LEARNING ABOUT AND UNDERSTANDING FRACTIONS AND THEIR ROLE IN THE HIGH SCHOOL CURRICULUM.

by

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Declaration

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ETIENNE PIENAAR 06 FEBRUARY 2014

ABSTRACT

Many learners, even at high school level, have difficulty with fractions and computations involving fractions. A report from the Department of Basic Education (DBE, 2012c: 15) has highlighted that the lack in basic fraction sense was one of the areas of concern that contributed to the low achievement in matriculation mathematics examinations in 2012. Fractions play an important role in our ever-advancing technological society. Many occupations today rely heavily on the ability to compute accurately, proficiently, and insightfully with fractions. High school learners' understanding or the lack thereof is carried over to their tertiary studies and workplaces. It is for that reason that in this dissertation, the learning and understanding of fractions and their role in the high school curriculum are studied through a critical literature review. Fractions are compound constructs and can therefore be interpreted in many different ways, depending on the area of study within mathematics. The concept of fractions consists of five sub-constructs, namely, part-whole, ratio, operator, quotient, and measure (Behr, Lesh, Post, & Silver, 1983; Kieren, 1980). This thesis starts with discussion of the background of the study and its importance. Thereafter the elements that assist in the understanding of the fraction concept is discussed. Then, the five different sub-constructs are elaborated on, and how these different sub-constructs are used in the high school curriculum is demonstrated. The conclusion offers some implications for classroom teaching and mathematics teachers' professional development.

OPSOMMING

Talle leerders, tot op hoërskool vlak, ervaar probleme met breuke en berekeninge met breuke nie. ‘n Verslag van die Departement van Basiese Onderwys (DBE, 2012c: 15) het beklemtoon dat die gebrek aan basiese breuk vaardighede een van die oorsake was wat daartoe gely het dat die prestasie in die 2012 matriek wiskunde eksamen so laag was. Breuke speel ‘n belangrike rol in ons voortdurende tegnologiese voor uitgaande samelewing. Talle beroepe vandag is grootliks afhanklik van die akkurate, bekwame en insiggewende berekeninge van breuke. Hoërskool leerders se begrip, of die gebrek daaraan word oorgedra na hul tersiêre studies en werksplekke. Dit is vir dié rede dat hierdie tesis die leer en begrip van breuke en hul rol in die hoërskool kurrikulum bestudeer deur middel van ‘n kritiese literatuur studie. Breuke is ‘n saamgestelde konsep en kan vir hierdie rede op verskillende wyses geïnterpreteer word, afhangende van die area van studie in wiskunde. Die konsep van ‘n breuk bestaan uit vyf sub-konstrukte, naamlik deel-van-‘n-geheel, ‘n verhouding, operateur, kwosiënt en meting (Behr, Lesh, Post, & Silver, 1983; Kieren, 1980). Hierdie tesis begin met ‘n bespreking oor die agtergrond van hierdie studie en die belangrikheid daarvan. Daarna word die faktore wat bydra tot die verstaan van die breuk konsep. Dit word gevolg deur ‘n uitbreiding op die vyf verskillende sub-konstrukte en waar hierdie verskillende sub-konstrukte in die hoërskool kurrikulum voorkom. Die bevinding bied ‘n paar implikasies vir onderrig. Hierdie studie fokus nie op die ontwerp van enige take of ander leermateriaal vir ‘n intervensie program nie, maar konsentreer op die belangrike kwessies rondom breuke. My hoop is dat die bevindinge van hierdie studie implikasies inhou vir wiskunde onderwysers se professionele ontwikkeling deur hul te motiveer om nuwe leerondersteuningsmateriaal te ontwikkel en die aanbieding van breuke in klaskamers aan te pas sodat die begrip van breuke by leerders ten volle ontwikkel kan word.

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ACRONYMS

ANA	Annual National Assessments
CAPS	Curriculum and Assessment Policy Statement
DBE	Department of Basic Education
FET	Further Education and Training
LTSM	Learning and Teaching Support Material
MALATI	Mathematics Learning and Teaching Initiative
MKT	Mathematical Knowledge for Teaching
NAEP	National Assessment of Educational Progress
NCS	National Curriculum Statement
NRC	National Research Council
PCK	Pedagogical Content Knowledge
RNCS	Revised National Curriculum Statement

CHAPTER 1: INTRODUCTION

1.1 Background

In this increasingly technical world, financial and educational success in contemporary global culture depends heavily on knowledge of mathematics. It is therefore critical that learner achievement in high school mathematics improves as this affects attainment at tertiary level (Siegler, Duncan, Davis-Kean, Duckworth, Claessens, Engel, et al., 2012).

In light of this, the South African Government has constructed a plan of action to ensure that the quality of education, specifically in mathematics and languages, in the country improve. The Department of Basic Education (DBE) plays a vital role in ensuring that the goals set by the National Government are achieved. The Annual National Assessments (ANA) was introduced for the first time for Grade 9 learners in 2012. ANA monitors the performance of learners in numeracy and literacy. All schools are required to write tests in mathematics and language (DBE, 2012a: 2). The ANA results for the last two years are of great concern because only 2.3% of learners in Grade 9 achieved above 50% for the mathematics test (DBE, 2012b: 6). This is unfortunately also true for the National Senior Certificate examination of 2012. Of the total number of candidates who wrote the examinations, only 44% wrote mathematics and 46% of these candidates did not achieve at least 30% to pass the subject at the end of Grade 12 (DBE, 2012b). These results are not isolated, as a study done by Steen (2007) in the United States shows similar concerns. In particular, “much contention occurs near the ends of elementary and secondary education, where students encounter topics that many find difficult and some find incomprehensible” (Steen, 2007: 9).

Areas of concern were fractions and ratios as learners found it difficult to comprehend these concepts (DBE, 2012c: 15). A study by the National Assessment of Educational Progress (NAEP), indicates that “students of age seventeen recurrently demonstrated a lack of proficiency with fraction concepts” (cited in Brown & Quinn, 2006: 28). In a study by Mullis, Dossey, Owen, and Phillips (cited in Brown & Quinn, 2006: 28), only 46% of high school learners understood the concept of fractions. Although this study was done in the United States, work done by Newstead and Murray (1999) suggests that the case is the same in South Africa. NAEP results (cited in Niemi, 1996: 6) indicate that many students “see fractions as purely symbolic entities not linked to concepts or principles". Hecht and Vagi (2010: 843) stated that “one of the most persistent problems for children with mathematical difficulties in solving problems involved fractions.” According to Kieren (cited in Niemi, 1996: 6), fraction knowledge forms a basis for understanding a wide range of related concepts, including ratio, proportion, decimals, percentages, and rational numbers.

The basic understanding of fractions is critical for any learner to be capable of coping with more advanced topics in the high school curriculum (Niemi, 1996: 6). Fractions are an integral part of school mathematics curriculum. Fractions have rich meaning and feature in mathematical areas such as algebra, geometry, probability and trigonometry. If learners have difficulty in understanding the many meanings of fractions, it is likely that they will also have difficulty in procedural competency in these areas. Consequently, it is our duty as teachers to ensure that we make a concerted effort to bring to light, through our teaching, the many different interpretations of fractions and the role they play in the mathematics curriculum. We are confronted with fractions on a daily basis, for example, weather reports, financial indicators, crime rates or the percentage gained in a class test. Despite these daily encounters with fractions, learners still have misunderstandings about the meaning of fractions. In my own teaching experience, I have come across multiple situations where learners cannot work with fractions. This gives me the impression that in these specific situations, the learners have some misguided idea of what fractions really are and how to solve problems involving fractions.

The concept of fractions consists of five sub-constructs, namely, part-whole, ratio, operator, quotient or measure (Behr, Lesh, Post, & Silver, 1983; Kieren, 1980). Associated with these sub-constructs are the computations $(+, -, \times, \div)$. If learners are taught about what are called the *sub-constructs* and how these relate to computations of fractions, there may be less confusion and more understanding of the meanings of fractions. The teacher's role in developing the concept of fractions and the understanding thereof in the early stages of a child's life (i.e. lower grades) is crucial because of their significance in the school curriculum from secondary through tertiary education.

1.2 Problem statement

Soon after I started teaching high school mathematics in 2009, I realised that my learners had great difficulty in comprehending fractions and operations involving fractions. It seemed to me that an inward fear or a mental block arose when the word *fractions* was mentioned or even when a sum containing a fraction was written on the board. This impression still pertains. The immediate reaction is almost always, "Oh! Why must everything always be so difficult?" I could never understand why this would be their response. I suspect that the root of the problems or difficulties with fractions lies in the rich meanings associated with fractions. Learners' difficulties with fractions stem from the different meanings or interpretations that fractions hold, depending on the tasks wherein the fractions appear and the teaching methods employed. It is therefore imperative that educators should, themselves, have a solid understanding of fractions and their meanings and the different areas in mathematics where they are used. Only then will the teacher be able to present fractions in context and develop a better conceptual understanding of fractions amongst learners. Addressing these issues in this study will shed light on why high school (FET) learners struggle with fractions and will, it is hoped, provide insight on the larger challenges concerning mathematics competency.

1.3 Aim and rationale

The overarching question posed in this thesis is "Why do learners in high school (grades 10-12: FET phase) struggle with fractions?" Specifically, I aim to enhance my own professional development as a mathematics teacher, as a deeper understanding of the content may improve my own understanding of fractions. To do this, I need to know the many meanings of fractions and the different areas in mathematics where they are used. In addition, I need to know why learners have difficulty with fractions and operations involving fractions. One way to discover this is to explore the literature that focuses on high school learners' difficulties with fractions.

I will investigate what factors contribute to learner's understanding of fractions and the limiting constructs. The lack of understanding of fractions by learners can be attributed to many factors. One of the most important is probably mathematical knowledge for teaching (MKT). There is some fundamental mathematical knowledge teachers should have and develop to improve their way of teaching so that it can have a positive effect on learners' understanding of fractions.

Lastly, an analysis will be done to explore the meanings of fractions in topics like algebra, geometry, probability, and trigonometry. This may assist the way in which fractions are represented so that fractions in these respective topics become more meaningful, which may lead to improve understanding of fractions by learners. After presenting a critical review on the research literature concerning what is called *sub-constructs* of fractions, I will suggest some ways in which fractions can be taught or represented. By doing this, I hope to make teachers, including me, aware of the different sub-constructs, where we use them, why we use them, and how they can be represented, which will help with our understanding of why learners have difficulty with fractions.

Although my study is not specifically focused on how teachers can better their teaching of fractions, the findings of this study may have implications for the professional development of mathematics teachers. The hope is that a better presentation of fractions will lead to better understanding and ultimately more effective learning in the mathematics classroom.

1.4 Framework for learning mathematics

In order to understand learners' misconception of fractions, a framework for learning mathematics needs to be established first. Olivier (1989:9) explained the importance of theory and compared it to a "lens through which one views the facts". The fact is that learners make mistakes in mathematics, but as Olivier stressed, if we do not know "why they make these mistakes, we are unable to do something about it" (1989:9). If we want to provide reasons for learners' mistakes, we need to look through a "lens" defined by a learning theory. There are two ways in which teachers can approach leaners' misconceptions: behaviouristic or constructivist.

Behaviourism assumes that a learner learns through a passive state, by which knowledge is being transferred from the knowledgeable/expert (teacher) to the clean slate (*tabula rasa*), that is the child (Olivier, 1989). Behaviourists therefore believe that external stimuli shape and construct knowledge within the child and that the child's current knowledge is obsolete and does not contribute to learning. Appropriate responses to stimuli are rewarded (positive reinforcement) and thus strengthened, whereas inappropriate responses are punished (negative reinforcement), which weakens the bonds. Learning is seen as a change in learners' behaviour. From this view, misconceptions or mistakes are insignificant and are 'punished' so they it will be wiped out of memory and in turn make space for the correct ones.

Constructivism assumes that learning takes place when a person interacts with his/her environment to construct his/her own knowledge. With constructivism, the learner is not a passive receiver of "ready-made" knowledge but an "active participant in the construction of his own knowledge" (Olivier, 1989, 2). A learner makes sense of (interprets/understands) new knowledge through existing knowledge. Constructivists often refer to the term *schemata*, which describes the child's previously constructed constructs, which are all interrelated. Learning is not viewed as a change in behaviour but as a change in learners' schemata. In the constructivist view, misconceptions are significant because learners make sense of new knowledge by tapping into existing knowledge. The danger here is that this can interfere with the construction of new knowledge and produce misconceptions, as I will discuss later in this thesis.

I personally lean towards the constructivist approach for learning mathematics, but acknowledge that rewarding good behaviour does motivate learners in some sense and they then are more willing to engage in the learning process.

1.5 Outline

To address the concerns raised above, a systematic non-empirical critical literature review of research studies on teaching and learning fractions in the case of high school curriculum, Grades 10-12, that is, the Further Education and training (FET) phase will be done. To do this, I will structure the rest of my thesis in two distinct chapters. One will concern discussing fractions in school mathematics and the other fractions in the school curriculum.

The next chapter contains a discussion on fractions in school mathematics in which I will evaluate how mathematics as a subject is viewed in the South African curriculum. From there, I investigate the importance of teachers' mathematical knowledge for teaching fractions and, furthermore, the factors contributing to the difficulty in learning fractions. This is followed by a analysis of where fractions fit into the set of real numbers, their importance, and the different ways in which fractions can be interpreted.

In the final chapter, an analysis is done to determine where fractions are used in the school curriculum. Fractions are found in algebra, geometry, probability and trigonometry. I investigate the link between fractions in the adding of like terms and linear equations in algebra. I go on to investigate the role that fractions play as a quotient in geometric similarity, before moving on to probability and, lastly, trigonometry. This thesis concludes with implications for school mathematics teaching.

CHAPTER 2: UNDERSTANDING THE FRACTION CONCEPT

2.1 Mathematical proficiency

Before I can explore the possible reasons for learners' misconceptions of fractions, I should first define what I view as being mathematically proficient, especially when working with fractions. To do this, I rely on the work done by the National Research Council (NRC), reported in *Adding it Up*. In this report, Kilpatrick, Swafford, and Findell described what they call "the five strands of mathematical proficiency" (NRC, 2001: 116). They believe that these five strands are "necessary for anyone to learn mathematics successfully" (NRC, 2001: 116). The five strands, namely, conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition, are "interwoven and interdependent in the development of proficiency in mathematics" (NRC, 2001: 116).

Conceptual understanding describes learners' comprehension of mathematical ideas, that is, fractions, operations, and relationships (NRC, 2001: 118). A learner is said to have conceptual understanding of fractions if s/he knows more than isolated facts and methods. Learners should not only be able to point out where the numerator and denominator are, but also what they represent in a fraction. Solving problems involving fractions (addition, subtraction, multiplication, and division) should be done without any formal knowledge (pre-set methods) given to learners through teaching. A learner with a good conceptual understanding of fractions is able to solve problems using multiple representations and understand which context is the most useful: "They may [even] attempt to explain the method" (NRC, 2001: 118). Carpenter and Lehrer called this "articulating what one knows" (1999: 22) and claimed that this is the "benchmark of understanding". Conceptual understanding is honed by applying teaching strategies and a meticulously designed sequence of activities that are inclusive and is aligned with the learning ability of the child.

Procedural fluency is defined as the "skill in carrying out procedures flexibly, accurately, efficiently, and appropriately" (NRC, 2001: 121). Procedural fluency does not stand opposed to conceptual understanding but they should support each other. To be fluent in a procedure, one should have a good conceptual understanding. Higher level concepts are better understood if the basic concepts are thoroughly grasped and practiced to such an extent that it is almost done automatically. The brain is a mysterious organ and can only work on a certain number of concepts at a time. The less effort is spent on basic elements of a problem, the more "brain power" is available for solving higher order problems. With procedural fluency, learners have the ability to recognise important aspects needed to solve problems in a logical and effective manner, and sometimes in a variety of ways. These alternative strategies for solving problems also provide another way to check their answers. To illustrate this, I will take an example from the Mathematics Learning and Teaching Initiative (MALATI) fractions materials called "Lisa Shares Chocolate" (Newstead, Van Niekerk, Lukhele, & Lebethe, 1999: 1). Lisa and Mary share seven chocolate bars equally amongst them and learners are asked to help them do it. This problem can be solved purely numerically, or learners can sketch the elements of the problem and share the whole chocolate bars, then divide the remaining between the two, or they can divide all the bars into two equal parts and share the pieces. If the learners work correctly (and with enough practice) they will arrive at an answer of three and a half chocolate bars.

The third strand, called *strategic competence,* is the ability to formulate, represent, and solve mathematical problems (NRC, 2001: 124). Kilpatrick and his colleagues stated that "this strand is similar to what is called problem solving" (NRC, 2001: 124). It is not enough for learners to merely solve problems but they should also be able to formulate their own. By doing this, learners demonstrate their knowledge of the topic. Learners gain enough exposure to a variety of different problems so they can create their own strategies to solve them as well as develop the skill to identify which strategy is the most useful in solving a specific problem. During the development of a range of problem-solving strategies, learners' procedural fluency also improves. An example of strategic competence is to ask leaners to represent fractions as part of a whole, using different representation models. Leaners should also be asked to create their own word problems and swop them with their peers to try and solve them.

Adaptive reasoning is the capacity for logical thought, reflection, explanation, and justification (NRC, 2001: 129). The importance of this is that as learners explain why they solve a specific problem in a certain way; they are demonstrating their understanding of the different representation models and that they feel at ease using it. One knows that a learner can reason adaptively if he/she is able to explain or "justify" his/her own thinking when solving problems involving, for example, fractions (NRC, 2001: 130). To return to "Lisa Shares Chocolate" again (Newstead et al., 1999:1): If Lisa and Mary share 7 chocolate bars and Lisa, Mary and Bingo share 7 chocolate bars, who will get the most and explain why? The answer one wishes to obtain from learners is that Lisa and Mary will get a bigger piece, not because it looks bigger when drawing a diagram, but rather that the same number of chocolate bars is divided between fewer people (2). The concept of a larger denominator creating smaller parts is tested. It is important for learners to reach this level of reasoning to enable them to move on to more challenging problems.

Productive disposition is the inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (NRC, 2001: 131). In short, productive disposition is the ability to see mathematics as meaningful. We, as mathematics teachers, know the negativity surrounding our subject, so it is our responsibility to instil into our leaners the motivation to study, sensibility towards and the importance, reality, and most importantly fun of mathematics in their daily lives. “The teacher of mathematics plays a critical role in encouraging students to maintain positive attitudes toward mathematics” (NRC, 2001:132).

In summary, let me emphasise some important facts of mathematical proficiency as a whole. Most important is that the strands of proficiency are interwoven and support one another (NRC, 2001: 133). All five strands must be used in collaboration with one another for successful learning to take place. (NRC, 2001: 133). Kilpatrick et al. also stated that proficiency is not simply all or nothing (present or absent) because mathematical ideas are understood at different levels and in different ways (NRC, 2001: 135). They went on to say that mathematical proficiency develops over time throughout learners’ school careers, and they “need to engage in activities around a specific mathematical topic if they are to become proficient in it” (NRC, 2001: 135). I would like to encourage the reader, while reading this thesis, to constantly try to relate what is being discussed to the five strands of mathematical proficiency identified here. In the next section, I will be discussing the concept of fractions in more detail.

2.2 Overview of fractions

The notion of a fraction being a compound concept consisting of different forms that are interlinked was first recognised by Kieren (cited in Charalambous & Pitta-Pantazi, 2007: 293). Naik and Subramaniam (2008: 1) referred to a fraction as being “complex since it consists of multiple sub-constructs.” Kieren (cited in Charalambous & Pitta-Pantazi, 2007:295) “proposed that the concept of fractions consists of four interrelated sub-constructs: a ratio (comparison of two quantities), an operator performed on a quantity, a quotient [the answer when one value is divided by another] and a unit of measure.” Kieren did not recognise the part-whole as a fifth sub-construct, but Behr (cited in Charalambous & Pitta-Pantazi, 2007: 295) later argued that the part-whole sub-construct is an essential part in understanding the other four sub-constructs. In the light of the above, I distinguish five sub-constructs of fractions: part-whole/partitioning, ratio, operator, quotient, and measure. Later in this chapter, I will discuss these sub-constructs or interpretations of fractions and their importance in more depth.

Fractions, in all their “forms” or sub-constructs, are essential concepts in the school curriculum and can be interpreted differently depending on the context in which they are used. Fractions are not only used in dividing the usual pizza into pieces. In trigonometry, fractions appear as ratios between sides of a right-angled triangle. In probability, fractions represent the possible outcomes. For learners to be successful in mathematics in higher grades, that is, understand the different meanings of fractions in topics like probability and trigonometry (conceptual knowledge), teachers should not overemphasise the part/whole construct or teach rules to solve problems involving fractions without first ensuring complete comprehension.

Teaching learners how to apply rules to solve fractions problems, without a sound understanding of why we are allowed to apply a certain rule, is detrimental. Instead of honing fraction proficiency, learners are taught to memorise and recall where necessary. Resnick (cited in Litwiller & Bright, 2002: 1) observed that many learners who once believed that they could make sense of mathematics in lower grades lose this belief as they progress to higher grades. This is especially true in the case of fractions. Far more beneficial for learners is that teachers model the fraction problems before the rules are taught. For instance, $\frac{1}{2}\ of\ \frac{3}{4}$ can very easily be given the rule, "Top times top, over bottom times bottom", which will yield the correct result of $\frac{3}{8}$, but why? This problem can be modelled by making a sketch of a rectangle and dividing it into four equal parts; each part will be a $\frac{1}{4}$. Three $\frac{1}{4}$ pieces are then shaded. To establish what a half of $\frac{3}{4}$ is, each $\frac{1}{4}$ block must be divided into two, resulting in a $\frac{1}{8}$. The shaded part then represents $\frac{3}{4}$, and half of the shaded part will be $\frac{3}{8}$. (see Figure 1 below:

$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Figure 1: Model for multiplying with fractions

(Source: Izsak, 2006: 367)

By placing the problem in context, teachers can promote problem-solving skills. Let us consider the same problem of $\frac{1}{2}\ of\ \frac{3}{4}$. This can be demonstrated as follows:

> Peter has a chocolate that is divided into four parts. He has already eaten one of the parts. How much does he have left? Susan comes along and Peter decides he wants to share his chocolate by giving her half of what is left. How much did Susan get of the whole chocolate?

The learner must be able to set up an algorithm so solve the problem of $\frac{1}{2}\ of\ \frac{3}{4}$. To merely offer a rule will not help in promoting comprehension. The illustration above can potentially assist in solving this problem. Learners can relate to such an example because, during school interval, this is often a reality for many friends sharing meals amongst each other. That is why rules without comprehension make mathematics, and specifically the topic of fractions, too abstract and difficult for many learners. That teachers choose to give rules rather than making use of mathematical modelling may be a result of the way in which mathematics as a subject (content and instruction) is viewed in the South African curriculum.

2.3 How is mathematics as a subject viewed in the South African curriculum?

One of the many reasons why learners struggle with fractions may be the way in which mathematics, as a subject, is viewed in the South African curriculum, that is, with the aim to create learners with mathematical knowledge for application and not necessarily to train mathematicians. Thus, “textbook-based teaching and rule-bound learning styles” (Adler, 1994: 104) are generally applied. In this section, I shall comment on the possible effect that this view has on instruction in the classroom and in the end on learners’ understanding of fractions.

Educational transformation in South Africa has been at the forefront of academic and societal debate since 1994. This focus on educational transformation has had an impact on the way in which mathematics as a subject is viewed in the South African curriculum. The general aim of the South African Curriculum is “to ensure that children acquire and apply knowledge and skills” (DBE, 2011:4). It also stresses specific aims and skills, in a mathematical context, for learners to obtain before exiting school.

One must ask why the change in curriculum policy was needed after 1994 and whether it is doing justice to our current learners. The curriculum devised during the apartheid era has been widely criticised by scholars. For example, Adler (1994:102) described the apartheid-era curriculum as “a system fundamentally scarred by racial inequality, absurd levels of fragmentation, authoritarianism, and a low skills-base”. After the African National Congress was elected as the ruling party in 1994, the education policy was reshaped to suit the “needs” of the country. This curriculum change also brought about changes in how mathematics is viewed as a subject. Critical learning through active participation, social transformation, and the development of skills were the focus of the National Curriculum Statement (NCS) (DBE, 2011: 4). The NCS was revised later (RNCS) and changes included some content and assessment strategies, but the predominant method of teaching stayed textbook based and assessment strategies mostly remained formal tests. In 2011, the Curriculum and Assessment Policy Statement (CAPS) was introduced, and its implementation started in 2012. CAPS is a single “comprehensive document that was developed for each subject to replace Subject Statements, Learning Programme Guidelines and Subject Assessment Guidelines.” (DBE, 2011: 3). However, CAPS was not to replace how the NCS or RNCS viewed mathematics as a subject in South Africa but rather to merely amend it.

CAPS defines mathematics "as a language that describes numerical, geometric and graphical relationships" (DBE, 2011: 8). It envisages specific aims and skills that mathematics learners should obtain in the Further Education and Training (FET) phase before moving on to the Higher/Tertiary Education phase (Department of Basic Education, 2011: 8-10). The ultimate goal is for learners taking mathematics as a subject at school, specifically in the FET phase, to "acquire a functioning knowledge of the Mathematics that empowers them to make sense of society and to ensure access to an extended study of the mathematical sciences and a variety of career paths" (DBE, 2011: 10).

Mathematics was thus reshaped from the apartheid-era curriculum to the CAPS vision, to enhance learners' understanding and application thereof. The reason for this was that education specialists believed the system at that time did not serve the learners adequately. The aim of the current education curriculum is to create learners with mathematical knowledge for application and not necessarily to produce mathematicians. Thus, understanding fractions and the different ways of interpreting them would be more beneficial to learners than having to solve highly abstract problems (involving fractions) outside of context and application. It is therefore important to note that the way in which school mathematics is viewed as a subject in the South African curriculum is somewhat different to the way mathematics is viewed in mathematics scholarly circles around the world. Moreover, teachers have a responsibility to ensure that through their teaching, they promote the different interpretations of fractions. By doing this, they will help to develop a better sense of the meaning of fractions and the role they play in learners' understanding in algebra, geometry, probability, and trigonometry and their real-life application.

School mathematics is a "special kind of mathematics" and should be viewed separately from the discipline of mathematics, according to Watson (2008: 3). She argued that school mathematics has "different warrants, authorities, forms of reasoning, core activities, purposes and unifying concepts, and necessarily truncates mathematical activity in ways that are different from those of the discipline" (Watson, 2008: 3). Watson's argument is strengthened by Julie's statement that "The mathematics occupying the minds of mathematics educators is not the same as that which occupies the mind of the mathematician" (Julie, 2002: 30). Julie went on to say that "school mathematics is structured by insights from learning theories, pedagogy, philosophy and history of mathematics" (Julie, 2002: 30). Teachers predominantly engage with reduced and summarised versions of mathematics and "seldom use original pieces of mathematics as the basis for their work" (Julie, 2002: 30). For this reason, "textbook-based teaching and rule-bound learning styles constitute learners' mathematical diet" (Adler, 1994: 104). "Tell and drill", as Adler (1994:104) called it, or "cognitive bullying" as Watson (2008: 3) referred to it, remains the most dominant teaching style in mathematics classrooms today. Julie (2002: 37) concluded that even though there is a "call for applications and modelling of mathematics" in school curriculum transition (Adler, 1994:101), mathematics teachers in South Africa are unacquainted with the fundamentals of mathematical modelling and, thus, this lack of mathematical knowledge "governs, guides and structures their way of working" (Julie, 2002: 37). This form of teaching does not develop nor stimulate an inquisitive, problem-solving attitude in learners. It is becoming increasingly apparent that teachers need to shift from these outdated practices and place more emphasis on application, skills, critical thinking and problem solving. As teachers, we should enhance our own understanding of fractions and modify our practices by which we present (instruction) them to our learners. This may result in learners having a better understanding of the concepts of fractions that will lead to better application.

2.4 Mathematical knowledge for teaching (MKT)

The only way in which teachers will be able to enrich instruction of fractions is if they broaden their own understanding thereof. Fractions are more complex than initially perceived, and therefore the teaching of fractions should also receive special attention. If teachers do not understand the intricacies of fractions themselves, they cannot effectively support the development of learners' understanding thereof (Izsák, 2008: 365). The hope is that a better presentation of fractions will lead to better understanding of fractions by learners. Hence, it is imperative that teachers should improve their 'mathematical knowledge for teaching'.

Before examining the concept of MKT, we need to look at the framework for "teaching knowledge". A framework for teaching knowledge refers to the background, experiences and content knowledge teachers draw upon when presenting a topic like fractions in the classroom (Ball, Thames, & Phelps, 2008). A study done by Lehrer and Franke (1992) found that a teacher with a rich teaching knowledge teaches "better" because problems are presented in context. Similarly, the results of many other studies (Heaton, 1992; Heid, Blume, Zbiek, & Edwards, 1999; Hill, Blunk, Charalambous, Lewis, Phelps, & Sleep, 2008) support the notion that there is a correlation between teachers' teaching knowledge and learner achievement (positive or negative). If teachers have a rich teaching knowledge about fractions, the concept of fractions can be taught more meaningfully to learners, who thus develop better conceptual understanding. Ben-Peretz (2011) claimed that teaching knowledge enables teachers to teach subject matter (e.g., fractions) using appropriate didactic principles and skills. Shulman (1986: 9) identified three types of teaching knowledge: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curriculum knowledge. These different knowledge types should help teachers to teach the sub-constructs of fractions in a variety of ways that can support learners in developing a better understanding of fractions. Incorporated into the notion of a framework of teaching knowledge is the concept of mathematical knowledge for teaching (MKT).

To offer a deeper understanding of what MKT entails, I will briefly refer to three knowledge types, as described by Shulman (1986:9): subject matter content knowledge, pedagogical content knowledge, and curriculum knowledge. Firstly, content knowledge refers to what the teacher knows about the content/topic, such as fractions, why we study fractions, and their importance. This has significance for learners. If the teacher cannot give a reason for why it is important to study fractions and their uses, learners may not have any interest in learning about fractions at all. Secondly, pedagogical content knowledge (PCK) involves going further than the subject matter itself. PCK involves the activity of teaching by making use of educational instructional methods. Feuerstein's theory describes this activity (i.e. teaching) as mediation, where the teacher selects and organises stimuli "considered most suitable to promote learning." (Guiying, 2005: 38). Ball et al. (2008: 3) describe PCK as knowledge that "bridges content knowledge and the practice of teaching". When applying PCK, the teacher decides on the most suitable way to present a specific topic (e.g., fractions) including what examples, diagrams and explanations to use. If teachers are successful at using PCK, they can potentially provide learners with a wide range/variety of ways to make sense of, in this case, fractions, because no two learners understand everything in the same way. Lastly, when a teacher possesses curriculum knowledge, s/he knows the requirements of the courses. Curriculum knowledge involves all aspects of the curriculum, for example, curriculum design and layout, the different topics, levels, range of learning and teaching support material (LTSM) available and which of these LTSMs are most suitable to use at a specific time (Shulman, 1986). Learners may benefit from having teachers with a better curriculum knowledge and become overwhelmed with facts but rather master the content gradually because the teachers organise their lessons in such a way that they build on each other and concepts are introduced at critical time slots and examined in depth. In this way, fractions can easily be understood and incorporated into all other areas in mathematics.

In the light of what has been discussed above, it is clear that MKT is a special type or subcategory of knowledge that is needed by teachers to perform their task of teaching mathematics (Ball et al., 2008: 5). MKT encapsulates the knowledge of the content matter, how it is presented, how it is perceived by learners (learning), and its effect on learner achievement. MKT "is the knowledge used to carry out the work of teaching mathematics" (Hill, Rowan, & Ball, 2005: 373) or put differently, MKT is the "mathematical knowledge that teachers need to carry out their work as teachers of mathematics" (Ball et al., 2008: 4). MKT also refers to the nature, depth and organisation of teacher knowledge that influences how, in this case, fractions are presented, the ability of teachers to answer any questions learners might have regarding fractions, and how skilfully pictures and diagrams are used to bring across the concept of fractions and procedures when doing calculations with them (Steele & Rogers, 2012: 159-160). A broader MKT knowledge base ultimately leads to better content knowledge and the necessary skills to effectively transfer that knowledge to learners (Leung & Park, 2002), which results in better learner achievement.

Studies investigating the relationship of teacher mathematical content knowledge on learner achievement are summarised in the volume *Knowledge Management and Dissemination* (2010). A study done by Hill, Rowan and Ball (2005) showed that student achievement correlates positively with their teachers' mathematical content knowledge. It is thus important for teachers to develop mathematical knowledge for teaching (MKT) fractions to ensure that when teaching learners they present fractions in a more meaningful way. This may have a positive impact on their teaching and consequently may lead to better understanding of fraction concepts in learners.

In summary, teaching involves a thorough knowledge of the content and how to teach that content, such as fractions, in such a way that it promotes a better understanding of fraction concepts by learners. To substantiate this claim, I return to the example used earlier, $\frac{1}{2}\ of\ \frac{3}{4}$. First of all, the teacher must be able to carry out this calculation. He or she must also be able to place this problem in context and answer any questions that learners might ask (content knowledge). Together with this, the teacher should be able to choose the best time to introduce a problem like this and how to structure the series of lessons to create a base of learner knowledge from which to address this problem (curriculum knowledge) and also what type of diagram or model can possibly be used to support the understanding of this type of problem (pedagogical content knowledge). As can be seen, MKT is not a single activity but involves various other knowledge types to assist the teacher in conveying a particular concept, for example, fractions. MKT is but one factor contributing to learners' understanding of fractions.

Other factors also contribute to learners' understanding of fractions in the mathematics classroom, for example, learner thinking and learning and learner informal knowledge, amongst others. In the next section, I will be looking at what factors contribute in the learning and understanding of fractions in learners.

2.5 Limiting constructions

Difficulties in understanding fractions do not solely lie in their compound construct nature, nor in the way mathematics teachers present fractions. There are many causes why learners have difficulties in learning fractions. A special kind of misconception of fractions amongst learners is called *limiting constructions*.

Knowledge is constructed from personal experiences. If these experiences provide learners with only a limited view of a particular concept, for instance, fractions, it may hinder further understanding of that concept (Murray & Le Roux, n.d.: 92). These "limited experiences have resulted in limiting constructions" (Murray & Le Roux, n.d.: 92). Lukhele, Murray and Olivier described limiting constructions as "ones' prior exposure to situations which give the learner a narrow view of the concept which hampers further thinking" (1999: 87).

Inevitably, classroom instruction, activities, and tasks are the cause for some of these limiting constructions but cannot always be prevented. Learners should be exposed to multiple problems involving various experiences and views to try and minimise these limiting views of fraction concepts (Murray & Le Roux, n.d.: 93). The following four limiting constructions are described in the literature consulted and have been identified by the research done by Pitkethly and Hunting (1996), Murray et al. (1996), and the Malati group in 1997-1999: whole number schemes, limited part-whole contexts, knowledge of half, and perceptual and visual representations.

2.5.1 Whole number schemes

The concept of whole numbers can interfere with learners' attempt to learn fractions (Pitkethly & Hunting, 1996: 10). Siegler, Fazio, Bailey, and Zhou (2013:15) agreed and reported that "children often view fractions exclusively in terms of part/whole relations" Learners perceive fraction symbols $\left(\frac{a}{b}\right)$ as two distinct whole numbers written on top of each other (Murray & Le Roux, n.d.: 103). This was evident in the responses in a pre-test given to learners during the MALATI programme, where, in one example, $\frac{7}{8}+\frac{7}{8}=\frac{14}{16}$ (Lukhele et al., 1999: 91), it was clear that the learners tried to solve this problem from a whole number perspective. The fraction is seen as two separate numbers on which whole number strategies are performed. This kind of error is common amongst leaners and well documented among researchers, for example, Hart (1989), Carpenter, Coburn, Reys, & Wilson (1976), Howard (1991), Streefland (1991), and Pitkethly and Hunting (1996). Carpenter et al. (1976:139) ascribed this to the "top times top over bottom times bottom" rule that is taught to learners when they are introduced to the multiplication of fractions.

2.5.2 Limited part-whole contexts

A very popular belief amongst learners is that fractions are only part of a whole. Similarly, they also believe that only circles or rectangles can be divided into equal parts. Learners struggle to come to grips with a problem such as sharing five pizzas among eight people or how to calculate a fraction of a collection of objects, for example $\frac{2}{3}$ of a box of Smarties containing 48 Smarties (Murray & Le Roux, n.d.: 104). According to Pitkethly and Hunting (1996: 11), there is an "overreliance on the continuous part-whole model which inhibits children's thinking of fractions as numbers and the development of other fraction interpretations".

2.5.3 Knowledge of half

A good example of this is a simple fraction bar or folding a piece of paper. The whole is repeatedly halved to make smaller fractions. The result is that the denominators increase exponentially (i.e., 2, 4, 8, 16, etc.). This type of halving creates the impression that uneven denominators cannot be created. Learners struggle to see how a whole can be subdivided into thirds, fifths, sixths, sevenths, and so on: "This powerful strategy inhibits the child's ability to develop partitioning schemes to create fractions that have odd number denominators, for example thirds" (Pitkethly & Hunting, 1996: 11). It is for this reason that that early experiences of equal-sharing activities should include other fractional parts like thirds and fifths.

2.5.4 Perceptual and visual representations

These is a widespread belief among teachers that fractions should be introduced using pictures and physical material (i.e., real chocolate bars or A4 size cardboard). This approach is problematic because it is mainly perceptual and figurative and learners do not learn to reason about fractions. It would be wiser for the teacher to give learners realistic problems, in which they are forced to create their own need for fractions. The context of the problem should demand of the learner to cut a whole into parts to be able to solve it, for example, dividing three chocolate bars between two friends. The concept of a fraction is then formed as a result of the learners' own reasoning.

2.6 Other factors contributing to learners' understanding of fractions

Siegler et al. (2013:15) identified a number of other factors that may contribute to learners' understanding or misunderstanding of fractions. Firstly, learners' knowledge of whole numbers "interferes" with their knowledge and understanding of fractions (Booth & Newton, 2012; Siegler et al., 2013; Vamvakoussi & Vosniadou, 2004). Secondly, the factors most commonly mentioned amongst researchers as influencing learners are the knowledge of concept (or conceptual knowledge), knowledge of procedures, factual knowledge and prior knowledge (Booth & Newton, 2012; Hecht, Close, & Santisi, 2003; Osana & Pitsolantis, 2011; Siegler et al., 2013; Vamvakoussi & Vasnaidou, 2004;). Lastly, Hecht et al. (2003: 278) mentioned a common factor overlooked by many. They maintained that misunderstanding of fractions is not only cognitive in nature but that "behaviour characteristics" also impact negatively on a learner's understanding of mathematics and, more specifically, fractions (Hecht et al. 2003: 278). I acknowledge that most of these studies were based on learners in earlier grades and my study is on why learners in the FET phase misunderstand fractions, but I believe the root of their misconceptions of fractions stems from their childhood. Thus, these studies are worth reviewing.

Hecht and his colleagues noted that "behavioural characteristics" also play a part in learning of fractions (Hecht et al., 2003: 278). Learners should be given ample time to practice fraction problems inside the class but, more importantly, at home. This encourages an intrinsic motivation for learners to want to work on their own. Inside the classroom, a class exercise can be well facilitated by a teacher if it is done by the learner. Take-home exercises are based on work done in class to hone their fraction skills (Hecht et al., 2003: 279). Here, classroom management, discipline, and culture of learning (positive learning environment/atmosphere) are important factors for which teachers are responsible, to promote learning. Teachers should make it easier for the learner to want to learn.

One of the factors mentioned by Siegler et al. (2003:15) concerning why learners have difficulty in understanding fractions is that learners make the "erroneous assumption that all properties of whole numbers are properties of all rational numbers" (Siegler et al., 2003: 15). Booth and Newton (2012: 248) confirmed this by stating that "children relate fractions to their knowledge of whole numbers". They go on to say that prior knowledge (such as of whole numbers) can in some cases "interfere" with learners' understanding of fractions (Booth & Newton, 2012: 249). Similarly, Vamvakoussi and Vosniadou (2004:456) posited that "prior knowledge of natural numbers stands in the way of understanding rational numbers." An example given by Vamvakoussi and Vosniadou to substantiate their claim is the idea that "the more digits a number has makes it bigger" and "multiplication always makes bigger" (2004: 456).

Obviously, both of these contentions are untrue because the more digits a decimal has, the smaller the number becomes and multiplying with a fraction (or scale factor) can create smaller quantities. The latter will be discussed in more depth later in this chapter. It is important to note that prior knowledge about whole numbers is not the enemy. Learners will always draw on prior knowledge to try and make sense of new concepts, but it the teacher's responsibility to facilitate and guide learners' thoughts and, through their own MKT, to help clarify misconceptions. Knowledge types also play an important role in the learning of fractions. I have already discussed the role that prior knowledge plays in the learning of fractions. The other knowledge type is known as *factional knowledge*.

Factional knowledge is also referred to as "simple arithmetic knowledge" by Hecht et al. (2003: 278). Here, the learner will retrieve "memorised facts involving arithmetic relations amongst numbers (Hecht et al., 2003: 278). An example of this is the multiplication table. Learners can apply this arithmetic knowledge when multiplying with fractions, to name just one. The last two knowledge types are *conceptual* and *procedural*.

Conceptual knowledge involves learners' understanding pertaining to the principles involved in fractions. If one looks at the different constructs of fractions, one can say that a learner has conceptual knowledge when he/she has the "understandings concerning what rational quantities represent" (Hecht et al., 2003: 278). Fractions can take different meanings, depending on the scenario: part-whole, ratio (magnitude), operator, quotient or measure (Kieren, 1980). So, if learners have acquired the concept of $\frac{3}{4}$ as a whole being divided into parts or the comparison of the number of boys to girls in a classroom, the ratio concept is being used (Osana & Pitsolantit, 2013: 30). Teachers should be careful not to overemphasise the part/whole construct in class as this also leads to learners' difficulty in learning fractions (Siegler et al., 2013: 15).

Procedural knowledge is the ability to recognise and use mathematical symbols correctly and the skill to "execute step-by-step action sequences to solve problems" (Osana & Pitsolantit, 2013: 30). An example given by Osana and Pitsolantis (2013:30) to explain procedural knowledge is the common way to find the equivalent fractions, that is, to multiply the numerator and the denominator by the same natural number. Factual knowledge is sometimes misused in applying procedural knowledge when solving problems. Learners confuse themselves by muddling fraction arithmetic procedures withsimple arithmetic knowledge. When calculating $\frac{2}{5}+\frac{1}{2}$, many leaners will write down $\frac{3}{7}$ and not $\frac{9}{10}$. The same happens with $\frac{1}{2} \div \frac{1}{4} = \frac{1}{2}$ instead of2, because they immediately access simple arithmetic knowledge (factual knowledge) instead of applying fraction arithmetic procedures (Siegler et al. 2013: 15).

As can be seen, many factors other than MKT contribute to the learning and ultimate understanding of fractions. Before continuing with the different constructs fractions can take, a general idea of where fractions fit into the larger set of real numbers is needed.

2.7 Fractions with respect to the set of real numbers.

In mathematics, numbers can be classified and defined in many ways. Knowing these different definitions and classifications is useful when it comes to fractions. A fraction in itself has no physical meaning without proper classification and definition. It is important to know what numbers constitute a fraction. By doing this, we are one step closer in making sense of the different sub-constructs of fractions (part-whole, ratio, operator, quotient and measure). The diagram below shows how numbers can be classified; the discussion follows.

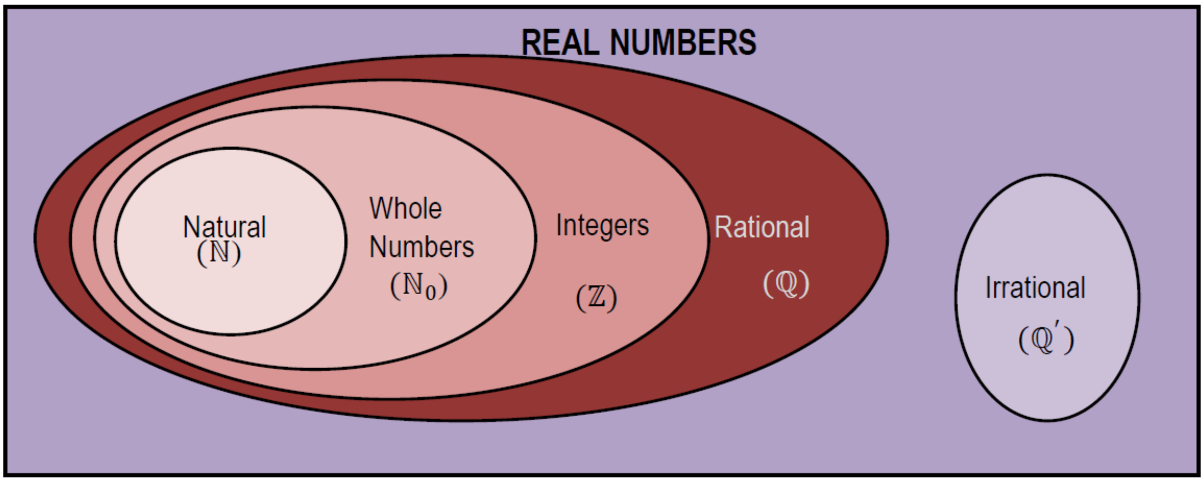


Figure 2: Classifying Numbers

(Source: Dorr, 2010:2)

Real numbers can briefly and simply be defined as all numbers on the number line. Within the real number set, we find natural, whole/counting numbers, integers (which includes positive whole numbers: 1, 2, 3…, zero: 0 and negative whole numbers: -1, -2, -3…) rational and irrational numbers. The rational number subset is of particular interest because, in the literature, these two words are used interchangeably. I, however, will throughout this thesis refer to "fraction" rather than "rational numbers" and in so doing, I am acknowledging that there are mathematical differences between the two, as I will explain later. Another term commonly associated with fractions is *quotient*.

A quotient is the result of a division problem. If a whole numbers (integer) is divided into another, the result is referred to as the *quotient* (Note that the divisor cannot be equal to zero). For example, $\frac{10}{2} = 5$, $\frac{10}{2}$ is the division problem of two integers (10 and 2). The result, 5, is referred to as the quotient of 10 and 2. Therefore, one is able to write a fraction as the result of a division sum (quotient) of two whole numbers (integers), where the denominator is not equal to zero, as in $\frac{3}{5}$, $\frac{1}{4}$ and so on.

A fraction is defined as a number that expresses part of a whole as a quotient of integers (where the denominator is not zero). Another way to describe a fraction is as a division expression where both the dividend or numerator (top number) and the divisor or denominator (bottom number) are integers, and the divisor (denominator) is not zero. We prefer using the terms *numerator* and *denominator* instead of *divisor* and *dividend* as this often confuses learners. Fractions can be written in different ways.

Proper, improper, mixed, equivalent, or complex fractions are all different forms that fractions can take. A fraction is proper when the value of the numerator is less than the value of the denominator, for example, $\frac{1}{2}$. When the numerator is greater than the denominator (e.g.,$\frac{3}{2}$), the fraction is called *improper.* A mixed fraction contains both whole numbers (positive or negative) and a proper fraction (e.g., $2\frac{3}{4}$). These are all thought of as simple fractions as opposed to complex fractions. Two fractions are equivalent when the ratio of the numerator to the denominator is the same for both of them. Both $\frac{2}{5}$ and $\frac{8}{20}$ are equivalent because $\frac{2\times4}{5\times4}=\frac{8}{20}$. There are infinite equivalent fractions because the ratio is preserved when multiplying the denominator and numerator with the same number (value). Complex fractions are also referred to as compound fractions because the numerator and denominator contain a fraction e.g. $\frac{\frac{2}{3}}{\frac{5}{7}}$.

The result of a division sum (where the denominator is not zero) of two integers will always be classified as a rational number, even if it results in a repeating or terminating decimal. In other words, any rational number can be written as a quotient of two integers. The first part of this definition describes any fraction. Every fraction is considered a rational number; however, not every rational number is a fraction.

All integers are rational numbers because they can be written as the "answer of a division sum" (a quotient), for example, $4=\frac{20}{5}$, $1=\frac{5}{5}$, $0{,}75=\frac{3}{4}$ and $0,\bar{2}=\frac{2}{9}$. However, not all integers are fractions. Although an integer may be written as a quotient, it is not a fraction. Our definition of a fraction states that it expresses *part of a whole*. An integer does not; it expresses *the whole*. A visual to help in understanding this is as follows:

The rectangle below is divided into four equal parts. Each piece is a $\frac{1}{4}$ (quarter) of the whole, written as $\frac{1}{4}$. If one were to remove one piece, there will be $\frac{3}{4}$ (three quarter) parts left. Therefore, each piece of the rectangle is a fraction. On the other hand, if one has all the pieces, it is one whole $\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{4}{4}=1\right)$.

$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Similarly, 1 can be written as $\frac{8}{8}$, but I still have all the pieces and not just some parts of the whole.

$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

In summary, fractions do not contain integers and rational numbers do. Therefore, fractions form a subcategory of rational numbers, but not all rational numbers are classified as fractions.

2.8 Are fractions important?

Almost every learner asks the question “Are fractions important and why?” The answer is very simply “Yes!” Siegler et al. (2012) concluded that: “Secondary school pupils' mathematics performance could be substantially improved if children gained a better understanding of fractions”. Fields such as architecture, medicine, chemistry, engineering, or technology all involve precise and accurate calculations. This means that understanding of the different meanings of fractions, including computations with fractions, is critical.

Learners experience fractions informally, for example, the sharing of sweets, dealing cards for a card game or when baking/cooking (using recipes). Even though learners are confronted with fractions in their everyday life, they only really experience abstract thinking when they are introduced to fractions in primary school. Basic arithmetic like addition, subtraction, multiplication and division of whole numbers is the main focus in lower grades to develop basic mathematics skills. In higher grades, fractions are found in topics such as algebra, geometry, probability, and trigonometry.

Fractions appear widely within the school curriculum, ranging from primary to high school. It is thus essential for learners to have a good understanding of the meanings of fractions if they wish to achieve well in higher grades. The only way in which learners will gain a better understanding of fractions is if they are exposed to them though instruction. Consequently, it is necessary for teachers to develop their own understanding of the rich meanings associated with fractions before teaching them to learners. The problem, however, is that many teachers are uncomfortable when having to teach fractions. The main reason for this, research suggests, is because of teachers’ lack of understanding of the multifaceted and interrelated sub-constructs of fractions. This directly influences their view of fractions and the way they teach them to learners. Even though some teachers feel threatened by fractions, they need to acknowledge their importance in making sense of the concepts associated with rational numbers in high school (and beyond). The mathematics curriculum is permeated by the idea of *part-whole* or partitioning. Algebra, geometry, trigonometry and probability all require a thorough understanding of the function of fractions in mathematics. Misunderstanding fractions does not only lead to poor thinking skills but also affects learners’ understanding and performance in all other areas in mathematics. It is clear that teachers themselves should first gain a better understanding of the different meanings of fractions and how they can be interpreted before attempting to present them to learners.

CHAPTER 3: FRACTIONS IN THE SCHOOL CURRICULUM

3.1 The different interpretations of fractions

As discussed earlier, MKT is the mathematical knowledge of teachers and has an impact on classroom instruction. A broader MKT knowledge base ultimately leads to better content knowledge and the necessary skills to effectively transfer that knowledge to learners (Leung & Park, 2002). MKT also "positively predicts student gains in mathematics achievement" (Hill, Rowan, & Ball, 2005: 399). One of the ways in which teachers can broaden their MKT is by studying the different interpretations (or sub-constructs) of fractions. The concept of fractions consists of five sub-constructs, namely, part-whole, ratio, operator, quotient, and measure (Kieren, 1980). Associated with these sub-constructs are the computations $(+, -, \times, \div)$.

It has been well documented (Ross & Bruce, 2009: 714), and observed in my own experience as a mathematics teacher, that learners struggle with mathematics, especially when it comes to fractions. This is supported by Charalambous and Pitta-Pantazi (2007), who noted that "fractions are among the most complex mathematical concepts that children encounter" The notion of a fraction being a compound concept, consisting of five interlinked sub-constructs (part-whole, ratio, operator, quotient, and measure), is one of the main reasons why fractions are seen as being complex. Kieren (cited in Behr et al., 1983: 92) argued that if one wants a complete understanding of fractions, one must also understand the sub-constructs and how they are interlinked. A fraction should not be viewed as a single concept but should rather be seen in all its meanings or forms, for instance, its relationship between the part and the whole, as the answer of a division sum (quotient), as an operation performed on a quantity, as a ratio, or as a unit of measure.

Kieren was the first to establish this notion of several interrelated sub-constructs, but originally he did not see part-whole as a distinct construct because he believed that the part-whole is the foundation of the other four (ratio, operator, quotient, and measure). Behr et al. (1983), however, "redefined" Kieren's work by placing part-whole as a separate "fundamental construct" and coupled partitioning with it, claiming that these two "are basic to learning other sub-constructs" (Behr et al., 1983: 99). Behr et al. (1983) went even further by "linking different interpretations of fractions to basic operations" (cited in Charalambous & Pitta-Pantazi, 2007: 295). They linked ratio to equivalence, stating that it is the "most natural" way to develop better understanding of the concept. The operator construct is viewed as helpful in developing understanding in multiplication of fractions. The measure construct is required to build proficiency in addition of fractions. Understanding all five sub-constructs is imperative in order to solve problems involving fractions. Below is a brief definition of each of the five sub-constructs:

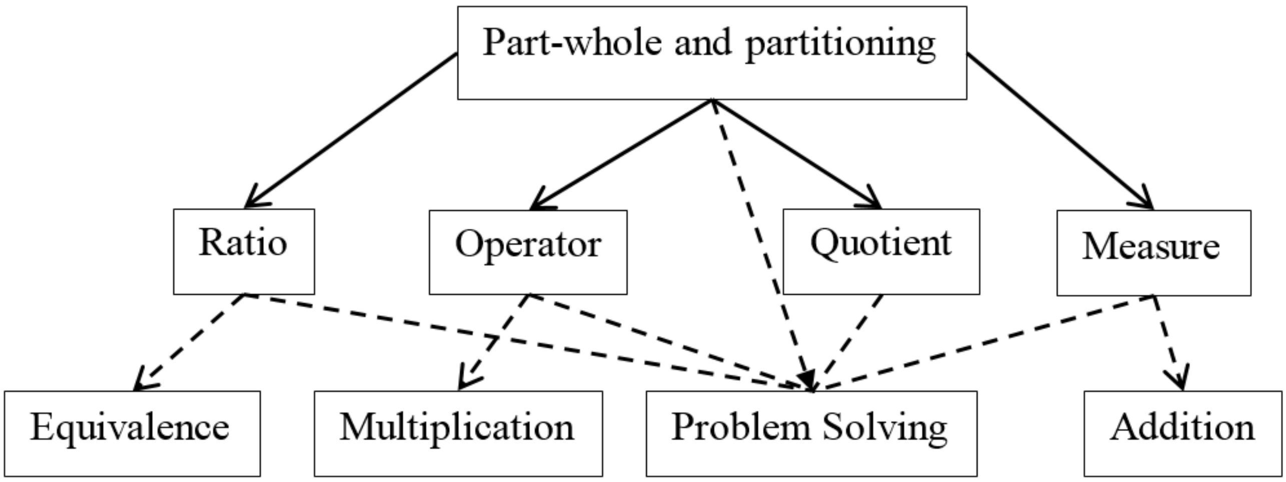


Figure 3: The theoretical model linking the five sub-constructs of fractions to different operations of fraction and to problem solving

(Source: Behr et al., 1983)

3.1.1 Fractions as a relationship between the part and the whole

The part-whole sub-construct of fractions is created when an object or a set of discrete objects are partitioned or divided into parts of equal size (Lamon, 1999; Marshall, 1993). This method is one of the most over-utilised in the school curriculum. For many leaners (and teachers), this is where it stops. The problem, though, is that part-whole and partitioning is supposed, by Kieren, to be the foundation to understanding the other sub-constructs. As defined here, the fraction represents an object cut into equal pieces and the *numerator* refers to how parts of the partitioned unit there are, whereas the *denominator* refers to the size of the pieces (parts in which the unit is partitioned). The bigger the denominator, the smaller the parts one is cutting the object into, resulting in more pieces. Here, it is important that the learner understands the relationship and meaning of the bottom number (denominator or divisor) and the top number (numerator or dividend). If teachers give learners problems involving fractions as a part-whole relationship, they need to mention that learners should first ask themselves, "What is the whole (shape or set)?" and "What is the part (of the shape or set)?". The result of the relationship is called the *quotient*; for example, $\frac{3}{4}$ shows that an object (whole unit) has been divided or cut into four equal parts and three of those parts are being considered. From this viewpoint, the numerator must be less than or equal to the denominator. Many models have been created to assist learners in the mastering of the part-whole sub-constructs.

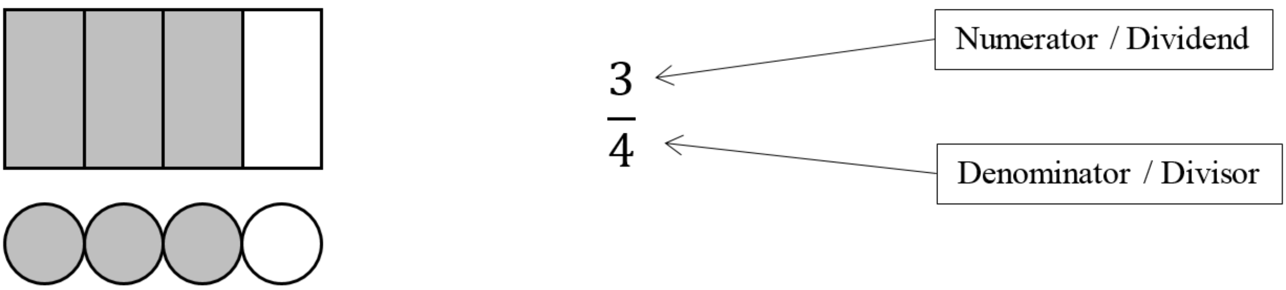


Figure 4: Using a model to demonstrate part-whole understanding of fractions

Note: When using a model to demonstrate part-whole understanding of fractions, we can say the rectangle is three-quarters shaded because there are four equal-sized parts and three of them are shaded. In the fraction "three-quarters ", the three shaded parts are referred to as the *numerator* and four of these equal-sized parts make up the whole shape, referred to as the *denominator*. The equal-sized parts could be called "quarters" (Wong & Evans., 2008: 598).

3.1.2 Fractions as ratios

Behr et al. (1983:94) had this to say about fractions as ratios:

> Ratio is a relation that conveys the notion of relative magnitude. Therefore, it is more correctly considered as a comparative index rather than as a number. When two ratios are equal they are said to be in proportion to one another. A proportion is simply a statement equating two ratios.

There might be some confusion between the differences between ratios, rates and proportions as they are dealt with at the same time in a single chapter at school. So, to clarify, if two quantities of the same unit are compared, it is called a *ratio*, whereas a *rate* is used to compare two quantities with different units. A *proportion* is composed of two equal ratios. Two quantities will be in proportion if the quantities are related in such a way that the size of one of the quantities affects the size of the other quantity. There is a direct proportion between two quantities if the both quantities increase or decrease at the same time in the same proportion. Two quantities are said to be in indirect proportion if when one quantity increases, the other decreases by the same proportion, or vice versa.

A ratio can be written in three different ways. A ratio must always be reduced to its simplest form but can have more than two numbers. However, usually only two quantities are compared. For example, if we want to compare the number 4 to the number 7, we would write it one of the following ways:

i. 4 to 7
ii. 4:7
iii. $\frac{4}{7}$

Below is an example of an application (problem-solving) on rate, ratios, and proportion from the Curriculum and Assessment Policy Statement for Grade 9:

Ratio and rate problems

- Include problems involving speed, distance and time. Learners should be familiar with the following formulae for these calculations:
  a) speed = $\frac{distance}{time}$
  b) distance = speed x time
  c) time = $\frac{distance}{speed}$
- Speed is usually given as constant speed or average speed.
- Make sure learners recognize and are able to convert correctly between units for time and distance.

Examples

a) A car travelling at a constant speed travels 60 *km* in 18 minutes. How far, travelling at the same constant speed, will the car travel in 1 hour 12 minutes?

b) A car travelling at an average speed of 100 *km/h* covers a certain distance in 3 hours 20 minutes. At what constant speed must the car travel to cover the same distance in 2 hours 40 minutes?

Direct and Indirect proportion

Learners should be familiar with the following relationships:

- x is directly proportional to y if $\frac{x}{y}$ = constant
- x and y are directly propotional if, as the value of x increases the value of y increases in the same proportion, and as the value of x decreases the value of y decreases in the same proportion
- The direct proportional relationship is represented by a straight line graph
- x is indirectly or inversely proportional to y if x x y = a constant. In other words $y = \frac{c}{x}$
- x and y are indirectly propotional if, as the value of x increases the value of y decreases and as the value of x decreases the value of y increases
- an indirect proportional relationship is represented by a non-linear curve

Figure 5: Rate, ratios and proportion

(Source: DBE, 2011: 120)

Fractions as ratios are regarded as "necessary for the development of the idea of fraction equivalence" (Marshell, 1993, cited in Charalambous & Pitta-Pantazi, 2007: 297).

3.1.3 Fractions as operators

In using fractions as operators, the rational numbers are "regarded as functions applied to some number, object, or set" (Behr et al., 1983; Marshall, 1993). Here a fraction multiplies a quantity and either increases it or decreases it. Behr et al. (1983: 96) referred to this as a "stretcher/shrinker". A function is applied to a quantity (number, object, or set) when trying to find a fraction of it. A new value is formed when the operation shrinks (reduces) or stretches (enlarges) the quantity (number, object or set). This operation is a combination of multiplication and division. For example, finding $\frac{3}{4}$ of a number, on a set of things, on a segment, or on a geometrical picture, the operation will be to divide by four and then to multiply by three, or to multiply by three and divide by four. The function transforms the discrete set into another set with fewer or more elements. Calculating $\frac{3}{4}\ of\ 16$ (sweets, centimetres, books, rands) can be done visually or numerically. Numerically, one would multiply 16 by 3 and then divide by 4: $(16 \times 3) \div 4 = 12$ (sweets, centimetres, books, rands). Alternatively, one can also calculate this by means of a visual representation:

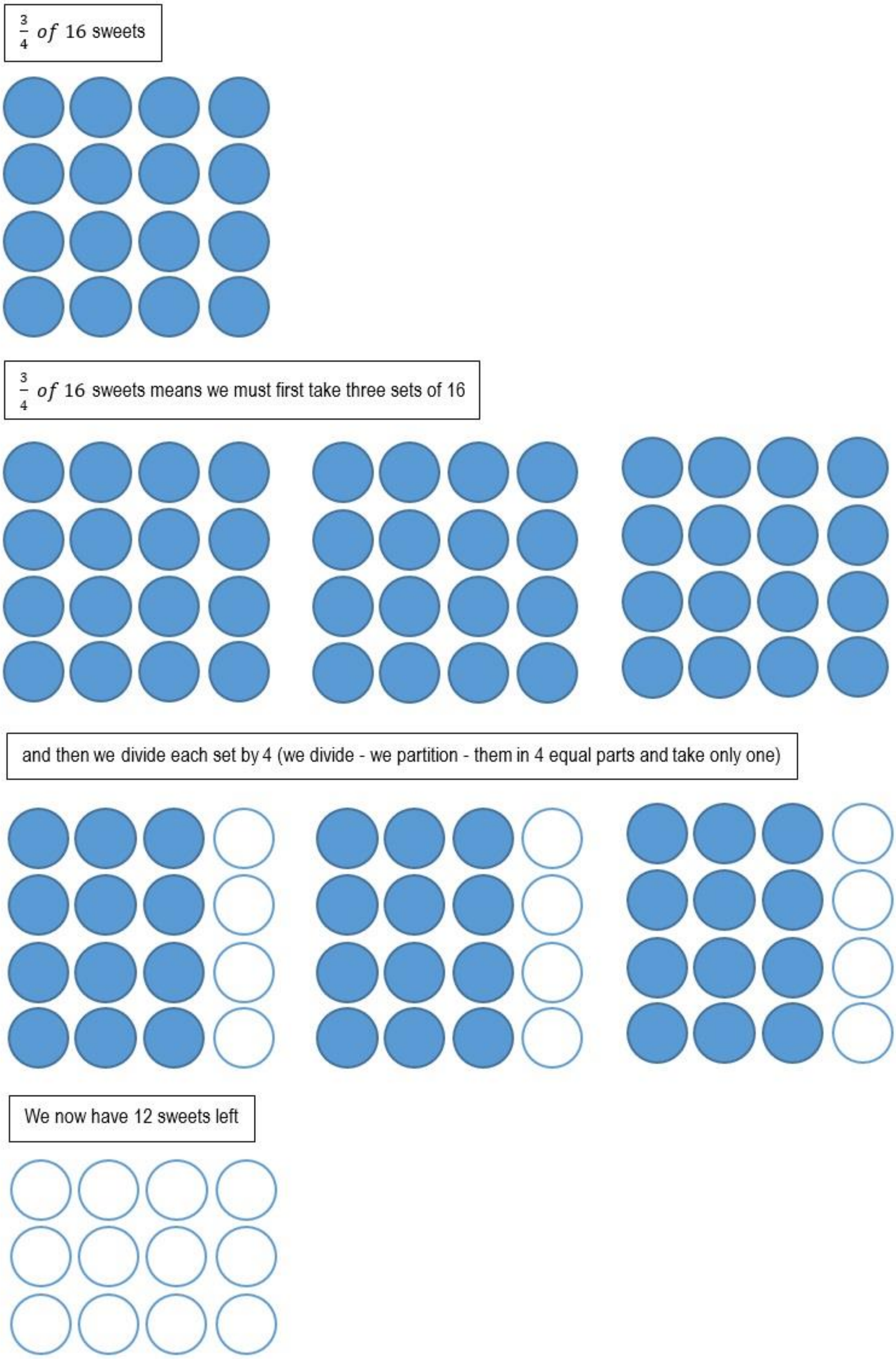


Figure 6: Fractions as operators applied to a set

(Source: Charalambous & Pitta-Pantazi: 2007)

The fraction-as-operator sub-construct is thought of as a function that dilates (transforms) a geometric shape to another that is similar. When a rectangle, with dimensions 2 x 6 units, is dilated by a scale factor of $\frac{1}{2}$, it "shrinks" the object. On the other hand, when the same shape is dilated by a scale factor of $\frac{3}{2}$, it "stretches".

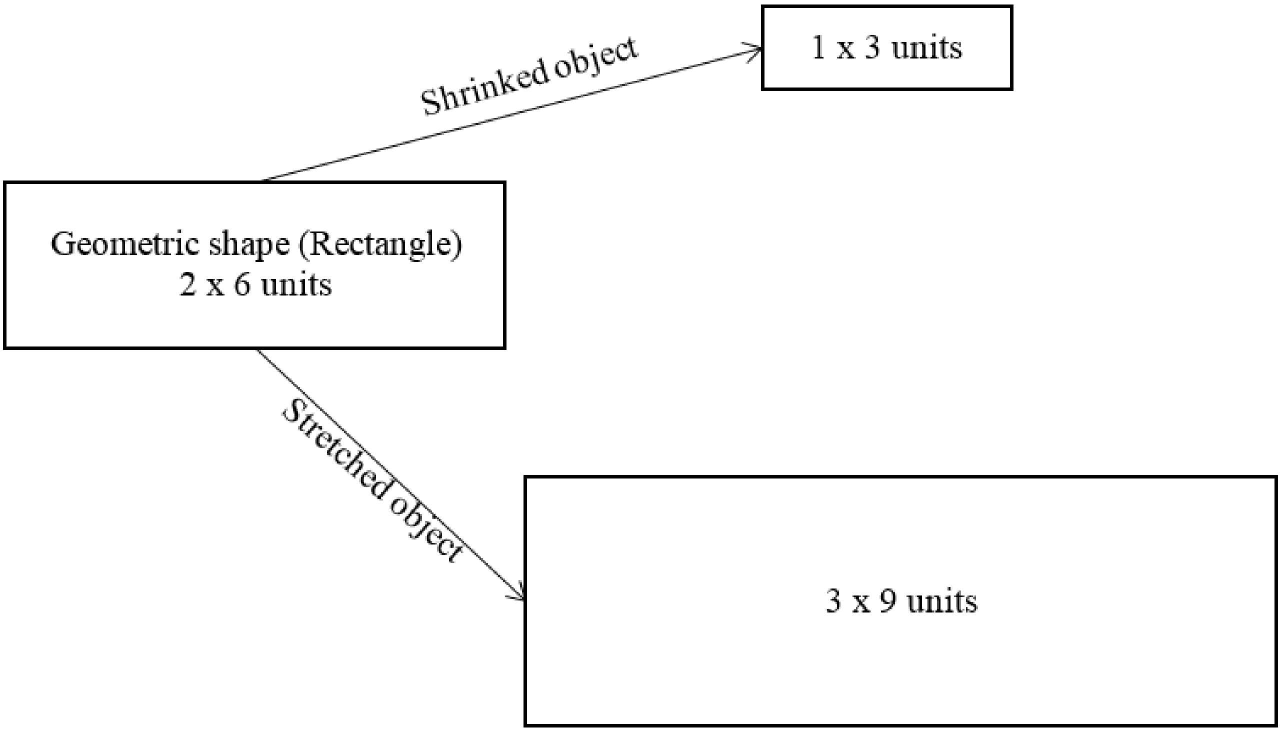


Figure 7: Fractions as operators applied to a geometrical shape.

(Source: Charalambous & Pitta-Pantazi, 2007)

The line segment below has a length of 10 units. This line segment is considered as a continuous object on which an operation will be performed. If one takes $\frac{2}{5}$ of the line, it is stretched to twice its original length. It is then shrunk by a scale factor of 5. Each piece is now equal to 4 units.

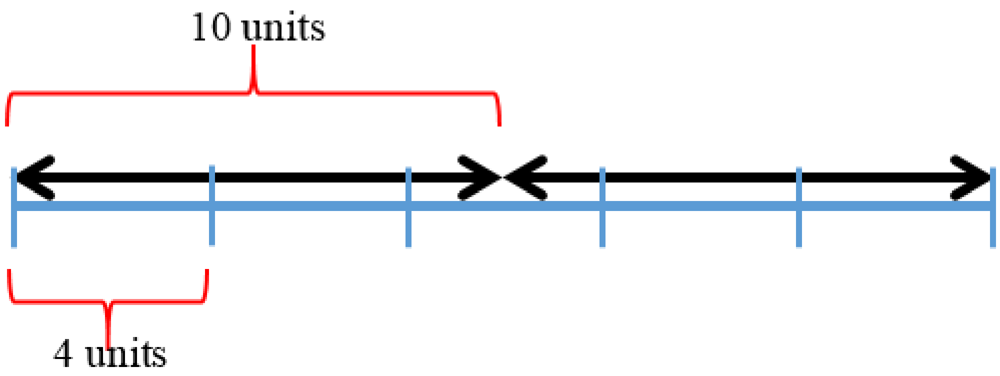


Figure 8: Fractions as operators applied to a line segment

(Source: Charalambous & Pitta-Pantazi: 2007)

When interpreting fractions as operators, it is useful to study them in the context of multiplication as ‘operation and fraction equivalence’. Fractions are used in this way in everyday life, for instance, in calculating the discount on merchandise, doubling a cake recipe to serve more people, and in replicas of scaled models of airplanes, boats, buildings, and so on.

3.1.4 Fractions as quotients

Fractions as the result of division are called the quotients. The quotient sub-constructs hold the idea that any fraction is the answer of a division sum. For example, $\frac{x}{y}$ is the “fair share” of each person when a pizza, x, is shared amongst y people, where both x and y are whole numbers (Streefland, 1993). The fraction $\frac{x}{y}$ can also be considered as $x \div y$. This interpretation also comes from a partitioning situation. In the quotient sub-constructs, unlike part-whole/partitioning, the numerator can be smaller than, bigger than, or equal to the denominator. Also the answer (quotient) after equal sharing may be smaller than, bigger than, or equal to the whole (unit). The reason for this is because two different measured spaces are considered, for instance, pizzas and people, and therefore the outcome obtained “refers to a numerical value and not the parts obtained by fair-sharing” (Charalambous & Pitta-Pantazi, 2007: 299). To place this idea in a South African context, I adapted the model used by Charalambous and Pitta-Pantazi by using South African currency rather than the usual pizza model. If one has R4,00 and wants to share it amongst 5 people, one could represent this mathematically by $4 \div 5$ or $\frac{4}{5}$ .Better yet is to represent this by means of a model:

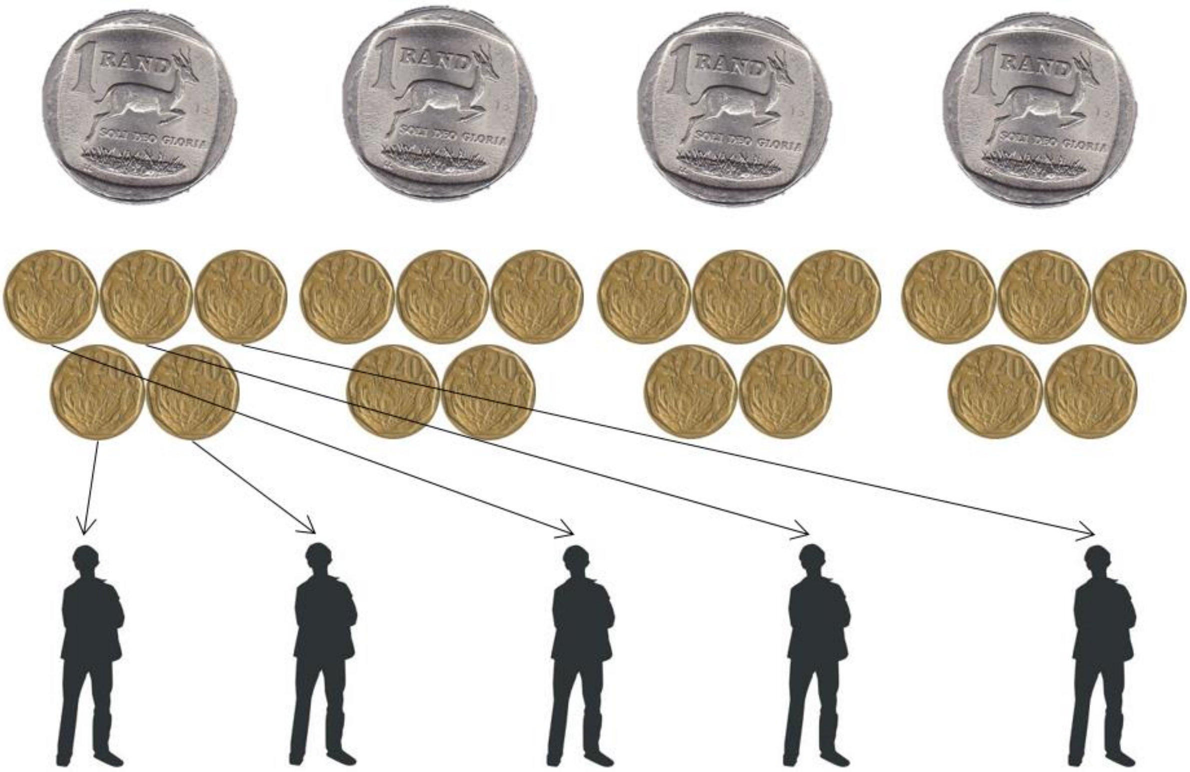


Figure 9: Model to demonstrate division of disproportionate items

Note: Subdivide each R1 coin into five equal parts. Then share each set equally amongst the five people. Each person will get four 20c pieces; therefore, each person will receive 80c when R4 is divided between five people. Similarly, each R1 coin can be subdivided into 10 equal parts. Each person will then get eight 10c pieces, making 80c. The second example illustrates how equivalence is represented in this model by sub-constructs (Adapted from Charalambous & Pitta-Pantazi, 2007).

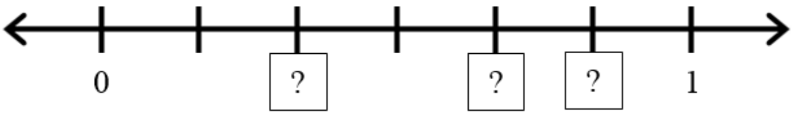
Making use of fractions as quotients involves a certain level of complexity. A learner must be able to see the division results in establishing equivalence as in this case, $4 \div 5 = \frac{4}{5} = \frac{400}{500} = \frac{80}{100} = 0{,}8,$ or $80c,$ in our case. However, as Behr et al., (1983: 95) pointed out, "This level of sophistication generally requires intellectual structures not available to middle school children because it relates rational numbers to abstract algebraic systems".

3.1.5 Fractions as measurement

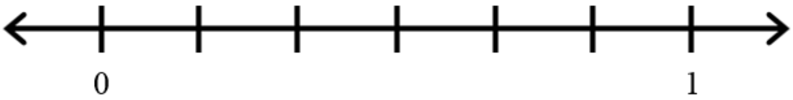
In the measurement sub-constructs, a fraction is thought of as a number, carrying with it a quantitative nature, that is, how big the fraction is in relation to other numbers on the number line. In using fractions as a measure, a length model is used. Here, we can compare number lines and physical materials on the basis of length. Hart (1981) observed that fractions as a measure can be used to extend the whole number system. If a standard object or unit of measure is subdivided into smaller equal parts, the parts are considered to be fractional units: "Geometric regions, sets of discrete objects, and the number line are the models most commonly used to represent fractions" (Behr, 1983: 93). When interpreting fractions as measure, or showing their quantitative nature, number lines are used as a mathematical model. Many learners have difficulty in finding fractions on number lines. The reasons for this may be because of a lack in understanding fractions as part-whole constructs and that learners are also unable to place fractions in the correct order of size to compare them. A number line is used by many learners as a tool to help them to visualise the size of a fraction, which also aids them in the representation and comparison of fractions. Fraction strips can help to develop the idea of equivalent fractions, while addition and subtraction of fractions can be visually represented by means of number lines.

There are two distinct skills learners should acquire when working with fractions using a number line. The first is filling in blank, prepositioned boxes along a number line. By doing this, learners get a sense of which fractions are smaller or bigger than others and they can also illustrate how a number line can be used to represent fractions of distance or length. The second is writing fractions in on a blank number line. This involves being able to measure (length) and to decide where to place a particular fraction in proportion to the length or distance marked.

a) Learners are required to identify the quantity represented on a number-line



b) Learners are required to locate a fraction, for example $\frac{1}{6}$, on a number line



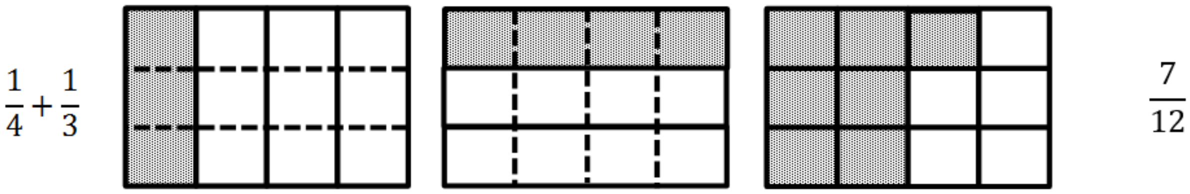


Figure 10: Adding fractions to subdivide each area of a geometric shape in equal parts

Note: The purpose is to write down how many one has (numerator) of that particular size (denominator). (Charalambous & Pitta-Pantazi, 2007)

According to Lamon (1999), while the fraction symbol is lacking in physical meaning and context, it has many interpretations, including part/whole, quotient, measure, ratio, and operator. The different interpretations should not be seen independently. The different interpretations help in seeing fractions from different perspectives, in making sense of them and in gaining a better understanding of the role of fractions in mathematics and in real-life situations (Kieren, 1980).

In this study, I have considered fractions as numbers, being a subset of rational numbers. However the terms *rational numbers* and *fractions* are used interchangeably in the literature. I refer to *fraction* rather than *rational number* for the simple reason that all fractions are rational numbers but not all rational numbers are fractions. Misunderstanding fractions does not only lead to poor thinking skills, but also affects learners' understanding in algebra, geometry, trigonometry and probability. Commenting on the results of a research study, Siegler et al., (2012) noted that "high school students' knowledge of fractions did correlate very strongly with their overall mathematics achievement.". A contributing factor to this, contradictory to what the South African curriculum for mathematics has envisaged for all learners, is that authorities look at the mass production of learners passing mathematics in matric without looking at the quality of their passes. Finally, learners' difficulty with fractions is also based on the notion proposed by Kieren (1980) of several interrelated sub-constructs. The way in which fractions can be interpreted is summarised by Clarke, Roche, and Mitchell (2007:208) in the following way:

- The part-whole interpretation depends on the ability to partition either a continuous quantity (including area, length, and volume models) or a set of discrete objects into equal sized subparts or sets.
- A fraction can represent a measure of a quantity relative to one unit of that quantity.
- A fraction $\left(\frac{a}{b}\right)$ may also represent the operation of division or the result of a division, such that $3 \div 5 = \frac{3}{5}$.
- A fraction can be used as an operator to shrink and stretch a number such as $\frac{3}{4} \times 12 = 9$ or $\frac{5}{4} \times 8 = 10$. The misconception that multiplication always 'makes bigger' and division always 'makes smaller' is common.
- Fractions can be used as a method of comparing the sizes of two sets or two measurements such as "the number of girls in the class is $3:5$ or $\frac{3}{5}$ the number of boys", that is,, a ratio.

3.2 Examples from the high school curriculum

3.2.1 Fractions in algebra

Algebra is one of the very first concepts a child learns in high school. Every other topic that follows has its roots in algebra. If learners can grasp algebraic concepts, and more so the role of fractions, their understanding of fractions in geometric similarity, probability and trigonometry will be improved. For this reason, I chose to start this chapter with fractions in algebra.

Fractions and algebra are two topics in school mathematics that are considered critical to the curriculum but difficult to learn (National Council of Teachers of Mathematics, 2000). Teachers know well that learners have difficulties when working with fractions and basic algebra. Empson and Levi (2010) blamed this poor performance in algebra on learners' misunderstanding of fractions. Wu (2001:1) backed this claim by stating that "the proper study of fractions provides a ramp that leads students gently from arithmetic to algebra." He went on to say that if there is a lack in understanding fraction concepts, "the ramp will collapse" (Wu, 2001: 1) and learning of algebra will suffer. In algebra, arithmetic is generalised and goes beyond the specific case, where, for example, 5 + 2 = 7, to equations that are true for all numbers all the time. Nevertheless, "Elementary algebra is built on a foundation of fundamental arithmetic concepts" (Brown & Quinn, 2007: 8). Therefore, if fraction algorithms are loosely defined and the overreliance on shortcuts and a list of step-by-step 'how to's' are forced on learners without real understanding, it will ultimately lead to vague algebraic concepts and procedures that will hinder performance in algebra (Brown & Quinn, 2007: 8).

Much of the content in basic algebra relies on the understanding of fractional concepts. For instance, combining *like* terms is a concept used while learning addition and subtraction of fractions.

Important to note at this stage is that it is very difficult for a first-time algebra learner to grasp the fact that variables are ‘placeholders’ for numbers in mathematics. At first glance, this notation looks foreign and, psychologically, the connection is still letters-to-words and not letters-to-variables (placeholders). Only when this bridge is built, can one work with algebra. In an algebraic expressions, *like* terms are defined as terms that contain the same variables raised to the same power. Only the coefficients of like terms are different. Only if those conditions are met, is one allowed to simplify the expression by adding them. If we look at the expression $3x + 4x$, the answer is simply $7x$. This should not be new, because in primary school we learnt that if I have three lollypops and I get another four lollypops, I have seven lollypops in total. The connections with an expression like that and $\frac{3}{10} + \frac{2}{5}$ which equals to $\frac{7}{10}$ is this: When adding fractions, the concept of “equal parts” is used. As in the algebraic expression above, the learner must keep in mind that the “x’s” are the “same”. Perhaps the figure below will help to demonstrate my point better.

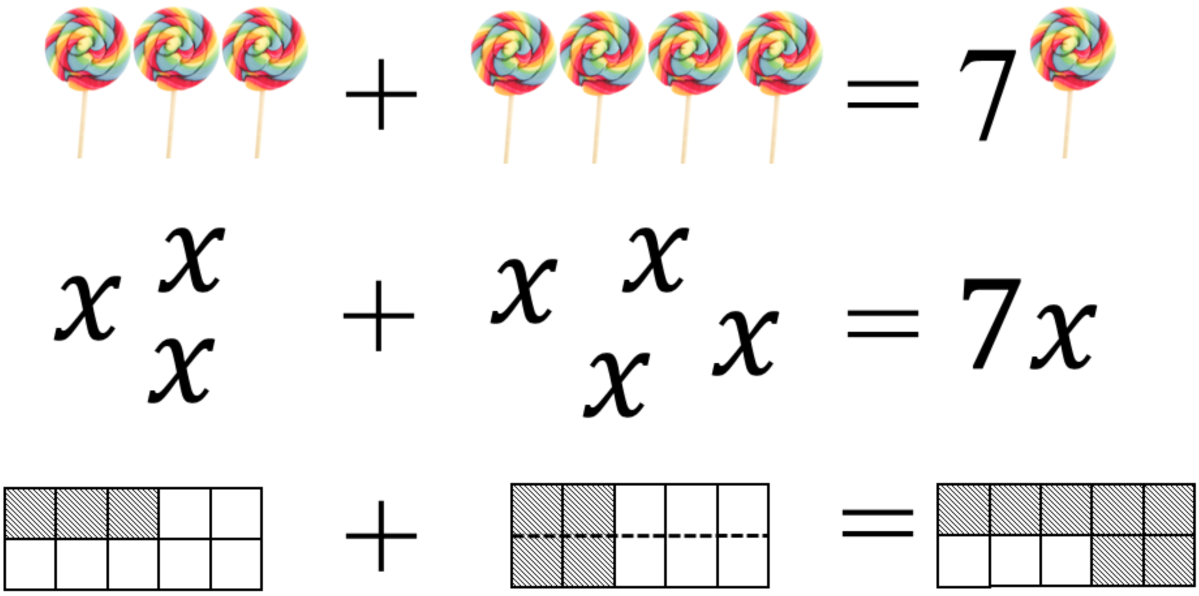


Figure 11: Adding the elements that are alike.

Note: In the case of the fractions, one must first make the parts the same size (2/5=4/10) before adding. In total, there are seven pieces, each the size of one-tenth. (Empson & Levi, 2010)

Brown and Quinn (2007: 9) claimed that “multiplying an equation by a constant to clear the denominators employs understanding of fraction concepts” and that “proportional equations use constructs that have their basis in equivalent fractions”. Solving a proportion means that one part of one of the fractions is missing, and one needs to solve for that missing value. Wu (2001: 5) referred to this as the “cross-multiplication algorithm” and gave the example of $\frac{a}{b} = \frac{c}{d} \leftrightarrow ad = bc$. The benefit of teaching fractions correctly, Wu (2009:18) claimed, is that learners will learn to prove this theorem and learn about a property true for all fractions. By carelessly cross multiplying, the process of reasoning is concealed (Thomas, 2010: 18). Instead, the learner who uses fraction reasoning to move from

$$\frac{a}{b} = \frac{c}{d}$$

(where a, b, c, d are whole numbers)

to conclude that

$$ad = bc$$

is applying simple reasoning: By equivalence of fractions, we have

$$\frac{ad}{bd} = \frac{bc}{bd}$$

and because the denominators are the same, $ad = bc$**.**

Wu (2001: 5) illustrates a good application of this algorithm by proving (in proportional equations):

$$\frac{ad}{bd} = \frac{bc}{bd} \text{ is the same as } \frac{a}{a+b} = \frac{c}{c+d}$$

Problem: If the ratio of boys to girls in an assembly of 224 students is 3:4, how many are boys and how many are girls? Given data that the ratio of boys to girls is 3:4 and if *B* is used to denote the number of boys and *G* to denote the number of girls in the audience, then:

$$\frac{B}{G} = \frac{3}{4}$$

From $\frac{a}{a+b} = \frac{c}{c+d}$ we know that $\frac{B}{B+G} = \frac{3}{3+4}$. We know that the total of boys and girls is 224 ($B + G = 224$), and $3 + 4 = 7$. Therefore:

$$\frac{B}{224} = \frac{3}{7}$$

By either multiplying the equation by a constant to clear the denominators or by the cross-multiplication algorithm, $B = 96$ and $\text{G} = 128$ $(G = 224 - B)$**.** Providing the proof of a statement such as $\frac{a}{a+b} = \frac{c}{c+d}$ is the kind of lesson that should be a regular part of the teaching of fractions. It is not just a useful but also exposes the learner to symbolic computation (Wu, 2001: 6).

A large portion of high school algebra deals with the solving of equations. This requires the ability to confidently compute with fractions (Wu, 2008: 4). Solving systems of linear equations is dependent on the ability to form equivalent equations and manipulate fractions, which often are part of the solution (Wu, 2007: 9). For example, to solve the equation (assuming that there is a solution for x that will satisfy the equation):

$$\frac{8}{11}x - 2 = \frac{1}{2}x + 68$$

So by adding $-\frac{1}{2}x$ and 2 to both sides, we isolate the x terms on the left and the constants on the right and find that

$$\frac{8}{11}x - \frac{1}{2}x = 68 + 2$$

By applying the distributive law that states that one arrives at the same answer when one multiplies a number by a group of numbers added together as one does when doing each multiplication separately, that is, $a(b + c) = ab + ac$, so to apply the above example, we are left with

$$\left(\frac{8}{11} - \frac{1}{2}\right)x = 70$$

$$\frac{5}{22}x = 70$$

Multiplying both sides by $\frac{22}{5}$ gives

$$x = \frac{22}{5} \times 70$$

$$x = 308$$

(Wu, 2009: 43)

By solving this equation step-by-step, as above, we have shown, successfully, that learners need to be comfortable with the arithmetic of positive and negative fractions in solving equations. Therefore, without the ability to compute fluently with fractions, learners have no hope of learning algebra (Wu, 2008: 4).

Finally, the entire study of linear equations is dependent on the slope of a line, a fraction representing the rate of change (Brown & Quinn, 2007: 9). The set of all solutions is a line. The set of all solutions of a graph or function is an ordered pair that satisfies the function. The solution set points of an equation is equivalent to the graph of an equation. In most cases, in school classes, the reason for this is never explained. The first reason is because teachers fail to define the gradient or slope of a line correctly. The gradient or slope (denoted as m) of a straight line is the rate at which the line rises (or falls) vertically for every unit across to the right. That is:

$$Gradient = \frac{Rise}{Run}$$

$$m = \frac{Change\ in\ y}{Change\ in\ x}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Given that the line is defined as $f(x) = mx + c$, let $P(p_x; p_y)$, $Q(q_x; q_y)$, be distinct points on the line.

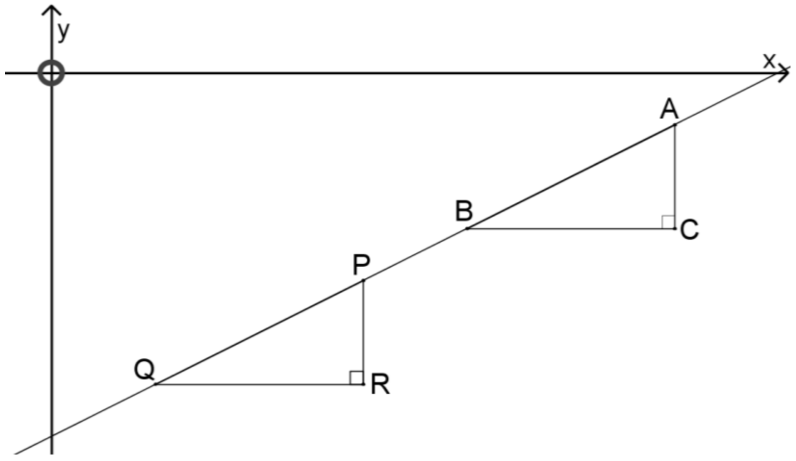


Figure 12: The slope of a line as linear equations and similarity concepts.

(Source: Wu, 2008)

The slope of the line is the defined as

$$\frac{p_y - q_y}{p_x - q_x}\left(= \frac{PR}{QR}\right)$$

Likewise, if we were to place points $A(a_x; a_y)$, $B(b_x; b_y)$ anywhere on the line (seeing that lines consists of an infinite number of points), we now find that the slope can also be defined as

$$\frac{a_y - b_y}{a_x - b_x}\left(= \frac{AC}{BC}\right)$$

By proving $\frac{p_y - q_y}{p_x - q_x} = \frac{a_y - b_y}{a_x - b_x}$ we conclude that $\frac{PR}{QR} = \frac{AC}{BC}$. The reason for this is because $\Delta PQR|||\Delta ABC$. To prove this requires similarity concepts, which are seldom taught properly in lower grades. Learners are simply told to memorise all the different "varieties" of the equation of a straight line: two-point form $\left[(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)\right]$, point-slope form $[(y - y_1) = m(x - x_1)]$, slope intercept form $[y = mx + c]$ and the standard form. Learners do not know that moving these points does not change the slope. This rote learning of topics in algebra has serious consequences in mathematics later on in its development. If we just look at the proof that $\frac{p_y - q_y}{p_x - q_x} = \frac{a_y - b_y}{a_x - b_x}$ or $\frac{PR}{QR} = \frac{AC}{BC}$ we can calculate the equation of any line, provided that a minimum of either two points that satisfy the function are given or the gradient is given with one other point that satisfies the function. Study the sketch below. For convenience, I chose to use $B(0; 5)$ and $Q(-10; 0)$ to be intercepts of the line $g(x) = mx + c$ and $B = P$. $A(x; y)$ is any random point on the graph.

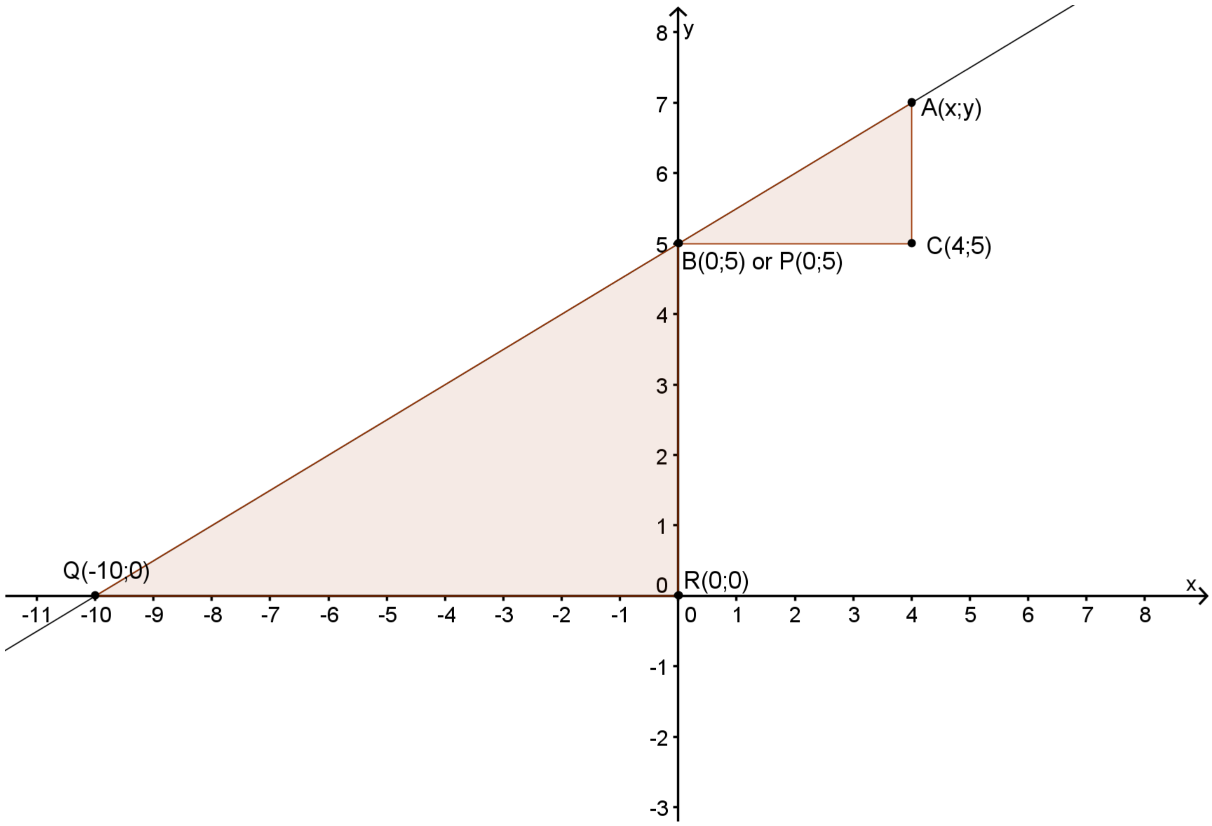


Figure 13: Using co-ordinates to prove similarity.

(Source: Wu, 2008)

Now,

$$\frac{a_y - b_y}{a_x - b_x} = \frac{y-5}{x-0} = \frac{y-5}{x} = \frac{AC}{BC} \quad \text{and} \quad \frac{p_y - q_y}{p_x - q_x} = \frac{5-0}{0-(-10)} = \frac{5}{10} = \frac{PR}{QR}$$

Remembering that

$$\frac{PR}{QR} = \frac{AC}{BC}$$

we find

$$\frac{5}{10} = \frac{y-5}{x}$$

$$5x = 10y - 50$$

$$-10y = -5x - 50$$

$$y = \frac{1}{2}x + 5$$

Alternatively, if given that the gradient is equal to $\frac{1}{2}$ and only one point is given, say $B(0;5)$, we can still plot any random point $A(x;y)$ and solve the equation as follows:

$$\frac{1}{2} = \frac{y-5}{x-0}$$

$$\frac{1}{2} = \frac{y-5}{x}$$

$$x = 2y - 10$$

$$-2y = -x - 10$$

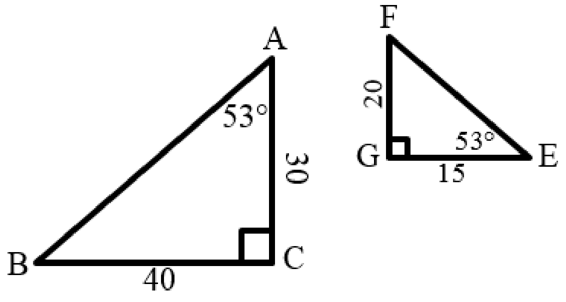
$$y = \frac{1}{2}x - 5$$

As can be seen, in this case, the y-intercept was given and I used it as a point, to illustrate that this method will work for *any* given point that will satisfy the function.

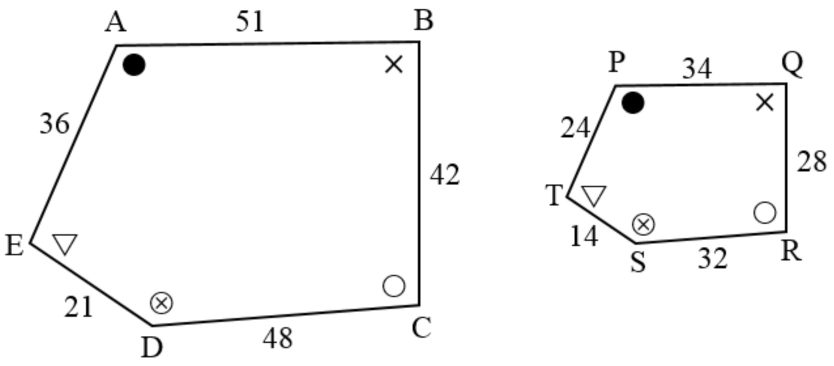
As is apparent, fractions play a very large role in the developing of concepts relating to algebra. Unfortunately, because many teachers take shortcuts when teaching fraction, generally because of contextual factors in our schools, learners learn major topics by rote. As Wu (2008:6) warned, "If we want learners to achieve in algebra, we cannot allow fractions to presented, as it is commonly done, as a collection of factoids held together only by hands-on activities and manipulatives" (Wu, 2008: 6).

3.2.2 Fractions in similarity.

If two quantities are compared, the result is a ratio. A *ratio* is the comparison of two numbers using division. The ratio of x to y can be written as a quotient $\frac{x}{y}$. Ratios are usually expressed in their simplest form. A statement that two ratios are equal is called a *proportion*. A proportion can be written as $\frac{a}{b} = \frac{c}{d}$. When two objects have the same shape, then the two objects are said to be geometrically similar and the ratio of any two linear dimensions of one object is similar for any geometrically similar objects. The ratio of similarity between any two similar figures is the ratio of any pair of corresponding sides. Simply stated, once it is determined that two figures are similar; all of their pairs of corresponding sides have the same ratio. For example:

SIMILAR POLYGONS		
WORDS	**DIAGRAM**	**CORRESPONDING PARTS**
For two polygons to be similar, corresponding angles must have equal measures, and the ratios of the lengths of the corresponding sides must be proportional.	 $\Delta ABC \vert\vert\vert \Delta EFG$	$B\hat{A}C = F\hat{E}G$ $A\hat{B}C = E\hat{F}G$ $B\hat{C}A = E\hat{G}F$ $\frac{AB}{EF} = \frac{BC}{FG} = \frac{AC}{EG} = \frac{2}{1}$

This will be true for irregular shapes to as long as the corresponding sides have the same ratio, for instance, as shown in Figure 14 below:

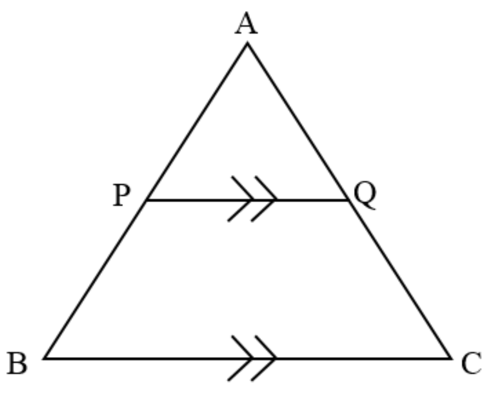


$\frac{AB}{PQ} = \frac{51}{34} = 1{,}5$	$\frac{BC}{QR} = \frac{42}{28} = 1{,}5$	$\frac{CD}{RS} = \frac{48}{32} = 1{,}5$	$\frac{DE}{ST} = \frac{21}{14} = 1{,}5$	$\frac{AE}{PT} = \frac{36}{24} = 1{,}5$

Figure 14: Similarity in geometric shapes

(Source: Master Maths, 2012)

The intercept theorem is closely related to similarity. In fact, it is equivalent to the concept of similar triangles, that is, it can be used to prove the properties of similar triangles, and similar triangles can be used to prove the intercept theorem. By matching identical angles, one can always place two similar triangles in one another, so that one gets the configuration in which the intercepts applies and vice versa the intercept theorem configuration always contains two similar triangles. The intercept theorem is important in high school geometry when dealing with the ratios of various line segments, which are created if two intersecting lines are intercepted by a pair of parallels. It is equivalent to the theorem about ratios in similar triangles. Traditionally, it is attributed to Greek mathematician Thales, which is the reason why it is named the Theorem of Thales in some languages.



If $PQ||BC$, then $\frac{AP}{PB} = \frac{AQ}{QC}$ or $\frac{AP}{AB} = \frac{AQ}{AC}$ or $\frac{PB}{AB} = \frac{QC}{AC}$

Figure 15: Midpoint Theorem

(Source: Master Maths, 2012)

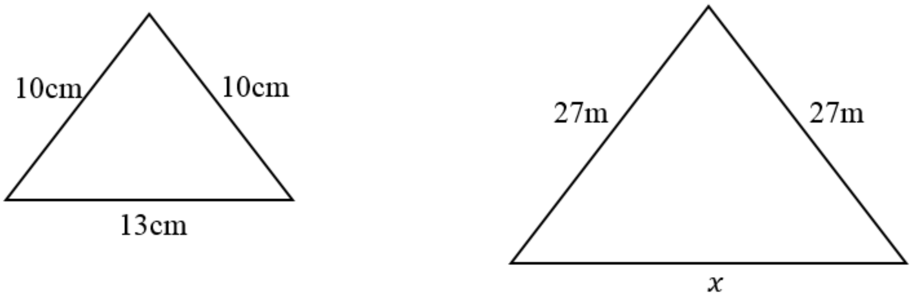
Learners are frequently confronted with the following (shown above). Given is ΔABC, such that PQ is parallel to BC. Two triangles are created, ΔABC and ΔAPQ. We can study the two triangles separately or we can apply the Intercept Theorem (Theorem of Thales).

Firstly, if we consider the two triangles separately, ΔABC and ΔAPQ will be similar. One way learners can solve this problem is to separate the two triangles and use proportions to solve for sides that are unknown. Using this method is beneficial because it can be applied to all of the three sides of the triangle. Learners should take caution when determining the length of the sides of the larger triangle as the original sketch may label it as two separate segments. Learners will have to add the two segments to get the total length.

A second way of solving this is to use the Intercept Theorem (Theorem of Thales) to set up proportions, but there is the limitation that this cannot be used to find the lengths of the parallel segments. In an ideal situation learners must be able to identify the appropriate method to fit the situation given, and apply it. The will only be possible if learners gain enough experience with these types of problem. It is best to have learners use the 'separate' method at first, and then after they have worked a few exercises on their own, they can use the Intercept Theorem as a shortcut in the appropriate situations. Some applications for similarity are used in architecture and in photography.

Example in architecture:

A souvenir model of the pyramid over the entrance of the Louvre in Paris has faces in the shape of a triangle. Two sides are each 10cm long and the base is 13cm long. On the actual pyramid, each triangular face has two sides that are each 27m long. To calculate the length of the base of the actual pyramid, a diagram can help to visualise the problem.



$$\frac{Side\ of\ small\ triangle}{Side\ of\ large\ riangle} = \frac{Base\ of\ small\ triangle}{Basie\ of\ large\ triangle}$$

$$\frac{10cm}{27m} = \frac{13cm}{x}$$
$$10x = (2700cm)(13cm)$$
$$10x = 35100cm$$
$$x = 3510cm$$
$$x = 35{,}1m$$

Figure 16: Similarity in real life

(Source: Holt, Rinehart & Winston, 2013)

In photography, to resize a $10cm \times 13cm$ long photograph so that it will fit into a space of $5cm$, the new size can be calculated as follows:

$$\frac{13cm}{x} = \frac{10cm}{5cm}$$
$$10x = (5cm)(13cm)$$
$$10x = 65cm$$
$$x = 6{,}5cm$$

This relationship will also be true for the perimeter and area of a shape. This brings us back to solving of equations, as explained in the section above, which has its own challenges, as previously mentioned.

3.2.3 Fractions in probability.

Probability refers to how likely something is to happen. A common term is *chance*. Probability can be expressed in fractions, decimals or percentages, or on a probability scale. Writing a probability as a fraction, decimal or a percentage does not change it. Percentages and decimals are just other ways to write fractions. A probability scale places the chance of an event happening alongside a scale of between 0 (impossible) and 1 (certain). If something has a low probability, it is unlikely to happen. If something has a high probability, it is likely to happen.

Teaching probability and statistics is important because of the popularity of the various applications of these subjects in our daily lives and is regarded as a particularly difficult concept as, unlike most areas of school mathematics, it deals with uncertainty. Using a fraction is the easiest way to express probability and this can be done by using the following formula:

$$Probability = \frac{Number\ of\ successful\ outcomes}{Total\ number\ of\ outcomes}$$

$$P(E) = \frac{n(E)}{n(S)}$$

Some examples would be the following:

(1) A bag contains 4 red balls, 7 blue balls, and 5 green balls. Write down the probability of picking a green ball from the bag. The number of successful outcomes is 5 (as you are taking a green ball), the total number of outcomes 4 + 7 + 5 = 16 (as this is the total number of balls in the bag). Therefore, the probability of picking a green ball is $\frac{5}{16}$ or 0, 3125 or 31, 25%.

(2) Jason rolls a standard, six-sided die and Rachel spins a spinner with three equal sections. What is the probability of rolling an even number (E) and spinning a B?

For the die: $P(E) = \frac{3}{6}$

For the spinner: $P(B) = \frac{1}{3}$

$$P(E \cap B) = P(E) \times P(B)$$
$$= \frac{3}{6} \times \frac{1}{3}$$
$$= \frac{3}{18}$$

This might be a bit confusing, so needs further analysis before moving on. This problem can be represented by either a table or a tree diagram.

Spinner	**Even number on die**					
	1	**2**	**3**	**4**	**5**	**6**
A	A, 1	A, 2	A, 3	A, 4	A, 5	A, 6
B	B, 1	***B, 2***	B, 3	***B, 4***	B, 5	***B, 6***
C	B, 1	C, 2	C, 3	C, 4	C, 5	C, 6

Using a tree diagram:

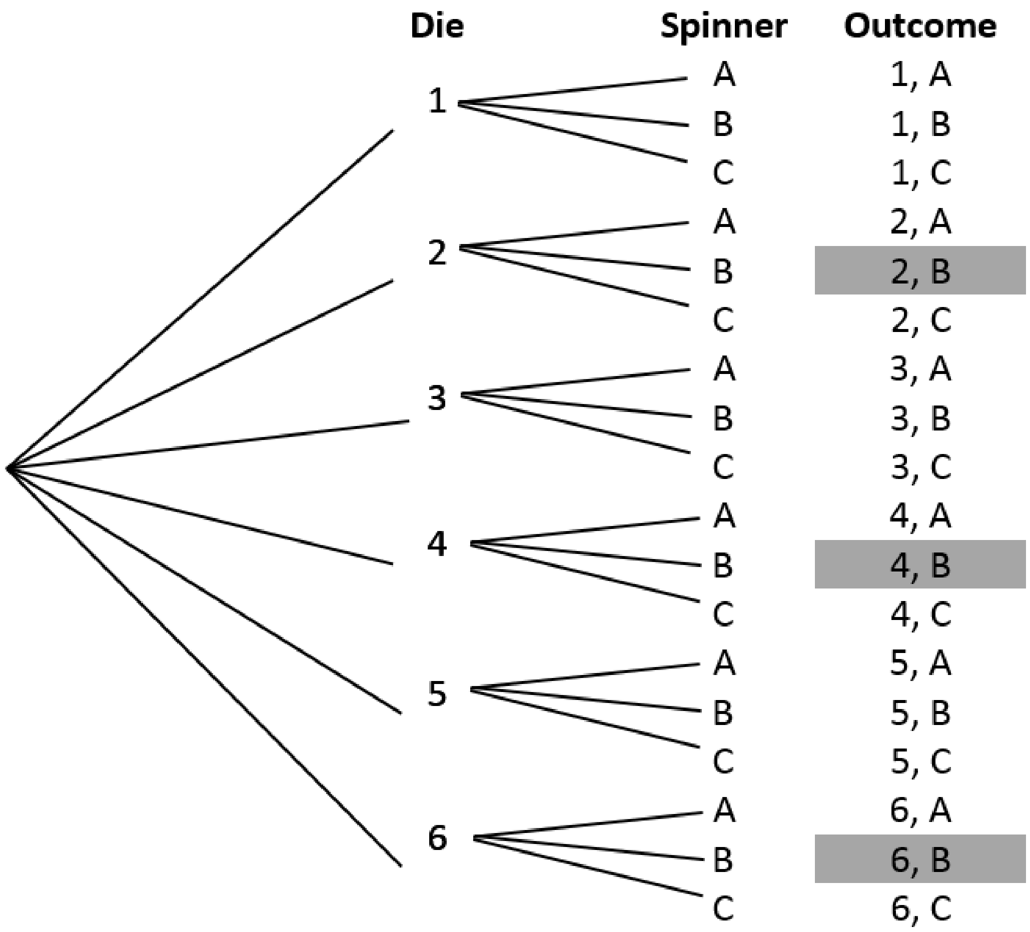


Figure 17: **Probability presented in table or a tree diagram**

(Source: McGraw-Hill Ryerson, 2013)

From the above, one can see that there are 18 possible outcomes, but only three of them "fit" the requirement of being even and B. Therefore the probability of rolling a B and an even number is $\frac{3}{18}$ or $0{,}1\overline{6}$ or $16{,}\overline{66}\%$.

Probability and data handling were characterised by Shaughnessy, Garfield, and Greer (1996) as "systematic study of uncertainty". They went on to say that it is "exactly this uncertainty which makes the study of statistics difficult but also important as it encourages the use of different kinds of reasoning and tools which are essential in mathematical modelling". Garfield and Ahlgren (1988: 47) identified poor learner understanding of ratios as one of the main underlying causes of poor learner performance when dealing with probability in schools.

3.2.4 Fractions in trigonometry.

Trigonometry is an important topic in the high school curriculum. Trigonometry developed from a need to calculate distances and to measure angles, especially in map making, surveying, and architecture, amongst others. Today, trigonometry is an indispensable tool in many applied problems in both science and technology. Trigonometry is one of the earliest mathematical topics that link algebraic, geometric, and graphical reasoning and therefore can serve as an important forerunner in the understanding of calculus (Weber, 2005: 1). Because trigonometry requires learners to relate pictures of triangles to numerical relationships and work with ratios, learning trigonometry can be difficult (Blackett & Tall, 1991: 1).

The word *trigonometry* is derived from the Greek words *trigono*, which means “triangle,” and *metron*, which means “measurement”. Thus, trigonometry is the “measurement of triangles”. Trigonometry is normally taught using five key representations. The first three are the right-angled triangle, the fourth is the unit circle, and the fifth is function representation. The unit circle combines with the function form to give the graphic representation. Lastly, the vector representation is a combination of the first three. I will briefly discuss the first three.

The definition of trigonometric functions in a right-angled triangle is based on properties of similar triangles. Earlier, we saw that corresponding sides of similar triangles are proportional. Consequently, in two similar triangles, the ratio of one side to another in one triangle will be the same as the ratio of the corresponding sides in the second triangle. According to Hart (cited in Blackett & Tall, 1991: 1), “Ratios prove to be extremely difficult for children to comprehend”. This gives us some insight into why learners have difficulties in understanding trigonometry. Another reason is given by Weber (2005: 1), who claimed that “Trigonometric functions are operations that cannot be expressed as algebraic formulae involving arithmetical procedures, and students have trouble reasoning about such operations and viewing these operations as functions”.

In a right-angled triangle, the hypotenuse is opposite the right angle and is the longest side. The other two sides are the opposite and the adjacent sides to their position from the acute angle θ in the triangle. The six trigonometric functions are sine (sin), cosine (cos), tangent (tan) and their inverses cosecant (cosec), secant (sec) and cotangent (cot), respectively. The sin and cos are the two most prominent trigonometric functions. All other trigonometric functions can be expressed in terms of sin and cos. In fact, sin and cos can be expressed in terms of each other. In the trigonometry of a right-angled triangle, sin and cos are the two ratios that involve the hypotenuse, whereas the tan involves the opposite and adjacent sides of the acute angle θ in the triangle.

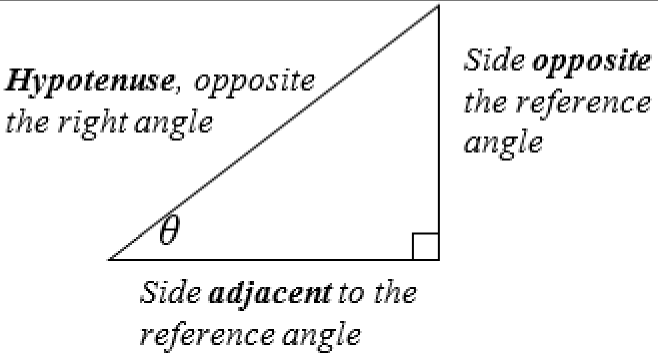
Ratio	In a triangle
	
$sin\theta$	$\frac{o}{h} = \frac{opposite}{hypotenuse}$
$cos\theta$	$\frac{a}{h} = \frac{adjacent}{hypotenuse}$
$tan\theta$	$\frac{o}{a} = \frac{opposite}{adjacent}$
$cosec\theta$	$\frac{h}{o} = \frac{hypotenuse}{opposite}$
$sec\theta$	$\frac{h}{a} = \frac{hypotenuse}{adjacent}$
$cot\theta$	$\frac{a}{o} = \frac{adjacent}{opposite}$

Figure 18: Trigonometric ratios

(Source: Fourie, 2012).

Consider the following sketch. Three right-angled triangles are drawn to coincide at vertex A. Since each triangle contains a right angle and angle A is common, the third angles (C, E, and G) will all be the same size (interior angles of a triangle). When three angles of a triangle are equal, the triangles are similar. The corresponding sides of similar triangles are in proportion Therefore

$$\frac{CB}{BA} = \frac{ED}{DA} = \frac{GF}{FA}$$

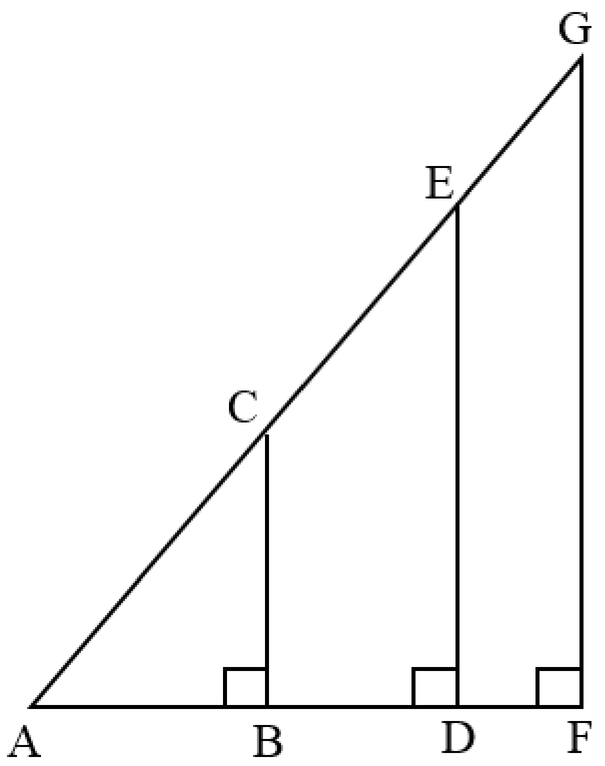


Figure 19: Similar triangles

(Source: Phillips, Basson & Botha, 2009a)

Many learners have difficulty rotating shapes in their minds or seeing individual polygons when they are overlapping. It is helpful to draw the triangles separately and orient them in the same direction. Blackett and Tall (1991: 1) believed that a child should conceptualise what happens as the right-angled triangle is enlarged (all three sides by the same scale factor) and the size of the angles remain the same.

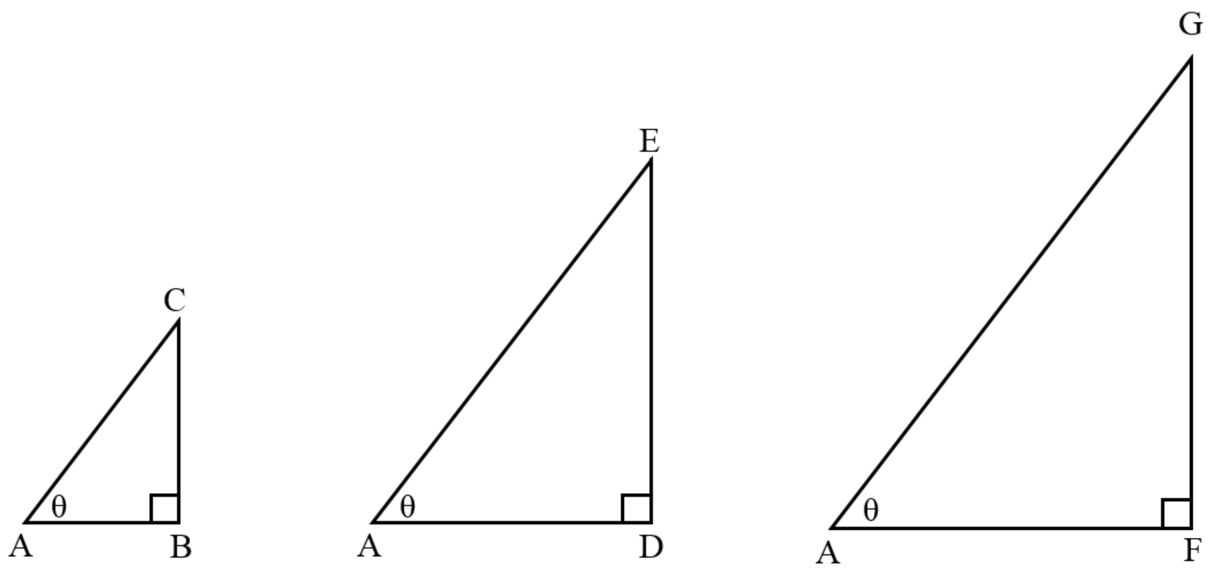


Figure 20: Similar triangles web-separated

(Source: Phillips, Basson & Botha, 2009a)

Right-angled triangles are very useful for exploring the trigonometric functions, but they have a very serious limitation: only acute angles can be put into right-angled triangles, but there are many angles that are not acute. To be able to work with the trigonometric functions of any angle, trigonometric functions can be defined by using the *unit circle*. The unit circle is a circle with its centre at the origin and a radius of 1. Being so simple, it is an ideal tool in learning and talking about lengths and angles. Its equation is $x^2 + y^2 = 1$. Trigonometric functions are defined in terms of co-ordinates of points on the unit circle. The point $A(1, 0)$ is called the *initial* point. $P(x, y)$ is called the *terminal* point and is the point on the unit circle as it moves in an anti-clockwise direction from the positive *x*-axis. As a position point moves around the unit circle, one needs to visualise a triangle moving along with it. If $P(x; y)$ is not on one of the axes, a line is dropped from $P(x; y)$, perpendicular to the x-axis, to form a right-angled triangle with the hypotenuse of the unit radius OP (1unit). The triangle thus formed is called a *reference triangle for θ*. The unit circle is an easy way to show the trigonometric ratios for sin, cos and tan at 30°, 45°, 60° and 90° (aka special triangles). The unit circle is also an easy way to show what happens to the trigonometric ratios when the reference angle is obtuse (from where the 'reduction formulae' are derived). When working with the unit circle, it is important to note that learners do not recognise that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Rationalising the denominators is only done in Grade 12 and therefore Grade 10 and Grade 11 learners lack this knowledge. This can be a stumbling block in the understanding of trigonometric functions, seeing that calculators are overused in classrooms and that they automatically rationalise the denominator for the learners. Learners will think they have made a mistake, when in fact they have not.

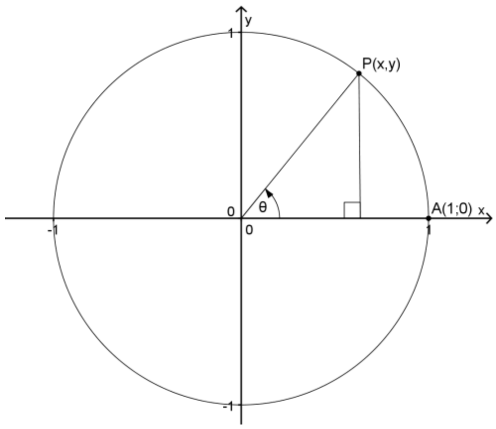


Figure 21: The unit circle

(Source: Heal, 2010)

The trigonometric functions can be extended beyond the unit circle and indeed to every point in the plane except the origin. Here $r = \sqrt{x^2 + y^2}$ is the distance to the origin and will always be positive. The trigonometric functions are defined in terms of x, y and r. This is called the *function representation* and is notated as shown below:

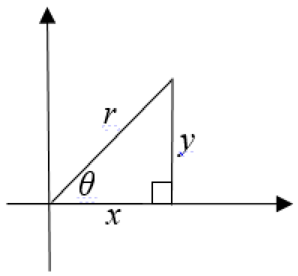
Ratio	On a system of axis
	
$sin\theta$	$\frac{y}{r}$
$cos\theta$	$\frac{x}{r}$
$tan\theta$	$\frac{y}{x}$
$cosec\theta$	$\frac{r}{y}$
$sec\theta$	$\frac{r}{x}$
$cot\theta$	$\frac{x}{y}$

Figure 22: Trigonometric ratios on a Cartesian plane

(Source: Phillips, Basson & Botha, 2009b).

Some other problems that learners experience with the abovementioned representations are as follows:

1. Learners will sometimes try to take the sine, cosine or tangent of the 90° angle in the right-angled triangle. They should soon see that something is wrong since the opposite side is the hypotenuse. Teachers should emphasise that the reference angle used in these cases to define trigonometric functions are between 0° and 90°. To explain angles equal or larger than 90°, it is better to use the unit circle.
2. The sine, cosine, and tangent are ratios that are associated with a specific angle. It is important to stress that there is a relationship between the reference angle and accompanying side lengths. Sine, cosine, and tangent are best described as *functions*. If there is a misunderstanding by learners about functions, there will also be a misunderstanding of the full function definition. On the other hand, if learners understand

this, they will have an easier time using the notation and understanding that the sine, cosine, and tangent for a specific angle are the same, no matter what right-angled triangle it is being used, because all right-angled triangles with that angle will be similar.

3. Many learners have trouble understanding that the sine, cosine, and tangent ratios is not depended on the size of the right-angled triangle. If it's proven that two triangles are similar, it can be deduced that the sides are in proportion. The ratios are written using two sides of one triangle and compared to the ratios of the corresponding sides in the other triangle. This is different but equivalent to the ratios that the students probably used to find missing sides of similar triangles in previous sections.
4. The application of trigonometric ratios sometimes involves the setting up of equations. As seen in the previous section, equations involving fractions come with their own challenges. However, learners can easily make small mistakes and not realise their errors. When solving an equation, the answer can be substituted back into the original equation, to be checked. The sine and cosine for acute angles fall within the range of -1 to 1, and thus are not very wide. It is extremely easy to mistakenly use the sine instead of the cosine in real-world problems.

IMPLICATIONS FOR TEACHING

My study of the various strands of literature related to my topic "Learning about and understanding fractions and their role in the high school curriculum" has a number of implications for teaching. I will organise these implications as follows and discuss each:

1. Commitment of teachers to their field
2. Attitude towards teaching
3. Requirements for applying to study teaching at universities
4. Acquiring high-calibre teacher-training students
5. Further learning and self-study for teachers
6. Approach to hands-on teaching
7. Developing the different sub-construct of fractions by learners

I will argue that all of the above are related to improving classroom teaching, where different interpretations of fractions come into play.

Of great concern to (high school) mathematics teachers is that only approximately 50% of Grade 12 learners who wrote Mathematics in 2012 could achieve the required 30% to pass the subject. The importance of Mathematics, and in particular fractions, for higher education areas of study such as engineering, chemistry, biology and medicine are indubitably a critical factor for graduating in these respective fields. It is time that we as teachers commit ourselves and realise that we cannot stay silent for much longer and that we will have become active if we want to give leaners the best possible education and provide them with an advantage in this competitive world. As mentioned, this study did not focus specifically on how teachers can improve their teaching of fractions, but I think it is important to suggest some ways in which we as teacher can improve our teaching on fractions.

If teachers do not have a passion for teaching, they will struggle to persuade their learners to become motivated and interested in mathematics. In addition to this, teachers will need to be supported financially or through other types of incentives to take time out of their busy teaching schedules to attend workshops, conferences, training, talks, seminars, and, meetings that focus on mathematics teaching for purposes of professional development and to the benefit of our learners. For example, I enriched my understanding of fractions through writing this mini-thesis and through networking with other mathematics educators at a recent national conference. The current state of the South African education system in is in need of teachers who are hardworking, motivated, and committed to making a positive impact in our schools. How this is to be achieved leads into the next point: the requirements for applying to study for a bachelor’s degree in education.

It is a concern that when applying for a medical degree, a candidate needs to obtain an aggregate of almost 80%, but to apply for an education degree, a mere aggregate of 55% is required. If we want to improve our quality of education in South Africa, we should have higher requirements for those who want to study education for the purposes of becoming high school mathematics teachers. In order to improve the quality of teacher training at tertiary institutions, those in authority must attract learners who attain high results in high school. I mention the above issues because they impact directly on the suggestions I make about how teachers can improve their teaching of fractions, stemming from my review of the various interpretations of fractions which can be related to different grade levels and to various topics in school mathematics.

In order for teachers to improve their teaching of fractions, they should analyse themselves in terms of their views on learning. For example, teachers can approach learners’ misconceptions of fractions in two ways: behaviouristic or constructivist. The constructivist “lens”, as Olivier (1989) puts it, should shape the way in which teachers teach. There are valid arguments for both views. I prefer a constructivist approach to teaching fractions but believe that there are merits in approaching some cases from a behaviourist perspective.

In order to enhance learners’ proficiency in fractions, teachers should be given opportunities to learn how to design their teaching, design their series of lessons, lesson activities and tests to embrace all the five strands of mathematical proficiency (Kilpatrick et al., in NRC, 2001: 116). It important to stress again that the five strands are interwoven and that they support one another in assisting in successful learning of fractions. It is therefore vital that teachers expand their own understanding of fractions as a compound concept, for example, $\frac{3}{5}$ can be a single chocolate bar divided in five equal parts of which I only have 3 pieces (part-whole construct), or it can be a relationship between two sides of a triangle (ratio construct). To develop such interpretations of fractions, teachers should develop their 'mathematical knowledge for teaching' (MKT) through adequate support from education authorities and other organisations concerned with the professional development of mathematics teachers.

I shall now expand on MKT, with special reference to fractions. MKT encompasses all three knowledge types described by Shulman (1983):(a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curriculum knowledge. A more recent study done by Copur-Gencturk (2012: 224) in which she examined the relationships between teachers' mathematical content knowledge, classroom teaching and the effect these have on learners' achievement, confirms what is echoed throughout the literature, viz., the importance of teachers' mathematical knowledge on the quality of lessons and its impact on learners' understanding of concepts such as fractions. She went on to say that "when teachers' mathematical knowledge increased, the teachers appeared to create an environment in which their students could make more sense of the concept being taught." (Copur-Gencturk, 2012: 224). Yasemin Copur-Gencturk is an advisor in the Degree Granting Institution, University of Illinois at Urbana-Champaign and the post-doctoral researcher for the Rice University School Mathematics Project, Houston, Texas.

I believe that many issues involving misconceptions of fractions can be dealt with if teachers comes to grip with, and master, these misconceptions themselves. For example, there are the following misconceptions. The bigger the denominator the bigger the fraction. Many learners think that $\frac{1}{8}$ is smaller than $\frac{1}{16}$. Another common misconception is when learners add fractions. They apply the 'multiplication rule' of fractions and then add the numerators together over the sum of the denominators. Many of these misconceptions and limiting constructs were discussed earlier.

It is quite important that every mathematics teacher engage in self-study through doing research and reading journal articles and mathematical magazines and publications, for the purposes of expanding their MKT with respect to fractions. Even if they teach lower grades, teachers should read up on how the work they are covering in that specific grade unfolds in the higher grades. More importantly, it is a good idea to become a member of a mathematics education association and to regularly attend talks and seminars or register for professional development courses offered. Another way in which teachers can improve their MKT is by attending local workshops and training held by the Education Department and nearby schools. Sometimes, these workshops cover content that the teacher may be familiar with, but she or he might find a different and more effective strategy to present it. Finding an informed mentor who can assist one is always a good idea. This person does not always have to be someone older or even more experienced. It needs to be a person dedicated and well informed in the field of mathematics education and to whom one can turn for assistance. I believe there is a major gap in the number of mathematical forums for teachers in South Africa. Teachers need to become more involved with one another and share ideas, learning material, and assessment tasks. Collaborations with university partners and other professional development forums provide for a greater understanding of continuous changing perspectives on important issues regarding the teaching and learning of mathematics, in particular in the area of fractions. These can contribute considerably to mathematics teaching and the improvement thereof.

The abovementioned suggestions are all very practical ways in which teachers can enhance their MKT. Papick maintains that ”mathematics teachers should deeply understand the mathematical ideas (concepts, procedures, and reasoning skills) that are central to the grade levels” (2011: 389). Papick is professor of mathematics at the University of Nebraska-Lincoln where he is involved in developing material and courses for teachers and prospective teachers that illustrate and demonstrate the critical linking of mathematical ideas to important concepts in the school mathematics curriculum. If teachers have a deep and rich understanding of mathematical concepts such as fractions, they are likely to understand how learners reason and how to address any misconceptions learners may have (Papick, 2011:389).

Finally, I will briefly discuss what activities or tasks might assist in the development of the interpretations or sub-constructs of fractions. When a single object, or a set of objects, is divided into equal parts, we refer to this as the part-whole sub-construct of fractions. Tasks to develop this sub-construct of fractions particularly involve sharing activities. The use of sharing situations amongst friends is the most commonly used; others include the fraction pie, shading of part-whole diagrams and fraction strips. Fractions strips are often used to introduce equivalent fractions or fractions as part of a collection or set of objects. The most common mistake teachers make when using fraction strips is that they are always arranged in ascending order. This is dangerous as the learners’ knowledge of whole numbers can interfere with the counting of these strips. The concept that a bigger denominator makes a smaller fraction is lost and a counting principle is enforced. To avoid this I suggest that when teachers use counting strips, they make sure that fraction parts are ‘mixed up’ or arranged randomly. It is important that when introducing the equal sharing principle, teachers not only make use of situations where one object is divided amongst friends, but also that sets are used.

A fraction as ratio is a relationship between two numbers of the same kind for example, learners or sweets. It is a comparison of two quantities of the same type that cannot be written as a single value. For learners to understand the concept of a ratio, they need to wrestle with the notion of *relative*, which means that the two values in the ratio change together, becoming either bigger or smaller, if they are multiplied with the same number (that is not equal to zero). This means that the value of the ratio stays the same. Activities normally include word sums involving recipes, ratios between different shapes' perimeter, ratio between the length of the sides of the same shape, going to the supermarket and other real-world situations. Here, it is important to first develop the notion of ratios using the sides of two or more different shapes and ratio of sides within the same shape, after which one can go on to introducing the notation of ratios as fractions. From there, a series of activities containing the reduction of ratios, equivalent ratios, comparison of ratios, and writing statements (sentences) as ratios (fractions) can be applied. For example, in the high school curriculum there are connections to trigonometric ratios. Problem-solving activities involve the setting up of an equation (proportion) to solve missing numbers to solve 'real-world' or everyday problems.

A fraction can act as an operator when it is applied to a shape, object or set by either increasing or decreasing it after multiplication. Initially when learners are introduced to this sub-construct their thinking and modelling is based on the part-whole sub-construct. Designing tasks should start with using a single object and move on to groups or sets of discrete objects. Drawing a grid around these objects provides a link between the discrete model and the area model. The area model supports the multiplicative thinking needed, using fractions as operators.

Other tasks, and probably the most popular, are dilation of a geometric shape (transformation geometry) by either enlarging, duplicating or reducing its dimensions and consequently its area, depending on the scale factor. It is best to use a grid to enlarge or reduce shapes as it makes it easier for leaners to see by means of counting the number of blocks for the perimeter and area form the object to the dilated image. From there, one can introduce multiplication as another way to arrive at the same answer. For example, the above can be used to show that fractions as operators can also be applied to line segments.

A quotient is the result of division exercise i.e. the division of two quantities such as two volumes, two areas or two numbers. The quotient sub-construct is similar to the part-whole sub-construct as the notion of fair sharing or dividing into equal parts is applicable here too. The difference, however, is that the quotient sub-construct embodies the idea of sharing of objects or sets of uneven quantity amongst even number of friends, people, and so on. Activities assessing whether learners understand this concept include those of the part-whole construct where learners are required to share objects amongst an even number of children to obtain a 'perfect' answer. For example, when 10 marbles are shared between 2 boys; $\frac{10}{2} = 5$ marbles for each boy.

The quotient construct yields answers that are not ‘perfect’; they can sometimes be fractions themselves or mixed numbers. An example of this is when 7 chocolate bars are shared between 3 friends. Each friend will get $2\frac{1}{3}$ of a chocolate bar.

The measurement understanding of fractions is the ability to identify the unit of measure as a distance from the start to the endpoint. Usually a unit of measure and an instrument are used, for example, centimetres on a ruler or meters on a measuring tape. It is a good idea to think of a fraction in terms of a number line, where the distance between zero and one can be divided into equal lengths. For instance, dividing it into six parts creates a unit of measure called one-sixth. Labelling the line starting at $\frac{0}{6}$ to $\frac{6}{6}$ means that each point is 0, 1, 2, 3, 4, 5, 6 units from zero, respectively. Activities designed using number lines help with finding fractions on a number line, equivalent fractions (when dividing the same number line into different unit of measure for example thirds or quarters), as well as reinforcing the understanding that a fraction is a number that has a value and can be located on a number line in relation to other numbers.

I conclude by stressing that the focus of this study was not on how teachers can improve their teaching of fractions, but I hope that through the study and its findings, the suggestions I make will motivate mathematics teachers to become involved in personal development programmes and design their own activities that will develop learners’ understanding of fraction constructs in its entirety, as I intend to do.

CONCLUSION

It is clear that a deep understanding of fractions in all their ‘forms’ or constructs is important for achievement in mathematics in high school and also at tertiary level. A report by the Department of Basic Education (DBE) shows a decline from 53% (2009) to 44% (2012) in candidates registered to write mathematics (DBE, 2012b). This means that 64% of candidates could not achieve a mere 40% in the subject. One area of concern was basic number operations, namely, addition, subtraction, multiplication, and division, which include working with fractions, percentages, equations involving fractions, and ratios. A suggestion made by the DBE was that teachers should strengthen learners’ ability to do the calculations that deal with percentages, fractions, and ratios by making use of self-designed worksheets and other formative assessment tasks. In light of the above, it is evident that high school learners’ mathematics performance could noticeably improve if they were to gain a better understanding of fractions. It is for that reason that I decided to investigate the learning and understanding of fractions and their role in the high school curriculum.

This study has some limitations worth noting. Firstly, there was no original data as this was a systematic non-empirical critical literature review. Secondly, I discussed the different sub-constructs of fractions and made some suggestions, with a few examples of actual activities, but how to investigate these by means of activities will need to be investigated in further research. A number of studies have been done in primary schools, but few address the problem of fraction understanding and how to address it in high school in South Africa. Most mathematical concepts are formed in the middle grades in primary school but the question of how to address the misconceptions that may still exist in high school needs to be addressed.

It is hoped that this study might help to draw attention to the multifaceted construct of fractions and its various interpretations, to gain a deeper understanding thereof, and by doing so, to alter the way in which teachers present the fractions in various areas throughout the school curriculum. Other important aspects addressed in this study concerned the mathematical knowledge needed in order to present fractions in a meaningful way and the factors contributing to learners' understanding of fractions. Furthermore, limiting constructs contributing to learners' misconceptions of fractions, the importance of fractions, and the role that fractions play in the high school curriculum were discussed.

Finally, I am hoping that the findings of this study may inform teachers about which instructional practices, activities, and tasks are best suited to develop each of the sub-constructs of fractions.

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