An investigation into the development of mathematical modelling competencies of Grade 7 learners

by
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Declaration

I, the undersigned, hereby declare that the work contained in this dissertation is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

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Abstract

Mathematical modelling is becoming a popular teaching and learning approach in mathematics education. There is however a need within the modelling domain to identify exactly what modelling competencies are and how these competencies develop. This study examines how mathematical modelling competencies develop in Grade 7 students working in groups.

Modelling is placed in the field of mathematics teaching and learning as a significant means of learning mathematics. Modelling competencies are identified and characterised from existing literature and explored through empirical generation and collection of data. The study is qualitative in nature and uses a mixed approach of design research and some aspects of grounded theory. Students' progress through a modelling program is documented while the modelling competencies of students stereotyped as weak and strong are also investigated. The findings firmly support earlier research that competencies do develop in students who are exposed to modelling. A comprehensive picture of the modelling situation is presented since this study merges competencies from other studies into a detailed analysis of the modelling situation - it presents an authentic modelling situation of students working in groups and furthers the discussion on modelling competencies.

The analysis of the data suggests that the development of modelling competencies is complex and interrelated but that competencies do develop progressively in groups involved in modelling tasks. Recommendations for additional studies include studies of a longer duration and a full investigation into the link between modelling and language ability.

Key words: mathematical modelling, modelling competencies, problem solving, mathematics teaching and learning, stereotyped (weak and strong) students.

Abstrak

Wiskundige modellering is besig om ‘n populêre onderrig- en studiebenadering in wiskundeonderwys te word. Daar is egter ‘n behoefte om die modelleringsbevoegdhede te identifiseer in hierdie veld en om te weet hoe hierdie bevoegdhede ontwikkel. Hierdie studie ondersoek watter bevoegdhede in wiskundige modellering by Gr.7 studente wat in groepe saamwerk ontwikkel.

Modellering is in die studieveld van wiskundeonderrig en -leer geplaas as 'n betekenisvolle leerwyse in wiskunde. Modelleringsbevoegdhede word vanuit bestaande literatuur en navorsing geïdentifiseer en beskryf deur empiriese generering en versameling van data. Die studie is kwalitatief van aard en gebruik ‘n gemengde benadering van ontwikkelingsondersoek en sekere aspekte van begronde teorie. Studente se vordering in die modelleringsprogram is gedokumenteer terwyl die modelleringsbevoegdhede van gestereotipeerde swak en sterk studente ook ondersoek is. Die resultate bevestig vroeëre navorsing dat bevoegdhede ontwikkel word deur studente wat blootgestel is aan modellering. ‘n Omvattende beeld van die modelleringsituasie is in hierdie studie aangebied waardeur modelleringsbevoegdhede, soos geïdentifiseer in ander studies, tot ‘n gedetailleerde analise van die modelleringsituasie saamgevoeg word. Dit verteenwoordig dus ‘n outentieke modelleringsituasie van studente wat in groepe saamwerk en bevorder so die gesprek oor modelleringsbevoegdhede.

Die analise van die data suggereer dat die ontwikkeling van modelleringsbevoegdhede kompleks en geïntegreerd is, en dat bevoegdhede progressief ontwikkel in groepe wat betrokke is by modelleringstake. Aanbevelings vir addissionele studies sluit langer ondersoektydperke in en 'n dieper ondersoek na die verband tussen modellering en taalvaardigheid.

Sleutelwoorde: Wiskundige modellering, modelleringsbevoegdhede, modelontlokkende aktiwiteite, probleemoplossing, wiskundeonderrig en leer, gesteriotipeerde (sterk en swak) studente.

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CONTENTS

List of Appendices

List of Tables

List of Figures

CHAPTER 1

BACKGROUND, OVERVIEW AND MOTIVATION

1. OVERVIEW

This chapter provides the background and outline of the study. It serves to orientate the reader to key features of the study and describe the nature of mathematical modelling in education. The chapter provides a brief theoretical and conceptual framework for the study. It also outlines essential aims, objectives and methodology for the study. The chapter provides some definitions and delineations of the study.

1.1 MOTIVATION AND ANALYSIS OF FIELD OF STUDY

Modelling is one of a number of reality-based learning activities (Maaß 2006: 114). Modelling involves simplifying a complex real situation, creating a model, working mathematically with it and interpreting the result in that real situation. It is a complex, multifaceted task designed for groups of students to solve (See Appendix A-C).

Modelling, as a task is often described as a 'model-eliciting activity' (Lesh & Doerr 2003a: 3) which is based on real life situations. Both modelling and problem solving activities advocate that there is not one procedure between question and answer. In problem solving, students may only need to move through one cycle from question to answer, while authentic modelling tasks involve students moving through multiple modelling cycles (Lesh & Harel 2003: 160) or modelling routes (Borromeo Ferri 2006: 91) that they continually review and revise. Very often, in problem solving students 'practice' a method or heuristic, while modelling situations require varying strategies to simplify and understand the problem before a solution (one of many) can be formulated.

Modelling also invariably involves groups of students or students working together. The collaborative effect of groups is valuable in mathematics problem solving. Artz and Armour-Thomas (1992: 164) found that small groups seemed to encourage spontaneous verbalization and allowed ideas to be criticized. This verbalization allows one to gauge student thinking to a greater degree. Since competencies of the group are likely to be greater than those of individuals (Zawojewski, Lesh & English 2003: 342) the group is seen as the unit in this study and their performance assessed as a unit. In authentic modelling situations, students are informed of why a solution strategy must be found, and who needs the solution strategy (Lesh & Harel 2003: 160). Niss (1992: 352) states that for knowledge and skills of the modelling process to be well-founded, it would be essential for students to gain experiences with the entire modelling cycle (see Fig 1.1) at each educational level.

Mathematics education has much to gain from including mathematical modelling activities in the classroom. According to Niss, Blum and Galbraith (2007:19) modelling can make 'fundamental contributions' to a student's development of mathematical competencies. These mathematical competencies, which are laid out by Blomhoj and Jensen (2007: 47), include mathematical modelling as one of the mathematical competencies. Modelling generates an environment where students can become mathematically literate since it calls for them to use their own mathematical constructs in solving real world problems. In this study mathematical literacy is taken from the PISA (OECD 2003: 24) definition as "an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen." Lingefjard (2006: 111) affirms that modelling is one of the main and most useful applications of mathematics.

The research of Lesh and Doerr (1998: 375) led to four key aspects in mathematical modelling. These key aspects reiterate the value of modelling to mathematics education. Firstly, the representational abilities needed in the process of mathematising problems are overlooked in traditional word problems. The result is that modelling tasks allow a broader range of capable students to emerge. Secondly, and of particular interest is that students are able to invent constructs that they need (Lesh, Hoover & Kelly in Lesh & Doerr 1998: 376). Thirdly, when students develop models they often use a rich assortment of interacting representations that involve personal systems that the students introduced themselves. Fourthly, student experiences in model-eliciting activities serve as exemplars for thinking about similar situations.

According to Lesh and English (2005: 487), industry advisors emphasise that industry needs people who are competent in making sense of complex systems, working within teams of diverse specialists, adapting to a variety of evolving conceptual tools, working on multistage projects and developing sharable conceptual tools. These very same proficiencies lie at the heart of modelling tasks. Schools are expected to prepare students for the future, which includes the workplace of the future. This would mean that schools need to assimilate modelling into mathematics classroom practices. Thus, by including modelling in mathematics education, the phenomenologically organised "big ideas" such as quantity, space and shape, change and relationships and uncertainty (OECD 2003: 35) are addressed in a more meaningful way than can be addressed by traditional teaching approaches (see 2.3). Another of the 'big ideas' in mathematics is proportional reasoning (see 3.3.1); and this pervades the tasks used in this study. These overarching ideas will go a long way in producing mathematically literate people of benefit to industry.

It will be necessary to place modelling tasks into the mathematics education arena. This means we have to distinguish between traditional word sums and the nature of the problem solving activities in this study and furthermore, to describe modelling as a perspective to problem solving. It will also be necessary to describe how modelling fits into and is accommodated by the problem-centred approach as a framework for teaching and learning in order to create cohesiveness in concepts such as 'solving problems, 'problem solving', 'modelling' and the 'problem-centred approach'. The degree to which the modelling cycle (see Fig 1.1) is addressed; and the extent to which students are involved in each phase of the cycle is indicative of the didactical approach in the classroom. Modelling should be seen as an umbrella term while traditional teaching and problem solving are subsets of the modelling cycle. The problem-centred approach, however, establishes the type of milieu and classroom norms necessary for modelling to take place.

The literature on mathematical modelling presents one with a variety of modelling cycles that explain and clarify the process of mathematical modelling. Most of these cycles have at their core, the translation from a real world problem into a mathematical problem. These modelling cycles represent the general processes involved in the modelling evolution. Many authors have labeled the different processes involved in the modelling cycle. The cycle of Stillman, Galbraith, Brown and Edwards (2007: 690) will be used as an example. Their cycle has the following components/processes:

- Understanding, structuring, simplifying, interpreting context
- Assuming, formulating, mathematising
- Working mathematically
- Interpreting mathematical output
- Comparing, critiquing, validating
- Communicating, justifying (for a satisfactory model)
- Revisiting the modelling process (for an unsatisfactory model)

The modelling cycle of Blum and Leiss (Borromeo Ferri 2006: 87) is accepted as a complete cycle of the modelling process. The phases of the cycle are reviewed as competencies and can be used for assessment criteria or reworded and termed 'modelling competencies'.

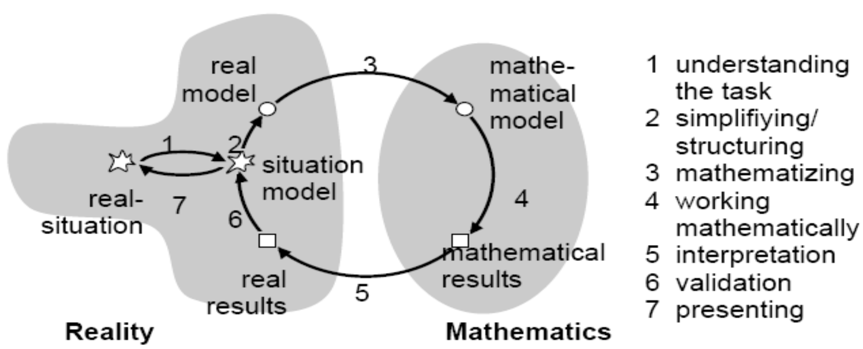


Fig 1.1: Modelling Cycle

It is possible to use each point in the modelling cycle as a reference that specifies a competency. These will be the foci of the study and the teaching experiment. Competency development in group situations will be evaluated. Part of the value of focusing on these competencies would be to 'sell' the idea of including modelling as classroom activity to teachers. Burkhardt (2006: 193) sees the role of assessment as an opportunity to promote change.

In an endeavour to specify and characterize modelling competencies, the following research is directly relevant. Maaß (2006: 136) categorises modelling competencies in the following way:

- Partial competencies for conducting the single phases of the modelling process
- Meta-cognitive modelling processes. (This is how Kaiser distinguishes between modelling competencies and modelling abilities. Modelling competencies include a willingness to work on the problems which adds a meta-cognitive component)
- Competencies to structure real problems
- Argumentation competencies
- Competencies to judge a solution

This categorization assists in establishing broad areas of competence and includes aspects that are not common in traditional mathematics classrooms such as argumentation and meta-cognition. Galbraith (2007: 83) also sees the challenge in measuring competencies as linking performance in the sub-processes of modelling to the overall modelling ability. This was considered as part of the three methods of assessing competencies in this study (see 4.6.1).

Although competencies have been specified by some authors, and a certain amount of measurement and documenting of these competencies has taken place, much can still be done in this field. Maaß (2006: 119) states that there are "few comprehensive studies about modelling competencies and their related weaknesses". The study of group modelling competencies is an area of need in the field of modelling in mathematics education.

Development and improvement of modelling competencies would also make the research exchangeable into a classroom situation since it would address teachers' needs to produce assessment for accountability and evidence of progression. Kaiser and Maaß (2007: 99) state that modelling situations only play a minor role in everyday teaching while Blum (2002: 150) regards the modelling status in classrooms as 'rare', it is necessary to study modelling competencies so as to make a valuable contribution to its implementation in the classroom.

From the above, it was decided to use the following competencies in this study. The development of the selected competencies will be documented from a qualitative approach. The competencies selected were from Blum and Leiβ's cycle (see Fig

1.1): *understanding* the problem, *simplifying* the problem, *mathematising*, *working mathematically*, *interpreting* the solution, *validating* the solution and *presenting* the solution process. During the entire process, the competency of *argumentation* will also be focused on. These constitute the cognitive competencies (see 3.5.2). Together with this, certain meta-cognitive competencies were selected: a '*sense of direction*' (Treilibs, Burkhardt & Low 1980: 52), *using informal knowledge* (Mousoulides, Sriraman & Christou 2007: 39) and *planning and monitoring* (see 3.5.4). From the affective domain, *student beliefs* were also considered as part of the competencies for this study (see 3.5.3). This resulted in twelve competencies. Although these may appear too many for a single study, these competencies cannot be isolated as this would not do justice to the complexity of a modelling situation and the complexity of group interaction.

Once competencies are identified, how can their development be documented and explained? Will the competencies that evolve represent a general improved way of thinking? How will students that are stereotyped as weak or strong in traditional mathematics classrooms respond to and develop in modelling situations? What instruments would be needed to document the development of competencies? Is the lack of modelling implementation in classrooms related to the problem of evaluating the development these competencies? This study intends to analyse modelling processes and their relevant competencies by considering the development of these competencies in groups. This will be done by examining what Borromeo Ferri (2006: 91) calls "modelling routes" and looking at how students develop transferability of modelling competencies. Niss (1992: 355) suggested that we ought to assess students' work on the entire process of modelling in all its phases. This suggestion of a need for a more holistic assessment of modelling is taken into consideration for this study in that three methods of assessing modelling competencies is explored and implemented (see 4.6.1). Although it is understood that individuals all contribute to the group functioning, the group is the unit of focus in this study, it is in the development of the group's competencies where the focus lies.

Since students are developing their own path from problem to solution while modelling and since much of successful modelling is based on meta-cognitive aspects, measuring meta-cognitive and affective competencies is complex.

Lingefjard and Holmquist (2005: 123) state that assessing mathematical modelling is a more complicated task than one anticipates before teaching it. According to Maaß (2006: 119), meta-cognitive modelling competencies have 'hardly been examined'. Consideration must also be given to the fact that authentic modelling is often performed in groups. This adds challenges to assessment and measuring aspects, especially in a classroom situation. Greer and Verschaffel (2007: 220) rightly state that authentic modelling lies outside the standard forms of assessment. Other forms of assessment need to be established and developed.

Hall (Houston: 2007: 250) named three areas in assessing modelling competencies. He itemized modelling competencies as 'content' competencies which involve technical aspects of modelling, 'presentation' competencies which involve writing of the report and 'drive' competencies which allow for originality and student management of the modelling process. The modelling marking scheme of Berry and Le Masurier (Houston 2007: 252) identified sixteen modelling competencies split into the following subsections: Abstract, Formulation, Initial model, Data, Revisions to the model and Conclusions. More recently, the Assessment Research Group (UK) listed 11 items used as assessment criteria for mathematical modelling that can be mapped to the various phases of a modelling cycle. These criteria were assessed using a scale from 0 to 4. The item scores can be added or used in other teacher determined ways. Maaß (2006: 116) elaborates on modelling competencies by giving a detailed list (from Blum & Kaiser 1997: 9) that corresponds to each part of the modelling cycle.

Tanner and Jones (2002: 146) also targeted assessing meta-cognitive competencies in modelling. Here, marks were allocated (3, 2, 1 or 0) for planning, monitoring, evaluating and reflecting; depending on the level achieved independently by students. They identified eleven broad targeted strategic skills for modelling. When these skills were measured, a significant improvement in modelling achievement was sustained even in delayed test circumstances.

Clatworthy (1989: 63) explored assessing modelling by using a rubric to measure modelling standards rather than a written examination. His idea was to use the rubric as feedback to students regarding their modelling competencies. The rubric consisted of four broad criteria related to modelling skills. He stated that it was possible to trace the development of a student over a period of time and to build up a comprehensive picture of student performance. It was also found that using a feedback system, student confidence and attitudes improved. The recommendation was to develop consistent methods for assessing modelling skills.

It has also been found that mathematical modelling allows for a broader range of understandings and abilities to surface, and as such, students or students with traditionally 'poor' mathematical abilities are being recognized as having extraordinary potential which includes students from populations that are underrepresented in fields such as mathematics, science and technology (Lesh & Yoon 2007: 162). Modelling research also shows that average ability students and those in mathematics remedial programs can develop powerful models for describing complex systems (Lesh & English 2005: 488; Lesh & Harel 2003: 175). Of particular interest in this study is the development of competencies in students stereotyped as "strong" or "weak". This led to decisions of using two weak groups and one strong group in the research strategy. The students are not labeled because of their ability but because of their achievement in traditional mathematics assessment.

This study will benefit mathematics education as it endeavours to document developing modelling competencies in group situations. By doing this, it will be possible to justify the meaningful place modelling should play in mathematics classrooms. It will also validate why teaching approaches should be moving away from traditional didactical methods to inquiry-based and modelling approaches. It will also be possible to develop assessment guidelines for teachers to use in modelling situations in their own classrooms. Extensive knowledge of students' group modelling competencies will also assist with better design of teacher development programs in mathematical modelling. If there is benefit to weak students, these tasks can be set and used at all schools across ability and teaching spectrums in the need to upgrade mathematics teaching and learning.

1.1.1 A South African analysis

Since this study takes place within a South African context and schooling system, a brief description of the South African modelling environment is discussed. South Africa was led through the work of Human, Olivier and Murray (1998), de Villiers

(1988, 1993) and Wessels, D.C.J. (2009) in the problem centred approach and problem solving via modelling. Wessels, H.M. (2006) documented the types and levels of representation when students were involved in statistical modelling. Although there is not a great volume of work and very few studies related specifically to modelling competencies, this study must be situated within this context (see 2.6) since modelling research is relatively new but builds on problem-centred learning.

The current South African curriculum (RNCS 2002: 5) is a learner-centred curriculum that advocates problem solving, learning with understanding and allows for modelling. However, this is not a main feature in South African classrooms where a traditional approach to teaching mathematics still dominates. This study through its design experiment, intends to advance the possibility of adopting a problem centred and modelling perspective in daily teaching in South Africa.

1.2 THEORETICAL AND CONCEPTUAL FRAMEWORK FOR MODELLING

Since this study acquires an understanding and awareness from a greater international community on mathematical modelling, a theoretical framework must be made explicit. As stated by Kaiser and Sriraman (2006: 302), there isn't a common understanding of modelling within an international discussion on modelling, but that certain perspectives can be found. This study has pedagogical, psychological and subject-related aims and adopts a contextual, educational and cognitive perspective to mathematical modelling. (See 3.5). This means that learning and teaching in context and student (group) competency development is the focus of the study.

1.3 DEFINITION OF TERMS

Since the terms problem solving, model, modelling, modelling task, modelling cycle, modelling environment and competency are used extensively in this study, it is important to provide definitions of these terms. While they are stated here succinctly they will be explored further in the study.

1.3.1 Problem solving is used in the umbrella sense of the word. There are many different forms of problem solving that take place in a classroom. It is the entire process of dealing with a problem in an effort to solve it (Blum & Niss 1989: 2).

1.3.2 Modelling in this study refers to the act of students solving a modelling task by moving through the phases of a modelling cycle. The phases are: understanding the problem, simplifying the problem, mathematising, working mathematically, interpreting the problem, validating and presenting the solution process. In this study modelling means that students construct their own model of a situation rather than apply a known model.

1.3.3 A model: the general definition of Lesh and Doerr (1998: 362) is used. A model is a scheme that describes a (real life) system. It assists in thinking about that system, making sense of it or making predictions. A model consists of elements, relationships, operations that describe how the elements interact and patterns or rules that apply to the preceding relations or operations. A model focuses on the underlying structural characteristics of a real life system being described. In a typical model eliciting task (Summer Jobs) students are asked to assist a theme park employer to re-hire vendors for the following season. Students are given the working hours and number of sales for each vendor as well as how many people attended during those hours for the previous season. Through modelling the problem, students build a model of the problem and provide a 'tool' (model) that the employer can use to hire the best people.

1.3.4 A modelling task describes a complex real-world task that enables students to produce a model.

1.3.5 A modelling environment is one where students are allowed to solve modelling tasks, in groups, under minimal teacher intervention. Students discuss, build and justify ideas and methods. The teacher facilitates and guides this discussion.

1.3.6 Competency refers to the student's ability to do something well. In mathematical modelling specifically it refers to the student's ability to perform each node of the modelling cycle.

1.3.7 Students/learners. These words are used synonymously and interchangeably in this study and since the term *student* is more common internationally while *learner* is more common in South Africa.

1.3.8 Task/problem. This study uses the term task with reference to the modelling tasks the groups were given to solve (see Appendix A-C), while other authors also use the term modelling problems. They can be used synonymously.

1.4 PROBLEM STATEMENT

1.4.1 Main research question

How do mathematical modelling competencies of students working in groups develop?

1.4.2 Sub-questions

1.4.2.1 How do student group modelling competencies develop through model-eliciting activities?
1.4.2.2 What differences in the development of modelling competencies are apparent in traditionally 'weak' and 'strong' groups of students?

1.5 AIMS OF THE INVESTIGATION

The following aims were formulated to answer the research questions:

1.5.1 Aims

1.5.1.1 To place modelling in the context of mathematical teaching approaches.

1.5.1.2 To characterize modelling competencies.

1.5.1.3 To examine the development of group modelling competencies during a model-eliciting program by a collective analysis of qualitative data.

1.5.1.4 To document the development of group modelling competencies.

1.5.1.5 To analyse the development of group competencies in students stereotyped as weak and strong in mathematics.

1.5.2 Objectives

The following objectives were set out to successfully meet the aims:

1.5.2.1 Describe the nature of mathematical modelling.

1.5.2.2 Identify and clarify mathematical modelling competencies from existing literature.

1.5.2.3 Characterize and define the competencies for each phase of the modelling cycle.

1.5.2.4 Collect qualitative data from observation notes, student written work, audio and video transcriptions, group presentations and informal interviews with students.

1.5.2.5 Examine the development of mathematical modelling competencies

1.5.2.6 Use data collected to document the development of group modelling competencies.

1.5.2.7 Use qualitative data sources to analyse the development of mathematical modelling competencies of traditionally weak and strong students.

1.6 RESEARCH METHODS AND DESIGN

This study merges aspects of two paradigms: design research and grounded theory in an attempt to answer the research question. It may be part of what Bruce (2007: 1) elaborates on as 'educational researchers attempt to address complex issues of "improvement" '. This study is essentially about understanding what modelling competencies are and how they develop and is conducted by a reflective practitioner. Since the elements in understanding student modelling are vast, this study does not undertake a full iterative cycle common to design research. The extensive pilot studies form part of an iterative cycle. Since the study is acutely focused on a teaching experiment design research is apt. The use of aspects of grounded theory focused on analysis of the data.

1.6.1 Research design analysis

According to Collins, Joseph and Bielaczyc (2004: 16) design research was developed to deal with issues essential to the study of learning. They further delimited the following aspects that fall in to the scope of design theory studies: theoretical questions about the nature of learning in context; the need for approaches to study learning in real situations; the need to go outside of narrow measures of learning and the need to obtain research findings from formative evaluation. This study aligns itself with these views as it stems from a strong theoretical base and studies modelling situations as they occur in context and the assessment of student modelling is wholly formative in nature. Furthermore, the view of Bakker (2004: 38) regarding design research is relevant to this study in that 'its strength comes from its explanatory power and grounding in experience'.

Design experiments (used synonymously with design research in this study) allow for varied approaches. The approach in this study, which suits the studying of modelling tasks in general is what Cobb and Steffe (in Cobb, Confrey, deSessa, Lehrer & Schauble (2003: 9) 'term a sequence of teaching sessions with a small number of students aiming to create a small-scale version of a learning ecology so that it can be studied in depth and detail'.

Grounded theory is 'the systematic generation of theory from systematic research' (Grounded Theory Institute) and according to Bruce (2007: 10) 'grounded theory studies are "grounded" in the data collected to develop or refine models of understanding through an inductive process'. Bruce (2007: 6) also tells us that the strategies followed in her research, which are the same in this study, supports grounded theory in that there is 'full acknowledgement that as the researcher I was interacting with the participants, the data and the literature as the study was co-constructed'. This study is a careful, systematic construction of literature, data and interaction with participants. Although it focuses on selected participants the theorizing that will take place will be abstracted, generalisable and transferable. Another aspect that is fundamental to grounded theory is the derivation of codes, concepts and categories (Allan 2003: 1) which is used extensively in the analyses part of the study.

In this way, by employing these two paradigms, this study moves towards Bruce's (2007: 5) suggestion that research can be both *plan-ful* and *emergent*. There is a definite teaching trajectory and 'Hypothetical Learning Trajectory' (HLT) (Simon 1995: 133) that is designed and followed in the study as well as flexible instruments to filter emerging data so that it can be categorized and abstracted. A hypothetical learning trajectory (HLT) – a thought experiment- 'consists of the goal for the students' learning, the mathematical tasks that will be used to promote student learning, and hypotheses about the process of the students' learning' (Simon in Simon & Tzur 2004: 93).

Since this is a qualitative study, validity and reliability needs to be addressed. Although Lincoln and Guba (in De Vos 2006: 346) propose four constructs (credibility, transferability, dependability and conformability) that reflect the assumptions of a qualitative study. The terms validity and reliability are used and are discussed in relation to this study in 4.5. While care was also taken to include the steps a researcher can take to improve the trustworthiness of a study (Niewenhuis 2008: 112). These include using multiple sources, keeping notes of research decisions, controlling for bias, avoiding generalizations and choosing quotes carefully. Several forms of data will be generated and a detailed inventory of data collection will be presented (see Table 4.1) to make the data collection transparent and to address validity and reliability.

1.6.2 Population and Sampling

Twelve Grade 7 students will be used and they will work in three distinct groups of four students each: a group of students considered mathematically 'strong' and two groups classified as mathematically 'weak'. These distinctions are based on the students' previous years' school mathematics results. It was decided to use two 'weak' groups in order to establish if results are consistent and if they can be 'generalized' to a greater degree. The development of modelling competencies in weak students is of particular interest and thus a second group is included to ensure the establishment of the results. If modelling competencies do develop in a positive manner in weak groups, then the significance of the study can be established. The homogenous grouping is for researcher observation purposes only. The study does not take place in a classroom environment where heterogeneous grouping is preferred. The students have had no prior experience in mathematical modelling and are exposed to a traditional classroom.

1.6.3 Measuring instruments

A number of measuring instruments to assist in documenting modelling competencies will be designed and used during the teaching experiment. The modelling competencies of all three groups will be gauged using the assessment instruments. The modelling competencies of all three groups will be documented throughout the teaching experiment as well as comparison of groups. Some of the instruments are sourced from existing literature so that the researcher could add new instruments with the knowledge that the reliability and validity of the sourced instruments are well established. The tasks are all taken from existing studies to ensure validity and reliability.

1.6.4 Teaching experiment and schedule

The teaching experiment consists of a series of modelling tasks (see Appendix A-C) based on proportional reasoning over a period of 3 months. The students will meet with the researcher once a week for 60 minute sessions. The tasks and research instruments will be subjected to a pilot study to verify their value in use. From the pilot study findings the instruments will be adjusted and the final tasks selected.

1.6.5 Data generation and data processing

Qualitative data will be obtained by audio recording the contact sessions and videoing the presentation sessions. As the assessment strategy relies on discussion and interaction, the audio and video sessions have to be transcribed before an assessment can be made and an 'index' given to the group's modelling competency. Each competency became a code to use for data analysis. According to Barbosa (2006: 297) a modelling milieu will produce student discourse; this will tell us something about their internal processes. This discussion will need to be translated by the researcher in terms of the students' development through modelling competencies. The group discussions then become the object of the research.

Barbosa (2006: 297) further classifies different types of discussions: mathematical (ideas belonging to a pure mathematics field), technological (techniques of building a model) and reflexive (refers to the nature of the model). These discourse classifications may be useful in describing and coding modelling competencies.

Other data will be obtained from researcher's weekly observation notes, student written work and documented informal interviews with students. Students will be required to document their progress as a group at certain intervals of each task. At the end of each task, a presentation session will take place and this is evaluated using a presentation quality assurance guide (Lesh & Clarke 2000: 145). An individual post-presentation question sheet (Schorr & Lesh 2003: 154) will also be used to gauge individual responses to the. The scope and variety of instruments should provide a comprehensive picture of modelling competencies and their development.

1.7 DELINEATION

The study assumes that modelling competencies will develop given the time and depth of tasks that require modelling. The aim is not to show an improvement of competencies, but to document the development of these competencies. Since two groups comprise "weak" students the competencies may be at a lower level and may develop more slowly. The tasks selected require proportional reasoning in developing solutions. Other areas of mathematics are not included in the study. The selection of students is for specific research purposes and is not intended for implementation in the classroom where heterogeneous grouping is preferred. Students are evaluated as a group and not as individuals as this will promote the development of competencies. This study is specific to Grade 7 learners although some results should be generalisable due to the theoretical basis of the study.

1.8 CHAPTER DIVISIONS

Chapter one aims to provide a background and motivation for the study. It sets out the context of mathematical modelling. The aims and objectives for the study are presented. Definitions and delineations are set out as well as the structure of the rest of the study.

Chapter two provides a context for modelling in mathematics education. It describes the nature of didactical situations generally. Three popular approaches to mathematics teaching and learning are described; traditional teaching, problem solving and the problem-centred approach. The problem-centred approach is established as the ideal environment for modelling. Several relevant theories to mathematical modelling are presented.

Chapter three presents the nature of mathematical modelling, and focuses on modelling competencies at each node of the modelling cycle. Cognitive, meta-cognitive and affective competencies are discussed as well as competencies in stereotyped students. Different strategies from literature in measuring modelling competencies are presented. The specific competencies for this study will be described and clustered around meaningful questions that can be used for reporting.

Chapter four describes the method and design of the qualitative data generation and collection program. It describes the choice and grouping of students, instruments used, tasks employed and the data to be generated from these.

Chapter five provides a discussion on the data collected, the processing and analyses of the data. The discussion centers on the competencies and clustered competencies. Evolving competencies are identified and their metamorphosis documented.

Chapter six draws several conclusions from the data collected and the resulting analysis. It also provides justification of whether the aims of the study were met and provides several important recommendations.

CHAPTER 2

MATHEMATICAL MODELLING AND TEACHING APPROACHES

2.1 INTRODUCTION

The aim of this chapter is to provide a comprehensive view of major teaching approaches in mathematics education from which to understand modelling. These teaching approaches are linked to ideologies about the nature of mathematics and to social and cultural conventions about mathematics. Theories that are useful in explaining or justifying modelling in mathematics education are discussed. An integration of 'what should be' and 'what is' is attempted in discussing these theories.

Society has often changed its views of mathematics education. In a need to reform mathematics teaching, focus was placed on meaningful learning and teaching and learning with understanding (Cobb, Wood, Yackel, & McNeal 1992: 574). To this end, problem solving, a problem-centred approach, and modelling are explored as alternatives to traditional mathematics teaching approaches. Each of these dimensions to mathematics teaching is discussed to reveal its essential features.

Modelling is seen as the full cycle (Fig 1.1) of carrying out applied mathematical problem solving (real world or experienced based). Traditional instruction and problem solving are subsets of this modelling perspective (Lesh & Doerr 2003a: 4). The problem-centred approach encompasses modelling as a classroom activity and provides an affinitive environment for modelling. Many teaching perspectives do not include modelling and its related competencies because of the perceived difficulty involved for both student and teacher. Preparation for achievement beyond school include the type of abilities developed by modelling such as constructing, describing, explaining, predicting and manipulating complex systems (Lesh & Doerr 2003a: 521). Traditional teaching and disjointed problem solving instruction is inadequate in preparing students for the 21st century while modelling and modelling competencies have a vital contribution to make to what students can learn and how students should learn in mathematics education.

2.2 THE NATURE OF THE DIDACTICAL SITUATIONS

The extent to which the students are free to pursue meaningful mathematical ideas indicates the didactic method of the classroom. In a traditional teaching situation, mathematical knowledge takes the form of axiomatic presentation (Brousseau 1997: 21). Brousseau explains that a suitable view is presented of established mathematical knowledge for teaching purposes. It promises the student and teacher a way of ordering concepts and activities so that the concepts can be accrued in the shortest possible time. The knowledge accumulated ties in more closely with expert knowledge and is neatly packaged for teacher and student. The teacher therefore takes up the responsibility for the essential part of the work in a mathematics classroom (Brousseau 1997: 25). This normally suits the teacher as he or she is in control of the learning that will take place and the learning can be efficiently assessed. Very often, the teacher is unable to withdraw from the obligation of teaching at all costs (Brousseau 1997: 27). Brousseau elaborates on the 'didactical contract' that exists in classrooms. This is the common relationship between student and teacher, where the teacher feels a social obligation to show/tell/demonstrate. This relationship suits both teacher and student as is come to be expected in mathematics classrooms.

The student, who feels unable to solve a given problem, asks the teacher to show/tell/demonstrate. But, the more the teacher gives in to the student, the more the teacher tells the student exactly what to do, the less the student is able to obtain the knowledge the teacher wanted the student to learn. The teaching of algorithms and recipes seems to develop under the pressure of such a didactical contract (Brousseau 1997: 39). What is needed, according to Brousseau, is not the teacher's communication of knowledge, but the devolution of problems in an adidactical situation. Learning takes place when the student adapts to this adidactical situation.

For learning to take place in an adidictical situation (see 2.2.1), the teacher needs to create this situation. It is not a simple matter of asking questions that students know the answers to, or teaching students what they should answer, as in a traditional classroom. The student, in adapting to an adidactical situation needs to make some accommodation to his/her conceptual system in order to deal with the problem at hand. The teacher therefore should propose a learning situation to the student in such a way that the student produces a personal answer to the question and modifies the answer to satisfy the constraints of the milieu (surrounding with didactical intentions that influences the student) and not simply to give the correct answer or please the teacher (Brousseau 1997: 228). In turn, the teacher expects the student to take responsibility for solving problems that the student knows has been selected to lead him/her to new knowledge. Teaching, from this view means students incorporate new learning in their conceptual systems. This happens by placing the students in appropriate situations where they will respond spontaneously by adaptations. Allowing students to take part in modelling tasks in an environment proposed by the problem-centred approach will result in this adidactical situation. The student's response to this situation will result in meaningful learning in mathematics.

2.2.1 Adidactical situations and problem devolution

Brousseau (1997: 30) affirms that a student will learn by adapting to a milieu. The milieu should generate contradictions, difficulties and disequilibria. The teacher, in an adidactical situation, does not attempt to tell the students all. The teacher has to provoke adaptation in the students by the choice of problems put to them. Blomhoj and Jensen (2007: 49) describe the key characteristic of the initial modelling process as 'learning to cope with the feeling of perplexity due to too many roads to take and no compass given'. The problems must also be such that the student accepts them; want to solve them; thereby generating motivation. The teacher refrains from suggesting the knowledge he/she is expecting or wanting to see. The teacher seeks to transfer part or all of the responsibility on the student. The student, in turn, knows that the problem was chosen to help him/her acquire new knowledge. The above is Brousseau's (1997: 230) explanation of the term 'devolution' of a problem. In general terms, to devolve a situation or a problem means to:

- Transfer power, responsibility and rights to another person
- Make it the duty or responsibility of somebody else
- Be decided by something or depend on something for validity
- Give to somebody under terms of legal instruction (Microsoft Encarta Dictionary: 2007)

The term denotes more than just a handing over of a problem to the student, but it carries with it the essence of entrusting the problem to the student. There is a strong sense here of a responsibility on behalf of the student and teacher for managing and for being accountable to the other person. It carries a trust between student and teacher, trust from the student that this atypical situation will lead him/her to new knowledge.

A traditional behaviourist approach to teaching mathematics does not produce a didactical situation where students learn autonomously and independently. Their learning is dependent and solely guided by the teacher. Problem solving in some forms approach conditions suitable to an adidactical situation, while the types of tasks set in open modelling activities necessitates a high degree of student engagement in problems that require overcoming a real obstacle. Blomhoj and Jensen (2007: 49) hold that a dilemma of teaching ‘directed autonomy’ needs to be overcome when attempting to develop mathematical modelling competencies. In their view the students need to be responsible for most of the decisions, but the decisions need to be the right ones. Modelling tasks encapsulate Brousseau’s concept of devolution – that the student must accept responsibility for the learning situation and must show responsibility for finding a solution to problems where the answers are not directly known (Brousseau 1997: 230).

2.3 TRADITIONAL MATHEMATICS TEACHING

According to the NRC (1989: 57) evidence shows that the least effective form of learning mathematics is the one that is most prevalent in classrooms: the mode of lecture and listen. In this mode, teachers explain or prescribe a method, while students carry out or transcribe these methods. This approach fitted into the conception of teaching and learning at a time when behaviourism explained how students learned. The approach is based on the assumption that knowledge can be effectively transmitted from one person to another (Human 1992: 15). The broad theory in place here is of an objectivist nature (Biggs 1996: 347). Teachers transmit knowledge, students receive and store it. It is not possible to develop competencies in higher order thinking such as mathematisation, validating or interpreting since students are hardly expected to perform any of these. These are left as the teacher’s task.

If mathematics is seen as a cycle that involves mathematisation, manipulation and interpretation, traditional mathematics instruction assumes that teaching mathematical manipulation is the essential part (Bell 1993: 6). The extreme focus on manipulation techniques means that mathematics is often detached and meaningless. The lessons follow a predictable format of explanation, demonstration and practice. It is assumed that once a certain level is mastered that more difficult levels can be presented and practiced. It is also assumed that computation skills are necessarily transferable to a real or applied situation.

Teaching follows a sequential, linear process and the teacher is the sole active player in this situation. According to Brousseau (1997: 25), a total collapse of the teaching situation occurs within when the teacher ends up taking responsibility for the essential part of the work in the classroom, and to receive a "correct" answer, the teacher will often choose easier and easier questions. The teacher is trying to achieve optimal learning for most students. Good teaching is equated with learning that takes place effortlessly. As a consequence, the teacher does most of the cognitive and meta-cognitive work and not the student.

What is of concern is that even with the development of new and contrary theories in mathematics teaching and learning, traditional teaching still has a stronghold in many mathematics classrooms. If one views mathematical literacy in terms of a student's ability to mathematise (OECD 2003: 28), then the traditional approach in classrooms will not engender mathematical literacy in students since mathematisation is largely absent. A full modelling cycle has at its core student integrated mathematisation to build the bridge between the real world and the world of mathematics. Mathematisation is further discussed in section 2.5.3.

The tasks students are required to perform in a traditional classroom are often called "problems", but they are routine exercises planned and controlled to provide practice on a particular mathematical technique, that has just been demonstrated to students (Schoenfeld 1992: 337). These problems are ultra structured and require neither representational translation nor mathematisation on the part of the student. In terms of the modelling cycle, traditional teaching leaves the student confined to one node (working mathematically) of the modelling cycle. Traditional classrooms do not foster the link between mathematics, the real world and student interaction with both.

It is not possible to develop modelling competencies in this environment. This milieu will fail to provide students with any opportunity to build a view of mathematics as a useful human activity that they have control over. They are left with perceptions and attitudes that make it very difficult to convince them of the usefulness of mathematics to themselves and in the real world.

2.3.1 The role of context in a traditional mathematics classroom

According to Freudenthal (1991: 73) context means a sphere of reality in which some particular learning process is unveiled to students in order to be mathematised. He adds that context is not clothing for pure mathematics rather the context is the message while mathematics is the means of decoding it. Traditional word problems represent simplified forms of decontextualized world based situations and student work is absent of heuristics or strategies (Mousoulides, Sriraman & Christou 2007: 24). A modelling approach requires students not only to make use of known heuristics, but to develop methods of their own to solve a real problem in a specific context. Traditional mathematics classrooms offer little in terms of real context or using mathematics to solve real problems. According to Palm (2003: 38), when students are offered word problems they often present solutions that are incompatible with the 'real' situations described in the tasks. Students attempt to solve the problem by focusing on the 'numbers' that may occur in the problem and then try to 'do' mathematics with the numbers. The problem is only seen in terms of numbers while very little attention is given to the meaning of the words. This is because of the exclusive emphasis on mechanical procedures in a traditional classroom.

Sometimes the tasks presented to the students have quasi-realness to them. They may have descriptions of where the numbers given may occur in the world out there, but they do not describe a real situation, nor do they require that a student integrate the context with the mathematical calculations that are required. Often, with language translation, these problems can be solved by students worldwide. Palm's study (2003: 39) investigated the hypothesis that the students would bring in 'realistic' responses more often if the word problems were more authentic.

The authenticity presented to students will determine the mathematisation level required to solve the problem. Mathematical modelling uses meaningful contexts that are 'experientially real' (Gravemeijer & Doorman 1999: 111) to the students, thereby giving students the avenue for growth in mathematisation skills (English 2006: 306). Often the information is undefined or needs structuring before anything meaningful can be mathematised, whereas the 'real' context in word problems often comes across as convenient and artificial or in the interpretation of Doerr and English (2003: 111), cover stories for procedural, irrelevant tasks.

Authentic tasks provide challenges for teachers and students in mathematics classrooms where a neatly packaged version of mathematics is presented. The authentic problems that are presented in mathematical modelling follow the definition of authenticity as described by Fitzpatrick and Morrison (Palm 2003: 40):

> … if a performance measure is to be interpreted as relevant to 'real life' performance, it must be taken under conditions representative of the stimuli and responses that occur in real life.

Steen, Turner and Burkhardt's (2007: 289) adaptation of Steen and Forman's definition of authenticity as portraying honest problems that employ realistic data often incomplete or inconsistent and reflect the integrity of mathematics and the domain of application allow one to concede that traditional word problems do not meet these criteria. Modelling tasks do meet these criteria and students will find a need to develop a model to solve the problem. When doing this, they display and develop competencies that are different and more complex to traditional computational skills.

2.3.2 The role of the teacher in a traditional mathematics classroom

The teacher expresses his or her authority in a definite manner in a traditional classroom. The focus of the lesson is on the instruction that takes place. He or she determines the content and order of the lesson and the activities. The teacher chooses the problem/question/stimulus and then formulates the answer (Keegan 1995: 7). As a source of knowledge, the teacher provides instruction, explanation and direction to students. He or she also determines the value of students' work and validates their thinking and their actions by the answers they produce. The teacher's main functions are those of demonstrating facts, rules and processes, monitoring student's repetition of these rules and correcting errors (Lesh & Doerr 2003a: 32). Teachers would see professional development in terms of improving their own content knowledge and improving ways of demonstrating or explaining procedures to students so that learning is faster and easier for students. It is (incorrectly) assumed by the teacher that a correct answer implies the correct corresponding thinking process in the student.

Modelling activities require a different role from the teacher. The teacher would need to present tasks and orientate students as to the background in terms of allowing them to discuss the implications of the context chosen. The teacher then assumes a subtle off-centre roll of facilitator, but only to focus or redirect student thinking, not to facilitate the solution path. There is no suggesting a solution procedure, but knowing enough of the problem and students current thinking to allow groups to function independently and autonomously. The teacher, in a problem-centred classroom possesses a wider knowledge of various approaches to the solution as well as ‘pedagogical content knowledge’ (Shulman 1986: 9) than the teacher in a traditional classroom. Teacher development lies at the heart of any real change to mathematics education. Understanding the role of a teacher in modelling tasks therefore could be accommodated by a design research approach (see 4.2).

2.3.3 The role of the student in a traditional mathematics classroom

Students have been conditioned to be “product orientated” (Kantowski 1981: 121). The aim for the student is to produce a correct answer that is ratified by the teacher. The student senses achievement if his or her actions are concurrent with the teacher’s expectations. The students overriding goal is to follow specific procedural instructions in order to be effective (Cobb et al. 1992: 585). Students believe that mathematics consists of set rules to be followed and applied. A student would consider him/herself successful if he/she can reproduce rules, methods and procedures quickly and accurately.

2.3.4 Traditional teaching and modelling

Traditional “word problems” that may be found in the traditional genre have certain characteristics. They are often a simple extension of the procedural work covered in the lesson preceding the word problems. Often students apply the same procedure to numbers that occur in the word problem without any real understanding of how or why the procedure is relevant to solving the particular problem. Within the traditional mathematics classroom it is possible to find a single learning trajectory while Doerr and English (2003: 122) found that this was clearly not the case in any classroom they worked in while researching mathematical modelling. This is why the Hypothetical Learning Trajectory in design research is appropriate (see 4.2).

According to Lesh and Doerr (2003a: 13) all information in word problems is presented in a homogenous format as written symbols without any use of other representational media (graphs, pictures, tables etc). Very little mathematisation takes place, very little translation between representational media is required and students do not need to create their own models to solve the word problems. Lesh and Doerr (2003c: 549) explain that traditional word problems require students to create meaning for symbolically described situations where model-eliciting activities require the opposite – for students to develop symbolic descriptions of meaningful situations.

The word problems used in the traditional didactical approach are used as an end to some other function. They serve to fulfill a predetermined course of action. The problems themselves are only important in that they enable students to practice set curricular skills and procedures. Modelling activities require and develop certain competencies that contribute to students' higher order cognitive and meta-cognitive abilities (see 3.5).

Important distinctions can be made between traditional teaching and mathematical modelling in areas such as the nature of quantities and operations. (English 2006: 305). The type of quantities used in modelling situations go beyond traditional raw numbers to include : accumulations, probabilities, frequencies, ranks and vectors while the operations needed that go beyond traditional operations are: sorting, organising, selecting, quantifying, weighting and transforming large data sets (Doerr & English 2003: 112, Lesh, Zawojewski & Carmona 2003: 220); whereas in solving traditional word problems, students engage in one or two processes of representing information in the problem or applying known procedures.

2.4 PROBLEM SOLVING

Within the field, problem solving means different things to different people so a consensus is not possible. Problem solving can also be seen as the process of getting from givens to goals when the path is not obvious (Lesh & Doerr 2003a: 31). In a traditional classroom, this often means that the givens and goals are specified by the teacher or textbook. Usually a linear path can be selected by students using already taught procedures or algorithms. The teacher is focussed on the most common solution processes and guides students to these.

Schroeder and Lester (1989: 32, 33) defined three main areas of problem solving that exist. In a traditional sense this is solving 'word' problems as an extension of routine computational exercises. This can be seen as teaching *for* problem solving – teaching of procedures takes place first and then problems related to the taught concepts are solved. In some progressive programs, students are taught *about* problem solving and are taught to employ various methods as options when faced with a problem (e.g. drawing a table or graph etc).

In other teaching and learning theories, students learn *via or through* problem solving. Problems are used to teach important mathematical concepts. By solving problems in their own way, with mental tools students already have available to them; students are able to learn important mathematical concepts. In the problem-centred approach, there is more to problem solving than just posing the correct type of problem. Very often students work in groups; are shown how to listen to others and how to justify their own stance. Discussion is facilitated by the teacher both within groups and between groups. In the problem-centred approach it is the environment that is very important. For this reason, the problem-centred approach can be seen as the most suitable didactical situation in which to place mathematical modelling.

In teaching for problem solving and teaching about problem solving, students are often given problems that are 'similar'. According to Brousseau (1997: 27), students then simply read the didactical indicators that they are to carry forward the solution that they have already been given to produce the answer that the teacher wants; and often then, their response to these types of problems does not indicate that they have become personally involved or that they understand what it is that they are doing. If students are able to become involved in what they are doing, then they have reached significantly high levels of mathematisation. This involves a much wider range of competencies.

Modelling in its full sense means that students follow an unpredictable path through the various phases of the cycle as their understanding develops and transforms. Lesh, Surber and Zawojewski (Doerr 1995: 9) propose that students map their perceptions to their cognitive models, transform their models and map back to the problem situation. Doerr further adds that a model is not the solution to a problem, but a tool that a student can use and re-use.

Mathematical modelling goes beyond problem solving by endeavouring to also get students to "create a system of relationships" (Doerr & English 2003: 110) from the given situation that can be generalised and reused. Often, in modelling, the goal has to be defined by the students before solving the problem. Zbiek and Conner (2006: 92) refer to modelling as unstructured problems where the purpose or mathematical entity emanating from the situation is never expressed explicitly. Although students are solving problems when modelling, a modelling approach requires that students manage problems with mathematics (Falsetti & Rodriguez 2005: 25), by structuring and describing the problem. Modelling requires that students produce a model of a situation. Creating the model is the central process or procedure that students need to be exposed to, not the learning of existing algorithms or methods. By creating and redefining models, certain competencies are evident and develop.

The extent of a person's mathematical literacy is seen in the way he or she uses mathematical knowledge and skills in solving problems (OCED 2003: 30). This should be seen in a global sense. It is possible to use problem solving in a fragmented sense. One-off, unconnected tasks dotted into the curriculum do not portray the full value of problem solving. Isolated problem solving activities often have a predetermined path that students will follow and often the intent is that these isolated experiences serve as a vehicle for emphasising problem solving processes (Lesh, Cramer, Doerr, Post & Zawojewski 2003: 58) and not necessarily an independent, autonomous use of mathematical knowledge and skills.

2.4.1 The role of realistic context in problem solving

Many problem solving approaches promote their tasks as being 'based on reality'. Resnick (Greer 1996: 187) highlighted a number of key differences between learning that takes place inside and outside of school. Important differences are that school learning is primarily individual and decontextualised, while out-of-school activities are most often carried out in groups in a situated context. According to Hatano (1996: 205), knowledge is situated in contexts in which the experience occurs and must be influenced by the features of the context. Niss (1992: 347) identified authentic situations ranging from pure mathematical problems that are dressed up in non-mathematical language to situations which are authentic to an area outside mathematics and a whole range of situations in between. Authentic situations according to Niss (1992: 353) are those that are embedded in a true existing practice or area outside mathematics and which deals with issues and objects that are genuine to this area. Using this definition, Niss concludes that authentic extra-mathematical situations do not appear in schools very often.

Gravemeijer and Doorman (1999: 111) describe the role of context in the Dutch approach known as Realistic Mathematics Education where contextual problems play a role from the beginning of learning. According to them, real meaning is that which is experientially real to students. Their idea is that contextual problems function as anchoring points for the re-invention of mathematics. The real context of the problem allows students to become part of the problem and seek their answer in the mathematical realm. Uprichard, Phillips and Soriano

(1984: 84) deduce that the extent to which one is a successful problem solver is based on specific prerequisite knowledge related to the problem context. They further add that solving mathematical word problem requires knowledge in mathematics and language. It is in the interaction between these that the student builds new cognitive pathways. Too often students are never given problems that require a union of these two. They are not required to find the mathematics in a context nor are they required to speak about the mathematics they are dealing with.

Freudenthal (1991: 18) saw traditional teaching as lacking in what he termed membrane permeability. The relation between context and form are separated and isolated. He made a strong move towards what he called "mathematics starting and staying in reality". A natural osmosis between the real world and mathematics must take place, just as osmosis works on the principle of strongest filters to weakest. At the start of a modelling problem the strongest sense of understanding lies in the real context of the problem. Students then have to filter this to the mathematical realm. When students have made sense of the mathematical structure of the problem, they then return to the real problem to integrate these two worlds. Reality should find a stronghold in mathematics while mathematics is anchored by relevance to reality.

When mathematics starts and ends in reality, (as in modelling problems) then a strong need for mathematising (see 2.5.3) is created. This mathematising is paramount in students becoming personally involved in their learning. Another important aspect of mathematising is that students reflect on their own activities (Freudenthal 1991: 36). Furthermore, he added that mathematising instigated a change in perspective in students, which could result in a student 'turning things upside down' and axiomatising on a global scale.

Modelling tasks are exclusively set in real contexts and reflect real problems. Model-eliciting tasks take the complexity of real life problems into account. There are no simple single answers. As defined by Kehle and Lester (2003: 97), mathematical modelling is a multi-phase process that begins when someone working in a complex real situation, poses a specific problem. This real world context is assumed to be both motivating to individual students and requires the input of the entire group to solve. Very often students are able to use their own informal knowledge of situations and experiences as a springboard into mathematising the problem. DaPueto and Parenti (1999: 7) affirm that situated teaching embeds concepts into significant situations and aims to develop students' skills in managing the interaction between a range of linguistic and semantic. This is done by 'exploiting potential cognitive' resources by both choice of context and teacher's interface between students.

Treffers (1987: 256) points out that regarding or disregarding contextual factors is the essence of modelling a problem and is a 'non-negligible part of mathematising'. A modelling task allows students to bring their own informal knowledge into the learning realm. In solving these tasks, students are required to bridge between their own real knowledge and school knowledge. This bridge relies largely on modelling competencies such as understanding, simplifying, validating and interpreting the problem.

2.4.2 Problem solving approaches and modelling

The problem solving strategies that are taught conventionally also comprise heuristic methods that are taught explicitly such as drawing a table or picture; trying a simpler problem etc. These are strategies to employ when one does not know how to solve the problem. This becomes the focus of problem solving lessons since the types of problem asked have a clear given and a clear goal. The student has to "search for a powerful procedure that links well-specified givens to well-specified goals" (Zawojewski & Lesh 2003: 318). In traditional teaching and learning, students use tools given by the teacher, problem solving requires students to find tools that can serve to unlock the problem and to build their own tools for moving between question and answer. Model-eliciting activities do not call for a procedure to be identified and applied, but rather for a procedure to be created for solving the problem (Zawojewski & Lesh 2003: 329). Many problems require a single cycle of getting from a given to a goal while other problems only require partial progress through the modelling cycle. This means that often the problem has already been mathematised from a real situation and students need only apply mathematics and interpret results. According to Kehle and Lester (2003: 98) problem solving research has focused on the activity students engage in during the calculating phase of the modelling process, on problems already cast in the mathematical context. Modelling tasks place students in the situation of significantly mathematising the problem first.

In problem solving approaches that teach heuristics as tools to solve problems, the focus is on the tools that students use and not as much on knowledge or abilities the student already has to bring structure to the problem so that the problem can be solved. Within the modelling cycle, known heuristics give the students a selection of tools to use, but there is no guarantee that the problem can be structured or simplified with these known tools. Often, what is needed in a modelling situation is for students to structure and organise the information before the problem can be mathematised. Teaching about problem solving may fall short in giving the students chance to structure a problem for themselves and in allowing students to use their own methods and may contribute to weaker autonomous modelling. In using taught 'tools' student competency to structure and simplify a problem is weakened as students have limited choices, while in modelling various options for mathematising the situation are available to students. Students develop solutions based on existing knowledge and abilities that they have.

Falsetti and Rodriguez (2005: 21) give more detail about the relationship that exists between problem solving and modelling. They distinguish a number of problem solving activities. The first activity we can relate to word problems is discussed in traditional teaching (see 2.3). In the second type of activity, the problem is typically mathematical and the tools to study it are well known. The third type of problem solving activity is one where a real problem is stated and strategies must be developed to solve these problems thereby developing mathematical thinking. These last two types describe problem solving approaches. The fourth type of problem needs a modelling process; this is when the nature of the problem is mathematical, but known mathematical procedures are not relied on in solving the problem – the problem must be solved by modelling, where mathematical procedures must be created. Often student procedures are informal and inelegant.

There remains an element of interaction with the modelling cycle with all these problem types, but it is the point at which creating a model becomes necessary that differs from problem to problem depending on the nature of the problem. The first three types of problem types address particular points of the modelling cycle and develop limited competencies while the fourth type embraces the development of all modelling competencies (see 3.5).

2.5 THE PROBLEM-CENTRED APPROACH

From a teaching and learning perspective, it is important to discuss the problem-centred approach, not only as a teaching and learning theory but for its comprehensive account of mathematical learning. It also depicts a suitable sustainable learning environment for groups to develop solutions to modelling tasks.

The foundation of this approach is that curriculum and instruction should begin with problems, dilemmas and questions for students (Hiebert, Carpenter, Fennema, Fuson, Human, Murray, Olivier & Wearne 1996: 12). These authors use the following fundamental features of Dewey's concept of reflective inquiry in instruction design i.e. problems are identified; problems are studied through active engagement and conclusions are reached. There is a clear difference in emphasis in this approach that does not make it simply an evolutionary descendent of other problem solving approaches.

This approach does not advocate simply adding interesting problems to the existing curriculum, but changing the entire method of classroom instruction – where solving problems is not the end product of a section of work, but the beginning of learning. This approach re-structures teaching and learning in that problems are presented from the outset and not as applications. It presents a reversal of a traditional approach (see 2.3). By setting problems from the outset, important mathematics is learnt by solving these when students use their own constructs and conceptions. Further reflection takes place when students compare their methods with their peers through skillful teacher facilitation.

In this approach, there is a clear move away from 'representational view of mind' (Cobb, Yackel &Wood 1992: 2) to one that views mathematics both as an individual construction and social communal activity. In the former view, learning is seen as a process of acquiring accurate mental representations that exist independently of individual and collective activity (Cobb, Yackel et al.1992: 27). As such, the authors explain that in the classroom, the expert (teacher) projects his/her internal representations into the learning environment of the student as mind independent external representations. The teacher takes these representations as natural and self evident. The student, then separated from the learning environment, with their own internal representations is left to grasp the more complex concepts (Cobb, Yackel et al. 1992: 7). The alternative view, proposed by these authors, is a problem-centred approach. This view emphasizes that the teaching learning process is interactive and involves the implicit and explicit negotiation of meaning (Cobb, Yackel et al. 1992: 10), where the process starts with the students' informal mathematisation of hypothetical situations (Cobb, Yackel et al. 1992: 13). Students therefore internalize and interiorize their own mathematical activities as they solve problematic situations.

The problem-centred approach provides a suitable environment for students to develop competencies in the modelling cycle and in developing proficiencies in autonomous, independent problem solving. This approach is acutely student centered, but this does not necessarily mean that they act entirely on their own. The effective but subtle role of the teacher is discussed in the next section.

The role of context is more than just bridging between real world and mathematics. The real problems allow students to develop their own methods and approaches to solving the problem. What transpires in this approach is that as students solve real problems using their own methods. Gravemeijer and Doorman (1999: 115), writing about Realistic Mathematics Education which shares many aspects with the problem-centred approach, state that mathematical ways of symbolizing will emerge in a natural way in these activities and that the formal mathematics produced will be experienced as an extension of student's own experiences. Treffers (1987: 275) discussing context, distinguished between 'embeddedness' and 'embodiment' in describing approaches that were organised in everyday life or imagined reality and those using an artificially created environment. Modelling tasks are embedded within what could be real to students and are not like traditional computations that have a context forced on them.

The guiding principles of the problem-centred approach, which can so aptly house modelling activities, are that students work with their own informal methods and these methods become progressively formal through reflective reasoning and meaningful interaction with peers and the teacher. This approach affirms strong benefits of social constructivism, activity theory, social interactionism and DNR (Duality, Necessity, Repeated reasoning) based instruction. (See 2.9.2; 2.9.4; 2.9.6; 2.9.7).

2.5.1 The role of the teacher in the problem-centred approach

The teacher in a problem-centred classroom is primarily there to create a classroom culture and norms for problem solving. More importantly than providing the tasks, the teacher supports and encourages students to tackle tasks in their own way, to listen and learn from each other and to share their approaches and methods. In a problem-centred environment as well as environments where model-eliciting tasks occur, the teacher plays the role of 'mediator' (Halloun 2007: 681). This role means that the teacher performs various functions at different times. According to Feuerstein and Jensen (Halloun 2007: 681), this means the teacher mediates, transforms, reorders, organizes, groups and frames the learning environment. The essence of these roles involves teacher feedback. The teacher would be involved in providing feedback in various situations. The teacher may need to provide feedback to individuals, small groups or the whole class. The teacher may need to act as a moderator, not in the sense of judging student ideas, but bringing students to discuss their own models and to refine them on their own. The teacher sets examples of how groups should function to perform optimally and reach higher modelling competency levels.

2.5.2 The role of the students in the problem-centred approach

According to Hiebert et al. (1996: 16), students share the responsibility of developing the classroom culture and community of students in the classroom. They outline the following student responsibilities - firstly, students must take responsibility for sharing, explaining and justifying their work. Secondly, students must recognise that learning means learning from others. This approach therefore bears socio-constructivist principles.

Students work in groups extensively in a problem-centred classroom and exclusively when working on a model-eliciting task. This group work is characterized by a process described by Lave (Derry, duRussel & O'Donnell 1998: 27) as 'negotiation'. They elaborate that negotiation takes place in groups because each member brings with them their own cognitive ideas and unique perspectives - this means that members understand and interpret the problems in significantly different ways. They further describe negotiation as being embedded in language and communication strategies and understanding of the group members. Part of the time consuming aspect of the problem-centred approach and modelling tasks is the negotiation that needs to take place to bring the group members into 'greater conceptual alignment with one another' (Derry et al. 1998: 30). It is however, precisely this quality of interaction that enables students to raise their modelling competencies and problem solving abilities.

Negotiation within the problem-centred environment or modelling environment provides other benefits for students' modelling competencies. While some teaching approaches have as sought after product a numerical answer, the problem-centred approach and modelling tasks in particular, have many other products that are shaped by students. Smith (Derry et al. 1998: 31) describes the classes of information that is created by groups involved in interdisciplinary teams. These class products extended to the modelling framework. Smith distinguishes three types of products: tangible, intangible and ephemeral. The tangible products that a group can produce include the target product – that which is expected of them at the end of the task. It also includes instrumental products; these are the products (data tables, notes) that were produced to support the group's work. The intangible products represent the individual thoughts – some of these thoughts were shared with other members of the group. The ephemeral products are things such as unrecorded conversations or temporary board sketches – these existed only briefly but contributed to the other products and are included in the other products. These products are important to students, teachers and researchers in the modelling process as it leaves 'auditable trails of documentation' (Lesh & Doerr 2003a: 31) that can be used to assess individual or group progress and modelling competencies.

What is significant is that a problem-centred environment and modelling enable students to develop products that would largely be undeveloped in other learning environments. Students would be made sensitive to the ephemeral and intangible products that this type of learning environment produces whereas they would not be exposed to this in a traditional setting. Romberg and Kaput (1999: 10) also maintain that students ought to develop an appreciation for producing evidence for a claim and grounds to evaluate the evidence by being involved in the 'genuine mathematical enquiry' (Romberg & Kaput 1999: 10). Genuine mathematical enquiry encompasses different competencies than pure computation; these are best achieved within a problem-centred approach and modelling task environment.

2.5.3 The role of mathematisation in learning mathematics

It is important to distinguish mathematisation that leads to representational mapping and mathematisation involved in the type of activity referred to in this study called modelling. It is also significant to note that mathematisation is not modelling, but a part of the modelling process. Modelling in this study does not refer to personal models or mental representations, but to a class of representational forms that include an assembly of particular properties (Lehrer & Schauble 2003: 61). Their postulation is that to mathematise the world effectively, students require mathematical resources beyond arithmetic, including spatial visualization, data structure, measure, and uncertainty and accordingly modelling should be given a central place in mathematics education from the earliest years.

In modelling situations, students have to decide if the model they created 'fits' the real world context. This would mean an underlying assumption that students compare and evaluate other models that could fit the real situation. Lehrer and Schauble (2003: 61), maintain that students can develop their own rival models with thoughtful instruction. This is where the problem-centred approach supports this need. Students are led, by skilful teaching, to compare solutions and approaches in light of being able to produce several suggestions or models on their own. It is this ability of students to multi-model the problem that is particularly valuable.

Lehrer and Schauble (2003: 61) also state that mathematisation not only helps students see things that they otherwise may not, but prepares students for 'disconfirmation' when presenting a solution. What transpires in a problem-centred classroom is that when representations are mathematised, the conjectures can be held to account (Lehrer & Schauble 2003: 61). What is most important according to them is that students do not simply use models to stand for other objects, but chose models that highlight the structure and relations embedded in the problem. Furthermore, when students are able to transfer these models to other situations they are working at a higher competency level since they will have to mathematise across structurally similar problems. Mathematisation precedes competencies such as working mathematically, interpreting, verifying and presenting and follows competencies of understanding and simplifying. As such, it lies as a gateway competency in the modelling cycle.

Mathematisation in the context of modelling needs to go beyond what the problem may look like in mathematics to how the problem functions in mathematics. Students are always at some point on the continuum between what Lehrer and Schauble termed 'form and function'. Models initially show some physical resemblance to the problem, but students need to be assisted to move along the continuum. This continuum can also been described in terms of Streefland's a 'model of' on one end and a 'model for' (Van den Heuvel-Panhuizen 2003: 14). Mathematisation takes a central role in modelling problems and assisting student move along the continuum of form and function. It is only when student competencies of understanding and simplifying a problem are adept that meaningful mathematisation can take place. In turn, mathematisation catapults the student into the realm of the other competencies of modelling such as interpreting, validating and presenting. Mathematising cannot be taught, it requires individuals and groups to continually negotiate Freudenthal's membrane between the real world and the world of mathematics until they are confident commuters within and between both.

2.5.4 The problem-centred approach and modelling

Hatano (1996: 213) proposed several guidelines for mathematics education that embody the conception of knowledge acquisition based on his characterization of knowledge i.e. that knowledge acquisition is constructed, restructured, constrained, domain specific and contextual. Some of his suggestions that relate directly to the problem-centred approach and modelling are to:

1. Pose interesting problems
2. Place the problem in a pragmatic or familiar context
3. Encourage students to use their prior skills and ideas
4. Suggest students use tools that enable them to do what they want to do
5. Use peer interaction for motivating students. Students may solve collectively a problem that individually they are unable to solve.
6. Intervene in peer interaction only when appropriate and as long as it does not endanger students' spontaneous construction of knowledge

The above principles are the framework of the problem-centred approach and thus provide an environment for modelling. Students invent, use and refine their own informal strategies for solving problems. The competencies needed for this are of a higher order and complex nature. Carpenter, Fennema, Fuson, Hiebert, Human, Murray, Olivier and Wearne (1999: 59) agree that the fundamental principle of the problem-centred approach is that students are given opportunities to construct new procedures for solving problems in contexts or real world problems in modelling terms. Mousoulides et al. (2007: 29) affirm that modelling requires a context in order to 'frame the problem and develop the mathematics'.

The confluence of the problem-centred environment and modelling tasks present mathematics education with a 'developmental space' for the learning of essential, meaningful mathematics. Within this space, students strengthen existing concepts and develop the 'radius of action' (Blomhoj & Jensen 2007: 51) for their mathematical knowledge. It is within this space that modelling competencies can improve and advance.

2.6 MODELLING AND THE SOUTH AFRICAN CONTEXT

Murray, Olivier, Human (1998: 184) in a culmination of their work on the problem-centred approach, identified three main pillars for primary school teaching. Firstly-well-planned number concept activities, including activities which promote the building of patterns and relationships. Secondly, well-planned problems are necessary and finally effective discussion is necessary for learning with understanding to take place. The last two pillars encapsulate model-eliciting tasks such as those used in this study. Modelling, in the South African context is an extension of the problem-centred approach. Wessels (2009: 319) gives a theoretical account of modelling and proposes it as a viable teaching approach in the South African school system.

De Villiers (1993: 3) distinguished three categories of modelling, that being: direct modelling, analogous modelling and creative modeling. These distinctions are useful since they can be used to categorize the style of teaching that takes place in most classrooms. Direct modelling can be associated with a traditional teaching approach since it involves an immediate recognition and use of a known procedure by students. Analogous modelling entails recognizing a model from a previous situation and applying it to the current or new situation. The teaching of problem solving heuristics can be seen as analogous modelling. Creative modelling on the other hand is more in line with the type of tasks proposed in this study, where students build a model consisting of new ideas and structures. As stated by De Villiers, this category is a powerful teaching strategy.

In other South African research in the field of modelling, Mudaly and De Villiers (2004: 8) further explored the relationship between modelling and proof and found that engaging in the modelling process 'evoked a strong desire for a proof' in students. Proof may be considered closely related to the competency 'validating' (see Fig 1.1). De Villiers also explored modelling and technology extensively (De Villiers 2007), while this study remains within a low technology environment which is typical of most South African classrooms.

Wessels (2006) in her thesis extended and integrated modelling into the statistics education domain and focused on primary school pupils' types and levels of data representation while modelling and she maintained that statistics is about modelling data (2006: 72).

Although these studies fall within the modelling field, they do not all directly have bearing on the modelling competencies. With the focus on grade 7 students' modelling competencies while working on proportional reasoning tasks, this study is advances the research on modelling in South Africa.

2.7 ASPECTS IN LEARNING MATHEMATICS

2.7.1 Cognitive aspects in learning mathematics

By dictionary definition (Britannica: 2009), cognition refers to all processes of consciousness by which knowledge is accumulated, such as perceiving, recognizing, conceiving, and reasoning.

According to Umland (2008: 101), mathematical thinking is a refinement of our everyday cognitive ability to reason about the world. In her view, the mind is not an all purpose thinking machine, but a synthesis of cognitive components. Cangelosi (2003: 166) identifies the cognitive domain as having to do something mentally like constructing a concept; discovering a relation; using simple knowledge; comprehension and communication; using algorithmic skill; application and creative thinking.

Cognitive aspects to learning mathematics can be seen as those mathematical mental processes such as those in Bloom's taxonomy: understanding, comprehending, analyzing, synthesizing and evaluating information. Freudenthal (1991: 18) states that cognition does not start with concepts, but that concepts are a result of cognition. At the heart of modelling lies mathematising which encompasses the cognitive domain of (among others) quantifying, categorizing, structuring and systematizing (Lesh & Doerr 2003a: 5). Cognitive competencies for this study are defined in section 3.5.2 as those competencies that will be necessary in completing a mathematical modelling cycle.

In problem solving a student will experience 'cognitive disequilibrium' (Halloun 2007: 681) at the outset. The problem presented by the teacher has an obvious gap between known and want to know. However, in a modelling situation, the students only reach the state of cognitive disequilibrium after they have provided and compared viable/non viable ideas regarding the structure of the problem and the output the task requires. Often, in modelling the output is not a numerical solution but a real-life based task such as making a decision who to hire or writing a letter of best options. This makes the cognitive demand of the task much higher.

2.7.2 Meta-cognitive aspects in learning mathematics

Meta-cognition is often used to describe students higher order thinking. Meta-cognitive activities encompass instances where selection, planning and monitoring are employed by students. Flavell and Vemunt (Masui & De Corte 1999: 519) categorized meta-cognitive activities. Flavell distinguished between knowledge of self, knowledge about the task and strategic knowledge while Vermunt categorized eight different activities. These are pre-activities (orienting and reflecting), control activities (planning and adjusting) and activities of monitoring (process-monitoring, self-testing, diagnosing, evaluating). Schoenfeld (1987: 190) focused and elaborated on three related but distinct meta-cognitive categories. These were set as questions directed at students: a) knowledge about own thought processes which can be answered by asking how accurate you are in describing your own thinking; b) Self-regulation can be answered by asking how well you keep track of what you are doing and how well you use the input from observations to guide your problem solving actions and c) beliefs and intuitions can be gauged by asking what ideas about mathematics you bring to your work in mathematics and how that shapes the way you do mathematics.

While it has been considered more appropriate to deal with these meta-cognitive aspects as they occur in problem solving situations, meta-cognitive aspects from a modelling perspective involve a somewhat different perspective. Lesh, Lester and Hjalmarson's interpretation (2003: 383) is that meta-cognition is closely related to concepts with specific concepts and situations and not as a traditionally treated content-independent type of thinking. So in terms of the modelling cycle, student competencies must be based on a holistic integration of cognitive and meta-cognitive competencies for a specific modelling task. Lesh, Lester et al. (2003: 385) also add that meta-cognitive processes vary from one stage of problem solving to another and from one problem to another. They remind us that in a traditional view, meta-cognitive processes are taught as if they can be equally applied across all problems. A modelling perspective holds that meta-cognitive competencies can only be developed by students being involved in solving many group-infused open modelling tasks.

Lesh, Lester et al. (2003: 384) furthermore suggest that a modelling perspective recognises the interdependence of higher order and lower order thinking. It is not only that the meta-cognitive thinking abilities influence lower order understanding and abilities, but that the lower order thinking will influence the effective functioning of meta-cognitive processes. In their view, meta-cognitive abilities do not precede or follow cognitive activities, but interact with the cognitive actions. This interaction of higher order and lower order thinking is best achieved by modelling in group situations where varying levels of thinking are observed and coordinated and since meta-cognitive competencies develop naturally during modelling activities.

Since meta-cognitive competencies appear to be a very powerful factor in learning mathematics competencies were also identified from this realm for the study (See 3.5.4).

2.8 A PERSPECTIVE FOR MODELLING

Kaiser and Sriraman (2006: 302) reveal that there isn't a common understanding of modelling within an international discussion on modelling, but that certain perspectives can be found. It is important to embed this study within or amid theoretical and conceptual frameworks that currently exist and to formulate a justification around these. The theoretical and conceptual framework adopted essentially reveals what it means to the researcher to 'learn mathematics by doing mathematics'. As we are reminded by Torner and Sriraman (in Arzarello, Bosch, Lenfant & Prediger 2007: 5), grounded assumptions associated with the nature of mathematical knowledge appear to be most deep-seated.

Since this study displays (in Kaiser & Sriraman's terms) pedagogical aims; psychological aims and subject-related aims it can be considered as an example of an 'integrative perspective' (Kaiser & Sriraman 2006: 302). This study integrates and advances 'contextual', 'educational' and 'cognitive' perspectives (Kaiser & Sriraman 2006: 304) to mathematical modelling. (See definitions below).

The everyday realities of teaching and learning in a mathematics classroom are part of the framework of the researcher and as such form a big part of the background within which this study is approached. In order to merge the teaching and learning aspects a contextual and educational perspective are equally important. According to Blomhoj (n.d.) a contextual perspective emphasizes model-eliciting problems starting from meaningful situations. This enables students to engage in meaningful problems that can be mediated within group situations. Learning of meaningful mathematics is an important aim. The context of the problem not only sets up an authentic situation, but competency development is interdependent with context. Lesh and Doerr, leading role players in promoting a contextual perspective in modelling, maintain that "a models and modelling perspective" brings together conceptual development and problem solving in one perspective (Lesh & Doerr 2003b: x) as a significant means of learning. Although this perspective is meaningful to modelling research, another scaffold with which to understand modelling as a mathematical activity is necessary.

An educational perspective focuses on the integration of mathematical modelling in mathematics teaching (Blomhoj n.d.). Niss, Blum and Galbraith (2007: 20), maintain that one of the most important objectives for students is to acquire modelling competencies and they consider the factors that would hamper or advance modelling abilities important. They also consider the teaching and learning practice of modelling as well as assessment of modelling. This study aims to integrate modelling not only into a research study, but to give important consideration and recommendations to modelling as a mathematical activity that can be realized in mathematics classrooms. The act of modelling itself is deemed an important part of mathematical learning and a substantial way of learning important mathematics.

A cognitive perspective forms the main focus of the empirical component of the study with the contextual and educational perspective underlying the decisions made and focus represented in the study. The modelling competencies of groups form the base of the empirical part of the study. According to Borromeo Ferri (2006: 91), an advocator of the cognitive perspective, one can only really speak of what is visible in terms of students' modelling routes to keep within a cognitive viewpoint. External representations on a micro level form the basis on which this study determines its findings. Borromeo Ferri (2006: 92) revised the modelling cycle from a cognitive perspective and added students' EMK (extra mathematical knowledge) to the cycle. While her study focused in 'individual modelling routes' (Borromeo Ferri 2006: 91), this study in the teaching experiment (see 4.4) focuses on group routes and competencies by using visible external representations of group modelling sessions.

Lingefjard (2006: 97) gives terms to Blum and Niss' (1989: 5) summary of five arguments why modelling belongs in a curriculum. These are formative, instrumental, practical, critical and cultural arguments. The formative and instrumental arguments have as their focus the student and his/her personal development in mathematics; the practical argument has as its focus the usefulness of student to society and vice versa; the critical argument has the ability of students to recognize and understand the uses of mathematics in society at its heart while the cultural argument has mathematics as a science in all its facets as its focus. According to Lingefjard, each argument fosters a different character. Some focus on the development of the individual, others on the usefulness to society or on mathematics as a science. This means that modelling is significant to mathematics education from a variety of viewpoints. He further adds that different weightings produce a different perspective on teaching and learning mathematics. So it is with a research perspective to mathematical modelling. Different weightings produce differing frameworks within which to embed a study. The weighting in this study is on the formative and instrumental arguments.

From the above it will then be considered necessary to give focus to the underlying theoretical perspectives that support the contextual, educational and cognitive perspectives that include a formative and instrumental argument for modelling. The theoretical perspectives such as those in section 2.9 support and sustain the perspectives adopted for modelling in this study.

2.9 RELEVANT THEORIES TO MODELLING

Theoretical perspectives are important in understanding the modelling situation. As teaching and learning in modelling are such complex activities, they cannot be 'explained' by one theory. Several theories shed light on different parts of the modelling stage. According to Cobb (2007: 31), theoretical perspectives provide insight and understanding into learning processes. Different perspectives will enable us to ask 'different types of questions'(Cobb 2007: 31) of that which is being studied and assist in developing a more critical understanding of that which we want to understand. The perspectives presented here follow a progressive development in the sense that they follow an evolutionary progress. They hold a symbiotic relationship with modelling in that they explain the theoretical benefits of modelling and assist in shedding light on what happens in practice, while the activity of student modelling allows us to understand the theory pervasively. Of interest is what Sriraman and Lesh (2006: 249) disclose, that modelling and modelling perspectives advanced out of research on concept development more so than out of research on problem solving. This may explain why there exists a rich and diverse theoretical background to look towards. We look toward theory so as to deepen our understanding of practice. Otherwise what may be observed in passing by a teacher or researcher will remain just that - and not be considered a significant factor in learning.

2.9.1 Constructivism as underlying theory for modelling situations

While behaviourism formed a popular educational theory that justified traditional modes of teaching that focussed on the actions of the teacher, constructivism advocates that children learn by constructing their own knowledge. Constructivism endeavored to develop instructional principles directly from a background cognitive theory (Cobb 2007: 5). Cobb also points out that an important aspect of constructivism is that not only is the resulting pedagogy based on ideology, but constructivism also proves to match the analyses of the process of how students learn.

In the classroom, the shift in focus is away from what the teacher does, to what the student does (Iran-Nejad 1995: 16). Each player in the classroom needs a new defining role. The teacher is no longer the active assistant and the student the passive participant. Learning is no longer thought of as the transfer of knowledge from teacher to student. This means that the didactical situation becomes more complex, moving towards Brousseau's adidactical situation where the teacher's position is more subtle and overtly less active. According to Van de Walle (1999: 3), constructivism requires tools. The tools that children use to construct their own knowledge are the ideas they already have. This means that students move from a state of known to unknown. Students use existing mathematical ideas to create new mathematical ideas. This means that their cognitive networks are strengthened and expanded.

Within the constructivist approach a strong understanding exists that students bring with them an amount of knowledge and experience. This knowledge and experience should be the starting point for teaching new concepts. Within a traditional teaching perspective, the students are seen as empty vessels that need to be filled. Here it is first decided not what they know, but what knowledge and skills the student *should* know at a particular age or level. Some problem solving approaches also try to fit or equip the student for the problem (teaching heuristics) and not the other way around.

From a modelling perspective, students are expected to structure, organise and describe a system or set of relationships, from the point of what they already know. The student has to make sense of the situation from his/her existing conceptual base. It is therefore important to take Lesh and Doerr's stance that not all the things that students learn are to be constructed (Lesh &Doerr 2003c: 532). According to them things like notational and procedural conventions are learned through other ways. Modelling and modelling perspectives focus on constructs that students build that can be used by them to interpret an experience. According to Lesh and Doerr, the focus is on the noun rather than on the verb. Furthermore, they add that modelling and modelling activities necessitate other process, not only constructing. Students need to work with their already existing systems by sorting, differentiating, reorganizing, refining, adapting or reflectively abstracting systems. Together with already existing conceptual systems, Lesh and Doerr (2003c: 533) also spell out that student learning, holistically has a behavioral aspect (basic facts and skills), a process aspect (habits of mind not connected to a particular mathematical construct) and an affective aspect (their attitudes, beliefs, feelings) when modelling. These are not necessarily constructed. Students bring all these aspects into play when constructing models to interpret a situation and the modelling competencies investigated in this study comprise all of these aspects (see 3.5).

Constructivism as a learning paradigm implies that teaching can no longer take the form of lecturing. If students have to construct their own mathematical concepts, then teaching involves creating an environment for this to take place. According to Falsetti and Rodriguez (2005: 16), problems will play an important role in an environment that takes constructivism into account as the underlying theory. Duckworth (Iran-Nejad 1995: 16) defined constructivism not as meaning that is given to us in our encounters, but meaning that is given by us, constructed by us in the situation (environment) according to how our understanding is currently organised. So we can infer that modelling tasks demand a high level of student self construction. This in turn will lead to a higher level of learning and understanding.

A constructivist classroom is a microcosm where students and teacher work together as a community. Students and teacher act in a manner that assumes that students have something to contribute. Students are able to negotiate their own understanding of mathematics.

Students are seen as active in their own endeavour to learn. This mental and physical activity of students increase the more a classroom resembles a problem-centred one, and is at its most prominent when students are involved in modelling activities.

According to De Corte, Greer and Verschaffel (1996: 497), constructivism as a theory is known as a first wave theory, which models teachers' and students' individual cognition, while a second wave theory, is one that will focus on affect, context and culture. It is in these second wave theories that modelling can also closely be affiliated to as modelling tasks and their solutions are strongly linked to student and teacher affect, problem context and a change in classroom culture. To this end, socio-cultural theory, activity theory, distributed cognition and symbolic interactionism as second wave theories are directly relevant as they shed light on some part of modelling and modelling competencies. Each second wave theory in turn allows us a glimpse into modelling as an activity from a particular theoretical side. The layered effect of using these theoretical lenses allows us a fuller picture of modelling.

2.9.2 Social-Constructivism and its relevance to modelling

Social constructivism as a philosophy of mathematics firmly rest on two principles of radical constructivism which are that:

1. Knowledge is not passively received but actually built up by the cognizing subject;

2. The function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality (von Glaserfeld in Ernest: 1996: 336).

These principles assist in understanding modelling as a process whereby students actively construct and build on their existing concepts and that they are continually making sense of, and organising their own experiential and conceptual world.

The principles of social constructivism are extended to include the existence of a social and physical reality (Ernest 1996: 342). Essentially social constructivism views mathematics as a valuable social, human construction. Modelling assumes that students' social interaction is viable and valuable to building their conceptual bases. According to Lesh and Doerr (2003c: 525, 545) although modelling is inherently a social enterprise, it goes beyond social constructivism in that modelling perspectives recognise at least four ways that models come to be adopted by individuals or groups. The first based on consistency of the new construct, the second; based on the usefulness of the new construct, the third; based on whether peers consider the construct justified or acceptable, and lastly; based on whether the construct is acceptable to teachers or other authorities. In modelling, students are brought into a social arena where a physical reality must be mathematised.

Socio-cultural theory investigates the participation of the individual in cultural practice (Cobb 2007: 22). From Cobb's point of view, what is important is that students' out-of-school practices (the real experiences they are involved with) are not seen as an obstacle in a mathematics classroom. This is evident in a traditional classroom where reality is avoided (see 2.3.1). Students' real experiences contribute to their structuring and analysing of modelling problems. Socio-cultural theory draws largely from the writing of Vygotsky which will be discussed in the next section.

2.9.3 Socio-cultural perspectives

The NCTM proposed that students be actively involved in mathematics by working in small groups where they negotiate and communicate ways to work with data in open-ended problems and explain their ideas to each other (Forman 1996: 115). According to Hiebert et al. (Steele 2001: 404) relational understanding of mathematical concepts develops through communication. Communication takes many forms but it is in the form of discussion that reflecting and refining concepts takes place. When students are involved in contexts that foster discussing concepts and ideas with others, they are then involved with reasoning and explaining their thoughts. This discussion develops competencies of simplifying, structuring and validating.

Vygotsky argued that thought and language have separate roots, and it is this merging of thought and language that practical and abstract intelligence develops (Vygotsky 1978: 24). One of the basic principles of the Vygotskian approach is the assumption that individual learning is dependent on social interaction (van Oers 1996: 93). Pirie (Steele 2001: 405) maintained that when students are mentally and physically active they are able to make connections from ordinary language and mathematical language. This type of activity derives from social interaction.

Vygotsky documented how student meaning is first constructed on a social level and then on an individual level (Vygotsky 1978: 27). Vygotsky's stance is that students first internalize concepts on an 'intermental' plane of social interaction and, overtime these concepts form part of the student's 'intramental' plane of individual thought (Cobb 2007: 22). Group dynamics not only allow for language communication to come to the fore, but for students to be in the presence of knowledgeable 'other' thereby fostering the ZPD (Zone of Proximinal Development) described by Vygotsky (Vygotsky 1978: 86). The more knowledgeable 'other' person (or 'significant other') does not always have to be the teacher, but can take the form of a group of peers. As pointed out by Hatano and Inagaki (Hatano 1996: 204), children learn from their peers, who, though not necessarily more capable, may propose innovative ideas at the critical moment. It is important to note that there is no transferring of a way of thinking from 'others' to student within this situation, there is still a need for the student to construct his/her own knowledge in the zone. The ZPD does not imply learning; it is a situation that is created to optimize learning and to allow for the development of meaningful learning. To create this Zone does not only require other people in the situation, but would necessitate problems at a higher level than what the student can solve independently

Group contact or interaction is largely avoided in a traditional classroom. Language is used only by the teacher in producing succinct explanations. Students are not involved with using language to construct meaning, neither are they placed in situations where they need to use language to express their thoughts to other students. Some reasons for this are that stimulus-response type methodology is relevant specifically to individuals. In a traditional paradigm, students who work together indicate some form of dishonest sharing of answers. Furthermore, it leaves the teacher in a predicament regarding assessment of the activity, when final answers are given prevalent significance. It is assumed that the final answer incorporates the concepts needed to reach it. This is what Confrey (1994: 7) calls the stark evaluative climate in mathematics classroom. Confrey further explains that constructivism has helped to redefine mathematics as an issue of communication and interpretation and not documentation and logical necessity.

Many problem solving strategies allow and engender group interaction. Hart (1993: 170), reports that three factors were found to facilitate students' problem solving - group collaboration, group monitoring and social norms in small-group problem solving. An open modelling problem, by its nature, necessitates group dynamics, group discussion and a creation of mathematical understanding through communication. A model-eliciting activity demands that the individuals participate actively in solving the group's problem. Modelling competencies can only be developed while students are modelling, in a group environment. The above discussion shows that group learning is not only a necessity for modelling, but that modelling competencies manifest and develop because of the group situation.

2.9.4 Activity Theory

Cognitive activity is also considered within a social context from an activity theory perspective. According to Crawford (1996: 138), activity theory added a theoretical center for investigating relationships between cognitive abilities and problem solving actions, essentially the link between thinking and acting. Modelling has a crucial 'action' component. Students are not only mentally active, but perform a number of other actions – measuring, making rough sketches, explaining, justifying, reading etc. These actions lead to the development and improvement of modelling competencies. Activity, according to Leont'ev (Crawford 1996: 136), includes conscious intellectual and other behaviour that is stimulated by a motive and subordinated to a goal or expectation. Activity theory, as explained by Leont'ev, reciprocates the stimulus-response theories in that the state of the subject is not determined by objects, but the state of the objects is determined by the subject. More specifically to modelling and observing modelling competencies, Lenot'ev's (1977) proposal that activity is 'external activity that unlocks the circle of internal mental processes that opens it up to the objective world' is useful. The link between Activity Theory and modelling situations can be paralleled when evaluating Davydov, Zinchenko and Talysina's explanation of activity. In their view, "activity" involves the following:

- A need that impels a subject to search for an object or goal.
- The discovery of an object or goal. Once the object is discovered, intellectual activity is guided by the subjective image of the object and not the object itself. The subjective image being the image of the object made by each individual.
- The generation of an image. This is seen as a bilateral process between subject and object.
- The conversion of 'activity' into objective results (Crawford 1996: 137).

According to Engestrom (Crawford 1996: 138), activity theory is extended to activity systems where a group of people act together to develop and use their expertise. The focus being on the system of people that act towards achieving a goal. Modelling can be seen as an activity system since this system involves Engestrom's components of community, rules of behaviour, division of labour, instruments, shared goals and outcomes. In the next section, a system is redefined and becomes the unit of analysis. The unit of analysis in this study is the group and the development of group modelling competencies.

2.9.5 Distributed Cognition as theory for modelling

This theory advanced a redefinition of the unit of analysis in teaching and learning. From a pure constructivist point, the individual is the focus of analyses. Through the development of second wave theories that sought to include the important role of language, social interaction, context and culture, Pea (Cobb 2007: 13) defined the unit of analysis as a system consisting of the individual, tools (physical and intellectual) and social context. This view assists us to conceptualise the individual more clearly and not to dismiss the individual from consideration. This approach, as does Socio-cultural theory, serves to move away from individualist cognitive science and incorporates aspects of Socio-cultural theory.

Distributed cognition, however focuses on the immediate environment and involves analyses on a specific group's activity (Cobb 2007: 25). The immediate environment becomes a resource for learning. Learning is not seen to only take place within the individual. In this sense, intelligence (preferred to cognition by Pea) is accomplished and not possessed (Pea 1993: 50). Consequently the student who is accomplishing intelligence is doing so in a social setting using mental and physical tools. Pea's view is that tools or artifacts not only advance and shape the activity, but carry intelligence (of the designers and creators) in themselves.

Pea (1993: 66) reconstructed the Polya 4-step process for problem solving, but stating that these steps were not necessarily constructed by the individual, nor are they confined to mental construction. Taken further, a modelling situation can be seen as the development of cognition where students are advancing in a socially distributed activity where artifacts and tools also carry distributed intelligence as described by Pea. This does not mean that there is a fixed quantity of intelligence that is distributed among mind, setting and tools but rather an expanding of intelligence, whereby the student may use tools to cope with the complexity of mental activities in order to be freed up to invent and innovate (Pea 1993: 77). Vygotsky (1978: 55) makes an important distinction between tools and signs. A tool serves as a means for a student to influence the object of the activity, while a sign is internally orientated and aimed at organising internal objects. He further adds that a higher psychological behaviour is when a combination of tool and sign occurs in a student's psychological activity. Student use of mental (tools and signs) and physical tools merge when modelling.

Cobb (2007: 26) also points out that distributed cognition theorists focus on cognition as it is evident in everyday or workplace activities and not as is evident in completing typical school based tasks. Modelling tasks provide opportunity to witness student cognition and development while they work on real tasks in group environments. Brown, Collins and Duguind (Cobb 2007: 26) argue that students will not develop application techniques after learning abstract concepts but rather, that students should engage in activities where those concepts and skills are actually used. This parallels the aims of modelling situations. Real world tasks are recognized as collaborative and require tools and resources beyond the individual's long term memory (Pea 1993: 74). This perspective, in Cobb's view, has greater potential than socio-cultural theory to inform the creation of designs at the classroom level. Modelling tasks give us a glimpse into real world tasks made apparent in a teaching and learning environment. They afford us an opportunity to focus on student collaborative coping skills and the scope of student tool and resource use in what may mirror the workplace of the future.

Pea explains that problem solving as a linear model is restructured by distributed cognition theory in that these typical stages (Read, Plan, Execute, and Check) are not considered to be constructions of the individual mind nor are these stages necessarily mental constructions. According to Pea (1993: 67), very often tools, artifacts and external representations mediate the stages of the problem situation. From this a modelling situation is starting to crystallise as a more complex problem solving task from the distributed cognition perspective. Different processes of the problem may be distributed in the environment or to other persons as happens in a modelling activity.

However, one point to note is that distributed cognition theorists still consider the individual as having primacy in activity. Intelligence is manifest in an activity that encompasses an individual, others and tools. It is however important to consider the cognition of the individual mind and the development and progression of competencies within the individual and within a group. A modelling task could be seen in terms of a learning design that is orientated towards distributed intelligence in the sense that the problem requires group pooling together of ideas, use of environmental knowledge and use of tools and signs to produce a model. Symbolic interactionism as discussed in the next section is an even more useful theory from which to view sense making during modelling.

2.9.6 Symbolic interactionism as theory

As explained by Yackel (2000: 2) symbolic interactionism is compatible with psychological constructivism and as pointed out by Voigt (1996: 30), emphasizes both the individual's sense making processes and the social processes. Furthermore, Yackel explains that the essence of social interactionism is the centrality given to the process of interpretation and interaction. According to Blumer (1969: 4), meaning arises from the process of interaction between people. Student interaction involves interpretation of the thoughts of others through their actions. In the same way students have to convey their thoughts and intentions to others through their actions. What is important to note, according to Yackel, is that neither the individual nor the social is taken as primary. There is a continual interpretation through inter-action. This means that meaning is seen as a social product (Yackel 2000: 3). The individual creates meaning by interpreting the actions of others.

Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti and Perlwitz (1991: 6) summarise mathematical learning as both acculturation and individual construction. Social interactionism provides a very sound lens through which one can view modelling tasks as they take place in groups where the need for interpretation and interaction is a vital component. Since this study focuses on the development of group modelling competencies, social interactionism explains the development of group meaning through individual actions within the group. The process of interpretation through inter-action comprises the domain of modelling competencies. This interaction allows modelling competencies to manifest and develop.

2.9.7 DNR based instruction in modelling

Harel (2008: 3) proposed a broad theoretical framework for mathematics instruction known as DNR. The initials D, N and R stand for three main principles of the framework, that being Duality, Necessity and Repeated Reasoning. This framework serves the foundational principles of modelling situations.

The duality principle is based on the mutual relationship between a student's way of understanding and way of thinking. Harel's definition of a way of understanding is a cognitive product of a mental act, while a way of thinking is a cognitive characteristic of a mental act (Harel 2008: 4). Harel explains that this means that students develop ways of thinking through the construction of ways of understanding and the ways of understanding produced are determined by the ways of thinking the students possess. Harel points out that traditional teaching of ways of thinking will have no effect on student's ways of understanding. Therefore, attention to ways of thinking is important. These ways of thinking can only be produced by solving problems independently and not being told how to solve problems or being given particular heuristics in solving these.

Modelling situations provide rich ground for developing the mutual ways of thinking/ways of understanding duality since both cognitive products and their characteristics are scrutinised. Modelling tasks foster a strong link between student ways of thinking and ways of understanding. Modelling tasks leave a 'trail' (Lesh & Doerr 2003a: 26) of audible and written evidence for the student, teacher and researcher to be mindful of changes to these ways of thinking and understanding. Studying modelling competencies may shed light on this duality.

Necessity as a principle is a prerequisite for the duality principle. Necessity indicates that the task satisfies an intellectual need in the student. This necessity also refers to student behaviour when they encounter an intrinsic problem (Harel 2008: 21). The problem must stimulate a desire in the student to solve it. This often involves construction of new knowledge that is meaningful to the student since this construction stemmed from a need to solve the problem. Real-life contextual problems found in modelling situations provide a strong foundation for the necessity principle. The modelling competencies that students need and develop are related to this necessity principle. The competencies are intrinsic and are developed out of a real need to solve the problem at hand. The type of problems that students are expected to solve while modelling, elicit an interest and underlying need to structure and solve them.

The Repeated Reasoning Principle according to Harel is not repeated practice in computation or problem solving heuristics, but the requirement to practice the type of reasoning involved in organising, internalizing and retaining ways of thinking and ways of understanding. This is not drill and practice type exercises. It is the reasoning that is to be repeated, not computation, recipes or algorithms. Repeated reasoning leads to students being able to apply their knowledge autonomously. The type of reasoning involved in solving modelling problems leads to improved ways of understanding and improved ways of thinking. Repeated reasoning links strongly with Lesh and Doerr's (2003a: 26, 28) view that understanding is not an 'all or nothing' affair and that the evolution of cognitive constructs is characterized by a gradual increase in local competence.

DNR based instruction is a suitable framework in which to place modelling and modelling activities. The three pronged scaffold allows the development of modelling as a task in mathematics education and the development of modelling competencies.

2.10 CONCLUSION

In traditional classrooms students do not use understanding to inform the procedures they use, nor do they understand the procedures that they memorise and execute (Hiebert et al. 1996: 17). The competencies they develop here are repetition and memorization. In Romberg and Kaput's view (1999: 15), society's perception of the scope of content that students are expected to understand is changing. The competencies that are then considered important are also changing. Memory and rote execution do not assist students in meeting the challenges of the 21st Century. The competencies that are needed to a greater extent are those represented in a full modelling cycle.

Brousseau's basic theoretical framework of didactical situations becomes the backbone for both a problem-centred approach and modelling tasks. A re-negotiation and re-definition of the didactical contract becomes important in creating an adidactical situation that will lead to a problem being devolved to the students and their acceptance of the responsibility of solving the problem. This milieu together with modelling tasks will enable the student to reveal the concepts, structures and processes that he/she possesses and can meaningfully employ to solve problems.

Since modelling is a complex, multi-faceted endeavour it is not possible to expect that any one theory will be sufficient to encompass modelling and model-eliciting activities. In search of a sound philosophical framework, modelling has to turn to a range of theoretical perspectives. These perspectives all say something, but not everything about modelling. Since modelling tasks produce an array activities, products and interactions between people it is necessary to seek some explanations in many theories and perspectives. In turn, modelling tasks in practice may provide practical aspects to many theoretical perspectives. This reflexivity provides a healthy interpretive framework for theory and practice.

Constructivism, amongst other theoretical ideas, is the starting point in finding theories that help us describe modelling. A theory that explains how learning takes place in groups of students is necessary. Within these groups the focus is on the activity, so activity theory is consulted. The type of problems that require students producing a model require students to work in real contexts with tools and artifacts, to this effect distributed cognition (without loss to the individual) is expansive. Symbolic interactionism sheds light on group dynamics by explaining that interpretation takes place through interaction of people while the DNR based instruction guides instruction towards improving ways of thinking and understanding. Modelling tasks and its related products and residue reflect all of these views. Instead of holistically embracing one particular theory, an approach with modifications of the theoretical perspectives provides a comprehensive framework for modelling.

Lesh and Doerr (2003c: 531) attest to the potential power of modelling by pointing out that the environment within which competencies are developed also create the most important characteristics employers are looking for: that being, the ability to work and communicate within a team, the ability to adapt to rapidly changing technological tools and the ability to plan, monitor and assess progress during projects that involve complex systems. Lege (2005b: 96) declares that there is an overabundance of reasons why modelling should be part of a mathematics curriculum or classroom. These reasons include developing sense-making, providing for realistic problem solving, injecting a social element into mathematics education and critical competence. So, implementing modelling at all levels of mathematics education will be visionary.

CHAPTER 3

MATHEMATICAL MODELLING COMPETENCIES

3.1 INTRODUCTION

The aim of this chapter is to provide some definitions for modelling from different perspectives and to establish modelling as a means of significant learning in mathematics education. Modelling tasks are defined in terms of model-eliciting activities. Various studies dealing with modelling competencies are highlighted. Modelling competencies are explored and discussed in three broad categories. Different approaches to assessing modelling competencies are also discussed. The chapter concludes with a discussion of group modelling competencies and around the success of students with varying mathematical abilities. This chapter aims at setting the scene for modelling as an environment, a task and a mathematical competence.

3.2 MODELLING AS MATHEMATICAL COMPETENCE

Goldin (1987: 138) in his work on problem solving competence based on cognitive representation describes competence as the capability to perform successfully on a class of tasks. This definition is directly relevant to this study, where competence is taken to develop over a series of tasks. Goldin further adds that competence does not necessarily imply that the student can perform successfully every time a task is encountered. This ties in with the concept of a gradual increase in local competence (Lesh & Doerr 2003a: 28). Goldin sought to describe competencies to help understand observed problem solving behaviour although not to directly describe performance. In his understanding, competency is 'not something real that exists in a problem solver, but as the researcher's way of representing classes of potentially observable behaviour' (1987: 138).

Looking more specifically to modelling, Blomhoj and Jensen (2007: 47) define competence as insightful readiness to act in response to the challenges of a given situation. According to them, it makes competence headed for action, and is based on, but not identical to, knowledge or skills. Another important aspect, in their view, is that competence development is a continuous process. Within their discussion, they explicitly formulated and exemplified a set of mathematical competencies that could be seen as independent dimensions extending and spanning the definition of mathematical competence. The mathematical competencies set out are: mathematical thinking competence, problem tackling competence, representing competence, symbol formalism competence, communicating competence, aids and tools competence, reasoning competence and modelling competence. Sriraman and Lesh (2006: 248), contend that reasoning in ratios, estimation and mathematical modelling are three most important types of mathematical thinking. Modelling has therefore already established itself as a mathematical competence.

Mathematical modelling competency is the ability to identify relevant questions, variables, relations or assumptions about a real world situation, to translate these into mathematics and to interpret and validate the solution (Niss et al. 2007: 12). These authors further add that modelling competency does not develop in isolation from other mathematical or general competencies such as representation or social competence or performing mathematical procedures. Their suggestion that modelling competencies may involve several domains led to the inclusion of cognitive, meta-cognitive and affective competencies for this study. By performing modelling tasks, modelling competencies and other related competencies in the field of mathematics or general fields are developed.

Modelling activities, according to Lingefjard (2006: 98), should allow students to express their mathematical competence and concurrently develop their competence even further. He is of the opinion that there are a number of competencies that students should develop and that are developed while modelling. These are competencies in doing mathematics, using everyday knowledge, performing the modelling process, validating mathematical models, reflecting on and critiquing models and explaining, describing and communicating mathematical models. Swan, Turner, Yoon and Muller (2007: 275, 276), affirm that modelling activities develop and promote mathematical thinking and learning in a powerful way. They further elaborate that a way of developing mathematical competencies is to establish a strong connection between mathematical knowledge and the real contexts where that knowledge can be used. They maintain that modelling promotes and supports the development of other mathematical competencies. Furthermore, they reveal that the University of Chicago School Mathematics Project is built on the premise that modelling helps develop competencies in both pure and applied mathematics.

Modelling competence, in turn is enhanced by capability in mathematical competencies. Modelling competencies reflect the complexity of mathematical application tasks. Stillman and Galbraith (Galbraith 2007: 84) identified these complexities as being of a conceptual, mathematical, linguistic, intellectual, representational and contextual nature. These complexities in some way reveal the growing complexity of societies' move away from industrial mode of being to informational mode of being (Lesh & Doerr 2003a: 15). Modelling mirrors this in that it requires that people are able to make sense of complex systems and not only perform complex calculations. A number of years ago, Lesh (1981: 250) indicated that the mathematics which is becoming increasingly necessary for informed citizenship does not require elaborate computational skills, and if complex calculations were required, a number of technological tools are available. Dealing with mathematical complexity is more important in modelling than dealing with mathematical calculation, although calculation is eminent.

Lesh (1981: 239) affirms that some of the most effective techniques for facilitating (mathematical) cognitive development focus on broadening and strengthening a students' existing conceptual base rather than accelerating the acquisition of isolated skills. Iversen and Larson (2006: 290) explained modelling as complex thinking using simple mathematics, as opposed to traditional teaching which involves simple thinking using complex mathematics. Lesh, Hoover, Hole, Kelly and Post (2000: 634) explained that modelling problems need not be computationally complex even though they are structurally rich. Niss and Blum (1991: 44) advised that modelling does not imply that mathematical knowledge and insight is less important or obsolete; they insist, that on the contrary, the more widely and extensively mathematics is used, the more genuine mathematical knowledge becomes necessary. Modelling and its related competencies allow students to develop their existing simple mathematical concepts and to develop complex and genuine thinking.

This symbiotic relationship and synergy between mathematical competence and modelling competence, (where growth in modelling competence is dependent on and develops mathematical competence while mathematical competence is dependent on and develops modelling competence) is put forward by Niss et al. (2007: 5) as a duality that exists when answering the question 'why' modelling should play a part in mathematics education. The first category of answers relates to modelling as a potential vehicle for student learning of mathematics and the second category includes arguments that learning mathematics develops competency in applying mathematics and building models. In traditional approaches to mathematics education, it has been accepted that learning mathematics will enable students to apply it to other areas, but Niss et al. (2007: 6) remind us that there is sufficient confirmation from classrooms and research that no automatic transfer takes place between pure theoretical mathematics and situations that need mathematising. Students can only improve their mathematising by being placed in situations where they can mathematise independently and meaningfully as in modelling tasks.

The duality mentioned above also makes sense in terms of Lesh and Doerr's (1998: 363) description of modelling as the interaction between three systems - a student's internal conceptual system, external systems and representational system. They explain that the representational system plays a linking role between externalizing the student's internal conceptual system and internalizing external systems that are experienced in the world or that are artefacts constructed by people. These interactions are brought into play when students construct models. Modelling enables students to bring their mathematical conceptual system into play and necessitates a representation of this system that can be shared with others. Modelling will therefore make student and teacher more aware of the student's internal conceptual system. Doerr (1995: 5) also states that model building provides students with an opportunity to express their own concepts and to learn through the process of representing their concepts. Gravemeijer's definitions of model building are particularly explanatory when discussing student modelling and will be discussed in the next section.

3.2.1 Emergent modelling

Gravemeijer (2002: 1) distinguishes between two forms of modelling – modelling as translation and modelling as organising. Modelling as translating better describes traditional word problems (see 2.3). Modelling as organising is not simply translating problem situations into mathematical expressions, but rather a 'process of mathematisation by which the situation is being structured in terms of mathematical relationships'. Gravemeijer's conception of 'emergent' (Gravemeijer 2002: 1) modelling is fundamental to understanding modelling as a task and a means of learning mathematics even though his conception of model is different to the modelling tasks in this study. The Realistic Mathematics Education (RME) approach to modelling is that concepts are reified through the process of acting and reasoning with a model while the approach in this study is that the mathematical structures or relationships presented in the given situation is captured by the students in their model

(Gravemeijer, Cobb, Bowers & Whitenack 2000: 242). Another distinction is that the model characterised by RME originates from students' current ways of thinking and acting rather than on the given starting point situation. So the RME conception includes student models of addition or subtraction while in this study it is about students' model and modelling of a given complex problem situation.

The RME conception of modelling is further elaborated in terms of a *model of* and a *model for* a situation (Gravemeijer 2002: 2) and is still relevant to these open modelling tasks. The transition occurs when the student moves away from thinking about the model of the specific situation towards a generalisable model for more formal mathematical reasoning. The formulation of a Hypothetical Learning Trajectory is therefore important for the design of the research strategy (see 4.2.2). Gravemeijer et al. (2000: 243) specify four levels of activity in the developmental progression from *model of* to *model for*. Initially the model is embedded in the activity setting, at the next level the model is grounded in the students understanding of the experientially real situation. The third level or general activity level students lose their dependency on the 'situation-specific imagery' and the model 'takes on a life of its own'. At the fourth level of thinking the model 'becomes an entity in its own right'. Terminology used in open modelling situations is 'situation model' which corresponds to the first two activity levels and 'generalisable model' which corresponds to the last two activity levels. A situation model is one that suits the conditions, parameters and structures of the given real problem while a generalisable model can be applicable to other situations. Since the generalisable model reflects a more formal way of mathematical thinking (Gravemeijer 2002: 2), this is considered when group models are assessed in this study (see 3.7 and 5.3.6). This change in student thinking is also in line with a move from horizontal mathematisation (from the real world to mathematics) to vertical mathematisation (working with increasingly more complex mathematics). Furthermore, the notion of *model of* and *model for* requires a research design where the emergence of these is part of the trajectory.

Modelling in this study is taken as an arena of mathematics where complex mathematics is developed using simple existing conceptual systems. To enable interaction of internal, external and representational systems, to facilitate the development of complex structures by simple concepts and to allow for emergent modelling- a specific type of task is necessary; this is known as a model-eliciting task/ thought revealing task or end-in-view task (Lesh, Hoover, Hole, Kelly & Post 2000: 608, English & Lesh: 2003: 298).

3.3 MODEL-ELICITING ACTIVITIES

Lesh, Cramer et al. (2003: 40) aimed at extending Dienes' principles by exploring the possibility of replacing activities involving concrete materials with activities in which the context are grounded in the everyday experiences of students. So, instead of using a series of three or more embodiments for a construct, they replaced the embodiments with experiences called model-eliciting activities. In their view these activities are simulations of real life problems that require a fair amount of time to complete and involve teams working together, often using powerful conceptual technologies.

A modelling approach shifts the focus from finding solutions to creating a system of relationships that can be generalised and reused (Doerr & English 2003: 110). As such, the problems given to students that are modelling problems will be of a different nature to traditional word sums or other problem solving tasks. Lege (2005a: 34) explains that modelling extends the task of problem solving beyond the Polya type as well as heuristic thinking. He reveals that a modelling activity is simultaneously a problem to solve and a circumstance that creates a need for students to learn more mathematics and to apply critical thinking to the situation. Barbosa (2006: 294) establishes the boundaries of a modelling activity as a problem (not an exercise) that has to be extracted from everyday or other sciences. He sees modelling as a learning milieu where students are invited to take a real problem and investigate it via mathematics.

Model-eliciting activities elicit from students a product that is not a short answer, but a product that can be used to describe, explain, manipulate or control a real system that has been converted into a mathematically significant system by the student (Lesh & Doerr 2003a: 3). According to Lesh and Doerr, the processes used to produce an answer are the most important components. They affirm that the process is the product in a model-eliciting activity. An example of a model-eliciting activity is taken from Lesh et al. (2000: 635) called the Sears catalogue problem (see Appendix B), where students are given a ten year old back to school catalogue, a local newspaper dating back ten years and a current newspaper. The goal being that students had to produce a newspaper article telling readers how much money would be needed to have the same buying power today that $200 had ten years earlier. This example serves to illustrate the product orientated process that needs to take place. It is also the type of problem that meets the six instructional design principles for model-eliciting activities.

The six principles of instructional design are provided by Lesh et al. (2000: 608) that enable one to transform or create activities into model-eliciting activities.

- The first principle is the personal meaningfulness principle and relates to the reality of the situation. The task must engage student's use of their own personal experiences or knowledge.
- The second principle – the model construction principle – regards the need for students to develop, construct and refine a model.
- The third principle requires that students are able to evaluate themselves; this means they need to know why and for whom they are constructing the model and if it is good enough.
- The fourth principle, the model-externalisation principle requires that students explicitly reveal their thinking about a situation.
- The fifth principle, the simple prototype principle for designing model-eliciting activities means that the problem situation is as simple as possible while still creating the need to develop a model.
- The last principle is the model generalization principle and relates to how adaptable the model will be to other situations.

Although these principles enable one to determine the extent to which a task is a modelling problem, they are further useful in assessing students' modelling competencies if they are re-worded for this purpose (see 3.7). With these principles as criteria for task design, it is understandable that these model-eliciting tasks take time to design and that many textbook problems or problem solving activities do not meet these criteria. A model-eliciting perspective is based on the premise that research should bear in mind findings from psychological concept development so that activities that are developed motivate students and allow them to develop the type of mathematics needed for these situations in a natural way (Mousoulides, Sriraman & Christou 2007: 26). This also brings Lesh et al. (2000: 608) personal meaningfulness principle for model-eliciting tasks to mind.

Model-eliciting or thought revealing activities are useful for assessment since the products that students generate reveal significant information about students' ways of thinking (Lesh et al. 2000: 593). They further add that since these activities also produce documentation that goes beyond the final results, important information about the process that contributed to the final result can be elicited. Not only do these activities assist in the documentation of development, they promote development (Lesh et al. 2000: 594). The authors admit that model-eliciting tasks developed out of a need to have students produce ways of thinking to advance practical research. In constructing models, students reveal much (on route and after the task) about the ways of thinking that created the models.

Furthermore, as explained by Lesh and Doerr (2003a: 5), model-eliciting activities can be designed to lead to significant forms of learning since these activities involve mathematising. They also explain that the conceptual systems that students develop are mathematically significant sense making systems and involve situated versions of powerful and deep elementary constructs that form the foundations of elementary reasoning. Since elementary reasoning peaks in proportional reasoning, the tasks used for this study involve proportional reasoning (see 3.3.1).

English and Lesh (2003: 298) also refer to modelling tasks as 'end-in-view' problems. This means these that both the products and the processes are different to traditional problems. A model-eliciting task or end in view problem makes it necessary for students to develop a model, to do this they need to make several turns of the modelling cycle (see Fig 1.1). At each point or node of the modelling cycle and between points of the cycle, certain competencies are necessary. These are called modelling competencies and are discussed in section 3.5. Modelling tasks enable students to display and develop modelling competencies. These tasks bring with them a nucleus of significant consequences and products to the teaching and learning of mathematics.

3.3.1 Proportional reasoning in model-eliciting tasks

It was decided that a specific section of mathematics would be selected for the mathematical context of the tasks for this study. The area of proportional reasoning was preferred as it would lend itself to being approached from various aspects and mathematical levels by students. Their proportional reasoning would be a result of their general mathematics experiences and not a result of a specific section of work taught at any stage.

The range of features and applications of proportional reasoning are fundamental tools that students should attain from middle grade instruction (Ben-Chaim, Fey, Fitzgerald, Benedetto & Miller 1997: 6), while McLaughlin (2003: 1) states that Piaget considered the ability to reason proportionally to be a primary indicator of formal operational thought. Proportional reasoning means that students can compare ratios or infer relationships between ratios. The ability to understand inversions and reciprocities stems from an understanding of proportional reasoning (Inhelder & Piaget 1958: 175). Proportional reasoning deals with mathematical relationships that are multiplicative in nature and not additive in nature (Ilany, Keret & Ben-Chaim 2004: 81). This makes proportional reasoning a very important part of mathematical learning. However as stated by Steinthorsdottir (2005: 225) students are slow to achieve mastery of these ideas and a large portion of society never acquires fluency in proportional thinking (Hoffer 1998: 285).

Van de Walle (2007: 355) indicated that students may need as much as three year's worth of experiences in multiplicative situations to develop proportional reasoning abilities while the premature use of rule application hampers this development. Lesh et al. (1988: 93) proposed that conceptual development in the area of proportional reasoning is characterized by a gradual increase in local competence and not by acquiring general all purpose strategies. They also view proportional reasoning as a pivotal concept in mathematics. According to them it is the 'capstone' of elementary school mathematics and the 'cornerstone' of all that follows. They further add that it is both an elementary higher order understanding and the highest levels of elementary understanding (Lesh, Post & Behr 1988: 96). Since proportional reasoning deals with the most common forms of structural similarity, these authors link proportional reasoning to some of the most important elementary but deep concepts at the foundational level of many areas of science and mathematics.

Inhelder and Piaget (1958: 314) verified through numerous experiments that proportional reasoning does not develop before substage III-A (between the ages of 11-12 and 14-15). They looked for an explanation for the late acquisition of proportional reasoning. They explained that the proportional schema has two aspects, one logical and one mathematical and that those logical proportions are derived from structures of inversions, reciprocities and lattices. These are acquisitions specific to formal operations (stage III). Lesh and Harel (2003: 161) summarized Inhelder and Piaget's stages of development for proportional reasoning. First stage reasoning is based on qualitative judgments and does not take all information into account. The second stage of development reasoning tends to be additive rather than multiplicative. Stage three students begin to reason about relationships between relations based on patterns and replications. They often use a build up strategy or tables but they may not be using reversibility of operations yet. The fourth stage is based on true multiplicative proportions.

Stenthorsdottir (2005: 226) elaborated on the multiplicative relationships ‘within’ and ‘between’ ratios in a proportional setting. The former being a relationship between elements in the same ratio and the latter being a relationship between corresponding parts of the two ratios. Stenthorsdottir in conjunction with other authors identified four levels of proportional reasoning. At the first level, students showed limited ratio knowledge; at the second level students considered a ratio as an indivisible unit and used repeated addition of the same ratio or multiplied the ratio by a whole number. They were unable to solve problems where the target ratio is a non-integer multiple of the original ratio. At the third level students could scale ratios by non-integers since students can think of the ratio as a reducible unit. Students combine reduction strategies and build up strategies for the ratio. At level four, students recognise a relation within the terms of each ratio and between corresponding terms of the ratio (Stenthorsdottir 2005: 228).

Lamon (1993: 46) used six levels to describe proportional reasoning in Grade six students. These were split into two strategy levels; non-constructive (avoiding, additive, pattern building) and constructive (pre-proportional, qualitative proportional and quantitative proportional). She also described relative thinking (Lamon 1993: 58) as a critical factor that separated additive reasoning from multiplicative reasoning, while using a ratio as a unit is a significant process that developed in sophistication. She considers it vital to provide students with problem situations where they could see the need for relative comparisons. She suggested that students be given time to explore multiplicative situations and to co-ordinate their additive and relative perspectives before going on to traditional symbolism (Lamon 1993: 60). Lo and Watanabe (1997: 234) suggested that the question as to what kinds of experiences would help students develop more sophisticated understandings of proportion needs to be answered. It appears that modelling may provide a suitable didactical advantage over traditional teaching approaches in the learning of proportional thinking.

Modelling tasks provide students with an informal way of using their concepts of proportion and ratio. Within a very real context these concepts are developed out of a need to solve a problem. Very often the ratios are not obvious and are not presented in neatly packaged fractions to be reduced or multiplied. The ratios are implicit within the task and students working at level two of the above discussion can make sense of the problem. The tasks selected for the study are based on proportional reasoning. This includes concepts such as finding quantities that are part of a whole, finding equality of ratios, considering relationships between quantities, considering relative size and relative importance. Furthermore, the students in this study are entering formal operational stage thinking not only as a result of age (see Table 5.6) but as a result of educational experience.

3.4 MODELLING AND MATHEMATICAL THINKING

The work of Doerr (1995: 13), revealed four major themes in student model building. Two being directly relevant to this study, the first that students pursued problems with far more diversity than the problem itself at first may have suggested, and second being that student models became progressively more complex as they worked through the problem. She further adds that student model building will foster a far greater diversity, creativity and richness of responses than can be found in traditional mathematics classrooms. Doerr argued that model building, as opposed to model using, allow students to express, represent and re-represent their own concepts. Therefore, they learn by defining relationships between concepts and representing the consequences of these relationships (Doerr 1995: 5).

Lithner's study (2008: 267) about mathematical reasoning used Haylock’s description of creative mathematical thinking as a twofold term. Firstly, it relates to a thinking process that is divergent and secondly to a product that is perceived as creative. The type of thinking that is generated by modelling activities is such that creative mathematical thinking as process and product is engendered. Often, the type of thinking that sets a defining moment in a group’s path to a model solution is of a very creative and divergent nature. The type of creative thinking described by Lithner is fostered by groups working on model-eliciting activities.

Galbraith (2007: 84) points out that in many countries, students omit answering application questions, this reveals an inadequacy in the confidence and competence with which students approach contextual problems. Vom Hofe, Kleine, Blum and Pekrun (2005: 62) remark that serious problems in mathematical development are caused by insufficient growth in modelling competencies and mental models during secondary school. They provide further evidence that insufficient modelling competencies are caused by deficits in the generation of these mental models (referred to as Grundvorstellungen) that carry the meaning of mathematical concepts.

It is important to place modelling in a judicious place in mathematics education. The sentiment of Lesh, Cramer et al. (2003: 42) is sound. They state that it would be foolish to abandon fundamentals. What is needed is a sensible mix of complexity in the form of applied problems and fundamentals, both must evolve in parallel and one does not come before or without the other. Earlier Lesh (1981: 245) admitted that not every idea must be taught through applications. He felt it was more beneficial that application be taught to develop breadth of understanding for a handful of conceptual models that are the most basic from which a maximum number of concepts can be derived and that can be applied to a maximum number of situations. What is important is that both applied problems and learning of basics take place in mathematics classrooms.

The implication for schools is that modelling situations will naturally begin to engage students in the phases of a modelling cycle (Sriraman & Lesh 2006: 253) to produce confident thinking that is complex, diverse and creative. These valuable dimensions of thinking emerge through the development of modelling competencies. The nature of student engagement with modelling tasks, the development of competencies and the sequence of the development lies at the centre of this study

3.5 MODELLING COMPETENCIES

3.5.1 Introduction

Adding to the discussion in 3.2, it is necessary to define modelling competencies in general and specifically those that will be focused on in this study. Henning and Keune (2007: 225) use Weinert's definition of competence as being the sum of available abilities and skills; the willingness of a student to solve problems and acting responsibly concerning the solution. According to Dark (2003: 282), competency as a definition in the workplace includes both ability and motivation. Furthermore, she adds that the ability to perform with continuous adaptation and dynamism lies at the heart of the skilfulness implied by competency. These definitions led the way to include meta-cognitive and affective competencies to the list of competencies for this study.

According to Zbiek and Conner (2006: 91), modelling must be conceptualized so that it allows one to explain how learning occurs. At each point of the modelling cycle, certain competencies are necessary for effective modelling. These competencies are often observable when students work in groups verbalising and representing their thinking. It is now necessary to discuss modelling competencies as they are the gateway to understanding student learning as it relates to mathematical modelling. Maaß (2006: 117) includes in her definition of modelling competence, skills and abilities to perform the modelling process as well as a willingness to put these into action. Since modelling competency covers broad areas, it is therefore considered appropriate to discuss competencies under three sections: cognitive competencies; affective competencies and meta-cognitive competencies.

3.5.2. Cognitive competencies

Cognitive competencies are those identified in the literature as occurring at each node of the modelling cycle. There are some variations, but many are consistently described by various authors. The competencies of Kaiser and Blum (Maaß 2006: 116) are used in this study since they are 'based on theoretical considerations'. They contain a detailed listing of the sub-competencies and they enable one to verify student progress through the modelling cycle. The cognitive competencies in their view are;

- Competencies to understand the real problem, which include sub-competencies of making assumptions, recognizing variables, constructing relations between variables and distinguishing between relevant and irrelevant information.
- Competencies to set up the mathematical model, which includes sub-competencies of mathematising, simplifying and making suitable representation of the situation. For the purposes of this study, mathematising is seen a separate competence as it occupies a crucial phase between the real world and setting up of the model. (The role of representation is also discussed more fully in 3.10 as a significant contributor to student competencies).
- Competencies to solve the mathematical questions within the model, including sub-competencies of using heuristics and mathematical knowledge.

- Competencies to interpret the results in the real situation, which include sub-competencies of generalizing solutions and viewing solutions using appropriate mathematical language and to communicate the solutions.
- Competencies to validate the solution, taking into account sub-competencies of critically checking and reflecting their solutions, reviewing, reflecting other possible solutions and to generally question the model.

The following eight are therefore taken as cognitive competencies and will be used in this study; understanding, simplifying, mathematising, working mathematically, interpreting, validating, presenting and arguing.

The theories exposed in section 2.9 present the development of cognition as a constructive, active, distributed and social endeavour. Modelling presents activities for students that take these views on learning into account. The cognitive competencies that are needed for modelling and that develop as a result of modelling reflect and reiterate these views.

Although the above are cognitive abilities that are needed to solve a modelling task, they do not work in isolation from affective competencies or meta-cognitive competencies. Maaß (2006: 117) indicated that meta-cognitive issues, beliefs and what she termed a 'sense of direction' (2006: 137) from Treilibs' (1979) work, are all important for the development of modelling competencies. Beliefs and meta-cognitive dimensions are discussed in 3.5.3 and 3.5.4.

3.5.2.1 Characterising cognitive competencies

It is important to establish a common understanding of some of the competencies that this study focuses on. These words are general and broad and some delineation must take place to set them in the modelling context. Each specified competency can further be detailed and described to fit the specific task formulation. These competencies allow themselves to be studied from a qualitative view. It is however, important to see these competencies not as static entities that students either have or do not have, but to understand their supple nature. As suggested by Sternberg (1998: 137), abilities are not static or fixed forms of developing expertise. Student action, by way of interaction with others and written work will allow a glimpse into the development of these cognitive competencies.

i) Understanding

Pirie (Pirie & Kieran 1994: 165) considered understanding as an entire dynamic and active process. Understanding means to know the nature of something or assume information that is implied. One may characterize understanding as narrow or simple. Narrow understanding in one situation may be characterized as more complex in another. Understanding can only be determined in conjunction with context and experience. Limited understanding may be characterized by students using only a few features of the task. Better understanding may include being able to make a generalization of the problem features. Understanding context or understanding the essence of the task may be better indicators of understanding. Understanding in modelling also means understanding the product that needs to be produced for the client. Various products are expected and understanding the features and requirements of a report, letter, toolkit etc are also important. Being able to understand the link between the specific model created for the task and the general model as an extension of the task is also important. Problem understanding may come before or after problem simplification and may be affected by an intuitive identification of needing to use proportional reasoning.

ii) Simplifying

According to Mousoulides et al. (2008: 298), problem specification or simplification includes students identifying a number of assumptions. This helps students to determine which information is relevant or irrelevant. Students may need to revise their simplifying attempts when they reach contradictions or inconsistencies in their initial models. Simplifying entails extracting the essential features of the problem. Students simplify the problem to make it easier to work with. This would also mean using a significant sample of the data suggested by the task. Selecting which information given in the task to include and which to exclude as well as the reasons for this are significant factors in simplifying a problem.

iii) Mathematising

Treffers (1987: 247) defines mathematising as an 'organising and structuring activity in which acquired knowledge and abilities are called upon in order to discover still unknown regularities, connections, structures'. It is set midway in the modelling cycle (see Fig 1.1) so it stands between the real world and mathematical world. Mathematising is translating from the real world to the mathematical world. Detecting features in the real world that corresponds to mathematical concepts. 'Students would identify variables and relationships within the mathematical world with connected variables and relationships with prior conditions and assumptions' (Mousoulides et al. 2008: 299). Significant mathematising would occur in tasks that are contextually suited to the student. It is within an authentic, real context that mathematical concepts can be identified and developed fully. In the realm of mathematising one would have to take into account the features of the tasks. How do students deal with problems that have no numbers (Task 1: Big foot)? How do they quantify the 'ideas' in these problems? On the other hand some tasks seem to have too many numbers to deal with at once (Task 2: Catalogue Problem). The presence or absence of numbers does not preclude mathematisation.

Mathematisation refers to the structuring of reality using mathematical ideas and concepts (Bodin &Villani 2001: 10). Treffers (1987: 247) defined horizontal and vertical mathematisation; horizontal mathematisation involves translating from a real situation into a mathematical one, while vertical mathematisation involves working within mathematics and using 'mathematical tools' (Bodin &Villani 2001: 10). It is envisaged that modelling tasks will springboard students into their own mathematical world, thereby allowing two expositions. The students' own mathematical world to an observer and to himself. It is further hoped that this interaction between real world and mathematical world will strengthen the bond between the two and encourage genuine understanding of mathematics.

According to Treffers (1993: 104) children do not automatically mathematise a problem. This is due to a number of factors that are necessary: personal reality, mathematical knowledge and context of the problem. He further states that 'mathematics can be developed from (personal) reality in natural manner: the formal rules and procedures can be derived from the informal working methods of the pupils' (Treffers 1993: 105). Dapeuto and Parenti (1999: 5) point out that modelling activities which include mathematisation can "involve assumptions, intentions, perceptions and intuitions which interlace mathematics' and other disciplines' aspects".

In an attempt to determine the difficulty level of each phase of the modelling cycle, Voskoglou (2007: 156) determined through a stochastic model of the modelling process that the stage of mathematising has the greatest 'gravity' for students. This study is in accord with their result so the suggestion of Blum and Leiß (2007: 222) of providing multiple and diverse opportunities for students to acquire modelling competencies is strongly accepted.

iv) Working mathematically

This refers to the ease with which the chosen mathematics is selected, applied and used. One must also pay attention to the 'type' of mathematics selected to solve the task and how it is employed. Some students will use simple mathematics in new and complex ways while other may try complex mathematics in superficial and routine ways. Since students do work with technological tools as in 'distributed cognition' (see 2.9.5), the focus is not on traditional computing accuracy, but on student understanding of the underlying features of the mathematics they have chosen to use. Activities such as: representing a relationship by means of a formula, proving regularities, refining or adjusting models, combining and integrating models and generalizing (Bodin & Villani 2001: 10) lie in this arena. Da Puerto and Parenti (1999: 3) use the term 'processing' a model to indicate that which occurs when the choice of elements in reality is associated with elements of mathematics (mathematisation).

v) Interpreting

Borromeo Ferri (2006: 91) gives Kaiser's description of the normative modelling process and explains interpreting as mathematical results that must be reinterpreted in the real situation. This entails taking the mathematical aspect of the model and re-evaluating it in terms of the real world problem. Does the model work for the real world situation that it was created for? Students may also start taking the generalizable aspects of the model into account. How can the model be adapted for other situations?

This means that a reverse mathematisation takes place. The mathematics that they have identified and worked with now has to be re-evaluated in terms of the real problem. Borromeo Ferri (2006: 93) describes interpretation as the transition from mathematical results to real results and confirms that this is often not done consciously by students. From the students' point it may mean that an initial estimate was made and should be checked.

The question in the task often assists students in seeing the need to interpret their work in terms of the real situation. Often when students are referred back to what the tasks required of them they will verify solutions or at least question them. Monaghan (2007: 65) found that students transformed the modelling task into connected but different tasks and did not end up solving the original task. Interpreting has a more advanced function that just understanding the answer in terms of the task but assumes a reading between the lines of both the problem and mathematical solution.

vi) Validating or Verifying

Caron and Belair (2007: 127) suggest that on a global level, validation depends on how well students understand the real life situation and the reason for the model. So here we see the strong link between understanding and validating. Verifying also involves checking the solution. It may mean that an initial estimate was made that must be justified. Verifying also entails comparing an estimated solution to the reached one as well as evaluating how well the computed solutions fits the problem situation as well as checking computation accuracy (Uprichard, Phillips & Soriana 1984: 83). Students may also verify that the model behaves in a consistent way. Here they are working on the mathematical aspects of the model.

Students must see the value in validating what they have done and validating their model. Students from a traditional classroom rarely spend too much time verifying their solutions since this has always been in the teacher's realm. Verifying of answers mean that students should see the link between what they do now and what would transpire later. Students may also verify that the model behaves in a consistent way. Not only should students ask if the model works for the real world situation that it was created for, but also whether it will work under different conditions. Students validate in their endeavour to make a model generalisable.

When students start considering the generalisable aspects of the model, they ask how the model can be adapted for other situations. Students with higher levels of validating competency will take into account the possibility of other and better solutions. Borromeo Ferri (2006: 93) distinguishes two ways of validating. Intuitive validation (more unconscious) and knowledge-based validation (more conscious), where knowledge base refers to students 'extra-mathematical knowledge'. Student's use of informal knowledge is discussed in section 3.5.4.

vii) Presenting

This competency involves communication, the interchange of ideas, information and instructions about the mathematical model (Mousoulides et al. 2008: 299). Students will need to refer to the trail of documentation they created. Students will have to consider a selection of their working to share with others. In communicating they will need to bring fellow students and the teacher into the dynamic domain of their group interaction. They will need to have considered other possible solutions and be able to justify their model. As this phase of the modelling process will shed light on many of the other competencies, much time is spent on this during the contact sessions in the design experiment. The Quality Presentation Assurance Guide (Lesh &Clarke: 2000: 145) is used later in this study to establish levels of presentation competency.

viii) Arguing

According to Douek (1999: 91) argumentation includes verbal argument, numerical data or drawings; and argumentation will consist of arguments connected by deduction, induction or analogy. Modelling requires skills of argumentation, and provides a germinating ground for the development of these skills. In a study by Veerman, Andriessen and Kanselaar (2002: 155) where it was envisaged to find principles for the design of educational tasks that provoke collaborative argumentation, the authors concluded that students are able to argue only in situations that provided them with some form of responsibility and if the goal was useful – such as developing a common product (Veerman et al. 2002: 183). They also found that triggering argumentation was not the main issue, but more importantly, the creation of environments in which it is rewarding to have meaningful arguments. What these authors have described is the basis of a problem centred approach and specifically modelling tasks. However, Schwarz and Linchevski (2007: 511) found that although argumentation is recognized in studies as responsible for learning gains in post hoc analysis, it is not treated as an independent variable. Their suggestion is to invite students in groups to discuss issues or to accommodate contradictory points of view. Modelling allows a framework of challenging discussion for students to develop arguing competency.

Argumentation can be seen as a competency on its own. It is the ability to reason from a starting point to an ending point, or to reason through a part of this. It involves the logical progression from one idea to another idea which will lead to higher levels of thinking. Argumentation, in this study is not necessarily the ability to convince another person of a particular way of thinking. Arguing is often referred to as reasoning where reasoning entails connecting thoughts to explain something to oneself or to others.

The process of arguing and not a finished product (an argument) is the focus of this competence. It is reasoning that is communicated to convince or explain. Student competency in arguing is in direct relation to their understanding of the task and their mathematising of the task. Once students have fully engaged with the task they are able to argue their understanding, their mathematisation, their sense of direction and their route forward to the rest of the group. Arguing becomes a major component when students are allowed to work in small groups.

ix) Summary

These cognitive competencies are necessary when students solve a model-eliciting task. Their manifestation is largely task and context orientated. They will only become more generalisable and transferable through students being extensively involved in modelling tasks within different contexts and experiences. Uprichard, Phillips and Soriano's (1984: 83) stance is that learning is an internal process affected by internal and external factors and that a student solves a problem by creating new cognitive pathways to reach their goal. Modelling allows students to display and develop a full repertoire of cognitive competencies.

3.5.3 Affective competencies

Blomhoj and Jensen (2007: 49) describe a key to modelling as learning to cope with feelings of 'perplexity due to too many roads to take and no compass given'. This type of perplexity is not easily overcome. It relates to student affective competencies: their beliefs about mathematics, the nature of tasks, how problems are solved and the value of mathematics in solving real problems. Mostly, these beliefs are directly related to experiences students have had in the mathematics classroom. As such, they develop anti-positive beliefs about mathematics. The term anti-positive is used as opposed to 'negative' since students do not harbour especially negative feelings that they are necessarily aware of, but their beliefs do not foster their modelling abilities.

Mandler's view on affective factors (McLeod 1992: 578) is that student emotions play a role when they are unable to carry out a schema. This according to McLeod leads to three aspects of affective experience. Firstly, students do hold certain beliefs about mathematics themselves, about how mathematics should be taught and about social context in which mathematical learning takes place (often solitary and competitive). Secondly, that interruptions and blockages are an inevitable part of learning mathematics and students will experience both positive and negative emotions – more noticeably during tasks that are novel. Thirdly, students will develop a positive or negative attitude based on a repetition of certain emotional responses. It is envisaged that since students are working on novel tasks that can be solved with basic mathematical knowledge and skills and they are working within a group environment, these blockages will be overcome or resolved. Students will encounter a positive emotional response that will lead to a more positive attitude to mathematics (see 5.2.1).

Maaß (2006: 138) found that there was a close connection between positive attitudes towards modelling or mathematics and corresponding student performance. She categorised four types of modellers based on their attitudes. A *reality distant* modeller she found had a positive attitude towards context free mathematics but not to real world contexts. *Mathematics distant modellers* preferred real world contexts but showed negative attitudes towards mathematics and low performance in mathematics lessons. The *reflective modeller* has positive attitudes towards mathematics and modelling. Maaß found these students did not show deficits in modelling competencies. The *uninterested modeller*, was neither interested in contextual mathematics or mathematics itself. There were deficits in modelling competencies.

These distinctions in attitudes are important to this study as the traditionally weaker students in the groups may show characteristics of the distant and uninterested categories. It will be of interest to take into consideration the views in 3.11 regarding traditionally weak students. Interest and motivation may be a deciding factor in modelling success. It is therefore imperative that the choice of tasks address this. It will be significant to describe the modelling processes and characteristics of traditionally weak students' more fully so that modelling practice in classrooms may be justified, guided and improved.

Deci (Carter 2004: 4) stated that a 'child's curiosity is an astonishing source of energy'. Interest and motivation are important factors in student engagement with a learning activity. Sternberg (1998: 130) discussed Mayer's term 'meta-skill'. Developing meta-skill involved developing student individual interest and involved teachers developing situational interest that would motivate students to think about their problem solving practices. Interest and motivation are closely linked. Stimulating interest will result in motivation to engage in the task. Very often the fact that the task product is so different to traditional exercises provides motivation in itself.

Zbiek and Conner (2006: 105) extended their discussion around modelling and motivation. They propose that modelling supports three different types of motivation. Firstly, that exploring a real world situation was appealing to students as was associating mathematics to the real world. Secondly, that students may realise the complexity of the real world and believe that mathematics may be a valuable tool to deal with the complexity. Together with this, when students interpret their solution, they may confirm that mathematics is applicable to the real world. A third type of motivation was a motivation to learn new mathematics. The student may find this he/she needs new mathematical knowledge and ideas to meet the demands of the task. It is important to realise that cognitive competencies are at the heart of students developing these forms of motivation. Their motivation further drives the development of cognitive competencies in modelling.

Model-eliciting activities, which have been designed around the six principles discussed in 3.3, foster motivation, which becomes a gateway for students to develop cognitive and meta-cognitive competencies. Lingefjard and Holmquist (2005: 131) found clear evidence that attitude (among other aspects) has a major impact on how students develop modelling competencies over time.

This study supports the view that motivation and interest are significant factors in modelling, but since they are implicit factors in competency development they are difficult to observe or assess. It was decided that student beliefs would be a central affective competency focussed on for this study.

3.5.4 Meta-cognitive competencies

The view of Lesh, Lester and Hjalmarson (2003: 383) that meta-cognitive competencies are situated in certain contexts and will differ from problem to problem is accepted for this study (see 2.7.2). Sternberg (1998: 131) also found that meta-cognition is content affected as did Tanner and Jones (2002: 146). This seems to be especially true at the beginning of a modelling problem. Lingefjard and Holmquist (2005: 123) described student meta-cognition as adopting an approach to learning. They added that the approaches students employ vary with the context of the problem. It is therefore appropriate to consider meta-cognition as changing from task to task. The categories of meta-cognition are taken from Masui and De Corte (1999: 519) to discuss meta-cognitive modelling competencies (see 2.7.2).

Student use of their informal knowledge (Mousoulides et al. 2007: 39) about contexts to shed light on the task is one competency. This means that the orienting aspect of meta-cognition will be different from one problem to another depending on the context of the problem. Students may orientate very quickly to the problem situation, or may need to be guided through the nuances of a situation that they are not too familiar with. If model-eliciting design guidelines are adhered to the reality context should be familiar to students.

Lesh and Zawojewski (1988: 54) explain that managerial decisions are needed to solve problems effectively and that these involve planning, monitoring and assessing the solution process as a whole. The need for students to plan their route through the problem is heightened in model-eliciting tasks. It may initially take the form of following any idea that seems plausible and returning to a new idea when the first one fails. It is more efficient if the student is able to adjust the plan as it unfolds. The more planning that takes place, the less adjusting students may need do. The level of student planning and adjusting is also closely linked to the context of the problem and student experience in problem solving and modelling. This ability to adjust is not likely to be embraced by students who are exposed to a traditional setting.

Students, who plan and adjust as they work, will find that they monitor the process more effectively than students who employ a ‘follow any route’ approach. Process monitoring is likely to be fostered in group situations that modelling requires, since the opinions of other group members are verbalized and discussed. This verbalization of the group’s opinions and thoughts sets a very good basis for students to develop their own meta-cognitive abilities.

Gray (Tanner & Jones 1994: 415) identified three broad areas of meta-cognition which support student modelling. These are planning, monitoring and evaluating. These do not take place sequentially; students are continually involved in all three activities and interactions between the activities. Even up to the last phase, when presenting their solution, these meta-cognitive aspects are necessary. Tanner and Jones (1994: 422) used a ‘stop-start-go’ approach to modelling tasks to guide students into planning, monitoring and evaluating: Silent reading of the problem, followed by small group discussion which led to a whole class brainstorming before returning to small groups. They added reporting back sessions at stages, this assisted groups to monitor their progress in anticipation of having to report back. Students were also requested, at their final presentation, to discuss what they would have done differently if they were to re-do the investigation. This focused their attention to the specific route they followed and allowed for explicit reflection to take place.

Proust (2007: 278) defines prediction as the essence of cognition as a mental function and saw a close association between meta-cognitive engagements that were ‘predictive’ or ‘retrodictive’. Retrodicitive action entails judging a situation after the fact. Prediction relies on this feedback from previous activities. This means that students predictive or retrodictive abilities can be developed through experience with modelling situations. By allowing students to present their products as the final phase in modelling allows them to develop the ability to evaluate their own and other products which develops predictive and retrodictive actions.

Curcio and Artzt (1997: 129) included two aspects in their assessment instrument for assessing student statistical problem solving in small groups. These aspects they considered to be neither cognitive nor meta-cognitive, but were regarded as important to assessing student in group situations. Watching and listening (Artzt &Armour-Thomas 1992: 141) was one aspect. This can be seen as a sign that students are interested in the problem and that they are gathering information in a need to be involved in either cognitive competency or meta-cognitive competency. Watching and listening, spans all three competence categories. In a similar way, they took note of 'off task behaviour' that may indicate student inability to take part in either cognitive or meta-cognitive activities of the group. This sort of group behaviour is best gauged by observation of groups in practice.

The term a 'sense of direction' was coined in the work of Treilibs et al. (1980: 52) where the authors characterised good modellers. They explained a 'sense of direction' as the ability to 'foresee the underlying structure of the solution method'. This, as they explained, was in contrast to poor modellers who generated variables and results randomly. Furthermore according to the authors this direction also entails the ability of students to stop the modelling process to validate their work. Poor modellers on the contrary, were reluctant to stop discussing the empirical situation and often ran out of time. Closely tied to this the authors stated that good modellers could 'partition' their treatment of the problem. This means that they can 'consider a section of the solution, complete it and not return to it other than to retrieve results'. This term 'sense of direction' is very valuable in describing modelling endeavours as it captures underlying features of student work. This is especially true when asking 'why' students do not have a sense of direction for a particular problem.

As stated by Sternberg (1998: 128), meta-cognition is diverse and interacts with many other aspects of students such as their abilities, personalities and learning styles. In this study, meta-cognitive competencies are used to inform overall modelling competencies and to inform the development of these competencies. The following three meta-cognitive aspects: use of informal knowledge, planning and monitoring, and a sense of direction are used as competencies in this study as they appear to be more critical. The design of the observation instruments is to focus researcher observation towards these competencies and to note anecdotal information.

3.5.5 Summary

Many competencies play a role in students being able to model successfully. The competencies that develop and are developed by modelling are numerous, complex and integrated and form ideal aims of mathematics education. It is not possible to cover all these in one study. It was therefore decided to focus on the following competencies and to document their development through a teaching experiment as laid out in Chapter 1 and 4. They are: cognitive competencies (understanding, simplifying, mathematising, working mathematically, verifying, interpreting, presenting and arguing), meta-cognitive competencies (using informal knowledge, planning and monitoring, a sense of direction) and affective competencies (beliefs about mathematics).

3.6 APPROACHES TO MEASURING MODELLING COMPETENCIES

Measuring mathematical modelling is a much more complicated task than anticipated before teaching of modelling is attempted as it involves not only the solution, but logical reasoning, linguistic competency, previous knowledge of the student and their attitudes (Lingefjard & Holmquist 2005: 123). Lesh and Harel (2003: 157), describe the similarities and differences between modelling cycles that students go through during a 60-90 minute activity and the stages of development that students go through during the natural development of constructs and the general stages of development that developmental psychologists have observed over several years.

Modelling competencies have been measured in studies using multiple choice questions (Kaiser 2007: 116). This study found that modelling competencies can be developed through modelling courses, but Kaiser remarks that testing sub-competencies do not test competencies which are necessary for a holistic conducting of the modelling process. There is a need to mesh the development of part competencies and holistic competencies. In another test-based study by Mousoulides et al. (2008: 297) a test was administered three times during the course of a six week modelling course. The results showed that the impact of the program on student modelling abilities was significant.

Clatworthy (1989: 62) developed an assessment rubric that was used in a modelling course to assess modelling competence and provided each student with a feedback sheet which assisted in developing positive attitudes. They moved away from a single assessment towards a dynamic growth model. This enabled them to 'track' a student and assess overall trends. They concluded that the development of reliable methods for assessing modelling remains a challenge. The move away from single examination type assessment towards rubric/ dynamic models is possible and desirable for the purpose of research. What will be more significant is the development of assessment protocols for modelling in the classroom situation that teachers can use.

Chamberlain and Moon (2005: 47) suggested the use of a 'Quality Assurance Guide' (Lesh & Clarke 2000: 145) and a 'Ways of Thinking' sheet to identify creatively gifted students using model-eliciting activities as a tool. The 'Quality Assurance Guide' is a rubric used to rate a student's product in responding to a model-eliciting activity, while the 'Ways of Thinking' sheet focused the researcher on describing student strategies, mathematical concepts used and effectiveness of student creativity. Lesh, Zawojewski et al. (2003: 212) point out that thought revealing activities (model-eliciting activities) for teachers often involve developing tools that teachers can use to make sense of student work. These tools include, observation forms, ways of thinking sheets, quality assessment guides and guidelines for conducting mock interviews based on abilities valued by employers.

In a study by English and Fox (2005: 325) a tool was developed for a single modelling problem to describe student modelling. The tool (in the form of a table) examined the nature of the factors the students considered for that problem, the operations they applied and the representations they used. This table assisted the researchers in their analysis of student model building. It would be interesting to document the use of the tool over several tasks so that the development of student model building could be examined.

English (2007: 279) addressed the cycles of mathematical development displayed by a group of students in a study at primary school level. She used a purely qualitative approach, transcribed audio and video tapes, and found that students elicited key mathematical ideas and processes from the problem as they worked through to model construction. Other methods of modelling assessment include the use of posters (Berry & Nyman 1998: 103) and action maps (Jurdak 2004: 68) that have proved successful.

Jensen (2007: 143) suggests a multidimensional approach to assessing mathematical modelling competencies. Three dimensions of competency were explored: the degree of coverage including the degree of autonomy with which this takes place; the radius of action, being the contexts and situations in which someone can achieve competency and the technical level that points towards the mathematics that the student uses and integrates into the activity. Jensen adds that these three dimensions provide vocabulary for discussing quality in performance and as such offer a more valid but less reliable alternative to mark-schemes (Jensen 2007: 147).

For the purposes of this study, it was decided to aim for qualitative data of the highest quality. Three methods of assessing modelling competencies would be documented. Firstly individual modelling competencies will be assessed and monitored using instruments that are descriptive and investigative in nature and that would do justice to the complexity of a modelling situation. There is however also a need to view student modelling from a holistic point of view. This entails combining or clustering certain competencies that allow one a new view on student modelling and to examine student modelling competencies from a different angle. It was therefore decided to include two approaches that allow a comprehensive view of modelling and approaches that provide vocabulary for discussing group modelling progress. Using the six instructional design principles for modelling provided a sound framework to view group modelling holistically while keeping the important aims of modelling in mind (see 3.7). The second holistic approach used in the study is Jensen's multidimensional approach (see 3.8). The multidimensional approach also allows one to compare modelling over a series of tasks. These two approaches are used in the analyses of the data later in the study (see 5.3 and 5.4) and will be discussed in the following sections.

3.7 THE SIX INSTRUCTIONAL DESIGN PRINCIPLES OF MODELLING FOR ASSESSING MODELLING

It will now be shown that the six principles can be reworded so that they become assessment criteria and assist in measuring milestones in group competency development. As discussed earlier (see 3.3), the principles for designing model-eliciting tasks can be used to cluster the competencies that are at the heart of the study. This means that each principle or question assimilates a number of individual cognitive, meta-cognitive or affective competencies. The principles in their original form are questions relating to the task design. Here they are changed to determine group competencies and performance when modelling especially in terms of emergent modelling. In this way the link between instructional principles for modelling and assessing modelling is also made. With this link made, the teaching and learning of modelling becomes a coherent whole in mathematics education.

According to Cohen (1987: 16) when instruction and assessment is aligned, results are greatly improved. The NCTM (2000: 19) affirms that ‘when assessment is an integral part of mathematics instruction, it contributes significantly to all students' mathematics learning’. They further add that ‘to ensure deep, high-quality learning for all students, assessment and instruction must be integrated so that assessment becomes a routine part of the ongoing classroom activity rather than an interruption’. It was therefore decided to use the same set of modelling design instructional principles (after reformulating them) to develop related assessment principles for modelling. Treffers also reminds us that ‘it almost goes without saying that a concrete goal description of what is intended with a certain piece of instruction can offer an important support for that constructive analysis and also at the same time for the planning, realisation and evaluation of that instruction’ (Treffers 1987: 135). Assessment for modelling has to be aligned with modelling instructional principles. The assessment of modelling should also parallel the aims of modelling. The six principles that are reworded provide for assessment based on a holistic approach of modelling. This means the entire process and products of modelling are included in the framework.

The re-worded principles can be used for assessing students over all tasks in a different way to rubrics. Rubrics need to be specified for each task and before a task commences. These re-worded principles also allow the teacher/researcher to engage fully with the task, to focus specifically on what responses they expect and set out an anticipated trajectory. The trajectory will be open enough to allow for unexpected/unusual student responses while focussing on the essence of modelling in a way that rubrics or multiple choice questions do not.

The first principle that will be discussed is the personal meaningfulness principle (Lesh, Hoover & Kelly 1992: 113 & Lesh et al. 2000: 608), which originally asks if the task situation will be encouraging enough to allow students to make sense of the problem based on their own personal knowledge and experiences. This question can be reworked and now becomes:

- To what extent does the group make sense of the real life situation? Clustered within this question are competencies of understanding, using informal knowledge, mathematising and interpreting.

The second principle, the model construction principle, asks if the task ensures that the students will recognize a need for a model. This can be changed to:

- To what extent does the group construct a model? This will cluster competencies of, mathematising, working mathematically, validating and presenting.

The third principle, the self-evaluation principle, asks if the task will enable students to judge their own responses. This principle can be used in this study in the form:

- To what extent does the group judge that its ideas, responses and models are good enough? This will cluster competencies of their understanding, 'sense of direction' and validating.

The fourth principle is the model-documentation principle. This asks if the task will require that students explicitly reveal how they are thinking. It now becomes:

- What is the quality of the documentation that the group produces when modelling? This clusters competencies of simplifying, mathematising, working mathematically and their sense of direction.

The fifth principle, the simple prototype principle asks whether the situation is simple enough while still needing students to set up a model. For this study it was decided to ask the following, which is slightly different to the original questions' intention:

- At what level is the group working on a continuum of simple to complex when modelling? Competencies of understanding, verifying and argumentation are pertinent here.

The last principle is the model generalization principle which asks if the task challenges the student to go beyond producing a single purpose model to produce a reusable, shareable modifiable model. The question in this study becomes:

- To what extent does the group develop a prototype, generalisable model? Competencies include arguing, validating and presenting.

This question can be further explored by using Gravemeijer et al. (2000: 243) levels of modelling activity in emergent modelling when students move from *model of* to *model for* (see 3.2.1). The four levels are not necessarily hierarchical. The first level involves the activity in the *task setting*; the second level is a *referential* level, the third level of *general activity* emerges as the students lose their reliance on the situation specific imagery while at the fourth level the model becomes 'an entity in its own right' and a 'means of mathematical reasoning'. It is this emergence of a generalisable model that reflects higher and more formal levels of mathematical thinking. According to Gravemeijer (1999: 160) 'formal mathematical thinking is a form of reasoning that builds on arguments that are located in the newly formed mathematical reality', in this case the generalisable model.

Using these questions, it will be possible to report back at the end of the study (see 5.3) in a more holistic and meaningful way. The questions focus on important aspects of modelling: reality, construction, reflection, representation, thinking and prediction.

3.8 A MULTIDIMENSIONAL APPROACH TO ASSESSING MODELLING

Blomhoj and Jensen (2007: 51) analysed different developmental projects and distinguished three dimensions of modelling competency. One dimension relates to the degree of coverage that students can cope with during a modelling activity. Another dimension relates to the technical level that students use in their activity; that being their use and flexibility of use of mathematics. The third dimension is the radius of action which indicates how flexible students are, in adapting to various modelling situations and contexts. This approach, as set out in Blomhoj and Jensen (2007: 55) and Jensen (2007: 143) allows one to gauge modelling competencies in a holistic way. It was suggested by these authors that these dimensions can be visualized geometrically. They used a 'volume' approach. As suggested by Jensen (2007: 145), one can use the multidimensional approach for assessment to recognize progression in competency. He adds further that we need to focus on all the dimensions when supporting modelling competency development and that we can use the multidimensional approach to identify progression in a modelling. Jensen (2007: 147) also called for more attention from the mathematics education community to assess competencies in this way. Maaß (2006: 117) was unable to use these dimensions due to the length of her study; she did however implicitly use them in the choice of tasks. This study, does however aim to document the development of competencies so that the results will make it possible to describe modelling competencies using these dimensions (see 5.4).

A brief description of each facet of the multidimensional approach is given.

i) The degree of coverage

This deals with the parts or nodes of the modelling process that the groups could deal with and the level of reflection involved (Blomhoj & Jensen in Jensen 2007: 144). Here the cognitive and meta-cognitive competencies specified for this study are directly relevant since Jensen (2007: 144) maintains that a student who can enter an 'internal dialogue regarding the validation of a modelling process' will have a higher degree of coverage. The degree of coverage also deals with how deeply the problem is dealt with. How many features of the problem do students deal with and how they justify their decisions? It is essentially about how deeply students cover the modelling process itself.

ii) The radius of action

This refers to the range of mathematical 'domains' (Jensen 2007: 144) that students are able to model successfully. This would require students to be given modelling tasks across mathematical fields and areas. Jensen reminds us that students will display differences in their modelling competencies depending on the context of the tasks. It was therefore also important that in this study the contexts of the three tasks were different although all three tasks (see Appendix A, B and C) deal with proportional reasoning. Individual competencies such as understanding, mathematising, working mathematically, using informal knowledge and sense of direction are pertinent here.

iii) The technical level

As proposed by Jensen (2007: 144) the technical level relates to the 'size and content of the mathematical toolbox'. This, according to Jensen, is the type of mathematics the students use and how flexibly they use mathematics. By focusing on students' technical levels it is possible to gauge to what extent they are able to use and apply mathematics taught. Since a modelling situation makes it possible for students to 'choose' their own mathematics, insight is gained into what type of mathematics students use confidently and flexibly. It also allows one to focus on the competencies of using informal knowledge, mathematising and working mathematically.

Since each dimension contains other complex dimensions and integrates individual competencies, perhaps the term multi-multidimensional is necessary. The multidimensional approach provides some challenges in terms of traditional assessment. Jensen (2007: 146) addresses two specifically: the wish for a single ranking system and the traditional dominance of the technical level. The multidimensional approach cannot be used in a quantitative way by a single ranking system, but rather gives us three mainstays in viewing modelling competence. Therefore the focus cannot solely be on technical mathematical performance as in traditional instruction. Furthermore, according to Jensen, the three dimensions provide vocabulary for discussing modelling performance. These dimensions are used to assess student modelling competencies later in the study (see 5.4).

3.9 NOVICE MODELLING BEHAVIOUR

Kaiser's study (2007: 117) revealed that students experience great difficulties in clarifying the goal of modelling as well as selecting a suitable model. Students showed significant stable progress in these areas that were initially deficient. The study by Mousoulides et al. (2008: 300) found a significant negative relation between students' initial achievement in the modelling abilities test and their rate of change. They concluded that the modelling program was more efficient for students with lower modelling abilities. They attributed this to the social interactions and absence of direct instruction that created a safe environment for students with low modelling abilities. Their study also served to guide the choice of participants for this study.

It has been observed (Hodgson & Harpster: 1997: 260) that students new to modelling activities exhibit linear behaviour. Students performing modelling activities stopped the modelling cycle when they perceived that they had answered the question. It was also found that these novice modellers did not often look back or revise their initial models. Lesh et al. (2000: 597) reveal that students' early work on model-eliciting problems consists of an assortment of disorganized and inconsistent ways of thinking about the problem, while Peter-

Koop (2004: 458) described it as chaotic and haphazard. Only later do students focus on relationships, patterns or trends and finally include conditional statements. (Lesh et al. 2000: 599).

The above reveals that apparent non-progress in terms of early behaviour in the teaching experiment phase of this study is to be expected. Zawojewski, Lesh and English (2003: 355) affirm that frustration is often the first reaction of students. They indicate that the best thing a teacher (researcher) can do is to listen as it is best to leave the problem solving to the students. They further advise interventions that do not remove the cognitive demand of the task are possible. If the tasks are model-eliciting or thought revealing, students will be able to progress through these initial stages of disorganisation and develop significant models using concepts they already have. It requires a new role from both teacher and student to allow this to happen (see 2.2.1).

3.10 THE ROLE OF REPRESENTATION IN MODELLING

The role of representation as an externalization or expression of internal conceptual system with the purpose of communicating with others or yourself (Johnson & Lesh 2003: 266) is understood to be significant in the development and documentation (from a research point of view) of modelling competencies. McKendree, Small and Stenning (2002: 59) show that representational ability is about being fluent in transforming information in a personally meaningful way. They contend that representation requires that students engage in collaboration with others in a deep way to impose a personal framework on a problem.

A representation is not a whole solution to a problem but achieves two goals, first by capturing assumptions as critical features and secondly allowing application of appropriate operations (McKendree et al. 2002: 62). This allows one to see that representation as not linked to any specific part of the modelling cycle. Since representation underlies all cognitive competencies it is not treated as a separate competency but as a sub-competency of the entire process. In understanding a problem students will resort to some form of representation. It therefore plays a considerable role when simplifying a problem. Lesh, Cramer et al. (2003: 58) say that it is useful for students to invent their own representations that express their own way of thinking. In mathematising, representation gives a new dimension to the problem while working mathematically will result in students using and switching between representational forms.

Lesh and Doerr (2003a: 12) emphasise that meanings associated with a conceptual system tend to be distributed across different representational media. They further add that representational fluency underlies important abilities of what it means to understand the conceptual system and that solution processes for model-eliciting activities involve a continual movement between representational systems. They propose the following as examples of representational media: spoken language; written symbols; experience based metaphors; diagrams or pictures; tables; graphs; equations and concrete models.

Aliprantis and Carmona (2003: 259) used the following types of representational systems when coding student modelling work: algebraic, charts, graphs, lists, pre-algebraic and text. They found that since students were not required to show any specific representational form, students came up with their own representations, used more than one representational system or used mixed representations. This, according to them (2003: 262) implied that students were able to map from one representation to another. This ability to translate from one representational system to another requires reinterpretation of an idea from one mode to another (Cramer 2003: 450). Model-eliciting tasks enable students to make and translate between representations that are meaningful to them and not to produce any specific representational form. Since students are working in teams, they are very often introduced to forms of representation that are created and understood by fellow group members and as such can be explained to them as meaningful for expressing and solving the problem.

Johnson and Lesh (2003: 275) extend the role of representation and derive that the ways of thinking that students put into a diagram or graph often leads to new ways of thinking. They declare that modelling processes involve multiple media and multiple representational mappings. Very often model-eliciting activities require the use of technology based media. This opens up the field of representational media available to students to facilitate the way they express themselves and the way they can interpret real life situations (Johnson & Lesh 2003: 267).

Janvier (1987: 27) explained translation as psychological processes involved in going from one mode of representation to another. Representation and a students' ability to translate between representational forms allows one insight into their thinking. As such it is considered to be very valuable determining the development of modelling competencies and related mathematical thinking. Since students express inner conceptual systems and describe real world systems using forms of representation significant advances in their thinking will be expressed by their representations. Kaput's (1987: 25) conclusion is that the idea of representation is continuous with mathematics itself.

In modelling situations, students are not prescribed any particular representation format to work with, or to present their findings in any particular format. Therefore their representational forms are inherently understood by them and indicate the level mathematics that they are able to deal with, with true understanding and meaning. Furthermore, what is important for this study is that student representational modes, and transfer between modes may improve with modelling competency development, or may indicate that modelling competencies are developing. Focus on representational fluency has been incorporated into research instruments (see Appendix H) not as a separate competency but as informing the quality of documentation that students produce (see 5.3.4).

3.11 MODELLING COMPETENCIES IN STEREOTYPED STUDENTS

In a study by Lege (2005b: 96) students who were scheduled to take a mathematics remediation course took part in a two week modelling course. It was found that some of the students who had done nothing all year, became involved and engaged in the modelling tasks. This study revealed that students experienced (in modelling) the opportunity to demonstrate their ability to think and reason in a way that did not correspond to their performance in a traditional classroom. Lesh and Harel (2003: 163) involved students in their study who were enrolled in remedial mathematics classes because of poor performance. These students were also from disadvantaged populations. They found that students, who were among the least advantaged, invented surprisingly powerful ideas. Lesh and Doerr (1998: 376) also established from their research that students who were able to invent powerful constructs often included those who were labelled as below average in ability. They explain that this labelling was based on their performance in a traditional mathematics education setting where a shallow range of abilities and a one-dimensional range of tasks and products is emphasised.

Peter-Koop (2004: 458) conducted a four year modelling study with grade three/four students. Two out of the four groups documented were so called 'low achievers'. It was found that groups, including low achievers, were highly successful in finding an appropriate solution to one of the Fermi type problems used for a modelling activity. It was also found that all groups (with the exception of a mixed ability group) were able to explain their solution process adequately. On the other end of the spectrum, Iverson and Larson (2006: 290) found that students who were high achievers in mathematics tended to force more advanced mathematics into their model development while low achievers used simple mathematics in sophisticated ways (Iverson & Larson 2006: 288).

A study by Aliprantis and Carmona (2003: 261) conducted research on economics problems for middle school students. Their groupings included a remedial group. They found that overall the students were able to develop their own mathematical and economical ideas that went beyond expectation. Using the above findings with consistent revelation of the success of traditionally described ‘weak’ students, this study aims to document the modelling competencies of students considered ‘weak’ in traditional classrooms. A significant implication will be to document the development of their modelling competencies. Modelling does not require a set response or answer, but students’ own concepts and representations to explain and express a complex system with no direct teaching involved. Often these problems are considered to be too difficult for average or weak students.

It is necessary, according to Lesh (1981: 239) not to try to teach average or below average students the processes used by gifted problems solvers, but to identify and strengthen processes which are accessible but not frequently developed in these students and for students to develop a whole new mode of thinking. It has subsequently been shown that model-eliciting activities allow for new ways of thinking with processes students already have. The weakness may not have been with student thinking or processes, but the quality of activity presented to them.

Maaß (2005: 71) proposed that modelling allows weak students to develop affective and cognitive access (on a long term basis) to mathematics since the formulation of the tasks allows students to develop solutions according to their own capabilities. She further added that the tasks enable weaker students to perceive mathematics as more useful, interesting and comprehensible which results in the unusual success for weaker students and that this in turn, allows affective access to mathematics and may positively contribute to the acquisition of mathematical competence.

Lesh et al. (2000: 633) explained an anomaly that presented itself at conferences, of low-achieving middle school students apparently outperforming college students. They formulated an answer in terms of the six principles they devised for model-eliciting activities. In their view, students can invent powerful constructs if: they try to make sense of the situation based on extension of their own knowledge and experiences; they recognise the need to develop the model; they feel the need to extend their first way of thinking – the need to go through multiple modelling cycles must be obvious; they externalize their ways of thinking so that they can examine and refine them; they recognise a need for a general solution rather than an isolated solution; the situation serves as a useful model for other structurally related situations.

This study intends to contribute meaningfully to this discussion and by doing so, advance mathematics learning for all students.

3.12 GROUP MODELLING COMPETENCIES

The unit of analysis in this study is the group and not the individual. The focus is on how group modelling competencies develop. The necessity and benefits of group interaction are discussed in section 2.9.2, 2.9.3, 2.9.5 and 2.9.6. The successful resolution of modelling tasks requires a group of students that interact and develop a model to reach a conclusion. Although individual students will make significant contributions to the group progress and students will develop ideas initiated by others, the focus of this study is the development of competencies in the group as a whole. According to Berry and Nyman (2002: 643) groups ensure that students get involved in their own learning and students realize that they are not the only ones who are having difficulty in understanding.

Zawojewski, Lesh and English (2003: 346) explain that making sense of learning and problem-solving behaviours needs a different perspective, not that of individual achievement in a group, but one that parallels the workplace – the final group product being the most important aspect of performance. This means that social dimensions of understanding are involved in the model development process. Berry and Nyman (2002: 643) further add that teamwork gives students the opportunity to speak mathematics thereby sharpening their skills and understanding.

Since model-eliciting tasks are designed with the objective of serving social functions such as producing constructions, descriptions and generalized plans (Zawojewski et al. 2003: 338), their assessment lends itself to group assessment strategies. Model-eliciting tasks require that students produce and document their final product, thereby revealing the group's way of thinking (Zawojewski et al. 2003: 340). These authors further add that although the external documentation is useful in assessing students, it does not provide a view into individual student's thinking and that each individual student may have a different interpretation of the group's final model. They conclude that "The final product is the model posed as the solution by the group, and provides opportunities to interpret the construct produced by the group as a unit" (Zawojewski et al. 2003: 340).

It is also important to remember that the type of tasks used in model-eliciting tasks have to be solved in groups since it would be unlikely for an individual to solve independently (Zawojewski et al. 2003: 341). While groups are solving these tasks it will be possible to determine and evaluate their developing competencies in modelling. For this reason the group and their 'collective mathematisation' (Gravemeijer, Cobb, Bowers & Whitenack 2000: 241) became the focus of the study. All developing competencies are group competencies.

3.13 CONCLUSION

Model-eliciting activities enable students to reveal, document and explain their thinking. As such it is possible to establish modelling competencies and it becomes necessary document the development of these competencies. This study, from the above discussion, assumes the following: that modelling is a significant task in developing mathematical competence; that modelling competencies can be developed through model-eliciting tasks; that competencies are diverse and complex and that students of varying ability can be successful modellers. This study focuses specifically on what modelling competencies are and how they develop when students solve model eliciting tasks.

The chapter has explored modelling competencies in the cognitive, affective and cognitive realm. The instructional design principles for modelling were explored and a framework to assess modelling was designed. A number of ways in which modelling has been assessed was presented which resulted in a selection of the multidimensional approach for this study. In the analyses of the data, modelling competencies will be assessed in three distinct but related ways (see 4.6.1).

Mousoulides et al. (2007: 39) identified needs in the area of modelling. They considered research in how ideas and abilities develop and how this development can be documented and assessed an area of necessity. They also suggested that 'capturing change and the effects of change' (Mousoulides et al. 2007: 35) can guide research in modelling while Kaiser, Blomhoj and Sriraman (2006: 82) also indicated that more research is needed to further understanding on a micro level where students are involved in modelling. This study aims to meet these needs. The study aims to focus specifically on the micro level processes that students are involved in while modelling and to document how these processes evolve. We still do not know enough about how modelling competencies unfold and how competencies reveal themselves in an authentic modelling situation where groups of students interact with each other. Furthermore, more needs to be exposed about how the competencies of students classified weak or strong manifest and develop. Since modelling is a complex activity a teaching experiment that includes a well considered learning trajectory that will expose students to modelling will be necessary to generate the data required for the study. The following chapter discusses the methods and strategies designed in meeting these needs and ultimately providing data to successfully answer the research questions and also to provide didactical direction to modelling in mathematics classrooms.

CHAPTER 4

RESEARCH DESIGN AND METHODOLOGY

4.1 INTRODUCTION

The purpose of this chapter is to outline the procedures and methods of the research strategy (paradigm) used in the generation of qualitative data for this study. The study set out to characterize modelling competencies and to examine the development of these competencies in groups during a model-eliciting teaching experiment by a collective analysis of qualitative data. Evidence for the development of competencies in traditionally weak and strong groups of students is also gathered. This chapter will document the process of generating data that assisted in reaching the aims of the study and drawing suitable conclusions.

A design research framework was followed while grounded theory was part of the data analyses. The study followed the three phases typical of design research that is: the planning phase, teaching experiment and retrospective analyses (Bakker 2004: 38). The planning phase included a study of the relevant literature which enabled the study to be well seated within the current modelling literature domain. The literature study facilitated a fundamental understanding of modelling and modelling competencies which guided the problem formulation and led to a judicious selection of tasks, groups and the preliminary design of instruments to place into pilot studies. Two pilot studies were run in whole class settings. One pilot study was to review the tasks selected so that a final selection could be made and the second pilot study was to review the instruments selected and designed. Aspects of validity and reliability were considered throughout the preparation, teaching experiment and data analyses and are discussed towards the end of the chapter.

The design experiment itself involved 12 students working in groups of four on three modelling tasks (Appendices A, B and C). Sessions of about one hour were held after school in weekly intervals. Each session in the design experiment was audio recorded or video recorded and transcriptions made for the retrospective analyses phase. Coding took place systematically and grounded theory became relevant in the analyses of the data. Data was analysed using three methods of assessment (see 4.6.1) and the findings detailed in Chapter 5.

4.2 RESEARCH DESIGN

A qualitative research design was followed since the spectrum of techniques includes observation, interviewing and documentary analysis (De Vos 2006: 333). Furthermore, it was deemed suitable in answering the research question and for documenting the development of the competencies identified. The research design integrated two paradigms in developing and enriching an area of interest. It can be classified as grounded theory in many of its elements, since according to Fouche (2006: 270) grounded theory involves a reciprocal relationship between data collection, analysis and theory. Together with this, the study includes many elements of a design research framework. This combining of methods is not contradictory, but amalgamation of significant and suitable methods to a modelling study. The basis of conclusions relied on observation and on deduction since according to Strydom (2006: 283) observation gives a full picture of the situation investigated and aims at a detailed analysis of the problem.

4.2.1 Grounded theory

Maaβ (2005: 65) incorporated the use of grounded theory in analyzing qualitative data from students involved in modelling tasks. The description detail was inadequate - so further explication of grounded theory became necessary for this study. As suggested by Moghaddam (2006: 53), grounded theory will answer the question of 'what is going on in an area?'. Modelling is an area that needs such an analysis because of its complex nature. The arena of modelling and students attempting modelling tasks is wide, and as such grounded theory becomes a very apt research paradigm for analysing data from the modelling environment. This study considered all the data generated and the findings are grounded in that data. Abstraction of the findings took place to add to the theoretical base of modelling.

Strauss and Corbin (Moghaddam 2006: 54) outline assumptions on which grounded theory is based:

- The need to go to the field to discover what is really going on.
- The relevance of the theory to the development of the discipline.
- The complexity and variability of phenomena and of human action.
- The assumption that persons act on the basis of meaning.

- The understanding that meaning is defined and redefined through interaction.
- The sensitivity to the evolving and unfolding nature of the process.
- An awareness of the interrelationships among conditions, actions and consequences.

The last assumption is central to validity and reliability in grounded theory studies since this awareness will make the researcher wary of bias and the sensitive role he/she plays in a qualitative study. Since grounded theory is mostly based on the researchers' interpretations it was decided to establish an extensive theoretical base in this study (see Ch 2 and Ch 3) from which these interpretations could be founded to avoid bias.

In terms of the coding used for the study, which is a main feature of grounded theory, the coding emerged from the literature study as well as the pilot studies that were run. The literature study resulted in researcher 'literature sensitivity' (Strauss & Corbin in Moghaddam 2006: 55). In this way the coding was 'readily (not forcibly) applicable to and indicated by the data under study' as well as 'meaningfully relevant to and able to explain the behavior under study' (Glaser & Strauss 1967: 3). Each of the competencies was allocated a colour and letter. Transcripts were coded using the colour in the text and letter in the margin. The sessions were analysed and compared in this way. In the analyses section of the study, *in vivo* coding is used which both adds to the validity of the study and allows the reader to see how the discussion is grounded in the data.

The view of Moghaddam (2006: 65) is accepted that 'the theory which is driven in this way answers process oriented questions, connecting the conditions that give rise to a certain complex, dynamic phenomenon.' With grounded theory the eventuality is to elevate the theory from time and person and place (Glaser 2002: 1). This means that researcher bias is minimized. This 'theory' that is formulated according to Thomas and James is about (a) inspiration involving patterning or accommodation and (b) explanation and prediction. In the first sense it is about bringing ideas together and in the second sense it remains to positivist and functionalist about explanation (Thomas & James 2006: 772). In terms of a design experiment (see 4.2.2) the theory that develops is 'relatively humble in that they target domain-specific learning processes' (Cobb et al. 2003: 9). Grounded theory echoes this need for theory that is 'suited to its supposed uses' (Glaser & Strauss 1967: 3). This study is seated in modelling as a specific domain of mathematics education and the theory generated is in and for that domain. The theory is developed as a result of answering the research question. It is essentially a theory of learning through modelling while the discussions in the analyses consider improvement for the practical field of teaching modelling.

4.2.2 Design research

Collins, Joseph and Bielaczyk (2004: 18) state that design research was developed as a 'way to carry out formative research, to test and refine educational designs based on theoretical principles derived from prior research'. This study is derived from prior research on modelling and is based on theoretical principles and is very much a formative study to refine a modelling design. The path followed is what Brown (1992: 141) describes as engineering an innovative educational environment and concurrently conduct experimental studies in this environment. This study created a suitable modelling environment through studying relevant literature in all modelling aspects, running pilot studies on many tasks before selecting the final three, revising and reworking the instruments before final implementation. Some instruments (Appendix K 'I think') and informal interviews were designed or considered towards the end of the teaching experiment period as the researcher considered necessary at that point.

Brown (1992: 142) sketched out some of the complex features of design experiments. In engineering a working environment, four aspects are interdependent.

- *Input* in design experiments includes classroom ethos, teacher as researcher, the curriculum and technology needed.
- *Output* is determined by accountability and the assessment of the right things. According to Brown, we need to assess aspects that the learning environment was supposed to foster.
- *Contributions to learning theory* is explained by Brown as a critical tension in the goals between contributing to a theory of learning, a theoretical aim that is a keystone of the experiment, and
- contributing to *practical feasibility (dissemination).*

This study is deliberated along these criteria in that there is a definite classrooms ethos that is fostered and maintained together with teacher as researcher. The output is assessed in three different ways all closely linked to the theoretical and practical considerations of modelling. A contribution to modelling theory is made by abstracting ideas from the data collected while the study can be transferred to a whole class environment. Gravemeijer (1999: 157) describes the theory that is generated by developmental research as 'local instruction theory that underlies this reconstructed instructional sequence' which he says is comparable to the notion of a Hypothetical Learning Trajectory (HLT). A local instruction theory is however more general than a HLT and encompasses a 'whole instructional sequence' (Gravemeijer 1999: 157).

Bakker (2004: 39) lists three phases of design research; a preparation and planning phase which led to a HLT or 'thought experiment' (Gravemeijer & Terwel 2000: 786), a teaching experiment and retrospective analyses. In this study, the planning and preparation phase involves a lengthy and in-depth literature study, pilot studies and planning for the teaching experiment. The teaching experiment involves the empirical data generation and a retrospective analysis involving the transcriptions, coding and analysing of the teaching experiment sessions.

Since design research is also termed developmental research (Bakker 2004: 37) a number of other concepts are relevant. The HLT is deliberated as part of developmental research. According to Bakker (2004: 39) the HLT is the 'link between instruction theory and a concrete teaching experiment.' The modelling tasks and activities are carefully deliberated along these facets since according to Gravemeijer (1994: 448) the researcher in developmental research will try to visualize how the 'teaching-learning' process will proceed. He describes this envisioning as a 'thought experiment' and furthermore describes the action of the researcher as one who 'tries to make sense of what is going on in the classroom against the background of the thought experiments that precede the instructional activities' (Gravemeijer 1994: 454). In terms of answering the research question, design research 'allows a designer to lay out a proposed developmental route for the classroom community' (Gravemeijer et al. 2000: 241). Furthermore these authors remind us that the developmental route allows students to model informally (*model of*) and then more formally (*model for*), also termed emergent modelling (see 3.2.1). In this way modelling competencies in this study could be elicited and fully explored.

The concept of ‘emergent models’ (see 3.2.1) is also considered in the assessment of student modelling in open tasks (see 5.3.6) since the teaching experiment allowed for the emergence of both ‘types’ of models. The assessment of the HLT is closely and widely monitored and three forms of assessment are presented emanating from a theoretical base while the teaching experiment is almost immediately transferable to classroom practice. This is largely true of many modelling studies, since the tasks used for research are classroom ready while much modelling research takes place in classroom settings (English 2006: 307, Lege 2005a: 91, Maaβ 2006: 120). This is largely due to the aims of modelling as a learning activity and modelling as research interest being so different to traditional teaching.

To facilitate research being transferable to classroom practice, the ‘intervention should be able to migrate from our experimental classroom to average classrooms operated by and for average students and teachers’ (Brown 1992: 143). The tasks used and adapted in this study are classroom ready and with minor changes of currency, names and units of measurement can be used in many contexts. Furthermore, since classrooms involve a community of students, Gravemeijer et al. (2000: 240) remind us that

> taken-as-shared models that emerge through negotiation is consistent with our approach of formulating hypothetical learning trajectories for the classroom community rather than for any particular student.

This study took place with a single researcher (teacher) and groups of students working at the same time in weekly sessions of similar length to timeframes in most schools. This will facilitate classroom implementation.

These ideals are also forthcoming from grounded theory where the function of theory is to enable prediction and explanation of behavior and to be useful in theoretical advancement and practical applications (Glaser & Strauss 1967: 3). In a similar vein is the conception of design research by van den Akker (1999: 4) who maintains that in this approach, knowledge growth is seen as equally important as product improvement. Within the research setting itself, it is important to maintain a clear view of the anticipated learning pathways and that understanding the ecology of learning is a theoretical target for research (Cobb et al. 2003: 12). These two aspects were clear and guiding beacons for this study since a clearer understanding of how modelling competencies develop was desired.

4.3. PLANNING AND PREPARATION PHASE

4.3.1. Literature study

A thorough study of existing modelling literature was undertaken in preparation for the empirical research strategy. Theoretical aspects to teaching and learning of mathematics were scrutinised. The link between theoretical aspects and practical research were deliberated. Decisions regarding the use of groups, choice of the group distinctions and using tasks that involved proportional reasoning led from a theoretical understanding of these ideas. Competency in general and modelling competencies in particular were methodically and systematically studied. Once these were well founded, research instruments could be designed.

4.3.2 Empirical design

The empirical part of the study took the form of a teaching experiment or design experiment. The research question was considered and deliberated on and the most suitable and design for answering the research question was sought. Design research in the form of a teaching experiment that would require an in-depth retrospective analysis was projected in terms of the research question. It was anticipated to be the most suitable framework in which to develop modelling teaching and learning theory. The teaching experiment would involve 12 students solving three tasks. These sessions would be audio/video recorded and transcribed by the researcher. Tasks and instruments had to be selected and/or designed so that the development of competencies (see 3.5) could be documented. Students had to be selected as per the criteria stated in the aims (see 1.5.1.5) in terms of 'weak' or 'strong' in a traditional setting. It was decided that pilot studies would be necessary in the 'progressive refinement' (Collins et al. 2004: 18) of the study, typical of design research. The pilot studies would be a link between the literature study and the teaching experiment.

4.3.2.1 Pilot studies

Since a qualitative study was undertaken, the integrity of the data collected would depend on the quality of the tasks and instruments themselves. It was therefore decided to run two pilot studies. The first would be to ascertain the quality and reliability of the tasks in allowing modelling competencies to be observable and to develop. It was necessary that the tasks were suited in context and difficulty level to the students. The discussion and results regarding the first pilot study are presented in 4.3.2.3. The second pilot study was set up to ensure that the research instruments could provide needed data. The discussion regarding the second pilot study is presented in 4.2.2.4. The pilot studies were valuable in determining which of the tasks were suited to the participants, and if the instruments were suited to quality data being collected. A number of changes to the research instruments had to be made after the second pilot study. This strengthened the reliability of the instruments. The entire process of running pilot studies contributed to decisions about time frames, instruments used and suitable equipment required for each task. It served as valuable orientation for the researcher regarding the practicalities of undertaking the main study. Since whole classes were involved it sharpened the researchers' sensitivity to consistency in data generation and analysis.

4.3.2.2 Selection of students

From previous studies in the field that obtained surprising results with students considered weak in a traditional mathematics setting (see 3.11), it was decided to make these students a focus in the study. But since modelling competencies are the central aspect of the study, another group, of higher ability students, was also considered. This purposive sampling (Strydom & Delport 2006: 327) maximized the data collected. An important aspect of this research is to improve practice. It is therefore considered appropriate to focus on how weak students gain modelling competencies in order to facilitate eventual teacher development programs and successful implementation of modelling problems in the classroom. It was decided to use two groups of traditionally weak students (8 students) and one group of more able students (4 students). This would mean that the quantity of data collected for the weak students was increased. Too often improved styles of teaching are not implemented because of the perceived difficulties involved for weaker students. Students in this study are not considered 'weak' in terms of ability, but weak in that their school results are average to below average. Weak students comprise what Kuhn and Udell termed a 'difficult to work with population' (Kuhn & Udell 2003: 1246). Their reasoning in using this populating is accepted for this study. The competencies that do develop are not the ones emphasised in traditional schoolwork or developing anyway as may be the case with academically stronger students. Another factor in selecting these groups is that 'negative cases' (Guba & Lincoln in Morse, Barrett, Mayan, Olson & Spiers 2002: 2) is fundamental to assuring validity and reliability in qualitative studies. The two weaker groups are an attempt of documenting the negative cases in a study.

A letter was forwarded to the Grade 6 Mathematics teacher (the year before these students became involved in the study) to assist with selection of students. A subsequent meeting was held with the teacher to explain the research study and to discuss the criteria for including students. Four students who perform well at mathematics and enjoy mathematics were selected, while eight students who perform at an average to below average level and who do not have any learning problems or social problems were selected. Six boys and six girls were selected and each group consisted of two boys and two girls thereby not widening the scope of the study to include gender issues. Students selected also reflected a diversity of cultural groups so that it was representative of the population at the school.

Students were invited to take part in the study by means of letters. These letters included written permission from their parents to take part in the study. The relevant permission was also granted from the Gauteng Department of Education (see Appendix D) and School Governing Body.

4.3.2.3 Selection of tasks

Pilot study 1

The first pilot study took place during November 2008 with a class of Grade 7 students. These students would not be involved in the teaching experiment sessions during 2009 since they would have left the school. One class of approximately 30 students took part. The class was split into seven groups of students' own choice. Each group in the class was given a different modelling task. This would ensure that no 'cross pollination' of ideas took place. The series of lessons took place during normal mathematics class time. Video recordings were made of certain events during these sessions. Students also used the Group Reporting Sheet (see Appendix I).

The purpose of the first pilot study was to ascertain the suitability of the tasks and to allow the researcher to become more acquainted with the possibilities and pitfalls of each task. A further need was to narrow down the tasks that could be used in the teaching experiment during the following year. The six tasks chosen for the pilot study were:

1) Big Foot (see appendix A)

2) The Catalogue problem (see appendix B)

3) Quilt problem (see appendix C)

4) Mapping the school – (idea taken from Lehrer and Schauble 2003: 65). Students were required to fit a map of a section of the school buildings and grounds on to a specific size paper.

5) Photograph problem (idea adapted from Maaβ 2006: 126). The photograph of the unveiling ceremony of a particularly large statue of Nelson Mandela was given with the students being asked to find out how big the statue is.

6) Blood alcohol (taken from Lesh, Hoover & Kelly 1992: 123). Students were asked to develop a tool to be used by a Drink and Drive Hotline to estimate the blood alcohol level of any caller based on the number of drinks consumed, time elapsed and body weight.

7) Emergency 911 (taken from http://www.flaguide.org/tools/math/reasoning/baycity.php) student were asked to develop a tool that could be used by a city to determine which of the two ambulances to call out for an emergency based on time of day and response time.

A number of valuable insights were gained from this pilot study. Several practical implications of the study arose (correct paper size, quality of recording etc). A number of changes were made to the Group Reporting Sheet to reflect and record group reasoning. Some tasks showed up as being more suitable than others. Emergency 911 and Blood Alcohol levels were less successful in terms of the proportional reasoning required from students. These tasks also had vast data that students had to contend with. The tasks themselves were very interesting to students and provided much opportunity to learn, but they did not meet the criteria of the study in terms of proportional reasoning. It was therefore decided to eliminate these tasks for this study. The task titled Mapping the school was also not considered suitable for practical reasons. The week this pilot study took place was a rainy one. The group dedicated to working on this task had to change to another task for the first session as they could not make the necessary measurements outside. It also became apparent that the task was not complex enough for Grade 7 students.

It was therefore decided, based on the pilot study, to use the following tasks:

1) Big Foot, 2) The Catalogue Problem, 3) The Quilt problem

This would mean approximately 12 contact sessions.

The tasks were selected and developed with two basic criteria in mind. Firstly they were open modelling tasks and secondly that they required some form of proportional reasoning in reaching a conclusion. The tasks selected for inclusion in the main study meet the six model-eliciting principles (Lesh et al. 2000: 608) and were used before in proportional reasoning research which strengthened the validity and reliability of the study. This study extends the use of these tasks to document competency development in group modelling. It was also decided to use Lesh and Clarke's (2000: 145) Presentation Quality Assurance Guide (see appendix J). This meant that each task answered a need by a particular client for a tool that could be used to solve more general problems. Changes to some tasks had to be made to reflect the client aspect of the task and to fit a South African situation.

The Big Foot problem (see Appendix A) was taken from Lesh, Hoover and Kelly (1992: 123). Their study sought to develop model-eliciting tasks. The current study seeks to describe the process of competence development. Additional information about animal prints was taken from Walker (1996: 20, 63, 91). Big Foot required the groups to come up with a model to calculate a person's size (understood as height by most groups) using their foot size.

The Catalogue problem (see Appendix B) was taken from Lesh et al. (2000: 635). Their study dealt with developing thought revealing activities. This task was adjusted to reflect South African context. Here groups had to compare prices from ten years ago with current prices and come up with a model to find the value of money (today) that would be worth the same as an amount 10 years ago. They were given catalogue items from 1999 and 2009 to base their decisions on. The current world wide economic recession made the context of this task particularly relevant.

The Quilt problem (see Appendix C) was taken from Lesh and Carmona (2003: 85). They used the task to describe modelling cycles that take place in modelling. Lesh and Harel (2003: 168) also used the same task to describe the local conceptual development that took place during this modelling task. The current study aims to use this task to document modelling competencies and document the development of these competencies. This task required the groups to find a method to scale up a printed quilt pattern to a specified "real" size. Additional information to the quilt problem was taken from www.engineering.purdue.edu/ENE/Research/SGMN. Additional photographs were sourced from www.flickr.com.

Since context plays an important role in mathematics problem solving (see 2.3.1, 2.4.1) the contexts of these tasks were carefully considered. The tasks are all ‘real life’ problems. They all meet the first modelling instructional design principle (see 3.3, 3.7) – the personal meaningfulness principle which requires that the task encourages students to use their own personal knowledge and experience. The tasks all start with a potentially real, complex problem. Nicol and Crespo (2005: 241) provide clarity on the concept of ‘real life’ tasks in mathematics. According to them the ‘quality of a task need not be judged by its relation to real life but in relation to how it engages students’ desires to think about and do the mathematics featured in the task’. So a task that is compelling and engaging (Nicol & Crespo 2005: 241) may also be what Treffers (1987: 61) considered a ‘problem that is inviting and that the solution contains a strong “Eureka” element which will lead to an important personal experience’. This is the case in Realistic Mathematics Education where students are presented with problems that they can imagine. The contexts can be organised with mathematics and the situations prompt students to mathematise (van den Heuvel-Panhuizen 1994: 334). All three tasks in this study have this real aspect that stimulates student interest in them. Furthermore they are complex and need to be structured and mathematised by student models. These three tasks have already been used in modelling research and this increases the validity and reliability of the study.

4.3.2.4 Development of research instruments

Pilot study 2

The second pilot study developed out of a need to simulate other conditions of the intended study. Each group was given the same (Photograph) problem. This meant that some time could be spent at the beginning of the session discussing the context of the task as a class which is integral in design research. Once again, students selected their own groups. During the first session, the Researcher Observation Guide (see appendix H) was used. This assisted the researcher in focusing on certain aspects of group interaction and strengthened the reliability of the instrument. A limiting factor was the size of the class which made it difficult to spend quality time with each group. It was found, however that this class did work through the task faster than the previous class (Pilot 1). The contextual discussion was attributed to this. Students in this pilot used the revised Group Reporting Sheet.

The pilot studies were valuable and contributed largely to the success of the study. Many practical implications had to be dealt with and the researcher could ascertain with surety that the tasks would lead to the data needed to document the development of modelling competencies. It also allowed the researcher a wider range of experience with the modelling problems and group dynamics. It also meant that the researcher could refine her position as researcher and facilitator for the main study.

Research instruments were designed with a need to document the development of modelling competencies. This meant a number of dimensions had to be taken into account. Focused observation was narrowed down to a few general areas only since the researcher was also the facilitator of the sessions. The data needed came from observation, transcripts of discussions, written work, presentations and interviews. A brief discussion on each instrument and how it contributes to documenting competency development follows.

i) Interview Questionnaire

(see Appendix F)

This was formulated with the view to documenting supplementary information regarding student beliefs about mathematics and their experiences regarding school mathematics. Their competencies in using informal knowledge could also be gauged from their answers. The questions related to their experience of mathematics and linking mathematics to the real world.

ii) The Group Reporting Sheet

(see Appendix I)

This sheet was put together to enable students to document their working sessions in a way that would enable the researcher to identify some modelling competencies. The recording sheet was given to each group to fill out during the last fifteen minutes of the weekly sessions. Competencies of planning, understanding, mathematising, working mathematically and their 'sense of direction' could be gauged from this.

iii) The Researcher Observation Guide and Field Notebook

(see Appendix H)

This was formulated using ideas from Maaß (2006: 119), Artzt and Armour-Thomas (1992: 141) and Curcio and Artzt (1997: 130). This sheet was designed to guide the researcher as to certain group competencies such as 'watching and listening' and a 'sense of direction' during each contact session. Although watching and listening were not set as competencies for this study, it was decided to determine if this had any effect on competency development. The Researcher Observation Guide was used when the transcripts of the recording were analyzed to ascertain the level of these competencies. Observation (visual and audio) was an effective data collection method since the researcher could become part of an unknown situation and since according to Forman (1996: 128), indicators that learning is taking place is as present in group discussion as it is in individual problem solving practice. Observation allowed modelling to be seen in its entirety and not only to focus on a few narrow areas. This focus allowed better generation of researcher field notes.

The Researcher Field Notebook was used during and after modelling session to take down any information that may have transpired to be relevant at a later stage. Extensive notes were also taken after sessions in period of researcher reflection. It was found that the Researcher Field Notebook was a better source of information than the Researcher Observation Guide since the researcher had to contend with being the facilitator during the sessions also. The guide may be very useful in situations where more than one person is gathering data on modelling competencies.

iv) The Written Work Guide

(see Appendix G)

This was designed for the researcher to assess the Group Reporting Sheets. Student written work was also viewed in conjunction with transcripts. Photographs of written work were also taken to support data collected.

v) The Presentation Quality Assurance Guide

(see Appendix J)

Lesh and Clarke's (2000: 145) rubric was used. This enabled the researcher to assess the presentations in a consistent way. Since this rubric exists in the field of modelling, its validity and reliability are strengthened. Competencies of presenting and arguing could be assessed using this rubric. The presentation sessions were video recorded so each aspect of the quality assurance guide could be focused on in detail.

vi) The Individual Response Guides

(see Appendix K)

This was designed to supplement information gathered in the interviews. This information was needed to enhance and compliment the main focus. Although group modelling competencies are the focus of the study, relevant and significant supplementary information could be gathered in this way. Students completed an individual response sheet at the end of each task and a sheet called 'I think' at the end of the contact sessions.

4.3.2.5 Indexing competencies from research instruments

The competencies specified were indexed based on certain level of achievement. The levels were Level 0 which indicated a pre-level, this meant that students were displaying a particular competency, but their cognitive level was such that it did not allow them to make any progress. Lin and Yang (2005: 100) give one of their operational definitions for a component of student modelling as a random idea without mathematical knowledge that is 'irrelevant, vague and only intuitive'. Level 0 was used in this sense. Level 1 to Level 3 indicates levels of achievement for the particular competency. The levels could then be graphed (see 5.2.13) and development of competencies could be summarized in a visual way. If a particular competency was not evident at all, it was not allocated any level. Although competencies were allocated indexes, the discussion and use of *in vivo* codes make up the main thrust of the analyses.

4.4 THE TEACHING EXPERIMENT

Twelve students were selected for the study (see 4.3.2.2) and met with the researcher weekly for sessions of one hour, beginning in February 2009 and finishing at the end of May 2009. The sessions took place after the school day had ended. A school holiday of three weeks fell into this period also. The first session was an orientation session. The following sessions were working sessions. When groups completed a task (after 3-4 sessions), presentation sessions were held. The researcher's role as facilitator of the sessions and also that of resource and guide was not to intervene too much or too deeply (Carpenter et al.1999: 46).

During the orientation session the task characteristics were discussed with the students, this being that the tasks are longer, not immediately solvable and require effective group collaboration. Each task also involved a client that needed a tool or product to be developed. Groups discussed rules for order and also what constituted good working relationships. Students were allowed to ask questions about the program. Students answered the written Interview Questionnaire (see Appendix F) at the end of the orientation session. To give students some idea of the tasks to come and to allow them to get to know their group members better, a short warm-up task was undertaken by the groups.

Students worked for 12 weeks. Three modelling tasks were completed and presented. During the working sessions (from the second contact session), students worked in their groups on the tasks. Each group's discussions were audio recorded for greater clarity and future transcriptions. Each group's presentation was video recorded. All written documentation was kept for future analysis. At the end of each session, audio transcripts, written work and researcher notes were merged into determining an index for each competency. These were graphed for a visual representation (see 5.2.13). The narrative characterisation of competency development accompanies the graphical representation.

Groups started with Big Foot (Task 1), followed by the Catalogue problem (Task 2) and Quilt problem (Task 3). The Group Reporting Sheets were filled in at the end of each session for the first task and then once during the next two tasks and again close to the end of the second and third tasks. It was found that filling them in each week only led to repetitive shallow answers. The Researcher Observation Guide (Appendix H) and the Written Work Guide (see Appendix G) were completed by the researcher for each session.

Once each task was completed, groups presented their solution process to the other groups. The Presentation Quality Assurance Guide from Lesh and Clarke (2000: 145) was used during these sessions. At the end of these sessions, each student spent time completing an Individual Assessment sheet (Appendix K) as discussed by Schorr & Lesh (2003: 154). In the interests of validity and reliability it was decided to use these already existing instruments. This amounted to a variety of sources from which to gather suitable and relevant evidence on the development of modelling competencies. The English teacher at the school was also interviewed and their language abilities specifically reading, comprehension and reasoning abilities were discussed.

The following table was drawn up to diagrammatically explain the exact steps that were taken during the study since according to Gravemeijer (1994: 456) 'fellow researchers must be able to retrace the learning process of the developmental researcher in order to enter into a discussion'.

Session	Activities	Instruments used	Data revealed by instruments	Session reference for Fig 5.1-5.5 and 5.9-5.14
1	Orientation session Completing of Pre-questionnaire (Appendix F) Getting to know group members Short warm up task	Appendix F (Pre-intervention questionnaire) used at the start of the session.	Information regarding student beliefs about mathematics, and themselves as students.	
2,3,4	Working session Task 1- Appendix A Completing of Group Report and Group Action Map as a group (Appendix I)	Appendix A; Appendix H; Appendix I. Transcriptions made of audio recordings. Appendix G used to view written work.	Competencies revealed	1,2,3
5	Presentation session Groups presented solution strategies for Task1 to researcher and to each other. (Researcher interview with English teacher after this session)	Video recording made. Appendix J used by researcher to assess presentations. Students completed Appendix K individually.	Competencies revealed especially presenting and arguing.	4
6, 7,8,9	Working Session Task 2 – Appendix B Completing of Group Report and Group Action Map as a group (Appendix I)	Appendix A; Appendix H; Appendix I. Transcriptions made of audio recordings. Appendix G used to view written work. (*Groups were not finished at the end of the 3rd session, so another week was allocated)*	Competencies revealed	5,6,7

10	Presentation session Groups presented solution strategies for Task 2 to researcher and to each other.	Video recording made. Appendix J used by researcher to assess presentations. Students completed Appendix K individually.	Competencies revealed especially presenting and arguing.	8
11,12, 13	Working session Task 3- Appendix C Completing of Group Report and Group Action Map as a group (Appendix I)	Appendix A; Appendix H; Appendix I. Transcriptions made of audio recordings. Appendix G used to view written work.	Competencies revealed	9,10,11
14	Presentation session Groups presented solution strategies for Task 3 to researcher and to each other. Researcher interview with English teacher after this session.	Video recording made. Appendix J used by researcher to assess presentations. Students completed Appendix K individually. Students complete Appendix F. (Post intervention questionnaire)	Competencies revealed especially presenting and arguing. Beliefs about mathematics and themselves as students as well as general success of the program from student perspective.	12

Table 4.1: The course of the teaching experiment

4.5 VALIDITY AND RELIABILITY

Validity and reliability in qualitative research take on different terms. Lincoln and Guba (De Vos 2006: 346) listed alternatives for the terms validity and reliability in qualitative studies. They are credibility, transferability, dependability and conformability. Van den Akker (1999: 10) mentioned two aspects in quality criteria for design research as practicality and effectiveness. It was decided to align this study with the existing terms of validity and reliability since 'there are a plethora of terms and criteria' (Morse et al. 2002: 5) for qualitative research which can lead to confusion. It was decided to further distinguish between internal and external validity and internal and external reliability (Bakker 2004: 46).

4.5.1 Internal validity

This revolves around the quality of the data collected and the unassailability of the reasoning that leads to the conclusions. (Bakker 2004: 46). This study augmented the validity by making use of a number of strategies. Using multiple data sources was identified by Niewenhuis (2008: 112) as one of a number of steps a researcher can take to enhance the trustworthiness of a study. This study used a 'multimethod' strategy by combining a number of techniques to obtain data (McMillan & Schumacher 2006: 316). The different strategies enabled the researcher to interact with the participants in different ways and so doing collect different data about the same phenomenon.

In the same way, the use of three assessment methods for analyzing the data can be considered as triangulation to ensure validity of the study since according to McMillan and Schumacher (2006: 325) triangulation can refer to the use of multiple perspectives to interpret the data. The three methods focused on the same set of data, but from a different perspective. The results from the three methods are similar in documenting the development of competencies. In design research multiple sources of data ensure that the retrospective analysis that is conducted will be rigorous and empirically grounded (Cobb et al. 2003: 12). The three methods of assessment were

1) analysing individual competencies (see 3.5.2, 3.5.3 and 3.5.4)

2) questions aligned with the instructional principles of modelling (see 3.7, 4.6.1) and

3) considering a multidimensional approach (see 3.8, 4.6.1).

Each can be considered a subset of the next. The first method of assessment involved focusing on individual competencies identified by the literature study. The second method involved analyzing the sessions in terms of the six aims for modelling instruction while the third method of assessment involved the extent of the group's universal modelling ability. Each method had a different focus on the modelling stage and although each method had a slightly different slant, there was a degree of correspondence that assisted with establishing credibility and crystallization. Richardson (in Nieuwenhuis 2008a: 80) argues that crystallization is a better alternative to triangulation for qualitative studies. Nieuwenhuis (2008a: 81) adds that crystallization affords us with an intricate, deeper understanding of a phenomenon. He adds further that a crystallized reality is credible in that the reader is able to see the emerging patterns.

Other steps listed by Niewenhuis and considered in this study for establishing validity are the keeping of researcher notes, avoiding generalizations, choosing quotes carefully and controlling for bias. The retrospective analyses that follows in Chapter 5 contains the exact transcript sections from recordings during the task sessions. McMillan and Schumacher (2006: 324) list participant verbatim language and mechanically recorded data as strategies to enhance validity of the study. Direct quotations from the transcriptions are used extensively in the retrospective analyses in Chapter 5.

There was also an acute awareness of the sensitive role of the researcher in this study. The researcher maintained a minimal intervention approach to the sessions so that this stance could be sustained throughout all tasks and sessions. This meant that groups were exposed to the same type of interaction from the researcher.

4.5.2 External validity

This refers to the 'generalizability of the results' (Bakker 2004: 47) and how they can be used in other contexts. The challenge is how to put forward the results so that others can use them in their own circumstances (Barab & Kirshner in Bakker 2004: 47). This study followed a teaching experiment (see 4.4) which can be adapted to other settings and contexts. The results generated from this study are similar to other studies conducted on modelling. (Lesh & Harel 2003). The tasks used are from other research on modelling and have already proven transferability as with some of the instruments. The new instruments designed were used over three tasks and the data generation was analysed by using concepts from the existing research. The coding was determined from the literature study which would make them universal to modelling and transferable to other modelling tasks.

4.5.3 Internal reliability

This attends to the reliability within the study itself (Bakker 2004: 46). The study was well planned and run consistently. The most important instruments such as tasks and guides used were sourced from existing literature. Transcriptions were made by the researcher immediately after each session so that this data could be used together with researcher field note book. Indexing the competencies required four levels (see 4.6) which meant that the scope for each level was much bigger than if more than four levels were allocated. The broader levels (0, 1, 2, and 3) allow for tighter alignment across tasks and across groups. More than four levels would have made this margin very close and the margin of discrepancy greater. The broader levels assisted in ensuring internal reliability of the study. The findings of the study are not contrary to the findings in other studies and as such the reasonableness of the inferences (Bakker 2004: 46) is continuous. The sessions were run in exactly the same way and researcher applied herself to behaving in a consistent manner. The pilot studies assisted in setting norms for achievement at each level for each task.

As stated by de Vos (2006: 346) to assume an unchanging social world is in contrast to qualitative assumptions. Each of the three groups in this study can be considered as a 'different setting'. Since the results of this study over the three groups are consistent, the study can be deemed dependable.

4.5.4 External reliability

This is an account of if the same results could be found by a different researcher. To ensure the external reliability of the study, a complete track of the sessions held in the teaching experiment is given in Table 4.1so that the exact sequence of events can be duplicated. The tasks are standard in modelling literature while the role of the researcher is consistently maintained and does not require any special relationship with the instruments or students.

4.6 RETROSPECTIVE ANALYSIS

There were six means of data collection. Audio recordings of the groups while working on their modelling tasks, video recordings of the presentations sessions, written work and written questionnaires for groups and individuals, individual and group interviews and researcher notes. This study by virtue of its research question is descriptive in nature, so what follows in Chapter 5 is an account of what happened during the teaching experiment sessions, not what should have happened or how it must happen. As stated by Glaser & Strauss (Allan 2003: 1) grounded theory investigates the actualities in the real world and analyses the data with no preconceived hypothesis.

All working sessions were transcribed by the researcher on the same day. The audio transcriptions, researcher observation and written work were assessed on a weekly basis. A table was drawn up that incorporated the competencies and levels. A score (0, 1, 2, 3) was determined from the data. Once an index was allocated the resulting graphs were drawn up. The score should be seen not as an absolute measurement but as an index of the competency (see 5.2.13). The word index is used as a manifestation or indicator and not as a fixed measurement. This enabled a visual clue to the development of group modelling competencies. Both written work and transcripts of recordings were coded per competency. Further informal interviews allowed researcher a fuller understanding of what groups were doing and thinking at certain times. The researcher posed non-directive questions (what are you doing? how will this help you? can you explain it?) from time to time while groups were working but did not interfere with the direction that the group was taking.

Competency development was first coded around each competency. Afterwards the six instructional design questions (see 3.7) for modelling were used to discuss modelling competency development in terms of important aims of modelling. Finally the multidimensional approach (see 3.8) was be used to further the description of the developments of student modelling competencies. The multidimensional approach was also used to assess more holistic features of modelling such as degree of coverage, radius of action and technical level. Although the study is vast in scope, one cannot feature modelling competency development in a study without taking all competencies into account and without reporting on the process as a whole. This being the case, an atomistic analysis together with a global view is important.

4.6.1 Three methods of assessing data

Data was collected on an ongoing basis. Each week the audio recordings, written documents and researcher notes were transcribed and sequenced. The specific **competencies** described in section 3.5 were used for coding of the data. The data was then clustered according to the **six questions** formulated in 3.7 and again using the **multidimensional approach**. This resulted in a large volume of systematically structured data in which to base findings and conclusions.

The competencies identified are:

Beliefs about mathematics; a sense of direction; planning and monitoring; using informal knowledge; understanding; simplifying; mathematising; working mathematically; interpreting; validating; presenting and arguing.

The six questions were those formulated in 3.7

To what extent did the group make sense of the real life situation?

To what extent did the group construct a model?

To what extent did the group judge that their own ideas, responses and models are good enough?

What was the quality of the documentation that the group produced when modelling?

At what level did the group working on a continuum of simple to complex?

To what extent did the group develop a prototype, generalisable model?

A final description of using the **multidimensional approach as formulated in 3.8** will be considered while analysing the data. That being:

The degree of coverage and autonomy the group displayed.

The radius of action indicates the spectrum of contexts and situations in which a group display this competency.

The technical level indicates the type and complexity of mathematics that is involved and integrated by the group.

4.7 CONCLUSION

In an attempt to answer the research questions (see 1.4.1) the study required a broad collection of data to ensure a comprehensive picture could be presented regarding the development of group modelling competencies. As such a design research framework employing multiple methods of data analyses was drawn upon. Modelling tasks take place in social, contextual settings. They require an integration of cognitive, affective and meta-cognitive competencies. To do justice to the extensive web of overt and underlying communications within the group, a large selection of competencies and data was needed. Although the complexity and volume was overwhelming at times, it provided valuable information that led to successful answering of the research questions. Together with this, there was enough information for important recommendations for modelling at a school level and further research areas.

CHAPTER 5

RESEARCH RESULTS: ORDERING, ANALYSING AND PROCESSING OF DATA

5.1 INTRODUCTION

This chapter sets out the most important and significant results of the study. Data that was collected by means of the strategy set out in the previous chapter is analysed and interpreted in this chapter. The individual competencies as the focus of the study are set out in the first section. Thereafter the six questions reworded in sections 3.7 and 4.6.1 is revisited. The chapter is concluded by using the multidimensional approach (see 3.8) in analysing modelling competencies.

For ease of understanding, the three groups are numbered. Group 1 consisted of two girls and two boys and were considered mathematically 'weak'. They are made up of student labelled L, M, O and N. Group 2 consisted of two girls and two boys and was considered mathematically 'strong'. They were made up of student labelled S, J, T and A. Group 3 consisted of two girls and two boys and the group was considered mathematically 'weak'. This group was made up of students labelled E, W, G and Mi. In the sections included from the transcriptions R stands for Researcher.

The research instruments allowed a collection of data. Recordings for each session were transcribed. The transcriptions were coded for each competency. Colour coding was used as this allowed the researcher to find specific codes and themes more quickly. Several sentences could be coded for more than one competency. This allowed clustered competencies to be focussed on. Student written work was also coded according to the reworded questions set out in 4.6.1.

For the reader to fully relate to the discussion that follows regarding the tasks, a précis of the tasks is given in the table below. For the full task see Appendices A-C.

Task 1: Big Foot <u>Instruction</u> You have been given an example showing one of the footprints. The person who made this footprint seems to be very big. Yet, to find this person, it would help if we could figure out how big the person really is. Your job is to make a "HOW-TO" TOOL KIT that the police can use to figure out how big people are—just by looking at their footprints. Your tool kit should work for footprints like the one that is shown here. However, it also should work for other footprints. Supporting Material: Example of footprint (size 24). Information sheets on local animals and their sizes and footprints.
Task 2: Catalogue Problem <u>Instruction</u> Hello, my name is Sipho and I need some help with a problem. My parents are really unreasonable. My sister, Karabo, is ten years older than me. When she was in Grade 7 her pocket money was R30 per month. I also get R30 per month. With R30 I cannot buy as much as she could ten years ago. To prove this I collected some information about prices now and ten years ago. What I need from you: Use my price information to determine how much pocket money today would be the same as R30 ten years ago. Write a report for me to give to my parents Describe your method and your conclusions Show that you accurately figured out how much money gives me the same spending power as R30 did ten years ago. Explain your method so other children in similar situations can use it to figure out what their allowances should be. Remember, my parents to not like emotional or illogical arguments. Thanks Sipho Supporting information: Catalogue prices from 1999 and 2009.
Task 3: Quilting Problem <u>Instruction</u> Quilts are often made from pictures that are found. This means that each type of shape must be cut from paper. Often quilters find beautiful pictures of patterns they would like to make and have to figure out the size of the pieces. The members of the *School's name* Quilt Club often have difficulties converting photographs like these into templates that are exactly the right size and shape so that they too can make the quilts. You have been asked to write a letter to the *School's name* Quilt Club to explain to them how make pattern pieces that are exactly the right size and shape. Also include in your letter the templates for the following quilting pattern. (Note: the quilt is for a double bed so should be 200cm x 236cm) Supporting information: Newspaper article, photographs of quilts.

Table 5.1: Précis of Tasks 1-3.

5.2 COMPETENCIES OF THE STUDY

5.2.1 Competency 1: Beliefs about mathematics

Beliefs are an individual's understandings and feelings that shape the way that person conceptualizes and engages in mathematical activities (Schoenfeld 1992: 358). Since students would be engaging in a task that is not typical of their experience of mathematics classrooms beliefs would be an important competency to analyse. The area of beliefs about mathematics relates to a number of aspects of this competency. The following three subunits of this competency are reported on.

5.2.1.1 Mathematics in everyday life

As a result of the Interview Questionnaires (see Appendix F), student beliefs about mathematics and the real world became apparent. When asked the question "How do you think we use mathematics in our everyday lives?" all twelve students' initial responses reflected typically traditional beliefs. There are subtle changes after being exposed to only 3 tasks over a period of 12 weeks.

How do you think we use mathematics in our everyday lives? 2009-02-03	How do you think we use mathematics in our everyday lives and What did you learn about mathematics during the program? 2009-06-05
• *Measuring, counting and costing* • *Distances travelled in a day and paying bills* • *Almost every subject you just don't know it* • *It is most important for multiplying, add, subtract and divide* • *Accounting, buying, selling and many more* • *For knowledge* • *I use it when baking, cooking, buying things* • *We use it when we go shopping* • *I think it is one of the important subjects you should also know* • *By calculating ingredients, time, amount and work*	• *Counting money, working out problems. I learned that mathematics is not only individual numbers but also it is a sociable subject.* • *Money, bills, taxes, profits and income. I learned that there is much more than adding etc.* • *That we could find the real answers in real life by adding, subtracting, scaling down/up etc.* • *I learned that maths could be used in different ways.* • *I learned that there are many different ways to use mathematics.* • *I learnt that maths can even be used when sewing a quilt together.* • *Mathematics is not always practiced but it is used in our everyday lives.*

Table 5.2: Changing beliefs in mathematics

It seems as if mathematics had been relegated to cooking, baking and shopping at best while many students could not find any real use for mathematics outside the classroom at the start of the program. Their responses after the program show signs of improvement, but not every student showed improved beliefs in mathematics. Student ideas about what constitutes a mathematical task also form part of their beliefs. After two contact sessions one of the members in Group 2 said that:

J: this is just general stuff, not mathematics

While working on Big Foot, a member of Group 2 asked 'I wonder if this story is true?' and the response from the other members 'I don't think it is true' and 'of course not!' Which could mean that they do not believe *any* problem in mathematics could be 'true' or that the context of this task was not real to South Africa? Added to this was a comment by Group 1 about the task, that: 'you don't get people who just do good things' in response to the situation in Task 1.

5.2.1.2 Teacher-student contract

Beliefs about the teacher-student contract (see 2.2) also became visible during initial phases of problem understanding. When groups were not sure how to proceed, they then scrutinised every word that the researcher said. They did this in order to find a clue in anything that may have been said to them. During the catalogue problem, Group 1 had decided they had reached a conclusion.

R: you must let the prices help you decide. If you can see that all the prices doubled then you could justify why his pocket money must double. The prices must help you decide.
N: so maybe you take ten years ago and you double the prices
R: what if it's in between double and triple?
N: if we doubled this price R1,98 plus R1,98 plus R1,55 plus R1,55 and it will come to R48.

Later on they remained focused on this idea:

O: why don't you say he sent us a letter showing us that prices have increased?
L: ok he sent us a letter that the prices were doubling up
N: ya, the prices have been double what your daughter had then, 10 years ago.

All groups adapted very well to the different role that the researcher played in these tasks. They gained confidence in working alone as a group and not being able to 'get the answer' from the researcher. The role of researcher was also that of facilitator for these tasks. By the third session, they knew that the researcher would act as a sounding board and guide, but that they were largely responsible for decisions and a way forward. They came to regard me as a resource, as a satellite group member and not as a verifier and evaluator of their attempts. Monaghan (2007: 55) revealed that although the adults overseeing a research study did not overtly direct students to follow any particular approach, adult intervention did seem to influence student work. In the above episode, the group decided to take one word out the researcher's sentence and 'hang' onto it. They never verified that the prices had actually doubled. It was only at the end of the next session that they decided that the lists did not show this.

5.2.1.3 The nature of mathematical tasks

Group 2 had the following discussion after a heated argument in Task 3 that can be related to beliefs about mathematics and the nature of mathematical tasks:

S: it's incorrect!
T: nothing is incorrect with this, ok
S: I am really good at maths
T: yes, but with these problems...

Where T is moving towards changing beliefs about maths problems, but S remains unchanged with 'I am really good at maths'.

An incident showing that students viewed mathematics as a neat, systematic endeavour is noted here. It is also mentioned as it directly affected Group 3's modelling competencies. It was noticed that Group 3 spent a lot of time rewriting a list of items from a catalogue for Task 2 (Catalogue Problem). They lost valuable time doing this. When questioned the following week about this decision:

R: why did you write the prices over? When they were here? (on the instruction sheet provided). Was there a reason?
Mi: (sheepishly) *we didn't want to mess the page.*

During Task 2, Group 2 noticed that they had reached an answer of a 'whole number'. This led them to think that they must be on the right track. They still assumed that neat, round numbers are more correct than decimal numbers:

J: but ours and yours adds up to R200 (exactly)*, there is a connection!*

By the end of Task 3, the following comments were made by Group 3 and this shows a gradual change in their beliefs about the nature of mathematical tasks:

G: the thing about the beginning of a task is that we are always so confused.
G: that's impossible. Look, I don't think we are going to get a direct answer at the very beginning of a task.
W: no, we'll never do that, unless we have a time machine.

5.2.1.4 Summary

Student change in beliefs gradually takes place when a task is devolved (see 2.2.1) and if they exposed to a problem-centred environment as described in 2.8.4. A change in what mathematics is the nature of mathematical tasks and perception of teacher's role all became apparent and incrementally improved. Even when certain beliefs did not change, at least student were talking about them. Since beliefs are not considered to be measurable in terms of weaker or better but rather as an indication of what experiences students have been exposed to, no index was allocated and as such no graph was drawn for this competency.

5.2.2 Competency 2: A sense of direction

A sense of direction can be gauged on two levels. Firstly, on the group knowing what they wanted to do next, and secondly, on the group knowing what they wanted to achieve at the end of the task. This area of competence was fragile in the two weaker groups, particularly Group 3. Often weaker groups become so immersed in the 'what to do next' or what they were currently doing, that they lost sight of what they needed to achieve to conclude the task successfully.

Group 2 displayed a greater sense of direction during the initial phases of the program and even more ably at the end of the program. These students immersed themselves in the tasks to a better degree than students in the weaker groups. They were also able to decide that they had reached a 'dead end' on their own. This happened fairly quickly. This group tried more (albeit unsuccessful) approaches than the other groups. They could gauge their own sense of direction. The other groups were not always sure that an idea was valuable and pursued an idea until it burnt itself out thereby wasting a great deal of time. The stronger group was able to use their mathematical 'toolbox' (Jensen 2007: 144) to a much greater degree than the weaker groups. This group also always kept the required end result of the task in mind.

During the first session of Big Foot (Task 1), both Group 1 and 3 asked the researcher 'what must we do next'. They seemed to be unsure how to handle multiple sources of information. The difference between the task instruction sheets and the supporting documentation were not clear to them and confused them.

Group 1
N: Maam when we are done with number four must we cut out the footprint?

Group 3
M: First... maam, maam must we first do the questions?
W: Must we write on the page or what?

During Task 2; Group 1 had a "what to do" moment, but this time it was resolved by the group itself.

N: Maam what do we have to find out about this?
M: we have to tell Sipho's parents how much pocket money to give him.

An early understanding is important, but significant mathematisation of the problem is equally important in the group's sense of direction. The competency to mathematise filters into and is comprised of other competencies. Mathematisation with reference to a sense of direction refers to groups giving careful consideration to the structure of the problem. Students considered the problem in relation to the possibilities of some mathematics they knew. The range of possibilities that each group comes up with for each problem gives one insight into their sense of direction in mathematising the problem and in finally developing a model for the problem situation.

Although Group 2 worked to an understanding of the problem in Task 1 early (and long before the other groups did), they did not progress further than this for a long time i.e. they did not mathematise the problem even though they understood it well. The first thing the group said after reading the task was:

S: Ok, so what they are saying is that we have to make our own method of measuring how tall or big people are out of a footprint.

Although they knew what they wanted in the end, they were not certain about what to do next. Their first recording sheet for Task 1 has under the heading "what do we want to do next":

To figure out the measurements of the man to see how tall he is.

This is not different to what they want to achieve by the end of the task. There competency in simplifying and mathematising the task affected their sense of direction. Thereafter, they were concerned with 'doing mathematics' with the measurements they had taken. Group 2 became very distressed about their own lack of progress during the second session of Big Foot, but what must be noted is that they were aware of their own lack of progress.

S: what are you doing?
J: 39 times 18 (foot length multiplied by shoe size)
S: 702... aaahhh.. That's even worse than we had before!
J: ok divide by 20, divide by 20.
S: 702 divided by 20. It doesn't go in. 35.1
J: 3m and 51 cm. That sounds reasonable. He laughs. *Doesn't that sound like a human size?*
T: Yes it does, actually.
S: 3m. Are you nuts! That's like a storey high!
J: dividing by 20 won't work. So then what is our back up on that?
S: I have no idea.
And later
S: this is a dead end.

In this case their sense was guided by their informal everyday knowledge about how tall people possibly can be.

Not long after this, this group ended up reading the supporting documentation in an attempt to find a solution path. They were the only group to return to the supporting documentation to find direction.

S: look here, over here, hello
J: what did we say is the relationship between...?
S: Hello. (reading) *What is the relationship between the animal's footsize and height*.

During Task 2, all three groups were developing more autonomy and independence. Group 3 was the only group to become 'stuck' with early understanding in Task 2, even though they understood what the task required of them.

W: we don't know what to do with Sipho's allowance.
W: if his sister could buy like ten things and we see that he can buy 4 things, what do we do after that? Do we times it?

They were the only group, after the second session that did not have a path forward. They were prompted several times by the researcher to communicate their thinking.

R: What does Sipho want from you?
M: to compare the prices from ten years before to now.
W: He wants us to write a paragraph, a report to his parents.
R: What must you tell his parents?

W: how much pocket money, how much more he should get. Like R30 ten years ago was a lot of money, now it's not a lot.

When examining their written work, a subtle change took place in Group 2. At the end of the first session of Task 1 this group wrote:

We need help with finding a method of telling how tall someone is by measuring the length and width of their foot.

While during Task 2, they wrote

We need help with making sure that our method works.

And in Task 3

We don't need any help at the moment.

Their focus went from finding a method to checking a method to not relying at all on approval for a method.

Although groups achieved understanding of the tasks to a greater or lesser degree; their progress was hampered by a struggle to mathematise the problem. This goes further than just translating the task into a mathematical format. As we are reminded by Freudenthal (Gravemeijer & Terwel 2000: 781) mathematising is organizing. In this way a sense of direction is closely linked to mathematising. Understanding and mathematising work together in creating the group's 'sense of direction'. These, in turn are strongly affected by language abilities. Furthermore, it is interesting to note that the groups suffered the same difficulty in their presentations in terms of tracing their steps backwards to explain their solution process. What may be of benefit is to study students tracing their modelling route backwards in their presentations. Often gaps in their understanding of the problem emerge during presentations (see 5.2.12). Group paths to a solution are indeed 'meandering or roving' with some ideas being focussed and others unfocussed as suggested by Davis and Summit (2003: 146). Language ability is an extensive factor in mathematical modelling (see 6.4). A group's use and skill in language pervades all competencies.

The following graphical representation for this competency gives some idea as to competency development over the three tasks

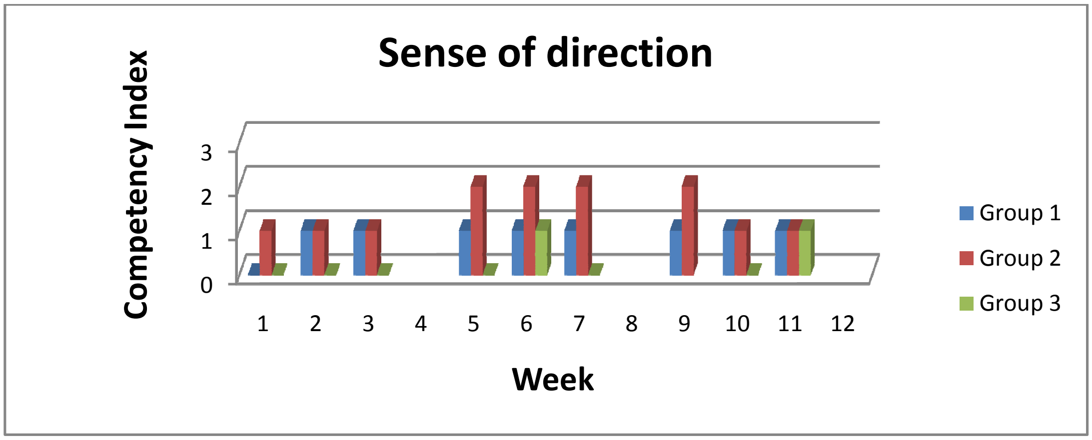


Fig 5.1: Sense of direction

Group 1 remained tentative throughout the tasks. They had some ideas but they were unsure and did not act on some of the better ideas. They allowed one group member to make these decisions only. Group 2 made progress in this competency by the second task and had largely 'solved' the task during the first session of Task 3. Group 3 had very little direction for all 3 tasks and took a long time to reach some consensus on what they wanted to do as a group. The groups' sense of direction will improve as their autonomy and confidence in solving these tasks improves.

5.2.3 Competency 3: Planning and monitoring

Planning and monitoring are meta-cognitive abilities (Masui & de Corte 1999: 520). Planning involves managing the problem by organising aspects of the solution process, while monitoring involves managing the problem by keeping track of the solution process. When looking for meta-cognitive transfer across tasks, students must recognise requirements for the new task, select skills to apply and monitor their application in solving a new task (Mayer & Wittrock 1996: 50). These authors also state that the students must be active participants in managing a problem. De Corte et al. (1996: 506) define meta-cognition as an executive structure that organises and guides learning. Two separate but combined aspects are highlighted, to organise and to guide. Although planning and monitoring are not the same skill, they both involve managing the problem and as such were combined into one competency. Both skills involve a stepping aside from the mathematical load of the problem to viewing the group's progress as a whole. There are many different aspects and sub-skills involved in 'organising' and 'guiding' (see 3.5.4), but underlying the whole of meta-cognition is planning and monitoring.

These competencies were weak during the beginning phases of the program. Students did not spend much time planning; they simply followed the suggestion that came up first. This was particularly true of the weaker groups. There was very little discussion about any planning and little real listening to the suggestions of others. Goos, Galbraith and Renshaw (2002: 193) found that unsuccessful problem solving in groups was characterised by poor meta-cognitive decisions and 'exacerbated by a lack of critical engagement with each others' thinking'. The groups seemed to be guided by a need to 'do something' in the beginning stages of their modelling experiences. Their general education experiences are also relevant here. The schooling they are exposed to is highly structured; teacher conducted and directed, in all areas, not only mathematics.

Group 1 worked in the most unplanned manner for the first two sessions although their presentation sheets were the most organised and well presented. During the first session of Task 1, they displayed only one incident of planning and none of monitoring. Group 2 managed to keep track of what they were doing over long periods of time without writing anything down. Group 3 only wrote things down as they made decisions and measurements but in no particularly planned manner.

During Task 1, Group 1 did very little to plan and monitor their process. The person who initiated the ideas took control of what they were going to do

M: I think we are going to have to do a survey to do this.

Group 2 displayed numerous incidents of continually planning and monitoring their progress. They returned to reading the task at times when they needed clarity or direction, while Group 1 and 3 avoided re-reading. The language and reading abilities of the members Group 1 and 3 are much weaker than those of Group 2. Task comments relating to Group 2s planning and monitoring:

S: how would you start this?
A: maybe we should find a pattern.
S: we are not doing anything.
T: we thought we are going somewhere, but we are going nowhere.
T: what are we going to do now?
J: this is a dead end.
S: why is it a dead end?

S: Ok, do your method again, let's just revise it.

Compared with fewer comments relating to planning and monitoring from Group 1 during Task 2.

M: let's check this price list first of all.
M: Do we have to convince the parents or tell them the amount to give him?

Group 3 was more dysfunctional than the other groups. They worked in sporadic bursts and a number of times suffered from a member not being present due to school absence. This justified the choice of two weak groups (see 4.3.2.2). What can be noted is that both the 'meta-cognitive' coaches in Group 1 and 2 were older students, both 13 years of age (see Table 5.6 in 5.3.3). Groups that comprise younger students should be supported in this regard. This is another reason why heterogeneous grouping should be preferred in classroom situations.

This competency was certainly related to the meta-cognitive ability of each individual in the group. It did not appear that this competency developed significantly over this series of tasks. It is likely that a longer period is necessary. What can be said is that there was a certain 'slowing down' of group members making impulsive statements or the group rush towards solving the problem. There was more discussion, especially with Task 2, which may have the best context match for these students.
During Task 2, both Group 1 and 2, had incidents where the group started acting as a control. Initially the researcher would ask students to explain their decisions, but by the second week of Task 2, the following occurred:

N: ok, we have to figure out how much money.
L: Sixty Rand.
N: don't estimate L. Remember it says here... (she reads).. *my parents do not like emotional or illogical arguments.*
M: Ok R60, why?

All groups had a member that on his/her own accord acted as planner and monitor. In Group 1 it was the same person who initiated many of the ideas, while in Group 2 this person controlled many ideas from the group and kept checking on the group progress and making comments about their progress or lack of progress. In this group, he called the group to order during times when they were off task. His role played a vital and generative factor in Group 2s success and confidence. He continually questioned and challenged their ideas.

Had groups been made aware of the need to plan and monitor in these tasks, they may have shown greater competency. What is noticed is that they can develop these competencies on their own but at a much slower rate than is desirable. Teacher guidance and intervention is essential in demonstrating planning and monitoring competencies. Student experience in these environments over a longer period of time will be beneficial to their competency development.

The following graph shows this competency development. Both planning and monitoring are considered for each task session then the graph that follows may support us in understanding the development of other competencies.

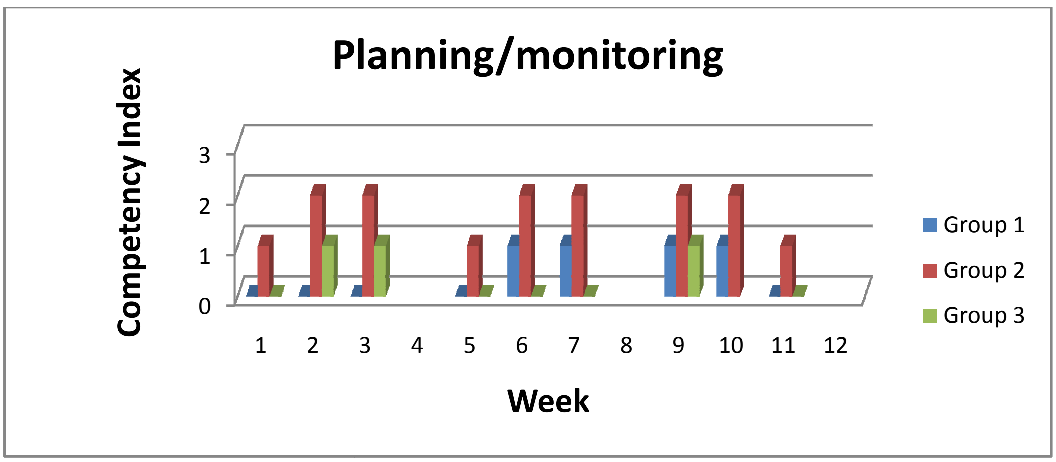


Fig 5.2: Planning/monitoring

Group 2 maintained a higher level of planning and monitoring, while competencies were starting to develop in Group 1. This was because the members of Group 1were starting to talk *to* each other in Task 3 to a greater degree and not *at* each other. Their discussions were becoming more 'woven' with more input from all group members.

M: ya, 27.4 times 9 (trial and error method) *equals 246*
N: it must be 236
M: Now I got 219 (multiplied by 8)
N: it has to be 236, remember that
M: 27.4 times 8.6 equals 235
L: close

Sternberg (1998: 133) affirms that many areas of competence directly involve meta-cognition and verifies that meta-cognition converges with other abilities necessary in developing expertise. The level of consistency with which this was happening during Task 3 is pleasing.

5.2.4 Competency 4: Understanding

Understanding is seen as the ability to find meaning in what has been read, to be able to communicate and reason about that meaning. It became apparent with these tasks that contain a large amount of written instructions and support material that understanding takes place on two levels. The level of understanding that relates to comprehending the written task is discussed here. A secondary (and higher) level of understanding - that of identifying the mathematics or structure within the task is considered as mathematising (5.2.6).

Group 1

Their initial low level of understanding of the first task led them to spend much time discussing ideas that were not directly relevant to the problem during the first session of Task 1. This group also spent much time discussing who Big Foot might be, where he came from, was he divorced, was he a criminal? There was almost a denying or delaying of introducing mathematics to the problem. It became clear that their inability to mathematise the problem led to a diminished understanding of the problem. They had to be continually sent back to read the task information. This delay however, may have contributed to a good understanding during the next session. One cannot give due any reflection that may have taken place during this seemingly undirected discussion.

Interestingly enough, this was the first group to start with any measuring activity. In their Group Reporting Sheet at the end of the first session, they were the only group to introduce the idea of 'examining our foot'. Their language use put them at a disadvantage when it came to communicating their ideas. One of their sentences reads: 'we decided to measure M's length, shoe size and width from head to toe'

Early in Task 1, they showed that they wanted to complete the task rather than understand what the task was about. This may be a result of their experiences in a classroom situation, where they (as weak students) seldom 'understand', but need to complete the task as soon as possible. Directly after reading the question:

L: Must we measure their tails?

And not much later:

N: Maam, when we are done with number four must we cut out the footprint? (Number four relates to a question in the supporting information)

R: What must you find out?
M: We must find out who is this person, and why did he do that? Where does he come from? Is he a person who escaped?
L: We must find out who is this person and why he did it. Is he divorced?
R: What must you find out about this person?
M: Who is he, why did he do it? How big is he?
N: The shoe is big.
M: Maybe he is just a handy man ...

Their weak language skills in reading, comprehension and reasoning were noticeable from the outset. To avoid a continual regression of understanding, the researcher assisted with reading placing emphasis on certain words. Only then was one child able to say that:

M: We are looking for HOW BIG the person is (confirming what he thought earlier)
L: I think he is nine foot something (does not really know what this means as measurements in South Africa are in meters not feet.)
R: You must be able to say why you think so. You must explain to the police how you got to that.
M: Here, here. Looking at the animal prints. You have a height and a footprint. Why can't we just compare it with a human?

For Task 2, Group 1 spent more time on discussing the problem. Although their discussion was largely repetitive, each group member had picked up a similar understanding of the problem. A sense that the problem was 'distributed' (see 2.9.6) emanated here. They seemed to have a deeper understanding of this task than the first one.

The context of the task being that of pocket money and price increases may be attributed to this. What is noteworthy, however, is that there was more discussion from each member of the group. A member who was largely silent throughout the first task joined the discussion a number of times and showed better understanding of the task than the others. She actually had the more sophisticated approach to the problem but she could not consolidate her idea with the other group members. Her ability to communicate convincingly was lacking.

By the second session in Task 2, their understanding of the problem was refined:

R: what is the problem about?
L: This boy whose name is Sipho has a problem about cash. His sister long time ago got R30 per month as her pocket money, now Sipho also gets R30 per month.
R: so what is the problem?
O: He doesn't get to buy as much as she could.
R: why do you say that?

M: Because the currency dropped, and so prices pick up and what the parents don't realise that spending doesn't pick up. So we have to prove to the parents, no, prove to him so he can prove to the parents so that he can get.... we are still working on the grand total.

During Task 3, after reading the task, one of the members started impulsively with:

L: are we going to draw patterns?

Much later, after lengthy discussion they were reading and thinking more carefully:

M: 200 by 236
R: what does that mean?
L: in Technology they do that
…
R: let's read. They read the instruction again.
M: so this is like Social Sciences, scaling up and scaling down - will that work?
R: discuss it

The context of the task that included visual clues and measuring contexts that they may be more familiar with together with the fact that they recognise some parts of it from other learning areas at school (Technology and Social Sciences) may have contributed to their recognising the structure (early mathematising) of the problem. Once students have reached understanding of the problem, they still find mathematising a particularly difficult part, since they have not been exposed to this regularly. This group is showing aspects of improved competency in understanding the problem because members have become more vocal than in the first task. This enabled them to explore more options than just the one suggestion in Task 1. They also returned to reading the task information without being told to:

L: ok last time we scaled it down by how much? 4. So I think we should. Reads*: you have been asked to write a letter to the School's name Quilt Club to explain to them, right how to make; it should be 200 cm times 236 cm. Calculator please.*
M: this has to be 236 and this has to be 200. (shows on the print- length and breadth)

Group 2

They understood Task 1 during the very first session. Their report, under the heading "what is the problem about"- had: 'the problem is about determining how big or small something is by its footprint'. But in their method section, they wrote about all the computations they performed with the measurements. They were not clear why they were performing specific computations. The fact that they are mathematically strong may have led them to understand that mathematics was needed for this problem, but they did see a clear link between their understanding of the problem and mathematics needed. They were the only group to ask about the scale of the paw prints given (see Appendix A). This group had a strong understanding of the problem and their strong mathematical skills, but the link (mathematising) was lacking during the first task.

Group 2 had a member who acted as their 'meta-cognitive coach' and their understanding and progress was guided by his questions. This meant that they checked understanding with each other. This example from Task 2:

S: Do you understand the problem J?
J: Yes.
S: What is the problem J?
J: The problem is, we must work out the difference, like how much he would need each day and then we plus it together for each month. Then you must compare it to the older times and see what the difference is.
T: And what you should get now.
A: And then we must write a report.

During Task 2, they still displayed evidence of wanting to apply any mathematics to the problem even though they understood the problem well, but they did this to a lesser degree than in Task 1.

R: what does Sipho want from you?
S: He wants a method.
R: to work out what?
T: To work out the value of...
J: He wants to be able to buy the same amount of stuff with R30 could buy then.
S: Ok how are we going to make a method?
A: Ok write method 1 and method 2.
S: Ok where were we last time, we tried timesing it and that wasn't accurate enough right?
S: What are you writing? We tried timesing it and then we minused it and that was awful.

They did not spend as much time as in the previous task pursuing pure calculation and after some discussion matched their understanding and mathematisation.

S: How would you compare all of these? (the catalogue items)

And later

J:The difference is R1-80. Maybe we should find an average. An average between them all.

This led to finding out how many pencils could be bought with R30 in 1999 and finding out how much would be needed in 2009 to buy the same number of pencils.

By Task 3, this group did not try to "do" mathematics with the problem, but rather:

They read and ask themselves without being prompted:

A: what do we have to do?
J: we have to make the template for these small pieces, all of them. We have to make a triangle, a square, that triangle and then this; this can just be strips of material.

When examining their Group Reporting Sheets, for the section "we decided to" for Task 1 this group have a long winded:

Measure the big foot to see what the width and the length is. Then we decided to add the length and the width and then we decided to multiply it. Then we tried to subtract and divide. The division and multiplication didn't work out, but we have decided to take addition further.

While for Task 3, the same section resulted in a more focused:

We found out what the scale of the picture was in relation to the size it should be. Now we can use that scale to work out the size of the templates in real life.

Group 3

They were the slowest in understanding with all the tasks and their understanding was fragile. After the first session of Task 1, they managed to determine that they wanted to find out how tall Big Foot was, but they did not understand the problem further than this. They did not see the link between the supporting materials nor were they able to bring themselves into the solution process as Group 1 had. Their reporting sheet was repetitive.

By Task 2, Group 3 displayed what Watson and Chick (2001: 139) termed "social disagreement" which has a negative effect on their progress in general. Perhaps social disagreement was experienced due to personal factors or not being under teacher supervision constantly. Their ages may be a contributing factor to this too (see Table 5.6). This confirms the justification of the choice of two weak groups for this study (see 1.6.2 and 4.3.2.2). If this was the only weak group, the results may have been very negative.

In examining their Group Reporting Sheets, subtle changes are evident. For Task 1, they wrote a confusing response to the question "what is this problem about?"

The problem is to solve the other animals' foot size and the big foot.

This shows that they confused the actual task instruction and the supporting material. It became evident that this group could not handle multiple forms of information very well.

Their understanding of Task 2 could not be stated in a single sentence but required prompting by the researcher

R: have you read the problem? (They nod). Before you carry on, what is it that Sipho wants from you?
W: To compare the prices from ten years before to now.
R: once you have compared them, what does he want from you?
M: To compare. Ten years ago his sister was getting R30 pocket money.
W: He wants us to write a paragraph, a report to his parents.
R: what must you tell his parents?
W: How much pocket money, how much more he should he get.
M: Like ten years ago thirty rand was a lot of money and now it's not a lot. He wants to get the same amount as his sister ten years ago.
R: same amount?
W: The same as you can buy all the things.

During the initial stages of Task 3, this group needed assistance reading the instruction sheet and supporting documentation. They, like Group1 had members who remained impulsive about the nature of the task:

G: are we making a quilt?

The researcher assistance with reading the task and discussing the context may have led to a fairly early understanding of the task instruction but not necessarily understanding that leads to mathematisation:

W: no, wait wait wait, we
G: we're scaling up, do you get it...
G: we scale down and then go up.

By Task 3 their recording sheet was more focussed although not yet ideal. For the same question "what is the problem about" they wrote:

Quilting. The measurements for the double bed.

Although they understood the task instruction, this does not guarantee that the nature of the task is understood or that they will mathematise effectively. This group went back to subtracting the given size (using additive proportional reasoning) from the final size and then tried their own scaling factor without consideration for given measurements.

E: 200 plus, no minus 27.3 equals, 172.7; no wait, it equals 172.
E: that can't be our answer.
G: it has to be something by something because it is a square.

They decided to multiply by 4 because it was suggested by one of the members. It was only much later while working with Task 3 that they tried to find the ratio between the given size and real size. They did, however, need assistance with this. Their proportional reasoning was weak. They could not even see that multiplication and division were inverses of each other even though it is 'taught' at school level.

The following figure shows a visual representation of competency development.

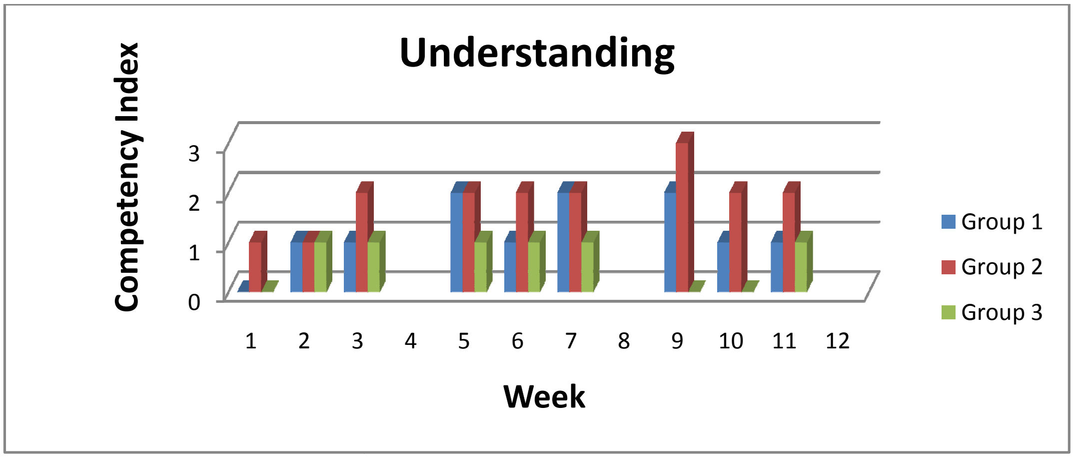


Fig 5.3: Understanding

The weaker groups took much longer to reach an adequate level of understanding for each task, while the stronger group who struggled through Task 1 maintained higher levels of understanding consistently for the next two tasks. Often groups spend time re-capping ideas from week to week and this leads to much more time needed to reach a level of understanding.

Although modelling requires students to go through the entire cycle (see Fig 1.1), it would appear that some competencies or nodes of the cycle are more critical than others; understanding and mathematising would appear to be two such critical competencies. Although understanding is set out at the beginning of the cycle, students continually revise and revisit their understanding of the problem. Their understanding directs and is directed by the other competencies in the modelling cycle.

This area of competency is difficult to assess across tasks. The essence of their development in this competency can be narrowed down to two significant areas. Firstly, all groups improved in their endeavours to understand. Some re-read the tasks or the supporting information; other groups spoke more to get to a level of understanding. Secondly, groups 'slowed down' their understanding phases. Their improved perception regarding the nature of the tasks also assisted in this slowing down.

Task understanding took place continually over the three sessions that students were involved with a task. This meant that task understanding grew as students worked with the tasks as in Task 1 for Group 1. Group 1 was allocated a higher index for the second task because the understanding was a common group understanding while in Task 1 it was largely the understanding of one member that they were relying on. Group 2 managed to 'solve' Task 3 within the first session, and was allocated a 'lower' index for the next two because they were dependent on that understanding to complete the task.

5.2.5 Competency 5: Simplifying

Task 1

Task 1 (Big Foot) required little mathematical simplifying initially since no numbers were given with the task. Groups needed to come to a decision about what it meant to work out 'how big' this person is. All groups decided that this meant finding out how tall Big Foot is. The decision by all groups to measure one (or more) of the group members was a significant step in simplifying the problem. When groups measured more than one person it showed an important move towards the solution. Only Group 3 did not consider measuring more than one person. The simplifying needed also required a sorting out of many avenues to take once the groups had made some measurements.

Task 2

During Task 2, only Group 1 tried to simplify the vast amount of information that they were given. The decided to use only prices that were less than R5 since according to them R30 was not a lot of money. They wanted to 'match' prices with pocket money (intuitive proportionality).

N: Ok, so now we say that he should get how much more than her because... say R80.
M: Look here, she could buy this at R1,50 and he has to buy it for R3,50. He has to pay more than his sister had to pay. How much more should he get? About R40 or R50?

Group 2 started off initially using just a few items, but then decided that an average price difference is what they wanted, so they decided to use all prices on the catalogues. Group 1s simplification may have been too severe and not intended as a meaningful simplification but as a way they could deal with the data. This meant that they ended up using only a small part of the data supplied (only prices less than R5 from the catalogue items). Lesh et al. (2000: 598) remind us that students' early interpretations of modelling problems often mean that they only focus on a subset of the information available.

Simplification is dependent on what the group plans based on their understanding and is part of their mathematisation abilities. Group 2 saw in the world of mathematics that calculating an average was relevant, while Group 1 only used adding and subtracting so they needed to simplify their list first. Group 3 also simplified their catalogue items to show only what R30 could buy. They did not filter their lists any further than this. This meant that they only used 8 items from the entire list. By the time they got to this point a number of unsuccessful cycles had emanated. Mathematising a problem successfully, necessitates 'formulating a problem situation in such a way that it is amenable to further analysis' (Rasmussen, Zandieh, King & Teppo 2000: 2). This means students must make progress in simplifying or structuring of the problem so that they can further their understanding of the problem and solution of the task. Although simplifying is a separate competency, it is integral in mathematising.

Task 3

During Task 3, simplification played yet another role. Here it was simplifying the given quilt pattern into a usable format and size. All groups identified scaling as the mathematics involved. They had recently been taught scaling in Social Science class when using maps. This scaling was more difficult since the measurements involved decimal numbers so the proportionality was not 'obvious'. Group 1 started by scaling down the finished measurement by 4 hoping that that would make the problem simpler.

O: why don't we divide by 4?
N: then that will be 236 divided by 4 is ...59. Then we have to make it very small, but that doesn't matter. 59 goes down.

This was unsuccessful so they decided to scale up:

N: maybe we should make it like ten times bigger?
L: ten times? That will be huge.
O: make it 4 times bigger and then measure it again.

It was only during the second session that they decided

R: you were working on a plan of how to get from 23 (pattern measurement) to 200 (real width). What happened to that plan?
M: it is not the right number.
R: what do you mean it is not the right number?
Shows on the calculator: *23 divided by 200. 0,115.*
R: is that the right way around?
M: oh, it's the other way around.

They do not pursue this idea, but rather preferred using trial and error to multiply 23 by 'something' to get as close to 200 as possible. They worked their way to 8.6

Group 2 mathematised the problem very early and found their ratio of 8.6 by dividing 200 by 23. They however decided to use the entire number (8 decimals) on the calculator to be accurate but this gave them enormous problems since the calculator did not help in later calculations since it could not process so many decimals, so they were left with manual calculations which frustrated them. It was only during the third session that they decided to use only one decimal place (8.6).

Group 3's first ideas were to simplify the task using additive reasoning.

E: what I am going to do is measure this (pattern width) *and then whatever it is, then minus it from 200* (the finished size) *and then we'll have our answer.*

Later, they tried to divide the entire quilt pattern up into the same size squares. This complicated their work and showed that they did not really understand how the quilt pieces are put together to form the pattern. They spent a great deal of time measuring and re-measuring. They wanted to scale up one of the pieces, but were not sure to what or why. They planned to scale up a 4cm square to 16 then 32. An incorrect measurement led them to see that each small square could be multiplied by 10 to get to the finished length. They picked up this idea and together with researcher assistance re-measured and worked similarly to Group 1.

The visual representation of competency development is displayed in Fig 5.

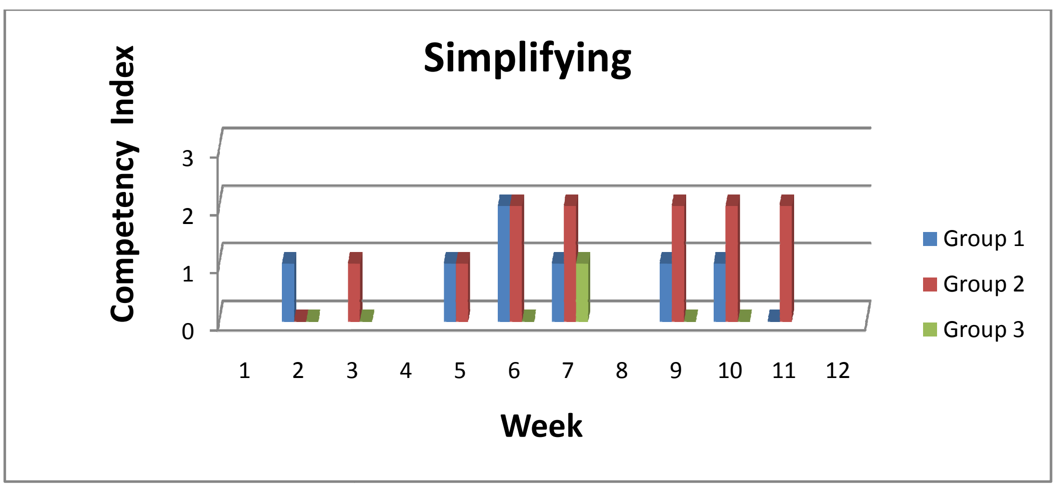


Fig5.4: Simplifying

What must be noted is that the improvement surprisingly comes after the first task and remains at a plateau for the remaining tasks. It would be interesting to see at what point the next improvement comes in over a longer period. Simplifying is used at different times during the task and for different reasons. Students revisit this competency a number of times during a task and not once as may be incorrectly inferred by the modelling cycle. Simplifying is not a once off, all or nothing competency. In some tasks it happens in small steps over the entire task.

5.2.6 Competency 6: Mathematising

Freudenthal's definition for mathematisation is to make a problem more mathematical (Gravemeijer & Terwel 2000: 781). Mathematising is a fundamental competency in modelling as with all problem solving since it involves the act of 'putting a structure onto a structure' or to 'establish the dependence of ideas on each other' (Wheeler 1982: 47). This means that mathematising involves order. To bring things to order means that students work through a substantial process of making interrelated and integrated decisions. This type of cognitive activity does not take place often in traditional classrooms. The teacher makes the ordering decisions or presents students with problems that have order or structure to them. In traditional classrooms mathematising is most neglected.

Coding group mathematising involved looking for more than just the mathematics groups chose to model the situation but looking for instances of their quantifying ideas or qualitative information in the real problem. In all groups the initial plan with Task 1 was to take measurements and then crunch out mathematics with the numbers. They added, subtracted, multiplied and divided their measurements randomly in order to try to find 'an answer'. Group 1 almost resisted any use of mathematics during the first session and kept returning to a creative descriptive analysis of the problem even after they said that the problem was about finding out how tall the person is and after taking some measurements.

N: How old do you think he is?
M: He is probably a handy man or repair man.
N: Is he divorced?

This was in line with Lamon's (1993: 46) first level of proportional thinking (see 3.3.1), that of avoiding. This may have something to do with students being unsure how to quantify qualitative information. But by the end of task, this group had created a fairly complex model that involved estimating a divisor until they reached the set quotient of 15. Although they had found a ratio between foot length and height by dividing, they could not use the inverse operation (multiplication) to find the missing height.

During Task 3, Group 1 realised that the problem involved scaling up or scaling down, but they spent a long time deciding about the ratio. They spent almost an hour trying various ratios and eventually estimated the number they needed to scale up from 23, 2 to 200. Although their ‘estimated’ methods are not too elegant, they understood what they were doing and what it was they wanted to achieve.

During Task 1, Group 2 went through a lengthy process of using all four operations in their search for something that would ‘result’ in the measurements they had taken in reality. They were intent on using some form of mathematics, but they could not find the link easily. It was right at the end of the third session that they decided to use their foot to height ratios and find the ‘average’; and even then their developing idea was unstable. They did however use the concept of an average to a greater and stronger degree during the second week of Task 2. Their first plan for Task 2 was to divide the page into the four areas and to apply the four operations to the list of item prices, but there were too many items. They did, however not stay on that cycle for very long before trying to come to a deeper understanding of the task. For Task 3 this group did not even pursue or write down a pre-determined course based on the four operations but gauged immediately after understanding the task requirements, that a scaling up and down was needed. It took them less than 20 minutes to calculate the factor of scaling. There was never a sense or need in this task to make the mathematics fit the problem, but rather they *found* the mathematics *in* the problem. They were gaining from their modelling experiences. It can be said that context played a role, but it is important to note that Task 3 was the furthest removed from their personal experiences.

The sentiment of Ahlfors et al. (1962: 190) is directly relevant here; that the most important activity in doing mathematics is to recognise or extract a mathematical concept from a concrete situation. This competency is central to successful modelling. This competency is also built upon their competency in language and fully understanding the task requirements. These students did not always see how the instruction, the end product and mathematics were all linked. Mathematising can develop within a task if students are given enough time and autonomy. Eventually, it is hypothesised, that with enough experience in modelling mathematising, students will develop the frame of reference needed to transfer concepts across tasks. The idea of local conceptual development (Lesh & Harel 2003: 161) is relevant in understanding modelling competency development.

There is evidence of increasingly complex mathematisation as the task develops. Transferability cannot easily be detected in only three tasks. What is very valuable in terms of improving mathematisation is that in group's moments of 'being stuck' they grasp at the mathematics that they do know and try to apply it. Then they decide if it does make sense and if it can be used. This may very well lead to more effective mathematising at a later stage.

The comparative competency indexes are shown in Fig 5.5:

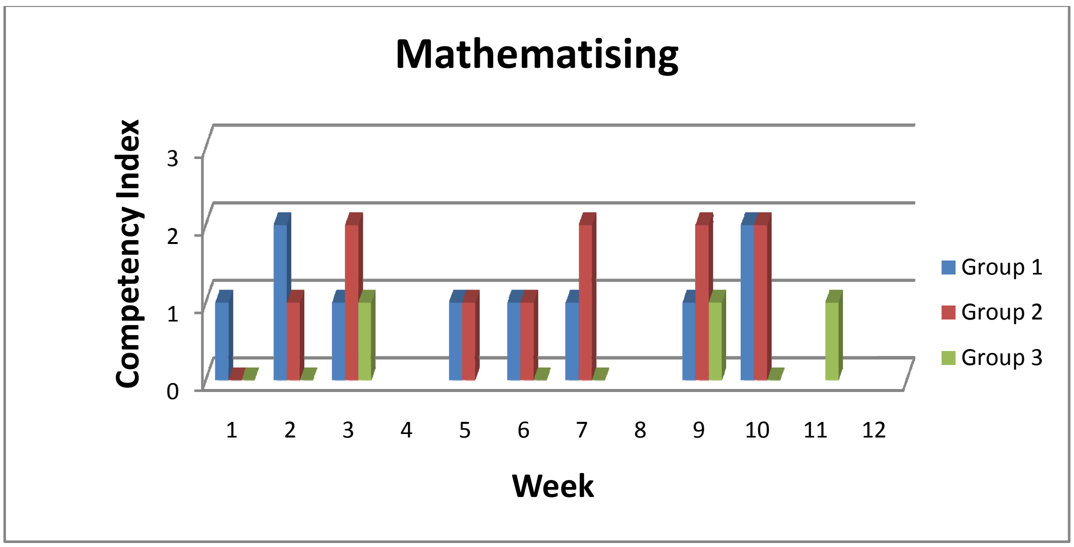


Fig 5.5: Mathematising

Mathematising competencies are not only task dependent but rely to a large degree on the mathematical 'toolbox' (Jensen 2007: 144) students have available to them and what experiences students have had in using their toolbox autonomously. Group 1 was not assigned an index for the last session of the third task since they had largely completed the task by the second session and were working on their presentation sheets in the third session. Group 3 was slow to mathematise the problem during Task 1, largely unable to mathematise the problem in its entirely in Task 2 and then the small improvement in Task 3 may be task and context dependent. The last task involved more visual, spatial work than the other two.

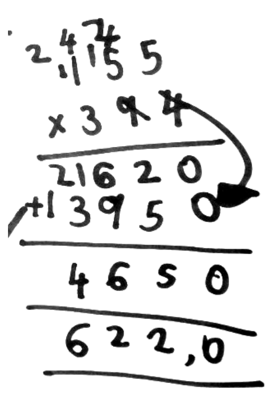
5.2.7 Competency 7: Working mathematically

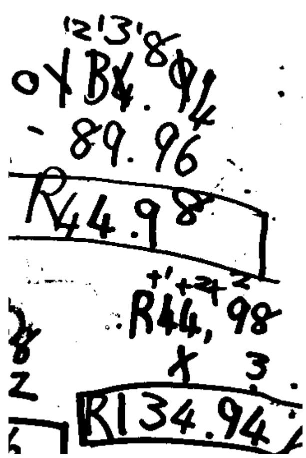
There are discrepancies in this competency, from some sophisticated uses of averages to being unable to use a measuring tape to measure foot lengths. Students seem to be so conditioned into measuring straight lines, desks or doors, that measuring something different to a straight line appeared to be a problem. Both Group 1 and 3 could not make sense of the tape measure with different units (cm and inches) while Group 2 knew which units they were using. The measurements made by Group 2 were more accurate (sometimes to the mm) than those of Group 1 and 3. Group 1 had many difficulties using a tape measure to measure accurately. One member of this group, in his post presentation questionnaire for this task wrote:

I learned how to measure.

By the third task, all groups showed great improvement in measuring abilities and accuracy in measuring. There was also a developing appreciation of accurate measurements.

During the first two sessions of Task 1, Group 1 and 3 used the calculator provided from the start but Group 2 performed all their calculations manually on paper. Group 2 seemed to prefer manual calculations for some aspects of their work throughout.





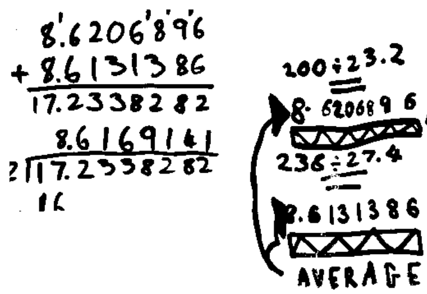


Fig5.6: Group2 Task 1   Fig 5.7: Group 2 Task 2   Fig 5.8: Group 2 Task 3

During Task 3, this group initially insisted on using the full decimal number on the calculator to be more accurate. This decision led to large numbers. Their comments during Task 2 and Task 3 on calculators:

S: calculators are nice, but they waste time

S: but calculators are stupid, they cannot calculate further than 8 digits.

During Task 1 Group 1 said that they would multiply by 100 to work out a percentage, but on further questioning they really did this to avoid a decimal. Percentages were not remotely part of their solution process. Sometimes students use mathematical words and terms in ways that are different to a teachers' use of the term. Further questioning is vital to negotiate meaning around words that are assumed to be set. Group 2 had many discussions around the meanings of words that are accepted as common to this age group such as length, width and thickness. However a large amount of negotiation still takes place. This has a significant impact on group understanding of the task. An incident from Task 1:

J: Because your shoe size is small and your length is a lot.
S: (correcting him) *Height*
J: What is wrong with length?
S: Well then we'll have to lie down. Long is actually bad when you compare it to a human being.
J: What is wrong with long?
S: You are very long; no- that just doesn't sound right.
J: You can't say something is long.

Using mathematical terms (which are also everyday words) loosely became a problem again in Task 3 for this group.

S: I am talking about length, not width.
J: I don't understand what is width and length because width and length of a square are exactly the same.
Later on
T: *that will be length and this will be the width.*
J: (addressing researcher) *she says that is the width*
R: maybe she is saying the thickness and not the width?
T: thank you.
J: why didn't you say that in the first place?
S: why didn't you say length in the first place?

Group 2 also 'created' their own meaning for words. They used words such as 'significance' and 'relation' for ratio in their own way.

J: So let's see what is the significance between how long the shoe.... (Task 1)

J: Now, let's see if we can see a difference, um, a significance between the two. (Task 2)
S: this is our relation to the real size! It is scaled down by 8 point 6. (Task 3)

Rounding off methods were weak in all groups. An answer on the calculator of 15,51 was estimated to be 15. The idea seemed to be to use the whole number and disregard decimals. Throughout the first task groups 'estimated' more than was necessary. A comment from Group 2:

J: but we are only estimating here …
J: I think it's estimation, you can't find this guy's height one hundred percent.

During Task 2, all groups remained within the realm of adding, subtracting and multiplying as the mathematics that they employed. They are comfortable and obviously using mathematics that they know. It was with disappointment that they were not using more sophisticated levels of mathematics that had been taught to them. Treilibs et al. (1980: 53) found that student abilities to apply mathematics to real situations lagged by at least three years after their first learning of it. Group 2 (based on their experience in Task 1) in one of their modelling cycles decided to use an average again. However, they used the average of the price difference of four items as a ratio and multiplied the R30 pocket money by this average. Shortly after this they moved into simple multiplicative reasoning:

J: No, look just wait, you know his sister got R30 and now he is trying to find out what they must get now, we must just see how much stuff, let's say 15 rubbers for R30,then if we buy 15 rubbers (today) for R135, then that is how much he must have. Then we check it it's the same for the other stuff.
S: Ok let's try. You write that method down.

Working with decimal numbers posed some problems to all groups. Group 1 used their calculator and rounded off incorrectly in Task 1. Group 2 had the following discussion about the decimal numbers they were working with:

T: but if you times 54.9, do you times it normally or do you times by ten or ten point zero?
J: I am confused; I don't know what you are doing.
S: It really doesn't matter; they are both the same number. You have to put the point in afterwards.

Group 3 could also not find any number between 6 and 7 in Task 1

W: I need to divide 9 or times 9 by something.
M: 9 times what equals 57? (9 being his foot length and 57 his height)
W: Nothing, nothing. I have tried 6 and I tried 7.
M: 9 times what equals 57?
W: Nothing, there is nothing. I am telling you. Try 6 and it is 54, try 7, because 7 is the next number, but 7 is 63. So there are no other numbers!

This discussion resulted in them multiplying to get the closest amount and adding the difference that they needed. By Task 3 when asked the same question by the researcher an improvement is noted:

R: what is in-between 8 and 9?
G: 8.5

When using the calculator for Task 3, they tried to calculate the ratio by:

E: 27.3 divided by 200, let's do that.
W: wow! (looking at the result on the calculator)
G: it can never be that!
E: you guys this is wrong, 27.3 divided by 200… look at that answer! (decimal result)

Task 2, with the context being money did not provide decimal problems for Group 1 or Group 2. Group 3 however did have some difficulty adding even when using a calculator:

M: did you put the point in? (to W who was adding on the calculator)
W: It doesn't work if you put it in.

There were gains all round in student mathematical working. This came from experiences in measuring and calculating. For Task 1, the type of mathematics employed by Group 1 was more sophisticated than Group 2 for Task 1. This supports earlier research which indicates that weak students can work mathematically to a greater extent than is expected (see 3.11). This was however not always the case and certainly not for Group 3. The weak students were however engaging with the tasks to a higher level than they do with their schoolwork. For Task 2 Group 2 used more sophisticated mathematics than Group 1, while Group 1 took much longer than Group 2 to find the ratio they were looking for in Task 3. The main difference seems to be that Group 1s concepts are more unstable than Group 2 and that they needed more time to call on their supporting meta-cognitive skills. During the Presentation sessions, some other teachers at the school who came in to observe were surprised that some of the weaker students were more confident than they had known them to be. As a whole view of this competency, the following figure is presented.

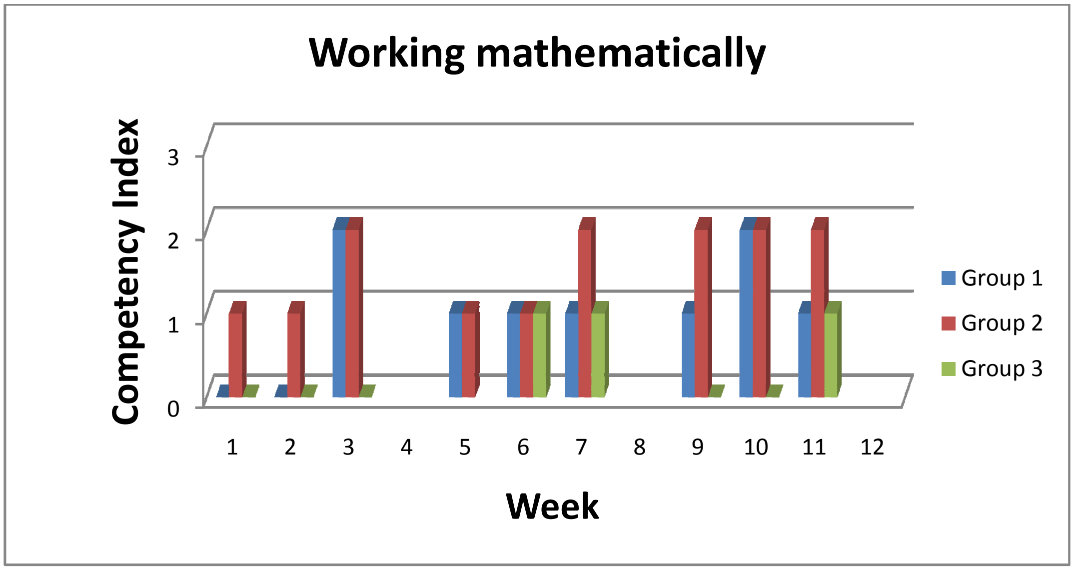


Fig 5.9: Working mathematically

The development of the competency to work mathematically will develop alongside students ‘school’ mathematics and their abilitiy to mathematise. As stated by Group 2 in Task 2:

S: HB pencil, R1,35, how would you change that to R3,15?

They knew what they wanted to achieve, but did not have explicit skills in doing so, so the task was to use mathematics that they did know to solve the problem. Their current understanding allowed them to use averages in a new way to understand proportional reasoning. Throughout groups and sessions a feeling that they were ‘using’ mathematics and not ‘doing’ mathematics emanated. The graph relfects groups matching mathematics they knew to the tasks and this improved for Group 1 and Group 2 within each task.

5.2.7.1 Proportional reasoning

As discussed in 3.2.1, proportional reasoning was the foundation of the tasks selected for the study. Up to this point of their schooling these students have had limited exposure to proportional reasoning. They have worked with fraction equivalence and simple rate problems. They had recently been exposed to scaling up and down in relation to maps. When using Lamon’s (1993: 46) proportional reasoning strategies, it was found that students shifted through all the strategies.

Group 1 showed ‘avoiding’ during Task 1. All groups used an additive strategy and pattern ‘noticing’ not ‘building’. Although the groups looked for and found patterns, they did not develop or build them to advance their solution progress.

O: (looking at the table they constructed) *it’s like a pattern, plus 1, plus 2...*

Group 2’s suggestion during Task 1:

S: Write down size 5 is 26cm. It’s not that far from 39? So Size 8 is 29. That’s a pattern.

J: But the bigger the feet the lower the number will be.
S: I see a pattern.

And Task 2:

T: How would you start this?
A: Maybe we should find a pattern.
J: Maybe we should look at the difference at the stationery store and see a pattern.
T: Is there a pattern?

All groups moved towards pre-proportional reasoning in Task 2 where they intuitively made sense of the proportionality in the problem. By Task 3, students in Group 1 and 2 had managed to find a ratio, but did not fully understand the implications of that ratio. The task context led all groups to recognize it as a scaling up and down type of problem. This may have facilitated them to find and use a ratio but this is still in a fragile state for the weaker groups. Their reflexivity between multiplication and division was still pre-proportional during across all tasks. The following discussion from Group 3 during Task 1

E: maybe we can divide it; we can do that and find the answer, and then times it again.
G: 22 divided by his height is 0,15 (The same ratio that Group 1 found)
R: how are you going to use 0,15?
No answer

During Task 1, Group 1was on the verge of multiplicative reasoning, although they could not relate a ratio as a quotient to a ratio as a product.

M: my feet were 9 and my height was 58. I divided it and timesed it by 100 equals 15. So now we have to do him (Big foot). *So it is 16* (Big foot’s shoe length) *divided by <u>something</u> times 100 equals 15.*

During Task 3, Group 1 was trying to scale a measurement of 23,2 up to 200. They proceeded, much the same as in Task 1:

M: so now we have to see what we have to times it to come to 200, so now (takes the calculator) *200 divided by* (guesses) *36.*
They narrowed it down by trial and error to 8.5
M: that's 197
L: close enough (they needed a finished measurement of 200)
N: so plus 3

It must be said that students need more exposure to real problems dealing with proportional reasoning such as those selected for the study. Weaker students have a poor intuitive concept of proportional ideas. Their low levels of proportional reasoning are directly related to weaker mathematising abilities, and this can be directly related to the type of experiences they have been exposed to.

Some members of Group 3 were still using additive reasoning in Task 3 when trying to find the scaling factor to multiply pattern length to the length the finished product should be:

E: what I am going to do is measure this (the width of the pattern piece) *and then whatever this it is, then minus is from 200, then we will have our answer. ...*

Later

E: or if we had a number that we can plus
G: we can times and then minus something …

R: how do you scale up? What is the next size? How do you decide?
W: like Social Sciences, scaling up or scaling down
G: I think you can add or multiply, but I think rather add so that I don't get it too big

And during the last session:

E: this measurement must become 200cm. 23.3 and we need to get 200cm. So let's say
G: how about times 10
After trial and error
G: try 8.6
E+G: 200!

Proportional reasoning development is a slow process but a subtle increase is evident in the groups who took part in this study although still unstable. With sufficient experience and contact with real tasks it is expected that this will develop and become more established. Proportional reasoning develops within a task and not necessarily across tasks over such a short period.

5.2.8 Competency 8: Interpreting

Interpretation involves translating the resulting mathematical solution in terms of the real problem. Comments students made that showed that they were interpreting their mathematical results were scrutinised for evidence of developing interpretation competencies. Very often the comments were accompanied by expressions of surprise and astonishment. This competency was indexed in terms of the group's interpretation of their solution within the context of the real problem. How much information and the nature of the information supplied to them for each task also needed to be taken into account.

Group 1

Group 1 did not check that their final solution for Task 1 provided a credible answer in terms of the real problem. They were sure that their mathematics was correct and assumed that their solution must be correct. They did not revisit the real problem to decide if their height calculation was possible or acceptable. They did continually check what their model would mean in terms of the problem.

M: so then he is about 75 inches tall.
L: so then he is 75 inches tall, what's that, like up to the roof?
M: He is 92 inches tall because my feet were 9 and my height was 58. I divided it and times by 100 to equal 15. So now we have to do him.

They came to measurements such as 92 inches or 98 inches, but never measured that out to check what that height meant in a real situation.

During Task 2, Group 1's interpretation was largely intuitive since their initial models were weak. The kept reminding themselves that they had to be 'fair'. They used this to interpret their solutions.

O: ok, but R60 won't be fair to his sister.

This led them to make sure that both brother and sister had the same amount of change after certain purchases. They did not consider that the same amount of change would not have the same value. Here, their interpretation of the situation shows their additive reasoning:

N: if he gets R50, he will get R12,34 (change), *but he is buying the same things as his sister. Now she gets R5,75* (change) *and he gets R12,34, it will be unfair.*

The context of this task strongly assisted in understanding the problem, but their mathematising of the problem was weak which led to weak models and interpretations. They remained within the real situation and worked with mathematics in a superficial manner. This naturally meant that their interpretation and validation of their working and model would be shallow.

During Task 3, this group's interpretation was reflexive since they had moved between the real measurements and the pattern measurements continually in their search for a scaling factor. Once they had found the scaling up factor, they felt so sure of their calculation that they felt that there was no need to interpret the ratio.

Group 2

Group 2 always tried to visualise their potential answers. At one point they calculated that Big Foot was over 5m tall. They decided this was not possible when they compared that the size of the building they were sitting in. They were not happy with an answer of 3m either as real height of a person. This group had the most well developed competency of interpreting.

This group's mathematisation of the second task meant that they had to interpret their solutions. They were not as concerned as Group 1 about fairness, but allowed themselves to be led by the price increases over the ten years. They used informal knowledge about pocket money and 13-year olds to assist in their interpretation. They moved more deeply into a mathematical realm in this task than Group 1 did and therefore had to return to the real situation to interpret what they had calculated.

S: times R30 gave me R135, which is a good amount.
J: but I think R150 is a reasonable amount.
A: but if his sister got R30, wouldn't you say…. (it's unfair?)
S: well then, shame. But you could buy pretty much the same things with 150 than you could with 30.

This group had serious problems when interpreting the 0, 5 cm edge that had to be added to the sewing pieces of the quilt in Task 3. They quickly and comfortably calculated the scaling factor for the quilt, but when deciding how to add on the edge for sewing, group members differed and they spent a long period of time sorting this out. This led them to re-define common terms such as length; width and thickness (see 5.2.7).

Group 3

Group 3, whose final solution for Task 1 was over 3m, did not consider that this may not be possible in a real situation. During Task 2, they used simple additive reasoning within the real situation and did not interpret their result as they remained within the real context of the problem only, while in Task 3 they interpreted their results twice which could be an indication of gradual competency development.

E: yes, but each block cannot be 172cm.

In the following session one group member was intuitively interpreting that their measurement for one block would give a correct measurement for the entire bed.

W: all right, this is how big the block is. There are 20 of these on the bed.
W: yes, there are 20 of these on the bed. (He counts them)

Task formulation has a strong influence on student interpretation competencies. This together with their depth in mathematising allows for more or less meaningful interpretation of their mathematical working in terms of the real world task. There was more interpreting of results for Task 1 than for the other tasks. This could be explained as a task context factor or the groups being more insecure about the mathematics employed for this task than in the other tasks. When groups are surer about the mathematics they have used, they less readily interpret their results.

The following graph shows the development of the groups' competency in interpreting their solutions. Some differences occur because groups move through the problem at different times. Some are ready to interpret during the second session of a task, while other groups only reach this during the third session. It is therefore suggested not to compare groups during sessions, but rather across sessions. All groups did not re-evaluate their solutions for Task 3 since they were so certain that their calculations were correct. The less students mathematise a problem, the less they are able to interpret the problem.

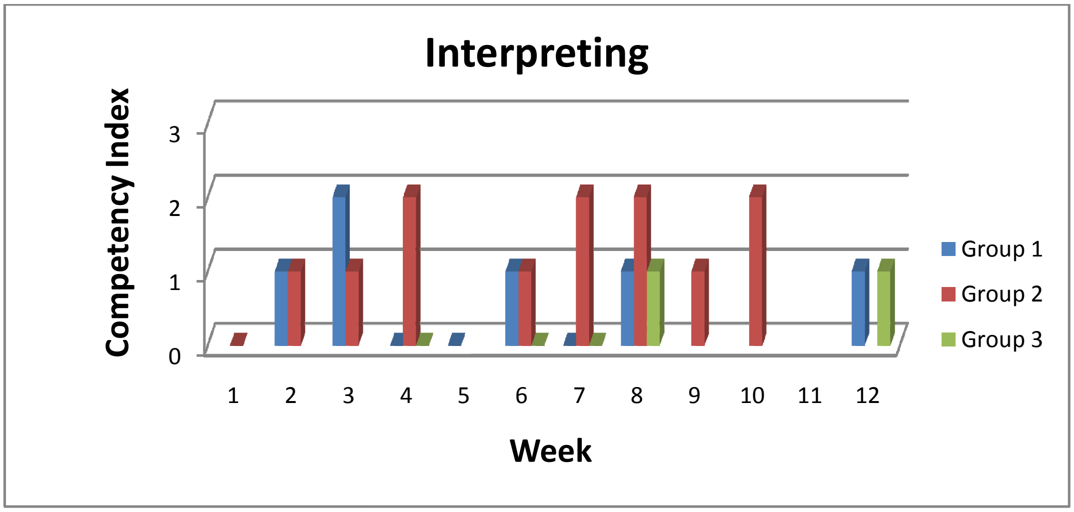


Fig 5.10: Interpreting

The only group that regularly and steadily interpreted was Group 2. Their language and mathematical ability assisted with this. Group 1 showed adequate interpreting for task 1 and then struggled during task 2 and 3. Their model for Task 1 was particularly good, which may have contributed to this. Group 3 was still struggling with this at the end of the third task since the competencies preceding this are not well developed either. Their low level of mathematisation directly affects their interpreting.

5.2.9 Competency 9: Validating

The context of the task and tasks being so different to traditional problems seemed to be a factor in students validating and verifying their models in generalisable terms. There were more excerpts of conversation relating to validation in Task 1 than in the other two tasks. It then must be concluded that task context influenced validation and not that the competency to validate did not improve. Task 1 explicitly requested that their model work for other shoe sizes whereas the 'generalisableness' of the models for Task 2 and Task 3 was implied only.

Group 1

Towards the end of the first task Group 1 developed a feeling that their model must apply to all feet sizes and heights and this became their verification guide to their working. They continued their investigations at home and with other students in order to validate their model.

M: I think we are going to have to do a survey to do this …
N: Let's go measure the other groups …
M: I also tried people at home and Beth at school. She was also 15.
M: Let's do N's foot to be sure

O: How come I am 20?
M: I don't know, maybe your feet are bigger than your… (height?)

When they discovered that two people shared the same foot to height ratio of 16:

M: You are 16 and N is 16. Then we must use 16.
R: why did you decide that N and I have the right measurements?
M: Because you are the most.

Group 2

Group 2 validated all possible solutions and models to a greater extent than the other groups.

The following comments during Task 1:

J: Its five meters! 5490. 5 meters. Gee!
S: You know that is a storey, a storey and a half!
J: Three meters should be the most.
S: Three metres are you nuts, that's like storey high!
J: But we know that a person cannot be 5 metres

S: This is a dead end.
A: Why?
S: Because it works for one and not the other.

S: Why don't we use that method because it works on other shoes?
J: It's better than 5 meters.

S: It looks like it is working for all the shoes!

T: What is the method?
S: Timesing 5.5 by the length of the shoe.
J: 2.1
S: 2.1 is a reasonable height.
J: I think that is reasonable.

Group 3

Group 3 needed to be prompted to consider validating their model. They validated that their method would work for one member of the group, since they had his height and foot length before they tried it out on Big foot.

G: I decided to times the length and the width again by the half length of the foot.
R: how do you know that that is correct?

E: It doesn't work on W's foot, he turns out to be 15.6 inches.
G: You have to do the same to W's foot and if it comes to his height it means we are correct with this one.
M: We are trying to do a sum to get his length, but we are first checking because we've got W's foot length and we want to check if it comes to his length.
E: So, if it is the answer, we have the height of Big foot.

Validating remained in the realm of what Borromeo Ferri termed intuitive validation (see 3.5.2.1) and seemed to be more prevalent during Task 1. The validation used in Task 2 was related to wanting to be fair and their use of extra mathematical knowledge (Borromeo Ferri 2006: 93). In Task 3, a sense of correct mathematics overrides the need to validate the solution. The following graph shows how the validating competency developed over the three tasks.

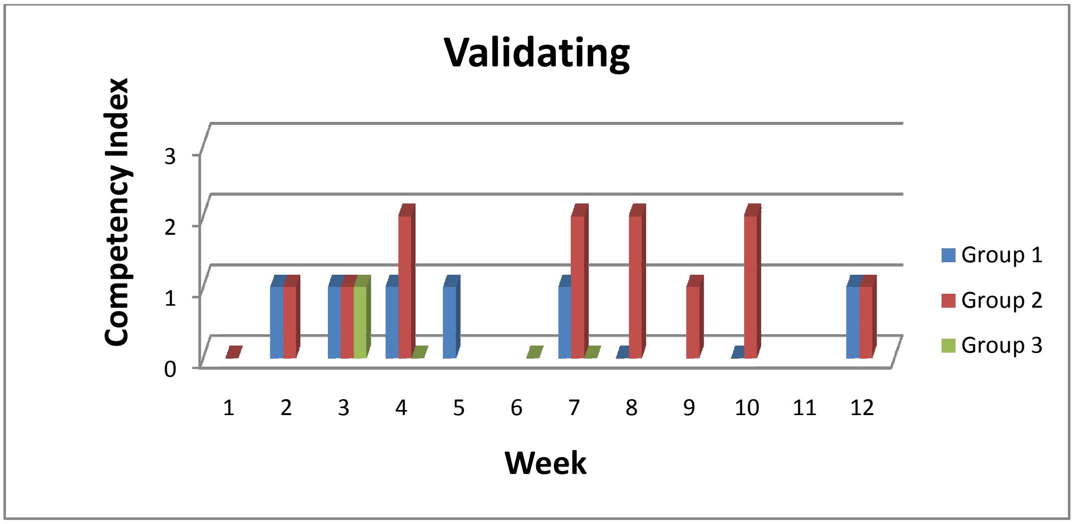


Fig 5.11: Validating

The graph for validating follows much the same trend as that of interpreting. Group 1 showing incidents sporadically especially during Task 1 while Group 2 more capably validating their work and models. Group 3 being too consumed in understanding and mathematising the task to engage in validating their work. Their confidence and experience in mathematics is a factor. A study of a much longer duration would assist in determining the development of this competency more accurately.

5.2.10 Competency 10: Arguing

This relates to a student's ability to link logical thoughts or ideas. In collaborative groups, often the next logical thought is provided by a different student than the previous logical thought. This enables one to clearly see what is described in section 2.9.3 regarding the benefits of group collaboration. Clearly, good language ability will influence an arguing competency. This is the reason why Group 2 could reason through many aspects of the problem.

In this study both single student reasoning and collaborative reasoning were found in group discussions. The longest sessions of reasoning and arguing came about when students were explaining their models. All the groups managed to sustain a longer period of argumentation on a particular feature of the problem as the tasks progressed. Group 3 particularly, did not pursue a line of thought for more than a sentence or two during Task 1, but this improved by Task 3.

Once again, the period of 3 month teaching experiment needs to be revised to allow for considerable competency development. The capacity to develop an argument seems to improve as understanding and mathematisation improves. Another aspect noted, was that by the third task more group members were involved in argument building, even if it was simply to state that they did not follow the argument or to ask other members to explain their reasoning or decisions. We are however reminded by Treilibs et al. (1980: 53) that poor modellers display discussion that is 'rambling and cyclic' and that they have shorter 'running memory spans' thereby generating longer and less efficient arguments than better modellers.

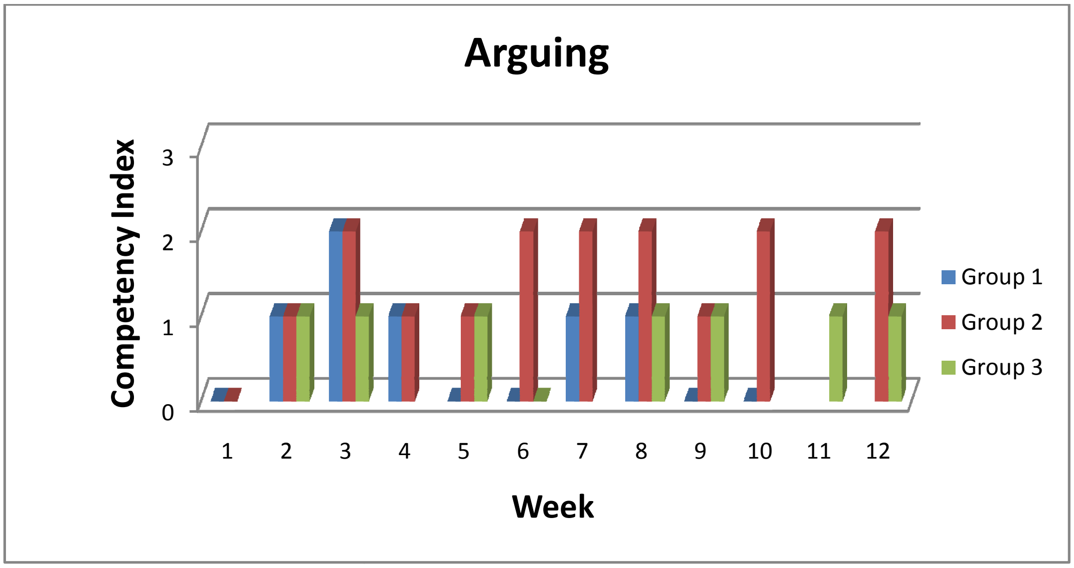


Fig 5.12: Arguing

The result of working within a group elicited a number of incidents of arguing since students had to be in touch with each other all the time and were attempting to understand each other. Group 2 achieved this to a much higher level than the other groups. Group 1 managed to maintain a level of arguing over the first two tasks, but was largely unsure as a group of how to proceed for the third task. Group 3 maintained a consistent level but were unable to plow this back into usable results for their work.

5.2.11 Competency 11: Using Informal Knowledge

Students did not understand the question on the post-presentation guide (see Appendix K) regarding "using outside knowledge". They do not compartmentalise or segregate what they know into distinct areas. This competency is also dependent on the task context. Students did add observations and annotations during the group interactions regarding their own knowledge about the task. It is not certain what effects these comments had on task resolution, but it is necessary to improve beliefs about mathematical tasks. Students should be encouraged to offer what they know in their real lives about the task. The groups' use of informal knowledge developed through the three tasks as follows:

Group 1 Task 1(making general comments)

M: I say he is tall, with a foot like that.
N: You get some people that are short and have big feet.
M: My aunty's friend has to bend down to get through the door.

M: He (big foot) *must be 90 to 96, because people are all different.*

While by Task 3 their informal knowledge was being put to use specifically for solving the task.

M: So this is like Social Sciences, scaling up and scaling down – will that work?
M: If we could draw this on computer and drag it bigger, it would be much easier.

Group 2 started with some strong use of informal knowledge to understand the task during Task 1:

T: So what about a small person that just happens to have a large foot?
J: But you know a person cannot be 5 meters.
T: Clothing comes in sizes, shoes could be the same.
S: Male feet are bigger than females.
S: You will obviously be taller than him, you are a female and you mature faster. You will stop growing in High School, we will stop growing after University

While in Task 3, Group 2 used their informal knowledge to assist them to make decisions about the more refined detail of adding on the sewing edge.

A: They have the little blocks and stitch them together. That's hard. That's a lot of work to put this together. Like embroidery.
T: But these are elderly women; I don't think an old lady want to work that out (adding the sewing allowance).

Group 3 made important progress in this competency. During Task 1 there is only one incident of using some informal knowledge to assist with the task, while during Task 3 they made more use of their informal knowledge.

While it is difficult to compare use of informal knowledge across tasks, what was noted was that all groups seem to be improving in this area. The following graph shows a visual representation of this competency index.

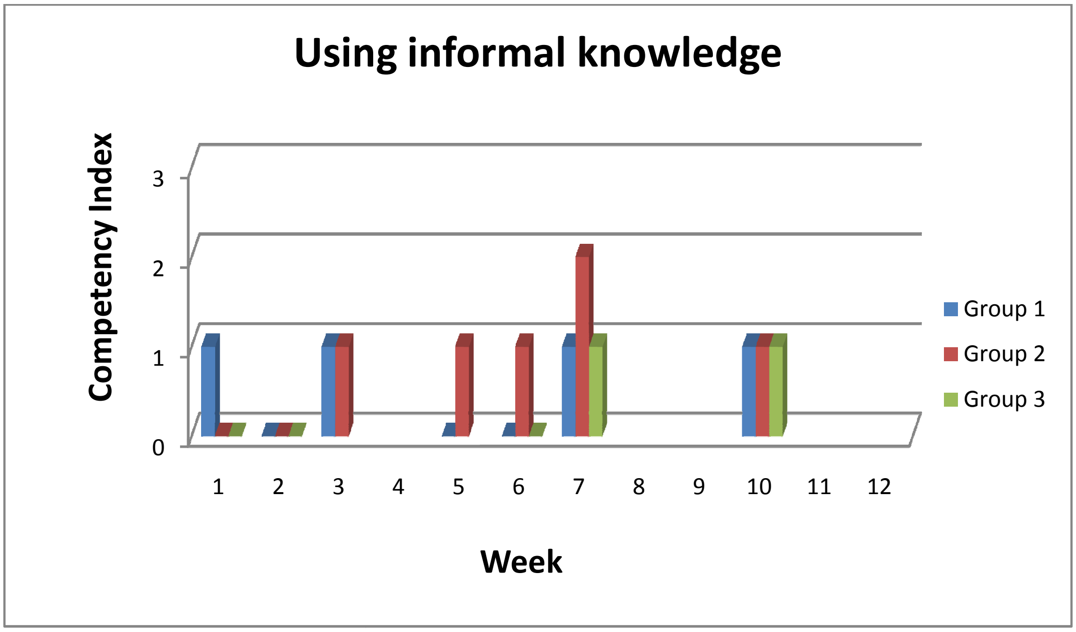


Fig 5.13: Using informal knowledge

It is clear from this graph that using informal knowledge is not something that students do readily in mathematics lessons. Group 2 used their informal knowledge to a greater degree. It appeared with Group 2 that their knowledge is more 'networked' and integrated than in the other groups. It is interesting to note the beginning of Task 3 where groups were using the same type of informal knowledge to the same degree.

5.2.12 Competency 12: Presenting

The first presentation session took place after 4 sessions of one hour. The first being the orientation session, while the other three sessions were working sessions on Task 1. Students were apprehensive about this session. All groups ran straight into the solution without taking the audience through their process as was requested. They did not seem to want to talk about their errors or dead-ends. Group 1 seemed most comfortable with their presentation. They even called a member of the audience to show how their model worked. Group 1 used a table in their presentation while the other groups had mostly calculations on their pages. It may be that this assisted them in structuring their talk and subsequent modelling of the situation to the audience.

The groups had problems in explaining their procedure or giving a historical account of what happened in their groups. Often they forgot how they arrived at a number or decision and had to be reminded about the intermediate phases of the solution process. The underlying purpose for presentations for them was to give and explain the answer. Deciding what to include and what to leave out in the presentations also requires practice and direction. The strong group, with stronger language capabilities, were able to explain their process and their model more fully.

Groups were allowed to question each other about their models. For all three tasks, it was mostly Group 2 that asked the other groups questions and criticised their models and to a greater extent by the third task. Students developed better questioning and criticizing skills during these sessions, although this competency falls out of the scope of this study.

Not only did the documentation they produced improve for presentation sessions, but Group 1 and 2 started considering the presentation sessions from early on in Task 3 which did not happen in the previous two tasks.

Group 1 (Task 3)

M: don't put a mark on that (N wants to write on the pattern piece). *We need it for our presentation. We do like the others did. Don't put a mark on it!*
M: we know exactly what to do. We will put the exact amount on the chart in our presentation. We put that amount over there.

Group 2

T: this is going to be your part of the speech, to explain. (at the presentation)
S: you know that paper is the one we are going to draw neatly on.

The following graph shows the results of student presentation sessions. These were dependent on the presentation of the solution, the generalisability of the model and extent to which the task instruction was met (see Appendix A-C). Week 4, 8 and 12 were dedicated to presentation sessions. Each group presented their solution and model to the other groups. Groups were allowed to ask each other questions and comment on the models. As the weeks progressed the groups became more involved in posing better questions and making more astute comments.

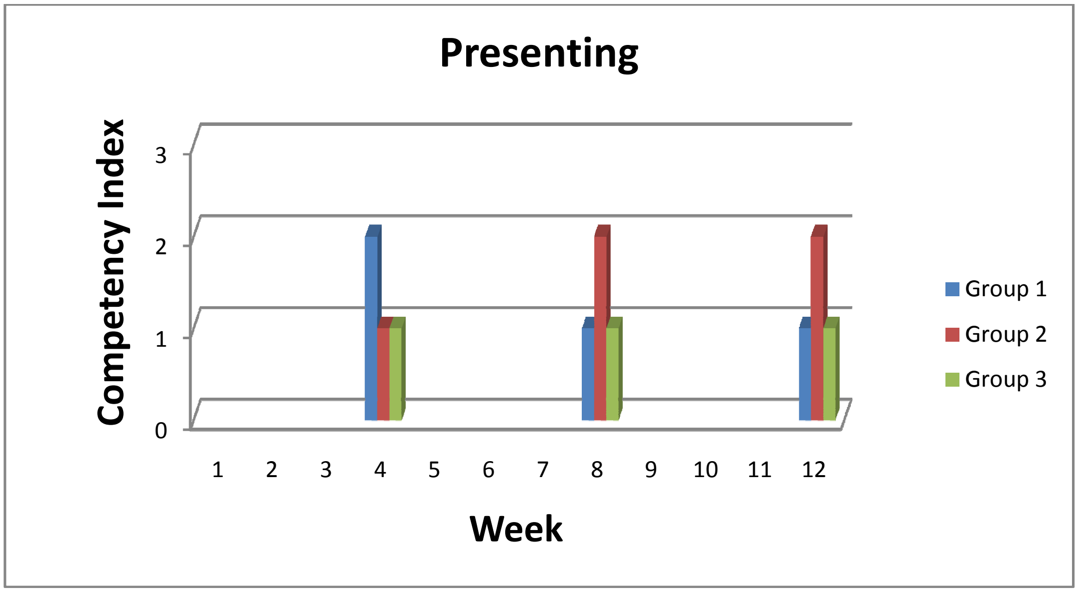


Fig 5.14: Presenting

Group 1 conducted a much better presentation for their first task, once again it must be stressed that they had a very good model. Group 2 managed to conduct much better presentations from the second task and became very involved in questioning and criticizing the other groups. Group 3's presentations reflected their 'incomplete' models. Groups need a great deal of time to prepare for presentation sessions if their contact sessions with the task are a week apart. It will be more effective if students worked over consecutive days on tasks.

5.2.13 Summary

Competency development in mathematical modelling is very complex. The competencies are interrelated, interdependent and non-linear. According to Sternberg (1998: 131) in the abilities domain, 'everything correlates with everything'. Student achievement in one competency affects many other competencies. Student competencies can and do develop when students are allowed to model. Weak students need more guidance, support, confidence and experience. Where heterogeneous grouping is allowed, group interaction and progress would be more flowing and more active. Homogenous grouping runs the risk of stagnant discussion and competency development.

Competency development in this study was slow and modest. The timeframe (3 months) must be revised and extended to allow for greater competency development. Further studies should take place within the mathematics classroom as it was difficult for some students to attend all the sessions after hours. Although development is slower than would be considered ideal, it is of a more substantial and authentic nature than the type of development that takes place in purely computational classrooms. The competencies identified for study are broad and multi-faceted which means that some aspects may show signs of improvement but may not be enough to suggest that the competency as a whole has improved. A further aspect expressed by van den Heuvel-Panhuizen (2003: 13) is that models bridge the gap between informal understanding of reality and understanding formal systems.

The following is a visual summary of competency development in this program. As stated in 4.3.2.5, the scores allocated are indexes of competence at that time and for that task and not specific measurements. It is of particular interest to note the difference in weak and strong groups as well as the difference between the two weak groups. If the graphs as seem as stitches in a tapestry, then Group 2 has a more fully stitched tapestry of competencies that seem stable across the tasks while Group 1 is developing competencies while Group 3 show vast areas of need. Competency development is not consistent or predictable, but it can be said that these groups have developed and grown by being part of the program. As mentioned by Lesh and Zawojewski (1988: 59), students had definitely not learnt about multiple reconceptualization cycles at school. Their competency development can be attributed to their exposure to this modelling program.

Key for Fig 5.15

u	understanding
s	simplifying/structuring
m	mathematising
wm	working mathematically
i	interpreting
v	validating
p	presenting
uik	using informal knowledge
p/m	planning/monitoring
sd	sense of direction
a	arguing

Table 5.3: Key for competency development graphs

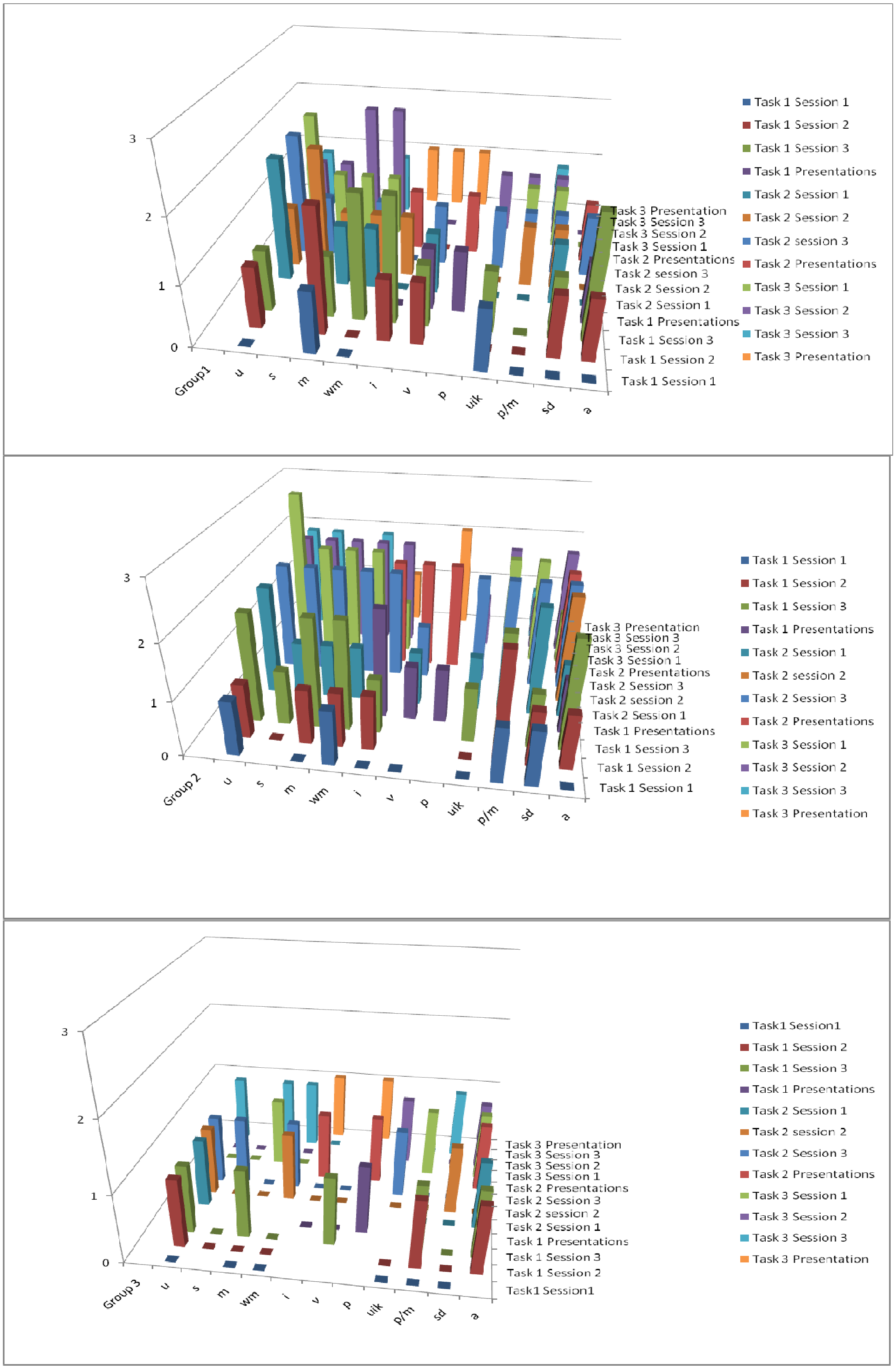


Fig 5.15: Comparison of group competencies

5.3 REVISITING THE SIX INSTRUCTIONAL DESIGN PRINCIPLES OF MODELLING

In order to gain an even broader picture of modelling competency development it was decided to not only focus on the individual competencies at each node of the modelling cycle, but to consider bigger fields of interrelated competence. In this section although individual competencies will affect the discussion, the groups' abilities to deal with broader modelling abilities is considered. The discussion relates to all three tasks and significant aspects that deal with each question are highlighted.

5.3.1 To what extent does the group make sense of the real life situation?

Context and experience are important factors in answering this question. Each group made better 'real life' sense of Task 2 than of Task 1. The supporting information in Task 1 was largely misinterpreted and led to more confusion than was necessary. When designing tasks, care must be taken to ensure that the amount and type of supporting information is justified. Group 2 and Group 3 did not see the link between animal footprint length and size of animal as having a correlation to the problem. Making sense of the real life situation did not necessarily improve their model construction or the complexity of mathematics that they used, but resulted in more group members being involved more often in discussions which led to more ideas.

Task 2 presented these students with a very close match to their real lives. Both Group 1 and Group 2 felt the need to be 'fair' to the sister. Group 1 understood that they were dealing with normal family relations and fairness was part of their solution. Group 2 mentioned fairness a few times but were guided more by their mathematical work in the end than by issues of fairness. For Task 3, all groups mentioned that they had seen a quilt before, but it was only Group 2 that commented on the examples of quilts as being 'cool' and 'awesome'. They also added that they thought making one would be very difficult. Group 2 gave much time to discuss the seam needed for sewing the pieces together, while the other groups did not consider this at all in their models. They spent too much time working on the mathematics needed for the shape and sizes of the pieces.

In analysing students' use of informal knowledge, two situations occur. One where they add information that they know about the situation but this information does not assist in solution progress and another situation where this informal knowledge does assist in solution progress.

M: You get some people that are short and have big feet.

which assisted them to understand and adjust their model while

T: nowadays you pay more for the popcorn than you do for the ticket.

shows an understanding of the real situation, but did not assist explicitly in solution progress.

Since students only had one task to deal with at a time, there was never a need to simply give any answer and move on to another task. Only once, (Task 1) did Group 1 suggest a 'senseless' answer that did not take the real world problem into account.

L: Just say he is 20m tall.

Groups did try to make sure that their answers were meaningful in terms of the real world question although they were not sure which aspects to consider. They all engaged with the task in the real world. This must be ascribed to the quality of the tasks and the real world context in which they have emanated. When looking at section 5.2.11, a gradual increase in students' use of informal knowledge is evident. Even though some instances do not directly help with task solution. An increase in crossing between the real world and the mathematics world is important. This ties in with mathematisation which involves making decisions to bring order to a problem. It is in this aspect that these students lack experience and confidence.

5.3.2 To what extent does the group construct a model?

Group 1 and Group 2 developed pleasing models for Task 1, as they understood the value in having a model to use in other situations also. Group 1's model for Task 1 was of a more complex nature than that of Group 2 but Group 1's was largely one student's idea while that of Group 2 was collaborative. Group 3's model involved additive proportional reasoning so it did not function as a model further than the one example they provided. The following table shows excerpts of each group's model development for Task 1.

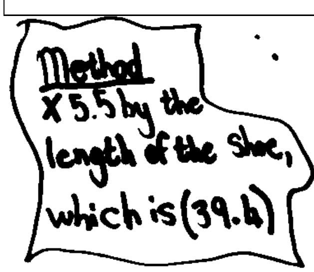
Group 1	Group 2	Group 3
He is 92 inches tall because my feet were 9 and my height was 58. I divided it and timesed it by 100, equals 15.	I can say 1.52 divided by 26.5 then I will get the exact answer	Why don't we say this number *(foot length)* plus what or times what equals his height
So now for him. So it is 16 (*Big foot's foot length)* divided by something times 100 equals 15.	5.7 (*the group multiplied by 100 to avoid the decimal)*	22.5 times 9 is too much
	Then if we times that *(foot length)* by 5.7 we might get his height.	22.5 times 6 is 136
	The bigger the feet the lower the number will be.	So, times 6 plus…
	What if we worked out an average?	You times it by 6 and then you plus 12
	Let's say the middle is 5.5, then it will work.	Where do you get 6 and 12 from?
		

Table 5.4: Group model development for Task 1

In terms of the Groups' development of models for Task 2, they remained within the realm of elementary mathematics (only Group 2 used an average, the other groups only added and subtracted). The weaker groups were not sure what to do with the large volume of information provided and their simplifying ideas were weak. This led to models that did not take the full situation into account.

It also appears that the more students give consideration to the *generalisable* model; the better the specific or situation model is that they construct. When groups had difficulty with the specific situation model, a generalisable model was not forthcoming. The specific and generalisable model work reflexively. The possible reason why students produced better models in Task 1 can be summarised as follows:

Task 1	There was a specific instruction to create a generalisable model Only a footprint provided	Remarks: Footprint to height ratio was easily seen to be generalisable
Task 2	There was a specific instruction to create a generalisable model Vast amount of information: lists, advertisements etc	Remarks: Prices and pocket money may be time dependent and less generalisable
Task 3	No specific task instruction on creating a generalisable model. Vast amount of supporting material	Remark: clearer task instruction needed

Table 5.5: Task instruction comparison

It may be valuable to set tasks up for novice modellers in such a manner that the need to create a generalisable model is stated more explicitly. The word "generalisable" is also too difficult for students with poor language abilities. The researcher suggested they find a method to calculate 'any' footprint or price to express the need for the group to generalise their model.

5.3.3 To what extent does the group judge that their own ideas, responses and models are good enough?

This is directly related to the meta-cognitive competencies of the group. Group 2 had one member who continually acted as the meta-cognitive 'coach'. He was not always instrumental in coming up with the ideas that led to a solution, but he was vital in getting the other members to talk, argue and refine what they were thinking. He repeated phrases like:

S: come on people, they want a method.
S: what is this going to help us?

Group 1started developing a feeling for being able to judge their own solutions during Task 2. It may have emanated from the researcher saying earlier in the task:

R: tell them the amount to give him, and why you came up with that amount.

M: he has to pay more than his sister had to pay. How much more should he get? About 40 or 50 rand.
N: That doesn't give us an answer!

While during the following session:

M: no, no, no, can you listen here. We are going to take R30 and buy as much stuff as we can of items below R5, ok? We take R30 and try to buy prices below R5 here (1999 list). *Then that much must be left with R60, so whatever you can buy with R30* (on the 1999 list)*, you must be able to buy with R60* (on the 2009 list).
N: Ya, that's a good idea.

Although they pursued this for some time, they quickly became sidetracked and explored other avenues that were not as useful. A feature of the weaker groups is that they became distracted and unfocussed by each other's input and ideas. They are largely unsure of what is a 'good' idea or a 'weak' idea.

Group 3 lacked students with suitable leadership abilities (Watson & Chick 2001: 168) to make suitable judgement on their progress or ideas. On closer examination of the groups, their ages were considered as a factor and were gauged as at the start of the program.

Group 1 Members: age in years:	Group 2 Members: age in years:	Group 3 Members: age in years.
11;12;13;13	12;12;12;13 years old	11; 11*; 12; 12 years old. *This member had turned 12 five days before the program.

Table 5.6: Group Ages

It may be a factor in modelling competency development; that Group 3 has the youngest members – two members being only 11 years old at the start of the program. This may have contributed their continual arguments about petty aspects of the task and their inability to progress without extensive assistance. Group 1 and Group 2 both had members who were already 13 years old. One of the 13 year olds in Group 1 was also the generator of most of the ideas in Group 1, while the 13 year old in Group 2 was their 'meta-cognitive coach'. In terms of proportional reasoning development (see 3.3.1) age is a significant factor. These younger students are only starting to enter Piaget and Inhelder's substage III.

The groups also used mathematical accuracy to judge their models. If they were sure that mathematically their working was correct, they assumed that their decisions were good. Group 2's discussion for Task 3:

S: Ok, I divided 200 by 23.2, right and it came out (8.6) *and then I timesed it by that* (8.6) *number and it turned out like this, look how close it is.* (199.52)
J: Yes, and then why did you times it by 8.6? Obviously it's going to be the same because you divided it by 23.2 (Laughs).
S: this is our relation to the real size! We are going to use that scale when we make the templates.
J: let's work out the average between the two.
J: yes, it works!
S: it is so close to the other ones.
A: at least its close, if it wasn't that would have been weird.

Group 3 also used 'numbers' to help them decide how useful their methods were:

E: 200 minus 27.3 equals 172.
G: is has to be something by something because it is a square.
E: yes, but each block cannot be 172cm.

and sometimes needed the researcher to judge their work:

W: is she right?
R: you should decide.

Groups did not always focus on what they are doing, but rather if the answer seemed correct then their method or process must be correct. This relates to their real understanding of the task and their ability to mathematise the problem. When groups were unable to judge their solutions, it implied weak understanding and mathematising.
This question leads one to review how reflective the groups were about their work and their own thinking. Reflection is not something that is developed with traditional instruction and as such these groups did not display much reflective thinking. To reflect means to give something back, to think seriously, to express carefully (American Heritage Dictionary) or to throw back without absorbing (Oxford Heritage Dictionary). These definitions guide us as to student behaviour regarding reflection. Students should be able to stand aside from their working or thinking and pass judgement on it. They need to think seriously and carefully about what they are doing. These are difficult tasks for students to undertake without guidance or leadership from a teacher initially. Students will cultivate reflective thinking only if it is required by their day to day learning activities. The task instruction also plays a role in how students will judge their ideas. In traditional instruction (see 2.3) or word problems, the need to reflect is negligible. Modelling tasks do elicit reflective behaviour in students, but harnessing and evaluating reflective thinking is difficult.

5.3.4 What is the quality of the documentation that the group produces when modelling?

In discussing the documentation that groups produced their forms of representation had to be examined. The following show the forms of representation used by each group for each task. It must be stated that these students have not been exposed to formal work on graphs at this point of their education. The 'graph' used by Group 2 in Task 1 was a primitive attempt to compare a footprint size and a height.

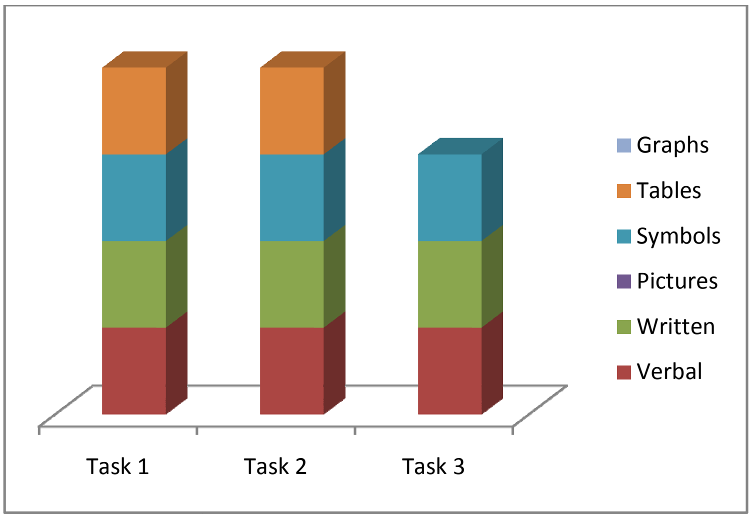


Fig 5.16: Forms of representation (Group 1)

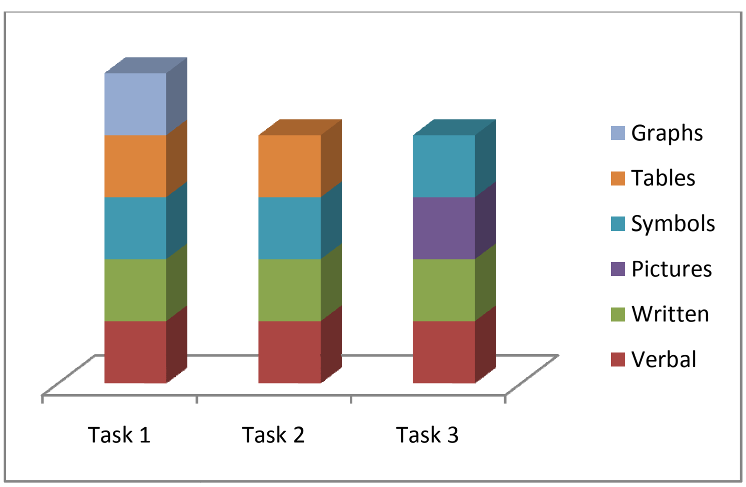


Fig 5.17: Forms of representation (Group 2)

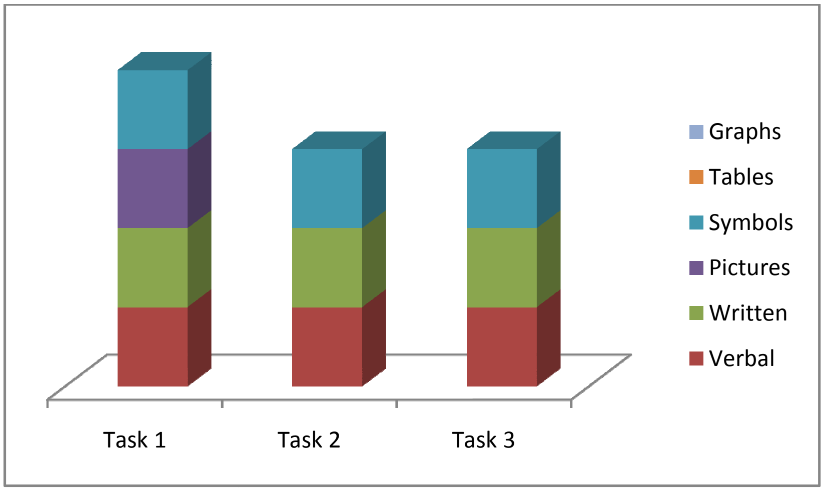


Fig 5.18: Forms of representation (Group 3)

While verbal and written forms are the basis of all representation used by the groups, other forms are clearly sporadic and lacking. Symbolic representation is allocated to each task as all groups used standard mathematical numerals and symbols (+,-, x, ÷) taught at school. None of the groups however 'invented' their own symbolism. This sole reliance on verbal and written representational forms is attributed to very little experience in solving open problems. Kaput (1987: 20) reminds us that the first eight years of a student's schooling is spent learning about base ten numbers and their properties and representational systems for rational numbers, while Lesh, Behr and Post (1987: 41) remind us of the dominant role written language; symbols and pictures play in textbook teaching.

It is interesting to note that Task 1 elicited the most forms of representations from all groups. Task 2 presented students with tables as part of the task, which may have prompted them to use tables here. Lesh, Post and Behr (1987: 38) state that good problem solvers are flexible in using a variety of representational systems - as can be seen with Group 2.

The rest of the discussion in this section considers all the documentation produced by the groups.

Groups had to produce working sheets and presentation sheets. Working sheets were the groups own working pages while solving the task. Presentation sheets were specifically generated by the group before their oral presentations. There were no prescriptions as to what had to be included in either sheets or the format for these sheets. All groups' working sheets in sessions were messy and haphazard. They seemed to think that only the final solution was important. Several times I had to ask groups to keep their working sheets and not to scribble over them. By Task 3 they even planned their presentation from the first session and wanted to improve their presentation:

M: don't put a mark on that (the pattern piece). *We need it for our presentation. We do like the others did. Don't put a mark on it!*

Group 2's working pages for the three tasks are taken as example. They covered pages with calculations for Task 1, that they sometimes couldn't find when they needed them:

J: we've got our shoe sizes and heights.
S: I don't feel like looking for it on this paper.

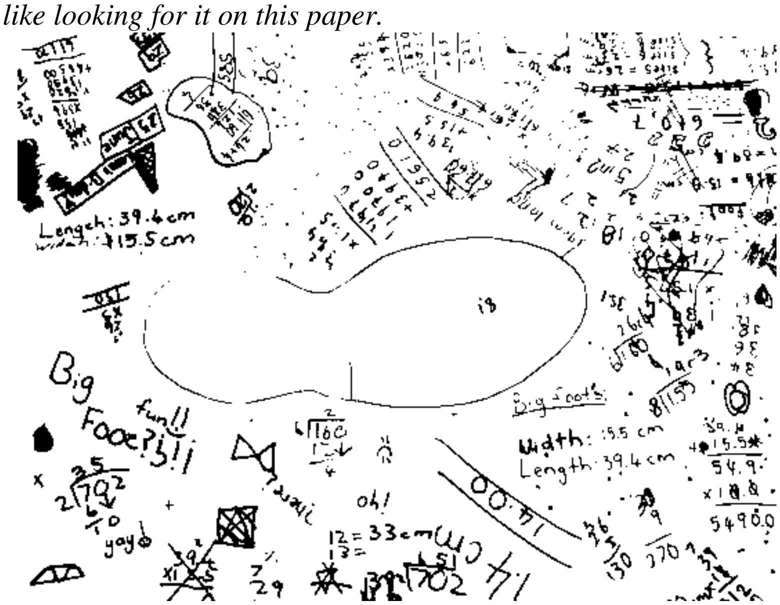


Fig 5.19: Working sheet - Group 2 Task 1

In the next task, their work was slightly more organised. This could be because the supporting documentation was more structured. Their work in Task 3 is better than Task 1, but not yet ideal. The Researcher did not make any suggestions to them during sessions to improve their written documentation as this would lead to an obvious supported improvement.

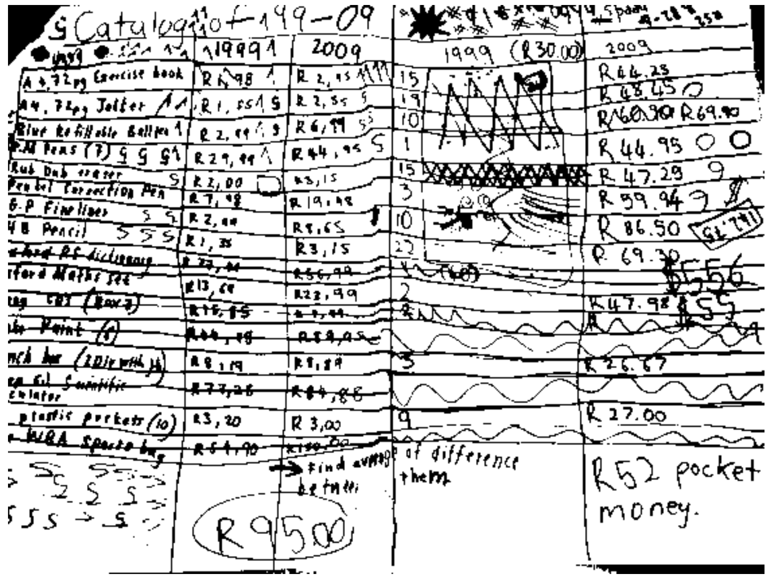


Fig 5.20: Working sheet - Group 2 Task 2

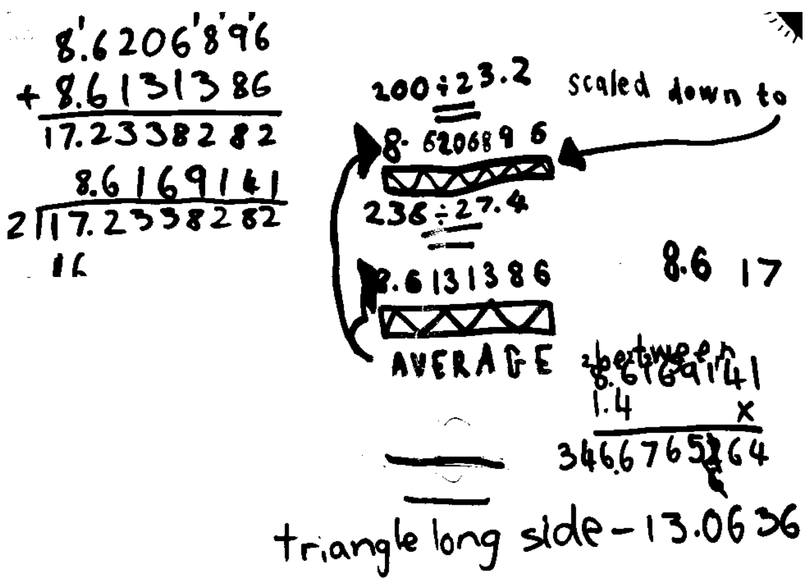


Fig 5.21: Working sheet- Group 2 Task 3

Working sheets did not always show the progression of group ideas nor did they contain sufficient evidence of some group competencies. It may be that students of this age group will need a more structured approach to their working sheets so that it contains more evidence of certain competencies. The presentation sheets only contained what the groups had filtered for use in presentation sessions. The progress of their competence was not always easy to find in their written documentation. Working sessions took place a week apart and this made it difficult for students to remember where and why they had written certain things down on their working sheets. In each task students were expected to create a different product and this added to the difficulty in comparing their written work.

5.3.5 At what level are groups working on a continuum of simple to complex?

This section is related to student competency in mathematising. All groups showed increasing understanding and complexity of understanding as each task developed which was not necessarily transferred to the next task. Very often their understanding or working on a task peaked during the second session and then the third session they consolidated these ideas. This study supports the results of Lesh and Harel (2003: 187) that thinking and 'apparent levels of development vary across tasks as well as across time within a given task'. Group 2 used more complex proportional reasoning such as an average for comparison, but even then the average was used for different reasons. There is very little task transfer that took place. The time-span of the program must be increased.

For an onlooker, it may be thought that the groups were using very simple ideas in their solution processes. These simple ideas were however, understood by the groups. Simplicity of computation must not be confused with simplicity of mathematising. It is far better that students mathematise at a simple level than perform complex routine calculations without understanding. As stated in 2.3 the products of traditional teaching is in a different paradigm to the products of modelling.

This question is best used and will be very valuable to assess a modelling program of longer length. It would not be an accurate assessment to ascertain that complexity of ideas and concepts can change in such a short time-span, but within each task the groups developed a more complex notion of the elements and relationships in the task.

5.3.6 To what extent does the group develop a prototype, generalisable model?

Student abilities to create a generalisable model mean that they are working at higher and most optimum levels. The ability to create a generalisable model suggests that students have fully understood the real situation, have mathematised the problem at a high level with the mathematical knowledge and concepts that they have at their disposal and are able to place and run their model in unfamiliar conditions.

For reasons stated earlier, Task 1 elicited this response from all groups to a much greater degree than the other tasks. Group 1 and 2 were especially able to create a generalisable model. The discussion in 5.3.2 is directly relevant here. Students needed to be directed to this end and the task instruction must state this need for generalisable model explicitly. In Task 2, the part of the instruction that is directly relevant was not taken up by groups as an essential part of their work. It needed to be explicitly re-stated by the researcher.

Explain your method so other children in similar situations can use it to figure out what their allowances should be.

The generalisable model relies greatly on the quality of the situation model. When these models are weak or not fully understood, then the task of creating a generalisable model becomes that much more difficult for the groups. Both Group 1 and Group 2 created strong models for Task 1 and were able to generalize their model. For Task 2, all groups worked so deeply in the situation model and were not confident about their situation models that this resulted in little or no attention being given to the generalisable model. For Task 3, the process of creating the situation model largely tied in with the process of creating a generalisable model. Wheeler (1982: 47) reminds us that 'Poincare pointed out that all mathematical notions are implicitly or explicitly concerned with infinity. The search for generalizability, for universality, for what is true "in all cases", is part of this thrust'.

If a generalisable model is desired in modelling tasks, then more focus and structure must be given to this part of the instruction especially for novice modelers. The teacher should also be aware of the problems around creating the generalisable model and propose pertinent questions while students are working on their situation model.

5.3.7 Summary

This study set out to determine the development of group modelling competencies. It was also shown, as suggested by Niss, Blum and Galbraith (2007: 13), that those competencies from several domains (mathematical, modelling, social and other), sustain and develop one another. Their complex interrelationship will need further studies. It must be stressed that the groups in this study were largely left to 'get on with' the tasks. There was only one adult present in each session and gave minimal suggestions as to methods or approaches. This may be a factor in the development. But the developments noted, are autonomous and authentic.

The questions discussed in the previous section allow one to gauge wider areas of impact. They have shown to be very useful in describing a broader area of competency permeation. These questions allow one to evaluate groups or students in a more general way. The questions point to the essence of modelling, they allow one to focus on what is important about student modelling.

The development of student modelling competencies reveals some expected results and some surprises. Improvement in student competencies in 'doing' modelling was noted, although this was not set out as a separate competency. The duration of the study affected the long term competency developments, but there were those competencies that developed even in the short period.

Students may reach further on the continuum in one task in one context but may be on a different part of the continuum for another task, since according to Lesh and Doerr (2003a: 29); conceptual development is far more situated than previously thought. The students modelling progress in one task cannot necessarily be applied to other tasks especially in novice and inexperienced modelers. As revealed by Lesh and Doerr, there appears to be a 'gradual increase in local competence'.

5.4 MULTIDIMENSIONAL APPROACH

Here the results of using the multidimensional approach (see 3.8) are presented. It was decided to present this section in visual form as in Fig 5.22 - 5.24. A 'volume' approach is suggested by the Blomhoj and Jensen but a different style was used for this study. It was decided that the same indexing system (0, 1, 2, and 3) that was used for assessing group individual competencies would be used here to remain consistent and to facilitate integrating individual competency development with the multidimensional approach. Jensen (2007: 147) rejects the use of assigning a grade to each dimension and then averaging out the three dimensions into a single grade. The assigning of an index is used so that the resulting visual representation would be facilitated. The three dimensions were still represented independently and no averaging out into a single grade/index took place. This is in trying to keep to what Jensen called respecting the complexity involved in assisting the development of modelling competencies. Being able to see each dimension individually is very important in presenting the multidimensional approach. The multidimensional approach allows one to step back from particular competencies and view modelling competency development for each group in its entirety. This approach also allowed one to gauge the effectiveness of the group over the entire task which enabled a global impression of the groups' competency development. The multidimensional approach also allows a glimpse into other factors involved in modelling such as the extent of a groups' thinking about the problem and the scope of their solution process. These factors may be missed in an individual single scale assessment of competencies.

5.4.1 Degree of coverage

This deals with the parts or nodes of the modelling process that the groups could deal with and the level of reflection involved (Blomhoj & Jensen in Jensen 2007: 144). Groups who were able to mathematise the open situation to a greater degree and create generalisable models are indexed higher in this aspect. Groups who were able to move through the modelling cycle autonomously and cover each aspect and competence well are working to a higher degree of coverage. It must be noted that groups were not 'taught' what modelling was prior to or during the program, nor what the modelling cycle entailed or which competencies were necessary for modelling. They were simply given a task to solve. Their process and navigation through the modelling cycle was authentic. The weaker groups needed assistance in staying on and progressing through the modelling cycle. The competencies of interpreting and validating were particularly weak which affected their degree of coverage.

5.4.2 Radius of action

This refers to the range of mathematical 'domains' (Jensen 2007: 144) that students are able to model successfully. This would require students to be given modelling tasks across mathematical fields and areas. Since this study only focused on proportional reasoning, the radius of action related to the number of different areas of mathematics that students resorted to in solving these tasks. Group indexes for this area remained unchanged for the first two tasks and only Group 2 reflected a change by the third task since it was possible to gauge their radius of action by then. Some groups remained in the realm of whole numbers; other worked on averages and encompassed decimal numbers comfortably in one task. Overall, it would seem that a curriculum focused on computational fluency does not allow students' mathematical flexibility. Students did not employ some of the more complex areas of mathematics that they have been 'taught'. Students must be provided with varied modelling experiences. They should solve problems where the situation context and mathematical context varies so to improve their radius of action.

5.4.3 Technical Level

This area addresses the type of mathematics the groups used. As proposed by Jensen (2007: 144) it relates to the 'size and content of the mathematical toolbox' that the groups had available to them. It is important to balance out what is available to students and what they are able to use comfortably. It was noted that groups started off each task with whole number operations before venturing to anything more complex while some groups remained at this level. Here, too, in allocating an index their level of proportional reasoning (see 3.3.1 and 5.2.7.1) is taken into account.

5.4.4 Analysis

The following table shows how indexes were allocated and a brief reason for each allocation.

	Index Task 1	Reason	Index Task 2	Reason	Index Task 3	Reason
Degree of Coverage						
Group 1	2	Interpreted and tested their model fairly soon.	2	Interpreted their calculations. Tried more group member suggestions.	2	Tried a number of modelling cycles.
Group 2	1	Spent a great deal of time working mathematically and not interpreting their working.	2	Interpreted and validated well. Spent more time understanding and mathematising than working mathematically.	2	Covered the modelling cycle effectively to solve problem.

Group 3	1	Were very unsure of what idea to follow or what to do next.	1	Took very long to move past understanding and simplifying the problem. Used only a small part of the data.	1	Spent a great deal of time simplifying which was not successful, it made the problem more difficult.
Radius of Action	Only Group 2 was allocated a 2 at the end of the program since they were able to comfortably move between modelling a variety of problem domains. This is attributed not only to their mathematical knowledge but their good use of meta-cognition, their own informal knowledge and good language ability.					
Technical Level						
Group 1	2	Used the four basic operations fluently.	1	Unable to use relative thinking. Used only small part of data.	2	Used trial and error to find ratio. Their reasoning more multiplicative in nature.
Group 2	2	Used the four basic operations fluently.	2	Considered two solution paths. Used summing up and average well.	2	Were able to find a ratio easily. Multiplicative reasoning evident.
Group 3	1	Proportional reasoning was additive in nature. They only used a small part of data available to them- only measured one person.	0	Lamon's 'avoiding' level of proportional reasoning evident. They were unable to take all information into account; reasoning was largely additive in nature.	1	Showed limited ratio understanding. Found a non-integer ratio difficult to work with.

Table 5.7: Multidimensional indexes

The following figures allow a visual interpretation of the multidimensional approach and may be more in line with Jensen's idea of this approach.

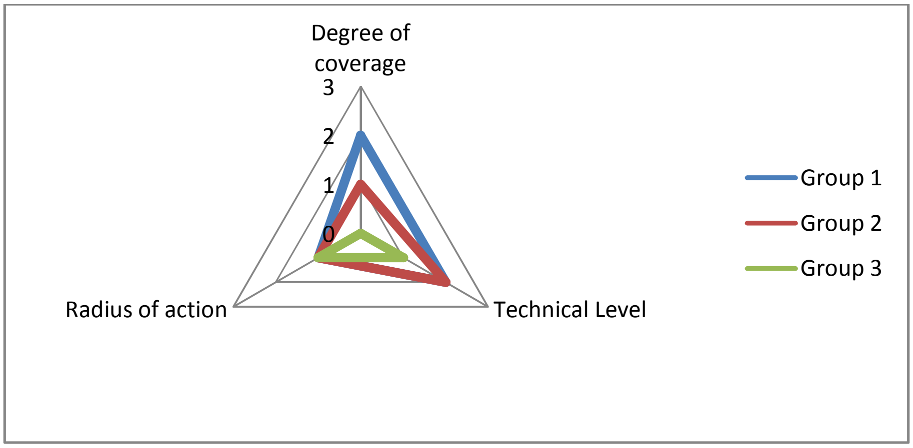


Fig 5.22: Task 1 Multidimensional approach

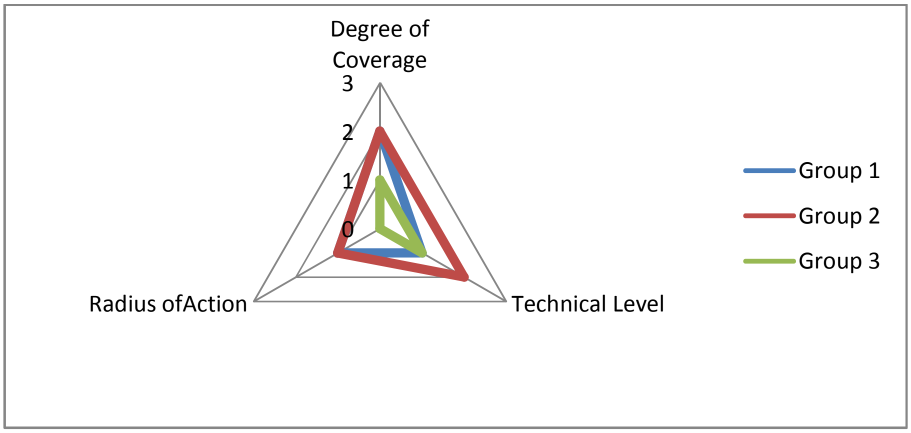


Fig 5.23: Task 2 Multidimensional approach

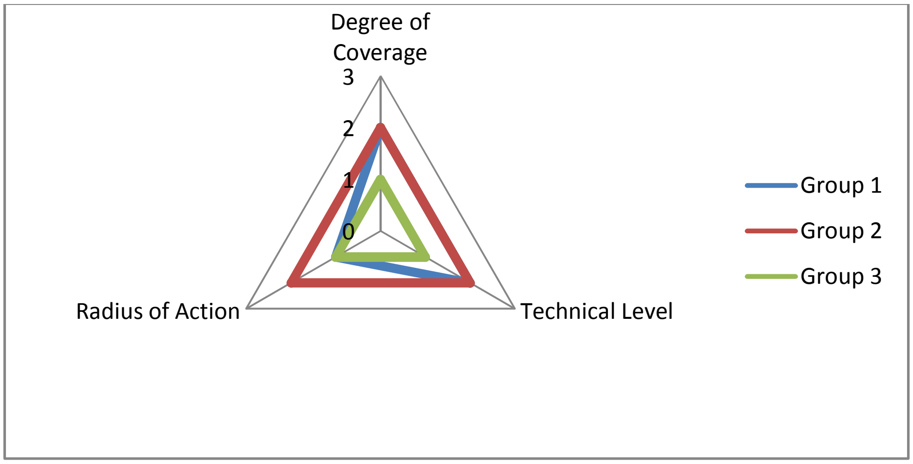


Fig: 5.24: Task 3 Multidimensional approach

5.5 CONCLUSION

The retrospective analysis in this chapter took into consideration a wide spectrum of modelling competencies and their development. It considered three methods of assessment of the developing modelling competencies. All three methods reflected that competency development in students working in groups is situated, complex and interrelated but gradually improving. Modelling competencies do develop in various degrees. These competencies develop because of the students' experiences with the modelling tasks and not as a result of traditional instruction. It may be said that traditional instruction hampers the development of modelling competencies.

The development of the competencies was documented in terms of the group as a whole. All groups showed evidence of competency development in some competency areas. The strong group commenced the program with greater initial competencies and the development was consistent. The weak groups also showed competency development, although to a lesser degree, certainly more than traditional mathematics may ever have developed in them. Weaker groups may simply need more experience with these tasks.

The process of documenting developing modelling competencies entails not only tracking a list of competencies but carefully monitoring how students organise and arrange these competencies. The three forms of assessing modelling competency development proved valuable in providing different perspectives from which to study changes in group modelling. Since the tasks had so many aspects to them, addressing different or clustering competencies allowed a much more holistic evaluation than would have been possible through pencil-and-paper or multiple choice tests.

The use of design research (see 4.2.2) in creating research that was 'plan-ful and emergent' (see 1.6) enabled the analyses of the data to be concurrent with the aims of the research. The use of pilot studies in the design cycle that led to careful consideration of tasks and instruments facilitated a meaningful teaching experiment and its suitable analyses in this chapter.

CHAPTER 6

CONCLUSION, LIMITATIONS AND RECOMMENDATIONS

6.1. CONCLUSIONS

This study set out to examine and document the development of modelling competencies of Grade 7 learners working in groups. This chapter concludes the study, sets out limitations and implications. It also proposes recommendations for further study.

Modelling is a sophisticated, well structured activity that places students and teacher in an adidactical situation which ensures that meaningful mathematical learning can take place through activity, collaboration and social interaction. It requires well functioning and communicating groups of students to collaborate to reach a meaningful conclusion to a real life complex problem. Modelling as a teaching and learning strategy was explored by comparing and contrasting it to other teaching approaches in mathematics education. A number of relevant and important theoretical perspectives were investigated and espoused to establish the relevance and importance of modelling as a teaching and learning activity in mathematics classrooms.

Modelling competencies were identified, characterized and significant ones selected for the focus in this study. Competencies were categorized as cognitive; meta-cognitive and affective. The nature of competence was explored both in terms of modelling competence and mathematical competence as well as the symbiosis between the two. The competencies identified and characterized in this study cannot be developed using traditional instruction and are essential to the development of mathematics as its own competency. Students were not guided or prompted towards displaying any particular competencies. The competencies identified for this study were necessary to the successful conclusion of tasks. These competencies are not the only ones that are evident or pertinent during typical group modelling sessions. Other factors can be identified and selected for studies (see 6.5) that affect modelling competency development.

Twelve students were identified for the study and were set into two groups of students with weak mathematical ability and one group of students with strong mathematical ability.

Students solved three modelling tasks over a period of four months. The tasks selected from the literature were shown to be superior by way of eliciting models from students and the corresponding modelling competencies. These tasks spontaneously elicited all the modelling competencies identified and allowed one a glimpse into student proportional reasoning. Modelling competencies elicited were indexed, documented and compared. The data collection strategy proved effective in generating relevant data for this study.

Competency development in mathematical modelling is complex, multifaceted and protean. Although some competencies showed noticeable improvement, some need a longer period of study. Many competencies are interrelated and interlinked and support or maintain each other. Modelling competency development is neither linear nor straightforward. Competency development cannot be stated in a concise or simple way. More importantly than seeking a linear or hierarchical description of competency development is ascertaining that competency development does take place, even over a short period of time with minimal instructor or researcher intervention. Modelling competency development is positive albeit incremental. This improvement in both weak and strong ability students allows great scope for implementing modelling tasks in all school types and to all ability students.

Groups did display increasingly more varied and complex ways of thinking and working. Many other aspects also improved such as group interaction and camaraderie. Most pleasing was that many students involved in the study have requested to continue working on modelling tasks on a voluntary basis after school hours.

The choice of groups in terms of being ‘weak’ and ‘strong’ worked well in terms of research purposes. It was evident that these distinctions are not always accurate in terms of modelling tasks where ‘weak’ students display strong modelling competencies. It was also observable that students in the weak groups would benefit from greater diversity in their groups. This is in line with Vygotsky’s zone of proximinal development (1978: 87). The way in which the distribution of knowledge and competencies takes place amongst group members is not clear and does not follow linear trend but more that of an outwardly spiraling ripple.

Competency development was different in the weak and strong groups but also between the weak groups differences emerged in competency development. Weak groups were often immobilized by weak basic mathematical concepts, lacking meta-cognitive skills, poor confidence and poor reading skills. Weak groups did however, display some very good use of informal knowledge and at times used some surprisingly complex ideas. The nature of modelling tasks allows these abilities to come to the fore. Although there were differences, all groups displayed improvements in cognitive and meta-cognitive competencies. The strong group displayed a higher level of frustration than weaker groups when their solution path was unclear. Students who are traditionally considered ‘weak’ at mathematics may be used to this sort of frustration.

Modelling competency development needs to be supported by judicious selection of tasks, well constructed heterogeneous grouping and effective teacher modelling knowledge. Both modelling-content and modelling-context knowledge is necessary for teachers and will lead to the type of support suggested in terms of section 2.5 and 2.9.6.

6.2 LIMITATIONS OF THE STUDY

This study used homogenous grouping for research purposes. This is not a desired state for mathematics classrooms where heterogeneous grouping is preferred (Linchevski & Kutscher 1998: 533). Homogenous grouping did affect the group’s overall area of action and depth of action. This is true for the weaker groups but not for the strong group. The strong group however, benefitted from their grouping.

The tasks selected for the study were from one realm of mathematics, that of proportional reasoning only although all groups used simple mathematics to solve these and no evidence of formally taught proportional reasoning strategies emanated from their working. It is not possible to generalize these results to other areas of mathematics.

This study was also specific to Grade 7 students at one school and cannot necessarily be generalized to other grades and ages. Groups were evaluated as a single functioning unit and individuals were not considered as their own entity. No indication of improvement of individual student’s modelling competencies can be given.

The use of a single researcher and coder must be stated as a limitation of the study although many other aspects of validity and reliability were considered which did enhance the validity of the study.

6.3 SUMMARY OF CONTRIBUTIONS

This study has made some contributions to theory and knowledge about modelling in general and modelling competencies in particular. This study forms part of an international need to research mathematical modelling competencies. It extends on previous studies of students of the same age group (Maaß 2006) but also focuses on the developing competencies in weak and strong groups of students. It provides foundation to the premise that modelling competencies do develop when students are exposed to modelling tasks that they solve in group situations. In this study modelling is presented in an open authentic situation. By presenting the development of competencies of students working in groups the study provides a base from which more detailed work on the individual competencies can take place. The identification of mathematisation as a key competency and how it is integral in other competencies is important for further research.

This study is supported by theories of constructivist-interactionist learning and problem-centred learning and extends these learning approaches by focusing on the specific competencies required by modelling situations. The study also demonstrates the educational benefits of problem-centred learning and how these theories can be applied in a classroom situation. The study has contributed to research specifically in South Africa on modelling and builds on previous work on problem-centred learning in South Africa. (see 2.6).

This study found that students can develop competencies when working with minimal teacher intervention although skillful intervention will enhance competency development even further. This insight is particularly useful when wanting to introduce modelling tasks where the teacher may be new or unsure of this teaching and learning activity. The study shows that the quality of tasks presented to students is a critical matter in classrooms.

This study also provides evidence that weak students can develop modelling competencies and certainly do benefit from open and context rich social activities. There should be no reservation as to whether modelling activities should be included in the activities and experiences of students in mathematics classrooms.

In terms of the contributions of this study to practical research and teaching, a number of recommendations can be made from this study. Firstly, the use of homogenous groups is not the ideal situation in mathematics classrooms and therefore not ideal for the development of modelling competencies in a classroom situation. It would be advantageous to document the development of modelling competencies in heterogeneous groups and to compare those results to these. Even students in the strong group may have benefitted from different ideas or approaches and by explaining to other students what they were doing or thinking.

Secondly, the time frame needed to successfully complete tasks and make needed documents for presentation sessions must not be underestimated. More time was needed to complete presentation sessions more successfully. Students required more time to reconstruct their thinking so as to be more fully prepared for presentation sessions. In terms of real classroom assessment that needs to take place, it is important to allow the time as more can be gained in terms of assessment from well prepared presentation sessions. The development of a single robust instrument that can be used for classroom assessment should be developed if modelling is to be implemented at a classroom level.

Thirdly, the tasks in this study dealt primarily with proportional reasoning, although students did not identify this as a linking factor in tasks. It will be beneficial to use modelling tasks that incorporate different areas of mathematics. Together with this, students should be exposed to new group members from time to time. Groups are subject to what Watson and Chick (2001: 136) have called factors that influence collaboration. These factors such as cognitive ability, misunderstanding, leadership and social disagreement were found to be particularly relevant in this study. The way in which knowledge is distributed through the group is also an area that will allow a fruitful study.

All groups found it difficult to remember their discussions and workings over weekly intervals. It would be more beneficial if groups were allowed to complete modelling tasks in one sitting or in consecutive days. This will also improve the quality of documentation that groups keep. All groups had difficulty remembering what they had discussed and where to find important parts of their workings over weekly periods.

6.4 RECOMMENDATIONS FOR FURTHER RESEARCH

The link between competency development in modelling and language ability is a substantial one and one that needs a further and fuller study. It was a significant factor in understanding tasks for the weak groups and contributed to poor completion of tasks for these groups.

Specifically for these tasks was the level of reading ability and reading comprehension. Reading not only enables students to understand the task instruction but enables students to make connections, visualize, infer and predict, as well as play a role in meta-cognition (Hyde 2007: 1). The weak groups consisted of students whose language and reading abilities were also weak while correspondingly students in the strong mathematics group also displayed strong language skills. Since this connection falls outside the scope of this study, it should be considered for a full study in its own right.

It may be beneficial to the field if a quantitative study of a similar nature is undertaken. Furthermore, a full study of each competency is needed. Although mathematisation was identified as an important competency, its influence and how it is influenced by other competencies would warrant a significant study. The comparative ‘weight’ of each competency should also be established. It must also be established if this ‘weighting’ is task or topic dependent.

Further teacher development in the field of modelling is necessary if we are to establish the impact of modelling to teacher professional development or in everyday classrooms. Specifically the role of developing a modelling perspective within the framework of further teacher development needs to be studied. Together with this is the possibility of studying how to introduce modelling into a school or curriculum and how to teach from a modelling perspective.

Finally, Lester and Kehle (2003: 510) remind us that ‘far too little is known about students learning in mathematically-rich environments for research on problem solving to end’.

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APPENDICES

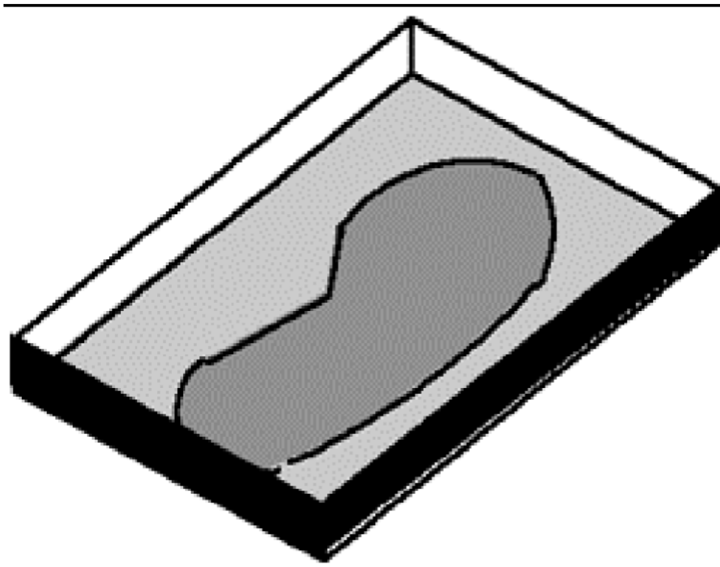
Appendix A: Task 1: Big Foot

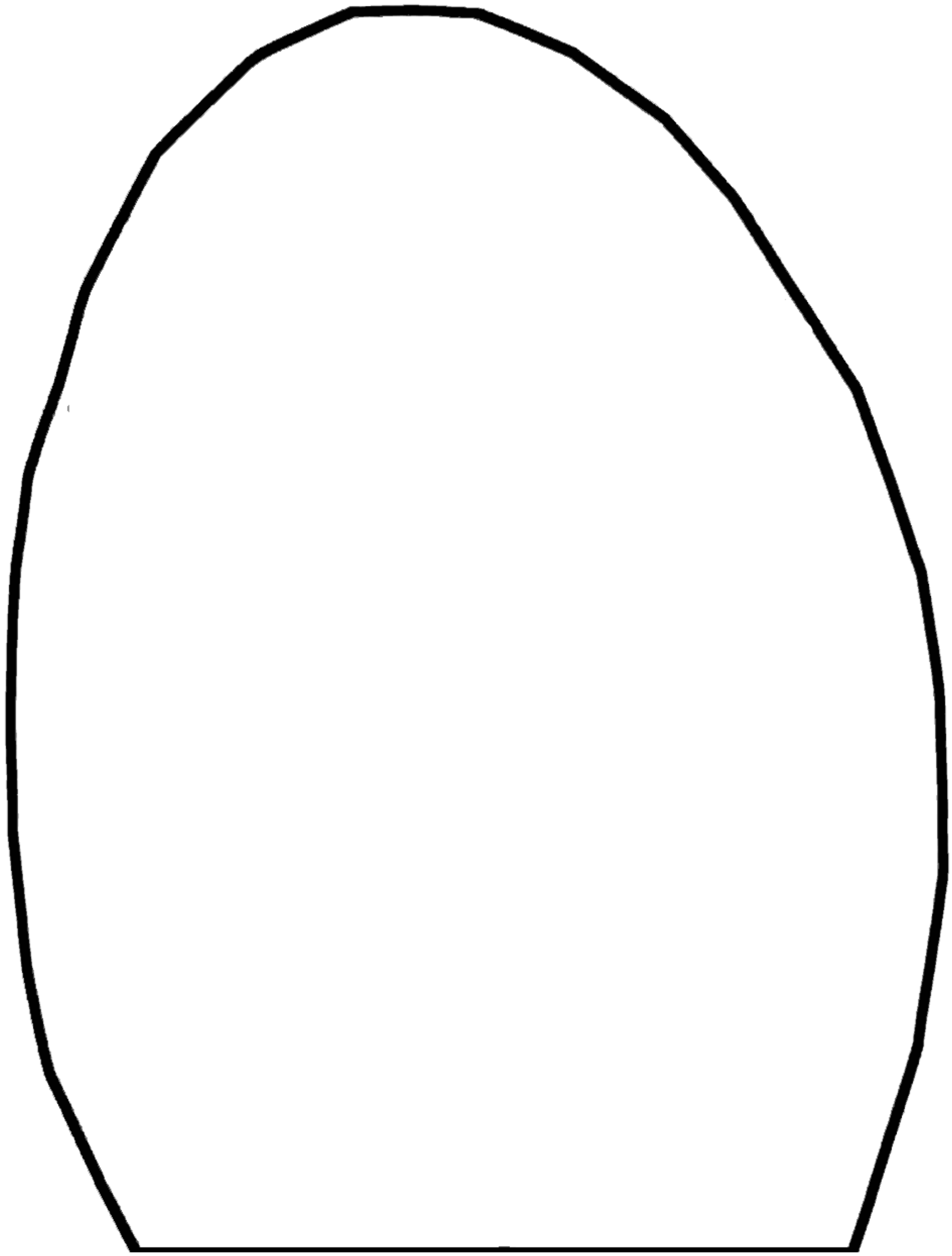
Task 1

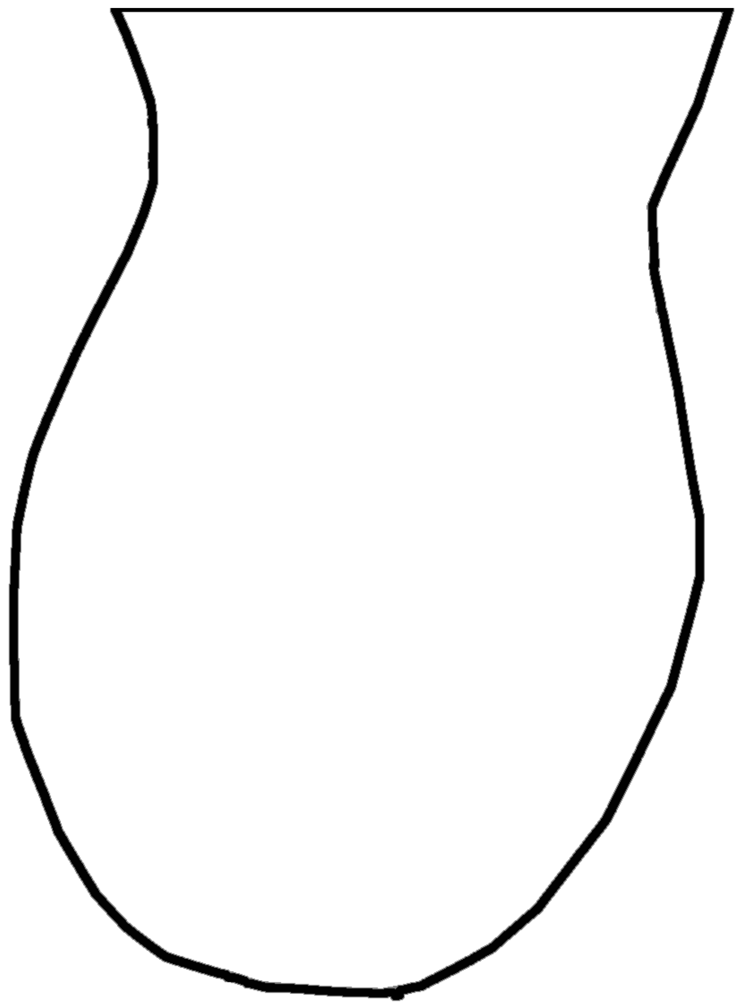
Big Foot

Early this morning, the police discovered that, sometime late last night, some nice people rebuilt the old brick drinking fountain in the park. The mayor would like to thank the people who did it. However, nobody saw who it was. All the police could find were lots of footprints. You have been given an example showing one of the footprints. The person who made this footprint seems to be very big. Yet, to find this person, it would help if we could figure out how big the person really is. Your job is to make a "HOW-TO" TOOL KIT that the police can use to figure out how big people are—just by looking at their footprints. Your tool kit should work for footprints like the one that is shown here.

However, it also should work for other footprints.







Shoe pattern and supporting information from: https://engineering.purdue.edu/ENE/Research/SGMM/CASESTUDIESKIDSWEB/casestudies/phone/CASESTUDIESKIDSWEB/bigfoot.htm

Enlarged to size 24

Supporting Information

Tom is a professional tracker. Tom began his journey as a tracker at a young age. While growing up in the Pine Barrens in eastern New Jersey, he spent many hours exploring the beauty and power of the natural world. It was there he met Rick, who became his best friend and fellow nature explorer. Rick introduced him to his grandfather, Stalking Wolf, an Apache tracker, who became Tom's mentor. He passed on his knowledge and skills to these two boys for many years.

The hours Tom, Rick, and Stalking Wolf spent in the wilderness taught Tom to understand and respect nature. One thing he learned about himself was his gift for tracking. Tom spent hours tracking a number of animals and learning about them through quiet, detailed observation. These hours spent learning to track gave Tom the ability to observe a track and make predictions about whom or what made it.

As he followed tracks, he would notice the pattern they took and the depth of them. These told a story about the animal's behaviour, for example a track that had a deeper toe indentation told Tom that the animal was running. Other clues that Tom would look for included the toeing marks, sideward ridges on the tracks, the length of the stride, and the size of the track. Being able to read the tracks and know the story they told came after observing the animals in similar situations.

Tom and Rick learned even more about tracks as they observed how weather affected them. They spent many hours watching a track they had made to see the changes that occurred to it as it rained or from freezing and thawing. Different kinds of soil also had an impact on the track.

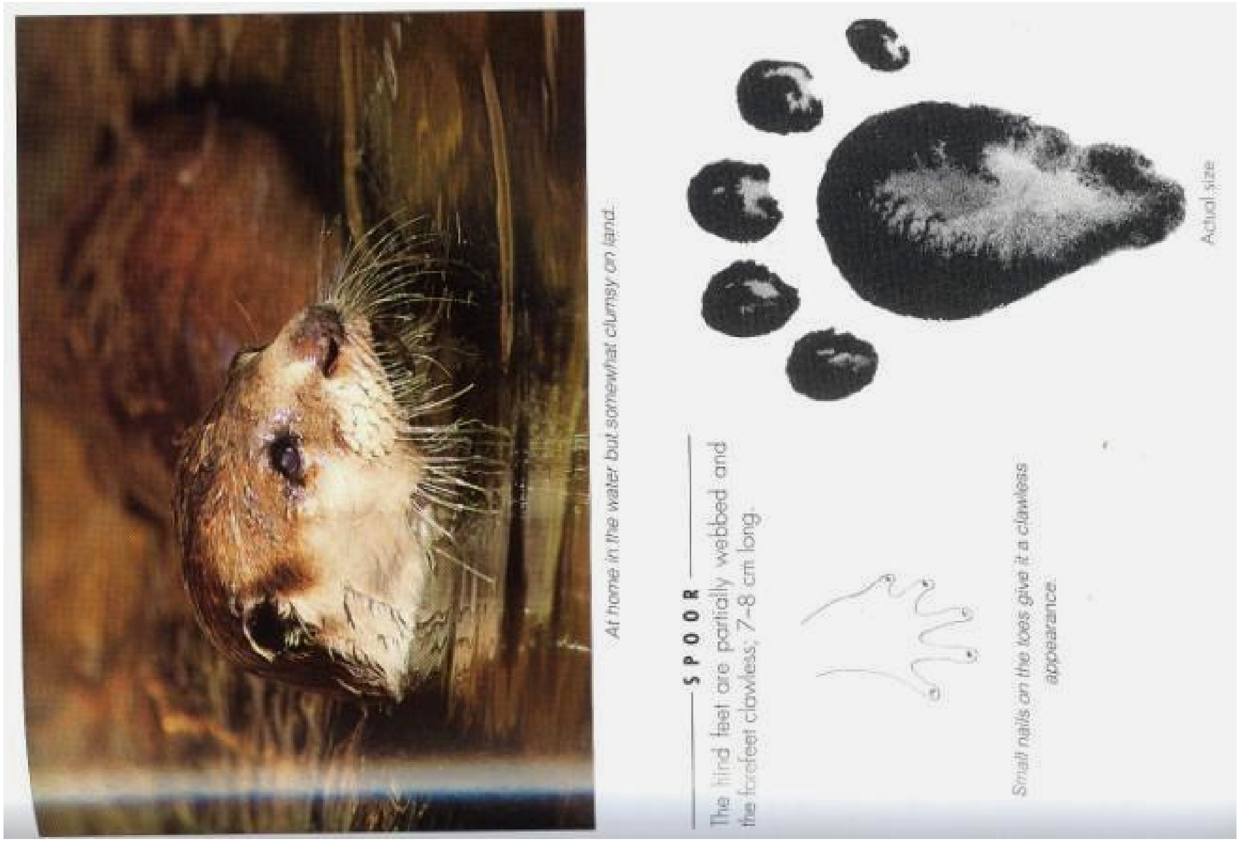
They became good not only at tracking animal footprints, but car tracks and human footprints also. As young boys, they helped a local search party find a missing five year old in the woods. The hours learning the art of tracking allowed these two boys to do with ease what a group of trained dogs and adults were unable to do. Tom used this ability many times as an adult also to track down people. The most notable incident for Tom was the time he found a mentally challenged adult who had been missing for a number of days. This time helped Tom to realize that his gift had value for others. In his book, The Tracker, Tom states, "he was there and alive and if my life ended in the next instant, all the years I had spent learning to track had been justified."

Tracker Readiness Questions

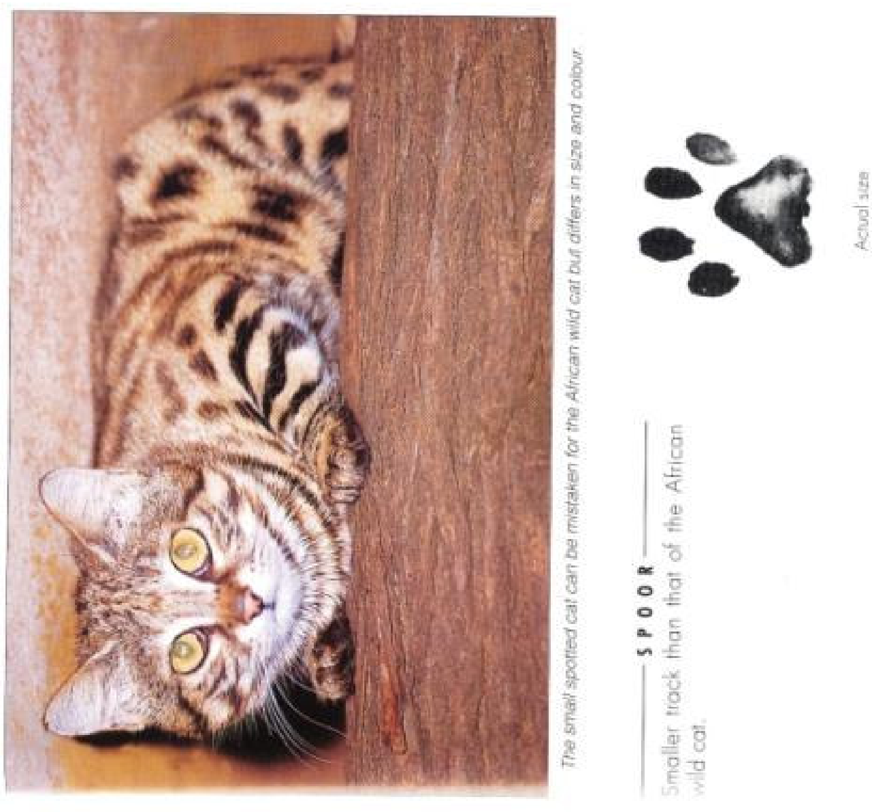
After reading the article and the data sheet, please answer the following questions.
1) What clues does Tom Brown look for in a track?
2) Complete the table below using the tracks on the data sheet:

Animal	Length	Width of widest part of print
Cape Clawless Otter		
Small spotted cat		
Brown Hyena		
Warthog		
Hedgehog		

4) What the relationship between the footprint and animal size?



Small Spotted Cat (left) Cape Clawless Otter (right)



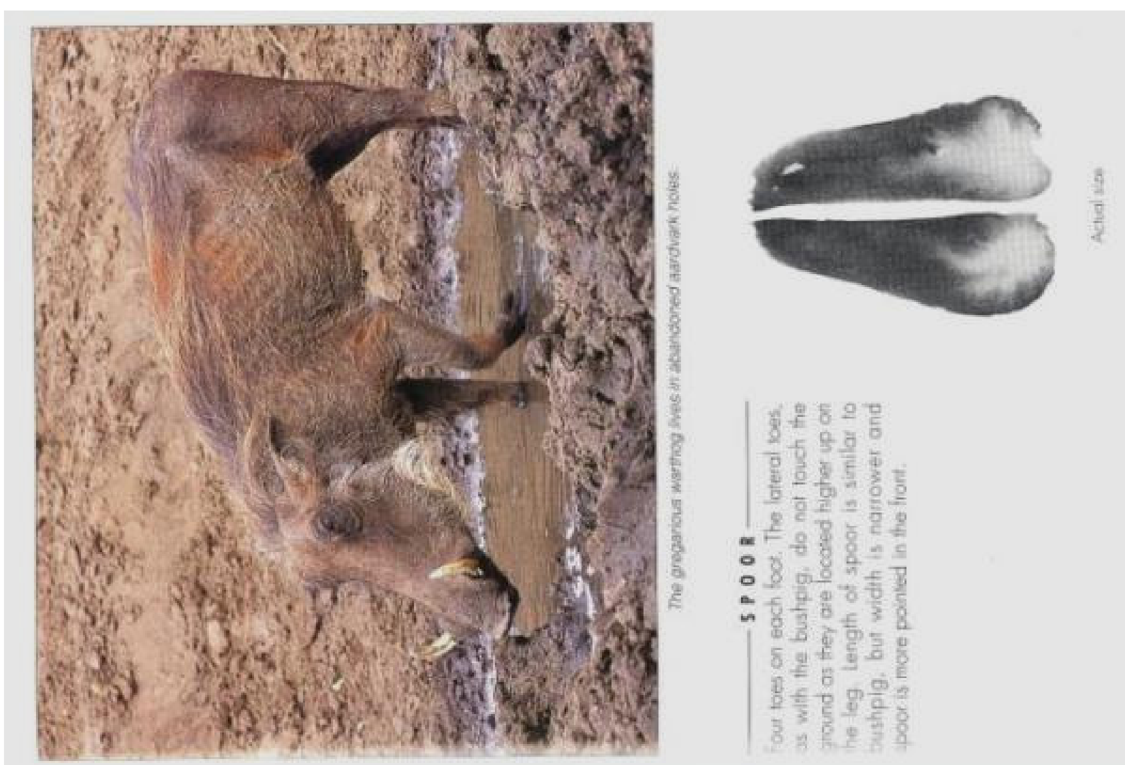

Brown hyaena have a distinct dark brown shaggy coat and erect, pointed ears.

SPOOR

Dog-like with short, blunt claws. Protected species, but more common than believed due to its silent nature and nocturnal habits.

Actual size

Warthog (left), Brown Hyena (right)


The gregarious warthog lives in abandoned aardvark holes.

SPOOR

Four toes on each foot. The lateral toes, as with the bushpig, do not touch the ground as they are located higher up on the leg. Length of spoor is similar to bushpig, but width is narrower and spoor is more pointed in the front.

Actual size

Hedgehog

Atelerix frontalis

Afrikaans: Krimpvarkie **Shona:** Shori **Ndebele:** Inhloni
Zulu: Nhloni **Siswati:** Ndsundvundswane **Venda:** Tshitoni
Tswana, Sotho: Tlhong **Nama/Damara:** ǁNoaros

DESCRIPTION

OVERALL LENGTH: 20 cm MASS: 0,4 kg GESTATION: 35–40 days.

Mainly nocturnal, the hedgehog is found singly, in pairs or in family groups. It has a good sense of smell and hides up in a variety of vegetation, in holes or amongst rocks. This animal is very inactive during the winter months. Amongst other predators, it is preyed upon by nocturnal birds of prey. It curls up if threatened. It issues loud shuffles and grunts when it searches for food.

DIET

Omnivorous; termites, insects, millipedes, centipedes, snails, frogs, lizards, small rodents, young birds, eggs, certain wild fruits and various vegetable matter. It is not dependent on water.

FAECES

Small cylindrical pellets. Food is well masticated, leaving small fragments of exoskeletons in faeces.

Actual size

SPOOR

1,5–2 cm in length; four-clawed toes.

Actual size

R. Smithers

Rare species

Hedgehogs grunt when searching for food.

Appendix B: Task 2: Catalogue Problem

Task 2

Catalogue Problem

Hello, my name is Sipho and I need some help with a problem. My parents are really unreasonable. My sister, Karabo, is ten years older than me. When she was in Grade 7 her pocket money was R30 per month. I also get R30 per month. With R30 I cannot buy as much as she could ten years ago. To prove this I collected some information about prices now and ten years ago. What I need from you:

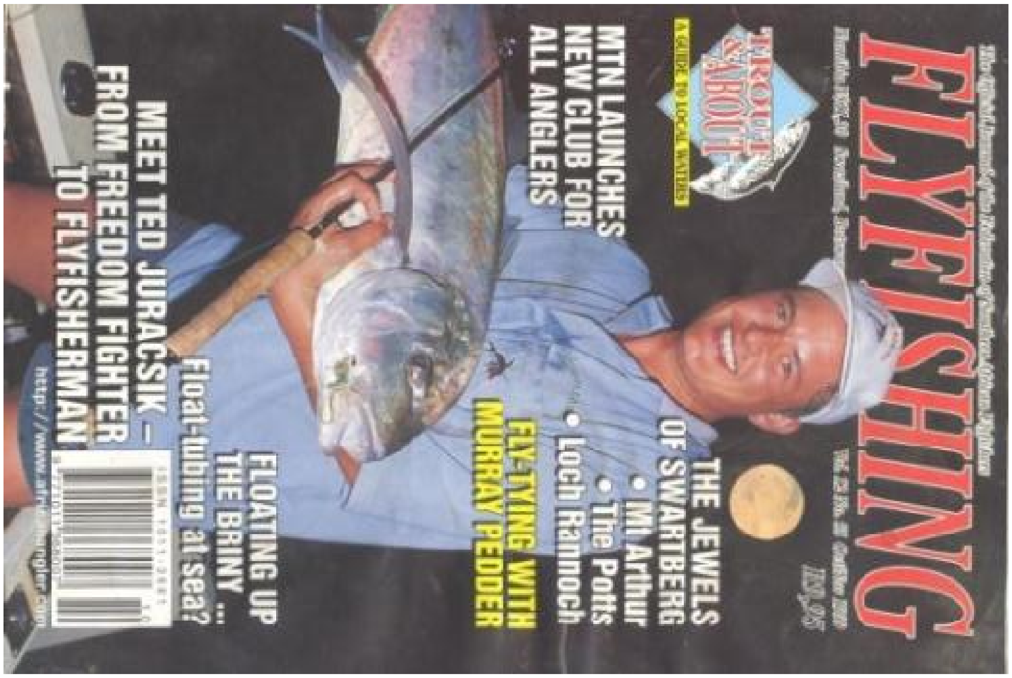
- Use my price information to determine how much pocket money today would be the same as R30 ten years ago.
- Write a report for me to give to my parents
  Describe your method and your conclusions
  Show that you accurately figured out how much money gives me the same spending power as R30 did ten years ago.
  Explain your method so other children in similar situations can use it to figure out what their allowances should be.
- Remember, my parents to not like emotional or illogical arguments.

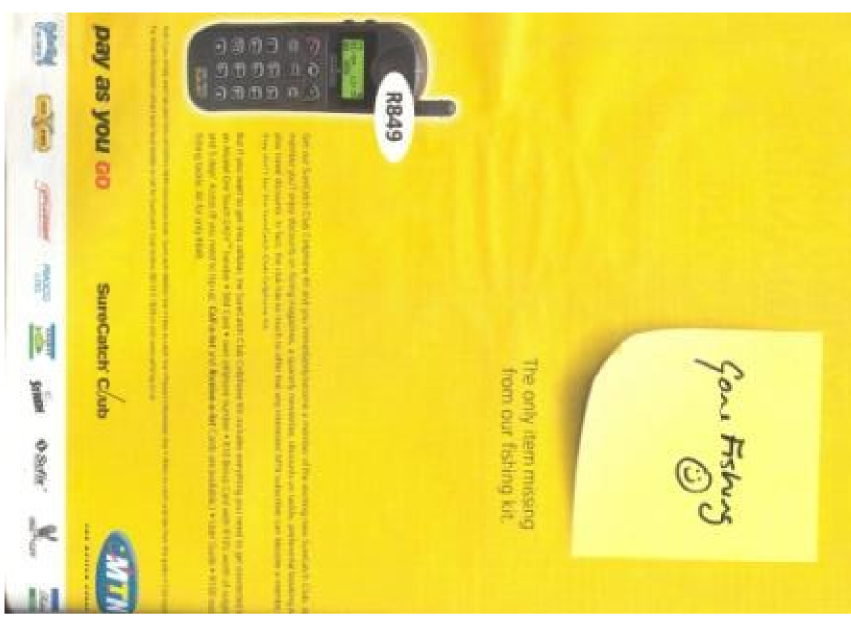
Thanks

Sipho

'SUPER STATIONERY STORE' CATALOGUE ITEMS 1999

A 4 72pg Exercise Book	R1,98
A 4 72pg Jotter	R1,55
Blue Refillable Ballpen	R2,99
Pental Milky Pens (7)	R29,99
Rub Dub Eraser	R2,00
Pentel Correction Pen	R7,98
Green Pental Fineliner	R2,99
HB Pencil	R1,35
Oxford Primary School Dictionary	R37,49
Oxford Maths Set	R13,69
Sony CDs (box 3)	R15,85
Plaka Paint (6)	R44,98
Lunch Box (2Div with juice bottle)	R 8,19
Sharp EL Scientific calculator	R77,28
A 4 plastic pockets (10)	R3,20
Blue WBA Sports Bag	R54,90


FLYFISHING
TROUT & ABOUT
A GUIDE TO LOCAL WATERS
MTN LAUNCHES NEW CLUB FOR ALL ANGLERS
THE JEWELS OF SWARTBERG
• Mt Arthur
• The Potts
• Loch Rannoch
FLY-TYING WITH MURRAY PEDDER
FLOATING UP THE BRINY ... Float-tubing at sea?
MEET TED JURACSIK – FROM FREEDOM FIGHTER TO FLYFISHERMAN


Gone Fishing
The only item missing from our fishing kit.
R849
pay as you GO
SureCatch Club
MTN

WALTONS BACK TO SCHOOL PRICES

2009

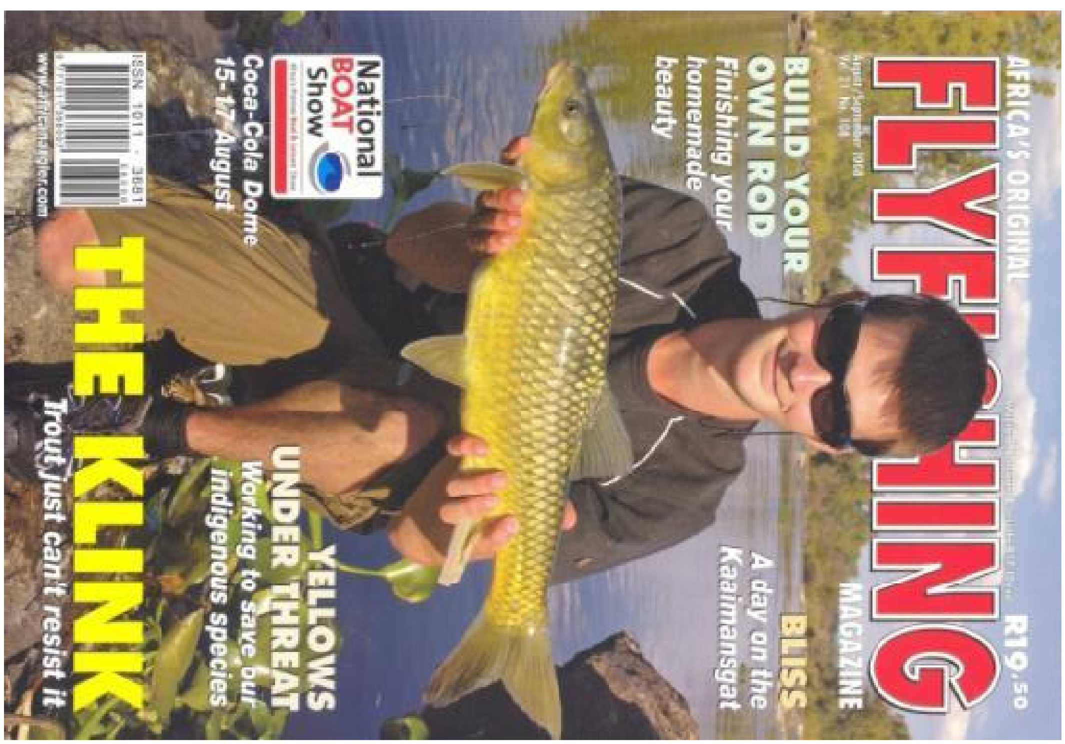
A 4 72pg Exercise Book	R2,95
A 4 72pg Jotter	R2,55
Blue Refillable Ballpen	R6,99
Pental Milky Pens (5)	R44,95
Rub Dub Eraser	R3,15
Pentel Correction Pen	R19,98
Green Pental Fineliner	R8,65
HB Pencil	R3,15
Oxford Primary School Dictionary	R56,99
Oxford Maths Set	R23,99
Sony CDs (box 3)	R4,99
Plaka Paint (6)	R89,95
Lunch Box (2Div with juice bottle)	R8,89
Sharp EL Scientific calculator	R84,88
A 4 plastic pockets (10)	R3,00
Blue WBA Sports Bag	R150,00

Spendy's Store

Prices from 1999 and 2009

Item	1999 Price	2009 Price
Children's Tracksuit	R95	R170
Levi Jeans	R350	R500
Reebok Trainers	R280	R600
Apples (1/2 kg)	R7	R16
Bananas (1/2 kg)	R1	R4
Eggs (1 doz)	R9	R14
Nintendo	R1600(Super Nintendo)	R1700 (Ds Lite)
Nintendo Games	R200	R400
Bread (loaf)	R4	R8
Coke (tin)	R4	R6
Leather Jacket	R990	R1700
Lego Set (200pc)	R90	R214
Board Game	R199	R292


PREPAID DEALS
handset deals under R900
Nokia 1208
R399
Samsung R100
R405
Nokia 1650
R515
handset deals above R900
Samsung E250
R929
Samsung M620
R1015
Nokia 3110c
R1309
You're paying too much for international calls from your cellphone.
Dial 087 94 12345 from your cellphone and follow the voice prompts to connect to your destination at the cost of a local call.
bottomline


AFRICA'S ORIGINAL
FLYFISHING
MAGAZINE
R19,50
BUILD YOUR OWN ROD
Finishing your homemade beauty
BLISS
A day on the Kaaimansgat
National BOAT Show
Coca-Cola Dome
15-17 August
YELLOWS UNDER THREAT
Working to save our indigenous species
THE KLINK
Trout just can't resist it
www.africanangler.com

Appendix C: Task 3: Quilt Problem

Task 3

Quilting problem

Quilters Sew Their Stuff

Last weekend, quilters from all over Gauteng came to Pretoria to display their best quilts. Quilts of various colors, sizes, and patterns demonstrated the talents and creativity of the quilters. In addition to being used as bed coverings, many quilts are also works of art. Many people will hang a quilt on a wall as they would a painting or photograph.

Many quilters will start with basic patterns, then modify and reorganize them in new ways to create original designs. Piecing together fabric in different ways is one method they use to develop new designs. Changing the colors can also affect how the product looks. Two quilts may use the same shapes in the same configuration, but look very different because of the way the colors were used in the design. This provides the quilter with a great deal of flexibility in their creation.

The quilting process, from pattern to final product, is very complex. Local quilter, Judy Richmond, explained it this way. "I start with the main pattern which gives the quilt a theme. Then I make a layout for my design. I figure out all the measurements for the different parts of the quilt. All of this is done before I start to cut any fabric

pieces so I don't waste any fabric and my pieces fit together very nicely."

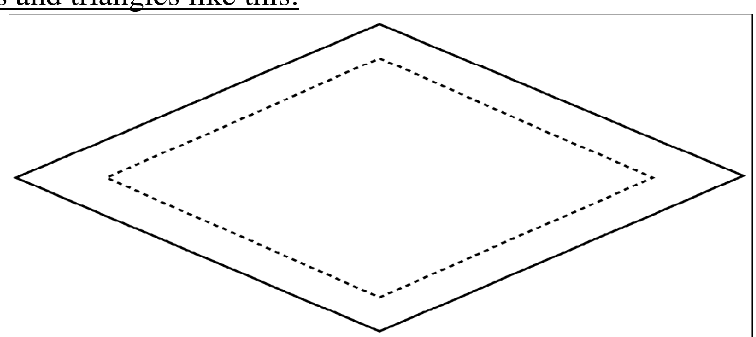
The next step is to make the templates the quilter will use to cut out the pieces. "Each shape in the quilt has its own template. I use the templates as patterns to cut out the pieces for my quilt. The template is the actual size of the piece for the final quilt plus ½ cm allowance for the seams. This allows the quilt pieces to fit together snuggly and smoothly," states Richmond.

Borders are an essential part to the quilt design. Most quilts have two outer borders of different sizes and colors. The outermost border is usually 15 cm and the inner border is about 5 cm. Borders also can be use to separate rows and columns of squares. These borders are often the same size as the inner border.

Quilters have to be very careful when making measurements for the layout and when cutting out the pieces from the templates so everything fits together correctly. Sewing the quilt pieces together must be completed in a particular order so the quilt looks right. When done properly, a quilt becomes a work of art that many families keep for many years to come.

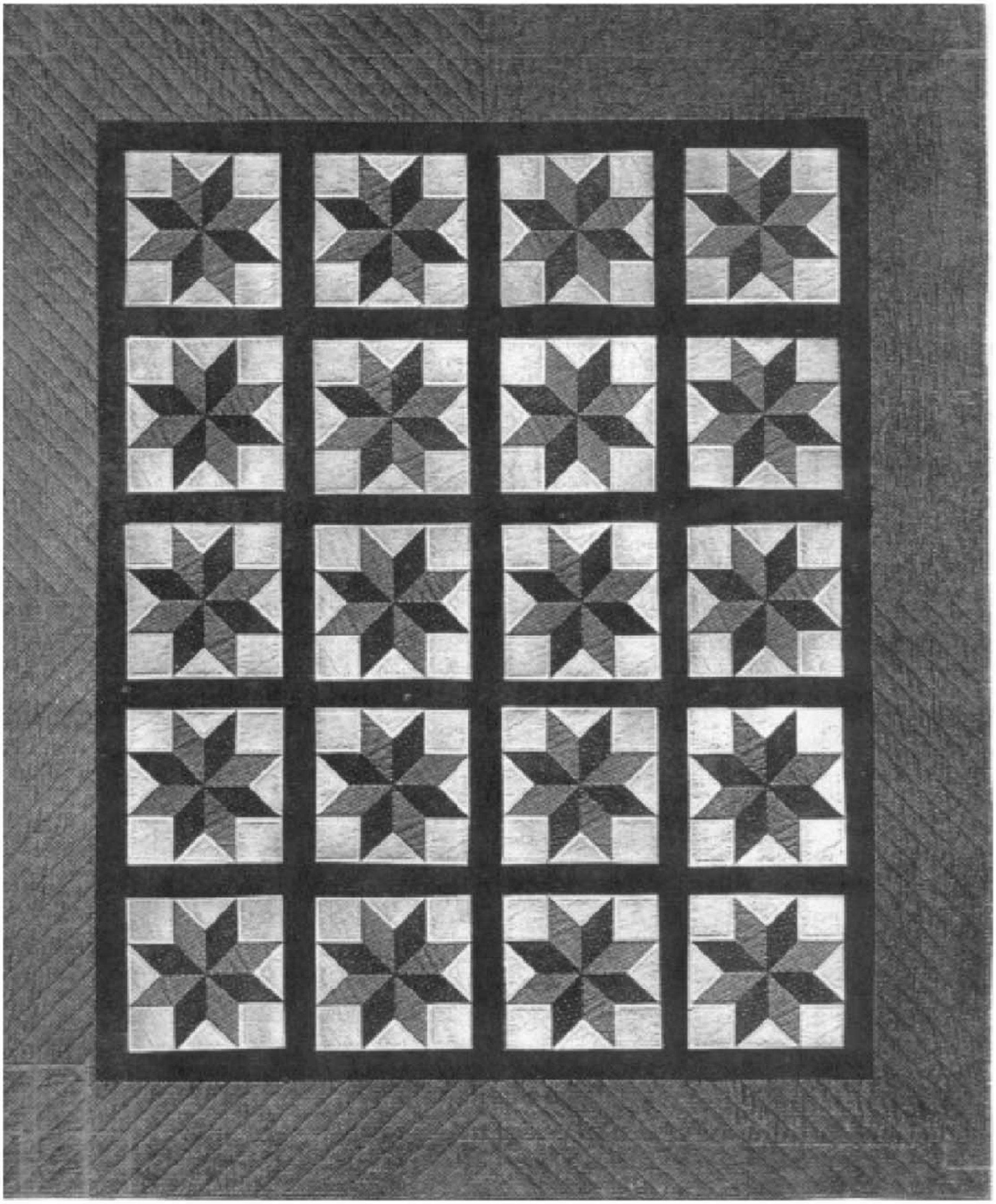
Here are some examples of other quilts:

Quilts are made by sewing small pieces of material together. The pieces are most often shapes such as squares, rectangles and triangles like this:



Quilts are often made from pictures that are found. This means that each type of shape must be cut from paper. Often quilters find beautiful pictures of patterns they would like to make and have to figure out the size of the pieces. The members of the *School's Name* Quilt Club often have difficulties converting photographs like these into templates that are exactly the right size and shape so that they too can make the quilts.

You have been asked to write a letter to the *School's Name* Quilt Club to explain to them how make pattern pieces that are exactly the right size and shape. Also include in your letter the templates for the following quilting pattern. (Note: the quilt is for a double bed so should be 200cm x 236cm)



Appendix D: Permission from Gauteng Department of Education

UMnyango WezeMfundo	**Lefapha la Thuto**
Department of Education	**Departement van Onderwys**

Enquiries : Shadrack Phele

Tel. No. : [+2711] 355 0285

Mrs. Biccard Piera
652 16th Avenue
Rietfontein
PRETORIA
0084

Dear Mrs. Biccard Piera

PERMISSION TO CONDUCT RESEARCH: PROJECT

The Gauteng Department of Education hereby grants permission to conduct research in its institutions as per application.

Topic of research : "An investigation into development of mathematical modelling competencies of Grade 7 students."

Nature of qualification : M.Ed. [Mathematics Education]

Name of institution : University of South Africa

Upon completion of the research project the researcher is obliged to furnish the Department with copy of the research report (electronic or hard copy).

The Department wishes you success in your academic pursuit.

Yours in Tirisano,

p.p. Shadrack Phele [MIRMSA]
TOM WASPE
CHIEF INFORMATION OFFICER
Gauteng Department of Education

Office of the DDG: IS & KM (CIO)
Room 1807, 111 Commissioner Street, Johannesburg, 2001 P.O.Box 7710, Johannesburg, 2000

Appendix E: Ethical Clearance Certificate:

UNIVERSITEIT·STELLENBOSCH·UNIVERSITY
jou kennisvennoot • your knowledge partner

5 March 2010

Tel.: 021 - 808-9183
Enquiries: Sidney Engelbrecht
Email: sidney@sun.ac.za

Reference No. 283/2010

Mrs P Biccard
Department of Curriculum Studies
University of Stellenbosch
STELLENBOSCH
7602

Mrs P Biccard

APPLICATION FOR ETHICAL CLEARANCE

With regards to your application, I would like to inform you that the project, *An investigation into the development of mathematical modelling competencies of Grade 7 learners,* has been approved on condition that:

1. The researcher/s remain within the procedures and protocols indicated in the proposal;
2. The researcher/s stay within the boundaries of applicable national legislation, institutional guidelines, and applicable standards of scientific rigor that are followed within this field of study and that
3. Any substantive changes to this research project should be brought to the attention of the Ethics Committee with a view to obtain ethical clearance for it.

We wish you success with your research activities.

Best regards

M Fouché

MRS. MALÈNE FOUCHÉ
Manager: Research Support

Afdeling Navorsingsontwikkeling • Division of Research Development
Privaat Sak/Private Bag X1 • 7602 Stellenbosch • Suid-Afrika/South Africa
Tel +27 21 808 9111 • Faks/Fax: +27 21 808 4537

<u>Appendix F: Questionnaire Schedules</u>

<u>Pre-Intervention Questionnaire</u>

Date: ________________________________

Name: _________________________________

Please answer these questions as fully and honestly as you can. There are no right or wrong answers. Your answers are confidential.

1. What do you believe mathematics is all about?

____

2. Are you very good at mathematics?

____

3. Why do you say so?

____

____

____

4. What do you enjoy about your mathematics lessons?

____

____

5. What do you not enjoy about mathematics lessons?

____

____

6. How do you think we use mathematics in our everyday lives?

____

____

7. What are your feelings about taking part in this programme?

____

____

Thank you! ☺

Post-Intervention Questionnaire

Date: ___________________________________
Student Code: ___________________________
Please answer these questions as fully and honestly as you can. There are no right or wrong answers. Your answers are confidential.

1. What do you believe mathematics is all about?

____
____

2. Are you very good at mathematics?

____

3. Why do you say so?

____
____

4. What do you enjoy about your mathematics lessons?

____
____

5. What do you not enjoy about mathematics lessons?

____
____

6. How do you think we use mathematics in our everyday lives?

____
____
____

7. Did you enjoy the activities in this program?

____

8. Why?

____
____

9. What did you learn about yourself during this program?

____
____

10. What did you learn about mathematics during this program?

____

____

11. Anything else you want to mention?

____

____

Thank you! ☺

Appendix G: Written Work Guide

Competency	Understanding				Simplifying				Mathematizing				Working mathematically				Interpreting				Validating				Sense of direction				UIK				Planning/monitoring				Arguing			
	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3

Forms of Representation																				
	V	W	V	W	V	W	V	W	V	W	V	W	V	W	V	W	V	W	V	W
	G	S	G	S	G	S	G	S	G	S	G	S	G	S	G	S	G	S	G	S
Group																				
Date																				
Task																				

Appendix H: Researcher Observation Guide

Researcher Observation Instrument

Date			
Task			
Group			
Watching Listening/Off task			
Questions asked by group- amount of assistance needed			
Sense of direction			
Notes			

Appendix I: Group Reporting Sheet

<table>
<tr><th colspan="4">Group Reporting Sheet</th></tr>
<tr><td>Date</td><td colspan="3"></td></tr>
<tr><td>Task</td><td colspan="3"></td></tr>
<tr><td>Group</td><td colspan="3"></td></tr>
<tr><td colspan="2">This problem is about</td><td colspan="2">What we want to achieve at the end</td></tr>
<tr><td colspan="2">What we need to find out before we can carry on</td><td>What we've worked out so far</td><td>Why we decided to work this out</td></tr>
<tr><td>What we want to do next</td><td>Why we want to do this</td><td>What we need help with</td><td>Why we want help with this</td></tr>
</table>

Group Action Map	
We decided to:	Because:
1	
2	
3	

Appendix J: Quality Presentation Guide

Presentation Quality Assurance Guide (from Lesh and Clarke 2000: 145)

Perfor-mance Level	***How useful is the product***	***What the client may say***	***Questions that may be asked***
Requires redirection	The product is on the wrong track. Working longer or harder won't work. The group will need teacher support	Start over. This won't work. Think about it differently.	Who is the client? What tool does the client need? What does the client need to do with the tool? How useful is the tool for the purposes of the client? What information does the tool take into account? Any technical errors? Does the tool meet the client's needs?
Requires major extensions or refinements	The product is a good start toward meeting the clients' needs, but a lot more work is needed to respond to all the issues	You're on the right track but this still needs a lot more work before it'll be in a form that's useful.	
Requires only minor editing	The product is nearly ready to be used. It still needs a few small modifications, additions or refinements.	This is close to what I need. You just need to add or change a few small things.	
Useful for this specific data given	No changes needed. It meets the client's needs.	This will work well as it is.	
Sharable or Reusable	The tool not only works for the immediate situation but would be easy for others to modify and use in similar situations	Excellent! This tool will be easy to modify to use in other similar situations	The tool created is not only useful to the group but to others as well. Students started thinking about the tool. Students considered strengths and weaknesses about alternatives.

Appendix K: Individual Response Guides

Post Presentation - Individual response
1. What were you able to contribute to the solution of the problem?
2. What 'outside' information did you use?
3. What kind of maths did you use while working on the problem?
4. What did you learn while working on this task?
Name:
Group:
Date:

I think…

Which problem did you find the most interesting?	Why?
Which problem did you find the most difficult?	Why?
If I had to choose one problem to do with the other grade 7s, which one should it be?	Why?
In which problem do you think your group used the most difficult mathematics?	
Which problem do you think your group did the best at?	Why?
Any other comments about the problems?	