Contributions to centralizers in matrix rings

by

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Declaration

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Dedications

Vir Mamma en Pappa

Abstract

Contributions to centralizers in matrix rings

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THE concept of a k-matrix in the full 2×2 matrix ring $M_2(R/\langle k \rangle)$, where R is an arbitrary unique factorization domain (UFD) and k is an arbitrary nonzero nonunit in R, is introduced. We obtain a concrete description of the centralizer of a k-matrix $\widehat{B}$ in $M_2(R/\langle k \rangle)$ as the sum of two subrings $\mathcal{S}_1$ and $\mathcal{S}_2$ of $M_2(R/\langle k \rangle)$, where $\mathcal{S}_1$ is the image (under the natural epimorphism from $M_2(R)$ to $M_2(R/\langle k \rangle)$) of the centralizer in $M_2(R)$ of a pre-image of $\widehat{B}$, and where the entries in $\mathcal{S}_2$ are intersections of certain annihilators of elements arising from the entries of $\widehat{B}$. Furthermore, necessary and sufficient conditions are given for when $\mathcal{S}_1 \subseteq \mathcal{S}_2$, for when $\mathcal{S}_2 \subseteq \mathcal{S}_1$ and for when $\mathcal{S}_1 = \mathcal{S}_2$. It turns out that if R is a principal ideal domain (PID), then every matrix in $M_2(R/\langle k \rangle)$ is a k-matrix for every k. However, this is not the case in general if R is a UFD. Moreover, for every factor ring $R/\langle k \rangle$ with zero divisors and every $n \geqslant 3$ there is a matrix for which the mentioned concrete description is not valid. Finally we provide a formula for the number of elements of the centralizer of $\widehat{B}$ in case R is a UFD and $R/\langle k \rangle$ is finite.

Uittreksel

Bydraes tot sentraliseerders in matriksringe

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Die konsep van 'n k-matriks in die volledige 2×2 matriksring $M_2(R/\langle k \rangle)$, waar R 'n willekeurige unieke faktoriseringsgebied (UFG) en k 'n willekeurige nie-nul nie-inverteerbare element in R is, word bekendgestel. Ons verkry 'n konkrete beskrywing van die sentraliseerder van 'n k-matriks $\widehat{B}$ in $M_2(R/\langle k \rangle)$ as die som van twee subringe $\mathcal{S}_1$ en $\mathcal{S}_2$ van $M_2(R/\langle k \rangle)$, waar $\mathcal{S}_1$ die beeld (onder die natuurlike epimorfisme van $M_2(R)$ na $M_2(R/\langle k \rangle)$) van die sentraliseerder in $M_2(R)$ van 'n trubeeld van $\widehat{B}$ is, en die inskrywings van $\mathcal{S}_2$ die deursnede van sekere annihileerders van elemente afkomstig van die inskrywings van $\widehat{B}$ is. Verder word nodige en voldoende voorwaardes gegee vir wanneer $\mathcal{S}_1 \subseteq \mathcal{S}_2$, vir wanneer $\mathcal{S}_2 \subseteq \mathcal{S}_1$ en vir wanneer $\mathcal{S}_1 = \mathcal{S}_2$. Dit blyk dat as R 'n hoofideaalgebied (HIG) is, dan is elke matriks in $M_2(R/\langle k \rangle)$ 'n k-matriks vir elke k. Dit is egter nie in die algemeen waar indien R 'n UFG is nie. Meer nog, vir elke faktorring $R/\langle k \rangle$ met nuldelers en elke $n \geqslant 3$ is daar 'n matriks waarvoor die bogenoemde konkrete beskrywing nie geldig is nie. Laastens word 'n formule vir die aantal elemente van die sentraliseerder van $\widehat{B}$ verskaf, indien R 'n UFG en $R/\langle k \rangle$ eindig is.

Contents

CHAPTER

1 Introduction

It is security, certainty, truth, beauty, insight, structure, architecture. I see mathematics, the part of human knowledge that I call mathematics, as one thing - one great, glorious thing.

— PAUL HALMOS

LET S_1 be a subgroup of a group S. The centralizer of an element $s \in S$ in S_1 is the set

$$\{c \in S_1 \mid cs = sc\} \tag{1.1}$$

which we denote by $\mathrm{Cen}_{S_1}(s)$. Note that $\mathrm{Cen}_{S_1}(s)$ is a subgroup of S and that if S_1 and S are rings, then $\mathrm{Cen}_{S_1}(s)$ is a subring of S_1 (with identity if S_1 has an identity). Regarding the work in this dissertation, S_1 and S will always be rings and s will always be an element of S_1. The concept of a centralizer is well-known and is used throughout the literature in ring theory. The results in [11] and [18] are, for instance, beautiful examples of where the structure of the centralizer of a certain element in a ring can be used to determine some information about the ring's structure. (The results in [18] were extended in [22].) Let us, for example, consider the following result in [18].

Theorem 1.1. ([18], p. 215, Theorem 3) Let R be a simple ring with unit such that for some element $a \in R$, a^n is in the center of R. If $\mathrm{Cen}_R(a)$ satisfies a polynomial identity of degree m, then R satisfies the standard polynomial identity of degree nm.

To illustrate the above result we consider $M_2(\mathbb{Q})$, the full 2×2 matrix ring over the real quaternions $\mathbb{Q}$.

Now, since $\mathbb{Q}$ is a division ring, it follows that $M_2(\mathbb{Q})$ is simple ([17], p. 39, Corollary 2.28). Because, (i) $B = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \in M_2(\mathbb{Q})$ such that B^2 is in the center of $M_2(\mathbb{Q})$; (ii) $\mathrm{Cen}_{M_2(\mathbb{Q})}(B) = M_2(\mathbb{C})$, where $\mathbb{C}$ is the field of complex numbers; and (iii) according to the Amitsur-Levitzki Theorem $M_2(\mathbb{C})$ satisfies the standard polynomial identity of degree 4 ([2], p .455, Theorem 1); it follows from Theorem 1.1 that $M_2(\mathbb{Q})$ satisfies the standard polynomial identity of degree 8.

The following result in [11] is another example of how the structure of the centralizer of an element in a ring can be used to determine whether the ring has some property which, in this case, is whether the ring itself is simple Artinian.

Theorem 1.2. ([18], p. 207-208) Let R be a ring with no nilpotent ideals and let $a \in R$ such that a^n is in the center of R. If $\mathrm{Cen}_R(a)$ is simple Artinian, then R is simple Artinian.

In this dissertation we will consider the centralizer of a matrix in $M_n(R)$, where R is a ring. Note that if R is a commutative ring with identity, then $M_n(R)$ is a prime example of a noncommutative central (i.e. the center of $M_n(R)$ is isomorphic to R) ring. It is a very difficult question in general to find a concrete description of the centralizer of an arbitrary matrix in $M_n(R)$. Most progress in this regard has been made with regard to the case when R is a field F. Let us discuss this case briefly.

First of all it is important to note that if $F[x]$ is the polynomial ring in the variable x over a field F, and if $B \in M_n(F)$, then

$$\{f(B) \mid f(x) \in F[x]\} \subseteq \mathrm{Cen}_{M_n(F)}(B).$$

(This statement in fact remains true if we replace F by any commutative ring.) Using the fact that $B \in M_n(F)$ is similar to a matrix D, called the rational canonical form of B, such that D is the direct sum of the companion matrices of the invariant factors of B ([13], p. 360-361, Corollary 4.7(i)); and that B only has one invariant factor if the minimum polynomial of B coincides with its characteristic polynomial ([13], p. 356-357, Theorem 4.2(i); [13], p. 367, Theorem 5.2(i)); we have the following concrete description of $\mathrm{Cen}_{M_n(F)}(B)$ in such a case.

Theorem 1.3. ([23], p. 23, Theorem 5) If B is an $n \times n$ matrix over a field F, then

$$\mathrm{Cen}_{M_n(F)}(B) = \{f(B) \mid f(x) \in F[x]\}$$

if and only if the minimum polynomial of B coincides with the characteristic polynomial of B.

Note that, by the converse statement of Theorem 1.3, the above mentioned description is not valid for any matrix of which the minimum polynomial does not coincide with its characteristic polynomial.

Since we will be working with 2×2 matrices in this dissertation and since the minimum polynomial and characteristic polynomial of a nonscalar 2×2 matrix always coincide (Lemma 2.6), the above theorem will play an important role in this dissertation.

Viewing $M_n(F)$ as an algebra over F, the following well-known result (due to Frobenius) gives us some information regarding the structure of $\mathrm{Cen}_{M_n(F)}(B)$ for an arbitrary $B \in M_n(F)$. However, a concrete description of $\mathrm{Cen}_{M_n(F)}(B)$ for the cases when the minimum polynomial of B is not equal to the characteristic polynomial of B is not yet known. Note that we denote the degree of a polynomial $f(x)$ by $\deg(f(x))$.

Theorem 1.4. ([14], p. 111, Theorem 19; [21], p. 331, Introduction and Preliminary Results) Let $B \in M_n(F)$, and suppose that $f_1, \ldots, f_l \in F[x]$ are the invariant factors of B, where f_i divides f_{i-1}, for $i = 2, \ldots, l$. Then the dimension of $Cen_{M_n(F)}(B)$ is given by

$$\sum_{i=1}^{l} (2i-1)(\deg f_i).$$

Keeping in mind (i) that the dimension of $\mathrm{Cen}_{M_n(F)}(B)$ over F is equal to the dimension of $\mathrm{Cen}_{M_n(\overline{F})}(B)$ over $\overline{F}$, where $\overline{F}$ is the algebraic closure of F ([23], p. 26, Lemma 5); (ii) that every matrix $B \in M_n(F) \subseteq M_n(\overline{F})$ is similar to its Jordan canonical form $J \in M_n(\overline{F})$, i.e. $SBS^{-1} = J$ for some $S \in M_n(\overline{F})$ ([13], p. 360, Corollary 4.7(iii)); (iii) that the dimension of the centralizer of similar matrices over the same ring is the same; and (iv) that matrices are similar if and only if they have the same invariant factors ([13], p. 361, Corollary 4.8(ii)); the above result can be obtained by proving it for an arbitrary Jordan canonical form $J \in M_n(\overline{F})$. This can in fact be done by finding a concrete description of $\mathrm{Cen}_{M_n(\overline{F})}(J)$ ([23], p. 25-28, Proposition 6, Lemma 4 and Theorem 6). If $F = \overline{F}$ then, of course, $\mathrm{Cen}_{M_n(F)}(B) = S^{-1}\mathrm{Cen}_{M_n(\overline{F})}(J)S$. Unfortunately $\overline{F} \neq F$ for every finite field F ([13], p. 267, Exercise 8). A result, analogous to Theorem 1.4, in which a formula for the dimension of $\mathrm{Cen}_{M_n(\mathbb{Q})}(B)$, for any $B \in M_n(\mathbb{Q})$, is given, is proved in [21].

If F is the complex field $\mathbb{C}$ (in this case note that $\overline{\mathbb{C}} = \mathbb{C}$) then a canonical basis for $\mathrm{Cen}_{M_n(\mathbb{C})}(J)$ is determined in [19] on p. 85-87. (This basis can be converted to a basis for $\mathrm{Cen}_{M_n(\mathbb{C})}(B)$, using the fact that B and its Jordan canonical form J are similar.) Furthermore it is shown that this basis is closed under nonzero products in the ring $M_n(\mathbb{C})$ ([19], p. 87, Lemma 4). It is also shown in [19] that the Jordan canonical forms of two matrices $A, B \in M_n(\mathbb{C})$ have the same canonical block structure ([19], p. 90, Definition 9) if and only if $\mathrm{Cen}_{M_n(\mathbb{C})}(A) \cong \mathrm{Cen}_{M_n(\mathbb{C})}(B)$ ([19], p. 91, Theorem 11). If F is the field of real numbers $\mathbb{R}$ (in this case note that $\overline{\mathbb{R}} = \mathbb{C}$) and the characteristic polynomial of $B \in M_n(\mathbb{R})$ is not separable over $\mathbb{R}$, then $J \in M_n(\mathbb{C}) \setminus M_n(\mathbb{R})$. A canonical basis for $\mathrm{Cen}_{M_n(\mathbb{R})}(B)$ is found ([19], p. 102, Theorem 24 and p. 104). Although this basis is not closed under nonzero products, a nonzero product of elements of this basis is ± 1 times another basis element.

Let S_1 and S_2 be subgroups of a group S and let $s \in S$. The set of all the elements in S_1 that commute with all the elements in $\text{Cen}_{S_2}(s)$ is called the centralizer in S_1 of the centralizer in S_2 of s and is denoted by $\text{Cen}_{S_1}(\text{Cen}_{S_2}(s))$. Note that $\text{Cen}_{S_1}(\text{Cen}_{S_2}(s))$ is a subgroup of S_1 and that if S_1, S_2 and S are rings, then $\text{Cen}_{S_1}(\text{Cen}_{S_2}(s))$ is a subring of S_1 (with identity if S_1 has an identity). Furthermore, it follows from the fact that $s \in \text{Cen}_S(s)$, that $\text{Cen}_S(\text{Cen}_S(s))$ can also be described as the center of $\text{Cen}_S(s)$. For an arbitrary $B \in M_n(F)$, a concrete description of $\text{Cen}_{M_n(F)}(\text{Cen}_{M_n(F)}(B))$ is known.

Theorem 1.5. ([23], p. 33, Theorem 7) Let $B \in M_n(F)$, then

$$\text{Cen}_{M_n(F)}(\text{Cen}_{M_n(F)}(B)) = \{f(B) \mid f(x) \in F[x]\}. \tag{1.2}$$

In order to prove Theorem 1.5 note that, by definition, B commutes with every element in its centralizer. Therefore it follows that we have the inclusion $\supseteq$ in (1.2). Since the dimension of $\{f(B) \mid f(x) \in F[x]\}$ is equal to the degree of the minimum polynomial of B, it is only necessary to show that the dimension of $\text{Cen}_{M_n(F)}(\text{Cen}_{M_n(F)}(J))$ is equal to the degree of the minimum polynomial of its Jordan canonical form J (which coincides with the minimum polynomial of B) to prove Theorem 1.5. This can again be done by finding a concrete description of $\text{Cen}_{M_n(F)}(\text{Cen}_{M_n(F)}(J))$.

Viewing Theorem 1.5 from a different perspective, considering $\text{Cen}_{M_n(F)}(B)$, we can also state this result as follows ([24], p. 106, Theorem 2):

Any matrix in $M_n(F)$ which commutes, not only with B, but also with every matrix which commutes with B, is a polynomial in B.

In [12] a concrete description is also found of

$$\text{Cen}_{M_n(F)}(\text{Cen}_{GL(n,F)}(B)) \qquad \text{and of} \qquad \text{Cen}_{GL(n,F)}(\text{Cen}_{GL(n,F)}(B)),$$

where $B \in M_n(F)$ and $GL(n, F)$ denotes the group of all $n \times n$ invertible matrices over the field F.

Although some other results regarding the centralizer of a matrix in a matrix ring over a ring are proved, the main goal of this dissertation is to find a concrete description of the centralizer of a so-called k-matrix in $M_2(R/\langle k\rangle)$, where R is a unique factorization domain (UFD) and $\langle k\rangle$ denotes the principal ideal generated by an arbitrary nonzero nonunit k in R.

In Sections 2.2 and 2.3 of Chapter 2 we apply Theorem 1.3 to 2×2 matrices in order to obtain an explicit description of the centralizer of a 2×2 matrix over a field or over an integral domain. Section 2.5 contains other preliminary results concerning the centralizer of an $n \times n$ matrix that will be used in the subsequent chapters, including Proposition 2.33 which may be considered as the inspiration behind this dissertation. In this proposition we show that the centralizer of an $n \times n$ matrix $\widehat{B}$ over a homomorphic image S of a commutative ring R contains the sum of two subrings $\mathcal{S}_1$ and $\mathcal{S}_2$ of $M_n(S)$, where $\mathcal{S}_1$ is the image of the centralizer in $M_n(R)$ of a pre-image of $\widehat{B}$, and where the entries in $\mathcal{S}_2$ are intersections of certain annihilators of elements arising from the entries of $\widehat{B}$. In addition we find a concrete description of the centralizer of a matrix unit in Section 2.1 and discuss some symmetric properties of the centralizer of a matrix in a matrix ring over a ring in Section 2.4.

We introduce the concepts of k-invertibility in a factor ring $R/\langle k \rangle$ of a UFD R in Section 3.1 and of a k-matrix in $M_2(R/\langle k \rangle)$ in Section 3.2 of Chapter 3. We show in Corollaries 3.7 and 3.18 that if R is a principal ideal domain (PID), then every element in $R/\langle k \rangle$ is k-invertible and every matrix in $M_2(R/\langle k \rangle)$ is a k-matrix. Examples 3.13 and 3.19(b) show that this is not true for UFD's in general. A characterization of the k-invertible elements in $R/\langle k \rangle$ is given in Corollary 3.14 in case k is a power of a prime and R is an arbitrary UFD. We conclude this chapter with Section 3.3 in which we consider the case when R is a UFD and $R/\langle k \rangle$ is finite. Analogous to the case when R is a PID, we prove in Corollaries 3.22 and 3.23 that if R is a UFD and $R/\langle k \rangle$ is finite, then every element in $R/\langle k \rangle$ is k-invertible and every matrix in $M_2(R/\langle k \rangle)$ is a k-matrix. In Remark 3.26 we also discuss the seemingly open problem, arising from these results, whether R is a PID if R is a UFD and $R/\langle k \rangle$ is finite.

Chapter 4, Section 4.1, contains the main result of the dissertation, namely Theorem 4.5, which provides a concrete description of the centralizer of a k-matrix in $M_2(R/\langle k \rangle)$ as the sum of the above mentioned two subrings, where R is a UFD and k is an arbitrary nonzero nonunit in R. In Section 4.2 we give necessary and sufficient conditions for when $\mathcal{S}_1 \subseteq \mathcal{S}_2$, for when $\mathcal{S}_2 \subseteq \mathcal{S}_1$ and for when $\mathcal{S}_1 = \mathcal{S}_2$.

Since every 2×2 matrix over a factor ring of a PID is a k-matrix, Theorem 4.5 applies to all 2×2 matrices over factor rings of PID's. In Example 4.9 we exhibit a UFD R, which is not a PID, a nonzero nonunit $k \in R$ and a matrix in $M_2(R/\langle k \rangle)$, which is not a k-matrix, for which Theorem 4.5 does not hold. In Example 4.10 we show that if R is a UFD and $k \in R$ is such that $R/\langle k \rangle$ is not an integral domain, then for every $n \geqslant 3$ there is a matrix B in $M_n(R)$ for which we have proper containment in Proposition 2.33.

The problem of enumerating the number of matrices with given characteristics over a finite ring has been treated extensively in the literature. Formulas have been found, for example, for the number of matrices with a given characteristic polynomial [20]; the number of matrices over a finite field that are cyclic [3] or symmetric [6]; and the number of matrices over the ring of integers $\mathbb{Z}$ modulo m,

$\mathbb{Z}_m$, that are nilpotent [4]. By using the results in [5], some of the above mentioned results, where the matrices over a finite field that satisfy some property are enumerated by rank, can be extended to matrices over certain finite rings that satisfy the property under consideration.

A question arising from the title of this dissertation and the above mentioned results is whether it is possible to enumerate the number of matrices in $\mathrm{Cen}_{M_n(R)}(B)$, denoted by $|\mathrm{Cen}_{M_n(R)}(B)|$, when R is a finite commutative ring and $B \in M_n(R)$. Using the fact that if R is a finite field F, then the dimension of $\mathrm{Cen}_{M_n(F)}(B)$ is known by Theorem 1.4, the answer is straightforward in such a case. For example, if $n = 2$, then the number of elements in $\mathrm{Cen}_{M_n(F)}(B)$ is $|F|^2$, if B is a nonscalar matrix, and it is $|F|^4$ if B is a scalar matrix. If $n = 2$ we can even easily determine the number of matrices with the same centralizer. Taking into account that the minimum polynomial always coincides with the characteristic polynomial of a nonscalar matrix $B \in M_2(F)$ (Lemma 2.6) and we therefore can apply Theorem 1.3 arriving at Corollary 2.7, it follows that $\mathrm{Cen}_{M_2(F)}(A) = \mathrm{Cen}_{M_2(F)}(B)$ for any nonscalar matrix $A \in M_2(F)$ if and only if $A \in \mathrm{Cen}_{M_2(F)}(B)$. Hence the number of matrices with the same centralizer as a matrix $B \in M_2(F)$ is $|F|$ (the number of scalar matrices in $M_2(F)$), if B is a scalar matrix, and $|F|^2 - |F|$ (the number of matrices in $\mathrm{Cen}_{M_2(F)}(B)$ minus the number of scalar matrices in $M_2(F)$), if B is a nonscalar matrix.

In Chapter 5 we define an equivalence relation on $M_2(R/\langle k\rangle)$ and we use this relation to obtain a formula for the number of matrices in $\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B})$ when R is a UFD and $R/\langle k\rangle$ is finite, k is a nonzero nonunit element in R and $\widehat{B} \in M_2(R/\langle k\rangle)$.

Chapter

2 Preliminary Results

The more I practice the luckier I get.

— Gary Player

The goals of this chapter are manifold. Firstly we easily find for any commutative ring R a concrete description of the centralizer of a scalar multiple of a matrix unit in $M_n(R)$ (Lemma 2.1, Section 2.1). Secondly we find a concrete description for the centralizer of an arbitrary 2×2 matrix in $M_2(R)$ when R is a field (Corollaries 2.9 and 2.10, Section 2.2) or when R is an integral domain (Corollary 2.12, Section 2.3). This chapter also contains a discussion of some symmetric properties of the centralizer of an $n \times n$ matrix over a not necessarily commutative ring (Section 2.4), as well as preliminary results that will be used repeatedly throughout this dissertation, in particular, in Chapter 4 (Section 2.5). We conclude with Proposition 2.33 (Section 2.5), which may be considered as the inspiration behind Chapter 4, and a discussion thereof.

2.1 The centralizer of a matrix unit in $M_n(R)$, R a ring

Throughout this dissertation we denote the matrix unit with 1 in position (i, j) and zeroes elsewhere by E_{ij}, and we use the notation

$$\begin{bmatrix} \mathcal{B} & \mathcal{C} \\ \mathcal{D} & \mathcal{E} \end{bmatrix}$$

to denote the set

$$\left\{ \begin{bmatrix} b & c \\ d & e \end{bmatrix} \,\middle|\, b \in \mathcal{B}, c \in \mathcal{C}, d \in \mathcal{D}, e \in \mathcal{E} \right\},$$

where $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are subsets of a ring R.

The set of all elements in a non-commutative ring R that annihilate a specific element b in R from the left (right) , i.e. the set $\{a \in R | \, ab = 0\}$ ($\{a \in R | \, ba = 0\}$), is called the left (right) annihilator of b in R. If $s \in R$ is in the left and right annihilator of $b \in R$ then $s \in \mathrm{Cen}_R(b)$. If R is a commutative ring then the left and right annihilator of an element obviously coincide. In such a case the set of all elements in R that annihilate a specific element $b \in R$ is called the annihilator of b in R and we denote it by $\mathrm{ann}_R(b)$ ([13], p. 417). If there is no ambiguity, we will sometimes simply write $\mathrm{ann}(b)$.

Lemma 2.1. Let R be a commutative ring and let $b \in R$. Then $\mathrm{Cen}_{M_n(R)}(bE_{rt}) =$

$$\left\{ \begin{array}{lll} a(E_{rr} + E_{tt}), & a \in R, & \text{if } r \neq t \\ aE_{rr}, & a \in R, & \text{if } r = t \end{array} \right. \quad + \quad \begin{array}{cc} & \begin{array}{ccccc} & & & \text{column } r \downarrow & \end{array} \\ \text{row } t \rightarrow & \begin{bmatrix} & & & \mathrm{ann}(b) & \\ & R & & & R \\ & & & \vdots & \\ \mathrm{ann}(b) & \mathrm{ann}(b) & \cdots & \mathrm{ann}(b) & \cdots & \mathrm{ann}(b) \\ & & & \vdots & \\ & R & & & R \\ & & & \mathrm{ann}(b) & \end{bmatrix} \end{array}.$$

Proof.

$$\begin{aligned} & Y = [y_{ij}] \in \mathrm{Cen}_{M_n(R)}(bE_{rt}) \\ \Leftrightarrow \; & [y_{ij}]bE_{rt} = bE_{rt}[y_{ij}] \end{aligned}$$

$$\Leftrightarrow \begin{array}{c} \text{columt } t \downarrow \\ \begin{bmatrix} & by_{1r} & \\ & by_{2r} & \\ & \vdots & \\ \bigcirc & by_{rr} & \bigcirc \\ & \vdots & \\ & by_{nr} & \end{bmatrix} \end{array} = \begin{array}{cc} & \\ \text{row } r \rightarrow & \begin{bmatrix} & & \bigcirc & & \\ & & & & \\ by_{t1} & \cdots & by_{tt} & \cdots & by_{tn} \\ & & & & \\ & & \bigcirc & & \end{bmatrix} \end{array}$$

$$\begin{aligned} \Leftrightarrow \; & by_{ir} = 0, \text{ for all } i \neq r, \text{ and } by_{ti} = 0, \text{ for all } i \neq t, \text{ and } b(y_{rr} - y_{tt}) = 0 \\ \Leftrightarrow \; & y_{ir} \in \mathrm{ann}(b), \text{ for all } i \neq r, \text{ and } y_{ti} \in \mathrm{ann}(b), \text{ for all } i \neq t, \text{ and } y_{rr} - y_{tt} \in \mathrm{ann}(b) \\ \Leftrightarrow \; & [y_{ij}] \in \left\{ \begin{array}{lll} a(E_{rr} + E_{tt}), & a \in R, & \text{if } r \neq t \\ aE_{rr}, & a \in R, & \text{if } r = t \end{array} \right. \quad + \end{aligned}$$

$$\begin{array}{cc} & \begin{array}{ccccc} & & \text{column } r & & \\ & & \downarrow & & \end{array} \\ \text{row } t \rightarrow & \left[\begin{array}{ccccc} & & \mathrm{ann}(b) & & \\ R & & & & R \\ & & \vdots & & \\ \mathrm{ann}(b)\ \mathrm{ann}(b) & \cdots & \mathrm{ann}(b) & \cdots & \mathrm{ann}(b) \\ & & \vdots & & \\ R & & & & R \\ & & \mathrm{ann}(b) & & \end{array}\right]. \end{array}$$

□

Example 2.2. Since $\mathrm{ann}(\hat{3}) = \langle\hat{4}\rangle$ in $\mathbb{Z}_{12}$ we have by Lemma 2.1 that

$$\mathrm{Cen}_{M_4(\mathbb{Z}_{12})}(\hat{3}E_{34}) = \left\{ \begin{bmatrix} \hat{0} & \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{a} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} & \hat{a} \end{bmatrix} \middle| \hat{a} \in \mathbb{Z}_{12} \right\} + \begin{bmatrix} \mathbb{Z}_{12} & \mathbb{Z}_{12} & \langle\hat{4}\rangle & \mathbb{Z}_{12} \\ \mathbb{Z}_{12} & \mathbb{Z}_{12} & \langle\hat{4}\rangle & \mathbb{Z}_{12} \\ \mathbb{Z}_{12} & \mathbb{Z}_{12} & \langle\hat{4}\rangle & \mathbb{Z}_{12} \\ \langle\hat{4}\rangle & \langle\hat{4}\rangle & \langle\hat{4}\rangle & \langle\hat{4}\rangle \end{bmatrix}.$$

2.2 The centralizer of a matrix in $M_2(R)$, R a field

The next well-known result will be used in Corollary 2.4.

Theorem 2.3. (THE DIVISION ALGORITHM) ([13], p. 158, Theorem 6.2) If f and g are polynomials over a field F and $g \neq 0$, then there exist unique polynomials q and r over F such that $f = qg + r$ and either $r = 0$ or $\deg r < \deg g$.

By using Theorem 1.3 and The Division Algorithm (Theorem 2.3) we arrive at the following result for the case when the minimum polynomial coincide characteristic polynomial of an $n \times n$-matrix.

Corollary 2.4. If B is a $n \times n$ matrix over a field F of which the minimum polynomial coincide with the characteristic polynomial, then

$$\mathrm{Cen}_{M_2(F)}(B) = \{a_{n-1}B^{n-1} + \cdots + a_1 B + a_0 I \mid a_i \in F\}.$$

Proof. Suppose B is an $n \times n$-matrix of which the minimum and characteristic polynomial coincide. Then it follows from Theorem 1.3 that

$$\mathrm{Cen}_{M_n(F)}(B) = \{f(B) \mid f(t) \text{ is a polynomial over F}\}.$$

Thus if we can prove that

$$\{f(B) \mid f(t) \text{ is a polynomial over F}\} = \{a_{n-1}B^{n-1} + \cdots + a_1 B + a_0 I \mid a_i \in F\}$$

then we are finished. Now, suppose that $f(x) \in F[x]$. Since $\deg(m(x)) = n$, where $m(x)$ is the minimum polynomial of B, it follows from The Division Algorithm (Theorem 2.3) that

$$f(x) = h(x)m(x) + r(x),$$

where $r(x)$ and $h(x)$ are polynomials over F, and $\deg(r(x)) \leqslant n - 1$. Thus

$$f(B) = \underbrace{m(B)h(B)}_{=0} + r(B) = r(B)$$

and therefore we are finished. □

The following result is well-known.

Theorem 2.5. (THE CAYLEY-HAMILTON THEOREM) ([13], p. 367, Theorem 5.2(ii)) An $n \times n$ matrix over a field satisfies its characteristic polynomial.

As a result of the next lemma, Corollary 2.4 is applicable to any 2×2 nonscalar matrix.

Lemma 2.6. The characteristic- and minimum polynomial of a nonscalar 2×2 matrix over a field coincide.

Proof. Let $B \in M_2(F)$ and let $q(x)$ be the characteristic polynomial of B. Since a characteristic polynomial is monic and, according to the Cayley-Hamilton Theorem (Theorem 2.5), $q(B) = 0$, we only have to prove that $\deg(q(x)) = \deg(m(x))$, where $m(x)$ is the minimum polynomial of B. Given that B is a 2×2 matrix, we have that $\deg(q(x)) = 2$. Since B is a nonscalar matrix, $B \neq tI$ for all $t \in F$ which implies that $sB + tI \neq 0$, for all $s, t \in F$. Therefore $\deg(m(x)) \geqslant 2$. Consequently $\deg(q(x)) = \deg(m(x))$. □

Since $\mathrm{Cen}_{M_n(F)}(B) = M_n(F)$ for any $n \times n$ scalar matrix B and any field F, using Corollary 2.4 and Lemma 2.6, we have the following result for the 2×2 case.

Corollary 2.7. If B is a 2×2 matrix over a field F, then

$$\mathrm{Cen}_{M_2(F)}(B) = \begin{cases} \{aB + bI \mid a, b \in F\} & \text{if } B \text{ is a nonscalar matrix} \\ M_2(F) & \text{if } B \text{ is a scalar matrix.} \end{cases}$$

Using the above result we can determine the centralizer of any 2×2 matrix over a field F in $M_2(F)$.

Example 2.8. Let F be the field of rational numbers $\mathbb{Q}$ and let $B = \begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix}$. By Corollary 2.7

$$\mathrm{Cen}_{M_2(\mathbb{Q})}(B) = \left\{ a \begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \middle| a, b \in \mathbb{Q} \right\} = \left\{ \begin{bmatrix} 2a + b & 3a \\ 4a & 8a + b \end{bmatrix} \middle| a, b \in \mathbb{Q} \right\}.$$

Corollary 2.7 can easily be written in the forms in Corollaries 2.9 and 2.10. We need both these forms in Chapter 4.

We will later in Corollary 2.17 prove that, for any $B \in M_n(F)$, $\mathrm{Cen}_{M_n(F)}(B) = (\mathrm{Cen}_{M_n(F)}(B^T))^T$. Knowing this, considering Corollary 2.9, we can for example, if the centralizer of a matrix in case (iv) is known, determine the centralizer of a matrix B in case (iii) by simply using $(\mathrm{Cen}_{M_n(F)}(B^T))^T$ as a formula.

Corollary 2.9. Let $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(F)$, F a field. Then

$$\mathrm{Cen}_{M_2(F)}(B) = \begin{cases} \text{(i) } M_2(F), \textit{ if } e = h,\ f = 0 \textit{ and } g = 0 \textit{ (i.e. } B \textit{ is a scalar matrix)} \\[2ex] \text{(ii) } \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a, b \in F \right\}, \textit{ if } e \neq h,\ f = 0 \textit{ and } g = 0 \\[2ex] \text{(iii) } \left\{ \begin{bmatrix} a & 0 \\ b & a - g^{-1}(e-h)b \end{bmatrix} \middle| a, b \in F \right\}, \textit{ if } f = 0,\ g \neq 0 \\[2ex] \text{(iv) } \left\{ \begin{bmatrix} a & b \\ f^{-1}gb & a - f^{-1}(e-h)b \end{bmatrix} \middle| a, b \in F \right\}, \textit{ if } f \neq 0. \end{cases}$$

Proof. Since the proofs of (i)–(iv) are similar, we only prove (iv).

(iv) Assume $f \neq 0$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Cen}_{M_2(F)}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) \Leftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{2.1}$$

By simplifying (2.1) the equation in position $(1, 1)$ is

$$ae + bg = ea + fc \Leftrightarrow bg = fc \Leftrightarrow c = f^{-1}gb \tag{2.2}$$

and the equation in position $(1, 2)$ is

$$af + bh = eb + fd \Leftrightarrow d = a - f^{-1}(e - h)b. \tag{2.3}$$

Thus it follows from (2.2) and (2.3) that

$$\mathrm{Cen}_{M_2(F)}(B) \subseteq \left\{ \begin{bmatrix} a & b \\ f^{-1}gb & a - f^{-1}(e - h)b \end{bmatrix} \middle| a, b \in F \right\}.$$

Since, direct verification shows that for arbitrary $a, b \in F$,

$$\begin{bmatrix} a & b \\ f^{-1}gb & a - f^{-1}(e - h)b \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ f^{-1}gb & a - f^{-1}(e - h)b \end{bmatrix}$$

we conclude that

$$\mathrm{Cen}_{M_2(F)}(B) = \left\{ \begin{bmatrix} a & b \\ f^{-1}gb & a - f^{-1}(e - h)b \end{bmatrix} \middle| a, b \in F \right\}.$$

□

We now give an alternative proof of Corollary 2.9. In this proof we explicitly show that Corollary 2.9 is equivalent to Corollary 2.7.

Alternative proof of Corollary 2.9. Again, since the proofs of (i)–(iv) are similar, we only prove (iv).

(iv) Assume $f \neq 0$. Then B is a nonscalar matrix, and so by Corollary 2.7,

$$\mathrm{Cen}_{M_2(F)}(B) = \left\{ s\begin{bmatrix} e & f \\ g & h \end{bmatrix} + t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \middle| s,t \in F \right\} = \left\{ \begin{bmatrix} se+t & sf \\ sg & sh+t \end{bmatrix} \middle| s,t \in F \right\}. \quad (2.4)$$

Let $\begin{bmatrix} se+t & sf \\ sg & sh+t \end{bmatrix}$ be an arbitrary matrix in (2.4). Now, put $a := se+t$ and $b := sf$. Then

$$sh+t = sh+se-se+t = -s(e-h)+a = -f^{-1}b(e-h)+a \qquad \text{and} \qquad sg = sff^{-1}g = f^{-1}gb.$$

Hence, $\begin{bmatrix} se+t & sf \\ sg & sh+t \end{bmatrix} = \begin{bmatrix} a & b \\ f^{-1}gb & a-f^{-1}(e-h)b \end{bmatrix}$ and we conclude that

$$\left\{ \begin{bmatrix} se+t & sf \\ sg & sh+t \end{bmatrix} \middle| s,t \in F \right\} \subseteq \left\{ \begin{bmatrix} a & b \\ f^{-1}gb & a-f^{-1}(e-h)b \end{bmatrix} \middle| a,b \in F \right\}. \quad (2.5)$$

Using direct verification, it follows that

$$\left\{ \begin{bmatrix} a & b \\ f^{-1}gb & a-f^{-1}(e-h)b \end{bmatrix} \middle| a,b \in F \right\} \subseteq \mathrm{Cen}_{M_2(R)}(B). \quad (2.6)$$

Thus the result follows from (2.4), (2.5) and (2.6). □

Corollary 2.10. Let $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(F)$, F a field. Then

$$\mathrm{Cen}_{M_2(F)}(B) = \begin{cases} \text{(i)} \left\{ \begin{bmatrix} a & (e-h)^{-1}f(a-b) \\ (e-h)^{-1}g(a-b) & b \end{bmatrix} \middle| a,b \in F \right\}, \text{if } e \neq h \\ \text{(ii)}\ M_2(F),\ \textit{if } e=h,\ f=0 \textit{ and } g=0 \textit{ (i.e. } B \textit{ is a scalar matrix)} \\ \text{(iii)} \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| a,b \in F \right\},\ \text{if } e=h, f\neq 0 \text{ and } g=0 \\ \text{(iv)} \left\{ \begin{bmatrix} a & 0 \\ b & a \end{bmatrix} \middle| a,b \in F \right\},\ \text{if } e=h, f=0 \text{ and } g\neq 0 \\ \text{(v)} \left\{ \begin{bmatrix} a & b \\ f^{-1}gb & a \end{bmatrix} \middle| a,b \in F \right\},\ \text{if } e=h, f\neq 0 \text{ and } g\neq 0. \end{cases}$$

Proof. Since the proofs of (i)–(v) are again similar, we only prove (i).

(i) Assume $e \neq h$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Cen}_{M_2(F)}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) \Leftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{2.7}$$

By simplifying (2.7) the equation in position (1,2) is

$$af + bh = eb + fd \Leftrightarrow b = (e-h)^{-1}f(a-d) \tag{2.8}$$

and the equation in position (2,1) is

$$ce + dg = ga + hc \Leftrightarrow c = (e-h)^{-1}g(a-d). \tag{2.9}$$

Thus it follows from (2.8) and (2.9) that

$$\mathrm{Cen}_{M_2(F)}(B) \subseteq \left\{ \begin{bmatrix} a & (e-h)^{-1}f(a-d) \\ (e-h)^{-1}g(a-d) & d \end{bmatrix} \middle| \, a, d \in F \right\}.$$

Since direct verification shows for an arbitrary $a, d \in F$ that

$$\begin{aligned} &\begin{bmatrix} a & (e-h)^{-1}f(a-d) \\ (e-h)^{-1}g(a-d) & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ = &\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & (e-h)^{-1}f(a-d) \\ (e-h)^{-1}g(a-d) & d \end{bmatrix}, \end{aligned}$$

the result follows. $\square$

There is an alternative proof of the above corollary similar to the alternative proof of Corollary 2.9.

2.3 The centralizer of a matrix in $M_2(R)$, R an integral domain

The following trivial result will be used repeatedly throughout this dissertation.

Lemma 2.11. Let S be a subring of a ring T and let $s \in S$. Then

$$\mathrm{Cen}_S(s) = S \cap \mathrm{Cen}_T(s).$$

Proof. $t \in \mathrm{Cen}_S(s) \Leftrightarrow t \in S \subseteq T$ and $ts = st \Leftrightarrow t \in S \cap \mathrm{Cen}_T(s)$. $\square$

Let $f_1, f_2, \ldots, f_m$ be arbitrary elements of a UFD. By writing $\gcd(f_1, f_2, \ldots, f_m)$, we mean an arbitrary greatest common divisor of $f_1, \ldots, f_m$.

Using Corollary 2.9 and Lemma 2.11, we have the following corollary from which we can determine the centralizer of a matrix in $M_2(R)$, where R is an integral domain.

Corollary 2.12. Let $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R)$, R an integral domain. Then $\mathrm{Cen}_{M_2(R)}(B)$

$$= \begin{cases} \text{(i) } M_2(R), \ \textit{if } e = h,\ f = 0 \textit{ and } g = 0 \textit{ (i.e. } B \textit{ is a scalar matrix)} \\ \\ \text{(ii) } \left\{ \begin{bmatrix} a & fbd^{-1} \\ gbd^{-1} & a-(e-h)bd^{-1} \end{bmatrix} \middle| \, a, b \in R \right\}, \ \textit{if at least one of } e-h, f \textit{ and } g \textit{ is nonzero,} \end{cases}$$

where d^{-1} is the inverse of $d = \gcd(e-h, f, g)$ in the quotient field of R.

Proof. Let F be the quotient field of R.

(i) The result follows from Corollary 2.9(i) and Lemma 2.11.

(ii) We distinguish between the following cases:

(a) $f = 0$, $g = 0$ and $e \neq h$;

(b) $f = 0$ and $g \neq 0$;

(c) $f \neq 0$.

(a) In this case $d = \gcd(e-h, 0, 0) = e-h$. Therefore it follows from Corollary 2.9(ii) and Lemma 2.11 that

$$\mathrm{Cen}_{M_2(R)}(B) \quad = \quad \left\{ \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \middle| \, a, c \in F \right\} \cap M_2(R)$$

$$
\begin{aligned}
&= \left\{ \begin{bmatrix} a & 0 \\ 0 & a-b \end{bmatrix} \middle| a, b \in F \right\} \cap M_2(R) \\
&= \left\{ \begin{bmatrix} a & 0 \\ 0 & a-b \end{bmatrix} \middle| a, b \in R \right\} \\
&= \left\{ \begin{bmatrix} a & 0b(e-h)^{-1} \\ 0b(e-h)^{-1} & a-(e-h)(e-h)^{-1}b \end{bmatrix} \middle| a, b \in R \right\} \\
&= \left\{ \begin{bmatrix} a & fbd^{-1} \\ gbd^{-1} & a-(e-h)bd^{-1} \end{bmatrix} \middle| a, b \in R \right\}.
\end{aligned}
$$

(b) It follows from Corollary 2.9(iii) and Lemma 2.11 that

$$
\begin{aligned}
\mathrm{Cen}_{M_2(R)}(B) &= \left\{ \begin{bmatrix} a & 0 \\ c & a-g^{-1}(e-h)c \end{bmatrix} \middle| a, c \in F \right\} \cap M_2(R) && (2.10) \\
&= \left\{ \begin{bmatrix} a & 0 \\ c & a-g^{-1}(e-h)c \end{bmatrix} \middle| a, c \in R \right\} \cap M_2(R). && (2.11)
\end{aligned}
$$

Let A be an arbitrary element of $\mathrm{Cen}_{M_2(R)}(B)$. It follows from (2.11) that

$$A = \begin{bmatrix} a & 0 \\ c & a-g^{-1}(e-h)c \end{bmatrix} \in M_2(R) \tag{2.12}$$

for some $a, c \in R$. We now show that

$$A = \begin{bmatrix} a & 0 \\ gbd^{-1} & a-(e-h)bd^{-1} \end{bmatrix} \tag{2.13}$$

for some $b \in R$. Since

$$\gcd(e-h, g) = \gcd(e-h, 0, g) = \gcd(e-h, f, g) := d, \tag{2.14}$$

it follows that

$$g = dg' \qquad \text{and} \qquad e-h = dl \tag{2.15}$$

for some $g', l \in R$ such that $\gcd(g', l) = 1$. Because $c(e-h)g^{-1} \in R$, by (2.12), it follows from (2.15) that

$$c(e-h)g^{-1} = cdl(dg')^{-1} = cl(g')^{-1} \in R. \tag{2.16}$$

Knowing that $\gcd(g', l) = 1$ it follows from (2.16) that $g'|c$, which implies that

$$c = bg' \tag{2.17}$$

for some $b \in R$. Hence, by using (2.16) and (2.17),

$$c(e-h)g^{-1} = cl(g')^{-1} = bg'l(g')^{-1} = bl = b(e-h)d^{-1} \in R. \tag{2.18}$$

Therefore, it follows from (2.15), (2.17) and (2.18) that

$$A = \begin{bmatrix} a & 0 \\ bg' & a - b(e-h)d^{-1} \end{bmatrix} = \begin{bmatrix} a & 0 \\ bgd^{-1} & a - b(e-h)d^{-1} \end{bmatrix} \in M_2(R). \tag{2.19}$$

Thus, by (2.10) and (2.19),

$$\begin{aligned} \mathrm{Cen}_{M_2(R)}(B) &\subseteq \left\{ \begin{bmatrix} a & 0 \\ bgd^{-1} & a - b(e-h)d^{-1} \end{bmatrix} \middle| a, b \in R \right\} \\ &= \left\{ \begin{bmatrix} a & 0 \\ bgd^{-1} & a - (e-h)g^{-1}(bgd^{-1}) \end{bmatrix} \middle| a, b \in R \right\} \\ &\subseteq \left\{ \begin{bmatrix} a & 0 \\ c & a - (e-h)g^{-1}c \end{bmatrix} \middle| a, c \in F \right\} \cap M_2(R) \\ &= \mathrm{Cen}_{M_2(R)}(B). \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} \mathrm{Cen}_{M_2(R)}(B) &= \left\{ \begin{bmatrix} a & 0 \\ gbd^{-1} & a - (e-h)bd^{-1} \end{bmatrix} \middle| a, b \in R \right\} \\ &= \left\{ \begin{bmatrix} a & fbd^{-1} \\ gbd^{-1} & a - (e-h)bd^{-1} \end{bmatrix} \middle| a, b \in R \right\}. \end{aligned}$$

(c) It follows from Corollary 2.9(iv) and Lemma 2.11 that

$$\mathrm{Cen}_{M_2(R)}(B) = \left\{ \begin{bmatrix} a & b \\ f^{-1}gb & a - f^{-1}(e-h)b \end{bmatrix} \middle| a, b \in F \right\} \cap M_2(R)$$

$$= \left\{ \begin{bmatrix} a & b \\ f^{-1}gb & a - f^{-1}(e-h)b \end{bmatrix} \middle| a, b \in R \right\} \cap M_2(R). \tag{2.20}$$

Let A be an arbitrary element of $\mathrm{Cen}_{M_2(R)}(B)$. Then it follows from (2.20) that

$$A = \begin{bmatrix} a & b \\ gbf^{-1} & a - (e-h)bf^{-1} \end{bmatrix} \in M_2(R) \tag{2.21}$$

for some $a, b \in R$. We now show that

$$A = \begin{bmatrix} a & fcd^{-1} \\ gcd^{-1} & a - (e-h)cd^{-1} \end{bmatrix}$$

for some $c \in R$. Now, let

$$d_1 := \gcd(f, g). \tag{2.22}$$

Then

$$f = d_1 f' \qquad \text{and} \qquad g = d_1 g' \tag{2.23}$$

for some $f', g' \in R$ such that $\gcd(f', g') = 1$. Since, by (2.21) and (2.23),

$$gbf^{-1} = d_1 g' b (d_1 f')^{-1} = g' b (f')^{-1} \in R \tag{2.24}$$

and $\gcd(f', g') = 1$, it follows that $f' | b$. Thus

$$b = f' b' \tag{2.25}$$

for some $b' \in R$. Hence, it follows from (2.24) and (2.25) that

$$gbf^{-1} = g' b (f')^{-1} = g' b' f' (f')^{-1} = g' b'. \tag{2.26}$$

Furthermore, it follows from (2.23) and (2.25) that

$$(e-h)bf^{-1} = (e-h) f' b' (d_1 f')^{-1} = (e-h) b' d_1^{-1} \tag{2.27}$$

and so, from (2.25), (2.26) and (2.27) that

$$A = \begin{bmatrix} a & b \\ gbf^{-1} & a - (e-h)bf^{-1} \end{bmatrix} = \begin{bmatrix} a & f'b' \\ g'b' & a - (e-h)b'd_1^{-1} \end{bmatrix} \in M_2(R). \tag{2.28}$$

Since $\gcd(d_1, e-h) = \gcd(\gcd(f,g), e-h) = \gcd(f,g,e-h) := d$, it follows that

$$d_1 = d_1'd \qquad \text{and} \qquad (e-h) = ld \tag{2.29}$$

for some $d_1', l \in R$ such that $\gcd(d_1', l) = 1$. Since, by (2.28) and (2.29),

$$(e-h)b'd_1^{-1} = ldb'(d_1'd)^{-1} = lb'(d_1')^{-1} \in R \tag{2.30}$$

and $\gcd(d_1', l) = 1$, it follows that $d_1'|b'$. Therefore

$$b' = cd_1' \tag{2.31}$$

for some $c \in R$. Thus, by (2.30) and (2.31),

$$(e-h)b'd_1^{-1} = lb'(d_1')^{-1} = lcd_1'(d_1')^{-1} = lc. \tag{2.32}$$

Hence it follows from (2.28), (2.31) and (2.32) that

$$A = \begin{bmatrix} a & f'd_1'c \\ g'd_1'c & a - lc \end{bmatrix}$$

so that it follows from (2.23) and (2.29) that

$$A = \begin{bmatrix} a & fd_1{}^{-1}d_1d^{-1}c \\ gd_1{}^{-1}d_1d^{-1}c & a-(e-h)d^{-1}c \end{bmatrix} = \begin{bmatrix} a & fcd^{-1} \\ gcd^{-1} & a-(e-h)cd^{-1} \end{bmatrix}.$$

Thus, it follows from (2.20) that

$$\begin{aligned}
\mathrm{Cen}_{M_2(R)}(B) &\subseteq \left\{ \begin{bmatrix} a & fcd^{-1} \\ gcd^{-1} & a-(e-h)cd^{-1} \end{bmatrix} \middle| a, c \in R \right\} \\
&= \left\{ \begin{bmatrix} a & fcd^{-1} \\ gf^{-1}\left(fcd^{-1}\right) & a-(e-h)f^{-1}\left(fcd^{-1}\right) \end{bmatrix} \middle| a, c \in R \right\} \\
&\subseteq \left\{ \begin{bmatrix} a & b \\ gf^{-1}b & a-(e-h)f^{-1}b \end{bmatrix} \middle| a, b \in R \right\} \cap M_2(R) \\
&= \mathrm{Cen}_{M_2(R)}(B).
\end{aligned}$$

Hence we conclude that

$$\mathrm{Cen}_{M_2(R)}(B) = \left\{ \begin{bmatrix} a & fbd^{-1} \\ gbd^{-1} & a-(e-h)bd^{-1} \end{bmatrix} \middle| a, b \in R \right\}.$$

$\square$

Example 2.13. Let R be the integral domain $\mathbb{Z}$ and let $B = \begin{bmatrix} 2 & 3 \\ 6 & 8 \end{bmatrix}$. It follows from Corollary 2.12 that

$$\mathrm{Cen}_{M_2(\mathbb{Z})}(B) = \left\{ \begin{bmatrix} a & \frac{3}{3}b \\ \frac{6}{3}b & a + \frac{6}{3}b \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\} = \left\{ \begin{bmatrix} a & b \\ 2b & a + 2b \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\}.$$

2.4 Symmetric properties of the centralizer of a matrix in $M_n(R)$, R a ring

For a set $\mathcal{C}$ and a function α with domain $\mathcal{C}$ we denote the set $\{\alpha(c) \mid c \in \mathcal{C}\}$ by $\alpha(\mathcal{C})$.

Lemma 2.14. Let R and S be (not necessarily commutative) rings and let $\alpha : R \to S$ be an isomorphism or anti-isomorphism. Then

$$\mathrm{Cen}_S(\alpha(r)) = \alpha(\mathrm{Cen}_R(r)).$$

Proof. Suppose α is an anti-isomorphism and $t \in \mathrm{Cen}_R(r)$. Then

$$\alpha(r)\alpha(t) = \alpha(tr) = \alpha(rt) = \alpha(t)\alpha(r).$$

Hence $\alpha(t) \in \mathrm{Cen}_S(\alpha(r))$. Therefore

$$\alpha(\mathrm{Cen}_R(r)) \subseteq \mathrm{Cen}_S(\alpha(r)).$$

Let $\alpha^{-1} : S \to R$ be the inverse map of α. Then, since

$$\alpha^{-1}(\alpha(r) + \alpha(s)) = \alpha^{-1}(\alpha(r + s)) = r + s = \alpha^{-1}(\alpha(r)) + \alpha^{-1}(\alpha(s))$$

and

$$\alpha^{-1}(\alpha(r)\alpha(s)) = \alpha^{-1}(\alpha(sr)) = sr = \alpha^{-1}(\alpha(s))\alpha^{-1}(\alpha(r)),$$

we have that α^{-1} also is an anti-isomorphism. Hence it follows that

$$\mathrm{Cen}_S(\alpha(r)) = \alpha(\alpha^{-1}(\mathrm{Cen}_S(\alpha(r)))) \subseteq \alpha(\mathrm{Cen}_R(\alpha^{-1}(\alpha(r)))) = \alpha(\mathrm{Cen}_R(r)).$$

The result for the case when α is an isomorphism is similar. □

We first discuss some symmetric properties of the centralizer of a matrix around the main diagonal.

We will use the concept of an opposite ring.

Definition 2.15. ([13], p. 122, Exercise 17(a)) The opposite ring, denoted by R^{op}, of a ring R is defined as follows. The underlying set of R^{op} is precisely the underlying set of R, and addition in R^{op} coincides with addition in R. Multiplication in R^{op}, denoted by $\circ$, is given by $a \circ b = ba$, where ba is the product in R.

Let $\beta : M_n(R) \to M_n(R^{op})$ be the map defined by taking the transpose of a matrix in $M_n(R)$. The matrix $\beta(B)$ is customarily denoted by B^T. If $\mathcal{B}$ is a set of matrices over R, then we denote the set $\{B^T \mid B \in \mathcal{B}\}$ by $\mathcal{B}^T$ and we call the set $\mathcal{B}^T$ the transpose of $\mathcal{B}$.

Using the fact that the map β is an anti-isomorphism ([13], p. 331, part of the proof of Theorem 1.4), the following result follows directly from Lemma 2.14.

Corollary 2.16. Let $B \in M_n(R)$, where R is a ring. Then

$$\mathrm{Cen}_{M_n(R^{op})}(B^T) = (\mathrm{Cen}_{M_n(R)}(B))^T.$$

Taking into account that if R is a commutative ring, then $R^{op} = R$, we have the following result.

Corollary 2.17. Let $B \in M_n(R)$, where R is a commutative ring. Then

$$\mathrm{Cen}_{M_n(R)}(B^T) = (\mathrm{Cen}_{M_n(R)}(B))^T.$$

In the next example we will see that Corollary 2.17 is not necessarily applicable if we replace R with a noncommutative ring.

Example 2.18. Let $\mathcal{Q}$ be the noncommutative ring of quaternions. Now, let

$$B = \begin{bmatrix} 0 & 0 & i \\ j & 0 & 0 \\ 0 & k & 0 \end{bmatrix} \in M_3(\mathcal{Q}).$$

Then direct verification shows that

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & i \\ k & 0 & 0 \end{bmatrix} \in \mathrm{Cen}_{M_3(\mathcal{Q})}(B),$$

but that

$$A^T \notin \mathrm{Cen}_{M_3(\mathcal{Q})}(B^T).$$

Furthermore, direct verification also shows that

$$A^T \in \mathrm{Cen}_{M_3(\mathcal{Q}^{op})}(B^T),$$

as is expected from Corollary 2.16.

In the next example we illustrate Corollary 2.17.

Example 2.19. Let $R = \mathbb{Z}$ and let $B = \begin{bmatrix} 2 & 3 \\ 6 & 8 \end{bmatrix}$. It follows from Corollary 2.12(ii), using Example 2.13, that

$$\begin{aligned} \mathrm{Cen}_{M_2(\mathbb{Z})}(B^T) &= \left\{ \begin{bmatrix} a & \frac{6}{3}b \\ \frac{3}{3}b & a + \frac{6}{3}b \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\} = \left\{ \begin{bmatrix} a & 2b \\ b & a + 2b \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ 2b & a + 2b \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\}^T = \left(\mathrm{Cen}_{M_2(\mathbb{Z})}(B)\right)^T, \end{aligned}$$

as is expected from Corollary 2.16.

According to the next corollary, the centralizer of a symmetric matrix over a commutative ring has the symmetric property that the transpose of each matrix which is in its centralizer, is again in its centralizer.

Corollary 2.20. Let $B \in M_n(R)$, where R is a commutative ring. If $B = B^T$, then

$$\mathrm{Cen}_{M_n(R)}(B) = (\mathrm{Cen}_{M_n(R)}(B))^T.$$

Proof. It follows from Corollary 2.17 that $\mathrm{Cen}_{M_n(R)}(B) = \mathrm{Cen}_{M_n(R)}(B^T) = (\mathrm{Cen}_{M_n(R)}(B))^T$. $\square$

We now discuss some symmetric properties of the centralizer of a matrix around the main skew-diagonal. First we have to define the following new concepts.

Definition 2.21. Let $b = [b_{ij}] \in M_n(R)$, where R is a ring.

We denote the matrix which is formed by rotating the entries of B around the horizontal axis, in other words by mapping the entry in position (i, j) to position $(n + 1 - i, j)$, by B^H.

The matrix which is formed by rotating the entries of B around the vertical axis, hence by mapping the entry in position (i, j) to position $(i, n + 1 - j)$, is denoted by B^V.

Lastly, we call the matrix which is formed by rotating the entries of B around the main skew-diagonal, which is the matrix formed by mapping the entry in position (i, j) to position $(n + 1 - j, n + 1 - i)$, the s-transpose of B. We denote this matrix by $B^{T'}$. If $B = B^{T'}$ then we call B s-symmetric.

Similarly to the transpose of a set of matrices $\mathcal{B}$, we denote the set $\{B^{T'} \mid B \in \mathcal{B}\}$ by $\mathcal{B}^{T'}$ and we call $\mathcal{B}^{T'}$ the s-transpose of $\mathcal{B}$.

Remark 2.22. Note that because the transpose of a matrix B is formed by mapping position (i, j) to position (j, i) it follows from the above definitions that $B^{HVT} = B^{T'}$.

Lemma 2.23. Let $B \in M_n(R)$, where R is a commutative ring. Then the map $\gamma : M_n(R) \to M_n(R)$ given by $\gamma(B) = B^{HV}$ is an isomorphism.

Proof. Let

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Then

$$B^{HV} = \begin{bmatrix} b_{nn} & b_{n,n-1} & \cdots & b_{n1} \\ b_{n-1,n} & b_{n-1,n-1} & \cdots & b_{n-1,1} \\ \vdots & \vdots & & \vdots \\ b_{1n} & b_{1,n-1} & \cdots & b_{11} \end{bmatrix} \text{ and } A^{HV} = \begin{bmatrix} a_{nn} & a_{n,n-1} & \cdots & a_{n1} \\ a_{n-1,n} & a_{n-1,n-1} & \cdots & a_{n-1,1} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{1,n-1} & \cdots & a_{11} \end{bmatrix}.$$

Since,

$$B^{HV} + A^{HV} = \begin{bmatrix} b_{nn} + a_{nn} & b_{n,n-1} + a_{n,n-1} & \cdots & b_{n1} + a_{n1} \\ b_{n-1,n} + a_{n-1,n} & b_{n-1,n-1} + a_{n-1,n-1} & \cdots & b_{n-1,1} + a_{n-1,1} \\ \vdots & \vdots & & \vdots \\ b_{1n} + a_{1n} & b_{1,n-1} + a_{1,n-1} & \cdots & b_{11} + a_{11} \end{bmatrix} = (B + A)^{HV}$$

it follows that γ preserves addition.

We now show that multiplication is also preserved. Without the loss of generality let us consider position $(n + 1 - i, n + 1 - j)$ of $B^{HV}A^{HV}$. The entry in this position is equal to the dot product of row $n + 1 - i$ of B^{HV} and column $n + 1 - j$ of A^{HV} which is equal to

$$b_{in}a_{nj} + b_{i,n-1}a_{n-1,j} + \cdots + b_{i1}a_{1j} = b_{i1}a_{1j} + \cdots + b_{i,n-1}a_{n-1,j} + b_{in}a_{nj}. \qquad (2.33)$$

But (2.33) is the dot product of row i of B and column j of A which is the entry of position (i, j) of BA. Because the entry of position (i, j) of BA is equal to the entry of position $(n+1-i, n+1-j)$ of $(BA)^{HV}$, we conclude that $B^{HV}A^{HV} = (BA)^{HV}$. Therefore γ preserves multiplication.

Now, suppose that $B^{HV} = A^{HV}$. Then the entries in position $(n+1-i, n+1-j)$ of B^{HV} and A^{HV} are equal, which implies that the entries in position (i,j) of A and B are equal. Since position (i,j) was chosen arbitrarily it follows that A and B are equal. Hence, γ is 1-1. Because $(B^{HV})^{HV} = B$ for all $B \in M_n(R)$, γ is also onto and therefore an isomorphism. □

Since the map $\beta : M_n(R) \to M_n(R^{op})$ defined by $\beta(B) = B^T$ is an anti-isomorphism, it follows from the above result that the map $\beta\gamma : M_n(R) \to M_n(R^{op})$, defined by taking the s-transpose of a matrix in $M_n(R)$ is also an anti-isomorphism. Therefore Corollaries 2.24 and 2.25 follows directly from Lemma 2.14.

Corollary 2.24. Let $B \in M_n(R)$, where R is a ring. Then

$$\mathrm{Cen}_{M_n(R^{op})}(B^{T'}) = (\mathrm{Cen}_{M_n(R)}(B))^{T'}.$$

Corollary 2.25. Let $B \in M_n(R)$, where R is a commutative ring. Then

$$\mathrm{Cen}_{M_n(R)}(B^{T'}) = (\mathrm{Cen}_{M_n(R)}(B))^{T'}.$$

Remark 2.26. Using $A, B \in M_3(\mathcal{Q})$ in Example 2.18, it follows by direct verification that

$$A^{T'} \notin \mathrm{Cen}_{M_3(\mathcal{Q})}(B^{T'}), \qquad \text{although} \qquad A \in \mathrm{Cen}_{M_3(\mathcal{Q})}(B).$$

It also follows in agreement with Corollary 2.24 that

$$A^{T'} \in \mathrm{Cen}_{M_3(\mathcal{Q}^{op})}(B^{T'}).$$

Therefore, similar to Corollary 2.16, Corollary 2.24 is not necessarily applicable if we replace R with a noncommutative ring.

In the next example we illustrate Corollary 2.25.

Example 2.27. Let $R = \mathbb{Z}_{12}$. It follows from Lemma 2.1, using Example 2.2, that

$$\begin{aligned} \mathrm{Cen}_{M_4(\mathbb{Z}_{12})}((\hat{3}E_{34})^{T'}) &= \mathrm{Cen}_{M_4(\mathbb{Z}_{12})}(\hat{3}E_{12}) \\ &= \left\{ \begin{bmatrix} \hat{a} & \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{a} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} & \hat{0} \end{bmatrix} \middle| \, \hat{a} \in \mathbb{Z} \right\} + \begin{bmatrix} \langle\hat{4}\rangle & \mathbb{Z}_{12} & \mathbb{Z}_{12} & \mathbb{Z}_{12} \\ \langle\hat{4}\rangle & \langle\hat{4}\rangle & \langle\hat{4}\rangle & \langle\hat{4}\rangle \\ \langle\hat{4}\rangle & \mathbb{Z}_{12} & \mathbb{Z}_{12} & \mathbb{Z}_{12} \\ \langle\hat{4}\rangle & \mathbb{Z}_{12} & \mathbb{Z}_{12} & \mathbb{Z}_{12} \end{bmatrix} \end{aligned}$$

$$= \left(\left\{ \begin{bmatrix} \hat{0} & \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{a} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} & \hat{a} \end{bmatrix} \middle| \hat{a} \in \mathbb{Z} \right\} + \begin{bmatrix} \mathbb{Z}_{12} & \mathbb{Z}_{12} & \langle \hat{4} \rangle & \mathbb{Z}_{12} \\ \mathbb{Z}_{12} & \mathbb{Z}_{12} & \langle \hat{4} \rangle & \mathbb{Z}_{12} \\ \mathbb{Z}_{12} & \mathbb{Z}_{12} & \langle \hat{4} \rangle & \mathbb{Z}_{12} \\ \langle \hat{4} \rangle & \langle \hat{4} \rangle & \langle \hat{4} \rangle & \langle \hat{4} \rangle \end{bmatrix} \right)^{T'}$$
$$= (\mathrm{Cen}_{M_4(\mathbb{Z}_{12})}(\hat{3}E_{34}))^{T'},$$

as expected from Corollary 2.25.

Similar to the transpose of a matrix over a commutative ring the centralizer of a s-symmetric matrix over a commutative ring has the symmetric property that the s-transpose of each matrix which is in its centralizer, is again in its centralizer.

Corollary 2.28. Let $B \in M_n(R)$, where R is a commutative ring. If $B = B^{T'}$, then

$$\mathrm{Cen}_{M_n(R)}(B) = (\mathrm{Cen}_{M_n(R)}(B))^{T'}.$$

Proof. It follows from Corollary 2.25 that $\mathrm{Cen}_{M_n(R)}(B) = \mathrm{Cen}_{M_n(R)}(B^{T'}) = (\mathrm{Cen}_{M_n(R)}(B))^{T'}$. □

2.5 Miscellaneous

The following results will be used repeatedly throughout this dissertation, and their proofs are straightforward.

Lemma 2.29. Let R be a commutative ring, $b, t \in R$, where t is invertible in R, and $B \in M_n(R)$. Then

$$(a) \quad \mathrm{Cen}_{M_n(R)}(B) = \mathrm{Cen}_{M_n(R)}(tB), \qquad (b) \quad \mathrm{Cen}_{M_n(R)}(B) = \mathrm{Cen}_{M_n(R)}(B + bI)$$

and

$$(c) \quad \mathrm{ann}_R(b) = \mathrm{ann}_R(tb),$$

Proof. Let $A \in M_n(R)$ and let $a \in R$. Then
(a)

$$\begin{aligned} A \in \mathrm{Cen}_{M_n(R)}(B) &\Leftrightarrow BA = AB \\ &\Leftrightarrow B(tA) = t(BA) = t(AB) = (tA)B \\ &\Leftrightarrow B \in \mathrm{Cen}_{M_n(R)}(tA), \end{aligned}$$

(b)

$$\begin{aligned} A \in \mathrm{Cen}_{M_n(R)}(B) &\Leftrightarrow AB = BA \\ &\Leftrightarrow A(B + bI) = AB + AbI = BA + bIA = (B + bI)A \\ &\Leftrightarrow A \in \mathrm{Cen}_{M_n(R)}(B + bI), \end{aligned}$$

(c)

$$a \in \mathrm{ann}_R(b) \Leftrightarrow ab = 0 \Leftrightarrow t(ab) = a(tb) = 0 \Leftrightarrow a \in \mathrm{ann}_R(tb).$$

□

For the remaining results in this section, let $\theta : R \to S$ be a ring epimorphism and $\Theta : M_n(R) \to M_n(S)$ the induced epimorphism, i.e. $\Theta([b_{ij}]) = [\theta(b_{ij})]$. For the sake of notation, we will sometimes denote $\theta(b)$ by $\hat{b}$ and $\Theta(B)$ by $\widehat{B}$. Also, if there is no ambiguity, we simply write $\mathrm{Cen}(B)$ instead of $\mathrm{Cen}_{M_n(R)}(B)$ and $\mathrm{Cen}(\widehat{B})$ instead of $\mathrm{Cen}_{M_n(S)}(\widehat{B})$ for $B \in M_n(R)$. If $r \in R$ and $A \subseteq R$, then rA denotes the set $\{ra \mid a \in A\}$.

Remark 2.30. Note that, given that θ is onto and preserves multiplication, it follows from the fact that R is a commutative ring, that S is also a commutative ring.

Lemma 2.31. Let R be an integral domain. If $0 \neq b \in R$, then

$$R \cap b^{-1} \ker\theta = \theta^{-1}(\mathrm{ann}(\hat{b})),$$

where b^{-1} is the inverse of b in the quotient field of R.

Proof. Let $a \in R$. Then

$$a \in b^{-1}\ker\theta \Leftrightarrow ba \in \ker\theta \Leftrightarrow \hat{b}\hat{a} = \hat{0} \Leftrightarrow \hat{a} \in \mathrm{ann}(\hat{b}) \Leftrightarrow a \in \theta^{-1}(\mathrm{ann}(\hat{b})).$$

□

In order to illustrate Lemma 2.31, let $R = \mathbb{Z}$, $S = \mathbb{Z}_{12}$, and $\theta : \mathbb{Z} \to \mathbb{Z}_{12}$ the natural epimorphism. Now, if $b = 2$ then

$$R \cap b^{-1}\ker\theta = \mathbb{Z} \cap \frac{1}{2}\langle 12\rangle = \langle 6\rangle$$

and

$$\theta^{-1}(\mathrm{ann}(\hat{b})) = \theta^{-1}(\mathrm{ann}(\hat{2})) = \theta^{-1}(\langle\hat{6}\rangle) = \langle 6\rangle.$$

Lemma 2.32. Let R be a commutative ring and let $B = [b_{ij}] \in M_n(R)$, then

$$(\Theta([b_{ij}]))^T = \Theta([b_{ij}]^T).$$

Proof. It follows from the definition of Θ that

$$(\Theta([b_{ij}]))^T = [\theta(b_{ij})]^T = [\theta(b_{ji})] = \Theta([b_{ji}]) = \Theta([b_{ij}]^T).$$

$\square$

The following result is the inspiration behind Chapter 4.

Proposition 2.33. Let R be a commutative ring and let $B = [b_{ij}] \in M_n(R)$. Then

$$\Theta(\mathrm{Cen}(B)) + [\mathcal{A}_{ij}] \subseteq \mathrm{Cen}(\widehat{B}),$$

where

$$\mathcal{A}_{ij} = \left(\bigcap_{k,\ k\neq j} \mathrm{ann}(\hat{b}_{jk})\right) \bigcap \left(\bigcap_{k,\ k\neq i} \mathrm{ann}(\hat{b}_{ki})\right) \bigcap \left(\mathrm{ann}(\hat{b}_{ii} - \hat{b}_{jj})\right).$$

Proof. We first prove that

$$\Theta(\mathrm{Cen}(B)) \subseteq \mathrm{Cen}(\widehat{B}). \tag{2.34}$$

Let $X \in \mathrm{Cen}(B)$. Then

$$\widehat{B}\Theta(X) = \Theta(B)\Theta(X) = \Theta(BX) = \Theta(XB) = \Theta(X)\Theta(B) = \Theta(X)\widehat{B},$$

which implies that $\Theta(X) \in \mathrm{Cen}(\widehat{B})$, i.e.

$$\Theta(\mathrm{Cen}(B)) \subseteq \mathrm{Cen}(\widehat{B}).$$

This proves (2.34). Now we show that

$$[\mathcal{A}_{ij}] \subseteq \mathrm{Cen}(\widehat{B}). \tag{2.35}$$

Let $[\hat{a}_{ij}] \in [\mathcal{A}_{ij}]$. Then it follows that position (r,t) of $\widehat{B}[\hat{a}_{ij}] - [\hat{a}_{ij}]\widehat{B}$ is equal to

$$\begin{aligned}&\hat{b}_{r1}\hat{a}_{1t} + \cdots + \hat{b}_{r,r-1}\hat{a}_{r-1,t} + \hat{b}_{rr}\hat{a}_{rt} + \hat{b}_{r,r+1}\hat{a}_{r+1,t} + \cdots + \hat{b}_{rn}\hat{a}_{nt} - \\ &(\hat{a}_{r1}\hat{b}_{1t} + \hat{a}_{r2}\hat{b}_{2t} + \cdots + \hat{a}_{r,t-1}\hat{b}_{t-1,t} + \hat{a}_{rt}\hat{b}_{tt} + \hat{a}_{r,t+1}\hat{b}_{t+1,t} + \cdots + \hat{a}_{rn}\hat{b}_{nt}).\end{aligned} \tag{2.36}$$

Since $\hat{a}_{lt} \in \operatorname{ann}(\hat{b}_{rl})$ for every l such that $l \neq r$, and $\hat{a}_{rq} \in \operatorname{ann}(\hat{b}_{qt})$ for every q such that $q \neq t$, by the definition of $[\mathcal{A}_{ij}]$, it follows that (2.36) is equal to

$$\hat{b}_{rr}\hat{a}_{rt} - \hat{a}_{rt}\hat{b}_{tt} = \hat{a}_{rt}(\hat{b}_{rr} - \hat{b}_{tt}). \tag{2.37}$$

Since $\hat{a}_{rt} \in \operatorname{ann}(\hat{b}_{rr} - \hat{b}_{tt})$, again by the definition of $[\mathcal{A}_{ij}]$, it follows that (2.37) is equal to $\hat{0}$. Thus position (r, t) of $[\hat{a}_{ij}]\widehat{B} - \widehat{B}[\hat{a}_{ij}]$ is $\hat{0}$. This proves (2.35). $\square$

We conclude this section with some results with regard to Proposition 2.33.

Lemma 2.34. The set $[\mathcal{A}_{ij}]$, as defined in Proposition 2.33, is a subring of $M_n(S)$ (not necessarily with identity).

Proof. Since $-A \in [\mathcal{A}_{ij}]$ if and only if $A \in [\mathcal{A}_{ij}]$, we only need to show that $[\mathcal{A}_{ij}]$ is closed under addition and multiplication.

Let $[\hat{x}_{ij}], [\hat{y}_{ij}] \in [\mathcal{A}_{ij}]$. The entry in an arbitrary position (s, t) of $[\hat{x}_{ij}] + [\hat{y}_{ij}]$ is $\hat{x}_{st} + \hat{y}_{st}$. Thus it follows from the definition of $[\mathcal{A}_{ij}]$ that

$$\hat{x}_{st}, \hat{y}_{st} \in \left(\bigcap_{k,\ k \neq t} \operatorname{ann}(\hat{b}_{tk}) \right) \bigcap \left(\bigcap_{k,\ k \neq s} \operatorname{ann}(\hat{b}_{ks}) \right) \bigcap \operatorname{ann}(\hat{b}_{ss} - \hat{b}_{tt}).$$

Since the annihilator of an element in R is an ideal in R, the intersection of ideals in R is an ideal in R and an ideal is closed under addition, it follows that

$$\hat{x}_{st} + \hat{y}_{st} \in \left(\bigcap_{k,\ k \neq t} \operatorname{ann}(\hat{b}_{tk}) \right) \bigcap \left(\bigcap_{k,\ k \neq s} \operatorname{ann}(\hat{b}_{ks}) \right) \bigcap \operatorname{ann}(\hat{b}_{ss} - \hat{b}_{tt}).$$

Therefore it follows again from the definition of $[\mathcal{A}_{ij}]$ that $[\hat{x}_{ij}] + [\hat{y}_{ij}] \in [\mathcal{A}_{ij}]$ and we conclude that $[\mathcal{A}_{ij}]$ is closed under addition.

The entry in an arbitrary position (s, t) of $[\hat{x}_{ij}][\hat{y}_{ij}]$ is

$$\sum_{l=1}^{n} \hat{x}_{sl}\hat{y}_{lt}.$$

Now, for an arbitrary l it follows that

$$\begin{aligned}\hat{x}_{sl} &\in \left(\bigcap_{k,\ k\neq l} \operatorname{ann}(\hat{b}_{lk})\right) \bigcap \left(\bigcap_{k,\ k\neq s} \operatorname{ann}(\hat{b}_{ks})\right) \bigcap \operatorname{ann}(\hat{b}_{ss} - \hat{b}_{ll}) \quad \text{and} \\ \hat{y}_{lt} &\in \left(\bigcap_{k,\ k\neq t} \operatorname{ann}(\hat{b}_{tk})\right) \bigcap \left(\bigcap_{k,\ k\neq l} \operatorname{ann}(\hat{b}_{kl})\right) \bigcap \operatorname{ann}(\hat{b}_{ll} - \hat{b}_{tt}).\end{aligned}$$

Similarly, because the annihilator of an element in R is an ideal in R, the intersection of ideals in R is an ideal in R and an ideal in R is closed under multiplication by any element in R, we have that

$$\hat{x}_{sl}\hat{y}_{lt} \in \left(\bigcap_{k,\ k\neq l} \operatorname{ann}(\hat{b}_{lk})\right) \bigcap \left(\bigcap_{k,\ k\neq s} \operatorname{ann}(\hat{b}_{ks})\right) \bigcap \operatorname{ann}(\hat{b}_{ss} - \hat{b}_{ll}) \quad \text{and} \tag{2.38}$$

$$\hat{x}_{sl}\hat{y}_{lt} \in \left(\bigcap_{k,\ k\neq t} \operatorname{ann}(\hat{b}_{tk})\right) \bigcap \left(\bigcap_{k,\ k\neq l} \operatorname{ann}(\hat{b}_{kl})\right) \bigcap \operatorname{ann}(\hat{b}_{ll} - \hat{b}_{tt}). \tag{2.39}$$

It follows from (2.38) that $\hat{x}_{sl}\hat{y}_{lt} \in \bigcap\limits_{k,\ k\neq s} \operatorname{ann}(\hat{b}_{ks})$ and from (2.39) that $\hat{x}_{sl}\hat{y}_{lt} \in \bigcap\limits_{k,\ k\neq t} \operatorname{ann}(\hat{b}_{tk})$.

Furthermore, since $\hat{x}_{sl}\hat{y}_{lt} \in \operatorname{ann}(\hat{b}_{ss} - \hat{b}_{ll})$, by (2.38), and $\hat{x}_{sl}\hat{y}_{lt} \in \operatorname{ann}(\hat{b}_{ll} - \hat{b}_{tt})$, by (2.39), it follows that

$$\hat{x}_{sl}\hat{y}_{lt}(\hat{b}_{ss} - \hat{b}_{tt}) = \hat{x}_{sl}\hat{y}_{lt}(\hat{b}_{ss} - \hat{b}_{ll} + \hat{b}_{ll} - \hat{b}_{tt}) = \hat{x}_{sl}\hat{y}_{lt}(\hat{b}_{ss} - \hat{b}_{ll}) + \hat{x}_{sl}\hat{y}_{lt}(\hat{b}_{ll} - \hat{b}_{tt}) = \hat{0} - \hat{0} = \hat{0}.$$

Therefore $\hat{x}_{sl}\hat{y}_{lt} \in \operatorname{ann}(\hat{b}_{ss} - \hat{b}_{tt})$ and so

$$\hat{x}_{sl}\hat{y}_{lt} \in \left(\bigcap_{k,\ k\neq t} \operatorname{ann}(\hat{b}_{tk})\right) \bigcap \left(\bigcap_{k,\ k\neq s} \operatorname{ann}(\hat{b}_{ks})\right) \bigcap \operatorname{ann}(\hat{b}_{ss} - \hat{b}_{tt}).$$

Since l was arbitrary chosen, we conclude that

$$\sum_{l=1}^{n} \hat{x}_{sl}\hat{y}_{lt} \in \left(\bigcap_{k,\ k\neq t} \operatorname{ann}(\hat{b}_{tk})\right) \bigcap \left(\bigcap_{k,\ k\neq s} \operatorname{ann}(\hat{b}_{ks})\right) \bigcap \operatorname{ann}(\hat{b}_{ss} - \hat{b}_{tt}).$$

which implies that $[\hat{x}_{ij}][\hat{y}_{ij}] \in [\mathcal{A}_{ij}]$. $\square$

Remark 2.35. Since $\operatorname{Cen}_{M_n(R)}(B)$ is a subring of $M_n(R)$ and Θ is a homomorphism, it follows that $\Theta(\operatorname{Cen}_{M_n(R)}(B))$ is also a subring of $M_n(S)$ (with identity, if R is a ring with identity).

Lemma 2.36. We have equality in Proposition 2.33 if $B = aE_{rt}$, $a \in R$.

Proof. First of all, note that $\widehat{B} = \hat{a}\widehat{E}_{rt} = [\hat{b}_{ij}]$, where $\hat{b}_{ij} = \hat{0}$ if $i \neq r$ or $j \neq t$, and $\hat{b}_{rt} = \hat{a}$. Firstly assume that $r \neq t$. Then

$$\begin{aligned}
\bigcap_{k,\ k\neq j} (\mathrm{ann}(\hat{b}_{jk})) &= \begin{cases} \mathrm{ann}(\hat{a}) & \text{if } j = r \\ S & \text{otherwise,} \end{cases} \\
\bigcap_{k,\ k\neq i} (\mathrm{ann}(\hat{b}_{ki})) &= \begin{cases} \mathrm{ann}(\hat{a}) & \text{if } i = t \\ S & \text{otherwise,} \end{cases} \\
\mathrm{ann}(\hat{b}_{ii} - \hat{b}_{jj}) &= S.
\end{aligned}$$

Therefore it follows from the definition of $[\mathcal{A}_{ij}]$ that

$$\mathcal{A}_{ij} = \begin{cases} \mathrm{ann}_S(\hat{a}) & \text{if } j = r \text{ or } i = t \\ S & \text{otherwise.} \end{cases}$$

If $r = t$, then it follows similarly that

$$\mathcal{A}_{ij} = \begin{cases} \mathrm{ann}_S(\hat{a}) & \text{if } i \neq j, \text{ and } j = r \text{ or } i = t \\ S & \text{otherwise.} \end{cases}$$

Now, since $\Theta(\mathrm{ann}_R(a)) \subseteq \mathrm{ann}_S(\hat{a})$, it followws that

$$\begin{aligned}
[\mathcal{A}_{ij}] &= \begin{array}{cc} \begin{array}{ccccc} & & \text{column } r & & \\ & & \downarrow & & \end{array} & \\ \left[\begin{array}{ccccc} & & \mathrm{ann}_S(\hat{a}) & & \\ & S & & S & \\ & & \vdots & & \\ \mathrm{ann}_S(\hat{a}) & \mathrm{ann}_S(\hat{a}) \cdots & W & \cdots & \mathrm{ann}_S(\hat{a}) \\ & & \vdots & & \\ & S & & S & \\ & & \mathrm{ann}_S(\hat{a}) & & \end{array}\right] & \leftarrow \text{row } t \end{array} \\
&= \Theta\left(\begin{array}{cc} \begin{array}{ccccc} & & \text{column } r & & \\ & & \downarrow & & \end{array} & \\ \left[\begin{array}{ccccc} & & \mathrm{ann}_R(a) & & \\ & R & & R & \\ & & \vdots & & \\ \mathrm{ann}_R(a) & \mathrm{ann}_R(a) \cdots & T & \cdots & \mathrm{ann}_R(a) \\ & & \vdots & & \\ & R & & R & \\ & & \mathrm{ann}_R(a) & & \end{array}\right] & \leftarrow \text{row } t \end{array} \right)
\end{aligned}$$

$$+\begin{array}{cc} & \begin{array}{ccc} & \text{column } r & \\ & \downarrow & \end{array} \\ & \left[\begin{array}{ccc} & \operatorname{ann}_S(\hat{a}) & \\ S & & S \\ & \vdots & \\ \operatorname{ann}_S(\hat{a})\ \operatorname{ann}_S(\hat{a})\ \cdots & W & \cdots\ \operatorname{ann}_S(\hat{a}) \\ & \vdots & \\ S & & S \\ & \operatorname{ann}_S(\hat{a}) & \end{array}\right] \begin{array}{c} \\ \\ \\ \leftarrow \text{row } t \\ \\ \\ \\ \end{array} \end{array},$$

where $T = R$ if $r = t$, $T = \operatorname{ann}_R(a)$ if $r \neq t$, $W = S$ if $r = t$ and $W = \operatorname{ann}_S(\hat{a})$ if $r \neq t$. Using Lemma 2.1 it follows that

$$\begin{aligned} \Theta\left(\operatorname{Cen}_{M_n(R)}(B)\right) + [\mathcal{A}_{ij}] &= \Theta\left(\operatorname{Cen}_{M_n(R)}(aE_{rt})\right) + [\mathcal{A}_{ij}] \\ &= \left\{\begin{array}{lll} \hat{c}(E_{rr} + E_{tt}), & \hat{c} \in S, & \text{if } r \neq t \\ \hat{c}E_{rr}, & \hat{c} \in S, & \text{if } r = t \end{array}\right. + [\mathcal{A}_{ij}] \\ &= \operatorname{Cen}_{M_n(S)}(\hat{B}). \end{aligned}$$

$\square$

Lemma 2.37. Using the notation of Proposition 2.33 it follows for $B \in M_n(R)$ that $\hat{a}E_{rt} \in \operatorname{Cen}(\widehat{B})$ if and only if $\hat{a}E_{rt} \in [\mathcal{A}_{ij}]$.

Proof. Let $\widehat{B} = [\hat{b}_{ij}]$. Then

$$\hat{a}E_{rt} \in \operatorname{Cen}(\widehat{B}) \Leftrightarrow [\hat{b}_{ij}]\hat{a}E_{rt} = \hat{a}E_{rt}[\hat{b}_{ij}]$$

$$\Leftrightarrow \begin{array}{c} \begin{array}{ccc} & \text{columt } t & \\ & \downarrow & \end{array} \\ \left[\begin{array}{ccc} & \hat{a}\hat{b}_{1r} & \\ & \hat{a}\hat{b}_{2r} & \\ & \vdots & \\ \bigcirc & \hat{a}\hat{b}_{rr} & \bigcirc \\ & \vdots & \\ & \hat{a}\hat{b}_{nr} & \end{array}\right] \end{array} = \begin{array}{c} \\ \\ \\ \text{row } r \rightarrow \\ \\ \\ \end{array} \left[\begin{array}{ccccccc} & & & \bigcirc & & & \\ \hat{a}\hat{b}_{t1} & \hat{a}\hat{b}_{t2} & \cdots & \hat{a}\hat{b}_{tt} & \cdots & \hat{a}\hat{b}_{tn} \\ & & & \bigcirc & & & \end{array}\right]$$

$$\Leftrightarrow \hat{a}\hat{b}_{t1}, \hat{a}\hat{b}_{t2}, \ldots, \hat{a}\hat{b}_{t,t-1}, \hat{a}\hat{b}_{t,t+1}, \ldots, \hat{a}\hat{b}_{tn}, \hat{a}\hat{b}_{1r}, \hat{a}\hat{b}_{2r}, \ldots, \hat{a}\hat{b}_{r-1,r}, \hat{a}\hat{b}_{r+1,r}, \ldots, \hat{a}\hat{b}_{nr} = \hat{0}$$
$$\text{and } \hat{a}(\hat{b}_{rr} - \hat{b}_{tt}) = \hat{0}$$

$$\Leftrightarrow\quad \hat{a}\in\left(\bigcap_{k,\ k\neq t}\operatorname{ann}(\hat{b}_{tk})\right)\bigcap\left(\bigcap_{k,\ k\neq r}\operatorname{ann}(\hat{b}_{kr})\right)\bigcap\operatorname{ann}(\hat{b}_{rr}-\hat{b}_{tt})\Leftrightarrow \hat{a}E_{rt}\in[\mathcal{A}_{ij}].$$

□

Example 2.38. Let $R=\mathbb{Z}$, let $B=\begin{bmatrix}0&3&1\\0&0&1\\0&0&0\end{bmatrix}$, $B'=\begin{bmatrix}4&3&1\\3&7&1\\0&0&10\end{bmatrix}$ and let $\theta:\mathbb{Z}\to\mathbb{Z}_{12}$ be the natural epimorphism. Using the notation of Proposition 2.33, we have ,using B and B', respectively, that

$$[\mathcal{A}_{ij}]=\begin{bmatrix}\hat{0}&\hat{0}&\mathbb{Z}_{12}\\\hat{0}&\hat{0}&\langle\hat{4}\rangle\\\hat{0}&\hat{0}&\hat{0}\end{bmatrix}\qquad\text{and}\qquad[\mathcal{A}_{ij}]=\begin{bmatrix}\hat{0}&\hat{0}&\langle\hat{4}\rangle\\\hat{0}&\hat{0}&\langle\hat{4}\rangle\\\hat{0}&\hat{0}&\hat{0}\end{bmatrix}.$$

Now, by Lemma 2.37

$$\begin{bmatrix}\hat{4}&\hat{0}&\hat{0}\\\hat{0}&\hat{0}&\hat{0}\\\hat{0}&\hat{0}&\hat{0}\end{bmatrix},\begin{bmatrix}\hat{0}&\hat{0}&\hat{0}\\\hat{4}&\hat{0}&\hat{0}\\\hat{0}&\hat{0}&\hat{0}\end{bmatrix},\begin{bmatrix}\hat{0}&\hat{0}&\hat{0}\\\hat{0}&\hat{0}&\hat{0}\\\hat{0}&\hat{0}&\hat{4}\end{bmatrix}\notin\operatorname{Cen}(\widehat{B}),\operatorname{Cen}(\widehat{B'}).$$

Note that the sum of the above three matrices, namely $\begin{bmatrix}\hat{4}&\hat{0}&\hat{0}\\\hat{4}&\hat{0}&\hat{0}\\\hat{0}&\hat{0}&\hat{4}\end{bmatrix}$, is an element of $\operatorname{Cen}(\widehat{B})$ and of $\operatorname{Cen}(\widehat{B'})$.

Corollary 2.39. If R is a commutative ring and $B=\begin{bmatrix}e&f\\g&h\end{bmatrix}\in M_2(R)$, then

$$\begin{aligned}\Theta(\operatorname{Cen}(B))+\begin{bmatrix}\mathcal{A}_{11}&\mathcal{A}_{12}\\\mathcal{A}_{21}&\mathcal{A}_{11}\end{bmatrix}&=\Theta(\operatorname{Cen}(B))+\begin{bmatrix}\hat{0}&\mathcal{A}_{12}\\\mathcal{A}_{21}&\mathcal{A}_{11}\end{bmatrix}\\&=\Theta(\operatorname{Cen}(B))+\begin{bmatrix}\mathcal{A}_{11}&\mathcal{A}_{12}\\\mathcal{A}_{21}&\hat{0}\end{bmatrix}\subseteq\operatorname{Cen}(\widehat{B}),\end{aligned}$$

where

$$\mathcal{A}_{11}=\operatorname{ann}(\hat{f})\cap\operatorname{ann}(\hat{g}),\ \ \mathcal{A}_{12}=\operatorname{ann}(\hat{g})\cap\operatorname{ann}(\hat{e}-\hat{h})\ \text{ and }\ \mathcal{A}_{21}=\operatorname{ann}(\hat{f})\cap\operatorname{ann}(\hat{e}-\hat{h}).$$

Proof. We will only prove that

$$\Theta(\operatorname{Cen}(B))+\begin{bmatrix}\mathcal{A}_{11}&\mathcal{A}_{12}\\\mathcal{A}_{21}&\mathcal{A}_{11}\end{bmatrix}=\Theta(\operatorname{Cen}(B))+\begin{bmatrix}\hat{0}&\mathcal{A}_{12}\\\mathcal{A}_{21}&\mathcal{A}_{11}\end{bmatrix}.\tag{2.40}$$

The proof that $\Theta(\mathrm{Cen}(B)) + \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{11} \end{bmatrix} = \Theta(\mathrm{Cen}(B)) + \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \hat{0} \end{bmatrix}$ is similar. Furthermore, it follows from Proposition 2.33 that

$$\Theta(\mathrm{Cen}(B)) + \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{11} \end{bmatrix} \subseteq \mathrm{Cen}(\widehat{B}).$$

We only have to prove the inclusion $\subseteq$ in (2.40). Because

$$\Theta(\mathrm{Cen}(B)) \subseteq \Theta(\mathrm{Cen}(B)) + \begin{bmatrix} \hat{0} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{11} \end{bmatrix},$$

it suffices to prove that

$$\begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{11} \end{bmatrix} \subseteq \Theta(\mathrm{Cen}(B)) + \begin{bmatrix} \hat{0} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{11} \end{bmatrix}.$$

Now, let

$$A = \begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} \in \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{11} \end{bmatrix}.$$

Then

$$A = \begin{bmatrix} \hat{a} & \hat{0} \\ \hat{0} & \hat{a} \end{bmatrix} + \begin{bmatrix} \hat{0} & \hat{b} \\ \hat{c} & \hat{d} - \hat{a} \end{bmatrix}.$$

Because $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in \mathrm{Cen}(B)$ and $\hat{d} - \hat{a} \in \mathcal{A}_{11}$, the result follows. □

Chapter

3

k-invertibility in $R/\langle k\rangle$ and k-matrices in $M_2(R/\langle k\rangle)$, R a UFD

Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room and it's dark, completely dark. One stumbles around, bumping into furniture, and gradually you learn where each piece of furniture is, and finally after six months or so you find the light switch. You can see exactly where you were.

— Andrew Wiles

The main purpose of this chapter is to introduce the concept of a k-matrix in $M_2(R/\langle k\rangle)$ (Section 3.2, Definition 3.16), where R is a UFD and k is a nonzero nonunit in R. To define this concept we need the concept of k-invertibility in $R/\langle k\rangle$ (Section 3.1, Definition 3.3). In Theorem 4.5, the main theorem of this dissertation, we will obtain a concrete description of the centralizer of a k-matrix in $M_2(R/\langle k\rangle)$. Since there is a seemingly open question regarding the case when $R/\langle k\rangle$ is finite (Remark 3.26), we discuss this case separately in Section 3.3. We will use the results in Section 3.3 in Chapter 5 where we will obtain a formula for the number of elements in the centralizer of a matrix in $M_2(R/\langle k\rangle)$, when R is a UFD and $R/\langle k\rangle$ is finite.

From here onwards, unless stated otherwise, we assume that R is a UFD and that $k \in R$, with k a nonzero nonunit. Let $\theta_k : R \to R/\langle k\rangle$ and $\Theta_k : M_2(R) \to M_2(R/\langle k\rangle)$ be the natural epimorphism and induced epimorphism respectively. We denote the image $\theta_k(b)$ of b ($b \in R$) by $\hat{b}_k$ and the image $\Theta_k(B)$ of B ($B \in M_2(R)$) by $\widehat{B}_k$. However, if there is no ambiguity, then we simply write θ, Θ, $\hat{b}$ and $\widehat{B}$ respectively.

3.1 k-invertibility in $R/\langle k\rangle$

The proofs of the following two results are straightforward. These results will be frequently used throughout this dissertation.

Lemma 3.1. An element $\hat{b} = \theta(b) \in R/\langle k\rangle$ is a zero divisor if and only if $\gcd(b, k) \neq 1$.

Proof. Assume $\gcd(b, k) = 1$. Then none of the primes in the prime factorization of k is in the prime factorization of b. Suppose there is an $\hat{a}_k \in R/\langle k\rangle$ such that $\hat{b}_k\hat{a}_k = \hat{0}_k$. Since ba is a pre-image of $\hat{b}_k\hat{a}_k = \hat{0}_k$, we have that $k|ba$. Now, suppose p is prime and p^n is in the prime factorization of k. Then, since $p|k$, it follows that $p|ba$ and therefore that $p|b$ or $p|a$. Because $\gcd(b, k) = 1$, it follows that $p \nmid b$, and thus, since $p^n|k$ and therefore since $p^n|ba$, that $p^n|a$. Consequently every power of a prime in the prime factorization of k, also divides a. Hence $k|a$ so that $\hat{a}_k = \hat{0}_k$. Thus $\hat{b}_k$ is not a zero divisor.

Conversely, suppose $\gcd(b, k) \neq 1$. Since $k = p_1^{n_1} \dots p_m^{n_m}$, with $p_1, \dots, p_m$ different primes and $n_1, n_2, \dots, n_m \geqslant 1$, it follows that there is a $p_i \in \{p_1, \dots, p_m\}$ such that $p_i|b$. But then it follows that

$$\begin{aligned} bp_1^{n_1} \dots p_{i-1}^{n_{i-1}} p_i^{n_i-1} p_{i+1}^{n_{i+1}} \dots p_m^{n_m} &= b' p_i p_1^{n_1} \dots p_{i-1}^{n_{i-1}} p_i^{n_i-1} p_{i+1}^{n_{i+1}} \dots p_m^{n_m} \\ &= b'k, \end{aligned}$$

where $b' = bp_i^{-1} \in R$ and p_i^{-1} is the inverse of p_i in the quotient field of R. Therefore it follows that $\hat{b}_k\hat{c}_k = \hat{0}_k$, where

$$c = p_1^{n_1} \dots p_{i-1}^{n_{i-1}} p_i^{n_i-1} p_{i+1}^{n_{i+1}} \dots p_m^{n_m}.$$

Since $k \nmid c$ it follows that $c \notin \langle k\rangle = \ker\theta_k$ and therefore that $\hat{c}_k \neq \hat{0}_k$. Thus we conclude that $\hat{b}_k$ is a zero divisor. □

A commutative ring R satisfies the Bézout identity if for any $a, b \in R$ there are $u, v \in R$ such that $ua + vb = \gcd(a, b)$. An integral domain that satisfies the Bézout identity is called a Bézout domain. It is trivial to show that a PID satisfies the Bézout identity. We will use this identity in the next lemma.

Lemma 3.2. Let R be a PID. An element $\hat{b} \in R/\langle k\rangle$ is invertible if and only if $\gcd(b, k) = 1$.

Proof. Since R is a PID, there are $u, v \in R$ such that $ub + vk = 1$. Thus

$$\hat{u}_k\hat{b}_k = \hat{u}_k\hat{b}_k + \underbrace{\hat{v}_k\hat{k}_k}_{=\hat{0}_k} = \hat{1}_k.$$

Therefore $\hat{b}_k$ is invertible with inverse $\hat{u}_k$.

Conversely, if $\hat{b}_k$ is invertible in $R/\langle k\rangle$, then there exists a $\hat{u}_k \in R/\langle k\rangle$ such that $\hat{b}_k\hat{u}_k = \hat{1}_k$ or equivalently, such that $bu = 1 + vk$, for some $v \in R$. Let $d := \gcd(b, k)$, then $d|bu$ and $d|vk$, which implies that $d|1$. Therefore d is a unit. Consequently $\gcd(b, k) = 1$. $\square$

Definition 3.3. A k-pre-image of an element $\hat{b} \in R/\langle k\rangle$ is a pre-image of $\hat{b}$ in R of the form $r\delta$, where $\gcd(r, k) = 1$ and $\delta|k$. We call r and δ the relative prime part and divisor part of $r\delta$ respectively. We call $\hat{b}$ k-invertible if $\hat{r}$ is invertible in $R/\langle k\rangle$ for at least one k-pre-image $r\delta$ of $\hat{b}$.

Remark 3.4. Since $1 \cdot k$ is a k-pre-image of $\hat{0}$, with relative prime part 1, we have that $\hat{0}$ is k-invertible for any UFD R and any nonzero nonunit $k \in R$.

The following lemma is trivial to prove.

Lemma 3.5. Let u be a unit in R, and let $b \in R$. Then $\hat{b}_k$ is k-invertible if and only if $\hat{b}_{uk}$ is uk-invertible.

Proof. Suppose $\hat{b}_k$ is k-invertible in $R/\langle k\rangle$. Hence it follows from definition that $\hat{b}_k$ has a k-pre-image of the form $r\delta$ in R, where $\hat{r}_k$ is invertible in $R/\langle k\rangle$, with inverse $\hat{r}'_k$, say, and $\delta|k$. Since, therefore $b = r\delta + ak = r\delta + au^{-1}uk$, for some $a \in R$, $rr' = 1 + ck = rr' + cu^{-1}uk$, for some $c \in R$, and $\delta|uk$, the result follows. $\square$

The proof of the next result is constructive.

Lemma 3.6. Every element in $R/\langle k\rangle$ has a k-pre-image.

Proof. Let $\hat{b} \in R/\langle k\rangle$. Since R is a UFD there exist different primes $p_1, \dots, p_m$ such that $k = p_1^{n_1} \dots p_m^{n_m}$, where $n_1, \dots, n_m \geqslant 1$. Since $k \neq 0$, there exists a nonzero pre-image b of $\hat{b}$ in R. Again, because R is a UFD, b can be expressed as $r_0p_1^{q_1} \dots p_m^{q_m}$, where $p_i \nmid r_0$, for $i = 1, \dots, m$, and $q_1, \dots, q_m \geqslant 0$. Therefore $\gcd(r_0, k) = 1$, and

$$\hat{b} = \hat{r}_0\widehat{p_1^{q_1}} \dots \widehat{p_m^{q_m}}.$$

Suppose we can show that each $\widehat{p_i^{q_i}}$ has a pre-image $r_i \cdot p_i^{t_i}$, where $\gcd(r_i, k) = 1$ and $t_i \leqslant n_i$. Then we have that

$$\hat{b} = \hat{r}_0(\widehat{r_1 p_1^{t_1}}) \ldots (\widehat{r_m p_m^{t_m}}) = \hat{r}_0\hat{r}_1 \ldots \hat{r}_m(\widehat{p_1^{t_1} \ldots p_m^{t_m}}) = \theta(rp_1^{t_1} \ldots p_m^{t_m}),$$

where $r = r_0 r_1 \cdots r_m$. Since $\gcd(r_i, k) = 1$ for $i = 0, 1, \ldots, m$, it follows that $\gcd(r, k) = 1$. Also, since $t_i \leqslant n_i$ for $i = 1, 2 \ldots, m$, we have that

$$\delta := p_1^{t_1} \cdots p_m^{t_m} | \underbrace{p_1^{n_1} \cdots p_m^{n_m}}_{=k},$$

implying that $r \cdot \delta$ is a k-pre-image of $\hat{b}$ with relative prime part r and divisor part δ.

Let us now prove that each $\widehat{p_i^{q_i}}$ has a pre-image $r_i \cdot p_i^{t_i}$, where $\gcd(r_i, k) = 1$ and $t_i \leqslant n_i$.

If $q_i \leqslant n_i$ then $p_i^{q_i} = 1 \cdot p_i^{q_i}$, where $t_i = q_i \leqslant n_i$ and $\gcd(r_i, k) = 1$, with $r_i = 1$. Thus we have the desired result.

Next we consider the case when $n_i < q_i$. Since

$$\widehat{p_i^{q_i}} = \widehat{p_i^{q_i} + k}$$

and

$$p_i^{q_i} + k = p_i^{q_i} + p_1^{n_1} \cdots p_m^{n_m} = p_i^{n_i}(p_i^{q_i - n_i} + p_1^{n_1} \cdots p_{i-1}^{n_{i-1}} p_{i+1}^{n_{i+1}} \cdots p_m^{n_m}),$$

it follows that $p_i^{n_i} \cdot r_i = r_i \cdot p_i^{n_i}$ is a pre-image of $\widehat{p_i^{q_i}}$, where

$$r_i = p_i^{q_i - n_i} + p_1^{n_1} \cdots p_{i-1}^{n_{i-1}} p_{i+1}^{n_{i+1}} \cdots p_m^{n_m}.$$

Since

$$p_i | p_i^{q_i - n_i} (q_i > n_i) \qquad \text{and} \qquad p_i \nmid p_1^{n_1} \cdots p_{i-1}^{n_{i-1}} p_{i+1}^{n_{i+1}} \cdots p_m^{n_m},$$

we have that $p_i \nmid r_i$. Furthermore, for all $j \in \{1, \ldots, i-1, i+1, \ldots, m\}$ it follows that

$$p_j \nmid p_i^{q_i - n_i} \qquad \text{and} \qquad p_j | p_1^{n_1} \cdots p_{i-1}^{n_{i-1}} p_{i+1}^{n_{i+1}} \cdots p_m^{n_m}$$

implying that $p_j \nmid r_i$. Thus r_i and k are relatively prime and $t_i = n_i \leqslant n_i$. $\square$

The next result follows directly from Lemma 3.2, Definition 3.3 and Lemma 3.6.

Corollary 3.7. If R is a PID, then every element in $R/\langle k\rangle$ is k-invertible.

The next example illustrates the constructive proof of Lemma 3.6.

Example 3.8. Let $R = \mathbb{Z}$. Then, since $12 = 2^2 \cdot 3$ and $10 = 2 \cdot 5$, using the procedure in the proof of Lemma 3.6, it follows that

(1) $\hat{9}_{12} = \theta_{12}(2^0 \cdot 3^2) = \theta_{12}(1(3^2 + 12)) = \theta_{12}(3(7)) = (\widehat{7 \cdot 3})_{12}$, where $\gcd(7, 12) = 1$ and $3|12$;

(2) $\hat{6}_{10} = \theta_{10}(3 \cdot 2 \cdot 5^0) = (\widehat{3 \cdot 2})_{10}$, where $\gcd(3, 10) = 1$ and $2|10$.

Since $\hat{7}_{12}$ and $\hat{3}_{10}$ are invertible in $\mathbb{Z}_{12}$ and $\mathbb{Z}_{10}$ respectively, it follows that $\hat{9}_{12}$ is 12-invertible and $\hat{6}_{10}$ is 10-invertible, as expected from Corollary 3.7.

Now, let $R = F[x, y]$, the polynomial ring in two variables x and y over the field F. Then, again using the procedure in the proof of Lemma 3.6, it follows that

(3) $\widehat{x^3}_{x^2y} = \theta_{x^2y}(x^3y^0) = \theta_{x^2y}(1(x^3 + x^2y)) = \theta_{x^2y}((x + y)x^2)$,
where $\gcd(x + y, x^2y) = 1$ and $x^2|x^2y$.

We will show in Example 3.13 that Corollary 3.7 does not hold for UFD's in general.

Proposition 3.10 and Corollaries 3.11 and 3.14 will help us to determine when an element in $R/\langle k\rangle$ is not k-invertible in case R is a UFD which is not a PID. In order to conclude that an element $\hat{b}$ in $R/\langle k\rangle$ is not k-invertible (using Definition 3.3), we have to show, for every k-pre-image $r\delta$ of $\hat{b}$, that $\hat{r}$ is not invertible in $R/\langle k\rangle$. However, if δ is of a specific form, then we will show in Proposition 3.10 that it suffices to show that $\hat{r}$ is not invertible in $R/\langle k\rangle$ for at least one k-pre-image $r\delta$ of $\hat{b}$.

We first establish a relationship between the divisor parts of the k-pre-images of an element in $R/\langle k\rangle$.

Lemma 3.9. Let R be a UFD, let $k = p_1^{n_1} \cdots p_m^{n_m} \in R$, where $p_1, \ldots, p_m$ are different primes in R and $n_1, \ldots, n_m \geqslant 1$, and let $\hat{b} \in R/\langle k\rangle$. Then δ is a divisor part of a k-pre-image of $\hat{b}$ if and only if $\gcd(b, k) = \delta$, i.e. the divisor parts of the k-pre-images of $\hat{b}$ are associates.

Proof. Suppose $r\delta$ is a k-pre-image of $\hat{b}$. Then $b = r\delta + sk$ for some $s \in R$. Now, since $\gcd(r, k) = 1$, it follows that $\gcd(b, k) = \gcd(r\delta + sk, k) = \gcd(\delta, k) = \delta$.

For the converse, note that since all the greatest common divisors of b and k are associates and every element in $R/\langle k\rangle$ has at least one k-pre-image, by Lemma 3.6, the result will follow if we can show that for an arbitrary unit t, $t\delta$ is also a divisor part of some k-pre-image of $\hat{b}$. Since $\widehat{rt^{-1}t\delta} = \widehat{r\delta} = \hat{b}$, $\gcd(rt^{-1}, k) = 1$ and $t\delta | k$, the result follows. □

Proposition 3.10. Let $k = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m} \in R$, with $p_1, \ldots, p_m$ different primes and $n_1, n_2, \ldots \ldots, n_m \geqslant 1$, and let $\hat{0} \neq \hat{b} \in R/\langle k\rangle$. Assume (using Lemma 3.9) that the divisor parts of the k-pre-images of $\hat{b}$ are of the form $u\delta$, where u is a unit in R. If $\delta = \gcd(b, k) = p_1^{q_1} p_2^{q_2} \cdots p_m^{q_m}$, where $0 \leqslant q_i < n_i$ for $i = 1, 2, \ldots, m$, then either $\hat{r}$ is invertible in $R/\langle k\rangle$ for every k-pre-image $r\delta$ of $\hat{b}$ or no such $\hat{r}$ is invertible in $R/\langle k\rangle$.

Proof. Since, by Lemma 3.6, there exists a pre-image $r\delta$ of $\hat{b}$ in R, with $\gcd(r, k) = 1$, all the pre-images, and in particular all the k-pre-images, of $r\delta$ are of the form

$$r\delta + c p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m} \tag{3.1}$$

for some $c \in R$. Because, according to Lemma 3.9, the divisor parts of all the k-pre-images of $\hat{b}$ are of the form $u\delta$, where u is a unit in R, it follows from (3.1) that the relative prime parts of all the k-pre-images of $\hat{b}$ are of the form

$$u^{-1} r + c u^{-1} p_1^{n_1 - q_1} \cdots p_m^{n_m - q_m} \tag{3.2}$$

for some $u, c \in R$, u a unit.

Now, suppose $\hat{r}$ is invertible in $R/\langle k\rangle$ with inverse $\hat{y}$. In other words

$$yr = 1 + d p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$$

for some $d \in R$. If we can show that the image under θ of the relative prime part of an arbitrary k-pre-image of $\hat{b}$ is invertible, then we are finished.

Let $u^{-1} r + c u^{-1} p_1^{n_1 - q_1} p_2^{n_2 - q_2} \cdots p_m^{n_m - q_m}$ be the relative prime part of an arbitrary k-pre-image of $\hat{b}$. Furthermore, let $l \in \mathbb{N}$ such that

$$2^l > \max\left\{ \frac{n_i}{n_i - q_i} \;\middle|\; i \in \{1, \ldots, m\} \right\} > 0. \tag{3.3}$$

For the sake of notation, let

$$s = d p_1^{q_1} \cdots p_m^{q_m} + c y \text{ and } t = p_1^{n_1 - q_1} p_2^{n_2 - q_2} \cdots p_m^{n_m - q_m}.$$

Then

$$\begin{aligned} & (u^{-1}r + cu^{-1}p_1^{n_1-q_1}p_2^{n_2-q_2}\cdots p_m^{n_m-q_m})yu(1-ts)(1+(ts)^{2^1})\cdots(1+(ts)^{2^{l-1}}) \\ = \; & (1+dp_1^{n_1}p_2^{n_2}\cdots p_m^{n_m} + cyp_1^{n_1-q_1}p_2^{n_2-q_2}\cdots p_m^{n_m-q_m})(1-ts)(1+(ts)^{2^1})\cdots(1+(ts)^{2^{l-1}}) \\ = \; & (1+ts)(1-ts)(1+(ts)^{2^1})\cdots(1+(ts)^{2^{l-1}}) \\ = \; & 1-(ts)^{2^l}. \end{aligned}$$

Let $1 \leqslant i \leqslant m$. Since $n_i > q_i$, it follows from (3.3) that

$$2^l(n_i - q_i) > \frac{n_i}{n_i - q_i}(n_i - q_i) = n_i,$$

and so

$$t^{2^l} = ap_1^{n_1}p_2^{n_2}\cdots p_m^{n_m}$$

for some $a \in R$. Therefore

$$\begin{aligned} & \theta\left((u^{-1}r + cu^{-1}p_1^{n_1-q_1}p_2^{n_2-q_2}\cdots p_m^{n_m-q_m})yu(1-ts)\ (1+(ts)^{2^1})\cdots(1+(ts)^{2^{l-1}})\right) \\ = \; & \theta\left(1-(ts)^{2^l}\right) \\ = \; & \hat{1}. \end{aligned}$$

Hence we conclude that

$$\theta\left(yu(1-ts)(1+(ts)^{2^1})(1+(ts)^{2^2})\cdots(1+(ts)^{2^{l-1}})\right)$$

is the inverse of the image under θ of the relative prime part of the arbitrary chosen k-pre-image of $\hat{b}$. □

Corollary 3.11. Let $\hat{0} \neq \hat{b} \in R/\langle k\rangle$. If $\gcd(b,k) = 1$, then $\hat{b}$ is k-invertible if and only if $\hat{b}$ is invertible in $R/\langle k\rangle$.

Proof. The pre-image $b \cdot 1$ is a k-pre-image of $\hat{b}$, with relative prime part b and divisor part 1. Now, suppose $k = p_1^{n_1}p_2^{n_2}\cdots p_m^{n_m}$, where $p_1, p_2, \ldots, p_m$ are different primes and $n_1, n_2, \ldots, n_m \geqslant 1$. Since $1 = p_1^0\cdots p_m^0$, the result follows from Proposition 3.10. □

Remark 3.12. Note that it follows from Lemma 3.1 and Corollary 3.11 that if $\hat{b}$ is an invertible element in $R/\langle k\rangle$, then $\hat{b}$ is k-invertible.

We are now in a position to give an example of a UFD R (which is not a PID), an element k in R and an element $\hat{b}$ in $R/\langle k\rangle$ which is not k-invertible.

Example 3.13. Let R be the polynomial ring in two variables $F[x, y]$ and let $k := x^2$. Consider the natural epimorphism $\theta : F[x, y] \to F[x, y]/\langle x^2\rangle$, and let $\hat{b} := \hat{y} = \theta(y)$. Since $\gcd(y, x^2) = 1$ and $\hat{y}$ is not invertible in $F[x, y]/\langle x^2\rangle$, we conclude from Corollary 3.11 that $\hat{y}$ is not x^2-invertible.

Note that if k is a power of a prime, then every pre-image of a nonzero $\hat{b} \in R/\langle k\rangle$ can be written in the form $r\delta$, where $\gcd(r, k) = 1$ and $\delta|k$. Therefore every pre-image of a nonzero $\hat{b} \in R/\langle k\rangle$ is a k-pre-image. In such a case we will sometimes refer to the divisor part and relative prime part of a pre-image of an element $\hat{b}$, instead of the relative prime part and divisor part of the k-pre-image of $\hat{b}$.

The following result follows almost directly from Proposition 3.10.

Corollary 3.14. Let $k = p^n \in R$, where p is prime, and let $\hat{0} \neq \hat{b} \in R/\langle k\rangle$. Then either the image under θ of the relative prime part of every pre-image of $\hat{b}$ is invertible or none is invertible.

Proof. Let b be an arbitrary pre-image of $\hat{b}$. Since R is a UFD, it follows that $b = cp^m$ for some $m \geqslant 0$ and some $c \in R$ such that $p \nmid c$, i.e. $\gcd(c, p^n) = 1$. Because $\hat{b} \neq \hat{0}$, it follows that $m < n$. Hence the result follows from Proposition 3.10. □

The following statement is an equivalent formulation of Corollary 3.14, and so we have a characterization for the nonzero k-invertible elements in $R/\langle k\rangle$, if k is a power of a prime.

Let $k = p^n \in R$, where p is prime, and let $\hat{0} \neq \hat{b} \in R/\langle k\rangle$. Then $\hat{b}$ is k-invertible if and only if the image under θ of the relative prime part of an arbitrary pre-image of $\hat{b}$ is invertible in $R/\langle k\rangle$.

Notice that we could also have concluded from Corollary 3.14 that $\hat{y}$ in Example 3.13 is not x^2-invertible.

Next we show that Proposition 3.10 does not hold in general if $q_i = n_i$ for some i.

Example 3.15. Let $R = \mathbb{Z}[x]$, and $k = 2x$ (with 2 and x primes in $\mathbb{Z}[x]$). Consider

$$\hat{0} \neq \hat{x} \in \mathbb{Z}[x]/\langle 2x\rangle.$$

Then $1 \cdot x$ and $3 \cdot x$ are $2x$-pre-images of $\hat{x}$ with relative prime parts 1 and 3 respectively, and $\hat{1}$ is invertible in $\mathbb{Z}[x]/\langle 2x\rangle$, but $\hat{3}$ is not.

3.2 k-matrices in $M_2(R/\langle k\rangle)$

Definition 3.16. We call a matrix $\begin{bmatrix} \hat{e}_k & \hat{f}_k \\ \hat{g}_k & \hat{h}_k \end{bmatrix} \in M_2(R/\langle k\rangle)$ a k-matrix if it satisfies the following conditions:

(i) At least one of the three elements $\hat{e}_k - \hat{h}_k$, $\hat{f}_k$ and $\hat{g}_k$ is k-invertible with a k-pre-image $r\delta$ that has divisor part δ; pick such an element, and call the remaining two elements $\hat{a}_k$ and $\hat{b}_k$, say.

(ii) δ is a unit in R, or at least one of the elements $\hat{a}_\delta$ and $\hat{b}_\delta$ is δ-invertible.

Lemma 3.17. If $\hat{e} - \hat{h}$, $\hat{f}$ or $\hat{g}$ is invertible in $R/\langle k\rangle$ then $\begin{bmatrix} \hat{e} & \hat{f} \\ \hat{g} & \hat{h} \end{bmatrix} \in M_2(R/\langle k\rangle)$ is a k-matrix.

Proof. Suppose $\hat{c}_k \in \{\hat{e}_k - \hat{h}_k, \hat{f}_k, \hat{g}_k\}$ is invertible in $R/\langle k\rangle$. Then it follows from Corollary 3.11 that $\hat{c}_k$ is k-invertible with a k-pre-image $c \cdot 1$ that has divisor part 1, and so (ii) in Definition 3.16 is satisfied. □

The following result follows directly from Corollary 3.7.

Corollary 3.18. If R is a PID, then every matrix in $M_2(R/\langle k\rangle)$ is a k-matrix.

We show that Corollary 3.18 does not hold for UFD's in general.

Example 3.19. Let $R = F[x, y]$ and let $k = x^2$. We exhibit (a) a matrix which is an x^2-matrix and (b) a matrix which is not an x^2-matrix.

(a) Let

$$\widehat{B}_{x^2} = \begin{bmatrix} \hat{y}_{x^2} & \hat{x}_{x^2} \\ \hat{x}_{x^2} & \hat{0}_{x^2} \end{bmatrix} \in M_2(F[x, y]/\langle x^2\rangle).$$

Since 1 is the relative prime part and x is the divisor part of the pre-image $1 \cdot x$ of $\hat{x}_{x^2}$ in $F[x, y]$, and $\hat{1}_{x^2}$ is invertible in $F[x, y]/\langle x^2\rangle$, it follows that $\hat{x}_{x^2}$ is x^2-invertible. Furthermore, $\hat{x}_x = \hat{0}_x$, which is x-invertible by Remark 3.4. Therefore we conclude from Definition 3.16 that $\widehat{B}_{x^2}$ is an x^2-matrix.

(b) Let

$$\widehat{B}_{x^2} = \begin{bmatrix} \widehat{(x+y)}_{x^2} & \hat{y}_{x^2} \\ \hat{x}_{x^2} & \hat{x}_{x^2} \end{bmatrix} \in M_2(F[x, y]/\langle x^2\rangle).$$

Regarding Definition 3.16(i), we consider the elements $\hat{y}_{x^2} = \widehat{(x+y)}_{x^2} - \hat{x}_{x^2}$, $\hat{y}_{x^2}$ and $\hat{x}_{x^2}$. We have already seen in (a) and Example 3.13, respectively, that $\hat{x}_{x^2}$ is x^2-invertible and $\hat{y}_{x^2}$ is not x^2-invertible.

Therefore the only possible choice of an x^2-invertible element in Definition 3.16(i) is $\hat{x}_{x^2}$, and the only remaining element is $\hat{y}_{x^2}$. By Lemma 3.9 all the divisor parts of the x^2-pre-images of $\hat{x}_{x^2}$ are of the form ux for some nonzero $u \in F$. Regarding Definition 3.16(ii) we must show that $\hat{y}_{ux}$ is ux-invertible for some $u \in F$ for $\widehat{B}_{x^2}$ to be an x^2-matrix. By Lemma 3.5 it suffices to show that $\hat{y}_x$ is not $\hat{x}$-invertible. Since $\hat{y}_x$ is not invertible in $F[x,y]/\langle x\rangle$ and $\gcd(y,x) = 1$, it follows from Corollary 3.11 that $\hat{y}_x$ is not x-invertible. Hence we conclude that $\widehat{B}_{x^2}$ is not an x^2-matrix.

The following result will be used in the proof of Theorem 4.5 and Theorem 4.11.

Corollary 3.20. Let $r\delta$ be a k-pre-image of $\hat{b} \in R/\langle k\rangle$, with relative prime part r and divisor part $\delta = \gcd(b,k)$ (by Lemma 3.9). Then it follows that

$$\langle t\rangle = \theta^{-1}(\mathrm{ann}(\hat{b})),$$

where $t = \delta^{-1}k \in R$, with δ^{-1} the inverse of δ in the quotient field of R.

Proof. Since, by Lemma 2.29(c), it follows that $\theta^{-1}(\mathrm{ann}(\hat{b})) = \theta^{-1}(\mathrm{ann}(\hat{r}\hat{\delta})) = \theta^{-1}(\mathrm{ann}(\hat{\delta}))$, the result is a special case of Lemma 2.31. □

3.3 The case when $R/\langle k\rangle$ is finite

The following results for the case when R is a UFD and $R/\langle k\rangle$ is finite are similar to the results in the previous sections in this chapter for the case when R is a PID.

Lemma 3.21. (see Lemma 3.2) Let $R/\langle k\rangle$ be finite. An element $\hat{b} \in R/\langle k\rangle$ is invertible if and only if $\gcd(b,k) = 1$.

Proof. Suppose $\gcd(b,k) = 1$. Since $R/\langle k\rangle$ is finite, $\hat{b}^n = \hat{b}^m$ for some $m, n \in \mathbb{N}$, $m \neq n$. Without loss of generality, suppose $m < n$. Then, since $\hat{b}$ is not a zero divisor by Lemma 3.1 we have that $\hat{b}^{n-m} = \hat{1}$. Hence $\hat{b}$ is invertible in $R/\langle k\rangle$.

The proof of the converse is the same as the proof of the converse of Lemma 3.2. □

The following result follows directly from Definition 3.3, Lemma 3.6 and Lemma 3.21.

Corollary 3.22. (see Corollary 3.7) If $R/\langle k\rangle$ is finite, then every element in $R/\langle k\rangle$ is k-invertible.

The next result follows directly from Definition 3.16 and Corollary 3.22.

Corollary 3.23. (see Corollary 3.18) If $R/\langle k\rangle$ is finite, then every matrix in $M_2(R/\langle k\rangle)$ is a k-matrix.

The following result is well-known.

Theorem 3.24. ([13], p. 132, Corollary 2.27) If $A_1, \dots, A_m$ are ideals in a ring S (not necessarily commutative or with a unit), then there is a monomorphism of rings

$$\phi : S/(A_1 \cap \cdots \cap A_m) \to S/A_1 \oplus \cdots \oplus S/A_m$$

defined by

$$\phi(s + (A_1 \cap \cdots \cap A_m)) = (s + A_1, \dots, s + A_m).$$

If $S^2 + A_i = S$ for all i and $A_i + A_j = S$ for all $i \neq j$, then ϕ is an isomorphism of rings.

The fact that ϕ and Φ in Corollary 3.25 are isomorphisms if R is a PID or if $R/\langle k\rangle$ is finite is an important property of these cases. This property will be used in Chapter 5.

Corollary 3.25. Let R be a PID or let $R/\langle k\rangle$ be finite, and let $k = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$, with $p_1, p_2, \dots, p_m$ different primes and $n_1, \dots, n_m \geqslant 1$. Then

(i) $$\phi : R/\langle k\rangle \to R/\langle p_1^{n_1}\rangle \oplus R/\langle p_2^{n_2}\rangle \oplus \cdots \oplus R/\langle p_m^{n_m}\rangle$$

defined by

$$\phi(\hat{r}) = (\theta_{p_1^{n_1}}(r), \theta_{p_2^{n_2}}(r), \cdots, \theta_{p_m^{n_m}}(r))$$

is an isomorphism.

(ii) $$\Phi : M_2(R/\langle k\rangle) \to M_2(R/\langle p_1^{n_1}\rangle) \oplus M_2(R/\langle p_2^{n_2}\rangle) \oplus \cdots \oplus M_2(R/\langle p_m^{n_m}\rangle)$$

defined by

$$\Phi([\hat{b}_{ij}]) = (\Theta_{p_1^{n_1}}([b_{ij}]), \dots, \Theta_{p_m^{n_m}}([b_{ij}]))$$

is an isomorphism.

Proof. (i) Since R satisfies the Bézout identity if R is a PID it follows that $\langle p_i^{n_i}\rangle + \langle p_j^{n_j}\rangle = R$ for every $i \neq j$, $1 \leqslant i, j \leqslant m$. Therefore the result follows directly from Theorem 3.24 for this case.

Suppose $R/\langle k\rangle$ is finite. Now, let $i \neq j$, $1 \leqslant i, j \leqslant m$ and let

$$\pi_i : R/\langle p_1^{n_1}\rangle \oplus \cdots \oplus R/\langle p_m^{n_m}\rangle \to R/\langle p_i^{n_i}\rangle$$

be the canonical projection. Since $\pi_i(\phi(\hat{b}_k)) = \hat{b}_{p_i^{n_i}}$ for every $\hat{b}_{p_i^{n_i}} \in R/\langle p_i^{n_i}\rangle$ it follows that

$$\pi_i\phi : R/\langle k\rangle \to R/\langle p_i^{n_i}\rangle$$

is an epimorphism (i.e. $\phi(R/\langle k\rangle)$ is a subdirect sum of the rings $R/\langle p_1^{n_1}\rangle, \ldots, R/\langle p_m^{n_m}\rangle$ (see [17], p. 52, Definition 3.5)). Hence $R/\langle p_i^{n_i}\rangle$ is also finite. Since $\gcd(p_i^{n_i}, p_j^{n_j}) = 1$ it follows from Lemma 3.21 that $\hat{p}_j^{n_j}$ is invertible in $R/\langle p_i^{n_i}\rangle$. Therefore there is an $\hat{a} \in R/\langle p_i^{n_i}\rangle$ such that $\hat{a}\hat{p}_j^{n_j} = \hat{1}$, or in other words, that $ap_j^{n_j} = 1 + cp_i^{n_i}$ for some $c \in R$. Hence $\langle p_i^{n_i}\rangle + \langle p_j^{n_j}\rangle = R$. The result therefore follows from Theorem 3.24.

(ii) The fact that Φ is onto, 1-1 and well-defined follows directly from (i). We now show that Φ is a homomorphism. Let $\widehat{A}, \widehat{B} \in M_n(R/\langle k\rangle)$. Then, since $\Theta_{p_i^{n_i}}$ is a homomorphism for all i, it follows that

$$\begin{aligned}
\Phi(\widehat{A}) \cdot \Phi(\widehat{B}) &= (\Theta_{p_1^{n_1}}(A), \ldots, \Theta_{p_n^{n_m}}(A)) \cdot (\Theta_{p_1^{n_1}}(B), \ldots, \Theta_{p_n^{n_m}}(B)) \\
&= (\Theta_{p_1^{n_1}}(AB), \ldots, \Theta_{p_n^{n_m}}(AB)) \\
&= \Phi(\widehat{A}\widehat{B})
\end{aligned}$$

It can be similarly shown that addition is preserved. □

Remark 3.26. A natural example to include in this section, if such an example exists, would be one of a UFD R, which is not a PID, and a nonzero nonunit $k \in R$, as we assume throughout this dissertation, such that $R/\langle k\rangle$ is finite. Unfortunately we could not find such an example. Neither have we been able to prove that if R is UFD and $k \in R$ is a nonzero nonunit such that $R/\langle k\rangle$ is finite, then R is a PID.

We could, though, find proofs for the following weaker results.

Proposition 3.27. Let F be a field. If R is a UFD that is a finitely generated F-algebra and $k \in R$ is a nonzero nonunit such that $R/\langle k\rangle$ is finite, then R is a PID.

Proposition 3.28. If R is a UFD and $R/\langle k\rangle$ is finite for all nonzero nonunit $k \in R$, then R is a PID.

To prove Proposition 3.27 and Proposition 3.28 we need the following preliminary definitions and results.

Definition 3.29. ([13], p. 372, Definition 1.2) A commutative ring R is Noetherian if R satisfies the ascending chain condition on ideals, i.e. for every chain $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ of ideals, there exists an $n \in \mathbb{N}$ such that $A_i = A_n$ for all $i > n$.

Remark 3.30. Note that a commutative ring R is Noetherian if and only if every ideal in R is finitely generated.

Definition 3.31. ([10], p. 6, Definition) In a ring R, the height of a prime ideal $\mathcal{P}$, denoted by height $\mathcal{P}$, is the supremum of all the integers n such that there exists a chain $\mathcal{P}_0 \subsetneqq \mathcal{P}_1 \subsetneqq \cdots \subsetneqq \mathcal{P}_n = \mathcal{P}$ of distinct prime ideals. We define the Krull dimension, denoted by $\dim R$, of R to be the supremum of the heights of all prime ideals.

Lemma 3.32 follows almost directly from Theorem 2 on page 4 in [7]. For the sake of completeness we prove the result from first principles.

Lemma 3.32. The minimal nonzero prime ideals in a UFD R are the ideals generated by the prime elements in R.

Proof. Suppose R is a UFD and $\mathcal{P}$ is a nonzero prime ideal in R. Since $\mathcal{P} \neq R$, $\mathcal{P}$ does not contain any units. Therefore, let $a = p_1^{n_1} \cdots p_m^{n_m}$, where $p_1, \ldots, p_m$ are primes in R, $n_i \geqslant 1$ for all i and $m \geqslant 1$, be an arbitrary nonzero element in $\mathcal{P}$. Then by the definition of prime ideals it follows that $p_i \in \mathcal{P}$ for some i, $1 \leqslant i \leqslant m$. Thus $\langle p_i \rangle \subseteq \mathcal{P}$.

Conversely, suppose $\mathcal{P}$ is a nonzero prime ideal contained in an ideal generated by a prime element p, $\langle p \rangle$. Since $\mathcal{P}$ does not contain any units, let $b = q_1^{s_1} \cdots q_l^{s_l}$, where $q_1, \ldots, q_l$ are primes in R, $s_i \geqslant 1$ for all i and $l \geqslant 1$, be an arbitrary element in $\mathcal{P}$. Then, again by the definition of prime ideals, $q_j \in \mathcal{P}$ for some j, $1 \leqslant j \leqslant l$. Since p is a divisor of every element in $\langle p \rangle$, it follows, given that $q_j \in \mathcal{P} \subseteq \langle p \rangle$, that $p | q_j$. Since q_j is a prime we therefore have that $p = uq_j$, for some unit u. Hence $\langle p \rangle = \langle q_i \rangle \subseteq \mathcal{P}$ and therefore $\langle p \rangle = \mathcal{P}$. Thus the result follows. □

To prove Lemma 3.34 we need the following result which is straightforward to prove.

Lemma 3.33. ([7], p. 4) A UFD satisfies the ascending chain condition on principal ideals, i.e. for every chain $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ of principal ideals, there exists an integer n such that $A_i = A_n$ for all $i > n$.

Lemma 3.34. A UFD with Krull dimension 1 is a PID.

Proof. Suppose R is a UFD with Krull dimension 1. In other words, using Lemma 3.32, the ideals generated by the prime elements in R are the maximal- and the minimal nonzero prime ideals. Since a maximal ideal in a ring with unity is a prime ideal, it follows that the maximal ideals in R are the ideals generated by the prime elements in R. Now, suppose a and b are nonunits in R such that $\gcd(a, b) = 1$, then there does not exist a prime element $p \in R$ such that $\langle a, b\rangle \subseteq \langle p\rangle$ (otherwise $a = sp$ and $b = tp$, for some $s, t \in R$ implying that $p|a$ and $p|b$ so that $\gcd(a, b) \neq 1$). Hence $\langle a, b\rangle = R$, which implies that there exist $c, d \in R$ such that $ca + db = 1$. Therefore R is a Bézout domain (R satisfy the Bézout identity). Hence every finitely generated ideal is a principal ideal.

Now, suppose R is not a PID. It follows from above that R is therefore not a Noetherian ring either (Remark 3.30). Hence there exists an infinite ascending chain of finitely generated ideals in R. Since every finitely generated ideal in R is principal, there exists an infinite chain of principal ideals in R. But since R is a UFD it satisfies the ascending chain condition on principal ideals by Lemma 3.33 and therefore we have a contradiction. Hence we conclude that R is a PID. $\square$

The following known result is important for the proof of Proposition 3.27.

Theorem 3.35. ([10], p. 6, Theorem 1.8A; [16], p. 92, Chapter 5, Section 14, Corollary 3) Let F be a field, and let R be an integral domain which is a finitely generated F-algebra. Then for any prime ideal $\mathcal{P}$ in R, we have

$$\text{height}\,\mathcal{P} + \dim R/\mathcal{P} = \dim R$$

The following three results are known and can be easily proved.

Lemma 3.36. ([17], p. 66, Exercise 4.9, no. 11(iii)) Let R be a commutative ring and let $k \in R$. Then there is a 1-1 correspondence between the prime ideals in R that contains k and the prime ideals in $R/\langle k\rangle$.

Lemma 3.37. ([17], p. 66, Exercise 4.9, no. 6) Let R be a commutative ring with unity and let $\mathcal{P}$ be a prime ideal in R. Then $R/\mathcal{P}$ is an integral domain.

Lemma 3.38. ([1], p. 94, Theorem 3.3.4) A finite integral domain is a field.

We are finally able to prove Proposition 3.27 and Proposition 3.28.

Proof of Proposition 3.27. Suppose p is a prime in the prime factorization of k. Then $\langle k\rangle \subseteq \langle p\rangle$. Therefore $R/\langle p\rangle$ is also finite. Since $\langle p\rangle$ is a prime ideal it follows from Lemma 3.37 and Lemma 3.38 that $R/\langle p\rangle$ is a field. Since $\langle 0\rangle$ is the only prime ideal in a field we have that $\dim R/\langle p\rangle = 0$. Furthermore, it follows from Lemma 3.32 that height $\langle p\rangle = 1$. Hence by Theorem 3.35 $\dim R = 1$. We therefore conclude from Lemma 3.34 that R is a PID. $\square$

Remark 3.39. It can similarly be shown that if R is a UFD with

$$\text{height}\ \mathcal{P} + \dim R/\mathcal{P} = \dim R$$

for all prime ideals P in R of height 1, we have that $R/\langle k\rangle$ finite, for some nonzero nonunit $k \in R$, implies that R is a PID.

Proof of Proposition 3.28. If we can show that there is no prime ideal in R that strictly contains an ideal generated by a prime element, then R has Krull dimension 1 by Lemma 3.32 and therefore, by Lemma 3.34, we are finished.

Let p be an arbitrary prime in R. Since there is a 1-1 correspondence between the prime ideals that contain $\langle p\rangle$, according to Lemma 3.36, and the prime ideals in $R/\langle p\rangle$, we only have to show that $R/\langle p\rangle$ does not contain any nonzero prime ideal. Since $R/\langle p\rangle$ is finite, according to assumption, and $\langle p\rangle$ is prime in R, it follows from Lemma 3.37 and Lemma 3.38 that $R/\langle p\rangle$ is a field. Therefore $R/\langle p\rangle$ does not contain any nonzero prime ideal. □

Remark 3.40. Using Proposition 3.28, Lemma 3.37 and Lemma 3.38, it follows that finding a UFD R, which is not a PID, that contains a nonzero nonunit $k \in R$ such that $R/\langle k\rangle$ is finite (if such a UFD exists), is the same as finding a UFD R, with primes p and q such that $R/\langle p\rangle$ is a finite field and $R/\langle q\rangle$ is an integral domain that is not a field.

Example 3.41. Since $F[x, y, z]$ is a UFD, which is a finitely generated F-algebra and not a PID, it follows from Proposition 3.27 that there is no nonzero nonunit $k \in F[x, y, z]$ such that $F[x, y, z]/\langle k\rangle$ is finite.

Since $\mathbb{Z}_n$ is finite for every $n \in \mathbb{Z}$, it follows from Proposition 3.28 that $\mathbb{Z}$ is a PID (as is already known).

Given that the Gaussian integers $\mathbb{Z}[i]$ is a UFD, we can prove that $\mathbb{Z}[i]$ is a PID as follows. Let $a + bi$ be an arbitrary nonzero nonunit element of $\mathbb{Z}[i]$. Then $a^2 + b^2 = (a + bi)(a - bi) \in \langle a + bi\rangle$. Since $a + bi$ is a nonzero nonunit we have that $a^2 + b^2 > 1$. Hence it follows that $\mathbb{Z}[i]/\langle a^2 + b^2\rangle = \mathbb{Z}_{a^2+b^2}[i]$ ([8], p. 604, Theorem 1) is finite. Because $\langle a^2 + b^2\rangle \subseteq \langle a + bi\rangle$ we have that $\mathbb{Z}[i]/\langle a + bi\rangle$ is also finite. Since we have chosen $a + bi$ arbitrary, it follows that $\mathbb{Z}[i]/\langle k\rangle$ is finite for all nonzero nonunit $k \in \mathbb{Z}[i]$. Therefore we conclude from Proposition 3.28 that $\mathbb{Z}[i]$ is a PID (as is already known).

CHAPTER

4

The centralizer of a k-matrix in $M_2(R/\langle k\rangle)$, R a UFD

We are what we repeatedly do. Excellence, then, is not an act but a habit.

— ARISTOTLE

THE purpose of this section is to obtain a concrete description of the centralizer of a k-matrix in $M_2(R/\langle k\rangle)$, R a UFD and k a nonzero nonunit in R, by showing that the converse containments $\supseteq$ hold in Proposition 2.33 and Corollary 2.39. This will be done in Section 4.1. Recalling Lemma 2.34 and Remark 2.35 this means that the centralizer of a k-matrix is the sum of two subrings. In Section 4.2 necessary and sufficient conditions will be given for when each of these subrings is contained in the other and for when these two subrings are equal.

In Section 4.1 we provide an example of a UFD, which is not a PID, and a non-k-matrix in $M_2(R/\langle k\rangle)$ (Example 4.9), as well as a universal example of a matrix in $M_n(R)$, where $n \geqslant 3$, for which the mentioned converse containment does not hold (Example 4.10). Note that we still assume that $\theta_k : R \to R/\langle k\rangle$ and $\Theta_k : M_2(R) \to M_2(R/\langle k\rangle)$ are the natural and induced epimorphism respectively.

4.1 A concrete description of the centralizer of a k-matrix

Lemma 4.1. Let R be a UFD, $k \in R$ and let $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R)$.

(a) If $\hat{f}$ is k-invertible (in $R/\langle k\rangle$), then

$$\begin{aligned} \mathrm{Cen}(\widehat{B}) &\subseteq \Theta(\mathrm{Cen}(B)) + \begin{bmatrix} \hat{0} & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) & \mathrm{ann}(\hat{f}) \end{bmatrix} \\ &= \Theta(\mathrm{Cen}(B)) + \begin{bmatrix} \mathrm{ann}(\hat{f}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) & \hat{0} \end{bmatrix}. \end{aligned} \tag{4.1}$$

(b) If $\hat{e} - \hat{h}$ is k-invertible, then

$$\begin{aligned} \mathrm{Cen}(\widehat{B}) &\subseteq \Theta(\mathrm{Cen}(B)) + \begin{bmatrix} \hat{0} & \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{bmatrix} \\ &= \Theta(\mathrm{Cen}(B)) + \begin{bmatrix} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{e}-\hat{h}) & \hat{0} \end{bmatrix}. \end{aligned} \tag{4.2}$$

(c) If $\hat{g}$ is k-invertible, then

$$\begin{aligned} \mathrm{Cen}(\widehat{B}) &\subseteq \Theta(\mathrm{Cen}(B)) + \begin{bmatrix} \hat{0} & \mathrm{ann}(\hat{g}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{g}) \end{bmatrix} \\ &= \Theta(\mathrm{Cen}(B)) + \begin{bmatrix} \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \hat{0} \end{bmatrix}. \end{aligned} \tag{4.3}$$

Proof. Let

$$\begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} \in \mathrm{Cen}(\widehat{B}). \tag{4.4}$$

(a) Since $\hat{f}$ is k-invertible, there is a k-pre-image $r\delta$ of $\hat{f}$, with $\hat{r}$ invertible in $R/\langle k\rangle$ and $\delta|k$. Notice that $\delta \neq 0$, since $k \neq 0$. Let $\hat{t}$ be the inverse of $\hat{r}$ in $R/\langle k\rangle$. Then

$$\hat{t}\hat{f} = \hat{t}\hat{r}\hat{\delta} = \hat{\delta}. \tag{4.5}$$

Now we will first show that

$$\begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} \in \Theta\left(\mathrm{Cen}\left(\begin{bmatrix} te & \delta \\ tg & th \end{bmatrix}\right)\right) + \begin{bmatrix} \hat{0} & \hat{0} \\ \mathrm{ann}(\hat{f}) & \mathrm{ann}(\hat{f}) \end{bmatrix}. \tag{4.6}$$

By Lemma 2.29(a) and (4.5),

$$\mathrm{Cen}\left(\begin{bmatrix} \hat{t}\hat{e} & \hat{\delta} \\ \hat{t}\hat{g} & \hat{t}\hat{h} \end{bmatrix}\right) = \mathrm{Cen}\left(\begin{bmatrix} \hat{e} & \hat{f} \\ \hat{g} & \hat{h} \end{bmatrix}\right), \tag{4.7}$$

and so by (4.4),

$$\begin{bmatrix} te & \delta \\ tg & th \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} te & \delta \\ tg & th \end{bmatrix} \in M_2(\langle k\rangle). \tag{4.8}$$

Considering the entries in position $(1,1)$ and $(1,2)$, we get $\delta c - btg$, $bt(e-h)+\delta(d-a) \in \langle k\rangle$, implying that $c = \delta^{-1}btg + \delta^{-1}v$ and $d = a - \delta^{-1}bt(e-h) + \delta^{-1}w$ for some $v, w \in \langle k\rangle$, with δ^{-1} the inverse of δ in the quotient field of R. Since $\delta | k$, we have that $\delta | v$ and $\delta | w$. Therefore $\delta^{-1}btg$, $\delta^{-1}v$, $a - \delta^{-1}bt(e-h)$, $\delta^{-1}w \in R$, and so

$$\begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} = \Theta\left(\begin{bmatrix} a & b \\ \delta^{-1}btg & a-\delta^{-1}bt(e-h) \end{bmatrix}\right) + \begin{bmatrix} \hat{0} & \hat{0} \\ \theta(\delta^{-1}v) & \theta(\delta^{-1}w) \end{bmatrix}. \tag{4.9}$$

Since $\delta^{-1}v$, $\delta^{-1}w \in R \cap \delta^{-1}\ker\theta$, it follows from Lemma 2.31, (4.5) and Lemma 2.29(c) that

$$\delta^{-1}v,\ \delta^{-1}w \in \theta^{-1}(\mathrm{ann}(\hat{\delta})) = \theta^{-1}(\mathrm{ann}(\hat{t}\hat{f})) = \theta^{-1}(\mathrm{ann}(\hat{f})). \tag{4.10}$$

We conclude from Corollary 2.9(iv), Lemma 2.11, (4.9) and (4.10) that

$$\begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} \in \Theta\left(\mathrm{Cen}\left(\begin{bmatrix} te & \delta \\ tg & th \end{bmatrix}\right)\right) + \begin{bmatrix} \hat{0} & \hat{0} \\ \mathrm{ann}(\hat{f}) & \mathrm{ann}(\hat{f}) \end{bmatrix} \tag{4.11}$$

which establishes (4.6).

We now distinguish between the following cases:

(i) $g \neq 0$;

(ii) $g = 0, e - h \neq 0$;

(iii) $g = 0, e - h = 0$.

(i) Let $A \in \text{Cen}\left(\begin{bmatrix} te & \delta \\ tg & th \end{bmatrix}\right)$. Since $g \neq 0$, we have $tg \neq 0$, and so considering

$$\begin{bmatrix} te & tg \\ \delta & th \end{bmatrix} = \begin{bmatrix} te & \delta \\ tg & th \end{bmatrix}^T,$$

it follows from Corollary 2.9(iv), Lemma 2.11 and Corollary 2.17 that

$$A = \begin{bmatrix} \alpha & (tg)^{-1}\delta\gamma \\ \gamma & \alpha - (tg)^{-1}\gamma t(e-h) \end{bmatrix} \in M_2(R) \tag{4.12}$$

for some $\alpha, \gamma \in R$, with $(tg)^{-1}$ the inverse of tg in the quotient field of R. By (4.5), $tf = \delta + mk$ for some $m \in R$ and so $(tg)^{-1}\delta\gamma = (tg)^{-1}(\delta + mk)\gamma - (tg)^{-1}mk\gamma = g^{-1}f\gamma - (tg)^{-1}mk\gamma$, from which we conclude that

$$A \;=\; \begin{bmatrix} \alpha & g^{-1}f\gamma \\ \gamma & \alpha - g^{-1}(e-h)\gamma \end{bmatrix} + \begin{bmatrix} 0 & -(tg)^{-1}mk\gamma \\ 0 & 0 \end{bmatrix}. \tag{4.13}$$

By (4.12) we have that $(tg)^{-1}\delta\gamma \in R$, and so $(tg)^{-1}mk\gamma \in R$, since $\delta|k$. Hence

$$\begin{bmatrix} \alpha & g^{-1}f\gamma \\ \gamma & \alpha - g^{-1}(e-h)\gamma \end{bmatrix} \in M_2(R),$$

which, again by Corollary 2.9(iv), Lemma 2.11 and Corollary 2.17, implies that

$$\begin{bmatrix} \alpha & g^{-1}f\gamma \\ \gamma & \alpha - g^{-1}(e-h)\gamma \end{bmatrix} \in \text{Cen}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) = \text{Cen}(B).$$

Next we deduce from Lemma 2.29(c) and Lemma 2.31 that

$$-(tg)^{-1}mk\gamma \in R \cap (tg)^{-1}\ker\theta = \theta^{-1}(\text{ann}(\hat{t}\hat{g})) = \theta^{-1}(\text{ann}(\hat{g})).$$

and so, by (4.13),

$$\Theta(A) \in \Theta(\text{Cen}(B)) + \begin{bmatrix} \hat{0} & \text{ann}(\hat{g}) \\ \hat{0} & \hat{0} \end{bmatrix}. \tag{4.14}$$

Thus combining (4.11) and (4.14), we have

$$\begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} \in \Theta\left(\text{Cen}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)\right) + \begin{bmatrix} \hat{0} & \text{ann}(\hat{g}) \\ \text{ann}(\hat{f}) & \text{ann}(\hat{f}) \end{bmatrix}. \tag{4.15}$$

(ii) Using Corollary 2.10(i) instead of Corollary 2.9(iv), similar arguments show that in this case,

$$\begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} \in \Theta\left(\mathrm{Cen}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)\right) + \begin{bmatrix} \hat{0} & \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) & \mathrm{ann}(\hat{f}) \end{bmatrix}. \tag{4.16}$$

(iii) If $e-h, g = 0$ and $A \in \mathrm{Cen}\left(\begin{bmatrix} te & \delta \\ tg & th \end{bmatrix}\right)$, then by Corollary 2.10(iii) and Lemma 2.11,

$$A = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix}, \tag{4.17}$$

for some $\alpha, \beta \in R$. Hence, again by Corollary 2.10(iii), Lemma 2.11 and (4.17), $A \in \mathrm{Cen}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)$, and so it follows from (4.11) that

$$\begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} \in \Theta\left(\mathrm{Cen}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)\right) + \begin{bmatrix} \hat{0} & \hat{0} \\ \mathrm{ann}(\hat{f}) & \mathrm{ann}(\hat{f}) \end{bmatrix}.$$

Consequently (4.15) holds again (as well as (4.16)).

We are finally in a position to prove that

$$\begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} \in \Theta\left(\mathrm{Cen}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)\right) + \begin{bmatrix} \hat{0} & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) & \mathrm{ann}(\hat{f}) \end{bmatrix}. \tag{4.18}$$

To this end, first note by (4.15) and (4.16)

$$\begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} = \widehat{X} + \begin{bmatrix} \hat{0} & \hat{q} \\ \hat{y} & \hat{z} \end{bmatrix} \tag{4.19}$$

for some $\widehat{X} \in \Theta(\mathrm{Cen}(B))$, $\hat{y}, \hat{z} \in \mathrm{ann}(\hat{f})$ and $\hat{q} \in \mathrm{ann}(\hat{g})$ (respectively $\hat{q} \in \mathrm{ann}(\hat{e}-\hat{h})$). By (2.34) in the proof of Proposition 2.33 $\widehat{X} \in \mathrm{Cen}(\widehat{B})$ and so it follows from (4.4) and (4.19) that

$$\begin{bmatrix} \hat{0} & \hat{q} \\ \hat{y} & \hat{z} \end{bmatrix} \in \mathrm{Cen}(\widehat{B}).$$

Thus
$$\begin{bmatrix} 0 & q \\ y & z \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} 0 & q \\ y & z \end{bmatrix} = M_2(\langle k\rangle). \tag{4.20}$$

Since $\hat{y}, \hat{z} \in \text{ann}(\hat{f})$, consideration of positions $(1,1)$ and $(1,2)$ in (4.20) shows that $\hat{q}\hat{g}$, $\hat{q}(\hat{e}-\hat{h}) = \hat{0}$, and so $\hat{q} \in \text{ann}(\hat{g}) \cap \text{ann}(\hat{e}-\hat{h})$. Thus, we conclude that

$$\begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} \in \Theta(\text{Cen}(B)) + \begin{bmatrix} \hat{0} & \text{ann}(\hat{g}) \cap \text{ann}(\hat{e}-\hat{h}) \\ \text{ann}(\hat{f}) & \text{ann}(\hat{f}) \end{bmatrix},$$

which establishes (4.18), and so we have proved (4.1).

(b) If one uses Corollary 2.10 instead of Corollary 2.9 and distinguishes between the following cases, then arguments analogous to those in the proof of (a) lead to (4.2):

(i) $g \neq 0$;

(ii) $f \neq 0$;

(iii) $g = 0$ and $f = 0$.

(c) Suppose $\hat{g}$ is k-invertible. By Corollary 2.17, Lemma 2.32 and (4.1),

$$\begin{aligned}
\text{Cen}\left(\begin{bmatrix} \hat{e} & \hat{f} \\ \hat{g} & \hat{h} \end{bmatrix}\right) &= \left(\text{Cen}\left(\begin{bmatrix} \hat{e} & \hat{g} \\ \hat{f} & \hat{h} \end{bmatrix}\right)\right)^T \\
&\subseteq \left(\Theta\left(\text{Cen}\left(\begin{bmatrix} e & g \\ f & h \end{bmatrix}\right)\right)\right)^T + \begin{bmatrix} \hat{0} & \text{ann}(\hat{f}) \cap \text{ann}(\hat{e}-\hat{h}) \\ \text{ann}(\hat{g}) & \text{ann}(\hat{g}) \end{bmatrix}^T \\
&= \Theta\left(\text{Cen}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)\right) + \begin{bmatrix} \hat{0} & \text{ann}(\hat{g}) \\ \text{ann}(\hat{f}) \cap \text{ann}(\hat{e}-\hat{h}) & \text{ann}(\hat{g}) \end{bmatrix},
\end{aligned}$$

i.e. (4.3) holds. □

Corollary 4.2. Let R be a UFD, $k \in R$ and let $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R)$. If at least one of the three elements $\hat{e}-\hat{h}$, $\hat{f}$ and $\hat{g}$ is equal to $\hat{0}$ and at least one of the remaining two elements is k-invertible, then

$$\begin{aligned}\operatorname{Cen}(\widehat{B}) &= \Theta(\operatorname{Cen}(B)) + \begin{bmatrix} \hat{0} & \operatorname{ann}(\hat{g})\cap\operatorname{ann}(\hat{e}-\hat{h}) \\ \operatorname{ann}(\hat{f})\cap\operatorname{ann}(\hat{e}-\hat{h}) & \operatorname{ann}(\hat{f})\cap\operatorname{ann}(\hat{g}) \end{bmatrix} \\ &= \Theta(\operatorname{Cen}(B)) + \begin{bmatrix} \operatorname{ann}(\hat{f})\cap\operatorname{ann}(\hat{g}) & \operatorname{ann}(\hat{g})\cap\operatorname{ann}(\hat{e}-\hat{h}) \\ \operatorname{ann}(\hat{f})\cap\operatorname{ann}(\hat{e}-\hat{h}) & \hat{0} \end{bmatrix} \\ &= \Theta(\operatorname{Cen}(B)) + \begin{bmatrix} \operatorname{ann}(\hat{f})\cap\operatorname{ann}(\hat{g}) & \operatorname{ann}(\hat{g})\cap\operatorname{ann}(\hat{e}-\hat{h}) \\ \operatorname{ann}(\hat{f})\cap\operatorname{ann}(\hat{e}-\hat{h}) & \operatorname{ann}(\hat{f})\cap\operatorname{ann}(\hat{g}) \end{bmatrix}. \end{aligned} \tag{4.21}$$

Proof. By Corollary 2.39 we only have to prove the containment $\subseteq$ in (4.21). We consider the following possibilities:

(i) $\hat{f}=\hat{0}$ and $\hat{e}-\hat{h}$ is k-invertible;

(ii) $\hat{f}=\hat{0}$ and $\hat{g}$ is k-invertible;

(iii) $\hat{e}-\hat{h}=\hat{0}$ and $\hat{f}$ is k-invertible;

(iv) $\hat{e}-\hat{h}=\hat{0}$ and $\hat{g}$ is k-invertible;

(v) $\hat{g}=\hat{0}$ and $\hat{f}$ is k-invertible;

(vi) $\hat{g}=\hat{0}$ and $\hat{e}-\hat{h}$ is k-invertible.

(i) Let $\begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} \in \operatorname{Cen}(\widehat{B})$. Since $\hat{f}=\hat{0}$ we have $\operatorname{ann}(\hat{f}) = R/\langle k\rangle$. Hence Lemma 4.1(b) implies that

$$\begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} = \widehat{X} + \begin{bmatrix} \hat{0} & \hat{x} \\ \hat{y} & \hat{z} \end{bmatrix} \tag{4.22}$$

for some $\widehat{X}\in\Theta(\operatorname{Cen}(B))$, $\hat{x}\in\operatorname{ann}(\hat{e}-\hat{h})$, $\hat{y}\in\operatorname{ann}(\hat{f})\cap\operatorname{ann}(\hat{e}-\hat{h})$ and $z\in\operatorname{ann}(\hat{f})\cap\operatorname{ann}(\hat{g})$. If we can show that $\hat{x}\in\operatorname{ann}(\hat{g})$, then we will have the containment $\supseteq$ in (4.21). By (2.34) in the proof of Proposition 2.33, $\widehat{X}\in\operatorname{Cen}(\widehat{B})$, and so we conclude from (4.22) that $\begin{bmatrix} \hat{0} & \hat{x} \\ \hat{y} & \hat{z} \end{bmatrix} \in \operatorname{Cen}(\widehat{B})$. Hence,

$$\begin{bmatrix} \hat{0} & \hat{x} \\ \hat{y} & \hat{z} \end{bmatrix}\begin{bmatrix} \hat{e} & \hat{0} \\ \hat{g} & \hat{h} \end{bmatrix} = \begin{bmatrix} \hat{e} & \hat{0} \\ \hat{g} & \hat{h} \end{bmatrix}\begin{bmatrix} \hat{0} & \hat{x} \\ \hat{y} & \hat{z} \end{bmatrix}. \tag{4.23}$$

Equating the entries in position $(1,1)$ we have $\hat{x}\hat{g}=\hat{0}$, whence $\hat{x}\in\operatorname{ann}(\hat{g})$ follows.

((ii)–(vi)) These five possibilities are treated similarly by using Lemma 4.1(a), Lemma 4.1(c), Corollary 2.17 and Lemma 2.32. □

We will also use the following result in the proof of Theorem 4.5.

Lemma 4.3. Let R be a UFD and let $b, k, \delta, \nu \in R$. If $k = \nu\delta$, then

$$\nu(\theta_\delta^{-1}(\mathrm{ann}_{R/\langle\delta\rangle}(\hat{b}_\delta)) \subseteq \theta_k^{-1}(\mathrm{ann}_{R/\langle k\rangle}(\hat{b}_k)).$$

Proof. Since $k \neq 0$, we have that $\nu \neq 0$. If $b = 0$ it follows that $\hat{b}_k = \hat{0}_k$ and $\hat{b}_\delta = \hat{0}_\delta$. Thus

$$\nu\theta_\delta^{-1}(\mathrm{ann}_{R/\langle\delta\rangle}(\hat{b}_\delta)) = \nu R \subseteq R = \theta_k^{-1}(\mathrm{ann}_{R/\langle k\rangle}(\hat{b}_k)).$$

If $b \neq 0$ and $s \in \theta_\delta^{-1}(\mathrm{ann}_{R/\langle k\rangle}(\hat{b}_\delta))$, then by Lemma 2.31, $s = b^{-1}t\delta$ for some $t \in R$, and so $\nu s \in b^{-1}\langle k\rangle$. Again, by Lemma 2.31, $\nu s \in \theta_k^{-1}(\mathrm{ann}_{R/\langle k\rangle}(\hat{b}_k))$. □

The next example illustrates Lemma 4.3.

Example 4.4. Let $b = 6 - 4x + 2x^2$, $k = 2x$, $\delta = x$ and $\nu = 2$ in $R = \mathbb{Z}[x]$. Then

$$\begin{aligned} 2\theta_x^{-1}\left(\mathrm{ann}_{\mathbb{Z}[x]/\langle x\rangle}\left(\theta_x\left(6-4x+2x^2\right)\right)\right) &= 2\langle x\rangle = \langle 2x\rangle \\ &\subsetneq \langle x\rangle = \theta_{2x}^{-1}(\langle \hat{x}_{2x}\rangle) \\ &= \theta_{2x}^{-1}\left(\mathrm{ann}_{\mathbb{Z}/\langle 2x\rangle}\left(\theta_{2x}\left(6-4x+2x^2\right)\right)\right). \end{aligned}$$

We are now able to prove our main result.

Theorem 4.5. Let R be a UFD, $k \in R$ and let $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R)$. If $\Theta : M_2(R) \to M_2(R/\langle k\rangle)$ is the natural epimorphism and $\Theta(B) = \widehat{B}$ is a k-matrix, then

$$\begin{aligned} \mathrm{Cen}(\widehat{B}) &= \Theta(\mathrm{Cen}(B)) + \begin{bmatrix} \hat{0} & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{bmatrix} && (4.24) \\ &= \Theta(\mathrm{Cen}(B)) + \begin{bmatrix} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \hat{0} \end{bmatrix} && (4.25) \\ &= \Theta(\mathrm{Cen}(B)) + \begin{bmatrix} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{bmatrix}. && (4.26) \end{aligned}$$

Proof. By Corollary 2.39 we only have to prove the containment $\subseteq$ in (4.24). Let $\begin{bmatrix} \hat{a}_k & \hat{b}_k \\ \hat{c}_k & \hat{d}_k \end{bmatrix} \in \mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k)$. Since $\widehat{B}_k$ is a k-matrix, we distinguish between the following cases:

(i) $\hat{f}_k$ is k-invertible and (δ is a unit or $\hat{e}_\delta - \hat{h}_\delta$ is δ-invertible or $\hat{g}_\delta$ is δ-invertible);

(ii) $\hat{e}_k - \hat{h}_k$ is k-invertible and (δ is a unit or $\hat{f}_\delta$ is δ-invertible or $\hat{g}_\delta$ is δ-invertible);

(iii) $\hat{g}_k$ is k-invertible and (δ is a unit or $\hat{f}_\delta$ is δ-invertible or $\hat{e}_\delta - \hat{h}_\delta$ is δ-invertible).

(i) Suppose $\hat{f}_k$ is k-invertible and (δ is a unit or $\hat{e}_\delta - \hat{h}_\delta$ is δ-invertible or $\hat{g}_\delta$ is δ-invertible). If δ is a unit, then $\hat{f}_k$ is invertible which implies that $\hat{0}_k = \mathrm{ann}(\hat{f}_k) = \mathrm{ann}(\hat{f}_k) \cap \mathrm{ann}(\hat{g}_k)$ and that $\hat{0}_k = \mathrm{ann}(\hat{f}_k) = \mathrm{ann}(\hat{f}_k) \cap \mathrm{ann}(\hat{e}_k - \hat{h}_k)$. Hence the result follows from Lemma 4.1(a). Thus suppose that δ is not a unit. By Lemma 4.1(a)

$$\begin{bmatrix} \hat{a}_k & \hat{b}_k \\ \hat{c}_k & \hat{d}_k \end{bmatrix} = \widehat{X}_k + \begin{bmatrix} \hat{0}_k & \hat{x}_k \\ \hat{0}_k & \hat{0}_k \end{bmatrix} + \begin{bmatrix} \hat{0}_k & \hat{0}_k \\ \hat{y}_k & \hat{z}_k \end{bmatrix},$$

where $\widehat{X}_k \in \Theta_k(\mathrm{Cen}(B))$, $\hat{x}_k \in \mathrm{ann}_{R/\langle k\rangle}(\hat{g}_k) \cap \mathrm{ann}_{R/\langle k\rangle}(\hat{e}_k - \hat{h}_k)$ and $\hat{y}_k, \hat{z}_k \in \mathrm{ann}_{R/\langle k\rangle}(\hat{f}_k)$. We will show that

$$\begin{bmatrix} \hat{0}_k & \hat{0}_k \\ \hat{y}_k & \hat{z}_k \end{bmatrix} \in \Theta_k(\mathrm{Cen}(B)) + \begin{bmatrix} \hat{0}_k & \mathrm{ann}_{R/\langle k\rangle}(\hat{g}_k) \cap \mathrm{ann}_{R/\langle k\rangle}(\hat{e}_k - \hat{h}_k) \\ \mathrm{ann}_{R/\langle k\rangle}(\hat{f}_k) \cap \mathrm{ann}_{R/\langle k\rangle}(\hat{e}_k - \hat{h}_k) & \mathrm{ann}_{R/\langle k\rangle}(\hat{f}_k) \cap \mathrm{ann}_{R/\langle k\rangle}(\hat{g}_k) \end{bmatrix}. \tag{4.27}$$

Then the containment $\subseteq$ in (4.24) will have been established.
By (2.34) and (2.35) in the proof of Proposition 2.33

$$\widehat{X}_k, \begin{bmatrix} \hat{0}_k & \hat{x}_k \\ \hat{0}_k & \hat{0}_k \end{bmatrix} \in \mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k).$$

Therefore

$$\begin{bmatrix} \hat{0}_k & \hat{0}_k \\ \hat{y}_k & \hat{z}_k \end{bmatrix} \in \mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k). \tag{4.28}$$

By Lemma 3.6, there is a k-pre-image $r\delta$ of $\hat{f}_k$, with relative prime part r and divisor part δ. By Corollary 3.20

$$\langle t\rangle = \theta_k^{-1}(\mathrm{ann}_{R/\langle k\rangle}(\hat{f}_k)),$$

where $t = \delta^{-1}k \in R$. Since $y, z \in \theta_k^{-1}(\mathrm{ann}_{R/\langle k\rangle}(\hat{f}_k))$, it follows that

$$y = mt \qquad \text{and} \qquad z = nt \tag{4.29}$$

for some $m, n \in R$. It follows from (4.28) that

$$\begin{bmatrix} \hat{0}_k & \hat{0}_k \\ \hat{y}_k & \hat{z}_k \end{bmatrix} \begin{bmatrix} \hat{e}_k & \hat{f}_k \\ \hat{g}_k & \hat{h}_k \end{bmatrix} = \begin{bmatrix} \hat{e}_k & \hat{f}_k \\ \hat{g}_k & \hat{h}_k \end{bmatrix} \begin{bmatrix} \hat{0}_k & \hat{0}_k \\ \hat{y}_k & \hat{z}_k \end{bmatrix}$$

and so

$$\begin{bmatrix} 0 & 0 \\ mt & nt \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} 0 & 0 \\ mt & nt \end{bmatrix} \in M_2(\langle k\rangle). \tag{4.30}$$

Considering positions $(1,1)$, $(1,2)$, $(2,1)$ and $(2,2)$ in (4.30), we obtain

$$fmt, fnt, emt + gnt - hmt \in \langle k\rangle,$$

which implies that

$$fm, fn, em + gn - hm \in \langle \delta\rangle.$$

This in turn implies that

$$\begin{bmatrix} 0 & 0 \\ m & n \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} 0 & 0 \\ m & n \end{bmatrix} \in M_2(\langle \delta\rangle)$$

or equivalently,

$$\begin{bmatrix} \hat{0}_\delta & \hat{0}_\delta \\ \hat{m}_\delta & \hat{n}_\delta \end{bmatrix} \begin{bmatrix} \hat{e}_\delta & \hat{f}_\delta \\ \hat{g}_\delta & \hat{h}_\delta \end{bmatrix} = \begin{bmatrix} \hat{e}_\delta & \hat{f}_\delta \\ \hat{g}_\delta & \hat{h}_\delta \end{bmatrix} \begin{bmatrix} \hat{0}_\delta & \hat{0}_\delta \\ \hat{m}_\delta & \hat{n}_\delta \end{bmatrix},$$

i.e.

$$\begin{bmatrix} \hat{0}_\delta & \hat{0}_\delta \\ \hat{m}_\delta & \hat{n}_\delta \end{bmatrix} \in \mathrm{Cen}_{M_2(R/\langle \delta\rangle)}(\widehat{B}_\delta). \tag{4.31}$$

Since $\hat{f}_k = (\widehat{r\delta})_k$ it follows that $f = r\delta + wk$ for some $w \in R$. Since $\delta | k$, it follows that $\delta \neq 0$, and so $f = r\delta + w\delta^{-1}k\delta \in \langle \delta\rangle$. Thus $\hat{f}_\delta = \hat{0}_\delta$. Since $\hat{e}_\delta - \hat{h}_\delta$ or $\hat{g}_\delta$ is δ-invertible, it follows from Corollary 4.2

and (4.31) that

$$\begin{bmatrix} \hat{0}_\delta & \hat{0}_\delta \\ \hat{m}_\delta & \hat{n}_\delta \end{bmatrix} \in \Theta_\delta(\mathrm{Cen}_{M_2(R)}(B)) + \begin{bmatrix} \hat{0}_\delta & \mathrm{ann}_{R/\langle\delta\rangle}(\hat{g}_\delta) \cap \mathrm{ann}_{R/\langle\delta\rangle}(\hat{e}_\delta - \hat{h}_\delta) \\ \mathrm{ann}_{R/\langle\delta\rangle}(\hat{f}_\delta) \cap \mathrm{ann}_{R/\langle\delta\rangle}(\hat{e}_\delta - \hat{h}_\delta) & \mathrm{ann}_{R/\langle\delta\rangle}(\hat{f}_\delta) \cap \mathrm{ann}_{R/\langle\delta\rangle}(\hat{g}_\delta) \end{bmatrix}.$$

Hence

$$\begin{bmatrix} \hat{0}_\delta & \hat{0}_\delta \\ \hat{m}_\delta & \hat{n}_\delta \end{bmatrix} - \Theta_\delta(A) = \begin{bmatrix} \theta_\delta(0) & \theta_\delta(\alpha) \\ \theta_\delta(\beta) & \theta_\delta(\gamma) \end{bmatrix}$$

for some

$$A \in \mathrm{Cen}_{M_2(R)}(B), \qquad \alpha \in \theta_\delta^{-1}(\mathrm{ann}_{R/\langle\delta\rangle}(\hat{g}_\delta) \cap \mathrm{ann}_{R/\langle\delta\rangle}(\hat{e}_\delta - \hat{h}_\delta)), \tag{4.32}$$

$$\beta \in \theta_\delta^{-1}(\mathrm{ann}_{R/\langle\delta\rangle}(\hat{f}_\delta) \cap \mathrm{ann}_{R/\langle\delta\rangle}(\hat{e}_\delta - \hat{h}_\delta)) \quad \text{and} \quad \gamma \in \theta_\delta^{-1}(\mathrm{ann}_{R/\langle\delta\rangle}(\hat{f}_\delta) \cap \mathrm{ann}_{R/\langle\delta\rangle}(\hat{g}_\delta)). \tag{4.33}$$

Thus

$$\begin{bmatrix} 0 & 0 \\ m & n \end{bmatrix} - A - \begin{bmatrix} 0 & \alpha \\ \beta & \gamma \end{bmatrix} \in M_2(\langle\delta\rangle), \tag{4.34}$$

i.e.

$$\begin{bmatrix} 0 & 0 \\ mt & nt \end{bmatrix} - tA - \begin{bmatrix} 0 & t\alpha \\ t\beta & t\gamma \end{bmatrix} \in M_2(\langle k\rangle). \tag{4.35}$$

Since $k = \delta t$, it follows from Lemma 4.3 that

$$t\alpha \in \theta_k^{-1}(\mathrm{ann}_{R/\langle k\rangle}(\hat{g}_k)) \cap \theta_k^{-1}(\mathrm{ann}_{R/\langle k\rangle}(\hat{e}_k - \hat{h}_k)) = \theta_k^{-1}(\mathrm{ann}_{R/\langle k\rangle}(\hat{g}_k) \cap \mathrm{ann}_{R/\langle k\rangle}(\hat{e}_k - \hat{h}_k)).$$

Using (4.33), one obtains similar results for $t\beta$ and $t\gamma$. By (4.32), $tA \in \mathrm{Cen}_{M_2(R)}(B)$, and so we conclude from (4.29) and (4.35) that

$$\begin{bmatrix} \hat{0}_k & \hat{0}_k \\ \hat{y}_k & \hat{z}_k \end{bmatrix} = \Theta_k(tA) - \begin{bmatrix} \hat{0}_k & \theta_k(t\alpha) \\ \theta_k(t\beta) & \theta_k(t\gamma) \end{bmatrix} \in \Theta_k(\mathrm{Cen}(B)) + \begin{bmatrix} \hat{0}_k & \mathrm{ann}_{R/\langle k\rangle}(\hat{g}_k) \cap \mathrm{ann}_{R/\langle k\rangle}(\hat{e}_k - \hat{h}_k) \\ \mathrm{ann}_{R/\langle k\rangle}(\hat{f}_k) \cap \mathrm{ann}_{R/\langle k\rangle}(\hat{e}_k - \hat{h}_k) & \mathrm{ann}_{R/\langle k\rangle}(\hat{f}_k) \cap \mathrm{ann}_{R/\langle k\rangle}(\hat{g}_k) \end{bmatrix}.$$

(ii) Invoking Lemma 4.1(b) instead of Lemma 4.1(a), the result follows as in case (i).

(iii) Suppose $\hat{g}_k$ is k-invertible and (δ is a unit or $\hat{f}_\delta$ is δ-invertible or $\hat{e}_\delta - \hat{h}_\delta$ is δ-invertible). Now, the result follows, similar to the proof of Lemma 4.1(c), from Corollary 2.17, (i) and Lemma 2.32, or similar to (i), using Lemma 4.1(c) instead of Lemma 4.1(a). □

The following result can simplify calculations regarding Theorem 4.5 and will be used in the proof of Theorem 4.11.

Lemma 4.6. Let R be a UFD and let $k, x, y \in R$, then

$$\operatorname{ann}(\hat{d}) = \operatorname{ann}(\hat{x}) \cap \operatorname{ann}(\hat{y})$$

in $R/\langle k\rangle$, with $\gcd(x, y) = d$.

Proof. By definition there are $u, v \in R$ such that $ud = x$ and $vd = y$ from which it follows that $\hat{u}_k\hat{d}_k = \hat{x}_k$ and $\hat{v}_k\hat{d}_k = \hat{y}_k$. Now, assume $\hat{l}_k \in \operatorname{ann}_{R/\langle k\rangle}(\hat{d}_k)$ so that $\hat{l}_k\hat{d}_k = \hat{0}_k$ which implies that $\hat{l}_k\hat{x}_k = \hat{l}_k\hat{u}_k\hat{d}_k = \hat{l}_k\hat{d}_k\hat{u}_k = \hat{0}_k\hat{d}_k = \hat{0}_k$ and that $\hat{l}_k\hat{y}_k = \hat{l}_k\hat{v}_k\hat{d}_k = \hat{l}_k\hat{d}_k\hat{v}_k = \hat{0}_k\hat{v}_k = \hat{0}_k$. Thus

$$\operatorname{ann}_{R/\langle k\rangle}(\hat{d}_k) \subseteq \operatorname{ann}_{R/\langle k\rangle}(\hat{x}_k) \cap \operatorname{ann}_{R/\langle k\rangle}(\hat{y}_k).$$

Conversely, assume $\hat{l}_k \in \operatorname{ann}_{R/\langle k\rangle}(\hat{x}_k) \cap \operatorname{ann}_{R/\langle k\rangle}(\hat{y}_k)$. Then $\hat{l}_k\hat{x}_k = \hat{0}_k$ and $\hat{l}_k\hat{y}_k = \hat{0}_k$ and so $\theta_k^{-1}(\hat{l}_k\hat{x}_k) = \ker\theta_k$ and $\theta_k^{-1}(\hat{l}_k\hat{y}_k) = \ker\theta_k$. Because $lx \in \theta_k^{-1}(\hat{l}_k\hat{x}_k)$ and $ly \in \theta_k^{-1}(\hat{l}_k\hat{y}_k)$ it follows that $lx \in \ker\theta_k$ and that $ly \in \ker\theta_k$. Since $d = \gcd(x, y)$, it follows from the Bézout identity that there are $u', v' \in R$ such that $u'x + v'y = d$ which implies that $u'lx + v'ly = ld$. Since $lx \in \ker\theta_k$ and $ly \in \ker\theta_k$ and $\ker\theta_k$ is an ideal in R, we have that $ld = u'lx + v'ly \in \ker\theta_k$. Thus $\theta_k(ld) = \hat{l}_k\hat{d}_k = \hat{0}_k$. Therefore $\hat{l}_k \in \operatorname{ann}_{R/\langle k\rangle}(\hat{d}_k)$ and we conclude that

$$\operatorname{ann}_{R/\langle k\rangle}(\hat{d}_k) = \operatorname{ann}_{R/\langle k\rangle}(\hat{x}_k) \cap \operatorname{ann}_{R/\langle k\rangle}(\hat{y}_k).$$

□

Example 4.7. Consider $B = \begin{bmatrix} y & x \\ x & 0 \end{bmatrix} \in M_2(F[x, y])$ and the x^2-matrix $\widehat{B} \in M_2(F[x, y]/\langle x^2\rangle)$ in Example 3.19(a). We use Theorem 4.5, (4.25), to obtain $\operatorname{Cen}(\widehat{B})$. According to Corollary 2.12(ii)

$$\operatorname{Cen}(B) = \left\{ \begin{bmatrix} h_1 & xh_2 \\ xh_2 & h_1 - yh_2 \end{bmatrix} \middle| h_1, h_2 \in F[x, y] \right\}. \tag{4.36}$$

Furthermore, $\operatorname{ann}(\hat{x}) = \langle\hat{x}\rangle$ and $\operatorname{ann}(\hat{x}) \cap \operatorname{ann}(\hat{y}) = \hat{0}$, and so it follows from (4.36) and Theorem 4.5, (4.25), that

$$\begin{aligned}
\mathrm{Cen}(\widehat{B}) &= \Theta\left(\left\{\left[\begin{array}{cc} h_1 & xh_2 \\ xh_2 & h_1 - yh_2 \end{array}\right]\middle|\, h_1, h_2 \in F[x,y]\right\}\right) + \left[\begin{array}{cc} \langle\hat{x}\rangle & \hat{0} \\ \hat{0} & \hat{0} \end{array}\right] \\
&= \left\{\left[\begin{array}{cc} \hat{h}_1 + \hat{h}_3\hat{x} & \hat{x}\hat{h}_2 \\ \hat{x}\hat{h}_2 & \hat{h}_1 - \hat{y}\hat{h}_2 \end{array}\right]\middle|\, \hat{h}_1, \hat{h}_2, \hat{h}_3 \in F[x,y]/\langle x^2\rangle\right\}.
\end{aligned}$$

Remark 4.8. Note that in the above example

$$\Theta(\mathrm{Cen}(B)) \not\subseteq \left[\begin{array}{cc} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e} - \hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e} - \hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{array}\right]$$

and that

$$\left[\begin{array}{cc} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e} - \hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e} - \hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{array}\right] \not\subseteq \Theta(\mathrm{Cen}(B)).$$

According to Corollary 3.18, Theorem 4.5 applies to all 2×2 matrices over factor rings $R/\langle k\rangle$, where R is a PID. In other words, we have equality in Proposition 2.33 for all 2×2 matrices over factor rings of PID's. This is not the case for all 2×2 matrices over factor rings $R/\langle k\rangle$, where R is a UFD, as the following example shows.

Example 4.9. Consider $B = \left[\begin{array}{cc} x+y & y \\ x & x \end{array}\right] \in M_2(F[x,y])$ and the non-x^2-matrix $\widehat{B} \in M_2(F[x,y]/\langle x^2\rangle)$ in Example 3.19(b). By Corollary 2.12(ii)

$$\mathrm{Cen}(B) = \left\{\left[\begin{array}{cc} h_1 & yh_2 \\ xh_2 & h_1 - yh_2 \end{array}\right]\middle|\, h_1, h_2 \in F[x,y]\right\}. \tag{4.37}$$

The second term in the righthand side of (4.25) is

$$\left[\begin{array}{cc} \mathrm{ann}(\hat{y}) \cap \mathrm{ann}(\hat{x}) & \mathrm{ann}(\hat{x}) \cap \mathrm{ann}(\hat{y}) \\ \mathrm{ann}(\hat{y}) \cap \mathrm{ann}(\hat{y}) & \hat{0} \end{array}\right] = \left[\begin{array}{cc} \hat{0} & \hat{0} \\ \hat{0} & \hat{0} \end{array}\right],$$

because $\mathrm{ann}(\hat{y}) = \hat{0}$. Therefore the righthand side of (4.25) is equal to

$$\left\{\left[\begin{array}{cc} \hat{h}_1 & \hat{y}\hat{h}_2 \\ \hat{x}\hat{h}_2 & \hat{h}_1 - \hat{y}\hat{h}_2 \end{array}\right]\middle|\, \hat{h}_1, \hat{h}_2 \in F[x,y]/\langle x^2\rangle\right\},$$

which does not contain the matrix $\begin{bmatrix} \hat{x} & \hat{x} \\ \hat{0} & \hat{0} \end{bmatrix}$. However, direct verification shows that

$$\begin{bmatrix} \hat{x} & \hat{x} \\ \hat{0} & \hat{0} \end{bmatrix} \in \mathrm{Cen}(\widehat{B}).$$

In the following example we will see that for every $n \geqslant 3$ and for any UFD R and $k \in R$ such that $R/\langle k\rangle$ is a ring with zero divisors, there is a matrix $B \in M_n(R)$ for which we do not have equality in Proposition 2.33.

Example 4.10. Let R be a UFD and let $k \in R$ such that $R/\langle k\rangle$ has zero divisors. Thus suppose that $\hat{d}, \hat{d}' \in R/\langle k\rangle$, $\hat{d}, \hat{d}' \neq \hat{0}$ and $\hat{d}\hat{d}' = \hat{0}$. Now let $B = \begin{bmatrix} 0 & d & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in M_3(R)$. Note that $d \neq 0$ since $\hat{d} \neq \hat{0}$. Because the characteristic polynomial of B is equal to the minimum polynomial of B it follows from Theorem 1.3 and Lemma 2.11 that $\mathrm{Cen}_{M_3(R)}(B) =$

$$\left\{ a\begin{bmatrix} 0 & 0 & d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & d & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; \begin{array}{l} a, b, c \text{ are elements} \\ \text{of the quotient} \\ \text{field of } R. \end{array} \right\} \cap M_3(R),$$

and so every matrix in $\Theta(\mathrm{Cen}(B))$ has $\hat{0}$ in position $(2,1)$. Furthermore, using the notation in Proposition 2.33 we have

$$[\mathcal{A}_{ij}] = \begin{bmatrix} \hat{0} & \hat{0} & R/\langle k\rangle \\ \hat{0} & \hat{0} & \langle \hat{d}'\rangle \\ \hat{0} & \hat{0} & \hat{0} \end{bmatrix}.$$

Hence every matrix in $\Theta(\mathrm{Cen}(B)) + [\mathcal{A}_{ij}]$ has $\hat{0}$ in position $(2,1)$. However, direct multiplication shows that

$$\begin{bmatrix} \hat{d}' & \hat{0} & \hat{0} \\ \hat{d}' & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{d}' \end{bmatrix} \in \mathrm{Cen}(\widehat{B}),$$

and so equality in Proposition 2.33 does not hold in this case. Now, again let R be a UFD and let $k \in R$

such that $R/\langle k\rangle$ has zero divisors. Let us consider the matrix

$$B' = \left[\begin{array}{ccc|c} 0 & d & 1 & \\ 0 & 0 & 1 & \bigcirc \\ 0 & 0 & 0 & \\ \hline & \bigcirc & & \bigcirc \end{array}\right] \in M_n(R).$$

Then

$$\mathrm{Cen}(B') \subseteq \left[\begin{array}{c|c} \mathrm{Cen}(B) & R/\langle k\rangle \\ \hline R/\langle k\rangle & R/\langle k\rangle \end{array}\right] \qquad \text{and} \qquad [\mathcal{A}_{ij}] \subseteq \left[\begin{array}{ccc|c} \hat{0} & \hat{0} & R/\langle k\rangle & \\ \hat{0} & \hat{0} & \langle \hat{d}'\rangle & R/\langle k\rangle \\ \hat{0} & \hat{0} & \hat{0} & \\ \hline & R/\langle k\rangle & & R/\langle k\rangle \end{array}\right].$$

Since

$$\widehat{A} := \left[\begin{array}{ccc|c} \hat{d}' & \hat{0} & \hat{0} & \\ \hat{d}' & \hat{0} & \hat{0} & \widehat{\bigcirc} \\ \hat{0} & \hat{0} & \hat{d}' & \\ \hline & \widehat{\bigcirc} & & \widehat{\bigcirc} \end{array}\right] \in \mathrm{Cen}(\widehat{B'}),$$

but clearly $\widehat{A} \notin \Theta(\mathrm{Cen}(B')) + [\mathcal{A}_{ij}]$, equality in Proposition 2.33, for these cases, does not hold.

It is interesting to note that it follows from Lemma 2.37 that

$$\left[\begin{array}{ccc|c} \hat{d}' & \hat{0} & \hat{0} & \\ \hat{0} & \hat{0} & \hat{0} & \widehat{\bigcirc} \\ \hat{0} & \hat{0} & \hat{0} & \\ \hline & \widehat{\bigcirc} & & \widehat{\bigcirc} \end{array}\right], \left[\begin{array}{ccc|c} \hat{0} & \hat{0} & \hat{0} & \\ \hat{d}' & \hat{0} & \hat{0} & \widehat{\bigcirc} \\ \hat{0} & \hat{0} & \hat{0} & \\ \hline & \widehat{\bigcirc} & & \widehat{\bigcirc} \end{array}\right], \left[\begin{array}{ccc|c} \hat{0} & \hat{0} & \hat{0} & \\ \hat{0} & \hat{0} & \hat{0} & \widehat{\bigcirc} \\ \hat{0} & \hat{0} & \hat{d}' & \\ \hline & \widehat{\bigcirc} & & \widehat{\bigcirc} \end{array}\right] \notin \mathrm{Cen}(\widehat{B'}).$$

4.2 Containment considerations regarding Section 4.1

Considering Remark 4.8, the following questions arise regarding the concrete description in Theorem 4.5:

(1) when is

$$\mathrm{Cen}(\widehat{B}) = \Theta(\mathrm{Cen}(B))?$$

(2) when is

$$\mathrm{Cen}(\widehat{B}) = \begin{bmatrix} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{bmatrix}?$$

(3) and when is

$$\Theta(\mathrm{Cen}(B)) = \begin{bmatrix} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{bmatrix}?$$

Theorem 4.11. Let R be a UFD, $k = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$, where $p_1, \ldots, p_m$ are different primes and $n_i \geqslant 1$ for all i, and let

$$B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R)$$

be such that $\widehat{B}$ is a k-matrix. Then

(a)

$$\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}) = \Theta_k(\mathrm{Cen}_{M_2(R)}(B)) \tag{4.38}$$

if and only if B is a scalar matrix or satisfies the following conditions for every i, $i = 1, 2, \ldots, m$:

(i) p_i is not a divisor of at least one of the elements $e - h$, f and g; pick such an element a, and call the remaining two elements b and c, say.

(ii) $\gcd(b, c, k) = 1$ or $\hat{a}_{\gcd(b,c,k)}$ is invertible in $R/\langle \gcd(b, c, k)\rangle$;

(b)

$$\mathrm{Cen}(\widehat{B}) = \begin{bmatrix} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{bmatrix} \tag{4.39}$$

if and only if $\hat{f} = \hat{0}$ and $\hat{g} = \hat{0}$;

(c)
$$\Theta(\mathrm{Cen}(B)) = \begin{bmatrix} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{bmatrix} \tag{4.40}$$
if and only if $\hat{f} = \hat{0}$, $\hat{g} = \hat{0}$ and ($\hat{e}-\hat{h}$ is invertible or $\hat{e}-\hat{h} = \hat{0}$).

Proof. (a) Since (4.38) follows trivially if B is a scalar matrix, we assume that B is a nonscalar matrix. Suppose that conditions (i) and (ii) are satisfied for every i, $i = 1, \ldots, m$. We now show that

$$\begin{bmatrix} \hat{0}_k & \mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_k(\gcd(g, e-h))) \\ \hat{0}_k & \hat{0}_k \end{bmatrix}, \begin{bmatrix} \hat{0}_k & \hat{0}_k \\ \mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_k(\gcd(f, e-h))) & \hat{0}_k \end{bmatrix}, \tag{4.41}$$

$$\begin{bmatrix} \hat{0}_k & \hat{0}_k \\ \hat{0}_k & \mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_k(\gcd(f, g))) \end{bmatrix} \subseteq \Theta_k(\mathrm{Cen}_{M_2(R)}(B)). \tag{4.42}$$

Since then, because $\Theta_k(\mathrm{Cen}_{M_2(R)}(B))$ is a ring (Remark 2.35), (4.38) follows from Theorem 4.5 and Lemma 4.6.

If $\mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_k(\gcd(g, e-h))) = \hat{0}_k$ then it follows trivially that

$$\begin{bmatrix} \hat{0}_k & \mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_k(\gcd(g, e-h))) \\ \hat{0}_k & \hat{0}_k \end{bmatrix} \subseteq \Theta_k(\mathrm{Cen}_{M_2(R)}(B)).$$

Thus suppose that $\mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_k(\gcd(g, e-h))) \neq \hat{0}_k$. Then $1 \neq \gcd(g, e-h, k) := \delta$ and, by Corollary 3.20,
$$\mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_k(\gcd(e-h, g))) = \langle(\widehat{k\delta^{-1}})_k\rangle. \tag{4.43}$$
To accomplish our objective, we show that for each $\hat{d}_k \in \mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_k(\gcd(g, e-h)))$ there is a $\hat{d}'_k \in \mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_k(\gcd(g, e-h)))$ such that $\hat{f}_k\hat{d}'_k = \hat{d}_k$, since then

$$\Theta_k\left(\begin{bmatrix} 0 & fd' \\ gd' & (e-h)d' \end{bmatrix}\right) = \begin{bmatrix} \hat{0}_k & \hat{d}_k \\ \hat{0}_k & \hat{0}_k \end{bmatrix},$$

so that we therefore can conclude from Corollary 2.12(ii) that

$$\begin{bmatrix} \hat{0}_k & \mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_k(\gcd(g, e-h))) \\ \hat{0}_k & \hat{0}_k \end{bmatrix} \in \Theta_k(\mathrm{Cen}_{M_2(R)}(B)).$$

Thus, let $\hat{d}_k$ be an arbitrary element in $\mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_k(\gcd(g, e-h)))$, i.e. suppose, using (4.43), that $\hat{d}_k := \hat{s}_k(\widehat{k\delta^{-1}})_k$ for some $\hat{s}_k \in R/\langle k\rangle$. Since by assumpsion $\delta := \gcd(e-h, g, k) \neq 1$, it follows from condition (i) that $\gcd(f, \delta) = 1$ and from condition (ii) that $\hat{f}_\delta$ is invertible in $R/\langle\delta\rangle$. Thus there is a $\hat{t}_\delta \in R/\langle\delta\rangle$ such that $\hat{t}_\delta\hat{f}_\delta = \hat{1}_\delta$ which implies that $tf = 1 + v\delta$ for some $v \in R$. Hence $ftd = (1 + v\delta)(sk\delta^{-1} + wk) = sk\delta^{-1} + (w + vs + v\delta w)k$, for some $w \in R$. In other words, if we set $\hat{d}'_k := (\widehat{td})_k$ then $\hat{f}_k\hat{d}'_k = \hat{f}_k(\widehat{td})_k = (\widehat{sk\delta^{-1}})_k = \hat{d}_k$.

The containment in $\Theta_k(\mathrm{Cen}_{M_2(R)}(B))$ of each of the other two sets in (4.41) and (4.42) can similarly be shown.

Conversely, suppose B does not satisfy both of the conditions (i) and (ii) for some i, $1 \leqslant i \leqslant m$. We distinguish between the following cases:

(a′) B does not satisfy (i) for some i, $i = 1, \ldots, m$, i.e. $\gcd(e - h, f, g, k) \neq 1$;

(b′) B satisfies (i) for every i, $i = 1, \ldots, m$, but for some i, $1 \leqslant i \leqslant m$, B satisfies (i) but not (ii).

(a′) Suppose there is a prime p_i in the prime factorization of k such that $p_i|e - h, f, g$. We distinguish between the following two cases:

(i′) $f = 0$ or $g = 0$;

(ii′) $f, g \neq 0$.

(i′) Since $p_i|e - h, f, g$, direct verification shows that

$$\begin{aligned}\widehat{A}_k &:= \begin{bmatrix} \hat{0}_k & \theta_k(p_1^{n_1}\cdots p_{i-1}^{n_{i-1}}p_i^{n_i-1}p_{i+1}^{n_{i+1}}\cdots p_m^{n_m}) \\ \theta_k(p_1^{n_1}\cdots p_{i-1}^{n_{i-1}}p_i^{n_i-1}p_{i+1}^{n_{i+1}}\cdots p_m^{n_m}) & \hat{0}_k \end{bmatrix} \\ &\in \mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k).\end{aligned}$$

Because $\theta_k(p_1^{n_1}\cdots p_{i-1}^{n_{i-1}}p_i^{n_i-1}p_{i+1}^{n_{i+1}}\cdots p_m^{n_m}) \neq \hat{0}_k$, it follows that the entries in position $(1, 2)$ and position $(2, 1)$ of $\widehat{A}_k$ only have nonzero pre-images in R. Since B is a nonscalar matrix, it follows from Corollary 2.12(ii) that every matrix in $\mathrm{Cen}_{M_2(R)}(B)$ has 0 in position $(1, 2)$ if $f = 0$ and 0 in position $(2, 1)$ if $g = 0$. Therefore $\widehat{A}_k \notin \Theta_k(\mathrm{Cen}_{M_2(R)}(B))$ if $f = 0$ or $g = 0$.

(ii′) Since $f, g \neq 0$ and $p_i|f, g$ it follows that

$$f = cp_i^r \qquad \text{and} \qquad g = dp_i^s \tag{4.44}$$

for some $s, r \geqslant 1$ and $c, d \in R$ such that $p_i \nmid c, d$. Now, $r \leqslant s$ or $s \leqslant r$. Let us first assume that $r \leqslant s$. Because $p_i | e-h, f, g$ direct verification shows that

$$\widehat{A}_k := \begin{bmatrix} \hat{0}_k & \hat{0}_k \\ \theta_k(p_1^{n_1}\cdots p_{i-1}^{n_{i-1}} p_i^{n_i-1} p_{i+1}^{n_{i+1}}\cdots p_m^{n_m}) & \hat{0}_k \end{bmatrix} \in \mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k).$$

We now show that $\widehat{A}_k \notin \Theta_k(\mathrm{Cen}_{M_2(R)}(B))$. Firstly note that the set of all the pre-images of $\widehat{A}_k$ is

$$\begin{bmatrix} \ker\theta_k & \ker\theta_k \\ p_1^{n_1}\cdots p_{i-1}^{n_{i-1}} p_i^{n_i-1} p_{i+1}^{n_{i+1}}\cdots p_m^{n_m} + \ker\theta_k & \ker\theta_k \end{bmatrix}.$$

Thus, if $\widehat{A}_k \in \Theta_k(\mathrm{Cen}_{M_2(R)}(B))$, then, by Corollary 2.9(iv) and Lemma 2.11, there is a pre-image

$$\begin{bmatrix} \kappa_1 & \kappa_2 \\ p_1^{n_1}\cdots p_{i-1}^{n_{i-1}} p_i^{n_i-1} p_{i+1}^{n_{i+1}}\cdots p_m^{n_m} + \kappa_3 & \kappa_4 \end{bmatrix} \in M_2(R)$$

of $\widehat{A}_k$, where $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in \ker\theta_k$, such that

$$\begin{bmatrix} \kappa_1 & \kappa_2 \\ p_1^{n_1}\cdots p_{i-1}^{n_{i-1}} p_i^{n_i-1} p_{i+1}^{n_{i+1}}\cdots p_m^{n_m} + \kappa_3 & \kappa_4 \end{bmatrix} = \begin{bmatrix} a & b \\ gf^{-1}b & a-(e-h)f^{-1}b \end{bmatrix}$$

in $M_2(R)$ for some $a, b \in R$. In other words, there are $a, b \in R$ such that $\kappa_1 = a$, $\kappa_2 = b$ and $p_1^{n_1}\cdots p_{i-1}^{n_{i-1}} p_i^{n_i-1} p_{i+1}^{n_{i+1}}\cdots p_m^{n_m} + \kappa_3 = gf^{-1}b$. But then, considering (4.44) and keeping in mind that $r \leqslant s$, that $gf^{-1}b \in R$, that $p_i^{n_i}|\kappa_2$ and that $p_i \nmid c, d$, we have that $gf^{-1}b = dp_i^s(cp_i^r)^{-1}\kappa_2 \in \langle p_i^{n_i}\rangle$, where $\langle p_i^{n_i}\rangle$ is the ideal generated by $p_i^{n_i}$ in R. Because $p_i^{n_i} \nmid p_1^{n_1}\cdots p_{i-1}^{n_{i-1}} p_i^{n_i-1} p_{i+1}^{n_{i+1}}\cdots p_m^{n_m} + \kappa_3$, it follows that $p_1^{n_1}\cdots p_{i-1}^{n_{i-1}} p_i^{n_i-1} p_{i+1}^{n_{i+1}}\cdots p_m^{n_m} + \kappa_3 \notin \langle p_i^{n_i}\rangle$, which implies that

$$p_1^{n_1}\cdots p_{i-1}^{n_{i-1}} p_i^{n_i-1} p_{i+1}^{n_{i+1}}\cdots p_m^{n_m} + \kappa_3 \neq gf^{-1}b.$$

We therefore have a contradiction. Therefore $\widehat{A}_k \notin \Theta_k(\mathrm{Cen}_{M_2(R)}(B))$.

If $s \leqslant r$ one can similarly show that

$$\widehat{A}_k := \begin{bmatrix} \hat{0}_k & \theta_k(p_1^{n_1}\cdots p_{i-1}^{n_{i-1}} p_i^{n_i-1} p_{i+1}^{n_{i+1}}\cdots p_m^{n_m}) \\ \hat{0}_k & \hat{0}_k \end{bmatrix} \in \mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k),$$

and that $\widehat{A}_k \notin \Theta_k(\mathrm{Cen}_{M_2(R)}(B))$, by using Corollary 2.17 and Corollary 2.9(iv), instead of Corollary 2.9(iv).

(b′) Suppose B satisfies (i) for every i, $i = 1, \dots, m$, but for some i, $1 \leqslant i \leqslant m$, B satisfies (i) but not (ii). Then at least one of the following cases is true:

(i′) $\gcd(e-h, f, g, k) = 1$, $1 \neq \gcd(e-h, g, k) := \delta$ and $\hat{f}_\delta$ is not invertible in $R/\langle\delta\rangle$;

(ii′) $\gcd(e-h, f, g, k) = 1$, $1 \neq \gcd(e-h, f, k) := \delta$ and $\hat{g}_\delta$ is not invertible in $R/\langle\delta\rangle$;

(iii′) $\gcd(e-h, f, g, k) = 1$, $1 \neq \gcd(f, g, k) := \delta$ and $\hat{e}_\delta - \hat{h}_\delta$ is not invertible in $R/\langle\delta\rangle$;

We now show that (4.38) does not follow in each of the above cases.

(i′) In this case Corollary 3.20 implies that

$$\operatorname{ann}_{M_2(R/\langle k\rangle)}(\theta_k(\gcd(g, e-h))) = \langle \widehat{(k\delta^{-1})}_k \rangle.$$

Note that since δ is not a unit, $\langle k\delta^{-1}\rangle \neq \langle k\rangle$, it follows, by Theorem 4.5, that

$$\widehat{A}_k := \begin{bmatrix} \hat{0}_k & \widehat{(k\delta^{-1})}_k \\ \hat{0}_k & \hat{0}_k \end{bmatrix} \in \operatorname{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}).$$

If we can show that $\widehat{A}_k \notin \Theta_k(\operatorname{Cen}_{M_2(R)}(B))$, then we are finished. Now,

$$\begin{bmatrix} \ker\theta_k & k^{-1}\delta + \ker\theta_k \\ \ker\theta_k & \ker\theta_k \end{bmatrix}$$

is the set of all the pre-images of $\widehat{A}_k$ in R. Furthermore, recall that $\gcd(e-h, f, g, k) = 1$. Therefore, if $\widehat{A}_k \in \Theta_k(\operatorname{Cen}_{M_2(R)}(B))$, it follows from Corollary 2.12(ii) that there is a pre-image

$$\begin{bmatrix} \kappa_1 & k\delta^{-1} + \kappa_2 \\ \kappa_3 & \kappa_4 \end{bmatrix} \in M_2(R)$$

of $\widehat{A}_k$, where $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in \ker\theta_k$, such that

$$\begin{bmatrix} \kappa_1 & k\delta^{-1} + \kappa_2 \\ \kappa_3 & \kappa_4 \end{bmatrix} = \begin{bmatrix} a & fb \\ gb & a-(e-h)b \end{bmatrix}$$

for some $a, b \in R$. Hence, $gb = \kappa_3$ and $(e-h)b = \kappa_1 - \kappa_4$, which implies, using the assumption that $\gcd(e-h, g, k) := \delta$, that $b = sk\delta^{-1}$ for some $s \in R$. But then, since $fb = k\delta^{-1} + \kappa_2$, we have that

$$fb = fsk\delta^{-1} = k\delta^{-1} + \kappa_2 \Leftrightarrow fs = 1 + t\delta \text{ for some } t \in R \Leftrightarrow \hat{f}_\delta\hat{s}_\delta = \hat{1}_\delta.$$

Since $\hat{f}_\delta$ is not invertible in $R/\langle\delta\rangle$, by assumption, we have a contradiction. Therefore

$$\widehat{A}_k \notin \Theta_k(\mathrm{Cen}_{M_2(R)}(B))$$

and so we conclude that $\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k) \not\subseteq \Theta_k(\mathrm{Cen}_{M_2(R)}(B))$.

((ii′) and (iii′)) In these cases it follows similarly to case (i′) that $\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k) \not\subseteq \Theta(\mathrm{Cen}_{M_2(R)}(B))$.

(b) Suppose $\hat{f}, \hat{g} = \hat{0}$. If B is a scalar matrix, then the result follows trivially. Thus suppose B is a nonscalar matrix. Now, $f, g \in \langle k\rangle$, and so by Corollary 2.12(ii)

$$\begin{aligned}
\Theta(\mathrm{Cen}(B)) &\subseteq \Theta\left(\left\{\left[\begin{array}{cc} a & fb \\ gb & a-(e-h)b \end{array}\right]\middle| a, b \in R\right\}\right) \\
&= \Theta\left(\left\{\left[\begin{array}{cc} a & 0 \\ 0 & a-(e-h)b \end{array}\right]\middle| a, b \in R\right\}\right) \\
&\subseteq \left[\begin{array}{cc} R/\langle k\rangle & \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{e}-\hat{h}) & R/\langle k\rangle \end{array}\right] \\
&= \left[\begin{array}{cc} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{array}\right].
\end{aligned}$$

Conversely, suppose

$$\Theta(\mathrm{Cen}(B)) \subseteq \left[\begin{array}{cc} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{array}\right].$$

Since $\left[\begin{array}{cc} \hat{a} & \hat{0} \\ \hat{0} & \hat{a} \end{array}\right] \in \Theta(\mathrm{Cen}(B))$ for every $\hat{a} \in R/\langle k\rangle$ it follows that $\mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) = R/\langle k\rangle$ which implies that $\mathrm{ann}(\hat{f}) = R/\langle k\rangle$ and that $\mathrm{ann}(\hat{g}) = R/\langle k\rangle$ and so $\hat{f}, \hat{g} = \hat{0}$.

(c) Using (b) and (a), it follows that

$$\Theta(\mathrm{Cen}(B)) = \left[\begin{array}{cc} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{array}\right]$$

$$\Leftrightarrow \quad \Theta(\mathrm{Cen}(B)) \subseteq \begin{bmatrix} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{bmatrix} \subseteq \Theta(\mathrm{Cen}(B))$$

$$\Leftrightarrow \quad \hat{f}, \hat{g} = \hat{0} \text{ and } \begin{bmatrix} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{bmatrix} \subseteq \Theta(\mathrm{Cen}(B))$$

$$\Leftrightarrow \quad \hat{f}, \hat{g} = \hat{0} \text{ and } (\hat{e}-\hat{h} \text{ is invertible in } R/\langle k\rangle \text{ or } \hat{e}-\hat{h} = \hat{0}).$$

□

Example 4.12. Let $R = F[x, y, z]$, $k = x^3y^2z$ and let

$$B = \begin{bmatrix} x^2y^2 & x+1 \\ x^2 & 0 \end{bmatrix}, \; B' = \begin{bmatrix} x^2y^2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B'' = \begin{bmatrix} 1+xyz & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that $\widehat{B}$, $\widehat{B}'$ and $\widehat{B}''$ are x^3y^2z-matrices. Since $\gcd(x^2y^2, x^2) = x^2$ and $(\widehat{x+1})_{x^2}$ is invertible in $R/\langle x^2\rangle$, it follows from Corollary 2.12(ii) and Theorem 4.11(a) that

$$\mathrm{Cen}(\widehat{B}) = \Theta(\mathrm{Cen}(B)) = \left\{ \begin{bmatrix} \hat{a} & (\widehat{x+1})\hat{b} \\ \widehat{x^2}\hat{b} & \hat{a} + \widehat{x^2y^2}\hat{b} \end{bmatrix} \middle| \; \hat{a}, \hat{b} \in F[x,y,z]/\langle x^3y^2z\rangle \right\}.$$

Furthermore, it follows from Theorem 4.11(b) that

$$\mathrm{Cen}(\widehat{B}') = \begin{bmatrix} R/\langle x^3y^2z\rangle & \langle\widehat{xz}\rangle \\ \langle\widehat{xz}\rangle & R/\langle x^3y^2z\rangle \end{bmatrix}$$

and, since $\theta_{x^3y^2z}(1+xyz)$ is invertible in $R/\langle x^3y^2z\rangle$, from Theorem 4.11(c) that

$$\begin{aligned} \mathrm{Cen}(\widehat{B}'') &= \Theta(\mathrm{Cen}(B'')) = \begin{bmatrix} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{bmatrix} \\ &= \begin{bmatrix} R/\langle x^3y^2z\rangle & \hat{0} \\ \hat{0} & R/\langle x^3y^2z\rangle \end{bmatrix}. \end{aligned}$$

Remark 4.13. Note that in Example 4.7, using the notation of Theorem 4.11(a), we have that $m = 1$. Now, for $i = 1$, B satisfies condition (i), since $x \nmid y$, but not condition (ii), since $\hat{y}_x$ is not invertible in $F[x, y]/\langle x\rangle$ as Remark 4.8 conveys.

Using Lemma 3.2, Corollary 3.7 and Corollary 3.18 we simplify Theorem 4.11(a) for the case when R is a PID.

Corollary 4.14. Let R be a PID and let $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R)$. Then

$$\mathrm{Cen}(\widehat{B}) = \Theta(\mathrm{Cen}(B))$$

if and only if B is a scalar matrix or $\gcd(e-h, f, g, k) = 1$.

Note that although Corollary 4.15 is not a characterization of the k-matrices for which question 1 at the beginning of this section is true, it is easier to verify if Corollary 4.15 applies to a specific matrix in $M_2(R)$ than to verify if Theorem 4.11(a) applies to a specific matrix in $M_2(R)$.

Corollary 4.15. Let R be a UFD, $k \in R$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R)$. If at least one of the three elements $\hat{e} - \hat{h}$, $\hat{f}$ and $\hat{g}$ is invertible in $R/\langle k\rangle$, then

$$\mathrm{Cen}(\widehat{B}) = \Theta(\mathrm{Cen}(B)).$$

Proof. Suppose $k = p_1^{n_1} \cdots p_m^{n_m}$, where $p_1, \ldots, p_m$ are different primes, $m \geqslant 1$ and $n_i \geqslant 1$ for all i. By Lemma 3.17 it follows that $\widehat{B}$ is a k-matrix. Without loss of generality, let us suppose that $\hat{f}$ is invertible in $R/\langle k\rangle$. Then, by Lemma 3.1, $\gcd(f, k) = 1$. Hence condition (i) in Corollary 4.11(a) is satisfied for every i, $i = 1, \ldots, m$. Now, suppose that $\gcd(e-h, g, k) = \delta$. If δ is a unit, then condition (i) as well as condition (ii) is satisfied for every i, $i = 1, \ldots, m$. Thus suppose that δ is not a unit. Then, since $\hat{f}_k$ is invertible in $R/\langle k\rangle$ and $\delta|k$, it follows that there is a $t \in R$ such that $tf = 1+sk = 1+s\nu\delta$ for some $s, \nu \in R$ which implies that $\hat{t}_\delta \hat{f}_\delta = \hat{1}_\delta$. Therefore condition (i) and condition (ii) in Corollary 4.11(a) is satisfied for every i, $i = 1, \ldots, m$. □

Chapter

5 The number of matrices in the centralizer of a matrix in $M_2(R/\langle k\rangle)$, R a UFD and $R/\langle k\rangle$ finite

Earth's crammed with heaven, and every common bush afire with God, but only he who sees takes off his shoes; the rest sit round it and pluck blackberries.

— Elizabeth Barrett Browning

In this chapter R will always be a UFD, unless stated otherwise, $k \in R$ will always be a nonzero nonunit such that $R/\langle k\rangle$ is finite and we will always denote the number of elements in a ring S by $|S|$. Note that we still assume that $\theta_k : R \to R/\langle k\rangle$ and $\Theta_k : M_2(R) \to M_2(R/\langle k\rangle)$ are the natural and induced epimorphism respectively. We, also, still denote the image $\theta_k(b)$ of b ($b \in R$) by $\hat{b}_k$ and the image $\Theta_k(B)$ of B ($B \in M_2(R)$) by $\widehat{B}_k$. However, if there is no ambiguity, then we simply write θ, Θ, $\hat{b}$ and $\widehat{B}$ respectively.

The purpose of this chapter is to determine the number of matrices in $\mathrm{Cen}_{M_2(R/\langle k\rangle)}(B)$, where R is a UFD, $R/\langle k\rangle$ is finite and $B \in M_2(R/\langle k\rangle)$.

To reach our goal, we first need some preliminary results.

Definition 5.1. Let $k \in R$, let $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R)$ and let $d := \gcd(e-h, f, g, k)$. We define the relation $\sim$ on $\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k)$ as follows: for $\widehat{A}_k, \widehat{C}_k \in \mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k)$,

$$\widehat{A}_k \sim \widehat{C}_k \qquad \text{iff} \qquad \widehat{A}_k - \widehat{C}_k \in M_2(\langle \widehat{(kd^{-1})}_k\rangle).$$

It follows immediately that $\sim$ is an equivalence relation.

We denote the equivalence class of $\widehat{A}_k$ by $\widehat{A}_k^*$ and the set

$$\{\widehat{A}_k^* \mid \widehat{A}_k \in (\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k))\}$$

of all equivalence classes by

$$(\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k))^*.$$

Since

$$M_2(\langle\widehat{(kd^{-1})_k}\rangle) \subseteq \begin{bmatrix} \mathrm{ann}(\hat{f}_k) \cap \mathrm{ann}(\hat{g}_k) & \mathrm{ann}(\hat{e}_k - \hat{h}_k) \cap \mathrm{ann}(\hat{g}_k) \\ \mathrm{ann}(\hat{e}_k - \hat{h}_k) \cap \mathrm{ann}(\hat{f}_k) & \mathrm{ann}(\hat{f}_k) \cap \mathrm{ann}(\hat{g}_k) \end{bmatrix},$$

it follows from Theorem 4.5 that $M_2(\langle\widehat{(kd^{-1})_k}\rangle) \subseteq \mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k)$. Therefore each equivalence class in $(\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k))^*$ has $|\langle\widehat{(kd^{-1})_k}\rangle|^4$ elements.

We define addition $\oplus$ and multiplication $\odot$ on $(\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k))^*$ by

$$\widehat{A}_k^* \oplus \widehat{C}_k^* = (\widehat{A}_k + \widehat{C}_k)^* \tag{5.1}$$

and by

$$\widehat{A}_k^* \odot \widehat{C}_k^* = (\widehat{A}_k \cdot \widehat{C}_k)^*. \tag{5.2}$$

It is easy to show that $\oplus$ and $\odot$ are well-defined and that the triple $\langle(\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k))^*, \oplus, \odot\rangle$ is a ring, which we sometimes, if the context is clear, denote by $(\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k))^*$.

We need the following trivial result in the next lemma.

Lemma 5.2. Let $S, S_1, \ldots, S_m$ be rings, $s \in S$ and let

$$\Gamma : S \to S_1 \oplus \cdots \oplus S_m \qquad \text{defined by} \qquad \Gamma(s) = (s_1, \ldots, s_m)$$

be an isomorphism. Then

$$t \in \mathrm{Cen}_S(s) \qquad \text{if and only if} \qquad t_i \in \mathrm{Cen}_{S_i}(s_i),$$

for all i.

Proof.

$$\begin{aligned} t \in \mathrm{Cen}_S(s) &\Leftrightarrow ts = st \\ &\Leftrightarrow (t_1s_1,\dots,t_ms_m) = (t_1,\dots,t_m)(s_1,\dots,s_m) = \Gamma(t)\Gamma(s) = \Gamma(ts) \\ &\quad\; = \Gamma(st) = \Gamma(s)\Gamma(t) = (s_1,\dots,s_m)(t_1,\dots,t_m) = (s_1t_1,\dots,s_mt_m) \\ &\Leftrightarrow t_is_i = s_it_i \quad \text{for all } i \\ &\Leftrightarrow t_i \in \mathrm{Cen}_{S_i}(s_i) \quad \text{for all } i. \end{aligned}$$

□

Lemma 5.3. Let $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R)$ and let $k \in R$. If $\gcd(e-h,f,g,k) = 1$, then

$$|\mathrm{Cen}(\widehat{B})| = |R/\langle k\rangle|^2.$$

Proof. Suppose $k = p_1^{n_1}p_2^{n_2}\cdots p_m^{n_m}$, where $p_1,\dots,p_m$ are different primes and $n_i \geqslant 1$ for all i. It follows from Lemma 3.25(ii) and Lemma 5.2 that

$$\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k) \cong \bigoplus_{i=1}^{m} \mathrm{Cen}_{M_2(R/\langle p_i^{n_i}\rangle)}(\widehat{B}_{p_i^{n_i}}).$$

Therefore,

$$|\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k)| = \prod_{i=1}^{m} |\mathrm{Cen}_{M_2(R/\langle p_i^{n_i}\rangle)}(\widehat{B}_{p_i^{n_i}})|.$$

If we can show that

$$|\mathrm{Cen}_{M_2(R/\langle p_i^{n_i}\rangle)}(\widehat{B}_{p_i^{n_i}})| = |R/\langle p_i^{n_i}\rangle|^2,$$

for all i it follows, again from Lemma 3.25(ii) and Lemma 5.2, that

$$|\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k)| = \prod_{i=1}^{m} |R/\langle p_i^{n_i}\rangle|^2 = |R/\langle k\rangle|^2.$$

Let p_i be an arbitrary prime in the prime factorization of k. Since $\gcd(e-h,f,g,k) = 1$, it follows that $p_i \nmid f$ or $p_i \nmid g$ or $p_i \nmid e-h$. Thus, by Lemma 3.21, at least one of $\hat{f}_{p_i^{n_i}}$, $\hat{g}_{p_i^{n_i}}$ or $\hat{e}_{p_i^{n_i}} - \hat{h}_{p_i^{n_i}}$ is invertible in $R/\langle p_i^{n_i}\rangle$.

If $\hat{f}$ is invertible in $R/\langle p_i^{n_i}\rangle$ with inverse $\hat{t}$, say, then given that $\gcd(e-h,f,g,p_i^{n_i}) = 1$, it follows from

Lemma 2.29(a), Corollary 4.15 and Corollary 2.12(ii) that

$$\begin{aligned}
\mathrm{Cen}_{M_2(R/\langle p_i^{n_i}\rangle)}(\widehat{B}) &= \mathrm{Cen}\left(\begin{bmatrix} \hat{e} & \hat{f} \\ \hat{g} & \hat{h} \end{bmatrix}\right) \\
&= \mathrm{Cen}\left(\begin{bmatrix} \hat{t}\hat{e} & \hat{1} \\ \hat{t}\hat{g} & \hat{t}\hat{h} \end{bmatrix}\right) \\
&= \Theta\left(\mathrm{Cen}\left(\begin{bmatrix} te & 1 \\ tg & th \end{bmatrix}\right)\right) \\
&= \left\{ \hat{a}\begin{bmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{1} \end{bmatrix} + \hat{b}\begin{bmatrix} \hat{0} & \hat{1} \\ \hat{t}\hat{g} & -\hat{t}(\hat{e}-\hat{h}) \end{bmatrix} \middle| \hat{a}, \hat{b} \in R/\langle p_i^{n_i}\rangle \right\}. \qquad (5.3)
\end{aligned}$$

It can be similarly shown that if $\hat{g}$ is invertible in $R/\langle p_i^{n_i}\rangle$ with inverse $\hat{t}$, say, then

$$\mathrm{Cen}_{M_2(R/\langle p_i^{n_i}\rangle)}(\widehat{B}) = \left\{ \hat{a}\begin{bmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{1} \end{bmatrix} + \hat{b}\begin{bmatrix} \hat{0} & \hat{t}\hat{f} \\ \hat{1} & -\hat{t}(\hat{e}-\hat{h}) \end{bmatrix} \middle| \hat{a}, \hat{b} \in R/\langle p_i^{n_i}\rangle \right\}; \qquad (5.4)$$

and if $\hat{e}-\hat{h}$ is invertible in $R/\langle p_i^{n_i}\rangle$ with inverse $\hat{t}$, say, then

$$\mathrm{Cen}_{M_2(R/\langle p_i^{n_i}\rangle)}(\widehat{B}) = \left\{ \hat{a}\begin{bmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{1} \end{bmatrix} + \hat{b}\begin{bmatrix} \hat{0} & -\hat{t}\hat{f} \\ -\hat{t}\hat{g} & \hat{1} \end{bmatrix} \middle| \hat{a}, \hat{b} \in R/\langle p_i^{n_i}\rangle \right\}. \qquad (5.5)$$

It is easy to see that the number of elements in the sets in (5.3), (5.4) and (5.5) are $|R/\langle p_i^{n_i}\rangle|^2$.

□

Lemma 5.4. Let $k \in R$, let $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R)$ and let $B' = \begin{bmatrix} d^{-1}(e-h) & d^{-1}f \\ d^{-1}g & 0 \end{bmatrix}$, where $d := \gcd(e-h, f, g, k)$, then

$$(\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k))^* \cong \mathrm{Cen}_{M_2(R/\langle kd^{-1}\rangle)}(\widehat{B'}_{kd^{-1}}).$$

Proof. It follows from Lemma 2.29(b) that

$$\begin{aligned}
& \widehat{A}_k^* \in (\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k))^* \\
\Leftrightarrow\ & \widehat{A}_k \in \mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k) \\
\Leftrightarrow\ & \widehat{A}_k \in \mathrm{Cen}_{M_2(R/\langle k\rangle)}\left(\begin{bmatrix} \hat{e}_k-\hat{h}_k & \hat{f}_k \\ \hat{g}_k & \hat{0}_k \end{bmatrix}\right) \\
\Leftrightarrow\ & A\begin{bmatrix} e-h & f \\ g & 0 \end{bmatrix} - \begin{bmatrix} e-h & f \\ g & 0 \end{bmatrix} A \in M_2(\langle k\rangle)
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \quad AB' - B'A \in M_2(\langle kd^{-1}\rangle) \\
&\Leftrightarrow \quad \widehat{A}_{kd^{-1}} \in \mathrm{Cen}_{M_2(R/\langle kd^{-1}\rangle)}(\widehat{B'}_{kd^{-1}})
\end{aligned}$$

and that

$$\begin{aligned}
& \widehat{A}^*_k = \widehat{C}^*_k \\
\Leftrightarrow \quad & \widehat{A}_k - \widehat{C}_k \in M_2(\langle \widehat{kd^{-1}}_k\rangle) \\
\Leftrightarrow \quad & A - C \in M_2(\langle k\rangle) + M_2(\langle kd^{-1}\rangle) \\
\Leftrightarrow \quad & A - C \in M_2(\langle kd^{-1}\rangle) \\
\Leftrightarrow \quad & \widehat{A}_{kd^{-1}} = \widehat{C}_{kd^{-1}}.
\end{aligned}$$

Hence $\Gamma : (\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k))^* \to \mathrm{Cen}_{M_2(R/\langle kd^{-1}\rangle)}(\widehat{B'}_{kd^{-1}})$, defined by

$$\Gamma(\widehat{A}^*) = \widehat{A}_{kd^{-1}},$$

is a well-defined function which is $1-1$ and onto. Since

$$\Gamma(\widehat{A}^*_k \odot \widehat{C}^*_k) = \Gamma((\widehat{AC})^*_k) = (\widehat{AC})_{kd^{-1}} = \widehat{A}_{kd^{-1}}\widehat{C}_{kd^{-1}} = \Gamma(\widehat{A}^*_k)\Gamma(\widehat{C}^*_k)$$

and

$$\Gamma(\widehat{A}^*_k \oplus \widehat{C}^*_k) = \Gamma((\widehat{A+C})^*_k) = (\widehat{A+C})_{kd^{-1}} = \widehat{A}_{kd^{-1}} + \widehat{C}_{kd^{-1}} = \Gamma(\widehat{A}^*_k) + \Gamma(\widehat{C}^*_k),$$

Γ is an isomorphism. □

We are finally able to determine the number of elements in the centralizer of a matrix in $M_2(R/\langle k\rangle)$, if R is a UFD and $R/\langle k\rangle$ is finite.

Theorem 5.5. Suppose R is a UFD, $k \in R$ is a nonzero nonunit such that $R/\langle k\rangle$ is finite, and

$$B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R),$$

then

$$|\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k)| = |R/\langle kd^{-1}\rangle|^2 \cdot |\langle (\widehat{kd^{-1}})_k\rangle|^4,$$

where d :=gcd$(e-h, f, g, k)$.

Proof. With B' as in Lemma 5.4, it follows from Lemma 5.3 that

$$|\mathrm{Cen}_{M_2(R/\langle kd^{-1}\rangle)}(\widehat{B'}_{kd^{-1}})| = |R/\langle kd^{-1}\rangle|^2.$$

Since each equivalence class in $(\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k))^*$ has cardinality $|\langle(\widehat{kd^{-1}})_k\rangle|^4$, it follows that

$$|\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k)| \quad = \quad |(\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k))^*||\langle(\widehat{kd^{-1}})_k\rangle|^4,$$

and so Lemma 5.4 implies that

$$|\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_k)| = |\mathrm{Cen}_{M_2(R/\langle kd^{-1}\rangle)}(\widehat{B'}_{kd^{-1}})||\langle(\widehat{kd^{-1}})_k\rangle|^4 = |R/\langle kd^{-1}\rangle|^2|\langle(\widehat{kd^{-1}})_k\rangle|^4.$$

□

Example 5.6. Let $R = \mathbb{Z}[i]$, $k = 12$ so that $R/\langle k\rangle = \mathbb{Z}_{12}[i]$ (see [8], p. 604, Theorem 1) and let

$$\widehat{B} = \begin{bmatrix} \widehat{4i} & \hat{3} + \widehat{6i} \\ \widehat{9i} & \hat{i} \end{bmatrix}.$$

Note that according to Corollary 2.12(ii), Corollary 3.18 and Theorem 4.5

$$\begin{aligned} \mathrm{Cen}_{M_2(\mathbb{Z}_{12}[i])}(\widehat{B}_{12}) &= \Theta_{12}\left(\left\{\begin{bmatrix} a & (1+2i)b \\ 3ib & a-3ib \end{bmatrix}\middle| a, b \in \mathbb{Z}[i]\right\}\right) + \begin{bmatrix} \langle\hat{4}\rangle & \langle\hat{4}\rangle \\ \langle\hat{4}\rangle & \hat{0} \end{bmatrix} \\ &= \left\{\begin{bmatrix} \hat{a} + \widehat{4c} & (\hat{1} + \widehat{2i})\hat{b} + \widehat{4m} \\ \widehat{3ib} + \widehat{4n} & \hat{a} - \widehat{3ib} \end{bmatrix}\middle| \hat{a}, \hat{b}, \hat{c}, \hat{m}, \hat{n} \in \mathbb{Z}_{12}[i]\right\}. \end{aligned}$$

Now, since $\gcd(3i, 3+6i, 9i, 12) = 3$, let $d = 3$ so that $kd^{-1} = 12 \cdot 3^{-1} = 4$. Since

$$|\mathbb{Z}[i]/\langle 4\rangle| = |\{a + ib \mid a, b \in \mathbb{Z}_4\}| = 16 \qquad \text{and} \qquad |\langle\hat{4}_{12}\rangle| = 9$$

it follows from Theorem 5.5 that

$$|\mathrm{Cen}_{M_2(\mathbb{Z}_{12}[i])}(\widehat{B}_{12})| = 16^2 \cdot 9^4 = 1679616.$$

For 2×2 matrices over a factor ring of $\mathbb{Z}$ we have the following result.

Corollary 5.7. Let $\widehat{B} = \begin{bmatrix} \hat{e} & \hat{f} \\ \hat{g} & \hat{h} \end{bmatrix} \in M_2(\mathbb{Z}_k)$, then

$$|\mathrm{Cen}(\widehat{B})| = (kd)^2,$$

where $d = \gcd(e-h, f, g, k)$.

Proof. According to Theorem 5.5

$$\begin{aligned} |\mathrm{Cen}_{M_2(\mathbb{Z}_k)}(\widehat{B}_k)| &= |\mathbb{Z}_{kd^{-1}}|^2 |\langle (\widehat{kd^{-1}})_k \rangle|^4 \\ &= (kd^{-1})^2 d^4 = (kd)^2. \end{aligned}$$

□

Example 5.8. Let $\widehat{B}_{12} = \begin{bmatrix} \hat{2}_{12} & \hat{2}_{12} \\ \hat{4}_{12} & \hat{8}_{12} \end{bmatrix}$. Since $\gcd(6, 2, 4, 12) = 2$, it follows that

$$|\mathrm{Cen}_{M_2(\mathbb{Z}_{12})}(\widehat{B}_{12})| = (12 \cdot 2)^2 = 24^2 = 576.$$

List of Symbols

$\mathcal{Q}$	the division ring of real quaternions , 3
$\mathbb{Z}_m$	the ring of integers modulo m , 8
$\mathbb{Z}[i]$	the ring of Gaussian integers , 51
UFD	unique factorization domain , 6
PID	principal ideal domain , 7
$\overline{F}$	the algebraic closure of the field F , 5
R^{op}	the opposite ring of the ring R , 23
$M_n(R)$	the full $n \times n$ matrix ring over the ring R , 3
$GL(n, F)$	the group of all $n \times n$ nonsingular matrices over the field F , 6
$R[x_1, \ldots, x_n]$	the polynomial ring over the ring R in the variables $x_1, \ldots, x_n$, 4
$\langle k \rangle$	the principal ideal generated by the element k , 6
$a\|b$	a is a divisor of b , 40
$a \nmid b$	a is not a divisor of b , 40
$\gcd(f_1, \ldots f_m)$	the greatest common divisor of $f_1, \ldots f_m$, 17
$\|R\|$	the number of elements in the ring R , 75
$\mathrm{ann}_R(b)$	the annihilator of the element b in the ring R , 10
$R/\langle k \rangle$	the ring R modulo the principal ideal $\langle k \rangle$, 6
$\mathrm{Cen}_{S_1}(s)$	centralizer of s in S_1 , 3
$\mathrm{Cen}_{S_1}(\mathrm{Cen}_{S_2}(s))$	the centralizer in S_1 of the centralizer of s in S_2 , 6
$f(X)$	the image of the set X under the map f , 23
$f^{-1}(X)$	the inverse image of the set X under the map f , 23
$\deg f(x)$	the degree of the polynomial $f(x)$, 5
fg	composite function of f and g , 27
$\ker f$	kernel of the homomorphism f , 29
$\pi_i : R_1 \oplus \cdots \oplus R_m \to R_i$	canonical projection of the i'th component of the direct sum $R_1 \oplus \cdots \oplus R_m$, 48
$\theta_k : R \to R/\langle k \rangle$	the natural epimomorphism from the ring R onto the ring $R/\langle k \rangle$, 37
$\Theta_k : M_2(R) \to M_2(R/\langle k \rangle)$	the natural induced epimomorphism from $M_2(R)$ onto $M_2(R/\langle k \rangle)$, 37
$\hat{b}_k$	image under θ_k of b , 37
$\widehat{B}_k$	image under Θ_k of B , 37
$[b_{ij}]$	the matrix with entry b_{ij} in position (i, j) , 10
E_{ij}	the matrix unit with a 1 in position (i, j) , 9
B^T	the transpose of the matrix B , 24
$\mathcal{B}^T$	the transpose of the set of matrices $\mathcal{B}$, 24
B^H	the matrix formed by rotating the entries of B around its horizontal axis , 25
B^V	the matrix formed by rotating the entries of B around its vertical axis , 25
$B^{T'}$	the s-transpose of the matrix B , 25
$\mathcal{B}^{T'}$	the s-transpose of the set of matrices $\mathcal{B}$, 26

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