A BRIEF INTRODUCTION TO BASIC MULTIVARIATE ECONOMIC STATISTICAL PROCESS CONTROL

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DECLARATION

I, the undersigned, hereby declare that the work contained in this assignment is my own original work and that I have not previously in it's entirety or in part submitted it at any University for a degree.

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ABSTRACT

Statistical process control (SPC) plays a very important role in monitoring and improving industrial processes to ensure that products produced or shipped to the customer meet the required specifications. The main tool that is used in SPC is the statistical control chart. The traditional way of statistical control chart design assumed that a process is described by a single quality characteristic. However, according to Montgomery and Klatt (1972) industrial processes and products can have more than one quality characteristic and their joint effect describes product quality. Process monitoring in which several related variables are of interest is referred to as multivariate statistical process control (MSPC). The most vital and commonly used tool in MSPC is the statistical control chart as in the case of the SPC. The design of a control chart requires the user to select three parameters which are: sample size, n, sampling interval, h and control limits, k. Several authors have developed control charts based on more than one quality characteristic, among them was Hotelling (1947) who pioneered the use of the multivariate process control techniques through the development of a T^2-control chart which is well known as Hotelling T^2-control chart.

Since the introduction of the control chart technique, the most common and widely used method of control chart design was the statistical design. However, according to Montgomery (2005), the design of control has economic implications. There are costs that are incurred during the design of a control chart and these are: costs of sampling and testing, costs associated with investigating an out-of-control signal and possible correction of any assignable cause found, costs associated with the production of nonconforming products, etc. The paper is about giving an overview of the different methods or techniques that have been employed to develop the different economic statistical models for MSPC.

The first multivariate economic model presented in this paper is the economic design of the Hotelling's T^2-control chart to maintain current control of a process developed by Montgomery and Klatt (1972). This is followed by the work done by Kapur and Chao (1996) in which the concept of creating a specification region for the multiple quality characteristics together with the use of a multivariate quality loss function is implemented to minimize total loss to both the producer and the customer. Another approach by Chou *et al* (2002) is also presented in which a procedure is developed that simultaneously monitor the process mean and covariance matrix through the use of a quality loss function. The procedure is based on

the test statistic $-2\ln L$ and the cost model is based on Montgomery and Klatt (1972) as well as Kapur and Chao's (1996) ideas. One example of the use of the variable sample size technique on the economic and economic statistical design of the control chart will also be presented. Specifically, an economic and economic statistical design of the T^2-control chart with two adaptive sample sizes (Faraz*et al*, 2010) will be presented. Faraz*et al* (2010) developed a cost model of a variable sampling size T^2-control chart for the economic and economic statistical design using Lorenzen and Vance's (1986) model.

There are several other approaches to the multivariate economic statistical process control (MESPC) problem, but in this project the focus is on the cases based on the phase II stadium of the process where the mean vector, μ and the covariance matrix, Σ have been fairly well established and can be taken as known, but both are subject to assignable causes. This latter aspect is often ignored by researchers. Nevertheless, the article by Faraz*et al* (2010) is included to give more insight into how more sophisticated approaches may fit in with MESPC, even if the mean vector, μ only may be subject to assignable cause.

Keywords: control chart; statistical process control; multivariate statistical process control; multivariate economic statistical process control; multivariate control chart; loss function.

OPSOMMING

Statistiese proses kontrole (SPK) speel ’n baie belangrike rol in die monitering en verbetering van industriële prosesse om te verseker dat produkte wat vervaardig word, of na kliënte versend word wel aan die vereiste voorwaardes voldoen. Die vernaamste tegniek wat in SPK gebruik word, is die statistiese kontrolekaart. Die tradisionele wyse waarop statistiese kontrolekaarte ontwerp is, aanvaar dat ’n proses deur slegs ’n enkele kwaliteitsveranderlike beskryf word. Montgomery and Klatt (1972) beweer egter dat industriële prosesse en produkte meer as een kwaliteitseienskap kan hê en dat hulle gesamentlik die kwaliteit van ’n produk kan beskryf. Proses monitering waarin verskeie verwante veranderlikes van belang mag wees, staan as meerveranderlike statistiese proses kontrole (MSPK) bekend. Die mees belangrike en algemene tegniek wat in MSPK gebruik word, is ewe eens die statistiese kontrolekaart soos dit die geval is by SPK. Die ontwerp van ’n kontrolekaart vereis van die gebruiker om drie parameters te kies wat soos volg is: steekproefgrootte, n, tussen-steekproefinterval, h en kontrolegrense, k. Verskeie skrywers het kontrolekaarte ontwikkel wat op meer as een kwaliteitseienskap gebaseer is, waaronder Hotelling wat die gebruik van meerveranderlike proses kontrole tegnieke ingelei het met die ontwikkeling van die T^2-kontrolekaart wat algemeen bekend is as Hotelling se T^2-kontrolekaart (Hotelling, 1947).

Sedert die ingebruikneming van die kontrolekaart tegniek is die statistiese ontwerp daarvan die mees algemene benadering en is dit ook in daardie formaat gebruik. Nietemin, volgens Montgomery and Klatt (1972) en Montgomery (2005), het die ontwerp van die kontrolekaart ook ekonomiese implikasies. Daar is kostes betrokke by die ontwerp van die kontrolekaart en daar is ook die kostes t.o.v. steekproefneming en toetsing, kostes geassosieer met die ondersoek van ’n buite-kontrole-sein, en moontlike herstel indien enige moontlike korreksie van so ’n buite-kontrole-sein gevind word, kostes geassosieer met die produksie van nie-konforme produkte, ens. In die eenveranderlike geval is die hantering van die ekonomiese eienskappe al in diepte ondersoek. Hierdie werkstuk gee ’n oorsig oor sommige van die verskillende metodes of tegnieke wat al daargestel is t.o.v. verskillende ekonomiese statistiese modelle vir MSPK. In die besonder word aandag gegee aan die gevalle waar die vektor van gemiddeldes sowel as die kovariansiematriks onderhewig is aan potensiële verskuiwings, in teenstelling met ’n neiging om slegs na die vektor van gemiddeldes in isolasie te kyk synde onderhewig aan moontlike verskuiwings te wees.

Een van die eerste meerveranderlike ekonomiese statistiese modelle wat voorgestel is, is díe van Montgomery en Klatt (1972), waarin 'n kostemodel vir Hotelling se T^2-kontrolekaart ontwikkel is. Dit is gevolg deur o.a. die werk wat deur Kapur en Chao (1996) gedoen is met die konsep van 'n spesifikasie omgewing vir die meerveranderlike kwaliteitseienskappe, tesame met die gebruik van 'n meerveranderlike verliesfunksie wat geïmplimenteer word om die totale verlies te minimeer, beide vir die produsent sowel as vir die kliënt. 'n Verdere prosedure is deur Chou *et al* (2002) ontwikkel wat beskryf hoe die prosesgemiddelde sowel as die kovariansiematriks gelyktydig gemoniteer word deur o.a. die gebruikmaking van 'n kwaliteitsverliesfunksie, L. Die prosedure is gebaseer op die toetsstatistiek, $-2\ln L$ sowel as op 'n samevoeging van die kostemodelle en idees van Montgomery en Klatt (1972) en Kapur and Chao (1996). 'n Voorbeeld van die veranderlike steekproefgrootte tegniek soos toegepas op die meerveranderlike ekonomiese en ekonomies statistiese ontwerp van die kontrolekaart word ook aangebied. Meer spesifiek, 'n meerveranderlike ekonomies statistiese ontwerp van die T^2-kontrolekaart met twee aanpassende steekproefgroottes (Faraz *et al*, 2010) word voorgelê. Faraz *et al* (2010) het 'n kostemodel ontwikkel vir die veranderlike steekproefgrootte T^2-kontrolekaart vir die meerveranderlike ekonomiese en ekonomies statistiese ontwerp, deur van Lorenzen and Vance se (1986) model gebruik te maak, dit aan te pas vir die meerveranderlike situasie en daaropvolgens te implementeer.

Daar bestaan verskeie ander benaderings tot die meerveranderlike ekonomies statistiese proses kontrole (MESPK), maar in hierdie projek is die fokus op fase II van die proses waar die gemiddelde vektor, μ en die kovariansiematriks, Σ reeds gestabiliseer het of in ieder geval bekend is, maar waar beide blootgestel is aan moontlike verskuiwing(s). Hierdie laaste aspek word dikwels deur navorsers geïgnoreer in 'n poging om die oplossing te vereenvoudig, aangesien die byvoeging van die kovariansiematriks in die hipotese van geen verskuiwing die probleem enigsins kompliseer. Desnieteenstaande is 'n opsomming van die artikel van Faraz *et al* (2010) ingesluit om meer insae te gee in hoe meer gesofistikeerde benaderings mag inpas by MESPK, self al is slegs die gemiddelde vektor, μ hier potensieel onderhewig aan 'n buite-kontrole sein.

Sleutelwoorde: kontrolekaart; statistiese proses kontrole; meerveranderlike statistiese proses kontrole; meerveranderlike ekonomiese statistiese proses kontrole; meerveranderlike kontrolekaarte; verliesfunksie.

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CHAPTER 1
Introduction

1.1 Introduction to SPC

In this modern day and age industry has been faced with a lot of competition. This has put pressure on the manufacturing industry to find ways to reduce costs of production while increasing production and improving the quality of products. The quality of products plays a very important role in customer decisions, therefore it is vital to look at the many ways of quality improvement methods that have helped industry to remain competitive. Statistical methods play a very important role in quality improvement in manufacturing industries (Woodall, 2000). The most commonly used statistical technique for quality improvement is the statistical process control technique (SPC).

"Statistical process control is a powerful collection of problem-solving tools useful in achieving process stability and improving process capability through the reduction of variability" (Montgomery; 2005). According to Woodall(2000), SPC is a statistical technique that consists of methods that helps industry understand, monitor and improve the performance of a process over time. Literature has proven that the use of SPC gives tremendous results in improving product quality and in reducing production costs. The main objective of the statistical process control is to monitor closely the production system so that any perturbations in the flow of the process can be detected quickly before a mass production of defective products takes place. This is usually done through the use of the statistical control chart technique. Historically the statistical control chart technique was developed in the 1920's by Dr Walter A. Shewhart and these types of control charts are now well known as the Shewhart control charts (Shewhart, 1931).

The main idea behind the development of the control chart is the theory of variability described by Shewhart. There are two distinct types of causes of process variation: common cause and assignable cause. The common causes are also referred to as the natural causes, and are assumed to work all the time and naturally become part of the system. The common/natural causes are usually unavoidable and a process that operates under the natural causes is said to be in statistical control. A process in statistical control is described as stable, predictable and as one that exhibits least inherent variability. The other type of causes of process variability is the assignable causes. These are the causes that were not part of the system as it was developed. The presence of the assignable causes in a process results in an unstable process. When a process is operating under the influence of these assignable causes it is said to be out of control. The ultimate purpose of control charts is to monitor the process and keep the assignable causes out of the process to ensure that the process is in a state of statistical control thereby improving product quality.

A control chart is a graphical display that shows whether sample statistics calculated from samples taken periodically from a process plots within the specified control limits. It is the most powerful tool in SPC which is used to monitor and maintain the process so that the process remains in statistical control. Figure 1.1 shows a typical control chart with an example of an out of control point.

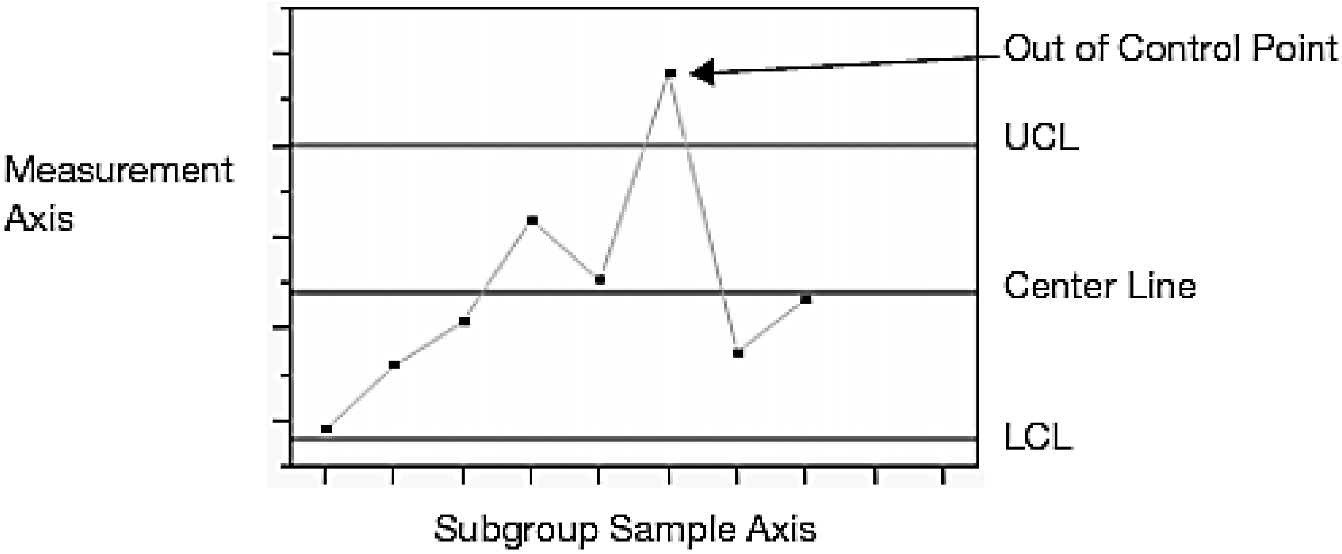

Figure 1.1: A typical control chart

It can be seen that the control chart is characterized by three lines, the upper control limit (UCL), the center line and the lower control limit (LCL). The upper and lower control limits are the highest and lowest expected values in a stable process respectively. The center line is the average or the mean of the selected samples. If a point falls outside the two control limits the process is said to be out of control and investigations and corrective measures are carried out before a mass production of defective products takes place.

1.2 SPC control chart design

The design of a control chart consists of selecting three design parameters, the sample size n, sampling frequency h, and the width of the control limit k. In the early years of control chart development the most common type of control chart design was the statistical design which uses statistical criteria to determine suitable control limits through putting some bounds on the average run length (ARL). The ARL is used to determine or to measure the effectiveness of the control chart. It is defined as the number of points plotted on the chart before an out-of-control state is reached while the system is still in control. The rate at which sampling is done is rarely determined by analytic methods. This therefore means that one has to consider such factors as the production rate, the expected frequency of shift to an out-of-control state, and the possible results of having such process shifts in selecting the sampling interval (Montgomery, 2005).

The design of a control chart has some economic implications. There are costs that researchers have become aware of in the design of a control chart. These are costs of sampling and testing, costs associated with investigating an out-of-control signal and possible correction of any assignable cause found, costs associated with the production of nonconforming products, etc. Due to these resulting costs it is only logical to take into consideration the design of a control chart from an economic point of view (Montgomery, 2005).

The main goal in the design of the economic control chart is to determine the optimal values for the three test parameters: sample size, n, sampling interval, h and control limits, k for the control chartto ensure that the expected costs of monitoring production process is minimized (Noorossana*et al*, 2002). The first economic design which was based on the ideas from Girshick and Rubin (1956) was developed by Duncan in 1956 (Duncan, 1956). The economic model was for the $\bar{x}$-control chart and was based on the assumption that there exists only a single out-of-control state. In his model Duncan included the cost of sampling and inspection, the cost of searching for an assignable cause, the costs of producing defective products, the costs of false alarms and the cost of correcting the process (Chou *et al*, 2002). Duncan's model was a cost function of the three test parameters: sample size, n, sampling interval, h and control limits, k. According to Montgomery (2005) the cost function can be defined as the expected loss per hour incurred by the process. The optimum values of the test parameters are obtained through adopting optimization procedures. According to Faraz*et* *al*(2010) the economic design of a control chart has some limitations. This is because it does not take into consideration the statistical performance and properties of the charts. Saniga (1989) developed a model for the economic statistical design of the control chart which puts statistical constraints on the optimal economic design.

1.3 Multivariate Statistical Process Control

A lot of attention has been given to the design of the control chart where only one quality characteristic is of interest (Montgomery and Klatt, 1972). However, according to the two authors industrial products and processes are characterized by more than one measurable quality characteristic and their joint effect describes product quality. Bersimis*et a*l (2007) also mentioned that there are many instances in the industry in which it is necessary to simultaneously monitor more than one quality characteristic on a product. Treating these quality characteristics as independent may give very misleading results. Montgomery and Klatt (1972) gave an illustration where different results were obtained when two quality characteristics where treated independently and their product computed. The results showed that both the quality characteristics, if monitored independently give an in-control state and, if their probabilities are multiplied they give an out-of-control-state. This is quite misleading and therefore the need to develop process control methods based on two or more related variables.

Process monitoring in which several variables are of interest is called multivariate statistical process control (MSPC). Hotelling (1947) was the first author to write about the MSPC. As with the univariate case the main tool used for monitoring MSPC is through the use of the quality control chart. MSPC procedure involves fulfilling four conditions:

1. One should be able to state if the process is in control or not.
2. Should be able to know if there was/is a false signal.
3. Should be able to know the relationship amount variables, attributes should also be taken into consideration.
4. If the process is out of control, one should know the reasons why it’s out of control (Bersimis*et a*l, 2007).

There are two phases of control charting practice and these are:

Phase 1: This phase involves the use of the control chart to test whether a process was in control at a time when the first sample was taken. The main purpose of this phase is to encourage the practitioner to depend on the control chart to ensure that the process is in statistical control. This is a stage in which the control chart parameters are established.

Phase 2: In this phase control charts are used to test whether a process remains in control after the initial stage. The main purpose of this phase is to make use of control charts to help the practitioner to monitor the process for any change from an in-control state. The practitioner in this phase monitors the process regardless of whether the parameters of the process, μ_0 and Σ_0 were initially known or estimated. During this stage μ_0 and Σ_0 are treated as given if possible. In this project the concentration is on phase two where μ_0 and Σ_0 are known or can be computed assuming a normal distribution together with the exponential distribution in between time arrivals of assignable causes.

Hotelling (1947) was the first author to develop a quality control chart for several related variables and the control chart is well known as the Hotelling T^2 control chart. The Hotelling T^2 control chart is rated as the most widely used multivariate control chart that deals with changes in the mean vector of p correlated quality characteristics (Aparisi and Haro, 2001). The control procedure for the T^2 control chart is based on the concept of statistical distance, which is a generalization of the T-statistic. The control chart is developed under the assumption that there exists a random vector $X(p\text{x}1)$ whose jth element is the jth quality characteristics. The distribution of X is then assumed to be the p-variate normal distribution and written as follows:

$$f(x) = 1/[(2\pi)^{p/2}\,|\Sigma|^{(1/2)}]\exp\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\}, \tag{1.1}$$

where $\mu = E(X)$ is the mean vector of the p quality characteristics and $\Sigma = Cov(X)$ is the $(p\times p)$ covariance of X. ($(^T)$ represents the transpose operation). It is important to note that μ and Σ are population mean and covariance matrices. In most cases these parameters are unknown, because rarely do scientists deal with the whole population, but rather with a sample. The sample means and the sample covariances are used instead. The control procedure of the control chart includes calculating the test statistic

$$T^2 = n(\bar{X} - \mu_0)^T S^{-1}(\bar{X} - \mu_0) \sim T^2_{p,n-p} \tag{1.2}$$

to test H$_0$: $\mu = \mu_0$ where μ_0 is a value of μ that corresponds to the in-control state. The process is declared out of control if $T^2 > T^2_{\alpha,p,n-p}$ where $T^2_{\alpha,p,n-p}$ is the α percentage point of the T^2 distribution. The Hotelling T^2 control charts monitor the process mean vector only (Chou *et al*, 2002).

Kapur and Chao (1996) presented another method proposed by Chen and Kapur, (1989) on the use of a multivariate quality loss function (MQLF) on dependent quality characteristics. MQLF is a method in which the bias and the variance of each quality characteristic, interaction between the biases and covariances between the quality characteristics are considered. This is because according to Kapur and Chao (1996) loss of product quality is minimized if the variance and the bias of the quality characteristic are reduced since the process mean can approximately be adjusted to the target value. Therefore the focus here is on finding ways to minimize or reduce the bias and the covariances of the expected loss. One technique to improve the quality of such a system is to develop and implement a specification region for the process, and truncate the distribution of the quality characteristics, which is assumed to be multivariate normal, by inspection based on the specification region. A specification region is the region that defines the joint intersection of the specification limits. This according to Kapur and Chao (1996) ensures that all products produced during the time of monitoring fall within the specification limits.

Chou *et al* (2002) developed the economic statistical design of a multivariate control chart for monitoring both the process mean vector and the covariance matrix of *p* qualitycharacteristics concurrently, using the test statistic $-2\ln L$. The statistical part of the design involves considering statistical constraints of the chart, i.e. the type-I and type-II probability errors and this is achieved by evaluating the distribution function of the test statistic $-2\ln L$ under the null and alternative hypotheses. The distributions are obtained through a set of steps which will be given in more detail in chapter five.

The design of a control chart for MSPC, as with SPC has economic implications. Montgomery and Klatt (1972) developed a cost model for the Hotelling T^2 control chart. The cost model is based on computing the expected cost per unit of sampling and carrying out the test procedure, the expected cost per unit associated with investigating and correcting the process when an out-of-control state is detected and finally, the expected cost per unit associated with producing defective products. The derivations of the model resulted in a number of probability vectors and so more detail will be given in chapter four on the development of the cost model as well as the probability vectors.

Kapur and Chao (1996) developed an optimization procedure for the development of a specification region based on the framework of MQLF. The optimization model consists of three types of quality loss: loss due to variability from the target value, loss due to inspection and loss due to scrap. The expected total loss (ETL) per unit product is obtained from summing the expected loss due to variability, scrap costs and inspection cost. More detailed information will be given on the loss function and the expected total loss per unit product in subsequent chapters. Chou *et al* (2002) developed an economic statistical cost model of the multivariate control chart by combining the cost function presented in Montgomery and Klatt (1972) and the multivariate loss function presented in Kapur and Chao (1996) to come up with a cost model that they used as the objective function of the design that needs to be minimised. Besides these mentioned, several other authors have given attention to the development of multivariate economic statistical process control (MESPC).

1.4 Organisation of the study

This paper presents a brief overview of the basic MESPC methods. The outline for the remainder of the thesis is as follows: In chapter two a review is given of the calculation of a cost function i.e. an economic approach for a single variable by Duncan (1956) as described by Montgomery (2005). This is followed in chapter three by the introduction to economic statistical process control for a single variable, but taking into account statistical properties such as the ARL, bounds on the probability of the type-I error as well as the probability of the type-II error etc. In chapter four the T^2 measure of Hotelling for the multivariate case is discussed briefly, this is then enhanced by a discussion on how Montgomery and Klatt (1972) introduced economic aspects to multivariate process control. This is then followed by a further discussion on an approach as advocated by Kapur and Chao (1996). The next two chapters i.e. chapter six and seven discuss the approach by Chao *et al* (2002) and Faraz*et al* (2010). Faraz's approach deviates somewhat from the goal that was initially set, but it is thought that it brings a nice angle to the problem and its solution. Thereafter some comment is made about solution techniques, conclusion and brief references to the approach by Love and Linderman (2003).

CHAPTER 2
Economic Design of a Control Chart

2.1 Control chart design

The design of a control chart requires the selection of three test parameters: the sample size, n, sampling frequency, h, and the control limit interval k. A control chart is defined as a graphical display that is used to monitor and maintain statistical control of a process. In the early years of control chart development, control chart design was centred on certain statistical criteria. This entails that the main objective of this type of design was to select the sample size and control limits that ensure that the capability of the chart, measured by the average run length (ARL) to detect a particular shift in the quality characteristics and the ARL of the system when the process is in control are equivalent to a specified value. This worked well for a number of years until researchers realised that there are costs that are involved in control chart design. These costs according to Montgomery, (2005) are: costs of sampling and testing, costs associated with investigating an out-of-control signal and possible correction, costs due to the production of non-conforming products. Due to these costs it was imperative to logically consider designing a control chart from an economic perspective. The remainder of this chapter is based mainly on Montgomery (2005) as initiated by Duncan (1956).

2.1.1 Assumptions:

1. The process is regarded as having only one in-control state μ_0 and $s \geq 1$ out-of-control states, with each out-of-control state associated with a specific type of assignable cause.
2. The assignable cause is assumed to occur according to a Poisson process.
3. When the process is out of control investigations and corrective measures are required to ensure that the process is in control.
4. Process shifts from one state to the other are abrupt.

There are three types of parameter costs associated with the design of a control chart. As mentioned before these are: the costs of sampling and testing, costs of investigating and possible correction of an assignable cause, costs of producing defective products. The main feature of economic model formulation is the use of a total cost function to find the relationships between the control chart design parameters and the cost parameters. The production, observations and modifications of the process can be perceived as a series of uncorrelated cycles over time. A cycle consists of stages, the first stage is when the process is in the in-control state, and continues until the process goes out of control. As soon as the process indicates an out-of-control state investigations and possible adjustments are done to the process to ensure that it goes back to its original state, i.e. the in-control state.

2.2 Model development

Define $E(T)$ as the expected mean length of a cycle and $E(C)$ as the expected total cost incurred during a cycle, then the expected cost per unit of time is computed as

$$E(A) = \frac{E(C)}{E(T)}. \tag{2.1}$$

Montgomery, (2005) pointed out that equation(2.1) has an unusual form in that both C and T are correlated random variables. It is a well-known fact that the expected value of a ratio is not equal to the ratio of expected values, therefore some explanations of the form of this equation seems justified. The justification is that the order of production-monitoring-adjustment, with accumulation of costs over the cycle, can be presented by a particular type of stochastic process called a renewal reward process. The stochastic processes of this type have the property that their average time cost is given by the ratio of the expected reward per cycle to the expected cycle length.

2.3 Brief literature review

Girshick and Rubin (1952) were the first researchers to suggest the expected cost per unit time procedure for the expected cost per unit in time equation(2.1) and thoroughly showed its relevance in this problem. All the proceeding work done by different researchers on the use of equation(2.1) was based on the early work done by Girshick and Rubin (1952). Bather (1963), Ross (1971) and Savage (1962) were among the researchers who tried to investigate the Girshick –Rubin model formulation. However the outcomes were very theoretical, such that they could not be easily implemented by practitioners. Another author like Weiler, (1952) suggested that for an $\bar{X}$ chart, the optimum sample size should minimize the total amount of inspection required to detect a specified shift. He went on to give different optimal sample sizes assuming that the shift is from an in-control state μ_0 to an out-of-control state $\mu_1 = \mu_0 + \delta\sigma$

$$n = \frac{12.0}{\sigma^2} \text{ when } \pm 3.09 \text{ -sigma control limits are used.} \tag{2.2}$$

$$n = \frac{11.1}{\sigma^2} \text{ when} \pm 3 \text{ -sigma control limits are used.} \tag{2.3}$$

$$n = \frac{6.65}{\delta^2} \text{ when } \pm 2.58 \text{ -sigma control limits are used.} \tag{2.4}$$

$$n = \frac{4.4}{\delta^2} \text{ when} \pm 2.33 \text{ -sigma control limits are used.} \tag{2.5}$$

The problem with Weiler (1952) is that he did not put into consideration the costs and this has implications in that once the total inspection is minimised it will also minimize the total costs. Taylor (1965) has shown that control procedures based on taking samples of constant size at fixed intervals is non-optimal. He suggested that sample size and sampling should be determined at each point in time based on the posterior probability that the process is in an out-of-control state. Although in his subsequent papers he develops the optimal control rule for a two-stage process with a normally distributed quality characteristics many practitioners are reluctant to used his rules but rather use the fixed sample size fixed sampling interval control rules which are easier to implement.

2.4 An Economic model of the $\bar{x}$ control chart

The economic models for the control charts have been devoted to the most used chart, the $\bar{x}$ chart. The first author to suggest an economic model for the optimum economic design of the $\bar{x}$ control chart was Duncan (1956). In his paper he showed how he developed the first full economic model of a Shewhart-type control chart as well as how he managed to incorporate formal optimization methodology into determining the control chart parameters. Duncan (1956) was motivated by the work done by Girshick and Rubin (1952), and used some of their ideas to develop the cost model for the $\bar{x}$ control chart. He made the following assumptions

Assumptions:

1. The process is characterised by an in-control state μ_0.
2. A single assignable cause of magnitude δ which occurs at random, results in a shift in the mean from μ_0 to either $\mu_0 + \delta\sigma$ or $\mu_0 - \delta\sigma$.
3. The process is monitored by an $\bar{x}$ chart with centre line μ_0 and upper and lower control limits $\mu_0 \pm k\left(\sigma/\sqrt{n}\right)$.
4. Samples are to be taken at intervals of h hours.
5. When a point exceeds a control limit, a search for assignable cause is initiated
6. The process is allowed to continue in operation during the search of an assignable cause.
7. Cost of adjusting or repairs is not charged against the net income from the process.
8. The assignable cause is assumed to occur according to a Poisson process with intensity of λ occurrences per hour and the time the process remains in an in-control state is an exponential variable with mean $\frac{1}{\lambda}$ h.

2.5 Duncan's model

Given that the process shifts to the out-of-control state between the jth and the $(j+1)$st samples, the expected time of occurrence within this interval is given by

$$E\left(t - jh\right) = \tau = \frac{\int_{jh}^{(j+1)h} \lambda(t - jh)e^{-\lambda t}dt}{\int_{jh}^{(j+1)h} \lambda e^{-\lambda t}dt} = \frac{1-(\lambda h+1)e^{-\lambda h}}{\lambda(1-e^{-\lambda h})}. \tag{2.6}$$

Proof

$$\int_{jh}^{(j+1)h} \lambda e^{-\lambda t} dt = \lambda[-\frac{1}{\lambda} e^{-\lambda t}]_{jh}^{(j+1)h} = [-e^{-\lambda t}]_{jh}^{(j+1)h} = e^{-\lambda jh} - e^{-\lambda(j+1)h}$$

Taking the numerator

$$\int_{jh}^{(j+1)h} \lambda(t - jh) e^{-\lambda t} dt = \int_{jh}^{(j+1)h} \lambda t e^{-\lambda t} dt - \int_{jh}^{(j+1)h} \lambda jh e^{-\lambda t} dt$$

Now

$$\int_{jh}^{(j+1)h} \lambda t e^{-\lambda t} dt = \lambda([-\frac{t}{\lambda} e^{-\lambda t}] + \frac{1}{\lambda} \int_{jh}^{(j+1)h} e^{-\lambda t} dt)$$

$$= -\left(j+1\right) h e^{-\lambda(j+1)h} + jh e^{-\lambda jh} + \frac{1}{\lambda} e^{-\lambda jh} - \frac{1}{\lambda} e^{-\lambda(j+1)h}$$

$$= -jh e^{-\lambda(j+1)h} - h e^{-\lambda(j+1)h} + jh e^{-\lambda jh} + \frac{1}{\lambda} e^{-\lambda jh} - \frac{1}{\lambda} e^{-\lambda(j+1)h}$$

and

$$\int_{jh}^{(j+1)h} \lambda jh e^{-\lambda t} dt = jh(e^{-\lambda jh} - e^{-\lambda(j+1)h}).$$

Therefore

$$E\left(t - jh\right) = \frac{-jh e^{-\lambda(j+1)h} - h e^{-\lambda(j+1)h} + jh e^{-\lambda jh} + \frac{1}{\lambda} e^{-\lambda jh} - \frac{1}{\lambda} e^{-\lambda(j+1)h} - jh e^{-\lambda jh} + jh e^{-\lambda(j+1)h}}{e^{-\lambda jh} - e^{-\lambda(j+1)h}}$$

$$= \frac{e^{-\lambda jh}[-h e^{-\lambda h} - (1 - e^{-\lambda h}) \frac{1}{\lambda}]}{e^{-\lambda jh}(1 - e^{-\lambda h})}$$

$$= \frac{1 - (\lambda h + 1) e^{-\lambda h}}{\lambda(1 - e^{-\lambda h})}.$$

When a shift takes place, the likelihood that it will be detected on any subsequent sample is

$$1 - \beta = \int_{-\infty}^{-k-\delta\sqrt{n}} \phi(z) dz + \int_{k-\delta\sqrt{n}}^{\infty} \phi(z) dz. \tag{2.7}$$

where $\phi(z) = (2\pi)^{-\frac{1}{2}} \exp(-z^2/2)$ is the standard normal density, $1-\beta$ is the power of the test and β is the probability of a type-II error. The probability of a false alarm is

$$\alpha = 2\int_{k}^{\infty} \phi(z)dz. \tag{2.8}$$

A production cycle starts with an in-control state and ends with the detection and elimination of an assignable cause. The production cycle consists of four periods

1. The in-control period - The expected length of the in-control period can be estimated by $1/\lambda$.
2. The out-of-control period - The expected length of the out-of-control signal is estimated by $h/(1-\beta)-\tau$, where $1/(1-\beta)$ is the expected number of samples needed to detect an out-of-control signal given that the process is out of control and τ is as defined in equation (2.6).
3. The time needed to take a sample and interpret the results is proportional to the sample size and is given by gn, where g is a constant.
4. The time needed to search for an assignable cause following an action signal is a constant D.

Merging all the four periods result in the formula for computing the expected length of a cycle i.e.

$$E(T) = \frac{1}{\lambda} + \frac{h}{1-\beta} - \tau + gn + D. \tag{2.9}$$

The expected total costs comprise of

1. The expected cost of sampling and testing: $(a_1 + a_2 n)\frac{E(T)}{h}$ where a_1 and a_2 are fixed and variable components of sampling cost and $\frac{E(T)}{h}$ is the expected number of samples taken within a cycle.
2. The expected number of false alarms produced during a cycle is

$$\alpha \sum_{j=0}^{\infty} \int_{jh}^{(j+1)h} je^{-\lambda t} dt = \alpha \sum_{j=0}^{\infty} j(e^{-\lambda jh} - e^{-\lambda(j+1)h}) = \alpha(1-e^{-jh}) \sum_{j=0}^{\infty} je^{-\lambda jh} = \frac{\alpha e^{-\lambda h}}{1-e^{-\lambda h}} \tag{2.10}$$

where $\sum_{j=0}^{\infty}\int_{jh}^{(j+1)h} je^{-\lambda t}dt$ is the expected number of samples taken before the shift. a_3^T is the cost of investigating a false alarm.

3. The cost of finding an assignable cause: a_3.
4. The net income per hour of operation in either the in-control state or the out-of-control state, denoted by V_0 and V_1 respectively.

Therefore the expected net income per cycle is

$$E(C)=V_0\frac{1}{\lambda}+V_1\left(\frac{h}{1-\beta}-\tau+gn+D\right)-a_3-\frac{a_3^T\alpha e^{-\lambda h}}{1-e^{-\lambda h}}-(a_1+a_2n)\frac{E(T)}{h}. \quad (2.11)$$

The expected net income per hour is found by dividing the expected net income per cycle by the expected cycle length i.e.

$$E(A)=\frac{E(C)}{E(T)}=\frac{V_0\frac{1}{\lambda}+V_1\left(\frac{h}{1-\beta}-\tau+gn+D\right)-a_3-\frac{a_3^T\alpha e^{-\lambda h}}{1-e^{-\lambda h}}-(a_1+a_2n)\frac{E(T)}{h}}{\frac{1}{\lambda}+\frac{h}{1-\beta}-\tau+gn+D}. \quad (2.12)$$

Assume $a_4=V_0-V_1$, then

$$E(A)=V_0-\frac{(a_1+a_2n)}{h}-\frac{a_4\left[\frac{h}{1-\beta}-\tau+gn+D\right]+a_3+\frac{a_3^T\alpha e^{-\lambda h}}{1-e^{-\lambda h}}}{\frac{1}{\lambda}+\frac{h}{1-\beta}-\tau+gn+D}, \quad (2.13)$$

i.e.

$$E(A)=V_0-E(L) \quad (2.14)$$

where

$$E(L)=\frac{(a_1+a_2n)}{h}-\frac{a_4\left[\frac{h}{1-\beta}-\tau+gn+D\right]+a_3+\frac{a_3^T\alpha e^{-\lambda h}}{1-e^{-\lambda h}}}{\frac{1}{\lambda}+\frac{h}{1-\beta}-\tau+gn+D}. \quad (2.15)$$

$E(L)$ denotes the expected loss per hour incurred by the process and it is a function of the control chart parameters n, k and h. The objective is to minimize $E(L)$ and by doing so maximizing $E(A)$. Duncan (1956) presented several approximations to develop an optimization procedure for this model. The optimization procedures were based on solving numerical approximations to the system for first partial derivatives of $E(L)$ with respect to n, k and h. A closed form solution for h is given using the optimal values of n and k. Various authors have presented optimization procedures for Duncan's model. According to Montgomery $E(L)$ can easily be minimized by using an unconstrained optimization or search technique together with a computer program for repeating the cost model. He said this method of optimization is commonly used. He used a FORTRAN program to optimize Duncan's model.

CHAPTER 3
An Easy and Low Cost Option for ESPC

3.1 Background information of economic statistical control charts for a single variable

Statistical quality control is a valuable and economically vital application used in operations research in manufacturing industries. The main objective in the use of statistical quality control is to monitor and maintain the industrial processes to ensure that product quality is improved or maintained in a cost resourceful way. The main tool used in statistical quality control is the statistical quality control chart technique. Historically the statistical control chart technique was developed in the 1920's by Dr Walter A. Shewhart and these types of control charts are now well known as the Shewhart control charts. A control chart is defined as a graphical display that shows the behaviour of the process, i.e. it gives information on whether a process is in-control or not. Immediate action is taken to investigate and correct the process once an out-of-control state in detected before a mass production of defective products.

After the introduction of the control chart technique by Shewhart, several researchers came up with different kinds of control charts. Among the charts was the $\bar{x}$-control chart which according to Saniga and Shirland (1977) became the most used control chart. The $\bar{x}$-control chart is normally used in cases where quality is measured on a continuous scale. The design of a control chart plays a very important role in determining the performance of a control chart. According to Montgomery (2005) the design of a control chart involves the selection of decision variables such as the sample size, n, sampling frequency, h, and lastly the control limits interval, k. Even though the design of control chart was mainly based on certain statistical principles, it has been discovered that the design of a control chart has economic implications. Montgomery (2005) presented the various costs involved in control chart design and these are: costs of sampling and testing, costs associated with investigating an out-of-control signal and possible correction of any assignable cause found, costs associated with the production of non-conforming products, etc. This has motivated researchers to develop control charts that take into consideration the mentioned costs.

The first economic design was presented by Girshick and Rubin (1952). Duncan (1956) adopted some of the ideas from Girshick and Rubin (1952) and developed the first economic model for the $\bar{x}$-control chart. His main objective was to determine the control chart parameters that minimize the expected net income per hour. Many other authors joined in in searching for the economic models for the different charts that had been developed since the introduction of the control chart technique. Lorenzen and Vance (1986) developed a unified approach to the economic and economic statistical design of the $\bar{x}$-control chart.Saniga (1989) developed the control chart based on the economic statistical design. His argument was that the effectiveness of the control chart can be enhanced by improving both the statistical and economic properties of the control charts. Montgomery *et al* (1995) describes the economic statistical design as a way of including statistical constraints such as the ARL or the average time to signal, ATS into the economic model to achieve a design that meets statistical requirements and at the same time minimizing the loss cost function.

The model for an economic statistical design is based on the following objective

Minimize	$F(n,h,k)$
Subject to	$ARL_0 \geq ARL_L$ $ARL_1 \leq ARL_U$

where

ARL_0 and ARL_1 stand for the average run lengths while the process is in control and when the process is outofcontrol respectively, and

ARL_L and ARL_U stand for the lower bound on the in-control state and upper bound on the out-of-control state respectively.

F is the loss cost function.

According to Van Deventer and Manna (2009) the economic statistical designs are determined through the use of non-linear constrained optimization techniques. In their study they realised that not many authors have adopted the optimization procedures when designing the $\bar{x}$-control chart. They pointed out that the main reason could be that cost models and their associated optimization techniques are too complex and not practically easy to apply. A few authors have attempted to adopt optimization procedures for determining the optimal parameters of the $\bar{x}$-control chart. However according to Van Deventer and Manna (2009) the methodologies presented by these authors are not easy to use in real life. This motivated them to modify one of the optimization procedures developed by Lorenzen and Vance (1986) on the unified approach to the economic and economic statistical designs of the $\bar{x}$-control chart. They used it to develop a more user friendly Excel program which is easy-to-use, easy-to-understand and easy-to-access. This Excel program computes the optimal parameter values that can be used to minimize the expected total loss.

3.2 Lorenzen and Vance (1986) process model and cost function

Model assumptions

The design of the economic and economic statistical design is based on the following assumptions:

1. The process begins in an in-control state μ_0 and standard deviation σ.
2. A single assignable cause of magnitude δ which occurs at random, results in a shift in the mean from μ_0 to either $\mu_0 + \delta\sigma$ or $\mu_0 - \delta\sigma$.
3. When a point exceeds a control limit, a search for assignable cause is initiated.
4. The assignable cause is assumed to occur according to a Poisson process with intensity of λ occurrences per hour and the time the process remains in an in control state is an exponential variable with mean 1/λ h.
5. Renewal reward process for the model is assumed.

Notation and symbols

τ	the expected time of occurrence of a shift between two samples while in control.
θ	the mean time between occurrences.
a	the fixed cost per sample.
δ	the shift in the size of the mean.
b	the cost per unit sampled.
W	the cost to identify and repair the assignable cause.
Y	the cost incurred per false alarm.
C_0	the quality cost per hour while in control.

C_1 cost incurred while process is outofcontrol.

g time to sample and chart one item.

T_0 the expected investigation time when a false alarm.

T_1 the expected time to identify the assignable cause.

T_2 the expected time to correct the process.

C the total cost per cycle.

L the total cost per time unit.

ATS average time to signal.

ATS_U upper bound of the average time to signal.

s Expected number of samples taken while the process was in control.

3.3 The mathematical model

The expected cycle time consists of combining the expected time until the occurrence of an assignable cause, the expected time between the occurrence of the assignable cause and the next sample, the time to analyse the sample and chart the results and lastly the expected time to detect a shift, identify the assignable cause and correct the process. It can be expressed mathematically as follows

$$E(T) = \frac{1}{\theta} + (1-\gamma_1)s\frac{T_0}{ARL_0} - \tau + ng + h(ARL_1) + T_1 + T_2 \quad (3.1)$$

where

$$\gamma_1 = \begin{cases} 1, \text{ if production carries on while searching} \\ 0, \text{ if productionis stopped while searching} \end{cases} \quad (3.2)$$

The expected cost per cycle includes costs for producing defective products, costs due to false alarms, costs due to investigating and possible correction of the assignable cause and lastly costs for sampling and testing. The expected cost per cycle can be expressed mathematically as

$$E(C) = \frac{C_0}{\theta} + C_1(-\tau + ng + h(ARL_1) + \gamma_1 T_1 + \gamma_2 T_2) + \frac{sY}{ARL_0} + W + (a+bn)\left(\frac{\frac{1}{\theta} - \tau + ng + h(ARL_1) + \gamma_1 T_1 + \gamma_2 T_2}{h}\right). \tag{3.3}$$

The expected cost per unit of time is found by dividing the expected cost per cycle by the expected cycle time, i.e.

$$E(L) = \frac{\frac{C_0}{\theta} + C_1(-\tau + ng + h(ARL_1) + \gamma_1 T_1 + \gamma_2 T_2) + \frac{sY}{ARL_0} + W}{\frac{1}{\theta} + (1-\gamma_1)s\frac{T_0}{ARL_0} - \tau + ng + h(ARL_1) + T_1 + T_2} + \frac{(a+bn)}{h}\left(\frac{\frac{1}{\theta} - \tau + ng + h(ARL_1) + \gamma_1 T_1 + \gamma_2 T_2}{\frac{1}{\theta} + (1-\gamma_1)s\frac{T_0}{ARL_0} - \tau + ng + h(ARL_1) + T_1 + T_2}\right). \tag{3.4}$$

3.4 The optimization procedure

$E(L)$ has three quality control chart parameters, i.e. the sample size, n, sampling frequency, h, and lastly the control limits interval, k. Lorenzen and Vance (1986) developed an algorithm for finding the most economic design based on Newton's method, the golden search and the Fibonacci search method. However according to Van Deventer and Manna (2009) the algorithm is complicated and so very few practitioners have adopted it. Therefore they presented a more user friendly Excel program that may be used to determine an economic or economic statistical design for the $\bar{x}$-control chart using Lorenzen and Vance's (1986) model. Firstly they re-expressed equation (3.4) according to Lorenzen and Vance (1986) as follows

$$E(L) = \frac{NUM_1}{DEN} + \frac{NUM_2}{DEN} \tag{3.5}$$

where

$$NUM_1 = \frac{C_0}{\theta} + C_1(-\tau + ng + h(ARL_1) + \gamma_1 T_1 + \gamma_2 T_2) + \frac{sY}{ARL_0} + W \tag{3.6}$$

$$NUM_2 = (a+bn)\left(\frac{\frac{1}{\theta} - \tau + ng + h(ARL_1) + \gamma_1 T_1 + \gamma_2 T_2}{h}\right) \tag{3.7}$$

and

$$DEN = \frac{1}{\theta} + (1-\gamma_1)s\frac{T_0}{ARL_0} - \tau + ng + h(ARL_1) + T_1 + T_2. \tag{3.8}$$

The expected number of samples taken while the process was in control, s, is computed as

$$s = \sum_{i=0}^{\infty} iP \;= \sum_{i=0}^{\infty} i(e^{\theta hi} - e^{\theta h(i+1)}) = \frac{1}{e^{\theta h}}. \tag{3.9}$$

where P is the probability that an assignable cause occurred between the i^{th} and $(i+1)$st sample. According to Montgomery (2005) the expected time of occurrence of the assignable cause between the ith and $(i+1)$st is given by

$$\tau = \frac{\int_{ih}^{(i+1)h} \theta(t-hi)e^{-\theta t}dt}{\int_{ih}^{(i+1)h} \theta e^{-\theta t}dt} = \frac{1-(1+\theta h)e^{-\theta h}}{\theta(1-e^{-\theta h})} = \frac{1}{\theta} - \frac{h}{e^{\theta h}-1}. \tag{3.10}$$

This is a more accurate result compared to Duncan's approximation approach.

The model consists of fixed and variable input parameters. The fixed parameters are: $\theta, \delta, a, b, Y, W, C_0, C_1, g, T_0, T_1, T_2, \gamma_1$ and γ_2 and the variable parameters are n, h, k .The Excel program developed by Van Deventer and Manna (2009) computes k and h for several sample sizes and also gives the corresponding values of the expected cost function $E(L)$. The minimized cost can then be found by direct implementation of the corresponding parameters incurred.

3.5 Conclusion

Van Deventer and Manna (2009) successfully implemented the optimization procedure in Excel and found that it is easy to use in finding optimal solutions to the design of both the economic and economic statistical $\bar{x}$-control charts. It has the advantages that it is easy to use, easy to understand and cheap since no expensive software is required and the procedure obtains an exact optimal design rather than the estimate designs as derived by Duncan (1956) and other subsequent researchers.

Chapter 4
Economic Design of T^2 Control Charts to Maintain Control of a Process

4.1 Introduction to multivariate economic design

Montgomery and Klatt (1972) mentioned that considerable attention had been given to the economic design of control charts under the assumption that only one quality characteristic was of interest. However, they argued that industrial products and processes possess more than one measurable quality characteristic and these quality characteristics put together give the overall product/process quality. In order to support their argument they gave an illustration on the production of synthetic fibre where the tensile strength (X_1) and the diameter (X_2) were taken as the quality characteristics of the synthetic fibre. The aim of the illustration was to show that if these two characteristics are assumed to be independent they both give a process in control. If the product of their probabilities is computed it gives an out-of-control state. This led to the conclusion that the application of independent $\bar{X}$ -charts distorts the whole control procedure and especially when the quality characteristics are not statistically independent.

A number of authors have worked on developing quality control procedures for correlated variables. Among them was Hotelling, (1947) who proposed the Hotelling T^2 control chart. Due to the economic implications of designing control charts, Montgomery and Klatt, (1972) developed an economic model for the Hotelling T^2 since no previous work was done towards the economic development of this chart. The economic model comprised of selecting suitable design parameters, the sample size, n, sampling frequency, h, and control limits widths, k, that minimize the expected cost of monitoring the process. For convenience, they made the following assumptions:

- Only one assignable cause of variation exists.
- Monitoring is done through taking successive fixed sample sizes at equal sampling intervals.

- When one sample produces an out of control state corrective measures are taken.

Below is a brief overview of the Hotelling T^2 control chart

4.2 The Hotelling T^2 control chart

The Hotelling T^2 control chart is rated as the most widely used multivariate control chart procedure that deals with changes in the mean vector of p correlated quality characteristics (Aparisi and Haro, 2001). It is developed under the assumption that there exists a random vector X of size $(p \times 1)$ whose jth element is the jth quality characteristic. The distribution of X is assumed to be the p-variate normal distribution which can be written as:

$$f(x) = 1/[(2\pi)^{p/2} |\Sigma|^{(1/2)}] \exp\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\} \tag{4.1}$$

where $\mu = E(X)$ is the mean vector of the p quality characteristics and $\Sigma = Cov(X)$ is the $(p \times p)$ variance-covariance of X. Note that μ and Σ are the population mean and variance - covariance matrices respectively. These parameters are unknown in most cases and are therefore estimated by the sample mean and sample variance-covariance matrix which are computed as follows

$$\text{Sample mean vector} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \tag{4.2}$$

$$\text{Sample covariance matrix} = S = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)\left(X_i - \bar{X}\right)^T \tag{4.3}$$

where X_i is the ith vector from a set of random vectors of size n contained in the sample matrix X.

The control procedure as presented by Hotelling (1947) and Jackson (1956 and 1959) is as follows: The in-control state is denoted by μ_0 (a value of μ that corresponds to the in-control state). The null hypothesis to test whether the process is in control or not is given by H_0: $\mu = \mu_0$ and the test statistic is given by

$$T^2 = n(\bar{X} - \mu_0)^T S^{-1}(\bar{X} - \mu_0) \sim T^2_{p,n-p}. \tag{4.4}$$

Equation(4.4)can be defined in words as the squared Mahalanobis distance between the observed sample mean and the hypothesised mean, μ_0. The process is confirmed to be out of control if $T^2 > T^2_{\alpha,p,n-p}$ where $T^2_{\alpha,p,n-p}$ is the α percentage point of the T^2 distribution. The α percentage points of the T^2 can be obtained from the cumulative F distribution since

$$F = \frac{n-p}{p(n-1)}T^2 \tag{4.5}$$

has an F distribution with p and $n-p$ degrees of freedom. From equation (4.5)

$$T^2_{\alpha,p,n-p} = \frac{p(n-1)}{n-p}F_{\alpha,p,n-p}. \tag{4.6}$$

The procedure for computing the test statistic is as follows

- Take samples of size n periodically.
- Compute $T^2 = n(\bar{X}-\mu_0)^T S^{-1}(\bar{X}-\mu_0) \sim T^2_{p,n-p}$.
- Plot T^2 against time.
- The process is out of control if $T^2 > T^2_{\alpha,p,n-p}$.
- Investigation of the assignable cause should begin if the process is out of control.

Under the alternative hypothesis, $H_1 : \mu \neq \mu_0$ the probability of a type-II error associated with the procedure depends on the distribution of T^2. Anderson (1958) indicated that under H_1 the generalized T^2 distribution with p and $n-p$ degrees of freedom i.e. $T^2 \sim T^{2'}_{\alpha,p,n-p}$. The random variable F is denoted by

$$F' = \frac{n-p}{p(n-1)}T^{2'} \tag{4.7}$$

which has a noncentral F distributionwith p and $n-p$ degrees of freedom and noncentral parameter $\tau = n(\mu-\mu_0)^T\Sigma^{-1}(\mu-\mu_0)$ for this particular test. If the variance-covariance is known T^2 will have a X^2 distribution with p and $n-p$ degrees of freedom.

4.3 A general cost model

Montgomery and Klatt (1972) presented a model for estimating the expected total cost per unit associated with a multivariate quality control procedure. The model is a multivariate extension of the work done by Knappenberger and Grandage (1969) on the development of an economic cost model for the univariate case. They made the following assumptions:

- When $\mu = \mu_0$ the process is in control.

- There exists only one out-of-control state in which the process mean is $\mu_1 = \mu_0 + \delta$, where δ a $(p\text{x}1)$ vector is known.
- The time when the process is in the in-control state before going to the out-of-control state is assumed to be an exponential random variable with mean λ^{-1} hours.
- When the process goes out of control it remains out of control until detected, i.e. until T^2 plots out of control.

The expected total cost per unit of a product associated with the test procedure consists of three components: the expected cost per unit of sampling and testing $E(C_1)$, expected cost per unit associated with investigating and correcting an out-of-control state, $E(C_2)$ and the expected cost per unit associated with producing non-conforming products, $E(C_3)$. Summing up all three gives the expected total cost per unit of a product associated with the test procedure as follows

$$E(C) = E(C_1) + E(C_2) + E(C_3). \tag{4.8}$$

The computation of $E(C_1)$ depends on the assumptions made by Cowden (1957) and Duncan (1956) that the cost of sampling and testing comprises of a fixed cost independent of the sample size and a cost per unit sampled, i.e.

$$E(C_1) = (a_1 + a_2 n) / k \tag{4.9}$$

where a_1 is the fixed cost per sample, a_2 is the per-unit cost of sampling and k is the number of units produced between successive samples.

Montgomery and Klatt (1972) used the assumption made by Knappenberger and Grandage (1969) that the cost of searching and possible repair of a process that has gone out of control is a random variable which can be denoted as V. The expected value of V is given by a_3 . Another random variable Z has been defined as

$$Z = \begin{cases} 1 \text{ if } T^2 > T^2_{\alpha,p,n\text{-}p} \\ 0 \text{ otherwise.} \end{cases} \tag{4.10}$$

If both V and Z are equal to zero, the process is in control, otherwise the process is out of control and the expected cost per unit of searching and correcting the assignable cause is calculated as

$$E(C_2) = \left\{ a_3 \, \mathrm{P}(Z=1) \right\} / k. \tag{4.11}$$

Equation (4.11) can be written in terms of vectors. Montgomery and Klatt (1972) defined the following vectors as follows

- ρ is a column vector of conditional probabilities ρ_i that the test procedure detects a true and a false alarm.
- β is a column vector of probabilities β_i, such that β_i is the probability that the process was either in control or out-of-control at the time the test was performed.

Equation (4.11) can therefore be expressed in terms of these vectors as follows

$$E(C_2) = (a_3 / k)(\rho_0 \beta_o + \rho_1 \beta_1) = (a_3 / k)\rho^T \beta. \tag{4.12}$$

To compute $E(C_3)$, let the cost of producing defective goods be given by a_4 and also define a random variable W as

$$W = \begin{cases} 1 & \text{if the unit is defective} \\ 0 & \text{otherwise.} \end{cases} \tag{4.13}$$

The cost per unit associated with concluding that the process is in control is given by

$$C_3 = a_4 W. \tag{4.14}$$

Equation (4.14) can also be written in terms of vectors. Let

- ϕ be the column vector of conditional probabilities ϕ_i that nonconforming products are produced given that the process is either in-control or out-of-control.
- γ be the column vector of probabilities γ_i that the process is either in-control or out-of-control at any given point in time.

Equation (4.14) can therefore be expressed as

$$E(C_3) = a_4 (\phi_0 \gamma_o + \phi_1 \gamma_1) = a_4 \phi^T \gamma. \tag{4.15}$$

Summing up all the three components gives us the expected total cost expressed in vector form as

$$E(C) = (a_1 + a_2 n) / k + (a_3 / k)\rho^T \beta + a_4 \phi^T \gamma. \tag{4.16}$$

The next section gives a brief summary of the development of the vectors by Montgomery and Klatt, (1972).

4.4 The development of the vectors ϕ, ρ, β and γ

Another assumption made by Montgomery and Klatt (1972) is that there exist two specification limit vectors l and u for each of the p quality characteristics. The vector l defines the lower specification limit while u defines the upper specification limit. Products that are produced when X lies within these specifications that is $l \leq X \leq u$ are called conforming products and products produced outside the specification limits are called non-conforming products. In the next sections Montgomery and Klatt (1972), discussed the development of each vector in detail.

4.4.1 The vector ϕ

The elements of this vector are the conditional probabilities of producing defective products given that the process is either in control or is out of control and they depend on the probability distribution of vector X (a p-variate normal distribution). The p-variate normal distribution has two population parameter vectors μ and Σ which are unknown in most cases, but can be estimated by $\bar{X}$ and S as mentioned in the previous section. The elements of ϕ are estimated by

$$\phi_0 = 1 - \int_{l_1}^{u_1}\int_{l_2}^{u_2}\cdots\int_{l_p}^{u_p} B\exp\left\{-\frac{1}{2}\left(x-\mu_0\right)^T S^{-1}\left(x-\mu_0\right)\right\}dx_1 dx_2 \ldots dx_p \tag{4.17}$$

and

$$\phi_1 = 1 - \int_{l_1}^{u_1}\int_{l_2}^{u_2}\cdots\int_{l_p}^{u_p} B\exp\left\{-\frac{1}{2}\left(x-\mu_1\right)^T S^{-1}\left(x-\mu_1\right)\right\}dx_1 dx_2 \ldots dx_p. \tag{4.18}$$

$$B = \left[(2\pi)^{p/2}\left|S\right|^{1/2}\right]^{-1}. \tag{4.19}$$

4.4.2 The vector ρ

The elements of vector ρ are the conditional probabilities that the test procedure indicates the process is out of control given that the process is either in control or out of control. These conditional probabilities are given by

$$\rho_0 = P\left\{T^2 > T^2_{\alpha,p,n-p}\right\} = \int_{T^2_{\alpha,p,n-p}}^{\infty} f(T^2)dT^2 \tag{4.20}$$

and

$$\rho_1 = P\left\{T^{2'} > T^2_{\alpha,p,n-p}\right\} = \int_{T^2_{\alpha,p,n-p}}^{\infty} f(T^{2'})dT^{2'} \tag{4.21}$$

where

$f\left(T^2\right)$isthe T^2 distribution with p and $n-p$ degrees of freedom and $f\left(T^{2'}\right)$ is the generalized T^2 distribution with p and $n-p$ degrees of freedom and approximate noncentrality parameter $\hat{\tau} = n\left(\mu_1 - \mu_0\right)^T S^{-1}\left(\mu_1 - \mu_0\right)$ for the corresponding noncentral F distribution. It can be seen that ρ_0 is the probability of a type-I error and ρ_1 is the power of the test.

4.4.3 The vector β

The elements of β are defined as the steady-stateprobabilities that the process was either in-control or out-of-control during the period of sampling. These elements are obtained through a transition probability matrix,say G. G consists of elements that represent the probability of the process shifting from one state to the other during the production of k units. The probability of the process remaining in control for t hours is given by

$$\begin{aligned}
&1-\int_0^t \lambda e^{-\lambda t} dz \\
&= 1-\lambda\left[-\frac{1}{\lambda}e^{-\lambda z}\right]_0^t \\
&= 1-\lambda\left[-\frac{1}{\lambda}e^{-\lambda t}+\frac{1}{\lambda}\right] \\
&= 1-\left[1-e^{-\lambda t}\right] \\
&= e^{-\lambda t}.
\end{aligned} \tag{4.22}$$

Assuming that R units per hour are produced and fractional units can be produced, then the probability of the process to remainincontrol during the production of k units is given by

$$P_0 = \exp\left(-\lambda k / R\right) \tag{4.23}$$

and the probability of out-of-control is given by

$$P_1 = 1-\exp\left(-\lambda k / R\right). \tag{4.24}$$

An assumption has also been made that the flow of output product is sufficiently large relative to the sample size so that the possibility of a shift occurring during the taking of a sample can be neglected.

The elements of G are defined as

$$G=\begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix}=\begin{bmatrix} P_0 & P_1 \\ \rho_1 P_0 & \rho_1 P_1+(1-\rho_1) \end{bmatrix} \tag{4.25}$$

where

- P_0 is the probability of producing k units while the process is in control.
- P_1 is the probability of shifting to an out-of-control state during the production of k units.
- $\rho_1 P_0$ is the probability that the out-of-control state is discovered at the mth sample multiplied by the probability of remaining in-control during the production of k units.
- $\rho_1 P_1+(1-\rho_1)$ is the probability of the process being out-of-control at the mth multiplied by the probability of the process going out-of-control again during the production of k units plus the probability of not detected the out-of-control state at the mth sample.

The vector β is obtained by noting that G is the transition matrix of an irreducible aperiodic positive recurrent Markov chain such that

$$\beta^T G=\beta^T \tag{4.27}$$

where $\beta^T=[\beta_0,\beta_1]$, and $\beta_0+\beta_1=1$

and

$$\beta_0=\rho_1 P_0/(P_1+\rho_1 P_0) \tag{4.28}$$

$$\beta_1=P_1/(P_1+\rho_1 P_0). \tag{4.29}$$

4.4.4 The vector γ

The elements for vector γ are defined as the steady state probabilities that the process is either in control or out of control at any given time. According to Duncan (1956) the average fraction of time that elapses before the shift occurs given that a shift has occurred between the mth and $(m+1)$th samples is given by

$$\Delta = \frac{1-\left(1+\frac{\lambda k}{R}\right)\exp(-\frac{\lambda k}{R})}{\left[1-\exp\left(-\frac{\lambda k}{R}\right)\right]\lambda k / R}. \tag{4.30}$$

Δ is defined as the conditional expectation of the occurrence of the assignable cause within an interval of sampling.The steady state probability that the process is in control at any point, γ_0 is computed as follows

$$\gamma_0 = \beta_0 P_0 + \Delta\beta_0 P_1. \tag{4.31}$$

while the steady state probability that the process is out-of-control at any point, γ_1 is computed as follows

$$\gamma_1 = \beta_1 P_0 + (1-\Delta)\beta_0 P_1 \tag{4.32}$$

4.5 A solution method and Example

Montgomery and Klatt (1972) discovered that finding a solution to the expected total cost per unit $E(C)$ presented in equation (4.16) is not simple. This is because

1. Determining the test parameters is not that simple.
2. There is no general solution that can be obtained since a particular set of values $(S, \delta, \mathrm{a}_i, \lambda, R, l \text{ and } u)$ must be specified.

To make it simple they made the following assumptions

$$K = \lambda k / R. \tag{4.33}$$

$$A_i = (a_i \lambda / R) / a_4,\ \ i = 1, 2, 3. \tag{4.34}$$

to give

$$E\left(C^{\#}\right) = \frac{a_1 + a_2 n}{K} + \left(\frac{a_3}{K}\right)\rho^T \beta + \phi^T \gamma. \tag{4.35}$$

The values of $n, K,$ and $T^2_{\alpha,p,n-p}$ which minimize $E\left(C^{\#}\right)$ were found using a two-stage grid search. The first stage is a coarse grid which investigates a wide range of test parameters and allows the general behaviour of the cost surface to be studied and the output of the first stage is then used to find the general area for the second stage search. This was done using an interactive FORTRAN program.

4.6 Conclusion

Montgomery and Klatt (1972) illustrated the use of this model using a numeric example. For more details of the numeric example refer to Montgomery and Klatt (1972). From the results of the numeric example it has been shown that the economic model of the T^2-control chart produces sample sizes that are higher and also sampling intervals which are larger than the similar economic models for $\bar{X}$-charts. The reasons cited are

1. In the economic model of the T^2-control chart a more complex situation is being modelled i.e. more parameters are being estimated.
2. The population variance-covariance matrix is assumed unknown in the T^2-control chart while in some economic models of $\bar{X}$-charts it is assumed to be a constant.
3. The two charts are not in reality comparable.

CHAPTER 5
Economic Design of the Specification Region for Multiple Quality Characteristics.

5.1 Introduction

In industry products are usually regarded as defective if they fail to meet the pre-set specification limits and this failure usually results in some certain amount of loss. On the other hand, products that meet the pre-set specification limits are referred to as non-defective and usually result in no losses. This type of classification where products are classified as either bad or good can be analysed by a system called a binary system for quality evaluation. According to Kapur and Chao (1996) this binary system does not sufficiently reflect customers' view point. They therefore developed a system that focusses only on the defective products and the loss incurred due to their production (called cost of non-conformance). The system is evaluated by a quality loss function. Many authors have presented different kinds of loss functions, however according to Phadke (1989), Taguchi (1986, 1987) and Taguchi *et al*. (1989) a simple quadratic loss function (QLF) may be reasonably used in many situations. QLF is said to be compatible with three types of quality characteristics: 'nominal the best' (N-type), 'smaller the better' (S-type) and 'larger the better' (L-type).

The loss is defined as the squared value of the difference between the target value and the value of quality characteristics for a product, where the target value denotes an ideal value for the characteristic from the customer's perspective. In other words loss occurs whenever the quality characteristic of a product departs from its target value. When a single characteristic of a product is of interest the expected quality loss can be obtained from computing the squared difference between the mean and the target value (bias) and adding the variance for the quality characteristic to it (Phadke, 1989, Taguchi, 1986, 1987 and Taguchi *et al*., 1989). The mean for the quality characteristic can easily be adjusted to its target value with a low cost therefore in order to shrink the quality loss for a product, the bias and the variance must be dealt with, i.e. reduced. A reasonable method to shrink the variance is to develop control limits for the process by minimizing the total loss to the customer and also to the producer.

Y_i

The next step will be to use the control limits to truncate the distribution of the quality characteristics by inspection (Kapur and Chao, 1996).

Several authors have worked on the development of these control limits based on a single quality characteristic. Among them was Tang (1988) who gave an overview of an economic model for selecting the most profitable control limits in a broad inspection plan for the situation where cost is a linear function of the control limits interval. However from the customer's perspective, products are usually evaluated on the basis of multiple quality characteristics. In a multiple quality characteristic situation the specification region is formed by sets of specification limits of each quality characteristic. A handful of authors have in recent years worked on cases where the products are described by multiple quality characteristics. Just to mention a few, Raiman and Case (1990) gave a discussion of multi-dimensional extensions of using the quadratic loss function for the purpose of monitoring product process improvement over time. They deliberated that losses occur from the process transition and that for a particular transition, total loss can be obtained by summing losses caused by each quality characteristic.

Tang and Tang(1989)suggested the economic product specifications for multiple quality characteristics for a complete inspection. Their assumption was that quality characteristics are uncorrelated. Kapur and Chao (1996) argued that real life situations have proven that it is not always that these multiple quality characteristics are independent. They however suggested that in cases where the multiple quality characteristics are correlated the multivariate quality loss function (MQLF) by Chen and Kapur, (1989) must be used. MQLF takes into consideration the bias, the variance of each characteristic, interactions between biases and covariances between the characteristics. As in the case of the single quality characteristic,loss is reduced by reducing the variances and covariancesof the multiple quality characteristics. According to Kapur and Chao (1996) it is difficult to reduce variances and covariances and therefore a similar procedure for truncating the distributions of the quality characteristics is used.

Notation

Y_i	random variable (r.v.) associated with quality characteristic i,where $i=1,\ldots,m$;
y_i	a specific value associated with the quality characteristic i ,where $i=1,\ldots,m$;

y	$= (y_1, \dots, y_m)^T$;
t_i	a target value associated with quality characteristics i,where $i = 1, \dots, m$
t	$= (t_1, \dots, t_m)^T$;
$L(y,t)$	measure of loss of the quality associated with y and t;
$H_L(t)$	Hessian matrix for $L(y,t)$;
k_i	loss coefficient associated with quality characteristics i,where $i = 1, \dots, m$
k_{ij}	loss coefficient associated between quality characteristics i and j,where i and $j = 1, \dots, m$ and $i \neq j$;
μ_i	mean for r.v. Y_i, where $i = 1, \dots, m$;
μ	$= (\mu_1, \dots, \mu_m)^T$
σ_i^2	variance for r.v. Y_i, where $i = 1, \dots, m$;
σ_{ij}^2	covariance between Y_i and Y_j, where i and $j = 1, \dots, m$ and $i \neq j$;
μ_{Ti}	mean of the truncated distribution;
σ_{Ti}	variance of truncated distribution for quality characteristic i and j, where i and $j = 1, \dots, m$ and $i \neq j$;
V	variance covariance matrix of quality characteristic i and j, where i and $j = 1, \dots, m$ and $i \neq j$;
$\|V\|$	determinant of matrix V;
ρ_{ij}	correlation coefficient between quality characteristic i and j, where i and $j = 1, \dots, m$ and $i \neq j$;
q	fraction of products actually shipped to the customer;
C_s	unit scrap cost;
C_1	unit inspection cost.

5.2 Multivariate quality loss function

Define $L(Y,t)$ as a twice differentiable function in the neighbourhood of t where Y is a $(1 \times m)$ vector of quality characteristics. If $L(Y,t)$ is expanded as a Taylor's series at $Y = t$ we have

$$L(Y,t) = L(t,t) + L'(t,t)(Y-t) + \frac{1}{2}(Y-t)^T L''(t,t)(Y-t) + ... \tag{5.1}$$

The target vector t must be developed to ensure that the total quality loss is a minimum at t and therefore $L'(t,t)=0$. If the higher terms in equation (5.1) are left out the equation reduces to

$$L(Y,t)=\frac{1}{2}(Y-t)^T H_L (Y-t) \tag{5.2}$$

where $H_L = L''(t,t)$ denotes a semi positive Hessian matrix for $L(Y,t)$ since $L(Y,t)$ attains its lowest value at $Y=t$. If m quality characteristics are considered then equation (5.2) and (5.3) are equal.

$$L(Y,t)=\sum_{i=1}^{m}\sum_{j=i}^{i} k_{ij}(Y_i-t_i)(Y_j-t_j) \tag{5.3}$$

where

$$k_{ii}=k_i=\frac{1}{2}\left(\frac{\partial^2 L}{\partial Y_i^2}\right) \tag{5.4}$$

$$k_{ij}=\left(\frac{\partial^2 L}{\partial Y_i \partial Y_j}\right)\Bigg|_{Y_i=t_i, Y_j=t_j, i\neq j=1,\ldots,m}. \tag{5.5}$$

The loss coefficient k_{ij} can be determined by using a regression method (Chen and Kapur, 1989 and Neter*et al.*, 1983).

The expected value for equation (5.3) is thus given by

$$E\left[L(Y,t)\right]=E\left[\sum_{i=1}^{m} k_i (Y_i-t_i)^2\right]+E\left[\sum_{i=2}^{m}\sum_{j=1}^{i\text{-}1} k_{ij}(Y_i-t_i)(Y_j-t_j)\right]. \tag{5.6}$$

The first term on the right side of the equation can be expressed in terms of μ_i and σ_i^2 as follows

$$E\left[\sum_{i=1}^{m} k_i (Y_i-t_i)^2\right]=\sum_{i=1}^{m} k_i\left[(\mu_i-t_i)^2+\sigma_i^{\,2}\right]. \tag{5.7}$$

The second term on the right side can be written in terms of μ_i and σ_{ij}^2 as follows

$$
\begin{aligned}
&E\left[\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}(Y_i-t_i)(Y_j-t_j)\right] \\
&=E\left[\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}\left(Y_iY_j-t_jY_i-t_iY_j+t_it_j\right)\right] \\
&=E\left[\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}\left(Y_iY_j\right)\right]-E\left[\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}t_jY_i\right]-E\left[\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}t_iY_j\right]+E\left[\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}t_it_j\right] \\
&=\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}E\left(Y_iY_j\right)-\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}t_jE(Y_i)-\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}t_iE(Y_j)+\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}t_it_j \\
&=\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}(\sigma_{ij}+\mu_i\mu_j)-\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}t_j\mu_i-\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}t_i\mu_j+\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}t_it_j \\
&=\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}[\sigma_{ij}+(\mu_i-t_i)(\mu_j-t_j)].
\end{aligned}
\tag{5.8}
$$

Summing up equation (5.7) and (5.8) results in

$$E\left[L(Y,t)\right]=\sum_{i=1}^{m}k_i\left[(\mu_i-t_i)^2+\sigma_i^{\,2}\right]+\sum_{i=2}^{m}\sum_{j=1}^{i\text{-}1}k_{ij}\left[\sigma_{ij}+(\mu_i-t_i)(\mu_j-t_j)\right]. \tag{5.9}$$

As mentioned in the previous sections the objective is to shrink the expected total loss and to achieve this, biases and variances for each quality characteristic and the cross products between the biases must be reduced. Now assuming that Y_i and Y_j $\forall\ i \neq j$ are independent

$$E\left[L(Y,t)\right]=\sum_{i=1}^{m}k_i\left[(\mu_i-t_i)^2+\sigma_i^{\,2}\right]+\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}(\mu_i-t_i)(\mu_j-t_j) \tag{5.10}$$

which can be simplified to

$$E\left[L(Y,t)\right]=\sum_{i=1}^{m}k_i\sigma_i^{\,2} \tag{5.11}$$

if $\mu_i=t_i\ \forall\ i.$

5.3 The multivariate distribution and specification region

As mentioned in the previous sections the main objective of developing and implementing specification/control limits is to decrease the variances and covariance of quality characteristics to ensure that quality loss is minimized supposing that each mean can be closely adjusted to its target value. The specification limits are defined as two distinct values which are referred to as the upper and lower specification limits or in short USL and LSL respectively. For illustration purposes the specification region for two quality characteristics is as in Figure 5.1

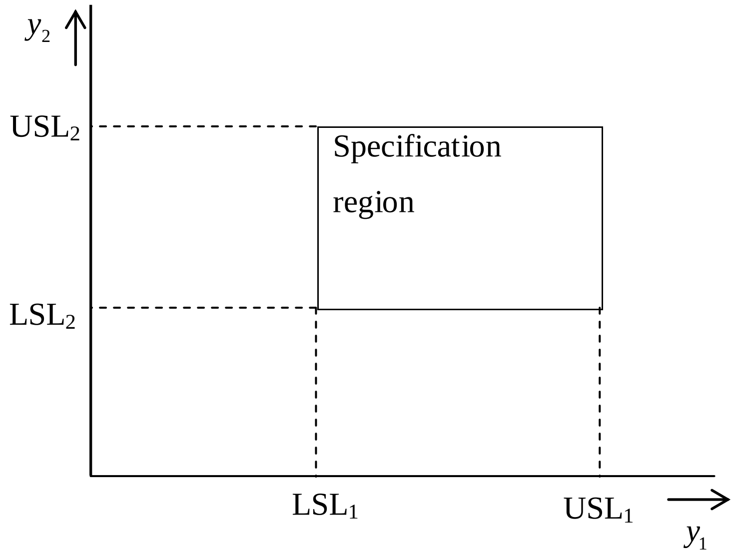

Figure 5.1: Specification region for two quality characteristics

The specification region for multivariate quality characteristics is the region defined by the intersection of the specification limits of all the quality characteristics. Putting it differently, it is an area, or volume created by truncating the multivariate distribution of the quality characteristics based of the specification limits. Kapur and Chao (1996) assumed a 100% screening with no inspection error. This implies that products produced under this assumption are non-defective. The actual distribution of the non-defective products when multiple quality characteristics are considered is the truncated multivariate distribution based on the specification limits. The truncated joint probability distribution function (p.d.f) of the quality characteristics $(Y_1,...,Y_m) \in A, \quad i = 1,\ldots,m$ assuming they are continuous random variables is given by

$$f_T(y_1,...,y_m) = \frac{f(y_1,...,y_m)}{\int_A \cdots \int f(y_1,...,y_m)dy_1...dy_m}. \tag{5.12}$$

Assuming that A denotes the specification region of interest, then the truncated joint probability distribution function (p.d.f) based on A is given by

$$f_T(y_1,...,y_m) = \frac{f(y_1,...,y_m)}{\int_{LSL_m}^{USL_m} \cdots \int_{LSL_1}^{USL_1} f(y_1,...,y_m)dy_1...dy_m} \tag{5.13}$$

where $\mathrm{LSL}_i \le y_i \le \mathrm{USL}_i, \quad i = 1,\ldots,m.$

5.4 The optimization model

The multinormal distribution usually is a good estimation for the behaviour of the N-type quality characteristics. Assume that $Y_1,...,Y_m$ have a multinormal distribution with p.d.f

$$f(y)=\frac{1}{(2\pi)^{m/2}|V|^{1/2}}\exp\left[-\frac{1}{2}(y-\mu)^T V^{-1}(y-\mu)\right] \tag{5.14}$$

where $-\infty < y_i < \infty$ for $i=1,...,m$ and V is a symmetric positive definite variance-covariance matrix. Let LSL$=\mu_i-\alpha_i\sigma_i$ and USL$=\mu_i+\beta_i\sigma_i$ for quality characteristics $i=1,...,m$. When the multivariate normal distribution based on the specification region with vertices $(\mu_1-\alpha_1\sigma_1,\mu_1+\beta_1\sigma_1),(\mu_2-\alpha_2\sigma_2,\mu_2+\beta_2\sigma_2),...,(\mu_m-\alpha_m\sigma_m,\mu_m+\beta_m\sigma_m)$ is truncated, the density function of the truncated multivariate normal distribution is given by

$$f(y)=\frac{1}{q(2\pi)^{m/2}|V|^{1/2}}\exp\left[-\frac{1}{2}(y-\mu)^T V^{-1}(y-\mu)\right] \tag{5.15}$$

where

$$q=\int_{\mu_1-\alpha_1\sigma_1}^{\mu_1+\beta_1\sigma_1}\cdots\int_{\mu_m-\alpha_m\sigma_m}^{\mu_m+\beta_m\sigma_m} f(y_1,...,y_m)dy_1...dy_m \tag{5.16}$$

and

$$\mu_i-\alpha_i\sigma_i\le y_i\le\mu_i+\beta_i\sigma_i,\ i=1,...,m$$

where q denotes the probability of producing conforming products. If $q=1$, it implies that all the products produced where conforming.

Kapur and Chao (1996) considered three types of quality loss: loss due to variability from the target value, loss due to inspection and loss due to scrap.The expected loss due to variability from a target value is given by

$$E\left[L(Y,T)\right]=\sum_{i=1}^{m}k_i\left[(\mu_{Ti}-t_i)^2+\sigma_{Ti}^2\right]+\sum_{i=2}^{m}\sum_{j=1}^{i-1}k_{ij}\left[\sigma_{Tij}+(\mu_{Ti}-t_i)(\mu_{Tj}-t_i)\right]. \tag{5.17}$$

It is imperative to note that inspection costs (when inspection is done) and scrap costs are incurred by the producer. It is assumed that all the quality characteristics of interest have the same inspection cost, denoted by C_1 and all products that fall outside the control limits are scrapped and the scrap cost is denoted by $C_s(1-q)$. Equation (5.17) can now be redefined taking into consideration the scrap cost and the inspection cost as follows:

$$ETL = \sum_{i=1}^{m} k_i \left[(\mu_{Ti} - t_i)^2 + \sigma_{Ti}^2 \right] + \sum_{i=2}^{m} \sum_{j=1}^{i-1} k_{ij} \left[\sigma_{Tij} + (\mu_{Ti} - t_i)(\mu_{Tj} - t_i) \right] + C_s(1-q) + C_1 \quad (5.18)$$

Where the mean, variance and covariance of the truncated multivariate normal distribution can be defined as follows:

$$\mu_{Ti} = \int_{\mu_1-\alpha_1\sigma_1}^{\mu_1+\beta_1\sigma_1} \cdots \int_{\mu_m-\alpha_m\sigma_m}^{\mu_m+\beta_m\sigma_m} \frac{1}{q} y_i f(y_1,\ldots,y_m) dy_m \ldots dy_1, \ i = 1,\ldots,m. \quad (5.19)$$

$$\sigma_{Ti}^2 = \left[\int_{\mu_1-\alpha_1\sigma_1}^{\mu_1+\beta_1\sigma_1} \cdots \int_{\mu_m-\alpha_m\sigma_m}^{\mu_m+\beta_m\sigma_m} \frac{1}{q} y_i^2 f(y_1,\ldots,y_m) dy_m \ldots dy_1 \right] - \mu_{Ti}^2, \ i = 1,\ldots,m. \quad (5.20)$$

$$\sigma_{Tij} = \left[\int_{\mu_1-\alpha_1\sigma_1}^{\mu_1+\beta_1\sigma_1} \cdots \int_{\mu_m-\alpha_m\sigma_m}^{\mu_m+\beta_m\sigma_m} \frac{1}{q} y_i y_j f(y_1,\ldots,y_m) dy_m \ldots dy_1 \right] - \mu_{Ti}\mu_{Tj}, \ i \neq j = 1,\ldots,m. \quad (5.21)$$

The multivariate normal probabilities have been obtained using IMSL subroutines (IMSL, 1980). Since the main objective of Kapur and Chao (1996) is to find the optimum specification region, the next section gives a brief overview of how this specification region can be found.

Assuming that $\mu_i, \sigma_i^2, \sigma_{ij}, k_i$ and t_i are known, then $\mu_{Ti}, \sigma_{Ti}^2, \sigma_{Tij}$, and q are all functions of α_i and β_i. This implies that α_i and β_i for $i = 1,\ldots,m$ are decision variables and their optimal values, α_i^* and β_i^* for $i = 1,\ldots,m$ can be obtained by minimizing the ETL. Denote y_{Li} and y_{Ui} as the lower and upper limits for the quality characteristic i respectively then for $i = 1,\ldots,m$, $\alpha_i \le (\mu_i - y_{Li}) / \sigma_i$ and $\beta_i \le (y_{Ui} - \mu_i) / \sigma_i$.

Kapur and Chao (1996) presented the optimization models for the optimum specification region for both the multivariate normal distribution as well as the bivariate normal distribution which are shown in the appendix.

5.5 A numerical example and computational results

Assume Y_1 and Y_2 are the two quality characteristics of interest of a product and that they jointly have a bivariate normal distribution with $\mu_1 = 10$, $\mu_2 = 20$, $\sigma_1^2 = 0.5$, $\sigma_2^2 = 0.8$ and $\rho = 0.6$. The target values for Y_1 and Y_2 are set at 9.2 and 19.4 respectively. If it is also known that $k_1 = 30$, $k_2 = 25$, $k_{12} = 10$, $C_s = 80$ and $C_1 = 2$ then the following questions need to be answered:

1. Is it necessary to perform an inspection on the products before they are shipped to the customer?
2. If yes, what should be the optimum specification region for the products so that the expected total loss can be minimized?

Now suppose it is given that $y_{L1} = 6.46,\ y_{L2} = 15.53,\ y_{U1} = 13.54$ and $y_{U2} = 24.47,$ then the expected loss can be computed from equation(5.17)if no inspection occurred, i.e.

$$E[(Y_1, Y_2, t_1, t_2)] = 30[(10-9.2)^2 + 0.5] + 25[(20-19.4)^2 + 0.8] + 10[(0.3795 + (10-9.2)(20-19.4)]$$
$$= 71.80.$$

If inspection was performed equation (5.18) can be used, i.e.

$$ETL = 30[(\mu_{T1} - 9.2)^2 + \sigma_{T1}^2] + 25[(\mu_{T2} - 19.4)^2 + \sigma_{T2}^2] + 10[\sigma_{T1}^2 + (\mu_{T1} - 9.2)(\mu_{T2} - 19.4)] + 80(1-q) + 2.$$

The optimum specification region is found by minimizing the ETL using the optimization model for the bivariate normal case. The optimum coefficients obtained for the specification region are $\alpha_1^* = 2.9, \beta_1^* = 0.9, \alpha_2^* = 2.6$ and $\beta_1^* = 0.9.$ This means that the optimum lower and upper specification limit for Y_1 and Y_2 are $(7.95, 10.64)$ and $(17.67, 20.81).$ This then implies that the vertices of the specification region are

$$(\mu_1 - \alpha_1^* \sigma_1, \mu_2 - \alpha_2^* \sigma_1) = (7.95, 17.67) \tag{5.22}$$

$$(\mu_1 - \alpha_1^* \sigma_1, \mu_2 + \beta_2^* \sigma_2) = (7.95, 20.81) \tag{5.23}$$

$$(\mu_1 + \beta_1^* \sigma_1, \mu_2 - \alpha_2^* \sigma_2) = (10.64, 17.67) \tag{5.24}$$

and

$$(\mu_1 + \beta_1^* \sigma_1, \mu_2 + \beta_2^* \sigma_2) = (10.64, 20.81). \tag{5.25}$$

The specification region formed by these vertices, gives a value of 57.71 for the ETL. From the results obtained one can conclude that inspection of the products before they are shipped to the customer is essential as it gives a remarkable reduction of the expected total loss from \$71.80 to \$57.71 (Kapur and Chao 1996).

5.6 Conclusion

There are three main losses that have been highlighted in the paper. The first one is loss due to variability from the target value (the loss incurred by the customer). This loss is obtained by using the multivariate quality loss function based on a Taylor series expansion. The second and third losses discussed are incurred by the producer and these are due to inspection and scrap respectively. It has been shown that the biases, variances and covariances of quality characteristics must be reduced so as to reduce the quality loss. It has been noted that the variances and covariances of the multiple quality characteristics are difficult to decrease and therefore an optimum specification region was developed to overcome this problem and the distribution underlying the quality characteristics was assumed to be the multinomial distribution. Two optimization models where presented, one for bivariate normal case and the other for multinomial case. The IMSL subroutines were used to evaluate the double integrals.

CHAPTER 6
Economic Statistical Design Of Multivariate Control Charts Using Quality Loss Function.

6.1 Introduction

Quality control costs time and money due to the costs involved in the physical sampling process, products taken from the production line during the sampling process, production of failed products during the time after the process has shifted and before the detection and correction of the shift etc. This implies that the frequency of sampling, the sample size, as well as the tolerance levels within which the process must operate are of prime importance in determining an optimal cost efficient economic-statistical design. Costs are thus involved in the sampling process itself as well as in the production of failed products at varying levels of importance/costs during an in- or out-of-control phase, whichever applies.

In a multivariate situation finding the optimum values of these parameters is complex as a number of factors influence the failure rate of the final product. Detecting a shift in the final product is one aspect of the problem, but finding the responsible variable(s) is another one. Taking into account the cost factor makes this a very complex problem. This has resulted in little attention being given in the literature initially to the economical aspect of multivariate quality control, despite the fact that it has received a lot of attention in the univariate case.

Duncan (1971) was the first person who paid formal attention to the cost aspect of quality control. Montgomery (1980) did a review and literature survey about the economic design of control charts and Vance (1983) also created a bibliography of related literature. Saniga (1977, 1989) proposed a uniform approach to economicstatistical control charts in the univariate case. All the mentioned publications were for the univariate case.

Chou *et al* (2002) wrote an article based on work done by Montgomery and Klatt (1972), Chen (1995) and Kapur and Chao (1996) in which a test statistic $-2\ln L$ is developed, L being the likelihood function. The initial loss cost function for the univariate case was given by

Montgomery and Klatt and expanded for the multivariate case by Kapur and Cho. Chou *et al*(2002) refers also to the related theory behind some of these functions using ideas from Chen (1995) and Nagarsenker and Pillai (1971). Furthermore one should not ignore the work done by Joyalemi and Berrettoni (1989).

Except for the proposal by Chou *et al* (2002) not much attention has been given to the probability of the type-I error as well as the ARL, which depends on the power of the test involved. The proposal of Chou *et al* (2002) can be seen as the starting point of a new field of research, especially if one takes into account that their proposal is subject to very restrictive constraints, i.e. all the quality variables are normally distributed, the mean vector as well as the covariance matrix are known, the sizes of the shifts/changes in each of the mean vector and the covariance are known and only one of either the mean vector with a shift or the covariance matrix with a change can happen at a time, not simultaneously. They even had to specify the upper limit on the Type I error and similarly an equal upper bound on the power of each type of the assignable causes.

Despite all these restrictions one has eventually to rely on extreme value techniques in order to obtain results, the more so when the samples are of size less than 50.

6.2 Model assumptions

1. The cost function is designed and then it is minimised subject to the constrained minimum value of power and maximum value of the type I error.
2. The quality of the process can be described by the mean vector and covariance matrix of p characteristics. This is monitored by a multivariate control chart using the test statistic $-2\ln L$.
3. The p quality characteristics all follow a multivariate normal distribution with given mean vector μ and given covariance matrix $\Sigma = \Sigma_0$.
4. It is assumed that the target vector is the same as $\mu = \mu_0$, i.e. in control. Two non-simultaneous assignable causes are considered, i.e. a shift to $\mu_1 = \mu_0 + \Delta_\mu$, and a change to $\Sigma_1 = \Sigma_0 + \Delta_\Sigma$ where the size of the second term in both cases is known.
5. The mean in-between time of arrivals of assignable causes is $1/\lambda$ according to the exponential distribution.
6. The process is stopped for investigating when an assignable cause occurs.

7. The cost for investigating real and false alarms is the same.
8. The upper bound on the type I error is taken as α and lower bounds on the powers for the two out-of-control states are both taken as $1-\beta$.

6.3 Derivation of the test statistic

The *i*-th random sampling vector is indicated by Y_i. With the usual notation the likelihood ratio criterion of testing the hypothesis $H_0 : \mu = \mu_0$ and $H_0 : \Sigma = \Sigma_0$ against the alternatives $H_1 : \mu = \mu_0 + \Delta_\mu$ and $H_1 : \Sigma = \Sigma_0 + \Delta_\Sigma$ is (Anderson, 1958) for a generic loss function *L* given as

$$L = \left(\frac{e}{n}\right)^{\frac{np}{2}} \left|(n-1) S \Sigma_0^{-1}\right|^{\frac{n}{2}} \exp\left\{-\frac{1}{2} Tr \Sigma_0^{-1} \left[(n-1) S + n\left(\bar{Y} - \mu_0\right)\left(\bar{Y} - \mu_0\right)^T\right]\right\} \qquad (6.1)$$

where*Tr*is the trace of the matrix. The null hypothesis should be rejected when $-2\ln L > UCL$ where *UCL* is the upper 100α th percentile of the distribution of $-2\ln L$, i.e. the upper control limit of the multivariate control chart which indicates that an assignable cause may exist in the process.

To consider the statistical constraints, i.e. the type I error and power constraints of the chart, the distribution functions of $-2\ln L$ under the different hypotheses should be evaluated. Under H_0 the upper control limit can be found for large samples (typically $n > 50$) using an approximate distribution, but when samples are smaller, direct expansions on some steps are needed. Note that the upper limit of the type I error is specified. Under H_1 of an assignable cause the power must be determined by means of a simulation procedure followed by a polynomial regression analysis.

6.4 The distribution function of $-2\ln L$ under H_0

According to Sugiura (1969) the distribution of $-2\ln L$ under H_0 is found using the five steps given below:

Step 1: The characteristic function of $-2\ln L$ under the null hypothesis is

$$\phi(t) = E\left\{e^{it(-2\ln L)}\right\}$$

$$= \frac{\left(2e/n\right)^{npit} \prod_{g=1}^{p} \left\{\Gamma\left(\frac{n(1-2it)-g}{2}\right)\right\}}{\left(1-2it\right)^{np(1-2it)/2} \prod_{g=1}^{p} \left\{\Gamma\left(\frac{n-g}{2}\right)\right\}}. \tag{6.2}$$

Step 2: Gamma in equation (6.2) is approximated through

$$\ln\Gamma(x+h) = \ln\sqrt{2\pi} + (x+h-1/2)\ln x - x - \sum_{r=1}^{w} \frac{(-1)^r B_{r+1}(h)}{r(r+1)x^r} + O\left(x^{-w-1}\right) \tag{6.3}$$

where $B_r(h)$ is the Bernoulli polynomial of degree r (see below). Substituting equation (6.3) in equation (6.2) and taking logarithms gives

$$\ln\phi(t) = \left(-\frac{p(p+1)+2p}{4}\right)\ln(1-2it) - \sum_{r=1}^{w} \frac{(-2)^r B_{r+1}}{r(r+1)x^r}\left[(1-2it)^{-r} - 1\right] + O\left(n^{-w-1}\right) \tag{6.4}$$

where $B_{r+1} = \sum_{g=1}^{p} B_{r+1}(-g/2)$.

Step 3: Applying exponentiation on equation (6.4) yields the characteristic function of $-2\ln L$ which can be expressed as the summation of a chi-square series (Montgomery, 1980).

Step 4: From Anderson (1984), if $Z = -2\ln L$, the density of Z is

$$f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itz} \phi(t)\, dt\,. \tag{6.5}$$

Step 5: The distribution function of $-2\ln L$ can now be obtained, taking w in equation (6.3) and equation (6.4) to be equal to 3 as

$$
\begin{aligned}
F(z) &= P(-2\ln L < z) = \int_0^z f(z)\,dz \\
&= P\left(\chi_f^2 < z\right) + B_2 n^{-1}[P\left(\chi_{f+2}^2 < z\right) - P\left(\chi_f^2 < z\right)] \\
&\quad + \frac{1}{6} n^{-2}[\left(3B_2^2 - 4B_3\right) P\left(\chi_{f+4}^2 < z\right) - 6B_2^2 P\left(\chi_{f+2}^2 < z\right) \\
&\quad + \left(3B_2^2 + 4B_3\right) P\left(\chi_f^2 < z\right)] + \frac{1}{6} n^{-3}[\left(4B_4 - 4B_2B_3 + B_2^3\right) P\left(\chi_{f+6}^2 < z\right) \\
&\quad + B_2\left(4B_3 - 3B_2^2\right) P\left(\chi_{f+4}^2 < z\right) \\
&\quad + B_2\left(4B_3 + 3B_2^2\right) P\left(\chi_{f+2}^2 < z\right) - \left(4B_4 + 4B_2B_3 + B_2^3\right) P\left(\chi_f^2 < z\right)] \\
&\quad + O\left(n^{-4}\right)
\end{aligned}
\tag{6.6}
$$

where

$$
f = p + \frac{p(p+1)}{2}; \quad B_2 = \frac{p\left(2p^2 + 9p + 11\right)}{24}; \tag{6.7}
$$

$$
B_3 = \frac{-p(p+1)(p+2)(p+3)}{32}; \quad B_4 = \frac{p\left(6p^4 + 45p^3 + 110p^2 + 90p + 3\right)}{480}. \tag{6.8}
$$

Note that there is a slight difference between equation (6.6) by Chou *et a*l (2002) and the original according to Sugiura (1969). The original is given below

$$
\begin{aligned}
F(z) &= P(-2\ln L < z) = \int_0^z f(z)\,dz \\
&= P\left(\chi_f^2 < z\right) + [1 + B_2 n^{-1}\{P\left(\chi_{f+2}^2 < z\right) - 1\} \\
&\quad + \frac{1}{6} n^{-2}\{\left(3B_2^2 - 4B_3\right) P\left(\chi_{f+4}^2 < z\right) - 6B_2^2 P\left(\chi_{f+2}^2 < z\right) \\
&\quad + \left(3B_2^2 + 4B_3\right)\} + \frac{1}{6} n^{-3}\{\left(4B_4 - 4B_2B_3 + B_2^3\right) P\left(\chi_{f+6}^2 < z\right) \\
&\quad + B_2\left(4B_3 - 3B_2^2\right) P\left(\chi_{f+4}^2 < z\right) \\
&\quad + B_2\left(4B_3 + 3B_2^2\right) P\left(\chi_{f+2}^2 < z\right) - \left(4B_4 + 4B_2B_3 + B_2^3\right)\}] \\
&\quad + O\left(n^{-4}\right).
\end{aligned}
\tag{6.9}
$$

The approximation by Chou *et al* (2002) however is sufficiently accurate for samples of size $n \geq 50$. For smaller samples the 5-step way in which the distribution was derived, is applied directly on the data and the functions in the second step will be expanded to incorporate more terms so that the big O accuracy level is sufficiently increased in terms of the type I error which must be specified, for an upper bound of 0.1 say. Analytically the distribution above will then be expanded to further terms of the form "B_{5+}" and combinations thereof etc. The expansion of the number of *B's* are directly related to the big *O* accuracy level. At this stage an *UCL* is found.

6.5 The distribution of $-2\ln L$ under H_1

Sugiura (1969) also developed the distribution of $-2\ln L$ under H_1. The approximated distribution still applies only for sample sizes of 50 and more. However for smaller sample sizes expansion of the function does not lead to further probability convergence as in the previous paragraph. In this case one has to resort to simulation and regression approaches and the lower bound on the powers must be specified.

The following 8 steps give a description of the simulation and regression procedures for the first out-of-control state:

Step 1: Choose a value for the *UCL*, called the *z*-value. This *z* must meet the statistical requirements. Chou et al (2002) chose values $n = 4$ and $p = 2$ from which the initial five steps lead to a 90th percentile of the distribution of $-2\ln L$ under H_0 of 14.386. For illustration purposes, take for example the probability of the type-I error as equal to 0.1, then the $UCL \geq 14.386$ i.e $z \geq 14.386 \approx 14.40$, which can be taken as the starting value of z.

For example, in the in-control case the level of accuracy is obtained by expanding the 3rd step function to enough terms. This will lead to an initial value for *n* and a lower level on the value of *z* = *UCL*.

Step 2: Generate n $p \times 1$ random vectors from the multivariate normal distribution with μ_1 and covariance matrix Σ_0.

Step 3: Using the results of step 1, calculate the value of $-2\ln L$.

Step 4: Let $m_j = 1$ if the calculated value of $-2\ln L$ is greater than the selected *z* value, otherwise let $m_j = 0$ where *j* denotes the index of the simulation.

Step 5: Repeat steps 2-4 10 000 times (say).

Step 6: Compute the simulated power for the selected *z* value as follows:

$$\text{Power1} = \frac{\sum_{j=1}^{10000} m_j}{10000}. \tag{6.10}$$

Step 7: Restart from step 1 with z increased by 0.5 and obtain the new simulated power. Carry on like this until the newly obtained simulated power becomes less than say 0.9 which corresponds to the minimum designed on $1-\beta$.

Step 8: Treat the set of z values as independent and the set of legal powers as the dependent variable. Then: Obtain a polynomial regression equation by using forward selection for a certain combination of n and p. This equation is used as a function to estimate the power for the corresponding UCL.

6.6 The cost model

Chou *et al* (2002) combined the work done by Montgomery and Klatt (1972) on the cost function with the work done by Kapur and Chao (1996) to develop a cost model. The cost model was used as the objective function of the design as well as the function they ought to minimize. Chou *et al* (2002) gave a brief description and derivations of the example cost model scenario given by Montgomery and Klatt (1972). They showed that the cost function developed by Montgomery and Klatt (1972) can be expressed as

$$E(C) = (a_1 + a_2 n)/k + \left(a_3 \sum_{i=0}^{2} \rho_i \alpha_i\right)/k + a_4 \sum_{i=0}^{2} \delta_i \gamma_i \tag{6.11}$$

where the a_i's are cost coefficients independent of the whole test procedure and ρ_i, δ_i, γ_i, and α_i are probability elements which will be described in more detail in the next section. k is the number of units produced between samples.

Even though Montgomery and Klatt (1972) showed how the probability elements of the cost function were derived, their derivations were based on the assumption that only shift in the process mean is of prime importance. Chou *et al* (2002), in addition to the shift in the process mean, have taken into consideration the shift in the process covariance matrix as well. This has resulted in a slight change in the definitions and derivations of the probability elements in order to accommodate the shift in the process covariance. Chou *et al* (2002) assumed that there exist three states, one in-control-state and two out-of-control-states, i.e. $i = 0,1,2$

withprobability α_i.The following section gives a brief summary of how Chou *et al* (2002) defined these probability elements:

The elements of vector ρ_i is defined as

$$\rho_0 = P(-2\ln L > z). \tag{6.12}$$

where z is assumed to be the upper control limit.

ρ_1 and ρ_2 are estimated respectively by

$$\text{Power1} = \frac{\sum_{j=1}^{10000} m_j}{10000} \quad \text{and} \quad \text{Power2} = \frac{\sum_{j=1}^{10000} m_j}{10000}.$$

The probability element δ_i is defined as the conditional probability of producing defective units given that the process is in state i. The elements of this probability element are obtained through the use of the following expressions:

$$\delta_0 = 1 - \int_{l_1}^{u_1} \cdots \int_{l_p}^{u_p} \left[(2\pi)^{p/2} |\Sigma_0|^{1/2} \right]^{-1} \exp\left[-\frac{1}{2} (Y - \mu_0)^T \Sigma_0^{-1} (Y - \mu_0) \right] dy_p \ldots dy_1. \tag{6.13}$$

$$\delta_1 = 1 - \int_{l_1}^{u_1} \cdots \int_{l_p}^{u_p} \left[(2\pi)^{p/2} |\Sigma_0|^{1/2} \right]^{-1} \exp\left[-\frac{1}{2} (Y - \mu_1)^T \Sigma_0^{-1} (Y - \mu_1) \right] dy_p \ldots dy_1. \tag{6.14}$$

$$\delta_2 = 1 - \int_{l_1}^{u_1} \cdots \int_{l_p}^{u_p} \left[(2\pi)^{p/2} |\Sigma_1|^{1/2} \right]^{-1} \exp\left[-\frac{1}{2} (Y - \mu_0)^T \Sigma_1^{-1} (Y - \mu_0) \right] dy_p \ldots dy_1. \tag{6.15}$$

The elements of α_i are defined as steady-state probabilities that the process is in state i during the sampling period. These elements are obtained through a transition probability matrix B. Assuming that Q units are produced per hour and fractional units can also be produced, then the probability of the process to remain incontrol during the production of k units is given by

$$P_0 = \exp(-\lambda k / Q) \tag{6.16}$$

where k/Q is the number of units produced per hour during the in-control state. The probability given to the two out-of-control states during the production of k units is given by

$$P_1 + P_2 = 1 - \exp\left(-\lambda k / R\right). \tag{6.17}$$

The formula to determine P_1 and P_2 was developed by Knappenberger and Grandage (1969) and is as follows

$$P_i = \frac{2!\left[1 - \exp(-\lambda k / Q)\theta^i \left(1-\theta\right)^{2-i}\right]}{i!\left(2-i\right)!\left[1-\left(1-\theta\right)^2\right]} \quad \text{for} \quad i = 1,2 \tag{6.18}$$

where θ is the parameter of the distribution and lies between 0 and 1. P_1 and P_2 can be determined by past experience in practise.

The elements of B can now be defined as follows

$$b_{00} = P_0, \;\; b_{0i} = P_i, \;\; b_{ii} = \rho_i P_i + (1-\rho_i) \text{ and } b_{ij} = \rho_i P_j \text{ for } j = 0,1,2 \text{ and } j \neq i,\; i = 1,2$$

where

- b_{00} is the probability that the process remains in control during the production of k units.
- b_{0i} is the probability of moving to the out-of-control state during the production of k units.
- b_{ii} is the probability of detecting out-of-control state during two consecutive sampling plus the probability of failing to detect it at the mth sample.
- b_{ij} is the probability of detecting an out-of control state at the mth sample multiplied by the probability of remaining in control for the production of k units.

B can be written in matrix form as

$$B = \begin{bmatrix} P_0 & P_1 & P_2 \\ \rho_1 P_0 & \rho_1 P_1 + (1-\rho_1) & \rho_1 P_2 \\ \rho_2 P_0 & \rho_2 P_1 & \rho_2 P_2 + (1-\rho_2) \end{bmatrix}. \tag{6.19}$$

The vector α is obtained by noting that B is the transition matrix of an irreducible aperiodic positive recurrent Markov chain such that

$$\alpha^T B = \alpha^T. \tag{6.20}$$

From this follows that

$$\alpha_0 = \rho_1\rho_2 P_0 / (\rho_1 P_2 + \rho_2 P_1 + \rho_1\rho_2 P_0). \tag{6.21}$$

$$\alpha_1 = \rho_2 P_0 / (\rho_1 P_2 + \rho_2 P_1 + \rho_1\rho_2 P_0). \tag{6.22}$$

$$\alpha_2 = \rho_1 P_0 / (\rho_1 P_2 + \rho_2 P_1 + \rho_1\rho_2 P_0). \tag{6.23}$$

The probability elements γ_i for $(i = 1 \text{ and } 2)$ consist of two parts. First is the probability that sampling was done while the process was in either of the two out-of-control states and secondly it is the probability that the process was out of control during the production of k units even though sampling was done while the process was in control. Mathematically it can be written as follows

$$\gamma_i = \alpha_i + \alpha_0 P_i (1-\tau) \quad \text{for } i = 1, 2. \tag{6.24}$$

The probability that the process is in control is given by

$$\gamma_0 = 1 - \gamma_1 - \gamma_2 = \alpha_0 P_0 + \alpha_0 P_1 \tau + \alpha_0 P_2 \tau \tag{6.25}$$

where τ is given by

$$\tau = \frac{1-(1+\lambda k/Q)\exp(-\lambda k/Q)}{\left[1-\exp(-\lambda k/Q)\right]\left(\lambda k/Q\right)}. \tag{6.26}$$

6.7 The loss function

Chou *et al* (2002) adopted the loss function developed by Taguchi and Wu (1985) called the Taguchi loss function. It is expressed mathematically as

$$L(y) = K(y-t)^2. \tag{6.27}$$

where $L(y)$ is the loss due to quality characteristic y and K is a constant relying on the cost at the control limits and control limit interval.

Kapur and Chao (1986) used equation (6.27) to develop a loss function for a multivariate case. They expressed the multivariate loss function as

$$L(y_1, y_2, \ldots, y_p) = \sum_{i=1}^{p}\sum_{j=1}^{i} K_{ij}(y_i - t_i)(y_j - t_j) \tag{6.28}$$

Where t_i is the target of the jth quality characteristic. K_{ij} is a constant that relies on the costat the control limits and control limit interval. For a full description of how the values of K_{ij} can be determined refer to Chen (1995).

Kapur and Chao (1996) proved that the expected value of the multivariate loss function in equation (6.28) can be written as

$$E\left[L(y_1, y_2, \ldots, y_p)\right] = \sum_{i=1}^{p} K_{ii}\left[(\mu_i - t_i)^2 + \sigma_i^2\right] + \sum_{i=2}^{p}\sum_{j=1}^{i=1} K_{ij}\left[(\mu_i - t_i)(\mu_j - t_j) + \sigma_{ij}\right] \quad (6.29)$$

where μ_i and σ_j^2 denote the mean and variance of y_i respectively and σ_{ij} is the covariance of y_i and y_j.

Chou *et al* (2002) finally combined equation (6.11) and (6.29) to develop the cost model that they used as the objective function. The result of joining the two equations is the average total loss (ATL) that includes costs due to testing and loss incurred due to the deviation of quality characteristics of products to their target value. The ATL can be written mathematically as

$$ATL = (a_1 + a_2 n)/k + \left(a_3 \sum_{i=0}^{2} \rho_i \alpha_i\right)/k + a_4 \sum_{i=0}^{2} \delta_i \gamma_i + a_5\left(1 - \sum_{i=0}^{2} \delta_i \gamma_i\right) \quad (6.30)$$

or

$$ATL = (a_1 + a_2 n)/k + \left(a_3 \sum_{i=0}^{2} \rho_i \alpha_i\right)/k + (a_4 - a_5)\sum_{i=0}^{2} \delta_i \gamma_i - a_5 \quad (6.31)$$

where a_5 is defined mathematically as

$$a_5 = E\left[L(y_1, \ldots, y_p)\right] - a_4. \quad (6.32)$$

It is the expected loss per unit product minus the loss incurred due to producing a defective unit of product. As stated in the introduction the objective of the economic statistical design of multivariate charts by considering quality loss is to determine the three test parameters frequency of sampling, h, the sample size, n, as well as the tolerance levels, k to ensure that ATL is minimised and that statistical constraints boundaries are met.

6.8 A two stage solution procedure by Chou *et al* (2002)

6.8.1 First Stage

There are three states:

State 0: It is called the in-control-state. When the process is in state zero the quality characteristics Y are assumed to follow a multivariate normal distribution with mean μ_0 and covariance matrix Σ_0. The aim is to determine if a particular sample size satisfies the statistical constraint $\rho_0 \le 0.1$. This can be determined if the feasible solution areas of UCL for the sample size can be found.

State 1: This is when the process mean vector shifts to μ_1. In this scenario the quality characteristics Y are assumed to follow a multivariate normal distribution with mean μ_1 and covariance matrix Σ_0. The aim here is to determine if a particular sample size satisfies the statistical constraint $\rho_1 \ge 0.9$ and this can be determined if a power estimation function is found. The power estimation function can be found through the use of the Power1 equation. and using this to determine the feasible solution area for the chart.

State 2: This is when the process mean vector shifts to Σ_1. In this scenario the quality characteristics Y are assumed to follow a multivariate normal distribution with mean μ_0 and covariance matrix Σ_1. The aim here is to use the Power2 function to determine the sample size, n, that satisfies statistical constraint $\rho_2 \ge 0.9$ and using this to determine the feasible solution area for the chart.

6.8.2 Second Stage

This is when a grid search is performed to obtain the three test parameters, sample size, n, sample frequency, k, and tolerance level, UCL that gives the minimum of ATL. This can be done through a grid search. Software in mathematics has been developed for this problem.

6.9 Conclusion

Even though the economic design of the control chart has been discussed at length by quite a number of authors, little attention has been given to multivariate control charts. Chou *et al* (2002) are one of the first authors who managed to develop a procedure to carry out an economic statistical design of the multivariate control chart. They achieved their objective by considering quality loss for monitoring the process mean vector and covariance matrix simultaneously. The procedure was developed through the use of the test statistic $-2\ln L$ and the cost model was developed from the combined work of Montgomery and Klatt (1972) and Kapur and Chao (1996). Chou *et al* (2002) gave an illustration of the design procedure and effects of cost parameters through a numerical example. The results showed that

1. There is a positive correlation between the fixed cost of taking samples and the sampling interval.
2. There was a decrease in both the sample size and the upper control limit as the inspection cost per unit increased.
3. There was a positive correlation between cost of investigating and correcting the process and both sample size and the upper control limit.
4. Increase in the penalty cost of producing a defective unit of a product results in the reduction of the sampling interval.

CHAPTER 7
Economic and Economic Statistical Design of a T^2 Control Chart with two Adaptive Sample Sizes.

7.1 Introduction

Control charts are important tools that are used to monitor a process to ensure that the process is in a state of statistical control and thus improving product quality. In the early years of control chart development the common practise in applying a control chart to monitor a process was to obtain samples of fixed sizes at fixed sampling intervals between successive samples. This kind of practise is known as the fixed ratio sampling (FRS) scheme. Faraz*et al* (2010) gave an example of the most common control chart for monitoring the mean of a single variable, the $\bar{x}$-control chart that uses a FRS scheme. In this type of control chart the user is required to select three design parameters: the sample size, n, sampling frequency, h and the width of the control limit, k. The design of an FRS $\bar{x}$-control chart involves selecting suitable sample sizes and sampling intervals. The control limits are determined statistically through specifying a value for the type-I error and /or the type-II error.

According to Faraz*et al* (2010) the FRS $\bar{x}$-control chart has proved to work very well in detecting large shifts in the process mean. However it is not a very good chart to use when detecting small to moderate shifts in the process mean. A procedure has been proposed by Faraz*et al* (2010) that improves the performance of the FRS $\bar{x}$-control chart. The procedure is known as the variable sample size (VSS) scheme. In VSS the region between the control limits and the origin is divided into two for the use of two different sample sizes, n_1 and n_2.If the present sample value falls in any one of the two regions, then the matching sample size will be applied for the successive sampling. Unlike in a FRS $\bar{x}$-control chart, VSS $\bar{x}$-control chart requires the user to select five design parameters: the small sample size, n_1, the large sample size, n_2, the sampling interval, h, the warning limit, w and the control limit, k.

The VSS has also been applied to multivariate quality control charts. Aparisi (1996) statistically designed the VSS T^2-control chart based on the average time to signal (ATS) while Faraz and Moghadam (2008) statistically designed the VSS T^2-control chart based on the adjusted average time to signal (AATS), (Faraz*et al*, 2010). The results of their work showed that applying VSS to the T^2-control chart significantly improves the efficiency of the FRS T^2-control chart.

Due to the economic implications of the design of the control chart, several authors have also come up with economic designs of the control chart based on the FRS scheme. Montgomery and Klatt (1972) developed a cost model for the FRS T^2-control chart. Even though research had proven the efficiency of the VSS scheme on the control chart, little has been done on applying this type of scheme to the economic design of the control chart. Faraz*et al* (2010) therefore considered developing an economic statistical design of the VSS T^2-control chart. Their methodology comprises using the T^2-control chart to build a model of a process that is controlled by a user and making sure that the model is optimized using a generic algorithm (GA).

7.2 VSS T^2-control scheme and the Markov chain approach

The traditional T^2-control chart was developed under the assumption that there exist quality characteristics with a joint p-variate normal distribution with an in-control mean vector $\mu_0^T = (\mu_{01}, \ldots, \mu_{0p})$ and variance-covariance matrix Σ. The FRS T^2-control procedure is presented well in Montgomery and Klatt, (1972). The statistic

$$T^2 = n(\bar{X} - \mu_0)^T S^{-1} (\bar{X} - \mu_0) \sim T^2_{p,n-p} \tag{7.1}$$

where $\bar{X}$ and S are the sample mean vector and sample variance-covariance matrix respectively is used to test whether the process is in control or not. The procedure is as follows:

- Samples of size n are taken periodically.
- $T^2 = n(\bar{X} - \mu_0)^T S^{-1} (\bar{X} - \mu_0) \sim T^2_{p,n-p}$ is used and
- T^2 is plotted against time.

- If $T^2 > T^2_{\alpha,p,n-p}$ reject the null hypothesis and conclude that the process is out of control,
- Investigation of the cause of the out of control should begin.

As can be seen the FRS T^2-scheme requires samples of a fixed size taken periodically. This is different from the VSS T^2-scheme which uses two sample sizes n_1 and n_2 where $n_1 < n_0 < n_2$ and in which the position of each sample point on the chart establishes the size of the next sample. The procedure for VSS T^2-scheme as given by Faraz*et al* (2010) is as follows:

- If $0 \le T^2 < w,$ then the next sample is taken with size n_1.
- If $w \le T^2 < k,$ then the next sample is taken with size n_2.
- If $T^2 \ge k,$ the process is out of control.

One of the objectives of Faraz*et al* (2010) is to compare the VSS and the FRS for different charts. The most recent statistical measure that is used to compare the different schemes' efficiency is the AATS, i.e. the mean time from the shift until a signal is observed. AATS determines the speed at which a control chart identifies a shift in the process mean and is computed as follows:

$$\text{AATS} = \text{ATC} - \frac{1}{\lambda} \tag{7.2}$$

where ATC is the average time of the cycle, i.e. the time between the start of the production process and the first signal after the process shift and $\frac{1}{\lambda}$ is the expected time-interval that the process remains in control. Due to the memory-less property of the exponential distribution ATC can be computed using the Markov chain approach. The Markov chainapproach has to do with mathematical models for stochastic systems whose states are governed by a transition probability. Faraz*et al* (2010) discusses the six transient states that are reached at each stage of sampling

state 1: $0 \le T^2 < w,$ the process is in-control; next sample size is n_1.

state 2: $w \le T^2 < k,$ the process is in-control; next sample size is n_2.

state 3: $T^2 \ge k,$ the process is in-control; next sample size is n_2, i.e. a false alarm.

state4: $0 \le T^2 < w,$ the process is out-of- control; next sample size is n_1.

state 5: $w \le T^2 < k,$ the process is out-of- control; next sample size is n_2.

state 6: $T^2 > k$, the process is out-of-control, an absorbing state is reached i.e. a true alarm.

The transition probability matrix P and its computations is given by Faraz*et al* (2010) as follows

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} \\ 0 & 0 & 0 & p_{44} & p_{45} & p_{46} \\ 0 & 0 & 0 & p_{54} & p_{55} & p_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.3}$$

where p_{ij} represents the transition probability, i is the prior state, and j is the current state. Define $F(x, p, \eta)$ as the cumulative probability distribution function of a non-central chi-square distribution with p degrees of freedom and a non-centrality parameter $\eta = n_i d^2$, where $d = (\mu_1 - \mu_0)^T \Sigma^{-1} (\mu_1 - \mu_0)$ then the p_{ij}'s can be computed as follows

$$p_1 = p_{11} = p_{21} = p_{31} = P(T^2 < w) \times e^{-\lambda h} = F(w, p, \eta = 0) \times e^{-\lambda h}$$

$$p_2 = p_{12} = p_{22} = p_{32} = P(w \le T^2 < k) \times e^{-\lambda h} = [F(k, p, \eta = 0) - F(k, p.\eta = 0)] \times e^{-\lambda h}$$

$$p_3 = p_{13} = p_{23} = p_{33} = P(T^2 \ge k) \times e^{-\lambda h} = [1 - F(k, p, \eta = 0)] \times e^{-\lambda h}$$

$$p_{14} = P(T^2 < w) \times (1 - e^{-\lambda h}) = F(w, p, \eta = n_1 d^2) \times (1 - e^{-\lambda h})$$

$$p_{15} = P(w \le T^2 < k) \times (1 - e^{-\lambda h}) = [F(k, p, \eta = n_1 d^2) - F(w, p, \eta = n_1 d^2)] \times (1 - e^{-\lambda h})$$

$$p_{16} = P(T^2 \ge k) \times (1 - e^{-\lambda h}) = [1 - F(k, p, \eta = n_1 d^2)] \times (1 - e^{-\lambda h})$$

$$p_{24} = p_{34} = P(T^2 < w) \times (1 - e^{-\lambda h}) = F(w, p, \eta = n_2 d^2) \times (1 - e^{-\lambda h})$$

$$p_{25} = p_{35} = P(w \le T^2 < k) \times (1 - e^{-\lambda h}) = [F(k, p, \eta = n_2 d^2) - F(w, p, \eta = n_2 d^2)] \times (1 - e^{-\lambda h})$$

$$p_{26} = p_{36} = P(T^2 \ge k) \times (1 - e^{-\lambda h}) = [1 - F(k, p, \eta = n_2 d^2) \times (1 - e^{-\lambda h})$$

$$p_{44} = P(T^2 < w) = F(w, p, \eta = n_1 d^2)$$

$$p_{45} = P(w \le T^2 < k) = F(k, p, \eta = n_1 d^2) - F(w, p, \eta = n_1 d^2)$$

$p_{46} = P(T^2 \geq k) = 1 - F(k, p, \eta = n_1 d^2)$

$p_{55} = P(w \leq T^2 < k) = F(k, p, \eta = n_2 d^2) - F(w, p, \eta = n_2 d^2)$

$p_{56} = P(T^2 \geq k) = 1 - F(k, p, \eta = n_2 d^2)$

The following is a well-known identity: The expected number of trials in each state to reach the absorbing state can be obtained from $b^T(I-Q)^{-1}$, where Q is the 5**x**5 matrix obtained from the transition matrix P whose elements that match those of the absorbing state have been deleted, I is the identity matrix of order 5, and $b^T = (p_1, p_2, p_3, 0, 0)$ is a vector of initial probabilities, with $\sum_{i=1}^{3} p_i = 1$. The steady state ATS is therefore computed as follows:

$$\text{ATC} = b^T(I-Q)^{-1}h \tag{7.4}$$

where h is a vector of sampling time intervals. (Faraz*et al* (2010) set the vector b to $(0,1,0,0,0)$ for convenience during the start-up time).

7.3 The cost model

Faraz*et al* (2010) adopted the cost model proposed by Lorenzen and Vance, (1986) which was based on the Markov chain approach to build their objective function. They made a number of assumptions that govern the development of their cost model.

7.3.1 Model assumptions

1. The process is controlled by a VSS T^2-control scheme that monitors p-correlated quality characteristics.
2. The p-correlated quality characteristics have a joint p-variate normal distribution with an in-control mean vector $\mu_0^T = (\mu_{01}, \ldots, \mu_{0p})$ and variance-covariance matrix Σ.
3. The process has an in-control state $\mu = \mu_0$.
4. Only a single assignable cause yields "step changes" in the process mean from $\mu = \mu_0$ to $\mu = \mu_1$ where the latter is known.
5. Σ is assumed constant.

6. The mean in-between time of arrivals of assignable causes is $1/\lambda$.
7. Once the process is out of control, it stays there until corrective measures are employed.
8. The quality cycle follows a renewal reward process.

7.4 The cost function

According to Duncan (1956) a quality cycle consists of four time-intervals: The in-control state, the out-of-control state, period of sampling and interpretation of results and lastly the period to investigate and possible correction of the assignable causes.

7.4.1 Quality cycle

Faraz*et al* (2010) expressed the expected total cycle time based on the events taking place in each of the four time-intervals. The diagram below describes the quality cycle

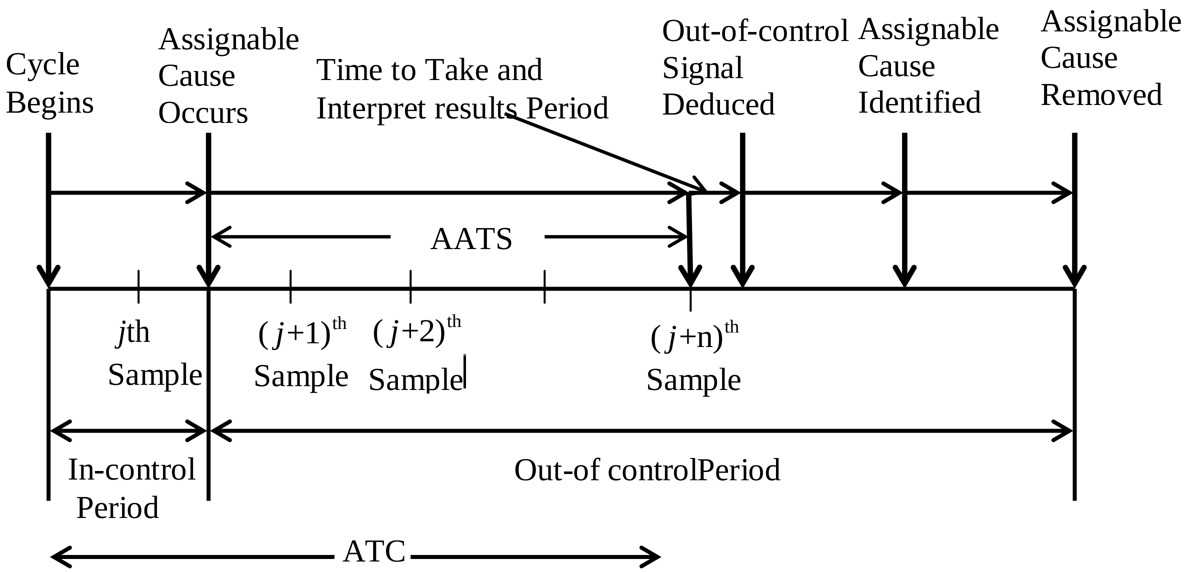

Figure 7.1: A quality cycle

7.4.2 The expected total cycle time

The expected time spent by the process in the in-control period before going out of control is computed as follows:

$$\text{In-control time} = \frac{1}{\lambda} + (1-\gamma_1)T_0\text{ANF} \tag{7.5}$$

where $\frac{1}{\lambda}$ is the expected time when the process is in control, given that the process was not halted due to false alarms and

$$\gamma_1 = \begin{cases} 1 & \text{if production was not stopped during investgations} \\ 0 & \text{otherwise.} \end{cases} \tag{7.6}$$

T_0 is the expected time taken to investigate a false alarm.

ANF refers to the average number of false alarms and can be calculated as follows

$$\text{ANF}=b^T(I-Q)^{-1}f \tag{7.7}$$

where $f=(0,0,1,0,0)$, i.e. taking the 3th element from the $b^T(I-Q)^{-1}$ vector.

Next the expected time spent by the process in an out-of-control period is calculated as follows:

$$\text{Out-of-control time} = AATS + \bar{n}E + T_1 + T_2 \tag{7.8}$$

where T_1 is the amount of time expected to find the assignable cause, and T_2 is the amount of time expected to do an overhaul of the process.

AATS is the average amount of time spent by the process out of control before it gives an out-of-control signal and $\bar{n}E$ is the total time to take and interpret a sample where

$$\bar{n} = n_1(p_{16}+p_{46}) + n_2(p_{26}+p_{36}+p_{56}), \tag{7.9}$$

and E is the proportionality constant.

The expected total cycle time then follows as the sum of the time spent in control and the time spend out of control, i.e.

$$\begin{aligned} E(T) &= \frac{1}{\lambda} + (1-\gamma_1)T_0\text{ANF+AATS+}\bar{n}E + T_1 + T_2 \\ &= \frac{1}{\lambda} + \text{AATS} + (1-\gamma_1)T_0\text{ANF+}\bar{n}E + T_1 + T_2 \\ &= \text{ATC} + (1-\gamma_1)T_0\text{ANF+}\bar{n}E + T_1 + T_2. \end{aligned} \tag{7.10}$$

7.4.3 Quality cycle cost.

To compute the quality cycle cost Faraz*et al* (2010) adopted the approach and symbols developed by Lorenzen and Vance (1986) in which they classified costs into four main components. These are: costs associated with producing defective products while the process is in control, costs associated with producing defective products while the process is out of control, costs associated with assessing alarms and lastly costs of sampling. The expected cost of producing defective products when the process is in control

$$= \frac{C_0}{\lambda} + C_1[\text{AATS} + \bar{n}E + \gamma_1 T_1 + \gamma_2 T_2] \tag{7.11}$$

where C_0 and C_1 represent the cost of producing defective products while the process is in-control and out-of-control respectively and

$$\gamma_2 = \begin{cases} 1 & \text{if production continues during repair of the process} \\ 0 & \text{otherwise.} \end{cases}$$

The expected cost of assessing false alarms and correcting the process is given by

$$a_3' \text{ANF} + a_3 \tag{7.12}$$

where a_3' is the cost of investigating false alarms and a_3 denotes to the cost of finding and correcting an assignable cause.

Finally the expected cost of sampling per cycle is given by

$$(a_1 \text{ANS} + a_2 \text{ANI}) + \frac{(a_1 + a_2 n_2)(\bar{n}E + \gamma_1 T_1 + \gamma_2 T_2)}{h} \tag{7.13}$$

where a_1 and a_2 are fixed and variable costs components of sampling and testing respectively. ANI denotes the expected number of examined items and ANS denotes the expected number of samples taken during the quality cycle. They are computed as follows

$$\text{ANI} = b^T (I - Q)^{-1} n \tag{7.14}$$

$$\text{ANS} = b^T (I - Q)^{-1} I \tag{7.15}$$

where $n^T = (n_1, n_2, n_2, n_1, n_2)$ and $1^T = (1,1,1,1,1)$.

The expected cost per cycle, $E(C)$ is therefore obtained from combining all three components given in equations (7.11), (7.12) and (7.13) as follows

$$\begin{aligned} E(C) = {} & \frac{C_0}{\lambda} + C_1[\text{AATS} + \bar{n}E + \gamma_1 T_1 + \gamma_2 T_2] + a_3' \text{ANF} + a_3 + (a_1 \text{ANS} + a_2 \text{ANI}) \\ & + \frac{(a_1 + a_2 n_2)(\bar{n}E + \gamma_1 T_1 + \gamma_2 T_2)}{h}, \end{aligned} \tag{7.16}$$

and the expected cost per hour is given by $E(A) = \dfrac{E(C)}{E(T)}$, due to the renewal reward assumption.

7.5 The optimization problem and GA approach

The objective of the economic design of the VSS T^2-control chart is to find the five chart parameters: the small sample size, n_1, the large sample size, n_2, the sampling interval, h, the warning limit, w and the control limit, k which minimize $E(A)$. $E(A)$ is composed of nine process parameters $(p, \lambda, d, T_0, T_1, T_2, \gamma_1, \gamma_2, E)$ and six cost parameters $(C_0, C_1, a_1, a_2, a_3, a_3')$. The chart parameters consist of two discrete variables, n_1 and n_2 such that $1 < n_1 < n_2$ and three continuous parameters, while $0 < w < k$. h is assumed to be at most 8, which is the number of working hours for each work shift. Faraz*et al*, (2010) finally gives the general optimization problem where $E(A)$ is minimized based on the following conditions:

1. $0 \le w < k$.
2. $1 \le n_1 < n_2$.
3. $k > 0$.
4. $0 < h \le 8$.
5. $n_1, n_2 \,\varepsilon\, Z^+$.

The optimization problem for the economic statistical design is obtained through adding statistical constraints, ANF $\le$ ANF and/or AATS $\le$ AATS$_1$. Obviously small ANF and AATS values are hoped for. The optimization problem can be solved using the genetic algorithm.

7.6 Economic design of the VSS T^2-control scheme

The purpose of this section is to design a VSS T^2-control chart that is comparable to the FRS T^2-control chart. To achieve this, the in-control time and cost must be assumed equal for both charts. This implies that ANF, ANS and ANI values should be the same during the in-control period. ANS and ANI are therefore calculated for the VSS scheme by setting $n^T = (n_1, n_2, n_2, 0, 0)$ and $1^T = (1,1,1,0,0)$. The VSS T^2-control chart is called an FRS T^2-control chart if $w = 0$ and when the sample size is the same at each sampling interval. In this case, the transition matrix elements equations change accordingly to suit the new specifications. The new elements of the transition matrix can be computed as follows

$p_{11} = p_{21} = p_{31} = p_{14} = p_{24} = p_{34} = p_{44} = p_{54} = 0.$

$p_{12} = p_{22} = p_{32} = P(T^2 < k) \times e^{-\lambda h} = F(k, p, \eta = 0) \times e^{-\lambda h}.$

$p_{13} = p_{23} = p_{33} = P(T^2 \geq k) \times e^{-\lambda h} = [1 - F(k, p, \eta = 0)] \times e^{-\lambda h}.$

$p_{15} = p_{25} = p_{35} = P(T^2 < k) \times (1 - e^{-\lambda h}) = F(k, p, \eta = n_0 d^2) \times (1 - e^{-\lambda h}).$

$p_{16} = p_{26} = p_{36} = P(T^2 \geq k) \times (1 - e^{-\lambda h}) = [1 - F(k, p, \eta = n_0 d^2) \times (1 - e^{-\lambda h}).$

$p_{45} = p_{55} = P(T^2 < k) = F(k, p, \eta = n_0 d^2).$

$p_{46} = p_{56} = P(T^2 \geq k) = 1 - F(k, p, \eta = n_0 d^2).$

By letting $n^T = (n_0, n_0, n_0, 0, 0)$ and $1^T = (1,1,1,0,0)$, the in-control ANS, ANI and ANF for the FRS T^2-control chart, taking into consideration the optimal parameters (k_0, h_0, n_0) are computed as follows

$$\text{ANS} = \frac{1}{1 - e^{-\lambda h_0}}. \tag{7.17}$$

$$\text{ANI} = n_0 x \frac{1}{1 - e^{-\lambda h_0}}. \tag{7.18}$$

$$\text{ANF} = \frac{p_{13}}{1 - e^{-\lambda h_0}} = \alpha \times \frac{e^{-\lambda n_0}}{1 - e^{-\lambda h_0}}. \tag{7.19}$$

Also considering the set of parameters (k, w, n_1, n_2, h) for the VSS T^2-control chart the values of n_1 and n_2 can be selected so as to get an average size n_0. Then n_2 is computed as follows

$$n_1 p_1 + n_2(1 - p_1) = \frac{n_0 \Delta}{1 - e^{-\lambda h_0}} \Rightarrow n_2 = \frac{n_0 - n_1 F(w, p, \eta = 0) \times e^{-\lambda h_0}}{1 - F(w, p, \eta = 0) \times e^{-\lambda h_0}}$$

where $\Delta = 1 - p_1 - p_2 - p_3 = 1 - e^{-\lambda h_0}$.

The goal of VSS T^2-control chart is to find the two chart parameters w and n_1 which cannot be equated to the parameters of the FRS T^2-control chart.

7.7 Conclusion

Faraz et al (2010) successfully developed an economic statistical model of a process controlled by a VSS T^2-control chart. They adopted the model developed by Lorenzen and

Vance (1986) to develop the model in which the expected total cost per hour is minimized using a genetic approach. In comparing the economic design of VSS and economic design of FRS charts, the results showed that VSS charts perform better than the FRS chart and they are also a very close competitor of the MEWMA chart when economic and statistical criterions are used.

CHAPTER 8
Concluding Remarks

8.1 Remarks

It is clear that the economic statistical aspects of quality control have barely been touched. A large gap in research exists. The following cases/situations represent themselves, and that is by far not exhaustive:

- the mean vector and covariance matrix are unknown.
- the sizes of the shift/change in the mean vector/covariance matrix are unknown.
- the problem of the power if the detail of H_1 is not available.
- non-normality of the qualitative variables.
- qualitative variables.
- application in the case where MCUSUM is appropriate.
- application where MEWMA is appropriate.

A possible robust approach is seemingly needed as the problem itself, in whatever form, seems analyticallyintractable, although this has not been proved. Thus it seems reasonable to findestimated densities using bootstrap and taking it from there.Clearly the field is wide open for further research in a field which may not be easy, but of which the results should be extremely rewarding for industry.As a last remark refer the Economic design of the Multivariate Exponentially WeightedMoving Average(MEWMA) as an example of further research.

8.2 Multivariate exponentially weighted moving average (MEWMA)

The MEWMA control chart has an advantage over the Shewhart type control charts in that they are very sensitive to small shifts in the process. It is therefore important to give a brief introduction to the economic design of this type of a control chart.

8.2.1 The economic design of the MEWMA

The design of the MEWMA control chart consists of a total of 13 parameters. These include the sample size, n, the upper control limit, UCL, the exponential smoothing constant, r and the time interval between samples, h, the expected time to sample and chart one item, E, the expected time spend searching for a false alarm, T_0, the expected time required to discover an assignable cause, T_1, the expected repair time, T_2, a parameter that indicates whether production continued during searches, γ_1, a parameter that indicates whether production continued during repairs, γ_2, costs of production when the process is in control, C_0, costs of production when the process is out of control, C_1, cost of investigating a false alarm, Y, cost of identifying and repairing an assignable cause, W and $a+bn$ which gives the cost of sampling and testing. A C program was developed by Linderman and Love (2000b) to minimize the expected cost per hour for the MEWMA chart. They achieved this by optimizing the design parameters: n, h, L and r. The cost function that was minimized is as follows

$$C(n,h,L,r) = \frac{E(Cost)}{E(Time)} \tag{1.1}$$

$$= \frac{\left\{C_0/\lambda + C_1\left[-\tau + nE + h(ARL1) + \gamma_1 T_1 + \gamma_2 T_2\right] + sY/ARL0 + W + \left[(a+bn)/h\right]\left[1/\lambda - \tau + nE + h(ARL1) + \gamma_1 T_1 + \gamma_2 T_2\right]\right\}}{\left\{1/\lambda + (1-\gamma_1)sT_0/ARL0 - \tau + nE + h(ARL1) + T_1 + T_2\right\}}$$

where $h-\tau$ is the expected time between a shift in the process and the next sample, and

$$\tau = \frac{\int_{jh}^{(j+1)h} e^{-\lambda h}(t-jh)dt}{\int_{jh}^{(j+1)h} e^{-\lambda h}dt} = \frac{1-(1+\lambda h)e^{-\lambda h}}{\lambda(1-e^{-\lambda h})} \tag{1.2}$$

and $s = e^{-\lambda h}/(1-e^{-\lambda h})$ is the expected number of samples while the process is in control.

8.3 Conclusion

Multivariate control charts are not as common as univariate control charts. The major setback of the use of multivariate control charts is that when an out-of-control state is detected it is difficult to determine the variable/s that caused the out-of-control state. Even though a number of approaches to solving this problem have been established by some authors, for example Woodall and Montgomery (1999), more complex operations are required to determine which variable/s caused a signal in a multivariate control chart. A major problem is that the solutions cannot be solved analytically. Some of the approaches employed to this problem include the use of principle component analysis, step down procedures, graphical techniques and starplots, but the field is still open for much research.

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APPENDIX

Table 1.Optimization model for multivariate normal case

Minimize

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$$ETL = \sum_{i=1}^{m} k_i \left[(\mu_{Ti} - t_i)^2 + \sigma_{Ti}^2 \right] + \sum_{i=2}^{m} \sum_{j=1}^{i-1} k_{ij} \left[\sigma_{Tij}^2 + (\mu_{Ti} - t_i)(\mu_{Tj} - t_j) \right] + C_s(1-q) + C_1$$

subject to

$$q = \int_{\mu_1-\alpha_1\sigma_1}^{\mu_1+\beta_1\sigma_1} \cdots \int_{\mu_m-\alpha_m\sigma_m}^{\mu_m+\beta_m\sigma_m} f(y_1 \ldots y_m) dy_m \ldots dy_1 :$$

$$\mu_{Ti} = \int_{\mu_1-\alpha_1\sigma_1}^{\mu_1+\beta_1\sigma_1} \cdots \int_{\mu_m-\alpha_m\sigma_m}^{\mu_m+\beta_m\sigma_m} \frac{y_i}{q} f(y_1 \ldots y_m) dy_m \ldots dy_1, \ i = 1, \ldots, m \ :$$

$$\sigma_{Ti}^2 = \left[\int_{\mu_1-\alpha_1\sigma_1}^{\mu_1+\beta_1\sigma_1} \cdots \int_{\mu_m-\alpha_m\sigma_m}^{\mu_m+\beta_m\sigma_m} \frac{y_i^2}{q} f(y_1 \ldots y_m) dy_m \ldots dy_1 \right] - \mu_{Ti}^2, \ i = 1, \ldots, m \ :$$

$$\sigma_{Tij} = \left[\int_{\mu_1-\alpha_1\sigma_1}^{\mu_1+\beta_1\sigma_1} \cdots \int_{\mu_m-\alpha_m\sigma_m}^{\mu_m+\beta_m\sigma_m} \frac{y_i y_j}{q} f(y_1 \ldots y_m) dy_m \ldots dy_1 \right] - \mu_{Ti}\mu_{Tj}, \ i \neq 1, \ldots, m \ :$$

$$f(y_i \ldots y_m) = \frac{1}{(2\pi)^{m/2} |V|^{1/2}} \exp\left[-\frac{1}{2}(y-\mu)^T V (y-\mu) \right]:$$

$$0 \leq \alpha_i \leq (\mu_i - y_{Li}) / \sigma_i, i = 1, \ldots, m \ :$$

$$0 \leq \beta_i \leq (y_{Ui} - \mu_i) / \sigma_i, i = 1, \ldots, m \ :$$

Table 2.Optimization model for bivariate normal case

Minimize

$$ETL = \sum_{i=1}^{2} k_i \left[(\mu_{Ti} - t_i)^2 + \sigma_{Ti}^2 \right] + k_{12} \left[\sigma_{T12}^2 + (\mu_{T1} - t_1)(\mu_{T2} - t_2) \right] + C_s(1-q) + C_1$$

subject to

$$q=\int_{\mu_1-\alpha_1\sigma_1}^{\mu_1+\beta_1\sigma_1}\int_{\mu_2-\alpha_2\sigma_2}^{\mu_2+\beta_2\sigma_2} f(y_1,y_2)dy_2dy_1 :$$

$$\mu_{Ti}=\int_{\mu_1-\alpha_1\sigma_1}^{\mu_1+\beta_1\sigma_1}\int_{\mu_2-\alpha_2\sigma_2}^{\mu_2+\beta_2\sigma_2} \frac{y_i}{q} f(y_1,y_2)dy_2dy_1,\ i=1,2\ :$$

$$\sigma_{Ti}^2=\left[\int_{\mu_1-\alpha_1\sigma_1}^{\mu_1+\beta_1\sigma_1}\int_{\mu_m-\alpha_m\sigma_m}^{\mu_m+\beta_m\sigma_m} \frac{y_i^2}{q} f(y_1,y_2)dy_2\ldots dy_1\right]-\mu_{Ti}^2,\ \ i=1,2\ :$$

$$\sigma_{T12}=\left[\int_{\mu_1-\alpha_1\sigma_1}^{\mu_1+\beta_1\sigma_1}\int_{\mu_2-\alpha_2\sigma_2}^{\mu_2+\beta_2\sigma_2} \frac{y_1y_2}{q} f(y_1,y_2)dy_2dy_1\right]-\mu_{T1}\mu_{T2} :$$

$$f(y_i,y_2)=\frac{1}{(2\pi)\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{y_1-\mu_1}{\sigma_1}\right)\left(\frac{y_2-\mu_2}{\sigma_2}\right)+\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2\right]\right]:$$

$$0\le\alpha_i\le(\mu_i-y_{Li})/\sigma_i, i=1,2\ :$$

$$0\le\beta_i\le(y_{Ui}-\mu_i)/\sigma_i, i=1,2\ :$$

Table 3.Optimization model for numeric example

Minimize

`

$$\text{ETL}=30\left[(\mu_{T1}-9.2)^2+\sigma_{T1}^2\right]+25\left[(\mu_{T2}-19.4)^2+\sigma_{T2}^2\right]+10\left[\sigma_{T12}+(\mu_{T1}-9.2)(\mu_{T2}-19.4)\right]+80(1-q)+2$$

subject to

$$q=\int_{10-\sqrt{0.5}\alpha_1}^{10-\sqrt{0.5}\beta_1}\int_{20-\sqrt{0.8}\alpha_2}^{20-\sqrt{0.8}\alpha_2} f(y_1,y_2)dy_2dy_1;$$

$$\mu_{Ti}=\int_{10-\sqrt{0.5}\alpha_1}^{10-\sqrt{0.5}\beta_1}\int_{20-\sqrt{0.8}\alpha_2}^{20-\sqrt{0.8}\alpha_2} \frac{y_i}{q} f(y_1,y_2)dy_2dy_1\ \ i=1,2\ ;$$

$$\sigma_{Ti}^2=\left[\int_{10-\sqrt{0.5}\alpha_1}^{10-\sqrt{0.5}\beta_1}\int_{20-\sqrt{0.8}\alpha_2}^{20-\sqrt{0.8}\alpha_2} \frac{y_i^2}{q} f(y_1,y_2)dy_2dy_1\right]-\mu_{Ti}^2,\ \ i=1,2\ ;$$

$$\sigma_{T12}=\left[\int_{10-\sqrt{0.5}\alpha_1}^{10-\sqrt{0.5}\beta_1}\int_{20-\sqrt{0.8}\alpha_2}^{20-\sqrt{0.8}\alpha_2} \frac{y_1y_2}{q} f(y_1,y_2)dy_2dy_1\right]-\mu_{T1}\mu_{T2};$$

$$f(y_i,y_2)=\frac{1}{(2\pi)\sqrt{0.5*0.8}\sqrt{1-0.36}}\exp\left[-\frac{1}{2(1-0.36)}\left[\left(\frac{y_1-10}{\sqrt{0.5}}\right)^2-2(0.6)\left(\frac{y_1-10}{\sqrt{0.5}}\right)\left(\frac{y_2-20}{\sqrt{0.8}}\right)+\left(\frac{y_2-20}{\sqrt{0.8}}\right)^2\right]\right]:$$

$$0\le\alpha_1\le5.0;$$

$$0\le\alpha_2\le5.0;$$

$$0\le\beta_1\le5.0;$$

$$0\le\beta_2\le5.0;$$

Table 4.Computational results under two different schemes

	Without inspection	With inspection
Mean (Y_1)	10	9.7289
Mean (Y_2)	20	19.6610
Bias (Y_1)	0.8	0.5289
Bias (Y_2)	0.6	0.2610
Variance (Y_1)	0.5	0.2867
Variance (Y_2)	0.8	0.4478
Covariance	0.3795	0.1593
q	1.0	0.7144
ETL per unit product	71.80	57.71