, \quad (5.1.14)$$

and thus we can write $\hat{\phi}\psi(z, w, z^*, w^*) \equiv \langle z, w | \hat{\phi}\hat{\psi} | z, w \rangle$.

Next we calculate the interaction term:

$$\begin{aligned} \langle z, w | \hat{A}_\lambda \hat{\star} \hat{\psi} | w, z \rangle &= \langle z, w | \sum_{n=0}^{\infty} \frac{(-\frac{i}{2} \theta^{\mu\nu})^n}{n!} ((\hat{\partial}_\mu)^n A_\lambda) \cdot \\ &\quad \left(\frac{1}{\pi^2} \int d^4 x' |z', w'\rangle \langle z', w'| \right) ((\hat{\partial}_\nu)^n \hat{\psi}) | z, w \rangle \\ &= \sum_{n=0}^{\infty} \frac{(-\frac{i}{2} \theta^{\mu\nu})^n}{n!} \langle z, w | ((\hat{\partial}_\mu)^n A_\lambda) | w, z \rangle e^{\overleftarrow{\partial}_z \overleftarrow{\partial}_{z^*}} e^{\overleftarrow{\partial}_w \overleftarrow{\partial}_{w^*}} \langle z, w | (\hat{\partial}_\nu)^n \hat{\psi} | z, w \rangle \\ &= A_\lambda(z, w) e^{\frac{i}{2} (\theta_V^{\mu\nu} - \theta_M^{\mu\nu}) \overleftarrow{\partial}_\mu \overleftarrow{\partial}_\nu} \psi(z, w). \end{aligned} \quad (5.1.15)$$

This differs from the non-commutative field theories that are conventionally written down, where either a Moyal or Vorus product is simply inserted between fields in the action as discussed previously in chapter one. Explicitly this product reads:

$$(\theta_V^{\mu\nu} - \theta_M^{\mu\nu}) = \begin{pmatrix} -i\theta^{03} & 0 & 0 & 0 \\ 0 & -i\theta^{12} & 0 & 0 \\ 0 & 0 & -i\theta^{12} & 0 \\ 0 & 0 & 0 & -i\theta^{03} \end{pmatrix}. \quad (5.1.16)$$

We denote the various star products as, the Moyal product $*_M$, the Vorus product $*_V$ and the product with the above matrix $*_{V-M}$. The Dirac equation in the coherent state basis is then:

$$(i\hat{\mathcal{D}} - q\hat{A} *_V *_M - m)\psi = 0. \quad (5.1.17)$$

Then using properties of $\hat{*}$ the action can be written as:

$$\begin{aligned} S &= \frac{1}{\pi^2} \int d^4x \bar{\psi}(z, w, z^*, w^*) *_V ((i\hat{\mathcal{D}} - q\hat{A} *_V *_M - m)\psi(z, w, z^*, w^*)) \\ &= \frac{1}{\pi^2} \int d^4x \bar{\psi}(z, w, z^*, w^*) *_V *_M ((i\hat{\mathcal{D}} - q\hat{A} *_V *_M - m)\psi(z, w, z^*, w^*)) \end{aligned} \quad (5.1.18)$$

Noting the symbols of:

$$\hat{a} \rightarrow z, \hat{a}^\dagger \rightarrow z^*, \hat{b} \rightarrow w, \hat{b}^\dagger \rightarrow w^*, \quad (5.1.19)$$

it easily follows from (5.1.3) that:

$$\hat{x}^1 \rightarrow x^1, \hat{x}^2 \rightarrow x^2, \hat{x}^3 \rightarrow x^3, \hat{x}^4 \rightarrow x^4. \quad (5.1.20)$$

We can summarize this by saying that the symbol of \hat{x}^μ is x^μ . The symbol of a Lorentz transformed state $\hat{J}_\omega \hat{\psi}$, focusing only on the spatial part, is then found to be:

$$\begin{aligned} \langle z, w | \hat{J}_\omega \hat{\psi} | w, z \rangle &= \langle z, w | \hat{x}^\nu \omega_\nu^\mu \hat{\partial}_\mu - \frac{i}{2} \omega_\nu^\lambda \theta_M^{\nu\mu} \hat{\partial}_\mu \hat{\partial}_\lambda \hat{\psi} | w, z \rangle \\ &= x^\nu \omega_\nu^\mu *_V \partial_\mu \psi - \frac{i}{2} \omega_\nu^\lambda \theta_M^{\nu\mu} \partial_\mu \partial_\lambda \psi \\ &= x^\nu \omega_\nu^\mu \partial_\mu \psi + \frac{i}{2} \omega_\nu^\lambda \theta_{V-M}^{\nu\mu} \partial_\mu \partial_\lambda \psi. \end{aligned} \quad (5.1.21)$$

Our abstract action consistently produces the Dirac equation. In the coherent state basis we produced a familiar integral action involving star products. However the star products involve a matrix θ_{V-M} which differs from the conventional construction of non-commutative quantum field theories. This matrix also occurs in the symbol of the Lorentz generator, but this is only due to our particular choice of basis, which due to its minimum uncertainty status, makes it the preferred basis if one wants to interpret this as an integral over space-time. Also note that although $\hat{*}$ effectively makes the \hat{A} , $\hat{\psi}$ commute, we still see the effects of non-commutativity in the action.

5.2 Constant Background Magnetic Field Revisited

Previously we have solved the Dirac equation in the presence of a constant magnetic field without any reference to a specific basis. We now repeat this

calculation in the coherent state basis. Using $(z, w | \hat{D} | \psi) = \langle z, w | \hat{D} \hat{\psi} | w, z \rangle$, the Dirac equation of (3.5.2) reads:

$$(i\hat{\phi} - \gamma^1 Bq(x^2 + \frac{i}{2}(-i\theta^{12})\partial_2) - m)\psi = 0. \quad (5.2.1)$$

Now A, ψ are functions of the real parameters x^μ . Setting $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ gives two equations:

$$(-i\sigma_j \partial_j + \sigma_1(Bqy + Bq\frac{1}{2}\theta^{12}\partial_2))\chi = (i\partial_0 - m)\phi, \quad (5.2.2)$$

$$(-i\sigma_j \partial_j + \sigma_1(Bqy + Bq\frac{1}{2}\theta^{12}\partial_2))\phi = (i\partial_0 + m)\chi. \quad (5.2.3)$$

Solving for one of the spinors yields:

$$(-i\sigma_j \partial_j + \sigma_1(Bqy + Bq\frac{1}{2}\theta^{12}\partial_2))^2 \phi = (-\partial_0^2 - m^2)\hat{\phi}. \quad (5.2.4)$$

Setting $\phi = e^{-ik^0 t + ik^1 x^1 + ik^3 x^3} \tilde{\phi}$, but dropping the $\tilde{}$ for notational convenience in what follows, we find after writing out the matrix explicitly:

$$\begin{pmatrix} k^3 & k^1 + Bqy + \frac{1}{2}Bq\theta^{12}\partial_2 - \partial_2 \\ k^1 + Bqy + \frac{1}{2}Bq\theta^{12}\partial_2 + \partial_2 & -k^3 \end{pmatrix}^2 \phi = ((k^0)^2 - m^2)\mathbb{I}_2\phi. \quad (5.2.5)$$

Again transform to the variable $\xi = \frac{1}{l_B}(y + \text{sgn}(q)l_B^2 k^1)$:

$$\begin{pmatrix} k^3 & \frac{1}{l_B} \left((\text{sgn}(q)\frac{\theta^{12}}{2l_B^2} - 1)\frac{d}{d\xi} + \text{sgn}(q)\xi \right) \\ \frac{1}{l_B} \left((\text{sgn}(q)\frac{\theta^{12}}{2l_B^2} + 1)\frac{d}{d\xi} + \text{sgn}(q)\xi \right) & -k^3 \end{pmatrix}^2 \phi = ((k^0)^2 - m^2)\mathbb{I}_2\phi. \quad (5.2.6)$$

After squaring the matrix this reads:

$$\begin{pmatrix} (k^3)^2 + \frac{1}{l_B^2} \left(\left(\left(\frac{\theta^{12}}{2l_B^2} \right)^2 - 1 \right) \frac{d^2}{d^2\xi} + \xi^2 + \frac{\theta^{12}}{l_B^2} \xi \frac{d}{d\xi} + \frac{\theta^{12}}{2l_B^2} \right) & \\ & \end{pmatrix} \mathbb{I}_2\phi - \frac{\text{sgn}(q)}{l_B^2} \sigma^3 \phi = ((k^0)^2 - m^2)\mathbb{I}_2\phi. \quad (5.2.7)$$

In order to solve this we have to consider three possibilities $(\frac{\theta^{12}}{2l_B^2})^2 < 1$, $(\frac{\theta^{12}}{2l_B^2})^2 = 1$, $(\frac{\theta^{12}}{2l_B^2})^2 > 1$. For the first case consider $\phi = e^{-b\xi^2} \tilde{\phi}$ then:

$$\begin{aligned} \frac{d^2}{d^2\xi} e^{-b\xi^2} \tilde{\phi} &= \frac{d}{d\xi} (-2b\xi e^{-b\xi^2} \tilde{\phi} + e^{-b\xi^2} \frac{d}{d\xi} \tilde{\phi}) \\ &= e^{-b\xi^2} (-2b + (2b\xi)^2 - 4b\xi \frac{d}{d\xi} + \frac{d^2}{d^2\xi}) \tilde{\phi}. \end{aligned} \quad (5.2.8)$$

Setting:

$$b = \frac{\frac{\theta^{12}}{2l_B^2}}{2 \left(\left(\frac{\theta^{12}}{2l_B^2} \right)^2 - 1 \right)}, \quad (5.2.9)$$

and inserting this into (5.2.7) gives:

$$\begin{aligned} & \frac{1}{l_B^2} \left(\left(\left(\frac{\theta^{12}}{2l_B^2} \right)^2 - 1 \right) \frac{d^2}{d^2\xi} - \frac{1}{\left(\left(\frac{\theta^{12}}{2l_B^2} \right)^2 - 1 \right)} \xi^2 \right) \mathbb{I}_2 \tilde{\phi} - \frac{\text{sgn}(q)}{l_B^2} \sigma^3 \tilde{\phi} \\ & = ((k^0)^2 - m^2) \mathbb{I}_2 \tilde{\phi}. \end{aligned} \quad (5.2.10)$$

Define again a new variable:

$$\tilde{\xi} = \frac{1}{\sqrt{1 - \left(\frac{\theta^{12}}{2l_B^2} \right)^2}} \xi, \quad (5.2.11)$$

then the equation reads:

$$\frac{1}{l_B^2} \left(-\frac{d^2}{d^2\tilde{\xi}} + \tilde{\xi}^2 \right) \mathbb{I}_2 \tilde{\phi} - \frac{\text{sgn}(q)}{l_B^2} \sigma^3 \tilde{\phi} = ((k^0)^2 - m^2) \mathbb{I}_2 \tilde{\phi}. \quad (5.2.12)$$

This is the same form as previously and we know the solutions for $\tilde{\phi}$, while the energies stay the same. Now consider $\tilde{\phi}_{+,n}$ using the same notation as previously:

$$\begin{aligned} \tilde{\phi}_{+,n} &= \begin{pmatrix} e^{-b\xi^2} F_{n+}(\tilde{\xi}) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{-b\xi^2} N_{n+} e^{-\frac{\tilde{\xi}^2}{2}} H_{n+}(\tilde{\xi}) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{\left(\frac{\theta^{12}}{2l_B^2} - 1 \right) \frac{\tilde{\xi}^2}{2}} N_{n-1} H_{n-1}(\tilde{\xi}) \\ 0 \end{pmatrix}. \end{aligned} \quad (5.2.13)$$

This has the same form as previously, but with a scaled Gaussian. For $\frac{\theta^{12}}{2l_B^2} = 1$ we return to (5.2.7):

$$\begin{aligned} & \left((k^3)^2 + \frac{1}{l_B^2} \left(\xi^2 + 2\xi \frac{d}{d\xi} + 1 \right) \right) \mathbb{I}_2 \phi - \frac{\text{sgn}(q)}{l_B^2} \sigma^3 \phi \\ & = ((k^0)^2 - m^2) \mathbb{I}_2 \phi. \end{aligned} \quad (5.2.14)$$

The solution in this case is:

$$\phi_{\pm,n} = N e^{-\frac{\xi^2}{4}} \xi^{\frac{\lambda}{2}}. \quad (5.2.15)$$

We see this state is only normalizable in $\xi = \sqrt{\frac{2}{\theta^{12}}}(y + \text{sgn}(q)\frac{k^1\theta^{12}}{2})$, if $\lambda = l_B^2((k^0)^2 - m^2 - (k^3)^2 \pm \frac{\text{sgn}(q)}{l_B^2}) - 1 > 0$. However if we look at the operator equation at this value of B we obtain a simple harmonic oscillator equation in the y direction, the $\text{sgn}(q)$ dictating what ordering is required to cancel the x terms. So the solutions must have quantized energy $\lambda = 2n_{\pm}$, and we have the correct spectrum. This is similar to the commutative case of mapping creation and annihilation operators onto $\frac{\partial}{\partial z}, z$ then using the same argument for annihilation operators on the vacuum states, giving zero since the energy must be bounded from below, we know that $z\partial_z$ has an eigenfunction z^n where n is an integer. This is one of the many subtleties that one can encounter when going to the coherent state basis, where knowledge of the underlying operator equation is required. Lastly if $\frac{\theta^{12}}{2l_B^2} > 1$, we follow the same steps as in the first case and obtain the same scaled Gaussian, but now $(\frac{\theta^{12}}{2l_B^2} - 1) > 1$. These states are normalizable in y which we will see when we discuss the inner product in the next section.

5.3 The Symbol Map

We now return to the map from operators to symbols, $m(\hat{\psi}) = (z, w|\psi) = \langle z, w|\hat{\psi}|w, z\rangle = \psi(z, z^*, w, w^*) = \psi(x^\mu) = \psi$. The inner product on the operator level is mapped to the following inner product on symbols, upto a factor $\frac{1}{\pi^2}$,:

$$(\phi|\psi) = \text{Tr}_c(\hat{\phi}^\dagger\hat{\psi}) \rightarrow \int d^4x\phi^* e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu}\psi. \quad (5.3.1)$$

So if our Hilbert space is the set of Hilbert-Schmidt operators then the space of symbols is not simply L^2 but the space where the above inner product is finite, which is a smaller subset of L^2 i.e. a subset of Schwartz class functions. In Fourier space this inner product becomes:

$$\begin{aligned} & \int d^4x d^4k d^4k' \phi^*(k') e^{-i(k')^\mu x_\mu} e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu} \psi(k) e^{ik^\mu x_\mu} \\ &= \int d^4k \phi^*(k) \psi(k) e^{\frac{1}{2}(\theta^{12}((k^1)^2 + (k^2)^2) + \theta^{03}((k^0)^2 + (k^3)^2))}. \end{aligned} \quad (5.3.2)$$

The finite Tr_c condition for operators is actually more restrictive than the square integrability of functions, it enforces a smoothness on length scales smaller than theta. Schwartz class functions have derivatives that decay rapidly enough on the length scale of theta. As we can see in the above equation, for the inner product to be finite, the high momentum modes must be suppressed because of the positive theta exponential. The suppression of high momentum modes implies, through the Fourier transform, a smoothness on small lengths scales.

Besides a modified inner product, the calculation of uncertainties must also be modified. Since the expectation value, following directly from the modified inner product, in the coherent state basis is different:

$$\langle O(z, z^*, w, w^*) \rangle = \int d^4x \psi^* *_V{}_{-M} O *_V{}_{-M} \psi. \quad (5.3.3)$$

We can easily find the symbol of an operator multiplied with position:

$$m(x^\mu \hat{\psi}) = (x^\mu + i \frac{\theta^{\mu\nu}}{2} \partial_\nu) \psi. \quad (5.3.4)$$

$x^\mu + i \frac{\theta^{\mu\nu}}{2} \partial_\nu$ are the so called Bopp shifted coordinates [5]. These satisfy the same commutation relations as the operator coordinates since $\frac{i}{2}([\theta_V^{\mu\nu} \partial_\nu, x^\alpha] + [x^\mu, \theta_V^{\alpha\beta} \partial_\beta]) = i\theta_M^{\mu\alpha}$. This must be the case, a particular representation of an operator must have its algebraic properties.

As an explicit example of finding a symbol directly from an operator, we can calculate the symbol of the free particle solution in 2 dimensions \hat{x}, \hat{y} :

$$\begin{aligned} (w|\psi) &= \langle w| N e^{i\kappa \hat{b}^\dagger} e^{i\kappa^* \hat{b}} |w\rangle \\ &= N e^{-|w|^2} \langle 0| e^{w^* \hat{b}} e^{i\kappa \hat{b}^\dagger} e^{i\kappa^* \hat{b}} e^{w \hat{b}^\dagger} |0\rangle \\ &= N e^{i\kappa w^* + i\kappa^* w}. \end{aligned} \quad (5.3.5)$$

Returning to the magnetic field solutions, as in chapter three we can integrate over k^1 , we relabel this as k_x , we parametrize the x Gaussian by $\tilde{\alpha}$ which is not necessarily the same as in chapter three. We can then also check normalizability as in chapter three, but we have to use the inner product (5.3.1). We do this because we did not calculate the symbols directly, instead we solved the symbolic Dirac equation which could possibly produce solutions from all L^2 . We express this in the Weyl representation, k_y is the Fourier variable for y . This is for the case $\frac{\theta^{12}}{2l_B^2} \neq 1$. Recalling (5.2.11) to write the Gaussians in terms of y .

$$\begin{aligned}
 & N_0 \int dx dy dk_x dk'_x e^{-\tilde{\alpha}^2 (k'_x)^2} e^{\frac{(\theta^{12})^2}{2l_B^2} - 1} \frac{(\tilde{\xi}')^2}{2} e^{-ik'_x x} e^{\frac{i}{2} \theta_{V-M}^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} e^{ik_x x} e^{-\tilde{\alpha}^2 k_x^2} e^{\frac{(\theta^{12})^2}{2l_B^2} - 1} \frac{\tilde{\xi}^2}{2} \\
 &= N_0 \sqrt{\frac{1}{\pi l_B^2}} \int dx dy dk_x dk'_x e^{-\tilde{\alpha}^2 (k'_x)^2} e^{-\tilde{\alpha}^2 k_x^2} e^{-i(k'_x - k_x)x} e^{\frac{i}{2} \theta^{12} k'_x k_x} \\
 &\quad e^{-\frac{(y + \text{sgn}(q) l_B^2 k'_x)^2}{l_B^2 2(\frac{\theta^{12}}{2l_B^2} + 1)}} e^{\frac{i}{2} \theta^{12} (\overleftarrow{\partial}_2 \overrightarrow{\partial}_2)} e^{-\frac{(y + \text{sgn}(q) l_B^2 k_x)^2}{l_B^2 2(\frac{\theta^{12}}{2l_B^2} + 1)}} \\
 &= N_0 \int dk_x dk_y e^{-2\tilde{\alpha}^2 k_x^2} e^{\frac{1}{2} \theta^{12} (k_x^2 + k_y^2)} |\psi(k_y)|^2 \\
 &= N_0 \int dk_x dk_y e^{-2\tilde{\alpha}^2 k_x^2} e^{\frac{1}{2} \theta^{12} (k_x^2 + k_y^2)} |e^{-\frac{1}{2} k_y^2 l_B^2 (\frac{\theta^{12}}{2l_B^2} + 1)} e^{ik_y \text{sgn}(q) l_B^2 k_x}|^2 \\
 &= N_0 \int dk_x dk_y e^{-2\tilde{\alpha}^2 k_x^2} e^{\frac{1}{2} \theta^{12} (k_x^2 + k_y^2)} e^{-k_y^2 l_B^2 (\frac{\theta^{12}}{2l_B^2} + 1)} \\
 &= N_0 \int dk_x dk_y e^{-(2\tilde{\alpha}^2 - \frac{\theta^{12}}{2}) k_x^2} e^{-k_y^2 l_B^2} \tag{5.3.6}
 \end{aligned}$$

For normalizability we must require $2\tilde{\alpha}^2 - \frac{\theta^{12}}{2} > 0$. It follows easily that for $\frac{\theta^{12}}{2l_B^2} = 1$ we obtain the same condition. This simply reflects the fact that the wave-functions must be smooth on short length scales in order to be valid symbols, i.e., the images of operators under the symbol map. The important point here is that only these solutions of the Dirac equation are acceptable.

We could have also calculated the symbols directly from the solution of the operator equation, although generally this is quite difficult. In our case we have (3.5.17), then we can write the \hat{y}^2 exponential as a Gaussian integral and

then we write \hat{x}, \hat{y} in terms of b, b^\dagger (2.1.3), and also use (5.1.3):

$$\begin{aligned}
 \langle w | \hat{\psi} | w \rangle &= \langle w | N_0 \int dk_x e^{-\alpha^2 k_x^2} e^{-\frac{\xi^2}{2}} e^{ik_x \hat{x}} | w \rangle \\
 &= \langle w | \frac{N_0}{\sqrt{\pi}} \int dk_x dve^{-v^2} e^{-\alpha^2 k_x^2} e^{i\sqrt{2}v\xi} e^{ik_x \hat{x}} | w \rangle \\
 &= \frac{N_0}{\sqrt{\pi}} \int dk_x dve^{-v^2} e^{-\alpha^2 k_x^2} e^{i\sqrt{2}v \frac{k_x(\text{sgn}(q)l_B^2 - \frac{1}{2}\theta^{12})}{l_B}} \langle w | e^{-v \frac{\sqrt{\theta^{12}}}{l_B} b^\dagger} e^{v \frac{\sqrt{\theta^{12}}}{l_B} b} \\
 &\quad e^{(-v \frac{\sqrt{\theta^{12}}}{l_B})^2 (-\frac{1}{2})[b^\dagger, -b]} e^{ik_x \frac{\sqrt{\theta^{12}}}{2} b^\dagger} e^{ik_x \frac{\sqrt{\theta^{12}}}{2} b} e^{(ik_x \frac{\sqrt{\theta^{12}}}{2})^2 (-\frac{1}{2})[b^\dagger, b]} | w \rangle \quad (5.3.7) \\
 &= \frac{N_0}{\sqrt{\pi}} \int dk_x dve^{-v^2} e^{-\alpha^2 k_x^2} e^{i\sqrt{2}v \frac{k_x(\text{sgn}(q)l_B^2 - \frac{1}{2}\theta^{12})}{l_B}} e^{-v^2 \frac{\theta^{12}}{2l_B^2}} e^{-k_x^2 \frac{\theta^{12}}{4}} e^{-w^* v \frac{\sqrt{\theta^{12}}}{l_B}} e^{wv \frac{\sqrt{\theta^{12}}}{l_B}} \\
 &\quad e^{ivk_x \frac{\theta^{12}}{\sqrt{2}l_B}} e^{iwk_x \sqrt{\frac{\theta^{12}}{2}}} e^{iw^* k_x \sqrt{\frac{\theta^{12}}{2}}} \\
 &= \frac{N_0}{\sqrt{\pi}} \int dk_x dve^{-v^2(1 + \frac{\theta^{12}}{2l_B^2})} e^{i\sqrt{2}v \frac{k_x(\text{sgn}(q)l_B^2 - \frac{1}{2}\theta^{12})}{l_B}} e^{-\alpha^2 k_x^2} e^{-k_x^2 \frac{\theta^{12}}{4}} e^{iv\sqrt{2} \frac{(y + \frac{k_x \theta^{12}}{2})}{l_B}} e^{ik_x x} \\
 &= N_0 \int dk_x e^{-\alpha^2 k_x^2} e^{-\frac{(y + \text{sgn}(q)l_B^2 k_x)^2}{l_B^2 2(\frac{\theta^{12}}{2l_B^2} + 1)}} e^{-k_x^2 \frac{\theta^{12}}{4}} e^{ik_x x}. \quad (5.3.8)
 \end{aligned}$$

We see this symbol matches with solutions from the symbolic Dirac equation, when $\frac{\theta^{12}}{2l_B^2} \neq 1$. The factor $e^{-k_x^2 \frac{\theta^{12}}{4}}$ relates $\alpha, \tilde{\alpha}$ and ensures that α is unrestricted as was found for the calculation done in chapter three on the operator level.

In this chapter we found an abstract action which produces the non-commutative Dirac equation from variation. We express this in the coherent state basis in terms of symbols, with a modified star product. Of course the coherent state is just one particular choice of basis, but with a good physical motivation, for every valid choice of basis we would obtain different wavefunctions composed with different star products. The different representations of positions operators (shifted coordinates) all must satisfy the same commutation relations. We solved the symbolic constant background magnetic field equation, and studied the solutions. An open question remains, how to confine these symbolic solutions. It is not clear how to define solvable boundary conditions in this space. In particular it is difficult to use the same naive approach as in chapter 3, which was only valid up to leading order in θ . An explicit construction would mean solving a symbolic equation that involves step functions Θ composed with star products which is difficult. So it remains to be seen how to count the number of states in a particular Landau level using coherent solutions. This is of interest since with a solid understanding of the subtleties of symbols, we can work with these instead of operator solutions, and then use our knowledge of function spaces such as approximation techniques to calculate physical quantities more easily.

Chapter 6

Conclusion and Outlook

In this conclusion we briefly summarize what has been achieved in this thesis and comment on the outlook on future work that can follow from this thesis.

We have found a consistent framework for relativistic non-commutative quantum mechanics. This was done in the operator formalism using Hilbert-Schmidt operators as the states. It follows closely the two dimensional non-relativistic non-commutative formalism. We found the generators of the Lorentz transformations on the space of Hilbert-Schmidt operators. In this formalism we found that the free equation retains the same spectrum as commutatively and the solutions have the same form. We also found the Lorentz symmetry of the free equation was conserved.

However, if potentials are introduced into the Dirac equation the same way as commutatively the Lorentz symmetry is broken. We restored this symmetry, for operators, by introducing a modified composition rule. Specifically if the potentials and states are combined with a operator star product then these transform in the same way as commutatively, and the Lorentz symmetry is restored. Using this star product we investigated the constant magnetic field, and found the same spectrum as commutatively. Approximating for the symbols we also obtained an effective magnetic field.

We also looked at confining the non-commutative Dirac equation. This was achieved with projection operators on configuration space. From this we found the boundary condition that needed to be satisfied. Unfortunately these can not be satisfied if the state vanishes completely outside the boundary. This not a problem of non-commutative physics however, but of relativistic quantum mechanics in general most likely since it is not a complete theory. The calculation was however consistent with non-relativistic non-commutative calculations for the two dimensional spherical well.

Finally an abstract operator valued action was found for the Dirac equation. Under variation this action produced the correct Dirac equations of motion. We investigated this action in the coherent state basis and obtained an action which was similar to those conventionally written down in non-commutative theories. However, we encountered a different star product, which was due to

our abstract action being constructed to maintain Lorentz invariance. Also in the coherent state basis the constant magnetic field was studied.

Our study was limited to relativistic non-commutative quantum mechanics, ideally in the future this formalism should be extended to non-commutative quantum field theories. This is the ultimate goal, to build a complete consistent non-commutative quantum field theory. This could also alleviate the problems encountered when studying confined Dirac particles. Then the thermodynamics of such systems can also be studied.

Our choice of commutations was motivated by the fact they made the extension from two dimensional non-relativistic non-commutative quantum mechanics easy. Also they emerge naturally from some string theory calculations. There are however other potential commutation relations like spin non-commutativity [44]. It is then also relevant to study the commutation relations in general, and find the ones best suited for either an intrinsic or effective theory. In particular in three dimensions it seems correct to use fuzzy sphere commutation relations. For non-commutative physics to be consistent there should be a non-relativistic limit where the three dimensional commutations relations emerge from the relativistic theory.

Appendices

Appendix A

Commutator of Generators of on Non-commutative Space Lorentz Transformations

In this appendix we verify that the generators of Lorentz transformations on non-commutative space, derived in section 2.2, do satisfy the standard commutation relations of the Lorentz algebra.

$$\begin{aligned}
 [\hat{J}_\omega, \hat{J}_{\omega'}] &= [\hat{x}^\nu \omega_\nu^\mu \hat{\partial}_\mu - \frac{i}{2} \omega_\nu^\lambda \theta^{\nu\mu} \hat{\partial}_\mu \hat{\partial}_\lambda, \hat{x}^{\nu'} (\omega')_{\nu'}^{\mu'} \hat{\partial}_{\mu'} - \frac{i}{2} (\omega')_{\nu'}^{\lambda'} \theta^{\nu'\mu'} \hat{\partial}_{\mu'} \hat{\partial}_{\lambda'}] \\
 &= [\hat{x}^\nu \omega_\nu^\mu \hat{\partial}_\mu, \hat{x}^{\nu'} (\omega')_{\nu'}^{\mu'} \hat{\partial}_{\mu'}] + [-\frac{i}{2} \omega_\nu^\lambda \theta^{\nu\mu} \hat{\partial}_\mu \hat{\partial}_\lambda, \hat{x}^{\nu'} (\omega')_{\nu'}^{\mu'} \hat{\partial}_{\mu'}] + \\
 &\quad [\hat{x}^\nu \omega_\nu^\mu \hat{\partial}_\mu, -\frac{i}{2} (\omega')_{\nu'}^{\lambda'} \theta^{\nu'\mu'} \hat{\partial}_{\mu'} \hat{\partial}_{\lambda'}].
 \end{aligned} \tag{A.0.1}$$

Calculating the terms individually, starting with the first, we have:

$$\begin{aligned}
 [\hat{x}^\nu \omega_\nu^\mu \hat{\partial}_\mu, \hat{x}^{\nu'} (\omega')_{\nu'}^{\mu'} \hat{\partial}_{\mu'}] &= \omega_\nu^\mu (\omega')_{\nu'}^{\mu'} (\hat{x}^{\nu'} [\hat{x}^\nu, \hat{\partial}_{\mu'}] \hat{\partial}_\mu + \hat{x}^\nu [\hat{\partial}_\mu, \hat{x}^{\nu'}] \hat{\partial}_{\mu'} + [\hat{x}^\nu, \hat{x}^{\nu'}] \hat{\partial}_\mu \hat{\partial}_{\mu'}) \\
 &= \hat{x}^\nu (\omega_\nu^\rho (\omega')_{\rho}^{\mu'} - (\omega')_{\nu}^{\rho} \omega_\rho^\mu) \hat{\partial}_\mu + i \omega_\nu^\mu (\omega')_{\nu'}^{\mu'} \theta^{\nu\nu'} \hat{\partial}_\mu \hat{\partial}_{\mu'}.
 \end{aligned} \tag{A.0.2}$$

The second term yields:

$$\begin{aligned}
 [-\frac{i}{2} \omega_\nu^\lambda \theta^{\nu\mu} \hat{\partial}_\mu \hat{\partial}_\lambda, \hat{x}^{\nu'} (\omega')_{\nu'}^{\mu'} \hat{\partial}_{\mu'}] &= -\frac{i}{2} \omega_\nu^\lambda (\omega')_{\nu'}^{\mu'} \theta^{\nu\mu} (\hat{\partial}_\lambda [\hat{\partial}_\mu, \hat{x}^{\nu'}] + [\hat{\partial}_\lambda, \hat{x}^{\nu'}] \hat{\partial}_\mu) \hat{\partial}_{\mu'} \\
 &= -\frac{i}{2} \omega_\nu^\lambda (\omega')_{\nu'}^{\mu'} \theta^{\nu\nu'} \hat{\partial}_\lambda \hat{\partial}_{\mu'} - \frac{i}{2} \omega_\nu^\rho (\omega')_{\rho}^{\mu'} \theta^{\nu\lambda} \hat{\partial}_\lambda \hat{\partial}_{\mu'}.
 \end{aligned} \tag{A.0.3}$$

From this the third term is easily calculated:

$$\begin{aligned}
 [\hat{x}^\nu \omega_\nu^\mu \hat{\partial}_\mu, -\frac{i}{2} (\omega')_{\nu'}^{\lambda'} \theta^{\nu'\mu'} \hat{\partial}_{\mu'} \hat{\partial}_{\lambda'}] &= \frac{i}{2} (\omega')_{\nu'}^{\lambda} \omega_{\nu'}^{\mu'} \theta^{\nu\nu'} \hat{\partial}_\lambda \hat{\partial}_{\mu'} + \frac{i}{2} (\omega')_{\nu}^{\rho} \omega_{\rho}^{\mu'} \theta^{\nu\lambda} \hat{\partial}_\lambda \hat{\partial}_{\mu'} \\
 &= -\frac{i}{2} \omega_\nu^\lambda (\omega')_{\nu'}^{\mu'} \theta^{\nu\nu'} \hat{\partial}_\lambda \hat{\partial}_{\mu'} + \frac{i}{2} (\omega')_{\nu}^{\rho} \omega_{\rho}^{\mu'} \theta^{\nu\lambda} \hat{\partial}_\lambda \hat{\partial}_{\mu'}.
 \end{aligned} \tag{A.0.4}$$

Putting these together:

$$\begin{aligned} [\hat{J}_\omega, \hat{J}_{\omega'}] &= \hat{x}^\nu (\omega_\nu^\rho (\omega')_\rho^\mu - (\omega')_\nu^\rho \omega_\rho^\mu) \hat{\partial}_\mu - \frac{i}{2} (\omega_\nu^\rho (\omega')_\rho^{\mu'} - (\omega')_\nu^\rho \omega_\rho^{\mu'}) \theta^{\nu\lambda} \hat{\partial}_\lambda \hat{\partial}_{\mu'} \\ &= \hat{J}_{\omega \times \omega'}, \end{aligned} \tag{A.0.5}$$

where $\omega \times \omega' = \omega_\nu^\rho (\omega')_\rho^\mu - (\omega')_\nu^\rho \omega_\rho^\mu$.

Appendix B

Lorentz Generator Acting on the Modified Interaction

$$\hat{J}_\omega(\hat{\psi}\hat{\star}\hat{\phi}) = \left[\hat{x}^\nu \omega_\nu^\mu \hat{\partial}_\mu - \frac{i}{2} \omega_\nu^\lambda \theta^{\nu\mu} \hat{\partial}_\mu \hat{\partial}_\lambda \right] (\hat{\psi} e^{-\frac{i}{2} \theta^{\mu\nu} \hat{\partial}_\mu \hat{\partial}_\nu} \hat{\phi}). \quad (\text{B.0.1})$$

Let us verify this explicitly by starting with the first term in (B.0.1):

$$\hat{x}^\nu \omega_\nu^\mu \hat{\partial}_\mu (\hat{\psi}\hat{\star}\hat{\phi}) = \hat{x}^\nu ((\omega_\nu^\mu \hat{\partial}_\mu \hat{\psi})\hat{\star}(\hat{\phi}) + (\hat{\psi})\hat{\star}(\omega_\nu^\mu \hat{\partial}_\mu \hat{\phi})). \quad (\text{B.0.2})$$

We use the following notation:

$$(\theta^{\mu\nu})^n (\hat{\partial}_\mu)^n \hat{\psi} (\hat{\partial}_\nu)^n \hat{\phi} = \theta^{\mu_1\nu_1} \theta^{\mu_2\nu_2} \dots \theta^{\mu_n\nu_n} \hat{\partial}_{\mu_1} \hat{\partial}_{\mu_2} \dots \hat{\partial}_{\mu_n} \hat{\psi} \hat{\partial}_{\nu_1} \hat{\partial}_{\nu_2} \dots \hat{\partial}_{\nu_n} \hat{\phi}. \quad (\text{B.0.3})$$

The following is useful,

$$\begin{aligned} \hat{x}^\rho (\hat{\psi}\hat{\star}\hat{\phi}) &= (\hat{x}^\rho \hat{\psi})\hat{\star}(\hat{\phi}) + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left(-\frac{i}{2} \theta^{\mu\nu}\right)^n (\hat{\partial}_\mu)^{n-1} \hat{\psi} (\hat{\partial}_\nu)^n \hat{\phi} (-\delta_\mu^\rho) \\ &= (\hat{x}^\rho \hat{\psi})\hat{\star}(\hat{\phi}) + \frac{i}{2} \theta^{\rho\nu} \hat{\psi} \hat{\star}(\hat{\partial}_\nu \hat{\phi}). \end{aligned}$$

Another way of writing this is:

$$\begin{aligned} \hat{x}^\rho (\hat{\psi}\hat{\star}\hat{\phi}) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{2} \theta^{\mu\nu}\right)^n ((\hat{\partial}_\mu)^n \hat{\psi}) \hat{x}^\rho ((\hat{\partial}_\nu)^n \hat{\phi}) + \\ &\quad \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{2} \theta^{\mu\nu}\right)^n [\hat{x}^\rho, (\hat{\partial}_\mu)^n \hat{\psi}] ((\hat{\partial}_\nu)^n \hat{\phi}) \\ &= (\hat{\psi})\hat{\star}(\hat{x}^\rho \hat{\phi}) + \frac{i}{2} \theta^{\rho\nu} (\hat{\partial}_\nu \hat{\psi})\hat{\star}\hat{\phi} + \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{2} \theta^{\mu\nu}\right)^n (i\theta^{\rho\lambda} \partial_\lambda (\hat{\partial}_\mu)^n \hat{\psi}) ((\hat{\partial}_\nu)^n \hat{\phi}) \\ &= (\hat{\psi})\hat{\star}(\hat{x}^\rho \hat{\phi}) + \frac{i}{2} \theta^{\rho\nu} (\hat{\partial}_\nu \hat{\psi})\hat{\star}\hat{\phi} + i\theta^{\rho\lambda} (\partial_\lambda \hat{\psi})\hat{\star}(\hat{\phi}). \end{aligned}$$

Putting these results into (B.0.2):

$$\begin{aligned} \hat{x}^\nu \omega_\nu^\mu \hat{\partial}_\mu (\hat{\psi} \hat{\star} \hat{\phi}) &= (\hat{x}^\nu \omega_\nu^\mu \hat{\partial}_\mu \hat{\psi}) \hat{\star} (\hat{\phi}) + (\hat{\psi}) \hat{\star} (\hat{x}^\nu \omega_\nu^\mu \hat{\partial}_\mu \hat{\phi}) + \\ &\frac{i}{2} (\omega_\nu^\mu \theta^{\nu\rho} + \omega_\nu^\rho \theta^{\nu\mu}) \hat{\partial}_\mu \hat{\psi} \hat{\star} \hat{\partial}_\rho \hat{\phi}. \end{aligned} \quad (\text{B.0.4})$$

Next consider the second term in (B.0.1):

$$\begin{aligned} -\frac{i}{2} \omega_\nu^\lambda \theta^{\nu\mu} \hat{\partial}_\mu \hat{\partial}_\lambda (\hat{\psi} \hat{\star} \hat{\phi}) &= -\frac{i}{2} \omega_\nu^\lambda \theta^{\nu\mu} \hat{\partial}_\mu ((\hat{\partial}_\lambda \hat{\psi}) \hat{\star} \hat{\phi} + \hat{\psi} \hat{\star} (\hat{\partial}_\lambda \hat{\phi})) \\ &= -\frac{i}{2} \omega_\nu^\lambda \theta^{\nu\mu} ((\hat{\partial}_\mu \hat{\partial}_\lambda \hat{\psi}) \hat{\star} \hat{\phi} + \hat{\psi} \hat{\star} (\hat{\partial}_\mu \hat{\partial}_\lambda \hat{\phi}) + (\hat{\partial}_\lambda \hat{\psi}) \hat{\star} (\hat{\partial}_\mu \hat{\phi}) + (\hat{\partial}_\mu \hat{\psi}) \hat{\star} (\hat{\partial}_\lambda \hat{\phi})) \\ &= -\frac{i}{2} \omega_\nu^\lambda \theta^{\nu\mu} ((\hat{\partial}_\mu \hat{\partial}_\lambda \hat{\psi}) \hat{\star} \hat{\phi} + \hat{\psi} \hat{\star} (\hat{\partial}_\mu \hat{\partial}_\lambda \hat{\phi})) - \frac{i}{2} (\omega_\nu^\lambda \theta^{\nu\mu} + \omega_\nu^\mu \theta^{\nu\lambda}) (\hat{\partial}_\mu \hat{\psi}) \hat{\star} (\hat{\partial}_\lambda \hat{\phi}). \end{aligned} \quad (\text{B.0.5})$$

Putting together (B.0.4) and (B.0.5):

$$\hat{J}_\omega (\hat{\psi} \hat{\star} \hat{\phi}) = (\hat{J}_\omega \hat{\psi}) \hat{\star} \hat{\phi} + \hat{\psi} \hat{\star} (\hat{J}_\omega \hat{\phi}). \quad (\text{B.0.6})$$

Appendix C

Properties of the $\hat{\ast}$ Product

Here we list some properties of the $\hat{\ast}$ product as derived from the Weyl representation, for general antisymmetric θ :

$$\begin{aligned}
 \hat{\phi} \hat{\ast} \hat{\psi} &= \int d^4 q \int d^4 k \phi(q) \psi(k) e^{iq_\rho \hat{x}^\rho} e^{-\frac{i}{2} \theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} e^{ik_\gamma \hat{x}^\gamma} \\
 &= \int d^4 q \int d^4 k \phi(q) \psi(k) e^{ik_\gamma \hat{x}^\gamma} e^{\frac{i}{2} \theta^{\mu\nu} q_\mu k_\nu} e^{iq_\rho \hat{x}^\rho} e^{-iq_\rho k_\gamma \theta^{\rho\gamma}} \\
 &= \int d^4 q \int d^4 k \phi(q) \psi(k) e^{ik_\gamma \hat{x}^\gamma} e^{-\frac{i}{2} \theta^{\mu\nu} q_\mu k_\nu} e^{iq^\rho \hat{x}_\rho} \\
 &= \int d^4 q \int d^4 k \phi(q) \psi(k) e^{ik_\gamma \hat{x}^\gamma} e^{\frac{i}{2} \theta^{\mu\nu} k_\mu q_\nu} e^{iq^\rho \hat{x}_\rho} \\
 &= \hat{\psi} \hat{\ast} \hat{\phi},
 \end{aligned} \tag{C.0.1}$$

after applying derivatives and changing the order of exponentials we can see objects composed through the star product commute. Next we consider the associativity of the $\hat{\ast}$ product:

$$\begin{aligned}
 \hat{\psi} \hat{\ast} (\hat{\phi} \hat{\ast} \hat{\chi}) &= \int d^4 q \int d^4 k \int d^4 p \phi(q) \psi(k) \chi(p) \cdot \\
 &\quad e^{iq_\rho \hat{x}^\rho} e^{-\frac{i}{2} \theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} (e^{ik_\gamma \hat{x}^\gamma} e^{-\frac{i}{2} \theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} e^{ip_\tau \hat{x}^\tau}) \\
 &= \int d^4 q \int d^4 k \int d^4 p \phi(q) \psi(k) \chi(p) e^{iq_\rho \hat{x}^\rho} e^{\frac{i}{2} \theta^{\mu\nu} q_\mu (k+p)_\nu} e^{\frac{i}{2} \theta^{\mu\nu} k_\mu p_\nu} e^{ik_\gamma \hat{x}^\gamma} e^{ip_\tau \hat{x}^\tau} \\
 &= \int d^4 q \int d^4 k \int d^4 p \phi(q) \psi(k) \chi(p) e^{\frac{i}{2} \theta^{\mu\nu} q_\mu k_\nu} e^{\frac{i}{2} \theta^{\mu\nu} (k+q)_\mu p_\nu} e^{iq_\rho \hat{x}^\rho} e^{ik_\gamma \hat{x}^\gamma} e^{ip_\tau \hat{x}^\tau} \\
 &= (\hat{\psi} \hat{\ast} \hat{\phi}) \hat{\ast} \hat{\chi},
 \end{aligned} \tag{C.0.2}$$

hence $\hat{\ast}$ is associative. Next we note:

$$\begin{aligned}
\text{Tr}(\hat{\psi}\hat{\ast}\hat{\phi}) &= \text{Tr}(\hat{\psi}e^{-\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu}\hat{\phi}) \\
&= \text{Tr}(\hat{\psi}e^{\frac{i}{2}\theta^{\mu\nu}\overrightarrow{\partial}_\mu\overleftarrow{\partial}_\nu}\hat{\phi}) \\
&= \text{Tr}(\hat{\psi}e^0\hat{\phi}) \\
&= \text{Tr}(\hat{\psi}\hat{\phi}),
\end{aligned} \tag{C.0.3}$$

which implies $\hat{\ast}$ doesn't change the inner product. From the following relation:

$$\begin{aligned}
\hat{x}^\rho\hat{\ast}\hat{x}_\rho &= \hat{x}^\rho\hat{x}_\rho - \frac{i}{2}\theta^\rho{}_\rho \\
&= \hat{x}^\rho\hat{x}_\rho - \frac{i}{2}g_{\rho\mu}\theta^{\rho\mu} \\
&= \hat{x}^\rho\hat{x}_\rho,
\end{aligned} \tag{C.0.4}$$

we conclude that the invariant interval s^2 doesn't change. Consider a Lorentz transformation by parameter ω such $\hat{x} \rightarrow \hat{x}'$ then:

$$\begin{aligned}
\hat{\psi}(\hat{x}')\hat{\ast}\hat{\phi}(\hat{x}') &= \hat{\psi}(\hat{x}')e^{-\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu}\hat{\phi}(\hat{x}') \\
&= \hat{\psi}(\hat{x}')e^{-\frac{1}{2}[(g_\mu^\lambda+\omega_\mu^\lambda)\hat{x}^\mu, (g_\nu^\rho+\omega_\nu^\rho)\hat{x}^\nu]\overleftarrow{\partial}_\lambda\overrightarrow{\partial}_\rho}\hat{\phi}(\hat{x}') \\
&= \hat{\psi}(\hat{x}')e^{-\frac{i}{2}(\theta')^{\lambda\rho}\overleftarrow{\partial}'_\lambda\overrightarrow{\partial}'_\rho}\hat{\phi}(\hat{x}') \\
&= \hat{\psi}(\hat{x}')\hat{\ast}'\hat{\phi}(\hat{x}'),
\end{aligned} \tag{C.0.5}$$

i.e. $\hat{\ast}$ doesn't change in different frames of reference. Finally we note:

$$(\hat{\phi}\hat{\ast}\hat{\psi})^\dagger = \hat{\phi}^\dagger\hat{\ast}\hat{\psi}^\dagger. \tag{C.0.6}$$

Appendix D

Uncertainty for the Constant Magnetic Field

We can check that these are, indeed, states in the Hilbert space H_q , i.e. $\text{Tr}_c(\psi^\dagger\psi) < \infty$, we consider the case ϕ_{-,n_-} in it's ground state, and check the first component, since each component should have finite Tr_c . Also since we have plane-waves in the z, t directions which are strictly not normalizable, however for z this can be overcome in the same way as commutative physics with the use of wave-packets, but for simplicity we only consider the x, y directions. Since k^1 is the only momentum we drop the ¹. We note the property of x, y eigenvalues $\langle x|y\rangle = \frac{1}{\sqrt{2\pi\theta^{12}}}e^{\frac{ixy}{\theta^{12}}}$ and also $2\pi\delta(p-q) = \int_{-\infty}^{\infty} dx e^{ix(p-q)}$.

$$\begin{aligned}
 & N_0 \int dx \langle x| \int dk' e^{-ik'\hat{x}} e^{-\alpha^2(k')^2} e^{-\frac{(\xi')^2}{2}} \int dy |y\rangle \langle y| \int dk e^{-\alpha^2 k^2} e^{-\frac{\xi^2}{2}} e^{ik\hat{x}} |x\rangle \\
 &= N_0 \int dk' dk dx dy e^{-\alpha^2(k')^2} e^{-\alpha^2 k^2} e^{-\frac{(\xi')^2}{2}} e^{-\frac{\xi^2}{2}} e^{i(k-k')x} |\langle x|y\rangle|^2 \\
 &= N_0 \frac{1}{\theta^{12}} \sqrt{\pi l_B^2} \int dk e^{-2\alpha^2 k^2} \\
 &= 1
 \end{aligned} \tag{D.0.1}$$

Now we can calculate $\langle x \rangle, \langle x^2 \rangle, \langle y \rangle, \langle y^2 \rangle$ and from this $\Delta x \Delta y$, in order to investigate how the lengths scales l_B and θ effect the uncertainty and possible localization of states, we only do this for one component of the spinor in the

ground state:

$$\begin{aligned}
 \langle x \rangle &= \frac{N_0}{\sqrt{\pi}} \int dk dk' dy dx e^{-\alpha^2 k^2} e^{-\alpha^2 (k')^2} e^{-\frac{(\xi')^2}{2}} \langle x|y \rangle \langle y|\hat{x} e^{-\frac{\xi^2}{2}}|x \rangle e^{i(k-k')x} \\
 &= \frac{N_0}{\sqrt{\pi}} \int dk dk' dy dx e^{-\alpha^2 k^2} e^{-\alpha^2 (k')^2} e^{-\frac{(\xi')^2}{2}} \langle x|y \rangle (i\theta^{12}) \frac{\partial}{\partial y} (e^{-\frac{\xi^2}{2}} \langle y|x \rangle) e^{i(k-k')x} \\
 &= \frac{iN_0}{\sqrt{\pi}} \int dk dk' dy dx e^{-\alpha^2 k^2} e^{-\frac{\xi^2}{2}} e^{-\alpha^2 (k')^2} e^{-\frac{(\xi')^2}{2}} \left(-\frac{1}{l_B^2} y - i\frac{x}{\theta^{12}}\right) e^{i(k-k')x} \\
 &= 0
 \end{aligned} \tag{D.0.2}$$

$$\begin{aligned}
 \langle x^2 \rangle &= \frac{N_0(\theta^{12})^2}{\sqrt{\pi}} \int dk dk' dy dx e^{-\alpha^2 k^2} e^{-\alpha^2 (k')^2} e^{-\frac{(\xi')^2}{2}} \left(-\frac{1}{l_B^2} - \frac{x^2}{(\theta^{12})^2} + \frac{y^2}{l_B^4}\right) e^{-\frac{\xi^2}{2}} e^{i(k-k')x} \\
 &= \alpha^2 + \frac{1}{2l_B^2} \left(\text{sgn}(q)l_B^2 - \frac{\theta^{12}}{2}\right)^2 + \frac{(\theta^{12})^2}{2l_B^2} \\
 &= \alpha^2 + \frac{l_B^2}{2} \left(\text{sgn}(q) - \frac{\theta^{12}}{2l_B^2}\right)^2 + \frac{(\theta^{12})^2}{2l_B^2}
 \end{aligned} \tag{D.0.3}$$

$$\begin{aligned}
 \langle y \rangle &= \sqrt{\frac{2\alpha^2}{\pi}} \int dk d\xi e^{-2\alpha^2 k^2} e^{-\xi^2} \left(l_B \xi - k \left(\text{sgn}(q)l_B^2 - \frac{1}{2}\theta^{12}\right)\right) \\
 &= 0
 \end{aligned} \tag{D.0.4}$$

$$\begin{aligned}
 \langle y^2 \rangle &= \sqrt{\frac{2\alpha^2}{\pi}} \int dk d\xi e^{-2\alpha^2 k^2} e^{-\xi^2} \left(l_B \xi - k \left(\text{sgn}(q)l_B^2 - \frac{1}{2}\theta^{12}\right)\right)^2 \\
 &= \frac{l_B^2}{2} + \frac{1}{4\alpha^2} \left(\text{sgn}(q)l_B^2 - \frac{1}{2}\theta^{12}\right)^2 \\
 &= \frac{l_B^2}{2} + \frac{l_B^4}{4\alpha^2} \left(\text{sgn}(q) - \frac{\theta^{12}}{2l_B^2}\right)^2
 \end{aligned} \tag{D.0.5}$$

Now we can compute the uncertainty $\Delta x \Delta y$ which we know by definition must be $\sqrt{\Delta x \Delta y} \geq \frac{\theta^{12}}{2}$:

$$\Delta x \Delta y = \left(\sqrt{\frac{(\theta^{12})^2}{4} + \dots} \right) \tag{D.0.6}$$

Since \dots is always positive, we can see that the uncertainty relation always holds as is implied by the finite trace.

Appendix E

Boundary Conditions

Starting from (4.1.13) first we look at each commutator in the matrices acting on some state \hat{f} . We note the following commutator identity: consider an operator \hat{O} such that $\hat{O}\hat{f} = [\hat{A}, \hat{f}]$ then,

$$\begin{aligned} [\hat{O}, \hat{B}]\hat{f} &= [\hat{A}, \hat{B}\hat{f}] - \hat{B}[\hat{A}, \hat{f}] \\ &= [\hat{A}, \hat{B}]\hat{f}. \end{aligned} \quad (\text{E.0.1})$$

The first entry in the commutator matrix acting on a test function \hat{f} gives:

$$\begin{aligned} [\hat{\partial}_0, \int_0^L dz \sum_{n=0}^M |z, n\rangle \langle z, n|]\hat{f} &= \int_0^L dz \sum_{n=0}^M -i(\theta^{-1})_{03}[\hat{z}, |z, n\rangle \langle z, n|]\hat{f} \\ &= 0. \end{aligned} \quad (\text{E.0.2})$$

The second entry gives:

$$\begin{aligned} [\hat{\rho}, \int_0^L dz \sum_{n=0}^M |z, n\rangle \langle z, n|]\hat{f} &= \int_0^L dz \sum_{n=0}^M \sqrt{\frac{2}{\theta_{21}}}[\hat{b}, |z, n\rangle \langle z, n|]\hat{f} \\ &= \int_0^L dz \sqrt{\frac{2}{\theta_{21}}} \left(\sum_{n=1}^M \sqrt{n} |z, n-1\rangle \langle z, n| - \sum_{n=0}^M \sqrt{n+1} |z, n\rangle \langle z, n+1| \right) \hat{f} \\ &= - \int_0^L dz \sqrt{\frac{2}{\theta_{21}}} (\sqrt{M+1} |z, M\rangle \langle z, M+1|) \hat{f}. \end{aligned} \quad (\text{E.0.3})$$

To simplify this we consider the conditions on the matrix elements of operators, when sandwiched between arbitrary states $|\xi, l\rangle$ and $|\xi', l'\rangle$ in configuration

space. From (4.1.14):

$$\begin{aligned}
 & \langle \xi, l | [\hat{\rho}, \int_0^L dz \sum_{n=0}^M |z, n\rangle \langle z, n|] \hat{f} | \xi', l' \rangle \\
 &= - \int_0^L dz \sqrt{\frac{2(M+1)}{\theta_{21}}} \delta_{lM} \delta(\xi - z) (\langle z, M+1 | \hat{f} | \xi', l' \rangle) \\
 &= -\Theta(\xi)\Theta(L - \xi) \sqrt{\frac{2(M+1)}{\theta_{21}}} \delta_{lM} (\langle \xi, M+1 | \hat{f} | \xi', l' \rangle). \tag{E.0.4}
 \end{aligned}$$

The third entry gives:

$$\begin{aligned}
 & [\hat{\rho}^\dagger, \int_0^L dz \sum_{n=0}^M |z, n\rangle \langle z, n|] \hat{f} = - \int_0^L dz \sum_{n=0}^M i \sqrt{\frac{2}{\theta_{21}}} [\hat{b}^\dagger, |z, n\rangle \langle z, n|] \hat{f} \\
 &= - \int_0^L dz \sqrt{\frac{2}{\theta_{21}}} \left(\sum_{n=0}^M \sqrt{n+1} |z, n+1\rangle \langle z, n| - \sum_{n=1}^M \sqrt{n} |z, n\rangle \langle z, n-1| \right) \hat{f} \\
 &= - \int_0^L dz \sqrt{\frac{2}{\theta_{21}}} (\sqrt{M+1} |z, M+1\rangle \langle z, M|) \hat{f}. \tag{E.0.5}
 \end{aligned}$$

For the following matrix element we obtain:

$$\begin{aligned}
 & \langle \xi, l | [\hat{\rho}^\dagger, \int_0^L dz \sum_{n=0}^M |z, n\rangle \langle z, n|] \hat{f} | \xi', l' \rangle \\
 &= - \int_0^L dz \sqrt{\frac{2(M+1)}{\theta_{21}}} \delta_{l(M+1)} \delta(\xi - z) (\langle z, M | \hat{f} | \xi', l' \rangle) \\
 &= -\Theta(\xi)\Theta(L - \xi) \sqrt{\frac{2(M+1)}{\theta_{21}}} \delta_{l(M+1)} (\langle \xi, M | \hat{f} | \xi', l' \rangle). \tag{E.0.6}
 \end{aligned}$$

Since $[\hat{t}, \hat{z}] = i\theta^{03}$ and $\hat{z}|z, n\rangle = z|z, n\rangle$ we can write for the action of \hat{t} on a state $|z, n\rangle$, $\hat{t}|z, n\rangle = i\theta^{03} \frac{\partial}{\partial z} |z, n\rangle$ as these operators satisfy the canonical commutation relations, then lastly we have:

$$\begin{aligned}
 & [\hat{\partial}_3, \int_0^L dz \sum_{n=0}^M |z, n\rangle \langle z, n|] \hat{f} = \int_0^L dz \sum_{n=0}^M -i(\theta^{-1})_{30} [\hat{t}, |z, n\rangle \langle z, n|] \hat{f} \\
 &= \int_0^L dz \sum_{n=0}^M -i(\theta^{-1})_{30} (i\theta^{03}) \left(\left(-\frac{\partial}{\partial z} |z, n\rangle \right) \langle z, n| + |z, n\rangle \left(\frac{\partial}{\partial z} \langle z, n| \right) \right) \hat{f} \\
 &= \int_0^L dz \sum_{n=0}^M \left(\left(-\frac{\partial}{\partial z} |z, n\rangle \right) \langle z, n| + |z, n\rangle \left(\frac{\partial}{\partial z} \langle z, n| \right) \right) \hat{f}. \tag{E.0.7}
 \end{aligned}$$

For the following matrix element we have:

$$\begin{aligned} \langle \xi, l | [\hat{\partial}_3, \int_0^L dz \sum_{n=0}^M |z, n\rangle \langle z, n|] \hat{f} | \xi', l' \rangle &= \int_0^L dz \frac{\partial}{\partial z} (\delta(\xi - z) \langle z, l | \hat{f} | \xi', l' \rangle) \\ &= \delta(\xi - L) \langle L, l | \hat{f} | \xi', l' \rangle - \delta(\xi) \langle 0, l | \hat{f} | \xi', l' \rangle. \end{aligned} \quad (\text{E.0.8})$$

There are three possibilities, $\xi = 0, \xi = L$ and $0 < \xi < L$, when ξ lies outside $[0, L]$ all the commutators are zero. This gives in total 12 boundary conditions since there are 4 spinor components and 3 sides of the cylinder. To avoid divergences in the conditions $\xi = 0, \xi = L$ and to extract the physical information, integrate ξ around a small interval ϵ and then take the limit $\epsilon \rightarrow 0$. For $\xi = 0$ integrate $\int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} d\xi$, then the terms involving $\delta(\xi - L)$ will clearly be zero. The terms involving the step functions ξ will be proportional to ϵ and when $\epsilon \rightarrow 0$ these will vanish, and the term involving $\delta(\xi)$ gives the matrix elements evaluated at $\xi = 0$. Similarly for $\xi = L$ only the $\delta(\xi - L)$ survives. For $\xi = c, c \in (0, L)$ the δ terms are zero and Θ terms are just the conditions at c since the step functions give 1 inside the interval. Now setting $\hat{\phi} = \begin{pmatrix} \hat{\phi}^{(1)} \\ \hat{\phi}^{(2)} \end{pmatrix}$ and $\hat{\chi} = \begin{pmatrix} \hat{\chi}^{(1)} \\ \hat{\chi}^{(2)} \end{pmatrix}$, the 4 conditions at $\xi = 0$ are:

$$\begin{aligned} \langle 0, l | \hat{\chi}_{in}^{(i)} | \xi', l' \rangle &= \langle 0, l | \hat{\chi}_{out}^{(i)} | \xi', l' \rangle, \\ \langle 0, l | \hat{\phi}_{in}^{(i)} | \xi', l' \rangle &= \langle 0, l | \hat{\phi}_{out}^{(i)} | \xi', l' \rangle, \end{aligned} \quad (\text{E.0.9})$$

and the 4 conditions at $\xi = L$ are:

$$\begin{aligned} \langle L, l | \hat{\chi}_{in}^{(i)} | \xi', l' \rangle &= \langle L, l | \hat{\chi}_{out}^{(i)} | \xi', l' \rangle, \\ \langle L, l | \hat{\phi}_{in}^{(i)} | \xi', l' \rangle &= \langle L, l | \hat{\phi}_{out}^{(i)} | \xi', l' \rangle. \end{aligned} \quad (\text{E.0.10})$$

Finally the 2 conditions at $c = \xi \in (0, L)$ and $l = M$ are:

$$\begin{aligned} \langle c, M + 1 | \hat{\phi}_{in}^{(1)} | \xi', l' \rangle &= \langle \xi, M + 1 | \hat{\phi}_{out}^{(1)} | \xi', l' \rangle, \\ \langle c, M + 1 | \hat{\chi}_{in}^{(1)} | \xi', l' \rangle &= \langle \xi, M + 1 | \hat{\chi}_{out}^{(1)} | \xi', l' \rangle, \end{aligned} \quad (\text{E.0.11})$$

and the 2 conditions at $c = \xi \in (0, L)$ and $l = M + 1$ are:

$$\begin{aligned} \langle c, M | \hat{\phi}_{in}^{(2)} | \xi', l' \rangle &= \langle \xi, M | \hat{\phi}_{out}^{(2)} | \xi', l' \rangle, \\ \langle c, M | \hat{\chi}_{in}^{(2)} | \xi', l' \rangle &= \langle \xi, M | \hat{\chi}_{out}^{(2)} | \xi', l' \rangle. \end{aligned} \quad (\text{E.0.12})$$

Appendix F

Symbols of Derivatives

In addition to the derivative operators defined in chapter four (4.1.12) we define two new ones:

$$\hat{\tau} = \hat{\partial}_0 + i\hat{\partial}_3, \hat{\tau}^\dagger = \hat{\partial}_0 - i\hat{\partial}_3. \quad (\text{F.0.1})$$

The action of these operators is given by, recalling (4.1.14):

$$\hat{\rho}\hat{\psi} = \sqrt{\frac{2}{\theta_{12}}}[\hat{b}, \hat{\psi}], \hat{\rho}^\dagger\hat{\psi} = -\sqrt{\frac{2}{\theta_{12}}}[\hat{b}^\dagger, \hat{\psi}], \hat{\tau}\hat{\psi} = \sqrt{\frac{2}{\theta_{03}}}[\hat{a}, \hat{\psi}], \hat{\tau}^\dagger\hat{\psi} = -\sqrt{\frac{2}{\theta_{03}}}[\hat{a}^\dagger, \hat{\psi}]. \quad (\text{F.0.2})$$

To calculate $\langle z, w | \hat{D}\hat{\psi} | z, w \rangle$ we first calculate the action of $\hat{\partial}_\mu$. To do this we first consider the operator $\hat{\tau}$:

$$\begin{aligned} \tau\psi(z, w, z^*, w^*) &= \sqrt{\frac{2}{\theta_{03}}} \langle 0, 0 | e^{-|z|^2} e^{-|w|^2} e^{z^*\hat{a}} e^{w^*\hat{b}} (\hat{a}\hat{\psi} - \hat{\psi}\hat{a}) e^{w\hat{b}^\dagger} e^{z\hat{a}^\dagger} | 0, 0 \rangle \\ &= \sqrt{\frac{2}{\theta_{03}}} \langle z, w | \hat{a}\hat{\psi} | w, z \rangle - z \langle z, w | \hat{\psi} | w, z \rangle \\ &= \sqrt{\frac{2}{\theta_{03}}} \frac{\partial}{\partial z^*} \langle z, w | \hat{\psi} | w, z \rangle. \end{aligned} \quad (\text{F.0.3})$$

Similarly for $\hat{\tau}^\dagger, \hat{\rho}, \hat{\rho}^\dagger$:

$$\begin{aligned} \tau^\dagger\psi(z, w, z^*, w^*) &= -\sqrt{\frac{2}{\theta_{12}}} \frac{\partial}{\partial z} \langle z, w | \hat{\psi} | w, z \rangle, \\ \rho\psi(z, w, z^*, w^*) &= \sqrt{\frac{2}{\theta_{03}}} \frac{\partial}{\partial w^*} \langle z, w | \hat{\psi} | w, z \rangle, \\ \rho^\dagger\psi(z, w, z^*, w^*) &= -\sqrt{\frac{2}{\theta_{03}}} \frac{\partial}{\partial w} \langle z, w | \hat{\psi} | w, z \rangle. \end{aligned} \quad (\text{F.0.4})$$

Using these results it can easily be verified that:

$$\hat{\partial}_0 \rightarrow \frac{\partial}{\partial x^0}, \hat{\partial}_1 \rightarrow \frac{\partial}{\partial x^1}, \hat{\partial}_2 \rightarrow \frac{\partial}{\partial x^2}, \hat{\partial}_3 \rightarrow \frac{\partial}{\partial x^3}, \quad (\text{F.0.5})$$

and thus we can write $\hat{\partial}\psi(z, w, z^*, w^*) \equiv \langle z, w | \hat{\partial}\hat{\psi} | z, w \rangle$ the operator derivative just becomes the usual derivative in the coherent state basis.

Bibliography

- [1] Seiberg, N.: Emergent spacetime. pp. 163–178, 2006. [hep-th/0601234](#).
- [2] Doplicher, S., Fredenhagen, K. and Roberts, J.E.: The Quantum structure of space-time at the Planck scale and quantum fields. *Commun.Math.Phys.*, vol. 172, pp. 187–220, 1995. [hep-th/0303037](#).
- [3] Snyder, H.S.: Quantized Space-Time. *Physical Review*, vol. 71, p. 1, 1947.
- [4] Seiberg, N. and Witten, E.: String theory and noncommutative geometry. *JHEP*, vol. 9909, p. 032, 1999. [hep-th/9908142](#).
- [5] Scholtz, F.G., Chakraborty, B., Gangopadhyay, S. and Hazra, A.G.: Dual families of non-commutative quantum systems. *Phys. Rev.*, vol. D71, p. 085005, 2005. [hep-th/0502143](#).
- [6] Bemfica, F.S. and Girotti, H.O.: The Noncommutative degenerate electron gas. *J. Phys.*, vol. A38, p. L539, 2005. [quant-ph/0506191](#).
- [7] Khan, S., Chakraborty, B. and Scholtz, F.G.: On the role of twisted statistics in the noncommutative degenerate electron gas. *Phys. Rev.*, vol. D78, p. 025024, 2008. [hep-th/0707.4410](#).
- [8] Chakraborty, B., Gangopadhyay, S. and Saha, A.: Seiberg-Witten map and Galilean symmetry violation in a non-commutative planar system. *Phys. Rev.*, vol. D70, p. 107707, 2004. [hep-th/0312292](#).
- [9] Li, K. and Dulat, S.: The Aharonov-Bohm effect in noncommutative quantum mechanics. *Eur. Phys. J.*, vol. C46, pp. 825–828, 2006. [hep-th/0508193](#).
- [10] Nair, V.P. and Polychronakos, A.P.: Quantum mechanics on the noncommutative plane and sphere. *Phys. Lett.*, vol. B505, pp. 267–274, 2001. [hep-th/0011172](#).
- [11] Chaichian, M., Sheikh-Jabbari, M.M. and Tureanu, A.: Hydrogen atom spectrum and the Lamb shift in noncommutative QED. *Phys. Rev. Lett.*, vol. 86, p. 2716, 2001. [hep-th/0010175](#).
- [12] Chair, N. and Sheikh-Jabbari, M.M.: Pair production by a constant external field in noncommutative QED. *Phys. Lett.*, vol. B504, pp. 141–146, 2001. [hep-th/0009037](#).

- [13] Liao, Y. and Dehne, C.: Some phenomenological consequences of the time-ordered perturbation theory of qed on non-commutative spacetime. *The European Physical Journal C - Particles and Fields*, vol. 29, no. 1, pp. 125–132, 2003. ISSN 1434-6044.
Available at: <http://dx.doi.org/10.1140/epjc/s2003-01208-3>
- [14] Ohl, T. and Reuter, J.: Testing the noncommutative standard model at a future photon collider. *Phys. Rev.*, vol. D70, p. 076007, 2004. [hep-ph/0406098](#).
- [15] Garcia-Compean, H., Obregon, O. and Ramirez, C.: Noncommutative quantum cosmology. *Phys. Rev. Lett.*, vol. 88, p. 161301, 2002. [hep-th/0107250](#).
- [16] Alexander, S., Brandenberger, R. and Magueijo, J.: Noncommutative inflation. *Phys. Rev.*, vol. D67, p. 081301, 2003. [hep-th/0108190](#).
- [17] Scholtz, F., Gouba, L., Hafver, A. and Rohwer, C.: Formulation, Interpretation and Application of non-Commutative Quantum Mechanics. *J.Phys.*, vol. A42, p. 175303, 2009. [math-ph/0812.2803](#).
- [18] Chandra, N., Groenewald, H.W., Kriel, J.N., Scholtz, F.G. and Vaidya, S.: Spectrum of the three dimensional fuzzy well. *J.Phys.*, vol. A47, no. 44, p. 445203, 2014. [hep-th/1407.5857](#).
- [19] Douglas, M.R. and Nekrasov, N.A.: Noncommutative field theory. *Rev. Mod. Phys.*, vol. 73, pp. 977–1029, 2001. [hep-th/0106048](#).
- [20] Szabo, R.J.: Quantum field theory on noncommutative spaces. *Phys. Rept.*, vol. 378, pp. 207–299, 2003. [hep-th/0109162](#).
- [21] Grosse, H. and Steinacker, H.: A Nontrivial solvable noncommutative ϕ^3 model in 4 dimensions. *JHEP*, vol. 08, p. 008, 2006. [hep-th/0603052](#).
- [22] Ballentine, L.E.: *Quantum Mechanics: A Modern Development*. Prentice Hall, 1989.
- [23] Sakurai, J.J. and Napolitano, J.: *Modern quantum physics*. Addison-Wesley Publishing Company, Inc., 1985.
- [24] Schawbl, F.: *Advanced Quantum Mechanics*. Springer, 2000.
- [25] Thom, J.D. and Scholtz, F.G.: Bound state energies and phase shifts of a non-commutative well. *J. Phys.*, vol. A42, p. 445301, 2009. [1012.0699](#).
- [26] Groenewold, H.: On the Principles of elementary quantum mechanics. *Physica*, vol. 12, pp. 405–460, 1946.
- [27] Basu, P., Chakraborty, B. and Scholtz, F.G.: A Unifying perspective on the Moyal and Voros products and their physical meanings. *J.Phys.*, vol. A44, p. 285204, 2011. [hep-th/1101.2495](#).
- [28] Srednicki, M.: *Quantum field theory*. Cambridge University Press, 2007.

- [29] Gomis, J. and Mehen, T.: Space-time noncommutative field theories and unitarity. *Nucl. Phys.*, vol. B591, pp. 265–276, 2000. [hep-th/0005129](#).
- [30] Bahns, D., Doplicher, S., Fredenhagen, K. and Piacitelli, G.: On the Unitarity problem in space-time noncommutative theories. *Phys. Lett.*, vol. B533, pp. 178–181, 2002. [hep-th/0201222](#).
- [31] Sinha, D., Chakraborty, B. and Scholtz, F.G.: Non-commutative Quantum Mechanics in Three Dimensions and Rotational Symmetry. *J.Phys.*, vol. A45, p. 105308, 2012. [hep-th/1108.2569](#).
- [32] Wess, J.: Deformed coordinate spaces: Derivatives. pp. 122–128, 2003. [hep-th/0408080](#).
- [33] Itzykson, C. and Zuber, J.B.: *Quantum Field Theory*. McGraw-Hill, 1980.
- [34] Sheikh-Jabbari, M.M.: C, P, and T invariance of noncommutative gauge theories. *Phys. Rev. Lett.*, vol. 84, pp. 5265–5268, 2000. [hep-th/0001167](#).
- [35] Bhattacharya, K.: Solution of the Dirac equation in presence of an uniform magnetic field. 2007. [hep-th/0705.4275](#).
- [36] DLMF: NIST Digital Library of Mathematical Functions. <http://dlmf.nist.gov/>, Release 1.0.9 of 2014-08-29. Online companion to [45]. Available at: <http://dlmf.nist.gov/>
- [37] Diener, J.P.W.: *Ferromagnetic Phase Transitions in Neutron Stars*. Ph.D. thesis, 2013. [nucl-th/1305.7346](#). Available at: <http://inspirehep.net/record/1236399/files/arXiv:1305.7346.pdf>
- [38] Scholtz, F.G., Chakraborty, B., Gangopadhyay, S. and Govaerts, J.: Interactions and non-commutativity in quantum Hall systems. *J. Phys.*, vol. A38, pp. 9849–9858, 2005. [cond-mat/0509331](#).
- [39] Scholtz, F., Chakraborty, B., Govaerts, J. and Vaidya, S.: Spectrum of the non-commutative spherical well. *J.Phys.*, vol. A40, pp. 14581–14592, 2007. [hep-th/0709.3357](#).
- [40] Menon, P. and Belyi, S.: Dirac particle in a box, and relativistic Zeno dynamics. *Physics Letters A*, vol. 330, pp. 33–40, 2004.
- [41] Thomas, A.W.: Chiral Symmetry and the BAG Model: A New Starting Point for Nuclear Physics. *Advances in Nuclear Physics*, vol. 13, p. 1, 1948.
- [42] Alberto, P., Fiolhais, V. and Gil, V.: Relativistic particle in a box. *European Journal of Physics*, vol. 17, pp. 19–24, 1996.
- [43] Klauder, J. and Skagerstam, B.: *Coherent States: Applications in Physics and Mathematical Physics*. World Scientific, 1985. ISBN 9789971966522.

- [44] Falomir, H., Gamboa, J., Loewe, M., Mendez, F. and Rojas, J.C.: Spin Non-commutativity and the Three-Dimensional Harmonic Oscillator. *Phys. Rev.*, vol. D85, p. 025009, 2012. [hep-th/1111.0511](https://arxiv.org/abs/hep-th/1111.0511).
- [45] Olver, F.W.J., Lozier, D.W., Boisvert, R.F. and Clark, C.W. (eds.): *NIST Handbook of Mathematical Functions*. Cambridge University Press, New York, NY, 2010. Print companion to [36].