NON-COMMUTATIVE QUANTUM MECHANICS: PROPERTIES OF PIECEWISE CONSTANT POTENTIALS IN TWO DIMENSIONS

By

Jacobus Daniël Thom

Dissertation presented for the degree of DOCTOR OF PHILOSOPHY at the University of Stellenbosch.

Promotor : Professor F G Scholtz

Co-promotor : Professor H B Geyer

December 2010

DECLARATION

By submitting this dissertation electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the authorship owner thereof (unless to the extent explicitly otherwise stated) and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

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ABSTRACT

The aim of this thesis is threefold. Firstly, I give an overview of non-commutative quantum mechanics and build up a description of non-commutative piecewise constant potential wells in this context. Secondly, I look at some of the stationary properties of a finite non-commutative well using the mathematical tools laid out in the first part. Lastly, I investigate how non-commutativity affects the tunneling rate through a barrier. Throughout this work I give the normal commutative descriptions and results for comparsion.

OPSOMMING

Die doel van hierdie tesis is drievoudig. Eerstens gee ek 'n oorsig van niekommutatiewe kwantummeganika en bou daarmee 'n beskrywing van niekommutatiewe deelswyskonstante potensiaal putte op. Tweedens kyk ek na 'n paar van die stasionêre eienskappe van 'n eindige niekommutatiewe potensiaal put deur die wiskunde te gebruik wat in die eerste deel uiteengesit is. Laastens ondersoek ek hoe niekommutatiwiteit die spoed van tonneling deur 'n potensiaal wal beïnvloed. Dwarsdeur die hierdie hele tesis gee ek die normale kommutatiewe beskrywings en resultate vir maklike vergelyking.

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Introduction

Background and Purpose

When quantum field theory was first developed in the 1920s, it suffered from nonsensical divergences when calculating even relatively simple quantities. The notion of non-commuting space-time coordinates was introduced by Snyder [1] more than sixty years ago. In his article, Snyder stated that these divergences stem from the point interactions between fields and particles. If one were to introduce a minimum length scale on space-time, some (or perhaps even all) of these divergences might be removed in this way. Snyder then went on to show that one can still have a Lorentz invariant space-time, which has such a natural unit of length, by defining the coordinates as operators. These operators, however, can no longer commute. They are also different from the non-commutative operators used in this thesis due to the fact that their commutation relations also depend on momentum [1, 2],

$$[\hat{x}_i, \hat{x}_j] = \ell^2 \frac{i}{\hbar} \left(\hat{x}_i \hat{p}_j - \hat{x}_j \hat{p}_i \right), \tag{1}$$

where ℓ is the abovementioned minimal length. The new algebra in $3+1$ dimensions, in terms of these operators, can be found by means of a dimensional reduction as angular momentum generators of the higher dimensional Lorentz group, SO(1,4). In this way the generators of Snyder's space-time are naturally invariant under Lorentz transformations [2].

Around the same time as when Snyder wrote his article, the procedure by which quantum field theory could be renormalised to remove the inherent divergences contained in it was developed. This, combined with the fact that a theory which has an uncertainty relation for position (as expressed by the non-commutativity) is per definition non-local, led to non-commutative theories receiving little attention until quite recently.

However, as efforts started going into finding a theory that included gravity on a quantum level, it became clear that this theory would have to be fundamentally different from its predecessors which dealt with quantising the electromagnetic, weak and strong forces. The main reason for this being that gravity modifies the very frame in which everything unfolds, namely space-time [3]. This is not the case with the other forces. When one tries to make a position measurement with an accuracy greater than that of the Planck length a tremendous amount of energy is required. This energy will itself modify the precise geometry of space-time. One sees that a quantum theory of gravity implies an intrinsic length beyond which cannot be probed [4]. This uncertainty of position might then be described by coordinates which no longer commute.

Although the non-commutativity used nowadays is still between space coordinates, it is not exactly the same as that introduced by Snyder. Whereas Snyder gave the non-commuting coordinates in a way that preserves Lorentz invariance, the most basic formulation of the commutation relations that define the non-commutativity in three dimensions is not rotationally invariant. It is therefore necessary to twist these relations in some way to restore Galilean [5] or Poincaré [6, 7] invariance.

In two dimensions, which is the focus of this thesis, however, no such twist is necessary. Indeed it can easily be shown that a rotation of the coordinates $\hat{x}$ and $\hat{y}$ has no effect on their commutation relation even in the presence of a non-commutativity parameter. I will show this explicitly at the end of the section on non-commutative matching conditions per sector of angular momentum, namely Section 2.3.1.

Work done in the last few decades has greatly added to the support for non-commutative space-time [8, 9]. This support in turn (re-)kindled activity studying the effects of non-commutativity in a wide variety of physics disciplines. These include the field that gave rise to the notion of spatial non-commutativity, namely quantum field theory [4]. The way that non-commutativity effects quantum mechanics and quantum many-body systems [5, 10–17], quantum electrodynamics [18–20], the standard model [21] and cosmology [22, 23] has also been studied since then.

Even though so much work was being done studying the effects of non-commutativity, it has only been in the last few years that any significant attention was paid to the formal aspects of a non-commutative quantum mechanical theory. Some of the ideas that make up such an interpretational framework were first used to give a precise definition of the non-commutative potential well and give the spectrum of an infinite well in such a non-commutative setting [24]. In a subsequent article this was expanded upon to give a consistent theoretical framework of non-commutative quantum mechanics [25].

The purpose of this thesis is threefold. First, to give an overview of this non-commutative quantum mechanics and use the mathematical technology it contains to describe how non-commutative piecewise constant wells can be constructed. Second, to look at some of the stationary properties of a finite non-commutative well. And third, to see how non-commutativity affects the tunneling rate through a barrier.

The second part depends greatly on investigating the effect that non-commutativity has on the matching conditions at the boundaries of such a well and the subsequent change to the coefficients of the wavefunction. These changes hold possible physical consequences which can hopefully guide one in the analysis and construction of three-dimensional theories, which are complicated by the fact that non-commutativity breaks rotational invariance. Although scattering, which forms a major part of the results given here, has been studied in detail in non-commutative quantum field theories [4, 26], much less work has been done in the context of non-commutative quantum mechanics in either two or three dimensions [27–29]. Furthermore, these studies use a first order expansion of the non-commutative parameter of the Schrödinger equation that has undergone either a Bopp-shift or alternatively written as a Moyal product. This approach works for analytic potentials, but is inadequate for something like the well. The commuting coordinates which are introduced in this way are also hard to give physical meaning to, which complicates the interpretation of the results. The formulation and mathematics of the non-commutative quantum mechanics given in this thesis on the other hand allows one to perform exact, if numerical, calculations of quantities such as bound state energies, phase shifts, differential and total scattering cross-sections.

The third part I mentioned explores a very simple case of time evolution in the non-commutative quantum mechanical framework used throughout this thesis. Tunneling in general is a non-perturbative phenomenon and quite sensitive to the parameters of a system. This lets one hope that such events might enhance any effect non-commutativity may have, making detection more likely. The specific case I explore here is that of a Gaussian wave packet tunneling through an annulus barrier. At the time of writing I could find no similar calculations in the literature and therefore present a simple numerical comparison of the probability that a particle (as given by the wave packet) has tunneled after a certain amount of time.

Organisation of Thesis

This thesis is divided into four chapters. The first chapter deals with the formalism of non-commutative quantum mechanics as well as some interpretational and technical aspects of it. A lot of the notation used throughout the rest of this thesis is also fixed here. I start off with a brief outline of how normal commutative quantum mechanics is formulated and then build up non-commutative quantum mechanics by analogy. In this chapter I also go into the detail of position measurement and probability and the differences (and similarities) it has with ordinary quantum mechanics. Finally I show that probability is still conserved under time evolution.

The second chapter contains some more mathematical concepts. Here I explain concepts such as the non-commutative piecewise constant potential and how such a potential is constructed. I then go on to introduce the object one thinks of as the wavefunction in a non-commutative setting. This section is followed by an important section which starts the explanation of how the various coefficients of the wavefunction are found for the various regions in a piecewise constant potential by means of non-commutative matching conditions which differ completely from commutative ones. This explanation is concluded in a section on how the basis in which the matching conditions are given (the Fock basis) is related to the one of wavefunctions (the position basis). This chapter is rounded off by a short illustration of how the non-commutative matching conditions affect the wavefunction.

The third chapter is the first one to contain results obtained from calculations using the mathematics and formalism of the first two chapters. It deals with the time-independent properties of a two-dimensional non-commutative potential well. As such it gives a detailed implementation of all the necessary mathematics and then gives the results of the various numerical calculations performed, such as bound state energies, phase shifts and cross-sections.

The fourth and final chapter takes a first look at how a non-commutative system behaves under time evolution. To this end a two-dimensional non-commutative annulus is placed inside an infinite non-commutative well. A Gaussian wave packet is constructed in the inner most region of the annulus and allowed to evolve in time. A comparison of commutative and non-commutative results is then made, which concludes the thesis.

Work Based on Thesis

I would like to make one final comment before the start of the thesis proper. When I started writing some of the preliminary material for this thesis I also started working on an article based on the results contained in Chapter 3. In May of 2009, I made a presentation of the work that would be contained in this article at the *International Workshop on Path Integrals, Coherent States and Non-commutative Geometry*. Subsequently, the article, *Bound state energies and phase shifts of a non-commutative well*, was published in Journal of Physics A in October 2009 [30].

CHAPTER 1
Non-commutative Quantum Mechanics

This work studies some of the properties of simple systems when the components of position no longer commute, as it does in normal quantum mechanics. It is only natural, therefore, to start this work with a chapter that details the formalism of this 'non-commutative' quantum mechanics. As such, I give here the basic concepts and notation used throughout the rest of the work, as well as the framework in which results should be interpreted. Most of the discussion here will be restricted to two dimensions, as it will be in the rest of this work.

1.1 Formalism

1.1.1 Review of Commutative Quantum Mechanics

To best understand where and how non-commutative quantum mechanics differs from commutative quantum mechanics (and also where they are the same), it is a good idea to review the basic tenets of the latter [25, 31, 32]. The basic postulates of quantum mechanics are:

- *Postulate 1.* For each classical observable A one has a corresponding operator $\hat{A}$ at the quantum level. The only possible values that one can measure in an experiment are the eigenvalues of the operator $\hat{A}$.

- *Postulate 2.* Each state of the system is uniquely represented by a so-called density operator $\hat{\rho}$. This operator must be Hermitian, non-negative and have a trace of unity.

- *Postulate 3.* If the system is described by such an operator $\hat{\rho}$, then the probability of finding an eigenvalue a of observable $\hat{A}$ when a measurement is done is given by $P(a) = \langle a| \hat{\rho} |a\rangle = \mathrm{tr}(\hat{\pi}_a \hat{\rho})$, where $\hat{\pi}_a = |a\rangle \langle a|$ is a Projection Valued Measure and $|a\rangle$ is the normalised eigenstate of $\hat{A}$ corresponding to the eigenvalue.

- *Postulate 4.* After a measurement the system is in the state defined by the specific (eigen)value that was obtained, $\hat{\rho} = |a\rangle \langle a|$, and a subsequent measurement immediately following the first will yield a with probability one. This postulate is sometimes known as von Neuman's projection postulate.

If one wants to describe a particle moving in two dimensions, one typically has R^2 as the configuration space and the wavefunction $\psi(\vec{x})$ is an element of the Hilbert space which is L^2, the space of square integrable functions. The inner product on L^2 is defined to be $(\phi, \psi) = \int \mathrm{d}^2\vec{x}\ \bar{\phi}(\vec{x})\psi(\vec{x})$, where the bar over ϕ indicates complex conjugation. A very useful commonly used shorthand is Dirac notation where one associates $|\psi\rangle$ with elements of the Hilbert space, $\psi(\vec{x})$, and $\langle\psi|$ with elements of the dual space, the linear functionals. These linear functionals map the elements of the Hilbert space onto complex numbers, $\langle\phi|\psi\rangle = (\phi, \psi)$.

Now that something has been said about the states of the quantum system, it is time to say something about the operators which act on these states, namely the observables. The Hermitian operators $\hat{x}_i$ and $\hat{p}_i$, acting on the elements of L^2, together form the abstract Heisenberg algebra,

$$\begin{aligned} [\hat{x}_i, \hat{x}_j] &= 0 \\ [\hat{p}_i, \hat{p}_j] &= 0 \\ [\hat{x}_i, \hat{p}_j] &= i\hbar\delta_{i,j}, \quad \text{where } i = 1, 2. \end{aligned} \tag{1.1}$$

An irreducible unitary representation of this algebra on L^2 is

$$\hat{x}_i\psi(\vec{x}) = x_i\psi(\vec{x}) \tag{1.2}$$

$$\hat{p}_i\psi(\vec{x}) = -i\hbar\frac{\partial\psi(\vec{x})}{\partial x_i}, \quad i = 1, 2, \tag{1.3}$$

which is the wave mechanical representation. Since by the Stone-von Neumann theorem all such irreducible unitary representations are unitarily equivalent [33], it does not matter which one one chooses to make calculations with.

1.1.2 Introducing Non-commutativity

Non-commutativity in the context of this work means mathematically that $[\hat{x}, \hat{y}] = 0$ no longer holds [24, 25], but is replaced by,

$$[\hat{x}, \hat{y}] = i\theta, \tag{1.4}$$

where θ can be taken as positive without loss of generality. This commutation relation lets one introduce some boson creation and annihilation operators and define $\hat{x}$ and $\hat{y}$ in terms of them,

$$\hat{x} = \sqrt{\frac{\theta}{2}}(b + b^\dagger) \tag{1.5}$$

$$\hat{y} = \frac{\sqrt{\theta}}{i\sqrt{2}}(b - b^\dagger), \tag{1.6}$$

where $[b, b^\dagger] = 1$ as usual. This means that the non-commutative configuration space, which is defined by $\hat{x}$ and $\hat{y}$, is isomorphic to boson Fock space. Therefore, if one denotes the configuration space by $\mathcal{H}_c$ (since Fock space is of course a Hilbert space itself), then one can define the non-commutative configuration space explicitly as,

$$\mathcal{H}_c = \overline{\text{span}}_c\{|n\rangle \equiv \frac{1}{\sqrt{n!}}(b^\dagger)^n |0\rangle\}_{n=0}^{n=\infty}, \tag{1.7}$$

where $\overline{\text{span}}_c$ is the closure of the span taken over all complex numbers and $|0\rangle$ the Fock vacuum state annihilated by b, i.e. $b|0\rangle = 0$.

With the non-commutative configuration space in place, one can move on to defining the space of quantum states. This so-called quantum Hilbert space, denoted by $\mathcal{H}_q$, is a generalisation of the L^2 space one has commutatively. If one considers a bounded operator acting on non-commutative configuration space, then $\mathcal{H}_q$ is just the set of all these operators with finite trace. In other words, the quantum Hilbert space is the set of Hilbert-Schmidt operators such that

$$\mathcal{H}_q = \{\hat{\psi}(\hat{x}, \hat{y}) : \hat{\psi}(\hat{x}, \hat{y}) \in \mathcal{B}(\mathcal{H}_c), \text{tr}_c \left(\hat{\psi}(\hat{x}, \hat{y})^\dagger, \hat{\psi}(\hat{x}, \hat{y})\right) < \infty\}, \tag{1.8}$$

where $\mathcal{B}(\mathcal{H}_c)$ is the set of bounded operators on $\mathcal{H}_c$ and tr_c denotes that the trace is taken over non-commutative configuration space. This definition also naturally leads one to define an inner product on the space, namely

$$(\hat{\phi}(\hat{x},\hat{y}),\hat{\psi}(\hat{x},\hat{y})) = \mathrm{tr}_c\left(\hat{\phi}(\hat{x},\hat{y})^\dagger\hat{\psi}(\hat{x},\hat{y})\right), \tag{1.9}$$

so that $\mathcal{H}_q$ is indeed a Hilbert space [34].

I will make a note on notation here. Following [24], the states in the non-commutative configuration space are given by $|\cdot\rangle$ in Dirac notation. The states of $\mathcal{H}_q$ on the other hand are denoted by round brackets, $\hat{\psi}(\hat{x},\hat{y}) = \left|\hat{\psi}\right)$, and, as commutatively, the elements of the dual space of $\mathcal{H}_q$ are $\left(\hat{\phi}\right|$. Combining one element of each space once again maps these onto complex numbers, $\left(\hat{\phi}|\hat{\psi}\right) = \left(\hat{\phi},\hat{\psi}\right) = \mathrm{tr}_c\left(\hat{\phi}^\dagger\hat{\psi}\right)$.

The next step is to find a unitary representation on $\mathcal{H}_q$ of the non-commutative Heisenberg algebra, which is given here by

$$\begin{aligned} [\hat{x},\hat{y}] &= i\theta \\ [\hat{p}_x,\hat{p}_y] &= 0 \\ [\hat{x},\hat{p}_x] &= i\hbar \\ [\hat{y},\hat{p}_y] &= i\hbar, \end{aligned} \tag{1.10}$$

where $\hat{x},\hat{y},\hat{p}_x$ and $\hat{p}_y$ are all Hermitian.
Using capital letters to distinguish operators acting on the quantum Hilbert space from those acting on non-commutative configuration space (following the notation in [24] once again), one defines such a representation in terms of $\hat{X}$, $\hat{Y}$, $\hat{P}_x$ and $\hat{P}_y$ as follows,

$$\hat{X}\hat{\psi}(\hat{x},\hat{y}) = \hat{x}\hat{\psi}(\hat{x},\hat{y}) \tag{1.11}$$

$$\hat{Y}\hat{\psi}(\hat{x},\hat{y}) = \hat{y}\hat{\psi}(\hat{x},\hat{y}) \tag{1.12}$$

$$\hat{P}_x\hat{\psi}(\hat{x},\hat{y}) = \frac{\hbar}{\theta}[\hat{y},\hat{\psi}(\hat{x},\hat{y})] \tag{1.13}$$

$$\hat{P}_y\hat{\psi}(\hat{x},\hat{y}) = -\frac{\hbar}{\theta}[\hat{x},\hat{\psi}(\hat{x},\hat{y})]. \tag{1.14}$$

Position is therefore defined to act by left multiplication of the corresponding coordinate and momentum acts adjointly. One thing to remember about the $\theta \to 0$ limit of these kinds of non-commutative operators is that it involves a careful process of going from the non-commutative description and mathematics to the commutative equivalents. As such, it is not simply the straightforward situation of setting *theta* to zero, where one would end up with $\hat{x} = 0$, and so forth.

I will make a note here on the use of the †-sign used to denote Hermitian conjugation in what follows and elsewhere in this work. Since capitalised operators and small operators are hardly ever used together, the †-sign on operators such as $\hat{P}^\dagger$ or $B^\dagger$ for example indicate Hermitian conjugation on the quantum Hilbert space, whereas if it is used on operators such as $\hat{x}^\dagger$ or $b^\dagger$ it indicates Hermitian conjugation with respect to the non-commutative configuration space.

Some further useful definitions are,

$$\hat{X} = \sqrt{\frac{\theta}{2}}(B + B^\dagger) \tag{1.15}$$

$$\hat{Y} = \frac{\sqrt{\theta}}{i\sqrt{2}}(B - B^\dagger) \tag{1.16}$$

$$\hat{P} = \hat{P}_x + i\hat{P}_y \tag{1.17}$$

$$\hat{P}^\dagger = \hat{P}_x - i\hat{P}_y, \tag{1.18}$$

where $\hat{P}^2 = \hat{P}_x^2 + \hat{P}_y^2 = \hat{P}^\dagger\hat{P} = \hat{P}\hat{P}^\dagger$.
If one defines the actions of B and $B^\dagger$ as

$$B\hat{\psi}(\hat{x}, \hat{y}) = b\hat{\psi}(\hat{x}, \hat{y}) \tag{1.19}$$

$$B^\dagger\hat{\psi}(\hat{x}, \hat{y}) = b^\dagger\hat{\psi}(\hat{x}, \hat{y}), \tag{1.20}$$

then

$$\hat{P}\hat{\psi}(\hat{x}, \hat{y}) = -i\hbar\sqrt{\frac{2}{\theta}}[b, \hat{\psi}(\hat{x}, \hat{y})] \tag{1.21}$$

$$\hat{P}^\dagger\hat{\psi}(\hat{x}, \hat{y}) = i\hbar\sqrt{\frac{2}{\theta}}[b^\dagger, \hat{\psi}(\hat{x}, \hat{y})]. \tag{1.22}$$

With the above in place and taking the commutative postulates to hold with the replacement of L^2 by $\mathcal{H}_q$ one has a solid interpretational framework.

1.1.3 Position Measurement

A note must be made on the measurement of position. Since the commutator of these variables no longer vanishes it is impossible to make simultaneous measurements of a particle's x- and y-coordinates. The best that one can do in this case is to construct a state of which the uncertainty in either direction is minimal. This will ensure that the notion of position is maintained in as much as that a particle is localised around a certain point. While there may be different ways of solving this problem [35], the procedure outlined here uses the notion of Operator Valued Measures.

In terms of the non-commutative configuration space, namely boson Fock space, the states of least uncertainty are well known. These are the normalised coherent states,

$$|z\rangle = e^{-z\bar{z}/2} e^{zb^\dagger} |0\rangle \, , \tag{1.23}$$

where $|z\rangle$ is shorthand for $|z, \bar{z}\rangle$ and $z = \frac{1}{\sqrt{2\theta}}(x + iy)$. The coherent states also provide an overcomplete basis for non-commutative configuration space. One can form an operator out of two of these states by way of an outer product,

$$|z) = |z\rangle \langle z| \, . \tag{1.24}$$

Since $\mathrm{tr}_c(|z\rangle \langle z|z\rangle \langle z|) < \infty$,

$$\begin{aligned} \mathrm{tr}_c(|z\rangle \langle z|z\rangle \langle z|) &= \sum_{n=0}^{\infty} \langle n|z\rangle \langle z|n\rangle \\ &= \sum_{n=0}^{\infty} e^{-|z|^2} \frac{z^n}{\sqrt{n!}} \frac{\bar{z}^n}{\sqrt{n!}} \\ &= e^{-|z|^2} \sum_{n=0}^{\infty} \frac{(|z|^2)^n}{n!} \\ &= 1 \end{aligned} \tag{1.25}$$

these operators are of Hilbert-Schmidt type and are elements of $\mathcal{H}_q$. These states also have the property that

$$B\,|z) = b\,|z\rangle \langle z| = e^{-z\bar{z}/2} [b, e^{zb^\dagger}]\, |0\rangle \langle z| = z e^{-z\bar{z}/2} e^{zb^\dagger} |0\rangle \langle z| = z\,|z) \, , \tag{1.26}$$

which leads one to naturally consider x and y, which make up z, to be coordinates. However, this also means that $|z)$ is an eigenstate of the operator B which is not Hermitian. Subsequently, they are not mutually orthogonal. As a matter of fact, since the overlap of two such states is $(z|w) = |\langle z|w\rangle|^2$ and

$$\begin{aligned}
\langle z|w\rangle &= \sum_{n=0}^{\infty} \langle z|n\rangle \langle n|w\rangle \\
&= \sum_{n=0}^{\infty} e^{-|z|^2/2}\frac{\bar{z}^n}{\sqrt{n}} e^{-|w|^2/2}\frac{w^n}{\sqrt{n}} \\
&= e^{-(|z|^2+|w|^2)/2+\bar{z}w} \\
\Rightarrow |\langle z|w\rangle|^2 &= e^{-|z-w|^2}, \qquad (1.27)
\end{aligned}$$

it is clear that $(z|w)$ will always be non-zero except when the two points in question are infinitely separated. This is what makes position measurement so difficult in non-commutative quantum mechanics. A way of dealing with this difficulty has been developed in the context of open quantum systems [36], namely Positive Operator Valued Measures.

In order to find such a measure that will allow one to give a consistent probabilistic interpretation one starts by finding a resolution of the identity in terms of these $|z)$. Whereas $|z\rangle$ provided an overcomplete basis for $\mathcal{H}_c$, the $|z)$ provide an overcomplete basis for $\mathcal{H}_q$. An identity operator on the quantum Hilbert space in terms of these states is given by,

$$1_q = \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\pi}\,|z)\,e^{\overleftarrow{\frac{\partial}{\partial \bar{z}}}\overrightarrow{\frac{\partial}{\partial z}}}\,(z|\,, \qquad (1.28)$$

where the arrows indicate the direction of differentiation. This identity can be proven by using an expansion for an operator in terms of the coherent states. This expansion works as follows. One starts with an expansion of some operator $\hat{a}$ in term of coherent states [37],

$$\hat{a} = \int \mathrm{d}z\,\mathrm{d}\bar{z}\,a(z,\bar{z})\,|z\rangle\,\langle z|\,. \qquad (1.29)$$

To find $a(z,\bar{z})$ one starts by looking at the diagonal elements of $\hat{a}$,

$$\langle w|\,\hat{a}\,|w\rangle = \int \mathrm{d}z\,\mathrm{d}\bar{z}\,a(z,\bar{z})\,\langle w|z\rangle\,\langle z|w\rangle\,, \tag{1.30}$$

where

$$\begin{aligned}\langle z|w\rangle &= e^{-|z|^2/2-|w|^2/2}\,\langle 0|\,e^{\bar{z}b}e^{wb^\dagger}\,|0\rangle \\ &= e^{-|z|^2/2-|w|^2/2}\sum_{n,m}\frac{1}{\sqrt{n!m!}}\bar{z}^n w^m\,\langle n|m\rangle \\ &= e^{-|z|^2/2-|w|^2/2}\sum_{n}\frac{1}{n!}(\bar{z}w)^n = e^{-|z|^2/2-|w|^2/2+\bar{z}w}.\end{aligned} \tag{1.31}$$

Therefore,

$$\begin{aligned}\langle w|\,\hat{a}\,|w\rangle &= \int \mathrm{d}z\,\mathrm{d}\bar{z}\,a(z,\bar{z})e^{-|z|^2/2-|w|^2/2+z\bar{w}-|z|^2/2-|w|^2/2+\bar{z}w} \\ &= \int \mathrm{d}z\,\mathrm{d}\bar{z}\,a(z,\bar{z})e^{-|z-w|^2} \qquad \mid \text{Let } v=z-w \\ &= \int \mathrm{d}v\,\mathrm{d}\bar{v}\,a(v+w,\bar{v}+\bar{w})e^{-|v|^2} \quad \mid \text{Use } f(x+a)=e^{a\frac{d}{dx}}f(x) \\ &= \left[\int \mathrm{d}v\,\mathrm{d}\bar{v}\,e^{-|v|^2}e^{v\frac{\partial}{\partial w}+\bar{v}\frac{\partial}{\partial \bar{w}}}\right]a(w,\bar{w}).\end{aligned} \tag{1.32}$$

Consider a typical term in the operator acting on $a(w,\bar{w})$,

$$\frac{1}{j!}\int \mathrm{d}v\,\mathrm{d}\bar{v}\,e^{-|v|^2}(v\frac{\partial}{\partial w}+\bar{v}\frac{\partial}{\partial \bar{w}})^j = \frac{1}{j!}\int \mathrm{d}v\,\mathrm{d}\bar{v}\,e^{-|v|^2}\sum_{\ell}\binom{j}{\ell}v^{j-\ell}\bar{v}^{\ell}(\frac{\partial}{\partial w})^{j-\ell}(\frac{\partial}{\partial \bar{w}})^{\ell}, \tag{1.33}$$

where we used the binomial theorem to expand $(v\frac{\partial}{\partial w}+\bar{v}\frac{\partial}{\partial \bar{w}})^j$. Setting $v=qe^{i\varphi}$ lets one do the integral over v and $\bar{v}$ easily,

$$\begin{aligned}\int \mathrm{d}v\,\mathrm{d}\bar{v}\,e^{-|v|^2}v^{j-\ell}\bar{v}^{\ell} &= \underbrace{\int_0^{2\pi}\mathrm{d}\varphi\,e^{i\varphi(j-2\ell)}}_{2\pi\delta_{j,2\ell}}\int_0^\infty \mathrm{d}q\,q^{j+1}e^{-q^2} \\ &= 2\pi\,\delta_{j,2\ell}\int_0^\infty \mathrm{d}q\,q^{2\ell+1}e^{-q^2} = \delta_{j,2\ell}\,\pi\ell!.\end{aligned} \tag{1.34}$$

Therefore one finds that,

$$\begin{aligned}\sum_j \frac{1}{j!}\int \mathrm{d}v\,\mathrm{d}\bar{v}\,e^{-|v|^2}(v\frac{\partial}{\partial w}+\bar{v}\frac{\partial}{\partial \bar{w}})^j &= \sum_\ell \frac{(2\ell!)}{(2\ell!)(\ell!)^2}\pi\ell!(\frac{\partial}{\partial w}\frac{\partial}{\partial \bar{w}})^\ell \\ &= \pi e^{\frac{\partial^2}{\partial w\partial\bar{w}}}. \end{aligned} \tag{1.35}$$

Finally by inverting equation (1.32) one is left with the following result,

$$a(w,\bar{w}) = \frac{1}{\pi}e^{-\frac{\partial^2}{\partial w\partial\bar{w}}}\left\langle w\right|\hat{a}\left|w\right\rangle. \tag{1.36}$$

Therefore, the expansion for a genereal operator $\hat{a}$ in terms of coherent states is,

$$\hat{a} = \frac{1}{\pi}\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\left(e^{-\frac{\partial^2}{\partial z\partial\bar{z}}}\left\langle z\right|\hat{a}\left|z\right\rangle\right)\left|z\right\rangle\left\langle z\right|. \tag{1.37}$$

One small step before applying this expansion is to note that $\left(z|\hat{\psi}\right) = \left\langle z\right|\hat{\psi}\left|z\right\rangle$,

$$\begin{aligned}\left(z|\hat{\psi}\right) &= \mathrm{tr}_c(\left|z\right\rangle\left\langle z\right|\hat{\psi}) \\ &= \sum_{n=0}^{\infty}\left\langle n|z\right\rangle\left\langle z\right|\hat{\psi}\left|n\right\rangle \\ &= \sum_n e^{\frac{-|z|^2}{2}}\frac{z^n}{\sqrt{n!}}\left\langle z\right|\hat{\psi}\left|n\right\rangle \\ &= \left\langle z\right|\hat{\psi}\sum_n e^{\frac{-|z|^2}{2}}\frac{z^n}{\sqrt{n!}}\frac{(b^\dagger)^n}{\sqrt{n!}}\left|0\right\rangle \\ &= \left\langle z\right|\hat{\psi}e^{\frac{-|z|^2}{2}}e^{zb^\dagger}\left|0\right\rangle \\ &= \left\langle z\right|\hat{\psi}\left|z\right\rangle. \end{aligned} \tag{1.38}$$

Then one uses the coherent state expansion to write,

$$\begin{aligned}\hat{\psi} &= \left|\hat{\psi}\right) = \int \mathrm{d}z\,\mathrm{d}\bar{z}\left|z\right\rangle\left\langle z\right|\left(\frac{1}{\pi}e^{-\frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial z}}\left\langle z\right|\hat{\psi}\left|z\right\rangle\right) \\ &= \int\frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\pi}\left|z\right)e^{-\frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial z}}\left(z|\hat{\psi}\right). \end{aligned} \tag{1.39}$$

Let me write $u(z,\bar{z}) = |z)$ and $v_n(z,\bar{z}) = \frac{\partial^n}{\partial z^n}\left(z|\hat{\psi}\right)$. Then the integral reads,

$$\int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\pi} u(z,\bar{z}) e^{-\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}} v_0(z,\bar{z}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\pi} u(z,\bar{z}) \frac{\partial^n}{\partial \bar{z}^n} v_n(z,\bar{z}). \tag{1.40}$$

Consider a general term in the sum and use integration by parts to write,

$$\begin{aligned}
\int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\pi} u(z,\bar{z}) \frac{\partial^n}{\partial \bar{z}^n} v_n(z,\bar{z}) &= -\int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\pi} \frac{\partial u}{\partial \bar{z}} \frac{\partial^{n-1} v_n}{\partial \bar{z}^{n-1}} \\
&= (-1)^2 \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\pi} \frac{\partial^2 u}{\partial \bar{z}^2} \frac{\partial^{n-2} v_n}{\partial \bar{z}^{n-2}} \\
&= (-1)^3 \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\pi} \frac{\partial^3 u}{\partial \bar{z}^3} \frac{\partial^{n-3} v_n}{\partial \bar{z}^{n-3}} \\
&\;\;\vdots \\
&= (-1)^n \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\pi} \frac{\partial^n u}{\partial \bar{z}^n} v_n(z,\bar{z}),
\end{aligned} \tag{1.41}$$

where the boundary terms are zero each time due to the Gaussian $e^{\frac{-|z|^2}{2}}$ term in both u and v_n. This means that

$$\begin{aligned}
\left|\hat{\psi}\right) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (-1)^n \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\pi} \frac{\partial^n u}{\partial \bar{z}^n} v_n(z,\bar{z}) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\pi} \frac{\partial^n u}{\partial \bar{z}^n} v_n(z,\bar{z}) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\pi} \left(\frac{\partial^n}{\partial \bar{z}^n} |z)\right) \left(\frac{\partial^n}{\partial z^n} \left(z|\hat{\psi}\right)\right) \\
&= \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\pi} |z)\, e^{\overleftarrow{\frac{\partial}{\partial \bar{z}}}\overrightarrow{\frac{\partial}{\partial z}}} \left(z|\hat{\psi}\right),
\end{aligned} \tag{1.42}$$

which proves the identity.

A more elegant way of proving this is by starting from a more general version of this resolution of identity. Using the fact that on non-commutative configuration space one can write the identity operator in terms of the coherent states as $1_c = \int \frac{\mathrm{d}^2 z}{\pi} |z\rangle \langle z|$ one sees that

$$1_q = \int \frac{\mathrm{d}^2 z \mathrm{d}^2 w}{\pi^2} |z,w)\,(z,w|\,, \tag{1.43}$$

where $|z,w) = |z\rangle\langle w|$, since,

$$\begin{aligned}\int \frac{\mathrm{d}^2z\mathrm{d}^2w}{\pi^2}|z,w)\left(z,w|\hat{\psi}\right) &= \int \frac{\mathrm{d}^2z\mathrm{d}^2w}{\pi^2}|z\rangle\langle w|\langle z|\hat{\psi}|w\rangle \\ &= \int \frac{\mathrm{d}^2z\mathrm{d}^2w}{\pi^2}|z\rangle\langle z|\hat{\psi}|w\rangle\langle w| \\ &= \hat{\psi}. \end{aligned} \tag{1.44}$$

The states $|z,w)$ still obey $B|z,w) = b|z\rangle\langle w| = z|z,w)$ and as such they are still states that can be used to describe position. However, they have another degree of freedom in w. Whether this extra degree of freedom has any physical meaning, or whether it is just useful mathematically remains to be seen.

The identity operator given just in terms of z can now essentially be obtained by performing the integration over w,

$$\begin{aligned} &\int \frac{\mathrm{d}^2z\mathrm{d}^2w}{\pi^2}|z,w)(z,w| \\ =& \int \frac{\mathrm{d}^2z\mathrm{d}^2v}{\pi^2}|z,z+v)(z,z+v| \\ =& \int \frac{\mathrm{d}^2z\mathrm{d}^2v}{\pi^2}|z\rangle\langle 0|e^{-\frac{|z+v|^2}{2}}e^{\bar{z}b+\bar{v}b}e^{-\frac{|z+v|^2}{2}}e^{zb^\dagger+vb^\dagger}|0\rangle\langle z| \\ =& \int \frac{\mathrm{d}^2z\mathrm{d}^2v}{\pi^2}|z\rangle\langle z|e^{\bar{v}b}e^{-(z\bar{v}+\bar{z}v+|v|^2)}e^{vb^\dagger}|z\rangle\langle z| \\ =& \int \frac{\mathrm{d}^2z\mathrm{d}^2v}{\pi^2}e^{-|v|^2}e^{zb^\dagger}|0\rangle\langle 0|e^{-z\bar{z}+\bar{z}b}e^{\bar{v}\overleftarrow{\frac{\partial}{\partial\bar{z}}}}e^{v\frac{\partial}{\partial z}}e^{-z\bar{z}+zb^\dagger}|0\rangle\langle 0|e^{\bar{z}b} \\ =& \int \frac{\mathrm{d}^2z\mathrm{d}^2v}{\pi^2}e^{-|v|^2}|z)\sum_{n,m=0}^{\infty}\frac{1}{n!m!}\bar{v}^n v^m\frac{\overleftarrow{\partial^n}}{\partial\bar{z}^n}\frac{\partial^m}{\partial z^m}(z| \\ =& \int \frac{\mathrm{d}^2z}{\pi^2}|z)\sum_{n,m=0}^{\infty}\frac{1}{n!m!}\int_0^{2\pi}\mathrm{d}\phi e^{i\phi(m-n)}\int_0^{\infty}\mathrm{d}re^{-r^2}r^{n+m+1}\frac{\overleftarrow{\partial^n}}{\partial\bar{z}^n}\frac{\partial^m}{\partial z^m}(z| \\ =& \frac{2}{\pi}\int \mathrm{d}^2z\,|z)\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\frac{\Gamma(n+1)}{2}\frac{\overleftarrow{\partial^n}}{\partial\bar{z}^n}\frac{\partial^n}{\partial z^n}(z| \\ =& \frac{1}{\pi}\int \mathrm{d}^2z\,|z)e^{\overleftarrow{\frac{\partial}{\partial\bar{z}}}\frac{\partial}{\partial z}}(z|. \end{aligned} \tag{1.45}$$

Going from the fourth last to the third last line involved writing v as $v = re^{i\phi}$ and reaching the second last line made use of the facts that $\int_0^{2\pi}\mathrm{d}\phi e^{i\phi(m-n)} = 2\pi\delta_{n,m}$ and $\int_0^{\infty}\mathrm{d}re^{-r^2}r^{2n+1} = \frac{1}{2}\Gamma(n+1)$.

With the identity operator on the quantum Hilbert space in place, one can now continue with finding an operator valued measure to use in constructing a probabilistic framework for position measurement. If one were to introduce the operator,

$$\hat{\pi}_z = \frac{1}{2\pi\theta}\,|z)\,e^{\overleftarrow{\frac{\partial}{\partial \bar{z}}}\frac{\partial}{\partial z}}\,(z|\,, \tag{1.46}$$

then it is immediately clear that

$$\int \mathrm{d}x\,\mathrm{d}y\,\hat{\pi}_z = \frac{1}{\pi}\int \mathrm{d}^2z\,|z)\,e^{\overleftarrow{\frac{\partial}{\partial \bar{z}}}\frac{\partial}{\partial z}}\,(z| = 1_q. \tag{1.47}$$

This operator $\hat{\pi}_z$ is self-adjoint on the quantum Hilbert space, positive and integrates to the identity. It therefore meets the criteria of being a Positive Operator Valued Measure as defined in [36]. A consistent probabilistic interpretation can now be given for finding a particle at a point (x, y) (remember $z = \frac{1}{\sqrt{2\theta}}(x + iy)$). One proceeds as one would commutatively, except that the objects in question now come from the quantum Hilbert space. In other words, if the probability of finding a particle at (x, y) is given by $P(x, y)$ and the system is in the state $\hat{\rho} = \left|\hat{\psi}\right)\left(\hat{\psi}\right|$, then

$$P(x, y) = \mathrm{tr}_q(\hat{\pi}_z\hat{\rho}) = \left(\hat{\psi}\right|\hat{\pi}_z\left|\hat{\psi}\right) \geq 0. \tag{1.48}$$

This definition also lets one quickly verify that the probabilities sum to one,

$$\int \mathrm{d}x\,\mathrm{d}y\,P(x, y) = \int \mathrm{d}x\,\mathrm{d}y\left(\hat{\psi}\right|\hat{\pi}_z\left|\hat{\psi}\right) = \left(\hat{\psi}|\hat{\psi}\right) = 1, \tag{1.49}$$

assuming that the states $\left|\hat{\psi}\right)$ are normalised.

Although the $\hat{\pi}_z$ are Hermitian on $\mathcal{H}_q$ they are not orthogonal as would usually be the case commutatively. Due to this fact, statements about the final state of a system no longer apply, since an immediate repetition of a measurement may give one a different result than the first measurement. The von Neumann projection postulate (Postulate 4) no longer applies as written down in the beginning of this chapter. Instead, as was done in [36], one can write the operators $\hat{\pi}_z$ as the product of another operator (so-called detection operators) and its Hermitian conjugate (on quantum Hilbert space), since they are positive,

$$\hat{\pi}_z = \hat{A}_z^\dagger \hat{A}_z, \tag{1.50}$$

where any $\hat{A}_z$ of the following form will do,

$$\hat{A}_z = \hat{U}_z \hat{\pi}_z^{\frac{1}{2}}, \tag{1.51}$$

and $\hat{U}_z$ is unitary.
The fourth postulate gets replaced under these circumstances by [36]

- *Postulate 4.* If the system starts in the pure state $\left|\hat{\psi}\right)$, then after a measurement the system will be in the state $\left|\hat{\phi}\right) = \frac{\hat{A}_z}{\sqrt{(\hat{\psi}|\hat{A}^\dagger \hat{A}|\hat{\psi})}}\left|\hat{\psi}\right)$. Similarly, if it was in the mixed state $\hat{\rho}$, then it will be in the state $\hat{\rho}_z = \frac{\hat{A}\hat{\rho}\hat{A}^\dagger}{\mathrm{tr}(\hat{A}^\dagger \hat{A}\hat{\rho})}$ afterwards.

As one can see, in both these cases the final state of the system depend on an unitary transformation over which one has no control. This is yet another way in which the inherent uncertainty associated with introducing non-commutative co-ordinates manifests itself.

1.1.4 Conservation of Probability

One thing that is important to verify in general, but also specifically for the chapter on time dependent tunneling, is that probability is still a conserved quantity. To check this one needs to show that the norm of a state does not change under time evolution of the system. Consider a Hamiltonian with some general potential,

$$\hat{H} = \frac{\hat{P}^2}{2\mu} + \hat{V}(\hat{X}, \hat{Y}), \tag{1.52}$$

where $\hat{P}^2 = \hat{P}_x^2 + \hat{P}_y^2 = \hat{P}^\dagger \hat{P} = \hat{P}\hat{P}^\dagger$ for easy reference.

Just as one requires the potential to be real in commutative quantum mechanics, one requires here that the potential is Hermitian when written in terms of operators acting on non-commutative configuration space,

$$\hat{V}^\dagger(\hat{x}, \hat{y}) = \hat{V}(\hat{x}, \hat{y}). \tag{1.53}$$

The time dependent Schrödinger equation is given by,

$$i\hbar\frac{\partial\hat{\psi}}{\partial t} = \hat{H}\hat{\psi}, \tag{1.54}$$

where $\hat{\psi} = \hat{\psi}(\hat{x}, \hat{y}, t)$ here.

Analogously to what one does in the commutative case, one multiplies (1.54) from the left by $\hat{\psi}^\dagger$ and the Hermitian conjugate of (1.54) by $\hat{\psi}$ from the right. Therefore, one obtains,

$$\begin{aligned} i\hbar\hat{\psi}^\dagger\frac{\partial\hat{\psi}}{\partial t} &= \hat{\psi}^\dagger\left[\frac{\hbar^2}{2\mu\theta^2}\Big([\hat{y},[\hat{y},\hat{\psi}]] + [\hat{x},[\hat{x},\hat{\psi}]]\Big) + \hat{V}(\hat{x},\hat{y})\hat{\psi}\right] & (1.55)\\ -i\hbar\frac{\partial\hat{\psi}^\dagger}{\partial t}\hat{\psi} &= \left[\frac{\hbar^2}{2\mu\theta^2}\Big([\hat{y},[\hat{y},\hat{\psi}^\dagger]] + [\hat{x},[\hat{x},\hat{\psi}^\dagger]]\Big) + \hat{\psi}^\dagger\hat{V}(\hat{x},\hat{y})\right]\hat{\psi}. & (1.56)\end{aligned}$$

Subtracting the second equation from the first gives one,

$$\frac{\partial}{\partial t}(\hat{\psi}^\dagger\hat{\psi}) = \frac{\hbar}{2i\mu\theta^2}\left[\hat{\psi}^\dagger\Big([\hat{y},[\hat{y},\hat{\psi}]] + [\hat{x},[\hat{x},\hat{\psi}]]\Big) - \Big([\hat{y},[\hat{y},\hat{\psi}^\dagger]] + [\hat{x},[\hat{x},\hat{\psi}^\dagger]]\Big)\hat{\psi}\right]. \tag{1.57}$$

Consider the following constructions,

$$\begin{aligned} [\hat{x},\hat{\psi}^\dagger[\hat{x},\hat{\psi}]] &= \hat{\psi}^\dagger[\hat{x},[\hat{x},\hat{\psi}]] + [\hat{x},\hat{\psi}^\dagger][\hat{x},\hat{\psi}] & (1.58)\\ [\hat{x},[\hat{x},\hat{\psi}^\dagger]\hat{\psi}] &= [\hat{x},[\hat{x},\hat{\psi}^\dagger]]\hat{\psi} + [\hat{x},\hat{\psi}^\dagger][\hat{x},\hat{\psi}] & (1.59)\\ \Rightarrow \hat{\psi}^\dagger[\hat{x},[\hat{x},\hat{\psi}]] &= [\hat{x},\hat{\psi}^\dagger[\hat{x},\hat{\psi}]] - [\hat{x},\hat{\psi}^\dagger][\hat{x},\hat{\psi}] & (1.60)\\ \Rightarrow [\hat{x},[\hat{x},\hat{\psi}^\dagger]]\hat{\psi} &= [\hat{x},[\hat{x},\hat{\psi}^\dagger]\hat{\psi}] - [\hat{x},\hat{\psi}^\dagger][\hat{x},\hat{\psi}] & (1.61)\end{aligned}$$

Using the above one sees that one can write equation (1.57) purely as the sum of commutators,

$$\begin{aligned} \frac{\partial}{\partial t}(\hat{\psi}^\dagger\hat{\psi}) &= \frac{\hbar}{2i\mu\theta^2}\Big[\Big([\hat{y},\hat{\psi}^\dagger[\hat{y},\hat{\psi}]] - [\hat{y},\hat{\psi}^\dagger][\hat{y},\hat{\psi}] + [\hat{x},\hat{\psi}^\dagger[\hat{x},\hat{\psi}]] - [\hat{x},\hat{\psi}^\dagger][\hat{x},\hat{\psi}]\Big)\\ &\quad - \Big([\hat{y},[\hat{y},\hat{\psi}^\dagger]\hat{\psi}] - [\hat{y},\hat{\psi}^\dagger][\hat{y},\hat{\psi}] + [\hat{x},[\hat{x},\hat{\psi}^\dagger]\hat{\psi}] - [\hat{x},\hat{\psi}^\dagger][\hat{x},\hat{\psi}]\Big)\Big]\\ &= \frac{\hbar}{2i\mu\theta^2}\Big[[\hat{y},\hat{\psi}^\dagger[\hat{y},\hat{\psi}]] - [\hat{y},[\hat{y},\hat{\psi}^\dagger]\hat{\psi}] + [\hat{x},\hat{\psi}^\dagger[\hat{x},\hat{\psi}]] - [\hat{x},[\hat{x},\hat{\psi}^\dagger]\hat{\psi}]\Big]\\ &= \frac{\hbar}{2i\mu\theta^2}\Big[[\hat{y},\hat{\psi}^\dagger[\hat{y},\hat{\psi}] - [\hat{y},\hat{\psi}^\dagger]\hat{\psi}] + [\hat{x},\hat{\psi}^\dagger[\hat{x},\hat{\psi}] - [\hat{x},\hat{\psi}^\dagger]\hat{\psi}]\Big]\\ &= \frac{\hbar}{2i\mu\theta^2}([\hat{y},\hat{j}_1] + [\hat{x},\hat{j}_2]), & (1.62)\end{aligned}$$

where

$$\hat{j}_1 = \hat{\psi}^\dagger[\hat{y},\hat{\psi}] - [\hat{y},\hat{\psi}^\dagger]\hat{\psi} \tag{1.63}$$

$$\hat{j}_2 = \hat{\psi}^\dagger[\hat{x},\hat{\psi}] - [\hat{x},\hat{\psi}^\dagger]\hat{\psi}. \tag{1.64}$$

The operators on the RHS of equation (1.62) are per construction of Hilbert-Schmidt type. Doing a trace of them over non-commutative configuration space therefore yields finite results per definition. Since the RHS of (1.62) consists of commutators and one can take cyclic permutations within a trace without changing its value, the trace of that side of the equation is zero. Therefore one finds that

$$\mathrm{tr}_c\left(\frac{\partial}{\partial t}(\hat{\psi}^\dagger\hat{\psi})\right) = \frac{\partial}{\partial t}\mathrm{tr}_c\left(\hat{\psi}^\dagger\hat{\psi}\right) = \frac{\partial}{\partial t}\left(\hat{\psi}|\hat{\psi}\right) = 0, \tag{1.65}$$

which indicates that the total probability of the system is conserved, as one would expect.

CHAPTER 2
Non-commutative Piecewise Constant Potentials

In this chapter I will discuss the constant potentials used in this work in their non-commutative form. There are two potentials used, namely the 'square' (or in this case circular) well and the annulus. I will also discuss related topics here, such as the form of the wavefunction and the way matching conditions are constructed.

2.1 Constructing Non-commutative Piecewise Constant Potentials

As soon as one chooses spatial coordinates to no longer commute with each other, one runs into the problem of defining a sharp boundary. The $x-$ and $y-$coordinates no longer commuting implies that for a potential depending on these coordinates, if one fixes, say, the x-coordinate, then one can never fix the y-coordinate, but rather it will have a certain spread determined by θ. However, it is exactly this additional feature that one wishes to investigate and therefore one requires a way of mathematically capturing this inherent fuzziness, while still allowing one to make relatively simple analyses.

A solution was proposed by Scholtz *et al.* [24, 25] by using projection operators to define the various regions of constant potential. While they only treated the non-commutative well in that paper, the ideas discussed there easily extend to any number of piecewise constant potentials. The basic idea is that one constructs a sum of states from the non-commutative configuration space (here Fock states). The range of states summed over in each projection operator corresponds to an area having the geometry of a circular annulus, all of these annuli being concentric with each other, which in turn specifies the different regions of the potential, beginning with an inner disk centered at the origin.

To make these statements somewhat more concrete, one now defines a general potential consisting of N_V piecewise constant sections,

$$\hat{V} = \sum_{i=1}^{N_V} V_i\, P_i \tag{2.1}$$

$$P_i = \sum_{n=N_{i-1}+1}^{N_i} |n\rangle\langle n|\,, \quad \text{with } N_0 = -1 \text{ and } N_{i=N_V} = \infty, \tag{2.2}$$

where V_i are constants and P_i is the projection operator that demarcates a region in the potential. From the definition of the P_i's it is easy to see that they are Hermitian on the non-commutative configuration space and that they obey the standard properties of projection operators, such as $P_i^2 = 1$ and $P_i\, P_j = 0$ when $i \neq j$. In general the P_i's cover all of the non-commutative configuration space, so that

$$\sum_{i=1}^{N_V} P_i = 1_c, \tag{2.3}$$

where 1_c denotes the identity operator on $\mathcal{H}_c$.

To give an example of a $\hat{V}$, for the non-commutative well one has two regions, $N_V = 2$, and

$$\begin{aligned} \hat{V} &= V_1\, P_1 + V_2\, P_2 \\ &= V_1 \sum_{n=0}^{N_1} |n\rangle\langle n| + V_2 \sum_{n=N_1+1}^{\infty} |n\rangle\langle n|\,. \end{aligned} \tag{2.4}$$

In order to connect the non-commutative potential to its commutative counterpart, the radial operator is introduced. Since I am working in two dimensions, this operator is

$$\hat{r}^2 = \hat{x}^2 + \hat{y}^2, \tag{2.5}$$

which can be written in terms of the creation and annihilation operators b and $b^\dagger$ as

$$\hat{r}^2 = \theta(2b^\dagger b + 1). \tag{2.6}$$

In the case of the non-commutative well, the boundary between the inner and outer regions is defined by what N_1 is. From the definition of the radial operator it is clear that the corresponding commutative radius should be

$$R^2 = \theta(2N_1 + 1). \tag{2.7}$$

The expression for the radius shows that in the departure from normal commutative quantum mechanics one picks up a quantisation of area in units of θ. The effects of this quantisation slowly vanish as $\theta \to 0$ and go over into commutative results. This was shown explicitly for the infinite well in [25] and one finds the same trend in the results presented in this work. The important thing to remember here is that as $\theta \to 0$ one needs the commutative potential to stay the same. That means that R should remain fixed. The correct way then of taking the $\theta \to 0$ limit is to increase N_1 as θ decreases.

2.2 Non-commutative Wavefunction

To find the non-commutative wavefunction one must solve the Schrödinger equation. Since one is working with piecewise constant potentials, one needs to solve the following,

$$\hat{P}^2\hat{\psi} = k^2\hbar^2\hat{\psi}, \tag{2.8}$$

where $k^2 = 2\mu(E-V)/\hbar^2$, V constant.

Although, $\hat{\psi}$ satisfies the Schrödinger equation, the object in non-commutative quantum mechanics that is closer to what one thinks of as the wavefunction in normal commutative quantum mechanics is $\left(z|\hat{\psi}\right) = \langle z|\,\hat{\psi}\,|z\rangle$, as discussed in Chapter 1. Using equation (2.8), one can find the partial differential equation obeyed by the wavefunction. One starts by taking equation (2.8) and forming the inner product with $\langle z|$ and $|z\rangle$,

$$\begin{aligned}\langle z|\,\hat{P}^2\hat{\psi}\,|z\rangle &= k^2\hbar^2\,\langle z|\,\hat{\psi}\,|z\rangle \\ &= k^2\hbar^2\left(z|\hat{\psi}\right). \end{aligned} \tag{2.9}$$

One now takes the LHS of equation (2.9) and replaces $\hat{P}$ with its action in terms of b and $b^\dagger$. In other words, one writes

$$\langle z|\,\hat{P}^2\hat{\psi}\,|z\rangle = \frac{2\hbar^2}{\theta}\,\langle z|\,[b^\dagger,[b,\hat{\psi}]]\,|z\rangle \tag{2.10}$$

$$= \langle z|\,b^\dagger[b,\hat{\psi}] - [b,\hat{\psi}]b^\dagger\,|z\rangle \tag{2.11}$$

If one keeps in mind that the coherent state $|z\rangle$ is shorthand for $e^{-z\bar{z}/2}e^{zb^\dagger}|0\rangle$ then with the help of the identity

$$[\hat{A}, f(\hat{B})] = c\frac{\partial f(\hat{B})}{\partial \hat{B}}, \tag{2.12}$$

where c must be a constant and $c = [\hat{A}, \hat{B}]$, it is easily shown that

$$b|z\rangle = be^{-z\bar{z}/2}e^{zb^\dagger}|0\rangle = e^{-z\bar{z}/2}[b, e^{zb^\dagger}]|0\rangle = z\, e^{-z\bar{z}/2}e^{zb^\dagger}|0\rangle = z|z\rangle\,. \tag{2.13}$$

Similarly,

$$\langle z| b^\dagger = \langle z| \bar{z}. \tag{2.14}$$

Therefore one finds that

$$\langle z| \hat{P}^2\hat{\psi}|z\rangle \quad = \quad \frac{2\hbar^2}{\theta}\langle z|\bar{z}[b,\hat{\psi}] - [b,\hat{\psi}]b^\dagger|z\rangle\,. \tag{2.15}$$

In order to write this equation completely as a PDE, one writes the actions of $b^\dagger$ on $|z\rangle$ and b on $\langle z|$ as partial derivatives. Consider

$$\begin{aligned}
\frac{\partial}{\partial z}|z\rangle &= \frac{\partial}{\partial z}e^{-z\bar{z}/2}e^{zb^\dagger}|0\rangle \\
&= \left(-\frac{1}{2}\bar{z} + b^\dagger\right)|z\rangle \\
\Rightarrow b^\dagger|z\rangle &= \left(\frac{\partial}{\partial z} + \frac{1}{2}\bar{z}\right)|z\rangle\,.
\end{aligned} \tag{2.16}$$

In a similar fashion, it can be shown that

$$\langle z| b = \langle z|\left(\frac{\overleftarrow{\partial}}{\partial \bar{z}} + \frac{1}{2}z\right), \tag{2.17}$$

where the arrow over the partial derivative indicates that it acts to the left.
The effect of the outer commutator in $\langle z| \hat{P}^2\hat{\psi}|z\rangle$ is then fully taken into account by

writing

$$\begin{aligned}
\langle z| \hat{P}^2\hat{\psi} |z\rangle &= \frac{2\hbar^2}{\theta}\left[\bar{z}\langle z|[b,\hat{\psi}]|z\rangle - \langle z|[b,\hat{\psi}]\left(\frac{\partial}{\partial z}+\frac{1}{2}\bar{z}\right)|z\rangle\right] \\
&= \frac{2\hbar^2}{\theta}\left[\bar{z}\langle z|[b,\hat{\psi}]|z\rangle - \left(\frac{\partial}{\partial z}+\bar{z}\right)\langle z|[b,\hat{\psi}]|z\rangle\right] \\
&= -\frac{2\hbar^2}{\theta}\frac{\partial}{\partial z}\langle z|[b,\hat{\psi}]|z\rangle , \qquad (2.18)
\end{aligned}$$

where one picks up an additional $-\frac{1}{2}\bar{z}$ when pulling the partial derivative past the $\langle z|$. The second commutator acts as follows,

$$\begin{aligned}
\langle z|[b,\hat{\psi}]|z\rangle &= \langle z| b\hat{\psi} - z\hat{\psi}|z\rangle \\
&= \langle z|\left(\frac{\overleftarrow{\partial}}{\partial \bar{z}}+\frac{1}{2}z\right)\hat{\psi}|z\rangle - z\langle z|\hat{\psi}|z\rangle \\
&= \left(\frac{\partial}{\partial \bar{z}}+z\right)\langle z|\hat{\psi}|z\rangle - z\langle z|\hat{\psi}|z\rangle \\
&= \frac{\partial}{\partial \bar{z}}\langle z|\hat{\psi}|z\rangle , \qquad (2.19)
\end{aligned}$$

where once again one picks up an additional $\frac{1}{2}z$ when switching the derivative from acting to the left to acting to the right.

Combining (2.18) and (2.19) gives us the PDE obeyed by the non-commutative wave-function,

$$\begin{aligned}
-\frac{2\hbar^2}{\theta}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\left(z|\hat{\psi}\right) &= k^2\hbar^2\left(z|\hat{\psi}\right) \\
\Rightarrow -\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\left(z|\hat{\psi}\right) &= \frac{\theta k^2}{2}\left(z|\hat{\psi}\right). \qquad (2.20)
\end{aligned}$$

If one were to write $\left(z|\hat{\psi}\right)$ as a function of z and $\bar{z}$ and then transform to coordinates x and y, where $z = (x+iy)/\sqrt{2\theta}$, one finds that equation (2.20) becomes

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi(x,y) = k^2\psi(x,y), \qquad (2.21)$$

which is immediately recognisable as the standard Schrödinger equation for a particle in a constant potential.

This derivation would seem to suggest that the commutative and non-commutative wavefunctions are the same and that introducing non-commutativity has changed nothing. However, while it is true that for a completely free particle non-commutativity does not have any physical effect [38], that is not the case when there is a potential, even if it is a piecewise constant one. This is due to the fact that, while the form of the wavefunction is indeed the same in all regions, non-commutativity enforces different boundary conditions than one uses commutatively. The different boundary conditions lead to different coefficients in the wavefunction, which in turn lead to the myriad of differences between commutative and non-commutative quantum mechanics.

In the next section I derive the matching conditions used in a general system of piecewise constant potentials.

2.3 Matching Conditions

This section deals with non-commutative matching conditions and how they differ from the commutative ones [25, 30]. This section is very important, since, as has been alluded in the previous section, the mathematics described here drive all the departures from standard commutative physics.

Though one ends up with different matching conditions, to derive them one starts analogously to the commutative case with the Schrödinger equation,

$$\frac{\hat{P}^2}{2\mu}\hat{\psi} + \sum_{i=1}^{N_V} V_i\, P_i \hat{\psi} = E\hat{\psi}. \tag{2.22}$$

Next, one constructs the solutions to the Schrödinger equation in each region separately with the same energy,

$$\frac{\hat{P}^2}{2\mu}\hat{\psi}_i + V_i P_i \hat{\psi}_i = E\hat{\psi}_i, \quad \text{where } \ i = 1..N_V. \tag{2.23}$$

Then one makes the ansatz that the solution to the full Schrödinger equation, $\hat{\psi}$, is the sum over all regions of the projection operator times the solution for a specific region,

$$\hat{\psi} = \sum_{i=1}^{N_V} P_i \hat{\psi}_i. \tag{2.24}$$

Substituting (2.24) into (2.22) gives one,

$$\frac{\hat{P}^2}{2\mu}\sum_{i=1}^{N_V} P_i\hat{\psi}_i + \sum_{i=1}^{N_V} V_i\, P_i \left(\sum_{j=1}^{N_V} P_j\hat{\psi}_j\right) = E\sum_{i=1}^{N_V} P_i\hat{\psi}_i. \tag{2.25}$$

The term containing the potential simplifies dramatically due to how the P_i's interact,

$$\begin{aligned}\sum_{i=1}^{N_V} V_i\, P_i \left(\sum_{j=1}^{N_V} P_j\hat{\psi}_j\right) &= \sum_{i,j} V_i\, P_i\, P_j\hat{\psi}_j & (2.26)\\ &= \sum_{i=1}^{N_V} V_i P_i\hat{\psi}_i. & (2.27)\end{aligned}$$

One can also multiply equation (2.23) by P_i from the left to obtain,

$$P_i\frac{\hat{P}^2}{2\mu}\hat{\psi}_i + V_i P_i\hat{\psi}_i = E\, P_i\hat{\psi}_i. \tag{2.28}$$

Comparing (2.28) and (2.25) for a specific value of i one sees that for the ansatz to hold without further comment, therefore, would require that

$$P_i\frac{\hat{P}^2}{2\mu}\hat{\psi}_i \stackrel{?}{=} \frac{\hat{P}^2}{2\mu}P_i\hat{\psi}_i, \tag{2.29}$$

which is obviously not the case, since $\hat{P}$ and the P_i's do not commute. However, one can force the ansatz to be consistent by doing the following. Consider,

$$\frac{\hat{P}^2}{2\mu}P_i\hat{\psi}_i = P_i\frac{\hat{P}^2}{2\mu}\hat{\psi}_i + \frac{1}{2\mu}[\hat{P}^2, P_i]\hat{\psi}_i. \tag{2.30}$$

By defining new operators

$$\Omega_i = [\hat{P}^2, P_i], \quad i = 1, 2, ..N_V, \tag{2.31}$$

and requiring that

$$\sum_{i=1}^{N_V} \Omega_i\hat{\psi}_i = 0, \tag{2.32}$$

one generates a consistency equation necessary for the ansatz to hold. This consistency equation takes the place of the standard matching of the wavefunction and its derivative that one has commutatively, since it forces certain conditions on the $\hat{\psi}_i$.

While equation (2.32) may be difficult to solve in general, the definition of the projection operators (2.2), as well as the momentum ((1.21), (1.22)), simplify matters greatly [24]. By using these definitions in (2.31) we obtain the following,

$$\begin{aligned} \Omega_i &= [\hat{P}^2, P_i] = (\hat{P}\,P_i)\hat{P}^\dagger + (\hat{P}^\dagger\,P_i)\hat{P} + (\hat{P}\hat{P}^\dagger\,P_i) \\ &= -i\hbar\sqrt{\frac{2}{\theta}}[b, P_i]\hat{P}^\dagger + i\hbar\sqrt{\frac{2}{\theta}}[b^\dagger, P_i]\hat{P} + \frac{2\hbar^2}{\theta}[b, [b^\dagger, P_i]]. \end{aligned} \tag{2.33}$$

Consider each term,

$$\begin{aligned} [b, P_i] &= \sum_{n=N_{i-1}+1}^{N_i} b\,|n\rangle\langle n| - \sum_{n=N_{i-1}+1}^{N_i} |n\rangle\langle n|\,b \\ &= \sum_{n=N_{i-1}+1}^{N_i} \sqrt{n}\,|n-1\rangle\langle n| - \sum_{n=N_{i-1}+1}^{N_i} \sqrt{n+1}\,|n\rangle\langle n+1| \\ &= \sum_{n=N_{i-1}}^{N_i-1} \sqrt{n+1}\,|n\rangle\langle n+1| - \sum_{n=N_{i-1}+1}^{N_i} \sqrt{n+1}\,|n\rangle\langle n+1| \\ &= \sqrt{N_{i-1}+1}\,|N_{i-1}\rangle\langle N_{i-1}+1| - \sqrt{N_i+1}\,|N_i\rangle\langle N_i+1| \end{aligned} \tag{2.34}$$

$$\begin{aligned} \left[b^\dagger, P_i\right] &= \sum_{n=N_{i-1}+1}^{N_i} \sqrt{n+1}\,|n+1\rangle\langle n| - \sum_{n=N_{i-1}+1}^{N_i} \sqrt{n}\,|n\rangle\langle n-1| \\ &= \sqrt{N_i+1}\,|N_i+1\rangle\langle N_i| - \sqrt{N_{i-1}+1}\,|N_{i-1}+1\rangle\langle N_{i-1}| \end{aligned} \tag{2.35}$$

$$\begin{aligned} \left[b, [b^\dagger, P_i]\right] &= [b, \sqrt{N_i+1}\,|N_i+1\rangle\langle N_i| - \sqrt{N_{i-1}+1}\,|N_{i-1}+1\rangle\langle N_{i-1}|] \\ &= (N_i+1)(|N_i\rangle\langle N_i| - |N_i+1\rangle\langle N_i+1|) \\ &\quad -(N_{i-1}+1)(|N_{i-1}\rangle\langle N_{i-1}| - |N_{i-1}+1\rangle\langle N_{i-1}+1|). \end{aligned} \tag{2.36}$$

Using equations (2.34)-(2.36) and looking at the matrix elements of $\sum_{i=1}^{N_V}\Omega_i\hat{\psi}_i$, it is then immediately clear that the only non-zero terms in the sum are of the form

$$\langle N_j| \sum_{i=1}^{N_V} \Omega_i \hat{\psi}_i \, |\ell\rangle \tag{2.37}$$

$$\text{and} \qquad \langle N_j + 1| \sum_{i=1}^{N_V} \Omega_i \hat{\psi}_i \, |\ell\rangle \, , \tag{2.38}$$

where $j = 1, 2, ...N_V - 1$.

However, since it is furthermore the case that $\langle N_j| \sum_{i=1}^{N_V} \Omega_i \hat{\psi}_i \, |\ell\rangle$ and $\langle N_j + 1| \sum_{i=1}^{N_V} \Omega_i \hat{\psi}_i \, |\ell\rangle$ are only non-zero when $i = j$ or $i = j + 1$ the large sum in the definition of the consistency equation (2.32) breaks up into $(N_V - 1)$ pairs of equations that must be simultaneously satisfied,

$$0 = \langle N_i| \, \Omega_i \hat{\psi}_i \, |\ell\rangle + \langle N_i| \, \Omega_{i+i} \hat{\psi}_{i+i} \, |\ell\rangle \tag{2.39}$$

$$0 = \langle N_i + 1| \, \Omega_i \hat{\psi}_i \, |\ell\rangle + \langle N_i + 1| \, \Omega_{i+i} \hat{\psi}_{i+i} \, |\ell\rangle \, , \tag{2.40}$$

where $i = 1, 2, ...N_V - 1$, $\forall \ell \geq 0$.

Since the Ω_i's were introduced to ease the writing of the consistency equation, one would like to eliminate them and write the above equations in terms of $\hat{\psi}_i$ only. This is easily done using once again the action of $\hat{P}$ and $\hat{P}^\dagger$ (equations (1.21) and (1.22)) and the various terms that make up a specific Ω_i, (2.34)-(2.36). For some specific i therefore,

$$\begin{aligned}
\langle N_i|\,\Omega_i\hat{\psi}_i\,|\ell\rangle &= -i\hbar\sqrt{\frac{2}{\theta}}\Big(\sqrt{N_{i-1}+1}\,\langle N_i|N_{i-1}\rangle\,\langle N_{i-1}+1|\,\hat{P}^{\dagger}\hat{\psi}_i\,|\ell\rangle \\
&\quad -\sqrt{N_i+1}\,\langle N_i|N_i\rangle\,\langle N_i+1|\,\hat{P}^{\dagger}\hat{\psi}_i\,|\ell\rangle\Big) \\
&\quad +i\hbar\sqrt{\frac{2}{\theta}}\Big(\sqrt{N_i+1}\,\langle N_i|N_i+1\rangle\,\langle N_i|\,\hat{P}\hat{\psi}_i\,|\ell\rangle \\
&\quad -\sqrt{N_{i-1}+1}\,\langle N_i|N_{i-1}+1\rangle\,\langle N_{i-1}|\,\hat{P}\hat{\psi}_i\,|\ell\rangle\Big) \\
&\quad +\frac{2\hbar^2}{\theta}\Big[(N_i+1)\,\langle N_i|\,(|N_i\rangle\,\langle N_i|-|N_i+1\rangle\,\langle N_i+1|)\hat{\psi}_i\,|\ell\rangle \\
&\quad -(N_{i-1}+1)\,\langle N_i|\,(|N_{i-1}\rangle\,\langle N_{i-1}|-|N_{i-1}+1\rangle\,\langle N_{i-1}+1|)\hat{\psi}_i\,|\ell\rangle\Big] \\
&= i\hbar\sqrt{\frac{2}{\theta}}\sqrt{N_i+1}\,\langle N_i+1|\,\hat{P}^{\dagger}\hat{\psi}_i\,|\ell\rangle+\frac{2\hbar^2}{\theta}(N_i+1)\,\langle N_i|\,\hat{\psi}_i\,|\ell\rangle \\
&= \frac{2\hbar^2}{\theta}(N_i+1)\,\langle N_i|\,\hat{\psi}_i\,|\ell\rangle-\frac{2\hbar^2}{\theta}\sqrt{N_i+1}\,\langle N_i+1|\,b^{\dagger}\hat{\psi}_i-\hat{\psi}_i b^{\dagger}\,|\ell\rangle \\
&= \frac{2\hbar^2}{\theta}\sqrt{N_i+1}\sqrt{\ell+1}\,\langle N_i+1|\,\hat{\psi}_i\,|\ell+1\rangle\,. \qquad (2.41)
\end{aligned}$$

Similarly,

$$\langle N_i|\,\Omega_{i+1}\hat{\psi}_{i+1}\,|\ell\rangle = -\frac{2\hbar^2}{\theta}\sqrt{N_i+1}\sqrt{\ell+1}\,\langle N_i+1|\,\hat{\psi}_{i+i}\,|\ell+1\rangle \qquad (2.42)$$

$$\langle N_i+1|\,\Omega_i\hat{\psi}_i\,|\ell\rangle = -\frac{2\hbar^2}{\theta}\sqrt{N_i+1}\sqrt{\ell}\,\langle N_i|\,\hat{\psi}_i\,|\ell-1\rangle \qquad (2.43)$$

$$\langle N_i+1|\,\Omega_{i+i}\hat{\psi}_{i+i}\,|\ell\rangle = \frac{2\hbar^2}{\theta}\sqrt{N_i+1}\sqrt{\ell}\,\langle N_i|\,\hat{\psi}_{i+i}\,|\ell-1\rangle\,. \qquad (2.44)$$

This means that for each i and for all $\ell > 0$ the matching conditions become,

$$\langle N_i+1|\,\hat{\psi}_i\,|\ell+1\rangle = \langle N_i+1|\,\hat{\psi}_{i+1}\,|\ell+1\rangle \qquad (2.45)$$

$$\langle N_i|\,\hat{\psi}_i\,|\ell-1\rangle = \langle N_i|\,\hat{\psi}_{i+1}\,|\ell-1\rangle\,. \qquad (2.46)$$

2.3.1 Matching Conditions per Sector of Angular Momentum

Due to how the projection operators are defined, the potential is radially symmetric. One can therefore find solutions of the Hamiltonian that correspond to a specific value of angular momentum, which I call $\hat{\psi}_m$ or $\hat{\psi}_{m,i}$ when referring to a specific region in the potential, and use them for an expansion of the full $\hat{\psi}$. Using the results of [25], it is a simple matter to show that the form of $\hat{\psi}_m$ given in [24] is indeed the correct one corresponding to an angular momentum state. Since elements of the quantum Hilbert space, such as $\hat{\psi}$, are operators which act on non-commutative configuration space, one can write $\hat{\psi}$ in its most general form as,

$$\hat{\psi} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{j,k}(b^\dagger)^j b^k. \tag{2.47}$$

In much the same way, one defines $\hat{\psi}_m$ to be the subset of such operators where the difference between j and k is always m. In other words,

$$\begin{aligned} \hat{\psi}_m &= \sum_{j=0}^{\infty} a_{j,j+m}(b^\dagger)^j b^{j+m} \quad \text{when } \ m \geq 0, \\ \hat{\psi}_m &= \sum_{j=0}^{\infty} a_{j+|m|,j}(b^\dagger)^{j+|m|} b^j \quad \text{when } \ m < 0. \end{aligned} \tag{2.48}$$

It is clear from writing $\hat{\psi}$ and $\hat{\psi}_m$ this way that

$$\hat{\psi} = \sum_{m=-\infty}^{\infty} \hat{\psi}_m. \tag{2.49}$$

In [25] the form of the angular momentum operator $\hat{L}_z$ was shown to be,

$$\begin{aligned} \hat{L}_z &= \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x + \frac{\theta}{2\hbar^2}\hat{P}^2 \\ &= \sqrt{\frac{\theta}{2}}(b+b^\dagger)\hat{P}_y + i\sqrt{\frac{\theta}{2}}(b-b^\dagger)\hat{P}_x + \frac{\theta}{2\hbar^2}\hat{P}^2, \end{aligned} \tag{2.50}$$

so that,

$$\hat{L}_z\hat{\psi}_m = -\frac{\hbar}{2}(b+b^\dagger)[b+b^\dagger,\hat{\psi}_m] + \frac{\hbar}{2}(b-b^\dagger)[b-b^\dagger,\hat{\psi}_m] + \hbar[b,[b^\dagger,\hat{\psi}_m]]. \tag{2.51}$$

The first two terms in the above equation combine to give,

$$\begin{aligned} & -\frac{\hbar}{2}(b+b^\dagger)([b,\hat{\psi}_m] + [b^\dagger,\hat{\psi}_m]) + \frac{\hbar}{2}(b-b^\dagger)([b,\hat{\psi}_m] - [b^\dagger,\hat{\psi}_m]) \\ = & -\hbar(b^\dagger[b,\hat{\psi}_m] + b[b^\dagger,\hat{\psi}_m]). \end{aligned} \tag{2.52}$$

The second term just obtained cancels with the first term coming from the double commutator,

$$\begin{aligned}\hat{L}_z\hat{\psi}_m &= -\hbar(b^\dagger[b,\hat{\psi}_m] + b[b^\dagger,\hat{\psi}_m]) + \hbar(b[b^\dagger,\hat{\psi}_m] - [b^\dagger,\hat{\psi}_m]b) \\ &= -\hbar(b^\dagger[b,\hat{\psi}_m] + [b^\dagger,\hat{\psi}_m]b) = -\hbar[b^\dagger b,\hat{\psi}_m]. \end{aligned} \tag{2.53}$$

If one considers a general term in $\hat{\psi}_m$ and using the identity in (2.12) we then find that

$$\begin{aligned} b^\dagger[b,(b^\dagger)^j b^{j+m}] + [b^\dagger,(b^\dagger)^j b^{j+m}]b &= b^\dagger[b,(b^\dagger)^j]b^{j+m} + (b^\dagger)^j[b^\dagger,b^{j+m}]b \\ &= -m(b^\dagger)^j b^{j+m} \quad \text{when } m \geq 0, \end{aligned} \tag{2.54}$$

$$\begin{aligned} b^\dagger[b,(b^\dagger)^{j+|m|} b^{j}] + [b^\dagger,(b^\dagger)^{j+|m|} b^{j}]b &= b^\dagger[b,(b^\dagger)^{j+|m|}]b^{j} + (b^\dagger)^{j+|m|}[b^\dagger,b^{j}]b \\ &= -m(b^\dagger)^{j+|m|} b^{j} \quad \text{when } m < 0. \end{aligned} \tag{2.55}$$

Putting everything together one obtains the result that

$$\hat{L}_z\hat{\psi}_m = \hbar m\psi_m, \tag{2.56}$$

so that $\hat{\psi}_m$ is an eigenstate of angular momentum and the definition of $\hat{\psi}_m$ was indeed correct.

I shall now make a few remarks regarding $\hat{\psi}$ and $\hat{\psi}_m$. Firstly, it is a straightforward matter to see that $(\hat{P}^2\hat{\psi})^\dagger = \hat{P}^2\hat{\psi}^\dagger$,

$$(\hat{P}^2\hat{\psi})^\dagger = \frac{2\hbar^2}{\theta}[b,[b^\dagger,\hat{\psi}]]^\dagger = -\frac{2\hbar^2}{\theta}[b^\dagger,[b^\dagger,\hat{\psi}]^\dagger] = \frac{2\hbar^2}{\theta}[b^\dagger,[b,\hat{\psi}^\dagger]] = \hat{P}^2\hat{\psi}^\dagger. \tag{2.57}$$

Therefore, if $\hat{\psi}$ is a solution of the constant potential Schrödinger equation, then so is $\hat{\psi}^\dagger$. This implies that one can always choose the solution $\hat{\psi}$ to be Hermitian. Making this choice brings about the property that $\hat{\psi}_m^\dagger = \hat{\psi}_{-m}$ as can be seen from the definitions of $\hat{\psi}$ and $\hat{\psi}_m$.

With $\hat{\psi}_m$ in hand, one can write (2.45) and (2.46) as

$$\langle N_i+1|\,\hat{\psi}_{m,i}\,|N_i+m+1\rangle = \langle N_i+1|\,\hat{\psi}_{m,i+1}\,|N_i+m+1\rangle \tag{2.58}$$

$$\langle N_i|\,\hat{\psi}_{m,i}\,|N_i+m\rangle = \langle N_i|\,\hat{\psi}_{m,i+1}\,|N_i+m\rangle\,, \tag{2.59}$$

where ℓ is fixed by the fact that $\hat{\psi}_m$ lowers the number in the $|\cdot\rangle$ by m. An important point to note here is that for negative m these matching conditions imply that $m \leq N_i$. This cutoff in negative angular momentum is, of course, completely absent in the commutative case (where negative and positive angular momentum cases are symmetrical). From this one has in turn that time reversal symmetry is broken (as discussed in [24]). This assymetry comes from the implied parity violation in the commutator of $\hat{x}$ and $\hat{y}$, due to the choice of the sign of θ and the fact that the commutator is not symmetrical under parity transformations.

These are the matching conditions that will be used later on when making calculations. In the next section I will describe how these matching conditions affect the coefficients of the wavefunction.

Before moving on to the next section, I would like to make a remark regarding the rotational invariance of a non-commutative quantum mechanical system in two dimensions, as I noted in the introduction. Consider the coordinates $\hat{x}$ and $\hat{y}$ rotated by an angle ϕ,

$$\hat{x} \;\rightarrow\; \hat{x}' = \hat{x}\cos(\phi) + \hat{y}\sin(\phi) \tag{2.60}$$

$$\hat{y} \;\rightarrow\; \hat{y}' = \hat{y}\cos(\phi) - \hat{x}\sin(\phi). \tag{2.61}$$

Therefore, the rotated commutation relation is,

$$\begin{aligned}
[\hat{x}', \hat{y}'] &= [\hat{x}\cos(\phi) + \hat{y}\sin(\phi), \hat{y}\cos(\phi) - \hat{x}\sin(\phi)] \\
&= [\hat{x}\cos(\phi), \hat{y}\cos(\phi)] - [\hat{y}\sin(\phi), \hat{x}\sin(\phi)] \\
&= [\hat{x}, \hat{y}]\cos^2(\phi) + [\hat{x}, \hat{y}]\sin^2(\phi) \\
&= [\hat{x}, \hat{y}],
\end{aligned} \tag{2.62}$$

so that one has rotational invariance.

2.4 Relating the Fock Basis to the Position Basis

In Section 2.2 it was shown that for a constant potential the form of the wavefunction does not change when introducing non-commutativity. This, of course, has many advantages. One can think about the wavefunction in much the same manner and many useful and well known mathematical tools still apply. The wavefunction is essentially a plane wave (or a superposition of plane waves) and one can therefore make partial wave expansions as has already been alluded to. It also allows calculations of phase shifts and cross-sections to go through with virtually no change in the way they are obtained. It guides one's thinking and permits the same intuitive understanding of the system.

However, this similarity also makes it so that the changes brought about by non-commutativity are very subtle and easy to miss. A clear knowledge of how and when to use everything that has been explained thus far is therefore needed. One crucial part is how the wavefunction's coefficient are modified by the non-commutative matching conditions.

In the previous section I worked with objects of the form $\langle n|\,\hat{\psi}_m\,|n+m\rangle$ and the wavefunction is given as $\langle z|\,\hat{\psi}_m\,|z\rangle$. To find the impact non-commutativity makes, therefore, a basis transformation is essentially needed. In order to derive this transformation, one again uses the coherent state expansion shown in Chapter 1, Section 1.1.3, namely,

$$\hat{a} = \frac{1}{\pi}\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\left(e^{-\frac{\partial^2}{\partial z \partial \bar{z}}}\,\langle z|\,\hat{a}\,|z\rangle\right)|z\rangle\,\langle z|\,, \tag{2.63}$$

for some general operator $\hat{a}$.
Looking at the operator in the exponential, one immediately recognises it as the one found in the Schrödinger equation (2.20). If one, therefore, substitutes $\hat{\psi}_m$ for $\hat{a}$, the exponential simply becomes a number, since $\langle z|\,\hat{\psi}_m\,|z\rangle$ is an eigenfunction of the operator $e^{-\frac{\partial^2}{\partial z \partial \bar{z}}}$. The expansion for $\hat{\psi}_m$ then reads,

$$\hat{\psi}_m = \frac{e^{\theta k^2/2}}{\pi}\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\langle z|\,\hat{\psi}_m\,|z\rangle\,|z\rangle\,\langle z|\,. \tag{2.64}$$

2.4.1 The Case When $E > V$

All that is left to do to relate the two bases to each other is to form the matrix elements of both sides using the Fock basis states,

$$\langle n|\,\hat{\psi}_m\,|\ell\rangle \;=\; \gamma\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\langle z|\,\hat{\psi}_m\,|z\rangle\,\langle n|z\rangle\,\langle z|\ell\rangle \tag{2.65}$$

In order to do the integral I write the wavefunction $\langle z|\,\hat{\psi}_m\,|z\rangle$ in radial coordinates by putting $z = re^{i\phi}$ and make use of $\langle n|z\rangle = e^{-|z|^2/2}\frac{z^n}{\sqrt{n!}}$. Assuming for the moment that $k^2 = 2\mu(E-V)/\hbar^2$ is positive, then the wavefunction is a linear combination of J_m and

Y_m,

$$
\begin{aligned}
\langle n|\,\hat{\psi}_m\,|\ell\rangle &= \frac{e^{\theta k^2/2}}{\pi}\int \mathrm{d}\phi\,\mathrm{d}r\,re^{im\phi}[A\,J_m(\sqrt{2\theta}kr)+B\,Y_m(\sqrt{2\theta}kr)] \\
&\quad\times e^{-r^2}\frac{r^n e^{in\phi}}{\sqrt{n!}}\frac{r^\ell e^{-i\ell\phi}}{\sqrt{\ell!}} \\
&= \frac{e^{\frac{\theta k^2}{2}}}{\pi\sqrt{n!}\sqrt{\ell!}}\underbrace{\int_0^{2\pi}\mathrm{d}\phi\,e^{i(m+n-\ell)\phi}}_{2\pi\delta_{\ell,n+m}}\int_0^\infty \mathrm{d}r\,r^{n+\ell+1}e^{-r^2} \\
&\quad\times[A\,J_m(\sqrt{2\theta}kr)+B\,Y_m(\sqrt{2\theta}kr)] \\
\Rightarrow \langle n|\,\hat{\psi}_m\,|n+m\rangle &= \frac{2e^{\frac{\theta k^2}{2}}}{\sqrt{n!}\sqrt{(n+m)!}}\int_0^\infty \mathrm{d}r\,r^{2n+m+1}e^{-r^2} \\
&\quad\times[A\,J_m(\sqrt{2\theta}kr)+B\,Y_m(\sqrt{2\theta}kr)], \qquad (2.66)
\end{aligned}
$$

where J_m and Y_m are the Bessel functions of the first and second kind (also called the Neumann function) respectively. This is a nice result on its own, since it also shows that the matrix elements of $\hat{\psi}_m$ are non-zero only when the values in the $\langle\cdot|$ and $|\cdot\rangle$ are separated by m when coming from the side of the wavefunction.

The integral is done by splitting it into two parts, namely the part containing J_m and the part containing Y_m,

$$
\begin{aligned}
I_1 &= \int_0^\infty \mathrm{d}r\,r^{2n+m+1}e^{-r^2}A\,J_m(\sqrt{2\theta}kr) \qquad (2.67)\\
I_2 &= \int_0^\infty \mathrm{d}r\,r^{2n+m+1}e^{-r^2}B\,Y_m(\sqrt{2\theta}kr). \qquad (2.68)
\end{aligned}
$$

The first integral, I_1, can be found directly in a handbook of integrals such as [39]:

$$
I_1 = \frac{n!}{2}e^{-w}w^{m/2}L_n^m(w), \qquad (2.69)
$$

where I have set $w=\frac{\theta k^2}{2}$ and $L_n^m(w)$ is an associated Laguerre polynomial.
Doing the second integral is slightly more complicated and proceeds via a small detour. In order to calculate it, I shall make use of the following identity for Neumann functions of real order,

$$
Y_\nu = \frac{\cos(\nu\pi)J_\nu - J_{-\nu}}{\sin(\nu\pi)}. \qquad (2.70)
$$

The calculation now continues by doing the integrals over J_ν and $J_{-\nu}$. One can find these integrals once again in [39],

$$
\begin{aligned}
\int_0^\infty \mathrm{d}r\, r^{2n+\nu+1} e^{-r^2} J_\nu(\sqrt{2\theta}kr) &= \frac{w^{\nu/2}}{2}\frac{\Gamma(n+\nu+1)}{\Gamma(\nu+1)} M(n+\nu+1,\nu+1,-w) && (2.71)\\
\int_0^\infty \mathrm{d}r\, r^{2n+\nu+1} e^{-r^2} J_{-\nu}(\sqrt{2\theta}kr) &= \frac{w^{-\nu/2}}{2}\frac{\Gamma(n+1)}{\Gamma(1-\nu)} M(n+1,1-\nu,-w), && (2.72)
\end{aligned}
$$

where $M(a,b,x)$ is the first solution of the confluent hypergeometric differential equation [40]. Using the following identities for $\cos(x)$ and $M(a,b,x)$,

$$
\begin{aligned}
\cos(x) &= \frac{e^{ix}+e^{-ix}}{2} && (2.73)\\
M(a,b,x) &= e^x M(b-a,b,-x), && (2.74)
\end{aligned}
$$

one finds that,

$$
\begin{aligned}
&\int_0^\infty \mathrm{d}r\, r^{2n+m+1} e^{-r^2} Y_\nu(\sqrt{2\theta}kr)\\
= &-\frac{w^{-\nu/2}}{2\pi}\Gamma(n+\nu+1)\Gamma(n+1)\frac{\pi}{\sin(\nu\pi)} e^{-w}\left[\frac{M(-n-\nu,1-\nu,w)}{\Gamma(n+\nu+1)\Gamma(1-\nu)}\right.\\
&\left.-\frac{e^{i\nu\pi}+e^{-i\nu\pi}}{2} w^\nu \frac{M(-n,\nu+1,w)}{\Gamma(n+1)\Gamma(\nu+1)}\right]\\
= &-\frac{w^{-\nu/2}}{4\pi}\Gamma(n+\nu+1)\Gamma(n+1)\frac{\pi}{\sin(\nu\pi)} e^{-w}\\
&\times\left[\frac{M(-n-\nu,1-\nu,w)}{\Gamma(n+\nu+1)\Gamma(1-\nu)} - e^{i\nu\pi} w^\nu \frac{M(-n,\nu+1,w)}{\Gamma(n+1)\Gamma(\nu+1)}\right.\\
&\left.+\frac{M(-n-\nu,1-\nu,w)}{\Gamma(n+\nu+1)\Gamma(1-\nu)} - e^{-i\nu\pi} w^\nu \frac{M(-n,\nu+1,w)}{\Gamma(n+1)\Gamma(\nu+1)}\right]\\
= &-\frac{w^{-\nu/2}}{2\pi}\Gamma(n+\nu+1)\Gamma(n+1)\\
&\times\left[\frac{1}{2}U(n+1,1-\nu,we^{i\pi}) + \frac{1}{2}U(n+1,1-\nu,we^{-i\pi})\right], && (2.75)
\end{aligned}
$$

where $U(a,b,x)$ is the second solution of the confluent hypergeometric differential equation. Here I applied (2.74) to both (2.71) and (2.72) to get the first line and used the analytic continuation given in [40] to obtain the last line. The notation I have used here is the same as that found in [40] where $we^{\pm i\pi}$ indicates evaluation just above (+) or just below (−) the negative real axis, which is the branch cut for the function U.

This result simplifies when w is real. In that case the sum of the two U's is simply the real part of $U(n+1, 1-\nu, -w)$. Furthermore, it is a property of U that it is defined even when ν happens to be integer [40], so that this result holds true also for I_2. In other words, the original integral (2.66) reduces to,

$$\begin{aligned}\langle n|\,\hat{\psi}_m\,|n+m\rangle &= A\frac{\sqrt{n!}}{\sqrt{(n+m)!}}w^{m/2}L_n^m(w)\\ &\quad -\frac{B}{\pi}\sqrt{n!(n+m)!}\,e^w w^{-m/2}\mathrm{Re}[U(n+1,1-m,-w)]. \qquad (2.76)\end{aligned}$$

2.4.2 The Case When $E < V$

One can do much the same derivation when k^2 is negative and $k^2 = -\kappa^2 = -2\mu(V - E)/\hbar^2$. In that case the wavefunction is a linear combination of I_m and K_m, where I_m and K_m and the modified Bessel functions of the first and second kind respectively. The integral one needs to solve is therefore,

$$\begin{aligned}\langle n|\,\hat{\psi}_m\,|\ell\rangle &= \frac{e^{\theta\kappa^2/2}}{\pi}\int \mathrm{d}\phi\,\mathrm{d}r r\, e^{im\phi}[A\, I_m(\sqrt{2\theta}\kappa r) + B\, K_m(\sqrt{2\theta}\kappa r)]\\ &\quad \times e^{-r^2}\frac{r^n e^{in\phi}}{\sqrt{n!}}\frac{r^\ell e^{-i\ell\phi}}{\sqrt{\ell!}}\\ \Rightarrow \langle n|\,\hat{\psi}_m\,|n+m\rangle &= \frac{2e^{\frac{\theta\kappa^2}{2}}}{\sqrt{n!}\sqrt{(n+m)!}}\int_0^\infty \mathrm{d}r\, r^{2n+m+1}e^{-r^2}\\ &\quad \times[A\, I_m(\sqrt{2\theta}\kappa r) + B\, K_m(\sqrt{2\theta}\kappa r)]. \qquad (2.77)\end{aligned}$$

Splitting the integral into two parts gives

$$I_3 = \int_0^\infty \mathrm{d}r\, r^{2n+m+1}e^{-r^2}A\, I_m(\sqrt{2\theta}\kappa r) \qquad (2.78)$$

$$I_4 = \int_0^\infty \mathrm{d}r\, r^{2n+m+1}e^{-r^2}B\, K_m(\sqrt{2\theta}\kappa r), \qquad (2.79)$$

where it is once again the first integral, I_3, which can be found most easily [39],

$$\begin{aligned}I_3 &= \frac{\Gamma(n+m+1)}{2\omega^{1/2}\Gamma(m+1)}\omega^{(m+1)/2}M(n+m+1, m+1, \omega)\\ &= \frac{1}{2}\frac{\Gamma(n+m+1)}{\Gamma(m+1)}\omega^{m/2}e^\omega M(-n, m+1, -\omega)\\ &= \frac{n!}{2}\omega^{m/2}e^\omega L_n^m(-\omega), \qquad (2.80)\end{aligned}$$

where $\omega = \frac{\theta\kappa^2}{2}$, I have once again used equation (2.74) to go to the second line and the identity $L_n^m(x) = \binom{n+m}{n} M(-n, m+1, x)$ to arrive at the last line. To do I_4 one follows the same procedure as for Y_m, namely doing the integral for a real order of K_ν first. In this case,

$$K_\nu = \frac{\pi}{2} \frac{I_{-\nu} - I_\nu}{\sin(\nu\pi)}. \tag{2.81}$$

The integral for I_ν is exactly like the one just shown. The integral over $I_{-\nu}$ is given by

$$\int_0^\infty e^{-r^2} r^{2n+\nu+1} I_{-\nu}(\sqrt{2\theta}\kappa r) = \frac{\Gamma(n+1)}{2\Gamma(1-\nu)} \omega^{-\nu/2} M(n+1, 1-\nu, \omega). \tag{2.82}$$

Using the solutions for the integrals over I_ν and $I_{-\nu}$ results in the following solution for the K_ν integral,

$$\begin{aligned}
&\int_0^\infty e^{-r^2} r^{2n+\nu+1} K_\nu(\sqrt{2\theta}\kappa r) \\
= \;& \frac{\pi}{4} \frac{1}{\sin(\nu\pi)} \\
& \times \left[\frac{\Gamma(n+1)}{\Gamma(1-\nu)} \omega^{-\nu/2} M(n+1, 1-\nu, \omega) - \frac{\Gamma(n+\nu+1)}{\Gamma(\nu+1)} \omega^{m/2} M(n+\nu+1, \nu+1, \omega) \right] \\
= \;& \frac{\omega^{-\nu/2}}{4} \Gamma(n+1)\Gamma(n+\nu+1) \frac{\pi}{\sin((1-\nu)\pi)} \\
& \times \left[\frac{M(n+1, 1-\nu, \omega)}{\Gamma(n+\nu+1)\Gamma(1-\nu)} - \omega^\nu \frac{M(n+\nu+1, \nu+1, \omega)}{\Gamma(n+1)\Gamma(\nu+1)} \right] \\
= \;& \frac{\omega^{-\nu/2}}{4} \Gamma(n+1)\Gamma(n+\nu+1) U(n+1, 1-\nu, \omega).
\end{aligned} \tag{2.83}$$

By the same caveat as for the case when $k^2 > 0$ this is also valid for K_m. In this case there is also no need to take the real part of U, since U is already real for positive arguments. Putting (2.80) and (2.83) together in (2.77) gives us the desired result when working with energies less than that of the potential,

$$\begin{aligned}
\langle n | \hat{\psi}_m | n+m \rangle \;=\;& A \frac{\sqrt{n!}}{\sqrt{(n+m)!}} e^{2\omega} \omega^{m/2} L_n^m(-\omega) \\
& + \frac{B}{2} \sqrt{n!(n+m)!}\, e^{\omega} \omega^{-m/2} U(n+1, 1-m, \omega).
\end{aligned} \tag{2.84}$$

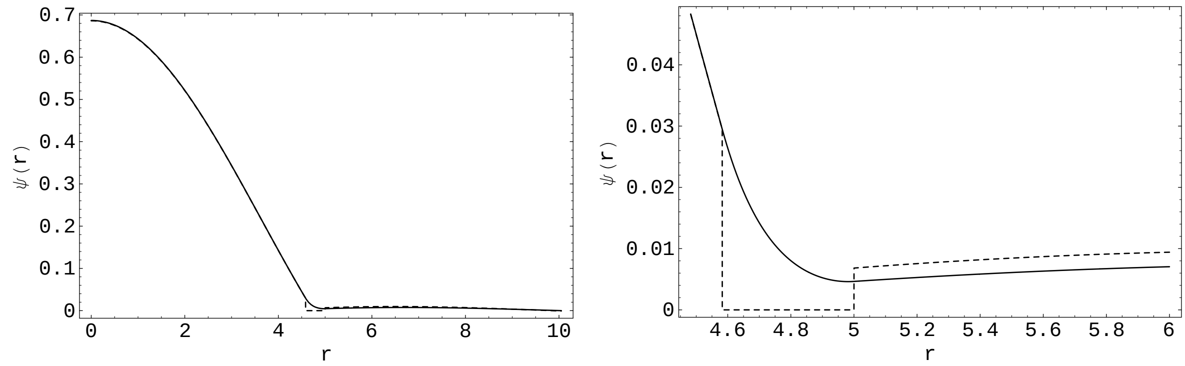
The matrix elements calculated here should strictly only be used when $m \geq 0$. However, when $m < 0$ very little actually changes due to the fact that $\hat{\psi}_m^\dagger = \hat{\psi}_{-m}$. Because of this the wavefunction $\langle z| \hat{\psi}_m |z\rangle$ that is used in the integral to calculate the matrix elements $\langle n| \hat{\psi}_m |\ell\rangle$ simply becomes,

$$
\begin{aligned}
\langle z| \hat{\psi}_m |z\rangle &= \langle z| \hat{\psi}_m^\dagger |z\rangle^* = \langle z| \hat{\psi}_{-m} |z\rangle^* = \langle z| \hat{\psi}_{|m|} |z\rangle^* &(2.85)\\
&= e^{-i|m|\phi}[\bar{A}\ J_{|m|}(\sqrt{2\theta}kr) + \bar{B}\ Y_{|m|}(\sqrt{2\theta}kr)] \quad \text{when} \;\; E > V, &(2.86)\\
&= e^{-i|m|\phi}[\bar{A}\ I_{|m|}(\sqrt{2\theta}\kappa r) + \bar{B}\ K_{|m|}(\sqrt{2\theta}\kappa r)] \quad \text{when} \;\; E < V. &(2.87)
\end{aligned}
$$

This makes the non-vanishing matrix elements have the form $\langle n-m| \hat{\psi}_m |n\rangle$ instead of $\langle n| \hat{\psi}_m |n+m\rangle$ as was the case for positive m. Aside from that all the analyses go through without alteration.

2.5 Comparison of Calculated Wavefunctions

In this chapter I have tried to give an overview of how non-commutative piecewise constant potentials should be implemented. To this end, I showed how these potentials were built up using projection operators on the non-commutative configuration space. I showed how the non-commutative wavefunction has the same form as the commutative one, but went on to explain that matching conditions for the non-commutative case are calculated in a different basis. Finally I showed how to connect the two bases in question so that the coefficients of the non-commutative wavefunction could be explicitly calculated. Now that all this mathematics is in place, I would like to give a numerical example of how the non-commutative matching conditions influences the wavefunction.



(a) (b)

Figure 2.1: Comparison of commutative and non-commutative wavefunctions. In the non-commutative case we chose $N_1 = 10$, $N_2 = 12$ and $N_3 = 50$ with corresponding commutative values of $R_1 = \sqrt{21}$, $R_2 = \sqrt{25}$ and $R_3 = \sqrt{101}$ for $\theta = 1$. Figure (a) shows the radial part of the ground state wavefunction in the case of an annulus inside an infinite well for the angular momentum, $m = 0$, sector. Figure (b) is a close up view of the region around the potential barrier of the annulus. The potential on the inside of the barrier annulus and outside the annulus up to the infinite wall is zero. The height of the annulus was chosen as $V = 20$ in units of $\hbar = 1$. The solid line indicates the commutative values, whereas the dashed line indicates the non-commutative ones.

It is clear from Figures 2.1(a) and 2.1(b) that, while the commutative and non-commutative wavefunctions have the same form in all the three regions of the potential, the non-commutative wavefunction is not continuous across potential boundaries. This is of course, because in the commutative case the wavefunction has matching conditions which ensure this continuity, whereas the non-commutative wavefunction obtains its matching conditions from a consistency equation which ensures that the full wavefunction is the sum of the wavefunctions in the various potential regions.

CHAPTER 3
Stationary Properties of the Non-commutative Well

The previous chapters laid the mathematical groundwork for making calculations in the context of non-commutative piecewise constant potentials with a concentric annular geometry. Here I turn my attention to the implementation of said mathematics and make a few numerical calculations designed to highlight some of the differences that non-commutativity introduces. This chapter, therefore, contains a discussion of some of the stationary properties of the non-commutative well, such as bound state energies, phase shifts and cross-sections.

3.1 Implementation of a Non-commutative Well

The previous chapter tried to keep things as general as possible with regards to the number of domains in the potential and subsequently the matching conditions. Here I would like to give a concrete example of how the various parts necessary for a calculation are implemented.

For the well there are of course only two regions, so that $N_V = 2$. The potential describing the well is therefore simply,

$$\hat{V} = V_1 P_1 + V_2 P_2, \tag{3.1}$$

$$P_1 = \sum_{n=0}^{N_1} |n\rangle \langle n| \quad \text{and} \quad P_2 = \sum_{n=N_1+1}^{\infty} |n\rangle \langle n| . \tag{3.2}$$

Since $N_V = 2$ one will also only have one set of equations in the matching conditions,

$$\langle N_1 + 1| \hat{\psi}_{m,1} |N_1 + m + 1\rangle = \langle N_1 + 1| \hat{\psi}_{m,2} |N_1 + m + 1\rangle \tag{3.3}$$

$$\langle N_1| \hat{\psi}_{m,1} |N_1 + m\rangle = \langle N_1| \hat{\psi}_{m,2} |N_1 + m\rangle . \tag{3.4}$$

Next one uses the relation between the position and Fock space representations of $\hat{\psi}$ given in Section 2.4 (equations (2.76) and (2.84)) to write these matrix elements in terms of a

Laguerre polynomial and a Kummer's function. Therefore, when $m \geq 0$,

$$\begin{aligned} & Aw^{m/2} L^m_{N_1+1}(w) \\ & -\frac{B}{\pi}(N_1+m+1)!\, e^w w^{-m/2} \mathrm{Re}[U(N_1+2, 1-m, -w)] \\ = \; & Ce^{\eta\omega_*}\omega_*^{m/2} L^m_{N_1+1}(\omega_*) \\ & +\frac{D}{\xi}(N_1+m+1)!\, e^{\omega_*}\omega_*^{-m/2} \mathrm{Re}[U(N_1+2, 1-m, -\omega_*)] \end{aligned} \tag{3.5}$$

$$\begin{aligned} & Aw^{m/2} L^m_{N_1}(w) \\ & -\frac{B}{\pi}(N_1+m)!\, e^w w^{-m/2} \mathrm{Re}[U(N_1+1, 1-m, -w)] \\ = \; & Ce^{\eta\omega_*}\omega_*^{m/2} L^m_{N_1}(\omega_*) \\ & +\frac{D}{\xi}(N_1+m)!\, e^{\omega_*}\omega_*^{-m/2} \mathrm{Re}[U(N_1+1, 1-m, -\omega_*)], \end{aligned} \tag{3.6}$$

where $w = \theta(E - V_1)$.

Similarly for $m < 0$,

$$\begin{aligned} & \bar{A}w^{|m|/2} L^{|m|}_{N_1-|m|+1}(w) \\ & -\frac{\bar{B}}{\pi}(N_1+1)!\, e^w w^{-|m|/2} \mathrm{Re}[U(N_1-|m|+2, 1-|m|, -w)] \\ = \; & \bar{C}e^{\eta\omega_*}\omega_*^{|m|/2} L^{|m|}_{N_1-|m|+1}(\omega_*) \\ & +\frac{\bar{D}}{\xi}(N_1+1)!\, e^{\omega_*}\omega_*^{-|m|/2} \mathrm{Re}[U(N_1-|m|+2, 1-|m|, -\omega_*)] \end{aligned} \tag{3.7}$$

$$\begin{aligned} & \bar{A}w^{|m|/2} L^{|m|}_{N_1-|m|}(w) \\ & -\frac{\bar{B}}{\pi}N_1!\, e^w w^{-|m|/2} \mathrm{Re}[U(N_1-|m|+1, 1-|m|, -w)] \\ = \; & \bar{C}e^{\eta\omega_*}\omega_*^{|m|/2} L^{|m|}_{N_1-|m|}(\omega_*) \\ & +\frac{\bar{D}}{\xi}N_1!\, e^{\omega_*}\omega_*^{-|m|/2} \mathrm{Re}[U(N_1-|m|+1, 1-|m|, -\omega_*)]. \end{aligned} \tag{3.8}$$

In both cases, if one sets $\mu = \hbar = 1$,

$$\eta = \begin{cases} 0, & E > V_2 \\ 2, & E < V_2 \end{cases} \tag{3.9}$$

$$\xi = \begin{cases} -\pi, & E > V_2 \\ 2, & E < V_2 \end{cases} \tag{3.10}$$

$$\omega_* = \begin{cases} \theta(E - V_2), & E > V_2 \\ -\theta(V_2 - E), & E < V_2 \end{cases} \tag{3.11}$$

These matching conditions can then be used, for example, to calculate energies in the case of bound states or to obtain the coefficients for when one wants to know the phase shifts.

3.2 Bound State Energies

The forms given in the previous section were general and certain restrictions due to boundary conditions still apply. This is one scenario where knowing that the non-commutative and commutative wavefunction have the same form helps a great deal. One knows that the wavefunction inside the well is generally a linear combination of a Bessel and a Neumann function. Since the Neumann function, Y_m, is singular at the origin, however, one sets its coefficient to be zero. Similarly in the case when the energy of the particle is less than that of the potential outside the well, and the wavefunction is a linear combination of I_m and K_m, one knows that I_m grows exponentially and therefore its coefficient is also set to zero. Looking back at equations (2.66) and (2.77) one sees that this implies that the coefficient of U is always zero inside the well as is the coefficient of L_n^m when looking at bound states. The same result can be found by looking at the commutative limit of the L_n^m's and the U's (see [24]). However, taking those limits are somewhat complicated whereas this explanation is much simpler and uses commutative results one is already familiar with. For bound states with $m \geq 0$ for example,

$$A\ w^{m/2} L_{N_1+1}^m(w) = \frac{D}{2}(N_1 + m + 1)!\, e^{\omega_1} \omega_1^{-m/2} U(N_1 + 2, 1 - m, \omega_1) \tag{3.12}$$

$$A\ w^{m/2} L_{N_1}^m(w) = \frac{D}{2}(N_1 + m)!\, e^{\omega_1} \omega_1^{-m/2} U(N_1 + 1, 1 - m, \omega_1), \tag{3.13}$$

where $w = \theta(E - V_1)$, $\omega_1 = \theta(V_2 - E)$ and I have set $\mu = \hbar = 1$.
By dividing (3.12) by (3.13) one can simplify this into one equation that one needs to solve to find the allowed energies for the bound states in a particular sector of angular momentum,

$$\frac{L^m_{N_1+1}(\theta(E-V_1))}{L^m_{N_1}(\theta(E-V_1))} = (N_1+m+1)\frac{U(N_1+2, 1-m, \theta(V_2-E))}{U(N_1+1, 1-m, \theta(V_2-E))}. \tag{3.14}$$

In the case of negative m a very similar equation applies,

$$\frac{L^{|m|}_{N_1-|m|+1}(\theta(E-V_1))}{L^{|m|}_{N_1-|m|}(\theta(E-V_1))} = (N_1+1)\frac{U(N_1-|m|+2, 1-|m|, \theta(V_2-E))}{U(N_1-|m|+1, 1-|m|, \theta(V_2-E))}. \tag{3.15}$$

Figure 3.1(a) gives one a good indication of the effect of non-commutativity on the bound state energies of a well. This graph shows the results of a numerical calculation for both negative and positive angular momentum for a chosen θ which is quite large, $\theta = \frac{20}{21}$. A few things are immediately visible. In the case of positive angular momentum, the bound state energy for a certain state is always lower than one finds commutatively. For negative angular momentum one is faced with the opposite being true. When θ is large, as it is here, this effect lets one easily find extra (fewer) bound states for positive (negative) angular momentum than one does commutatively. While this is a striking result, in reality the value of θ is most likely extremely small and consequently so is the splitting in energy between positive and negative angular momentum. Finding such a gain or loss in bound states would therefore require a fantastically precise tuning of the well height. This can already be seen in Figure 3.1(c) where it is virtually impossible to distinguish between commutative and non-commutative energies at that scale. Figure 3.1(c) has the smallest θ-value of the three figures, namely $\theta = \frac{20}{2001}$, and typically the non-commutative energies differ from the commutative ones here by less than one percent. Furthermore, one also notices that even in the case when θ is almost 1 the non-commutative energies obtained for very small values of angular momentum match commutative energies quite closely. In contrast, as m approaches N_1 commutative and non-commutative results diverge more and more, leading, as has already been mentioned, to cases where there are even extra bound states (e.g. $m = 9, 10, 11$) or none at all (e.g. $m = -9, -10, -11$). Figure 3.1(b) lies between the two extremes depicted in Figures 3.1(a) and 3.1(c). Its value of θ is $\frac{20}{201}$

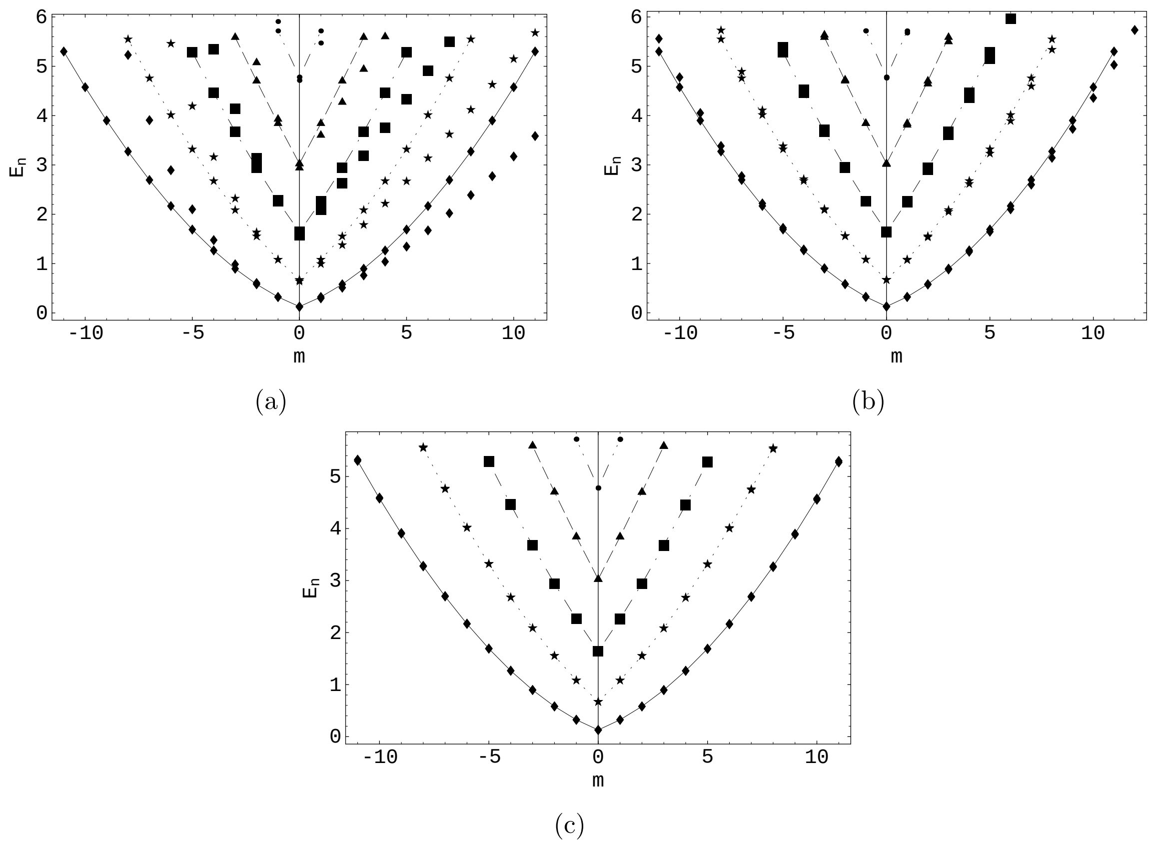


Figure 3.1: The figures show the commutative and non-commutative bound state energies for a well of depth $V_2 - V_1 = 6$ ($V_1 = 0$) with a radius of $\sqrt{20}$. In each figure, however, the value of N_1 corresponding to the radius is different. For (a), $N_1 = 10$ and therefore $\theta = \frac{20}{21} \approx 1$. For (b) $N_1 = 100$, $\theta = \frac{20}{201} \approx 0.1$. For (c) $N_1 = 1000$, $\theta = \frac{20}{2001} \approx 0.01$. Symbols which are the same indicate the same energy level, i.e. diamonds indicate the ground state, stars the first excited state, and so forth. Symbols connected with some kind of line are the commutative energies and unconnected ones indicate the non-commutative energies.

and as such forms an illustrative link in the chain going from highly non-commutative to much less so. Indeed it is only for the larger values of angular momentum and energy where a significant difference can be seen.

These figures highlight the two significant differences one finds non-commutatively when comparing with the commutative case, namely the breaking of time reversal symmetry, as seen in the energy splitting between positive and negative angular momentum, and the related cutoff in negative angular momentum.

3.3 Scattering

This section deals with non-commutative time-independent scattering. Finding the modified coefficients of the non-commutative wavefunction and using them to compute the phase shift due to scattering from the well is the goal. With that said, one is working in a region where the energy of the particle is greater than that of the potential outside the well. In this case, unlike for the bound states, no further restriction apply to the wavefunction on the outside and the equations (3.5) to (3.8) reduce to

$$\begin{aligned} Aw^{m/2}L^{m}_{N_1+1}(w) &= C\omega_2^{m/2}L^{m}_{N_1+1}(\omega_2) \\ &\quad -\frac{D}{\pi}(N_1+m+1)!\, e^{\omega_2}\omega_2^{-m/2}\mathrm{Re}[U(N_1+2,1-m,-\omega_2)] \quad (3.16) \\ Aw^{m/2}L^{m}_{N_1}(w) &= C\omega_2^{m/2}L^{m}_{N_1}(\omega_2) \\ &\quad -\frac{D}{\pi}(N_1+m)!\, e^{\omega_2}\omega_2^{-m/2}\mathrm{Re}[U(N_1+1,1-m,-\omega_2)], \quad (3.17) \end{aligned}$$

for $m \geq 0$, where $\omega_2 = \theta(E - V_2)$.
For $m < 0$ one has,

$$\begin{aligned} \bar{A}w^{|m|/2}L^{|m|}_{N_1-|m|+1}(w) &= \bar{C}\omega_2^{|m|/2}L^{|m|}_{N_1-|m|+1}(\omega_2) \\ &\quad -\frac{\bar{D}}{\pi}(N_1+1)!\, e^{\omega_2}\omega_2^{-|m|/2}\mathrm{Re}[U(N_1-|m|+2,1-|m|,-\omega_2)] \quad (3.18) \\ \bar{A}w^{|m|/2}L^{|m|}_{N_1-|m|}(w) &= \bar{C}\omega_2^{|m|/2}L^{|m|}_{N_1-|m|}(\omega_2) \\ &\quad -\frac{\bar{D}}{\pi}N_1!\, e^{\omega_2}\omega_2^{-|m|/2}\mathrm{Re}[U(N_1-|m|+1,1-|m|,-\omega_2)]. \quad (3.19) \end{aligned}$$

3.3.1 Phase Shifts

Calculating the phase shifts in a non-commutative setting proceeds analogously to the commutative case. Let me therefore give a brief outline of the commutative argument [31]. If one considers elastic scattering from a potential then the total amount of probability before and after scattering must clearly be the same. The only effect the potential can have then is to introduce a phase shift to the scattered wavefunction. Furthermore, since one assumes a finite extent for the potential (which is obviously the case for a well), far away from it the particle being scattered may be considered essentially free. Therefore, if one looks at the full solution of the Schrödinger equation far from the potential and compares it with a free particle that has undergone a phase shift, then one has a way of writing the phase shift in terms of the coefficients of the full solution.

To put these statements in mathematical terms one needs to know the large argument asymptotic forms of J_m and Y_m, of which a linear combination comprise the full solution of the Schrödinger equation on the outside of the well,

$$J_m(kr) \sim \sqrt{\frac{2}{\pi kr}}\cos(kr - \frac{m\pi}{2} - \frac{\pi}{4}) \quad (3.20)$$

$$Y_m(kr) \sim \sqrt{\frac{2}{\pi kr}}\sin(kr - \frac{m\pi}{2} - \frac{\pi}{4}). \quad (3.21)$$

If the full solution outside the well is given by $C\ J_m(kr) + D\ Y_m(kr)$ then for large r the asymptotic form of the wavefunction is given, when written in terms of exponentials, by

$$C\ J_m(kr) + D\ Y_m(kr) \sim (\tilde{C} + i\tilde{D})\frac{e^{-ikr}}{\sqrt{r}} + (\tilde{C} - i\tilde{D})\zeta\frac{e^{ikr}}{\sqrt{r}}, \quad (3.22)$$

where $\zeta = e^{-i(m\pi+\pi/2)}$ and I have absorbed $\frac{1}{\sqrt{2\pi k}}e^{i(m\pi/2+\pi/4)}$ into C and D to form $\tilde{C}$ and $\tilde{D}$ which cancel in the final result. Similar to the three dimensional case, $\frac{e^{-ikr}}{\sqrt{r}}$ are incoming spherical waves and $\frac{e^{ikr}}{\sqrt{r}}$ are outgoing spherical waves.
In the case of our free particle which acquired a phase on its outgoing part, δ_m,

$$A\ J_m(kr) \sim A\frac{1}{\sqrt{2\pi k}}e^{i(m\pi/2+\pi/4)}\left(\frac{e^{-ikr}}{\sqrt{r}} + \zeta e^{2i\delta_m}\frac{e^{ikr}}{\sqrt{r}}\right). \quad (3.23)$$

Comparing (3.22) and (3.23) one sees that,

$$\begin{aligned} C + iD &= A \\ C - iD &= A\, e^{2i\delta_m} \\ \Rightarrow \frac{C - iD}{C + iD} &= e^{2i\delta_m}. \end{aligned} \quad (3.24)$$

Since the coefficients that one calculates are real for real values of energy, we can simplify this result,

$$\begin{aligned}
\cos(2\delta_m) + i\sin(2\delta_m) &= \frac{1-i\frac{D}{C}}{1+i\frac{D}{C}}\frac{1-i\frac{D}{C}}{1-i\frac{D}{C}} \\
&= \frac{1-\left(\frac{D}{C}\right)^2}{1+\left(\frac{D}{C}\right)^2} + i\frac{-2\frac{D}{C}}{1+\left(\frac{D}{C}\right)^2} \\
\Rightarrow \cos(2\delta_m) = \frac{1-\left(\frac{D}{C}\right)^2}{1+\left(\frac{D}{C}\right)^2} \quad &\text{and} \quad \sin(2\delta_m) = \frac{-2\frac{D}{C}}{1+\left(\frac{D}{C}\right)^2}. \qquad (3.25)
\end{aligned}$$

Therefore, using the $\sin(2\delta_m)$ and $\cos(2\delta_m)$ terms to form $\tan(2\delta_m)$ and using the half-angle formula for $\tan(\cdot)$ one arrives at,

$$\begin{aligned}
\tan(2\delta_m) &= \frac{2\tan(\delta_m)}{1-\tan^2(\delta_m)} = \frac{2\left(-\frac{D}{C}\right)}{1-\left(-\frac{D}{C}\right)^2} \\
\Rightarrow \tan(\delta_m) &= -\frac{D}{C}, \qquad (3.26)
\end{aligned}$$

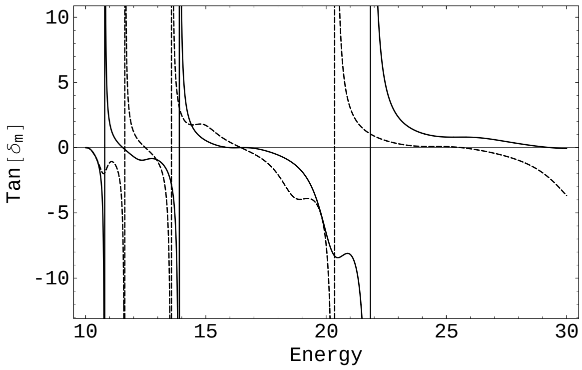
which is the same as the standard three dimensional result [41].

As I have mentioned a few times before, since it bears emphasising, the form of the commutative and non-commutative wavefunctions are the same. As a consequence, all the arguments leading up to equation (3.26) go through unchanged and the same formula holds non-commutatively. All that remains to be done is calculate the coefficients C and D using the non-commutative matching conditions (given in Section 3.3 for scattering states) and one has the non-commutative phase shift.

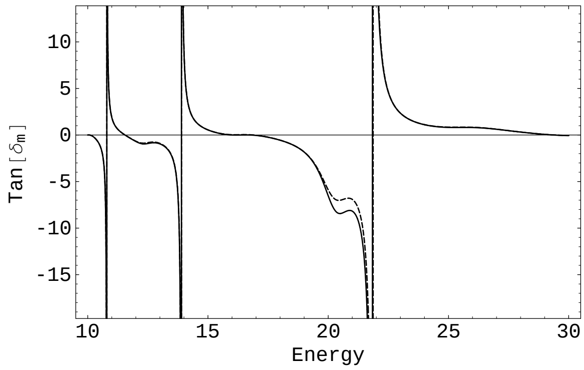
3.3.1.1 Numerical Phase Shifts

The simple formula for calculating the phase shifts in both the commutative and non-commutative situations allows for easy comparison of this quantity.

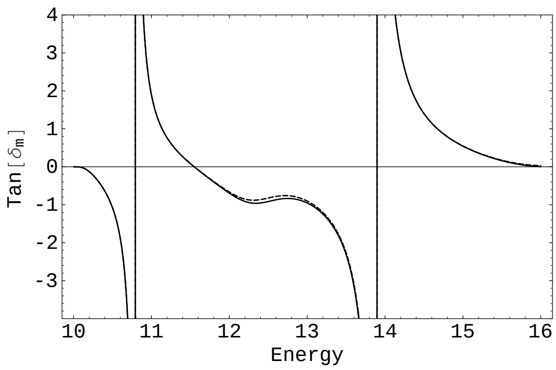
Figure 3.2 shows the results of a numerical calculation doing exactly that. In the first of the four figures, namely Figure 3.2(a), one sees a comparison of the phase shifts over the first 20 units of energy ($\hbar = \mu = 1$) above the well height where the angular momentum has been chosen $m = 4$ for a large value of θ ($\theta \approx 1$). Not too surprisingly for such a large value of the non-commutative parameter, there is almost no correspondence beyond the general form of the graph. The next figure, Figure 3.2(b), uses the same commutative phase shifts as Figure 3.2(a), but the non-commutative results were obtained using a



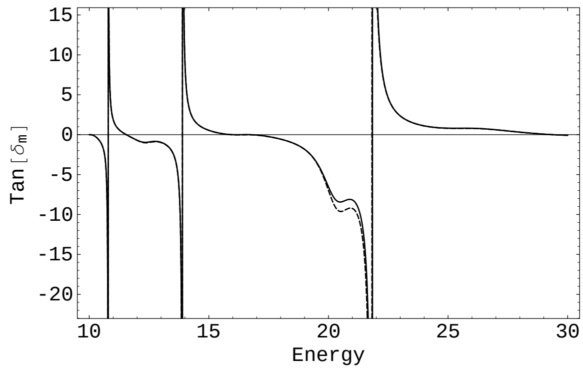
(a)



(b)



(c)



(d)

Figure 3.2: Comparison of commutative and non-commutative phase shifts. The first three figures depict the tangent of the phase shifts in the $m = 4$ angular momentum sector and the fourth figure is in the $m = -4$ case. The commutative system used is given by a well height of $V_2 - V_1 = 10$ ($V_1 = 0$) and a radius of $R = \sqrt{20}$. Solid lines indicate commutative results, while dashed lines indicate non-commutative ones. Figure (a) uses the smallest value of N, namely $N = 10$ and therefore $\theta = \frac{20}{21}$. Figures (b) and (c) use the same $N = 1000$ and $\theta = \frac{20}{2001}$. Figures (a) and (b) therefore show how the non-commutative phase shifts tend towards the commutative ones as θ becomes smaller. Figure (c) is simply a close up of (b) over the first six units of energy. Figure (d) is the same as (b) except now for negative angular momentum, $m = -4$.

much smaller θ-value, namely $\theta \approx 0.01$. Already for this value of θ the two cases are nearly indistinguishable, except near certain values of energy, such as near $E = 13$ (visible in Figure 3.2(c)) and $E = 21$ (although these are by far not the only places where this phenomenon can be seen). Since the commutative and non-commutative results match up so well elsewhere already, it would seem to suggest that, as one performs a $\theta \to 0$ limit, there are points where convergence is slower. However, the exact physical origin of this is not yet understood. The final figure, Figure 3.2(d), uses all the same parameters as Figure

3.2(b), except that the angular momentum is now negative. As one can see, whereas in the positive angular momentum case the non-commutative values were slightly above the commutative case, the opposite is true in the case of negative angular momentum.

3.3.2 Cross-Sections

With phase shifts in hand, it is a simple matter to calculate the total and differential cross-sections. The derivation of the formulas for the various cross-sections in two dimensions proceed much as they do for three dimensions [42]. The following are the commutative ones,

$$f_k(\phi) = \sqrt{\frac{2}{\pi}} \sum_{m=-\infty}^{\infty} \cos(m\phi) e^{i\delta_m} \sin(\delta_m) \tag{3.27}$$

$$\frac{d\sigma}{d\phi} = \frac{1}{k} |f_k(\phi)|^2 \tag{3.28}$$

$$\sigma = \int_0^{2\pi} \sigma(\phi) \mathrm{d}\phi = \frac{4}{k} \sum_{m=-\infty}^{\infty} \sin^2(\delta_m), \tag{3.29}$$

where of course $k^2 = 2\mu(E - V_2)/\hbar^2$, since this is for scattering energies. Furthermore, ϕ is the scattering angle, $f_k(\phi)$ is the scattering amplitude, $\frac{d\sigma}{d\phi}$ is the differential cross-section and σ is the total cross-section. Once again, due to the commutative and non-commutative wavefunctions having the same form, these formulas hold unaltered in the non-commutative case with the caveat that the phase shifts used be the non-commutative ones.

Figures 3.3(a) and 3.3(b) show the total scattering cross-section for the same system used in the previous section for the phase shifts. In the case of Figure 3.3(a) one is working with a large non-commutative parameter and as such the two graphs agree only in form. Figure 3.3(b) in contrast uses a relatively small value of θ. Agreement in this case is already very good, as it was for the phase shifts. In the previous section I also pointed out two places where the non-commutative phase shift differed from the commutative one more than over the rest of the considered range of energy. Figures 3.3(c) and 3.3(d) therefore show the effect that this difference has by plotting the contribution to the total cross-section from the $m = 4$ angular momentum sector only. Figure 3.3(c) shows the effect of the difference near $E = 13$ and Figure 3.3(d) is a close up of Figure 3.3(c) near $E = 21$ illustrating the effect there. In both these cases the contribution to the cross-section is slightly lower in the non-commutative case. Figures 3.3(e) and 3.3(f) are the same as Figures 3.3(c) and 3.3(d) except that the angular momentum is now negative, namely $m = -4$. Here the contribution to the total cross-section is slightly higher non-commutatively, in contrast to the positive angular momentum case.

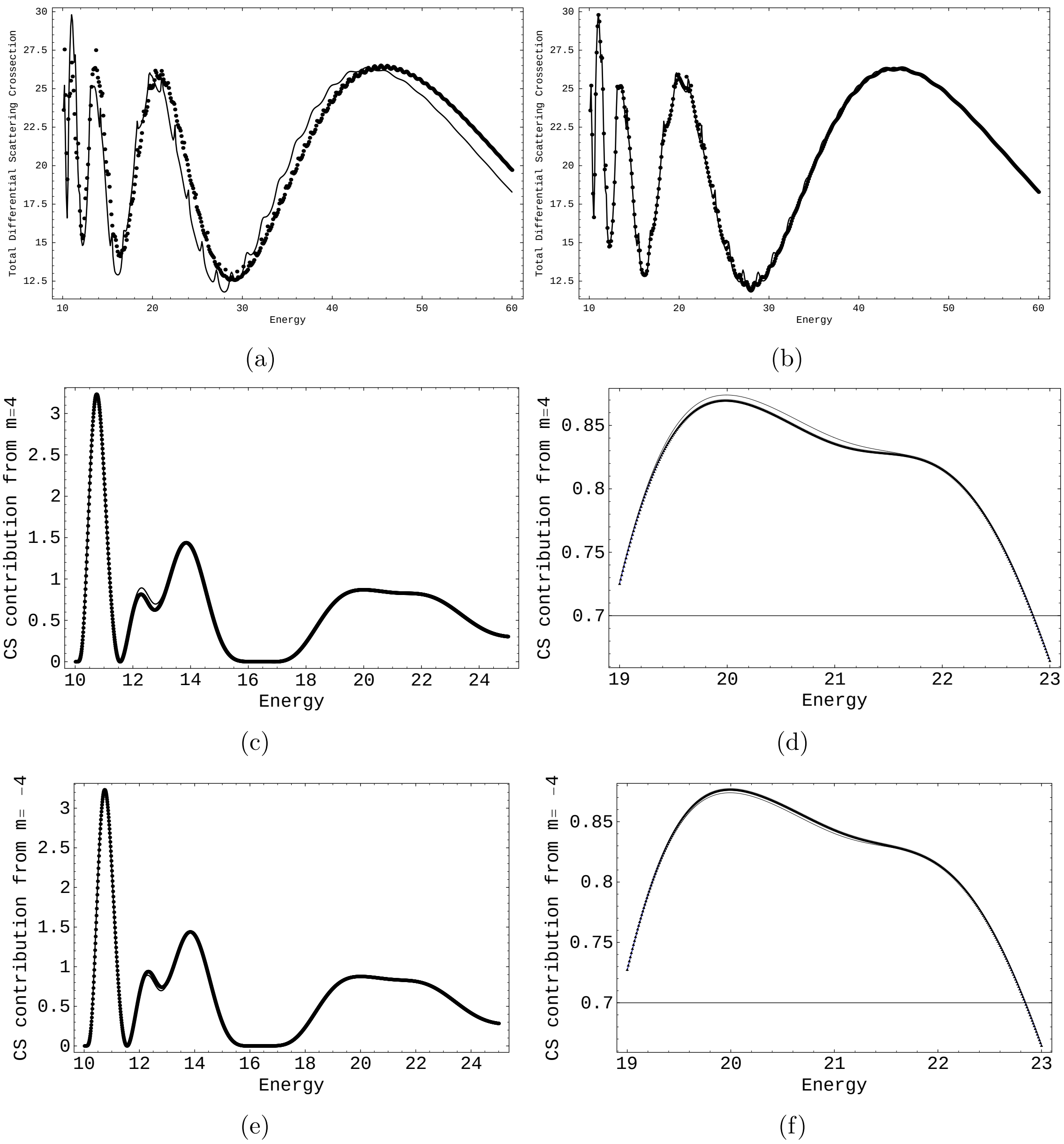


Figure 3.3: Comparison of commutative and non-commutative scattering cross-sections. In all six figures the commutative data comes from the same system, namely well height is $V_2 - V_1 = 10$ ($V_1 = 0$) and the radius is $R = \sqrt{20}$. These are the same values used in Figure 3.2. Figure (a) uses $N = 10$, $\theta = \frac{20}{21}$ for its non-commutative parameters. Figure (b) uses $N = 1000$, $\theta = \frac{20}{2001}$. Figures (c) and (d) plot the contribution to the total cross-section from angular momentum $m = 4$ only when $N = 1000$. Figure (d) is a close up of Figure (c) around energy $E = 21$. Figures (e) and (f) are the contribution when the angular momentum is $m = -4$ and $N = 1000$ once again. As for Figure (d), Figure (f) is a close up of Figure (e) near $E = 21$. In all these figures the solid line indicates commutative results, while dots or triangles indicate non-commutative ones.

CHAPTER 4
Time Dependent Tunneling

In this final chapter I will take a brief look at quantum tunneling in a non-commutative setting. To do this I set up a Gaussian wave packet inside an annulus which is contained in an infinite well. The system is then allowed to evolve in time. After a certain amount of time I look how much of the wave packet has leaked out through the annulus potential barrier and compare the results from commutative and non-commutative calculations.

4.1 The Non-commutative Annulus Inside an Infinite Well

As opposed to the previous chapter where one had two regions defining the potential, here one has four. These are the inside of the annulus, the barrier region, outside the annulus and outside the infinite well entirely. Just as the wavefunction commutatively is zero in a region of infinite potential, so it is also non-commutatively. One is left therefore, strictly speaking, with three regions which in turn implies two sets of matching condition equations to fix the coefficients of the wavefunction. The matching condition at the infinite well boundary only serves to fix the energy.

In mathematical terms, therefore,

$$\hat{V} = V_1 P_1 + V_2 P_2 + V_3 P_3 + V_4 P_4 \tag{4.1}$$

$$P_1 = \sum_{n=0}^{N_1} |n\rangle \langle n| \tag{4.2}$$

$$P_2 = \sum_{n=N_1+1}^{N_2} |n\rangle \langle n| \tag{4.3}$$

$$P_3 = \sum_{n=N_2+1}^{N_3} |n\rangle \langle n| \tag{4.4}$$

$$P_4 = \sum_{n=N_3+1}^{\infty} |n\rangle \langle n|, \tag{4.5}$$

where V_1 to V_3 are constants and $V_4 = \infty$. Typically I chose $V_1 = V_3 = 0$.

As I have mentioned, to calculate the coefficients of the wavefunction one uses the matching conditions around N_1 and N_2. Since one works here with a problem where the energies involved can span the range from lower to higher than V_2, however, this leads to double the amount of equations that need to be solved, namely one set when $E < V_2$ and one set when $E > V_2$. This is also the case commutatively where one has a linear combination of I_m and K_m as the wavefunction in the barrier region below V_2 and J_m and Y_m above. I will therefore not write down all sixteen equations (taking into account positive and negative m), but rather just give the equations for positive angular momentum and only the $\langle N+1|\,\hat{\psi}_m\,|N+m+1\rangle$ form. These are,

$$\begin{aligned} & Aw^{m/2}L^m_{N_1+1}(w) \\ = \; & Ce^{2\omega}\omega^{m/2}L^m_{N_1+1}(\omega) \\ & +\frac{D}{2}(N_1+m+1)!\,e^{\omega}\omega^{-m/2}\mathrm{Re}[U(N_1+2,1-m,-\omega)] \end{aligned} \tag{4.6}$$

$$\begin{aligned} & Aw^{m/2}L^m_{N_1+1}(w) \\ = \; & C\omega^{m/2}L^m_{N_1+1}(\omega) \\ & -\frac{D}{\pi}(N_1+m+1)!\,e^{\omega}\omega^{-m/2}\mathrm{Re}[U(N_1+2,1-m,-\omega)] \end{aligned} \tag{4.7}$$

$$\begin{aligned} & Ce^{2\omega}\omega^{m/2}L^m_{N_2+1}(\omega) \\ & +\frac{D}{2}(N_2+m+1)!\,e^{\omega}\omega^{-m/2}\mathrm{Re}[U(N_2+2,1-m,-\omega)] \\ = \; & Ew^{m/2}L^m_{N_2+1}(w) \\ & -\frac{F}{\pi}(N_2+m+1)!\,e^{w}w^{-m/2}\mathrm{Re}[U(N_2+2,1-m,-w)] \end{aligned} \tag{4.8}$$

$$\begin{aligned} & C\omega^{m/2}L^m_{N_2+1}(\omega) \\ & -\frac{D}{\pi}(N_2+m+1)!\,e^{\omega}\omega^{-m/2}\mathrm{Re}[U(N_2+2,1-m,-\omega)] \\ = \; & Ew^{m/2}L^m_{N_2+1}(w) \\ & -\frac{F}{\pi}(N_2+m+1)!\,e^{w}w^{-m/2}\mathrm{Re}[U(N_2+2,1-m,-w)], \end{aligned} \tag{4.9}$$

where w is still $w = \theta E$ and $\omega = \theta(E - V_2)$. Equations (4.6) and (4.8) are for when the energy is below that of the potential and equations (4.7) and (4.9) are for when the opposite holds true.

There is still one further equation that is required to find the energy which one gets from the matching condition at the boundary of the infinite well. As explained in the previous chapter, for the finite well one would typically have only the hypergeometric U function on the outside of the well for energies lower than that of the potential. Similar to that case one would then have an equation of the form,

$$\frac{(\cdot)L^m_{N_3+1} + (\cdot)U_{N_3+1}}{(\cdot)L^m_{N_3} + (\cdot)U_{N_3}} = \frac{(\cdot)U_{N_3+1}}{(\cdot)U_{N_3}}, \tag{4.10}$$

to calculate the energies from. Here, however, the potential is infinite on the RHS and for large arguments the function U tends to zero. The whole RHS is therefore zero (U_{N_3+1} goes to zero faster than U_{N_3}) and the equation reduces to finding the zeros of,

$$0 = Ew^{m/2}L^m_{N_3+1}(w) - \frac{F}{\pi}(N_3+m+1)!\, e^w w^{-m/2}\mathrm{Re}[U(N_3+2, 1-m, -w)]. \tag{4.11}$$

4.2 Solving the Time Dependent Problem

As I said at the start of this chapter, I will be using a Gaussian as my initial state of the system and let it evolve in time from there. Here I solve the time dependent problem in two different ways. In the commutative case I solve the time dependent Schrödinger equation directly with an initial condition which is the Gaussian. Consider the Schrödinger equation in two dimensions and in polar coordinates,

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right] + V(r)\psi(r,\phi,t), \tag{4.12}$$

where $\hbar = \mu = 1$ once again. Since the potential is radially symmetric one can factor out the angular part of $\psi(r,\phi,t) = e^{im\phi}R(r,t)$. The second derivative with respect to ϕ is then simply $-\frac{m^2}{r^2}\psi(r,\phi,t)$. If one now looks at just the radial term and set $R(r,t) = \frac{u(r,t)}{\sqrt{r}}$ then

$$\begin{aligned}
-\frac{1}{2r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\frac{u}{\sqrt{r}}\right) &= -\frac{1}{2r}\frac{\partial}{\partial r}\left(r\frac{\sqrt{r}\frac{\partial u}{\partial r} - u(r,t)\frac{1}{2}\frac{1}{\sqrt{r}}}{r}\right) \\
&= -\frac{1}{2r}\left(\frac{1}{2}\frac{1}{\sqrt{r}}\frac{\partial u}{\partial r} + \sqrt{r}\frac{\partial^2 u}{\partial r^2} - \frac{1}{2}\frac{1}{\sqrt{r}}\frac{\partial u}{\partial r} + \frac{1}{4}\frac{1}{r^{3/2}}u(r,t)\right) \\
&= -\frac{1}{2\sqrt{r}}\frac{\partial^2 u}{\partial r^2} - \frac{1}{\sqrt{r}}\frac{1}{8r^2}u(r,t).
\end{aligned} \tag{4.13}$$

Substituting all this into the Schrödinger equation gives,

$$
\begin{aligned}
i\frac{1}{\sqrt{r}}\frac{\partial u}{\partial t} &= -\frac{1}{2\sqrt{r}}\frac{\partial^2 u}{\partial r^2} - \frac{1}{\sqrt{r}}\frac{1}{8r^2}u(r,t) + \frac{m^2}{2r^2}\frac{u(r,t)}{\sqrt{r}} + V(r)\frac{u(r,t)}{\sqrt{r}} \\
\Rightarrow i\frac{\partial u}{\partial t} &= -\frac{1}{2}\frac{\partial^2 u}{\partial r^2} + \left(V(r) + \frac{m^2 - \frac{1}{4}}{2r^2}\right)u(r,t),
\end{aligned} \tag{4.14}
$$

which is the form of the standard $1+1$ dimensional Schrödinger equation with an effective potential of $V(r) + \frac{m^2-\frac{1}{4}}{2r^2}$. Now that the Schrödinger equation is in this form I use a numerical method to solve the PDE that preserves unitarity [43]. One final point here is, as seen from the expansion of the Gaussian I perform below, that only the zero angular momentum sector plays a role, which means one does not have to solve the Schrödinger equation for any other value.

In the non-commutative case I directly expand the Gaussian,

$$\hat{\psi}_G = e^{-\alpha \hat{r}^2}, \tag{4.15}$$

in terms of the energy eigenstates of the Hamiltonian. α is simply a parameter that controls the width of the Gaussian and was chosen as 1 most of the time. Unlike the commutative case, one only has a finite number of energy eigenstates for a certain value of angular momentum as is clear from equation (4.11) (see also [24]). From said equation we have $N3 + 1$ solutions, due to the nature of the Laguerre polynomial. This makes this expansion doable for numerical calculation in the first place. The expansion is given by,

$$\hat{\psi}_G = \sum_{n=0}^{N_3} \left(\hat{\psi}_{E_n}|\hat{\psi}_G\right)\hat{\psi}_{E_n}, \tag{4.16}$$

where $\hat{\psi}_{E_n}$ is an energy eigenstate of the system and N_3 is the same N_3 found in the definitions of P_3 and P_4 in this chapter. The inner product in the above equation is the trace of the operators over non-commutative configurations space, as said in Chapter 1. Closer study of this inner product reveals that it is only non-zero for angular momentum

$m = 0$,

$$\begin{aligned}
\left(\hat{\psi}_{E_n}|\hat{\psi}_G\right) &= \mathrm{tr}_c(\hat{\psi}^\dagger_{E_n}\hat{\psi}_G) \\
&= \sum_{\ell=0}^{\infty} \langle\ell|\,\hat{\psi}_G\hat{\psi}_{E_n}\,|\ell\rangle \\
&= \sum_{\ell=0}^{\infty}\sum_{j=0}^{\infty} \langle\ell|\, e^{-\alpha\theta(2b^\dagger b+1)}\,|j\rangle\,\langle j|\,\hat{\psi}_{E_n}\,|\ell\rangle \\
&= \sum_{\ell=0}^{\infty}\sum_{j=0}^{\infty} e^{-\alpha\theta(2j+1)}\delta_{j,\ell}\,\langle j|\,\hat{\psi}_{E_n}\,|\ell\rangle \\
&= \sum_{j=0}^{\infty} e^{-\alpha\theta(2j+1)}\,\langle j|\,\hat{\psi}_{E_n}\,|j\rangle\,. \qquad (4.17)
\end{aligned}$$

$$(4.18)$$

Here I have used the Hermiticity of $\hat{\psi}_{E_n}$ (see Section 2.3 just before Section 2.4) and the cyclic permutability of the trace in the second line. Thinking of $\hat{\psi}_{E_n}$ as an expansion of angular momentum states and remembering that the only non-zero matrix elements of the angular momentum states are of the form $\langle n|\,\hat{\psi}_m\,|n+m\rangle$ immediately shows that only the states in the inner product with angular momentum zero contribute. Furthermore, since one is working here inside an infinite well, $\langle j|\,\hat{\psi}_m\,|j\rangle = 0$ when $j > N_3$. One can, therefore, write the expansion of $\hat{\psi}_G$ as

$$\hat{\psi}_G = \sum_{n=0}^{N_3}\sum_{j=0}^{N_3} e^{-\alpha\theta(2j+1)}\,\langle j|\,\hat{\psi}_{E_n,m=0}\,|j\rangle\,\hat{\psi}_{E_n,m=0}. \qquad (4.19)$$

Finally, acting on both sides with the time evolution operator, $e^{-i\hat{H}t}$ gives,

$$e^{-i\hat{H}t}\hat{\psi}_G = \sum_{n=0}^{N_3}\sum_{j=0}^{N_3} e^{-\alpha\theta(2j+1)}\,\langle j|\,\hat{\psi}_{E_n,m=0}\,|j\rangle\,e^{-iE_nt}\hat{\psi}_{E_n,m=0}, \qquad (4.20)$$

since the $\hat{\psi}_{E_n,m=0}$'s are per definition eigenstates of the Hamiltonian.

While working with the expansion of $\hat{\psi}_G$ and the subsequent time evolution, I have stayed in the Fock basis throughout. Since I am not so much interested in calculating the probability of finding the particle at a specific position, but rather want to know simply the probability of finding the particle outside the annulus there is no need to switch to the position basis here. Similar to how positional probabilities were derived in Section 1.1.3, one can introduce a new operator $\hat{\pi}_{k,\ell}$,

$$\hat{\pi}_{k,\ell} = |k,\ell)\,(k,\ell|\,, \tag{4.21}$$

where k and ℓ are integers here.

As was done in Section 1.1.3 one can show that the following is a resolution of unity,

$$1_q = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} |k,\ell)\,(k,\ell|\,. \tag{4.22}$$

Since $\left(k,\ell|\hat{\psi}\right) = \langle k|\,\hat{\psi}\,|\ell\rangle$,

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} |k,\ell)\left(k,\ell|\hat{\psi}\right) &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} |k\rangle\,\langle\ell|\,\langle k|\,\hat{\psi}\,|\ell\rangle \\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} |k\rangle\,\langle k|\,\hat{\psi}\,|\ell\rangle\,\langle\ell| \\ &= \hat{\psi},\end{aligned} \tag{4.23}$$

which shows that $\sum_{k}^{\infty}\sum_{\ell=0}^{\infty} |k,\ell)\,(k,\ell|$ is indeed an identity operator on $\mathcal{H}_q$. Since $\hat{\pi}_{k,\ell}$ is clearly Hermitian on quantum Hilbert space, positive and sums to unity ($\sum_{k,\ell}\hat{\pi}_{k,\ell} = 1_q$) it can indeed be considered a Positive Operator Valued Measure. The total probability of the system at any given time is then given by

$$\eta_G = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\left(e^{-i\hat{H}t}\hat{\psi}_G\right|\hat{\pi}_{k,\ell}\left|e^{-i\hat{H}t}\hat{\psi}_G\right) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\left(\hat{\psi}_G\right|\hat{\pi}_{k,\ell}\left|\hat{\psi}_G\right) = \left(\hat{\psi}_G|\hat{\psi}_G\right), \tag{4.24}$$

which is not 1, since I have not normalised $\hat{\psi}_G$. The first step follows, of course, since total probability is conserved (see Section 1.1.4).

While the above is formally correct, it does not take any specifics of the system into account. Taking a closer look at the sum reveals that it is limited by two things,

$$\begin{aligned}\eta_G &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\left(\hat{\psi}_G(t)|k,\ell\right)\left(k,\ell|\hat{\psi}_G(t)\right)\\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}|\langle k|\,\hat{\psi}_G(t)\,|\ell\rangle\,|^2. \qquad (4.25)\end{aligned}$$

From (4.20) one knows that $\hat{\psi}_G(t) = e^{-i\hat{H}t}\hat{\psi}_G$ is a sum over angular momentum states of which $m = 0$. $\hat{\psi}_G(t)$ is therefore zero if $k \neq \ell$,

$$\eta_G = \sum_{\ell=0}^{\infty}|\langle \ell|\,\hat{\psi}_G(t)\,|\ell\rangle\,|^2. \qquad (4.26)$$

Furthermore, since the system is bounded by the infinite well potential, $\langle \ell|\,\hat{\psi}_G(t)\,|\ell\rangle$ is zero when $\ell > N_3$,

$$\eta_G = \sum_{\ell=0}^{N_3}|\langle \ell|\,\hat{\psi}_G(t)\,|\ell\rangle\,|^2. \qquad (4.27)$$

The probability of finding the particle outside the annulus is therefore simply,

$$P(\text{outside}) = \frac{1}{\eta_G}\sum_{\ell=N_2+1}^{N_3}|\langle \ell|\,\hat{\psi}_G(t)\,|\ell\rangle\,|^2. \qquad (4.28)$$

4.3 Numerical Results

With a way of calculating the time evolution of the Gaussian wave packet and subsequently calculating the amount of probability that has escaped through the annulus barrier, one has everything one needs to obtain some results numerically.

The first figure in this section displays the main result of this chapter. The Gaussian in question is allowed to evolve in time for ten time steps (in units of $\hbar = 1$). At each step the total probability outside the annulus is measured. Commutatively this is found, of course, by integrating the modulus squared of the (normalised) wavefunction on the outside of the annulus, as found from the PDE. In the non-commutative case I rely on equation (4.28). These values are what one sees in Figure 4.1.

For Figure 4.1 commutative values are plotted using a solid line and diamonds, whereas

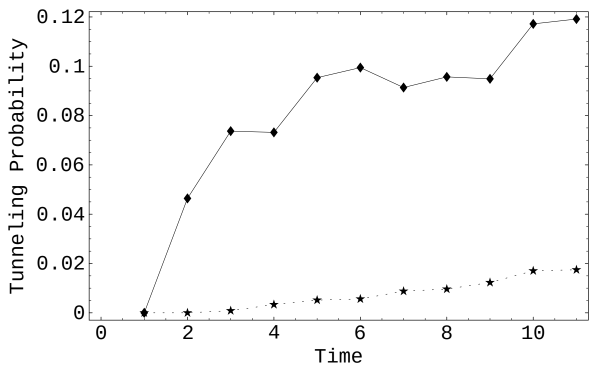
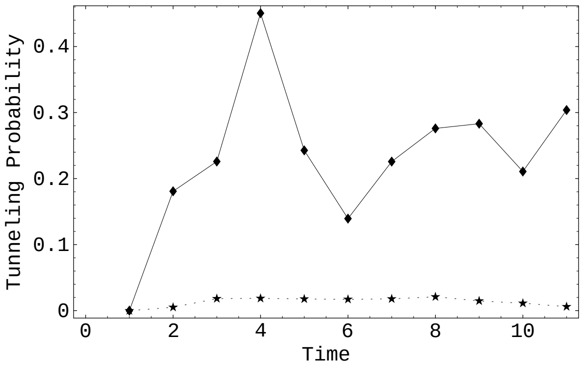


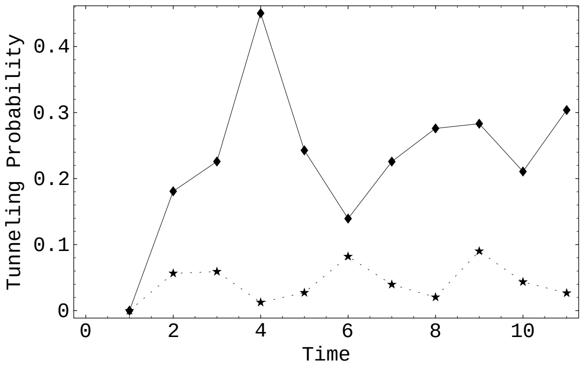
Figure 4.1: Comparison of commutative and non-commutative tunneling probabilities. This figure uses the following values, namely $N_1 = 10$, $N_2 = 12$ and $N_3 = 50$. Furthermore, the value of θ was chosen to be 1. This results in commutative values for the radii which are $R_1 = \sqrt{21}$, $R_2 = \sqrt{25}$ and $R_3 = \sqrt{101}$.

non-commutative values are denoted by a dashed line with stars. The barrier in this system is quite thin. Non-commutatively the difference between the inner N_1 and outer N_2 is only two on a total system size of $N_3 = 50$ units. Commutatively the barrier is only about $R_2 - R_1 \approx 0.42$ units thick. These values allow for a fairly steady flow of probability out of the annulus in the commutative case and the non-commutative parameter θ is very large ($\theta = 1$) to make the differences in the two cases clear. Not surprisingly, given the large value of θ, the difference between commutative and non-commutative results is large. One sees that the particle has a much higher chance to tunnel through the barrier in the commutative case than the non-commutative one. As alluded to in previous chapters it is not possible to define a sharp boundary in a non-commutative setting. This difference between the commutative and non-commutative tunneling probabilities can then be understood intuitively from the inherent fuzziness of the non-commutative potential, where the particle experiences forces over a wider region than it would commutatively.

Figures 4.2(a) and 4.2(a) attempt to show that the tunneling probabilities also tend towards the commutative values when $\theta \to 0$. Unfortunately due to the fact that as one makes θ smaller, N_3 needs to become larger to compensate if one wishes to keep R_3 fixed as one does here. This, however, leads to more bound states in the non-commutative system (remember that the maximum number of bound states in an infinite non-commutative well is N_3+1) and subsequently the number of terms in the expansion of $\hat{\psi}_G(t)$ blow up due the dependence on N_3 in both sums. Furthermore, more bound states mean higher energies



(a)



(b)

Figure 4.2: Comparison of commutative and non-commutative tunneling probabilities when the commutative system remains the same, but θ is different in each figure. For the commutative system (same in both (a) and (b)), $R_1 = \sqrt{5}$, $R_2 = \sqrt{7}$ and $R_3 = \sqrt{21}$. In (a): $N_1 = 2$, $N_1 = 3$, $N_1 = 10$ and $\theta = 1$. In (b): $N_1 = 17$, $N_1 = 24$, $N_1 = 73$ and $\theta = \frac{1}{7}$.

and calculating the Laguerre polynomials and Kummer's functions numerically for higher arguments become problematic and require drastic increases in the amount of precision used. The values used in the non-commutative system in Figure 4.2(a) are therefore such that when they are changed to those of Figure 4.2(b) one sees a large decrease in θ (θ goes from 1 to $\frac{1}{7}$) while still staying within the realm of relative computational ease. This results in a commutative system which is tiny and a barrier which is extremely thin. The wave packet, therefore, easily tunnels out of the annulus and bounces back from the infinite well wall, causing probability outside the annulus to oscillate. Even so, one sees that the non-commutative tunneling probabilities in the case when $\theta = \frac{1}{7}$ are closer to those of the commutative system on average. These figures show once again that as θ becomes smaller one starts to retrieve the commutative system.

Conclusion

The primary goal of this thesis was to explore some of the consequences of non-commutative quantum mechanics in a simple setting, such as the well, in order to gain an understanding of where and how it deviates from the normal commutative theory we are all familiar with.

To this end I first gave an overview of the formalism of non-commutative quantum mechanics. By using the formalism of commutative quantum mechanics as a starting point and building up non-commutative quantum mechanics side by side with this, it becomes very clear what the similarities and what the differences are between the mathematical structures underpinning these theories. One of the first differences one noticed is that the Hilbert space of physical states, which is L^2 in the commutative case, has been replaced with the space of Hilbert-Schmidt operators acting on non-commutative configuration space, which means that the states themselves are now operators. The greatest difference, however, is how position measurements are made. Position states are no longer orthogonal due to non-commutativity. This requires one to use positive operator valued measures to calculate positional probabilities, which is a significant change from the commutative case. Fortunately, even though the way these probabilities are calculated are different, showing that probability is conserved under time evolution is still rather simple.

The mathematical background of non-commutative quantum mechanics given in the first chapter allowed me to continue smoothly into a discussion of how non-commutative piecewise constant potentials are constructed. This is no longer as easy a task as it is commutatively and requires the use of projection operators to demarcate the various regions of the potential. Furthermore, these projection operators are written in terms of harmonic oscillator states, which span the non-commutative configuration space, and subsequently the matching conditions that are to be satisfied by the wavefunction are also given in this basis. To find the coefficients of the wavefunction, which is given in position basis per definition, an additional step is necessary, in the non-commutative case, to connect the two bases in question. The fact that the non-commutative matching conditions are so different from the ones one has commutatively has a significant impact on these coefficients. The main reason for this difference is that the matching conditions in the commutative case are constructed to ensure smoothness of the wavefunction accross regions of different potential. In the non-commutative case on the other hand one uses the matching conditions to ensure that the full wavefunction is the sum of the wavefunction in each zone of potential and therefore the wavefunction is not smooth from one area to the next.

All this mathematical background was necessary in order to obtain the results of the second half of this thesis. The third chapter dealt with time-independent properties of the non-commutative potential well. Even in such a simple setting one already finds some interesting deviations from the commutative case. Firstly, one sees that non-commutativity introduces a splitting in the bound states energies for positive and negative angular momentum, with positive (negative) angular momentum states having a lower (higher) energy than one finds commutatively. This can lead to more (in the positive case) or less (in the negative case) bound states than one finds commutatively for large values of the non-commutative parameter, θ. In particular one finds that one never has negative angular momentum states with angular momentum (as an absolute value) greater than N, where N is an indication of how finely space is quantised for a given area. Whether these statements hold for positive or negative angular momentum depends, of course, on what the sign of θ is. The sign of θ was chosen positive in this work, but one could easily have chosen it negative resulting in swapping the results and conclusions for positive and negative angular momentum. Convergence of non-commutative results to commutative ones as $\theta \to 0$ appear to be quite rapid. Finding evidence of non-commutativity in this way would either require a great deal of luck or some extreme scenario utilising the cut-off in negative angular momentum. Non-commutative scattering data also seems to flow over into commutative form fairly quickly. However, in this case there appear to be certain points where convergence is slower. The physical origin of this is not completely understood, but further investigation might turn up some more solid clues for where to find non-commutativity in experiment.

The final chapter saw a first look into non-commutative tunneling. This delivered the interesting result that the rate of tunneling seems to be slowed in the non-commutative case. If one were to consider that non-commutativity introduces an uncertainty in the exact position of the potential, this would make intuitive sense, since it would make the potential appear slightly fuzzy. This is clearly felt by the tunneling particle, which sees a thicker barrier than it does commutatively. In this way non-commutativity may affect the life times of resonances or quasi-bound state.

Non-commutative quantum mechanics as used in this thesis offers a systematic way of performing calculations and interpreting the results. It also naturally incorporates the idea of a minimum length required for a quantum theory of gravity. This makes one think that a non-commutative theory might be how we end up describing gravity at the smallest of scales. To this end, one needs an understanding of how non-commutativity affects the physics that a commutative theory already explains well as well as finding some place where one can see whether non-commutativity is really how nature works. It is, therefore, in these last two parts where my thesis aims to play a small role.

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