Neighbourhood Operators on Categories

by

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Declaration

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Abstract

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While the notions of open and closed subsets in a topological space are dual to each other, they take on another meaning when "points" and "complements" are no longer available. Closure operators have been extensively used to study topological notions on categories. Though this has recovered a fair amount of topological results and has brought an economy of effort and insight into Topology, it is thought that certain properties, such as convergence, are naturally associated with neighbourhoods. On the other hand, it is interesting enough to investigate certain notions, such as that of closed maps, which in turn are naturally associated with closure by means of neighbourhoods.

We propose in this thesis a set of axioms for neighbourhoods and test them with the properties of connectedness and compactness.

Uittreksel

Omgewingsoperatore op kategorieë

("Neighbourhood operators on categories")

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Al is die twee konsepte van oop en geslote subversamelings in 'n topologiese ruimte teenoorgesteldes van mekaar, verander hul betekenis wanneer "punte" en "komplemente" nie meer ter sprake is nie. Die gebruik van afsluitingsoperatore is alreeds omvattend in die studie van topologiese konsepte in kategorieë, toegepas. Alhoewel 'n redelike aantal topologiese resultate, groeiende belangstelling en groter insig tot Topologie die gevolg was, word daar geglo dat seker eienskappe, soos konvergensie, op 'n natuurlike wyse aan omgewings verwant is. Nietemin is dit van belang om sekere eienskappe, soos geslote afbeeldings, wat natuurlik verwant is aan afsluiting, te bestudeer.

In hierdie proefskrif stel ons 'n aantal aksiomas oor omgewings voor en toets dit gevolglik met die eienskappe van samehangendheid en kompaktheid.

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Dedications

ho fahatsiarovana an'i nenibe Rasoa.
to the memory of my grandmother, Rasoanirina.

Contents

Introduction

The use of neighbourhoods in defining spaces in Topology is as old as the field itself. According to Willard [63], it was Weyl who, in his *Die Idee der Riemannschen Fläche* [62], suggested studying abstract spaces in terms of neighbourhood systems. This might have been then the prevalent point of view of the Göttingen school. Apparently the idea of neighbourhoods already began with Hilbert in 1902 before even the works of Fréchet (cf. [37]):

"*Die Ebene ist ein System von Dingen, welche Punkte heißen. Jeder Punkt A bestimmt gewisse Teilsysteme von Punkten, zu denen er selbst gehört und welche Umgebungen des Punkten A heißen. [...]*"

("*The plane is a system of things that are called points. Every point A determines certain sub-systems of points to which the point itself belongs and that are called neighbourhoods of the point A. [...]*") [31, 37]

If Hausdorff is credited with having axiomatised the notion of spaces [28] and hence paving the way to modern Topology, this axiomatisation was then achieved by appealing to neighbourhood systems.

The present thesis seeks to further study the concept of neighbourhoods in a category. This has been motivated by three independent lines of works. The first and main one is the introduction of the notion of convergence via *closure operators*, by Giuli and Šlapal [25, 26] and which has triggered an independent study of neighbourhoods in [34]. Closure operators were introduced in 1987 by Dikranjan and Giuli on categories [19] to encompass all the known notions of closure, bringing seemingly disparate concepts and different fields under the same banner. The second one sprang from the introduction of the so-called *interior operators* on categories by Vorster [59], which were then thought to provide a dual notion to that of closure operators. Interior operators were subsequently studied by Castellini and Ramos [6, 8], and are also considered in [39]. These concepts naturally become interesting when, roughly speaking, the power object $\mathcal{P}(X)$, for an object X, seen as a partially ordered set does not provide complements (*Boolean Algebra*). The last one is the generalisation of topologies and neighbourhood systems undertaken by Á. Császár and which culminated in [18]. Császár's works attenuate the axioms of topologies and neighbourhoods on a given set.

These developments have shown the necessity of taking the axioms of neighbourhoods as a starting point in certain situations. The question to which we have hopefully provided an answer in this thesis is: how far can one go with these axioms and how many of the results in classical Topology could be recovered?

Let us briefly recall that the use of closure operator as a tool to study Topology was initiated by Kuratowski [38]. It considers a closure operator on a set X as an endomap

$$\mathcal{P}(X) \to \mathcal{P}(X)$$

on the power object. In the same way, neighbourhood systems on the same set are just given by a map

$$\mathcal{P}(X) \to \mathcal{P}(\mathcal{P}(X))$$

Here, the neighbourhoods are introduced on a category where a notion of substructures (subsets, subgroups, subspaces, etc.) or *subobjects* is available. Thus for a subobject denoted by m, we assign a collection $\nu(m)$ of subobjects which are in some sense "bigger" than m. A notion of *continuity* then follows in a natural way and opens the door to special types of maps: *initial* and *final* maps, *closed* and *open* maps, which in turn provide sufficient a tool to investigate topological notions such as *connectedness*, *convergence* and *compactness*.

To be able to deal smoothly with the *images* and *pre-images* of subobjects, which are of importance, the category on which we work is provided with a *factorisation system*, a notion which shall be introduced in Chapter 1. This is not really a restrictive assumption since many familiar categories, a few instances of which are the category **Top** of topological spaces and continuous functions, the category **Grp** of groups and group homomorphisms, or the category **Gph** of directed graphs and graph homomorphisms, admit at least one meaningful factorisation system.

Neighbourhood operators are defined in Chapter 2. If neighbourhoods were to be maps $\mathcal{P}(X) \to \mathcal{P}(\mathcal{P}(X))$, one would take into account the fact that $\emptyset \in \mathcal{P}(X)$. This is particularly important when one consider interior operators which are soon proved to be a special kind of neighbourhood operator. Chapter 3 provides the essential tools that one shall eventually need. They are the building blocks of all that is required in studying topological constructions and they are the reason why most standard results on subspaces, products, images, etc., are true. As dually opposed to closure operators, it is shown here that maps are open (resp. closed) when the neighbourhood operator commutes with the images (resp. pre-images.) These tools are used to investigate connectedness and compactness in Chapter 4 and 5. We have adopted two approaches in treating these topological properties: the one is "internal" and the other "external", they are found in two distinct sections. We leave to the reader the task of judging which of the two approaches is convenient or suitable. In both cases the external approach seems to show that the main hypotheses and results are akin to that of closure operators [10, 13]. What is not present with these two approaches is the treatment of these topological notions at the morphisms level, i.e., by considering slice categories (see [13]).

The last chapter is the result of an attempt to define *uniform neighbourhoods*, as suggested by the title, and is relatively independent. At its early stage, we only sketch the main points of the theory.

The structure of the thesis is very simple. The formal statements, i.e. propositions, lemmas, definitions, remarks, etc. are numbered according to their order of appearance in a section. For instance Proposition $m.n.p$ is the p-th item in the n-th section in Chapter m. Chapters and sections follow also the same tree-like rule. In each chapter, we have tried to give a reasonable amount of relevant references regarding the background of the topic. We have also tried to give a relatively axiomatic development. Hence when a proposition (resp. lemma, theorem, etc.) is credited to any other author but the proof itself is unreferenced, it means that the proof was independently done to keep the flow of the development by using prior results. In some cases, the proof is trivial and is not given. In any case, we do not pretend to offer better proofs nor more insight. Section 4.1 might be the only exception to that rule since this is essentially the paper [51], and so at the beginning of this section we already credit everything to that paper.

Chapter 1

Preliminaries

Throughout the text, we will be mainly dealing with a fixed category $\mathbf{C}$. Our terminologies follow [1] as well as [42]. Thus for example, for any object X of $\mathbf{C}$ and any arrow f of $\mathbf{C}$ we shall respectively write "$X \in \mathbf{C}$" and "f in $\mathbf{C}$". On the question regarding foundations, one can assume the existence of a universe $\mathcal{U}$ whose members are called "sets" or adopt the Gödel-Bernays Theory and assume that classes are sets when they belong to another class. We just mention that we call *set* or *small set* a member of $\mathcal{U}$ or a class that cannot belong to another one. In general we shall use the term *class* for both sets and classes that are large, i.e., classes that are not members of $\mathcal{U}$ or that cannot belong to another class. These are just conventions, the reader is left with a foundation that fits his/her tastes.

Our assumption on $\mathbf{C}$ is that it be finitely complete, i.e. it has finite limits, only supposing the existence of general limits, such as arbitrary products, when the need arises. In particular, we are given a terminal object, denoted by $\mathbf{1}$, and pullbacks. We furthermore assume that $\mathbf{C}$ is provided with a suitable class $\mathcal{M}$ which plays the role of embeddings and that a *factorisation system* is given unto $\mathbf{C}$ so that one is able to efficiently manipulate *images* and *pre-images*. The seeds of the notion of factorisation system can be traced back to [40, 41] and [35]. According to the monograph by Dikranjan and Tholen [20] the generally accepted form of such a system came only more than a decade later in [24] with Freyd and Kelly. We base our notion of factorisation system on [20]. However, instead of introducing this system directly through assumptions on the class $\mathcal{M}$, we first define *cover relations*, introduced by Z. Janelidze in [36], and then slowly proceed to factorisation systems. Cover relations provide insight to and shed light on the very nature of the class $\mathcal{M}$. We think that they play a non-negligible role in categories on which a factorisation system is provided.

1.1 Cover relations

Definition 1.1.1. [36] A *cover relation* on $\mathbf{C}$ is a binary relation $\sqsubseteq$ on the class of morphisms of $\mathbf{C}$, which is defined for morphisms having the same codomain, and which satisfies the following properties:

(L) If $f \sqsubseteq g$ in $\mathbf{C}$, then $hf \sqsubseteq hg$ for any h in $\mathbf{C}$ and whenever the compositions hf and hg make sense;

(R) If $f \sqsubseteq g$ and the composition fe exists for any other morphism e in $\mathbf{C}$, then $fe \sqsubseteq g$.

[36] Property (L) is called the *left preservation property* and property (R) is called the *right preservation property*.

Examples 1.1.2. (a) [36] In the category **Set** of sets and functions, the relation $\sqsubseteq$ defined by

$$f \sqsubseteq g \text{ iff } Ran(f) \subseteq Ran(g),$$

where $Ran(-)$ gives the range of a function, is a cover relation.

(b) [36] Generally, on a given category $\mathbf{C}$, the relation defined by

$$f \leq g \text{ if and only if there is a morphism } h \text{ such that } f = gh,$$

is a cover relation which is reflexive and transitive.

A binary relation $\sqsubseteq$ which satisfies only one of the properties (L) and (R) in Definition 1.1.1 is called *precover relation* [36].

Definition 1.1.3. [36] Given a precover relation $\sqsubseteq$, a morphism g is said to be a $\sqsubseteq$-*covering*, or simply *covering*, when there is no confusion, if $f \sqsubseteq g$ for any morphism f that has the same codomain as g. The class of all $\sqsubseteq$-coverings in $\mathbf{C}$ is denoted by $Cov_{\sqsubseteq}$.

In Examples 1.1.2 (a) a function f is a covering if and only if it is a surjective function.

Definition 1.1.4. [36] Let $\sqsubseteq$ be a precover relation. A morphism m is said to be a $\sqsubseteq$-image of a morphism g, if $m \sqsubseteq g$ and if for any other morphism f, the relation $f \sqsubseteq g$ implies the existence of a unique morphism h such that $f = mh$. The class of all $\sqsubseteq$-images in $\mathbf{C}$ is denoted by $\mathcal{M}_{\sqsubseteq}$.

Example 1.1.5. In Examples 1.1.2 (a), the image of a given function $f : X \to Y$ is the range $Ran(f)$ of f. Indeed, considering the natural inclusion $Ran(f) : f(X) \to Y$, we have trivially $Ran(Ran(f)) \subseteq Ran(f)$. If $g \sqsubseteq f$ for any other function $g : Z \to Y$, i.e. $Ran(g) \subseteq Ran(f)$, then we also have the trivial composition $g = Ran(f).ne$, where n is the natural inclusion $g(Z) \to f(X)$ and e the surjection $Z \to g(Z)$.

Lemma 1.1.6. *[36] Let $\sqsubseteq$ be a precover relation having the property (R). Then:*

(a) $\mathcal{M}_{\sqsubseteq}$ *is a class of monomorphisms;*

(b) A morphism m *is a* $\sqsubseteq$*-image of itself if and only if* m *is a monomorphism and for any* f *in* $\mathbf{C}$ *the relations* $f \leq m$ *and* $f \sqsubseteq m$ *are equivalent.*

Proof. (a) Let m be a $\sqsubseteq$-image of a morphism g and let u and v be morphisms such that $mu = mv$. By assumption, $mu \sqsubseteq g$. Thus there is a unique h such that $mu = mh$.

(b) Suppose that m is a $\sqsubseteq$-image of itself. From (a) above, m is a monomorphism. If $f \leq m$, then there is a morphism h such that $f = mh$. By assumptions on $\sqsubseteq$ and m, we have $f = mh \sqsubseteq m$. Now, if $f \sqsubseteq m$ then by the universality of images there is a unique morphism h such that $f = mh$, i.e. $f \leq m$.

Conversely, since $m \leq m$ we have $m \sqsubseteq m$. If there is a morphism f such that $f \sqsubseteq m$, then $f \leq m$ and so there is a morphism h such that $f = mh$. The uniqueness of such h follows from the fact that m is a monomorphism. □

Corollary 1.1.7. *On a category* $\mathbf{C}$*, the* $\leq$*-images are exactly the monomorphisms.*

Lemma 1.1.8. *[36] Let* $\sqsubseteq$ *be a precover relation and* $f : X \to Y$ *a* $\sqsubseteq$*-covering. Then the following equivalent conditions hold:*

(a) $1_Y \sqsubseteq f$*;*

(b) 1_Y *is a* $\sqsubseteq$*-image of* f*.*

The converse is true if $\sqsubseteq$ *has the property (R).*

Proof. It is clear from the definition that $1_Y \sqsubseteq f$. Every morphism $g \sqsubseteq f$ can be trivially and uniquely factorised through 1_Y since $g = 1_Y.g$. Conversely assume that $\sqsubseteq$ has the property (R) and that condition (a) is true. Since $g = 1_Y.g$ for any morphism g having the same codomain as f, we have $g \sqsubseteq f$. □

As suggested by previous examples and results, we intuitively think of the $\sqsubseteq$-coverings as surjections and the $\sqsubseteq$-images as embeddings.

1.2 Subobjects, Images and Pre-Images

Consider a cover relation $\sqsubseteq$ and its images $\mathcal{M}_{\sqsubseteq}$. The *subobjects* of an object $X \in \mathbf{C}$ are given by the collection

$$\mathcal{M}_{\sqsubseteq}/X := \{m : M \to X \mid m \in \mathcal{M}_{\sqsubseteq}\}.$$

When there is no confusion, we drop the subscript $\sqsubseteq$ and simply write $\mathcal{M}/X$.

As we, most of the time, consider a class of monomorphisms as subobjects – i.e. substructures of a structure – Corollary 1.1.7 and Lemma 1.1.6 indicate that the most natural way to order $\mathcal{M}_{\sqsubseteq}/X$ is by using the order relation $\leq$. We shall see that under suitable conditions every morphism in $\mathcal{M}_{\sqsubseteq}$ is a $\sqsubseteq$-image of itself.

For any pair of images $m : M \to X$ and $n : N \to X$, the relations $m \leq n$ and $n \leq m$ imply $M \cong N$, which we denote by $m \cong n$. It is clear that $\cong$ is an equivalence relation. Therefore, instead of manipulating the collection $\mathcal{M}_{\sqsubseteq}/X$ directly, we use the equivalence classes $[m] := \{n \mid n \cong m\}$. The collection of these equivalence classes can be naturally ordered:

$$[m] \leq [n] \text{ if and only if } m \leq n.$$

When the collection $\{[m] \mid m \in \mathcal{M}_{\sqsubseteq}/X\}$ can be indexed by a set for each $X \in \mathbf{C}$, we say that $\mathbf{C}$ is $\mathcal{M}_{\sqsubseteq}$-*wellpowered* or simply *wellpowered* ([20]) when there is no confusion. In most of the examples that we shall study $\mathbf{C}$ is wellpowered. For convenience we shall consider both $\mathcal{M}_{\sqsubseteq}/X$ and the class $\{[m] \mid m \in \mathcal{M}_{\sqsubseteq}/X\}$ as subobjects of X and we shall loosely write $m = n$ for $m \cong n$.

The following result is established in [36] and provides a bridge between special cover relations and the so-called right $\mathcal{M}$-factorization systems ([20]).

Theorem 1.2.1. *[36] There is a one-to-one correspondence between cover relations which are reflexive, transitive and admit images, and classes $\mathcal{M}$ of monomorphisms satisfying the following conditions:*

(CI) $\mathcal{M}$ is closed under composition with isomorphisms;

(D) For any f in $\mathbf{C}$, there are $m \in \mathcal{M}$ and g in $\mathbf{C}$ such that $f = mg$, and whenever one has a commutative diagram

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "g"'] & \cdot \arrow[dd, "n"] \\
\cdot \arrow[d, "m"'] \arrow[ur, dashed, "h"] & \\
\cdot \arrow[r, "v"'] & \cdot
\end{tikzcd}$$

with $n \in \mathcal{M}$, then there is a unique morphism h such that $nh = vm$ and $u = hg$.

For each reflexive and transitive cover relation $\sqsubseteq$ admitting images, one associates to $\sqsubseteq$ the class $\mathcal{M}_{\sqsubseteq}$. To each class $\mathcal{M}$ satisfying (CI) and (D) is associated the precover relation $\sqsubseteq^{\mathcal{M}}$. This precover relation is a cover relation ([36]) and is defined as follows:

$$f \sqsubseteq^{\mathcal{M}} g \text{ in } \mathbf{C} \text{ if and only if for all } m \in \mathcal{M},\ f \leq m \text{ whenever } g \leq m.$$

Any factorisation $f = mg$ that satisfies properties (CI) and (D) is called a *right $\mathcal{M}$-factorisation* of f and the property (D) is called the *diagonalisation property* of the factorisation ([20]). The factorisation $f = mg$ is essentialy unique. Indeed if $f = mg = ne$, then, by replacing v with the identity and u with e in the previous diagram, one obtains an isomorphism i such that $m = ni$ and $e = ig$.

Example 1.2.2. Let $\mathbf{C}$ be the category of sets with bijective functions and non-injective functions. By Corollary 1.1.7, every function which is not surjective has no $\leq$-image. On the other hand $\mathbf{C}$ admits a right *All*-factorisation system, where *All* $:= \{f \mid f \text{ in } \mathbf{C}\}$. Thus in any right $\mathcal{M}$-factorisation system, the class $\mathcal{M}$ need not be a class of monomorphisms.

Proposition 1.2.3. *[36] Let $\mathcal{M}$ be a class of monomorphisms such that $\mathcal{M}$ is part of a right $\mathcal{M}$-factorisation system. Then m is a $\sqsubseteq^{\mathcal{M}}$-image of a morphism f if and only if m is the $\mathcal{M}$-part of the right $\mathcal{M}$-factorisation of f.*

Proof. [36] Since $\sqsubseteq^{\mathcal{M}}$ is reflexive, $f \sqsubseteq^{\mathcal{M}} f$ and so $f = mp$ for some morphism p. Conversely, if $f = me$ then $f \leq m$ and so $m \sqsubseteq^{\mathcal{M}} f$. If $g \sqsubseteq^{\mathcal{M}} f$, then by definition $f \leq m$ implies $g \leq m$. □

The image of a subobject under a morphism is given as follows.

Definition 1.2.4. Let $f : X \to Y$ be a morphism in $\mathbf{C}$ and $\sqsubseteq$ a cover relation. Let $m \in \mathcal{M}_{\sqsubseteq}/X$. The image $f[m]$ of m under the morphism f is given by the $\sqsubseteq$-image of the composition fm.

Proposition 1.2.5. *Let $\mathcal{M}$ be a class of monomorphisms such that it is part of a right $\mathcal{M}$-factorisation system. Then for any morphism $f : X \to Y$, the process of forming images $f[-]$ gives a functor from $\mathcal{M}_{\sqsubseteq}/X$ to $\mathcal{M}_{\sqsubseteq}/Y$, i.e., $f[-]$ is order-preserving.*

Proof. Suppose that $m \leq n$ in $\mathcal{M}_{\sqsubseteq}/X$. Thus $m \sqsubseteq^{\mathcal{M}} n$ and so $fm \sqsubseteq^{\mathcal{M}} fn$. We then have $f[m] \sqsubseteq^{\mathcal{M}} fm \sqsubseteq^{\mathcal{M}} fn$ and since $\sqsubseteq^{\mathcal{M}}$ is transitive, $f[m] \sqsubseteq^{\mathcal{M}} fn$. Therefore there is a unique h such that $f[m] = f[n]h$. □

In the proof one could have proceeded by using the diagonalisation property (D) ([20]), thus giving a general proof which would be of interest especially when $\mathcal{M}$ is not a class of monomorphisms.

Definition 1.2.6. Let $\sqsubseteq$ be a precover relation. A morphism m is said to be $\sqsubseteq$-reflecting, or simply *reflecting* when there is no confusion, if for any pair f, f' of morphisms the relation $mf \sqsubseteq mf'$ implies $f \sqsubseteq f'$.

The following fact, which follows trivially from the universality of images, establishes the uniqueness of images up to isomorphisms.

Proposition 1.2.7. *Let $\sqsubseteq$ be a cover relation. For any morphism f, its $\sqsubseteq$-image is essentially unique. Furthermore, if two subobjects m and n are isomorphic, then m is $\sqsubseteq$-reflecting if and only if n is $\sqsubseteq$-reflecting.*

Proposition 1.2.8. *Let $\sqsubseteq$ be a reflexive and transitive cover relation. Then $\mathcal{M}_{\sqsubseteq}$ is stable under pullback. Furthermore the pullback of a $\sqsubseteq$-image is a $\sqsubseteq$-image of itself.*

Proof. If $\mathcal{M}_{\sqsubseteq}$ is empty, then the statement is vacuously true. Let $m \in \mathcal{M}_{\sqsubseteq}$ and consider the pullback $fn = mg$

$$\begin{tikzcd}
\cdot \arrow[dr, dashed, "i"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "h"'] & & \\
& \cdot \arrow[r, "g"] \arrow[d, "n"'] & \cdot \arrow[d, "m"] \\
& \cdot \arrow[r, "f"'] & \cdot
\end{tikzcd}$$

We shall prove that n is a $\sqsubseteq$-image of itself. We have $n \sqsubseteq n$ by reflexivity. Let u be a morphism such that m is the $\sqsubseteq$-image of u and let h be a morphism such that $h \sqsubseteq n$. Then $fh \sqsubseteq fn$. Since $fn = mg$ and $mg \sqsubseteq m \sqsubseteq u$, we have $fh \sqsubseteq u$. There is then a unique p such that $fh = mp$. The pullback property implies the existence of a unique morphism i such that $ni = h$ and $gi = p$. The same pullback property implies that i is unique such that $ni = h$. □

Proposition 1.2.8, together with Corollary 1.1.7, implies the classical well-known result that the class of monomorphisms is pullback stable.

Corollary 1.2.9. *If $\sqsubseteq$ is reflexive and transitive then every $\sqsubseteq$-image is a $\sqsubseteq$-image of itself.*

Corollary 1.2.10. *With the conditions of Proposition 1.2.8, if $mn \in \mathcal{M}_{\sqsubseteq}$ and m is a monomorphism, then $n \in \mathcal{M}_{\sqsubseteq}$ (see also [20].)*

Proof. Since m is a monomorphism, the following diagram is a pullback diagram:

$$\begin{tikzcd}
\cdot \arrow[r, "1"] \arrow[d, "n"'] & \cdot \arrow[d, "mn"] \\
\cdot \arrow[r, "m"'] & \cdot
\end{tikzcd}$$

Thus $n \in \mathcal{M}_{\sqsubseteq}$. □

Definition 1.2.11. Let $\sqsubseteq$ be a reflexive and transitive cover relation and $f : X \to Y$ a morphism in $\mathbf{C}$. The *pre-image* $f^{-1}[m]$ of a subobject $m \in \mathcal{M}_{\sqsubseteq}/Y$ is given by the pullback of m along f.

It is clear, from the property of pullback, that $f^{-1}[-]$ is order-preserving and that $f^{-1}[m]$ is essentially unique for any $m \in \mathcal{M}_{\sqsubseteq}/Y$.

Proposition 1.2.12. *In a category* $\mathbf{C}$ *with a right* $\mathcal{M}$*-factorisation system,* $f[-]$ *and* $f^{-1}[-]$ *are adjoint to each other, with* $f^{-1}[-]$ *being the right adjoint.*

Proof. It suffices to prove that for any appropriate subobjects m and n, the relations $f[m] \leq n$ and $m \leq f^{-1}[n]$ are equivalent. Consider the following diagram:

$$\begin{tikzcd}
\cdot \arrow[rrr, "f'"] \arrow[ddr, bend right, "f^{-1}[n]"'] & & & \cdot \arrow[ddl, bend left, "n"] \\
& \cdot \arrow[ul, dashed, "j"'] \arrow[r, "e"] \arrow[d, "m"'] & \cdot \arrow[ur, dashed, "i"] \arrow[d, "f[m]"'] & \\
& \cdot \arrow[r, "f"'] & \cdot &
\end{tikzcd}$$

Suppose that i exists. Since $nf' = f.f^{-1}[n]$ is a pullback, the arrow j exists. Conversely, the existence of i follows from the diagonalisation property (D) applied to $f[m]e = nf'j$. □

Remark 1.2.13. The fact that $f[-]$ and $f^{-1}[-]$ are order-preserving is not of importance here. In fact, these follow from adjointness. The results below, which are straightforward, confirm this statement.

Lemma 1.2.14. *[20, 13] Let* $f : X \to Y$ *be a morphism in* $\mathbf{C}$ *and assume that* $f[-]$ *and* $f^{-1}[-]$ *exist and are adjoint to each other. Then:*

(i) For any subobject m *of* X *and* n *of* Y, $f[f^{-1}[m]] \leq m$ *and* $f^{-1}[f[n]] \leq n$*;*

(ii) Let $\{m_i \mid i \in I\} \subseteq \mathcal{M}/X$, *if the supremum* $m = \sup\{m_i \mid i \in I\}$ *exists then necessarily* $f[m] = \sup\{f[m_i] \mid i \in I\}$. *Similarly if* $\{n_i \mid i \in I\} \subseteq \mathcal{M}/Y$ *and the subobject* $n = \inf\{n_i \mid i \in I\}$ *exists then* $f^{-1}[n] = \inf\{f^{-1}[n_i] \mid i \in I\}$*;*

(iii) For any morphism $g : Y \to Z$ *in* $\mathbf{C}$, *we have the natural equivalences:*

$$g[-] \circ f[-] \cong (gf)[-] \text{ and } f^{-1}[-] \circ g^{-1}[-] \cong (gf)^{-1}[-].$$

Closedness under limits of the class $\mathcal{M}$ is of interest to us. For example, we would like to know when the product or the intersection of substructures – for instances subgroups, subspaces, subgraphs, etc. – is again a substructure.

Proposition 1.2.15. *[20] Let* $\mathcal{M}$ *be a class of monomorphisms and such that* $\mathcal{M}$ *is a part of a right* $\mathcal{M}$*-factorisation system. Let* $\mathbf{D}$ *be a type,* $H : \mathbf{D} \to \mathbf{C}$ *and* $K : \mathbf{D} \to \mathbf{C}$ *two diagrams. For any natural transformation* $\mu : H \to K$, *the limit* $k : \lim H \to \lim K$ *belongs to* $\mathcal{M}$ *provided that every* μ_d *belongs to* $\mathcal{M}$ *for every* $d \in \mathbf{D}$.

We point out that types play the role of “set of index” in categories, they are but categories.

Proof. [20] Consider the $\mathcal{M}$-factorisation $k = me$ and the following commutative diagram.

$$\begin{tikzcd}
\lim H \arrow[r, "h_d"] \arrow[d, "e"'] & H_d \arrow[dd, "\mu_d"] \\
M \arrow[ur, dashed, "t_d"] \arrow[d, "m"'] & \\
\lim K \arrow[r, "k_d"'] & K_d
\end{tikzcd}$$

The diagonalisation property (D) implies the existence of t_d. By the universality of $\lim H$, there is a unique $i : M \to \lim H$ such that $h_d i = t_d$ for every $d \in \mathbf{D}$. The same universal property implies that $ie = 1_{\lim H}$. Now, we have $k_d mei = k_d m$ for every $d \in \mathbf{D}$. Since the k_d's are jointly monomorphic (or since $\lim K$ is universal), we have $mei = m$. Thus $ei = 1_M$. Since $\mathcal{M}$ satisfies (CI), $k \cong m$ belongs to $\mathcal{M}$. $\square$

Corollary 1.2.16. *[20] Let $\mathcal{M}$ be a class such that it is closed under $\mathbf{D}$-limits for every diagram $\mathbf{D}$, then:*

(i) $\mathcal{M}$ is closed under direct products;

(ii) $\mathcal{M}$ is closed under multiple pullbacks, i.e. for every multiple pullback diagram

$$\begin{tikzcd}
 & M_i \arrow[dr, "m_i"] & \\
M \arrow[ur, "j_i"] \arrow[rr, "m"'] & & X
\end{tikzcd}$$

m belongs to $\mathcal{M}$ provided that every m_i belongs to $\mathcal{M}$.

Proof. [20] (i) is clear. (ii) Consider the diagram $p_i : X_i \to X$ where $p_i = 1_X$ and $X_i = X$ for every $i \in I$. In the following limit diagram

$$\begin{tikzcd}
M \arrow[r, "j_i"] \arrow[d, "m"'] & M_i \arrow[r, "m_i"] \arrow[d, "m_i"] & X \arrow[d, "1_X"] \\
X \arrow[r, "1_X"'] & X_i \arrow[r, "p_i"'] & X
\end{tikzcd}$$

The top and bottom arrows are multiple pullback diagrams with respective limits M and X. The vertical arrows m_i and also 1_X form a natural transformation that induces the limit arrow $m : M \to X$. Since $m_i \in \mathcal{M}$ for every $i \in I$, $m \in \mathcal{M}$. □

Topological considerations, such as existence of supremum and infimum for subobjects, suggest that we should consider a class of monomorphisms.

Proposition 1.2.17. *[20] If $\mathcal{C}$ is a class of morphisms that is closed under multiple pullbacks, then $\mathcal{C}$ is a class of monomorphisms.*

Proof. The proof follows the sketch outlined in [20]. Let $n : N \to X$ be in $\mathbf{C}$ and $u, v : H \to N$ be morphisms such that $nu = nv$. Let I index the class of morphisms of $\mathbf{C}$. Let $m_i = n$ and $M_i = N$ for every $i \in I$ and let m be the pullback of all the m_i's. Consider the following diagram

$$\begin{tikzcd}
 & & M_i \arrow[dr, "m_i"] & \\
H \arrow[urr, "z_i"] \arrow[r, "\alpha(i)"'] & M \arrow[ur, "j_i"] \arrow[rr, "m"'] & & X
\end{tikzcd}$$

Let $K = \{h : H \to M \mid \text{for all } i \in I \; j_i h \in \{u, v\}\}$. $K \neq \emptyset$ because of the universality of the limit m and since $m_i u = m_j v$ for all $i, j \in I$. Now since $K \subseteq I$, there is a surjective map $\alpha : I \to K$. We define the family $\{z_i \mid i \in I\}$ as follows

$$z_i = \begin{cases} u & \text{if } j_i.\alpha(i) = v \\ v & \text{if } j_i.\alpha(i) = u \end{cases}$$

For any $i, j \in I$, $m.\alpha(i) = m.\alpha(j)$. We then have a morphism $m.\alpha(i) : H \to X$ and morphisms $z_i : H \to M_i$ such that $m.\alpha(i) = m_i.z_i$ for every $i \in I$. Because of the universality of m, we must have $|K| = 1$ and $j_i.\alpha(i) = z_i$ for every $i \in I$. Therefore it should be the case that $u = v$. □

Propositions 1.2.15 and 1.2.17 imply the following characterization.

Corollary 1.2.18. *In a right $\mathcal{M}$-factorisation system, the class $\mathcal{M}$ is a class of monomorphisms if and only if it is closed under multiple pullbacks.*

In the presence of multiple pullbacks, the limit $m : M \to X$ plays the role of intersection of all the m_i's. Thus we write $m = \bigwedge\{m_i \mid i \in I\}$. This implies the existence of the join $\bigvee$ for subobjects and in particular the existence of a least subobject $0_X : O_X \to X$ for every $X \in \mathbf{C}$. The intersection $m \wedge n$ of two subobjects m and n is thus essentially given by the arrow

$$m \wedge n = m.m^{-1}[n] = n.n^{-1}[m].$$

We note that the pre-image $f^{-1}[-]$ when it exists, always preserves the intersection $\bigwedge\{m_i \mid i \in I\}$ since limits commute with limits. Therefore Lemma 1.2.14 (ii) is trivially true.

Composition of images is also of interest to us as we would want the subspaces of subspaces to also be subspaces. This however does not always hold as evidenced for instance by the fact that the normal subgroups of a normal subgroup need not be normal. It requires some conditions.

Proposition 1.2.19. *[36] The following statements are equivalent for a class $\mathcal{M}$ of monomorphisms which is a part of a right $\mathcal{M}$-factoriaztion system:*

(i) $\mathcal{M}$ is closed under composition and contains isomorphisms;

(ii) Every morphism in $\mathcal{M}$ is $\sqsubseteq^{\mathcal{M}}$-reflecting;

(iii) Every morphism has a $\sqsubseteq^{\mathcal{M}}$-reflecting image.

Proof. By virtue of Proposition 1.2.7 and Corollary 1.2.9 (ii) and (iii) are already equivalent. We shall prove that (i) implies (ii) and (iii) implies (i).

Assume that (i) is true. Let $m \in \mathcal{M}$ and $f, f' \in \mathbf{C}$ such that $mf \sqsubseteq^{\mathcal{M}} mf'$. Suppose that $f' \leq k$ for $k \in \mathcal{M}$. Noting that $f' = kp$ for some morphism p, we have $mf \sqsubseteq^{\mathcal{M}} (mk)p$. By assumption $mk \in \mathcal{M}$ and by Proposition 1.2.3 it is a $\sqsubseteq^{\mathcal{M}}$-image of $(mk)p$. Hence for some morphism p' in $\mathbf{C}$, $mf = mkp'$. Thus $f = kp'$ or equivalently $f \leq k$.

Now, suppose that (iii) is true and let $m, n \in \mathcal{M}$. Let f be a morphism such that n is the image of f. We shall prove that mn is the image of mf. It is clear that $mn \sqsubseteq^{\mathcal{M}} mf$. Let $g \sqsubseteq^{\mathcal{M}} mf$. Since $mf \sqsubseteq^{\mathcal{M}} m$, we have $g \sqsubseteq^{\mathcal{M}} m$ by transitivity. Therefore $g = mp$ for some morphism p. Since m is reflecting, $p \sqsubseteq^{\mathcal{M}} f$. By the universality of the image, $p = nk$ for some morphism k. Thus $g = (mn).k$ and k is unique for this equality. □

Note that in particular if $m : M \to X \in \mathcal{M}$, then for any $k : N \to M \in \mathcal{M}$ the equality $mk = 0_X$ implies $k = 0_M$.

Definition 1.2.20. Suppose that a class of monomorphisms $\mathcal{M}$ is part of a right $\mathcal{M}$-factorisation system. We say that a morphism $f : X \to Y$ *reflects* 0 if $f^{-1}[0_Y] = 0_X$ or equivalently the equality $f[m] = 0_Y$ implies $m = 0_X$ for any appropriate $m \in \mathcal{M}$.

A particular case of the characterization in Theorem 1.2.1 follows:

Theorem 1.2.21. *[36] There is a one-to-one correspondence between cover relations which are reflexive, transitive, reflecting and admitting images and pairs $(\mathcal{E}, \mathcal{M})$ of classes of morphisms that satisfy:*

(CC) *$\mathcal{M}$ is closed under composition and contains isomorphisms;*

(P) *Every morphism f has a factorisation $f = me$, where $m \in \mathcal{M}$ and $e \in \mathcal{E}$, and every morphism $e \in \mathcal{E}$ is orthogonal to every morphism $m \in \mathcal{M}$, i.e., for any commutative diagram*

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "e"'] & \cdot \arrow[d, "m"] \\ \cdot \arrow[r, "v"'] \arrow[ur, dashed, "h"] & \cdot \end{tikzcd}$$

there is a unique morphism h such that $mh = v$ and $he = u$.

The equivalence between (CC) and (P) for a right $\mathcal{M}$-facorisation system is established in [20].

Proof. We follow the proof in [20]. For the commutative diagram above, we write $e \perp m$. Given a class $\mathcal{M}$ which is part of a right $\mathcal{M}$-factorisation system, we define the class $\mathcal{E}$ as

$$\mathcal{M}^{\perp} := \{e \text{ in } \mathbf{C} \mid \text{ for all } m \in \mathcal{M},\ e \perp m\}.$$

It suffices to prove that in the right $\mathcal{M}$-factorisation $f = me$ of f, one has $e \in \mathcal{E}$. Let $e = nd$ where $n \in \mathcal{M}$ and consider the following diagram

$$\begin{tikzcd} \cdot \arrow[r, "d"] \arrow[d, "e"'] & \cdot \arrow[dd, "mn"] \\ \cdot \arrow[d, "m"'] \arrow[ur, dashed, "k"'] & \\ \cdot \arrow[r, "1"'] & \cdot \end{tikzcd}$$

Since $\mathcal{M}$ satisfies (D), there is a unique morphism k such that $ke = d$ and $mn.k = m$. Note that m and n are monomorphism. We have $nk = 1$ and so n is an isomorphism. Now if we have the following diagram

$$\begin{tikzcd} \cdot \arrow[rr, "u"] \arrow[dr, "d"] \arrow[dd, "e"'] & & \cdot \arrow[dd, "p"] \\ & \cdot \arrow[ur, dashed, "t"'] \arrow[dl, "k"] & \\ \cdot \arrow[rr, "v"'] & & \cdot \end{tikzcd}$$

where u and v are arbitrary morphisms and $p \in \mathcal{M}$, there is a unique t such that $pt = vk$ and $u = td$. The morphism tn is unique such that $p(tn) = v$ and $u = (tn)e$. Thus $e \in \mathcal{E}$. Property (CC) holds by assumption.

Now, assume we have a pair $(\mathcal{E}, \mathcal{M})$ that satisfies (CC) and (P). We first show that the class $\mathcal{M}$ coincides with the class

$$\mathcal{E}_{\perp} := \{m \text{ in } \mathbf{C} \mid \text{ For all } e \in \mathcal{E},\ e \perp m\}.$$

The property (P) implies that $\mathcal{M} \subseteq \mathcal{E}_{\perp}$. Conversely, let $m \in \mathcal{E}_{\perp}$ and consider the factorisation $m = k.e$ with $k \in \mathcal{M}$ and $e \in \mathcal{E}$. We have the following diagram

$$\begin{tikzcd} \cdot \arrow[r, "1"] \arrow[d, "e"'] & \cdot \arrow[d, "m"] \\ \cdot \arrow[r, "k"'] \arrow[ur, dashed, "t"] & \cdot \end{tikzcd}$$

Since $e \perp m$, there is a unique t such that $te = 1$ and $mt = k$. On the other hand, since $k \in \mathcal{M}$ and m is a monomorphism, $t \in \mathcal{M}$. Thus t is an isomorphism and so is e. Therefore $m \cong k \in \mathcal{M}$ and the classes $\mathcal{E}$ and $\mathcal{M}$ determine each other uniquely through (P). This also implies the stability of $\mathcal{M} = \mathcal{E}_{\perp}$ under composition. $\square$

Definition 1.2.22. [20, 13] A pair $(\mathcal{E}, \mathcal{M})$ which satisfies (CC) and (P) is called a $(\mathcal{E}, \mathcal{M})$-*factorisation system* or simply a *factorisation system*, when there is no confusion, and the property (P) is called the *diagonalisation property of* the $(\mathcal{E}, \mathcal{M})$-factorisation system.

It is clear from the property (P) that a $(\mathcal{E}, \mathcal{M})$-factorisation of a morphism is essentially unique.

The (Iso, All)-factorisation system, where Iso is the class of all isomorphisms, is a $(\mathcal{E}, \mathcal{M})$-factorisation system where the class $\mathcal{M}$ is not a class of monomorphisms.

We shall now give our attention to the class $\mathcal{E}$.

1.3 Pullback stability of the class $\mathcal{E}$

As is expected, for a given cover relation $\sqsubseteq$ satisfying the conditions in Theorem 1.2.21 the two classes $Cov_{\sqsubseteq}$ and $\mathcal{E}$ coincide.

Proposition 1.3.1. *[36] Let $\mathcal{M}$ be a class of monomorphisms such that it is part of a $(\mathcal{E}, \mathcal{M})$-factorisation system. Then necessarily $\mathcal{E}$ is the class of all $\sqsubseteq$-coverings.*

Proof. [36] By Lemma 1.1.8 a morphism $f : X \to Y$ is a $\sqsubseteq$-covering if and only if 1_Y is a $\sqsubseteq$-image of f. By Proposition 1.2.3, this is the case if and only if in any $(\mathcal{E}, \mathcal{M})$-factorisation $f = me$ of f, one has $m = 1_Y$. $\square$

The class $\mathcal{E}$ satisfies the following dual properties of the class $\mathcal{M}$ through the property (P) when $\mathcal{M}$ is a class of monomorphisms.

Proposition 1.3.2. *The class $\mathcal{E}$ is closed under composition. If $\mathcal{M}$ is a class of monomorphisms, then $\mathcal{E}$ is closed under colimits.*

A fair number of topological results depend on the pullback stability of the class $\mathcal{E}$. Unfortunately, in any given factorisation system, this is not always the case. Consider for instance the category $\mathbf{C} = \mathbf{Haus}$ of Hausdorff spaces with continuous maps. The pair $(Dense\ Maps, Closed\ Embeddings)$ forms a $(\mathcal{E}, \mathcal{M})$-factorisation system. Consider the following pullback diagram

$$\begin{tikzcd} \emptyset \arrow[r, "i'"] \arrow[d, "e'"'] & \mathbb{R}\setminus\mathbb{Q} \arrow[d, "e"] \\ \mathbb{Q} \arrow[r, "i"'] & \mathbb{R} \end{tikzcd}$$

Both e and i are dense maps, however their pullbacks e' and i' are not dense maps.

Characterization of classes $\mathcal{E}$ that are pullback stable has been established in [12, 27] and [20]. A class $\mathcal{E}$ is pullback stable if and only if it satisfies a condition which is an instance of the Beck-Chevalley Property. Throughout the text we shall refer to this condition as (BCP). We first consider the following lemma.

Lemma 1.3.3. *In the following diagram*

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "u"'] & \cdot \arrow[r, "b"] \arrow[d, "v"] & \cdot \arrow[d, "w"] \\ \cdot \arrow[r, "c"'] & \cdot \arrow[r, "d"'] & \cdot \end{tikzcd}$$

if the right square $wb = dv$ is a pullback, then the left square $va = cu$ is a pullback if and only if the rectangle $w(ba) = (dc)u$ is a pullback square.

Proposition 1.3.4. *[27] The following conditions are equivalent:*

(i) $\mathcal{E}$ is pullback stable;

(ii) Every pullback diagram satisfies (BCP), i.e., for any pullback diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and for any $m \in \mathcal{M}/X$ we have $g[a^{-1}[m]] \cong b^{-1}[f[m]]$.

Proof. [13] If (ii) is true, then by taking $m = 1_X$, $f[1_X] = 1_Y$ implies $g[1_A] = 1_B$. Conversely assume that (i) is true and consider the following diagram for any subobject $m \in \mathcal{M}/X$

$$\begin{tikzcd}
\cdot \arrow[rrr, "g'"] \arrow[dr, "a^{-1}[m]"] \arrow[ddd, "a'"'] & & & \cdot \arrow[dl, "b^{-1}[f[m]]"'] \arrow[ddd, "b'"] \\
& A \arrow[r, "g"] \arrow[d, "a"'] & B \arrow[d, "b"] & \\
& X \arrow[r, "f"'] & Y & \\
M \arrow[ur, "m"] \arrow[rrr, "e"'] & & & f[M] \arrow[ul, "f[m]"']
\end{tikzcd}$$

The top arrow g' of the outer square exists because of the pullback property of the square $f[m].b' = b.b^{-1}[f[m]]$. We are done if $g' \in \mathcal{E}$, since in that case $b^{-1}[f[m]].g'$ is a $(\mathcal{E}, \mathcal{M})$-factorisation of $g.a^{-1}[m]$ and therefore $g[a^{-1}[m]] = b^{-1}[f[m]]$.

Since the inner square and the left square are a pullbacks, their composition is a pullback. Since the diagram commutes, the composition of the right square with the outer square is a pullback as well. But since the right square is a pullback, by Lemma 1.3.3 the outer square is a pullback. Thus since $e \in \mathcal{E}$ we have $g' \in \mathcal{E}$. $\square$

A weaker condition which is a consequence of the previous proposition is known as the *Frobenius Reciprocity Law*, which says that the pullback of any morphism in $\mathcal{E}$ along an $\mathcal{M}$-morphism belongs to $\mathcal{E}$.

Corollary 1.3.5. *A morphism f in $\mathcal{E}$ is stable under pullback along a morphism p in $\mathcal{M}$ if and only if for any appropriate subobject $m \in \mathcal{M}$ the following holds*

$$f[f^{-1}[p] \wedge m] = p \wedge f[m]$$

Proof. Consider the following pullback diagrams

$$\begin{tikzcd}
\cdot \arrow[r, "m'"] \arrow[d, "p''"'] & \cdot \arrow[r, "f'"] \arrow[d, "p'"'] & \cdot \arrow[d, "p"] \\
\cdot \arrow[r, "m"'] & \cdot \arrow[r, "f"'] & \cdot
\end{tikzcd}$$

Suppose that $f' \in \mathcal{E}$. We have $f[p'] = p$. By virtue of Proposition 1.3.4 we have $f'[m'] = p^{-1}[f[m]]$. Thus

$$f[p' \wedge m] = f[p'.m'] = f[p'].f'[m'] = p.p^{-1}[f[m]] = p \wedge f[m].$$

Conversely, by choosing m to be the identity on the domain of f, we have $f[p'] = p$, hence $f' \in \mathcal{E}$. $\square$

Proposition 1.3.6. *The Frobenius Reciprocity Law holds if and only if for any morphism f in $\mathbf{C}$ and for any adequate subobjects m and p the following holds*

$$f[f^{-1}[p] \wedge m] = p \wedge f[m]$$

Proof. The above equality obviously implies the Frobenius Reciprocity Law. To show that the latter implies the above equality it is enough to show that every subobject $m \in \mathcal{M}$ satisfies the above equality since every morphism admits a $(\mathcal{E}, \mathcal{M})$-factorisation system. We have

$$m[m^{-1}[k] \wedge l] = m[l.l^{-1}[m^{-1}[k]]] = (ml).(ml)^{-1}[k] = ml \wedge k$$

for any appropriate subobjects k and l. Now, let $f = me$ be such that e satisfies the above equality, then

$$(me)[e^{-1}[m^{-1}[k]] \wedge l] = m[m^{-1}[k] \wedge el] = k \wedge (me)[l].$$

$\square$

Given a class of morphisms $\mathcal{F}$ in $\mathbf{C}$, we denote by $\mathcal{F}^*$ the largest pullback stable class contained in $\mathcal{F}$ and by $\mathcal{F}'$ the largest class contained in $\mathcal{F}$, which are stable under pullback along morphisms in $\mathcal{M}$. We shall then write $\mathcal{F} = \mathcal{F}^*$ when it is pullback stable and $\mathcal{F} = \mathcal{F}'$ when it is stable under pullback along morphisms in $\mathcal{M}$. In particular the condition (BCP) and the Frobenius reciprocity Law are respectively expressed by the relations $\mathcal{E} \subseteq \mathcal{E}^*$ and $\mathcal{E} \subseteq \mathcal{E}'$.

In the sequel, we shall assume that $\mathbf{C}$ is provided with a $(\mathcal{E}, \mathcal{M})$-factorisation system and that our class of embeddings $\mathcal{M}$ is a class of monomorphisms.

Chapter 2

Structure of Neighbourhoods

By assigning to a subobject $m \in \mathcal{M}$ a suitable collection of subobjects, we define a functor that captures the essential features of neighbourhoods, including the condition of continuity. This way of conceiving an operator as a functor can already be seen in the paper [56] in which Tholen defines a *closure operator* as a suitable endofunctor defined on the class $\mathcal{M}$. All the basic notions concerning neighbourhoods are introduced in the first section. Then, we proceed by showing that *interior operators* as studied in [59, 6, 8] and [39] are special neighbourhood operators. We end the chapter by describing different ways of generating closure from neighbourhoods and vice versa.

2.1 Rasters and neighbourhoods

Rasters are introduced in [25, 26] and investigated in [34] and [53] as tools to study convergence. The difference between rasters and filters is that the former are not necessarily closed under finite meets. Our definition of rasters differs from that introduced earlier. It is largely motivated by the fact that the least subobject might be a neighbourhood of itself. For example, in **Top**, the subset $\emptyset$ of any set X is a neighbourhood of itself. Therefore its set of neighbourhoods is given by the power set $\mathcal{P}(X)$.

Definition 2.1.1. Let $X \in \mathbf{C}$. A collection $\mathcal{G} \subseteq \mathcal{M}/X$ is called a *raster* if:

(a) For any $g, g' \in \mathcal{G}$, if $g \leq g'$ and $g \in \mathcal{G}$, then $g' \in \mathcal{G}$;

(b) There is $k \in \mathcal{M}$, called a *center*, such that $k \leq g$ for all $g \in \mathcal{G}$.

We admit the raster $\mathcal{M}/X$, $X \in \mathbf{C}$, as the only raster having 0_X as one of its members and its only center. When $\mathcal{G} = \mathcal{M}/X$ then $\mathcal{G}$ is said to be *degenerate*.

Note that the class $\mathcal{M}$ is a subcategory of the category of arrows $\mathbf{C}^{\mathbf{2}}$, where $\mathbf{2} = \{\bullet \to \bullet\}$. Thus there is an arrow $m \to n$ if there is a pair (g, f) of arrows in $\mathbf{C}$

such that the following diagram commutative

$$\begin{tikzcd} \cdot \arrow[r, "g"] \arrow[d, "m"'] & \cdot \arrow[d, "n"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

By factorizing g as $g = pe$, with $p \in \mathcal{M}$ and $e \in \mathcal{E}$, one has $f[m] = np$. Hence, we have $f[m] \leq n$. Conversely, if there is an arrow f in $\mathbf{C}$ such that $f[m] \leq n$, then clearly there is an arrow $(g, f) : m \to n$ in $\mathcal{M}$. In particular for any f in $\mathbf{C}$ and two subobjects m and n, arrows $f^{-1}[n] \to n$ and $m \to f[m]$ always exist when $f^{-1}[n]$ and $f[m]$ make sense.

Now, denote by $Ras(\mathcal{M})$ the class of all rasters on $\mathcal{M}$. It is a category with the following consideration: for any rasters $\mathcal{G} \subseteq \mathcal{M}/X$ and $\mathcal{K} \subseteq \mathcal{M}/Y$, with $X, Y \in \mathbf{C}$, there is an arrow $\mathcal{G} \to \mathcal{K}$ if there is a morphism $f : X \to Y$ such that $f^{-1}[\mathcal{K}] \subseteq \mathcal{G}$ or equivalently – because of the adjunction between image and pre-image – $\mathcal{K} \subseteq f[\mathcal{G}]$, where:

$$f[\mathcal{G}] := \{k \mid \text{For some } g \in \mathcal{G},\ k \geq f[g]\}$$

and

$$f^{-1}[\mathcal{K}] := \{l \mid \text{For some } k \in \mathcal{K},\ l \geq f^{-1}[k]\}.$$

The arrow $\mathcal{G} \to \mathcal{K}$ shall be denoted by $f^{\triangleleft}$. Note that for any raster $\mathcal{C} \subseteq \mathcal{M}/Z$, where $Z \in \mathbf{C}$, and any morphism $g : Y \to Z$, we have

$$(gf)[\mathcal{G}] = g[f[\mathcal{G}]] \text{ and } (gf)^{-1}[\mathcal{C}] = f^{-1}[g^{-1}[\mathcal{C}]].$$

These follow from the naturality observed in Lemma 1.2.14 (iii). In particular we have $g^{\triangleleft} \circ f^{\triangleleft} = (gf)^{\triangleleft}$.

Definition 2.1.2. A *neighbourhood operator* ν of $\mathcal{M}$ in $\mathbf{C}$ is given by a functor $\nu : \mathcal{M} \to Ras(\mathcal{M})$ such that for any $m \in \mathcal{M}$, m is a center for $\nu(m)$, and such that $\nu(g, f) = f^{\triangleleft}$ for any pair of arrows (g, f).

Remark 2.1.3. The condition that one should have $m \leq n$ for any $n \in \nu(m)$ could be generalized by replacing the relation $m \leq n$ with an arrow $m \to n$. In this case the neighbourhoods are not required to be “bigger” than the subobject, but instead to be “continuously related” to it. This definition also implies that $\nu(m)$ is always non-empty (unless one admits empty rasters and requires condition (b) of the Definition 2.1.1 only for non-empty rasters.)

If there is an arrow $m \to n$ in $\mathcal{M}$, i.e. $f[m] \leq n$ for some morphism f in $\mathbf{C}$, then:

$$f^{-1}[\nu(n)] \subseteq \nu(m) \text{ and equivalently } \nu(n) \subseteq f[\nu(m)].$$

In particular, if $p \leq q$ for two subobjects p and q, then $\nu(q) \subseteq \nu(p)$.

Combining these observations, one has the following characterization of continuity:

Proposition 2.1.4. *Let ν be a neighbourhood operator and $f : X \to Y$ be a morphism. Let $m \in \mathcal{M}/X$ and $n \in \mathcal{M}/Y$. We have the following relations:*

(i) $\nu(f[m]) \subseteq f[\nu(m)]$;

(ii) $f^{-1}[\nu(n)] \subseteq \nu(f^{-1}[n])$;

(iii) $f^{-1}[\nu(f[m])] \subseteq \nu(m)$ *and* $\nu(n) \subseteq f[\nu(f^{-1}[n])]$.

Proof. The results follow from the fact that we have arrows $m \to f[m]$ and $f^{-1}[n] \to n$ in $\mathcal{M} \subseteq \mathbf{C^2}$. □

The formulae in Proposition 2.1.4 are expressed as follow at the subobjects level:

(i) For any $p \in \nu(f[m])$, there is $q \in \nu(m)$ such that $f[q] \leq p$;

(ii) If $k \in \nu(n)$ then $f^{-1}[k] \in f^{-1}[\nu(n)]$;

(iii) If $k \in \nu(f[m])$ then $f^{-1}[k] \in \nu(m)$.

We shall refer to these formulae as *ν-continuity* or simply by *continuity*. By definition, these formulae are equivalent in expressing ν-continuity.

In the sequel, we shall most of the time write $\nu_X(m)$ and $\nu_Y(n)$ instead of $\nu(m)$ and $\nu(n)$ as in the proposition to avoid confusion. Thus a neighbourhood operator ν of $\mathcal{M}$ in $\mathbf{C}$ gives a family of maps $\{\nu_X \mid X \in \mathbf{C}\}$, with $\nu_X : \mathcal{M}/X \to Ras(\mathcal{M}/X)$ and $Ras(\mathcal{M}/X) \subseteq Ras(\mathcal{M})$, satisfying the condition of ν-continuity. Therefore ν is also a neighbourhood operator in the sense of [34, 33, 51]. The converse is trivially true: a neighbourhood operator in the sense of [34, 33, 51] is a neighbourhood operator of $\mathcal{M}$ in $\mathbf{C}$.

Further conditions on ν are also considered:

(O) For any $X \in \mathbf{C}$ and $\mathcal{C} \subseteq \mathcal{M}/X$. For a subobject $m \in \mathcal{M}$, if $m \in \nu(c)$ for any $c \in \mathcal{C}$, then $m \in \nu(\bigvee \mathcal{C})$;

(F) If $p, q \in \nu(m)$, then $p \wedge q \in \nu(m)$ for any $m \in \mathcal{M}$. In other words ν is factored through the full subcategory $Fil(\mathcal{M}) \subseteq Ras(\mathcal{M})$ of filters on $\mathcal{M}$;

(I) For any $m \in \mathcal{M}$, if $p \in \nu(m)$ then there is $q \in \nu(q)$ such that $m \leq q \leq p$.

Note that in the presence of the axiom (O), for all $m \in \mathcal{M}/X$ and $X \in \mathbf{C}$, one has $m \in \nu(0_X)$ by taking $\mathcal{C}$ to be the empty class. In particular $0_X \in \nu(0_X)$.

We denote by $NBH(\mathbf{C}, \mathcal{M})$ the class (possibly large) of all neighbourhood operators on $\mathbf{C}$. It is naturally ordered:

$$\nu \leq \nu' \text{ in } NBH(\mathbf{C}, \mathcal{M}) \text{ if and only if for all } m \in \mathcal{M},\ \nu(m) \subseteq \nu'(m).$$

This ordering is a natural transformation. Indeed consider a morphism $f : X \to Y$ such that $f[m] \leq n$, with $m \in \mathcal{M}/X$ and $n \in \mathcal{M}/Y$. The following diagram is commutative

$$\begin{tikzcd}
\nu'_X(m) \arrow[r, "(1_X)^{\triangleleft}"] \arrow[d, "f^{\triangleleft}"'] & \nu_X(m) \arrow[d, "f^{\triangleleft}"] \\
\nu'_Y(n) \arrow[r, "(1_Y)^{\triangleleft}"'] & \nu_Y(n)
\end{tikzcd}$$

Hence $NBH(\mathbf{C}, \mathcal{M})$ is embedded in the functor category $Func(\mathcal{M}, Ras(\mathcal{M}))$.

For a fixed $X \in \mathbf{C}$ and $\nu, \nu' \in NBH(\mathbf{C}, \mathcal{M})$, we shall write $\nu_X \leq \nu'_X$ whenever $\nu_X(m) \subseteq \nu'_X(m)$ for any $m \in \mathcal{M}/X$.

Considering the axioms (O), (F) and (I), one has different types of neighbourhood operators and consequently different types of subcategories of $NBH(\mathbf{C}, \mathcal{M})$:

- $RegNBH(\mathbf{C}, \mathcal{M})$: the class of all *regular neighbourhood operators*, those whose objects satisfy (O);
- $NBHF(\mathbf{C}, \mathcal{M})$: the class of all *neighbourhood filter operators*, those whose objects satisfy (F);
- $INBH(\mathbf{C}, \mathcal{M})$: the class of all *idempotent neighbourhood operators*, those whose objects satisfy (I).

A variety of subcategories can be thus obtained by taking intersections of these subcategories, namely: $RegNBHF(\mathbf{C}, \mathcal{M})$, $INBHF(\mathbf{C}, \mathcal{M})$, $IRegNBH(\mathbf{C}, \mathcal{M})$ and $IRegNBHF(\mathbf{C}, \mathcal{M})$.

Lemma 2.1.5. *Let $\{\nu_i \mid i \in I\} \subseteq NBH(\mathbf{C}, \mathcal{M})$. If each ν_i satisfies the axioms (O) (resp. (F)), then so is the neighbourhood operator ν_* defined by:*

$$\nu_*(m) := \bigcap\{\nu_i(m) \mid i \in I\} \text{ for all } m \in \mathcal{M}.$$

If each ν_i satisfies (O) and (I), then ν_* satisfies also (I).

Proof. ν_* trivially satisfies (O) (resp. (F)) provided that each ν_i satisfies (O) (resp. (F)).

Now assume that each ν_i satisfies (O) and (I), and let $p \in \nu_*(q)$. For each $i \in I$, there is $q_i \in \nu_i(q_i)$ such that $q \leq q_i \leq p$. In the presence of (O), if $r = \bigvee\{q_i \mid i \in I\}$, then we have $r \in \nu_i(r)$ and $q \leq r \leq p$. $\square$

Lemma 2.1.6. *If in a family $\{\nu_i \mid i \in I\} \subseteq NBH(\mathbf{C}, \mathcal{M})$, every member ν_i satisfies (I), then so is the neighbourhood operator ν^* defined by*

$$\nu^*(m) := \bigcup\{\nu_i(m) \mid i \in I\} \text{ for all } m \in \mathcal{M}.$$

As one would have expected, ν_* and ν^* are respectively the infimum and supremum of the family $\{\nu_i \mid i \in I\}$ in $NBH(\mathbf{C}, \mathcal{M})$. It is easily checked (cf. Lemma 2.1.5) that ν_* also gives the infimum in the following subcategories: $RegNBH(\mathbf{C}, \mathcal{M})$, $NBHF(\mathbf{C}, \mathcal{M})$, $RegNBHF(\mathbf{C}, \mathcal{M})$, $IRegNBH(\mathbf{C}, \mathcal{M})$ and $IRegNBHF(\mathbf{C}, \mathcal{M})$. The neighbourhood operator ν^* is a supremum in $NBH(\mathbf{C}, \mathcal{M})$ and $INBH(\mathbf{C}, \mathcal{M})$ (2.1.6).

Lemma 2.1.7. *Let $\{\nu_i \mid i \in I\} \subseteq NBHF(\mathbf{C}, \mathcal{M})$, then the neighbourhood operator $\hat{\nu}$ defined by*

$$\hat{\nu}(m) := \{k \geq p_{i_1} \wedge p_{i_2} \wedge \cdots \wedge p_{i_n} \mid n \in \mathbb{N} \text{ and } p_{i_j} \in \nu_{i_j}(m)\} \text{ for all } m \in \mathcal{M},$$

satisfies (I) provided that each ν_i satisfies (I).

Proof. Let $p \in \hat{\nu}(m)$. For some $n \in \mathbb{N}$ and $r_{i_j} \in \nu_{i_j}(m)$, $j \leq n$, we have

$$m \leq r_{i_1} \wedge r_{i_2} \wedge \cdots \wedge r_{i_n} \leq p.$$

There are $q_{i_j} \in \nu_{i_j}(q_{i_j})$ for $j \leq n$ such that $m \leq q_{i_j} \leq r_{i_j}$. And for any $j \leq n$

$$\nu_{i_j}(q_{i_j}) \subseteq \hat{\nu}(q_{i_j}) \subseteq \hat{\nu}(q_{i_1} \wedge q_{i_2} \wedge \cdots \wedge q_{i_n}).$$

By definition, $q_{i_1} \wedge q_{i_2} \wedge \cdots \wedge q_{i_n} \in \hat{\nu}(q_{i_1} \wedge q_{i_2} \wedge \cdots \wedge q_{i_n})$. $\square$

The neighbourhood operator $\hat{\nu}$ is the supremum of the family $\{\nu_i \mid i \in I\}$ in $NBHF(\mathbf{C}, \mathcal{M})$. It also gives the formula of the supremum in $INBHF(\mathbf{C}, \mathcal{M})$.

Definition 2.1.8. For a neighbourhood operator ν, a subobject m is said to be *ν-idempotent* or simply *idempotent* when $m \in \nu(m)$.

The choice of the word *idempotent* shall be made clear later on.

Corollary 2.1.9. *Suppose that joins of idempotent subobjects are always idempotent. Then $\hat{\nu}$, as defined in Lemma 2.1.7, belongs to $RegNBH(\mathbf{C}, \mathcal{M})$ whenever each member of the family $\{\nu_i \mid i \in I\}$ satisfies (I).*

Proof. Suppose that $p \in \hat{\nu}(g)$ for all $g \in \mathcal{G}$ with $\mathcal{G} \subseteq \mathcal{M}/X$ and $X \in \mathbf{C}$. For some $p_i^g \in \hat{\nu}(p_i^g)$, $i \leq k$, $k \in \mathbb{N}$, we have

$$g \leq p_1^g \wedge p_2^g \wedge \cdots \wedge p_k^g \leq p.$$

Set $\alpha^g = p_1^g \wedge p_2^g \wedge \cdots \wedge p_k$. By Lemma 2.1.7 $\alpha^g \in \hat{\nu}(\alpha^g)$ and

$$\bigvee \mathcal{G} \leq \bigvee \{\alpha^g \mid g \in \mathcal{G}\} \leq p.$$

By assumption, if $q = \bigvee \{\alpha^g \mid g \in \mathcal{G}\}$, then $q \in \hat{\nu}(q)$. □

Definition 2.1.10. We say that a neighbourhood operator ν is *complete* if any join of its ν-idempotent subobjects is again idempotent and a family $\{\nu_i \mid i \in I\}$ is said to be complete if their supremum $\hat{\nu}$ in $NBHF(\mathbf{C}, \mathcal{M})$ is complete.

Therefore when $\nu \in NBHF(\mathbf{C}, \mathcal{M})$ and is complete, the class of ν-idempotent subobjects behave like a topology. The motivation of the term "complete" is not only the idea of completion as for partially ordered sets but also because $\hat{\nu}$ gives the supremum in $IRegNBHF(\mathbf{C}, \mathcal{M})$ under the assumption of completeness, and so it offers a "constructive" description of the filter $\hat{\nu}(m)$ for any $m \in \mathcal{M}$. The interest in the class $IRegNBHF(\mathbf{C}, \mathcal{M})$ is that it captures the features of neighbourhood systems in a topological space.

Lemmas 2.1.5, 2.1.6 and 2.1.7 are about preservation of infima and suprema. Embeddings that preserve infima (resp. suprema) are reflections (resp. coreflections). These are captured in the diagram below, where reflections are labelled by r and coreflections by c.

$$\begin{tikzcd}
RegNBHF(\mathbf{C},\mathcal{M}) \arrow[rrr, "r"] \arrow[dr, "r"] & & & NBHF(\mathbf{C},\mathcal{M}) \arrow[dl, "r"'] \\
& RegNBH(\mathbf{C},\mathcal{M}) \arrow[r, "r"] & NBH(\mathbf{C},\mathcal{M}) & \\
& IRegNBH(\mathbf{C},\mathcal{M}) \arrow[r, "r"'] \arrow[u, "r"] & INBH(\mathbf{C},\mathcal{M}) \arrow[u, "c"'] & \\
IRegNBHF(\mathbf{C},\mathcal{M}) \arrow[rrr, "r"'] \arrow[uuu, "r"] \arrow[ur, "r"'] & & & INBHF(\mathbf{C},\mathcal{M}) \arrow[uuu, "c"'] \arrow[ul, "c"]
\end{tikzcd}$$

$NBHF(\mathbf{C}, \mathcal{M}) \to NBH(\mathbf{C}, \mathcal{M})$ and $RegNBH(\mathbf{C}, \mathcal{M}) \to NBH(\mathbf{C}, \mathcal{M})$ are the inclusions which are important to us as they more reflect the behaviour of the neighbourhood system in a topological space. Their respective left adjoints shall be denoted by θ and ρ.

2.2 Interior operators or the notion of openness

We introduce interior operators as they were defined in [6] and [8]. It is clear that they could be viewed as endofunctors on $\mathcal{M}$.

Definition 2.2.1. An interior operator i of $\mathcal{M}$ in $\mathbf{C}$ is given by a family of maps $i := (i_X)_{X\in\mathbf{C}}$ such that $i_X : \mathcal{M}/X \to \mathcal{M}/X$ for each $X \in \mathbf{C}$ and:

(I1) $i_X(n) \leq n$ for all $n \in \mathcal{M}/X$;

(I2) If $m \leq n$ in $\mathcal{M}/X$, then $i_X(m) \leq i_X(n)$;

(I3) For any $f : X \to Y$ and $n \in \mathcal{M}/Y$ we have $f^{-1}[i_Y(n)] \leq\ i_X(f^{-1}[n])$.

(I3) is referred to as *i-continuity*. The class of all interior operators of $\mathcal{M}$ in $\mathbf{C}$ is denoted by $INT(\mathbf{C},\mathcal{M})$ and it is ordered as follows

$$i \leq i' \text{ if and only if } i_X(m) \leq i'_X(m) \text{ for any } X \in \mathbf{C} \text{ and } m \in \mathcal{M}.$$

$INT(\mathbf{C},\mathcal{M})$, viewed as a category with this ordering, is essentially the same as $RegNBH(\mathbf{C},\mathcal{M})$. Therefore any notion that deals with openness can be studied with neighbourhoods.

Proposition 2.2.2. *[33] $RegNBH(\mathbf{C},\mathcal{M})$ and $INT(\mathbf{C},\mathcal{M})$ are isomorphic with inverse assignments $\nu \mapsto i^\nu$ and $i \mapsto \nu^i$ given by*

$$i^\nu_X(m) = \bigvee\{p \mid m \in \nu_X(p)\} \textit{ and } \nu^i_X(m) = \{p \mid m \leq i_X(p)\}.$$

Proof. The proof follows that in [33]. It is straightforward to see that i^ν and ν^i satisfy all the required basic conditions. What we need to prove is the i-continuity and ν-continuity and also that the above assignments are inverse to each other and order-preserving.

Let $i \in INT(\mathbf{C},\mathcal{M})$ and let $f : X \to Y$ be in $\mathbf{C}$. If $k \in \nu^i_Y(n)$ in $\mathcal{M}/Y$, then $n \leq i_Y(k)$ and so $f^{-1}[n] \leq f^{-1}[i_Y(k)] \leq i_X(f^{-1}[k])$. Hence $f^{-1}[k] \in \nu^i_X(f^{-1}[n])$. Conversely let $\nu \in NBH(\mathbf{C},\mathcal{M})$ and $n \in \mathcal{M}/Y$. Because of the axiom (O), one has $n \in \nu_Y(i^\nu_Y(n))$. Since f is ν-continuous, we have $f^{-1}[n] \in \nu_X(f^{-1}[i^\nu_Y(n)])$. In other words $f^{-1}[i_Y(n)] \leq\ i_X(f^{-1}[n])$.

Now if $i \in INT(\mathbf{C},\mathcal{M})$ and $m \in \mathcal{M}/X$, then

$$i^{\nu^i}_X(m) = \bigvee\{n \mid m \in \nu^i_X(n)\} = \bigvee\{n \mid n \leq i_X(m)\} = i_X(m).$$

Conversely if $n \in \nu_X^{i^\nu}(m)$, then $m \leq i_X^\nu(n)$ and therefore by the axiom (O), we have $n \in \nu_X(i_X^\nu(n)) \subseteq \nu_X(m)$. On the other hand if $n \in \nu_X(m)$, then $m \leq i_X^\nu(n)$ by definition and so $n \in \nu_X^{i^\nu}(m)$.

Finally assume that $i \leq i'$ in $INT(\mathbf{C}, \mathcal{M})$ and let $k \in \nu_X^i(m)$. By definition we have $m \leq i_X(k) \leq i'_X(k)$. So $k \in \nu_X^{i'}(m)$ and $\nu^i \leq \nu^{i'}$. On the other hand suppose that $\nu \leq \nu'$ in $NBH(\mathbf{C}, \mathcal{M})$ and let $p \in \mathcal{M}$ such that $p \leq i_X^\nu(m)$. By definition $m \in \nu_X(p) \subseteq \nu'_X(p)$, so $p \leq i_X^{\nu'}(m)$. Therefore $i^\nu \leq i^{\nu'}$. $\square$

Proposition 2.2.3. *[6] Given a family $\{i_k \mid k \in K\} \subseteq INT(\mathbf{C}, \mathcal{M})$, its infimum i_* is defined as follows: for each $X \in \mathbf{C}$ and $m \in \mathcal{M}/X$*

$$(i_*)_X(m) = \bigwedge\{(i_k)_X(m) \mid k \in K\}$$

If any join of subobjects commutes with pullbacks, i.e., for any arrow f and any appropriate family $\{k_i \mid i \in I\}$ of subobjects $f^{-1}[\bigvee\{k_i \mid i \in I\}] = \bigvee\{f^{-1}[k_i] \mid i \in I\}$, then its supremum i^* is defined as follows: for each $X \in \mathbf{C}$ and $m \in \mathcal{M}/X$

$$i_X^*(m) = \bigvee\{(i_k)_X(m) \mid k \in K\}$$

Proof. [6] The basic conditions (I1) and (I2) are trivially satisfied by i_* and i^*. We shall prove the i-continuity. Let $f : X \to Y$ be in $\mathbf{C}$ and $m \in \mathcal{M}/Y$. As pre-images commute with pullbacks, we have

$$f^{-1}[(i_*)_Y(m)] = \bigwedge\{\, f^{-1}[(i_k)_Y(m)] \mid k \in K\} \leq (i_*)_X(f^{-1}[m]).$$

The situation is similar when pre-images commute with join of subobjects:

$$f^{-1}[(i^*)_Y(m)] = \bigvee\{\, f^{-1}[(i_k)_Y(m)] \mid k \in K\} \leq (i^*)_X(f^{-1}[m]).$$

$\square$

Proposition 2.2.4. *Let $\{\nu_k \mid k \in K\} \subseteq RegNBH(\mathbf{C}, \mathcal{M})$. Let us denote by i_k the interior operator i^{ν_k} for each $k \in K$ and let $\overline{\nu}$ be the supremum of the ν_k's in $RegNBH(\mathbf{C}, \mathcal{M})$. If joins of subobjects commute with pullbacks, then for any $m \in \mathcal{M}$, $\overline{\nu}(m)$ is generated by*

$$\{\bigvee{}_J i_j(p) \mid p \in \nu_j(m), J \subseteq K\},$$

where $\bigvee_J i_j(p)$ denotes the family $\bigvee\{i_j(p) \mid i \in J\}$.

Proof. It is enough to show that the raster generated by the above family is the supremum of the ν_k's. By construction, it satisfies the basic requirements for a neighbourhood operator. In particular it satisfies (O): if $\mathcal{G} \subseteq \mathcal{M}/X$ for some $X \in \mathbf{C}$ and

$$g \leq \bigvee_{J^g} i_j(p) \leq p \text{ for all } g \in \mathcal{G} \text{ and for some } J^g \subseteq K,$$

then

$$\bigvee \mathcal{G} \leq \bigvee_J i_j(p) \leq p, \text{ with } J = \bigcup \{J^g \mid g \in \mathcal{G}\}.$$

Now let $f : X \to Y$ be in $\mathbf{C}$ and $k, m \in \mathcal{M}$ such that $m \leq \bigvee_J (i_j)_Y(k) \leq k$, $J \subseteq K$. By assumption (or also from Proposition 2.2.3)

$$f^{-1}[m] \leq \bigvee_J f^{-1}[(i_j)_Y(k)] \leq \bigvee_J (i_j)_X(f^{-1}[k]) \leq f^{-1}[k].$$

Finally, suppose that for some $\mu \in RegNBH(\mathbf{C}, \mathcal{M})$ we have $\nu_k \leq \mu$ for any $k \in K$. Let $p, m \in \mathcal{M}$ such that $m \leq \bigvee_J i_j(p) \leq p$ for some $J \subseteq K$. By virtue of Proposition 2.2.2 one has $i_k \leq i^\mu$ for each $k \in K$, hence

$$m \leq \bigvee_J i_j(p) \leq i^\mu(p) \leq p,$$

Thus the family considered generates the supremum of the ν_k's. □

We warn that the word "generated" in the previous proposition is not to be understood as "basis". In other words the joins $\bigvee_J i_j(p)$ might not belong to $\overline{\nu}(m)$. In any case, this will cause no harm as it will not be used, since Proposition 2.2.3 already provides a good expression for the supremum of interior operators.

The condition that the joins of subobjects commute with pullbacks, as we shall see in the next chapter, is important for interior operators. It seems that the preservation of the join $\bigvee$ under the operations considered, as is the case for frames, is an indication that one deals with the notion of openness.

Proposition 2.2.5. *(i) Let ν be a neighbourhood operator. If ν satisfies (I), then $i \cong \rho(\nu)$ is idempotent in a sense that for any $m \in \mathcal{M}$, $i(i(m)) = i(m)$. If ν satisfies (F), then the same i preserves finite meets;*

(i) Let i be an interior operator. If i preserves meets then ν^i satisfies (F) and if i is idempotent then ν^i satisfies (I).

Proof. (i) We note that the assignment $\nu \mapsto i^\nu$ in Proposition 2.2.2 is just the restriction of the functor ρ to $RegNBH(\mathbf{C}, \mathcal{M})$. We have

$$i(i(m)) = \bigvee \{p \in \mathcal{M} \mid i(m) \in \rho(\nu)(p)\} \text{ for all } m \in \mathcal{M}.$$

But since $\rho(\nu)$ satisfies (O) and ν satisfies (I), we respectively have $i(m) \in \rho(\nu)(i(m))$ and $i(m) = \bigvee \{p \in \mathcal{M} \mid p \in \nu(p) \text{ and } p \leq m\}$. Thus $i(m) \leq i(i(m))$.

Now assume that ν satisfies (F). For any $m, n \in \mathcal{M}$, we always have

$$i(m \wedge n) \leq i(m) \wedge i(n).$$

If $k \leq i(m) \wedge i(n)$, then $m \in \nu(k)$ and $n \in \nu(k)$. By assumption $m \wedge n \in \nu(k)$. Hence $k \leq i(m \wedge n)$.

(ii) If the interior operator i preserves finite meets, then by definition the relation $i(m \wedge n) = i(m) \wedge i(n)$ implies that ν^i satisfies (F). Also if i is idempotent, we have $i(m) \in \nu^i(i(m))$ and so ν^i satisfies (I). □

Remark 2.2.6. If ν is a regular neighbourhood operator and $i = \rho(\nu)$, then a subobject m is ν-idempotent if and only if $i(m) = m$.

A similar result is proved in [33]. The proposition implies that the embeddings $IRegNBH(\mathbf{C}, \mathcal{M}) \to INBH(\mathbf{C}, \mathcal{M})$, $RegNBHF(\mathbf{C}, \mathcal{M}) \to NBHF(\mathbf{C}, \mathcal{M})$ and also $IRegNBHF(\mathbf{C}, \mathcal{M}) \to INBHF(\mathbf{C}, \mathcal{M})$ have left adjoints – which are restrictions of ρ denoted by r^* – that make the following diagram commute

$$\begin{tikzcd}
RegNBHF(\mathbf{C},\mathcal{M}) \arrow[d, "r"'] & NBHF(\mathbf{C},\mathcal{M}) \arrow[l, "r_1^*"] \arrow[d, "r"] \\
RegNBH(\mathbf{C},\mathcal{M}) & NBH(\mathbf{C},\mathcal{M}) \arrow[l, "\rho"'] \\
IRegNBH(\mathbf{C},\mathcal{M}) \arrow[u, "r"] & INBH(\mathbf{C},\mathcal{M}) \arrow[l, "r_2^*"'] \arrow[u, "c"'] \\
IRegNBHF(\mathbf{C},\mathcal{M}) \arrow[u, "r"] & INBHF(\mathbf{C},\mathcal{M}) \arrow[l, "r_3^*"'] \arrow[u, "c"']
\end{tikzcd}$$

2.3 From neighbourhoods to closure and back

We shall first recall the definition of a closure operator. The monograph [20] gives a detailed study of closure operators.

Definition 2.3.1. [56] A closure operator c of $\mathcal{M}$ in $\mathbf{C}$ is an endofunctor $c : \mathcal{M} \to \mathcal{M}$ with $I \leq c$ and $cod \circ c = cod$; where I is the identity functor of $\mathcal{M}$, and $cod : \mathcal{M} \to \mathbf{C}$ the codomain functor $(g, f) \to f$.

The class of all closure operators of $\mathcal{M}$ in $\mathbf{C}$ is denoted by $CL(\mathbf{C}, \mathcal{M})$.

Note that the relation $I \leq c$ means $m \leq c(m)$ for all $m \in \mathcal{M}$. Also the relation $cod \circ c = cod$ means that $c(g, f) = (h, f)$ for some h in $\mathbf{C}$. Now, if f is in $\mathbf{C}$ and m and n are subobjects such that $f[m] \leq n$, then we have $f[c(m)] \leq c(n)$. Since we always have arrows $m \to f[m]$ and $f^{-1}[n] \to n$, the following relations hold

$$f[c(m)] \leq c(f[m])$$

and

$$f[c(f^{-1}[n])] \leq c(n) \text{ or equivalently } c(f^{-1}[n]) \leq f^{-1}[c(n)].$$

In particular, if f is the identity arrow, i.e. $m \leq n$, then $c(m) \leq c(n)$.

If f is an arrow from an object X to an object Y, then we shall write $c_X(m)$ and $c_Y(n)$ respectively instead of $c(m)$ and $c(n)$ to avoid confusion. The same applies to $c(f^{-1}[n])$ and $c(f[m])$; we shall respectively write $c_X(f^{-1}[n])$ and $c_Y(f[m])$.

There are several ways to establish correspondences between $NBH(\mathbf{C}, \mathcal{M})$ (or $RegNBH(\mathbf{C}, \mathcal{M})$) and $CL(\mathbf{C}, \mathcal{M})$. However, none of these correspondences are known to give a satisfactory relation between the two classes, though they are equivalent when the subobject lattices are Boolean algebras. In this section, we shall describe these relations which are defined at the level of subobjects. The unfortunate behaviour of these correspondences does not imply that we cannot treat the notion of closedness via neighbourhoods. As we shall see in the following chapter, rather starting with the notion of closed morphisms than closed subobjects gives a much more natural notion of closedness.

We mainly follow [33, 34]. Let ν be a neighbourhood operator and $X \in \mathbf{C}$. Let $m \in \mathcal{M}/X$ and consider the collection

$$c_X(m) := \bigvee\{n \in \mathcal{M} \mid (\forall n' \leq^+ n), m \wedge \nu_X(n') > 0_X\},$$

where the relation $m \wedge \nu_X(n') > 0_X$ means that for any $k \in \nu_X(n')$ we have $m \wedge k > 0_X$ and the relation $n' \leq^+ n$ means $0_X < n' \leq n$.

Proposition 2.3.2. *[34] If every morphism reflects* 0*, then the family* $(c_X)_{X \in \mathbf{C}}$ *as defined above gives rise to a closure operator* c^ν *of* $\mathcal{M}$ *in* $\mathbf{C}$ *and the assignement* $\nu \mapsto c^\nu$ *is order reversing.*

Proof. It is clear that for any $m \in \mathcal{M}$ we have $m \leq c^\nu(m)$. It is enough to prove that for any arrow $f : X \to Y$ and two subobjects m and n such that $f[m] \leq n$ we have $f[c^\nu_X(m)] \leq c^\nu_Y(n)$. To achieve this, let $k \leq f[c_X(m)]$. Since $f[-]$ commutes with join, we have

$$k \leq \alpha := \bigvee\{f[p] \mid (\forall p' \leq^+ p), m \wedge \nu_X(p') > 0_X\}.$$

Since f reflects 0, we have $f[m] \wedge f[\nu_X(p')] > 0_Y$. On the other hand, $f[m] \leq n$ and $\nu_Y(f[p']) \subseteq f[\nu_X(p')]$. Therefore $n \wedge \nu_Y(f[p']) > 0_Y$ for every $p' \leq^+ p$. By definition $\alpha \leq c^\nu_Y(n)$. $\square$

Given a neighbourhood operator ν, the closure operator obtained as in Proposition 2.3.2 shall be denoted by $\mathfrak{cl}^\nu_1$.

The closure operator defined above arises from the study of convergence. A rather simple and direct way to define closed subobjects with respect to ν is to consider the following collection for every $X \in \mathbf{C}$

$$\{m \in \mathcal{M} \mid (\forall l \in \mathcal{M}), \text{ if } m \wedge \nu_X(l) > 0_X \text{ then } m \wedge l > 0_X\}.$$

Under the assumption that $\mathcal{E} \subseteq \mathcal{E}'$, one obtains a closure operator by performing some operation on the above collection; the way to obtain a closure operator from a subclass of $\mathcal{M}$ is shown later on. In this case the closure operator is denoted by $\mathfrak{cl}_2^\nu$.

It is also possible to define a closure operator by restricting ourselves to the classes $RegNBH(\mathbf{C}, \mathcal{M})$ or $INT(\mathbf{C}, \mathcal{M})$ as follows.

Definition 2.3.3. [33] Let i be an interior operator and $X \in \mathbf{C}$. A subobject m of X is:

(i) A^i-closed if for any $n \in \mathcal{M}$, $i_X(m \vee n) \leq m \vee i_X(n)$;

(ii) B^i-closed if the relation $m \vee n = 1_X$ implies $m \vee i_X(n) = 1_X$ for any $n \in \mathcal{M}$;

(iii) C^i-closed if m is pseudocomplemented and $\overline{m} = i_X(\overline{m})$.

A pseudocomplement of m is a subobject $\overline{m}$ such that for any $n \in \mathcal{M}$, the relations $n \leq \overline{m}$ and $m \wedge n = 0_X$ are equivalent.

It is clear that these three notions are equivalent in the point-set setting. Relations between them can be derived under some restrictions (cf. [33]). One obtains three different types of closure operators by performing the procedure that follows.

[33] Given a class $\mathcal{F} \subseteq \mathcal{M}$ we form the smallest pullback stable class containing $\mathcal{F}$ by considering the collection $\overline{\mathcal{F}} := \{f^{-1}[k] \mid k \in \mathcal{F},\ f \text{ in } \mathbf{C}\}$. We consider for any $m \in \mathcal{M}/X$ and $X \in \mathbf{C}$ the following assignment

$$c_X^{\mathcal{F}}(m) := \bigwedge\{n \in \overline{\mathcal{F}} \mid m \leq n\}.$$

It is straightforward to see that the family $(c_X^{\mathcal{F}})_{X \in \mathbf{C}}$ gives rise to a closure operator $c^{\mathcal{F}}$ of $\mathcal{M}$ in $\mathbf{C}$. $c^{\mathcal{F}}$ is universal in a sense that for any $m \in \mathcal{F}$ and a closure operator c, if $m = c(m)$, then $c \leq c^{\mathcal{F}}$. The fact that $\mathcal{F}$ is not pullback stable means that the property that we want to capture is not pullback stable. This is sometimes inconvenient as we have to add some restrictions on the category $\mathbf{C}$. In the case of the closure operator $\mathfrak{cl}_2^\nu$ as mentioned earlier, $\mathcal{F}$ is pullback stable when $\mathcal{E} \subseteq \mathcal{E}'$.

Definition 2.3.4. [33] Let i be an interior operator. We denote by α^i, β^i and γ^i the closure operators obtained by considering respectively for the class $\mathcal{F}$ the class of A^i-closed, B^i-closed and C^i-closed subobjects.

[33] The assignments $i \mapsto \beta^i$ and $i \mapsto \gamma^i$ are order-reversing while the assignment $i \mapsto \alpha^i$ does not preserve nor reverse the order in general. Indeed assume that morphisms reflect 0, for any $m \in \mathcal{M}/X$ and $X \in \mathbf{C}$, let i be the interior operator such that $i(m) = 0_X$ and i' be the interior operator such that $i'(m) = m$. Now for any $p \in \mathcal{M}$, the relations

$$i_X(p \vee m) \leq p \vee i_X(m) \text{ and } i'_X(p \vee m) \leq p \vee i'_X(m)$$

hold trivially. Therefore any subobject $p \in \mathcal{M}$ is A^i-closed and $A^{i'}$-closed and so $\alpha^i(m) = m$ and $\alpha^{i'}(m) = m$. However, in the category **Top** of topological spaces and continuous functions, if j is the usual interior operator that gives the largest open subspace of a space, then $i \leq j \leq i'$. However, $\alpha^j \neq \alpha^i$ and $\alpha^j \neq \alpha^{i'}$.

The fact that the first assignment mentioned above does not preserve order in any sense is an indication that it is not a part of a Galois connection.

Now, given a closure operator c of $\mathcal{M}$ in $\mathbf{C}$, we shall associate to c a neighbourhood operator ν.

Definition 2.3.5. [33] Let c be a closure operator and $X \in \mathbf{C}$. A subobject m of X is said to be

(i) $\mathfrak{A}^c$-open if for any $n \in \mathcal{M}$, $c_X(m \wedge n) \geq m \wedge c_X(n)$;

(ii) $\mathfrak{B}^c$-open if the relation $m \wedge n = 0_X$ implies $m \wedge c_X(n) = 0_X$ for any $n \in \mathcal{M}$;

(iii) $\mathfrak{C}^i$-open if m is pseudocomplemented and $\overline{m} = c_X(\overline{m})$.

[33] As for interior operators, one can perform an operation on a given subclass $\mathcal{F}$ of $\mathcal{M}$ and define a neighbourhood operator. We form the pullback stabilization $\overline{\mathcal{F}}$ of $\mathcal{F}$ and consider the following collection for every $m \in \mathcal{M}$:

$$\nu^{\mathcal{F}}(m) := \{n \in \mathcal{M} \mid (\exists p \in \overline{\mathcal{F}}), m \leq p \leq n\}.$$

It is clear that $\nu^{\mathcal{F}} \in NBH(\mathbf{C}, \mathcal{M})$ and it is universal in a sense that for any neighbourhood operator μ, if $m \in \mu(m) \bigcap \nu^{\mathcal{F}}(m)$ then $\nu^{\mathcal{F}} \leq \mu$. Furthermore, because of the way it is defined, $\nu^{\mathcal{F}}$ always satisfies (I).

Note that in order for $\nu^{\mathcal{F}}$ to be regular, the following condition must hold:

$$\text{For any } \mathcal{G} \subseteq \mathcal{F}^*, \bigvee \mathcal{G} \in \mathcal{F}^*$$

Definition 2.3.6. [33] Given a closure operator c, we denote by $\mathfrak{a}^c$, $\mathfrak{b}^c$ and $\mathfrak{c}^c$ the neighbourhood operators obtained by considering respectively for the class $\mathcal{F}$ the class of $\mathfrak{A}^i$-open, $\mathfrak{B}^i$-open and $\mathfrak{C}^i$-open subobjects.

[33] It is clear that the assignments $c \mapsto \mathfrak{b}^c$ and $c \mapsto \mathfrak{c}^c$ are order-reversing while – as was the case for α^i – the assignment $c \mapsto \mathfrak{a}^c$ is does not preserve nor reverse the order. Under the assumption that subobject lattices in $\mathbf{C}$ are Boolean algebras, the pairs $(\alpha, \mathfrak{a})$, $(\beta, \mathfrak{b})$ and $(\gamma, \mathfrak{c})$ establish nice correspondences between $NBH(\mathbf{C}, \mathcal{M})$ and $CL(\mathbf{C}, \mathcal{M})$.

The notion of $\mathfrak{C}^c$-open subobjects is used in [25, 26] and [59] to respectively generate neighbourhoods and induce interior operators, while the notion of $\mathfrak{A}^c$-open subobjects is considered in [27].

2.4 A few Examples

Example 2.4.1. (a) On the category **Top** of topological spaces. Consider the $(\mathcal{E}, \mathcal{M})$-factorisation system formed by *continuous surjections* and *embeddings*. We have the following neighbourhood operator $\mathcal{N}$: for any $(X, \tau_X) \in$ **Top** and $A \subseteq X$

$$\mathcal{N}_X(A) = \{B \mid A \subseteq C \subseteq B \text{ for some } C \in \tau_X\}.$$

In the sequel we shall denote $\mathcal{N}$ by τ and hence $\mathcal{N}_X$ by τ_X for simplification. We will refer to it as the *usual neighbourhood operator* on **Top**.

(b) In general, if we have a closure operator k on **Top**, then one can form a neighbourhood operator ϑ^k by saying that $N \in \vartheta^k_X(M)$ for a space X and a subspace $M \subseteq X$, if there is a set O such that $M \subseteq O \subseteq N$ and $O \bigcap k_X(X \setminus N) = \emptyset$. Therefore if σ is for example the sequential closure operator on **Top** [20, 21], then $N \in \vartheta^\sigma_X(M)$ if and only if there is O such that $M \subseteq O \subseteq N$ and such that for any sequence (x_n) in $X \setminus O$ if $x = \lim(x_n)$, then $x \in X \setminus O$.

(c) Even in the realm of topological spaces, pseudocomplements might not be available. Consider the category **Haus** with *dense maps* and *closed embeddings*. For any space X, the subobjects are essentially the closed subspaces and hence do not in general admit pseudocomplement. If we consider the neighbourhood operator $\mathcal{N}$ as defined in (a) above, then $C \in \mathcal{N}_X(C)$ if and only if C is clopen (open and closed.) Here $\mathcal{N}$ is not idempotent in general, while it is the case in **Top**. In the concrete case where $X = \mathbb{R}$, we have $[0; 1] \in \mathcal{N}_{\mathbb{R}}(x)$ for any $x \in (0; 1)$. However if $K = \bigvee\{x \mid 0 < x < 1\}$, then $K = [0; 1]$ and $K \notin \mathcal{N}_{\mathbb{R}}(K)$. Therefore $\mathcal{N}$ is not regular here.

(d) [6, 8] On **Top**, one defines the interior operators b and q given by

$$b_X(M) := \bigcup\{C \mid C \text{ closed and } C \subseteq M\}$$

and

$$q_X(M) := \bigcup\{C \mid C \text{ clopen and } C \subseteq M\}$$

For any space X and $M \subseteq X$. These interior operators are obtained from the neighbourhood operators η^b and η^q – i.e. $\rho(\eta^b) = b$ and $\rho(\eta^q)$ – given by

$$\eta^b_X(M) = \{N \mid M \subseteq C \subseteq N \text{ for some closed } C \subseteq X\}$$

and

$$\eta^q_X(M) = \{N \mid M \subseteq C \subseteq N \text{ for some clopen } C \subseteq X\}$$

These neighbourhood operators themselves are obtained from the classes of closed subobjects and clopen subobjects by performing the operation described after Definition 2.3.5. Thus η^b and η^q are idempotent and from Proposition 2.2.5, b and q are idempotent [6].

Example 2.4.2. (a) (Cf. [6]) Consider the category **Grp** of groups with *injective homomorphisms* and *surjective homomorphisms*. We define the following neighbourhood operator $\mathfrak{n}$: for any group G and a subgroup H

$$\mathfrak{n}_G(H) = \{K \mid H \leq N \leq K \text{ for some } N \triangleleft G\}$$

This can be refined to obtain the following neighbourhood operator $\mathfrak{n}'$:

$$\mathfrak{n}'_G(H) = \{K \mid H \leq N \leq K \text{ for some } N \triangleleft G \text{ with } G/N \text{ abelian}\}.$$

$\mathfrak{n}$ and $\mathfrak{n}'$ are also obtained from subclasses (normal subgroups) of $\mathcal{M}$.

(b) [51] Let us consider the category **Ab** of abelian groups with *injective homomorphisms* and *surjective homomorphisms*. For any group G, denote by $\mathfrak{t}(G)$ the torsion subgroup of G. We recall that $\mathfrak{t}(G) = \{g \mid (\exists n \in \mathbb{Z}), ng = 0\}$. Let $\mathbf{Tor}^{\mathfrak{t}} = \{G \in \mathbf{Ab} \mid \mathfrak{t}(G) = G\}$ and $\mathbf{Fr}^{\mathfrak{t}} = \{G \in \mathbf{Ab} \mid \mathfrak{t}(G) = \{0_G\}\}$. Then one can define the neighbourhood operator $\mu^{\mathfrak{t}}$ as follow

$$\mu^{\mathfrak{t}}_G(H) = \{K \mid H \leq N \leq K \text{ for some } N \text{ with } G/N \in \mathbf{Tor}^{\mathfrak{t}}\}$$

The interior operator $\rho(\mu^{\mathfrak{t}})$ is described in [6].

(c) A *directed graph* is a set X unto which is given a binary relation $\to$. The elements of X are called *vertices* and the *edges* of X are pairs $(x, y) \in X \times X$ where $x \to y$. A morphism of graphs is a function $f : X \to Y$ that preserves $\to$, i.e., if $x \to y$ in X then $f(x) \to f(y)$ in Y. The category of directed graphs with graph homomorphisms is denoted by **Gph** (cf. [20]). It is provided with a $(\mathcal{E}, \mathcal{M})$-factorisation system by taking the class of embeddings as $\mathcal{M}$ and the class of surjective graph homomorphisms as $\mathcal{E}$. One then defines the following two neighbourhood operators on a graph X

$$\upsilon^*_X(M) = \{N \mid (\forall x \in M), (\forall y \in X \setminus N),\ \text{there is no edge } x \to y\}$$

and

$$(\upsilon_*)_X(M) = \{N \mid (\forall x \in M), (\forall y \in X \setminus N),\ \text{there is no edge } y \to x\}$$

The interior operator defined in [59] is the one which is associated to the meet $\upsilon^* \wedge \upsilon_*$. These neighbourhood operators can be obtained from the so-called *up-closure* and *down-closure* [20] by using complementation as in Example 2.4.1 (a) and (b).

(d) In general, the procedure in Example 2.4.1 (a), (b) and also in (c) above could be extended to a topos provided with a universal closure operator (cf. [13]). The subobject lattices are Heyting Algebras and hence pseudocomplements are provided. Note that in [39], interior operators are introduced on a Grothendieck topos.

Chapter 3

A Quartet or the four Classes of Morphisms

Looking back to the formulations of ν-continuity in Proposition 2.1.4, one could ask, when do we have equalities instead of inclusions? In such cases, special ν-continuous morphisms make their appearance. These are ν-closed, ν-initial, ν-open and ν-final morphisms. Since the primary object of Topology is the study of continuous functions, these four notions of maps become essential if one wants to understand topological constructions. Indeed it is remarkable that many of the standard constructions in Topology such as the Tychonoff topology on a product, the formation of subspaces, quotients, etc. rely on these four notions which offer them a universal character.

These notions are well-known for closure operators. For example the papers [10, 17, 12, 27] and also the monograph [20] make use of these notions and similar ones. On the other hand, at a more general level these are concepts that one could not avoid when considering topological categories. An exhaustive study of these four types of morphisms would exceed the main purpose of this chapter. We mainly give here basic descriptions of these morphisms, establish axioms and define a few notions that are closely related to them and that shall be used to study compactness and connectedness in the following chapters.

3.1 When neighbourhoods preserve limits

Given a particular functor it is natural to ask whether it preserves limits and what it means for this specific functor to preserve limits. We know from Chapter 2, Proposition 1.2.15 the way limits are expressed in $\mathcal{M}$ and how they behave. Because of the ways arrows are formed in $Ras(\mathcal{M})$, limits in this category depend somehow on the arrows of $\mathbf{C}$.

Let I be a set and $\{\mathcal{G}_i \mid i \in I\} \subseteq Ras(\mathcal{M})$. Let us assume that it admits a product $\mathcal{G}$ with projection arrows $p_i^{\triangleleft} : \mathcal{G} \to \mathcal{G}_i$, where $p_i : X \to X_i$, for any $i \in I$. It is clear by definition that $\mathcal{U} \subseteq \mathcal{G}$, where

$$\mathcal{U} = \bigcup\{p_i^{-1}[\mathcal{G}_i] \mid i \in I\}.$$

Our purpose is to show that if $\mathcal{U}$ were a raster then $\mathcal{U} = \mathcal{G}$. Suppose that there are natural arrows $g_i^{\lhd} : \mathcal{K} \to \mathcal{G}_i$, where $g_i : Y \to X_i$, for any $i \in I$. If X is the product of the X_i's, then there is an arrow $h : Y \to X$ unique such that $f_i h = g_i$ for any $i \in I$. Since we have arrows $f_i^{\lhd} : \mathcal{U} \to \mathcal{G}_i$, where $f_i = p_i$, for any $i \in I$, $h^{\lhd}$ is unique such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{K} \arrow[rr, "h^{\lhd}"] \arrow[dr, "g_i^{\lhd}"'] & & \mathcal{U} \arrow[dl, "p_i^{\lhd}"] \\
& \mathcal{G}_i &
\end{tikzcd}$$

Therefore $\mathcal{U} = \mathcal{G}$ and $p_i^{\lhd} = f_i^{\lhd}$. It is clear that under the same assumptions, $\mathcal{U}$ provides the expression of a limit for general diagrams $D : I \to Ras(\mathcal{M})$.

Proposition 3.1.1. *Let $\nu \in NBH(\mathbf{C}, \mathcal{M})$. Let $\{m_i : M_i \to X_i \mid i \in I\} \subseteq \mathcal{M}$ and $m : M \to X$ be its limit with natural arrows $f_i : X \to X_i$, $i \in I$. If ν preserves the limit, i.e. $\nu(m) = \lim \nu(m_i)$, then*

$$\nu_X(m) = \bigcup\{f_i^{-1}[\nu_{X_i}(f_i[m])] \mid i \in I\}.$$

Proof. Let $\mathcal{K} = \bigcup\{f_i^{-1}[\nu_{X_i}(f_i[m])] \mid i \in I\}$. By the property of pullback $m \leq f_i^{-1}[m_i]$ or equivalently $f_i[m] \leq m_i$ for each $i \in I$. We then have

$$f^{-1}[\nu_{X_i}(m_i)] \subseteq f_i^{-1}[\nu_{X_i}(f_i[m])] \text{ for every } i \in I.$$

As ν preserves limits, $\nu_X(m) = \bigcup\{f_i^{-1}[\nu_{X_i}(m_i)] \mid i \in I\}$, and so $\nu_X(m) \subseteq \mathcal{K}$.

Now, since every f_i is ν-continuous, we have $f^{-1}[\nu_{X_i}(f_i[m])] \subseteq \nu_X(m)$, which implies $\mathcal{K} \subseteq \nu_X(m)$. $\square$

Remark 3.1.2. The converse of Proposition 3.1.1 is true under the assumption that the natural arrows $M \to M_i$ for any $i \in I$, belong to the class $\mathcal{E}$ since in that case $m_i = f_i[m]$ for any $i \in I$.

Now, let $f : X \to Y$ be in $\mathbf{C}$ and $n \in \mathcal{M}/Y$. Consider the pullback square

$$\begin{tikzcd}
\cdot \arrow[r, "g"] \arrow[d, "m"'] & \cdot \arrow[d, "n"] \\
\cdot \arrow[r, "f"'] & \cdot
\end{tikzcd}$$

This pullback can be viewed as a pullback in $\mathcal{M} \subseteq \mathbf{C^2}$ (cf. Corollary 1.2.16 (ii) and [20]) by considering the following cube:

$$\begin{tikzcd}
 & \cdot \arrow[rr, "f"] \arrow[dd, dashed, "1_X"] & & \cdot \arrow[dd, "1_Y"] \\
\cdot \arrow[ur, "m"] \arrow[rr, "g"] \arrow[dd, "m"'] & & \cdot \arrow[ur, "n"] \arrow[dd, "n"] & \\
 & \cdot \arrow[rr, dashed, "f"] & & \cdot \\
\cdot \arrow[ur, dashed, "1_X"] \arrow[rr, "f"'] & & \cdot \arrow[ur, "1_Y"'] &
\end{tikzcd}$$

where $m = \lim(n, 1_X, 1_Y)$. Thus if ν preserves pullbacks (finite limits), then

$$\nu(m) = f^{-1}[\nu(n)] \bigcup 1_X^{-1}[\nu(1_X)] \bigcup f^{-1}[\nu(1_Y)].$$

This amounts to writing

$$\nu(f^{-1}[n]) = f^{-1}[\nu(n)].$$

In case f is a monomorphism, we have

$$\nu(m) = \nu(f^{-1}[f[m]]) = f^{-1}[\nu(f[m])].$$

For a neighbourhood operator ν, preserving finite limits is already a strong condition as evidenced by the following definitions.

Definition 3.1.3. Let $\nu \in NBH(\mathbf{C}, \mathcal{M})$. We say that a morphism in $\mathbf{C}$ is *ν-closed* if any pullback along f is preserved by ν, i.e. for any $n \in \mathcal{M}/Y$

$$\nu_X(f^{-1}[n]) = f^{-1}[\nu_Y(n)].$$

Definition 3.1.4. A morphism $f : X \to Y$ is said to be *ν-initial* for a given neighbourhood operator ν if for any $m \in \mathcal{M}/X$

$$\nu_X(m) = f^{-1}[\nu_Y(f[m])].$$

Definition 3.1.5. A neighbourhood operator ν is said to be *hereditary* if every morphism in $\mathcal{M}$ is ν-initial and it is said to be *productive* if it preserves products.

One can easily check (cf. for example [21]) that this notion of closedness coincides with the classical notion of closed continuous maps in **Top** and that the notion of initiality coincides with the notion of *initial topology*.

Given a neighbourhood operator ν, we denote respectively by $\mathcal{K}(\nu)$ and $\mathcal{I}(\nu)$ the class of ν-closed morphisms and the class of ν-initial morphisms. They behave almost the same way:

Proposition 3.1.6. *Let $\nu \in NBH(\mathbf{C}, \mathcal{M})$. The following statements hold:*

(i) $\mathcal{I}(\nu)$ contains isomorphisms and is closed under composition;

(ii) If $gf \in \mathcal{I}(\nu)$, then necessarily $f \in \mathcal{I}(\nu)$;

(iii) If $gf \in \mathcal{I}(\nu)$ and $f \in \mathcal{E}'$, then $g \in \mathcal{I}(\nu)$.

Proof. (i) is straightforward. (ii) Let m be an appropriate subobject. Since $gf \in \mathcal{I}(\nu)$, we have $\nu(m) = f^{-1}[g^{-1}[\nu((gf)[m])]]$. By ν-continuity of g, we have

$$f^{-1}[g^{-1}[\nu((gf)[m])]] \subseteq f^{-1}[\nu(f[m])] \subseteq \nu(m).$$

(iii) For an appropriate subobject m, we have

$$f^{-1}[\nu(m)] \subseteq \nu(f^{-1}[m]) \subseteq (gf)^{-1}[\nu((gf)[f^{-1}[m]])].$$

Since $f[-] \circ f^{-1}[-] \cong f[f^{-1}[-]] \cong 1[-]$, one has $\nu(m) \subseteq g^{-1}[\nu(g[m])]$. $\square$

Proposition 3.1.7. *Let $\nu \in NBH(\mathbf{C}, \mathcal{M})$. The following statements are true:*

(i) $\mathcal{K}(\nu)$ contains isomorphisms and is closed under composition;

(ii) If $gf \in \mathcal{K}(\nu)$ and g is a monomorphism, then $f \in \mathcal{K}(\nu)$;

(iii) If $gf \in \mathcal{K}(\nu)$ and $f \in \mathcal{E}'$, then $g \in \mathcal{K}(\nu)$.

Proof. (i) is clear. (ii) Since g is a monomorphism, for any appropriate subobject n, $n \cong g^{-1}[g[n]]$. Hence

$$\nu(f^{-1}[n]) = \nu(f^{-1}[g^{-1}[g[n]]]) = f^{-1}[g^{-1}[\nu(g[n])]] \subseteq f^{-1}[\nu(n)].$$

(iii) For an appropriate subobject n, we have

$$f^{-1}[\nu(g^{-1}[n])] \subseteq \nu(f^{-1}[g^{-1}[n]]) \subseteq f^{-1}[g^{-1}[\nu(n)]].$$

Applying $f[-]$ on the left and the right gives us $\nu(g^{-1}[n]) \subseteq g^{-1}[\nu(n)]$. $\square$

Further results are also observed:

Proposition 3.1.8. *The following statements hold for any $\nu \in NBH(\mathbf{C}, \mathcal{M})$:*

(i) Every section (or split monomorphism) is ν-initial;

(ii) $\mathcal{M} \bigcap \mathcal{K}(\nu) \subseteq \mathcal{M}_{\leq} \bigcap \mathcal{K}(\nu) \subseteq \mathcal{I}(\nu)$ and $\mathcal{I}(\nu) \bigcap \mathcal{E}' \subseteq \mathcal{K}(\nu)$;

(iii) If ν is regular, then any subobject $m \in \mathcal{K}(\nu) \bigcap \mathcal{M}$ is $\mathfrak{cl}_2^{\nu}$-closed;

(iv) If $\mathcal{E} \subseteq \mathcal{E}'$, then any $f \in \mathcal{F}(\nu)$ preserves ν-closed subobject, i.e., for any subobject $m \in \mathcal{F}(\nu) \bigcap \mathcal{M}$, $f[m] \in \mathcal{F}(\nu) \bigcap \mathcal{M}$ when $f[m]$ makes sense.

We recall from Corollary 1.1.7 that $\mathcal{M}_{\leq}$ is the class of all monomorphisms in $\mathbf{C}$.

Proof. (i) follows from Proposition 3.1.6 (ii). (ii) Let $m \in \mathcal{F}(\nu) \bigcap \mathcal{M}_{\leq}$, then for any appropriate subobject p

$$\nu(p) = \nu(m^{-1}[mp]) = m^{-1}[\nu(mp)].$$

And if $f \in \mathcal{I}(\nu) \bigcap \mathcal{E}'$ then

$$\nu(f^{-1}[p]) = f^{-1}[\nu(f[f^{-1}[p]])] = f^{-1}[\nu(p)].$$

(iii) Let $m : M \to X \in \mathcal{K}(\nu) \bigcap \mathcal{M}$. Suppose that $m \wedge \nu_X(l) > 0_X$, for a given $l \in \mathcal{M}/X$. We have $\nu_M(m^{-1}[l]) = m^{-1}[\nu_X(l)]$ and so $0_M < k$ for any $k \in \nu_M(m^{-1}[l])$. If ν is regular, then $m^{-1}[l] > 0_M$ and $m \wedge l > 0_X$.

(iv) follows from Proposition 3.1.7 (i) and (iii). □

It is not known whether or not in general the class $\mathcal{K}(\nu) \bigcap \mathcal{M}$ is pullback stable. However, we can always consider the class $\mathcal{K}(\nu)^*$, which is the largest pullback stable class in $\mathcal{F}(\nu)$, and consider the class $\mathcal{K}(\nu)^* \bigcap \mathcal{M}$ as the class of *closed embeddings*. We shall come back to this particular class in the next chapter. For now, we note that one can define a closure operator from $\mathcal{K}(\nu)^* \bigcap \mathcal{M}$ by following the procedure for generating closure operators from a subclass of $\mathcal{M}$ in Chapter 2. There is unfortunately no natural way of ordering $\mathcal{K}(\nu)^*$ and $\mathcal{K}(\nu')^*$ if $\nu \leq \nu'$ in $NBH(\mathbf{C}, \mathcal{M})$.

3.2 Finality and openness

Dual notions of closedness and initiality follow naturally. It is worth pointing out that the class $\mathcal{M}$ is not closed under colimit in general. For example it is not closed under pushouts in **Set** and **Grp**. Therefore the question of preservation of colimits by a neighbourhood operator is not of importance. However, since the process of taking images is adjoint to that of taking pullbacks in $\mathcal{M}$ – hence limits – the dual notions of closed morphisms and initial morphisms are obtained by assuming that neighbourhoods commute with images, i.e., for any f in $\mathbf{C}$ and an appropriate subobject m

$$\nu(f[m]) = f[\nu(m)]$$

As one might have expected, this equality captures the notion of *open morphism.*

Definition 3.2.1. Let $\nu \in NBH(\mathbf{C}, \mathcal{M})$. We say that a morphism $f : X \to Y$ in $\mathbf{C}$ is *ν-open* if ν preserves any image by f, i.e., for any $m \in \mathcal{M}/X$

$$\nu_Y(f[m]) = f[\nu_X(m)].$$

When the morphism f belongs to the class $\mathcal{E}'$, we have the notion of *final morphism.*

Definition 3.2.2. Let $\nu \in NBH(\mathbf{C}, \mathcal{M})$. We say that a morphism $f : X \to Y$ in $\mathbf{C}$ is *ν-final* if for any $n \in \mathcal{M}/Y$

$$\dot{n} \bigcap f[\nu_X(f^{-1}[n])] = \nu_Y(n),$$

where $\dot{n} = \{k \in \mathcal{M} \mid k \geq n\}$.

Definition 3.2.3. A neighbourhood operator ν is said to be *cohereditary* if every morphism in $\mathcal{E}$ is ν-final.

We note that when $f : X \to Y$ is ν-open, it amounts to saying that for any subobject $m \in \mathcal{M}/X$, if $k \in \nu_X(m)$ then necessarily $f[k] \in \nu_Y(f[m])$. We recall that we always have the relation $\nu_Y(f[m]) \subseteq f[\nu_X(m)]$ by ν-continuity.

In the definition of *finality*, one cannot write $f[\nu_X(f^{-1}[n])] = \nu_Y(n)$ because of the condition that $m \leq p$ for any $p \in \nu(m)$, unless $f \in \mathcal{E}'$. Examples in **Top** show that this does not hold.

Given a neighbourhood operator ν, we denote respectively by $\mathcal{O}(\nu)$ and by $\mathcal{F}(\nu)$ the class of all ν-open morphisms and the class of all ν-final morphisms. As their duals, they also behave in almost the same way:

Proposition 3.2.4. *Let $\nu \in NBH(\mathbf{C}, \mathcal{M})$. The following statements hold:*

(i) $\mathcal{F}(\nu)$ contains isomorphisms and is closed under composition;

(ii) If $gf \in \mathcal{F}(\nu)$, then necessarily $g \in \mathcal{F}(\nu)$;

(iii) If $gf \in \mathcal{F}(\nu)$ and g is a monomorphism, then $f \in \mathcal{F}(\nu)$.

We note first that for any $f : X \to Y$ in $\mathbf{C}$ and $n \in \mathcal{M}/Y$, we have

$$\dot{n} \bigcap f[\nu_X(f^{-1}[n])] = \{k \geq n \mid f^{-1}[k] \in \nu_X(f^{-1}[n])\}.$$

Proof. (i) is straightforward. Let $f : X \to Y$ and $g : Y \to Z$ be in $\mathbf{C}$. (ii) Let $n \in \mathcal{M}/Z$. Suppose that $g^{-1}[k] \in \nu_Y(g^{-1}[n])$. Since f is ν-continuous, we have $(gf)^{-1}[k] \in \nu_Y((gf)^{-1}[n])$. If $k \geq n$ then $k \in \nu_Z(n)$ since gf is ν-final.

(iii) Let $k \geq n$ in $\mathcal{M}/Y$ and assume that $f^{-1}[k] \in \nu_X(f^{-1}[n])$. Note that we have $k \cong g^{-1}[gk]$ and $n \cong g^{-1}[gn]$ since g is a monomorphism. Replacing k and n, we have $(gf)^{-1}[gk] \in \nu_X((gf)^{-1}[gn])$. Since $gk \geq gn$ and gf is ν-final, $gk \in \nu_Z(gn)$. Finally since g is ν-continuous, $k \in \nu_Y(n)$. □

Proposition 3.2.5. *Let $\nu \in NBH(\mathbf{C}, \mathcal{M})$. The following statements are true:*

(i) $\mathcal{O}(\nu)$ contains isomorphisms and is closed under composition;

(ii) If $gf \in \mathcal{O}(\nu)$ and g is a monomorphism, then $f \in \mathcal{O}(\nu)$;

(iii) If $gf \in \mathcal{O}(\nu)$ and $f \in \mathcal{E}'$, then $g \in \mathcal{O}(\nu)$.

Proof. (i) is straightforward. Now, let $f : X \to Y$ and $g : Y \to Z$ be in $\mathbf{C}$.

(ii) Let $m \in \mathcal{M}/X$. Since gf is ν-open, we have

$$(gf)[\nu_X(m)] = \nu_Z((gf)[m]) \subseteq g[\nu_Y(f[m])].$$

Applying $g^{-1}[-]$ on the left and on the right gives us $f[\nu_X(m)] \subseteq \nu_Y(f[m])$.

(iii) Let $m \in \mathcal{M}/Y$. We have

$$g[\nu_Y(m)] = g[\nu_Y(f[f^{-1}[m]])] \subseteq (gf)[\nu_X(f^{-1}[m])].$$

By assumption we have $(gf)[\nu_X(f^{-1}[m])] = \nu_Z((gf)[f^{-1}[m]]) = \nu_Z(g[m])$. □

We also have the following observations.

Proposition 3.2.6. *The following statements hold for a neighbourhood operator ν:*

(i) Every retraction (or split epimorphism) is ν-final;

(ii) $\mathcal{E}' \bigcap \mathcal{O}(\nu) \subseteq \mathcal{F}(\nu)$ and $\mathcal{F}(\nu) \bigcap \mathcal{M}_{\leq} \subseteq \mathcal{O}(\nu)$;

(iii) If $\mathcal{E} \subseteq \mathcal{E}'$, then any ν-open subobject is preserved by any ν-open morphism, i.e., for any $m \in \mathcal{O}(\nu) \bigcap \mathcal{M}$ and $f \in \mathcal{O}(\nu)$, $f[m] \in \mathcal{O}(\nu) \bigcap \mathcal{M}$ when it makes sense.

Proof. (i) follows from Proposition 3.2.4 (ii). (iii) follows from Proposition 3.2.5 (i) and (iii).

(ii) Note that $f[f^{-1}[-]] \cong 1_Y[-]$ if $f : X \to Y \in \mathcal{E}'$. Hence if $f^{-1}[k] \in \nu_X(f^{-1}[m])$ in $\mathcal{M}/Y$ then $k \in \nu_Y(m)$. If $m : M \to X \in \mathcal{F}(\nu) \bigcap \mathcal{M}_{\leq}$, then for any $n \in \mathcal{M}/M$, the relations

$$k \in \nu_M(n) \text{ and } mk \in \nu_X(mn)$$

are equivalent to each other. □

The relationships between the previous four classes can still be further studied and separately investigated. Though these are of relative importance, we have only mentioned a few here, namely those that present obvious symmetries and which will be needed in the sequel.

3.3 Pullback Ascent and Descent

In a number of cases, we are mostly interested in the pullback stability of the four classes of morphisms which have just been defined. Under conditions that are surprisingly very similar to that assumed for closure operators (cf. [27, 12] and [17]), these classes are pullback stable. However, it is clear that these assumptions are sometimes stronger then necessary.

Theorem 3.3.1. ***Pullback Ascent and Descent****. Let $\nu \in NBH(\mathbf{C}, \mathcal{M})$ and let us consider the following pullback diagram*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

If (BCP) holds, i.e. $\mathcal{E} \subseteq \mathcal{E}^*$, then the following statements are true:

(i) If $a \in \mathcal{I}(\nu)$, then $g \in \mathcal{I}(\nu)$ (resp. $\mathcal{K}(\nu)$, $\mathcal{F}(\nu)$, $\mathcal{O}(\nu)$) provided that $f \in \mathcal{I}(\nu)$ (resp. $\mathcal{K}(\nu)$, $\mathcal{F}(\nu)$, $\mathcal{O}(\nu)$);

(ii) If $b \in \mathcal{F}(\nu)$, then $f \in \mathcal{I}(\nu)$ (resp. $\mathcal{K}(\nu)$, $\mathcal{F}(\nu)$, $\mathcal{O}(\nu)$) provided that $g \in \mathcal{I}(\nu)$ (resp. $\mathcal{K}(\nu)$, $\mathcal{F}(\nu)$, $\mathcal{O}(\nu)$).

Proof. (i) Assume that a is ν-initial.

If f is ν-initial then so is g by Proposition 3.1.6 (ii). If $f \in \mathcal{K}(\nu)$ and $n \in \mathcal{M}/B$, then

$$\begin{aligned} \nu_A(g^{-1}[n]) &= a^{-1}[\nu_X(a[g^{-1}[n]])] \quad \text{(Initiality of } a) \\ &= a^{-1}[\nu_X(f^{-1}[b[n]])] \quad \text{(BCP)} \\ &= a^{-1}[f^{-1}[\nu_Y(b[n])]] \quad \text{(Closedness of } f) \\ &= g^{-1}[b^{-1}[\nu_Y(b[n])]] \subseteq g^{-1}[\nu_B(n)]. \quad \text{((BCP) and } \nu\text{-continuity.)} \end{aligned}$$

If $f \in \mathcal{O}(\nu)$, then for any $n \in \mathcal{M}/A$

$$\begin{aligned} g[\nu_A(n)] &= g[a^{-1}[\nu_X(a[n])]] \quad \text{(Initiality of } a) \\ &= b^{-1}[f[\nu_X(a[n])]] \quad \text{(BCP)} \\ &= b^{-1}[\nu_Y(f[a[n]])] \quad \text{(Openness of } f) \\ &\subseteq \nu_B(b^{-1}[f[a[n]]]) \quad (\nu\text{-continuity}) \\ &\subseteq \nu_B(g[a^{-1}[a[n]]]) \subseteq \nu_B(g[n]) \quad \text{((BCP) and } \nu\text{-continuity.)} \end{aligned}$$

Now assume that $f \in \mathcal{F}(\nu)$ and let $k \geq m$ in $\mathcal{M}/B$ such that $g^{-1}[k] \in \nu_A(g^{-1}[m])$. Thus $g^{-1}[k] \in a^{-1}[\nu_X(a[g^{-1}[m]])] = a^{-1}[\nu_X(f^{-1}[b[m]])]$. There is $l \in \nu_X(f^{-1}[b[m]])$ such that $a^{-1}[l] \leq g^{-1}[k]$. By assumption $f[l] \in \nu_Y(b[m])$ and so $b^{-1}[f[l]] \in \nu_B(m)$. But $b^{-1}[f[l]] = g[a^{-1}[l]] \leq g[g^{-1}[k]] \leq k$.

(ii) Assume that b is ν-final. If g is also ν-final then so is f by Proposition 3.2.4 (ii).

- Let $g \in \mathcal{K}(\nu)$ and $n \in \mathcal{M}/Y$. let $p \in \nu_X(f^{-1}[n])$. We have

$$a^{-1}[\nu_X(f^{-1}[n])] \subseteq \nu_A(a^{-1}[f^{-1}[n]]) = \nu_A(g^{-1}[b^{-1}[n]]) \subseteq g^{-1}[\nu_B(b^{-1}[n])].$$

Thus if $p \in \nu_X(f^{-1}[n])$, then there is $l \in \nu_B(b^{-1}[n])$ such that $g^{-1}[l] \leq a^{-1}[p]$. Since b is ν-final $b[l] \in \nu_Y(n)$, hence $f^{-1}[b[l]] \in f^{-1}[\nu_Y(n)]$. But

$$f^{-1}[b[l]] = a[g^{-1}[l]] \leq a[a^{-1}[p]] \leq p.$$

- Suppose now that $g \in \mathcal{O}(\nu)$ and let $k \in \nu_X(n)$. We have $g[a^{-1}[k]] \in \nu_B(g[a^{-1}[n]])$. But $g[a^{-1}[k]] = b^{-1}[f][k]$ and $g[a^{-1}[n]] = b^{-1}[f][n]$. By assumption $f[k] \in \nu_Y(f[n])$.

- Finally assume that g is ν-initial and let $m \in \mathcal{M}/X$ with $k \in \nu_X(m)$. We have

$$a^{-1}[\nu_X(m)] \subseteq \nu_A(a^{-1}[m]) = g^{-1}[\nu_B(g[a^{-1}[m]])] = g^{-1}[\nu_B(b^{-1}[f[m]])].$$

There is $l \in \nu_B(b^{-1}[f[m]])$ such that $g^{-1}[l] \leq a^{-1}[k]$. Since b is ν-final, $b[l] \in \nu_Y(f[m])$. This implies that $k \in f^{-1}[\nu_Y(f[m])]$. $\square$

Corollary 3.3.2. *Let $\nu \in NBH(\mathbf{C}, \mathcal{M})$.*

(i) In the following pullback diagram,

$$\begin{tikzcd} f^{-1}[N] \arrow[r, "f'"] \arrow[d, "n'"'] & N \arrow[d, "n"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where $n \in \mathcal{M}$, if ν is hereditary or n' is a section then the restriction $f' \in \mathcal{F}(\nu)$ (resp. $\mathcal{O}(\nu)$, $\mathcal{K}(\nu)$, $\mathcal{I}(\nu)$) provided that $f \in \mathcal{F}(\nu)$ (resp. $\mathcal{O}(\nu)$, $\mathcal{K}(\nu)$, $\mathcal{I}(\nu)$.)

(ii) In the following pullback diagram,

$$\begin{tikzcd} \cdot \arrow[r, "f'"] \arrow[d, "p'"'] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

where $p \in \mathcal{E}$, if ν is cohereditary or p is a retraction then the morphism $f \in \mathcal{F}(\nu)$ (resp. $\mathcal{O}(\nu)$, $\mathcal{K}(\nu)$, $\mathcal{I}(\nu)$) provided that the restriction $f' \in \mathcal{F}(\nu)$ (resp. $\mathcal{O}(\nu)$, $\mathcal{K}(\nu)$, $\mathcal{I}(\nu)$.)

3.4 New neighbourhoods from old ones

We know from the previous chapter that the class $NBH(\mathbf{C}, \mathcal{M})$, as a large complete lattice, has arbitrary joins and arbitrary meets. Thus for any family $\{\nu_i \mid i \in I\}$, the supremum is provided by

$$\nu^*(m) = \bigcup\{\nu_i(m) \; |i \in I\}, \quad \text{for all } m \in \mathcal{M},$$

and the infimum is given by

$$\nu_*(m) = \bigcap\{\nu_i(m) \; |i \in I\}, \quad \text{for all } m \in \mathcal{M}.$$

In this section, we make a further step and generalise this way of generating a new neighbourhood operator. The consequences of this generalisation are left for the next chapter.

Let $\nu \in NBH(\mathbf{C}, \mathcal{M})$ and let $m \in \mathcal{M}/Y$ with $Y \in \mathbf{C}$. We consider the following two collections:

$$\omega_Y^{X,\nu}(m) = \{k \mid (\exists f : Y \to X)(\exists n \in \nu_X(f[m])), \; k \geq f^{-1}[n]\}$$

and

$$\phi_Y^{X,\nu}(m) = \{k \geq m \mid (\forall f : X \to Y),\ f^{-1}[k] \in \nu_X(f^{-1}[m])\}.$$

We note that if $Hom(Y, X) := \{f : Y \to X \mid f \text{ in } \mathbf{C}\}$, then

$$\omega_Y^{X,\nu}(m) = \bigcup\{f^{-1}[\nu_X(f[m])] \mid f \in Hom(Y, X)\}.$$

The above procedure gives rise to two special neighbourhood operators.

Proposition 3.4.1. *[51] For any neighbourhood operator ν, the two families*

$$\{\omega_Y^{X,\nu} \mid Y \in \mathbf{C}\} \text{ and } \{\phi_Y^{X,\nu} \mid Y \in \mathbf{C}\}$$

give rise to two neighbourhood operators $\omega^{X,\nu}$ and $\phi^{X,\nu}$.

Proof. It suffices to prove the ν-continuity since the two families generate rasters by definition. Let $u : U \to V$ be in $\mathbf{C}$ and $m \in \mathcal{M}/V$. Let $p \in \omega_V^{X,\nu}(m)$. For some morphism $f : V \to X$ in $\mathbf{C}$ and $l \in \nu_X(f[m])$, one has $f^{-1}[l] \leq p$. Note that $\nu_X(f[m]) \subseteq \nu_X((fu)[u^{-1}[m]])$. By definition, we have $(fu)^{-1}[l] \in \omega_U^{X,\nu}(u^{-1}[m])$. Thus $u^{-1}[p] \in \omega_U^{X,\nu}(u^{-1}[m])$. The proof for $\phi^{X,\nu}$ is similar. $\square$

Definition 3.4.2. [51] Given a neighbourhood operator ν and an object $X \in \mathbf{C}$, $\omega^{X,\nu}$ is called the *initial neighbourhood operator* associated with ν and induced by X and $\phi^{X,\nu}$ is called the *final neighbourhood operator* associated with ν and generated by X.

Because of the ν-continuity we always have: $\omega^{X,\nu} \leq \nu \leq \phi^{X,\nu}$. These agree on the object X, i.e., for any $m \in \mathcal{M}/X$, $\omega_X^{X,\nu}(m) = \nu_X(m) = \phi_X^{X,\nu}(m)$.

Proposition 3.4.3. *For any object $X \in \mathbf{C}$, the procedures of forming initial and final neighbourhood operators give rise to functors $\omega^{X,-} : NBH(\mathbf{C}, \mathcal{M}) \to NBH(\mathbf{C}, \mathcal{M})$ and $\phi^{X,-} : NBH(\mathbf{C}, \mathcal{M}) \to NBH(\mathbf{C}, \mathcal{M})$. Furthermore $\omega^{X,-}$ preserves suprema and $\phi^{X,-}$ preserves infima.*

Proof. It is clear to see that if $\nu \leq \nu'$ in $NBH(\mathbf{C}, \mathcal{M})$, then $\omega^{X,\nu} \leq \omega^{X,\nu'}$ and similarly $\phi^{X,\nu} \leq \phi^{X,\nu'}$. For a family $\{\nu_i \mid i \in I\}$, we have

$$\omega^{X,\nu^*} \leq \sup\{\omega^{X,\nu_i} \mid i \in I\},$$

and

$$\inf\{\phi^{X,\nu_i} \mid i \in I\} \leq \phi^{X,\nu_*}.$$

$\square$

On the other hand, we can also vary the object X in many ways and obtain different notions:

Definition 3.4.4. For any full subcategory $\mathcal{B} \subseteq \mathbf{C}$ and ν in $NBH(\mathbf{C}, \mathcal{M})$, the initial neighbourhood operator associated to ν and induced by $\mathcal{B}$ is given by:

$$\omega^{\mathcal{B},\nu} = \sup\{\omega^{X,\nu} \mid X \in \mathcal{B}\},$$

and the final neighbourhood operator generated by $\mathcal{B}$ is given by:

$$\phi^{\mathcal{B},\nu} = \inf\{\phi^{X,\nu} \mid X \in \mathcal{B}\}.$$

We can also define initial neighbourhood operators induced by a sink and generated by a source.

Definition 3.4.5. [51] Let ν be a neighbourhood operator. Let $\mathcal{S} = (f_i : X \to X_i)_{i\in I}$ be a source and $\mathcal{T} = (g_i : X_i \to X)_{i\in I}$ be a sink. The *initial neighbourhood operator induced by* $\mathcal{S}$ is defined by

$$\omega_Y^{\mathcal{S},\nu}(m) = \{k \mid (\exists g : Y \to X)(\exists i \in I)(\exists n \in \nu_{X_i}((f_i g)[m]),\ k \geq (f_i g)^{-1}[n]\},$$

and the *final neighbourhood operator generated by* $\mathcal{T}$ is given by

$$\phi_Y^{\mathcal{T},\nu}(m) = \{n \geq m \mid (\forall g : X \to Y)(\forall i \in I),\ (gf_i)^{-1}[n] \in \nu_{X_i}((gf_i)^{-1}[m])\}.$$

where $m \in \mathcal{M}/Y$ and $Y \in \mathbf{C}$.

Remark 3.4.6. (a) $\omega_X^{\mathcal{S},\nu}$ is the least "ν-structure" on X for which the natural arrows $f_i : X \to X_i$, $i \in I$, are ν-continuous.

(b) We have $\omega^{X,\nu} = \omega^{\{1_X\},\nu}$ and $\phi^{X,\nu} = \phi^{\{1_X\},\nu}$ for any $X \in \mathbf{C}$.

(c) A morphism $f : X \to Y$ is ν-initial (resp. ν-final) if and only if for any subobject $m \in \mathcal{M}/X$, $\nu_X(m) = \omega_X^{\{f\},\nu}(m)$ (resp. for any $n \in \mathcal{M}/Y$, $\nu_Y(n) = \phi_X^{\{f\},\nu}(n)$).

Definition 3.4.7. Let $\mathcal{S} = (f_i : X \to X_i)_{i\in I}$ be a source and $\mathcal{T} = (g_i : X_i \to X)_{i\in I}$ a sink. We say that $\mathcal{S}$ is *jointly ν-initial* or simply *ν-initial* if $\nu_X(m) = \omega_X^{\mathcal{S},\nu}(m)$ for every $m \in \mathcal{M}/X$ and we say that $\mathcal{T}$ is *jointly ν-final* if $\nu_X(n) = \phi_X^{\mathcal{T},\nu}(n)$ for every $n \in \mathcal{M}/X$.

The motivation for this definition comes from the following proposition.

Proposition 3.4.8. *Let $\mathcal{S} = (f_i : X \to X_i)_{i \in I}$ be a source and assume that the X_i's admit a product P with natural arrows $p_i : P \to X_i$, $i \in I$. If $\mathcal{S}$ is jointly ν-initial for $\nu \in NBH(\mathbf{C}, \mathcal{M})$, then the arrow $h : X \to P$, unique such that $p_i h = f_i$ for any $i \in I$, belongs to $\mathcal{I}(\nu)$. The converse is true if the projection arrows themselves are jointly ν-initial.*

Proof. The proof is essentially the same as in Proposition 3.1.6 (ii) and follows from Proposition 3.4.3. □

It is clear that Proposition 3.2.4 (ii) provides a dual of Proposition 3.4.8.

Examples 3.4.9. [51]

(i) In the category **Top** with the usual neighbourhood operator τ, if we consider the one-point space $X = (\{*\}, \tau_X)$, then $\omega^{X,\tau}$ is the neighbourhood operator that gives the indiscrete topology. Indeed for all topological spaces (Y, τ_Y)

$$\omega_Y^{X,\tau}(\emptyset) = \{\emptyset, Y\} \text{ and } \omega_Y^{X,\tau}(A) = \{Y\} \text{ if } A \neq \emptyset.$$

(ii) In the same category **Top** and the same neighbourhood operator τ, if we consider the empty space $(\emptyset, \{\emptyset\})$, then $\phi^{\emptyset,\tau}$ is the neighbourhood operator that gives the discrete topology:

$$\phi_Y^{\emptyset,\tau}(A) = \{B \mid A \subseteq B \subseteq Y\}.$$

These examples motivate the following definitions.

Definition 3.4.10. [51] The initial neighbourhood operator associated to a neighbourhood operator ν and induced from the terminal object $\mathbf{1}$ is called the *coarse neighbourhood operator* associated to ν. It is denoted by $\mathscr{C}(\nu)$. The final neighbourhood operator associated to a neighbourhood operator ν and generated by the initial object (if it does exist) I is called the *fine neighbourhood operator* associated to ν. It is denoted by $\mathscr{F}(\nu)$.

If I is an initial object in $\mathbf{C}$, then every unique arrow $\iota_X : I \to X$, where $X \in \mathbf{C}$ can be factored as follows

$$\begin{tikzcd} I \arrow[rr, "\iota_X"] \arrow[dr, "e"'] & & X \\ & \mathbf{0}_X \arrow[ur, "0_X"'] & \end{tikzcd}$$

Hence if $m \in \mathcal{M}/X$, then $\iota_X^{-1}[m] = e^{-1}[0_X]$. On the other hand, since I is an initial object $0_I = 1_I$. Therefore $\mathscr{F}(\nu)(m) = \dot{m}$ for all $\nu \in NBH(\mathbf{C}, \mathcal{M})$. The axioms with which $\mathbf{C}$ is provided do not guarantee that I exists and in the case there is no initial object, the relation $\mathscr{F}(\nu)(m) = \dot{m}$ is vacuously true. In the sequel, we shall simply write $\mathscr{F}(m)$ instead of $\mathscr{F}(\nu)(m)$.

3.5 Remarks on initial interior operators

There is clearly a difference between assuming that a morphism $f : X \to Y$ is ν-initial and forming $\omega^{\{f\},\nu}$ for a regular neighbourhood operator ν. If $f \in \mathcal{I}(\nu)$, then the structure ν_X satisfies (O) by definition, while $\omega_X^{\{f\},\nu}$ might not. In other words the functor $\omega^{\{f\},-}$ restricted to $RegNBH(\mathbf{C}, \mathcal{M})$ may fail to be a subfunctor. In general, for any source $\mathcal{S}$ we want to know if $\omega^{\mathcal{S},\nu}$ satisfies (I) and (F) if ν satisfies these axioms. Let us first observe that the neighbourhood operator $\omega^{X,\nu}$, for $X \in \mathbf{C}$, formed in Proposition 3.4.1, is obtained from the collection

$$\{\omega^{\{f\},\nu} \mid cod(f) = X\}.$$

The expressions for $\omega^{\mathcal{S},\nu}$ and $\omega^{X,\nu}$ are then motivated by the expression of the supremum in $NBH(\mathbf{C}, \mathcal{M})$, which we have denoted by ν^*. Depending on which of the properties (I), (O) and (F) ν satisfies, $\omega^{\mathcal{S},\nu}$ and $\omega^{X,\nu}$ will have different expressions, namely they are given by the expression of the supremum and should be formally defined as

$$\omega^{\mathcal{S},\nu} = \sup\{\omega^{\{f\},\nu} \mid f \in \mathcal{S}\}$$

and

$$\phi^{\mathcal{S},\nu} = \inf\{\phi^{\{f\},\nu} \mid f \in \mathcal{S}\},$$

where the supremum and infimum are respectively taken in $RegNBH(\mathbf{C}, \mathcal{M})$, $INBH(\mathbf{C}, \mathcal{M})$ and $NBHF(\mathbf{C}, \mathcal{M})$.

In any case, we shall still denote by $\omega^{\mathcal{S},\nu}$ the initial neighbourhood operator induced by the source $\mathcal{S}$ irrespective of the properties of ν and irrespective of the formula that it should have.

For the cases where ν satisfies (I) and/or (F) the supremum can be explicitly written thanks to Lemma 2.1.7 and Corollary 2.1.9. On the other hand Proposition 2.2.3 and 2.2.4, (also Corollary 2.1.9) give partial answers for the case where ν satisfies (O). These however for the most part assume that pre-images commute with joins of subobjects and this condition as we shall see in the next theorem is close to unavoidable. Nonetheless this situation is not as unfortunate as it looks. Indeed since our role model for $\mathbf{C}$ is **Top**, even without having recourse to that condition, the structure $\omega_X^{\mathcal{S},\nu}$ for any space X, is obtained as in Corollary 2.1.9 since τ and $\omega^{\mathcal{S},\tau}$ satisfy (I) and are complete. On the other hand, throughout the thesis we are mostly concerned with results about $NBH(\mathbf{C}, \mathcal{M})$. Also we keep using the formula of the suprema and infima in $NBH(\mathbf{C}, \mathcal{M})$. In the sequel whenever we need to focus on $RegNBH(\mathbf{C}, \mathcal{M})$, we shall make a brief remark.

The case for $\phi^{\mathcal{S},\nu}$ does not give any problem because the meet of a family of regular neighbourhood operators is formed as in $NBH(\mathbf{C},\mathcal{M})$, i.e. it is given by ν_*. The main problem is then to evaluate any single $\phi^{\{f\},\nu}$. In the following theorem we establish an equivalence regarding the building blocks $\omega^{\{f\},\nu}$ and $\phi^{\{f\},\nu}$.

Theorem 3.5.1. *(i) If $\omega^{\{f\},\nu}$ is regular for any morphism $f : X \to Y$ in $\mathbf{C}$ and any regular neighbourhood operator ν, then every pre-image of a morphism in $\mathcal{E}'$ commutes with joins of subobjects.*

(ii) If $\phi^{\{f\},\nu}$ is regular for any morphism f in $\mathbf{C}$ and any regular neighbourhood operator ν, then every pre-image of a morphism in $\mathcal{M}$ commutes with joins of subobjects.

(iii) If every pre-image of a morphism commutes with joins of subobjects, then both $\omega^{\{f\},\nu}$ and $\phi^{\{f\},\nu}$ are regular provided that ν is regular.

Proof. (i) Let $e : X \to Y$ be in $\mathcal{E}'$ and assume that $\omega^{\{f\},\nu}$ is regular for any regular neighbourhood operator ν and for any morphism f. Consider the fine neighbourhood operator $\mathscr{F}$. $\mathscr{F}$ is regular and for any $m \in \mathcal{M}/X$ we have

$$\omega_X^{\{e\},\mathscr{F}}(m) = e^{-1}[\mathscr{F}_Y(e[m])] = \{k \mid k \geq e^{-1}[e[m]]\}.$$

Now, if $\{n_i \mid i \in I\} \subseteq \mathcal{M}/Y$, then (cf. notation in Proposition 2.2.4)

$$\omega_X^{\{e\},\mathscr{F}}(\textstyle\bigvee_I e^{-1}[n_i]) = \{k \mid k \geq e^{-1}[e[\bigvee_I e^{-1}[n_i]]]\}.$$

But $e[\bigvee_I e^{-1}[n_i]] = \bigvee_I e[e^{-1}[n_i]] = \bigvee_I n_i$. Hence

$$\omega_X^{\{e\},\mathscr{F}}(\textstyle\bigvee_I e^{-1}[n_i]) = \{k \mid k \geq e^{-1}[\bigvee_I n_i]\}.$$

On the other hand, since $\omega^{\{e\},\mathscr{F}}$ is regular by assumption,

$$\textstyle\bigvee_I e^{-1}[n_i] \in \omega_X^{\{e\},\mathscr{F}}(\bigvee_I e^{-1}[n_i])$$

because $e^{-1}[n_i] \in \omega_X^{\{e\},\mathscr{F}}(e^{-1}[n_i])$ for each $i \in I$. Therefore we have

$$\textstyle\bigvee_I e^{-1}[n_i] \geq e^{-1}[\bigvee_i n_i].$$

(ii) The proof is essentially the same as in (i). Let $m : M \to X$ be in $\mathcal{M}$. Consider again the particular neighbourhood operator $\mathscr{F}$ and let $\{n_i \mid i \in I\} \subseteq \mathcal{M}/X$. For any $p \in \mathcal{M}/X$ we have

$$\phi_X^{\{m\},\mathscr{F}}(p) = \{l \geq p \mid m^{-1}[l] \geq m^{-1}[p]\}.$$

Note that $m^{-1}[m[\bigvee_i m^{-1}[n_i]]] = \bigvee_I m^{-1}[n_i]$ and that $m^{-1}[n_i] \in \mathscr{F}_M(m^{-1}[n_i])$ for any $i \in I$. Thus we have

$$m[\bigvee_I m^{-1}[n_i]] \in \phi_X^{\{m\},\mathscr{F}}(n_i) \text{ for each } i \in I.$$

By assumption $m[\bigvee_I m^{-1}[n_i]] \in \phi_X^{\{m\},\mathscr{F}}(\bigvee_I n_i)$ which means

$$\bigvee_I m^{-1}[n_i] \in \mathscr{F}_M(m^{-1}[\bigvee_I n_i]) \text{ or } m^{-1}[\bigvee_I n_i] \leq \bigvee_i m^{-1}[n_i].$$

(iii) The result follows from a straightforward verification. □

We point out that in Theorem 3.5.1 (i), if the hypothesis $e \in \mathcal{E}'$ is removed, then one trivially obtains the following relation

$$\bigvee_i e^{-1}[n_i] \geq e^{-1}[\bigvee_I e[e^{-1}[n_i]]],$$

which is in fact an equality, instead of $\bigvee_I e^{-1}[n_i] \geq e^{-1}[\bigvee_i n_i]$.

Also assuming that every pre-image $m^{-1}[-]$ induced by a morphism m in $\mathcal{M}$ commutes with joins of subobjects is the same as assuming that any subobject lattice $\mathcal{M}/X$ is a frame for any $X \in \mathbf{C}$ since the relations

$$m^{-1}[\bigvee_I n_i] = \bigvee_i m^{-1}[n_i] \text{ and } m \wedge (\bigvee_I n_i) = \bigvee_I (m \wedge n_i)$$

are equivalent.

One might be tempted to say that a morphism $f : X \to Y$ is i-initial for an interior operator i if for any $m \in \mathcal{M}/X$ the following holds for any $m \in \mathcal{M}/X$

$$i_X(m) = f^{-1}[i_Y(f[m])].$$

This might be also motivated by the fact that initial morphisms with respect to a closure operator and neighbourhood operators are defined with similar formulae. However, as is already investigated in [39], the above formula does not give the right notion for initiality. [39] If the pre-image $f^{-1}[-]$ commutes with join, then it admits a right adjoint $f_*[-] : \mathcal{M}/X \to \mathcal{M}/Y$ (as the right adjoint of a frame homomorphism.) f is then said to be i-initial whenever for any $m \in \mathcal{M}/X$

$$i_X(m) = f^{-1}[i_Y(f_*[m])].$$

Indeed denote by $\nu = \sigma(i)$ the regular neighbourhood operator associated to i. If $m \leq f^{-1}[i_Y(f_*[k])]$ for some $m, k \in \mathcal{M}/X$, then we have $f[m] \leq i_Y(f_*[k])$. Therefore $f_*[k] \in \nu_Y(f[m])$ and so $f^{-1}[f_*[k]] \in f^{-1}[\nu_Y(f[m])]$. But $f^{-1}[f_*[k]] \leq k$, hence

$k \in f^{-1}[\nu_Y(f[m])]$. Conversely if $k \in f^{-1}[\nu_Y(f[m])]$, then there is $p \in \nu_Y(f[m])$ such that $f^{-1}[p] \leq k$. Since $f[m] \leq i_Y(p) \leq i_Y(f_*[f^{-1}[p]])$, by adjunction we have

$$m \leq f^{-1}[i_Y(f_*[f^{-1}[p]])] \leq f^{-1}[i_Y(f_*[k])].$$

The same remark applies for finality though this notion is not clearly found in [39]. The same morphism f is final if for any $n \in \mathcal{M}/Y$ one has

$$f_*[i_X(f^{-1}[n])] = i_Y(n).$$

One observes that $m \leq f_*[i_X(f^{-1}[n])]$ if and only if $f^{-1}[m] \leq i_X(f^{-1}[n])$.

We noted at the beginning of this section that there is a difference between assuming that f is initial and forming the initial neighbourhood operator $\omega^{\{f\},\nu}$. Here considering neighbourhoods presents a little advantage as one could consider initiality without always referring to the right adjoint $f_*[-]$. On the other hand it is clear that the presence of this right adjoint opens new perspectives.

Chapter 4

Connectedness

Certainly there are varied ways to study connectedness in a category. In Section 4.1, we have tried to define this notion using dualities, namely connectedness and its dual notion disconnectedness are *left- and right- constant subcategories*. Using the so-called *coarse and fine objects* (cf. [10, 51]) we place these constant subcategories in adjunctions with $NBH(\mathbf{C}, \mathcal{M})$. All of these are described by initial and final morphisms. In the process we shall introduce the two separation axioms T_0 and T_1. The first section is thus essentially what is already in the paper [51].

In the second section we give an analogue of the notion of *monotone morphisms* considered in [54] for closure operators. We define at this stage what are *dense objects* with respect to a neighbourhood operator. The advantage here when compared to the former (cf. [54]) is that the notions of denseness and finality allow fewer restrictions.

4.1 Constant morphisms via constant objects

The idea of using *left- and right- constant subcategories* to describe classes of connectedness and disconnectedness in the sense of [2] goes back to the works of Preuß and Herrlich [47, 48, 29], and is also considered in [16, 7] and [10] for closure operators. This is done thanks to constant morphisms. Studies of torsion and torsion-free subcategories in abelian categories have helped in establishing a notion of constant morphisms which is self-dual (cf. [9]). This smooth transition to the dual category $\mathbf{C}^{op}$ makes this notion of constant morphisms more convenient.

[9] In an abelian category, constant morphisms are exactly the zero morphisms. In other words, they are those morphisms that can be factored through a zero object, say $\mathbf{0}$. Considering a class $\mathscr{K}$ of objects that imitates $\mathbf{0}$, one says that a morphism f is constant if it can be factored through a *constant object*, i.e. a member of $\mathscr{K}$.

Our choice of constant object is mainly motivated by the "size" of the objects that should be in $\mathscr{K}$.

Let us consider the following collection:

$$\mathscr{K} = \{X \in \mathbf{C} \mid t_X^{-1}[t_X[m]] = m \text{ for all } m \in \mathcal{M}/X\},$$

where t_X is the terminal morphism for any $X \in \mathbf{C}$. $\mathscr{K}$ fulfills all the conditions required for constant objects in [9] only when $\mathcal{E} \subseteq \mathcal{E}'$.

Lemma 4.1.1. *(i) If $K \in \mathscr{K}$ and $m : M \to K \in \mathcal{M}$, then $M \in \mathscr{K}$;*

(ii) If $K \in \mathscr{K}$ and $e : K \to P \in \mathcal{E}'$ then $P \in \mathscr{K}$.

Proof. The statement (i) follows from the fact that any $m \in \mathcal{M}/M$ satisfies also $m^{-1}[m[k]] = k$ for all $k \in \mathcal{M}/M$;

(ii) Consider the following commutative diagram:

$$\begin{tikzcd}
K \arrow[rr, "e"] \arrow[dr, "t_K"'] & & P \arrow[dl, "t_P"] \\
& \mathbf{1} &
\end{tikzcd}$$

and let $p \in P$. We have $e^{-1}[p] \in \mathcal{M}/K$ and

$$t_K^{-1}[t_K[e^{-1}[p]]] \cong e^{-1}[p].$$

By observing that $t_K^{-1}[-] \cong e^{-1}[t_P^{-1}[-]]$ and that $e[e^{-1}[l]] = l$ for any subobject $l \in \mathcal{M}/P$, we have $t_P^{-1}[t_P[p]] = p$. $\square$

Let $f : X \to Y$ be in $\mathbf{C}$ and assume that it can be factored through $K \in \mathscr{K}$. Hence $f = lh$ for some $h : X \to K$ and $l : K \to Y$ in $\mathbf{C}$, with $K \in \mathscr{K}$. Taking respectively the $(\mathcal{E}, \mathcal{M})$-factorisation of f, h and l one has the following commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "e_h"] \arrow[ddrr, "e"'] & & K \arrow[rr, "l"] \arrow[dr, "e_l"] & & Y \\
& \cdot \arrow[ur, "m_h"] & & \cdot \arrow[ur, "m_l"] & \\
& & f[X] \arrow[ur, dashed, "k"] \arrow[uurr, "m"'] & &
\end{tikzcd}$$

By the property (P) – the diagonalisation property – there is a morphism k unique such that it makes the above diagram commute. Under the assumption in Lemma 4.1.1 (ii), $f[X] \in \mathscr{K}$. Hence, assuming only that f is factored through $f[X] \in \mathscr{K}$ does not always imply that the domain of $f[1_X]$ be in $\mathscr{K}$. This is inconvenient in some situations. To remedy that situation, we ask that the factorisation of a constant morphism through constant objects be an $(\mathcal{E}, \mathcal{M})$-factorisation as follows:

Definition 4.1.2. A map $f : X \to Y$ is $\mathscr{K}$*-constant* or simply *constant* if in the $(\mathcal{E}, \mathcal{M})$-factorisation

$$X \xrightarrow{e} f[X] \xrightarrow{m} Y$$

of f, one has $f[X] \in \mathscr{K}$.

Under the additional condition that $\mathcal{E} \subseteq \mathcal{E}'$, the above definition is trivially equivalent to the one introduced by Clementino in [9]. The following properties of $\mathscr{K}$-constant morphisms, which are also found in [9], show that they are nonetheless well-behaved.

Proposition 4.1.3. *(i) If mf is constant and $m \in \mathcal{M}$, then f is constant;*

(ii) Let $e \in \mathcal{E}$. Any arrow f is constant if and only if fe is constant;

(iii) If $\mathcal{E} \subseteq \mathcal{E}'$, then hfg is constant provided that f is constant.

Proof. (i) Let $f = m_f e_f$ with $m_f \in \mathcal{M}$ and $e_f \in \mathcal{E}$. Let k be the $\mathcal{M}$-part of the composition mf. Since $mf = (mm_f)e_f$ we have $k \cong m.m_f$ and hence m_f and k have isomorphic domains.

(ii) As in the previous proof, let $f = m_f e_f$ and let $m'e'$ be the $(\mathcal{E}, \mathcal{M})$-factorisation of fe. Consider the following diagram:

$$\begin{tikzcd}
\cdot \arrow[rr, "e"] \arrow[dd, "e'"'] & & \cdot \arrow[dl, "e_f"'] \arrow[dd, "f"] \\
& \cdot \arrow[dr, "m_f"] & \\
\cdot \arrow[ur, dashed, "h"] \arrow[rr, "m'"'] & & \cdot
\end{tikzcd}$$

The arrow h arises from the property (P). Since $he' = e_f e \in \mathcal{E}$ and $m' = m_f h \in \mathcal{M}$ we have $h \in \mathcal{E} \bigcap \mathcal{M}$. As a consequence of the property (P) h is an isomorphism.

(iii) follows from Definition 4.1.2 and the observation that precedes it. □

Thus the composition of a morphism with a constant one is a constant morphism. In some sense, the class of constant morphisms behaves like an *ideal*.

Now we introduce the category $S(\mathbf{C})$ of all full subcategories of $\mathbf{C}$, it is a complete large class ordered by inclusion. The reverse order on its dual $S(\mathbf{C})^{op}$ shall be denoted by $\preceq$ to avoid confusion. In the same manner, the reverse order on $NBH(\mathbf{C}, \mathcal{M})^{op}$ is denoted by $\preceq$.

Definition 4.1.4. Given a class $\mathcal{D}$ in $S(\mathbf{C})$, we say that an object X is $\mathcal{D}$-connected if every morphism $f : X \to Y$ in $\mathbf{C}$, with $Y \in \mathcal{D}$, is constant. Dually, we say that X is $\mathcal{D}$- disconnected if every morphism $f : Y \to X$, with $Y \in \mathcal{D}$, is constant.

The following result follows directly from Proposition 4.1.3 (i) and (ii).

Proposition 4.1.5. *(i) Let $f : X \to Y$ be in $\mathbf{C}$ and $\mathcal{D}$ in $S(\mathbf{C})$ and suppose that $f \in \mathcal{E}$. If X is $\mathcal{D}$-connected then so is Y;*

(ii) Let $m : M \to X$ be in $\mathcal{M}$. If X is $\mathcal{D}$-disconnected then so is M.

The Galois connection introduced in [48] is described as follows:

For any $\mathcal{A}$ in $S(\mathbf{C})$ consider:

$$\Delta(\mathcal{A}) = \{X \in \mathbf{C} \mid \text{ If } f : Z \to X \text{ is in } \mathbf{C} \text{ and } Z \in \mathcal{A} \text{ then } f \text{ is constant}\}$$

and

$$\triangledown(\mathcal{A}) = \{X \in \mathbf{C} \mid \text{ If } f : X \to Z \text{ is in } \mathbf{C} \text{ and } Z \in \mathcal{A} \text{ then } f \text{ is constant}\}$$

It is straightforward to see that $\mathcal{A} \subseteq \Delta(\triangledown(\mathcal{A}))$ and $\mathcal{A} \subseteq \triangledown(\Delta(\mathcal{A}))$. Thus $\Delta : S(\mathbf{C}) \to S(\mathbf{C})^{op}$ and $\triangledown : S(\mathbf{C})^{op} \to S(\mathbf{C})$ form a pair of Galois connection where $\triangledown$ is the right adjoint and Δ the left adjoint. Proposition 4.1.5 shows that $\triangledown(\mathcal{A})$ is closed under $\mathcal{E}$-images and that $\Delta(\mathcal{A})$ is closed under $\mathcal{M}$-subobjects.

[7, 10, 8] Subcategories of $S(\mathbf{C})$ can be described by neighbourhood operators by considering *indiscrete* and *discrete objects*. Following the terminologies introduced in [10], we shall respectively term them *coarse* and *fine objects*.

Definition 4.1.6. Given a neighbourhood operator ν in $NBH(\mathbf{C}, \mathcal{M})$, we say that an object $X \in \mathbf{C}$ is ν *-fine* if $\mathscr{F}_X \leq \nu_X$. It is said to be ν*-coarse* if $\nu_X \leq \mathscr{C}(\nu)_X$.

[51] Since for any ν in $NBH(\mathbf{C}, \mathcal{M})$ $\mathscr{C}(\nu)_X \leq \nu_X \leq \mathscr{F}_X$, the above relations are actually equalities. $\mathscr{C}(\nu)$ is not regular in general. For example if we consider the category **Grp** with the regular neighbourhood operator $\mathfrak{n}$, then for all $G \in \mathbf{Grp}$, $\mathscr{C}(\mathfrak{n})_G(0_G) \cong \{G\}$. Therefore there would be no $\mathfrak{n}$-coarse groups. One can assume that an object X is ν-coarse if $\nu_X \leq \rho\mathscr{C}(\nu)_X$ where ρ is the reflector from $NBH(\mathbf{C}, \mathcal{M})$ to $RegNBH(\mathbf{C}, \mathcal{M})$. However, as we shall see in Example 4.1.24, this is no more convenient.

Given a neighbourhood operator ν, the class of all ν-coarse objects is denoted by $\mathfrak{I}(\nu)$ and that of ν-fine objects by $\mathfrak{D}(\nu)$.

$\mathfrak{D}$ can be viewed as a functor $\mathfrak{D} : NBH(\mathbf{C}, \mathcal{M})^{op} \to S(\mathbf{C})^{op}$. Indeed if $\nu \leq \nu'$ in $NBH(\mathbf{C}, \mathcal{M})$ then $\mathfrak{D}(\nu) \subseteq \mathfrak{D}(\nu')$. In the paper [51], the assignment $\mathfrak{I}$ is claimed to be a functor. This is not entirely true unless one has $\nu_{\mathbf{1}} = \mathscr{F}_{\mathbf{1}}$ for all neighbourhood operators ν, i.e. the structure on the terminal object is unique. All the essentials results in [51] already consider this assumption. In this case the functor $\mathscr{C}$, like $\mathscr{F}$, becomes a constant. This is more natural and provides a convenient dual to $\mathscr{F}$. This assumption is not very restrictive since all the examples considered naturally satisfy this condition.

In the sequel we shall assume that $\nu_{\mathbf{1}} = \mathscr{F}_{\mathbf{1}}$ for all neighbourhood operators ν whenever we need the monotonicity of $\mathfrak{I}$. Also we still use the notation $\mathscr{C}(\nu)$ instead of $\mathscr{C}$ in order to show which results rely on this assumption.

Proposition 4.1.7. *$\mathfrak{D}$ admits a right adjoint $\mathfrak{D}_*$. If $\nu_{\mathbf{1}} = \mathscr{F}_{\mathbf{1}}$ for any neighbourhood operator ν, then the assignment $\mathfrak{I} : NBH(\mathbf{C}, \mathcal{M})^{op} \to S(\mathbf{C})$ is monotone and admits a left adjoint $\mathfrak{I}^*$.*

Proof. It is sufficient to show that $\mathfrak{I}$ preserves infima and $\mathfrak{D}$ preserves suprema. Given a family $\mathcal{V} = \{\nu_i \mid i \in I\} \subseteq NBH(\mathbf{C}, \mathcal{M})$, let $\nu^* = \sup \mathcal{V}$ and $\nu_* = \inf \mathcal{V}$.

Is is clear that for any $X \in \mathbf{C}$ $\mathscr{F}_X \leq (\nu_i)_X$ for all $i \in I$ if and only if $\mathscr{F}_X \leq (\nu_*)_X$. Thus $\mathfrak{D}$ preserves any supremum in $NBH(\mathbf{C}, \mathcal{M})^{op}$.

Now, by assumption $\mathfrak{I}$ is monotone, hence $\mathfrak{I}(\nu^*) \subseteq \bigcap\{\mathfrak{I}(\nu_i) \mid i \in I\}$. On the other hand, suppose that $(\nu_i)_X \leq \mathscr{C}(\nu_i)_X$ for all $i \in I$. We have

$$\nu^*_X = (\sup \mathcal{V})_X \leq (\sup\{\mathscr{C}(\nu_i) \mid i \in I\})_X \leq \mathscr{C}(\nu^*)_X.$$

Therefore $\nu^*_X \leq \mathscr{C}(\nu^*)_X$ and X is ν^*-coarse. □

The above result, together with the Galois connections previously described can be pictured in the diagram below (cf. [8]):

$$\begin{tikzcd}
S(\mathbf{C}) \arrow[rr, shift left, "\Delta"] \arrow[dr, shift left, "\mathfrak{I}^*"] & & S(\mathbf{C})^{op} \arrow[ll, shift left, "\nabla"] \arrow[dl, shift left, "\mathfrak{D}_*"] \\
& NBH(\mathbf{C}, \mathcal{M})^{op} \arrow[ul, shift left, "\mathfrak{I}"] \arrow[ur, shift left, "\mathfrak{D}"] &
\end{tikzcd}$$

Proposition 4.1.8. *Suppose that in Proposition 4.1.7 we consider $RegNBH(\mathbf{C}, \mathcal{M})$ and suppose that for all ν in $RegNBH(\mathbf{C}, \mathcal{M})$ coarse objects are defined as follows:*

$$\mathfrak{I}(\nu) = \{X \mid \nu_X \leq \rho\mathscr{C}(\nu)_X\},$$

and that $\nu_{\mathbf{1}} = \mathscr{F}_{\mathbf{1}}$. Then $\mathfrak{I}$ admits a left adjoint $\mathfrak{I}^*$.

Proof. Given a family $\mathcal{V} = \{\nu_i \mid i \in I\} \subseteq RegNBH(\mathbf{C}, \mathcal{M})$, let $\hat{\nu}$ be its supremum in $RegNBH(\mathbf{C}, \mathcal{M})$. And let $X \in \mathbf{C}$ such that $(\nu_i)_X \leq \rho\mathscr{C}(\nu_i)_X$ for all $i \in I$. We have $\hat{\nu}_X \leq (\sup\{\rho\mathscr{C}(\nu_i) \mid i \in I\})_X$ where the supremum is taken in $RegNBH(\mathbf{C}, \mathcal{M})$. But since ρ is a left adjoint, we have:

$$(\sup\{\rho\mathscr{C}(\nu_i) \mid i \in I\})_X = (\rho(\sup\{\mathscr{C}(\nu_i) \mid i \in I\}))_X \leq \rho\mathscr{C}(\sup \mathcal{V})_X$$

where the last two suprema are taken as in $NBH(\mathbf{C}, \mathcal{M})$. But $\rho\mathscr{C}(\sup \mathcal{V}) \leq \rho\mathscr{C}(\hat{\nu})$ since we always have $\sup \mathcal{V} \leq \hat{\nu}$. Therefore $\hat{\nu}_X \leq \rho\mathscr{C}(\hat{\nu})_X$. □

[51] The above proposition is one of the rare cases where the commutativity of the join with pullback is not involved. However if one wants to reconcile this proposition with Proposition 4.1.7, then it is likely that the latter condition is used. If this is the case $\mathscr{C}(\nu)$ becomes regular and so $\rho\mathscr{C}(\nu) = \mathscr{C}(\nu)$. Therefore all the results about $NBH(\mathbf{C}, \mathcal{M})$ will be true for $RegNBH(\mathbf{C}, \mathcal{M})$ provided that this condition is true, and this provides a generalization of [8] for interior operators.

Lemma 4.1.9. *For any $\nu \in NBH(\mathbf{C}, \mathcal{M})$, $\mathfrak{I}(\nu)$ is closed under $\mathcal{E}'$-images and $\mathfrak{D}(\nu)$ is hereditary, i.e. closed under $\mathcal{M}$-subobjects.*

Proof. It is straightforward to see that $\mathfrak{D}(\nu)$ is hereditary. Let $f : X \to Y$ be in $\mathcal{E}'$ and assume that $\nu_X \leq \mathscr{C}(\nu)_X$. If $k \in \nu_Y(m)$ in $\mathcal{M}/Y$, then

$$f^{-1}[k] \in \nu_X(f^{-1}[m]) \subseteq \mathscr{C}(\nu)_X(f^{-1}[m]).$$

Consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "t_X"'] & & Y \arrow[dl, "t_Y"] \\ & \mathbf{1} & \end{tikzcd}$$

For some $r \in \nu_{\mathbf{1}}(t_X[f^{-1}[m]])$ we have $f^{-1}[m] \leq t_X^{-1}[r] \leq f^{-1}[k]$. But $t_X^{-1}[r] = f^{-1}[t_Y^{-1}[r]]$ and $t_X[f^{-1}[m]] = (t_Y f)[f^{-1}[m]] = t_Y[m]$. Thus $r \in \nu_{\mathbf{1}}(t_Y[m])$ and from the last inequalities, we have $m \leq f[t_X^{-1}[r]] \leq k$ and so $m \leq t_Y^{-1}[r] \leq k$. Hence $\nu_Y(m) \subseteq \mathscr{C}(\nu)_Y(m)$. □

Lemma 4.1.10. *For any ν in $NBH(\mathbf{C}, \mathcal{M})$ we have $\mathfrak{D}(\nu) \bigcap \mathfrak{I}(\nu) \subseteq \mathscr{K}$. The converse is true if $\nu_{\mathbf{1}} = \mathcal{F}_{\mathbf{1}}$.*

Proof. Let $X \in \mathfrak{D}(\nu) \bigcap \mathfrak{I}(\nu)$ and consider the terminal map $t_X : X \to \mathbf{1}$. Since $m \in \nu_X(m)$ for any $m \in \mathcal{M}/X$ and $\nu_X \leq \mathscr{C}(\nu)_X$ there is $s \in \nu_{\mathbf{1}}(t_X[m])$ such that

$$m \leq t_X^{-1}[t_X[m]] \leq t_X^{-1}[s] \leq m.$$

Thus $m = t_X^{-1}[t_X[m]]$. □

Corollary 4.1.11. *Let $f : X \to Y$ be in $\mathbf{C}$ and ν in $NBH(\mathbf{C}, \mathcal{M})$. Suppose that $\mathcal{E} \subseteq \mathcal{E}'$. If $X \in \mathfrak{I}(\nu)$ and $Y \in \mathfrak{D}(\nu)$ then f is necessarily constant.*

Proof. Consider the following $(\mathcal{E}, \mathcal{M})$-factorisation of f:

$$X \xrightarrow{e} f[X] \xrightarrow{m} Y .$$

From Lemma 4.1.9 and Lemma 4.1.10, it follows that $f[X] \in \mathcal{K}$. □

Considering the class of fine objects $\mathfrak{D}(\nu)$ with respect to $\nu \in NBH(\mathbf{C}, \mathcal{M})$, which we recall, plays the role of *discrete objects*, one can define the corresponding class of connectedness:

Definition 4.1.12. Given a neighbourhood operator ν we say that an object X is *ν-connected* or simply *connected* with respect to ν if $X \in \bigtriangledown(\mathfrak{D}(\nu))$.

This definition allows us to also regard the concept of connectedness with respect to a neighbourhood operator as a functor

$$\chi = \bigtriangledown\mathfrak{D} : NBH(\mathbf{C}, \mathcal{M})^{op} \to S(\mathbf{C}).$$

The Galois correspondences introduced in [48] shed light on the relation between different types of disconnectedness and the weak separation axioms (cf. [2]). These were further investigated by means of closure operators (cf. [15],[10] and [7] for instance). As is observed in the point-set setting, T_0 and T_1 spaces are both disconnected subclasses, i.e., there are $\mathcal{A}_0$ in $S(\mathbf{Top})$ and $\mathcal{A}_1$ in $S(\mathbf{Top})$ such that $\{T_0\text{-spaces}\} = \Delta(\mathcal{A}_0)$ and $\{T_1\text{-spaces}\} = \Delta(\mathcal{A}_1)$(cf. [2]). In particular, $\mathcal{A}_0$ is the class of spaces with indiscrete topology and X is T_0 if and only if $X \in \Delta(\mathcal{A}_0)$. Our notions of T_0 and T_1 are motivated by these facts.

Definition 4.1.13. Given a neighbourhood operator ν, we say that an object $X \in \mathbf{C}$ is T_0 if and only if $X \in \Delta(\mathfrak{I}(\nu))$.

Thus, the notion of T_0 can be also seen as a functor:

$$T_0 = \Delta\mathfrak{I} : NBH(\mathbf{C}, \mathcal{M})^{op} \to S(\mathbf{C})^{op}.$$

We have the following observations:

Proposition 4.1.14. *For any subclass $\mathcal{B}$ of objects of $\mathbf{C}$, if $\mathcal{E} \subseteq \mathcal{E}'$ then*

$$\mathfrak{I}\mathfrak{D}_*(\mathcal{B}) \subseteq \bigtriangledown(\mathcal{B}) \text{ and } \mathfrak{D}\mathfrak{I}^*(\mathcal{B}) \subseteq \Delta(\mathcal{B}).$$

Proof. For the first inclusion, let $Y \in \mathfrak{I}\mathfrak{D}_*(\mathcal{B})$. We have $\mathfrak{D}\mathfrak{D}_*(\mathcal{B}) \preceq \mathcal{B}$ by adjunction and so $\mathcal{B} \subseteq \mathfrak{D}\mathfrak{D}_*(\mathcal{B})$. Therefore any object in $\mathcal{B}$ is fine with respect to $\mathfrak{D}_*(\mathcal{B})$. Let $f : Y \to Z$ be in $\mathbf{C}$ such that $Z \in \mathcal{B}$. By virtue of Corollary 4.1.11 f is constant.

The second inclusion is similarly proven: suppose that $X \in \mathfrak{D}\mathfrak{I}^*(\mathcal{B})$ and $f : Z \to X$ is in $\mathbf{C}$ with $Z \in \mathcal{B}$. Since $\mathcal{B} \subseteq \mathfrak{I}\mathfrak{I}^*(\mathcal{B})$ by adjunction, Z is $\mathfrak{I}^*(\mathcal{B})$ -coarse and so f must be constant. □

Corollary 4.1.15. *If $\mathcal{E} \subseteq \mathcal{E}'$ then for any ν in $NBH(\mathbf{C}, \mathcal{M})$:*

$$\mathfrak{I}(\nu) \subseteq \chi(\nu) \text{ and } \mathfrak{D}(\nu) \subseteq T_0(\nu).$$

Proof. For the first inclusion we have $\nu \preceq \mathfrak{D}_*\mathfrak{D}(\nu)$. By the previous proposition:

$$\mathfrak{I}(\nu) \subseteq \mathfrak{I}(\mathfrak{D}_*\mathfrak{D}(\nu)) \subseteq \bigtriangledown\mathfrak{D}(\nu)$$

Similarly since $\mathfrak{I}^*\mathfrak{I}(\nu) \preceq \nu$, we have $\Delta\mathfrak{I}(\nu) \preceq \mathfrak{D}(\mathfrak{I}^*\mathfrak{I}(\nu)) \preceq \mathfrak{D}(\nu)$. □

We recall that in a topological space that satisfies the axiom T_1, every subspace is the intersection of all open sets containing it.

Definition 4.1.16. Given a neighbourhood operator ν, we say that an object $X \in \mathbf{C}$ is T_1 with respect to ν if for all $m \in \mathcal{M}/X$, $m = \inf \nu_X(m)$.

This gives us a functor $T_1 : NBH(\mathbf{C}, \mathcal{M})^{op} \to S(\mathbf{C})^{op}$ defined by:

$$T_1(\nu) = \{X \in \mathbf{C} \mid \text{ For all } m \in \mathcal{M}/X \ m = \inf \nu_X(m)\}$$

Lemma 4.1.17. *Let $f : X \to Y$ be in $\mathbf{C}$ and ν in $NBH(\mathbf{C}, \mathcal{M})$. If $Y \in T_1(\nu)$ then for any $n \in \mathcal{M}/Y$ $f^{-1}[n] = \inf \nu_X(f^{-1}[n])$.*

Proof. This follows from the fact that $f^{-1}[-]$ is a right adjoint. □

In particular if $m \in \mathcal{M}/X$ then $f^{-1}[f[m]] = \inf \nu_X(f^{-1}[f[m]])$ and if $f \in \mathcal{M}$, then $m \cong f^{-1}[f[m]]$. Therefore the property of being T_1 is hereditary. The reason for its being hereditary is that T_1 objects are special fine objects as shown by the following proposition.

Proposition 4.1.18. *Every T_1 object with respect to a neighbourhood operator ν is a ν^1 -fine object for some ν^1 in $NBH(\mathbf{C}, \mathcal{M})$.*

Proof. We look for an endofunctor $(-)^1 : NBH(\mathbf{C}, \mathcal{M})^{op} \to NBH(\mathbf{C}, \mathcal{M})^{op}$ that makes the following diagram commutes:

$$\begin{tikzcd} NBH(\mathbf{C},\mathcal{M})^{op} \arrow[rr, "(-)^1"] \arrow[dr, "T_1"'] & & NBH(\mathbf{C},\mathcal{M})^{op} \arrow[dl, "\mathfrak{D}"] \\ & S(\mathbf{C})^{op} & \end{tikzcd}$$

We define $(-)^1$ as follow: for any $X \in \mathbf{C}$ and $m \in \mathcal{M}/X$:

$$\nu^1_X(m) = \{k \mid \text{ There is } p \in \mathcal{M}/X \text{ with } m \leq p \leq k \text{ and } p = \inf \nu_X(p)\}.$$

By the procedure described in Chapter 2, Section 2.3, ν^1 is obtained from the pullback stable (cf. Lemma 4.1.17) subclass $\{p \in \mathcal{M} \mid p = \inf \nu(p)\}$, therefore ν^1 is an idempotent neighbourhood operator.

It is straightforward to see that $\mathfrak{D}(\nu^1) = T_1(\nu)$. □

We note that since $\mathfrak{D}(\nu) \subseteq T_1(\nu)$, one has $\mathfrak{D}(\nu) \subseteq \mathfrak{D}(\nu^1)$.

A strengthening of Lemma 4.1.10 follows:

Lemma 4.1.19. *For any ν in $NBH(\mathbf{C}, \mathcal{M})$ $T_1(\nu) \bigcap \mathfrak{I}(\nu) \subseteq \mathscr{K}$. Also, equality is achieved when $\nu_{\mathbf{1}} = \mathscr{F}_{\mathbf{1}}$.*

Proof. Let $X \in T_1(\nu) \bigcap \mathfrak{I}(\nu)$ and consider the terminal map $t_X : X \to 1$. For any $m \in \mathcal{M}/X$ we have:

$$m = \inf \nu_X(m) = \inf \mathscr{C}(\nu)_X(m).$$

For any $k \in \mathscr{C}(\nu)_X(m)$, there is $s \in \nu_{\mathbf{1}}(t_X[m])$ such that

$$m \leq t_X^{-1}[t_X[m]] \leq t_X^{-1}[s] \leq k.$$

Hence $t_X^{-1}[t_X[m]] \leq \inf \mathscr{C}(\nu)_X(m)$. □

We could define for each ν in $NBH(\mathbf{C}, \mathcal{M})$ the class $\mathscr{A}(\nu)$ of all ν -absolutely connected objects (we recall that absolutely connected spaces are those which cannot be written as the union of a non-trivial family of closed subsets) - by considering the value of the following functor at ν:

$$\mathscr{A} = \bigtriangledown T_1 : NBH(\mathbf{C}, \mathcal{M})^{op} \to S(\mathbf{C})$$

We have the following result:

Proposition 4.1.20. *Suppose that $\mathcal{E} \subseteq \mathcal{E}'$. Then for any neighbourhood operator ν in $NBH(\mathbf{C}, \mathcal{M})$, we have:*

$$\mathfrak{I}(\nu) \subseteq \mathscr{A}(\nu) \subseteq \chi(\nu) \quad \textit{and} \quad T_0(\nu) \preceq T_1(\nu) \preceq \mathfrak{D}(\nu).$$

Proof. We already have $T_1(\nu) \preceq \mathfrak{D}(\nu)$ from construction. On the other hand this relation implies $\bigtriangledown T_1(\nu) \subseteq \bigtriangledown \mathfrak{D}(\nu)$. So $\mathscr{A}(\nu) \subseteq \chi(\nu)$.

Now let $X \in T_1(\nu)$ and $f : Y \to X$ be in $\mathbf{C}$ such that $Y \in \mathfrak{I}(\nu)$. Consider the following $(\mathcal{E}, \mathcal{M})$-factorisation of f:

$$Y \xrightarrow{e} f[Y] \xrightarrow{m} X\ .$$

By Lemma 4.1.9 and Lemma 4.1.17 we have $f[Y] \in \mathfrak{I}(\nu) \bigcap T_1(\nu) \subseteq \mathscr{K}$. Therefore $X \in T_0(\nu)$.

Finally if $X \in \mathfrak{I}(\nu)$ and $f : X \to Y$ is in $\mathbf{C}$ with $Y \in T_1(\nu)$, then as in the previous case if $f[X]$ is the domain of $f[1_X]$, then $f[X] \in T_1(\nu) \bigcap \mathfrak{I}(\nu) \subseteq \mathscr{K}$. So $X \in \mathscr{A}(\nu)$. □

[51] For any full subcategory $\mathcal{B} \subseteq \mathbf{C}$, the class $\Delta(\mathcal{B})$ (resp. $\bigtriangledown(\mathcal{B})$) can be seen as a class of ν-fine objects (resp. ν-coarse objects) for some neighbourhood operator ν under mild condition. This is achieved if one of the equations $\Delta = \mathfrak{D}\mathfrak{I}^*$ or $\bigtriangledown = \mathfrak{I}\mathfrak{D}_*$ holds.

We observe first that

Theorem 4.1.21. *For any $\mathcal{B}$ in $S(\mathbf{C})$ and $\mathcal{A}$ in $S(\mathbf{C})^{op}$ we have:*

$$\mathfrak{I}^*(\mathcal{B}) = \phi^{\mathcal{B},\, \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))} \quad \textit{and} \quad \mathcal{D}_*(\mathcal{A}) = \omega^{\mathcal{A},\, \mathscr{F}}.$$

Recall that $\mathscr{C}(\mathfrak{I}^*(\mathcal{B})) = \mathscr{C}$ is a constant. We shall however write $\mathscr{C}(\mathfrak{I}^*(\mathcal{B}))$ in order to provide a clearer proof.

Proof. Suppose first that $k \in \mathfrak{I}^*(\mathcal{B})_X(m)$ for some $X \in \mathbf{C}$ and let $f : Y \to X$ be in $\mathbf{C}$ such that $Y \in \mathcal{B}$. Thanks to ν-continuity and the adjunction relation $\mathcal{B} \subseteq \mathfrak{I}(\mathfrak{I}^*(\mathcal{B}))$, we have:

$$f^{-1}[k] \in \mathfrak{I}^*(\mathcal{B})_Y(f^{-1}[m]) \subseteq \mathscr{C}\mathfrak{I}^*(\mathcal{B})_Y(f^{-1}[m]).$$

Since this is true for any such map f we have $k \in \phi_X^{\mathcal{B},\, \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))}(m)$ and

$$\mathfrak{I}^*(\mathcal{B}) \leq \phi^{\mathcal{B},\, \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))}.$$

The inequality follows trivially if such a map f does not exist.

Now let $X \in \mathcal{B}$ and $k \in \phi_X^{\mathcal{B},\, \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))}(m)$ in $\mathcal{M}/X$. For all $f : Y \to X$ in $\mathbf{C}$ with $Y \in \mathcal{B}$ we have $f^{-1}[k] \in \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))_Y(f^{-1}[m])$. In particular $k \in \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))_X(m)$. Therefore X is $\phi^{\mathcal{B},\, \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))}$-coarse since by the previous inequality

$$\mathscr{C}(\mathfrak{I}^*(\mathcal{B}))_X(m) \subseteq \mathscr{C}(\phi^{\mathcal{B},\, \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))})_X(m).$$

Thus

$$\mathcal{B} \subseteq \mathfrak{I}.\phi^{\mathcal{B},\, \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))} \quad \text{and} \quad \mathfrak{I}^*(\mathcal{B}) \preceq \phi^{\mathcal{B},\, \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))}.$$

The second equality is similarly proved. $\square$

Theorem 4.1.22. *For all $\mathcal{B}$ in $S(\mathbf{C})$ and $\mathcal{A}$ in $S(\mathbf{C})^{op}$, we have:*

$$\mathfrak{D}\mathfrak{I}^*(\mathcal{B}) \preceq \Delta(\mathcal{B}) \quad \textit{and} \quad \bigtriangledown(\mathcal{A}) \subseteq \mathfrak{I}\mathfrak{D}_*(\mathcal{A}).$$

Proof. It is enough to prove the first inequality. The second is similarly treated.

Let $X \in \Delta(\mathcal{B})$. We want $X \in \mathfrak{D}(\mathfrak{I}^*(\mathcal{B})) = \mathfrak{D}.\phi^{\mathcal{B},\, \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))}$. Let $Y \in \mathcal{B}$ and $f : Y \to X$ in $\mathbf{C}$. f must be constant and so in the $(\mathcal{E}, \mathcal{M})$-factorisation of f:

$$Y \xrightarrow{e} R \xrightarrow{r} X$$

we have $R \in \mathscr{K}$. For any $m \in \mathcal{M}/X$ we want $m \in \phi_X^{\mathcal{B},\, \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))}(m)$, that is

$$f^{-1}[m] \in \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))_Y(f^{-1}[m]) \text{ for any such } Y \text{ and } f.$$

By ν-continuity it is enough that we have $r^{-1}[m] \in \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))_R(r^{-1}[m])$ since $f^{-1}[m] \cong e^{-1}[r^{-1}[m]]$. But if $\mathscr{C}(\nu)_{\mathbf{1}} = \mathscr{F}_{\mathbf{1}}$ for any ν in $NBH(\mathbf{C}, \mathcal{M})$, then by Lemma 4.1.10 $\mathscr{C}(\nu)_R = \mathscr{F}_R$ since for any $s \in \mathcal{M}/R$ $t_R^{-1}[t_R[s]] = s$. Therefore

$$r^{-1}[m] \in \mathscr{F}_R(r^{-1}[m]) \subseteq \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))_R(r^{-1}[m]).$$

Thus $X \in \mathfrak{D}.\phi^{\mathcal{B},\, \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))}$. $\square$

Corollary 4.1.23. *Suppose that $\mathcal{E} \subseteq \mathcal{E}'$, then we have equalities in Proposition 4.1.14 and Theorem 4.1.22.*

Thus for any $\mathcal{B}$ in $S(\mathbf{C})$, an object X is $\mathcal{B}$-disconnected if and only if X is $\phi^{\mathcal{B},\, \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))}$-fine, that is $\mathscr{F}_X \leq \phi_X^{\mathcal{B},\, \mathscr{C}(\mathfrak{I}^*(\mathcal{B}))}$ and it is $\mathcal{B}$-connected if and only if it is $\omega^{\mathcal{B},\mathscr{F}}$-coarse, that is $\omega_X^{\mathcal{B},\mathscr{F}} \leq \mathscr{C}(\omega^{\mathcal{B},\mathscr{F}})_X$. These inequalities become interesting when one, as in [8], considers limits of connected spaces (resp. colimits of disconnected spaces.)

Let us consider a few examples.

Examples 4.1.24. [51]

(1) On **Top**, we define the neighbourhood operator δ: for any $X \in \mathbf{Top}$ and $A \subseteq X$:

$$\delta_X(A) = \{B \subseteq X |\ A \subseteq C \subseteq B \text{ where } C = \bigcap_{i\in I} O_i, (O_i)_{i\in I} \subseteq \tau_X \text{ and } O_i \neq C\}.$$

We have $\mathfrak{I}(\delta) = \{X \mid (\forall x \in X),\ \overline{\{x\}} \neq \{x\}\}$ and $\mathfrak{D}(\delta) = T_1(\delta) = T_1(\tau)$. Note that for τ the usual notions of connectedness, disconnectedness, T_1 and T_0 are recovered.

(2) [8] Consider the neighbourhood operator η^q and its "regular hull" q from Example 2.4.1 (d). We have $\mathfrak{D}(q) = \mathfrak{D}(\eta^q) = \mathfrak{D}(\tau)$. On the other hand, since $\mathscr{C}(q)_X = \{\emptyset, \{X\}\}$ for any space X, we have $\mathfrak{I}(q) = \mathfrak{I}(\eta^q) = \chi(\tau)$ (i.e. the class of connected spaces.) Now since $\mathscr{K} = \{\emptyset\} \bigcup \{X \mid X \cong \{*\}\}$, $\mathscr{K}$-constant morphisms take their usual meaning. We then have $T_0(q) = T_0(\eta^q) = \{\text{Totally disconnected spaces}\}$. We recall that totally disconnected spaces are those whose components are single points. This shows that considering coarse neighbourhood operators is natural and general enough to encompass the notion of indiscreteness investigated in [8].

(3) [8] Now consider the neighbourhood operators η^b and its reflection b again from Example 2.4.1 (d). We have $\mathfrak{D}(\eta^b) = \mathfrak{D}(\tau)$ since $X \in \mathfrak{D}(\eta^b)$ if and only if every subset of X is closed. On the other hand $\mathfrak{D}(b) = T_1(\tau)$. Indeed, if $X \in \mathfrak{D}(b)$ if and only if every subset of X is the union of closed subsets, or dually every subset of X is the intersection of open sets. Therefore b-fine spaces are T_1-spaces. We also have $\chi(b) = \bigtriangledown(\mathfrak{D}(b)) = \mathscr{A}(\tau)$ (cf. [2]), hence b-connected spaces are absolutely connected spaces. On the other hand $\mathfrak{I}(b) = \mathfrak{I}(\eta^b) = \mathfrak{I}(\tau)$ and so $T_0(b) = T_0(\eta^b) = T_0(\tau)$.

(4) Consider the category **Haus** with the neighbourhood operator $\mathcal{N}$ from Example 2.4.1 (c). $X \in \mathfrak{D}(\mathcal{N})$ if and only if X is a Hausdorff zero-dimensional (i.e. with clopen subsets as basis [21].) Suppose that $X \in T_1(\mathcal{N})$. Let $M \subseteq X$ (closed) and $x \notin M$. There is $C \in \mathcal{N}_X(M)$ (also closed) such that $x \notin C$. But then there is an open set U such that $M \subseteq U \subseteq C$ and so

$$x \in X \setminus C \subseteq X \setminus U \subseteq X \setminus M.$$

Therefore X is a Hausdorff T_3-space or regular space (cf. [21].) Conversely if X is a Hausdorff regular space, then $X \in T_1(\mathcal{N})$. On the other hand $\mathfrak{I}(\mathcal{N}) = \mathscr{K}$ since no non-trivial open sets would contain any singletons.

(5) Consider the neighbourhood operator $\mathfrak{n}$ on **Grp**. First assume that $\mathfrak{I}(\mathfrak{n}) = \{G \mid \mathfrak{n}_G \leq \rho\mathscr{C}(\mathfrak{n})_G\}$. Then $\mathfrak{I}(\mathfrak{n}) = \{G \mid G \text{ simple}\}$ and on the other hand $\mathfrak{D}(\mathfrak{n}) = \{G \mid (\forall H \leq G),\ H \triangleleft G\}$. But $T_0(\mathfrak{n}) = \{G \mid G \cong \{*\}\} = \mathscr{K}$. Therefore

$\mathfrak{D}(\mathfrak{n}) \nsubseteq T_0(\mathfrak{n})$ which violates Corollary 4.1.15. This is so because commutativity of join with pullback is not satisfied in **Grp**.

Now assume that $\mathfrak{I}(\mathfrak{n}) = \{G \mid \mathfrak{n}_G \leq \mathscr{C}(\mathfrak{n})_G\}$. Thus $\mathfrak{I}(\mathfrak{n}) = \emptyset \subseteq \mathbf{Grp} = T_0(\mathfrak{n})$. Similar observations can be made for $\mathfrak{n}'$.

(6) Consider the category **Ab** of abelian groups with *injective homomorphisms* and *surjective homomorphisms*.

It is a fact that $\Delta(\mathbf{Tor}^{\mathfrak{t}}) = \mathbf{Fr}^{\mathfrak{t}}$ and $\bigtriangledown(\mathbf{Fr}^{\mathfrak{t}}) = \mathbf{Tor}^{\mathfrak{t}}$ (Cf. [9] for instance). There are then neighbourhood operators ν and ν' such that $\mathfrak{D}(\nu) = \mathbf{Fr}^{\mathfrak{t}}$ and $\mathfrak{I}(\nu') = \mathbf{Tor}^{\mathfrak{t}}$. In particular, we have $\nu = \phi^{\mathbf{Fr}^{\mathfrak{t}}, \mathscr{C}}$ and $\nu' = \omega^{\mathbf{Tor}^{\mathfrak{t}}, \mathscr{F}}$.

(7) Consider now the category **Gph** from Example 2.4.2 (c). A υ^*-fine graph X is a *discrete graph* or a graph whose edges are only loops. The same applies for υ_*. If $\upsilon = \upsilon^* \wedge \upsilon_*$, then a υ-fine graph is a also a discrete graph. Note that $\mathscr{K}$ consists of the empty graph with an empty edge and one point graph with a loop. Therefore $X \in \chi(\upsilon)$, i.e. X is a υ-connected graph, if and only if for any $x, y \in X$, we have $x \to y$ or $y \to x$. On the other hand we have $X \in \mathfrak{I}(\upsilon^*)$ if and only if for any $x, y \in X$ we have $x \to y$. Thus $\mathfrak{I}(\upsilon^*) = \mathfrak{I}(\upsilon_*) = \mathfrak{I}(\upsilon) \subseteq \chi(\upsilon)$.

Let us consider a family $\mathcal{G} = \{X_i \mid i \in I\}$ of objects of $\mathbf{C}$ and assume that it admits a product $X \in \mathbf{C}$. We would want to estimate the behaviour of ν_X on that product. Our interest in that is in proving facts which are related to constructions that we perform with the family $\mathcal{G}$, in particular we are concerned with the product X.

Proposition 4.1.25. *Let $\mathcal{S} = (f_i : X \to X_i)_{i \in I}$ be a source in $\mathbf{C}$ and ν a neighbourhood operator. Assume that $\nu_{X_i} \leq \mathscr{C}(\nu)_{X_i}$ for any $i \in I$. Then $\nu_X \leq \mathscr{C}(\nu)_X$ if and only if $\mathcal{S}$ is ν-initial.*

Proof. For any $i \in I$, we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "f_i"] \arrow[dr, "t_X"'] & & X_i \arrow[dl, "t_{X_i}"] \\ & \mathbf{1} & \end{tikzcd}$$

If $\mathcal{S}$ is ν-initial, then $\nu_X = \omega_X^{\mathcal{S},\nu}$. Proposition 3.1.6 (i) implies that $\omega_X^{\mathcal{S},\nu} \leq \mathscr{C}(\nu)_X$. Conversely assume that $\nu_X \leq \mathscr{C}(\nu)_X$. If $k \in \mathscr{C}(\nu)_X(m)$ in $\mathcal{M}/X$, then there is $s \in \nu_{\mathbf{1}}(t_X[m])$ such that $t_X^{-1}[s] \leq k$. Thus $s \in \nu_{\mathbf{1}}(t_{X_i}[f_i[m]])$ for every $i \in I$ and so $t_{X_i}^{-1}[s] \in \nu_{X_i}(f_i[m])$. But $t_X^{-1}[s] = f_i^{-1}[t_{X_i}^{-1}[s]]$ so $k \in \omega_X^{\mathcal{S},\nu}(m)$. $\square$

A similar result is obtained in Chapter 5, Proposition 5.1.4. The necessary condition for $RegNBH(\mathbf{C}, \mathcal{M})$ is considered in Proposition 5.1.5.

[51] The above result implies that a limit of ν-coarse objects is ν-coarse if and only if the natural arrows of the limit are ν-initial. In the case $\triangledown(\mathcal{B}) = \mathfrak{I}(\nu)$ for some $\mathcal{B}$ in $S(\mathbf{C})^{op}$ and ν in $NBH(\mathbf{C}, \mathcal{M})$ – for example in a situation where the conditions in Corollary 4.1.23 are met – the above proposition is enough to prove that a certain limit of $\mathcal{B}$-connected objects is $\mathcal{B}$-connected. Indeed if in the source $\mathcal{S}$ we have $X_i \in \triangledown(\mathcal{B}) = \mathfrak{I}(\omega^{\mathcal{B},\mathscr{F}})$ for all $i \in I$ and if $\mathcal{S}$ is $\omega^{\mathcal{B},\mathscr{F}}$-initial then the proposition would imply that $X \in \triangledown(\mathcal{B}) = \mathfrak{I}(\omega^{\mathcal{B},\mathscr{F}})$. Therefore it is enough to know that $\mathcal{S}$ is $\omega^{\mathcal{B},\mathscr{F}}$-initial – i.e. $\omega_X^{\mathcal{B},\mathscr{F}} \leq \omega_X^{\mathcal{S},\omega^{\mathcal{B},\mathscr{F}}}$ – to conclude that X is $\mathcal{B}$-connected. The case where $\mathcal{B} = \mathfrak{D}(\nu)$ is of interest in that respect as it would prove that a suitable limit of ν-connected objects is ν-connected.

4.2 Monotone morphisms via partitions

Following [54] a partition will mean a pair $(p, \overline{p})$ of subobjects, where – we recall – $\overline{p}$ is the pseudocomplement of p. Since the relations $m \wedge p = 0$ and $m \leq \overline{p}$ are equivalent, $\overline{p}$ is uniquely determined. We say that the partition $(p, \overline{p})$ is ν-*open* or simply *open*, for a neighbourhood operator ν, if both $p \in \nu(p)$ and $\overline{p} \in \nu(\overline{p})$ are true, and we say that it is *trivial* if $p = 0$ or $\overline{p} = 0$. Thus the notion of connectedness considered in this section is directly translated from classical Topology. The same is true for *monotone morphisms*: we say that a morphism f is *monotone* if, essentially, it has connected fibres. These shall be formally defined.

Definition 4.2.1. [54] An object X is said to be ν-*connected* or simply *connected* for $\nu \in NBH(\mathbf{C}, \mathcal{M})$, if any ν-open partition $(p, \overline{p})$ of $\mathcal{M}/X$ is trivial.

The class of ν-connected objects, for a fixed ν is denoted by $\pi(\nu)$.

Definition 4.2.2. [54] A morphism $f : X \to Y$ is said to be ν-*monotone* or simply *monotone* if for any $n > 0_Y$, there exists $q \leq^+ n$ such that the domain of $f^{-1}[q]$ is in $\pi(\nu)$.

We note the following useful observation.

Lemma 4.2.3. *If a morphism $f : X \to Y$ reflects 0 and $\mathcal{E} \subseteq \mathcal{E}'$, then it preserves pseudocomplements.*

Proof. [34, 54] Let $p \in \mathcal{M}/Y$. By adjunction, and by definition of pseudocomplements, the following relations are equivalent: $m \leq f^{-1}[\overline{p}]$, $f[m] \leq \overline{p}$ and also $f[m] \wedge p = 0_Y$. On the other hand, since f relects 0 and $\mathcal{E} \subseteq \mathcal{E}'$ the following relations are equivalent: $f[m] \wedge p = 0_Y$, $f[m \wedge f^{-1}[p]] = 0_Y$ and $m \wedge f^{-1}[p] = 0_X$. Therefore $f^{-1}[\overline{p}] = \overline{f^{-1}[p]}$. □

On the other hand subobjects always reflect pseudocomplements without assuming the condition $\mathcal{E} \subseteq \mathcal{E}'$.

Lemma 4.2.4. *If $m : M \to X \in \mathcal{M}$, then m reflects pseudocomplements.*

Proof. Let $(p, \overline{p})$ be a partition in $\mathcal{M}/X$. Suppose that $k \wedge m^{-1}[p] = 0_M$ for a subobject $k \in \mathcal{M}/M$. Thus $mk \wedge p = 0_X$. Indeed since $m[a \wedge b] \cong ma \wedge mb$ for any $a, b \in \mathcal{M}/M$, one has

$$m[k \wedge m^{-1}[p]] = mk \wedge (m \wedge p) = mk \wedge p.$$

Therefore $mk \wedge p = 0_X$ and so $mk \leq \overline{p}$. Hence $k \leq m^{-1}[\overline{p}]$. Conversely assume that $k \leq m^{-1}[\overline{p}]$. Then $mk \leq \overline{p}$ and so $mk \wedge p = 0_X$. Therefore $k \wedge m^{-1}[p] = 0_M$ since m reflects 0 by Proposition 1.2.19. □

Proposition 4.2.5. *If $f : X \to Y \in \mathcal{E}'$ is monotone and reflects 0, then it takes partitions to partitions.*

Proof. The second part of this proof is taken from [54, Theorem 9]. Let $(p, \overline{p})$ be a partition on $\mathcal{M}/X$. If $l \leq \overline{f[p]}$, then $l \wedge f[p] = 0_Y$ and so $f^{-1}[l] \wedge p = 0_X$. Hence $f^{-1}[l] \leq \overline{p}$ and so $l = f[f^{-1}[l]] \leq f[\overline{p}]$. Therefore $\overline{f[p]} \leq f[\overline{p}]$.

Conversely assume that $f[p] \wedge f[\overline{p}] > 0_Y$ (Otherwise $f[\overline{p}] \leq \overline{f[p]}$ and we are done.) Since f is monotone, there is $c \leq^+ f[p] \wedge f[\overline{p}]$ such that the domain C of $f^{-1}[c]$ is connected. We have $f^{-1}[c] \wedge p > 0_X$ and $f^{-1}[c] \wedge \overline{p} > 0_X$. By Lemma 4.2.4 the pair $((f^{-1}[c])^{-1}[p], (f^{-1}[c])^{-1}[\overline{p}])$ is a partition of C. Since the latter is connected, we would have $f^{-1}[c] \wedge p = 0_X$ or $f^{-1}[c] \wedge \overline{p} = 0_X$. Therefore it should be the case that $f[p] \wedge f[\overline{p}] = 0_Y$. □

Definition 4.2.6. A subobject $m : M \to X$ is said to be ν-*dense* or simply *dense* if for any $n > 0_X$ one has $m \wedge \nu_X(n) > 0_X$. A morphism $f : X \to Y$ is said to be ν-dense if its $\mathcal{M}$-image $m = f[1_X]$ is ν-dense.

We recall that the relation $m \wedge \nu_X(n) > 0_X$ means that for any $k \in \nu_X(n)$, we have $m \wedge k > 0_X$.

Examples 4.2.7. (i) In **Top**, τ-denseness takes its usual meaning.

(ii) In **Gph** with the neighbourhood operator υ. A subgraph $M \subseteq X$ is υ^*-dense if for any $x \in X$, there is $y \in M$ such that $x \to y$. M is υ_*-dense if for any $x \in X$, there is $y \in M$ such that $y \to x$.

(iii) Consider **Ab** with the neighbourhood operator $\nu = \phi^{\mathbf{Fr}^{\mathrm{t}}, \mathscr{C}}$. Assume that $H \leq G$ is ν-dense. If $K \leq G$ is a free torsion subgroup, then considering the natural embedding $K \to G$, we have $K \in \mathbf{Fr}^{\mathrm{t}}$ so that $L \in \nu_G(P)$ if and only if $K \leq L$. Therefore we have $H \bigcap K \neq \{0\}$. Thus if $g \in G$, then for some $n \in \mathbb{Z}$, we have $ng \in H$ or simply $G/H \in \mathbf{Tor}^{\mathrm{t}}$.

Lemma 4.2.8. *Let $m : M \to X \in \mathcal{M}$ such that m is ν-dense. Then $X \in \pi(\nu)$ provided that $M \in \pi(\nu)$.*

Proof. Let $(q, \overline{q})$ be an open partition of $\mathcal{M}/X$. By Lemma 4.2.4 and by ν-continuity of m, $(m^{-1}[q], m^{-1}[\overline{q})]$ is an open partition of $\mathcal{M}/M$. Hence $m^{-1}[q] = 0_M$ or $m^{-1}[\overline{q})] = 0_M$. Therefore $m \wedge q = 0_X$ or $m \wedge \overline{q} = 0_X$. These cannot be allowed if $q > 0_X$ and $\overline{q} > 0_X$ since m is dense and the partition $(q, \overline{q})$ is open. □

Proposition 4.2.9. *Suppose that ν is hereditary and the following condition on sub-objects holds for an object $X \in \mathbf{C}$:*

$$\textit{If } m \wedge \bigvee_I m_i > 0_X \textit{ then there is } i \in I \textit{ such that } m \wedge m_i > 0_X.$$

If $m : M \to X \in \mathcal{M}$ then $\mathfrak{cl}_1^\nu(M) \in \pi(\nu)$ provided that $M \in \pi(\nu)$.

We recall from Chapter 2 that $\mathfrak{cl}_1^\nu(m) = \bigvee\{n \mid (\forall n' \leq^+ n), m \wedge \nu_X(n') > 0_X\}$.

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
r^{-1}[P] \arrow[r, "r'"] \arrow[d, "{r^{-1}[p]}"'] & P \arrow[d, "p"] \\
M \arrow[r, "r"] \arrow[dr, "m"'] & K \arrow[d, "k"] \\
& X
\end{tikzcd}$$

where $K = \mathfrak{cl}_1^\nu(M)$ and $k = \mathfrak{cl}_1^\nu[m]$. According to Lemma 4.2.8, it suffices to show that r is dense. If $p > 0_K$, then there is by assumption $l \in \mathcal{M}/X$ such that $kp \wedge l > 0_X$ and for any $l' \leq^+ l$, $kp \wedge \nu_X(l') > 0_X$. But then $m \wedge \nu_X(kp \wedge l) > 0_X$ and so $m \wedge \nu_X(kp) > 0_X$ since $\nu_X(kp) \subseteq \nu_X(kp \wedge l)$. Because ν is hereditary and $k^{-1}[m] = k^{-1}[kr] \cong r$, one has $r \wedge \nu_K(p) > 0_K$. □

Proposition 4.2.10. *[54, Proposition 8] Let $f : X \to Y$ be in $\mathcal{E}'$ and assume that it reflects 0. If $X \in \pi(\nu)$ then $Y \in \pi(\nu)$.*

Proof. If $(q, \overline{q})$ is a partition on $\mathcal{M}/Y$, then by Lemma 4.2.3 $(f^{-1}[q], f^{-1}[\overline{q}])$ is a partition on $\mathcal{M}/X$. Therefore $f^{-1}[q] = 0_X$ or $f^{-1}[\overline{q}] = 0_X$, this is equivalent to saying $q = 0_Y$ or $\overline{q} = 0_Y$. □

Thanks to Lemma 4.2.8 it is enough to assume that $f[1_X] : f[X] \to Y$ is ν-dense.

Definition 4.2.11. A morphism f is called *ν-quotient* or simply *quotient* if $f \in \mathcal{F}(\nu) \bigcap \mathcal{E}$.

It will happen sometimes that we need $f \in \mathcal{E}'$, and so require that $f \in \mathcal{F}(\nu) \bigcap \mathcal{E}'$. However in most of the cases, and especially in this section, the condition that $\mathcal{E} \subseteq \mathcal{E}'$ will be needed and therefore the stability of f in $\mathcal{E}$ is already incorporated in the definition.

Proposition 4.2.12. *[54, Theorem 9] Let $f : X \to Y$ be in $\mathcal{F}(\nu) \bigcap \mathcal{E}'$. Assume that f reflects 0 and that it is monotone. Then $X \in \pi(\nu)$ provided that $Y \in \pi(\nu)$.*

Proof. By proposition 4.2.5, if $(q, \overline{q})$ is a partition of $\mathcal{M}/X$, then $(f[q], f[\overline{q}])$ is partition of $\mathcal{M}/Y$. [54] If $q < f^{-1}[f[q]]$, then by taking $k = \overline{q} \wedge f^{-1}[f[q]] > 0_X$ one has

$$f[k] = f[q] \wedge f[\overline{q}] = 0_Y.$$

It should be the case that $q \cong f^{-1}[f[q]]$ and similarly $\overline{q} \cong f^{-1}[f[\overline{q}]]$. Thus the partition $(q, \overline{q})$ is open if and only if the partition $(f[q], f[\overline{q}])$ is open. The result follows from the fact that f reflects 0. $\square$

In the proof, one can also replace the condition $f \in \mathcal{F}(\nu)$ by $f \in \mathcal{O}(\nu)$.

Proposition 4.2.13. *[54, Corollary 10] Let $f : X \to Y$ be in $\mathcal{E}'$ and assume that ν is hereditary. In addition assume that f is monotone and reflects 0. Let $n : N \to Y \in \mathcal{M}/Y$ and denote by $g : f^{-1}[N] \to N$ the restriction of f. If $f \in \mathcal{F}(\nu) \bigcup \mathcal{O}(\nu)$ and $N \in \pi(\nu)$, then $f^{-1}[N] \in \pi(\nu)$.*

Proof. Note that g is also monotone and reflects 0. The result follows from Proposition 4.2.12 and the Pullback Ascent and Descent Theorem 3.3.1 which ensures that we also have $g \in \mathcal{F}(\nu) \bigcup \mathcal{O}(\nu)$. $\square$

Chapter 5

Compactness

Though *compactness* only appears in this chapter, it was probably the first topological notion to be treated via neighbourhood operators. It does not come as a surprise since it is closely related to the notion of *convergence*. In the previous works [53, 34, 55] there were attempts to give a version of the Tychonoff-Čech theorem. However certain notions of complement (Boolean Algebra) and point (atom) were considered. Also compactness via neighbourhood operators and via their closure operators $\mathfrak{cl}_1$ were compared. We give here two versions of the Tychonoff-Čech theorem via two different notions of compactness: the first one is based on convergence as developed in [34, 55, 53] but in a slightly different way and the second one is based on closed maps as considered in [57, 13]. Though there are clearly minimal conditions that would make these two notions equivalent, we shall not study this relationship here. Also, the notion of separation which has not received a formal treatment - though it is defined in previous works on neighbourhood operators - is not studied here. One reason for this is because the theorem under consideration is not linked to separation.

Before introducing the two types of compactness we bring the notion of *Tychonoff objects* as a motivation for the chapter. With suitable conditions, they can be seen as subspaces of products of compact spaces.

5.1 A general overview

We define *Tychonoff* or *completely regular objects* as objects that bear initial structures induced from special objects. Though Tychonoff spaces are exactly subspaces of compact spaces, this is a consequence of both the Tychonoff-Čech theorem and the fact that each space X is Tychonoff if and only if it has the initial topology induced by the family $\mathcal{C}^*(X, \mathbb{R})$ of bounded real-valued continuous functions defined on X. Hence for each $f \in \mathcal{C}^*(X, \mathbb{R})$, there is a closed interval I_f such that the source

$$\{f : X \to I_f \mid f \in \mathcal{C}^*(X, \mathbb{R})\}$$

is jointly initial (cf. Definition 3.4.7.) Such sources and subspaces of the product $\prod\{I_f \mid f \in \mathcal{C}^*(X, \mathbb{R})\}$ are essentially the same. Since all the I_f's are isomorphic, and in particular are isomorphic to $I = [0;1]$, one could see a Tychonoff space as a subspace of a cube $I^{\mathfrak{m}}$ where $\mathfrak{m}$ is a cardinal. If one replaces I with any topological space E, one studies the so-called E-completely regular spaces pioneered by Engelking and Mrówka ([22].) We follow such an approach and we use the notion of initiality as a starting point in defining Tychonoff objects.

Let us fix $\nu \in NBH(\mathbf{C}, \mathcal{M})$.

Definition 5.1.1. Let $E \in \mathbf{C}$. An object X is said to be *E-Tychonoff* if $\nu_X = \omega_X^{E,\nu}$.

The class of all E-Tychonoff objects shall be denoted by $\mathscr{T}(E)$.

Examples 5.1.2. (i) if $E = \mathbf{1}$, then $\mathscr{T}(E) = \mathfrak{I}(\nu)$ is the class of ν-coarse objects (cf. Chapter 4.);

(ii) [60] In **Top** with the usual neighbourhood operator τ, $\mathscr{T}(\mathbb{R})$ is the class of all subspaces of real-compact spaces. If E is the two-point discrete space, then $\mathscr{T}(E)$ is the class of all zero-dimensional spaces (cf. [3].)

In the examples above we did not specify the nature of the source $Hom(X, E)$ that induces the initial structure. From now on, we agree that these sources are monosources ([1]) or joint monomorphisms. We find it convenient to recall in this section the definition of joint monomorphisms.

Definition 5.1.3. A source $\{f_i : X \to X_i \mid i \in I\}$ is said to be a joint *monomorphism* or a *monosource* if for any pair of arrows (u, v), we have $u = v$ whenever $f_i u = f_i v$ for all $i \in I$.

In **Set** and **Top**, a monosource is a source whose arrows separate points, i.e., those sources $\{f_i : X \to X_i \mid i \in I\}$ such that for all $x, y \in X$ with $x \neq y$, there is $i \in I$ such that $f_i(x) \neq f_i(y)$.

We are naturally interested in the constructions that exist in $\mathscr{T}(E)$. The following result follows directly from the fact that $\mathcal{I}(\nu)$ is closed under composition (cf. Proposition 3.1.6 (i)).

Proposition 5.1.4. *Let $E \in \mathbf{C}$ and let $\{f_i : X \to X_i \mid i \in I\}$ be a ν-initial source. If $X_i \in \mathscr{T}(E)$ for every $i \in I$, then $X \in \mathscr{T}(E)$.*

When we consider the regularity of ν to be a priority, then the result still holds under the condition that joins of subobjects commute with pullbacks:

Proposition 5.1.5. *Let $E \in \mathbf{C}$ and let $\{f_i : X \to X_i \mid i \in I\}$ be a ν-initial source for a regular neighbourhood operator ν. Assume that joins of subobjects commute with pullbacks. If $X_i \in \mathscr{T}(E)$ for every $i \in I$, then $X \in \mathscr{T}(E)$.*

Proof. Let $k \in \nu_X(m)$ in $\mathcal{M}/X$. By Proposition 2.2.3, if $i = \rho(\nu)$ then

$$m \leq \bigvee\{(f_i)^{-1}[i_{X_i}((f_i)_*)[k]] \mid i \in I\} \leq k.$$

If $\{f_l : X_i \to E \mid l \in L_i\}$ is the source associated to each X_i, then

$$m \leq \bigvee\{(f_l f_i)^{-1}[i_E((f_l f_i)_*)[k]] \mid l \in L_i,\ i \in I\} \leq k.$$

Therefore $\nu_X = \omega_X^{E,\nu}$. $\square$

Theorem 5.1.6. *If ν preserves products and is hereditary, then for any $E \in \mathbf{C}$, the class $\mathscr{T}(E)$ is closed under products and subobjects.*

Let us now fix $E \in \mathbf{C}$. If $X \in \mathscr{T}(E)$, then there is a unique monomorphism $h : X \to E^{\mathfrak{m}}$, with $|Hom(X, E)| = \mathfrak{m}$. h is ν-initial by construction (Proposition 3.4.8) and this is enough for h to be an embedding in **Top** with the neighbourhood operator τ. Following the theory of E-compact spaces, we denote by $\kappa(E)$ the class of closed subobjects of powers of E for a convenient closure operator. We want to "densely embed" each object X in $\mathscr{T}(E)$ into an object in $\kappa(E)$ and hence provide X with a *compactification*.

Lemma 5.1.7. *If $\mathfrak{cl}_1^\nu$ exists then it is idempotent provided that the following condition holds (cf. Proposition 4.2.9)*

$$\textit{If } m \wedge \bigvee\nolimits_I m_i > 0 \textit{ then there is } i \in I \textit{ such that } m \wedge m_i > 0.$$

Proof. Let $X \in \mathbf{C}$ and $m \in \mathcal{M}/X$. Suppose that $\mathfrak{cl}_1^\nu(m) \wedge \nu_X(n') > 0_X$ for some $n \in \mathcal{M}/X$ and any $n' \leq^+ n$. By construction of $\mathfrak{cl}_1^\nu$ and by assumption, for each $l \in \nu_X(n')$ there is $k \leq^+ \mathfrak{cl}_1^\nu(m)$ such that $l \wedge k > 0_X$ and $m \wedge \nu_X(l') > 0_X$ for any $l' \leq^+ l \wedge k \leq k$. In particular since $k \wedge l \leq l$ and hence $\nu_X(l) \subseteq \nu_X(k \wedge l)$, we have $m \wedge \nu_X(l) > 0_X$. Therefore $n \leq \mathfrak{cl}_1^\nu(m)$ and so $\mathfrak{cl}_1^\nu(\mathfrak{cl}_1^\nu(m)) \leq \mathfrak{cl}_1^\nu(m)$. $\square$

Proposition 5.1.8. *Assume that $\mathfrak{cl}_1^\nu$ exists. Assume in addition that $\kappa(E)$ is closed under $\mathfrak{cl}_1^\nu$-closed subobjects and that $\mathfrak{cl}_1^\nu$ is idempotent. Then for any $X \in \mathscr{T}(E)$, there is an object $C \in \kappa(E)$ and a monomorphism $p : X \to C$ which is ν-dense and ν-initial.*

Proof. Since the source $Hom(X, E)$ is jointly initial, there is an initial monomorphism $h : X \to E^{\mathfrak{m}}$ with $\mathfrak{m} = |Hom(X, E)|$. Consider the following diagram

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, "h"'] & h[X] \arrow[r, "r"] \arrow[d, "m"] & \mathfrak{cl}_1^\nu(h[X]) \arrow[dl, "k"] \\
& E^{\mathfrak{m}} &
\end{tikzcd}$$

where $h = me$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$ and $\mathfrak{cl}_1^{\nu}(m) = k$. Since $\mathfrak{cl}_1^{\nu}$ is idempotent $\mathfrak{cl}_1^{\nu}(h[X]) \in \kappa(E)$. On the other hand $h = k.(re) \in \mathcal{I}(\nu)$, so $re \in \mathcal{I}(\nu)$ (Proposition 3.1.6 (ii)). Now, re is a monomorphism since e is a monomorphism. From Proposition 4.2.9 r is ν-dense. The morphism $p = re$ is what we needed. $\square$

Examples 5.1.9. In **Top** the closure operator $\mathfrak{cl}_1^{\tau}$ and $\mathfrak{cl}_2^{\tau}$ coincide with the Kuratowski closure operator.

(i) If $E = [0;1]$, then we obtain the class $\mathscr{T}(E)$ of the classical Tychonoff spaces. The compactification of a space X is given by its closure βX;

(ii) [60] If $E = \mathbb{R}$, then $\mathscr{T}(E)$ is the class of Tychonoff spaces and $\kappa(E)$ is the class of real-compact spaces. The compactification of a space X is provided by υX;

(iii) [60, 3] If E is the two-point discrete space, then $\kappa(E)$ is the class of compact zero-dimensional spaces. The compactification of a zero-dimensional space X is provided by ζX which is a quotient of βX;

(iv) If E is the Sierpinski space, then $\mathscr{T}(E)$ is the class of T_0-spaces and the compactification of a T_0-space X is the closure of the embedding $X \to E^{|\sigma_X|}$, where σ_X is the topology on X.

5.2 Compactness via ultrafilter

The notion of filter in a partially ordered set is naturally extended to our setting.

Definition 5.2.1. Given a neighbourhood operator ν, we say that a raster $\mathcal{G} \subseteq \mathcal{M}/X$ converges to a subobject $m > 0_X$ if $\nu_X(m) \subseteq \mathcal{G}$. It is said to *converges tightly* to m if $\nu_X(n) \subseteq \mathcal{G}$ for any $n \leq^+ m$.

Proposition 5.2.2. *Let $f : X \to Y$ be in* **C**. *The ν-continuity of f is equivalent to the following statement: for any raster $\mathcal{G} \subseteq \mathcal{M}/X$ and $m \in \mathcal{M}/X$, if $\nu_X(m) \subseteq \mathcal{G}$, then $\nu_X(f[m]) \subseteq f[\mathcal{G}]$.*

Proof. For the converse, take $\mathcal{G} = \nu_X(m)$. $\square$

The same statement holds for tight convergence when $\mathcal{E} \subseteq \mathcal{E}'$.

Let us fix a neighbourhood operator ν.

Definition 5.2.3. We say that an object X is *ultrafilter compact* with respect to ν if for any ultrafilter $\mathcal{U} \subseteq \mathcal{M}/X$, there is $l \in \mathcal{M}/X$ such that $\mathcal{U}$ converges tightly to l.

In this section, we shall assume the Ultrafilter Lemma

Axiom 5.2.4. ***(Ultrafilter Lemma)*** *Every filter can be embedded into an ultrafilter.*

In the presence of suitable separation axioms [34, 55], the limit l is unique. In particular in the point-set setting it is reduced to a point (singleton). For now, we shall develop the theory without any separation axiom.

The class of all ultrafilter compact objects with respect to ν shall be denoted by $\mathfrak{U}(\nu)$.

Lemma 5.2.5. *If $\mathcal{E} \subseteq \mathcal{E}'$ then for any morphism f and for any appropriate subobjects p and q, if $q \wedge f[p] > 0$ then $f^{-1}[q] \wedge p > 0$. The two relations are equivalent when every morphism in $\mathcal{E}'$ reflects 0.*

Proof. By assumption we have $f[f^{-1}[q] \wedge p] = q \wedge f[p]$. □

Proposition 5.2.6. *(i) Let $f : X \to Y$ be in $\mathbf{C}$ and $\mathcal{U}$ an ultrafilter on X. If $f \in \mathcal{E}'$, then $f[\mathcal{U}]$ is an ultrafilter on Y;*

(ii) Let $f : X \to Y$ be in $\mathbf{C}$ and $\mathcal{U}$ a filter on X. Assume that the $\mathcal{E}$-part of f belongs to $\mathcal{E}'$. If $\mathcal{U}$ converges (resp. tightly) to $l \in \mathcal{M}/X$ then $f[\mathcal{U}]$ converges (resp. tighly) to $f[l]$;

(iii) Let $f : X \to Y$ be in $\mathbf{C}$ such that f reflects 0 and let $\mathcal{G}$ be a filter on X. If $f[\mathcal{G}]$ converges (resp. tightly) to $f[l]$ for some $l \in \mathcal{M}/X$, then $\mathcal{G}$ converges (resp. tightly) to l provided that f is ν-initial.

Proof. (i) Let $\mathcal{V}$ be a filter such that $f[\mathcal{U}] \subseteq \mathcal{V}$. Let $v \in \mathcal{V}$. For any $u \in \mathcal{U}$, we have that $f[u] \wedge v = f[u \wedge f^{-1}[v]]$. By Lemma 5.2.5 $u \wedge f^{-1}[v] > 0_X$ for any $u \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, it is the filter generated by $f^{-1}[v]$ and itself. Therefore $f^{-1}[v] \in \mathcal{U}$ and so $v \in f[\mathcal{U}]$.

(ii) Suppose that $\mathcal{U}$ converges tightly to l and let $p \leq^+ f[l]$. By assumption, $p = f[f^{-1}[p] \wedge l]$ and $f^{-1}[p] \wedge l \leq l$. By virtue of Lemma 5.2.5 $f^{-1}[p] \wedge l \leq^+ l$. By assumption $\nu_X(f^{-1}[p] \wedge l) \subseteq \mathcal{U}$ and by continuity:

$$\nu_Y(p) \subseteq f[\nu_X(f^{-1}[p] \wedge l)] \subseteq f[\mathcal{U}].$$

(iii) If $p \leq^+ l$, then $f[p] \leq^+ f[l]$. Hence $\nu_Y(f[p]) \subseteq f[\mathcal{U}]$. Since f is ν-initial, we have:

$$\nu_X(p) = f^{-1}[\nu_Y(f[p])] \subseteq f^{-1}[f[\mathcal{U}]] \subseteq \mathcal{U}$$

as desired. □

Proposition 5.2.7. *(i) Let $f : X \to Y$ be in $\mathcal{E}'$:*

a) If $X \in \mathfrak{U}(\nu)$, then $Y \in \mathfrak{U}(\nu)$;

b) Suppose that f is ν-initial and that f reflects 0. If $Y \in \mathfrak{U}(\nu)$, then $X \in \mathfrak{U}(\nu)$.

(ii) Let $m : M \to X$ be in $\mathcal{M}$ such that m is $\mathfrak{cl}_1^\nu$-closed or $\mathfrak{cl}_2^\nu$-closed. If $X \in \mathfrak{U}(\nu)$, then $M \in \mathfrak{U}(\nu)$.

Proof. (i)(a) Given an ultrafilter $\mathcal{G}$ on Y, since $f \in \mathcal{E}'$, $f^{-1}[\mathcal{G}]$ is a filter on X. If $\mathcal{U}$ is an ultrafilter containing $f^{-1}[\mathcal{G}]$, then $\mathcal{U}$ converges tightly to a subobject l. Hence $f[\mathcal{U}]$ converges tightly to $f[l]$. But since $\mathcal{G}$ is an ultrafilter $\mathcal{G} = f[f^{-1}[\mathcal{G}]] \subseteq f[\mathcal{U}] \subseteq \mathcal{G}$.

(i)(b) If $\mathcal{F}$ is an ultrafilter on $\mathcal{M}/X$, then $f[\mathcal{F}]$ is an ultrafilter that converges tightly to a subobject $l = f[f^{-1}[l]]$. Then $\mathcal{F}$ converges tightly to $f^{-1}[l]$.

(ii) If $\mathcal{F}$ is a filter on $\mathcal{M}/M$, then $\nu_X(p) \wedge m[\mathcal{F}] > 0_X$ for some $l > 0_X$ and any $p \leq^+ l$. Since m is closed with respect to $\mathfrak{cl}_1^\nu$ or $\mathfrak{cl}_2^\nu$, $m^{-1}[l] > 0_M$. If $k \leq^+ m^{-1}[l]$, then $\nu_X(m[k]) \wedge m[\mathcal{F}] > 0_Y$. Since m is ν-initial $\nu_M(k) \wedge \mathcal{F} > 0_M$. □

To prove the Tychonoff-Čech theorem, we shall use the result that follows.

Lemma 5.2.8. ***(Alexander Subbase)*** *Given a filter $\mathcal{F} \subseteq \mathcal{M}/X$, with $X \in \mathbf{C}$, if $\mathcal{F}$ converges tightly to a subobject $l > 0_X$ with respect to ν, then $\mathcal{F}$ converges tightly to l with respect to $\theta(\nu)$.*

We recall that $\theta(\nu)$ is obtained by taking finite meets. This lemma implies the following one:

Lemma 5.2.9. *Let $\mathcal{F}$ be a filter and let $\{\nu_i \mid i \in I\} \subseteq NBH(\mathbf{C}, \mathcal{M})$. If $\mathcal{F}$ converges tightly to a subobject $l > 0_X$ with respect to each ν_i, then $\mathcal{F}$ converges tightly to l with respect to ν^* and with respect to $\hat{\nu}$.*

Proof. Since θ is a left adjoint $\hat{\nu} = \theta(\nu^*)$. □

Theorem 5.2.10. ***(Tychonoff-Čech)*** *Let I be a set and $X : I \to \mathbf{C}$ a diagram that admits a limit denoted also by X. Assume that the natural projections $p_i : X \to X_i$, $i \in I$, belong to $\mathcal{E}'$ and reflect 0. Assume in addition that for any diagram $D : I \to \mathbf{C}$ that admits a limit D, there is an index $i_0 \in I$ such that $D \to D_{i_0} \in \mathcal{E}$ and that ν preserves I-products. Then $X \in \mathfrak{U}(\nu)$ provided that $X_i \in \mathfrak{U}(\nu)$ for each $i \in I$.*

Proof. For a given ultrafilter $\mathcal{U}$ on $\mathcal{M}/X$, $p_i[\mathcal{U}]$ is an ultrafilter on X_i for each $i \in I$. There is $m_i > 0_{X_i}$ such that $p_i[\mathcal{U}]$ converges tightly to m_i for each $i \in I$. Let $m : M \to X = \lim(m_i)$. By Proposition 1.2.15, $m \in \mathcal{M}$. From our assumption, there is $i_0 \in I$ such that the natural projection $M \to M_{i_0}$ belongs to $\mathcal{E}$. In the following diagram

$$\begin{tikzcd}
X \arrow[r, "p_{i_0}"] & X_i \\
M \arrow[u, "m"] \arrow[r] & M_{i_0} \arrow[u, "m_{i_0}"']
\end{tikzcd}$$

we have $p_{i_0}[m] = m_{i_0}$. Therefore $m > 0_X$ and $p_i[m] > 0_{X_i}$ for each $i \in I$. Hence, since $p_i[m] \leq^+ m_i$, each $p_i[\mathcal{U}]$ converges tightly to each $p_i[m]$. Thus $\mathcal{U}$ converges tightly to m with respect to each $\omega^{\{p_i\},\nu}$ (Proposition 5.2.7.) But since $\nu_X = (\sup\{\omega^{\{p_i\},\nu} \mid i \in I\})_X$, by Lemma 5.2.9, $\mathcal{U}$ converges tightly to m with respect to ν. □

Theorem 5.2.10 is already sufficient enough to produce the classical Tychonoff-Čech theorem. If we restrict ourselves to $RegNBH(\mathbf{C}, \mathcal{M})$ then it is likely that the proof involves "points" or "complements" (cf. [34, 55].)

Proposition 5.2.11. *Let $E \in \mathfrak{U}(\nu)$. Assume the following conditions:*

(i) Every morphism in $\mathbf{C}$ reflects 0, $\mathcal{E} \subseteq \mathcal{E}'$ and the following condition holds

$$\textit{If } m \wedge \bigvee_I m_i > 0 \textit{ then there is } i \in I \textit{ such that } m \wedge m_i > 0;$$

(ii) For every cardinal $\mathfrak{m}$, ν preserves $\mathfrak{m}$-products and for every functor diagram $D : \mathfrak{m} \to \mathbf{C}$ there is $n \in \mathfrak{m}$ such that $\lim(D) \to D(n)$ belongs to $\mathcal{E}$.

Then for any $X \in \mathscr{T}(E)$, there is an object $C \in \mathfrak{U}(\nu)$ and a monomorphism $p : X \to C$ which is ν-dense and ν-initial.

Proof. Note that the natural projections $E^{\mathfrak{m}} \to E$ are retractions (split epimorphisms). The result follows from Lemma 5.1.7. Proposition 5.1.8, Proposition 5.2.7 (ii) and Theorem 5.2.10. □

In **Top** the class $\mathfrak{U}(\tau)$ is the class of ultrafilter compact spaces, which, under choice-like assumptions, coincides with the class of compact spaces.

5.3 Compactness via closed morphisms

The idea of defining the notion of *separation* and that of *compactness* by requiring the diagonal morphism $X \to X \times X$ and the terminal morphism $X \to \mathbf{1}$ to be conveniently closed can be traced back to Penon [44], Manes [43] and also to the work of Herrlich, Salicrup and Strecker (cf. [57, 32].) This can be done by providing $\mathbf{C}$ with a class of morphisms that behaves like $\mathcal{K}(\nu)$ (Proposition 3.1.7.) The consequence is that a great deal of results in classical Topology can be carried to this setting. References include the monograph [13] and the papers [57, 32].

It is legitimate to think that $\mathcal{K}(\nu)$ is just a special case of such classes of closed morphisms and that it would not bring anything new. On the other hand neighbourhoods were primarily introduced to provide a suitable setting for convergence [25, 26, 34]. Our main goal in considering $\mathcal{K}(\nu)$ is to provide a less restricted approach to the Tychonoff-Čech theorem by taking the Kuratowski-Mrówka theorem as a starting point. But we want also to reinforce further the analogy between the ways topological notions are treated by closure and neighbourhood operators. This definitely helps us to determine which notions are likely to lend themselves to further "categorification".

Definition 5.3.1. (Kuratowski-Mrówka [21]) An object X is *ν-compact* or simply *compact* if for any object $Y \in \mathbf{C}$, the second projection $p_Y : X \times Y \to Y$ is closed.

This definition is responsible for the existence of an "open rectangle" in a product whose factors include compact objects. It is sometimes referred to as the *Tube Lemma*: *A space X is compact if and only if for any space Y and $y \in Y$, if O is an open set containing $X \times \{y\}$, then there is a neighbourhood N of y, such that*

$$X \times \{y\} \subseteq X \times N \subseteq O.$$

We recall from Chapter 1 that the largest pullback stable class containing $\mathcal{K}(\nu)$ is denoted by $\mathcal{K}(\nu)^*$. Also one can take the class of all pullbacks of $\mathcal{K}(\nu)$ as its approximation [57]. The morphisms in $\mathcal{K}(\nu)^*$ are called *proper morphisms* [13]. Alternatively we say that an object X is compact if the terminal morphism $t_X : X \to \mathbf{1}$ belongs to $\mathcal{K}(\nu)^*$. Indeed any projection $p_Y : X \times Y \to Y$ is a pullback of such a map:

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & \mathbf{1} \end{tikzcd}$$

Lemma 5.3.2. *The class $\mathcal{K}(\nu)^*$ is closed under composition (cf. [13]).*

Proof. Let $f, g \in \mathcal{K}(\nu)^*$ and consider the following pullback diagram.

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d] & \cdot \arrow[r, "g"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, "f'"'] & \cdot \arrow[r, "g'"'] & \cdot \end{tikzcd}$$

Both $f' \in \mathcal{K}(\nu)$ and $g' \in \mathcal{K}(\nu)$. The result follows from Proposition 3.1.7 (i). $\square$

Proposition 5.3.3. *[13]*

(i) If $f : X \to Y \in \mathcal{K}(\nu)^*$ and Y is compact, then so is X;

(ii) If $f : X \to Y$ is in $\mathcal{E}^*$ and X is compact, then so is Y;

(iii) The full subcategory of ν-compact objects in $\mathbf{C}$ is closed under finite products and under embeddings in $\mathcal{K}(\nu)^*$.

Proof. [13] (i) By universality $t_X = t_Y.f$ and follows from Lemma 5.3.2.

(ii) Follows from Proposition 3.1.7 (iii).

(iii) For two compact objects X and Y, and for any object Z the composition

$$(X \times Y) \times Z \to X \times (Y \times Z) \to Y \times Z \to Z$$

is closed by virtue of 3.1.7 (i). Finally if $m : M \to X$ is in $\mathcal{K}(\nu)^*$ then (i) implies that M is compact. $\square$

We also have the following Lemma.

Lemma 5.3.4. ***(Alexander Subbase for closed morphisms)*** *If $f : X \to Y$ is ν-closed, then it is $\theta(\nu)$-closed.*

Proof. If $l \in \theta(\nu)_X(f^{-1}[n])$ then $f^{-1}[n] \leq l_1 \wedge l_2 \wedge \cdots \wedge l_k \leq l$ where $l_i \in \nu_X(m)$, $k \in \mathbb{N}$. For each $i \in I$, $l_i = f^{-1}[n_i]$ where $n_i \in \nu_Y(n)$. But $n_1 \wedge n_2 \wedge \cdots \wedge n_k \in \theta(\nu)_Y(n)$ and $l_1 \wedge l_2 \wedge \cdots \wedge l_k = f^{-1}[n_1 \wedge n_2 \wedge \cdots \wedge n_k]$. $\square$

Theorem 5.3.5. ***(Tychonoff-Čech for closed morphisms)*** *Let I be a set and let $X = \prod_I X_i$ be a product. Assume that:*

(i) The natural projections $\{p_{X_i} : X \to X_i \mid i \in I\}$ belong to $\mathcal{E}^*$;

(ii) ν preserves I-products and any finite product.

If every X_i is ν-compact, then so is X.

Proof. Let $Y \in \mathbf{C}$. For each $i \in I$, we have the following diagram:

$$\begin{tikzcd}
X \arrow[d, "p_{X_i}"'] & X \times Y \arrow[l, "p_X"'] \arrow[r, "p_Y"] \arrow[d, "p_{X_i} \times 1_Y"] & Y \arrow[d, "1_Y"] \\
X_i & X_i \times Y \arrow[l, "p^i"] \arrow[r, "p_2"'] & Y
\end{tikzcd}$$

We denote by $p_i = p_{X_i}.p_X$ and $f = p_{X_i} \times 1_Y$ for convenience. Let $k \in \nu_{X \times Y}(p_Y^{-1}[n])$ where $n \in \mathcal{M}/Y$. We have $k \in p_Y^{-1}[\nu_Y(p_Y[p_Y^{-1}[n]])]$ or there is $i \in I$ such that $k \in p_i^{-1}[\nu_i(p_i[p_Y^{-1}[n]])]$. Since $p^i.f = p_i$ and $p_2.f = p_Y$, we have:

$$k \in f^{-1}[(p^i)^{-1}[\nu_i((p^i f)[p_Y^{-1}[n]])]]$$

or

$$k \in f^{-1}[p_2^{-1}[\nu_Y((p_2 f)[p_Y^{-1}[n]])]]$$

In either case, by continuity, there is $l \in \nu_{X_i \times Y}(f[p_Y^{-1}[n]])$ such that $f^{-1}[l] \leq k$. Since $p_{X_i} \in \mathcal{E}^*$ its pullback $f \in \mathcal{E}'$, so $f[p_Y^{-1}[n]] = f[f^{-1}[p_2^{-1}[n]]] = p_2^{-1}[n]$, and since p_2 is closed by compactness of X_i,

$$\nu_{X_i \times Y}(f[p_Y^{-1}[n]]) = \nu_{X_i \times Y}(p_2^{-1}[n]) = p_2^{-1}[\nu_Y(n)].$$

Therefore $f^{-1}[l] \in f^{-1}[p_2^{-1}[\nu_Y(n)]] = p_Y^{-1}[\nu_Y(n)]$ and so $k \in p_Y^{-1}[\nu_Y(n)]$. □

We notice that the condition (ii) in the theorem is very similar to what was called *finite structure property of products* (cf. [11] and [13, Formula (F9)]). Here this condition just says that the structures on the finite products are also initial. It is known that the Tychonoff-Čech theorem in **Top** is logically equivalent to the Axiom of Choice. Unless it is the case that the assumption according to which the natural projections belong to $\mathcal{E}^*$, requires some choice conditions (cf. [11] where this is discussed in details for closure operators), the above theorem provides a choice-free form of the Tychonoff-Čech theorem.

Because of the Alexander Subbase Lemma and since θ commutes with the suprema in $NBH(\mathbf{C}, \mathcal{M})$ and $NBHF(\mathbf{C}, \mathcal{M})$ Theorem 5.3.5 is also enough to produce the classical Tychonoff-Čech theorem. A similar compactification to the one in Proposition 5.2.11 can be obtained from Proposition 5.1.8.

Chapter 6

Towards Uniform Neighbourhoods

We suppose here that the neighbourhoods are induced from a certain uniformity. The idea is to exhibit the behaviour of the neighbourhoods when a notion of "closeness" is present to allow one to define Cauchy-like properties. We present in this chapter a few results about the notion of *completeness* that are natural consequences of such properties, namely that suitable limits of complete objects are complete.

Looking at the lattices $\mathcal{M}/X$, $X \in \mathbf{C}$, it is natural to think of *uniform covers* [58, 63] in defining uniformities. However, this will soon reveal that distributivity-like properties will be needed. We rather follow the essential ideas presented by Bourbaki in [4] and use the *surroundings* or *entourages*. The entourages themselves owe their origin to the axioms established by one of its early members in [61]: to each point x in a space X is given a system $\{V_\alpha(x) \mid \alpha \in \Lambda\}$ of neighbourhoods that satisfy a few axioms. This is to say that there are suitable maps $\{V_\alpha \mid \alpha \in \Lambda\}$, with $V_\alpha : X \to \mathcal{P}(X)$ for each $\alpha \in \Lambda$, that define the structure. This can be naturally extended to maps $\mathcal{P}(X) \to \mathcal{P}(X)$ on the power set. Thus a uniformity on an object $X \in \mathbf{C}$ is given by a family of endomaps on $\mathcal{M}/X$.

This approach is already found in [23] for frames and similar axioms to those presented in [14] in a different framework are also found here. For frames, the investigation has led to the *Weil uniformities* (cf. [46]). Early references to frame uniformities include [49] and [50]. Here, the approach is different when it comes to the expression of symmetry. Nevertheless they coincide in the point-set setting.

6.1 Uniformities

We denote by $Func(\mathcal{M}/X)$ the category of endofunctors on $\mathcal{M}/X$ for each $X \in \mathbf{C}$. We recall that for two monotone maps $f, g : P \to Q$ between two posets, regarded as functors, there is a natural transformation $\eta : f \Rightarrow g$ if and only if for any $m \in P$ we have $f(m) \leq g(m)$. Thus $Func(\mathcal{M}/X)$ is "pointwise" ordered and we shall simply denote η by $\leq$.

Definition 6.1.1. A base $\mathcal{B}_X$ for a *uniformity* $\mathcal{D}_X$ on X is a full subcategory $\mathcal{B}_X$ of

$Func(\mathcal{M}/X)$ that satisfies the following properties:

(D1) For any $D \in \mathcal{B}_X$, $1_X[-] \leq D$;

(D2) For any $E \in \mathcal{B}_X$, there is $D \in \mathcal{D}_X$ such that $D \circ D \leq E$;

(D3) If $m > 0_X$ and $n > 0_X$ in $\mathcal{M}/X$, with $m \leq D[n]$, then there is $n' \leq^+ n$ such that $n' \leq D[m]$.

The uniformity $\mathcal{D}_X$ is said to be *additive* if in addition it satisfies the following:

(FU) For any $D, E \in \mathcal{D}_X$, there is $C \in \mathcal{D}_X$ such that $C \leq D$ and $C \leq E$.

Definition 6.1.2. A uniformity of $\mathcal{M}$ in $\mathbf{C}$ is a family $\mathcal{D} = \{\mathcal{D}_X \mid X \in \mathbf{C}\}$ such that each $\mathcal{D}_X$ is a uniformity on X and such that for any morphism $f : X \to Y$ in $\mathbf{C}$ and $E \in \mathcal{D}_Y$, there is $D \in \mathcal{D}_X$ such that $f[-] \circ D \leq E \circ f[-]$.

Definition 6.1.3. A morphism $f : X \to Y$ is said to be $\mathcal{D}$-initial for a uniformity $\mathcal{D}$ if for any $D \in \mathcal{D}_X$, there is $E \in \mathcal{D}_Y$ such that $f^{-1}[-] \circ E \circ f[-] \leq D$. f is said to be $\mathcal{D}$-final if for each $D \in \mathcal{D}_Y$ and $m, n \in \mathcal{M}/Y$, we have $m \leq D[n]$ provided that there is $E \in \mathcal{D}_X$ such that $1_Y \leq f[-] \circ E \circ f^{-1}[-] \leq D$

We are mainly interested in subobjects and products in this chapter regarding constructions (limits). Hence we shall not investigate further the notion of $\mathcal{D}$-final morphisms.

Now, given a morphism $f : X \to Y$ and a uniformity $\mathcal{D}$ the collection $\{D_f \mid D \in \mathcal{D}_Y\}$, where

$$D_f = f^{-1}[-] \circ D \circ f[-],$$

gives rise to a base for a uniformity $\mathcal{D}_{X_f}$ on X when $\mathcal{E} \subseteq \mathcal{E}'$ and morphisms in $\mathcal{E}$ reflect 0. Indeed we have:

$$1_X[-] \leq f^{-1}[-] \circ f[-] \leq D_f.$$

Also if $D^2 \leq E$ in D_Y, then $D^2 \circ f[-] \leq E \circ f[-]$, and so

$$(D_f)^2 \leq (D^2)_f \leq E_f.$$

Finally if $m \leq D_f[n]$ with $m > 0_X$ and $n > 0_X$, then $f[m] \leq D[f[n]]$. There is $q \leq^+ f[n]$ such that $q \leq D[f[m]]$. We have $f^{-1}[q] \wedge n \leq^+ n$ and

$$f^{-1}[q] \wedge n \leq f^{-1}[n] \leq D_f[m].$$

A set-union of uniformities is of course not a uniformity in general since the additivity axiom (FU) is not always satisfied for example (cf. [63].) However, the supremum exists and it is given by finite meets from the member of the union. In our case, a union would already suffice. The axiom (FU) is measured by an adjunction which is similar to the one present between $NBH(\mathbf{C}, \mathcal{M})$ and $NBHF(\mathbf{C}, \mathcal{M})$.

Definition 6.1.4. A source $\mathcal{S} = \{f_i : X \to X_i \mid i \in I\}$ is said to be $\mathcal{D}$-initial if for any $D \in \mathcal{D}_X$, there is $i \in I$ and $E \in \mathcal{D}_{X_i}$ such that

$$(f_i)^{-1}[-] \circ E \circ f_i[-] \leq D.$$

Now, for any $X \in \mathbf{C}$ and $m \in \mathcal{M}/X$, consider the following collection:

$$\nu_X(m) = \{l \mid (\exists D \in \mathcal{D}_X), l \geq D[m]\}.$$

The family $\{\nu_X \mid X \in \mathbf{C}\}$ gives rise to a neighbourhood operator ν of $\mathcal{M}$ in $\mathbf{C}$. The neighbourhood operators induced in that way are called *uniform neighbourhood operators*.

If ν is a neighbourhood operator induced by a uniformity $\mathcal{D}$ then a map (resp. source) which is $\mathcal{D}$-initial is ν-initial. (We point out that even if τ-final morphisms in **Top** are described by some ν-final morphisms, then $\mathcal{D}$-finality and ν-finality are not related at all. The book [4] shows interesting examples of this case.)

For the sequel we fix a neighbourhood operator ν and we suppose that it is induced by a uniformity $\mathcal{D}$.

6.2 Completeness

For this section, we assume also the Ultrafilter Lemma.

Definition 6.2.1. A filter $\mathcal{F} \subseteq \mathcal{M}/X$ is said to be a *$\mathcal{D}$-Cauchy filter* or simply a *Cauchy filter* if for any $D \in \mathcal{D}_X$, there is $p \in \mathcal{F}$ such that for all $q \leq^+ p$, $p \leq D[q]$.

When a filter $\mathcal{F}$ converges tightly to a subobject l, then we shall write $l = \lim(\mathcal{F})$. It is sometimes convenient to assume the following condition for a limit l of $\mathcal{F}$:

(L) If $D \in \mathcal{D}_X$ and $q \in \nu_X(m)$, where $m \leq^+ l$, then $D[l] \leq D^2[q]$.

In the relation $D[l] \leq D^2[q]$, one can consider a fixed number $n \in \mathbb{N}$ and assume $D[l] \leq D^n[q]$. The idea is that taking neighbourhoods of the neighbourhoods yields “bigger" neighbourhoods and that the ν-structure on the limit l is not so complicated (cf. Chapter 4, Section 4.1 where the ν-structure on $\mathbf{1}$ is assumed to be unique.)

Proposition 6.2.2. *Let $\mathcal{F} \subseteq \mathcal{M}/X$ be a convergent filter and let $l = \lim(\mathcal{F})$. If l satisfies (L), then $\mathcal{F}$ is a Cauchy filter.*

Proof. Let $D, E \in \mathcal{D}_X$ with $E^3 \leq D$. There is $p \in \mathcal{F}$ such that $p \leq E[l]$. If $q \leq^+ p$ then there is $r_q \leq^+ l$ such that $r_q \leq E[q]$. We have:

$$p \leq E[l] \leq E^2[r_q] \leq E^3[q] \leq D[q].$$

□

Lemma 6.2.3. *Let $\mathcal{F}$ be a Cauchy filter on X and let $\mathcal{U}$ be an ultrafilter such that $\mathcal{F} \subseteq \mathcal{U}$. Then $\mathcal{U}$ is a Cauchy filter. Furthermore if $l = \lim(\mathcal{U})$, then $l = \lim(\mathcal{F})$.*

Proof. The first statement follows from the definition. Let $l = \lim(\mathcal{U})$ and let $E \in \mathcal{D}_X$. Let $D \in \mathcal{D}_X$ such that $D \circ D \leq E$. If $m \leq^+ l$ then there is $u \in \mathcal{U}$ such that $u \leq D[m]$. Since $\mathcal{F}$ is Cauchy, there is $p \in \mathcal{F}$ such that $p \leq D[q]$ for any $q \leq^+ p$. Since $\mathcal{U}$ is a filter, $p \wedge u > 0_X$ and so $p \leq D[p \wedge u] \leq D[u] \leq D^2[m] \leq E[m]$. □

Definition 6.2.4. X is said to be *ν-precompact* with respect to $\mathcal{D}$ or simply *precompact* if for any filter $\mathcal{F}$, there is an ultrafilter $\mathcal{U}$ containing $\mathcal{F}$ such that $\mathcal{U}$ is Cauchy. X is said to be *$\mathcal{D}$-complete* or simply *complete* if for every Cauchy filter on X, there is $l > 0_X$ such that $l = \lim(\mathcal{F})$.

Proposition 6.2.5. *(i) If an object X is precompact and complete, then $X \in \mathfrak{U}(\nu)$;*

(ii) Assume that the condition (L) is satisfied for any filter $\mathcal{F} \subseteq \mathcal{M}/X$ for which it makes sense. If $X \in \mathfrak{U}(\nu)$, then it is precompact and complete.

Proof. (i) Follows from the definition.

(ii) We shall first prove that X is precompact. For a filter $\mathcal{F}$ on X, let $\mathcal{U}$ be an ultrafilter such that $\mathcal{F} \subseteq \mathcal{U}$. There is $l > 0_X$ such that $l = \lim(\mathcal{U})$ and by Proposition 6.2.2, $\mathcal{U}$ is Cauchy.

Now, let $\mathcal{F}$ be a Cauchy filter. If $\mathcal{U}$ is an ultrafilter such that $\mathcal{F} \subseteq \mathcal{U}$, then $\mathcal{U}$ is Cauchy. Since $X \in \mathfrak{U}(\nu)$, there is $l > 0_X$ such that $l = \lim(\mathcal{U})$. The result follows from Lemma 6.2.3. □

Lemma 6.2.6. *(i) Let $f : X \to Y$ be in $\mathbf{C}$ such that f reflects 0. Assume in addition that $\mathcal{E} \subseteq \mathcal{E}'$. If $\mathcal{F} \subseteq \mathcal{M}/X$ is a Cauchy filter, then so is $f[\mathcal{F}]$;*

(ii) If $m : M \to X \in \mathcal{M}$ and $\mathcal{F} \subseteq \mathcal{M}/M$ is a Cauchy filter, then so is $m[\mathcal{F}]$.

Proof. (i) If $D \in \mathcal{D}_Y$, then there is $E \in \mathcal{D}_X$ such that $E \leq f^{-1}[-] \circ D \circ f[-]$. There is $p \in \mathcal{F}$ such that for any $q \leq^+ p$:

$$p \leq E[q] \leq f^{-1}[D[f[q]]], \text{ or } f[p] \leq D[f[q]].$$

Since $k \leq^+ f[p]$ and $f^{-1}[k] \wedge p \leq^+ p$,

$$f[p] \leq D[f[f^{-1}[k] \wedge p]] \leq D[k \wedge f[p]] = D[k].$$

(ii) Every subobject m reflects 0 and satisfies the condition $m[m^{-1}[a] \wedge b] = a \wedge mb$ for any appropriate subobjects a and b (cf. Proof of Proposition 1.3.6.) $\square$

Thanks to Lemma 6.2.6, by replacing the definition "*every ultrafilter converges tightly*" with "*every Cauchy ultrafilter converges tightly*", one obtains essentially the same results as in Chapter 5, Section 5.2 (i.e. Proposition 5.2.7 (i)(b) and (ii), and Theorem 5.2.10) about ultrafilter compactness.

Proposition 6.2.7. *(i) Suppose that f is ν-initial and that f reflects 0. Assume that $\mathcal{E} \subseteq \mathcal{E}'$. If Y is complete, then so is X.*

(ii) Let $m : M \to X$ be in $\mathcal{M}$ such that m is $\mathfrak{cl}_1^\nu$-closed or $\mathfrak{cl}_2^\nu$-closed. If X is complete, then so is M.

Theorem 6.2.8. *Let I be a set and $X : I \to \mathbf{C}$ a diagram that admits a limit denoted also by X. Assume that the natural projections $p_i : X \to X_i$, $i \in I$, belong to $\mathcal{E}'$, reflect 0 and are $\mathcal{D}$-initial. Assume in addition that for any diagram $D : I \to \mathbf{C}$ that admits a limit D, there is an index $i_0 \in I$ such that $D \to D_{i_0} \in \mathcal{E}$. Then X is complete provided that each factor X_i is complete.*

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