EFFECTIVE FIELD THEORIES FOR DISORDERED SYSTEMS FROM THE LOGARITHMIC DERIVATIVE OF THE WAVE-FUNCTION.

By

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DECLARATION

I, the undersigned, hereby declare that the work contained in this dissertation is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

Signature   Date

Abstract

In this dissertation, we give an overview of disordered systems, where we concentrate on the theoretical calculation techniques used in this field. We first discuss the general properties of disordered systems and the different models and quantities used in the study of these systems, before describing calculation techniques used to investigate the quantities introduced. These calculation techniques include the phase formalism method used one dimension, as well as the scaling approach and field theoretic approaches leading to non-linear σ-models in higher dimensions. We then introduce a complementary effective field theoretic approach based on the logarithmic derivative of the wave-function, and show how the quantities of interest are calculated using this method. As an example, the effective field theory is applied to one dimensional systems with Gaussian disorder. The average density of states, the average 2-point correlator and the conductivity are calculated in a weak disorder saddle-point approximation and in strong disorder duality approximation. These results are then calculated numerically and in the case of the density of states compared to the exact result.

Opsomming

In hierdie tesis, gee ons 'n oorsig van sisteme met wanorde, waar ons konsentreer op teoretiese berekeningsmetodes wat in die veld gebruik word. Eerstens bespreek ons die algemene eieskappe van sisteme met wanorde en verskillende modelle en hoeveelhede wat gebruik word in die studie van hierdie sisteme, voordat ons die berekeningsmetodes beskryf wat gebruik word om die boge-noemde hoeveelhede te ondersoek. Hierdie berekeningstegnieke sluit in die fase formalisme wat in een dimensie gebruik word, asook die skalingsbenadering en veldteoretiese metodes wat lei tot nie-lineêr σ-modelle in hoër dimensies. Ons voer in 'n komplementere effektiewe veldeteorie gebaseer op die logaritmiese afgeleide van die golffunksie, en wys hoe hoeveelhede van belang met hierdie metode bereken word. As 'n voorbeeld, word die effektiewe veldetoerie toegepas op 'n een dimensionele sisteem met 'n Gauss verdeling. The gemiddelde digtheid van toestande, die gemiddelde 2-punt korrelator en die gemiddelde geleidingsvermoë word bereken in 'n swak wanorde saalpunt benadering en in 'n sterk wanorde duale benadering. Hierdie resultate word dan numeries bereken, en in die geval van die digtheid van toestande vergelyk met die eksakte resultaat.

To my grandmother

"At times like these,
all you want to do is sneeze."

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APPENDICES

List of figures

Acknowledgments

Inspiring people,
golden leaves, setting sun
– appreciation.

In the long years it has taken to reach the completion of this work, there have been many people who have been at my side; who have guided, motivated, inspired and even entertained me. I am deeply indebted to them all.

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So finally, *finis coronat opus*.

Chapter 1
Introduction

The theory of disordered systems has developed extensively since the initial works of Mott [Mot49], Dyson [Dys53], Schmidt [Sch57] and Anderson [And58]. Early work on one dimensional (1D) disordered systems concentrated on calculating the average density of states [Dys53, Sch57, Fri60] with some works including the study of the spectral densities and the electric conductivity [Hal65, Hal66a, Hal66b]. Dramatic progress was made by Berezinskii [Ber74] who utilised diagrammatic techniques to prove that all states are localised in 1D disordered systems, although this is generally difficult to extend to higher dimensions. Abrahams *et al.* [Abr79] introduced a scaling theory of localisation, based on the non-interacting electron model, which predicts that a metal-insulator transition occurs in dimensions greater than two, although there seems to be experimental evidence for a transition in two dimensions [Kra96]. Making use of the replica trick [Edw75], the problem was mapped onto a non-linear σ-model [Weg79b, Sch80, Efe80], which gave quantitative confirmation of the scaling approach. Efetov's supersymmetry approach [Efe83, Efe97] introduced a mathematically more rigorous alternative to the replica trick, which he used to prove, amongst other things, a conjecture of Gor'kov and Eliashberg [Gor65] that random matrix theory [Dys62, Meh91] can be applied to the energy level statistics of particles in disordered systems.

Notwithstanding the considerable amount of work that has gone into the investigation of disordered systems, there are still many outstanding problems, for instance the lack of an order parameter [McK81] to describe the 2nd order metal-insulator phase transition, as well as the questions concerning the value of the upper critical dimension which would allow one to introduce mean field theories [Har81]. Also, finding an analytically tractable description of the localisation problem, especially for strongly disordered systems, of which there has been little progress, would lead to a better understanding of disordered based phenomena, such as the quantum Hall effect [vK80, Moo01]. For this reason, any additional approaches for studying disordered systems, possibly leading to new insights, are useful.

In general, we would like to calculate disordered averages of observables that depend on a random potential $V(x)$. These disordered averages can be calculated when the exact dependence of the observable on the random potential is known. However, when this dependence is not known, as for example in the density of states and correlators of the wave-function, other methods of averaging these observables over the disorder are needed. Usually, the disorder averages of advanced or retarded Green's functions, $G^{\pm}(E) = (E - H \pm i\epsilon)^{-1}$, are calculated since their dependence on $V(x)$ is known. These averages are then related to the averages of the observable. Thus, one would calculate the average of the advanced Green's function and then relate it to the density of states using

$$\langle \rho(E) \rangle = - \lim_{\epsilon \to 0} \frac{1}{\pi L^d} \mathrm{Im} \mathrm{Tr} \langle G(E) \rangle, \tag{1.1}$$

where the angle brackets denote averaging over the disorder. Both of the main field theoretic techniques for investigating disordered systems, the supersymmetry [Efe83, Efe97] and replica [Edw75] methods, are based on calculating the averages of products of Green's functions using

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a generating function and then extracting physical observables from the result.

In this dissertation we would like to propose a complementary approach for calculating disorder averages. This approach entails a transformation where we change from the random potential $V(x)$ to a new set of random variables, which can be related to the logarithmic derivative of the wave-function and energy of a particle moving in the random potential. Our motivation for introducing this formalism is based on the fact that we are working with fields that are directly related to the random quantities that appear in the Schrödinger equation. Using this formalism thus allows us to calculate directly averages of the density of states and correlations of the wave-function and its absolute value. Also, since the field theory that is introduced is of a more conventional type, it is possible to carry out a duality transformation to obtain a dual field theory which would allow one to investigate strongly disordered systems. It is hoped that this complementary approach will give additional insight and a better understanding of disordered systems.

This dissertation is structured as follows : in Chapter 2, we give an overview of the general properties of disordered systems and the quantities studied when investigating disordered systems. Although this work is mainly focused on the calculation techniques utilised in disordered systems, there is a brief discussion of some experimental results. In the later sections of the chapter, we concentrate on the techniques introduced when one-dimensional disordered systems were initially investigated, and then on the methods used for studying higher dimensional disordered systems, namely the scaling approach and the non-linear σ-model approaches based on the replica or supersymmetric field theory methods. Finally, at the end of the chapter, we discuss some open problems of the current calculation techniques, which hopefully the new formalism introduced in this work will give insight in overcoming.

In Chapter 3, we introduce the formalism, both for one-dimensional systems and for higher dimensions, and show how disordered averaged observables are calculated within this framework. Since the one dimensional system with Gaussian disorder is probably the best studied disordered system, with a variety of well known results available [Hal66a, Lif88], it is ideal for testing and developing approximation techniques within our formalism with the ultimate aim of extending these techniques to higher dimensions, and possibly also to the case of a magnetic field. Therefore, in Chapter 4, we focus on the one dimensional Gaussian disordered system in order to illustrate how the formalism can be applied, using standard approximation techniques, to recover known results for the density of states [Hal65], and to obtain results for the 2-point correlator of the absolute value of the wave-function [Lif88] as well as the conductivity [Ber74]. In Chapter 5 we numerically calculate and generate plots of the main results obtained in Chapter 4, and compare these results with known results. Finally, in Chapter 6, we conclude this work and discuss some further possible developments of the formalism.

Chapter 2
Disordered systems : An overview

In this chapter we shall give a general overview of the development of disordered systems. We shall be using the reviews of Halperin [Hal66a], Thouless [Tho74], Lee and Ramakrishnan [Lee85], Kramer and MacKinnon [Kra93], and Belitz and Kirkpatrick [Bel94], as well as the detailed text of Lifshits *et al.* [Lif88] as a general guideline.

2.1   General properties of disordered systems

Disordered systems fall into two main categories, which can be arrived at by noting that perfect crystals are characterised by two general symmetries. Firstly, there is translational order, where the atoms of the crystal are arranged with geometrical regularity, and secondly, there is compositional order where the atoms of different species in the crystal are arranged in a regular pattern.

Disordered systems can thus be classified by determining which of the two symmetries above are broken in the system. The first category occurs when the symmetry of compositional order is broken so that the different species that make up the crystal are no longer regularly arranged. This type of disorder is known as compositional or substitutional disorder, with a disordered substitutional alloy being a simple example of this type of disorder. The other category of disordered systems are characterised by the lack of translational order and is known as structural or topological disorder. This is the type of disorder that is found in amorphous, liquid and gaseous media. Also, some systems may be a combination of both categories, where the system lacks both compositional and translational order. From this classification of disordered systems, we see that it is natural to construct various models of disorder using the model of an ideal crystal as a starting point. A schematic diagram which shows the difference between these different models of disorder and the ideal crystal can be found in Fig. 2.1.

Within the framework of the non-interacting electron picture all these systems of atoms in disordered configurations can be represented by a random potential wherein the electrons move according to the one-particle Schrödinger equation[1]

$$-\nabla^2\psi + U(\vec{r})\psi = E\psi. \tag{2.1}$$

Thus, we find that to investigate disordered systems we need to study the properties of electronic wave-functions where there is a given statistically defined random potential. However, this random potential need not represent the full one-electron problem, as there are several approximations that can be made to construct simpler mathematical models which still capture the essence of the disordered system. The first approximation that can be made is the effective mass approximation where the difference between the disordered system and the pure crystal background is described by the random potential. In the situation where the disorder is caused by impurities, for instance, each impurity is represented by a model potential that describes the

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[1]Note that we use a system of units in which $\hbar^2 = 2m = 1$.

difference between the impurity and the host atom. Thus

$$-\nabla^2 \psi + V(\vec{r})\psi = E\psi, \tag{2.2}$$

where $V(\vec{r}) = U(\vec{r}) - W(\vec{r})$, with $W(\vec{r})$ describing the deterministic background atomic structure.

Another approximation that is also quite common is the one-band approximation. In this approximation, the continuous Schrödinger equation is reduced to a discrete lattice model equivalent. This reduction is based on the tight binding approximation from solid state theory [Pei55]. To obtain the discrete model, the wave-function is expanded in terms of a complete set of orthonormal Wannier functions localised at each atomic site [Tho74],

$$\psi(\vec{r}) = \sum_{i,n} a_i^{(n)} \phi^{(n)}(\vec{r} - \vec{R}_i) \tag{2.3}$$

where $\vec{R}_i$ is the position of the atom i, and n labels the orbitals (bands). The Schrödinger equation (2.1) becomes a matrix equation for the amplitudes $a_i^{(n)}$. Taking into account only one orbital, we obtain

$$\epsilon_i a_i + \sum_j V_{ij} a_j = E a_i, \tag{2.4}$$

where ϵ_i is the atomic energy level on site i and V_{ij} is the matrix element of the Hamiltonian
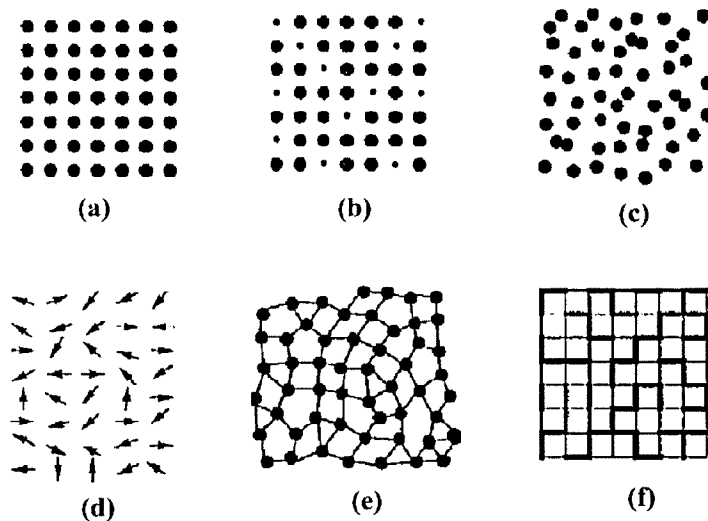
Figure 2.1: **Schematic diagram of different models of disorder based on the model of an ideal crystal. a) Ideal crystal; b) compositional disorder; c) structural disorder; d) orientational disordered system; e) topological disorder; f) regular lattice of identical atoms with random hopping matrix elements. Taken from [Kra93].**

between the sites i and j. The corresponding Hamiltonian is

$$\hat{H} = \sum_i \epsilon_i |i\rangle\langle i| + \sum_{ij} V_{ij}|i\rangle\langle j|. \tag{2.5}$$

The Hamiltonian thus consists out of diagonal terms ϵ_i in the electron position, and off-diagonal "transfer" terms V_{ij} that connect nearby sites. The coefficients, ϵ_i and V_{ij} are taken to be random in order to model the disordered system. The simplest discrete model is where only nearest neighbours contribute to the transfer terms.

Hamiltonians of the type given in (2.5) may be used to describe the vibrational properties [Dea72], as well as the electronic properties of amorphous semiconductors [Kra79], alloys [Ell74] and spin glasses [Edw75]. The orbital model, where the orbitals are not neglected in (2.5), was used by Wegner to establish the scaling theory of localisation [Weg79b, Weg79a].

2.1.1   Structure of calculated quantities

Since the systems contain randomness, the energy levels and the corresponding states $\psi(\vec{r})$ will also be random. Usually, one is not interested in calculating the spectra or states, but rather combinations of these quantities. To calculate thermodynamic properties of disordered systems, one must know the density of states in the macroscopic limit, $V \to \infty$, which is given by

$$\rho_V(E) = V^{-1}\sum_n \delta(E - E_n)$$
$$= V^{-1}\operatorname{Tr}\delta(E - \hat{H}) \tag{2.6}$$

as $V \to \infty$. Note that the density of states does not carry information about the structure of the states of the system.

More information about the structure of the states are provided by the spectral density

$$A_V(\vec{k}, E) = V^{-1}\sum_n \delta(E - E_n)\left|\int_V \psi_n(\vec{r})\,e^{i\vec{k}\cdot\vec{r}}d\vec{r}\right|^2$$
$$= V^{-1}\int_V \langle\vec{r}|\,e^{i\vec{k}\cdot\hat{r}}\delta(E - \hat{H})\,e^{-i\vec{k}\cdot\hat{r}}\,|\vec{r}'\rangle d\vec{r}d\vec{r}' \tag{2.7}$$

where $\hat{r}$ is the position operator. The spectral density can be used to determine the characteristics of inelastic scattering of neutrons as well as optical absorption properties [Hal65].

The density of states and the spectral density are both one-particle quantities. An example of more complex quantities that are also useful are two particle quantities, e.g. the electrical conductivity in a variable electric field of frequency ω in an isotropic system :

$$\operatorname{Re}\sigma(\omega, T, \mu) = \frac{\pi e^2}{\omega}\lim_{V\to\infty}\int\left[\eta_F(E) - \eta_F(E + \omega)\right]F_V(E, E + \omega)dE \tag{2.8a}$$

where $\eta_F(E) = (1 + e^{(E-\mu)/T})^{-1}$ is the Fermi distribution, μ the chemical potential and

$$F_V(E, E') = V^{-1}{\sum_{i,j}}'|p_{ij}|^2\delta(E - E_i)\delta(E' - E_j) \tag{2.8b}$$

where the summation accent denotes that diagonal elements are not included, and p_{ij} is the matrix element of the momentum. This formula for the conductivity is known as the Kubo formula [Kub57, Mah90] and it gives the linear response of the system to a current induced by an external electric field.

Many other quantities can be constructed for investigation. In general, quantities can be obtained by taking quantities that are additive over the volume, normalised to the unit volume, which are calculated by way of their microscopic definitions [Lif88]. All such quantities can be expressed in terms of the energy levels, E_n, and the states, $\psi(\vec{r})$.

All of the above quantities can be expressed in terms of the Green's function of the Schrödinger equation, $\hat{G}(E) = (E - \hat{H})^{-1}$, using the relationship

$$\delta(E - \hat{H}) = \lim_{\epsilon \to 0} \pi \operatorname{Im} \hat{G}(E - i\epsilon). \tag{2.9}$$

Using the above relationship, we have

$$\rho_V(E) = \lim_{\epsilon \to 0} (\pi V)^{-1} \int_V d\vec{r}\, G(\vec{r}, \vec{r}; E - i\epsilon) \tag{2.10a}$$

$$A_V(\vec{k}, E) = \lim_{\epsilon \to 0} (\pi V)^{-1} \int_V d\vec{r}d\vec{r}'\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \operatorname{Im} G(\vec{r}, \vec{r}'; E - i\epsilon) \tag{2.10b}$$

$$F_V(E, E') = \lim_{\epsilon \to 0} (\pi^2 V)^{-1} \int_V d\vec{r}d\vec{r}' \left(\frac{\partial}{\partial r_1{}^\alpha} - \frac{\partial}{\partial r_1'{}^\alpha} \right) \left(\frac{\partial}{\partial r_2{}^\alpha} - \frac{\partial}{\partial r_2'{}^\alpha} \right)$$
$$\times \operatorname{Im} G(\vec{r}_1, \vec{r}_1'; E - i\epsilon) \operatorname{Im} G(\vec{r}_2, \vec{r}_2'; E - i\epsilon)\big|_{\vec{r}_1=\vec{r}_2'=\vec{r};\, \vec{r}_2=\vec{r}_1'=\vec{r}'} . \tag{2.10c}$$

The above quantities are random, since the potential $U(\vec{r})$ used in determining these quantities is random. These quantities must be averaged over the randomness, i.e over all impurity configurations at a fixed impurity level. In doing this averaging, we need to ensure that the average values of the physical quantities obtained in this manner differ only slightly from the sample values. This is the case when the potential satisfies general conditions of spatial homogeneity in the mean and the absence of correlations between the potential at one point and the potential at a point which is infinitely separated from the original.

The assumption that the disordered systems are spatially homogeneous in the mean reflects the simple fact of translational invariance in the mean, which is present in all macroscopically large disordered systems. Thus, for the Schrödinger equation with a random potential, $U(\vec{r})$, this property requires that all averages of the type $\langle U(\vec{r}_1)U(\vec{r}_2)\ldots U(\vec{r}_n)\rangle$ be invariant under translations of all the $\vec{r}_i$ by the same vector $\vec{a}$. Thus

$$\langle U(\vec{r}_1)U(\vec{r}_2)\ldots U(\vec{r}_n)\rangle = \langle U(\vec{r}_1 + \vec{a})U(\vec{r}_2 + \vec{a})\ldots U(\vec{r}_n + \vec{a})\rangle. \tag{2.11}$$

The second condition that there are no statistical correlations between points that are infinitely separated can be formulated as a condition for the factorisation of averages of the type

$$\lim_{|a|\to\infty} \langle U(\vec{r}_1 + \vec{a})\ldots U(\vec{r}_n + \vec{a})U(\vec{r}_1')\ldots U(\vec{r}_n')\rangle = \langle U(\vec{r}_1)\ldots U(\vec{r}_n)\rangle\langle U(\vec{r}_1')\ldots U(\vec{r}_n')\rangle. \tag{2.12}$$
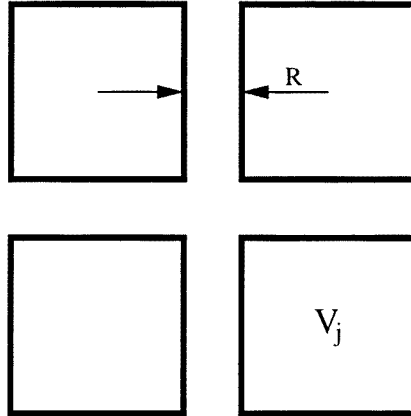
Figure 2.2: Schematic diagram of partitioning of the volume into smaller macroscopic subvolumes

These two conditions lead to the Birkoff ergodic theorem [Doo53]

$$\lim_{V \to \infty} \int_V f[U(\vec{r} + \vec{a})] d\vec{a} = \langle f[U] \rangle, \tag{2.13}$$

which expresses the fact that the spatial mean values coincide with the phase mean values.

2.1.2 Self averaged quantities

The consequence of the assumption of spatial homogeneity and lack of correlations at infinity is that all specific extensive physical quantities are self averaged. These are quantities, built from the eigenvalues and eigenfunctions of the disordered system under consideration, that tend to nonrandom limits in the macroscopic limit, $V \to \infty$.

The first proof of the property of self averaging was given by Lifshits [Lif42] for the polarisation vector of a disordered lattice and for the density of states by Rofe-Beketov [RB60]. Kohn and Luttinger [Koh57] gave a proof for a certain class of random quantities where the potential is of the form $U(\vec{r}) = \sum_j u_j(\vec{r} - \vec{r}_j)$, of which the one dimensional version was studied by Frisch and Lloyd [Fri60].

An explanation of how the property of self averaging occurs based on the proofs of Pastur [Pas71] and Slivnyak [Sli66], can be found in [Lif88], and goes as follows :

Every specific extensive quantity F_V becomes additive when the volume V is macroscopically large. Thus if we partition V into smaller but still macroscopic subvolumes, V_j, separated by "corridors" of width R (see Fig. 2.2) then we can write the quantity $V^{-1}F_V$ in the form of the arithmetic mean of its values in the subvolumes

$$V^{-1}F_V = \sum_j \frac{V_j}{V} V_j^{-1} F_j, \tag{2.14}$$

where we neglect the contribution of the surface terms which disappear in the macroscopic limit.

Since we assume that there is spatial homogeneity in the system, the statistical properties of

all the F_j are the same. If R is large, we can assume the F_j's are statistically independent for different j's due to the property that there are no correlations between well separated volumes. Thus the F_j's can take on all possible values independently.

Thus, if the number of subvolumes, V/V_j, is large, then in the sum over j, there will be V_j volumes with practically all possible values of F_j. This implies that the summation is equivalent to the summation over all possible realizations of some one F_j. Thus for macroscopic volumes, $V^{-1}F_V$ coincides with the average of one $V_j^{-1}F_j$ over all realizations. But this latter average is a non–random quantity and for large V_j is equal to the average of the initial quantity $V^{-1}F_V$ over all the realizations. Thus we find that the quantity $V^{-1}F_V$ is self averaging.

Another method to prove self averaging is to take the Laplace transform of a quantity in its energy variable, and then writing the kernel of the Laplace transform in terms of the Kac-Feynman functional integral representation [Kac57, Fey65] of Brownian motion trajectories. Since this representation contains $U(\vec{r})$ explicitly, an expression can be obtained for the quantity under consideration whose self averaging is simply proved using the Birkoff ergodic theorem (2.13). For more detail of this type of proof applied to the density of states, the spectral density and the electrical conductivity, see [Lif88]. Other techniques to prove the self averaging of various quantities can also be found in [Gus77] and [Pas78].

2.1.3 Independence of boundary conditions

Note that the non–random limits of self averaging quantities do not depend on the boundary conditions placed on the system. This is once again due to the properties of spatial homogeneity and the lack of correlations at infinity as well as the locality of the equations, which always lead to situations where changes in the boundary conditions induces changes to the specific extensive quantity by an amount that is of the order of the ratio of the sample's surface to its volume [Fel71]. Thus, in one dimension for instance, varying the boundary conditions leads to the quantity under consideration to differ from its original value by an amount of order L^{-1}, leading to the same result in the macroscopic limit $L \to \infty$.

2.1.4 Properties of the disordered system spectra

2.1.4.1 Density of states and spectrum boundaries

The property of self averaging of the density of states implies that in the macroscopic limit, the density of states is the same for all typical realizations of the random potential. Thus the density of states, although not a random quantity, can still provide information about the structure of the spectrum for all realizations simultaneously. Since the spectra of the realizations exist at points where the nonrandom density of states is non–zero, we can conclude that the spectra of all the typical realizations must coincide. Thus there are boundaries in the energy scale where the spectrum contains no states on one side, with non–zero states on the other. The deterministic nature of the density of states implies that these genuine spectrum boundaries are nonrandom, and occur at points where the density of states becomes zero.

It is possible to classify the spectrum boundaries into two types, namely stable boundaries and fluctuation boundaries. Stable boundaries are those boundaries in which, in the vicinity of the boundary, the spectrum is generated by any part of the potential realization. An example of such

a boundary is the high energy region of the Schrödinger equation. In the region of fluctuation boundaries, however, the spectrum occurs only as a result of highly improbable fluctuations of the random potential, and is realized by states localised at these fluctuations.

Thus the position of the fluctuation boundary depends on the nature of the disordered system (the shape of the impurity potential, statistics of the position of impurities in the Schrödinger equation, the values of atomic masses, etc.), while the stable boundary remains unchanged under any variations of the random parameters of the system.

As a rule, genuine spectrum boundaries are singular points in the density of states. The study of spectra near the vicinity of singular points is of interest, since it is in the region of these singular points that the quantum states and the systematics undergo changes.

2.1.4.2 Qualitative picture of the spectrum

Since a typical realization of a disordered system does not have translational invariance, a macroscopically large number of localised states must be present. These states, unlike Bloch states, are concentrated in a finite region of space. This characteristic of the states, which is one of the main differences between ordered and disordered systems, leads to a change in the kinetic properties of disordered systems at low temperatures. (Note, that at higher temperatures, the thermal fluctuations become significant and thus dominate over the disordered properties of the system, which are then negligible.)

In an infinite system, the discrete spectrum corresponds to localised states and the continuous spectrum to extended states [2]. In general, due to the differences in the type of spectra, the localised states and continuous states do not exist at the same energy in the spectrum. This means that there are nonrandom points in the energy axis that separate the energy levels where localised states occur and the energy levels where continuous states occur. These points of separation are known as mobility edges.

We now give a widely accepted picture of the spectrum of disordered states. We first describe what takes place for one semi–infinite band, where we assume that the spectrum fills up the interval from E_g, the fluctuation boundary, to the stable boundary at ∞.

In the three dimensional case, at high energies, the motion of a particle is quasi-classical. The particle is scattered by isolated impurities by small angles, thus a significant change in the particle's motion is only possible after a large number of collisions, with the effect that the initial phase of the wave-function is completely "lost". We can therefore describe the motion of the particle in terms of classical kinetic theory. Since the energy is assumed to be considerably higher than the height of the maximum height of the potential (see Fig. 2.3), this type of motion occurs in the classically allowed region, implying that the states of the particle in this energy region are extended and have a continuous spectrum.

However, in the region neighbouring the fluctuation boundary, E_g, which results from highly improbable fluctuations of the potential, the corresponding states are localised in the region of these fluctuations and thus the spectrum is discrete.

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[2]Note that this statement rests upon the subtle assumption of the Markov property for the random potential, so this condition may not be sufficiently general. See [Gin96] for an alternative condition for identifying localised states based on the Molchanov theorem.[Mol53]
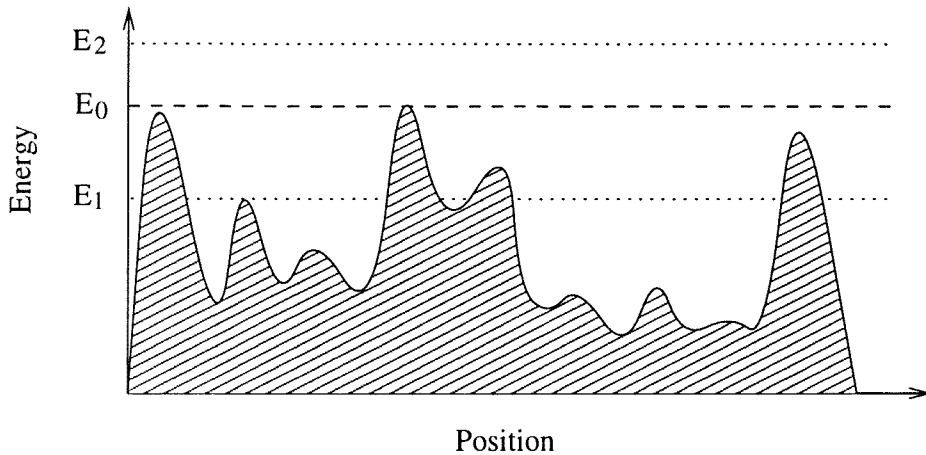
Figure 2.3: Schematic diagram of a random potential. The energy $E_1 < E_0$ is near the fluctuation boundary, while the energy $E_2 > E_0$ is in the classically allowed region.

The spectrum of a three dimensional disordered system thus generally consists of a continuous part $E > E_C$, and a discrete part $E_g < E < E_C$, separated by the mobility edge E_C. The amount of disorder in the system determines the relative proportions of the continuous and discrete components of the spectrum.

If a band structure is present, then both ends of the spectrum may be separated by a finite interval, and assuming that both boundaries are fluctuation boundaries, there will be two mobility edges $E_{C1} < E_{C2}$ in the band, with the extended states having energies which lie between the two mobility edges. As the amount of disorder increases, the mobility edges move further into the band until they finally coincide, so that the spectrum no longer has extended states. This transition from a metallic state to an insulator state was first predicted by Anderson [And70].

In the one dimensional case, the situation is different. Here, the particle's momentum either stays the same or changes to the opposite direction after an elastic scattering by an impurity. The scattering can therefore not be considered as weak, and the motion of the particle is not quasi–classical even at high energies, thus a detailed study of multiple–scattering effects is required. As a result, all states of one dimensional systems prove to be localised even when a weak random potential is applied. Note however, that in the vicinity of the stable boundary $E \to \infty$, the localisation is exceedingly small.

2.1.4.3   Localisation criteria

In order to study disordered systems which contain a macroscopically large number of localised states, it is necessary to investigate quantities that will enable us to detect and understand the discrete part of the spectrum. In this subsection we introduce a few of those criteria.

Firstly, when the volume of the system is finite, the entire spectrum is discrete. However, the states of the system with finite volume that tend to the states in the discrete and continuous spectrum behave differently in the macroscopic limit $V \to \infty$. Those that become states in the discrete spectrum are practically independent of V, while the others, that tend to the continuous spectrum, behave like $V^{-1/2}$. This makes it possible to construct quantities that have different

limits in the macroscopic limit which depend on whether the states at a given energy E are localised or not.

Consider the function

$$\sum_n \delta(E - E_{nV}) |\psi_{nV}(\vec{r})|^\alpha , \qquad (2.15)$$

where E_{nV} is an energy level of the finite volume system and α is an arbitrary exponent.

When $\alpha = 2$, then (2.15) provides an expression for the total density of states at energy E, which includes the sum of the discrete spectrum as well as the integral over the continuous spectrum as $V \to \infty$. However, if $\alpha > 2$, the states of the continuous spectrum contribute nothing to (2.15) in the limit $V \to \infty$, since the respective terms are of the order V^{-1}, which is insufficient for forming an integral out of the sum.

Thus, if (2.15) is nonzero for $\alpha > 2$ in the macroscopic limit, then all the states at energy E are localised, since the localised and extended states do not coexist at the same energy, and that the continuous states do not contribute. The energy E is then part of the discrete spectrum. If (2.15) is zero when $\alpha > 2$ then all the states with energy E are extended, and E is part of the continuous spectrum, or else there are no states present at that energy.

We can thus introduce the following quantities

$$p(\vec{r}, \vec{r}'; E) = \sum_n \delta(E - E_n) \left| \psi_n(\vec{r}) \psi_n(\vec{r}') \right|^2 \qquad (2.16a)$$

$$p_{E_1, E_2}(\vec{r}, \vec{r}') = \int_{E_1}^{E_2} p(\vec{r}, \vec{r}'; E) dE \qquad (2.16b)$$

$$p(\vec{r}, \vec{r}') = \int_{-\infty}^{\infty} p(\vec{r}, \vec{r}'; E) dE = \sum_n \left| \psi_n(\vec{r}) \psi_n(\vec{r}') \right|^2 \qquad (2.16c)$$

which are positive when a given realization of the states are localised at energy E for (2.16a), in the interval E_1 to E_2 for (2.16b), or in the entire spectrum for (2.16c). Note that the summation in (2.16) is over the discrete levels in the $V \to \infty$ limit only.

Equations (2.16a) and (2.16c) can also be written as [And58, Coh71]

$$p(\vec{r}, \vec{r}'; E) = \lim_{\epsilon \to 0} \frac{\epsilon}{\pi} \left| G(\vec{r}, \vec{r}'; E - i\epsilon) \right|^2 \qquad (2.17a)$$

$$p(\vec{r}, \vec{r}') = \lim_{T \to 0} \frac{1}{T} \int_0^T \left| G(\vec{r}, \vec{r}'; t) \right|^2 dt \qquad (2.17b)$$

where $G(\vec{r}, \vec{r}'; E)$ and $G(\vec{r}, \vec{r}'; t)$ are the Green's functions of the stationary and time dependent Schrödinger equations respectively. The random quantity $p(\vec{r}, \vec{r}')$ in (2.17b) is the probability that a particle at an initial moment, near the point $\vec{r}$, will be at a point $\vec{r}'$ after an infinitely long time interval. This quantity was introduced by Anderson [And58] as the indicator of whether or not spin diffusion is possible in a disordered system. If diffusion is possible, then $p(0,0) = 0$ or else $p(0,0) > 0$. Thus the absence of diffusion $p(0,0) > 0$ is directly related to the presence of a discrete spectrum (and thus localised states).

The quantities introduced in (2.16) and (2.17) are for a specific realization of the disorder,

since the quantum mechanical states were random. These quantities are self averaging, thus their average values, as well as the quantities themselves corresponding to all typical realizations are simultaneously zero or non-zero. This enables us to formulate localisation criteria that describe the nature of the states in all typical realizations. One such criterion which determines if the states in the region of energy E are localised or extended, depending on whether the function is positive or zero, is

$$
\begin{aligned}
p(r, E) &= \langle p(\vec{r} + \vec{r}', \vec{r}'; E) \rangle \\
&= \left\langle \sum_n \delta(E - E_n) |\psi_n(\vec{r})\psi_n(0)|^2 \right\rangle .
\end{aligned}
\tag{2.18}
$$

This averaged quantity implies that not only is the entire spectrum nonrandom, but also the discrete part is nonrandom, and thus the mobility edge introduced in the previous section is a nonrandom characteristic of the spectrum. Thus, for the semi-infinite band model used in previous section we have

$$
p(\vec{r}, E) > 0; \quad \text{if} \quad E < E_{\text{C}}
$$
$$
p(\vec{r}, E) = 0; \quad \text{if} \quad E > E_{\text{C}}.
$$

It is also possible to introduce a nonrandom quantity that allows us to determine more about the spectrum. Thus, if we know $p(\vec{r}, E)$ for all $\vec{r}$, we can introduce

$$
p(E) = \int p(\vec{r}, E) d\vec{r} = \left\langle \sum_n \delta(E - E_n) |\psi_n(0)|^2 \right\rangle .
\tag{2.19}
$$

Thus, comparing the right hand side with the expression for the density of states, (2.6), we see that $p(E)$ is the density of discrete energy levels. Since we assume that the discrete and continuous spectra cannot coexist, then if $p(E) = 0$ the spectrum at E is purely continuous, or else $p(E) = \rho(E)$ and the spectrum at E of the infinite system is purely discrete in each realization.

An alternative criterion for investigating what type of states are in the system, proposed by Halperin [Hal67], is

$$
h(E) = V^{-1} \left\langle \sum_n \delta(E - E_n) w_n^2 \right\rangle < \infty
\tag{2.20}
$$

as $V \to \infty$. Here

$$
w_n = \int (\vec{r} - \vec{r}_n)^2 |\psi_n(\vec{r})|^2 d\vec{r}
\tag{2.21}
$$

and

$$
\vec{r}_n = \int \vec{r} |\psi_n(\vec{r})|^2 d\vec{r}.
\tag{2.22}
$$

If $\vec{r}_n$ and w_n are finite in a macroscopic large sample, then the states ψ_n are concentrated primarily around their "centres", $\vec{r}_n$. Thus the meaning of the criterion $h(E) < \infty$ is that the mean square of the coordinate of a particle must be finite if the states are localised. (Note that this criterion requires that the particle mobility be zero, and that the static conductivity be zero [Hal73].)

Finally, if we know the behaviour of a system of finite dimensions, we can use a criterion introduced by Thouless [Tho74] to distinguish between localised and extended states. This criterion is based on the shift in the energy levels when the boundary conditions change as compared to the level separations. The energy of a localised state is expected to be insensitive to the form of the boundary conditions, provided the localisation center is not too close to the surface of the sample. On the other hand, for extended states the difference between the energy levels when the boundary conditions are changed must be of the same order of magnitude as the separation between energy levels. Thus if the ratio of the shift to separation gets smaller as the size of the sample increases, the corresponding states are localised, and if the ratio grows with the dimensions, the states are extended.

A similar criteria is useful in numerical and Monte Carlo simulations of disordered systems, since in these cases it is sufficient to follow a sequence of systems of finite dimensions and consider only the eigenvalues in order to decide if states are localised or not [Lic78]. Additionally, simple qualitative considerations make it possible to relate the average level shift to the direct current (DC) conductivity [Tho74], as well as the ideas of scaling [Abr79, Tho77].

2.1.4.4   Localisation and conductivity

The localisation criteria discussed in the previous section, while seeming natural from a quantum mechanical point of view, are not linked to any concrete kinetic characteristic of the system. Yet, due to the different properties of the localised states and extended states, there is a distinct change to the kinetic properties of the disordered system. For instance, if only localised states exist in a given energy interval, and the Fermi surface of the system lies in this interval, then the DC conductivity must be zero at $T = 0$ [Mot61, Mot67]. Thus we have another condition for localisation, this time based on the kinetic properties of the system, that

$$\sigma_{\mathrm{DC}}|_{T=0} = \lim_{\omega \to 0} \sigma(\omega)\Big|_{T=0} = 0. \tag{2.23}$$

Equation (2.8a) for the conductivity implies that (2.23) is equivalent to the condition that the limiting function $\langle F_V(E, E') \rangle$ is zero at $E = E'$:

$$\lim_{E' \to E} \lim_{V \to \infty} \langle F_V(E, E') \rangle \big|_{E=E_{\mathrm{F}}} = 0. \tag{2.24}$$

As an example, for free electrons (where σ_{DC} is infinite), we have

$$\lim_{V \to \infty} F_V(E, E') = 4E\delta(E - E')\rho_0(E), \tag{2.25a}$$

where $\rho_0(E)$ is the density of states for a d-dimensional Schrödinger equation with zero potential,

$$\rho_0(E) = \frac{(4\pi)^{-d/2} E^{d/2-1}}{\Gamma(d/2)}. \tag{2.25b}$$

In one dimensional disordered systems, the DC conductivity at zero temperature is zero for every position of the Fermi level, corresponding to the localisation of all the states in one dimensional disordered systems [Bor63, Byc74]. In the three dimensional case however, the DC conductivity is zero at $T = 0$ only if the Fermi level lies between the fluctuation boundary and the corresponding mobility edge. Thus, in the semi-infinite, band model introduced before where there is only one mobility edge,

$$\sigma_{\mathrm{DC}}|_{T=0} = 0 \quad \text{if} \quad E_{\mathrm{g}} < E_{\mathrm{F}} \leq E_{\mathrm{C}} \tag{2.26}$$

$$\neq 0 \quad \text{if} \quad E_{\mathrm{C}} < E_{\mathrm{F}}. \tag{2.27}$$

Note that in general, for non-zero temperatures, as mentioned earlier, when $E_{\mathrm{F}} \leq E_{\mathrm{C}}$ electron transport processes occur due to thermal activation [Shk79], while for $E_{\mathrm{F}} > E_{\mathrm{C}}$ they are primarily of a band nature.

A number of calculations and experiments have been conducted to determine how the conductivity tends to zero at the mobility edge. Mott initially suggested [Mot61] that σ_{DC} must vanish as ω^2, but Berezinskii showed [Ber74], using a rigorous diagrammatic technique, that in one dimension the conductivity goes to zero continuously, that is

$$\operatorname{Re}\sigma(\omega) \propto \omega^2 \ln^2 \omega \tag{2.28}$$

as $\omega \to 0$. Additionally, Abrahams *et al.* [Abr79], using a scaling argument theory developed by Thouless [Tho77, Tho79, Tho82], and assuming that only one parameter is needed along the lines of Renormalisation Group theory, found that in two dimensions all states must be localised[3], and that in three dimensional disordered systems the conductivity must vanish continuously when the system is in the localised phase. Similar assertions were also made earlier by Wegner [Weg76] and Schuster [Sch76]. These ideas were confirmed by both calculation of quantum corrections to classical kinetic quantities [Alt83] and in a number of experiments [Dyn82, Ros80, Zab84].

2.1.5 Interacting vs. non-interacting electron models of disorder

In the previous sections we gave an overview of properties of disordered systems, as well as introduced quantities that are useful in investigating these properties. However, in these sections, we made the assumption that we neglect the electron-electron interactions when modelling the disordered system. This is the approach that was originally taken by Anderson [And58] in order to simplify the problem and localisation due to non-interacting effects is generally known as Anderson localisation . Although this work concentrates on disordered systems of non-interacting electrons, it is important to point out that localisation can also occur due to electron-electron interactions. Mott [Mot49] showed that repulsion between electrons can produce transitions to

---

[3]Possibly by a power law instead of an exponential law [Kav81].

localised states in systems without disorder where the interactions are large compared to the kinetic energy bandwidth.

Recent experiments [Kra96] on high mobility Silicon MOSFETS have given evidence for extended states in two dimensions, in contrast with the scaling theory results of Abrahams *et al.* [Abr79]. These results of these experiments are thought to be due to the combination of disorder and strong electron-electron interaction effects. Unfortunately, the theoretical description of what happens under these circumstances are at present a central unsolved problem, especially when the electron-electron interactions are strong [Abr99].

2.2   Experimental results

In this section, we give a brief overview of some experimental methods of detecting localised states in disordered systems. For detailed reviews see [Ros80, Sar95, Paa91].

One of the main problems with trying to show localisation effects in disordered systems in experiments is that localised states exists only at temperatures very close to zero, which is difficult to achieve in experiments. Thus, while not possible to measure localised states directly, it is possible to obtain hints of localisation with experiments.

Mott investigated the temperature behaviour of the conductivity in disordered systems at low temperatures [Mot79] and found that there was a $T^{-1/4}$ dependence, i.e.

$$\sigma(T) = \sigma_0 \exp\left[-(T_0/T)^{1/4}\right].\tag{2.29}$$

This behaviour of the conductivity can be understood when assuming that the transport of the electrons is mediated by the phonon assisted hopping processes between localised states. The localised states are spatially localised in a finite volume characterised by a localisation radius ξ_r. The phonon assisted hopping is possible because localised states can be close to each other in the energy spectrum, if their centres are separated by distances much larger than the ξ_r.

Mott's $T^{-1/4}$ law was confirmed in early measurements of conductivity of amorphous silicon [Bey74]. Thus, the observation of Mott's law showed that electronic states can be localised by disorder. Note, that this type of localisation is known as strong localisation since the localisation radius is smaller than the system size.

Another hint of localisation can be found in weakly disordered metallic films at low temperatures. These films showed a logarithmic increase in the resistance when the temperature was decreased. (See Fig. 2.4). Although the system had metallic behaviour with large conductance, the conductance was lower than its classical value. This effect, referred to as weak localisation, is due to the phenomenon of enhanced backscattering due to quantum interference [Gor79], leading to corrections of classical transport results. Although the localisation length in weak localisation is larger than the system size, the effect is considered a precursor of strong localisation.

To experimentally reach the regime of strong localisation, where the localisation radius is less than the system size (where the system size is less than the phase coherence length), one has to use strongly disordered systems, or work in energy regions with very low density of states. Although localisation-delocalisation transitions have been observed in several experiments [Sar95], it is difficult comparing the results with theoretical models. This is due to the fact that in the low
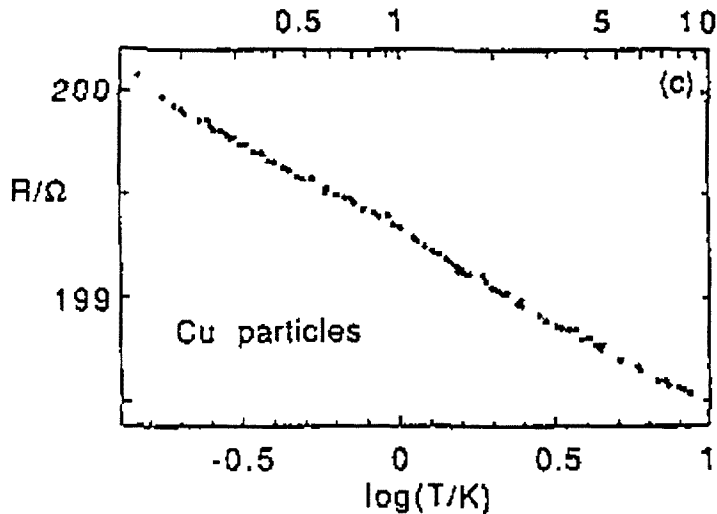
Figure 2.4: The resistance of a thin disordered film of coupled Cu particles as a function of the logarithm of the temperature.[Kob80]

density of states regime Coulomb interactions between electrons become essential, while most theoretical models are based on the non-interacting electron picture.

One of the clearest experiments for localisation-delocalisation transitions is the quantum Hall effect [vK80, Huc95]. The quantum Hall effect that occurs in two dimensional electron gases in the presence of strong perpendicular magnetic fields is characterised by step-function behaviour of the Hall conductivity σ_H as a function of the filling factor, and by a vanishing dissipative conductivity σ in the Hall regions. All states in the quantum Hall system are strongly localised, except for the states at critical fillings where the localisation radius is larger than the system size (See Fig. 2.5).

The systems showing quantum Hall effects are well suited to study properties of critical states. In a recent experiment [Cob96] for instance, it was possible to extract the whole conductivity distribution at criticality for a truly mesoscopic quantum Hall system, where the main finding was that the distribution is independent of system size. This means that the conductance fluctuations are of the same order as the average, and is much stronger that one would expect from classical transport theory.

Finally the localisation phenomenon in random media has also been observed experimentally in other, non-electronic, wave phenomena. Localisation has been observed in light scattering experiments [vA91], and there are indications that light waves may even be strongly localised if the scattering is strong enough [Eco90].

2.3 One dimensional disordered systems

In this section, we concentrate on one dimensional systems. There are several reasons for doing so [Hal66a]. Firstly, studying one dimensional systems allows one to obtain a qualitative picture of the solution to the disordered problem, which one hopes will carry over to the three dimensional case. Secondly, the qualitative form of the solutions in the one dimensional case may
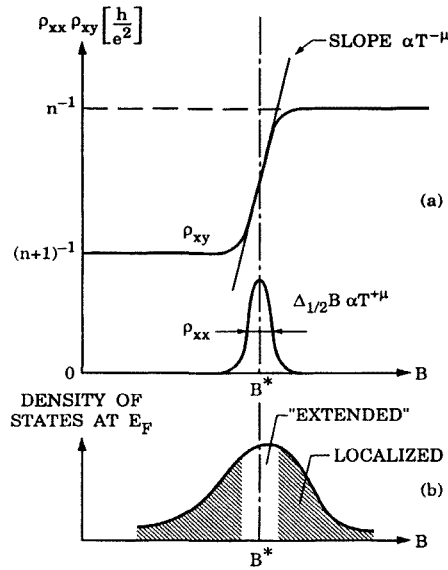
Figure 2.5: **The resistivity and the density of states in the quantum Hall effect : (a) sketch of the experimentally measured ρ_{xx} and ρ_{xy} as functions of the applied magnetic field; (b) density of states at the Fermi energy as a function of B. From [Pru88]**

suggest approximations which can also be applied to the three dimensional problem. Thirdly, if one has conceived of an approximate solution in the three dimensional case, it is possible to test these approximations, by applying it to the one dimensional case, and then comparing the approximate result to the exact result. Finally, there are also possible applications of the one dimensional solutions to the study of linear polymers, where similar techniques are used [dC90].

2.3.1 Density of states

2.3.1.1 Calculating the trace of the Green's function

Although our basic objective is to study the Schrödinger equation with a random potential (2.1), we shall first concentrate on its discrete analogue, since the initial attempts at studying disordered systems [Dys53, Sch57] were devoted to this model. We consider the tight binding version of (2.5)

$$\hat{H} = \sum_i U_i |i\rangle\langle i| + \sum_j H_{ij} |i\rangle\langle j|, \qquad (2.30a)$$

with

$$H_{i,i-1} = H_{i-1,i} = \begin{cases} H_i, & i > 0 \\ H_{i-1}, & i \leq 0 \end{cases} \qquad (2.30b)$$

where U_i and H_i are mutually independent random variables. From (2.9) and the self averaging property of the density of states, it follows that

$$\rho(E) = \lim_{\epsilon \to 0} \frac{N}{\pi L} \text{Im}\langle G_{00}(E - i\epsilon) \rangle \tag{2.31}$$

where $G_{00}(E) = (E - \hat{H})_{00}^{-1}$ is the diagonal matrix element of the Green's function of (2.30) and N and $L = Na$ is the number of sites and the length of the system respectively.

In the region of large negative energies, the spectrum of $\hat{H}$ is absent, so that $\langle G_{00}(E) \rangle$ is analytic and uniquely defined for all complex values of E, and thus also defined on the spectrum. Thus, to calculate $\rho(E)$ it is sufficient to find $\langle G_{00}(E) \rangle$ for values of E lying to the left of the spectrum [Dys53].

By expanding the Green's function in a perturbation series, it is possible to show that

$$G_{00}(E) = (E - U_0 - H_1^2 g_1 - H_{-1}^2 g_{-1})^{-1} \tag{2.32a}$$

where the g_m satisfies the equation

$$g_m = \begin{cases} (E - U_m - H_{m+1}^2 g_{m+1})^{-1}, & m \geq 0 \\ (E - U_m - H_{m-1}^2 g_{m-1})^{-1}, & m < 0 \end{cases} \tag{2.32b}$$

These results are obtained from general perturbation equations of G, which are valid in a space with any dimensionality and any interaction radius, and then using the fact that the problem is one dimensional. Note that in higher dimensions, one would arrive a chain of recurrence formulae of ever growing complexity instead of the compact form of (2.32).

From (2.32b) it follows that the quantities $u_{\pm 1} = H_{\pm 1}^2 g_{\pm 1}$ are independent of U_0 and of each other, and are identically distributed. Thus, denoting the probability densities of u and U by $P(u)$ and $Q(U)$, using (2.32a) we find that

$$\langle G_{00}(E) \rangle = \int \frac{P(u)P(u')Q(U)}{E - U - u - u'} du\, du'\, dU \tag{2.33}$$

Also, from (2.32b), we see that u_{m+1} does not depend on U_m and H_m. Since the distribution functions of U_m and H_m do not depend on m, the same is true of u_m. Thus, one can obtain the recursion equation of $P(u)$,

$$P(u) = \int K_E(u, u')P(u')du' \tag{2.34a}$$

where

$$K_E(u, u') = u^{-2} \int Q\left(E - u' - \frac{h}{u}\right) R(h)h\, dh \tag{2.34b}$$

with $R(h)$ the probability density of the random quantity H_m^2. Equation (2.34a) was first found by Dyson [Dys53] by averaging and detailed analysis of the terms in the perturbation series for the Green's function.

Using equations (2.34a), (2.33) and (2.31) the problem of finding the density of states can
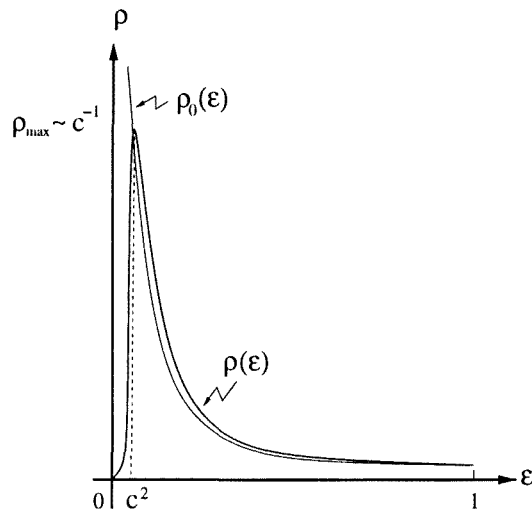
Figure 2.6: Density of states of the model of repulsive point scatters. [Lif88]

be solved in principle. To do so, it is necessary to solve the integral equation in (2.34a) for the probability density $P(u)$, which depends on E as a parameter, use $P(u)$ to compute $\langle G_{00}(E) \rangle$ according to (2.33), then to continue the energy analytically in the function obtained into the neighbourhood of the spectrum, which can now be used to calculate the density of states using (2.31).

Note that in deriving (2.32), the one dimensional nature of the problem, the nearest neighbour nature of the interaction, and the discreteness of the problem was used. The requirement of discreteness, however, is not essential, and the above scheme carries over to the continuous case.

As an example of results obtained with the above method, we consider the density of states of the disordered system with the random point scatters,

$$U(x) = \sum_j \kappa_0 \delta(x - x_j). \tag{2.35}$$

In the case of the repulsive point scatterers, $\kappa_0 > 0$, the density of states, using the above scheme, is given by

$$\rho(E) = \frac{\pi}{2\kappa\bar{r}} \sum_{n=1}^{\infty} n f\left(\frac{n\pi}{\kappa}\right), \qquad \kappa^2 = E > 0 \tag{2.36}$$

where we have assumed that the scatterer concentration $c = (\kappa_0 \bar{r})^{-1}$ is low and $f(y)$ is the probability density of the distances y between neighbouring scatters. In the case of the Poisson distribution $f(y) = \bar{r}^{-1} e^{-y/\bar{r}}$, the result is [Byc66]

$$\rho(E) = \frac{4\pi c^2}{\kappa_0 \varepsilon^{3/2}} \frac{\exp(-2\pi c/\sqrt{\varepsilon})}{[1 - \exp(-2\pi c/\sqrt{\varepsilon})]^2}, \qquad \varepsilon = \frac{4E}{\kappa_0^2}. \tag{2.37}$$
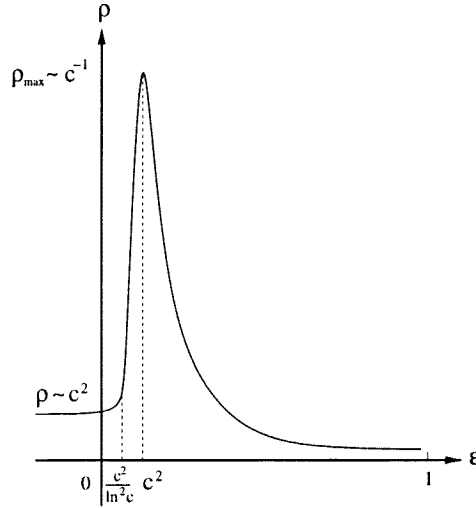
Figure 2.7: Density of states of the model of attractive point scatters.[Lif88]

In the limiting cases this is

$$
\rho(E) = \begin{cases} \frac{4\pi c^2}{\kappa_0 \varepsilon^{3/2}} \exp(-2\pi c/\sqrt{\varepsilon}), & \varepsilon \ll c^2 \\ (2\pi\sqrt{\varepsilon})^{-1}, & c^2 \ll \varepsilon \ll 1 \end{cases} \tag{2.38}
$$

This result is shown in Fig. 2.6. The small disorder ($c \ll 1$) leads to the situation where $\rho(\varepsilon)$ differs from the density of states $\rho_0(\varepsilon)$ of the ideal system only in the narrow neighbourhood $\Delta\varepsilon \sim c^2$ of the true fluctuation boundary of the spectrum.

For the model of attractive scatterers ($\kappa_0 < 0$) the boundary spectrum is at $-\infty$, but for low concentrations ($c \ll 1$) it is possible to examine the behaviour of the density of states in the neighbourhood of the boundary of the initial spectrum at $E = 0$.

The following expression is obtained [Gre76] for the density of states in the region $|\varepsilon| \ll 1$:

$$
\rho(E) \sim \begin{cases} \frac{8c^2}{3|\kappa_0|} \left(\frac{1+i\sqrt{\varepsilon}}{1-i\sqrt{\varepsilon}}\right)^{-ic/\sqrt{\varepsilon}}, & \varepsilon < 0 \\ \frac{4c^2\left[\tan^{-1}\sqrt{\varepsilon} - \frac{\sqrt{\varepsilon}}{1+\varepsilon} - \pi(1-\exp(\frac{2\pi c}{\sqrt{\varepsilon}}))^{-1}\right]}{|\kappa_0|e^{2c}\varepsilon^{3/2}(1-\exp(-\frac{2\pi c}{\sqrt{\varepsilon}}))}, & \varepsilon > 0 \end{cases} \tag{2.39}
$$

In extremely low concentrations ($\ln|c| \gg 1$) this simplifies to

$$
\rho(E) \sim \begin{cases} \frac{8c^2}{3|\kappa_0|}, & \begin{cases} \varepsilon < 0, |\varepsilon| \ll 1 \\ 0 < \varepsilon \ll \frac{c^2}{\ln^2 c} \end{cases} \\ 4\pi c^2 \varepsilon^{-3/2} \exp(-2\pi c/\sqrt{\varepsilon}), & \frac{c^2}{\ln^2 c} \ll \varepsilon \ll c^2 \\ (2\pi\sqrt{\varepsilon})^{-1}, & c^2 \ll \varepsilon \ll 1 \end{cases} \tag{2.40}
$$

This is shown in Fig. 2.7.

Thus the main difference between the density of states on the region $|\varepsilon| \ll 1$ in the system with attractive scatterers and the system with repulsive scatterers is the non-zero density of

states in the region left of $\varepsilon_0 \sim c^2 / \ln^2 c$.

2.3.1.2   Integral density of states based on the phase formalism

In the previous section we used the relationship between $\rho(E)$ and the Green's function to calculate the density of states. However, in one dimension, there exists a closer relationship between the density of states and the solutions of the random Schrödinger equation. For continuous models this can be formulated as : The number of states with an energy not exceeding E in a system that occupies the interval $(0, L)$ can be at the most greater by unity than the number of zeros on $(0, L)$ possessed by the solution of the corresponding Schrödinger equation with a given logarithmic derivative at one of the ends. This is known as the node counting theorem [Cou53, Man72]. The connection between the spectrum and the zeros of the solution of the disordered Schrödinger equation was first used by Schmidt [Sch57].

We define $\mathcal{N}(E_1, E_2)$ as the number of states of the Schrödinger equation with energies lying between E_1 and E_2 as

$$\mathcal{N}_L(E_1, E_2) = \int_{E_1}^{E_2} \rho_L(E')dE'. \tag{2.41}$$

Also, we introduce the non reduced phase $\alpha(x)$ of the wave-function

$$\cot(\alpha(x)) = \frac{\psi'(x)}{\psi(x)}, \tag{2.42}$$

where we require that $\alpha(x)$ be a continuous function of x. Using (2.42) in the Schrödinger equation, we find that $\alpha(x)$ satisfies

$$\alpha' = \Phi_E(U, \alpha), \qquad \Phi_E(U, \alpha) = \cos^2(\alpha) + [E - U(x)]\sin^2(\alpha) \tag{2.43}$$

with the initial condition $\alpha(0) = \alpha_0$, with $\cot(\alpha_0) = \left(\frac{\psi'}{\psi}\right)_{x=0} = 0$ where $0 \leq \alpha_0 \leq \pi$. Note that the phase is a monotonic function of the energy, which implies that the number of eigenvalues, $\mathcal{N}_L(E_1, E_2)$, of the Schrödinger equation in the interval $(0, L)$, with boundary conditions $\cot(\alpha(0)) = \cot(\alpha_0)$ and $\cot(\alpha(L)) = \cot(\alpha_L)$, coincides with the number of values of E that belong to the interval (E_1, E_2) and for which $\alpha(L, E) = \alpha_L + m\pi$ where m is an integer, thus

$$\mathcal{N}_L(E_1, E_2) = \frac{1}{\pi L} \left[\alpha(L, E_2) - \alpha(L, E_1)\right]. \tag{2.44}$$

In the macroscopic limit, using the fact that the number of states is a self averaged quantity, we find that

$$\mathcal{N}_L(E_1, E_2) = \lim_{L \to \infty} \frac{1}{\pi L} \left\langle \alpha(L, E_2) - \alpha(L, E_1)\right\rangle. \tag{2.45}$$

Integrating over (2.43), we can rewrite (2.45) as

$$\mathcal{N}(E_1, E_2) = \frac{1}{\pi} \left(\langle\Phi_{E_2}(U, \alpha)\rangle_{ST} - \langle\Phi_{E_1}(U, \alpha)\rangle_{ST}\right). \tag{2.46}$$

where on the right hand side we have averages over the joint stationary distribution of the random potential $U(x)$ and phase $\alpha(x)$ reduced to the interval $(0, \pi)$.

The formulas (2.45) and (2.46) are universal and valid for any random potential. They show that to calculate $\mathcal{N}(E_1, E_2)$ and thus $\rho(E)$ we need to know the joint probability distribution of the potential and reduced phase, or the probability distribution of the non-reduced phase when $x \to \infty$. In many cases these functions satisfy certain integral equations (of the Smoluchowski type) or differential equations (of the Fokker-Planck type). These integral or differential equations describe the probabilistic evolution of $U(x)$ and $\alpha(x)$ and can be derived on the basis of (2.43) with allowance for the probabilistic properties of the random potential. If one has additional information about the random potential, simpler expressions for $\mathcal{N}(E_1, E_2)$ can be obtained. For instance, if we know that $U(x)$ has no delta-like singularities (2.46) can be reduced to

$$\mathcal{N}(E_1, E_2) = P_{E_2}(0) - P_{E_1}(0) \tag{2.47}$$

where $P_E(\alpha)$ is the stationary probability of the reduced phase only, thus allowing one to obtain the density of states directly from $P_E(\alpha)$.

In the following sections we summarise the results obtained for various disordered cases where the above relation (2.47) is used, without going into detail on how $P_E(\alpha)$ is obtained in the different situations.

2.3.1.3 Gaussian potential

This is the simplest case, where $U(x)$ satisfies

$$\langle U(x) \rangle = 0, \qquad \langle U(x)U(x') \rangle = 2D\delta(x - x'). \tag{2.48}$$

In this case $P_E(\alpha)$ is solved using a Fokker-Planck differential equation [Fri60, Hal65].

In this case, for $z = \cot(\alpha)$, the stationary probability distribution $P(z)$ is given by

$$\frac{DP(z)}{J(E)} = \exp[-\Phi(z)] \int_{-\infty}^{z} \exp[\Phi(t)]dt \tag{2.49a}$$

with

$$J^{-1}(E) = D^{-1/3} \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} dy \exp[\Phi(yD^{1/3}) - \Phi(xD^{1/3})] \tag{2.49b}$$

where $\Phi(x) = \frac{x^3}{3D} + \frac{Ex}{D}$. The first to obtain this result were Frisch and Lloyd [Fri60], who started from a system of random repulsive point scatterers, and proceeded to the Gaussian potential by way of the limiting process $\kappa_0 \to 0$, $\bar{r} \to 0$, with $\kappa_0^2 \bar{r}^{-1} \sim 1$. Halperin [Hal65] also discussed the calculation of the number of states for the Gaussian potential, and additionally discussed the spectral density and the conductivity.

From (2.49b), we see that $J(E)$ tends to zero as $E \to -\infty$, thus $J(E) = \mathcal{N}(E)$, where $\mathcal{N}(E) = \mathcal{N}(E, -\infty)$. Equation (2.49b) can thus be rewritten to obtain an exact result for the

number of states [Fri60]

$$\mathcal{N}(E) = \sqrt{\frac{D}{\pi}} \left[\int_0^\infty dx\, x^{-1/2} \exp\left(-\frac{x^3}{12} - \frac{Ex}{D^{2/3}} \right) \right]^{-1} \tag{2.50}$$

which can also be written [Hal65] in terms of Airy functions [Gra94]

$$\mathcal{N}(E) = \frac{D^{1/3}}{\pi} \left[\text{Ai}^2\left(-\frac{E}{D^{2/3}} \right) + \text{Bi}^2\left(-\frac{E}{D^{2/3}} \right) \right]^{-1}. \tag{2.51}$$

The asymptotic results in the positive and negative energy regions are thus

$$\mathcal{N}(E) \approx \begin{cases} \pi^{-1} |E|^{1/2} \exp\left(-\frac{4}{3} \frac{|E|^{3/2}}{D} \right) \left[1 + \mathrm{O}\left(\frac{D}{|E|^{3/2}} \right) \right], & E < 0, |E| \gg D^{2/3} \\ \pi^{-1} E^{1/2} \left(1 + \frac{5}{32} \frac{D^2}{E^3} + \mathrm{O}\left(\frac{D^4}{E^6} \right) \right), & E > 0, E \gg D^{2/3} \end{cases} \tag{2.52}$$

2.3.1.4 Model of rectangular barriers of random length

In this model [Ben69], a generalisation of the Kronig-Penny model, the potential has the form $U_0 s(x)$, with U_0 positive, while the random function $s(x)$ takes the value of 0 or 1 on intervals whose lengths are independent random quantities with probability distributions $f_0 = a_0^{-1} e^{-y/a_0}$ and $f_1 = a_1^{-1} e^{-y/a_1}$ respectively.

A potential of this type may serve as a model potential for describing a one-dimensional binary alloy with compound concentrations $a_0(a_0 + a_1)$ and $a_1/(a_0 + a_1)$ where the interatomic separation is small compared to the electron wavelength.

In this case, for $P_0(\alpha)$, the stationary distribution where $s = 0$, satisfies

$$\left(1 - \frac{U_0}{E} \sin^2(\alpha) \right) P_0' + \left[(a_0\sqrt{E})^{-1} \left(1 - \frac{U_0}{E} \sin^2(\alpha) \right) + (a_1\sqrt{E})^{-1} \right] P_0 = (a_1 E)^{-1} J(E) \tag{2.53}$$

where $J(E)$ is a parameter obtained from the normalisation of P_0. In the asymptotic regions, the number of states is given by

$$\mathcal{N}(E) \approx \frac{\sqrt{E}}{\pi} \left[1 - \frac{a_0}{2(a_0 + a_1)} \frac{U_0}{E} + \mathrm{O}\left(\frac{U_0^2}{E^2} \right) \right], \qquad E \to \infty \tag{2.54a}$$

on the right end of the spectrum, and on the left end of the spectrum

$$\mathcal{N}(E) \approx \frac{1}{a_0 + a_1} \frac{\exp(-\pi\sqrt{E}/a_0)}{H^2(a_0^{-1} U_0^{-1/2}, a_1^{-1} U_0^{-1/2})}, \qquad E \to 0 \tag{2.54b}$$

where $H(x, y) = e^{-x} \int_0^\infty e^{-t}(1 + 2x/t)^{-y/2} dt$.

Various limits of the above results can be obtained in which the limiting cases where the parameters a_0 and a_1 are very large or small are taken. These limits correspond to the situation where either the wells and barriers, or both, become very wide or very narrow, since a_0 and a_1 are the average lengths of the wells and barriers.

The above examples, as well as more examples of other models, where the number of states is calculated using the phase formalism, are discussed in detail in the text of Lifshits *et al.* [Lif88].

2.3.2   Localisation and conductivity

2.3.2.1   The Lyaponov exponent and localisation

In this section we consider localised states, and show that it is possible to extract information about the envelope of bounded wave-functions corresponding to localised states.

Using the idea of the phase formalism introduced earlier, it is possible to introduce the idea of polar coordinates for states that satisfy the random Schrödinger equation, with

$$\psi = r \sin \alpha, \qquad \psi' = r \cos \alpha \qquad (2.55)$$

where $\alpha(x)$ is the nonreduced phase introduced in (2.43). We thus see that $\alpha(x)$ is responsible for the oscillations of the wave-function and that the amplitude $r(x)$ is responsible for its growth or decay. To decide if the states are localised or not, we need to be able to extract the envelope of the wave-function and to do this we therefore need to investigate the growth or decay term. For random potentials that are spatially homogeneous, it can be shown [Ose68, Mil71] that there exists the limits

$$\lim_{x \to \pm\infty} \frac{\ln r(x)}{x} = \gamma_\pm(E, \alpha, [U]) \qquad (2.56)$$

for each fixed energy and representation of the potential $U(x)$. We thus see that $\gamma_\pm(E, \alpha, [U])$ will provide information about the growth or decay of the wave-function in a specific realization of the disorder. For a fixed energy and α, in accordance with (2.56), the wave-function behaves like either $e^{\gamma x}$ or $e^{-\gamma x}$ depending on the realization of the random potential. However, in the decaying case, the corresponding energy is an eigenvalue of the Schrödinger equation on an semi-infinite interval, and since the set of discrete levels is extremely mobile (even small variations in the potential result in their motion), the probability that a fixed value of E falls into this set is zero [Lif88]. Thus for a given energy and α, the corresponding solution increases exponentially as we move away from the point where the solution's logarithmic derivative is fixed (due to the boundary conditions on α).

To see how the envelope forms for localised states, Borland [Bor63] showed that solutions that grow from the left and right ends of the one dimensional system (see Fig. 2.8) can be matched at a certain discrete energy eigenvalue, and that the matched solutions form a localised state in the bulk of the system. Note that in a pure system, this matching cannot occur, since the energies where this occurs lie in the forbidden bands of the infinite ordered crystal.

Since $\gamma_\pm(E, \alpha, [U])$ is a self averaging quantity, we note that

$$\lim_{x \to \pm\infty} \frac{\langle \ln r(x) \rangle}{x} = \gamma(E) \qquad (2.57)$$

where $\gamma(E)$ is a non-random positive quantity. We can thus use $\gamma(E)$ to determine the average growth rate from which we can obtain the localisation length, ξ.

The Lyaponov exponent, $\gamma(E)$, can also be expressed in terms of the probability density of
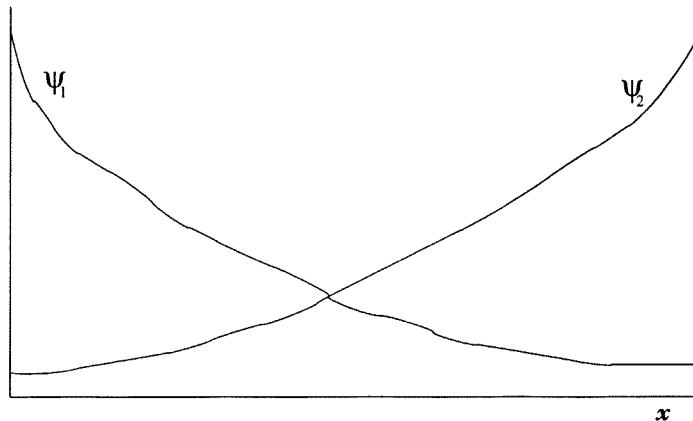
Figure 2.8: Envelope of the solutions that are localised. From [Bor63].

$\alpha(x)$ introduced in section 2.3.1.2, and is given by

$$\gamma(E) = \int P(z)z\,dz \tag{2.58}$$

where $z = \psi'/\psi = \cot\alpha$. If we know the form of $P(z)$, then we can obtain a closed expression for $\gamma(E)$. For the Gaussian disordered system, $P(z)$ was obtained by Frisch and Lloyd [Fri60], and is given in (2.49). This allows us to obtain

$$\gamma(E) = \mathcal{N}(E)\frac{\sqrt{\pi}}{2}\int_0^\infty dy\,\sqrt{y}\exp\left(-\frac{y^2}{12} - \frac{Ey}{D^{2/3}}\right) \tag{2.59}$$

with $\mathcal{N}(E)$ given by (2.50) or (2.51).

In the asymptotic limits, we thus obtain

$$\gamma(E) = \begin{cases} |E|^{1/2}, & E \to -\infty \\ \frac{D}{4E} & E \to \infty \end{cases} \tag{2.60}$$

If we define the localisation length as $\xi = 1/2\gamma(E)$, then the envelope of the wave-function decays as $\psi \sim \exp[-|x|\gamma(E)] = \exp(-|x|/\xi)$. From (2.60) we see, as expected, that the localisation length ξ is small for large negative energies and large for large positive energies. This confirms the statement made earlier that near the stable boundary ($E \to \infty$) the effects of the localisation is small.

2.3.2.2 Low frequency conductivity

As mentioned in the general discussion in a previous section, the DC conductivity of one dimensional disordered systems at zero temperature is zero. Since the long-range electron hopping is suppressed in the localised system, the system is unable to polarise or absorb energy, as states with similar energies are well separated spatially. Thus, the zero frequency electrical conductivity at zero temperature is expected to vanish. Mott argued [Mot67, Mot79] that the

real part of the conductivity goes as

$$\operatorname{Re} \sigma \simeq \omega^2 \ln^2 \omega. \tag{2.61}$$

Mott's qualitative argument goes as follows : for a photon with energy ω to be absorbed, a transition of an electron between two localised states with energy difference ω should take place. This energy splitting ω, is proportional to the overlap integral of these states. The transition probability, however, is proportional to the overlap integral squared. The probability for the photon to be absorbed is mainly determined by the transition probability, but involves the dipole matrix element rather than the overlap integral. The absorption is therefore $\sim \omega^2 \bar{x}^2$, where $\bar{x}$ is a typical spatial distance between the states with energy difference ω. The overlap integral, which is proportional to ω, is exponentially small in $\bar{x}$ since $\psi \sim \exp(-|x|/\xi)$, so that $x \sim \xi \ln(1/\omega)$. Thus the result is obtained that $\sigma \sim \omega^2 \ln^2 \omega$.

As a summary of the more quantitative results, Bychkov [Byc74] has rigorously shown, based on the Fokker-Planck equations for the conductivity by Halperin [Hal65], that $\sigma(0) = 0$. The complexity of Halperin's equations made it intractable to determine the limiting form of $\sigma(\omega)$ as $\omega \to 0$. Berezinskii [Ber74] invented a diagrammatic technique which allowed him to calculate the low temperature behaviour of $\sigma(\omega)$,

$$\sigma(\omega) = \sigma_0[2\zeta(3)\varepsilon - \varepsilon^2(\ln^2 \varepsilon + (2\mathcal{C} - 3)\ln \varepsilon - c)] \tag{2.62}$$

where σ_0 is the Drude conductivity, $\varepsilon = -2i\omega\tau$, τ is the elastic scattering time, $\mathcal{C}$ is the Euler constant and $c = O(1)$ is another constant. This result thus confirms Mott's qualitative result.

2.4 General approaches to higher dimensional disordered systems

2.4.1 Scaling theory

As mentioned in Section 2.1.4.3, one of the criteria to study localisation, the insensitivity of the boundary conditions, was first introduced by Thouless and coworkers [Edw72, Tho74, Tho77]. They noted that the degree of localisation of an eigenstate is closely related to the sensitivity of the energy eigenvalue to the boundary conditions of a finite system. The energy of a localised state in a large system should be insensitive to the boundary conditions[4] since the wave-function is exponentially small at the boundary. An extended state would be very sensitive to the boundary conditions as these states have a non-zero amplitude throughout the system.

Thouless *et al.* calculated the eigenenergies for a particular system using periodic boundary conditions, then repeated the calculation using anti-periodic boundary conditions. They then calculated the energy difference ΔE in the eigenvalues obtained for the different boundary conditions, and then investigated the dependence of ΔE on the size of the system. Thouless argued that the dependence of the conductive properties on the system size enters through only one parameter, namely an energy ratio $\Delta E/\delta W$, where δW is the average spacing between the energy levels. The energy ratio will be small if the eigenstates are localised and will approach zero as the size of the system increases, since the localised states are insensitive to the boundary

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[4]This is assuming that the center of localisation is not near the boundary.

conditions. If the states are extended, then the energy ratio will tend to a finite value as the size of the system increases.

Thouless also argued that the energy ratio is linearly related to the dimensionless conductance $g(L) = G(L)\hbar/e^2$ where $G(L)$ is the conductance of the system. It was subsequently shown by Anderson and Lee [And79] that the dimensionless conductance g is proportional to $(\Delta E/\delta W)^2$.

The scaling approach was developed further by Abrahams, Anderson, Licciardello and Ramakrishnan [Abr79] when they constructed a scaling theory where they combined perturbation theory and the ideas of Thouless, after replacing the energy ratio with the conductance as the controlling parameter.

2.4.1.1   One parameter scaling.

In the scaling approach introduced by Abrahams *et al.*, one tries to understand localisation by considering the behaviour of the conductance g as a function of the system size L.

The conductance $g(L)$ is taken as a microscopic measure of the disorder, since it is small when the disorder is large and large when the disorder is small. The asymptotic behaviour of $g(L)$ when the system size L is much larger than the mean free path of the electron has different limits depending on the degree of microscopic disorder.

When the random potential is small, the electron wave-function is extended and nearly plane wave like. The mean free path l between the collisions is large when compared to the atomic spacing or the Fermionic wave length k_{F}^{-1}. Under these conditions the Ohm's Law regime holds. For a d-dimensional hypercube with $L \gg l$ we can obtain the conductance in terms of the conductivity by considering Ohm's Law,

$$\frac{V}{R} = I = jA = \sigma EA = \frac{\sigma A}{L}V, \tag{2.63}$$

where the area for the hypercube is given by $A = L^{d-1}$. Since the conductance is given by $g = \frac{1}{R}$, we see that

$$g(L) = \sigma L^{d-2}. \tag{2.64}$$

However, when the randomness is large the states are localised and the relevant length scale is given by the localisation length ξ, which is in general larger than the mean free path l. In this regime, for $L \gg \xi$, the conductance is proportional to the transition amplitude, thus

$$g(L) \propto |\langle L|0\rangle|^2 \propto e^{-L/\xi}. \tag{2.65}$$

This is clearly non-ohmic behaviour.

For a particular disorder, $g(L)$ evolves smoothly from the initial conductance g_0 as L increases beyond l, going over to either of the asymptotic forms (2.64) or (2.65). The limiting behaviour reached depends on the dimensionality d, as well as the microscopic disorder, that is the initial conductance g_0 at the mean free path length scale l. The dependence on the dimension is significant, since all states in one-dimension are known to be localised with the localisation length ξ of the order of the mean free path. In this case there is no sizeable length scale where

ohmic behaviour occurs and the only asymptotic form is given by (2.65).

Abrahams *et al.* argued that the logarithmic derivative

$$\beta(g) = \frac{d\ln g}{d\ln L}$$
$$= \frac{L}{g}\frac{dg}{dL} \tag{2.66}$$

is a function of the conductance g alone and can thus be used as a scaling function. The idea is that the change in effective disorder when the system size increases is determined by the value of the effective disorder at the previous length scale, where the only measure of the effective disorder is the conductance.

2.4.1.2 The scaling function.

Since $g(L)$ has different asymptotic behaviours, we know that the scaling function, $\beta(g)$, will also have regions where the behaviour of the function differs from other regions. We now discuss these different regions.

Large conductance, g $\gg$ g_c.   Here g_c is the characteristic dimensionless conductance that is of the order π^{-2}. Ohm's Law is valid in this regime, thus the conductance is described by (2.64). This leads to the asymptotic form

$$\beta(g) = (d-2), \tag{2.67}$$

for $g \gg g_c$. In two dimensions this tends to zero, reflecting the fact that the conductance and the conductivity have the same physical dimension for a planar system. Thus the conductance of a square does not depend on the size of the system.

Small conductance, g $\ll$ g_c.   In this region the electronic states are localised, thus the scale dependence of $g(L)$ is described by (2.65). In this case $\beta(g)$ is given by

$$\beta(g) = \ln\left(\frac{g}{g_c}\right), \tag{2.68}$$

where $\beta(g)$ is independent of the dimension. The scale function $\beta(g)$ is negative, corresponding to the fact that for localised states g decreases as the length scale increases.

Perturbative region.   For weak disorder, that is for systems where $(k_F l)^{-1} \ll 1$, it is possible to calculate corrections to the Boltzmann transport theory result of σ given by

$$\sigma = \frac{e^2}{\hbar}\frac{n}{k_F^2}, \tag{2.69}$$

where n is the electron density.

It is found that to higher orders in $(k_F l)^{-1}$ there are significant corrections to the conductivity.
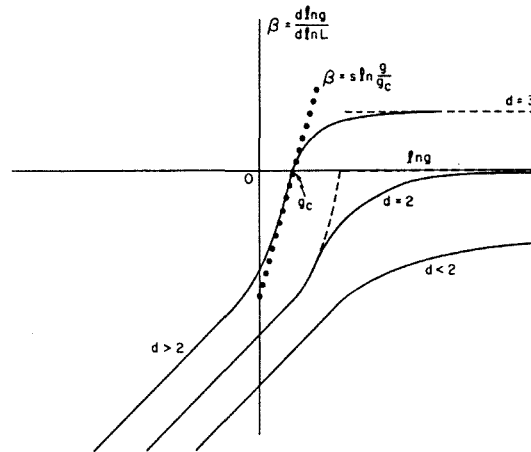
Figure 2.9: Plot of the scaling function $\beta(g)$ as a function of the conductance for $d = 1, 2, 3$. From [Abr79]

These terms contribute a correction of the order g^{-1} to the scaling function, thus for large g

$$\beta(g) = (d - 2) - \frac{a}{g}, \qquad (2.70)$$

where a is a constant that depends on the system.

The scaling function $\beta(g)$ is thus always less than the Ohm's Law value given by (2.67). Conduction in a disordered electronic system is thus never quite ohmic.

A scaling curve can be constructed using (2.70) for large g and (2.68) for small g, and the assumptions that $\beta(g)$ is continuous and monotonic. The assumption of continuity is valid since $\beta(g)$ describes how the conductance of a *finite* system evolves as a function of the scale size.

The scaling function $\beta(g)$, constructed for $d = 1, 2$ and 3 is shown in Fig. 2.9 as a function of the conductance.

2.4.1.3 Consequences of scaling theory.

Three dimensions. As can be seen from Fig. 2.9, in three dimensions the scaling function $\beta(g)$ is a positive value equal to unity at large g; is negative for very small g; and passes through zero at a certain conductance, say g_c.

Suppose the amount of microscopic disorder in the system is such that the initial conductance g_0 at the mean free path length scale is larger than g_c. Thus one would start on the positive part of the $\beta(g)$ curve, the exact position depending on the value of g_0. When the length scale is increased, g also increases and one moves up in the β curve. At asymptotically large length scales, the limit $\beta(g) = 1$ is reached and the system is an Ohm's Law conductor.

If the initial conductance is smaller than g_c, then $\beta(g)$ is negative. Increasing the length scale from l, decreases g and one moves downward on the $\beta(g)$ curve. At large length scales, $\beta(g)$ will thus correspond to the scaling curve of localised states. Thus, a system with $g_0 \gg g_c$ at the microscopic length scale l is a metal, while systems with $g_0 \ll g_c$ are insulators.

It can be seen that the critical point $\beta(g) = 0$, characterised by the conductance g_c at the length scale l, is an unstable fixed point. This means that for any small fluctuation in the

conductance away from g_c, the scaling can lead to qualitatively different regimes. The scaling trajectories thus move away from the point $\beta(g) = 0$, which marks the change between the localised and ohmic regimes. The critical point thus gives an indication of the mobility edge. This is so since g_0 is the conductance of the moving electrons with Fermi energy E_F. If the disorder is kept fixed and E_F is varied, then g_0 will change smoothly. When the Fermi energy E_F is equal to the mobility edge energy E_C, then g_0 will coincide with the critical conductance g_c. Thus by studying the predictions of scaling theory for g_0 close to g_c, one can investigate the behaviour of the Anderson transition near the mobility edge. To study the scaling behaviour near the critical point one uses renormalisation group techniques.

Two dimensions. In two dimensions, $\beta(g) < 0$ always, thus only localised behaviour is possible. There is no critical conductance g_c and thus no mobility edge. If the system is weakly disordered with a large initial conductance g_0 at length l, then one moves down the scaling curve as the length scale increases until asymptotically $\beta(g) \sim \ln\left(\frac{g}{g_c}\right)$. During this decrease of $\beta(g)$, there is a crossover from logarithmic behaviour to exponential behaviour, which experimentally may resemble a sharp mobility edge.

Scaling theory thus predicts that there are no truly extended states in two dimensions, since the electron states are localised at large enough length scales, even for small microscopic disorder.

One can estimate the localisation length by integrating (2.70) between the length scales l and L to obtain

$$g(L) = g_0 - \frac{e^2}{\hbar\pi^2} \ln\left(\frac{L}{l}\right), \tag{2.71}$$

where g_0 is the conductance at the lower cutoff l. In conventional Boltzmann transport theory $g_0 = (\frac{e^2}{2\pi\hbar})(k_F l)$. The conductance thus decreases logarithmically with size. The scale-dependent term becomes comparable to the Boltzmann term at a length scale $L = \xi$, given by

$$\xi \simeq l e^{k\pi^2 g_0/e^2} = l e^{\frac{\pi}{2} k_F l}. \tag{2.72}$$

At this length scale the scale dependent term cancels with the Boltzmann term to give a zero conductance, thus the perturbative estimate of the localisation length is given by ξ.

This can be understood by considering the electronic wave-functions as a function of the length scale. As we increase L, then as long as $L < \xi$, we have a non-zero conductance as the wave-functions are extended over the whole system. When $L \geq \xi$, then the conductance will vanish as the wave-functions are now localised with the estimate of the localisation length given by ξ.

One dimension. In one dimension, $\beta(g)$ is less than unity and decreases further with decreasing conductance, rapidly going over to the localised regime. Using (2.70) one finds that in one dimension, the scale-dependent corrections to g become comparable to the Boltzmann transport term, $g_0 = (\frac{e^2}{2\pi\hbar})(k_F l)$ at length scales of order l. This is the perturbative estimate of the localisation length, which corresponds to the result stated earlier that in one dimension the states are localised with a localisation length of the order of the mean free path.

2.4.2 Field theoretic methods

The close relation between the localisation problem and the problem of critical phenomena suggests that a mapping of the localisation problem to a field theory is possible. This mapping was obtained by Wegner [Weg79b] which led to a considerable body of field theoretic approaches, culminating in the non-linear sigma model.

As mentioned in Section 2.1.1, the quantities of interest can be written in terms of Green's functions of the Schrödinger equation. We thus need a theory that will allow us to calculate the disorder average of these Green's functions or the average of products of these Green's functions. One of the most practical ways of calculating Green's functions is found in field theory where one has a generating functional from which Green's functions, or products thereof, can be obtained by differentiation.[5]

We introduce the generating functional $\mathcal{Z}$, with fields $\psi, \psi^\dagger$ and sources $\eta, \eta^\dagger$:

$$\mathcal{Z} = \int [d\psi^*] \, [d\psi] \, e^{i\left(\psi^\dagger(E-H+i\epsilon)\psi + \eta^\dagger\psi + \psi^\dagger\eta\right)}. \tag{2.73}$$

Here the fields ψ depend on continuous position variables, or if it is a discrete model, on a discrete position index. It may, however, also refer to the index of any basis state chosen as basis for the Hilbert space, i.e. we construct a matrix representation. Of course, in this case the index does not have to be discrete either, it may also be continuous.

We carry out the integration in (2.73) by completing the square in the exponent to obtain

$$\mathcal{Z} = \frac{1}{\det\left[E - H + i\epsilon\right]} e^{-i\eta^\dagger(E-H+i\epsilon)^{-1}\eta}. \tag{2.74}$$

We can obtain $(E - H + i\epsilon)^{-1}_{ij}$, and thus the Green's function, by taking the derivatives with respect to η and $\eta^\dagger$ and then setting the sources to zero :

$$i \left. \frac{\partial^2 \mathcal{Z}}{\partial \eta_i^\dagger \partial \eta_j} \right|_{\eta_i^\dagger, \, \eta_j = 0} = \frac{1}{\det\left[E - H + i\epsilon\right]} \left(\frac{1}{E - H + i\epsilon}\right)_{ij} = \frac{1}{\det\left[E - H + i\epsilon\right]} G_{ij}. \tag{2.75}$$

Since we wish to find the disorder average of the Green's function, we average (2.75) over the disorder. Unfortunately this averaging process will not enable us to determine the disorder average of the Green's function since the right hand side of (2.75) contains a determinant that is dependent on the disorder.

There are various tricks for normalising the generating functional so that the disorder dependent determinant does not occur. Two methods are the replica trick, introduced by Edwards and Anderson [Edw75], and Efetov's supersymmetric method [Efe83, Efe97].

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[5]The general field theory methods are given in the standard texts found in references [Bai86, Itz89, ZJ89]

2.4.2.1 Replica Approach

We wish to find the average one-particle Green's function $\langle G \rangle$, where we have averaged over the disorder and the matrix elements of G are given by

$$G_{ij} = \left(\frac{1}{E - H + i\epsilon} \right)_{ij} \tag{2.76}$$

$$= \det \left[E - H + i\epsilon \right] \times \left. \frac{i\partial^2 \mathcal{Z}}{\partial \eta_i^\dagger \partial \eta_j} \right|_{\eta_i^\dagger, \eta_j = 0}$$

$$= \left. \frac{1}{\mathcal{Z}} \right|_{\eta_i^\dagger, \, \eta_j = 0} \times \left. \frac{i\partial^2 \mathcal{Z}}{\partial \eta_i^\dagger \partial \eta_j} \right|_{\eta_i^\dagger, \, \eta_j = 0}$$

$$= \left. \frac{i\partial^2 \ln \mathcal{Z}}{\partial \eta_i^\dagger \partial \eta_j} \right|_{\eta_i^\dagger, \, \eta_j = 0} . \tag{2.77}$$

We thus see that to calculate the average of the Green's function, we need to find the average of a logarithm, which can be calculated by introducing the replica trick. The replica trick is based on the mathematical equality

$$\ln x = \lim_{n \to 0} \frac{(x^n - 1)}{n}, \tag{2.78}$$

where n is a continuous variable.

If we take x to be the generating functional $\mathcal{Z}$, then we see that we can write the averaged value of the Green's function as

$$\langle G \rangle = \lim_{n \to 0} i \frac{\partial^2}{\partial \eta_i^\dagger \partial \eta_j} \frac{(\langle \mathcal{Z}^n \rangle - 1)}{n}, \tag{2.79}$$

where the disorder is introduced by averaging over $\mathcal{Z}^n$. For any integer n, $\mathcal{Z}^n$ is the generating functional of n identical non-interacting replicas of the original system. These replicas involve the same random potential linearly in the action, which allows for the explicit averaging over a Gaussian distribution. If we analytically continue n so that the limit $n \to 0$ can be taken we obtain information about the original system.

We thus have a method of averaging the one-particle Green's function over the disorder. Unfortunately, this method has many disadvantages. Usually the Hamiltonians obtained by the replica procedure are very complicated and cannot be solved exactly. Standard approximations which are valid when n is an integer often fail in the $n \to 0$ limit. The procedure of analytical continuation from integer n to $n = 0$ is not simple. There is also the added complication of replica symmetry breaking. These difficulties can lead to meaningless results when the method is applied. The lack of a justified procedure for the analytical continuation of $n \to 0$ means that the results obtained are uncertain, unless there is a check by an independent method [Kam99].

2.4.2.2 Supersymmetric method

The supersymmetric method is based on the use of functional integrals over both commuting and non-commuting or Grassmann variables, which makes it possible to represent physical

quantities in a form which enables one to average over the disorder at the very beginning of calculations. After averaging, the problem is reduced to a field theoretical model containing bosons and fermions which has supersymmetry, that is, there is a symmetry under the group of transformations that mix bosons and fermions. The models that are obtained can be studied by traditional methods that are applied to regular models in solid state physics and field theory.

The supersymmetry method has distinct advantages over the replica trick. It is free of the analytical continuation problem and the effective Hamiltonians are simpler than those obtained in the replica method since there are fewer variables.

Once again we wish to calculate $\langle G \rangle$. We introduce Grassmann variables χ and $\chi^\dagger$ into the generating functional (2.73), thus

$$\mathcal{Z} = \int [dS^*]\,[dS]\,[d\chi^*]\,[d\chi]\,\mathrm{e}^{\mathrm{i}\left(S^\dagger(E-H+\mathrm{i}\epsilon)S + \chi^\dagger(E-H+\mathrm{i}\epsilon)\chi + \eta^\dagger S + S^\dagger \eta\right)}, \tag{2.80}$$

where we have changed notation from ψ to S for the commuting fields.

If we complete the square and integrate over both the conventional variables and the Grassmann variables we obtain

$$\mathcal{Z} = \frac{1}{\det\left[E - H + \mathrm{i}\epsilon\right]}\,\det\left[E - H + \mathrm{i}\epsilon\right]\mathrm{e}^{-\mathrm{i}\eta^\dagger(E-H+\mathrm{i}\epsilon)^{-1}\eta}$$
$$= \mathrm{e}^{-\mathrm{i}\eta^\dagger(E-H+\mathrm{i}\epsilon)^{-1}\eta}. \tag{2.81}$$

We can now calculate the one-particle Green's function by taking the second derivative of (2.81) with respect to the sources $\eta, \eta^\dagger$, so that

$$\left.\frac{\mathrm{i}\partial^2 \mathcal{Z}}{\partial\eta_i^\dagger \partial\eta_j}\right|_{\eta_i^\dagger,\,\eta_j=0} = \left(\frac{1}{E - H + \mathrm{i}\epsilon}\right)_{ij} = G_{ij}(E + \mathrm{i}\epsilon). \tag{2.82}$$

Since there is no disorder dependent determinant in (2.82), we can average over the disorder without problems, thus

$$\langle G_{ij}(E + \mathrm{i}\epsilon)\rangle = \left\langle \left.\frac{\mathrm{i}\partial^2 \mathcal{Z}}{\partial\eta_i^\dagger \partial\eta_j}\right|_{\eta_i^\dagger,\,\eta_j=0}\right\rangle = \left.\frac{\mathrm{i}\partial^2 \langle\mathcal{Z}\rangle}{\partial\eta_i^\dagger \partial\eta_j}\right|_{\eta_i^\dagger,\,\eta_j=0}, \tag{2.83}$$

where the averaged generating functional is given by

$$\langle\mathcal{Z}\rangle = \int P(H)dH[dS^*]\,[dS]\,[d\chi^*]\,[d\chi]\,\mathrm{e}^{\mathrm{i}\left(S^\dagger(E-H+\mathrm{i}\epsilon)S + \chi^\dagger(E-H+\mathrm{i}\epsilon)\chi + \eta^\dagger S + S^\dagger \eta\right)} \tag{2.84}$$

and $P(H)$ is the probability distribution of the random disorder.

It is possible to introduce supervectors[6] $\Phi = (S, \chi)$, which allows us to write (2.80) as

$$\mathcal{Z} = \int [d\Phi^*]\,[d\Phi]\,\mathrm{e}^{\mathrm{i}\left(\Phi^\dagger[(E-H)+\mathrm{i}\epsilon]\Phi + \eta^\dagger S + S^\dagger \eta\right)}. \tag{2.85}$$

---

[6]For a detailed overview of supervectors and supermathematics, consult [Efe97].

The advanced G^+ and retarded G^- Green's functions can thus be written as

$$G^{\pm}(\vec{r}_1, \vec{r}_2; E) = \pm i \int [d\Phi^*] [d\Phi] \, S(\vec{r}_1) S^{\dagger}(\vec{r}_2) \exp \left(i \int d\vec{r}' \Phi^{\dagger}(\vec{r}') [\mp (E - H) + i\epsilon] \Phi(\vec{r}') \right) \quad (2.86)$$

where we have explicitly written the implied integral in the exponent which was assumed when multiplying fields.

For the study of kinetic quantities such as the conductivity ((2.8a) and (2.10c)) we need to average products of Green's functions. Since the averages of the product of advanced and retarded Green's functions are the main difficulty, we concentrate on these. In order to write the product of two Green's functions, we need to introduce the double set of fields Φ_1 and Φ_2, so that the functional integral for the product $G^+ G^-$ has the form

$$G^+(\vec{r}, 0; E - \frac{\omega}{2}) G^-(0, \vec{r}; E + \frac{\omega}{2}) = \int [d\Phi_1^*] [d\Phi_1] [d\Phi_2^*] [d\Phi_2] \, S_1(\vec{r}) S_1^{\dagger}(0) S_2(0) S_2^{\dagger}(\vec{r}) e^{-\mathcal{L}} \quad (2.87a)$$

with

$$\mathcal{L} = i \int d\vec{r}' \left[\Phi_1^{\dagger} \left[\left(E - \frac{\omega}{2} - H \right) - i\epsilon \right] \Phi_1 - \Phi_2^{\dagger} \left[\left(E + \frac{\omega}{2} - H \right) + i\epsilon \right] \Phi_2 \right] \quad (2.87b)$$

where we use the indices 1 and 2 to denote the fields corresponding to the advanced and retarded Green's functions.

All the calculations and formulae can be made more compact of we introduce the super vector $\Psi = (S_1, \chi_1, S_2, \chi_2)$, allowing us to write the (2.86) as

$$G^+(\vec{r}, 0; E - \frac{\omega}{2}) G^-(0, \vec{r}; E + \frac{\omega}{2}) = \int [d\Psi] [d\Psi^*] \, S_1(\vec{r}) S_1^{\dagger}(0) S_2(0) S_2^{\dagger}(\vec{r}) e^{-\mathcal{L}} \quad (2.88a)$$

with

$$\mathcal{L} = i \int d\vec{r}' \Psi^{\dagger} \Lambda^{1/2} \left[E - \left(\frac{\omega}{2} + i\epsilon \right) \Lambda - H \right] \Lambda^{1/2} \Psi \quad (2.88b)$$

where $\Lambda = \text{diag}\{1, 1, -1, -1\}$.

It is now possible to calculate averages of the form $\langle G^+(\vec{r}, 0; E - \frac{\omega}{2}) G^-(0, \vec{r}; E + \frac{\omega}{2}) \rangle$ by averaging (2.88) over the disorder. Usually a Gaussian distribution is used to characterise the disorder. Averaging over the Gaussian disorder (where standard deviation of the Gaussian potential is $1/2D$) leads to functional integral with the same form as (2.88a), with the Lagrangian

$$\mathcal{L} = i \int d\vec{r}' \left(\Psi^{\dagger} \Lambda^{1/2} \left[E - \left(\frac{\omega}{2} + i\epsilon \right) \Lambda - H_0 \right] \Lambda^{1/2} \Psi + (\Psi^{\dagger} \Psi)^2 \right), \quad (2.89)$$

where H_0 is the deterministic part of the Hamiltonian. Note that, once again, this averaging is possible due to the absence of the Hamiltonian dependent normalisation since we are using a combination of commuting and anti-commuting fields. Also, averaging of products of advanced or products of retarded Green's functions leads to a result similar to (2.89), except that Λ is $\pm I$, where I is the identity matrix.

Reduction to the non-linear σ-model.   It is not possible to calculate any correlation functions with the Lagrangian (2.89) exactly. It is however possible to approximate this functional integral in the limit of large D. Also, it is possible to reduce the Lagrangian in (2.89) to the non-linear σ-model by introducing variables $Q = \langle \Psi^\dagger \Psi \rangle$, and making the approximation that the Ψ^4 interaction term can be decoupled. Integrating out the fast-electron modes allows us to reduce the problem to a model with the Lagrangian describing the slow-diffusion modes [Efe83, Efe97]. Taking a saddle-point approximation, and neglecting the spatial variation in Q (the zero mode approximation), we have the correlator that can be used to calculate the ground state conductivity

$$\left\langle G^+(\vec{r}, 0; \omega)\left(G^-(0, \vec{r}; 0) - G^+(0, \vec{r}; 0)\right)\right\rangle = (\pi\nu)^2 \int [dQ]\, Q_{\alpha\beta}^{12}(\vec{r})\Lambda_{\beta\beta}Q_{\beta\alpha}^{21}(0)\exp(-F[Q]) \quad (2.90a)$$

where α and β are arbitrary indices that are not summed over, ν is the density of levels on the Fermi surface and

$$F[Q] = \frac{\pi\nu}{8} \int d\vec{r}\, [\mathcal{D}(\Delta Q)^2 + 2i\omega\Lambda Q], \quad (2.90b)$$

where STr is the supertrace [Efe97] and $\mathcal{D}$ is the diffusion coefficient. Systems with the free energy (2.90b) belongs to the class of non-linear σ-models. Equation (2.90b) is equivalent to the non-linear σ-model obtained via the replica method [Sch80].

2.4.2.3   Summary of some results from the non-linear sigma model

Renormalisation.   The non-linear σ-model is renormalisable with two renormalisation constants [Bré76]. The two renormalisation constants that are needed are a field renormalisation constant Z, and a coupling constant renormalisation constant, Z_g. Z_g connects the bare resistivity G (which can be expressed in terms of the coupling constant) with its renormalised counterpart g, via

$$G = \kappa^{2-d}Z_g g, \quad (2.91)$$

where κ is the arbitrary momentum scale of the RG. Also, Z_g determines the β function

$$\beta(g) = \frac{dg}{d\ln\kappa} = \frac{d-2}{1 + g(d\ln Z_g/dg)}. \quad (2.92)$$

The β function has been calculated to four loop order by Wegner [Weg89] and to five loop order by Hikami [Hik90], who noticed that the non-linear σ-model in question is closely related to certain classes of string theories. Thus, to this order, the result for the generic non-linear σ-model (where there are no external symmetry breaking fields) is

$$\beta(\ln g) = (d-2) - 2g^{-1} - 12\zeta(3)g^{-4} + \frac{27}{2}\zeta(4)g^{-5} + O(g^{-6}). \quad (2.93)$$

Thus, the field theoretic approach confirms that the perturbation theory result from the scaling theory (2.70) is correct up to the lowest order in the expansion.

Two-level correlation functions. The two-level correlation functions $R(\omega)$ determine the probability for an energy level to be at the distance ω from another level. These two-level correlation functions have been calculated by Dyson [Dys62, Meh63] using a phenomenological approach. The two-level correlation function can be calculated with the non-linear σ-model, and is [Mir00]

$$R(\omega) = 1 - \frac{\sin^2(\omega)}{\omega^2} + \left(\frac{\cos(\omega)}{\omega} - \frac{\sin(\omega)}{\omega^2} \right) \mathrm{Si}(\omega), \tag{2.94a}$$

where

$$\mathrm{Si}(\omega) = \int_0^\omega \frac{\sin(y)}{y} dy \tag{2.94b}$$

This corresponds to the Wigner-Dyson distribution [Wig51, Meh91] that is obtained in Random Matrix Theory.

There has been considerable research in studying the energy level statistics of disordered systems and the relation to Random Matrix Theory and quantum chaos. These topics have been discussed extensively in the reviews of Beenakker [Bee97], Janssen [Jan98], and Mirlin [Mir00], as well as in the text of Mehta [Meh91].

Conductance fluctuations. The non-linear σ-model has been applied for direct calculations of the conductivity, where it was found that the conductivity is given in terms of a local operator and a non-local operator [Cas87]. The explicit calculation [Cas87, Pru89] recovers the result obtained from the renormalisation of the non-local σ-model.

The direct calculation has been extended to study conductance fluctuations within the framework of the non-linear σ-model [Alt86, DS91], confirming earlier results obtained from many-body perturbation theory [Lee87]. These works established that the distribution of the fluctuations of the conductance and the density of states is not purely Gaussian, but has log-normal tails that get stronger with increasing disorder. For a review of these works, see [Alt91].

2.5 Open problems

To conclude this chapter, we remark on a few open problems that exist in disordered systems. Firstly, as already mentioned in section 2.1.5 the techniques discussed above mainly concentrated on the non-interacting electron model. Although the non-linear σ-model has been generalised to include electron-electron interactions using the Keldish formalism [Kam99], this formalism is fairly complicated and suffers from some of the same drawbacks as the non-interacting version of the non-linear σ-model.

As pointed out, the non-linear σ-model is derived either from the supersymmetric formalism or the replica trick. The replica trick analysis, however relies on a formidable analytical continuation procedure, which complicates practical applications. Also one needs to take into account replica symmetry breaking, and thus all possible saddle-points when reducing the initial replica field theory to the non-linear σ-model [Alt00]. The derivation using the supersymmetric method on the other hand is unproblematic, but cannot be used to introduce electron-electron interactions.

For this one must use the Keldish version of the σ-model.

There are still drawbacks of the non-linear σ-model itself. It contains an approximation where the non-diffusive modes are integrated out, and a saddle point expansion is made in terms of the coupling constant g^{-1}, and thus only holds for weakly disordered systems. It can thus not help when trying to investigate strongly disordered systems.

An additional open problem in localisation investigations is the lack of an order parameter [McK81], in the sense of conventional phase transitions, as well as the unknown value of the upper critical dimension [Har81]. If an order parameter existed then the transition could be investigated using the standard Landau-Ginzburg [Lan65] techniques. Some of the reasons for the difficulty in obtaining an order parameter is the methodology used to calculate quantities. Since the field theoretic techniques (and many of the one dimensional techniques) are based on the averaging of the Green's functions, most quantities that are to be calculated are first expressed in terms of the Green's function, before the average is calculated. This additional layer in some sense obscures the possible order parameter due the extra complication in the theoretical methods.

In the following chapter we shall introduce an alternative field theoretic approach to studying disordered systems, which is not meant to replace the current field theoretic methods such as the non-linear σ-model, but rather to complement them. Since the approach is based on calculating quantities expressed directly in terms of the wave-function and eigenenergies, it is hoped that this more direct method will allow for the possible identification of a suitable order parameter for localisation. Furthermore, since the field theory does not assume that there is only weak localisation present when introducing it (unlike the non-linear σ-model), it is possible to introduce a duality transformation to obtain a field theory which would allow investigation of strongly disordered systems. Finally, since the theory is of a more conventional type, it is possible to utilise the vast array of calculation techniques available in standard field theory literature, and thus obtain complementary understanding of disordered system, particularly in higher dimensions.

Chapter 3
Formalism

We consider a particle moving on a d-dimensional torus, and in a periodic, random potential caused by impurities in a system. We wish to calculate observed quantities of this particle, which under the assumption of self-averaging implies the averaging of these observables over the different realizations of the random potential, i.e.,

$$\langle \widehat{O} \rangle = \tilde{Z}^{-1} \int [dV] \, \widehat{O}[V] P[V], \tag{3.1}$$

where $\tilde{Z} = \int [dV] \, P[V]$ and $P[V]$ is the probability distribution describing the random potential. If we assume that the impurities are quenched, then the movement of the particle is described by the time independent Schrödinger equation. We impose periodic boundary conditions and for the moment assume that time reversal symmetry is not broken, so that the wave-function can always be chosen real.

To introduce our formalism, we consider the logarithmic derivative of the wave-function [7] instead of the wave-function, and correspondingly change from the Schrödinger equation to the equation of motion of this new variable. There are several advantages to this transformation, particularly from a functional integration point of view. Firstly, in contrast to the Schrödinger equation in which the random potential multiplies the wave-function, the equation of motion governing the new variable is a non-linear stochastic differential equation in which the random potential simply plays the role of a random source. Secondly, the physically irrelevant normalisation of the wave-function is eliminated. Lastly, as is well known [Tho74], the localisation length cannot be extracted from the correlations of the wave-function as these are always short ranged due to the random phase cancellations. Instead, the correlations of the absolute values of the wave-function should be computed. The current formulation is ideal for this purpose, as will be illustrated later.

Although the strategy is identical, there are subtle differences in the introduction of the formalism for one-dimensional systems, where we transform from the scalar wave-function to a scalar variable, and higher dimensions, where the transformation is from the scalar wave-function to a vector variable. For this reason, we first introduce the formalism for one-dimensional systems and then afterwards consider the more general theory in higher dimensions.

3.1 One dimensional formalism

In one dimension, we introduce the following real valued field related to the logarithmic derivative of the wave-function,

$$\phi = \frac{\psi'}{\psi}, \tag{3.2}$$

---

[7]The transformation to the logarithmic derivative of the wave-function was introduced in one-dimensional disordered systems by Frisch and Lloyd [Fri60].

where we use the notation that $\phi \equiv \phi(x)$, unless the argument is specifically stated. Using ϕ in the Schrödinger equation, we obtain the first order Ricatti equation,

$$V(x) = E + \phi' + \phi^2. \tag{3.3}$$

where we work in units of $\frac{\hbar^2}{2m}$. Note that in these units, V and E have the dimensions (length)$^{-2}$, while ϕ has the dimensions (length)$^{-1}$.

Using (3.2), we can write the unnormalised wave-function as

$$\psi(x) = \exp\left[\int_0^x dx' \phi(x')\right]. \tag{3.4}$$

Up to a global phase factor, all information that can be obtained from the wave-function can also be obtained from (3.4), including information about the phase, which we need to consider carefully when computing the localisation length so as to avoid obtaining incorrect results due to random phase cancellation.

The phase of the wave-function changes as the wave-function changes sign, thus from (3.2), we see that ϕ must diverge at these points, indeed it jumps from $-\infty$ to ∞ across these points. We thus need a prescription to calculate the integral in the exponent of (3.4) at these points, since the result must be finite. The prescription we use is to integrate over a contour from 0 to x, where the contour avoids the positions on the real axis where there are singularities in $\phi(x)$ by moving around them in the upper complex plane with a semi-circle of radius ϵ.

This contour integral can be written in terms of the principle value of the integral plus a phase which depends on the number of times a singularity occurs in the interval $[0, x]$. Using this prescription, we are able to separate the phase from the integral over ϕ. Thus we obtain for the normalised wave-function (the notation $\oint_0^x$ is used for the contour as described above)

$$\psi(x) = \sqrt{N[\phi]} \exp\left[\oint_0^x dx' \phi(x')\right]$$
$$= \sqrt{N[\phi]} \exp\left[P\int_0^x dx' \phi(x') - i\pi \sum_j \text{Res}[\phi(x_j)]\right] \tag{3.5}$$

where x_j are the positions of the singularities and

$$N[\phi]^{-1} = \int dx \exp\left[2P\int_0^x dx' \phi(x')\right]. \tag{3.6}$$

We can now use the assumed periodicity of the wave-function, as well as the above equations to determine the conditions on ϕ. If we consider $\psi(x + nL)$, where n is an integer, then using (3.5) we have

$$\psi(x + nL) = \sqrt{N[\phi]} \exp\left[\oint_0^{x+nL} dx' \phi(x')\right] = \psi(x) \exp\left[\int_x^{x+nL} dx' \phi(x')\right]. \tag{3.7}$$

The periodicity of the wave-function, $\psi(x) = \psi(x + nL)$, implies that

$$\int_x^{x+nL} dx' \phi(x') = 0 \quad \forall \, x. \tag{3.8}$$

This condition is satisfied if we constrain the field $\phi(x)$ to be a non-constant periodic function, where $\phi(x) = \phi(x + nL)$.

Let us now consider (3.1). In principle, if we knew the functional dependence of the observable $\hat{O}$ on V, we could compute the desired quantities directly from (3.1). However, except in extremely trivial cases, we do not know this dependence and, in particular, we do not know the functional dependence of eigenvalues and eigenfunctions on V, which are the averages we would like to compute. On the other hand, the functional dependence of these, and many other observables, on the variable ϕ is fairly easy to determine (see Section 3.4). It therefore seems like a good strategy to change integration variables in (3.1) from V to ϕ using the simple relation (3.3). Doing so will shift the complexity of (3.1) from the observables to the action (probability distribution) for ϕ, which will generally be highly non-linear, even though $P[V]$ may be simple, i.e. Gaussian. The latter problem is, however, more amenable to treatment through the arsenal of perturbative and non-perturbative field theoretic techniques than the original problem as stated in (3.1).

To facilitate the change of variables (3.3) in (3.1), we introduce the identity [8]

$$\begin{aligned}
1 &= N^{-1} \int [d\phi] \, d\bar{E} \, \delta\left(\int dx \phi(x)\right) \delta[V - \bar{E} - \phi' - \phi^2] |J| \\
&\equiv N^{-1} \int [d\phi] \, d\bar{E} \, \delta[V - \bar{E} - \phi' - \phi^2] |J|,
\end{aligned} \tag{3.9}$$

where the functional integral is over all possible non-constant periodic configurations of ϕ, the Jacobian is given by $|J| = \left|\det\left(\frac{d}{dx} + 2\phi\right)\right|$, where the zero mode is excluded in this determinant, and N is the total number of states (dimension of the Hilbert space). Note that the role of $\bar{E}$ under our change of variables is to replace the integration over the constant mode of V, which cannot be included with the ϕ integration, as a shift in the constant mode of the potential leaves the wave-function, and thus ϕ unaffected, while it will induce a shift in the energy.

Using the conditions imposed on ϕ one can formally argue that, the determinant $J = \det\left(\frac{d}{dx} + 2\phi\right)$ can be transformed to $\det\left(\frac{d}{dx}\right)$ through a similarity transformation. To see this, we introduce auxiliary Grassmann valued fields u and $\bar{u}$ so that the determinant is given by

$$\det\left(\frac{d}{dx} + 2\phi\right) = \int du \, d\bar{u} \, \exp\left[\int dx \, \bar{u} \left(\frac{d}{dx} + 2\phi\right) u\right]. \tag{3.10}$$

where it is understood that the zero mode is excluded from the functional integral. We make a

---

[8]We use $\delta[\cdot]$ to denote a functional Dirac delta, while $\delta(\cdot)$ denotes a conventional Dirac delta.

gauge transformation by setting $v = e^{-\Phi} u$ and $\bar{v} = e^{\Phi} \bar{u}$, so that

$$
\begin{aligned}
\det\left(\frac{d}{dx} + 2\phi\right) &= \int dv\, d\bar{v}\, \exp\left[\int dx\, \bar{v} e^{-\Phi}\left(\frac{d}{dx} + 2\phi\right) e^{\Phi} v\right] \\
&= \int dv\, d\bar{v}\, \exp\left[\int dx\, \bar{v}\left(\frac{d}{dx} + \Phi' + 2\phi\right) v\right] \\
&= \int dv\, d\bar{v}\, \exp\left[\int dx\, \bar{v}\left(\frac{d}{dx} + \Phi' + 2\phi\right) v\right] = \det\left(\frac{d}{dx}\right),
\end{aligned}
\tag{3.11}
$$

if we choose $\Phi' \equiv -2\phi$. Note that this argument assumes that the determinants arising from the change of variables $v = e^{-\Phi} u$ and $\bar{v} = e^{\Phi} \bar{u}$ cancels. This would be correct if the function e^{Φ} is invertible, i.e., if Φ contains no singularities, which will always be the case for topologically trivial configurations (see later). However, for topologically non-trivial configurations this is not the case and e^{Φ} will in general either vanish or diverge at certain points, invalidating the above argument. Therefore it is appropriate to keep this determinant until an appropriate saddle point has been identified, then the topological trivial fluctuations around this saddle point may be gauged away as above.

Inserting the identity (3.9) into the averages of (3.1), and then completing the integration over the disorder, allows us to obtain a field theory, formulated in terms of the variable ϕ, for the disordered average of observables

$$
\begin{aligned}
\langle \widehat{O} \rangle &= \tilde{Z}^{-1} \int [dV]\, \widehat{O}[V] P[V] \\
&= Z^{-1} \int [dV] \int [d\phi]\, d\bar{E}\, \delta[V - \bar{E} - \phi' - \phi^2]\, \widehat{O}[V] P[V] |J| \\
&= Z^{-1} \int [d\phi]\, d\bar{E}\, \widehat{O}[\bar{E}, \phi] P[\bar{E} + \phi' + \phi^2] |J|,
\end{aligned}
\tag{3.12a}
$$

where

$$
Z = \int [d\phi]\, d\bar{E}\, P[\bar{E} + \phi' + \phi^2] |J| = N\tilde{Z}.
\tag{3.12b}
$$

It should be noted that although we considered only a random potential when constructing this field theory, it is also possible to obtain the field theory when there is both a random potential $V(x)$ and a deterministic potential $W(x)$. In this case the result is similar to (3.12a), except the energy is now shifted by $W(x)$,

$$
\langle \widehat{O} \rangle = Z^{-1} \int [d\phi]\, d\bar{E}\, \widehat{O}[\bar{E}, \phi] P[\bar{E} - W + \phi' + \phi^2] |J|.
\tag{3.13}
$$

3.2   Higher dimensions

In higher dimensions, we introduce a real valued vector field related to the gradient of the logarithm of the wave-function,

$$
\vec{A} = \frac{\vec{\nabla}\psi}{\psi}.
\tag{3.14}
$$

Since the wave-functions are assumed to be of class C^2, we note by direct computation that $\vec{\nabla} \times \vec{A} = 0$. Also, periodicity of ψ demands that $\vec{A}$ is periodic and does not contain a constant mode.

Using $\vec{A}$ in the higher dimensional Schrödinger equation, we obtain

$$V(\vec{x}) = E + \vec{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{A}. \tag{3.15}$$

As in the one-dimensional case, we can introduce an identity to implement a change of variables, based on (3.15), between $V(\vec{x})$ and the field $\vec{A}$, constrained as described above:

$$1 = N^{-1} \int [d\vec{A}] \, d\bar{E} \, \delta[V - \bar{E} - \vec{\nabla} \cdot \vec{A} - \vec{A} \cdot \vec{A}] \delta[\vec{\nabla} \times \vec{A}] |J|, \tag{3.16}$$

with $\int$ having the same meaning as in (3.9) and $|J| = \left| \det(-\vec{\nabla} - 2\vec{A}) \right|$, where it is again understood that the zero modes are excluded.

Using the identity (3.16) in (3.1) and integrating over the disorder gives the corresponding field theory for the disordered average of observables in higher dimensions,

$$\langle \widehat{O} \rangle = Z^{-1} \int [d\vec{A}] \, d\bar{E} \, \widehat{O}[\bar{E}, \vec{A}] P[\bar{E} + \vec{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{A}] |J| \delta[\vec{\nabla} \times \vec{A}], \tag{3.17a}$$

with

$$Z = \int [d\vec{A}] \, d\bar{E} \, P[\bar{E} + \vec{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{A}] \delta[\vec{\nabla} \times \vec{A}] |J|. \tag{3.17b}$$

We can of course solve the constraint $\vec{\nabla} \times \vec{A} = 0$ by setting $\vec{A} = \vec{\nabla}\phi$ so that the resulting theory can be expressed as a scalar theory. However, the form (3.17) is particularly useful for the inclusion of a magnetic field $\vec{B}$, since one only has to treat $\vec{A}$ as a complex field with the constraint $\vec{\nabla} \times \vec{A} = 0$ replaced by $\vec{\nabla} \times \vec{A} = ie\vec{B}/c$.

In the absence of a magnetic field one can therefore write $\vec{A} = \vec{\nabla}\phi$, in which case (2.17) reduces to

$$\langle \hat{O} \rangle = Z^{-1} \int [d\phi] \, d\bar{E} \hat{O}(\phi, \bar{E}) P[\bar{E} + \nabla^2\phi + (\vec{\nabla}\phi)^2] |J| \tag{3.18}$$

with

$$Z = \int [d\phi] \, d\bar{E} P[\bar{E} + \nabla^2\phi + (\vec{\nabla}\phi)^2] |J| \tag{3.19}$$

where $|J| = \left| \det[(-\vec{\nabla} - 2\vec{\nabla}\phi) \cdot \vec{\nabla}] \right|$. The Jacobian appearing above can be handled in the usual way by introducing an integral over auxiliary Grassmann valued fields

$$|J| = \int [d\bar{u}] [du] \exp\left(\int d^d x \left[\vec{\nabla}\bar{u} \cdot \vec{\nabla}u - 2\bar{u}(\vec{\nabla}\phi) \cdot \vec{\nabla}u \right] \right). \tag{3.20}$$

3.3 Translational invariance

Central to our analysis will be the translational invariance of the effective action, which stems from the assumed translational invariance of the probability distribution, $P[V]$, and implies that

translational invariance is restored after averaging over the disorder. The translational invariance of the action leads to the appearance of an implied integration over a collective coordinate in the functional integral, corresponding to integration over the moduli space associated with the translational symmetry. It is appropriate to make this integration over the collective coordinate explicit to ensure that is correctly taken into account. To do this, we use a method inspired by the Faddeev-Popov [Fad67] quantisation method of gauge theories[9]. For simplicity we consider here the one-dimensional case, and indicate below how to extend to higher dimensions.

We introduce the identity

$$1 = c \int_{-L/2}^{L/2} dx_0 F'[\phi^{x_0}]\delta(F[\phi^{x_0}] - \nu), \tag{3.21}$$

where $\phi^{x_0} \equiv \phi(x + x_0)$, F is an arbitrary functional of ϕ which is not translationally invariant, and $F'[\phi^{x_0}]$ denotes the derivative with respect to x_0. Unless explicitly stated otherwise, the integration is over the interval $[-\frac{L}{2}, \frac{L}{2}]$. The proportionality constant c can in some cases be divergent, due to the existence of the Gribov ambiguity [Gri78] for certain choices of F. Under these conditions, one needs to be careful to extract the spurious divergent term after completing the functional integration, as this term cancels with a similar term in the normalisation.

As is usually done in gauge theories, it is more convenient to implement the identity (3.21) after integrating both sides with $c^{-1} \int d\nu f(\nu)$, where f is an arbitrary function, so that

$$c^{-1} \int d\nu f(\nu) = \int dx_0 F'[\phi^{x_0}]f(F[\phi^{x_0}]), \tag{3.22}$$

where the left hand side is independent of ϕ. Multiplying both the numerator and denominator of (3.12a) by the left hand side of (3.22) gives

$$\langle \widehat{O} \rangle = Z_F^{-1} \int [d\phi] \, d\bar{E} dx_0 \, F'[\phi^{x_0}]f(F[\phi^{x_0}])\widehat{O}[\bar{E}, \phi]P[\bar{E} + \phi' + \phi^2]|J|, \tag{3.23a}$$

where the partition function is given by

$$Z_F = \int [d\phi] \, d\bar{E} dx_0 \, F'[\phi^{x_0}]f(F[\phi^{x_0}])P[\bar{E} + \phi' + \phi^2]|J|. \tag{3.23b}$$

Note that the functional $F[\phi]$ acts in a similar fashion to gauge fixing terms in conventional gauge field theories. Since the choice of $F[\phi]$ is arbitrary, we can choose $F[\phi]$ so that our calculations can be simplified. As we shall see later, this choice will depend on what observables we wish to average. Also, any proportionality constants that appear due to the use of the Faddeev-Popov quantisation method cancel out since we use the identity in both the numerator and the denominator (although, as mentioned earlier, extra care is needed if Gribov copying occurs).

To extend to higher dimensions, we introduce a vector-valued functional $\vec{F}[\vec{A}^{\vec{x}_0}]$ and the associated Faddeev-Popov determinant $\Delta[\vec{A}] = \det \frac{\partial F_i[\vec{A}]}{\partial x_j}$. The results obtained above can then

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[9]See Appendix A for a brief introduction to the Faddeev-Popov method of quantisation as well as a note concerning the Gribov problem.

be generalised to higher dimensions through the replacement $F[\phi^{x_0}] \to \vec{F}[\vec{A}^{\vec{x}_0}]$ and $F'[\phi] \to \Delta[\vec{A}]$, the latter denoting the corresponding Jacobian.

3.4 Disordered averaged observables.

Instead of writing the observables as a functional of V, we would like to obtain them directly as functionals of the fields ϕ or $\vec{A}$ and the energy, $\bar{E}$. It is possible to do this for observables like the density of states, correlators of the wave-function, and the conductivity.

3.4.1 Density of states

The density of states at energy E is defined as

$$\rho(E) = \frac{1}{L^d} \sum_m \delta(E - E_m), \tag{3.24}$$

where E_m are eigenvalues of the Schrödinger equation, and d denotes the dimension of a system of size L. However, if we consider the identities, (3.9, 3.16), we see that the functional integral can be considered as a sum over all possible solutions of the Ricatti equations (3.3, 3.15), i.e all possible eigenstates with all possible eigenvalues for the corresponding Schrödinger equation, and is thus the total number of states. However, when fixing the integral over the energy at E, the functional integral yields only the number of eigenstates at E, and is therefore proportional to the density of states, $\rho(E)$ for a particular configuration of the disorder. After averaging over V with respect to the probability distribution $P[V]$, this yields the disordered averaged density of states. Thus, we obtain via inspection the formula for the average density of states, normalised by the total number of states, for one dimensional systems,

$$\langle \rho(E) \rangle = \frac{1}{ZL} \int [d\phi] \, P[E + \phi' + \phi^2]|J|, \tag{3.25a}$$

and for higher dimensional systems, in the absence of a magnetic field:

$$\langle \rho(E) \rangle = \frac{1}{ZL^d} \int [d\phi] \, P[E + \nabla^2\phi + (\nabla\phi)^2]|J|. \tag{3.25b}$$

3.4.2 Correlators of the wave-function

To obtain the observable related to correlations of the wave-function, $\psi(x)\psi(y)$, we use the relation between the wave-function and the ϕ field given in (3.5). To avoid the problems associated with the random phase cancellation when computing the localisation length, we need to calculate the correlator between the absolute values of the normalised wave-functions [Tho74]. Using (3.5) with (3.23), we obtain an equation for calculating the disordered average of the 2-point correlator of the wave-function at fixed energy E,

$$\langle |\psi_E(x)| \, |\psi_E(y)| \rangle = Z_F^{-1} \int [d\phi] \, dx_0 \, F'[\phi^{x_0}] f(F[\phi^{x_0}]) N[\phi(x)] P[E + \phi' + \phi^2]|J|$$

$$\times \exp\left[P \int_0^x dx' \phi(x') \right] \exp\left[P \int_0^y dx' \phi(x') \right]. \tag{3.26}$$

Since $F[\phi^{x_0}]$ and f are arbitrary, we make a choice which simplifies the calculation of the correlator by cancelling out the normalisation factor, $N[\phi]$. To do this, we choose $f = 1$, and

$$F[\phi^{x_0}] = -\int dz \exp\left(2P \int_0^z dx' \phi(x' + x_0)\right), \tag{3.27a}$$

so that

$$F'[\phi^{x_0}] = 2\phi(x_0) \int dz \exp\left[2P \int_0^z dx' \phi(x' + x_0)\right]$$

$$= 2\phi(x_0) \exp\left[2P \int_{x_0}^0 dx' \phi(x')\right] N[\phi(x)]^{-1}. \tag{3.27b}$$

Using (3.27b) in (3.26) we have

$$\langle |\psi_E(x)| \, |\psi_E(y)| \rangle = Z_F^{-1} \int [d\phi] \, dx_0 \, \phi(x_0) P[E + \phi' + \phi^2] |J|$$

$$\times \exp\left[P \int_{x_0}^x dx' \phi(x') + P \int_{x_0}^y dx' \phi(x')\right]. \tag{3.28}$$

Note that the integrand in (3.28) can be written as a total derivative to x_0, and thus the integral over x_0 gives a result of zero. This is an artifact of the choice of the gauge in the Faddeev-Popov identity, which is zero due to Gribov copying. Note, however, that the ratio in (2.25) is finite since an identical vanishing term appears in the denominator. Therefore, to obtain the correct result for the averaged observable, it is necessary, after completing the functional integral, to extract the terms that give zero using some form of regularisation, and cancel them out with similar terms that occur in the normalisation. What remains is the correct result for the disordered average of the observable. Indeed, one can see this mechanism at work in the derivation of the following central result. If we make the following change of variables in $\phi \to \varphi'$ in the numerator and denominator of (3.28), we note that we can factorise from the functional integral in the numerator and denominator the term $\int dx_0 \exp\left(-2\varphi(x_0)\right) \varphi'(x_0)$, which vanishes as it is a total derivative. Cancelling these terms gives for the correlator

$$\langle |\psi_E(x)| |\psi_E(y)| \rangle = Z^{-1} \int [d\varphi] \exp\left(\varphi(x)\right) \exp\left(\varphi(y)\right) P[E + \varphi'' + \varphi'^2] |J[\varphi]| \tag{3.29a}$$

where

$$Z = \int [d\varphi] \, d\bar{E} \int dx \exp\left(2\varphi(x)\right) P[\bar{E} + \varphi'' + \varphi'^2] |J[\varphi]| \tag{3.29b}$$

The denominator is just the averaged normalisation of the wave-function. This suggests that the correlation function of the normalised wave-function can be obtained from that of the un-normalised wave-function in the higher order theory where the averaged norm has to be divided out. This is a very useful result as the latter is much easier to compute. Note also that the above argument is independent of the choice of the function f, but only relies on a specific choice of the gauge. The only disadvantage of this argument is the appearance of the ill-defined ratio of two identically vanishing quantities. It is, however, clear from the context how this quantity

should be regularised. An independent proof of this result would, however, be useful. We shall use either of the forms (3.28) or (3.29) to compute the correlator, depending on which is the most convenient.

Assuming that the probability distribution is translationally invariant, we note that (3.28) is also translationally invariant. To show this, we transform (3.28) into a form where the symmetries of the system are more explicit. We first translate ϕ by y so that

$$\langle |\psi_E(x)| |\psi_E(y)| \rangle = Z_F^{-1} \int [d\phi] \, dx_0 \, \phi(x_0 - y) P[E + \phi' + \phi^2] |J|$$
$$\times \exp\left[P \int_{x_0}^{x} dx' \phi(x' - y) + P \int_{x_0}^{y} dx' \phi(x' - y) \right]. \tag{3.30a}$$

Making the change of variables $x_0 - y \to x_0$ and simplifying the integrals in the exponent, we find that

$$\langle |\psi_E(x)| |\psi_E(y)| \rangle = Z_F^{-1} \int [d\phi] \, dx_0 \, \phi(x_0) P[E + \phi' + \phi^2] |J|$$
$$\times \exp\left[P \int_{x_0}^{x-y} dx' \phi(x') + P \int_{x_0}^{0} dx' \phi(x') \right]. \tag{3.30b}$$

Similarly, we can translate ϕ by x, and then make the change of variables $x_0 - x \to x_0$, to obtain

$$\langle |\psi_E(x)| |\psi_E(y)| \rangle = Z_F^{-1} \int [d\phi] \, dx_0 \, \phi(x_0) P[E + \phi' + \phi^2] |J|$$
$$\times \exp\left[P \int_{x_0}^{y-x} dx' \phi(x') + P \int_{x_0}^{0} dx' \phi(x') \right], \tag{3.30c}$$

leading to the conclusion that $\langle |\psi_E(x)| |\psi_E(y)| \rangle$ is a function of $|x - y|$ only, that is

$$\langle |\psi_E(x)| |\psi_E(y)| \rangle = Z_F^{-1} \int [d\phi] \, dx_0 \, \phi(x_0) P[E + \phi' + \phi^2] |J|$$
$$\times \exp\left[P \int_{x_0}^{|x-y|} dx' \phi(x') + P \int_{x_0}^{0} dx' \phi(x') \right]. \tag{3.30d}$$

It is also possible to write the correlator in a form that displays a reflection symmetry around $|x - y| = \frac{L}{2}$. To do so, we use the condition that ϕ cannot contain a constant so that we can write (3.30d) as

$$\langle |\psi_E(x)| |\psi_E(y)| \rangle = Z_F^{-1} \int [d\phi] \, dx_0 \, \phi(x_0) P[E + \phi' + \phi^2] |J|$$
$$\times \exp\left[P \int_{x_0}^{|x-y|} dx' \phi(x') + P \int_{x_0}^{L} dx' \phi(x') \right]. \tag{3.30e}$$

Once again translating ϕ by $|x - y|$, and then make the change of variables $x_0 - |x - y| \to x_0$,

we obtain the correlator where the reflection symmetry is explicit,

$$\langle |\psi_E(x)| \, |\psi_E(y)| \rangle = Z_F^{-1} \int [d\phi] \, dx_0 \, \phi(x_0) P[E + \phi' + \phi^2]|J|$$

$$\times \exp\left[P\int_{x_0}^{L-|x-y|} dx' \phi(x') + P\int_{x_0}^{0} dx' \phi(x') \right]. \tag{3.30f}$$

We need to ensure that when any approximations are made to calculate the averages of observables such as the correlator, that these translation and reflection symmetries are respected.

The above considerations can be generalised to higher dimensions, with the integral in the exponent of (3.5) being replaced by the line integral $\oint_{\vec{0}}^{\vec{x}} \vec{A} \cdot d\vec{s}$ along any path connecting $\vec{0}$ and $\vec{x}$. Due to the constraint $\vec{\nabla} \times \vec{A} = 0$ this integral is path independent.

We note that if the wave-function changes its sign ($\vec{A}$ becomes singular) along a certain path connecting $\vec{0}$ and $\vec{x}$, it must do so along any other path, which implies that the associated singularity in $\vec{A}$ must appear in all possible paths connecting $\vec{0}$ and $\vec{x}$. This in turn implies that the singularity in $\vec{A}$ occurs on a surface separating $\vec{0}$ and $\vec{x}$ into disconnected regions. Any path connecting $\vec{0}$ and $\vec{x}$ may therefore cross a singularity and a prescription to handle this singularity is required. We can do this in the same way as the one-dimensional case : if t parameterises the path, we can avoid the singularity by a detour in the complex plane. In this way the absolute value and random phase of the wave-function can again be separated, with the principle value of the line integral determining the absolute value.The normalisation of the wave-function can again be cancelled by an appropriate (non-unique) choice of $\vec{F}$ so that the correlation in higher dimensions is given by

$$\langle |\psi_E(\vec{x})| \, |\psi_E(\vec{y})| \rangle = Z_F^{-1} \int [d\vec{A}] \, d\vec{x}_0 \Delta[\vec{A}^{\vec{x}_0}] f(\vec{F}[\vec{A}^{\vec{x}_0}]) P[E + \vec{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{A}]|J|\delta[\vec{\nabla} \times \vec{A}]$$

$$\times \exp\left[P\int_{\vec{x}_0}^{\vec{x}} \vec{A}(\vec{x}') \cdot d\vec{x}' + P\int_{\vec{x}_0}^{\vec{y}} \vec{A}(\vec{x}') \cdot d\vec{x}' \right]. \tag{3.31}$$

Introducing the variable $\vec{A} = \vec{\nabla}\phi$ the vanishing integral over x_0 can again be factorised from the functional integral in the numerator and denominator to cancel which yields for the correlator

$$\langle |\psi_E(x)||\psi_E(y)| \rangle = Z^{-1} \int [d\phi] \, \exp\left(\phi(x)\right) \exp\left(\phi(y)\right) P[E + \nabla^2\phi + (\vec{\nabla}\phi)^2]|J| \tag{3.32}$$

where

$$Z = \int [d\phi] \, d\bar{E} \int d^d x \exp\left(2\phi(x)\right) P[\bar{E} + \nabla^2\phi + (\vec{\nabla}\phi)^2]|J|. \tag{3.33}$$

3.4.3 Conductivity

The conductivity of a system of non-interacting fermions is given by the Kubo formula [Nak56]. For our purposes it is convenient to integrate the Kubo formula by parts and use the periodic boundary conditions on the wave-functions, so that the real part of the conductivity is then given by [Hal65]

$$\mathrm{Re}\sigma(\omega) = -\int dE \frac{\partial f(E)}{\partial E} \Phi(E, \omega), \tag{3.34}$$

where

$$\Phi(E,\omega) = \frac{4e^2}{\hbar L} \sum_{\alpha,\beta} \left[\int dx \, \frac{d\psi_\alpha}{dx} \psi_\beta \right]^2 \delta(E - E_\alpha)\,\delta(E + \hbar\omega - E_\beta).$$ (3.35)

Here $f(E)$ is the Fermi-function, and all energies are measured in units of $\frac{\hbar^2}{2m}$. Without loss of generality, we can focus on the disorder average of the quantity $\Phi(E,\omega)$, which is of course just the contribution to the conductivity of a particle at energy E.

We are able to obtain the form of the observable for the disorder average of Φ by using a technique similar to the one used to obtain the expression for the density of states. The only difference is that there is a summation over two eigenvalues in (3.35), and so we need to introduce two identities of the form (3.9), where the integration over ϕ is replaced by integrations over ϕ_α and ϕ_β which correspond to two solutions of the Schrödinger equation with the same random potential. We can write the average of Φ as

$$\langle \Phi(E,\omega) \rangle = \frac{4e^2}{\hbar L} Z^{-1} \int [d\phi_\alpha]\,[d\phi_\beta]\,\delta[\hbar\omega - \phi_\alpha' - \phi_\alpha^2 + \phi_\beta' + \phi_\beta^2] P[E + \phi_\alpha' + \phi_\alpha^2]\,|J_\alpha|\,|J_\beta|$$
$$\times N[\phi_\alpha]N[\phi_\beta] \left(\int dx \, \phi_\alpha(x) \exp\left[\oint_0^x dx'\phi_\alpha(x') + \oint_0^x dx'\phi_\beta(x') \right] \right)^2$$ (3.36a)

where

$$Z = \int dE d\bar{E} \int [d\phi_\alpha]\,[d\phi_\beta]\,\delta[\bar{E} - \phi_\alpha' - \phi_\alpha^2 + \phi_\beta' + \phi_\beta^2] P[E + \phi_\alpha' + \phi_\alpha^2]\,|J_\alpha|\,|J_\beta|.$$ (3.36b)

with $|J_\alpha| = \frac{d}{dx} + 2\phi_\alpha$ and $|J_\beta| = \frac{d}{dx} + 2\phi_\beta$. Completing the integral over ϕ_β, we have

$$\langle \Phi(E,\omega) \rangle = \frac{4e^2}{\hbar L} Z^{-1} \int [d\phi_\alpha]\, dx d\bar{x}\, P[E + \phi_\alpha' + \phi_\alpha^2]\,|J_\alpha|\, N[\phi_\alpha]N[\tilde{\phi}_\beta]\phi_\alpha(x)\phi_\alpha(\bar{x})$$
$$\times \exp\left(\oint_0^x dx'[\phi_\alpha(x') + \tilde{\phi}_\beta(x')] \right) \exp\left(\oint_0^{\bar{x}} dx'[\phi_\alpha(x') + \tilde{\phi}_\beta(x')] \right),$$ (3.37)

where $\tilde{\phi}_\beta(x)$ is a functional of ϕ_α and is determined by

$$\hbar\omega - \phi_\alpha' - \phi_\alpha^2 + \tilde{\phi}_\beta' + \tilde{\phi}_\beta^2 = 0.$$ (3.38)

We can now use the Faddeev-Popov method, where we introduce the same choice of gauge for the ϕ_α and $\tilde{\phi}_\beta$ fields as in the previous section, which allow us to cancel out the $N[\phi_\alpha]$ and $N[\tilde{\phi}_\beta]$ normalisation factors respectively, so that

$$\langle \Phi(E,\omega) \rangle = \frac{4e^2}{\hbar L} Z_F^{-1} \int [d\phi_\alpha]\, dx d\bar{x} dx_0 d\bar{x}_0 \phi_\alpha(x)\phi_\alpha(\bar{x})\phi_\alpha(x_0)\tilde{\phi}_\beta(\bar{x}_0) P[E + \phi_\alpha' + \phi_\alpha^2]\,|J_\alpha|$$
$$\times \exp\left[\oint_{x_0}^x dx'\phi_\alpha(x') + \oint_{\bar{x}_0}^x dx'\tilde{\phi}_\beta(x') + \oint_{x_0}^{\bar{x}} dx'\phi_\alpha(x') + \oint_{\bar{x}_0}^{\bar{x}} dx'\tilde{\phi}_\beta(x') \right]$$ (3.39a)

with

$$Z_F = \int dE d\bar{E} \int [d\phi_\alpha] \, dx_0 d\bar{x}_0 \, P[E + \phi'_\alpha + \phi^2_\alpha] \, |J_\alpha| \, F'[\phi^{x_0}_\alpha] F'[\tilde{\phi}^{\bar{x}_0}_\beta]. \qquad (3.39b)$$

In higher dimensions we use the same strategy as above to obtain

$$\langle \Phi(E, \omega) \rangle = \frac{4e^2}{\hbar L^{2-d}} Z^{-1}_{\vec{F}} \int [d\vec{A}_\alpha] \, d\vec{x}_0 \, d\vec{y}_0 \, \Delta[\vec{A}_\alpha] P[E + \vec{\nabla} \cdot \vec{A}_\alpha + \vec{A}_\alpha \cdot \vec{A}_\alpha] |J_\alpha|$$

$$\times \vec{A}_\alpha(\vec{x}_0) \int d\vec{x} \, \vec{A}_\alpha(\vec{x}) \exp \left[\oint_{\vec{x}_0}^{\vec{x}} \vec{A}_\alpha(\vec{x}') \cdot d\vec{x}' + \oint_{\vec{y}_0}^{\vec{x}} \vec{A}_\beta(\vec{x}') \cdot d\vec{x}' \right]$$

$$\times \vec{A}_\alpha(\vec{y}_0) \int d\vec{y} \, \vec{A}_\alpha(\vec{y}) \exp \left[\oint_{\vec{x}_0}^{\vec{y}} \vec{A}_\alpha(\vec{x}') \cdot d\vec{x}' + \oint_{\vec{y}_0}^{\vec{y}} \vec{A}_\beta(\vec{x}') \cdot d\vec{x}' \right], \qquad (3.40)$$

where $\vec{A}_\beta \equiv \vec{A}_\beta[\vec{A}_\alpha]$ is the solution of

$$\hbar\omega - \vec{\nabla} \cdot \vec{A}_\alpha - \vec{A}_\alpha \cdot \vec{A}_\alpha + \vec{\nabla} \cdot \vec{A}_\beta + \vec{A}_\beta \cdot \vec{A}_\beta = 0. \qquad (3.41)$$

Here we are working with the wave-function, rather than its absolute value as before. The phase of the wave-function is now important and kept track of through the integration contour prescription we are using. Changing to the variables $\varphi' = \phi$ and $\vec{\nabla}\phi = \vec{A}$ is now no longer possible as the information on the phase of the wave-function will be lost. Thus the simplifications that occurred earlier in the computation of correlators of the absolute value of the wave-function can no longer be achieved here. This illustrates again the point that the current formalism is more appropriate to the computation of correlators of absolute values rather than correlators of wave-function itself or current-current correlators as above. Thus, although formulae for these quantities can be derived they are of less practical value and cumbersome to compute in practice. Fortunately if one is interested in the statistics of eigenvalues and eigenfunctions, the correlators of absolute values of wave-functions are normally sufficient. Since the current formalism is more geared towards the latter it is on this aspect that we will focus from here on.

Chapter 4
One dimensional systems with Gaussian disorder

In this chapter we consider one dimensional Gaussian disordered systems. We do so to illustrate how the formalism described in Chapter 3 can be applied, using standard approximation schemes, to recover known results for the density of states [Hal65, Lif88] and the conductivity [Ber74].

Gaussian disorder is characterised by the 1– and 2–point correlation functions, such that

$$\langle V(x) \rangle = 0, \qquad \langle V(x)V(y) \rangle = \frac{1}{2l}\delta(x-y), \tag{4.1}$$

where the constant l has a dimension of (length)^3 in our units. The probability distribution $P[V]$ thus has the form

$$P[V] = \exp\left[-l \int_{-L/2}^{L/2} dx\, V(x)^2\right]. \tag{4.2}$$

Note that the constant l characterises the width of the distribution, and thus gives an indication as to the amount of disorder present in the system.

Using (4.2) in (3.12), we obtain a φ^4 field theory for calculating the disorder averages of fixed energy observables:

$$\langle \widehat{O} \rangle = Z^{-1} \int [d\phi]\, \widehat{O}[\phi, E] \exp\left(-S[\phi, E]\right) |J|, \tag{4.3a}$$

where the action is given by

$$S[\phi, E] = l \int_{-L/2}^{L/2} dx (E + \phi' + \phi^2)^2, \tag{4.3b}$$

and $|J| = \left|\det\left(\frac{d}{dx} + 2\phi\right)\right|$. Here, and in what follows, we set $\widehat{O}[\bar{E}, \phi] = \widehat{O}[\phi, E]\delta(E - \bar{E})$, since we are normally interested in calculating disorder averages of observables at a fixed energy, E. Additionally, we can insert (4.2) into (3.23), which is the gauge fixed equivalent of (3.12), to obtain a gauge fixed form of (4.3). However, since the initial discussion holds for all gauges, we shall concentrate on (4.3), with the understanding that Z and $\widehat{O}$ denotes the normalisation and observable in a generic gauge and we omit the subscript F of (3.23).

It is not possible to calculate the functional integral in (4.3) exactly, so we shall use perturbative approximations in order to calculate the disordered averages.

Often it is useful, particularly to discuss the saddle points and perturbative corrections around them, to represent the Jacobian in (4.3) in terms of ghost (Grassmann valued) fields. Then (4.3) reads

$$\langle \widehat{O} \rangle = Z^{-1} \int [d\phi]\, [d\bar{u}]\, [du]\, \widehat{O}[\phi, E] \exp\left(-S[\phi, E] + S_{\text{gh}}\right), \tag{4.4a}$$

where the ghost action is given by

$$S_{\text{gh}} = \int_{-L/2}^{L/2} dx (\bar{u}u' + 2\phi\bar{u}u) \tag{4.4b}$$

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and the integral excludes the zero mode.

4.1 Microscopic realization of the model

It is useful to have a microscopic realization of the Gaussian model introduced above that relates the parameters to microscopic quantities. For this purpose, we consider the model discussed in Chapter 2 of N delta scatterers [Fri60, Gre76] placed randomly on a ring, with the potential given by

$$V(x) = a \sum_{i=1}^{N} \delta(x - x_i) - \frac{aN}{L}. \tag{4.5}$$

Here a is a dimensionful constant (units of $(\text{length})^{-1}$) that determines the strength of the scatters, and the subtracting term is chosen so that $\langle V \rangle = 0$. Applying our formalism, the disordered average of some observable $\hat{O}$ is now given by

$$\langle \hat{O} \rangle = Z^{-1} \int_{-L/2}^{L/2} \prod_{i=1}^{N} \frac{dx_i}{L} \int [d\phi] \, d\bar{E} \, \hat{O}(\bar{E}, \phi) \, |J| \, \delta \left[\bar{E} + \phi' + \phi^2 - a \sum_{i=1}^{N} \delta(x - x_i) + \frac{aN}{L} \right] \tag{4.6}$$

where

$$Z = \int_{-L/2}^{L/2} \prod_{i=1}^{N} \frac{dx_i}{L} \int [d\phi] \, d\bar{E} \, |J| \, \delta \left[\bar{E} + \phi' + \phi^2 - a \sum_{i=1}^{N} \delta(x - x_i) + \frac{aN}{L} \right]. \tag{4.7}$$

Introducing a Fourier representation for the functional Dirac-delta, we can write (4.6) as

$$\langle \hat{O} \rangle = Z^{-1} \int [d\phi] \, [d\Lambda] \, d\bar{E} \, \hat{O}(\bar{E}, \phi) \, |J| \, \exp \left[N \log \int \frac{dx}{L} e^{-ia\Lambda} + i \int dx \Lambda \left(\bar{E} + \phi' + \phi^2 + \frac{aN}{L} \right) \right], \tag{4.8}$$

with a similar expression for Z. Assuming that the scatters are weak (a small) and the system size L is large, we expand to lowest order in a and $1/L$:

$$\langle \hat{O} \rangle = Z^{-1} \int [d\phi] \, [d\Lambda] \, d\bar{E} \hat{O}(\bar{E}, \phi) \, |J| \exp \left[-\frac{Na^2}{2L} \int dx \, \Lambda^2 + i \int dx \, \Lambda [\bar{E} + \phi' + \phi^2] \right]. \tag{4.9}$$

Performing the Λ integration, we have:

$$\langle \hat{O} \rangle = Z^{-1} \int [d\phi] \, d\bar{E} \hat{O}(\bar{E}, \phi) \, |J| \exp \left[-\frac{L}{2Na^2} \int dx \, (\bar{E} + \phi' + \phi^2)^2 \right] \tag{4.10}$$

with

$$Z = \int [d\phi] \, d\bar{E} \, |J| \, \exp \left[-\frac{L}{2Na^2} \int dx \, (\bar{E} + \phi' + \phi^2)^2 \right]. \tag{4.11}$$

We can therefore identify the parameter l of our Gaussian model with $l = \frac{L}{2} N a^2 = \tilde{l}/2a^2$, with $\tilde{l}$ the mean free path length.

4.2   Weak disorder limit

4.2.1   Saddle point approximation

Before embarking on a detailed analysis of the saddle points of (4.3) it is useful to consider (4.3) more generally. We note that the action in (4.3) is a complete square. Thus the global minimum of the action is obtained when the quantity under the square vanishes. However, we note from (3.3) that the configurations and energies that satisfy this condition are precisely the solutions of the free Schrödinger equation. Thus we note that if the energy value lies in the region where the density of states of the pure system is non-vanishing, the solutions of the free Schrödinger equation at or closest to that energy value will provide a global minimum of the action. Thus we can expect these configurations to be dominant in the functional integral when the energy lies, in the language of quantum mechanics, in the 'classically allowed' region, where the density of states of the pure system is non-vanishing. When the energy lies in a region where the density of states of the pure system vanishes, the global minimum can no longer be obtained. This corresponds to the 'classically forbidden' region. In this case we expect tails to develop that will be governed by the analogue of quantum mechanical tunnelling.

For large l, compared to the size and energy of the system, the functional integral in (4.3) describes a weakly disordered system. In this case, we calculate (4.3) perturbatively in a loop-wise expansion using a saddle point approximation. There are actually two approaches we can take here. In the first we write down and solve the saddle point equations without taking into account the presence of the observable. The observable is then taken into account at a later stage, when performing the Gaussian integrations around the saddle point, through its linear contribution to the action which will still be present as the observable was not included in the original saddle point equations. This is the approach we will follow here as it is the most convenient. The second, more conventional, approach is to take the observable into account when writing and solving the saddle point equations. Then there will no longer be a linear term present in the action when performing the Gaussian integration over the fluctuations around the saddle point. However, the result is the same as in the previous approach as we show and the end of this subsection. Following the first approach the saddle-point equation is given by

$$\frac{\delta S[\phi, E]}{\delta \phi(x)}\bigg|_{\phi_c} = l\left(-\frac{d}{dx} + 2\phi_c\right)\left(E + \phi_c' + \phi_c^2\right)$$
$$= l\left(-\phi_c'' + 2E\phi_c + 2\phi_c^3\right) = 2\bar{u}_c u_c, \tag{4.12a}$$

with the saddle-point equations for the ghosts

$$u_c' + 2\phi_c u_c = \bar{u}_c' - 2\phi_c \bar{u}_c = 0 \tag{4.12b}$$

where we must remember that the boundary conditions are that ϕ_c must be periodic and cannot be a constant, and that the zero mode is excluded from the ghost integration. The latter implies that $\bar{u}_c = u_c = 0$ so that the equation for ϕ_c reduces to

$$-\phi_c'' + 2E\phi_c + 2\phi_c^3 = 0. \tag{4.13}$$

Following a procedure similar to the one used in Zinn-Justin [ZJ89], we expand the action $S[\phi, E]$ around the classical solution $\phi = \phi_c + \frac{1}{\sqrt{l}}\eta$, so that

$$S[\phi, E] = S[\phi_c, E] + \int_{-L/2}^{L/2} dx' \eta(x')\Delta^{-1}\eta(x') + \frac{4}{\sqrt{l}}\int_{-L/2}^{L/2} dx' \phi_c(x')\eta^3(x') + \frac{1}{l}\int_{-L/2}^{L/2} dx' \eta^4(x')$$

(4.14a)

where

$$\Delta^{-1} = -\frac{d^2}{dx^2} + 2E + 6\phi_c^2.$$

(4.14b)

If we consider the classical action $S[\phi_c, E]$, then we see that

$$\begin{aligned} S[\phi_c, E] &= l \int_{-L/2}^{L/2} dx \left(E + \phi_c' + \phi_c^2\right)^2 \\ &= l \int_{-L/2}^{L/2} dx \left[(\phi_c')^2 + 2(E + \phi_c^2)\phi_c' + (E + \phi_c^2)^2\right] \end{aligned}$$

(4.15)

Multiplying the saddle point equation, (4.12), through with ϕ_c', and then simplifying, we find that $\phi_c' = \pm(E + \phi_c^2)$, so that

$$S[\phi_c, E] = 2l \int_{-L/2}^{L/2} dx[(\phi_c')^2 \pm (\phi_c')^2] = 2l \int_{-L/2}^{L/2} dx[(E + \phi_c^2)^2 \pm (E + \phi_c^2)^2],$$

(4.16)

where the sign that is chosen depends on the sign of the classical solution.

Using (4.14) in (4.3), we obtain the saddle-point approximation for the disordered averaged observable,

$$\begin{aligned} \langle \widehat{O} \rangle &\approx Z^{-1} \sum \int [d\eta] \, \widehat{O}[\phi_c + \frac{1}{\sqrt{l}}\eta, E] \exp\left(-S[\phi_c, E]\right) |J_c| \\ &\times \exp\left[-\int_{-L/2}^{L/2} dx' \left(\eta(x')\Delta^{-1}\eta(x') - \frac{4}{\sqrt{l}}\phi_c(x')\eta^3(x') - \frac{1}{l}\eta^4(x')\right)\right], \end{aligned}$$

(4.17a)

where the summation is over all possible saddle point solutions,

$$Z = \sum \int [d\eta] \exp\left(-S[\phi_c, E]\right) |J_c| \exp\left[-\int_{-L/2}^{L/2} dx' \left(\eta(x')\Delta^{-1}\eta(x') + \frac{4}{\sqrt{l}}\phi_c(x')\eta^3(x') + \frac{1}{l}\eta^4(x')\right)\right],$$

(4.17b)

and $|J_c| = \left|\det\left(\frac{d}{dx} + 2\phi_c\right)\right|$, where the fluctuations in $|J|$ have been gauged away as discussed in the previous chapter.

Expanding in a loop expansion to one loop order we obtain the approximation that we shall use in the weak disorder limit,

$$\langle \widehat{O} \rangle \approx Z^{-1} \sum \int [d\eta] \, \widehat{O}[\phi_c + \frac{1}{\sqrt{l}}\eta, E] \exp\left(-S[\phi_c, E]\right) |J_c| \exp\left[-\int_{-L/2}^{L/2} dx' \eta(x')\Delta^{-1}\eta(x')\right]. \quad (4.18)$$

Later in this chapter, after examining the saddle-point solutions, as well as how to calculate the propagator, we will discuss under what conditions this approximation is valid by investigating the relative contributions of the second loop terms.

Let us return to the issue raised at the beginning of this subsection regarding the two ways of handling the observable in the saddle point approximation. We could have written the full saddle point equations for ϕ, including the observable as:

$$l(-\tilde{\phi}_c'' + 2E\tilde{\phi}_c + 2\tilde{\phi}_c^3) = J; \quad J \equiv -\frac{\delta}{\delta\phi_c}\ln\hat{O}(\phi_c, E) \tag{4.19}$$

Here we have already used the fact that the ghosts vanish and we use $\tilde{\phi}_c$ to distinguish this saddle point from the earlier one. We can now proceed in the standard way of solving this equation through a series expansion in $1/l$ with the result:

$$\tilde{\phi}_c = \phi_c - \frac{1}{l}\Delta J + O(1/l^2) \tag{4.20}$$

Here ϕ_c satisfies the homogeneous equation (4.13) and Δ is the propagator introduced in (4.14b). Writing $\phi = \tilde{\phi}_c + \frac{1}{\sqrt{l}}\tilde{\eta}$ and expanding the action to second order in the fluctuations we have:

$$S[\phi, E] - \ln\hat{O}[\phi, E] = S[\phi_c, E] - \ln\hat{O}[\phi_c, E] + \int_{-L/2}^{L/2} dx\tilde{\eta}\Delta^{-1}\tilde{\eta} + O(1/l) \tag{4.21}$$

Let us now consider the classical term. Expanding this we have:

$$S[\tilde{\phi}_c, E] - \ln\hat{O}(\tilde{\phi}_c, E) = S[\phi_c, E] - \ln\hat{O}(\phi_c, E) + \frac{1}{l}J\Delta J \tag{4.22}$$

Here we have used the equation of motion for ϕ_c and the definition of J. We note that after integration over the fluctuations this result agrees exactly with our previous result if we were to expand the observable to linear order and perform the Gauss integration over the fluctuations, the $J\Delta J$ term arising from this integration.

4.2.2 Saddle point solutions

In order to obtain the classical solution, we need to solve the saddle point equation given in (4.12). If we consider (4.12) as a dynamical equation of motion with a potential given by $V(x, E) = 2E + \phi_c(x)^2$, then we note that the solutions of the equation for positive and negative energies will have different physical behaviour as the potential changes from a parabolic form for positive energies to a double well potential for negative energies. We thus need to consider the positive and negative energy regions separately when obtaining the saddle point solutions.

4.2.2.1 Positive energy region

In the positive energy region, the possible saddle point solution is given by

$$\phi_c(x) = -\sqrt{E}\tan\left(\sqrt{E}x\right). \tag{4.23}$$

However, since ϕ_c must satisfy periodic boundary conditions, i.e.,

$$\phi_c(x) = \phi_c(x + L) \quad \text{and} \quad \phi'_c(x) = \phi'_c(x + L), \tag{4.24}$$

we find that the energy must be quantised and that the only exact solution of (4.12) for positive energies is obtained when $E = \frac{m\pi}{L}$, with m an positive integer, such that

$$\phi_c^m(x) = -\frac{m\pi}{L}\tan\left(\frac{m\pi}{L}x\right), \quad \forall E \geq 0. \tag{4.25}$$

Additionally, from (3.2), and the periodic boundary conditions placed on the Schrödinger wave-function, we see that m must be even. Note that for these values of E this saddle point gives the global minimum of the action and will dominate the functional integral.

If we wish to find solutions of (4.12) that are satisfied for all positive energies, we need to include approximate saddle point solutions that satisfy the saddle point equation to leading order in L^{-1}. Thus, if we choose m to be the nearest even integer to $\sqrt{E}L/\pi$, we find that (4.25) satisfies (4.12) when L is large, that is

$$-(\phi_c^m)'' + 2E\phi_c^m + 2(\phi_c^m)^3 = 0 + O(\frac{1}{L}). \tag{4.26}$$

Note that the approximate saddle point solution, ϕ_c^m, satisfies the saddle point equation exactly for all positive energies in the thermodynamic limit. Also, note that choosing any other positive, even integer different to m will also approximately satisfy the saddle point approximation. Thus we find that there is a whole class of saddle point solutions that approximately satisfy the saddle point equation to leading order in L^{-1}. These solutions are topologically distinct for different m, as each solution has a different number of singularities. Since the number of singularities are related to the number of nodes of the wave-function, each saddle-point solution corresponds to wave-functions with a different number of nodes. To be more precise, when compactifying the real line to S^1 by identifying $+\infty$ and $-\infty$, we note that ϕ is a mapping from S^1 to S^1, where the mappings are classified according to winding numbers. It is simple to see that the solution ϕ_c^m has winding number m.

There are in fact more topological considerations coming in here than meets the eye. Let us consider the term quadratic in the action $\int_{-L/2}^{L/2} dx\eta\Delta^{-1}\eta$. We note that Δ^{-1} contains the square of the classical solution which has singularities. Thus we see that any fluctuation that does not vanish at the singularities of the classical solution will yield an infinite action and will therefore not contribute to the functional integral. Thus, effectively the presence of the singularities projects down onto a smaller space of fluctuation over which one integrates in the functional integral, namely only those fluctuations with zeros at the singularities of the classical solution are allowed (see also Section 4.2.4). From a topological perspective this means that we are inserting topological obstructions at the points where the singularities of the classical solution occur. Such obstruction, when inserted in a one dimensional space, will effectively cut the real line at those points. Thus, from a topological point of view the singularities of the classical solutions lead to a slicing of the interval $[-\frac{L}{2}, \frac{L}{2}]$ into m smaller intervals of length $\frac{L}{m}$. The implications of this, as we shall see after computing the propagators, is that the functional integral will factorise

into m products of functional integrals each performed over fluctuations defined on an interval of length $\frac{L}{m}$ and with vanishing boundary conditions. These considerations also play an important role in answering the following question: given that the saddle point solutions are approximate, how will the linear term left over in the action effect us? We return to this question after discussing the propagators and show that this term does indeed vanish.

Let us now proceed to calculate the classical action for this approximate solution. Since the classical solution does not satisfy the saddle point equation exactly, we cannot use (4.16) to calculate the classical action, but should rather use (4.15). Using (4.26) in (4.15) we obtain the classical action in the positive energy region

$$S[\phi_{\mathrm{c}}^m, E] = l \int_{-L/2}^{L/2} dx \left[E - \left(\frac{m\pi}{L} \right)^2 \right]^2 \equiv lL \left(\Delta E \right)^2 \tag{4.27}$$

We note that the different approximate solutions yield a classical action differing by $O(1/L)$. Thus, to obtain a reliable approximation to the functional integral (4.18), it is necessary to sum over all the topologically different sectors. Additionally, since the fluctuations, η, are smoothly varying functions around the saddle point solutions, we see that all the information about the phase (which is determined by the number of singularities in ϕ) is contained in the classical solutions, so that the localisation length (which is extracted from the absolute value of the wave-function) is purely determined by the fluctuations, η.

4.2.2.2 Negative energy region

In the negative energy region, the possible saddle point solutions are given by

$$\phi_{\mathrm{c}} = \pm\sqrt{|E|} \qquad \text{or} \tag{4.28a}$$

$$\phi_{\mathrm{c}} = \pm\sqrt{|E|} \tanh \left(\sqrt{|E|}x \right) \tag{4.28b}$$

The constraint that ϕ has no constant mode does not allow us to obtain constant saddle point solutions situated at the minima of the double well potential, thus the first solution is disallowed (Note, however, that it would also have given the global minimum value). The remaining solutions, (4.28b), are known as the kink and anti–kink instanton solutions [Raj82, Kas97, Col85, Gil77, Ger75b, Ger75a], and are analogous to tunnelling solutions. These represent the field having an initial value of $\mp\sqrt{|E|}$ at a large negative position, and later, at a position far away from the initial position, having a value of $\pm\sqrt{|E|}$. Although these instanton solutions satisfy the constraint that the field cannot be a constant, they do not satisfy the periodic boundary conditions imposed on ϕ. We can, however, construct approximate saddle point solutions that satisfy the saddle point equation to leading order in L^{-1}, as well as the boundary conditions, by forming combinations of the kink and anti–kink instantons. Thus, the simplest approximate solution that can be obtained is

$$\phi_{\mathrm{c}}^{\pm}(x) = \pm\sqrt{|E|} \tanh \sqrt{|E|}(x - \bar{x}_0) \tanh \sqrt{|E|}(x + \bar{x}_0), \quad \forall E < 0 \tag{4.29}$$

where we assume a periodic continuation of $\phi_c^\pm(x)$ outside $[-\frac{L}{2}, \frac{L}{2}]$. In this solution, the field has an initial value of $-\sqrt{|E|}$ at a large negative position, and then has a value of $\sqrt{|E|}$ for positions near zero, and then finally returns to the initial value of $-\sqrt{|E|}$ for large positive positions, thus satisfying the periodic boundary condition. This is analogous to a quantum mechanical particle moving in a double well potential, and tunnelling takes place from the one well to the other and then back to the original. Other approximate solutions can be obtained by combining higher powers of the kink and anti–kink instantons, which is equivalent to having solutions where the to and fro tunnelling occurs multiple times [Raj82], corresponding to topologically different sectors over which one has to sum. However, since there is an exponential decay associated with each tunnelling process, the solutions, (4.29), where the tunnelling occurs once will dominate the functional integral.

Note that in (4.29), the nodes of the kink and anti-kink solutions occur at $\pm\bar{x}_0$, and it is assumed that the relative separation $2\bar{x}_0$ between the instanton and the anti-instanton pair is large [Kas97]. We shall see below that the constraint that ϕ contains no constant mode implies $2\bar{x}_0 = \frac{L}{2}$ so that this condition is automatically fulfilled. Under these conditions, it is possible to use a dilute gas approximation, where one assumes that the two instantons are independent, thus allowing us to write (4.29) as

$$\phi_c^\pm(x) \approx \pm\sqrt{|E|} \tanh\sqrt{|E|}(x - \bar{x}_0)\Theta(x) \mp \sqrt{|E|} \tanh\sqrt{|E|}(x + \bar{x}_0)\Theta(-x) \qquad (4.30)$$

Using (4.30) in (4.15) we find that the classical action in the negative energy region is given by

$$S[\phi_c^\pm, E] = 4l|E|^2 \int_{-L/2}^{0} dx\, \text{sech}^4\left(\sqrt{|E|}(x + \bar{x}_0)\right) = \frac{16}{3}l|E|^{\frac{3}{2}}, \qquad (4.31)$$

where we used the result $\bar{x}_0 = \frac{L}{4}$ which we show later, as well as the assumption that $L \gg 0$ in order to obtain the final result.

4.2.3 Zero modes

There is a complication in the saddle point approximation, since the approximation contains a zero mode, which is a manifestation of the translational invariance of the action (4.3b). This zero mode occurs when one of the eigenvalues of the differential operator in the action for the fluctuations in (4.18) is zero, that is

$$\Delta^{-1}\eta_0(x) = \left(-\frac{d^2}{dx^2} + 2E + 6{\phi_c}^2\right)\eta_0(x) = 0, \qquad (4.32)$$

where the zero mode, η_0, is the corresponding eigenfunction. It is possible to obtain the zero mode in terms of the classical solution by taking the derivative of the saddle point equation, (4.12), and comparing the result with (4.32). The derivative of (4.12) is

$$\left(-\frac{d^2}{dx^2} + 2E + 6{\phi_c}^2\right)\phi_c'(x) = 0, \qquad (4.33)$$

and thus we obtain that $\eta_0 \propto \phi'_c$, where the normalisation of the zero mode is calculated using (4.16).

When a zero mode occurs, the functional integral which includes an integration over the zero mode is not well defined as there is no longer a damping term provided by the action to keep the result of the integration over the zero mode finite. To circumvent this problem associated with the zero mode, we make the translational symmetry explicit in the form of a collective coordinate [Gil77, Ger75b, Ger75a] which we introduce by using the Faddeev-Popov method discussed in the previous chapter.

Inserting the identity (3.21) into (4.3) with the choice $F[\phi^{x_0}] = \int dx\,\eta_0(x)\phi(x + x_0)$ in order to project out the zero mode, and using the saddle point approximation to one loop order, (4.18), we obtain for the positive energy region

$$\langle \widehat{O} \rangle \approx Z^{-1} \sum_m \int dx_0 \int [d\eta]\, \widehat{O}[\phi_c^m(x - x_0) + \frac{1}{\sqrt{l}}\eta(x - x_0), E]\exp\left(-S[\phi_c^m, E]\right)|J_c|$$

$$\times \left(\delta(x_0)\delta_{m,0} + (1 - \delta_{m,0})F'[\phi_c^m + \frac{1}{\sqrt{l}}\eta]\right)\delta(F[\phi_c^m + \frac{1}{\sqrt{l}}\eta])\exp\left(-\int_{-L/2}^{L/2} dx\,\eta\Delta_m^{-1}\eta\right),$$

$$(4.34)$$

It is important to note that the $m = 0$ sector contains no zero mode and is excluded from the above procedure so that no x_0 integral appears in this sector. The propagator Δ_m^{-1} is given by (4.14b), with the classical solution ϕ_c^m. Here we noted that the constraint $\delta(\int dx\,\phi_c^m)$ is trivially satisfied so that the constraint $\delta(\int dx\,\phi)$ in (4.3) simply becomes the constraint that the η integration is over all non-constant modes. Using the definition of F, we find that

$$F[\phi_c^m + \frac{1}{\sqrt{l}}\eta] = \int dx\,\eta_0(x)\left(\phi_c^m(x) + \frac{1}{\sqrt{l}}\eta(x)\right)$$

$$= \int dx\left(\frac{d\phi_c^m}{dx}\phi_c^m + \frac{1}{\sqrt{l}}\eta_0(x)\eta(x)\right)$$

$$= \int dx\left(\frac{1}{2}\frac{d(\phi_c^m)^2}{dx} + \frac{1}{\sqrt{l}}\eta_0(x)\eta(x)\right)$$

$$= \int dx\left(\frac{1}{\sqrt{l}}\eta_0(x)\eta(x)\right),$$

$$(4.35)$$

Thus, we see that the constraint that $F[\phi_c^m + \frac{1}{\sqrt{l}}\eta] = 0$, projects out the integral over the zero mode in the functional integral in (4.34), leaving an integral over the collective coordinate and a functional integral over all modes orthogonal to the zero mode.

Also, the determinant from the Faddeev-Popov procedure is given by

$$F'[\phi^{x_0}] = \frac{d}{dx_0} \int dx\, \eta_0(x) \left(\phi_c(x + x_0) + \frac{1}{\sqrt{l}} \eta(x + x_0) \right)$$

$$= -\int dx\, \frac{d\eta_0}{dx} \left(\phi_c(x + x_0) + \frac{1}{\sqrt{l}} \eta_0(x)\eta(x + x_0) \right)$$

$$\Rightarrow F'[\phi_c^m + \frac{1}{\sqrt{l}}\eta] = -\int dx\, \frac{d\eta_0}{dx} \left(\phi_c^m(x) + \frac{1}{\sqrt{l}} \eta_0(x)\eta(x) \right)$$

$$= \int dx\, \eta_0 \left(\frac{d\phi_c^m}{dx} + \frac{1}{\sqrt{l}} \frac{d\eta}{dx} \right) = \int dx \left(\eta_0^2 + \frac{1}{\sqrt{l}} \eta_0 \frac{d\eta}{dx} \right) \tag{4.36}$$

Using (4.27), (4.35) and (4.36) in (4.34), making the approximation that $F'[\phi_c^m + \frac{1}{\sqrt{l}}\eta] \approx F'[\phi_c^m] = \int dx \eta_0^2$ and then integrating over the zero mode, we obtain

$$\langle \widehat{O} \rangle \approx Z^{-1} \sum_m \int dx_0 \int [d\eta]' \exp\left(-lL(\Delta E)^2\right) |J_c| \left(\delta(x_0)\delta_{m,0} + (1 - \delta_{m,0})F'[\phi_c^m] \right)$$

$$\times \widehat{O}[\phi_c^m(x - x_0) + \frac{1}{\sqrt{l}}\eta(x - x_0), E] \exp\left(-l\int_{-L/2}^{L/2} dx\, \eta\Delta_m^{-1}\eta\right), \quad \forall E \geq 0 \tag{4.37}$$

where the accent denotes that the zero mode is excluded when calculating the functional integral. Note that $F'[\phi_c^m]$ is a divergent constant (the zero mode is not normalisable), that may depend on m. However, requiring that the disorder average gives the correct result in the pure limit, we find that $F'[\phi_c^m]$ is a constant independent of m, which should be regularised in such a way that it coincides with the normalisation of the $m = 0$ sector. Thus, it is a global factor, weighing all sectors equally, and can be absorbed in the normalisation.

In the negative energy region, in addition to the exact zero mode which exists due to the translational invariance of the system, the saddle point solution (4.29) allows another quasi-symmetry to exist [Gil77]. This quasi-symmetry is due to the fact that for large system sizes, local translations are possible that only changes the action by terms of order $\exp(-cL)$. To be specific, for large separations in (4.29), a translation in $\bar{x}_0$ has an exponentially small effect on the action, so that to leading order in $\frac{1}{L}$ it is a symmetry of the action, which in the $L \to \infty$ limit becomes an exact symmetry. Associated with this approximate symmetry there is an additional zero mode which also needs be considered when calculating the functional integral.

To circumvent the problems associated with the zero modes, we once more make the translational symmetries explicit in the form of collective coordinates [Gil77, Ger75b, Ger75a] which we introduce by using the Faddeev-Popov method discussed earlier. However, since there are two collective coordinates which we wish to introduce simultaneously, we need to modify the identity in (3.21) so that [Gil77]

$$1 = c \int_{-L/2}^{L/2} dx_0 d\bar{x}_0 \Delta[\phi] \delta(F[\phi^{x_0,\bar{x}_0}]) \delta(\bar{F}[\phi^{x_0,\bar{x}_0}]) \tag{4.38a}$$

where the determinant is given by

$$\Delta[\phi] = \begin{vmatrix} \frac{\partial F}{\partial x_0} & \frac{\partial \bar{F}}{\partial x_0} \\ \frac{\partial F}{\partial \bar{x}_0} & \frac{\partial \bar{F}}{\partial \bar{x}_0} \end{vmatrix}. \tag{4.38b}$$

In order to project out the zero modes, we choose

$$F[\phi^{x_0, \bar{x}_0}] = \int dx \eta_0(x, \bar{x}_0)\phi(x + x_0) \tag{4.39a}$$

$$\bar{F}[\phi^{x_0, \bar{x}_0}] = \int dx \bar{\eta}_0(x, \bar{x}_0)\phi(x + x_0) \tag{4.39b}$$

where the zero mode η_0 and quasi zero mode $\bar{\eta}_0$ are given by

$$\eta_0(x, \bar{x}_0) = \frac{\partial \phi_c^{\pm}}{\partial x}, \qquad \bar{\eta}_0(x, \bar{x}_0) = \frac{\partial \phi_c^{\pm}}{\partial \bar{x}_0}. \tag{4.39c}$$

Using (4.38) and (4.39) in (4.3), changing the variables $\phi(x + x_0) \to \phi(x)$ in the functional integral and then writing $\phi = \phi_c^{\pm} + \frac{1}{\sqrt{l}}\eta$, we have

$$\langle \widehat{O} \rangle = Z^{-1} \int dx_0 d\bar{x}_0 \int [d\eta] \, \widehat{O}[\phi_c^{\pm}(x - x_0) + \frac{1}{\sqrt{l}}\eta(x - x_0), E] \exp\left(-S[\phi_c^{\pm}, E]\right) |J_c| \left| \int dx \frac{\partial \phi_c^{\pm}}{\partial \bar{x}_0} \right|^{-1}$$

$$\times \Delta[\phi_c^{\pm} + \frac{1}{\sqrt{l}}\eta] \delta(\bar{x}_0 - \frac{L}{4}) \delta(F[\phi_c^{\pm} + \frac{1}{\sqrt{l}}\eta]) \delta(\bar{F}[\phi_c^{\pm} + \frac{1}{\sqrt{l}}\eta]) \exp\left[-\int_{-L/2}^{L/2} dx' \eta(x') \Delta_{\pm}^{-1} \eta(x') \right], \tag{4.40}$$

where the propagator is given by (4.14b) using the saddle point solution $\phi_c^{\pm}$. Also, we have handled the constraint $\delta(\int dx(\phi_c^{\pm} + \eta))$ by restricting the η integration to be over non-constant modes, leaving the constraint $\delta(\int dx \phi_c^{\pm})$, which can be written as $\delta(\int dx \phi_c^{\pm}) = \delta(\bar{x}_0 - \frac{L}{4})/|\int dx \frac{\partial \phi_c^{\pm}}{\partial \bar{x}_0}|$. Note that the constraint on $\bar{x}_0$ forces the instanton and anti-instanton pair of (4.29) to be separated by $\frac{L}{2}$, which for large L justifies the use of the dilute gas approximation (4.30) with $\bar{x}_0 = \frac{L}{4}$.

Using (4.39) and the dilute gas approximation (4.30), we can calculate the elements of the matrix used to obtain the determinant (4.38b) in the approximation where the fluctuations are neglected. Thus

$$\frac{\partial F}{\partial x_0} = \int dx \, \eta_0 \frac{\partial \phi_c^{\pm}}{\partial x_0} = \int_{-L/2}^{0} dx \, \eta_{0-}^2 + \int_{0}^{L/2} dx \, \eta_{0+}^2 \propto |E|^{3/4}, \tag{4.41a}$$

where $\eta_{0-} = |E|\text{sech}^2\left(\sqrt{|E|}(x - \frac{L}{4})\right)$ and $\eta_{0+} = |E|\text{sech}^2\left(\sqrt{|E|}(x + \frac{L}{4})\right)$. Further,

$$\frac{\partial \bar{F}}{\partial x_0} = \int dx \bar{\eta}_0 \frac{\partial \phi_c^{\pm}}{\partial x_0} = -\int_{-L/2}^{0} dx \, \eta_{0-}^2 + \int_{0}^{L/2} dx \, \eta_{0+}^2 = 0 \tag{4.41b}$$

and

$$\frac{\partial \bar{F}}{\partial \bar{x}_0} = \int dx \frac{\partial \bar{\eta}_0}{\partial \bar{x}_0} \phi_c^{\pm} = -\int_{-L/2}^{0} dx\, \eta_{0-} \phi_{c-}^2 - \int_0^{L/2} dx\, \eta_{0+} \phi_{c+}^2 \propto |E|^{3/4}, \tag{4.41c}$$

where $\phi_{c-} = -\sqrt{|E|}\tanh\sqrt{|E|}(x - \frac{L}{4})$ and $\phi_{c+} = \sqrt{|E|}\tanh\sqrt{|E|}(x + \frac{L}{4})$. Using (4.41) in (4.38b) we find that $\Delta[\phi_c^{\pm}] \propto |E|^{3/2}$. Also, we have

$$\int_{-L/2}^{L/2} dx \frac{\partial \phi_c^{\pm}}{\partial \bar{x}_0} = -\int_0^{L/2} dx\, \eta_{0-} - \int_{-L/2}^{0} dx\, \eta_{0+} \propto \sqrt{|E|}, \tag{4.42}$$

while from (4.31) we have $S[\phi_c^{\pm}, E] = \frac{16}{3}|E|^{3/2}$. Thus, after integrating over $\bar{x}_0$ in equation (4.40), and using the dilute gas results [Gil77] above, we obtain the saddle-point approximation for the negative energies

$$\langle \widehat{O} \rangle \approx Z^{-1}|E|\exp\left(-\frac{16}{3}l|E|^{3/2}\right)|J_c|$$
$$\times \int [d\eta]' dx_0 \, \widehat{O}[\phi_c^{\pm}(x - x_0) + \frac{1}{\sqrt{l}}\eta(x - x_0), E]\exp\left(-\int_{-L/2}^{L/2} dx\, \eta\Delta_{\pm}^{-1}\eta\right) \tag{4.43}$$

where the notation $[d\eta]'$ denotes that the zero modes are excluded in the functional integral.

4.2.4 Propagators and determinants

To calculate the functional integrals in (4.37) and (4.43), we need to be able to calculate the propagator, Δ, and the determinant of its inverse, $\det(\Delta^{-1})$. This involves solving the eigenvalue equation

$$\left(-\frac{d^2}{dx^2} + 2E + 6\phi_c^2(x)\right)\Psi_n = \lambda_n \Psi_n \tag{4.44}$$

with periodic boundary conditions over the interval $[-\frac{L}{2}, \frac{L}{2}]$.

In the positive energy region ϕ_c is given by (4.25), so that (4.44) is

$$\left(-\frac{d^2}{dx^2} + 2\left(\frac{m\pi}{L}\right)^2 + 6\left[\frac{m\pi}{L}\tan\left(\frac{m\pi}{L}x\right)\right]^2\right)\Psi_n = \left[\lambda_n + 2\left(\frac{m\pi}{L}\right)^2 - 2E\right]\Psi_n \tag{4.45}$$

Let us first consider the case $m \neq 0$. The differential equation in (4.45) can be solved exactly using the method of generalised ladder operators [Sch92, Jaf97] so that

$$\lambda_n = (n^2 + 6n + 3)\left(\frac{m\pi}{L}\right)^2 + 2E \quad \forall\, n \geq 0 \tag{4.46a}$$

and

$$\Psi_0 \propto \left(\cos\left[\frac{m\pi}{L}x\right]\right)^3$$
$$\Psi_n \propto \prod_{i=0}^{n-1}\left(-\frac{d}{dx} + (i+3)\frac{m\pi}{L}\tan\left(\frac{m\pi}{L}x\right)\right)\left(\cos\left[\frac{m\pi}{L}x\right]\right)^{n+3} \quad \forall\, n > 0. \tag{4.46b}$$

Note that these functions have the properties we would expect. They vanish at the singularities

of the classical solution and they are periodic on the interval $\frac{L}{m}$. To perform the functional integral the fluctuations have to be expanded in terms of these functions in each subinterval of length $\frac{L}{m}$. The fluctuations in each subinterval is of course completely independent so that the functional integral becomes an integral over the independent expansion coefficients in each subinterval. Thus the functional integral will factorise into a product of m functional integrals, each defined over fluctuations in the interval $[-\frac{L}{2m}, \frac{L}{2m}]$ with vanishing boundary conditions. This again reflects the remarks made earlier on the topological implications of the singularities in the classical solution. They lead to a slicing of the interval $[-\frac{L}{2}, \frac{L}{2}]$ into m subintervals, each containing independent fluctuations.

Unfortunately, although we have the exact solution, we do not have the eigenfunctions in a closed form. This makes the calculation of the propagator difficult. We thus wish to make an approximation to (4.45) that allows us to calculate the propagator. The dominant property of the classical solution (4.25) is that it contains singularities which appear periodically with period $\frac{L}{m}$. As mentioned several times before the eigenfunctions of (4.45) must be zero at the points where these singularities occur, as is also seen in (4.46b). Additionally, the periodicity of the classical solution leads to Bloch characteristics of the eigenfunctions, so that the eigenfunctions are also periodic over the interval $\frac{L}{m}$, which implies that we only need to solve (4.45) over the interval $[-\frac{L}{2m}, \frac{L}{2m}]$.

We wish to make an approximation for the propagator that captures the essential characteristics of the original eigenfunctions. As a first approximation, we treat the $\tan^2$ potential in (4.45) to lowest order in perturbation theory, where we capture the singularities of the potential by imposing vanishing boundary conditions at $\pm \frac{L}{2m}$. We thus need to solve the eigenvalue equation

$$(-\frac{d^2}{dx^2} + 2E)\Psi_n = \lambda_n \Psi_n \tag{4.47}$$

with the boundary condition that $\Psi(\pm \frac{L}{2m}) = 0$.

Excluding the constant mode as required by (4.37), the solution of (4.47) is

$$\lambda_n = \left(\frac{nm\pi}{L}\right)^2 + 2E \quad \forall\, n > 0 \tag{4.48a}$$

with

$$\Psi_n = \sqrt{\frac{2m}{L}} \cos\left(\frac{nm\pi}{L}x\right) \quad \forall \text{ odd } n > 0$$
$$= \sqrt{\frac{2m}{L}} \sin\left(\frac{nm\pi}{L}x\right) \quad \forall \text{ even } n > 0. \tag{4.48b}$$

As expected (4.48a) and (4.46a) are in good agreement for large n.

The propagator on the space of functions defined on the interval $[-\frac{L}{2m}, \frac{L}{2m}]$ with vanishing

boundary conditions is now obtained in the energy representation as

$$
\Delta_m(x, y) \equiv \left(-\frac{d^2}{dx^2} + 2E \right)^{-1}
$$

$$
= \frac{2m}{L} \sum_{n=1}^{\infty} \left(\frac{\cos([2n-1]\frac{m\pi}{L}x)\cos([2n-1]\frac{m\pi}{L}y)}{([2n-1]\frac{m\pi}{L})^2 + 2E} + \frac{\sin(2n\frac{m\pi}{L}x)\sin(2n\frac{m\pi}{L}y)}{(2n\frac{m\pi}{L})^2 + 2E} \right) \quad (4.49)
$$

while the determinant can be calculated with various techniques[10], i.e. the identity 1.143.1 found in Gradshteyn and Ryzhik[Gra94], so that

$$
\left| \det(\Delta_m^{-1}) \right| = C \frac{|m|}{\sqrt{2EL}} \sinh\left(\frac{\sqrt{2E}L}{|m|} \right), \quad (4.50)
$$

where C is an energy independent constant, but not necessarily independent of m. We determine C by requiring that the density of states (see later) be the correct solution in the pure limit. Obtaining the pure solution (up to a global normalisation constant), requires that (4.50) must be a constant independent of m. However, since $E = \left(\frac{m\pi}{L} \right)^2$ in this limit, and C is independent of E, we find that C must also be independent of m. Thus we see that C is a constant that is independent of E and m.

Finally we summarise the results for the $m = 0$ sector, for which the solution of the eigenvalue equation (4.32) is trivial:

$$
\lambda_n = \left(\frac{2n\pi}{L} \right)^2 + 2E; \quad n \neq 0 \quad (4.51a)
$$

$$
\psi_n = \frac{1}{\sqrt{L}} \exp\left(\frac{2\pi i n x}{L} \right) \quad (4.51b)
$$

$$
\left| \det \Delta_0^{-1} \right| = \frac{C}{L} \sqrt{\frac{2}{E}} \sinh\left(L\sqrt{\frac{E}{2}} \right) \quad (4.51c)
$$

$$
\Delta_0(x, y) = \frac{2}{L} \sum_{n=1}^{\infty} \left(\frac{\cos(\frac{2n\pi}{L}(x-y))}{(\frac{2n\pi}{L})^2 + 2|E|} \right). \quad (4.51d)
$$

Let us finally return to the question posed earlier, namely, how will the linear term left over in the action effect us, given that our saddle point solutions are approximate. The linear term can easily be calculated to be

$$
\text{Linear term} = \int_{-L/2}^{L/2} dx\, \phi_c^m \eta \quad (4.52)
$$

As remarked before the quadratic term in the fluctuations forces the fluctuation η to vanish at the singularities of ϕ_c^m, thus we must expand the fluctuations in each interval in terms of the eigenfunctions of the fluctuation operator (4.48b). We note, however, from (4.45) that changing the value of the energy E that appears in (4.45) will only shift the eigenvalues and will not effect the eigenfunctions at all. Thus we can just as well obtain these eigenfunctions by replacing the energy appearing in (4.44) and (4.45) by $(\frac{m\pi}{L})^2$. However, since the classical

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[10]See Appendix B for various methods in calculating the determinant

solution satisfies the classical equation of motion (4.26) exactly at these values of the energy, we find after multiplying (4.48b) with ϕ_c^m, integrating over x and performing a partial integration that $\lambda_n \int_{-L/2}^{L/2} dx\, \phi_c^m \Psi_n = 0$. Since all these eigenvalues are non-vanishing, it implies that all these eigenfunctions are orthogonal to ϕ_c^m and thus that the linear term must vanish.

In the negative energy region, ϕ_c is given by (4.30) where we are using the dilute gas approximation. Thus the equation (4.44) becomes

$$\left(-\frac{d^2}{dx^2} + 4|E| + \bar{\phi}_0\right)\Psi_n = \lambda_n \Psi_n, \tag{4.53a}$$

where the quasi zero mode $\bar{\phi}_0$ is given by

$$\bar{\phi}_0 = \operatorname{sech}^2\left(\sqrt{|E|}(x - \frac{L}{4})\right)\Theta(x) + \operatorname{sech}^2\left(\sqrt{|E|}(x + \frac{L}{4})\right)\Theta(-x). \tag{4.53b}$$

If we integrate over (4.53a), and use the condition that $\bar{\phi}_0$ is orthogonal to Ψ_n, we find that the constraint that the eigenfunction cannot contain a zero mode is satisfied for all eigenfunctions except the one corresponding to $\lambda_n = 4|E|$. The eigenfunction with this eigenvalue must be explicitly checked to see if the constraint is satisfied.

We first calculate the eigenfunctions and eigenvalues of (4.53) in the subinterval $[-\frac{L}{2}, 0]$ or $[0, \frac{L}{2}]$ using the method of generalised operators [Sch92, Jaf97] or via the solution of a hypergeometric equation [Mor53], and find that the eigenvalues consist of two discrete eigenvalues, which are $\lambda_0 = 0$ and $\lambda_1 = 3|E|$, and a continuum of eigenvalues $\lambda_k = (k^2 + 4)|E|$. The corresponding eigenfunctions are

$$\Psi_0^\pm \propto \operatorname{sech}^2\left(\sqrt{|E|}(x \mp \frac{L}{4})\right)\Theta(\pm x), \qquad \Psi_1^\pm \propto \frac{\tanh\left(\sqrt{|E|}(x \mp \frac{L}{4})\right)}{\cosh\left(\sqrt{|E|}(x \mp \frac{L}{4})\right)}\Theta(\pm x) \tag{4.54a}$$

and

$$\Psi_k^\pm \propto \left[3\tanh^2\left(\sqrt{|E|}(x \mp \frac{L}{4})\right) - 1 - k^2\right]e^{i\sqrt{|E|}kx}\Theta(\pm x)$$

$$- 3ik\tanh\left(\sqrt{|E|}(x \mp \frac{L}{4})\right)e^{i\sqrt{|E|}kx}\Theta(\pm x). \tag{4.54b}$$

We can now calculate the eigenfunctions for the full region $[-\frac{L}{2}, \frac{L}{2}]$ by matching the eigenfunctions in (4.54) at $x = 0$, i.e. requiring that $\Psi^+(0) = \Psi^-(0)$ and $\frac{d\Psi^+}{dx}(0) = \frac{d\Psi^-}{dx}(0)$. The

eigenfunctions are

$$\Psi_0 \propto \pm \mathrm{sech}^2\left(\sqrt{|E|}(x+\frac{L}{4})\right)\Theta(-x) + \mathrm{sech}^2\left(\sqrt{|E|}(x-\frac{L}{4})\right)\Theta(x) \tag{4.55a}$$

$$\Psi_1 \propto \pm \frac{\tanh\left(\sqrt{|E|}(x+\frac{L}{4})\right)}{\cosh\left(\sqrt{|E|}(x+\frac{L}{4})\right)}\Theta(-x) + \frac{\tanh\left(\sqrt{|E|}(x-\frac{L}{4})\right)}{\cosh\left(\sqrt{|E|}(x-\frac{L}{4})\right)}\Theta(x) \tag{4.55b}$$

$$\Psi_k \propto \pm(2-k^2)\mathrm{e}^{\mathrm{i}\sqrt{|E|}k(x+\frac{L}{2})}\Theta(-x) \pm \frac{3\Theta(-x)}{\sqrt{|E|}}\frac{\partial}{\partial x}\left[\tanh\left(\sqrt{|E|}(x+\frac{L}{4})\right)\mathrm{e}^{\mathrm{i}\sqrt{|E|}k(x+\frac{L}{2})}\right]$$

$$+ (2-k^2)\mathrm{e}^{\mathrm{i}\sqrt{|E|}k(x-\frac{L}{2})}\Theta(x) + \frac{3\Theta(x)}{\sqrt{|E|}}\frac{\partial}{\partial x}\left[\tanh\left(\sqrt{|E|}(x-\frac{L}{4})\right)\mathrm{e}^{\mathrm{i}\sqrt{|E|}k(x-\frac{L}{2})}\right] \tag{4.55c}$$

where $k \neq 0$ since the corresponding eigenfunction does not satisfy the constraint that there are no constant modes. Also, the periodicity requirements on the eigenfunctions imply that k must satisfy the equation

$$\mathrm{e}^{\mathrm{i}\sqrt{|E|}kL} = \left[\frac{2-k^2+3\mathrm{i}k}{2-k^2-3\mathrm{i}k}\right]^2. \tag{4.56}$$

Note that there are degenerate solutions for each eigenvalue, since the eigenfunctions can be constructed as a symmetric or an anti-symmetric solution.

As in the positive energy region, we do not have the eigenvalues and subsequently also not the eigenfunctions in a closed form, which makes calculation of the propagator and determinant difficult. We thus once again wish to make an approximation that will enable us to calculate the propagator and determinant. We note that the right hand side of (4.56) is approximately unity, allowing us to obtain

$$k = \frac{2n\pi}{L\sqrt{|E|}}, \qquad \forall n \in \mathcal{Z}, n \neq 0 \tag{4.57}$$

for large values of k. The eigenfunctions Ψ_k, can then be approximated by a plane wave, and is given by

$$\Psi_k \propto \pm \mathrm{e}^{\mathrm{i}\sqrt{|E|}k(x+\frac{L}{2})}\Theta(-x) + \mathrm{e}^{\mathrm{i}\sqrt{|E|}k(x-\frac{L}{2})}\Theta(x) \tag{4.58}$$

with corresponding eigenvalue $\lambda_n = \left(\frac{2n\pi}{L}\right)^2 + 4|E|$. This implies, as is to be expected, that all the higher lying scattering states can be very well approximated by free particle states and that this only breaks down for the lowest lying scattering states, where the potential is important. Note, however, that even the spectrum of the lower lying scattering states is well approximated by a free particle spectrum as the right hand side of (4.56) is approximately unity also in this case. The eigenfunctions are, however, distorted away from plane waves due to the presence of the potential.

It is now possible to calculate the determinant using the above approximation so that

$$|\det(\Delta_{\pm}^{-1})| = (3|E|)^2\left[\prod_{n=1}^{\infty}\left(\left(\frac{2n\pi}{L}\right)^2 + 4|E|\right)\right]^4 \propto \sinh^4\left(\sqrt{E}L\right), \tag{4.59}$$

where we have once again used the identity 1.143.1 from Gradshteyn and Ryzhik [Gra94]. Note that the quadratic term in (4.59) is due to the doubly degenerate bound state (from the symmetric and anti-symmetric eigenfunctions), while the product over the continuum eigenvalues is raised to the fourth power since there is a fourfold degeneracy in the continuum eigenfunctions (from the symmetric and anti-symmetric states, as well as the right moving and left moving plane waves).

In calculating the propagator, we neglect the contribution of the bound state Ψ_1, which gives only a small contribution to the propagator as it involves the product of two very well localised eigenfunctions evaluated a points which are well separated. Taking into account only the scattering states, (4.58), we find for the propagator

$$\Delta_\pm(x,y) = \frac{2}{L} \sum_{n=1}^\infty \left(\frac{\cos(\frac{2n\pi}{L}(x-y))}{(\frac{2n\pi}{L})^2 + 4|E|} \right). \tag{4.60}$$

Using these approximate results for the propagator and determinant for positive energies, (4.49) and (4.50), and negative energies, (4.60) and (4.59), with the respective formulations for disordered averages, (4.37) and (4.43), now allows us to calculate the average value of observables to leading order in L.

4.2.5 Validity of one loop approximation

As mentioned earlier, this approximation is valid when l is large in a sense determined by the other two scales, namely E and L. To find the precise criterion one has to estimate the higher order loop corrections [Ami84] to (4.37) and (4.43) using (4.17). Let us do this in the large L limit, which is the one of interest to us. Consider the generic form (4.14a). We note that in the large L limit the only configurations that can contribute to the functional integral are those yielding a finite action. This implies that all terms appearing in (4.14a) must be independent of L in the large L limit. Stated differently, in the $L \to \infty$ limit only two scales remain, l and E and all terms depend only on these. Since l is a global prefactor (scale factor) for the action, the integrals appearing in (4.14a) can in fact only depend on the scale E. Using this we can now easily determine the scale (energy) dependence of each term in (4.14a) from dimensional considerations. For example, the classical action must be of the form $cl|E|^{3/2}$, with c a pure number, which can only be determined if we know the true saddle point configuration. The quadratic term is already dimensionless so that it has no scale dependence on E, the cubic term will scale like $1/|E|^{3/4}$ and the quartic term like $1/|E|^{3/2}$. In this way we identify the dimensionless coupling constant $1/(l|E|^{3/2})$, which should be small for our approximation to be valid. Thus we conclude $l|E|^{3/2} \gg 1$.

4.2.6 Density of states

We now apply the saddle-point approximation discussed above to the disorder averaged density of states given by (3.25). For positive energies, this amounts to computing (4.37) with the observable $\hat{O} = 1$. Upon integrating over the functional integral to obtain the factor $\det(\Delta_m^{-1})^{-1/2}$,

we have

$$\langle \rho(E) \rangle = N_+ \sum_m \exp\left(-lL \left[E - \left(\frac{m\pi}{L} \right)^2 \right]^2 \right) |J_c| \left[\det \Delta_m^{-1} \right]^{-1/2}, \tag{4.61}$$

where N_+ is an unknown normalisation factor, independent of the energy, that needs to be fixed in some manner.

The explicit form of the determinant was given in (4.50), but we actually do not need it. We note the appearance of the ratio of determinants

$$\frac{|J_c|}{\sqrt{\det \Delta^{-1}}} = \frac{\det\left(\frac{d}{dx} + 2\phi_c \right)}{\left[\det\left(-\frac{d^2}{dx^2} + 2E + 6\phi_c^2 \right) \right]^{1/2}} \tag{4.62}$$

where the zero modes are excluded in both determinants. Now we note from the classical solution (4.25) that the determinant in the denominator can be factorised as

$$\det\left(-\frac{d^2}{dx^2} + 2E + 6\phi_c^2 \right) = \det\left[\left(-\frac{d}{dx} + 2\phi_c \right) \left(\frac{d}{dx} + 2\phi_c \right) + 2\Delta E \right]$$

$$\approx \det\left[\left(-\frac{d}{dx} + 2\phi_c \right) \left(\frac{d}{dx} + 2\phi_c \right) \right] \tag{4.63}$$

Now we note that the square root of this determinant is precisely the numerator and that they cancel. The best way of seeing this is to write:

$$\left[\det\left(-\frac{d}{dx} + 2\phi_c \right) \left(\frac{d}{dx} + 2\phi_c \right) \right]^{-1/2} = \int [d\eta] \exp\left[\eta \left(-\frac{d}{dx} + 2\phi_c \right) \left(\frac{d}{dx} + 2\phi_c \right) \eta \right]$$

$$= \left| \det\left(\frac{d}{dx} + 2\phi_c \right) \right|^{-1} \tag{4.64}$$

where we have made the change of variables $\tilde{\eta} = \left(\frac{d}{dx} + 2\phi_c \right) \eta$ to obtain the final result. The absolute value of the square root of the determinant then appears through the Jacobian of this change of variables.

Thus we obtain the final result for the density of states in the positive energy region

$$\langle \rho(E) \rangle = N_+ \sum_m \exp\left(-lL \left[E - \left(\frac{m\pi}{L} \right)^2 \right]^2 \right). \tag{4.65}$$

For negative energies, the saddle-point approximation of (3.25) leads to (4.43) with $\hat{O} = 1$. After integrating over the resulting functional integral, we obtain

$$\langle \rho(E) \rangle = N_- |E| \exp\left[-\frac{16}{3} l |E|^{3/2} \right], \tag{4.66}$$

where N_- is another energy independent normalisation factor. Here we have again cancelled the determinant $|J_c|$ and $\left(\det \Delta_{\pm}^{-1} \right)^{-1/2}$. The argument is as above, the only difference is that we must use the saddle point solution in the dilute gas approximation (4.30) in which case the Δ^{-1}

can again be factorised as above, indeed

$$
-\frac{d^2}{dx^2} + 2E + 6\phi_c^2
$$

$$
= -\frac{d^2}{dx^2} + 2E + 6\,|E|\left(\tanh^2(\sqrt{|E|}(x-x_0))\theta(x) + \tanh^2(\sqrt{|E|}(x+x_0))\theta(-x)\right)
$$

$$
= \left(-\frac{d}{dx} + 2\sqrt{|E|}\left(\tanh(\sqrt{|E|}(x-x_0))\theta(x) + \tanh(\sqrt{|E|}(x+x_0))\theta(-x)\right)\right)
$$

$$
\times \left(\frac{d}{dx} + 2\sqrt{|E|}\left(\tanh(\sqrt{|E|}(x-x_0))\theta(x) + \tanh(\sqrt{|E|}(x+x_0))\theta(-x)\right)\right) \tag{4.67}
$$

where we have used the fact that the instantons are well separated to cancel the terms arising from the derivatives of the step functions.

The result obtained in (4.66) agrees with the asymptotic negative energy expansion of the exact result obtained by Halperin [Hal65] through different means.

4.2.7 2-point correlators

If we consider the disorder average of the 2-point correlator given by (3.28), we note the exponential terms can be written as

$$
\exp\left[P\int_z^x dx'\phi(x') + P\int_z^y dx'\phi(x')\right]
$$

$$
= \exp\left[P\int_z^x dx'\phi_c(x') + P\int_z^y dx'\phi_c(x')\right]\exp\left[\int_z^x dx'\eta(x') + \int_z^y dx'\eta(x')\right], \tag{4.68a}
$$

where we have relabeled the x_0 integration in (3.28) to z. The linear terms of η in the exponential can be written as an integral over the interval $[-\frac{L}{2}, \frac{L}{2}]$ by using a combination of step functions,

$$
\int_z^x dx'\eta(x') + \int_z^y dx'\eta(x') = \int_{-L/2}^{L/2} dx'\,S(x,y,z|x')\eta(x'), \tag{4.68b}
$$

where $S(x,y,z|x')$ can be considered as a source term for the η fields and is given by

$$
S(x,y,z|x') = \theta(x-x') + \theta(y-x') - 2\theta(z-x'). \tag{4.68c}
$$

Furthermore, a periodic continuation is understood outside $[-\frac{L}{2}, \frac{L}{2}]$.

Applying the positive energy saddle point approximation (4.37) to (3.28), and making the change of variables $z \to z + x_0$, we have

$$
\langle |\psi_E(x)|\,|\psi_E(y)|\rangle
$$

$$
= Z^{-1}\sum_m \int dx_0 dz(\delta(x_0)\delta_{m,0} + (1-\delta_{m,0}))\frac{\partial}{\partial z}\int [d\eta]'\exp\left(-lL(\Delta E)^2\right)|J_c|\exp\left(-\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\,\eta\Delta_m^{-1}\eta\right)
$$

$$
\times \exp\left[2P\int_z^0 dx'\phi(x') + P\int_{x_0}^x dx'[\phi_c^m(x'-x_0) + \frac{1}{\sqrt{l}}\eta] + P\int_{x_0}^y dx'[\phi_c^m(x'-x_0) + \frac{1}{\sqrt{l}}\eta]\right]. \tag{4.69}
$$

Before proceeding it is important to make a few general remarks regarding (4.69). We note that in the pure limit $l \to \infty$, the fluctuations vanish and the correlator becomes proportional

to the "pure correlator" which is just $\int_{-L/2}^{L/2} dx_0 \, |\psi(x - x_0)| \, |\psi(y - x_0)|$, i.e.,the product of the absolute value of the wave-function averaged over all positions. This correlator contains, of course, no information on the disorder as this information is contained in the fluctuations around the classical configurations. In this sense the fluctuations describes the "envelope" of the wave-function, while the classical configuration describe the oscillatory part of the wave-function, which occurs on shorter length scales [Mir00]. To extract information of the envelope, or correlations of the fluctuations, one should then, as always, subtract the uncorrelated part, which is the pure correlator in this case. In the pure limit this quantity vanishes, as it should, since there is no envelope in this case.

Keeping the above in mind, let us consider (4.69) carefully in the light of the remarks made earlier about the factorisation of the functional integral due to the singularities in the classical configuration. Recall that the presence of these singularities implies that the functional integral should be interpreted as an integral over independent fluctuations in m subintervals ($m \neq 0$), leading to the factorisation of the functional integral into a product of m independent integrals each defined over fluctuations on an interval of length $\frac{L}{m}$. What this means in terms of propagation is that in the mth sector propagation cannot occur on length scales larger then $\frac{L}{m}$. This immediately implies that the part of the correlator (4.69) that involves the fluctuations will factorise into a completely uncorrelated product for all sectors for which $\frac{L}{m} \leq |x - y|$. Translational invariance then implies that this product will be independent of x and y, so that in these sectors the correlator is just proportional to the "pure correlation" function, which contains no information on the disorder and thus the envelope.

To see how this comes about explicitly from (4.69) one first has to realize (1) that the integral of the fluctuations over one period of length $\frac{L}{m}$ vanishes and (2) that the singularities occur at $x = (2n + 1)\frac{L}{2m} + x_0$. From this one then sees that the only insertions that need to concern us are (assuming $x > y$)

$$\exp(2 \int_{x_0 + n_y L/m}^{y} dx' \eta(x') + \int_{x_0 + n_x L/m}^{x} dx' \eta(x'))$$

where n_x and n_y label the intervals in which x and y lie and are simply determined from the expression for the positions of the singularities. Now, if $|x - y| > \frac{L}{m}$, these two insertions lie in two completely uncorrelated intervals and the functional integral factorises into their products, each evaluated in its own subinterval. To see the translational invariance in each subinterval and thus that the result is independent of x and y one has to take into account the insertion $\int_z^0 dx' \eta(x')$. By shifting the integration variable so that the upper bound becomes $x_0 + n_y L/m$ (the shift in the lower bound is absorbed by the z-integral) this can be combined with the y-dependent insertion to yield an insertion with lower bound z. The end result is therefore two factorised insertions, each containing an integral over the lower bound which makes each translationally invariant and therefore independent of x and y. A more direct way of obtaining this result is to make the change of variables $\phi = \varphi'$ in the functional integral (3.30f). Then the insertions become insertions at single point and the result follows immediately.

We have now established that all sectors with $m \geq L/|x - y|$ contribute to the pure or uncorrelated part of the correlator (4.69) and should therefore be subtracted. This leads us to

conclude that we must restrict the sum in (4.69) to those sectors only which satisfy the criterion $m < L/|x - y|$.

Finally we note that as a result of the gauge that we chose to cancel the normalisation of the wave-functions giving rise to (3.28), equation (4.69) contains a total derivative with respect to z, which gives a result of zero when completing the integration over z. However, since we have a ratio of total derivatives, we should obtain a non-zero result if we use a consistent regularisation method. With this in mind, we integrate over z in the $m \neq 0$ sectors, then the fluctuations and finally complete the integration over z in the $m = 0$ sector, allowing us to cancel the result from the z integrations with a similar term in the denominator of (4.69). We thus obtain

$$\frac{\langle |\psi_E(x)| \, |\psi_E(y)| \rangle}{\langle |\psi_E(0)| \, |\psi_E(0)| \rangle} \approx \frac{C(x,y)}{C(0)} \tag{4.70}$$

where

$$
C(x,y) = \sum_{m=0}^{N(x-y)} \int dx_0 \left|\cos\left(\frac{m\pi}{L}(x - x_0)\right)\right| \left|\cos\left(\frac{m\pi}{L}(y - x_0)\right)\right| \exp\left(-lL(\Delta E)^2\right) |J_c|
$$
$$
\times \left(\det(\Delta_m^{-1})\right)^{-1/2} \left(\delta(x_0)\delta_{m,0} \exp\left[\frac{1}{4l}F_0(x,y)\right]\right.
$$
$$
\left. + (1 - \delta_{m,0}) \exp\left[\frac{1}{4l}\int dx' dx'' \, S(x,y,x_0|x')\Delta_m(x',x'')S(x,y,x_0|x'')\right]\right) \tag{4.71}
$$

and $N(x - y)$ is the largest even integer smaller than $L/|x - y|$. After using (4.49) for the propagator and (4.50) for the determinant, we have

$$
C(x,y) = \sum_{m=0}^{N(x-y)} \int dx_0 \left|\cos\left(\frac{m\pi}{L}(x - x_0)\right)\right| \left|\cos\left(\frac{m\pi}{L}(y - x_0)\right)\right| \exp\left(-lL(\Delta E)^2\right)
$$
$$
\times \left(\delta(x_0)\delta_{m,0} \exp\left[\frac{1}{4l}F_0(x,y)\right] + (1 - \delta_{m,0}) \exp\left[\frac{1}{4l}F_m(x,y,x_0)\right]\right), \tag{4.72}
$$

where we have again cancelled the determinants and

$$
F_m(x,y,x_0) = \frac{2m}{L} \sum_{n=1}^{\infty} \frac{1}{D_{2n-1}} \left[\cos\left((2n-1)\frac{m\pi}{L}(x-x_0)\right) + \cos\left((2n-1)\frac{m\pi}{L}(y-x_0)\right) - 2\right]^2
$$
$$
+ \frac{2m}{L} \sum_{n=1}^{\infty} \frac{1}{D_{2n}} \left[\sin\left(2n\frac{m\pi}{L}(x-x_0)\right) + \sin\left(2n\frac{m\pi}{L}(y-x_0)\right)\right]^2, \quad m \neq 0 \tag{4.73a}
$$

with

$$
D_n = \left[\left(\frac{nm\pi}{L}\right)^2 + 2E\right] \times \left(\frac{nm\pi}{L}\right)^2. \tag{4.73b}
$$

For $m = 0$ we have

$$
F_0(x,y) = \frac{4}{L} \sum_{n=1}^{\infty} \frac{1}{D_n} \left[\cos\left(\frac{2n\pi}{L}|x-y|\right) - 1\right] \tag{4.74a}
$$

and

$$
D_n = \left[\left(\frac{2n\pi}{L}\right)^2 + 2E\right] \times \left(\frac{2n\pi}{L}\right)^2. \tag{4.74b}
$$

Let us consider what can be learned from this result without a detailed numerical computation, which we postpone until later. On length scales of the size of the system it is clear that only the lowest sectors contribute, and we can extract information about the envelope from them. Considering just the $m = 0$ sector already provides useful insight. In this limit one finds for the correlator

$$C(x, y) \propto \exp\left(\frac{4}{l(8E)^{3/2}}\frac{\cosh\left(\sqrt{2E}\left(\frac{L}{2} - |x - y|\right)\right)}{\sinh\left(\sqrt{2E}\frac{L}{2}\right)} + \frac{|x - y|^2}{8lEL} - \frac{|x - y|}{8lE}\right). \qquad (4.75)$$

To evaluate $F_0(x, y)$ we have noted that by first differentiating to $|x - y|$ twice the sum can be brought into the form of a standard identity [Gra94]. After performing the sum with the aid of this identity we integrated twice. The integration constants were fixed by imposing periodic boundary conditions, as the original sum was periodic, and requiring that at $x = y$ its value should coincide with that of the original sum. We clearly see an exponentially decaying behaviour, which allows us to read off the localisation length $\xi_{\text{loc}} = 8El$. This agrees with the results of [Lif88], although the precise numerical coefficient does not agree. This coefficient, however, depends strongly on the precise definition of the localisation length and there are varies conventions in use [Tho74, Kra93]. This result has interesting implications. It suggests that the behaviour of the envelope, and thus the localisation length, can be extracted from the $m = 0$ sector, but this is equivalent to perturbation theory around the unstable saddle point $\phi = 0$. This gives us a very simple approach to compute the localisation length and opens up the possibility of improving our results through a self-consistent calculation.

Going down to short length scales one observes that due to the Gaussian factor in (4.71) there is a length scale at which the cutoff of $N(x - y)$ we have imposed on the sum becomes irrelevant. It is easily found that this length scale is $\xi_{\text{del}} \equiv 2\pi\sqrt{ElL}/(1 + 2E\sqrt{lL}) \approx \pi/\sqrt{E}$. At this length scale one sees that the $m = \sqrt{E}L$ sector becomes predominant and since the variance are of the order $1/L$ the other sectors are severely suppressed. Here one therefore expects the correlator to behave like that of a pure system, showing oscillations on a scale of $1/\sqrt{E}$. Thus, on these scales the wave-function will appear to be delocalised. From our condition for the validity of our approximation $lE^{3/2} \gg 1$ one observes that $\xi_{\text{loc}} \gg \xi_{\text{del}}$, which is consistent with the picture emerging from the correlator, namely that on length scales much shorter than the localisation length the wave-function becomes delocalised. Note that in the extreme limit of strong disorder (in which our approximation becomes questionable) this lower length scale will disappear, and the wave-function will appear to be localised on all length scales.

The picture that emerges is therefore the following: On long length scales of the size of the system the wave-function has an exponentially decaying behaviour, the envelope playing the dominant role. On short length scales the wave-function becomes delocalised and oscillates on a scale of $1/\sqrt{E}$. If we assume the system system size to be much larger than the localisation length, i.e. $L \gg El$ (the other limit is not really interesting as the wave-function will be delocalised on a scale of the size of the system), then we have the hierarchy of scales $L \gg \xi_{\text{loc}} \gg \xi_{\text{del}}$ and the question arises what happens on the intermediate length scales, ξ_{loc}. From (4.71) one notes that on these scales many different sectors will contribute to the correlator giving rise to a complicated

structure involving many different length scales (recall that each sector has its own characteristic length scale of $\frac{L}{m}$). This behaviour is, of course, not unexpected in the crossover region.

In the negative energy region, the saddle point approximation (4.43) is obtained using the dilute gas approximation. However, we find that the approximate saddle point solution (4.29) in a dilute gas approximation, (4.30), breaks the symmetries of the system, thus we first need to write the exponential terms in (3.28) in a form where the symmetries are explicit. We do this by using (3.30) so that

$$
2\exp\left[P\int_{x_0}^{x}dx'\phi(x') + P\int_{x_0}^{y}dx'\phi(x')\right]
$$
$$
\rightarrow \exp\left[P\int_{x_0}^{L-|x-y|}dx'\phi(x') + P\int_{x_0}^{0}dx'\phi(x')\right] + \exp\left[P\int_{x_0}^{|x-y|}dx'\phi(x') + P\int_{x_0}^{0}dx'\phi(x')\right]. \quad (4.76)
$$

Using this form for the exponential insertions in (3.28), along with saddle point approximation for negative energies (4.43), and then integrating over the fluctuations, gives

$$
C(x,y) = \int dx_0 \exp\left[P\int_{x_0}^{L-|x-y|}dx'\phi_c^{\pm}(x'-x_0) + P\int_{x_0}^{0}dx'\phi_c^{\pm}(x'-x_0) + \frac{1}{4l}F(L-|x-y|,x_0)\right]
$$
$$
+ \int dx_0 \exp\left[P\int_{x_0}^{|x-y|}dx'\phi_c^{\pm}(x'-x_0) + P\int_{x_0}^{0}dx'\phi_c^{\pm}(x'-x_0) + \frac{1}{4l}F(|x-y|,x_0)\right], \quad (4.77a)
$$

where

$$
F(x,x_0) = \frac{4}{L}\sum_{n=1}^{\infty}\frac{1}{D_n}\left[\cos\left(\frac{2n\pi}{L}x\right) - 2\cos\left(\frac{2n\pi}{L}(x-x_0)\right) - 2\cos\left(\frac{2n\pi}{L}x_0\right) + 3\right] \quad (4.77b)
$$

and

$$
D_n = \left[\left(\frac{2n\pi}{L}\right)^2 + 4|E|\right]\left(\frac{2n\pi}{L}\right)^2. \quad (4.77c)
$$

Note that in obtaining (4.77), we once again relabeled the integration variable in (3.28) from x_0 to z, translated $z \rightarrow z + x_0$, and then, as above, cancelled the z dependent terms giving rise to a total derivative with respect to z. Note from the saddle point solution (4.30) that to leading order this correlation function decays or grows like $\exp(\pm\sqrt{E}x)$, confirming the result of Lifshits *et al* [Lif88] that the localisation length is proportional to $1/\sqrt{E}$ for large negative energies.

4.2.8 Conductivity

To calculate the disorder average of the conductivity given by (3.39) in a saddle point approximation, we first need to be able to solve for $\tilde{\phi}_\beta \equiv \phi_\beta[\phi_\alpha]$ using (3.38). To do this, we make the ansatz that ϕ_β consists of a classical and a fluctuating term, i.e. $\tilde{\phi}_\beta = \phi_\beta^c + \frac{1}{\sqrt{l}}\chi$. Using this ansatz, as well as the expansion $\phi_\alpha = \phi_\alpha^c + \frac{1}{\sqrt{l}}\eta$ in (3.38), and neglecting the coupling terms between the classical and fluctuating terms, we find that $\chi = \eta$ and ϕ_β^c must satisfy the saddle point equation with the energy shifted by $\hbar\omega$,

$$
\hbar\omega - (\phi_\alpha^c)^2 - \phi_\alpha^{c\,'} + (\phi_\beta^c)^2 + \phi_\beta^{c\,'} = 0, \quad (4.78)
$$

with the positive energy solution given by

$$\phi_\beta^c(x) = -\frac{q\pi}{L}\tan\left(\frac{q\pi}{L}x\right). \tag{4.79}$$

To satisfy the periodic boundary conditions, we see that q must be the nearest integer to $(m^2 + \hbar\omega L^2/\pi^2)^{1/2}$.

Applying the saddle point approximation in the positive energy region (4.37) (where we relabel x_0 to z) to (3.39), and making the change of variables $x \to x + z$, $\bar{x} \to \bar{x} + z$, $x_0 \to x_0 + z$ and $\bar{x}_0 \to \bar{x}_0 + z$ allows us to integrate over z (since the integrand is now independent of z) to obtain

$$\frac{\langle\Phi(E,\omega)\rangle}{\langle\Phi(E,\omega_0)\rangle} = \frac{\sum_m \exp(-lL(\Delta E)^2)|J_c|\,\tilde{\Phi}(\omega)}{\sum_m \exp(-lL(\Delta E)^2)|J_c|\,\tilde{\Phi}(\omega_0)}, \tag{4.80a}$$

with

$$\tilde{\Phi}(\omega) = \int [d\eta]'\,dx\,d\bar{x}\,dx_0\,d\bar{x}_0\,\exp\left[\oint_0^x dx'\phi_\alpha^m(x') + \oint_0^{\bar{x}} dx'\phi_\alpha^m(x') + \oint_0^x dx'\phi_\beta^m(x') + \oint_0^{\bar{x}} dx'\phi_\beta^m(x')\right]$$

$$\times \left[\phi_\alpha^m(x) + \frac{1}{\sqrt{l}}\eta(x)\right]\left[\phi_\alpha^m(\bar{x}) + \frac{1}{\sqrt{l}}\eta(\bar{x})\right]\exp\left[-\int_{-\frac{L}{2}}^{\frac{L}{2}} dx'\left[\eta(x')\Delta_m^{-1}\eta(x') - \frac{1}{\sqrt{l}}S(x')\eta(x')\right]\right]$$

$$\times \frac{\partial}{\partial x_0}\exp\left[\oint_{x_0}^0 dx'(\phi_\alpha^m(x') + \frac{1}{\sqrt{l}}\eta(x'))\right]\frac{\partial}{\partial \bar{x}_0}\exp\left[\oint_{\bar{x}_0}^0 dx'(\phi_\beta^m(x') + \frac{1}{\sqrt{l}}\eta(x'))\right], \tag{4.80b}$$

where $S(x') \equiv S(x,\bar{x},0|x')$ and the classical solutions in the positive energy region are now denoted by the superscript m. Note that the total derivatives that appear are due to the original Faddeev-Popov method used to cancel out the normalisations of the wave-functions. As before, we can cancel them with similar terms in the denominator of (4.80a).

Introducing source terms for the $\eta(x)$ and $\eta(\bar{x})$ terms, integrating over the fluctuations and then integrating by parts, we have

$$\tilde{\Phi}(\omega) = \int dx\,d\bar{x}\,\exp\left(\frac{1}{l}F(x,\bar{x})\right)[\det(\Delta_m)]^{-\frac{1}{2}}$$

$$\times \left[\frac{q\pi}{L}\sin\left(\frac{q\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right) - \frac{m\pi}{L}\sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{q\pi}{L}x\right)\right]$$

$$\times \left[\frac{q\pi}{L}\sin\left(\frac{q\pi}{L}\bar{x}\right)\cos\left(\frac{m\pi}{L}\bar{x}\right) - \frac{m\pi}{L}\sin\left(\frac{m\pi}{L}\bar{x}\right)\cos\left(\frac{q\pi}{L}\bar{x}\right)\right], \tag{4.81a}$$

where

$$F(x,\bar{x}) = \frac{2}{L}\sum_{n=1}^\infty \frac{1}{D_{2n}}\left[\sin\left(2n\frac{m\pi}{L}x\right) + \sin\left(2n\frac{m\pi}{L}\bar{x}\right)\right]^2$$

$$+ \frac{2}{L}\sum_{n=1}^\infty \frac{1}{D_{2n-1}}\left[\cos\left((2n-1)\frac{m\pi}{L}x\right) + \cos\left((2n-1)\frac{m\pi}{L}\bar{x}\right) - 2\right]^2 \tag{4.81b}$$

with D_n given in (4.73b).

Once again we can cancel the determinants so that the final result for the conductivity is

$$\frac{\langle \Phi(E, \omega) \rangle}{\langle \Phi(E, \omega_0) \rangle} = \frac{\sum_m \exp(-lL(\Delta E)^2) \tilde{\Phi}(\omega)}{\sum_m \exp(-lL(\Delta E)^2) \tilde{\Phi}(\omega_0)}, \tag{4.82a}$$

with

$$\tilde{\Phi}(\omega) = \int dx d\bar{x} \left[\frac{q\pi}{L} \sin\left(\frac{q\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) - \frac{m\pi}{L} \sin\left(\frac{m\pi}{L}x\right) \cos\left(\frac{q\pi}{L}x\right) \right]$$
$$\times \exp\left(\frac{1}{l}F(x, \bar{x})\right) \left[\frac{q\pi}{L} \sin\left(\frac{q\pi}{L}\bar{x}\right) \cos\left(\frac{m\pi}{L}\bar{x}\right) - \frac{m\pi}{L} \sin\left(\frac{m\pi}{L}\bar{x}\right) \cos\left(\frac{q\pi}{L}\bar{x}\right) \right], \tag{4.82b}$$

where $F(x, \bar{x})$ is given in (4.81b).

Note that in the limit $\omega \to 0$, the real part of the conductivity is zero as expected, since $q = m$.

4.2.9   Self-consistent renormalisation of energy

In the previous sections, we derived equations for averages of observables in a saddle-point approximation up to one loop order. It is possible, however, to improve the results obtained by summing up certain higher order loop corrections by renormalising the action using a one loop "mass renormalisation" [Nie90].

We introduce a renormalisation variable, the self-energy Σ, which has the energy E as its bare value. The dressed propagator is then given by (4.14b) with the energy replaced by Σ, so that

$$\Delta^{-1} = -\frac{d^2}{dx^2} + 2\Sigma + 6\phi_{\rm c}^2. \tag{4.83}$$

If we expand (4.83) in terms of $\delta = \Sigma - E$, then to first order

$$\Delta = \Delta_0 - 2\Delta_0^2 \delta \tag{4.84}$$

where Δ_0 is the bare propagator.

Using Feynman diagrams, it is possible to express the dressed propagator in terms of the bare propagator. If we consider the lowest order loop corrections arising from the η^4 term in the action (4.14a), we have

$$\Delta = \Delta_0 - \frac{3}{2l} \Delta_0^2 \Delta_0(0). \tag{4.85}$$

Comparing (4.85) with (4.84), we can read off that $\delta = \frac{3}{4l} \Delta_0(0)$ so that

$$\Sigma = E + \frac{3}{4l} \Delta_0(0). \tag{4.86}$$

If the renormalisation is done self-consistently then

$$\Sigma = E + \frac{3}{4l} \Delta(0). \tag{4.87}$$

We can calculate $\Delta(0)$ for positive Σ using (4.49), and by using (4.60) for negative Σ. Thus we

obtain

$$\Sigma = E + \frac{3m}{2lL} \sum_{n=1}^{\infty} \left[(2n-1)^2 \left(\frac{m\pi}{L} \right)^2 + 2\Sigma \right]^{-1} \qquad \Sigma > 0 \qquad (4.88a)$$

$$\Sigma = E + \frac{3}{2lL} \sum_{n=1}^{\infty} \left[\left(\frac{2n\pi}{L} \right)^2 + 4\,|\Sigma| \right]^{-1} \qquad \Sigma < 0. \qquad (4.88b)$$

In the macroscopic limit $L \to \infty$ we have

$$\Sigma = E + \frac{3}{8l} \frac{1}{\sqrt{2\Sigma}} \qquad \Sigma > 0 \qquad (4.89a)$$

$$\Sigma = E + \frac{3}{16l} \frac{1}{\sqrt{|\Sigma|}} \qquad \Sigma < 0. \qquad (4.89b)$$

Once having solved for Σ, we can obtain the renormalised results for the disordered average observables, by replacing the bare energy E with the self-energy Σ in the action, and thus in the results for the density of states, the 2-point correlators and the conductivity.

4.3   Strong disorder limit

In the discussion thus far the validity of our approximation has hinged on the smallness of the dimensionless coupling constant $1/(lE^{3/2})$. However, as we decrease either l or E this becomes large and our approximation starts to break down. In particular one is interested in a fixed amount of disorder at low energies, where our approximation is bound to fail. In this region the appropriate thing to do is to try and establish a perturbative expansion not based on $1/(lE^{3/2})$, but rather on $lE^{3/2}$ itself. Clearly the corresponding "dual" theory will involve different degrees of freedom and may posses a completely different action. One fairly standard way of establishing such a dual theory is through a Hubbard-Stratanovich transformation [Str57, Hub59]. This is what we will use now in an attempt to derive such an effective theory for the strong disorder (strong coupling) limit. The Hubbard-Stratanovich transformation works on the principle that a Gaussian factor can be represented with an auxiliary field, i.e.

$$\exp\left[-\alpha^2 \int dx (A(x))^2 \right] = \int [d\Lambda] \exp\left[-\int dx \left(\frac{1}{\alpha^2} \Lambda^2 + 2iA\Lambda \right) \right]. \qquad (4.90)$$

Thus writing the action in (4.3b) as

$$S[\phi, E] = l \int_{-L/2}^{L/2} dx [(\phi')^2 + (E + \phi^2)^2], \qquad (4.91)$$

where we noted that the cross terms are zero due to the periodicity of ϕ, we see that we can represent the $l \int dx (E + \phi^2)^2$ with an auxiliary field. Thus, using (4.90) in (4.3) we obtain

$$\langle \widehat{O} \rangle = Z^{-1} \int [d\Lambda] \, \widehat{O}[\Lambda, E] \exp\left(-\frac{1}{l} \int dx \, [\Lambda^2 + 2iEl\Lambda] \right), \qquad (4.92a)$$

where the Λ dependent observable is given by

$$\widehat{O}[\Lambda, E] = \int [d\phi]\,[d\bar{u}]\,[du]\,\,\widehat{O}[\phi, E] \exp\left(-\int dx\left[l\,\phi'^2 + 2i\phi^2\Lambda - \bar{u}u' - 2\phi\bar{u}u\right]\right) \qquad (4.92b)$$

and

$$Z^{-1} = \int [d\Lambda] \exp\left(-\frac{1}{l}\int dx\left[\Lambda^2 + 2iEl\Lambda\right]\right)$$
$$\times \int [d\phi]\,[d\bar{u}]\,[du]\,\,\exp\left(-\int dx\left[l\,\phi'^2 + 2i\phi^2\Lambda - \bar{u}u' - 2\phi\bar{u}u\right]\right). \qquad (4.92c)$$

Here we have represented the Jacobian in terms of ghost fields. It is also appropriate to remark here on our procedure of a "partial Hubbard-Stratanovich" transformation versus one in which we represent the complete quadratic action through a Hubbard-Stratanovich transformation. The reason for doing so is that with the partial transformation we still have a conventional kinetic energy term for the ϕ field which will not be the case for the full transformation. This fact facilitates the integration of the ϕ field to obtain the effective action for the Λ field.

We now approximate (4.92) by expanding (as described in Zinn-Justin [ZJ89]) up to first order in the loop corrections. We do this by first splitting the integral over Λ into an integral over the constant mode, Λ_0, and non-constant modes η. This is achieved by inserting the identity $\int d\Lambda_0 d\eta\delta[\Lambda - (\Lambda_0 + \sqrt{l}\eta)]\delta(\int dx\eta)$ into the numerator and denominator of (4.92a). Integrating over Λ, we have

$$\langle\widehat{O}\rangle = Z^{-1}\int d\Lambda_0\int [d\eta]\,\widehat{O}[\Lambda_0 + \sqrt{l}\eta, E]\exp\left[-\int dx\,\eta^2\right]\exp\left[-\frac{\Lambda_0^2 L}{l} - 2iE\Lambda_0 L\right], \qquad (4.93)$$

where the observable $\widehat{O}[\Lambda_0 + \sqrt{l}\eta, E]$ is given by (4.92b) with $\Lambda = \Lambda_0 + \sqrt{l}\eta$.

We now want to integrate the ϕ-field and ghost fields from (4.92b) to obtain the appropriate observable to be used in (4.92a). Clearly this cannot be done exactly, instead we do it in a 1-loop approximation. Thus we neglect the $\sqrt{l}\phi^2\eta$ and $\phi\bar{u}u$ coupling terms in (4.92b) upon which the η and ghost integrations can be done to yield for the disordered average

$$\langle\widehat{O}\rangle = Z^{-1}\int d\Lambda_0\widehat{O}(\Lambda_0, E)\exp\left[-\frac{\Lambda_0^2 L}{l} - 2iE\Lambda_0 L\right] \qquad (4.94a)$$

with the observable given by

$$\widehat{O}(\Lambda_0, E) = \int [d\phi]\,\widehat{O}[\phi, E]\exp\left[-\int dx(l\,\phi'^2 + 2i\phi^2\Lambda_0)\right]. \qquad (4.94b)$$

The justification of this approximation is based on the fact that the $\phi^2\eta$ coupling is accompanied by a factor of $\sqrt{l}$. Arguing as we did in the weakly disordered case, we note that the dimensionless coupling constant that accompanies this term is indeed $\sqrt{l}E^{3/2}$, thus neglecting this term at low energies is justified. Furthermore, at low energies one expects the dominant sector to be $m = 0$. In this case the ghost and ϕ-fields can be decoupled as illustrated before. This argument is, however, not very rigorous.

Assuming that $\widehat{O}[\phi, E]$ is real, we note that $\widehat{O}^*(\Lambda_0, E) = \widehat{O}(-\Lambda_0, E)$, which allows us to integrate over positive Λ_0 in (4.94a) if we take the real part of the integrand, thus

$$\langle\widehat{O}\rangle = Z^{-1} \int_0^\infty d\Lambda_0 \, \text{Re} \, \widehat{O}(\Lambda_0, E) \exp\left[-\frac{\Lambda_0^2 L}{l} - 2iE\Lambda_0 L\right]. \tag{4.95}$$

Here we considered only the lowest order approximation where we totally neglected the contribution from the $\phi^2\eta$ term, but it is possible to extend (4.94) to include the contribution of the quadratic terms in η arising from the $\phi^2\eta$ coupling, upon which the η integral can still be done to yield a determinant, which will give higher order corrections to (4.95).

4.3.1 Density of states

For the average density of states, the observable (4.94b) is

$$\widehat{O}(\Lambda_0, E) = \left[\det\left(-\frac{d^2}{dx^2} + \frac{2i\Lambda_0}{l}\right)\right]^{-1/2}$$
$$\propto \frac{(1+i)\sqrt{\Lambda_0}L}{2\sqrt{l}} \, \text{cosech}\left[\frac{(1+i)\sqrt{\Lambda_0}L}{2\sqrt{l}}\right] \tag{4.96}$$

which we can use in (4.95) to obtain

$$\langle\rho(E)\rangle = \int_0^\infty d\Lambda_0 \text{Re}\left[(1+i)\text{cosech}\left(\frac{(1+i)\sqrt{\Lambda_0}L}{2\sqrt{l}}\right)\sqrt{\Lambda_0} \exp\left(-\frac{\Lambda_0^2 L}{l} - 2iE\Lambda_0 L\right)\right]. \tag{4.97}$$

4.3.2 2-point correlations

The observable used to calculate the correlator in the dual region can be obtained from (3.28). Explicitly writing the total derivative and using the approximation in (4.94b), we integrate over the ϕ field to obtain

$$\widehat{O}(\Lambda_0, E) = \int dx_0 \frac{\partial}{\partial x_0} \frac{(1+i)\sqrt{\Lambda_0}}{2\sqrt{l}} \, \text{cosech}\left(\frac{(1+i)\sqrt{\Lambda_0}L}{2\sqrt{l}}\right)$$
$$\times \exp\left[\frac{1}{4l}\int dx'dx'' S(x,y,x_0|x')\Delta(x',x'')S(x,y,x_0|x'')\right], \tag{4.98}$$

where $\Delta(x, x'') = \left(-\frac{d^2}{dx^2} + \frac{2i\Lambda_0}{l}\right)^{-1}$.

Using this in (4.95), extracting the terms that are x_0 independent and then cancelling the x_0 integral with a similar term in the normalisation, we have

$$\frac{\langle|\psi_E(x)|\,|\psi_E(y)|\rangle}{\langle|\psi_E(0)|\,|\psi_E(0)|\rangle} = \frac{C(x,y)}{C(0,0)} \tag{4.99a}$$

where

$$C(x,y) = \int_0^\infty d\Lambda_0 \, \text{Re} \frac{(1+i)\sqrt{\Lambda_0}}{2\sqrt{l}} \, \text{cosech}\left(\frac{(1+i)\sqrt{\Lambda_0}L}{2\sqrt{l}}\right)$$
$$\times \exp\left[-\frac{\Lambda_0^2 L}{l} - 2iE\Lambda_0 L\right] \exp\left[\frac{1}{4l}F(|x-y|)\right], \tag{4.99b}$$

with

$$F(x) = \frac{2}{L} \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2n\pi}{L}x\right) - 1}{\left[\left(\frac{2n\pi}{L}\right)^2 + \frac{2i\Lambda_0}{l}\right]\left(\frac{2n\pi}{L}\right)^2}. \tag{4.99c}$$

Chapter 5
Numerical results

In this chapter we numerically calculate the main results that we obtained in the previous chapter and generate plots from these calculations in order to obtain a better understanding of the results.

Before we do so, it is convenient to work numerically in terms of dimensionless quantities. As the quantity $lE^{3/2}$ is a natural quantity appearing in the theory, separating the weak and strong disorder limits, we introduce a dimensionless energy $\bar{E} = El^{2/3}$, and correspondingly a dimensionless length $\bar{L} = Ll^{-1/3}$. Thus we see that $\bar{E} \to \infty$ corresponds to the high energy limit or the weak disorder limit, which both describe a system where disorder has very little effect. Conversely, when $\bar{E} \sim 0$, then the energy is weak or the disorder is very strong, which once again is in the regime where the disorder plays a prominent role. Using these new parameters, we note that the weak approximation limit holds when $|\bar{E}| \gg 1$ and the strong approximation limit holds when $|\bar{E}| \ll 1$.

5.1 Density of states

We concentrate on the results obtained for the density of states given by equations (4.65) and (4.66) in the weak disorder limit and (4.97) in the strong disorder limit. We shall first consider the weak disorder approximation in the macroscopic limit as exact results for the density of states with a Gaussian distribution were obtained in the thermodynamic limit by Frisch and Lloyd [Fri60], which allows us to compare our results with theirs. We then show the results for the positive energy weak disorder limit with finite length scales and lastly, we discuss the results obtained for the strong disorder approximation.

5.1.1 Weak disorder approximation in the macroscopic limit

In the macroscopic limit $L \to \infty$, the sum in the positive energy result, (4.65), becomes a Riemann integral, while the Gaussian exponent becomes a Dirac delta. Taking the limit and computing the integral, we obtain for positive energies

$$\langle \rho(\bar{E}) \rangle \propto \frac{1}{\sqrt{\bar{E}}}, \qquad \forall \bar{E} > 0. \tag{5.1}$$

In the negative energy region, the result for the density of states (4.65) is already the leading order result in L, with the corrections being of order $1/L$. Thus the result is the same in the macroscopic limit and in terms of the dimensionless energy is given by

$$\langle \rho(\bar{E}) \rangle \propto |\bar{E}| \exp\left[-\frac{16}{3} l \, |\bar{E}|^{3/2} \right], \qquad \forall \bar{E} < 0. \tag{5.2}$$

These results are shown in Fig. 5.1a along with the exact result from Frisch and Lloyd, obtained by taking the derivative of (2.51) to energy and using the translation to our quantities that $ED^{-2/3} \to 4^{2/3}\bar{E}$. Note that, as expected, the results start to deviate from the exact result in the region where the saddle point approximation no longer holds $|\bar{E}| < 1$.
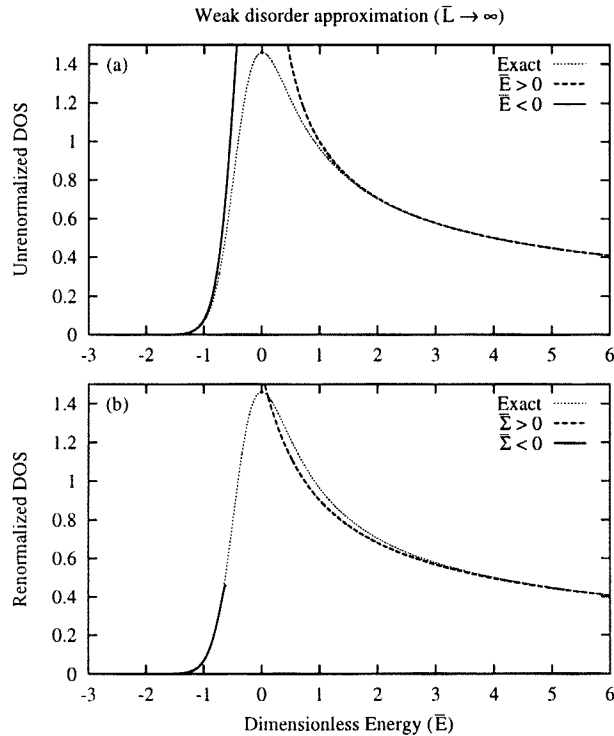
Figure 5.1: Density of states in the weak approximation for $\bar{L} \to \infty$. In (a) the unrenormalised approximations for the density of states for positive and negative dimensionless energies, $\bar{E}$, are shown against the exact result of Frisch and Lloyd [Fri60]. In (b) the renormalised equivalent of (a) is plotted where the dimensionless self energy $\bar{\Sigma}$ is used instead of $\bar{E}$.

The result obtained for the positive energy weak disorder approximation in the macroscopic limit, (5.1), is just the density of states for a pure system, and thus has the singularity that one expects for pure systems at $\bar{E} = 0$. To improve on the result (and show that the singularity does not exist in disordered systems), we need to use the self-consistent renormalisation method discussed in Section 4.2.9.

To use the method in the weak disorder approximation, we first need to solve the self-consistent equation (4.89a) for the self-energy in the positive energy region, and (4.89b) for the self-energy in the negative region. The solution of these equations, in terms of the dimensionless self-energy, $\bar{\Sigma} = \Sigma l^{2/3}$, are given by

$$\bar{\Sigma} = \frac{2^{1/3} 32 \bar{E}^2 + 16 \bar{E} x + 2^{1/3} x^2}{24x}$$
$$x = (243 - 256 \bar{E}^3 + 9\sqrt{3}\sqrt{243 - 512 \bar{E}^3})^{1/3} \quad \forall \bar{\Sigma} > 0 \tag{5.3a}$$

and

$$\bar{\Sigma} = \frac{64 \bar{E}^2 + 16 \bar{E} x + x^2}{24x}$$
$$x = (-243 - 512 \bar{E}^3 + 9\sqrt{3}\sqrt{243 + 1024 \bar{E}^3})^{1/3} \quad \forall \bar{\Sigma} < 0 \tag{5.3b}$$
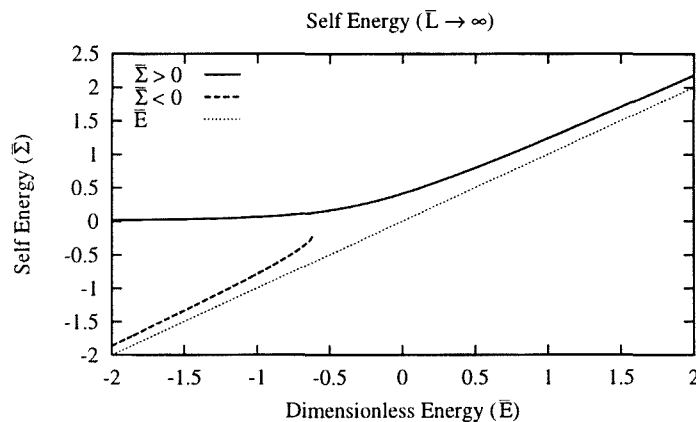
Figure 5.2: **Solutions of the self-consistent equations, (4.89a) and (4.89b), for the dimensionless self-energy $\bar{\Sigma}$ as a function of the dimensionless energy $\bar{E}$ in the macroscopic limit $\bar{L} \to \infty$.**

These solutions are plotted in Fig. 5.2. Note that the $\bar{\Sigma} > 0$ solution exists for all energies, while the $\bar{\Sigma} < 0$ solution exists only for negative energies less than $-(243/1024)^{1/3}$. Also, both solutions tend to the energy in the asymptotic limits where $|\bar{E}| \to \infty$.

As mentioned in Section 4.2.9, the self-consistent renormalisation method is equivalent to replacing the energy $\bar{E}$ in the results obtained with the self energy $\bar{\Sigma}$. We do this with 5.1 and 5.2, and plot the results in Fig. 5.1b. The result for the negative self-energy solution matches the result perfectly almost up until the energy value where the solution for the self-energy no longer exists. The positive self-energy solution, although not matching as well as the negative self-energy solution, does better than the unrenormalised density of states in the positive energy region. The renormalised solution under estimates the exact solution in the energy region $[0.5, 3]$, but gives a better result for lower energies. Additionally, there is no longer a singularity at zero energy as found in the unrenormalised positive energy solution. This is the expected result for disordered systems as the disorder averages out any singularities that would have existed in the pure system [Nie90].

5.1.2 Weak disorder approximation for finite length

It is also possible to calculate the density of states in the weak approximation for finite size systems using (4.65) and (4.66). The negative energy result (4.66) is rather insensitive to the length L as these are higher order corrections of order $1/L$. Thus the result for a finite size system, at least for system sizes that are not too small, is approximately the same as the result presented above where the macroscopic limit $\bar{L} \to \infty$ has been taken. On the other hand, a finite system size has a prominent effect in the positive energy region. The reason for is that there is a finite level spacing at finite system sizes, which are expected to resolve at high energy or weak disorder. For this reason, we only consider the positive energy result (4.65) in this section.

In the positive energy region the result in terms of the dimensionless quantities $\bar{E}$ and $\bar{L}$ is
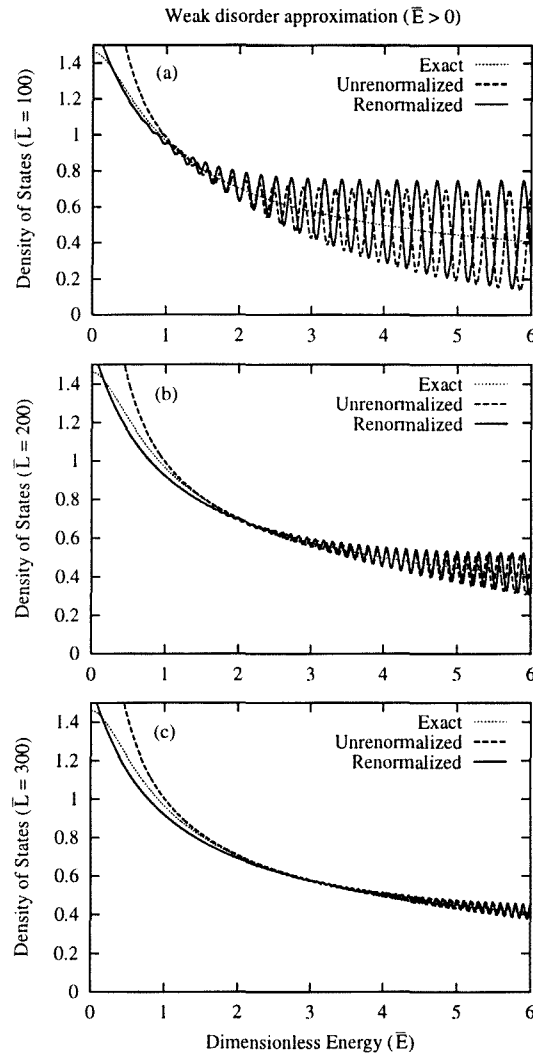
Figure 5.3: Density of states in the weak disorder approximation for positive dimensionless energy $\bar{E}$ with dimensionless system sizes : (a) $\bar{L} = 100$, (b) $\bar{L} = 200$ and $\bar{L} = 300$. Also plotted is the renormalised density of states for positive dimensionless self-energy $\bar{\Sigma}$ at the same dimensionless system sizes.

given by

$$\langle \rho(\bar{E}) \rangle \propto \sum_{m=0}^{\infty} \exp\left(-\bar{L}\left[\bar{E} - \left(\frac{m\pi}{\bar{L}}\right)^2\right]^2\right). \tag{5.4}$$

To calculate (5.4) numerically, we need to introduce an upper cutoff to the sum over m. The result of the numerical calculations are shown in Fig. 5.3 for various dimensionless system sizes. Also shown in Fig. 5.3 are renormalised density of states calculated using the self-energy given by (5.3a). Technically, we should actually solve for the self-energy using the finite size equations given in (4.89), but this leads to very small higher order corrections of the result.

As can be seen in Fig. 5.3 oscillations appear in the tail of the density of states. These
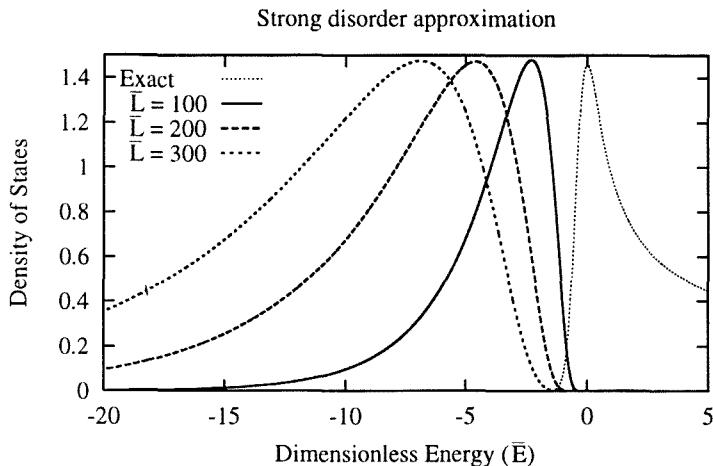
Strong disorder approximation

Figure 5.4: Density of states in the strong disorder approximation for finite dimensionless system sizes: $\bar{L} = 100$, $\bar{L} = 200$ **and** $\bar{L} = 300$.

oscillations shift to higher energies as the size of the system is increased. The origin of these oscillations is the finite level spacing at finite L. The level spacing scales as $\sqrt{E}/L$ and is thus larger at higher energies. This explains why these oscillations become more prominent at higher energy, while they disappear completely at very low energies where the level spacing can no longer be resolved. As the system size is increased, the level spacing reduces and the energy at which the level spacing can no longer be resolved is higher, explaining the shift of the oscillation to higher energy upon the increase of the system size. Effectively, this result again illustrates the point that at higher energies the disorder has very little effect and that the system behaves as a pure system. The density of states of the pure system will have Dirac-delta peaks at energies $E_n = \left(\frac{2n\pi}{L}\right)^2$. The disordered system resembles this behaviour more and more at higher energies. At low energies, on the other hand, a broadening occurs due to the disorder. When this broadening exceeds the level spacing, the level spacing can no longer be resolved and all remnants of the pure spectrum are lost.

5.1.3 Strong disorder approximation

In the strong disorder limit the density of states is given by (4.97), which in terms of the dimensionless quantities is

$$\langle \rho(\bar{E}) \rangle = \int_0^\infty d\Lambda_0 \mathrm{Re}\left[(1+\mathrm{i})\sqrt{\Lambda_0}\, \mathrm{cosech}\left((1+\mathrm{i})\sqrt{\Lambda_0}\bar{L}\right) \exp\left(-\Lambda_0^2\bar{L} - 2\mathrm{i}\bar{E}\Lambda_0\bar{L}\right)\right]. \tag{5.5}$$

Using an upper cutoff in the integral which is determined by the width of the Gaussian term in the integrand, we calculate (5.5) numerically for various system sizes, with the result shown in Fig. 5.4.

Unfortunately, the result that is obtained is not the one that is expected. We see that as the system size is increased the peak of the strong disorder density of states shifts to the left while the width increases. Another problem with the result is that the density of states in this

approximation is zero for positive energies, which is incorrect. We expected the peak of the density of states to be at zero energies, even in the macroscopic limit. If this was the case, we could have been able to obtain an approximation for the density of states for all energy regions, by matching the weak disorder approximation results in the positive and negative energy regions with the strong disorder result.

The incorrect result for the density of states in the strong disorder approximation is obtained because we neglected the contribution from the Jacobian in (4.92), since we dropped the $\phi\bar{u}u$ in the action. For the density of states, we find that this energy dependent term, which cancelled the determinant from the integration over the fluctuations in the weak disorder approximation, needs to be taken into consideration. Unfortunately, keeping the $\phi\bar{u}u$ term, and thus the integration over the ghosts, makes the resulting functional integral difficult to calculate. We thus find that the present strong disorder approximation is inadequate for obtaining results for the density of states around $E = 0$.

5.2   2 point correlators

To calculate the 2-point correlators, we use (4.70), with (4.75) and (4.77a) for the positive and negative weak disorder approximations and (4.99b) for the strong disorder approximation. In terms of the dimensionless quantities these equations are

$$\frac{\langle |\psi_E(d)|\,|\psi_E(0)|\rangle}{\langle |\psi_E(0)|\,|\psi_E(0)|\rangle} \approx \frac{C(d)}{C(0)} \tag{5.6a}$$

where

$$C(d) \propto \exp\left(\frac{4}{(8\bar{E})^{3/2}}\frac{\cosh\left(\sqrt{2\bar{E}}\bar{L}\left(\frac{1}{2}-d\right)\right)}{\sinh\left(\sqrt{2\bar{E}}\frac{\bar{L}}{2}\right)} + \frac{d^2\bar{L}}{8\bar{E}} - \frac{d\bar{L}}{8\bar{E}}\right) \tag{5.6b}$$

for the weak disorder approximation in the positive energy region,

$$C(d) \propto \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_0 \exp\left[P\int_{x_0}^{1-d} dx'\,\phi_c^\pm(x'-x_0) + P\int_{x_0}^0 dx'\,\phi_c^\pm(x'-x_0) + F(1-d,x_0)\right]$$

$$+ \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_0 \exp\left[P\int_{x_0}^{d} dx'\,\phi_c^\pm(x'-x_0) + P\int_{x_0}^0 dx'\,\phi_c^\pm(x'-x_0) + F(d,x_0)\right]$$

$$F(d,x_0) = \bar{L}^3 \sum_{n=1}^{\infty} \frac{[\cos(2n\pi d) - 2\cos(2n\pi(d-x_0)) - 2\cos(2n\pi x_0) + 3]}{[(2n\pi)^2 + 4\bar{E}\bar{L}^2](2n\pi)^2}, \tag{5.6c}$$

for the weak disorder approximation in the negative energy region, and

$$C(x,y) \propto \int_0^\infty d\Lambda_0\,\mathrm{Re}\left[(1+i)\sqrt{\Lambda_0}\,\mathrm{cosech}\left((1+i)\sqrt{\Lambda_0}\bar{L}/2\right)\exp\left(-\Lambda_0^2\bar{L} - 2i\bar{E}\Lambda_0\bar{L} + F(d)\right)\right]$$

$$F(d) = \frac{\cosh\left((1+i)\sqrt{\Lambda_0}\bar{L}/2\right) - \cosh\left((1+i)\sqrt{\Lambda_0}\bar{L}(1/2-d)\right)}{16(i-1)\Lambda_0^{3/2}\sinh\left((1+i)\sqrt{\Lambda_0}\bar{L}/2\right)} + \frac{d^2\bar{L}}{16i\Lambda_0} - \frac{d\bar{L}}{16i\Lambda_0}, \tag{5.6d}$$
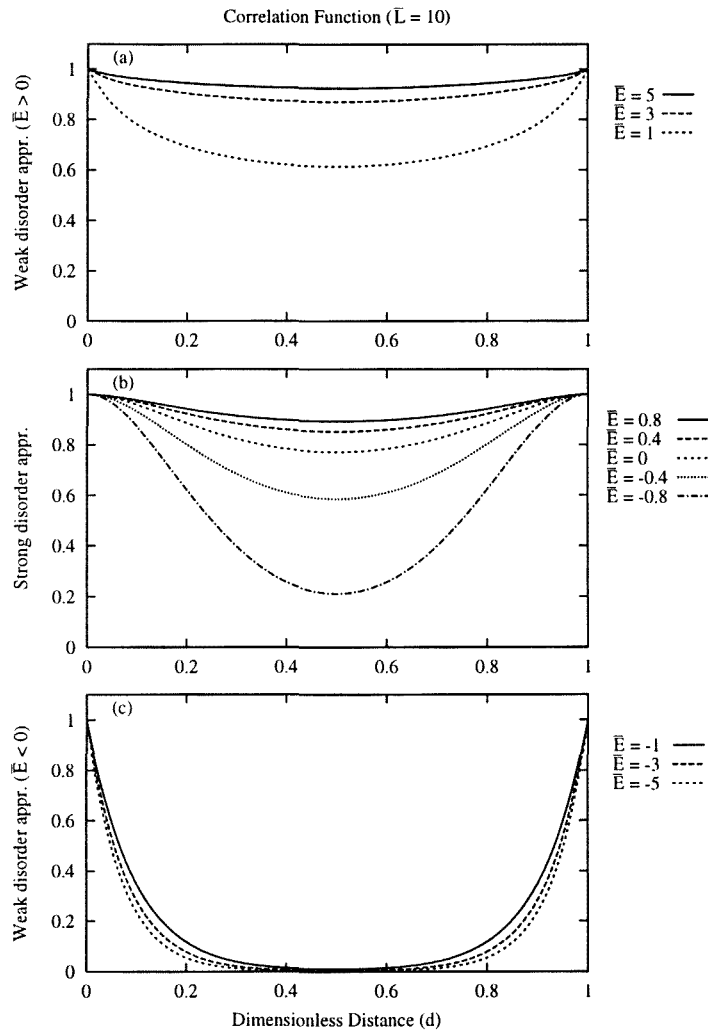
Correlation Function ($\bar{L} = 10$)

Figure 5.5: **Correlation function for (a) the weak disorder approximation in the positive energy region, (b) the strong disorder approximation and (c) the weak disorder approximation with negative energies. The dimensionless system size is $\bar{L} = 10$**

for the strong disorder approximation. Here we have also introduced the dimensionless distance $d \equiv \frac{|x-y|}{L}$.

The above results were calculated numerically with the appropriate cutoffs for summation and integral to infinity, and are shown in Fig. 5.5 for various energy values. The correlation function decreases as the distance is increased until $d = 0.5$ after which it increases again. This is of course due to our ring topology.

At high energies, as plotted in Fig. 5.5a, the decay is very flat, which is expected, since at these energies, the result is dominated by nearly extended solutions, since the localisation length is of the same order as the system size. Note, however that the system is still localised, even though the states are very weakly localised.

As the energy is decreased into the energy region where the weak disorder approximation
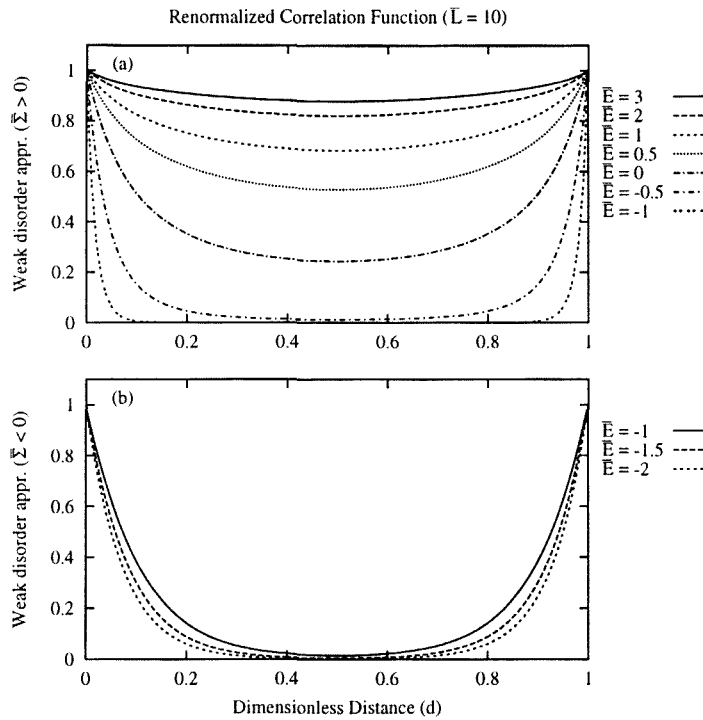
Figure 5.6: **Renormalised correlation function using the self-consistent self-energy $\bar{\Sigma}$ for (a) the weak disorder approximation in the positive energy region, and (b) the weak disorder approximation in the negative energy region. The dimensionless system size is $\bar{L} = 10$.**

breaks down ($\bar{E} = 1$), we use the strong disorder approximation in Fig. 5.5b, which is valid in this energy region. The correlation function decays even more, indicating that at these energies the states are becoming more strongly localised.

As the energy becomes even more negative, the strong disorder approximation no longer holds when $\bar{E} = -1$, so we once again start using the weak disorder approximation in Fig. 5.5c , this time with the negative energy results. In this energy region the correlation function decays exponentially since the states are now strongly localised. This exponential decay is due to the formation of bound states caused by rare fluctuations in the random potential.

It is also possible to calculate the renormalised correlation function using the self-consistent self-energy obtained in (5.3a). Once again, since we are calculating the correlation function for a given system size, we should technically be using the solution of the finite size self-consistent equation for the self-energy, (4.88), but as this gives higher order corrections we shall continue using the infinite solution. We note that since the self-energy is positive for all energy values, we can calculate the correlation function for all energies when using the self-energy, (5.3a), in the weak disorder approximation for positive energies, (5.6a). The result of the renormalised correlation function in the weak disorder limit for positive self-energies is shown in Fig. 5.6a. As can be seen, there is a continuous change over from the weakly decaying correlation functions at high positive energies, where the states are weakly localised to the exponentially decaying

correlation functions at high negative energies where the states are strongly localised.

The amount of decay for the correlation function in the energy regions below $E = 1$ is over estimated when compared to similar values obtained in the unrenormalised correlation function (Fig. 5.5). This over estimate becomes more pronounced at negative energies, since the exponential decay in the renormalised correlation function decays much faster than the unrenormalised equivalent. Notwithstanding the over estimation that occurs with the renormalised correlation function, we are still able to get a qualitative understanding of how the states change from being weakly localised at positive energies to strongly localised at negative energies. Instead of using the positive self-energy solutions for the density of states for all energies, we should use the negative self-energy solution for those negative energies where solutions of the negative self-energy equation exists. These results are shown in Fig. 5.6b. These results are not over estimated, and are almost unchanged from the unrenormalised results for the negative energy that is plotted in Fig. 5.5c.

We thus find that the formalism obtained in Chapter 4 compares well with the asymptotic tails of the exact density of states, while the correlation function gives a qualitative understanding of how the states change behaviour from being weakly localised at high positive energies to strongly localised at high negative energies. The only shortcoming of the formalism is the poor approximation that was utilised in the strong disorder regime, which is unfortunately difficult to improve without using more complicated field theoretic techniques.

Chapter 6
Conclusion and future developments

In Chapter 2 of this dissertation we gave an overview of the general properties of disordered systems and the quantities that are studied when investigating these systems. We focused mainly on the calculation techniques used in studying disordered systems, and discussed the methods used in one dimension, such as the phase formalism and the diagrammatic methods, as well as the methods used in higher dimensions, namely the scaling approach and the non-linear σ-model approaches based on the replica or supersymmetric field theory methods. We pointed out that the current field theoretic techniques do have some open problems, in that they are based on calculating averages of Green's functions which are then related to the average quantities that are actually being investigated, thus creating an additional layer which in some sense obscures further insight into the disordered problem, due to the extra complication in the theoretical methods. Also, we noted that the non-linear σ-model theories cannot be used to study strongly disordered systems. Using this as a motivation, we introduced a complementary effective field theory which is based on calculating quantities expressed directly in terms of the wave-function and eigenenergies.

The effective field theory formalism, which we introduced in Chapter 3, is based on changing variables from the random potential describing the disordered system to a new set of random variables related to the logarithmic derivative of the wave-function. This allows a more direct computation of certain disordered averages, such as the density of states or observables that explicitly depend on the wave-function. In particular we showed how to calculate the disorder averages of the density of states (3.25), the 2-point correlators of the wave-function (3.28, 3.31), as well as the real part of the conductivity (3.39, 3.40).

As an illustration of how the formalism works, we considered one dimensional Gaussian disordered systems. We were able to obtain results for the weak disorder and strong disorder limits for the density of states, (4.65) and (4.66), and the 2-point correlators, (4.69) with (4.75), (4.77) and (4.99). Unfortunately we were only able obtain results of the conductivity in the weak disorder limit (4.82), as there is a complication in the perturbative expansion of the strong disorder limit when using the Hubbard-Stratanovich transformation on (4.82), which we have as yet been unable to resolve.

In Chapter 5, we numerically calculated the results obtained for the one-dimensional Gaussian disordered system. For the average density of states, the weak disorder saddle-point approximation reproduced the results of Frisch and Lloyd [Fri60] in the high energy tails, especially when the self-consistent renormalisation of the energy is utilised. The strong disorder approximation, however, does not give very good results when compared to the exact result. This poor approximation is due to the neglect of the Jacobian term in the approximation, which should not actually be ignored. The calculation of the 2-point correlator shows that in one dimension the states are localised. At high positive energies, the system is weakly localised as the localisation length is of the same order as the system size, while at high negative energies the system is strongly localised due to the formation of strongly bound states. We thus find that the formalism

reproduces results obtained in the weak disorder limit, but a better approximation is needed in the strong disorder dual field theory.

There is still much work to be done on the formalism that we have introduced. Firstly, now that the preliminary investigation to discover if the theory is viable has been completed, there needs to be a more thorough investigation of the formalism in higher dimensions to see if signs of a metal insulator transition can be found. Also, the approximation method in the strong disorder limit has to be improved so that results for the density of states can be obtained for all energy ranges. A more speculative goal would be to see if the density of states can be solved exactly using the effective field theory method, as we know that an exact result exists as obtained by Frisch and Lloyd [Fri60].

Another outstanding problem that needs to be addressed is the calculation of the conductivity in the strong disorder limit, which currently is an intractable problem. One option which is currently being explored is to introduce a higher derivative field theory, which seems to simplify many of the calculations presented in this work. This higher derivative field, $\tilde{\phi}$ is equivalent to the introduction of a field equal to the logarithm of the wave-function. Using this field, the insertions used to calculate the 2-point correlators are considerably simpler as they are now only the exponent of the field, $e^{\tilde{\phi}}$, instead of the insertion given by (3.4). This new approach is still in the early stages, but it offers to provide a simpler theory which would allow one to solve disordered problems in any dimension using the same method, and would also make it cleaner to include deterministic fields, such as external magnetic fields.

In conclusion, although the study of disordered systems has progressed much in the last fifty or so years, there is still much that needs to be done in the field, with analytically tractable approaches needed especially in higher dimensions, and for strong disordered systems. This dissertation has introduced one such theory, and has led to the possibility of an even more viable field theory for the study of disordered systems.

Appendix A
The Faddeev-Popov quantisation method

Suppose we have an integral $\int_{R^2} d^2x \exp\left[f(x^2)\right]$, where $f(x)$ is a function of distance only, and not the angle θ.

We can transform to polar coordinates and then integrate out the unphysical degree of freedom, θ. Thus

$$\int_{R^2} d^2x \exp\left[f(x^2)\right] = \int r\,dr \int d\theta \exp\left[f(r^2)\right] = 2\pi \int_0^\infty r\,dr \exp\left[f(r^2)\right].$$ (A.1)

We know the transformation to polar coordinates and can thus integrate out the θ degree of freedom. However, we usually have a manifold $M \subset R^n$ and we do not know how to transform from R^n to M. We need to have a procedure that will allow us to integrate out the unphysical degrees of freedom without explicitly knowing the exact transformation necessary to do so.

Choose $g(x)$ so that $g(x(\theta)) = 0$ has only one solution, say θ_0, where $x(\theta)$ is obtained by rotating x through an angle θ. Thus

$$\int_0^{2\pi} d\theta \delta(g(\theta)) = \frac{1}{\left|\frac{dg}{d\theta}\right|}\bigg|_{\theta=\theta_0}.$$ (A.2)

This implies that

$$\left|\frac{dg}{d\theta}\right|_{\theta=\theta_0} \int_0^{2\pi} d\theta \delta(g(\theta)) = 1.$$ (A.3)

If we have the integral $\int_{R^2} d^2x \exp\left[f(x^2)\right]$, then we can insert (A.3) to obtain

$$\begin{aligned}
\int_{R^2} d^2x \exp\left[f(x^2)\right] &= \int_{R^2} d^2x \left|\frac{dg}{d\theta}\right|_{\theta=\theta_0} \int_0^{2\pi} d\theta\, \delta(g(x(\theta))) \exp\left[f(x^2)\right] \\
&= \int_{R'^2} d^2x' \left|\frac{dg}{d\theta}\right|_{\theta=\theta_0}(x') \int_0^{2\pi} d\theta\, \delta(g(x')) \exp\left[f(x'^2)\right] \\
&= \int_0^{2\pi} d\theta \int_{R'^2} d^2x' \left|\frac{dg}{d\theta}\right|_{\theta=\theta_0}(x')\, \delta(g(x')) \exp\left[f(x'^2)\right],
\end{aligned}$$

where we have changed from the x coordinates to the x' coordinates by setting $x' = x(\theta)$. Since $\delta\big(g(x')\big)$ is independent of θ, we can integrate out the θ degree of freedom to give

$$\int_{R^2} d^2x \exp\left[f(x^2)\right] \propto 2\pi \int_{R'^2} \underbrace{d^2x' \left|\frac{dg}{d\theta}\right|_{\theta=\theta_0}(x')\, \delta(g(x'))}_{measure} \exp\left[f(x'^2)\right].$$ (A.4)

We have managed to integrate out the θ degree of freedom without knowing what transformation is needed. However, we had to introduce a more complex measure.

A.1 The Gribov problem

In the derivation of the Faddeev-Popov procedure, it was assumed that $g(x(\theta)) = 0$ has only one solution. There is however, *no* continuous and single valued function of x with this property.

Proof 1 *We assume there is a continuous and single valued function, $h(\theta) = f(x(\theta))$, then $h(0) = h(2\pi)$. If we plot any function $h(\theta)$ with the above conditions, it is easy to see that the function must cut the θ axis an even number of times. This implies that there is no continuous and single valued function with only one zero point.*

This fundamental error in the Faddeev-Popov procedure is usually bypassed by using perturbation theory to expand around one of the zero points. If the perturbation is small enough (i.e. the convergence radius is small) then the other zero points can be ignored in the procedure.

Appendix B
Methods for calculating determinants

In this appendix, we review several methods used to calculate determinants of positive definite second order differential operators, i.e.

$$\det\left(-a^2\frac{d^2}{dx^2}+b^2\right)=\det\left(\left[-a\frac{d}{dx}+b\right]\left[a\frac{d}{dx}+b\right]\right) \tag{B.1}$$

The methods discussed are the heat kernel method, the method of generalised ladder operators, as well as methods to obtain the determinant directly from identities.

B.1 Heat kernel method

To calculate determinants of the above form, we introduce the heat kernel defined as

$$h(t)=\operatorname{Tr}\exp\left(-tD^\dagger D\right), \tag{B.2}$$

where $D=a\frac{d}{dx}+b$.

Taking the Mellin transform of the heat kernel, we obtain the zeta function,

$$\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}h(t)\\
&= \frac{1}{\Gamma(s)}\sum_n\int_0^\infty dt\, t^{s-1}\exp\left(-t|\lambda_n|^2\right)\\
&= \sum_n|\lambda_n|^{-2s},
\end{aligned} \tag{B.3}$$

where λ_n is the eigenvalue of D.

Taking the derivative of the zeta function, and then the limit $s\to 0$, we have

$$\zeta'(s)\big|_{s=0}=-\sum_n\ln(|\lambda_n|^2)=-\ln\det(D^\dagger D). \tag{B.4}$$

Thus, we see that calculating the Mellin transform of the heat kernel is equivalent to calculating the determinant of the differential operator.

B.1.1 Calculating the zeta function

If we assume that the differential operator D has periodic boundary conditions excluding constant functions[11], then we see from (B.2) that the heat kernel can be written as

$$\begin{aligned}
h(t) &= {\sum_m}'\exp\left(-t\left[\frac{4a^2\pi^2}{L^2}m^2+b^2\right]\right)\\
&= \sum_m\exp\left(-t\left[\frac{4a^2\pi^2}{L^2}m^2+b^2\right]\right)-\exp\left(-tb^2\right),
\end{aligned} \tag{B.5}$$

---

[11]These boundary conditions are chosen to be the same as those generally used in this dissertation.

where the accent indicates that $m = 0$ is excluded from the sum.

Using the Jacobi identity [Ere53],

$$\sum_{m=-\infty}^{\infty} e^{-\alpha m^2} = \sqrt{\frac{\pi}{\alpha}} \sum_{l=-\infty}^{\infty} e^{-\pi^2 l^2/\alpha} \tag{B.6}$$

we can write the heat kernel as

$$h(t) = \frac{L}{2a\sqrt{\pi t}} \sideset{}{'}\sum_{l} \exp\left(-\frac{L^2}{4a^2 t}l^2 - tb^2\right) - \exp\left(-tb^2\right). \tag{B.7}$$

Using (B.7) in (B.3), we obtain

$$\zeta(s) = \frac{L}{2a\sqrt{\pi}\Gamma(s)} \sum_{l} \int_0^\infty dt\, t^{s-3/2} \exp\left(-\frac{L^2}{4a^2 t}l^2 - tb^2\right) - \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \exp\left(-tb^2\right)$$

$$= \frac{L\Gamma(s-\frac{1}{2})}{2|a|\sqrt{\pi}\Gamma(s)}|b|^{1-2s} - \mu^s|b|^{-2s}$$

$$+ \frac{L}{2|a|\sqrt{\pi}\Gamma(s)} \sideset{}{'}\sum_{l} \left(\frac{|l|L}{2|a||b|}\right)^{s-1/2} \int_0^\infty dy\, y^{s-3/2} \exp\left(-\frac{|l||b|L}{2|a|}[y + \frac{1}{y}]\right), \tag{B.8}$$

where μ is a dimensionful constant.

We are only interested in calculating $\zeta(s)$ to first order in s as all higher terms will give zero after taking the derivative to s, and then the limit $s \to 0$. Furthermore, since $\Gamma(s) \sim 1/s$, we can take the limit $s \to 0$ in the integral in (B.8), so that we obtain

$$\zeta(s) = -\frac{|b|L}{|a|}s + s\sideset{}{'}\sum_{l} \frac{1}{|l|} \exp\left(-\frac{|l||b|L}{|a|}\right) - \mu^s|b|^{-2s} + O(s^2)$$

$$= -\frac{|b|L}{|a|}s - 2s\ln\left[1 - \exp\left(-\frac{|b|L}{|a|}\right)\right] - \left(\frac{\mu}{|b|^2}\right)^s + O(s^2) \tag{B.9}$$

Using (B.4), we have

$$\ln\det(D^\dagger D) = -\left.\zeta'(s)\right|_{s=0}$$

$$= \frac{|b|L}{|a|} + 2\ln\left[1 - \exp\left(-\frac{|b|L}{|a|}\right)\right] - \ln\left(\frac{b^2}{\mu}\right), \tag{B.10}$$

thus

$$\left|\det\left(-a^2\frac{d^2}{dx^2} + b^2\right)\right| = \frac{\mu}{b^2}\left[\exp\left(\frac{|b|L}{2|a|}\right) - \exp\left(-\frac{|b|L}{2|a|}\right)\right]^2$$

$$= \frac{4\mu}{b^2}\sinh^2\left(\frac{|b|L}{2|a|}\right). \tag{B.11}$$

B.2 Generalised ladder operators

The method of using the generalised ladder operators [Sch92] is a method where the solutions of Hamiltonians which can be represented as "absolute squares" are constructed algebraically.

Thus if one has the Hamiltonian H^0

$$H^0 = -\frac{d^2}{dx^2} + V^0(x), \tag{B.12}$$

where V^0 is chosen, without any loss of generality, so that the ground state energy is zero. The ground state ψ_0 then satisfies the eigenvalue equation

$$H^0\psi_0 = \left[-\frac{d^2}{dx^2} + V^0(x)\right]\psi_0 = 0 \tag{B.13}$$

from which we obtain

$$V^0(x) = \frac{\psi_0''}{\psi_0} \tag{B.14}$$

and thus

$$H^0 = -\frac{d^2}{dx^2} + \frac{\psi_0''}{\psi_0}. \tag{B.15}$$

Introducing the ladder operators Q^+ and Q^-,

$$Q^\pm = \pm\frac{d}{dx} - \frac{\psi_0'}{\psi_0} \tag{B.16}$$

we have that

$$Q^\pm Q^\mp = -\frac{d^2}{dx^2} \mp \frac{\psi_0''}{\psi_0} + (1 \pm 1)\left(\frac{\psi_0'}{\psi_0}\right)^2 \tag{B.17}$$

If we define

$$V^1(x) = -V^0(x) + 2\left(\frac{\psi_0'}{\psi_0}\right)^2 \tag{B.18}$$

and

$$H^1 = -\frac{d^2}{dx^2} + V^1(x) \tag{B.19}$$

then we see that $H^0 = Q^-Q^+$ and $H^1 = Q^+Q^-$. Since $Q^- = (Q^+)^\dagger$, we have represented H^0 and its supersymmetric partner, H^1, as absolute squares.

Using the above formulation, we can now see how to algebraically construct the solution of the eigenvalue equation

$$H^0\psi_n^0 = E_n^0\psi_n^0 \tag{B.20}$$

Multiplying (B.20) by Q^+, we have

$$Q^+H^0\psi_n^0 = Q^+Q^-Q^+\psi_n^0 = H^1Q^+\psi_n^0 = E_n^0Q^+\psi_n^0 \tag{B.21}$$

Thus $Q^+\psi_n^0$ is an eigenstate of H^1 with an eigenvalue E_n^0, except for the ground state ψ_0^0.

Similarly, if ψ_n^1 is an eigenstate of H^1 with an eigenvalue E_n^1, then

$$H^1\psi_n^1 = E_n^1\psi_n^1, \tag{B.22}$$

and after multiplying (B.22) by Q^-, we have

$$Q^-H^1\psi_n^1 = Q^-Q^+Q^-\psi_n^1 = H^0Q^-\psi_n^1 = E_n^1Q^-\psi_n^1, \tag{B.23}$$

and thus $Q^-\psi_n^1$ is an eigenstate of H^0 with eigenvalue E_n^1.

We thus see that the spectra of the two Hamiltonians can be derived from one another. Also, the eigenfunctions of H^1 can be obtained by applying the up ladder operator Q^+ to the eigenfunctions of H^0, while the eigenfunctions of H^0 can be obtained from the eigenfunctions of H^1 by applying the down ladder operator, Q^-. These properties are useful if one of the eigenvalue problems is exactly solvable or more easily approximated than the other, since after solving the easier problem, the solutions of the other problem is obtained by applying the relevant ladder operator.

Note that the procedure may be extended by introducing another set of ladder operators which factorise H^1, allowing one to introduce an additional Hamiltonian, H^2. Thus one could introduce a set of ladder operators that are applied successively in order to obtain a Hamiltonian problem which can be solved easier. After solving that eigenvalue equation, the corresponding eigenfunctions of the original problem can be obtained by applying the ladder operators in the opposite direction.

B.3 Direct method from identities

To calculate the determinant in (B.1), we consider the following

$$\frac{d}{db}\ln\det\left(-a^2\frac{d^2}{dx^2}+b^2\right) = \sum_{m=1}^{\infty}\frac{4b}{\frac{4a^2\pi^2}{L^2}m^2+b^2}$$

$$= \lim_{x\to 0}\frac{bL^2}{a^2\pi^2}\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2+\alpha^2}, \tag{B.24}$$

where we have assumed periodic boundary conditions and $\alpha = \frac{bL}{2a\pi}$.

Using the identity [Gra94]

$$\sum_{m=1}^{\infty}\frac{\cos(nx)}{n^2+\alpha^2} = \frac{\pi}{2\alpha}\frac{\cosh(\alpha\pi-\alpha x)}{\sinh(\alpha\pi)} - \frac{1}{2\alpha^2} \tag{B.25}$$

we have that

$$\frac{d}{db}\ln\det\left(-a^2\frac{d^2}{dx^2}+b^2\right) = \frac{L}{a}\frac{\cosh\left(\frac{bL}{2a}\right)}{\sinh\left(\frac{bL}{2a}\right)} - \frac{2}{b} \tag{B.26}$$

Integrating over b and taking the exponent, we obtain

$$\det\left(-a^2\frac{d^2}{dx^2}+b^2\right)=\frac{C(a)}{b^2}\sinh^2\left(\frac{bL}{2a}\right),$$ (B.27)

where $C(a)$ is an integration constant that may depend on a.

Another direct method is to write the determinant as a product of the eigenvalues, so that

$$\det\left(-a^2\frac{d^2}{dx^2}+b^2\right)=\prod_{n=-\infty}^{\infty}{}'\left[\left(a\frac{2\pi n}{L}\right)^2+b^2\right]$$

$$=\left(\prod_{n=1}^{\infty}\left[\left(a\frac{2\pi n}{L}\right)^2\right]\right)^2\left(\prod_{n=1}^{\infty}\left[1+\left(\frac{bL}{2a}\right)^2\frac{1}{\pi^2 n^2}\right]\right)^2$$ (B.28)

Using the identity [Gra94]

$$\sinh(x)=x\prod_{n=1}^{\infty}\left[1+x^2\frac{1}{\pi^2 n^2}\right]$$ (B.29)

we find that

$$\det\left(-a^2\frac{d^2}{dx^2}+b^2\right)=\left(\prod_{n=1}^{\infty}\left[\left(a\frac{2\pi n}{L}\right)^2\right]\right)^2\left(\frac{2a}{bL}\right)^2\sinh^2\left(\frac{bL}{2a}\right)$$

$$=\frac{C(a)}{b^2}\sinh^2\left(\frac{bL}{2a}\right)$$ (B.30)

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