A REPORT ON THE DEVELOPMENT OF THE CONTROL CHART

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DECLARATION:

I, the undersigned, hereby declare that the work contained in this assignment is my own original work and that I have not previously in its entirety or in part submitted it at any University for a degree.

.... 9 March 2007

Signature Date

ABSTRACT:

The goal of quality control as stated by Feigenbaum (1961) is to provide a product or service into which quality is designed, built, marketed and maintained at the lowest economical cost which simultaneously allows for full customer satisfaction. Statistical process control techniques, specifically control charts, are widely employed to achieve this goal.

Walter A. Shewhart developed the control chart in 1924 in order to differentiate between random causes of variation and assignable causes of variation. In situations where assignable causes occur, a corrective action should be taken to return the process to the in-control state. A process should be able to operate in the in-control state for a relatively long period before an assignable cause will come about. The heuristic design of Shewhart however, was not guaranteed to be economically optimal.

In 1956 Duncan proposed that the design parameters of control charts should be chosen in a manner that minimises the economic costs associated. Since then various developments in the economic design of control charts have taken place. Some research has been done to increase the power of the control chart and consequently the statistical design of control charts has been presented. The statistical design of control charts is designed in a manner as to place constraints on the control chart.

Saniga in (1989) proposed to combine the economic and statistical designs in the economic statistical design in an attempt to minimise the economic costs of the control charts under some statistical constraints. Results obtained by various authors show that the economic statistical designs perform best in the sense of achieving the desired statistical properties while simultaneously minimising the associated costs.

Various cost models have been developed under different distributions and the expressions for the expected cycle time and expected cycle cost have been derived. In 2005 Yang and Rahim

developed a cost model for the economic statistical design of control charts for a process with multiple quality characteristics under a Weibull shock model. The development of this model will be discussed in detail.

OPSOMMING:

Die doel van kwaliteitsbeheer soos voorgestel deur Feigenbaum (1961) is om 'n produk of diens te lewer wat van so aard is dat kwaliteit in die produk se ontwerp, samestelling, bemarking ingesluit is en volgehou word teen die laagste ekonomiese koste wat steeds volle verbruikers tevredenheid verseker. Statistiese kwaliteitsbeheer tegnieke, met spesifieke verwysing na kontrole kaarte, word ingespan ten einde hierdie doel te bereik.

Walter A. Shewhart het in 1924 die kontrole kaart ontwikkel met die doel om tussen kans oorsake en aanwysbare oorsake van variasie te onderskei. In omstandighede waar 'n aanwysbare oorsaak teenwoordig is, word 'n korrektiewe aksie vereis om die proses terug te bring na die in-kontrole toestand. 'n Proses behoort vir 'n relatiewe lang periode in die in-kontrole toestand te funksioneer voor die voorkoms van 'n aanwysbare oorsaak. Die heuristiese ontwerp van Shewhart waarborg egter nie dat die kontrole kaart ekonomies optimaal sal funksioneer nie.

In 1956 stel Duncan voor dat die seleksie van parameters vir die ontwerp van kontrole kaarte op so manier geskied dat dit die ekonomiese koste minimeer. Sedertdien is verskeie verbeterings in die ekonomiese ontwerp van kontrole kaarte gemaak. Navorsing is ook gedoen ten einde die onderskeidingsvermoë van die kontrole kaart te verbeter en sodoende is die statistiese ontwerp van kontrole kaarte voorgestel. Die statistiese ontwerp plaas statistiese beperkings op die kontrole kaart.

In 1989 stel Saniga voor dat die ekonomiese en statistiese ontwerpe gekombineer word om die ekonomiese statistiese ontwerp te vorm. Die doel van die ekonomiese statistiese ontwerp is om die ekonomiese koste te minimeer gegewe sekere statistiese beperkings. Resultate toon dat die ekonomiese statistiese ontwerp beter presteer en die statistiese beperkinge kan hierdeur behaal word terwyl die toepaslike kostes terselftertyd geminimeer word.

Verskeie koste modelle is ontwikkel onder verskillende verdelings en uitdrukkings vir die verwagte siklus tyd en verwagte siklus koste is afgelei. In 2005 ontwikkel Yang and Rahim 'n koste model vir die ekonomies statistiese ontwerp van kontrole kaarte vir 'n proses met meerveranderlike kwaliteitseienskappe onder 'n Weibull skok model. Die ontwikkeling van hierdie spesifieke model sal in besonderhede bespreek word.

DEDICATION:

This paper is dedicated to my Father, who loves me unconditionally.

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NOTATION

For the purposes of this paper the notation proposed by Yang and Rahim (2005) will be used throughout. This notation remains consistent with notation used by Banerjee and Rahim (1988) to define the cost model that will be developed in this paper. All other notation used will be given alongside the specified models.

Z_0: expected search time associated with the false alarm.

Z_1: expected time to discover the assignable cause.

Z_2: expected time to repair the process.

a: fixed sample cost

b: cost per unit sampled.

Y: cost per false alarm.

W: cost to locate and repair the assignable cause.

D_0: cost per hour while the process is in control.

D_1: cost per hour while the process is out if control.

α: Pr(test result has an alarm | the process is in control), $\alpha = \Pr\left(T^2 > \chi^2_{\alpha,k} \middle| \mu = \mu_0\right)$, where $T^2 = n(\bar{X} - \mu_0)\Sigma^{-1}(\bar{X} - \mu_0)^T$.

β: Pr(test result has no alarm | the process is out of control, $\mu = \mu_1$,

$\beta = \Pr\left(T^2 < \chi^2_{\alpha,k} \middle| \mu = \mu_1\right)$, where $T^2 = n(\bar{X} - \mu_1)\Sigma^{-1}(\bar{X} - \mu_1)^T$.

h_j: the length of the j^{th} sample interval where j can take on the values 1,2,...; and $h_0 = 0$.

θ: is the scale parameter of the Weibull distribution, where $\theta \geq 1$.

λ: is the shape parameter of the Weibull distribution, where $\lambda > 0$.

ω_j: the time until the j^{th} sample is taken; $\omega_j = \sum_{i=1}^{j} h_i = j^{1/\theta} h_1$, where $j = 1,2,\ldots$ and $\omega_0 = 0$. The choice of h will be discussed in detail in Chapter 6 describing The Cost Model.

T_j: the residual time in the cycle beyond time ω_j given that the process is in the in-control state at time ω_j.

T_0: total time until an assignable cause occurs from the beginning.

p_j: the conditional probability that the process is out of control in the j^{th} sampling interval, given that the process is still in control before time ω_{j-1}, that is $p_j = Pr(T < \omega_j | T_0 > \omega_{j-1})$. Since p_i is independent of j, so let $p_j = p$.

q_j: the probability that the process will be out of control during the j^{th} sampling interval; $q_j = Pr(\omega_{j-1} < T_0 < \omega_j)$.

τ_j: the expected in control time in the j^{th} sampling interval, given that the shock occurred in the j^{th} sampling intervall; $\tau_j = E(T - \omega_{j-1} | \omega_{j-1} < T < \omega_j)$.

τ: the unconditional expected in control time in a sampling interval; $\tau = \sum_{j=1}^{\infty} q_j \, \tau_j$.

$E(C)$: the expected cycle cost.

$E(C_j)$: the expected cycle cost associated with the j^{th} sampling interval.

CHAPTER ONE: *INTRODUCTION*

1.1 BACKGROUND AND PROBLEM STATEMENT:

In a competitive marketplace it is imperative for companies to provide products that are reliable at competitive prices. The poor quality of output does not only influence the customers' choice between similar products, but also leads to increased operational costs. A company can achieve quality improvement through quality control techniques that aim to minimise these costs and result in an improvement in productivity. "The goal of competitive industry, as far as product quality is concerned, can be clearly stated: It is to provide a product and service into which quality is designed, built, marketed, and maintained at the most economical costs which allow for full customer satisfaction" (Feigenbaum; 1961: 5).

Statistical process control (SPC) techniques have been applied widely in the manufacturing industry. Specifically the control chart is applied for the purpose of monitoring a production process. "Statistical process control is an effective approach for improving product quality and saving production costs for a firm. Since 1924, when Dr Shewhart presented the first control chart, various control chart techniques have been developed and widely applied as a primary tool in statistical process control" (Chou, Chen, Liu, Huang; 2003).

According to Duncan (1956) control charts are usually used for two purposes – those that are employed to bring a system under statistical control and those that are used to maintain control within a system. When control charts are employed to measure the performance of the process, a process is said to be operating in statistical control when the only variation present in the process is inherent to the process. A corrective action is necessary if variation on the quality of the product is due to some assignable cause, and the process "must be brought back into statistical control by detection and elimination of the assignable causes of variation" (Rahim and Costa; 2000). In some cases it could potentially be advantageous to take a proactive approach to prevent the occurrence of an out-of-control state. The process could be adjusted in a preventive manner to decrease the number of non-conforming products that are produced. "Relative to this objective, engineering process control (EPC) techniques have been developed with the aim of constantly adjusting the process so that it is always kept on target" (Xie, Goh and Cai; 2001).

According to Kapur and Cho (1996) the problem that arises from this concept of quality is two-fold. Firstly, what definition of the cost of quality is to be used, and secondly, the exact form of the quality loss function that is to be used to evaluate the quality of the product is typically not known. Kapur and Cho (1996) define the cost of quality as the cost of non-conformance. This implies that all costs associated with a non-conforming unit should be included when comparing processes. They continue to argue that quality loss "should be evaluated from the viewpoint of both the producer and the customer. The loss to the customer is due to variability from the target value. To the producer, losses are incurred due to inspection and scrap" (Kapur and Cho; 1996).

As noted by Feigenbaum (1961: 394) controlling the quality of raw materials, batches, components, and assemblage during the course of manufacturing is probably the most popular method of SPC. "Since it was first born as a specific discipline in the 1920s, quality control has taken its place as a central activity in the industrial system" (De-Vor, Chung, Sutherland; 1992: 3). It is clear that the process of SPC is not an isolated system, but rather an integrated process that incorporated "the quality-development, quality-maintenance, and quality-improvement efforts of the various groups in an organization so as to enable marketing, engineering, production, and service at the most economical levels which allow for full customer satisfaction." (Feigenbaum; 1961: 6) This idea of an integrated total quality control process is echoed by Betterley, Mettrick, Sweeney and Wilson (1994: 169): "Checking and measurement extends to all aspects of industrial processes: in other words, to people, materials, methods, facilities and the environment. The term 'process control' distinguishes this improvement activity from traditional *quality control*, which tended to focus on the result of processes in the context of a fixed specification."

The design of the control chart plays a critical role in its application to a process. "The design can affect the cost, statistical properties, and ultimately user confidence. Cost considerations are important for obvious reasons. Statistical criteria such as the magnitude of the false alarm rate and power as well as the length of time needed to detect undesired shifts can have significant practical consequences on the effective implementation and continued use of control charts" (Zhang and Berardi; 1997). Typically, a design can be developed in order to minimise the costs associated with the production process, to control certain statistical criteria that are imposed on the process, or to minimise the costs while applying some statistical bounds to the process.

The economic design of a control chart was first proposed by Duncan (1956), the design held some advantages, but it was argued that its application may be limited in some instances. "Designers of economic control charts are simply not able to determine a priori what the control chart statistical properties will actually be. In addition, they cannot determine how sensitive the cost is to the improvement of these properties." (Al-Oraini and Rahim; 2002) Due to such arguments the statistical design of control charts was consequently developed to address these shortcomings. In 1989 Saniga proposed an economic statistical design that aims to harness the advantages of the previous models. The economic statistical designs are presented as the constrained version of economic designs. "The principle of economic statistical design is fully consistent with the objective of statistical quality control of simultaneously reducing costs and maintaining high quality" (Zhang and Berardi; 1997).

1.2 OBJECTIVE OF THE STUDY:

The objective of this study is to give a review of the literature available on the various design models that have been proposed to date, taking into account statistical as well as economical aspects. Comparisons will be drawn between the economic, statistical and economic statistical designs of control charts. Due to the critical role that costs play in the competitive industries, the development of various cost models will also be discussed in this study.

1.3 SCOPE OF THE STUDY:

The study aims to describe the development of several different cost models that have been designed. The development of recent cost models based on the non-normal distributions will be discussed. The exponential and Weibull distributions will be considered as potential distributions that describe the distribution of the assignable cause.

1.4 ORGANISATION OF THE STUDY:

The study is organised and presented in seven chapters. Chapter one provides a brief introduction to the field of SPC, an overview of the objective of the study and the scope of the study.

Control charts can be classified into four general categories: heuristic; economic; statistical; and economic statistical. The second chapter provides some historical background to control charts, including the heuristic design as initially developed by Shewhart in 1924.

Chapter three describes the economic design of control charts. A cost model is derived for the economic design of a control chart. Possible variations on the preventative replacement of components are also discussed briefly, including a discussion on the development of the model.

Chapter four discusses the statistical design of control charts. The ability of the statistical design to overcome the shortcomings of the economic design is also mentioned.

Chapter five discusses the economic statistical design proposed by Saniga (1989). The motivation behind the choice of the distribution of the assignable causes is included here.

Chapter six shows the development of the various cost models that have been proposed to date.

Chapter seven presents the conclusions drawn from this study as well as some remarks.

The order of discussion is motivated by the chronological order in which the development of the design of control charts has taken place.

It is noted here that for the sake of simplicity, the term 'system' or 'process' will be used to generally refer to the various systems or processes where control charts are applied to, without specifying the context in which the system is operating in.

CHAPTER TWO:
THE HISTORICAL CONTROL CHART

In most production processes, some amount of variation in the quality of products is inevitable. Statistical quality control aims to analyse small samples taken at regular time intervals from the output of the production process in an attempt to minimise this variation. Walter A. Shewhart developed the control chart in 1924 in order to differentiate between random causes of variation and assignable causes of variation in a process (Shewhart; 1931). An assignable cause is defined to be variation in the process due to a non-random factor(s). He argues that a certain amount of variability in the process is however unavoidable. In these cases where a process experiences some chance causes of variation, the process is said to be in statistical control and no action should be taken.

On the other hand, the occurrence of large variability requires that the assignable cause should be found and eliminated. A process that is subject to an assignable cause is said to be in an out-of-control state. According to Rahim and Costa (2000) there can be several different explanations for the occurrence of a change in the process mean or a failure to meet specifications. Such an assignable cause can be one of the following: a sudden increase or decrease in stress or temperature, or some human error, such as fatigue. Rahim and Costa (2000) also specify causes that lead to changes in the process variance specifically, namely, worn out bearings, poor quality raw materials, a loose tool part, vibration, or an irregular flow of lubrication to a machine. Due to the instability of such a process in the long run, it could possibly operate in the in-control state for a relatively long period, but eventually an assignable cause will come about. The deterioration of a process often starts gradually due to some ageing effects of wear, corrosion, or fatigue.

According to Zhang and Berardi (1997) the Shewhart's heuristic design of the control chart is not guaranteed to be economically optimal. There are costs associated with the process functioning in the in-control state. These costs are due to sampling the process, the production of nonconformities, and the occurrence of false alarms (Lorenzen and Vance; 1986). In the out-of-control state a larger proportion of the output from the process will not conform to requirements. Once the process moves from the in-control to the out-of-control state, it is assumed that it shifts to a known state. The process cannot return to an in-control state without external intervention. The costs associated with the out-of-control state are also due to sampling and the increased percentage of non-conformaties produced. According to Kapur and Cho (1996) a product is classified as non-conforming if the quality characteristic of interest fails to meet the specification limits that are predetermined. There are also costs associated with searching for the assignable cause, repairing the process and possible downtime during the search and repair of the process (Lorenzen and Vance; 1986). The main objective of statistical quality control is to detect the occurrence of an assignable cause as soon as possible in order to correct the process before a large number of non-conforming items are produced (Montgomery; 2005: 150). The control chart as developed by Shewhart (1931) has been widely used for this purpose.

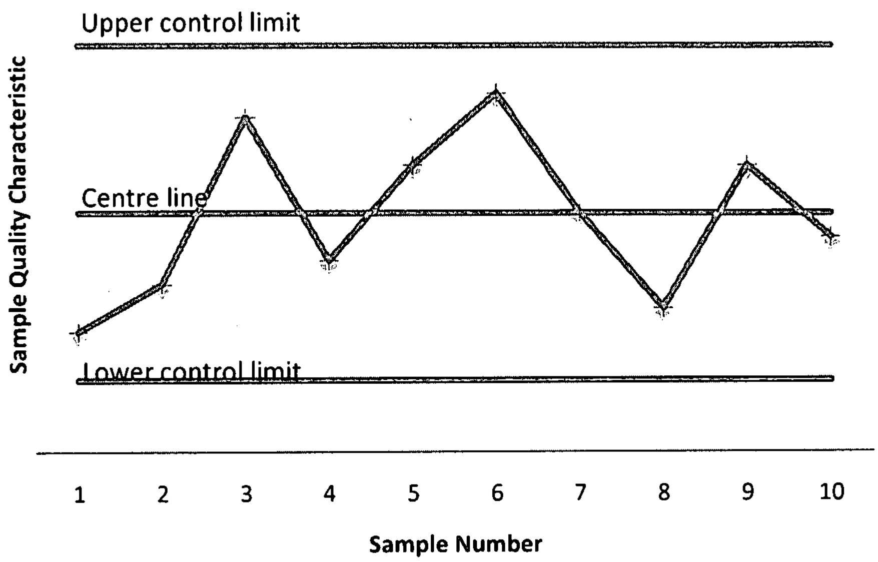
Other uses of the control chart include:

- estimating the parameters of a production process;
- using these parameters to determine the process capability; and
- providing information regarding possible improvements to the process.

According to Lorenzen and Vance (1986) the methodology behind constructing control charts consists of sampling from a process over time and then charting the process measurement that is of interest. The mean of a continuous process and the percentage of non-conforming items in a Bernoulli process are examples of such a process measurement. The control chart gives a graphical representation of the quality characteristic as measured over time. The centre line of the control chart indicates the average value of the quality characteristic of interest that corresponds to the situation where only random causes occur (the in-control state). The two other horizontal lines are the specification limits, called the upper control limit (UCL) and the lower control limit (LCL), respectively. When the process is in an in-control state most of the sample values are expected to fall within these two limits. However, if sample statistics plot outside these specified limits, the control chart suggests that there is evidence that the process is out of control. This indicates that a larger proportion of output will lie outside the specification limits. The out-of-control state of the process requires a corrective action in order to find and eliminate the assignable cause that was responsible for the shift. In Figure 2.1 below, a graphical depiction of a typical control chart is given.

Figure 2.1.

A typical control chart.



In some instances the sample statistics could all plot within the specification limits, but a systematic or non-random pattern is observed. This behaviour could be indicative of the process being out of control and an investigation into the process should take place. Alternatively, the points plotted for a process that is currently in the in-control state is expected to exhibit a relatively random pattern.

Woodall (1985) defines the run length of a control procedure as the number of samples required before an out-of-control signal is presented. The average run length (ARL) is used to measure the performance of a control procedure. According to Woodall (1985) a good control procedure has a fittingly large ARL when the system is functioning in the in-control state and a small ARL if not.

Hypothesis testing procedures can be useful in statistical quality control problems. Most of the statistical control techniques are also based on hypothesis testing. The use of hypothesis tests is dependent on the assumptions of independence and on the randomness of the observations.

Note that the $\bar{X}$-control chart is an example of such an independent statistic, the Cumulative Sum (CUSUM) chart, however, is not independent from one sample to the next.

The first step in hypothesis testing is determining the parameter values that are specified on the null and alternative hypotheses, respectively. According to Montgomery (2005; 97:98) there are three methods to determine these values: First, the values can be obtained from results obtained from past experience or knowledge. This is a popular method used in statistical process control, where past information is used to specify the parameter values corresponding to a state of control, and then periodically testing the hypothesis that the parameter value has not changed. Secondly, the values may be obtained from some model of the process or from theory. Finally, the parameters may be the result of contractual or design specifications. Statistical hypothesis testing methods are used to verify whether the process parameters conform to the specified values. Alternatively, hypothesis testing can also be used to modify a process until the specified values are obtained.

A random sample is taken from the process that is being observed, an appropriate test statistic is calculated and a decision is made to either reject or fail to reject the null hypothesis, denoted by H_0. Two possible errors can be committed when conducting a hypothesis test: If H_0 is rejected when it is true, a Type I error has been committed. The probability of a Type I error is denoted by $\alpha = P\{Type\ I\ error\} = P\{reject\ H_0|H_0\ is\ true\}$. A Type II error occurs when the null hypothesis is not rejected when it is in fact false. The probability of a Type II error is denoted by $\beta = P\{Type\ II\ error\} = P\{fail\ to\ reject\ H_0|H_0\ is\ false\}$. According to Chou, Chen, Liu, Huang (2003) the control chart technique can be considered to be a graphical expression and operation of statistical hypothesis testing.

Control charts are extensively used to ascertain and maintain statistical process control. The power of a test, in control chart terms, is the probability of correctly identifying a shift in the process mean when one does exist. According to Zhang and Berardi (1997) the power provides a performance measure of the capability of a control chart to detect shifts that are undesirable. "They [control charts] are also effective devices for estimating process parameters, particularly process-capability studies" (Al-Oraini and Rahim; 2002). Upton and Cook (2004) define process capability analysis as a method used to determine the degree to which the long term performance of an industrial production process complies with the requirements and goals set out by the engineers and managers of the process. Montgomery (2005; 326:327) notes that the control chart is a simple and effective method to examine the process capability. The $\bar{X}$ and R charts provide information about the capability of the system. The process capability ratio (PCR), denoted by C_p, can be used to express the process capability for a quality characteristic with both upper and lower specification limits. The C_p is given by:

$$C_p = \frac{UCL - LCL}{6\sigma}.$$

The 6σ spread of the process gives the basic definition of process capability. According to Montgomery (2005; 203) this implies that the natural tolerance limits in the system, namely 3σ above and below the mean, are inside the lower and upper control limits. Accordingly, only a small number of non-conforming units will be produced. The problem that arises from this definition is that σ is seldom known. We replace σ, by an estimate $\hat{\sigma} = \overline{\mathrm{R}}/\mathrm{d}_2$, resulting in an estimate $\hat{C}_p$ of C_p. An alternative way of interpreting the C_p is given by the quantity:

$$P = \left(\frac{1}{C_p}\right) 100\%.$$

This is the percentage of the specification band that is used up by the process. An estimate for P is given by $\hat{P}$. This quantity only gives an indication of whether the variation falls within the specified limits or not, no conclusions can be drawn regarding the locality of the mean.

Process capability analysis can be a useful tool throughout the manufacturing cycle, this analysis includes "quantifying the process capability, ... analyzing this variability relative to product requirements or specifications, and ... assisting development and manufacturing in eliminating or greatly reducing this variability" Montgomery (2005: 327). Process capability defines the uniformity of the process. The term "uniformity" refers to the measure of the consistency of the production output. Some of the major uses of process capability analysis as given by Montgomery (2005; 328) are:

1. The prediction of how well the system will hold the tolerances.
2. To assist the product developers and designers in the selection and modification of the system.
3. To assist in the establishment of an interval between sampling for the monitoring of the process.

4. The specification of performance requirement for new equipment installed in the system.
5. To enable management to make decisions regarding the selection of suppliers and other aspects of supply chain management.
6. The planning of the sequence of production systems when an interactive effect of processes on tolerances is present.
7. The reduction of variability in the process.

Once the process capability has been determined, the control chart can be applied to the process. Shewhart $\bar{X}$-control charts are used for monitoring the process mean and R-charts are used for controlling the process variance when a single quality characteristic of the product is monitored (Yang and Rahim; 2005). However, when more than one quality characteristic is of interest, as stated by Kapur and Cho (1996), a natural extension from the univariate Shewhart $\bar{X}$-control chart is the Hotelling multivariate control chart (Hotelling; 1947). This is often the case as products may have more than one quality characteristic. Chou et al. (2003) provide an example where two or more measurable characteristics are of importance – in the production of synthetic fiber, both the tensile strength and the diameter are equally important quality characteristics. Kapur and Cho (1996) give an example of a situation where the overall quality of a metal cutting tool includes a number of qualities such as the cutting force, cutting speed, and metal removal rate.

Tang and Tang (1989) assumed that the quality characteristics are independent of each other. In this case, Raiman and Case (1990) argue that the total loss can be calculated by adding the losses associated with each quality characteristic. In real life, these quality characteristics may however be dependent. Noorossana, Woodall and Amiriparian (2002) consider the correlation between the multiple characteristics and argue the importance of not ignoring these correlations in practice. "These characteristics are jointly distributed random variables and cannot appropriately be controlled by independently applying a control chart to each variable" (Chou et al.; 2003). Using separate univariate control chart procedures can be inefficient and possibly misleading.

In 1947 Hotelling developed the Hotelling T^2-control charts to monitor the process mean vector in the multivariate case. In 1972 Montgomery and Klatt were the first to develop the Hotelling T^2-control chart from an economic perspective. Montgomery (2005) presents two versions of the Hotelling T^2-chart: the first for sub grouped data, and the second, for individual observations.

For sub grouped data, it is supposed that two quality characteristics, X_1 and X_2, are jointly distributed according to a bivariate normal distribution. The mean values of the quality characteristics are denoted by μ_1 and μ_2, and the standard deviations by σ_1 and σ_2, respectively. The covariance between the two quality characteristics, X_1 and X_2, is denoted by σ_{12}. The standard deviations and covariance are assumed to be known. Now, the sample means, $\bar{x}_1$ and $\bar{x}_2$ can be calculated for a sample size n, then the test statistic

$$\chi_0^2 = \frac{n}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}[\sigma_2^2(\bar{x}_1 - \mu_1)^2 + \sigma_1^2(\bar{x}_2 - \mu_2)^2 - 2\sigma_{12}(\bar{x}_1 - \mu_1)(\bar{x}_2 - \mu_2)]$$

will come from a χ^2-distribution with 2 degrees of freedom. When the process is functioning in the in-control state, i.e. if the process means remain at μ_1 and μ_2, the values of the statistic χ_0^2 should be smaller than the UCL, where the UCL $= \chi_{\alpha,2}^2$. If one of the process means shifts to a new value (this value is assumed to be out of control), then the probability that χ_0^2 exceeds UCL $= \chi_{\alpha,2}^2$ will increase.

Montgomery (2005: 494) extends these results to the case where k related quality characteristics are controlled. The assumption is made that the joint probability of the k quality characteristics is the k-variate normal distribution. Now the sample means for each of the k quality characteristics needs to be computed, using a sample size n. The sample means are represented by a k x 1 vector as follows:

$$\overline{\boldsymbol{x}} = \begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \dots \\ \bar{x}_k \end{Bmatrix}.$$

The test statistic used to control the process is extended to:

$$\chi_0^2 = n(\overline{\boldsymbol{x}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\overline{\boldsymbol{x}} - \boldsymbol{\mu})$$

where $\boldsymbol{\mu}' = [\mu_1, \mu_2, \dots, \mu_k]$ is the vector of the means when the process is functioning in the in-control state and $\boldsymbol{\Sigma}$ is the corresponding covariance matrix. The UCL of the control chart is given by $\chi^2_{\alpha,k}$. In practice, the estimation of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ is obtained from the analysis of preliminary samples of size n. The vector $\bar{\bar{\boldsymbol{x}}}$ is used as an estimate for the in-control mean values for the process, and $\boldsymbol{S}$ is used to estimate $\boldsymbol{\Sigma}$. The test statistic now becomes

$$T^2 = n(\bar{\boldsymbol{x}} - \bar{\bar{\boldsymbol{x}}})'\boldsymbol{S}^{-1}(\bar{\boldsymbol{x}} - \bar{\bar{\boldsymbol{x}}}).$$

When used in this form, the procedure is called the Hotelling T^2-control chart. The selection of the control limits for the Hotelling T^2-control chart is dependent on how the chart is being utilised (Montgomery; 2005: 496). A control chart can be used to establish the state of control for the process, or it can be used to monitor future production. For the first phase, the control limits for the T^2-control chart are given by:

$$UCL = \frac{k(m-1)(n-1)}{mn-m-k+1}F_{\alpha,k,mn-m-k+1}$$

$$LCL = 0.$$

where m is the number of preliminary samples available for the estimation of $\bar{\bar{\boldsymbol{x}}}$ and $\boldsymbol{S}$. When the control chart is used for monitoring production in the future, the control limits are given by:

$$UCL = \frac{k(m+1)(n-1)}{mn-m-k+1}F_{\alpha,k,mn-m-k+1}$$

$$LCL = 0.$$

In many industrial settings, specifically in the chemical and process industries, the subgroup size is naturally $n = 1$. Suppose that m samples on individual observations ($n = 1$) are available and that k quality characteristics are observed in each sample. In this case, the Hotelling T^2 test statistic becomes:

$$T^2 = n(x - \overline{x})'S^{-1}(x - \overline{x}).$$

Montgomery (2005: 501) argues that the control limits used to control future production should be as follows:

$$UCL = \frac{k(m+1)(m-1)}{m^2 - mk} F_{\alpha,k,m-k}$$

$$LCL = 0.$$

In order to apply a univariate or multivariate control chart to monitor a process the following parameters must be specified by the user:

- Sample size, n
- Sampling interval, h
- Control limits, L (the number of standard deviations above or below the centre line)

The process of selecting these parameters is known as the design of the control chart. In the design of the Shewhart $\bar{X}$-control chart all three these parameters are assumed to be known. (Yang and Rahim; 2005)

"The control limits of an $\bar{X}$-control chart are set at $\pm L$ standard deviations off the target mean." (Banerjee and Rahim; 1988). At an interval of every h hours a sample of size n is taken from the process. The mean obtained from the sample is then plotted on the control chart. The occurrence of an assignable cause will lead to a shift in the process mean from μ_0 to $\mu_0 \pm \delta\sigma$, where

- μ_0 is the process mean,
- σ is the process standard deviation, and
- δ is the shift parameter.

Once the sample mean is plotted outside these control limits, the process is deemed to be out of control. When an out-of-control state is signalled by the control chart, corrective action is required in order for the process to return to an in-control state.

Typical types of control charts used:

1. $\bar{X}$-charts - used to control a continuous process.
2. p-charts - used to control a Bernoulli process.
3. u-charts - used to control the number of defects per unit.
4. R-charts – used to control the natural variation.

For these basic control charts some rules of thumb have been developed for the selection of the parameters n, h and L. Some of the proposed designs for the $\bar{X}$ chart are: n = 5, h = 8, L = 3 (Ishikawa; 1976); n = 5, h = 1, L = 3 (Feigenbaum; 1961). For the p chart, some of the proposed designs are: n > 50 with 3 < $n\bar{p}$ < 4, h = 8, L = 3 (Ishikawa; 1976); n = 25, h = 1 or 8, L = 3 (Feigenbaum; 1961). Parameter selection proposed for the u chart: n = 2 or 3, h = 8, L = 3 (Ishikawa; 1976). From the proposals above it is clear that with the exception of setting L at 3, there seems to be little consensus on the choice of parameters. The question that is raised is: How should the design parameters of a control chart be chosen?

Duncan (1956) was the first to attempt to answer this question, stating that general guidelines should be replaced by a process specific design. He argues that the most natural criterion is a design that minimises the net sum of all the costs involved. Banerjee and Rahim (1988) define an optimal process as follows: "The objective is to determine these parameters to minimise the expected total cost per unit time." Duncan (1956) developed the economic design of an $\bar{X}$-control chart to control normal process means.

In 1973 Ladany developed the economic design of the p-chart. To date, however, the economic design of the u-chart has not been published. Lorenzen and Vance (1986) argue that the application of these economic designs to different charts become trivial. They present a general method for determining the economic design of control charts. A method is developed that applies to all control charts, irrespective of the statistic used. Lorenzen and Vance (1986) argue that it is only necessary to compute the ARL of the statistic under the assumption that the process is in control and also that the process is out of control in some specified fashion. These assumptions are valid when the statistics plotted are independent. A number of problems have arisen in the development of economic designs. The first being that papers on the economic design of control charts had not been developed in a systematic way. Secondly, a wide range of methods existed regarding the continuation of production during search and repair times, as well as whether income was to be maximised or cost minimised. Finally, as mentioned above, techniques for all control charts had not been developed. Lorenzen and Vance (1986) attempted to unify the notation that had been used to date.

In the next chapter the economic design of control charts will be discussed in greater detail.

CHAPTER THREE: *THE ECONOMIC DESIGN OF CONTROL CHARTS*

3.1 INTRODUCTION:

The heuristic design proposed by Shewhart (1931) became very popular, but according to Yang and Rahim (2005) this design was not economically optimal. In 1956 Duncan was the first to propose an economic design of the $\bar{X}$-control chart. Duncan (1956) proposes an optimal economic design of $\bar{X}$-control charts for the signal occurrence of an assignable cause. The proposed cost model includes:

- the cost of sampling and inspection,
- the cost of defective products,
- the cost of false alarms,
- the cost of searching for the assignable cause, and
- the cost of corrections to the system.

The assumption of a single assignable cause may not be suitable for a production process which is affected by two or more assignable causes. Several assignable causes could occur in a discrete part manufacturing system. Some examples are improper machine adjustments, operating errors, or flawed raw materials, etc.

Duncan (1971) generalises his original economic model to the situation in which there are *s* assignable causes, with different causes shifting the mean by different amounts. He assumes once an assignable cause occurs, the system continues to function in the out-of-control state and no additional assignable cause will occur until the out-of-control state is detected. According to Chen and Yang (2002) the Weibull distribution can be used to reproduce various situations by varying its shape and scale parameters. They extended the time of occurrence of assignable causes in Duncan's multiplicity-cause model from the exponential distribution to the Weibull distribution. Results obtained by Chen and Yang (2002) show that the proposed multiplicity-cause model has smaller lost-cost value than the single-cause model proposed by Banerjee and Rahim (1988).

3.2 THE ECONOMIC COSTS:

According to Saniga (1989) the objective in the economic design of control charts is to find the sample size, control limits width and sampling frequency that minimises the loss in profit that accrues to the company due to the production of poor quality products. The loss in profit is associated with the costs of producing products that do not lie within the specification limits, the costs of detecting an assignable cause responsible for the poor quality products, the costs related to false alarms, and the costs of seeking and removing the assignable cause.

Lorenzen and Vance (1986) define a cost cycle as the time between the beginnings of successive periods where the process is in-control. The cost associated with the in-control state includes the cost of sampling the process, the cost of producing non-conforming units, and costs arising from false alarms. Lorenzen and Vance (1986) assume that an out-of-control process cannot return to the in-control state without intervention. The costs associated with the out-of-control state are represented by costs due to sampling, an increased number of non-conforming units produced, searching for the assignable cause, as well as repairing costs and possible downtime (if production is ceased during the search for an assignable cause).

The cycle time is defined to be the sum of (i) the time until the next assignable cause occurs, (ii) the time until the next sample is taken, (iii) the time to analyse the sample and chart the obtained results, (iv) the time until the chart gives an out-of-control signal, and (v) the time it takes to discover the assignable cause and to repair the process. Note that, according to this description, the repair time is defined to be equal to zero.

Duncan (1956) attempted to control the normal process means under the assumption that the occurrence time of a single assignable cause followed an exponential distribution. The designs of exponential charts proposed by Chan, Xie and Goh (2000) and Xie, Goh and Ranjan (2002) were based on pure statistical considerations. Zhang, Xie and Gho (2005) noted that "the implementation of an exponential chart has significant economic impact as it involves various costs, such as the cost incurred by the occurrence of the event, cost of false alarms, cost of locating and repairing the assignable cause and cost of allowing the system to operate in an out-of-control state." It therefore seems reasonable to take the economic concerns into account when designing control charts. The economic objectives are often considered to be the most important considerations for the company.

3.3 THE DEVELOPMENT OF THE ECONOMIC DESIGN:

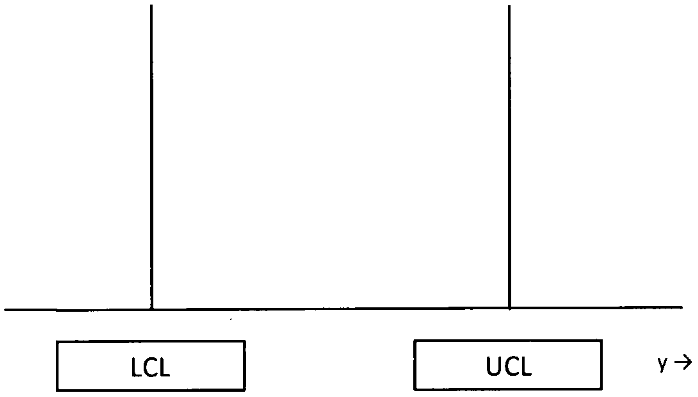
Some more recent developments in the area of the economic design of control charts include the application of the economic design approach to adaptive control charts (Costa and Rahim; 2001, De Magalhaes, Costa and Epprecht; 2002, Ohta, Kimura and Rahim; 2002, Park, Lee and Kim; 2004), to control charts for correlated and/or non-normal data (Chou, Chen and Liu; 2000,2001), to multivariate control charts (Kapur and Cho; 1996, Molnau, Montgomery and Runger; 2001, Chou, Liu, Chen and Huang; 2002, Noorossana, Woodall and Amiriparian; 2002).

Kapur and Cho (1996) argue that the process mean can be approximately adjusted to the target value, but that it is more complicated to effect the variances and covariances. They propose to enhance the quality of such a system by developing and executing a specification region for the process and truncating the distribution of the quality characteristics by inspection based on the specification region.

Customers typically evaluate a product on the basis of a number of quality characteristics. The relationship between these quality characteristics is often dependent and therefore the total loss is not equal to the sum of the losses caused by each characteristic. The multivariate model developed by Kapur and Cho (1996) is motivated by these factors mentioned above. Figure 3.1 illustrates the specification limits for a single quality characteristic.

Figure 3.1.

Specification limits for a single quality characteristic.

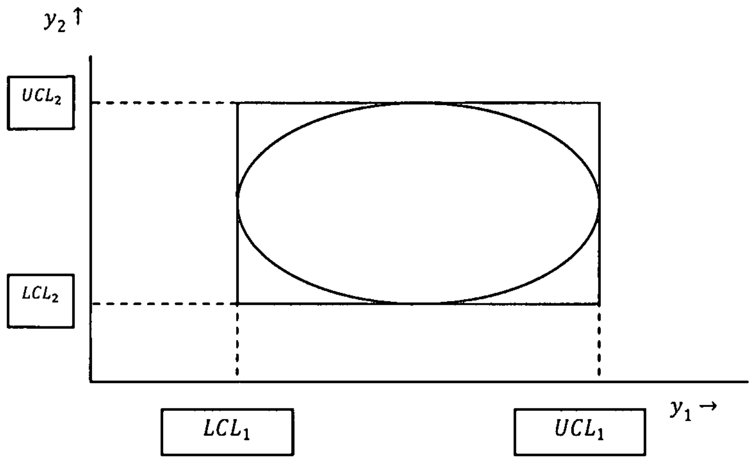


Source: Kapur and Cho (1996)

For multiple quality characteristics, the specification region is the region intersected jointly by the control limits for each quality characteristic. Kapur and Cho (1996) truncate the multivariate distribution of the characteristics based on the specification region. In the study they consider the situation where all items are inspected with no inspection error. This is a perfect situation and implies that all the items distributed to the customers fall between the control limits. Let LCL_i and UCL_i be the LCL and UCL for quality characteristic i, where $i = 1, \ldots, m$. If two quality characteristics are considered, the specification region can be formed by two sets of control limits, as shown in Figure 3.2, for $\sigma_{12} = 0$.

Figure 3.2.

Specification region for two quality characteristics (where $\sigma_{12} = 0$).

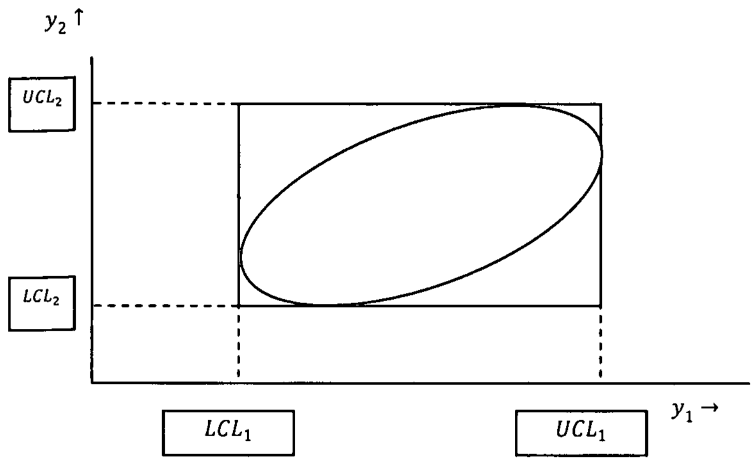


Source: Montgomery (2005)

The specification limits for multiple quality characteristics is given in Figure 3.3, for $\sigma_{12} \neq 0$.

Figure 3.3.

Specification region for two quality characteristics (where $\sigma_{12} \neq 0$).



Source: Montgomery (2005)

Zhang, Xie and Goh (2005) first attempted to apply the economic design approach to the exponential chart. They consider a system in which the occurrence of some discrete event that is of interest can be modeled by a homogenous Poisson process – where an event is defined by Zhang et al. (2005) to be the occurrence of something indicative in a process, such as the production of a non-conforming item.

For a process that is operating in the in-control state, the rate of occurrence of the event of interest – such as the production of a non-conforming item – will be lower. Once the process moves to the out-of-control state, due to some assignable cause, the event of interest will occur at a higher rate. Zhang et al. (2005) assume that there is a single assignable cause that could occur in the process. They further also assume that the occurrence of the assignable cause follows a homogeneous Poisson distribution. The rate of occurrence of the assignable cause however, is much lower than the rate of occurrence of the event.

3.4 THE ECONOMIC COST MODEL:

For the development of the economic model for the design of exponential charts, the following notation and definitions are given by Zhang et al. (2005):

Event: the occurrence of something indicative in the system, such as the production of a non-conforming item or a defect.

T_L: the lower control limit of the control chart.

T_U: the upper control limit of the control chart.

M: expected number of observed events before the assignable cause occurs.

N: expected number of false alarms.

G: expected number of events observed from the occurrence of the assignable cause until the time an out-of-control is signalled.

L: expected length of an operational cycle.

P: expected profit from an operational cycle.

I: expected profit per hour in an operational cycle.

ATS_0: the in-control average time to signal; defined as the average time between two consecutive alarms when the process is operating in the in-control state. An alarm is defined to be a warning signal generated by a control chart for possible shifts occurring in the system.

ATS_1: the out-of-control average time to signal; defined as the average time from the occurrence of the assignable cause until the control chart signals an out-of-control state.

α: probability of a Type I error.

β: probability of a Type II error.

λ_0: the rate of occurrence of the event when the process is operating in-control.

λ_1: the rate of occurrence of the event when the process is operating out-of-control.

λ_a: the rate of occurrence of the assignable cause ($\lambda_a \ll \lambda_0 < \lambda_1$).

V_0: average profit per hour when the process is operating in-control.

V_1: average profit per hour when the process is operating out-of-control.

c: average cost associated with one observed event.

A_0: average cost associated with one false alarm.

A_1: average cost associated with locating and removing an assignable cause.

t_5: expected time to locate and remove an assignable cause.

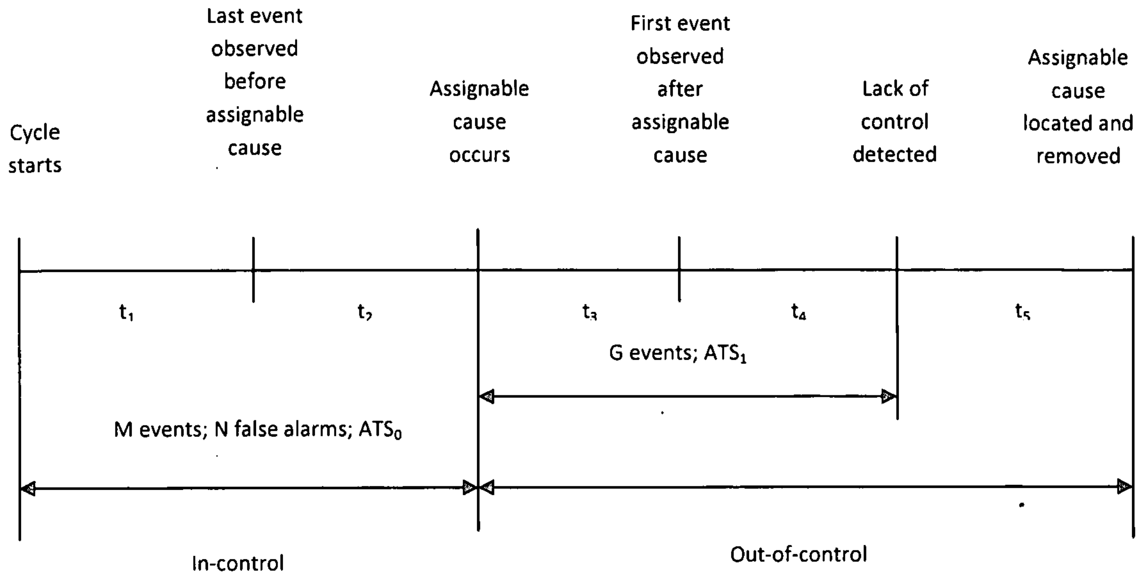
G_0: goal value imposed on ATS_0 in the economic-statistical model.

G_1: goal value imposed on ATS_1 in the economic-statistical model.

For the economic model, Zhang et al. (2005) define the operational cycle of a process as follows: "…the time period from the startup or re-instatement of the system until the location and removal of an assignable cause." A diagram of an operational cycle is given in Figure 3.4 below. From Figure 3.4, the five time components t_1, t_2, t_3, t_4 and t_5 of the operational cycle are clear.

Figure 3.4.

Diagram of an operational cycle.



Source: Zhang et al. (2005)

In developing the model, Zhang et al. (2005) make the following assumptions:

1. The process is assumed to start from an in-control state.
2. An assignable cause occurs leading to a shift of the system from an in-control state to an out-of-control state. The occurrence of this assignable cause can then be modeled by a homogenous Poisson process.
3. The process continues to operate during the search for possible assignable causes. This includes the search period during a false alarm and for a true assignable cause.

4. The occurrence of an event can be modeled by a homogenous Poisson process with a rate of occurrence of λ_0 when the process is operating in an in-control state and λ_1 when the process is operating in the out-of-control state.

From a practical point of view, the LCL of the control chart is relatively more important (Zhang et al.; 2005). Consequently, in the study done by Zhang et al. (2005) only T_L is considered. This simplifies the design of an exponential design chart considerably to the determination of T_L, which has a one-to-one correspondence to the probability of a false alarm, namely, α. This correspondence is described by the following expression:

$$\alpha = 1 - e^{-\lambda_0 T_L} \text{ or } T_L = -\frac{\ln(1-\alpha)}{\lambda_0}.$$

Under the assumption that the assignable cause occurs between the i^{th} and the $(i+1)^{\text{th}}$ event, then, similar to Duncan (1956), the expected time of occurrence of the assignable cause, t_2, is:

$$t_2 \approx \frac{\int_{i/\lambda_0}^{(i+1)/\lambda_0} \lambda_a e^{-\lambda_a y}(y - {}^{i}/_{\lambda_0})dy}{\int_{i/\lambda_0}^{(i+1)/\lambda_0} \lambda_a e^{-\lambda_a y} dy} = \frac{1}{\lambda_a} - \frac{1}{\lambda_0 (e^{\frac{\lambda_a}{\lambda_0}} - 1)},$$

independent of i.

From Figure 3.4, it can be seen that

$$t_1 + t_2 = \frac{M}{\lambda_0} + t_2 = \frac{1}{\lambda_a}.$$

Combining the above equations gives:

$$M = \left(\frac{1}{\lambda_a} - t_2\right)\lambda_0 = \frac{1}{e^{\lambda_a/\lambda_0 - 1}}.$$

The expected number of false alarms is now given as:

$$N = \alpha M = \frac{\alpha}{e^{\lambda_a/\lambda_0} - 1} = \frac{1 - e^{\lambda_0 T_L}}{e^{\lambda_a/\lambda_0} - 1}.$$

And ATS_0 is computed as:

$$ATS_0 = \frac{1}{\alpha}E(T) = \frac{1}{\alpha\lambda_0}.$$

Zhang et al. (2005) explain this expression for ATS_0 in the following manner: When a process is operating in the in-control state, on average, for every $1/\alpha$ events that are observed, there will be one event that will plot below the LCL. A false alarm will thus be signalled. If the process is operating in control, the mean of T equals $1/\lambda_0$. Once the assignable cause occurs; it is assumed that the occurrence of the event follows a homogenous Poisson distribution with parameter λ_1. It is known, due to the memoryless characteristic of the exponential distribution, that $t_3 = 1/\lambda_1$. Now let G be the expected number of events observed between the occurrence of the assignable cause and the detection of the out-of-control state, then

$$t_4 = \frac{G-1}{\lambda_1},$$

and ATS_1, the out-of-control average time to signal, is given by:

$$ATS_1 = \frac{G}{\lambda_1}.$$

Zhang et al. (2005) assume that the expected time that it takes to locate and then remove the cause of the shift, i.e. t_5, is known. Let the random variable T_a refer to the time the assignable cause occurs, which is assumed to follow an exponential distribution with parameter λ_a. Similarly, let the random variable T_2 denote the time from the last event before the assignable cause occurred, to the incidence of the assignable cause. Let T_3 represent the time from the occurrence of the assignable cause to the observation of the first event after the occurrence of the assignable cause (refer to Figure 3.4).

Further Zhang et al. (2005) argue that due to the memoryless characteristic of the exponential distribution, T_3 follows an exponential distribution with parameter λ_1. They let T_x be the first sample time between the events observed after the incidence of the assignable cause, i.e. $T_x = T_2 + T_3$. The distribution of T_2 cannot be derived exactly due to the fact that it is a randomly truncated exponential random variable. Zhang et al. (2005) give approximate formulas for these derivations:

$$Pr\{T_2 \leq t\} = \sum_{i=0}^{\infty} Pr\left\{T_2 \leq t, \frac{i}{\lambda_0} \leq T_a \leq \frac{(i+1)}{\lambda_0}\right\}$$

$$\approx \sum_{i=0}^{\infty} Pr\left\{\frac{i}{\lambda_0} \leq T_a \leq \left(\frac{i}{\lambda_0} + t\right)\right\}$$

$$= \sum_{i=1}^{\infty} \left(e^{-i\lambda_a/\lambda_0} - e^{-i\lambda_a/\lambda_0 - \lambda_a t}\right) = \frac{1-e^{-\lambda_a t}}{1-e^{-\lambda_a/\lambda_0}}.$$

$$Pr\{T_x \le x\} = Pr\{T_2 + T_3 \le x\} = \int_0^x Pr\{T_2 + T_3 \le x | T_3 = t\}\lambda_1 e^{-\lambda_1 t} dt$$

$$= \int_0^x Pr\{T_2 \le x - t\}\lambda_1 e^{-\lambda_1 t} dt \approx \int_0^x \frac{1 - e^{-\lambda_a (x-t)}}{1 - e^{-\lambda_a/\lambda_0}} \lambda_1 e^{-\lambda_1 t} dt$$

$$= \frac{\lambda_a(1-e^{-\lambda_1 x}) - \lambda_1(1-e^{-\lambda_a x})}{(\lambda_a - \lambda_1)\left(1-e^{-\frac{\lambda_a}{\lambda_0}}\right)}.$$

Let p_x be a power of T_x, i.e.,

$$p_x = Pr\{T_x \le T_L\} \approx \frac{\lambda_a(1-e^{-\lambda_1 T_L}) - \lambda_1(1-e^{-\lambda_a T_L})}{(\lambda_a - \lambda_1)(1-e^{-\frac{\lambda_a}{\lambda_0}})}.$$

Once the first sample has been taken, the power of each subsequent sample, represented by $(1-\beta)$, is:

$$1 - \beta = 1 - e^{-\lambda_1 T_L}.$$

The expected number of events observed from the incidence of the assignable cause to the detection of an out-of-control state, i.e. G, is calculated as

$$G = p_x + (1 - p_x)\,[2(1-\beta) + 3(1-\beta)\beta + 4(1-\beta)\beta^2 + \cdots]$$

$$= p_x + (1 - p_x)\left(1 + \frac{1}{1 - e^{-\lambda_1 T_L}}\right)$$

$$\approx 1 + \frac{\lambda_a e^{-\lambda_1 T_L} - \lambda_1 e^{-\lambda_a T_L} - (\lambda_a - \lambda_1)e^{-\lambda_a/\lambda_0}}{(\lambda_a - \lambda_1)(1-e^{-\frac{\lambda_a}{\lambda_0}})(1-e^{-\lambda_1 T_L})}.$$

The expected length of an operational cycle, L, equals

$$\frac{1}{\lambda_a} + t_3 + t_4 + t_5 = \frac{1}{\lambda_a} + \frac{G}{\lambda_1} + t_5.$$

The expected profit from an operational cycle, P, equals

$$\frac{V_0}{\lambda_a} + V_1\left[\frac{G}{\lambda_1} + t_5\right] - A_0\left[\frac{1-e^{-\lambda_0 T_L}}{e^{\lambda_a/\lambda_0}-1}\right] - A_1 - c\left[\frac{1}{e^{\frac{\lambda_a}{\lambda_0}}-1} + G\right].$$

According to Zhang et al. (2005) the expected profit per hour in an operational cycle can be expressed as the ratio of the expected profit to the expected length of the cycle, i.e. $I = P/L$. When a pure economic design is considered, the expected profit per hour I, is maximised as below, while for an economic statistical design, two constraints are imposed on this statistical performance of the exponential chart.

$$I = P/L$$

$$ATS_0 \geq G_0$$

$$ATS_0 \leq G_1.$$

where G_0 and G_1 are the goal values imposed on ATS_0 and ATS_1, respectively. The economic-statistical design of control charts will be discussed in more detail in Chapter 5.

3.5 A PRODUCTION SYSTEM WITH INCREASING FAILURE RATE AND EARLY REPLACEMENT:

Rahim and Banerjee (1993) present an extension and generalisation of the model developed by Banerjee and Rahim (1988). They consider a general distribution of the in-control period and introduce the concept of the salvage value of the equipment. The proposed model permits the age-dependent replacement of equipment before failure occurs. Rahim and Banerjee (1993) claim that the replacement of a component before failure is only meaningful when such a replacement leads to some economic benefit. Instinctively, the residual life of a component beyond a certain age for a process involving increasing hazard rate shock models will be fairly short.

Rahim and Banerjee (1993) suggest that more frequent sampling could possibly be necessary after a process reaches a given age. Due to more frequent sampling of the process, an increase in the operational cost will be expected. This increase in operational cost could possibly be countered by the termination of the process at some time beyond the given age. A truncated production cycle is defined by Rahim and Banerjee (1993) to be "a production cycle which terminates after the detection of a failure or at a certain prespecified age, whichever comes first."

The Markovian shock model does not allow for the question of replacement before failure due to its memoryless property.

The model proposed by Rahim and Banerjee (1993) differs from Duncan's original economic model in the following:

1. In Rahim and Banerjee's model, the duration of the in-control period is assumed to follow a random probability density, $f(t)$, having an increasing hazard rate, $r(t)$, and $F(T)$ as the cumulative distribution function. Duncan assumes an exponential probability distribution, leading to a constant hazard rate for all t.

2. Random samples of size n are drawn from the process at times h_1, $(h_1 + h_2)$, $(h_1 + h_2 + h_3), \ldots$ in order to monitor the process (As introduced by Banerjee and Rahim (1988)). Additionally, h_j satisfies the following conditions: (i) $h_1 \geq h_2 \geq h_3, \ldots$, and (ii) $\lim_{m\to\infty} F(\omega_m) = 1$, where $\omega_m = \sum_{j=1}^{m} h_j$. This feature differs from Duncan's model where he considered $h_j = h$ for all $j = 1,2, \ldots m$.
3. The proposed model is developed under the assumption that a production cycle ends either with a true alarm or at a time ω_m, whichever occurs first. A process that has not signalled a true alarm at time ω_{m-1}will be allowed to continue for an additional time h_m. The process is stopped at time ω_m and the old component is replaced by a new one. Based on this assumption there is no cost incurred during the m^{th} sampling interval. Rahim and Banerjee (1993) introduce m as a design parameter along with n, h and L. For Duncan's model m is assumed to be infinity and the production cycle will only be stopped by a true alarm.
4. Rahim and Banerjee (1993) assume, for simplicity of mathematical calculations, that the production process ceases during search and repair.

A truncated production cycle is defined by Rahim and Banerjee (1993) in the following way: It starts with the installation of a new component and ends with a repair to the process, or after a prespecified number of sampling intervals, m, whichever occurs first. A renewal occurs at the end of every truncated production cycle.

The expected cycle time, $E(T)$, consists of the following periods: (i) in-control time during production which includes stoppages for false alarms when the process is in fact in control, (ii) the time between the shift to out of control and the first sample point that plots outside the control limits, and (iii) the time to search for an assignable cause and repair the process.

The expected cycle cost, $E(C)$, is made up by the following: (i) the cost associated with the production of non-conformaties in the in-control as well as the out-of-control states, (ii) the costs associated with false alarms, including search costs and the cost of downtime if production is stopped during the search, (iii) the cost associated with the location of the assignable cause and the repair of the process, including the cost of downtime, (iv) the cost associated with sampling and testing, and minus (v) the salvage value for the working machine of age x.

In addition to these assumptions and notations, they use the following notations to obtain the main results: Z_0: the expected search time associated with a false alarm; Z_1: the expected search time to discover the assignable cause and repair the process; a: the fixed sampling cost; b: the sampling cost per unit sampled; Y: the cost per false alarm; W: the cost to locate and repair the assignable cause; D_0: the quality cost per hour while production is in control; D_1: the quality cost per hour while producing out of control; α: the Pr{exceeding the control limits| the process is in control}; β: Pr{not exceeding the control limits| the process is out of control}; $\omega_j = \sum_{i=1}^{j} h_i$, j = 1,2, ..., m; $\omega_0 = 0$; $\nabla F\left(\omega_j\right) = F\left(\omega_j\right) - F\left(\omega_{j-1}\right)$, j = 1,2, ... , m; $F\left(\omega_j\right) = 1 - F\left(\omega_j\right)$; $S(x)$: the salvage value of for a working equipment of age x; ECT: the expected cost per cycle.

3.5.1 THE EXPECTED CYCLE LENGTH, THE EXPECTED CYCLE COST AND THE INSPECTION INTERVALS:

The expressions for $E(C)$ and $E(T)$ are derived by Rahim and Banerjee (1993) under the assumption that $m \geq 2$, since the case where $m = 1$ is trivial. The following theorem is needed:

Under the assumptions 1, 2 and 3 as stated in Section 3.5 above, the following is true:

$$E(T) = \sum_{j=1}^{m} h_j \bar{F}\left(\omega_{j-1}\right) + \alpha Z_0 \sum_{j=1}^{m-1} \bar{F}\left(\omega_j\right) + \beta \sum_{j=1}^{m-1} \nabla F\left(\omega_j\right) \sum_{i=j+1}^{m} h_i \beta^{i-j-1} + Z_1.$$

The expected cycle is defined as the expected time for inspection intervals when the process is functioning in the in-control state, the expected time for searching during false alarms and the expected time for detecting an assignable cause and the repair time. Also

$$E(C) = D_0 \sum_{j=1}^{m} h_j \bar{F}\left(\omega_{j-1}\right) + \alpha Y \sum_{j=1}^{m-1} \bar{F}\left(\omega_j\right) + (D_0 + D_1) \int_0^{\omega_m} x f(x) dx +$$
$$(D_1 - D_0) \sum_{j=1}^{m} \omega_j \nabla F\left(\omega_j\right) + D_1 \beta \left[\sum_{j=1}^{m-1} \nabla F\left(\omega_j\right) \sum_{i=j+1}^{m} h_i \beta^{i-j-1}\right] + (a + bn)\left[1 + \sum_{j=1}^{m-2} \bar{F}\left(\omega_j\right) +\right.$$
$$\left.\beta \sum_{j=1}^{m-2} \nabla F\left(\omega_j\right)\left\{(1-\beta) \sum_{i=1}^{m-1-j} i \beta^{i-1} + (m-1-j)\beta^{m-1-j}\right\}\right] + W - \bar{F}(\omega_m) S(\omega_m).$$

The expected cost per cycle can be expressed as the expected cost of operating while in control with no alarm, the expected cost associated with false alarms, the expected cost of operating while the process was initially in control and then went out of control, therefore triggering a true alarm, the expected cost of sampling and the repair cost, minus the salvage value for a working piece of equipment of age ω_m.

The proof of this theorem is presented in the Appendix A. Rahim and Banerjee (1993) continue to derive the optimal decision variables n, h_j, L with j = 1,2, ... , m, by minimising the ratio $E(C)/E(T)$. They considered several truncated and nontruncated schemes. The derivation of $E(T)$ and $E(C)$ for the exponential and Gamma distributions are also given in Appendix A. The numerical examples given in their paper address the proposal that it could potentially be economically beneficial to retire a component or machine prior to its failure.

The development of the statistical design of control charts will be discussed in the following chapter.

CHAPTER FOUR: *THE STATISTICAL DESIGN OF CONTROL CHARTS*

A statistical limit of the heuristic control charts is represented by their poor ability to detect out-of-control conditions when small shifts in the mean of the parameter of interest occur. Celano, Costa and Fichera (2006) suggest a possible to improve the statistical behaviour of the control chart by restricting the control interval. This solution does tend to lead to an excessive number of false alarms, which could in turn lead to a too large number of interventions.

According to Al-Oraini and Rahim (2002), similarly, a major limitation of the economic design is that the rate of the Type I error may be too high for many situations and will lead to a large number of false alarms. In addition to the economic design of control charts, another approach to the design of control charts is known as the statistical design. Control charts designed under this second approach place constraints on the control limits, which in turn determine the probability of the Type I error, α, and the power, $1-\beta$. These constraints are established in advance and are then used to determine the sample size, and if the average time to signal (ATS) is also specified, the sampling interval. The economic design of control charts is developed with the goal of minimising the expected total cost. Although the economic design does guarantee the lowest cost, it typically performs poorly because the statistical properties are ignored. Alternatively, ignoring the statistical properties of a control chart may result into too many defective products (Chen and Cheng; 2007)

It is known that the optimal economic design is very sensitive to the size of the shift in the mean, $\delta > 0$, which is expected to occur when the process shifts to the out-of-control state. Woodall (1985) notes that the performance of the control chart may not be suitable if δ is not close to δ_1 - the smallest shift considered to be significant. In some instances the statistical design of control charts may be more appropriate than the economic design, for example "if the time interval between samples is predetermined and any shift $\delta > \delta_1$ is to be detected regardless of its frequency of occurrence" (Woodall; 1985). An additional advantage of the statistical design cited by Woodall (1985) stems from fact that economically designed models consider the total cost associated with false alarms to be proportional to the number of false alarms.

Economic designs could cause a very low in-control average run length (ARL) as the calculated cost does not take into deliberation the fact that a disproportionate number of false alarms introduces further variability into the system and undermines confidence in the control procedure. Woodall (1986) argues that these effects are not accounted for by the economic model, since the economic model assumes the total cost of false alarms is proportional to the number of false alarms.

The cost calculations that prove the optimality of a control procedure obtained from the economic design models are misleading according to Woodall (1986), since it is likely that the control chart will be ignored in practice due to the unreasonable number of false alarms that occur. Economic models balance the cost associated with sampling and repairing the process with the cost of producing poor quality products. Woodall (1986) claims that for this reason the economic design often is unsuccessful in producing charts with the ability to detect small shifts in quality quickly before considerable losses arise. He continues to argue that the sensitivity of an economic design to a range of shifts in quality depends a great deal on the specified expected shifts. These shifts in the process often correspond to a sizeable profit loss and are therefore larger than the smallest shift that should be considered to be significant enough to be detected quickly.

Woodall (1986) suggests that if small shifts could be detected quickly, it might avoid the potential large loss of profit associated with the process functioning in the out-of-control state. He criticises the model proposed by Lorenzen and Vance (1986) for assuming the cost and time parameters to be beyond the control of the managers of the process. The argument follows that companies who are determined to improve profitability and remain competitive will aim to decrease these costs, leading to a reduction in the expected out-of-control shift and an increase in the time before the shift is expected to occur. Woodall (1986) argues that if these parameters are considered to be fixed as in the case of the economic model, it could be a hamper to the improvement of the process.

Montgomery (1980) argues that the economic design of control charts is seldom employed in practice due to its intricate nature and the imprecise values of the input variables. Woodall (1985) claims that the statistical design models are much simpler to use to mold a control chart to a particular application. However, he notes that the resulting procedure will not be optimal in the sense of minimising the total cost.

Woodall (1985) suggests it is often not practical to detect shifts from the target value immediately, if these shifts are too small to be of practical significance. He argues that the choice of an appropriate control procedure to be applied to a specific problem is dependent on the selection of the in-control and out-of-control regions of parameter values. Although he focuses on the symmetric two-sided procedures designed to detect shifts in the mean in either direction, he notes that the approach can easily be extended to other types of control charts. Chan et al. (2000) and Xie et al. (2002) both develop the designs of exponential charts based on pure statistical considerations.

According to Saniga (1989) the application of individual control charts, the statistical design is fairly simple. For control charts that are employed together, for example a $\bar{X}$ and R control charts, he suggests the control chart be designed statistically for power of 0.95 and probability of Type I error of 0.0026 on both charts using required sample-size isodynes as specified in the particular study done by Saniga (1984).

An example of two typical constraints that are imposed on this statistical performance of the exponential chart are given by:

$$ATS_0 \geq G_0$$

$$ATS_0 \leq G_1.$$

where G_0 and G_1 are the goal values imposed on ATS_0 and ATS_1, respectively.

Al-Oraini and Rahim (2002) show that even though the statistical design of control charts give control charts with low Type I error rate and high power, it may lead to higher costs than the economic designs. "The economic statistical design was first proposed by Saniga (1989) to combine the benefits of both pure statistical and economic designs while minimising their weaknesses." (Al-Oraini and Rahim; 2002) The economic statistical design of control charts will be discussed in the next chapter.

CHAPTER FIVE: *THE ECONOMIC STATISTICAL DESIGN OF CONTROL CHARTS*

5.1 INTRODUCTION:

Since the results obtained from the economic design of control charts may result in poor statistical characteristics, Saniga (1989) presents an economic statistical model in which the cost function is minimised subject to the constraints on the minimum value of the power as well as the maximum value of the probability of a Type I error. The economic statistical designs are generally costlier than economic designs because of the added statistical constraints. According to Zhang and Berardi (1997) product quality can be improved by lowered process variability due to tight limits imposed on the statistical properties.

No general rules have been proposed to govern the selection of the design parameters and bounds for the economic statistical model. Zhang and Berardi (1997) agree with Duncan (1956) that these should be chosen considering the specific problem at hand. The relevant cost information, economic and statistical outcomes for each individual situation are to be judged. According to Zhang and Berardi (1997) the sensitivity analysis is an useful tool in assisting designers in making these decisions.

"In many cases of analyzing data, one is confronted with finding the appropriate distribution to describe the pattern of variation of the empirical data" (Berrettoni, 1964). If the process measurements are truly normally distributed, the statistic $\bar{X}$ will also be normally distributed. Based on the central limit theorem, if the measurements are asymmetrically distributed, the statistic $\bar{X}$ will be approximately normally distributed when the sample size n is sufficiently large (Chou, Li and Wang; 2001, Chen and Cheng; 2007). Practically, the sample size n is not always sufficiently large due to the cost associated with sampling from the process. "Therefore, if the measurements are not normally distributed, the traditional way of designing the control chart may reduce the ability of a control chart to detect the assignable causes" (Chou et al.; 2001).

Thorough research has been conducted to find the appropriate distribution. Most development of the economic statistical design of control charts has assumed a Poisson process; however this assumption is not always valid. Duncan (1956) assumes the average time for the occurrence of an assignable cause to be exponentially distributed. The first to consider a failure mechanism that obeys a Weibull model was Hu (1984). Rahim and Banerjee (1993) propose an economic design with Weibull and Gamma failure mechanism. The work was developed by Zhang and Berardi (1997) to provide an economic statistical design model under the Weibull failure mechanism. Chou et al. (2001) developed an economic statistical design using the Burr distribution. Al-Oraini and Rahim (2002) consider the Gamma distribution for the development of an economic statistical design.

Duncan (1956) assumed that the average time for occurrence of an assignable cause is an exponentially distributed random variable with parameter λ. This implies that the number of periods for which the process remains in the in-control state has the memoryless property associated with the Poisson process. Duncan's assumption of a Poisson process-failure mechanism leads to a simpler model, but it is not necessarily always appropriate.

5.2 THE WEIBULL DISTRIBUTION:

Hu (1984) first considered a failure mechanism that obeys a Weibull model. He assumes that the length of the sampling intervals is fixed throughout the process. He further concludes that the exponential model can be used in the place of the Weibull model without having any significant effect. Yang and Rahim 2005 argue that the assumption that the process failure mechanism follows an exponential distribution, leading to the use of constant sampling intervals, does not seem to be appropriate for most processes. They claim that for a system having a Weibull distributed failure mechanism with increasing failure rates, it would not be suitable to keep sampling intervals constant at all times.

Banerjee and Rahim (1988) noted that the idea of keeping sampling intervals constant at all times for a process having a failure mechanism that is Weibull distributed, is counter-intuitive. They propose a cost model in which the length of the sampling interval varies with time. The increasing hazard rate of the Weibull distribution is used, so that the probability of a shift occurring in an interval, given no shift had occurred until the start of the interval, is constant for all intervals (Zhang and Berardi, 1997).

The advantages of using the Weibull distribution are numerous. The choice of applying the Weibull distribution allows for the aging and wear-effects on the failure mechanisms of the system. "...the Weibull plot is very sensitive in showing heterogeneous and/or mixed distributions" (Berrettoni, 1964). The Weibull distribution can be used for most processes with an increasing, constant or even decreasing failure rate. The exponential distribution is a special case of the Weibull distribution, in which $\theta = 1$. "Weibull distributions are often used in cases of what is called censored sampling or time truncated tests, where some components may be withdrawn from testing before they have failed. Difficulties then arise concerning how to combine the information about those which were observed until they failed with the information about those which were withdrawn (censored) but were still working when last seen" (Betteley, Mettrick, Sweeney and Wilson; 1994:122).

Zhang and Berardi (1997) argue that an economic statistical control chart design under a Weibull failure mechanism will give a flexible model that can be utilised for many industrial applications. Kapur and Cho (1994) consider a Weibull distribution for a quality characteristic due to the flexibility in terms of shape and the ability to model more realistic situations. The Weibull distribution

> "...gives designers the capability to access the impact of their decisions on several critical aspects that are either not available or not apparent via other design methods. Using economic statistical design with an appropriate sensitivity analysis, designers can readily observe the impact on cost, sample size, sampling interval, and control limits due to the constraints on the statistical error

> rates and control chart responsiveness, and the specification of the failure distribution parameters." (Zhang and Berardi, 1997)

Berrettoni (1964) highlights the wide-spread application of the Weibull distribution to empirical data. He mentions the following applications for this purpose:

- Corrosion resistance of magnesium alloy plates,
- return goods classified by number of weeks after shipment,
- number of down times per shift,
- leakage failure of dry batteries,
- life expectancy of ethical drugs,
- reliability of step motors, and
- reliability of solid tantalum capacitors.

Betteley et al. (1994:122) give an in-depth discussion of the probability density function of the Weibull distribution. The probability density function is given as

$$f(t) = -\frac{dR(t)}{dt} = \lambda\theta t^{\theta-1}\exp\left(-\lambda t^{\theta}\right),$$

with corresponding hazard function

$$h(t) = \frac{f(t)}{R(t)} = \lambda\theta t^{\theta-1},$$

and cumulative probability function

$$F(t) = 1 - exp^{(-\lambda t)\theta}.$$

The parameter θ is the most important parameter for determining the shape of the distribution of times to failure. When $\theta < 1$, the hazard rate decreases with time and θ must be greater than 0 and is usually at least 0.5. When $\theta = 1$, the distribution simplifies to the exponential model with a constant hazard rate that represents random failures, and when $\theta > 1$, the hazard rate increases with time. A large θ, determined from reliability test data, often indicates the existence of some particular flaw that "kills" virtually all the parts by the time they reach a particular age.

Often, the Weibull distribution is expressed in a slightly different way so that the reliability might be expressed in the form

$$R(t) = exp\left[-\left(\frac{t-\gamma}{\eta}\right)^{\beta}\right]$$

In this formula:

β: is called the shape parameter or the form parameter.

γ: is called the location parameter. It represents the shortest possible lifetime and is sometimes designated t_0.

η: is the scale parameter or characteristic of life.

The probability density function and the hazard function become

$$f(t) = \left(\frac{\beta}{\eta}\right)\left(\frac{t-\gamma}{\eta}\right)^{\beta-1} exp\left[-\left(\frac{t-\gamma}{\eta}\right)^{\beta}\right],$$

$$h(t) = \left(\frac{\beta}{\eta}\right)\left(\frac{t-\gamma}{\eta}\right)^{\beta-1}.$$

5.2.1 THE MODEL DEVELOPMENT:

Saniga (1989) presents a method to design control charts that have bounds on the probability of the Type I and Type II errors, the average time to signal (ATS) an expected shift and are economical. He argues that control charts which are designed in such a manner will be in agreement with industrial demand for low-process variability and long-term quality, as well as having various other advantages. Saniga (1989) claims: "These designs can be viewed as improvements to statistical designs, since they consider economic factors while achieving desirable statistical properties."

Although the economic statistical design of control charts will be more costly than purely economic designs, these charts will also protect against the expected shifts other than those used to develop an economic design.

Saniga (1989) defines the economic statistical design of control charts as the design in which the "...economic-loss cost function is minimized subject to a constrained minimum value of power and maximum value of the Type I error probability and ATS an expected shift in process parameters." He allows for the specification of desired levels of power at shift levels apart from the expected shift level as in the case of a pure statistical design.

The economic loss function, F, with power p at the expected shift level, the probability of a Type I error of α, and ATS at the expected shift level, is considered. If h is the sampling frequency and we are able to assume that the statistic of interest is independent from one sample to the next, then ATS= h/p. Saniga (1989) defines ps_i $(i = 1,2,\ldots,m)$ to be the power for selected shifts at levels apart from the expected shift. Let D be the design vector including the sample size, the width of the control limits, and the sampling frequency. The objective is to find D to minimise $F(D)$ subject to the following constraints:

$$\alpha \leq \alpha_u;\ p \geq p_1;\ ATS \leq ATS_u;\ ps_i \geq ps_{1i}\,, i = 1,2,\ldots,m.$$

where α_u, p_1, and ATS_u are the desired bounds on the probability of the Type I error, the power and the ATS, respectively, at the expected shift level. The value ps_{1i} represents the lower bound on power for the ith selected shift apart from the expected shift. The constraints on ps_i permit the same amount of flexibility as would have been permitted by the pure statistical design. Saniga (1989) states that, in general, any number of constraints could be added for selected individual or joint shifts in the variance and the mean, this assures the sensitivity to shift apart from the expected shift.

Alternatively, all constraints can be formulated as a function of the average run length (ARL) in a similar manner as proposed by Woodall (1985) for a pure statistical design. A control chart is designed to have specified ARL values at two shifts in the process, the first a shift of relatively little practical significance and the second, as shift that is significant enough to justify early recognition. Yet another proposal by Saniga (1989) is to restrict the ARL value corresponding to the no shift case to be high and to further restrict the ARL values corresponding to several other shifts of importance to be low. Saniga (1989) develops methods to model and solve any of these alternatives easily.

The cost of the economic statistical design will always exceed or at least be equal to the cost of the economic design. The inclusion of constraints cannot reduce the value of the objective function. The cost of the economic statistical design may, in fact, be less costly than the economic design if the expected shift is not the shift that actually takes place. The cost increase associated with the economic statistical design may be a disadvantage, but the economic statistical design has several advantages when compared to the heuristic, statistical and economic designs. According to Saniga (1989) these advantages include improved assurance of long-term product quality and maintenance and even the reduction of the variance of the distribution of the quality characteristic that is of interest. The economical statistical design has circumvented many of the disadvantages that are associated with the heuristic, statistical and economic designs.

The goal of long-term product quality is attained more easily by the pure statistical design with constraints on power or ATS (or h) than with the heuristic or economic design. This is policy for some firms; but required by law for others, such as the pharmaceutical industry. The main disadvantage of the pure statistical design is that the economic factors are not considered, when statistical process control (SPC) will always have economic costs. The economic design of control charts are developed to take these considerations into account and are generally characterised by high power (Saniga, 1977),it is however easy to construct control regions of cost and other parameters for the situations where the power is low and the ATS is high.

A pure statistical design has parameters as close as possible to $\alpha = \alpha_u$, $p = p_1$, ATS= ATS_u, and $ps_i = ps_{1i}$, and it is clear that these values may not necessarily minimise the cost represented by F. Saniga (1989) points out that by constraining the probability of the Type I error, there are less frequent false alarms. Not all the cost associated with false alarms can be accounted for by the economic design of control charts, since some of these costs are intangible. For example, a manager will be less likely to shut down the production line when a large number of false alarms have occurred in the past. This is typical for an industry where the quantity of production is as important as the production quality. Woodall (1986) argues that a high probability of Type I error can also lead to undue adjustment of the production process. This in turn, will cause an increase in the variance of the distribution of the quality characteristic.

Economic models developed before 1989 were designed under the assumption that the assignable causes are found and all the adjustments made to the process are perfect. The economic statistical design of Saniga (1989) is based on the same assumption, but the probability of a false alarm and a potentially imperfect adjustment can be kept relatively small. The economic statistical design differs from the economic design in the respect that the frequency of needless adjustments to the system can be controlled efficiently.

5.2.2 THE COST MODEL:

Chiu (1975) developed a more generalised cost model for the model proposed by Duncan (1956). This model is specifically designed for a system monitored by a control chart that allows for production to cease during the search for the assignable cause and to include the cost and time that it takes to repair the process. Saniga (1989) develops the economic statistical model using the notation proposed by Chiu (1975):

λ: mean rate of occurrence of the assignable cause, assuming the time to shift is exponentially distributed.

t_0: expected time to search after a false alarm.

t_1: expected time to search and adjust for an assignable cause after a true alarm.

A_0: expected cost of a false alarm.

A_1: expected cost of finding and removing an assignable cause.

V_0: profit per hour when the process is functioning in the in-control state.

V_1: profit per hour when the a shift has occurred.

b: fixed cost associated with sampling.

c: variable cost associated with sampling and testing.

n: sample size.

h: time interval between two samples.

α: probability on a single sample of the UCL being exceeded in the absence of an assignable cause.

P: probability on a single sample of the UCL being exceeded in the presence of an assignable cause.

τ: average time between the sample taken just before the occurrence of an assignable cause and the occurrence itself.

F: average loss-cost per hour.

The model is defined in terms of the cost per hour, F, where Saniga (1989) defined F to be:

$$F = \frac{\lambda M B_1 + T B_0 + \lambda W + (b+cn)(1+\lambda B_1)/h}{1+\lambda B_1 + t_0 B_0 + \lambda t_1},$$

with $\tau = \frac{\{1-(1+\lambda h)\exp(-\lambda h)\}}{[\lambda - \lambda \exp(-\lambda h)]}, B_0 = \frac{\alpha(1-\lambda\tau)}{h}, B_1 = \frac{h}{p} - \tau, M = V_0 - V_1, T = A_0 + V_0 t_1,$ and $W = A_1 + V_0 t_1$.

The joint design of $\bar{X}$- and R-control charts involves determining the values of the sample size n, the sampling frequency h, the control chart coefficient for the $\bar{X}$-control chart, L_1, and the LCL and UCL for the R-chart, L_{2l} and L_{2u}, respectively. For the $\bar{X}$-control chart the control limits are defined as:

$$UCL = \mu_0 + L_1\sigma_0/n^{1/2} \text{ and } LCL = \mu_0 - L_1\sigma_0/n^{1/2}.$$

For the R-chart:

$$LCL = L_{2l}\sigma_0 \text{ and } UCL = L_{2u}\sigma_0.$$

Saniga (1989) sets the LCL on the R-chart to zero. He argues that this has to be done for existing economic models since the expected shifts are assumed to always correspond to a decrease in quality. He continues to state that an LCL for the R-chart can be obtained in an economic statistical design in which the power of the test can be stipulated for any shift of interest, not only for the expected shift.

The costs associated with the placement of statistical constraints on the economic model are presented by Saniga (1989). These costs on an economic model will be larger for the detection of smaller shifts, since the statistical constraints placed on the control chart will force tighter control than would be economically optimal in the short term. Saniga (1989) notes that the relaxation of the statistical constraints yields designs that are less costly. These designs will typically have smaller samples, higher probability of Type I error, a higher ATS, and similar power to a design with tighter constraints. In the study presented by Saniga (1989), he found that the results of his experiment shows that pure statistical designs (i.e. designs with tight statistical constraints at the expected process shift) are the most economic statistical designs in 24 out of 64 cases with small shift sizes and in 12 out of 64 cases with larger shift sizes. He found that this usually occurs when the difference between operating in an out-of-control state and an in-control state is at the smaller of the two levels.

These results imply that a pure statistical design could be made to be more cost effective without sacrificing their statistical properties.

Saniga (1989) continues to compare the economic statistical design to the economic design and shows that the economic statistical design has twice as large sample and larger control-limit coefficient. However, the economic design has a larger sampling interval. Another advantage of the economic statistical design relative to the economic design is in terms of power and a reduction in the probability of the Type I error. He states that the economic statistical design has approximately an 8% increase in power and about a fivefold reduction in Type I error probability. Furthermore, the economic statistical design is able to detect shifts at the level of the expected shift that is almost twice as fast as the economic design. This improvement can be ascribed to the constraints placed on the ATS. These advantages do however come at an economical cost to the company, as explained by Saniga (1989): "These advantages are costly at the level of the expected shift, with the economic statistical design about 25% more costly than the economic design."

The first attempt to harness these advantages and improve the economic performance of the economic statistical design was developed by Saniga in 1989. He claims to have developed a method to obtain the most economical statistical design for the Shewhart-type control charts and has applied it to the joint determination of the parameters of $\bar{X}$- and R- charts. He notes that the method developed can be thought of as an improvement to the pure statistical design, due to the fact that it leads to designs that are at least as good as statistical designs in terms of statistical properties but "are also generally less costly and never more costly under the assumptions of the economic model" (Saniga; 1989). These economic statistical designs are more costly than economic designs but provide an increased ability to detect shifts over a wider range, and they have some additional advantages as well.

The study performed by Zhang and Berardi (1997) present some situations where the statistical performance of control charts are improved substantially with only a minor increase in cost by using economic statistical design instead of an economic design. Zhang and Berardi (1997) show that the cost increase is to be relatively insensitive to the improvement in the Type I error rate and power throughout the investigated range. From this it follows that it could be relatively inexpensive for practitioners to implement control charts with lower false alarm rates and with a higher probability for detecting a process shift when the process is out of control. Zhang and Berardi (1997) warn that the bound on ATS should not be set too low because of the sensitivity of the expected cost to small ATS bounds. They suggest that a reasonable bound on ATS may be found from the actual ATS as found in the economic design. For larger shifts, Zhang and Berardi (1997) found this approach to be an inexpensive technique to achieve significant improvements in the statistical properties of the control charts. The ability of achieving the detection of extremely small shifts in the process mean, is however, an expensive proposition for $\bar{X}$-control charts.

The specification of the Weibull shape and scale parameters has a significant impact on the cost and responsiveness in the economic statistical design. For this reason the distribution parameters need to be estimated adequately to reflect the process characteristics. According to Zhang and Berardi (1997) the economic design obscures this important result and actually may be misleading in the direction of its cost change. The sensitivity analysis provided in their study is valuable to the designers of control charts in making trade-off decisions between the expected costs and the statistical constraints desired by management.

Although the Weibull distribution is generally considered to be a better model for component lifetime, the exponential distribution assumption is widely used in practice. Xie, Kong and Goh (2000) argue that the exponential distribution may still be appropriate for components that undergo regular replacement or maintenance, regardless of the fact that the original failure rate function could potentially be far from constant. “This is because regular maintenance or replacement will tend to reduce or even eliminate the chance for components to enter their wear-out life period during which the failure rate increases rapidly” (Xie et al.; 2000).

If the exponential distribution can be considered to be an approximation of reality, the potential effects of this approximation need to be tested in order to obtain an indication of the applicability of the approximation in practice. The approximation is considered to be acceptable if the difference between the approximate reliability and the actual reliability is relatively small. A large difference however, indicates that an adjustment to the approximate reliability is needed. "This difference not only affects the estimated operational reliability, but also influences the subsequent decisions regarding the optimal maintenance time or spare allocation problem" (Xie et al.; 2000).

The results obtained by Xie et al. (2000) compare the actual reliability to the approximate reliability when the average failure rate is used as the failure rate of the exponential model. They note that the approximation is accurate and even slightly conservative. The conservative nature of the approximation holds some practical advantages as well.

Zhang and Berardi (1997) note that in many cases, the economic statistical design matches the statistical control chart design. This implies that cost effective designs are possible without decreasing the statistical ability of the control chart.

CHAPTER SIX: *THE DEVELOPMENT OF THE COST MODEL*

Although Duncan's model was first introduced in 1956 and many more recent models have been developed, Duncan's model remains the first of the typical cost models developed. Most of the more recent models still make use of the assumptions and concepts as proposed by Duncan. A brief review of the model will be given.

6.1 A REVIEW OF DUNCAN'S COST MODEL:

The cost model proposed by Duncan (1956) includes the following components:

- the cost of an out-of-control condition,
- the cost of the occurrence of a false alarm,
- the cost of finding an assignable cause, and
- the cost of sampling, inspection, evaluation and plotting.

Duncan (1956) makes use of the assumption that the system starts in an in-control state and is subject to random shifts in the process mean that are inherent to the system. Once a shift takes place, the system functions in the out-of-control state until corrective action is taken. He defines the cycle length as the total time from when the process begins in the in-control state, shifts to the out-of-control state after the occurrence of the assignable cause, has the out-of-control state signalled, and results in the identification of the assignable cause.

Once the average cycle length is determined, Duncan converts the cost components to a 'per hour of operation' basis. He noted that given associated cost and time parameters, the optimal values for the three design parameters (sample size, n, sampling interval, h, and control limits, L) are calculated using optimisation techniques.

Chou, Li and Wang (2001) discuss the four average cycle length components in Duncan's cost model in turn, as follows:

1. Assuming that the system starts off in the in-control state, the time interval the system remains functioning in the in-control state is an exponential random variable with mean $1/\lambda$, which is the average process in-control time.

2. With the occurrence of an assignable cause, the process mean shifts to $\mu + \delta\sigma$, and the probability of detecting this out-of-control condition on any subsequent sample is given by the power of the chart, i.e. $1 - \beta$. The expected number of subgroups taken before a shift is detected will therefore be given by $1/(1 - \beta)$. Given an occurrence of the shift in the interval between the j^{th} and $(j + 1)^{\text{th}}$ subgroups, the average time of occurrence in the interval between these subgroups is $\tau = \frac{1-(1+\lambda h)e^{-\lambda h}}{\lambda(1-e^{-\lambda h})}$. Chou, Li and Wang (2001) note that the expected length of time the process will function in the out-of-control state will be given by $h/(1 - \beta^{-\tau})$.

3. The average sampling, inspecting, evaluating, and plotting time for each sample is a constant g proportional to the sample size n, now the delay in plotting a subgroup point on the $\overline{\text{X}}$-control chart is given by gn.

4. The time it takes to find the assignable cause following a false alarm signal is a constant D.

According to Duncan (1956) the expected length of a cycle, denoted by $E(T)$ is

$$E(T) = \frac{1}{\lambda} + \frac{h}{1-\beta} - \tau + gn + D.$$

and the expected cost per hour, $E(C)$, is

$$E(C) = \frac{a_1 + a_2 n}{h} + \frac{a_4 \left[E(T) - \left(\frac{1}{\lambda}\right)\right] + a_3 + \alpha a_5 e^{-\lambda h}/(1 - e^{-\lambda h})}{E(T)}$$

where a_1 and a_2 are the fixed and variable components of the cost of sampling, respectively, a_3 is the cost associated with the discovery of an assignable cause, a_4 represents the penalty associated with the continuation of production during the out-of-control state, and a_5 stands for the cost associated with the investigation of a false alarm. When considering the economic design of an $\bar{X}$-control chart, the aim is to determine the values of n, h and L that minimise the function $E(C)$.

6.2 FURTHER DEVELOPMENTS:

Lorenzen and Vance (1986) proceed to modify Duncan's original cost model. The following features of the model by Lorenzen and Vance (1986) were considered by Banerjee and Rahim (1988):

1. The time that the process remains in the in-control state follows a Weibull distribution. For the purpose of this article, the probability density function of the Weibull distribution is given by

   $f(t) = \lambda k t^{(k-1)} exp\{-\lambda t^k\}$, for $t > 0,\ k \geq 1,\ \lambda > 0$.

2. Random samples of size n are drawn from the process at times h_1, $(h_1 + h_2)$, $(h_1 + h_2 + h_3), \ldots$ in order to monitor the process. This feature differs from Duncan's model where he considered $h_j = h$ for all $j = 1,2, \ldots$.

3. The time taken to sample and chart a single item is insignificant.

4. The production comes to an end during both the search and repair processes.

5. The length of the sampling intervals h_j are defined to keep the probability of a shift in an interval, given no shift up to its start, constant for all intervals. Banerjee and Rahim

(1988) claim that this can be achieved by setting the length of the intervals h_j $(j = 1,2,\ldots)$ as follows:

$$h_j = \left[j^{1/k} - (j-1)^{1/k}\right] h_1.$$

The basic requirements for h_j are all met, namely: (i) $h_1 \geq h_2 \geq h_3 \geq \cdots$, and (ii) $\lim_{m\to\infty} \sum_{j=1}^{m} h_j = \infty$. Also, it is true that $h_j \equiv h_1$ for all j where $k = 1$.

Banerjee and Rahim (1988) consider features 3 and 4 to be self-explanatory, but note that features 1, 2 and 5 merit some clarification. To justify these features, the practical applications of the Weibull distributions related to the process-failure mechanism are studied by Berrettoni (1964). He establishes that the distribution of time to leakage failure of dry cell batteries can be approximated by a Weibull distribution. Consider the example given by Berrettoni (1964) where a typical production process with several mechanical and electrical control devices are regarded. Some of the control devices require the continuous direct current output of the dry cell batteries. The process drifts to the out-of-control state if a battery fails. Banerjee and Rahim (1988) give further examples of Weibull probability models. These examples include the occurrence of shocks in a chemical process due to an increased lubricant viscosity, the wear that tools are subjected to and the fatigue life of steel and certain fibers.

Banerjee and Rahim (1988) argue that for a process with an increasing hazard rate, constant sampling intervals for Markovian shock models present a constant integrated hazard over each interval. By manipulation of the sampling interval, a constant hazard rate per interval can be obtained. Thus it would be realistic to choose the length of the sampling intervals as a function of the process's age.

After considering these factors, Banerjee and Rahim (1988) proposed to choose the length of the sampling intervals in such a manner as to maintain a constant integrated hazard rate over each sampling interval for the Weibull shock models. They claim that this is equivalent to the probability of a shift in an interval being constant, given that no shift had taken place until the start of the interval. Refer to Lorenzen and Vance (1986) for the complete description of the model. Banerjee and Rahim (1988) introduced the notations as used in most of this paper.

The renewal theorem approach proposed by Banerjee & Rahim (1988) will be applied in order to obtain the expected cycle time $E(T)$ as well as the expected cost $E(C)$. The following notation is used in sections 6.2.1 and 6.2.2:

Z_0: expected search time associated with the false alarm.

Z_1: expected time to discover the assignable cause.

Z_2: expected time to repair the process.

a: fixed sample cost

b: cost per unit sampled.

Y: cost per false alarm.

W: cost to locate and repair the assignable cause.

D_0: cost per hour while the process is in control.

D_1: cost per hour while the process is out if control.

α: Pr(test result has an alarm | the process is in control), $\alpha = \Pr\left(T^2 > \chi^2_{\alpha,k} \middle| \mu = \mu_0\right)$, where $T^2 = n(\bar{X} - \mu_0)\Sigma^{-1}(\bar{X} - \mu_0)^T$.

β: Pr(test result has no alarm | the process is out of control, $\mu = \mu_1$,

$\beta = \Pr\left(T^2 < \chi^2_{\alpha,k} \middle| \mu = \mu_1\right)$, where $T^2 = n(\bar{X} - \mu_1)\Sigma^{-1}(\bar{X} - \mu_1)^T$.

h_j: the length of the j^{th} sample interval where j can take on the values 1,2,…; and $h_0 = 0$.

θ: is the scale parameter of the Weibull distribution, where $\theta \geq 1$.

λ: is the shape parameter of the Weibull distribution, where $\lambda > 0$.

ω_j: the time until the j^{th} sample is taken; $\omega_j = \sum_{i=1}^{j} h_i = j^{1/\theta} h_1$, where $j = 1,2,\ldots$ and $\omega_0 = 0$. The choice of h will be discussed in detail in Chapter 6 describing The Cost Model.

T_j: the residual time in the cycle beyond time ω_j given that the process is in the in-control state at time ω_j.

T_0: total time until an assignable cause occurs from the beginning.

p_j: the conditional probability that the process is out of control in the j^{th} sampling interval, given that the process is still in control before time ω_{j-1}, that is $p_j = Pr(T < \omega_j | T_0 > \omega_{j-1})$. Since p_i is independent of j, so let $p_j = p$.

q_j: the probability that the process will be out of control during the j^{th} sampling interval; $q_j = Pr(\omega_{j-1} < T_0 < \omega_j)$.

τ_j: the expected in control time in the j^{th} sampling interval, given that the shock occurred in the j^{th} sampling intervall; $\tau_j = E(T - \omega_{j-1} | \omega_{j-1} < T < \omega_j)$.

τ: the unconditional expected in control time in a sampling interval; $\tau = \sum_{j=1}^{\infty} q_j \, \tau_j$.

$E(C)$: the expected cycle cost.

$E(C_j)$: the expected cycle cost associated with the j^{th} sampling interval.

6.2.1 THE EXPECTED CYCLE TIME:

Yang and Rahim (2005) make use of the renewal theorem as proposed by Banerjee and Rahim (1988) to obtain an expression for the expected cycle time, denoted by $E(T)$. Lorenzen and Vance (1986) define a quality cycle as the time between the starts of successive in-control periods. Banerjee and Rahim (1988) define a quality cycle to begin when a new component is installed and ends after a shift – due to some component failure – is detected and the process is returned to an in-control state by replacing the component that had failed. The cycle is divided into three different components:

1. the in-control period,
2. the time to obtain a true alarm given that the process is out-of-control, and
3. the time to search for and repair the assignable cause.

The possible states at the end of the first sampling interval are studied in order to derive the expected cycle length $E(T)$. The expected residual time and the expected residual costs are calculated depending on the state of the system after the first interval. The renewal equation can then be formulated with these values together with the associated probabilities.

Yang and Rahim (2005) describe the possible states at the end of the first interval as follows:

S_{11}: the process is out-of-control and the test result has a true alarm;
S_{12}: the process is out-of-control but the test result has no true alarm;
S_{13}: the process is in-control and the test result has no false alarm;
S_{14}: the process is in-control but the test result has a false alarm.

Table 6.1 below indicates all the possible states of the system at the end of the initial sampling and testing, the expected residual cycle time, as well as the probability associated with each representative state at the end of the initial sampling and testing.

Table 6.1.

The expected residual time and probability for each state

State	Expected Residual Cycle Time	Probability
S_{11}	$Z_1 + Z_2$	$p(1-\beta)$
S_{12}	$Z_1 + Z_2 + (1-\beta)\sum_{i=1}^{\infty}(\omega_{i+1} - h_1)\beta^{i-1}$	$p\beta$
S_{21}	$E(T_1)$	$(1-p)(1-\alpha)$
S_{22}	$Z_0 + E(T_1)$	$(1-p)\alpha$

Source: Yang and Rahim (2005)

These associated probabilities in Table 6.1 above are defined as follows by Banerjee and Rahim (1988), let p_j with $(j = 1,2,3,\ldots)$ be defined as the conditional probabilities that the component unit used in the system will fail during the j^{th} sampling interval, on the condition that the process was functioning in the in-control state at the beginning of the j^{th} sampling interval. The beginning of the j^{th} sampling interval is denoted by ω_{j-1}. Such that, for $j = 1,2,3,\ldots$:

$$p_j = \frac{\int_{\omega_{j-1}}^{\omega_j} f(t)dt}{\int_{\omega_{j-1}}^{\infty} f(t)dt}$$

The unconditional probability that the component will fail during the j^{th} sampling interval, denoted by, q_j, can be obtained by:

$$q_j = \int_{\omega_{j-1}}^{\omega_j} f(t)dt.$$

The renewal theorem yields the following expression for $E(T)$:

$$E(T) = h_1 + p(1-\beta)(Z_1+Z_2) + p\beta[Z_1+Z_2+(1-\beta)\sum_{i=1}^{n}(\omega_{i+1}-h_1)] + (1-p)(1-\alpha)E(T_1)$$
$$+\alpha(1-p)[Z_0+E(T_1)]$$

The main results given by Banerjee and Rahim (1988), as well as the associated proofs are given in Appendix B.

The equation above can now be simplified to

$$E(T) = h_1 + p(Z_1+Z_2) + \alpha Z_0(1-p) + p\sum_{i=1}^{\infty} h_{i+1}\beta^i + (1-p)E(T_1)$$

where $E(T_1)$ is given by:

$$E(T_1) = h_1 + p(Z_1+Z_2) + \alpha Z_0(1-p) + p\sum_{i=1}^{\infty} h_{i+2}\beta^i + (1-p)E(T_2)$$

and

$$E(T_{j-1}) = h_j + p(Z_1+Z_2) + \alpha Z_0(1-p) + p\sum_{i=1}^{\infty} h_{i+j}\beta^i + (1-p)E(T_j)$$

for $j = 2,3,\ldots$.

Yang and Rahim (2005) provide a set of recursive systems for $E(T), E(T_1), E(T_2), \ldots$, etc. Solving this system leads to an expression for $E(T)$. Since it is known that all $p_j = p$, an expression for $E(T)$ is obtained as follows:

$$E(T) = Z_1 + Z_2 + \frac{\alpha Z_0(1-p)}{p} + \sum_{i=1}^{\infty} h_i(1-p)^{i-1} + p\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} h_{i+j}(1-p)^{j-1}\beta^i$$

The calculation of $E(T)$ follows directly when all the parameters are given. The approximate values for the two infinite series $\sum_{i=1}^{\infty} h_i(1-p)^{i-1}$ and $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} h_{i+j}(1-p)^{j-1}\beta^i$ need to be calculated. For the algorithm used to calculate an approximate value for $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} h_{i+j}(1-p)^{j-1}\beta^i$ see Yang and Rahim (2005). A similar approach can be used to solve the third infinite series $\sum_{i=1}^{\infty} h_i(1-p)^{i-1}$.

6.2.2 THE EXPECTED CYCLE COST:

An expression for the expected cycle cost, $E(C)$, can be obtained by dividing the cycle cost into two components (Banerjee and Rahim; 1988):

1. the cost incurred in the first sampling and testing, and
2. the expected residual cost beyond time ω_1, given that the process is in control at time ω_1.

Table 6.2 below indicates the possible states of the system, as defined for the expected cycle time, the expected costs incurred during the initial sampling and testing, as well as the expected residual costs.

Table 6.2.
The expected residual cost for each state.

State	Expected Cost during h_1	Expected Residual Cost
S_{11}	$a+bn+D_0\tau_1+D_1(h_1-\tau_1)$	W
S_{12}	$a+bn+D_0\tau_1+D_1(h_1-\tau_1)$	$W+\dfrac{a+bn}{(1-\beta)}+D_1(1-\beta)\sum_{i=1}^{\infty}\omega_{j+1}-h_1)\beta^{i-1}$
S_{21}	$a+bn+D_0h_1$	$E(C_1)$
S_{22}	$a+bn+D_0h_1$	$Y+E(C_1)$

Source: Yang and Rahim (2005)

The renewal equation for the expected cost can be expressed recursively as:

$$E(C) = [a + bn + D_0\tau_1 + D_1(h_1 - \tau_1) + W](1 - \beta)p$$
$$+ \left[a + bn + D_0\tau_1 + D_1(h_1 - \tau_1) + W + \frac{a + bn}{(1 - \beta)}\right.$$
$$\left. + D_1(1 - \beta)\sum_{i=1}^{\infty}\left(\omega_{j+1} - h_1\right)\beta^{i-1}\right]\beta p \; + [a + bn + D_0h_1 + E(C_1)](1 - \alpha)(1 - p)$$
$$+ [a + bn + D_0h_1 + Y + E(C_1)](1 - p)\alpha.$$

Note that $E(C)$ is dependent on $E(C_1)$. Simplifying gives,

$$E(C) = a + bn + (D_0 + D_1)\tau_1 p + Wp + (a + bn)\beta p/(1 - \beta) + D_1 p\sum_{i=1}^{\infty} h_{i+1}\beta^i + D_0h_1(1 - p) + \alpha Y(1 - p) + D_1h_1p + (1 - p)E(C_1).$$

Now for $j = 2,3, \dots$,

$E(C_{j-1}) = a + bn + (D_0 + D_1)\tau_1 p + Wp + (a + bn)\beta p/(1 - \beta) + D_1 p\sum_{i=1}^{\infty} h_{i+1}\beta^i + D_0h_1(1 - p) + \alpha Y(1 - p) + D_1h_1p + (1 - p)E\left(C_j\right)$ (See Appendix B)

The system can be solved recursively, given an expression for $E(C)$:

$E(C) = (a + bn)\left[{}^{1}/_{p} + \frac{\beta}{1-\beta}\right] + (D_0 + D_1)\tau + W + D_1 p\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} h_{i+j}(1 - p)^{j-1}\beta^i + D_0\sum_{i=1}^{\infty} h_i(1 - p)^i + \frac{\alpha Y(1-p)}{p} + D_1 p\sum_{i=1}^{\infty} h_i\,(1 - p)^{i-1}$ (See Appendix B)

The expected cost per unit time $E(V)$ can be derived using the ratio of the expected cycle cost $E(C)$ and the expected cycle time $E(T)$:

$$E(V) = \frac{E(C)}{E(T)}$$

This result is obtained by applying the property of the renewal reward process (Ross; 2000: 378). Where $E(V)$ is a function of the design parameters n, h_1, and $\chi^2_{\alpha,k}$ and the aim is to find values of these parameters that minimises $E(V)$. Note that $E(V)$ is χ^2-distributed with k (the number of quality characteristics) degrees of freedom.

Two types of errors could occur during the process of hypothesis testing. The first, called the Type I error, the rejecting of the null hypothesis when the process is functioning in the in-control state. According to Yang and Rahim (2005) the costs that are generally associated with a Type I error are the costs of unnecessary investigation and potential loss of production if the process is ceased during the search for the assignable cause. The second, called the Type II error, which is the failure to reject the null hypothesis when the process is functioning in an out-of-control state. Yang and Rahim (2005) state that the costs associated with a Type II error include the costs due to an increase in the number of defective units produced when the process is functioning in the out-of-control state. They also argue that both these costs can be decreased by increasing the sample size while decreasing the sampling interval. Inevitably, however, this reduction-in-error cost would lead to an increase in the cost associated with sampling and testing. The costs associated with the Type I error can also be reduced by decreasing the critical region. Due to the inverse relationship that exists between the Type I and Type II error this decrease of the critical region will lead to an increase in the probability of the Type II error.

The average run length (ARL) is the average number of samples taken before the control chart signals a true out-of-control condition (ARL_1) or signals a false alarm (ARL_0). This gives an indication of the statistical effectiveness of the control scheme.

According to Woodall (1985) there are two basic approaches to designing control charts, economic design and statistical design. "The aim of economic-statistical design is to minimise the cost function while simultaneously imposing constraints on the statistical performances." (Yang and Rahim; 2005) These constraints are applied to the Type I and Type II errors.

According to Yang and Rahim (2005) the optimal design parameters of the proposed T^2-control chart with statistical properties can be obtained by minimising the above cost function subject to the constraints $\alpha = \alpha_0$ and $\beta = \beta_0$, where α_0 and β_0 are the required upper bounds of α and β, respectively. The constraint $\alpha = \alpha_0$ corresponds to the constraint that the max of $ARL_0 = 1/\alpha$, similarly the constraint $\beta = \beta_0$ corresponds to the constraint that the min of $ARL_1 = 1/(1-\beta)$.

Zhang and Berardi (1997) argue that there is no general rule for the selection of the constraints on the economic statistical control charts. The constraints should be chosen based on the specific conditions of each specific problem, the relevant cost information, as well as the economic and statistical consequences. The designer is assisted by the sensitivity analysis is making these decisions.

CHAPTER SEVEN: *CONCLUDING REMARKS*

7.1 CONCLUSION:

The design of control charts can be classified into four general categories: heuristic; economic; statistical; and economic statistical. A brief discussion of the historical background and the development of the heuristic design of Shewhart were given. It was noted that the heuristic design does not take the economic costs associated with control charts into account. Duncan suggests a design for control charts that aims to minimise the costs. A cost model was developed by Duncan that is used by researchers as a basis for most of the later cost models.

It was noted that the pure economic design of control charts does have some shortcomings and the pure statistical design of control charts was developed with the aim of overcoming these shortcomings. Saniga (1989) was the first to attempt to combine these two methods of design in order to harness the advantages and eliminate the disadvantages of the respective methods. The economic statistical design places statistical constraints on the pure economic model. The advantages of the economic statistical design over the economic and statistical designs are discussed briefly.

The distribution of the occurrence of the assignable cause is assumed by Duncan to follow an exponential distribution. Various other distributions have also been suggested, for example, the Weibull distribution. Rahim and Banerjee (1993) introduce the concept of early replacement for three different distributions, namely, the Weibull, the Gamma and the exponential distribution. Arguments for the exponential approximation to the Weibull distribution were also presented.

The development of the cost model of Yang and Rahim (2005) under the Weibull shock model was discussed and expressions for the expected cycle time and expected cycle cost were derived.

Regardless of the large amount of research done to develop different models for various situations, the field of quality control still remains open for future research. One area that still needs to be researched is the development of a multivariate cost model based on the Gamma distribution. Some research has been done by Al-Oraini and Rahim (2002) who consider the Gamma distribution, which allows for increasing hazard rates.

The Gamma distribution also has important practical applications, i.e. considering a standby redundant process with two components having a perfect switch. While the first component is on, the second is off, until component 1 fails, when the switch turns on component 2. If the life of each component can be described by an exponential distribution with parameter λ then the process life has a Gamma distribution with scale parameter λ and shape parameter $\nu = 2$. A Gamma distribution of control periods with an increasing hazard rate is assumed and comparisons between the results of the economic statistical design with those of a pure economic model are drawn. Al-Oraini and Rahim (2002) show that the statistical performance of control charts can be improved dramatically with only a slight increase in the cost relative to the economic model. They also conclude for the economic statistical designs, the accurate specification of λ is required given that different values of λ have a noteworthy impact on the expected cost. Additional research on this distribution could prove to be valuable.

The use of a generalised lamda distribution to model the distribution of the occurrence of the assignable causes is also still open to research as well. There also remains scope for the use of a Bayesian approach to control charts.

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APPENDIX

APPENDIX A:

The proof of the theorem providing the expressions for $E(T)$ and $E(C)$ in Rahim and Banerjee (1993) will be given along with the derivation of $E(T)$ and $E(C)$ for the exponential and Gamma shock model.

Let $E(T_j)$, j = 0,1,2, ..., $m-1$, be the expected residual time in the cycle beyond time ω_j given that the process was functioning in the in-control state at time ω_j. If there has been a false alarm at time ω_j, $E(T_j)$ has to be equal to the expected residual time in the cycle beyond ω_j plus the time searching for the false alarm. Define $E(T_0) = E(T)$. Let $E(R_j)$ be the expected residual time in the cycle beyond time ω_j given that the process was functioning in the out-of-control state at time ω_j and a true alarm has not been signalled so far. It follows that $E(R_{m-1}) - Z_1 = h_m$, and for j = 1,2, ... , $m-2$,

$$E\left(R_j\right) - Z_1 = \mathrm{h}_{\mathrm{j}+1} + \beta\left\{\mathrm{E}\left(\mathrm{R}_{\mathrm{i}=\mathrm{j}+1}\right) - \mathrm{Z}_1\right\} = \sum_{\mathrm{i}=\mathrm{j}+1}^{\mathrm{m}} \beta^{\mathrm{i}-\mathrm{j}-1}\mathrm{h}_{\mathrm{i}}.$$

Further, let $p_0 = 0$ and p_j be the conditional probability that the process shifts to the out-of-control state during the time interval (ω_{j-1}, ω_j) given that the process was in control at time ω_j for j = 1,2, ... , m. In other words,

$$p_j = \frac{\nabla \mathrm{F}(\omega_j)}{\overline{\mathrm{F}}(\omega_{j-1})}.$$

We are now able to derive the following identities:

$$\nabla F\left(\omega_j\right) = p_j \prod_{i=0}^{j-1}(1-p_i) \text{ and } \bar{F}\left(\omega_j\right) = \prod_{i=1}^{j}(1-p_i).$$

Once the possible states of the system at the end of the sampling interval has been considered, an expression for $E(T)$ can be obtained. For each possible state the expected residual time in the cycle as well as the associated probabilities will be presented in Table A.1.

Table A.1.

The expected residual time

State	Expected Residual Cycle Time	Probability
In control and no alarm	$E(T)$	$(1-p_1)(1-\alpha)$
In control and a false alarm	$Z_0 + E(T_1)$	$(1-p_1)\alpha$
Out of control but no alarm	$E(R_1)$	$p_1\beta$
Out of control and true alarm	Z_1	$p_1(1-\beta)$

Source: Rahim and Banerjee (1993)

Now, an expression for $E(T)$ is derived, namely

$$E(T) = h_1 + (1-\beta)Z_1p_1 + p_1\beta E(R_1) + \alpha Z_0(1-p_1) + (1-p_1)E(T_1).$$

Similarly, we have, for $j = 1,2, \dots, m-2$,

$$E\left(T_j\right) = h_{j+1} + (1-\beta)Z_1 p_{j+1} + p_{j+1}\beta E\left(R_{j+1}\right) + \alpha_0 Z_0\left(1 - p_{j+1}\right) + \left(1 - p_{j+1}\right) E\left(T_{j+1}\right).$$

Since no sampling takes place during the last sampling interval, the expression for $E(T_{m-1})$ is given by:

$$E(T_{m-1}) = h_m + Z_1.$$

Now,

$$E(T) = \sum_{j=1}^{m} h_j \bar{F}\left(\omega_{j-1}\right) + Z_1\left\{\sum_{j=1}^{m-1} \nabla F\left(\omega_j\right) + \bar{F}(\omega_{m-1})\right\} + \beta \sum_{j=1}^{m-1} \nabla F(\omega_j)\left\{E\left(R_j\right) - Z_1\right\} + \alpha Z_0 \sum_{j=1}^{m-1} F(\omega_j).$$

Simplifying gives

$$E(T) = \sum_{j=1}^{m} h_j \bar{F}\left(\omega_{j-1}\right) + \alpha Z_0 \sum_{j=1}^{m-1} \nabla F\left(\omega_j\right) + \beta \sum_{j=1}^{m-1} \nabla F(\omega_j) \sum_{i=j+1}^{m} h_i \beta^{i-j-1} + Z_1.$$

Now to obtain $E(C)$ we define $E(C_j)$ for j = 1,2, … , $m-2$, to be the expected residual cost beyond time ω_j, given that the process is in the in-control state at time ω_j. Now let τ_j be the conditional expected in-control duration within the time interval (ω_{j-1}, ω_j) given that the shift to the out-of-control state occurred during the sampling interval (ω_{j-1}, ω_j), i.e.

$$\tau_j = \left[\int_{\omega_{j-1}}^{\omega_j} \left(x - \omega_{j-1}\right) f(x) dx\right] / \left[\nabla F(\omega_j)\right].$$

Let Q_i be the expected number of samples conducted after time ω_j given that the process is at out-of-control state at time ω_j and a true alarm has not yet been observed at time ω_j. Then

$$Q_{m-1} = 0,$$

$$Q_{m-2} = 1$$

and for all $j = 1,2, \ldots, m-3$,

$$Q_j = \sum_{i=1}^{m-1-j} i(1-\beta)\beta^{i-1} + (m-1-j)\beta^{m-1-j}.$$

Now the expected residual cost beyond time h_1 for each possible state of the system at the end of the first sampling interval is determined,

$$E(C) = a + bn + p_1[D_0\tau_1 + D_1(h_1 - \tau_1)] + p_1(1-\beta)W + p_1\beta[W + D_1\{E(R_1) - Z_1\} + (a + bn)Q_1] + (1-p_1)E(C_1) + (1-p_1)\alpha Y + (1-p_1)D_0h_1.$$

And for $j = 1,2, \ldots, m-2$,

$$\begin{aligned} E\left(C_j\right) = a + bn + p_{j+1}\left[D_0\tau_{j+1} + D_1(h_{j+1} - \tau_{j+1})\right] + p_{j+1}(1-\beta)W \\ + p_{j+1}\beta\left[W + D_1\left\{E\left(R_{j+1}\right) - Z_1\right\} + (a+bn)Q_{j+1}\right] + \left(1 - p_{j+1}\right)E\left(C_{j+1}\right) \\ + \left(1 - p_{j+1}\right)\alpha Y + \left(1 - p_{j+1}\right)D_0h_{j+1} \end{aligned}$$

and

$$E(C_{m-1}) = p_m[D_0\tau_m + D_1(h_m - \tau_m) + W] + (1-p_m)[D_0h_m + W - S(\omega_m)].$$

Equivalently,

$$E(C) = (a+bn) + \left[1 + \sum_{j=1}^{m-2} \prod_{i=1}^{j}(1-p_i) + \beta \sum_{j=1}^{m-1} p_j Q_j \prod_{i=0}^{j-1}(1-p_i)\right] + (D_0 - D_1)\left[\sum_{j=1}^{m} p_j \left(\tau_j - h_j\right) \prod_{i=0}^{j-1}(1-p_i)\right] + D_0\left[\sum_{j=1}^{m} h_j \prod_{i=0}^{j-1}(1-p_i)\right] + W\left[\sum_{j=1}^{m} p_j \prod_{i=0}^{j-1}(1-p_i) + (1-p_m)\prod_{i=1}^{m-1}(1-p_i)\right] + \beta D_1\left[\sum_{j=1}^{m-1} p_j \left\{E\left(R_j\right) - Z_1\right\} \prod_{i=0}^{j-1}(1-p_i)\right] + \alpha Y\left[\sum_{j=1}^{m-1} \prod_{i=1}^{j}(1-p_i)\right] - S(\omega_m)\prod_{i=1}^{m}(1-p_i).$$

Simplifying gives

$$E(C) = D_0 \sum_{j=1}^{m} h_j \bar{F}(\omega_{j-1}) + \alpha Y \sum_{j=1}^{m-1} \bar{F}\left(\omega_j\right) + (D_0 - D_1)\int_0^{\omega_m} x f(x)dx + D_1\beta\left[\sum_{j=1}^{m-1} \nabla F(\omega_j) \sum_{i=j+1}^{m} h_i \beta^{i-j-1}\right] + (a+bn)\left[1 + \sum_{j=1}^{m-2} \bar{F}\left(\omega_j\right) + \beta \sum_{j=1}^{m-2} F\left(\omega_j\right)\left\{(1-\beta)\sum_{i=1}^{m-1-j} i\beta^{i-1} + (m-1-j)\beta^{m-1-j}\right\}\right] + W - \overline{F}(\omega_m)S(\omega_m).$$

The expected cycle time and expected cost for the exponential shock model:

The expression for $E(T)$ and $E(C)$ for the exponential shock model under a uniform sampling scheme is given by

$$E(T) = \frac{hexp(-\lambda h)}{1-\exp(-\lambda h)} + \alpha Z_0 \exp(-\lambda h)/(1 - exp(-\lambda h\backslash)) + h_1/(1-\beta) + Z_1$$

and

$$E(C) = D_0\left(h + \frac{hexp(-\lambda h)}{1-exp(-\lambda h)} + \frac{\alpha Y exp(-\lambda h)}{1-exp(-\lambda h)} + \frac{D_0 - D_1}{\lambda} + (D_1 - D_0)\frac{h}{1-exp(-\lambda h)} + \frac{D_1 h\beta}{1-\beta} + (a-bn)(\frac{1}{1-\beta} + exp(-\lambda h)/(1-exp(-\lambda h))\right) + W.$$

The expected cycle time and expected cycle cost for the Gamma shock model:

$$E(T) = h_1 + (\alpha Z_0 + h_2)\frac{exp(-\lambda h_2)}{1 - exp(-\lambda h_2)}\left[1 + \lambda h_1 + \frac{\lambda h_2 exp(-\lambda h_2)}{1 - exp(-\lambda h_2)}\right] + h_2\frac{\beta}{1-\beta} + Z_1$$

and

$$E(C) = (a + bn + \alpha Y + D_1 h_2)\frac{exp(-\lambda h_1)}{1-exp(-\lambda h_2)}\left[1 + \frac{\lambda h}{exp(-\lambda h)}\right] + \frac{a+bn}{1-\beta} + \frac{2}{\lambda}D_0 + D_1\left[\frac{h}{1-\beta} - \frac{2}{\lambda}\right] + W.$$

APPENDIX B:

The theorem that provides the expressions for $E(T)$ and $E(C)$ in Banerjee and Rahim (1988) will be given as well as the two lemmas on which the proof is dependent. Some of the groundwork associated with the Weibull distribution is presented by Banerjee and Rahim (1988) in the following lemma.

Lemma 1.1. Let $f(t)$ be defined by

$$f(t) = \lambda\theta t^{(k-1)} exp\{-\lambda t^{\theta}\}, \text{ for } t > 0,\ \theta \geq 1,\ \lambda > 0$$

and the h_j's be defined by

$$h_j = \left[j^{1/\theta} - (j-1)^{1/\theta}\right] h_1.$$

Also, let $A(x) = \sum_{v=0}^{\infty}(v+1)^{1/\theta}x^{v}$ for $|x| < 1$. Then the following is true:

$$p_j = 1 - \exp\left(-\lambda h_1^{\theta}\right) \text{ for all } j = 1,2, \ldots.$$

Therefore, let $p_j = p$ for all $j = 1,2, \ldots$

$q_1 = p$ and $q_{j+1} = (1-p)^j p$ for all $j = 1,2, \ldots$.

$$1 + \sum_{j=1}^{\infty}(1-p)^j = {}^1/_p.$$

$$\tau = \sum_{j=1}^{\infty} q_j \tau_j = \left({}^1/_\lambda\right)^{1/\theta} \Gamma\left(1 + {}^1/_\theta\right) - \mathrm{h}_1(1-\mathrm{p})\mathrm{pA}(1-\mathrm{p}).$$

$$h_1 p + \sum_{j=2}^{\infty} h_j p(1-p)^{j-1} = h_1 p^2 A(1-p).$$

$$\sum_{j=1}^{\infty} q_j \sum_{i=j+1}^{\infty} h_i \beta^{i-j-1} = \frac{h_1 p[pA(1-p)-(1-\beta)A(\beta)]}{(1-p-\beta)}.$$

The proof of this lemma follows directly from the substitution of the values of p_j and h_j. The following theorem will provide an expression for $E(C)$ and $E(T)$.

Theorem B.1:

The following is true:

$$E(T) = Z_1 + Z_2 + \frac{\alpha Z_0(1-p)}{p} + h_1 pA(1-p) + \frac{\beta h_1 p[pA(1-p)-(1-\beta)A(\beta)]}{(1-p-\beta)}.$$

$$E(C) = (a+bn)\left[\frac{\beta}{1-\beta}+\frac{1}{p}\right] + \frac{\alpha Y(1-p)}{p[D_0-D_1]\left(\frac{1}{\lambda}\right)^{\frac{1}{\theta}}\Gamma\left(1+\frac{1}{\theta}\right)} + D_1h_1p(1-p)A(1-p) + \beta h_1D_1p[pA(1-p) -$$

$$(1-\beta)A(\beta)] \div [1-p-\beta] + D_1h_1p^2A(1-p) + W,$$

where $A(x) = \sum_{v=0}^{\infty}(1+v)^{1/\theta}x^{v}$ and $p = 1-\exp\left(-\lambda h_1^{\theta}\right)$.

The procedure proposed by Banerjee and Rahim (1988) views the process at the end of the first sampling interval. Systems of equations can be formulated by taking into account all possible states of the process, the residual times in the cycle, and the probabilities associated with these states. Suppose we consider the beginning of the second sampling interval and the unit being used in the process is in the operating state. Let T_1 be the residual time in the cycle beyond time h_1, and let C_1 be the residual cost in the cycle that will be incurred beyond time h_1 given that the process had not failed at time h_1. $T_2, T_3, \dots$ and $C_2, C_3, \dots$ are defined in a similar fashion. For $j = 0,1,2,\dots$, T_j is the residual time in the cycle beyond time ω_j and C_j is the residual cost beyond time ω_j given that the process is in the in-control state at time ω_j. It is clear that $T_0 = T$ and $C_0 = C$. The proof of the Theorem B.1 depends on the following two lemmas.

Lemma B.1:

The following statements are true:

$$E(T) = h_1 + (Z_1+Z_2)p + p\sum_{i=1}^{\infty}h_{i+1}\beta^{i} + \alpha Z_0(1-p) + (1-p)E(T_1). \qquad \text{(B.1)}$$

$$E(T_1) = h_2 + (Z_1+Z_2)p + p\sum_{i=1}^{\infty}h_{i+2}\beta^{i} + \alpha Z_0(1-p) + (1-p)E(T_2). \qquad \text{(B.2)}$$

For $j = 2,3,\dots$

$$E\left(T_{j-1}\right) = h_j + (Z_1 + Z_2)p + p\sum_{i=1}^{\infty} h_{i+j}\beta^i + \alpha Z_0(1-p) + (1-p)E(T_j). \qquad \text{(B.3)}$$

Proof:

We view the process at the end of the first sampling interval. Table 6.1 gives a list of the states that are all possible at the end of the first sampling interval; it also provides the corresponding expected value of the residual time in the cycle, as well as the associated probabilities. Therefore,

$$E(T) = h_1 + \{Z_1 + Z_2\}(1-\beta)p + \left\{Z_1 + Z_2 + (1-\beta)\sum_{i=1}^{\infty}(\omega_{i+1} - h_1)\beta^{i-1}\right\}p\beta + \{E(T_1)\}(1-p)(1-\alpha) + \{Z_0 + E(T_1)\}(1-p)\alpha.$$

Simplifying, we have

$$E(T) = h_1 + (Z_1 + Z_2)p + \left\{\sum_{i=1}^{\infty} h_{i+1}\beta^i\right\}p + \alpha Z_0(1-p) + (1-p)E(T_1).$$

This proves equation B.1, similar arguments will lead to B.2 and B.3.

Lemma B.2:

The following statements are true:

$$E(C) = a + bn + (D_0 - D_1)\tau_1 p + Wp + \frac{(a+bn)\beta p}{1-\beta} + D_1\left\{\sum_{i=1}^{\infty} h_{i+1}\beta^i\right\}p + D_0 h_1(1-p) + \alpha Y(1-p) + (1-p)E(C_1) + D_1 h_1 p. \qquad \text{(B.4)}$$

For $j = 2,3,\ldots,$

$$E\left(C_{j-1}\right) = a + bn + (D_0 - D_1)\tau_1 p + Wp + \frac{(a+bn)\beta p}{1-\beta} + D_1\left\{\sum_{i=1}^{\infty} h_{i+1}\beta^i\right\}p + D_0 h_j(1-p) + \alpha Y(1-p) + (1-p)E\left(C_j\right). \qquad \text{(B.5)}$$

Proof:

Once again, consider the state of the process at the end of the first sampling interval. Table 6.2 provides the possible states and the total expected cost per cycle for each state. The associated probabilities are the same as stated in Table 6.1. Therefore,

$$E(C) = \{a + bn + D_0\tau_1 + D_1(h_1 - \tau_1) + W\}(1-\beta)p + \left\{a + bn + D_0\tau_1 + D_1(h_1 - \tau_1) + W + \frac{a+bn}{1-\beta} + D_1(1-\beta)\sum_{i=1}^{\infty}(\omega_{i+1} - h_1)\beta^{i-1}\right\}\beta p + \{a + bn + D_0 h_1 + E(C_1)\}(1-\alpha)(1-p) + \{a + bn + D_0 h_1 + Y + E(C_1)\}\alpha(1-p).$$

Simplifying, leads to

$$E(C) = a + bn + (D_0 - D_1)\tau_1 p + Wp + \frac{(a+bn)\beta p}{(1-\beta)} + D_1\left\{\sum_{i=1}^{\infty} h_{i+1}\beta^i\right\}p + D_0 h_1(1-p) + \alpha Y(1-p) + (1-p)E(C_1) + D_1 h_1 p.$$

Further simplification gives equation (B.4). In a similar fashion, equation (B.5) can be proven.

Proof of Theorem B.1:

Lemma B.1 gives a set of recursive systems in the forms of $E(T)$, $E(T_1)$, $E(T_2)$, etc. The system can be solved to obtain an expression for $E(T)$. Lemma B.1 implies that

$$E(T) = [h_1 + (1-p)h_2 + (1-p)h_3 + \cdots] + (Z_1 + Z_2)\,[p + (1-p)p + (1-p)^2 p + \cdots] + p\sum_{i=1}^{\infty} h_{i+1}\beta^i + (1-p)p\sum_{i=1}^{\infty} h_{i+2}\beta^i + (1-p)^2 p\sum_{i=1}^{\infty} h_{i+3}\beta^i + \cdots + \alpha Z_0\,[(1-p) + (1-p)^2 + (1-p)^3 + \cdots].$$

Results from Theorem B.1 imply that

$$E(T) = h_1 pA(1-p) + Z_1 + Z_2 + \frac{\beta p h_1[(1-\beta)A(\beta) - pA(1-p)]}{[\beta-(1-p)]} + \alpha Z_0(1-p)/p.$$

This proves

$$E(T) = Z_1 + Z_2 + \frac{\alpha Z_0(1-p)}{p} + h_1 pA(1-p) + \frac{\beta h_1 p[pA(1-p) - (1-\beta)A(\beta)]}{(1-p-\beta)}.$$

Proceeding in a similar fashion, the expression for $E(C)$ can be obtained by using Lemma B.2, which leads to

$$E(C) = (a+bn)\,[1 + (1-p) + (1-p)^2 + \cdots] + (D_0 - D_1)\,[p\tau_1 + (1-p)p\tau_2 + (1-p)^2 p\tau_3 + \cdots] + [W + (a+bn)\beta/(1-\beta)] \times [p + (1-p)p + (1-p)^2 p + \ldots] + D_1\left[p\sum_{i=1}^{\infty} h_{i+1}\beta^i + (1-p)p\sum_{i=1}^{\infty} h_{i+2}\beta^i + \cdots\right] + D_0\,[h_1(1-p) + h_2(1-p)^2 + h_3(1-p)^3 + \cdots] + \alpha Y\,[(1-p) + (1-p)^2 + \cdots] + D_1\,[h_1 p + h_2(1-p)p + h_3(1-p)^2 p + \cdots].$$

Results from Theorem B.1 imply that

$$E(C) = \frac{a+bn}{p} + (D_0 + D_1)\tau + W + \frac{(a+bn)\beta}{1-\beta} + \frac{D_1\beta p h_1[(1-\beta)A(\beta) - pA(1-p)]}{[\beta-(1-p)]} + D_0 h_1 p(1-p)A(1-p) + \frac{\alpha Y(1-p)}{p} + D_1 p^2 h_1 A(1-p).$$

Further simplification proves

$$E(C) = (a+bn)\left[\frac{\beta}{1-\beta} + \frac{1}{p}\right] + \frac{\alpha Y(1-p)}{p[D_0 - D_1]\left(\frac{1}{\lambda}\right)^{\frac{1}{k}}\Gamma\left(1+\frac{1}{k}\right)} + D_1 h_1 p(1-p)A(1-p) + \beta h_1 D_1 p[pA(1-p) - (1-\beta)A(\beta)] \div [1-p-\beta] + D_1 h_1 p^2 A(1-p) + W.$$