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Pricing and Hedging Asian Options Using Monte Carlo and Integral Transform Techniques

by

Trust Chibawara

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Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and has not previously, in its entirety or in part, been submitted at any university for a degree.

Trust Chibawara

Date

Abstract

In this thesis, we discuss and apply the Monte Carlo and integral transform methods in pricing options. These methods have proved to be very effective in the valuation of options especially when acceleration techniques are introduced. By first pricing European call options we have motivated the use of these methods in pricing arithmetic Asian options which have proved to be difficult to price and hedge under the Black−Scholes framework. The arithmetic average of the prices in this framework, is a sum of correlated lognormal distributions whose distribution does not admit a simple analytic expression. However, many approaches have been reported in the academic literature for pricing these options. We provide a hedging strategy by manipulating the results by Geman and Yor [42] for continuous fixed strike arithmetic Asian call options. We then derive a double Laplace transform formula for pricing continuous Asian call options following the approach by Fu et al. [39]. By applying the multi-Laguerre and iterated Talbot inversion techniques for Laplace transforms to the resulting pricing formula we obtain the option prices. Finally, we discuss the shortcomings of using the Laplace transform in pricing options.

Opsomming

In hierdie tesis bespreek ons Monte Carlo- en integraaltransform metodes om die pryse van finansiële opsies te bepaal. Hierdie metodes is baie effektief, veral wanneer versnellingsmetodes ingevoer word. Ons bepaal eers die pryse van Europese opsies as motivering, voordat ons die bostaande metodes gebruik vir prysbepaling van Asiatiese opsies met rekenkundige gemiddeldes, wat baie moeiliker is om te hanteer in die Black−Scholes raamwerk. Die rekenkundige gemiddelde van batepryse in hierdie raamwerk is 'n som van gekorreleerde lognormale distribusies wie se distribusie nie oor 'n eenvoudige analitiese vorm beskik nie. Daar is egter talle benaderings vir die prysbepaling van hierdie opsies in die akademiese literatuur. Ons bied 'n verskansingsstrategie vir Asiatiese opsies in kontinue tyd met 'n vaste trefprys aan deur die resultate van Geman en Yor [42] te manipuleer. Daarna volg ons Fu et al. [39] om 'n dubbele Laplace transform formule vir die pryse af te lei. Deur toepassing van multi-Laguerre en herhaalde Talbotinversie tegnieke vir Laplace transforms op hierdie formule, bepaal ons dan die opsiepryse. Ons sluit af met 'n bespreking van die tekortkominge van die gebruik van die Laplace transform vir prysbepaling.

Dedication

To my grandfather (late) and grandmother who nurtured me to grow into a hardworking person and inspired me to appreciate the beauty and the fulfillment of a life well lived,

Mr. and Mrs. Nkoma.

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Chapter 1

Introduction

Options[1] have become extremely popular; so popular that in many cases more money is invested in them than in the underlying assets. They are extremely attractive to investors both for speculation and for hedging and this may largely be owing to the fact that there is a systematic way to determine how much they are worth and hence they can be bought and sold with some confidence.

Explicit analytic formulas are available for the fair price of standard European call and put options written on a stock whose price is modeled by a geometric Brownian motion. However, for more complicated derivatives there is no closed form analytic formula for pricing these options. Such derivatives are usually priced by Monte Carlo simulation or by numerical methods. These options have nonstandard features and almost unlimited flexibility in the sense that they can be tailored to the specific needs of any investor. An important class of such options is the class of Asian options.

Asian options have a wide variety of application in commodities, currency, energy, interest rates, equity and insurance markets. The name 'Asian' option emerged in 1987 when a Banker's Trust Tokyo office used it for pricing average options on crude oil contracts. Unlike vanilla options, Asian options are path dependent options whose payoff is based on the average of the underlying asset price over an interval of time. If the average is computed using a finite sample of asset price observations (usually taken at a set of regularly spaced

[1]An option is a contract between two parties in which one party has the right but not the obligation to buy or sell some underlying asset. The underlying asset could be anything from a commodity, to equity or currency.

time points) we have a discrete Asian option, as opposed to continuous Asian options that are obtained by the average via the integral of the price path over an interval of time.

There are two main classes of Asian options: Floating strike and Fixed strike. The floating strike Asian option pays the difference between the average and the spot price of the underlying asset while the fixed strike pays the difference between the average of the underlying asset and the pre−specified strike price. Moreover, the average can be either arithmetic or geometric, however the geometric type of averaging is relatively uncommon and not used in practice.

The payoff at time T of an arithmetic Asian option is given by $X = (A - K)^+$ while that of the geometric Asian option is given by $X = (G - K)^+$, where Y^+ means $\max(Y, 0)$,

$$A = \begin{cases} \frac{1}{N}\sum_{i=1}^{N} S_{t_i} \\ \\ \frac{1}{T}\int_0^T S_t dt \end{cases} \qquad G = \begin{cases} \left(\prod_{i=1}^{N} S_i\right)^{\frac{1}{N}} \\ \\ \exp\left(\frac{1}{T}\int_0^T \ln(S_t)\, dt\right). \end{cases}$$

Note that when we are pricing the floating strike Asian option, we can simple replace K (the fixed strike) by the price of the asset at the expiration date T (the spot price, i.e. $K = S_T$) on the above price settings.

1.1 The General Framework

Throughout this thesis, we shall use the Black−Scholes type model [7]. In that framework, we consider a financial market with finite time horizon T which consists of a riskless asset B_t with deterministic interest rate whose dynamics are modeled by $dB_t = rB_t dt$ and a risky asset with a positive price process S_t. We represent the randomness of the economy by the probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_t, \mathbb{P})$, where $(\mathcal{F}_t)_t$ is the filtration of the available information available and $\mathbb{P}$ is the "real world" probability measure. We shall assume that the dynamics of the risky asset are modeled by a stochastic process $\{S_t\}_{t\geq 0}$ satisfying the stochastic differential equation given by

$$dS_t = \mu S_t dt + \sigma S_t d\widetilde{W}_t$$

where μ and σ are constants representing the drift and the volatility of the stock price respectively and $\{\widetilde{W}_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion. Introducing the concept of no arbitrage[2], we know from the Girsanov transform (Appendix A) that there exists a risk neutral measure $\mathbb{Q}$ under which the dynamics of S_t becomes

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{1.1}$$

where $\{W_t\}_{t\geq 0}$ is a $\mathbb{Q}$-Brownian motion and r is the constant risk-free interest rate. Using the risk neutral valuation formula, the values at time t of any option of financial contract maturing at time T is given by

$$V_{t,T}(S_t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t), \tag{1.2}$$

where the price of the underlying asset is given by the dynamics in equation (1.1), X is the random variable which gives the option payoff at maturity and $\mathbb{E}_{\mathbb{Q}}(\cdot)$ is the expectation taken under the risk neutral measure $\mathbb{Q}$. The payoff function is what makes each financial contract unique.

A major element in deriving the price of any contingent claim or derivative security is the construction of the hedging or replicating portfolio. Contingent claim or derivative security meaning any financial instrument whose pay$-$off is contingent upon or derived from the behavior of some other underlying asset. In fact, the theoretical price of any claim exists precisely because the claim can be replicated.

A hedging strategy $\Psi = (\psi_t, \phi_t)$ consisting of $\{\psi_t\}$ shares of riskless asset and $\{\phi_t\}$ shares of risky asset held in the portfolio at time t, will be defined as a measurable process adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Consequently, the value or the wealth of the portfolio at time t will be given by

$$V_t(\Psi) = \psi_t B_t + \phi_t S_t.$$

We denote the discounted price of the risky asset (which is a martingale under $\mathbb{Q}$) by $\widetilde{S}_t = e^{-rt}S_t$ and the discounted wealth process by $\widetilde{V}_t(\Psi) = e^{-rt}V_t(\Psi)$. We now state and prove below some of the results that would be utilized in this thesis.

Proposition 1.1.1. *Let $\{\psi_t\}_{0\leq t\leq T}$ and $\{\phi_t\}_{0\leq t\leq T}$ be predictable processes satisfying*

$$\int_0^T |\psi_t|\, dt + \int_0^T |\phi_t|^2\, dt < \infty \quad a.s.$$

[2]That is, there is no trading strategy that requires the investment of no capital and yields free money without risk.

Then $(\psi_t, \phi_t)_{0\le t\le T}$ *defines a self-financing strategy if and only if*

$$\widetilde{V}_t(\Psi) = \widetilde{V}_0(\psi_0, \phi_0) + \int_0^t \phi_u d\widetilde{S}_u \quad a.s \text{ for all } t \in [0,T].$$

Proof. Suppose that the portfolio $(\psi_t, \phi_t)_{0\le t\le T}$ is self financing, then

$$\begin{aligned}
d\widetilde{V}_t &= -re^{-rt}V_t dt + e^{-rt}dV_t \\
&= -re^{-rt}\left(\psi_t e^{rt} + \phi_t S_t\right)dt + e^{-rt}\psi_t d\left(e^{rt}\right) + e^{-rt}\phi_t dS_t \\
&= \phi_t\left(-re^{-rt}S_t dt + e^{-rt}dS_t\right) \\
&= \phi_t d\widetilde{S}_t.
\end{aligned}$$

□

Definition 1.1.2. *An option is replicable if its value at time* T *is equal to the value* $V_T(\Psi) = \psi_T B_T + \phi_T S_T$ *of an admissible strategy* Ψ. *Thus in the sense of no arbitrage, the value of the option must be the same as the cost of constructing the replicating portfolio.*

In the Black−Scholes model the option prices can be obtained via a partial differential equation known as the Black−Scholes equation [7] and is given as

$$-rV(t,x) + \frac{\partial V}{\partial t}(t,x) + rx\frac{\partial V}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2}(t,x) = 0 \tag{1.3}$$

with the terminal conditions $V(T,x) = X$, where X is the random variable which gives the option payoff at maturity. The solution to this partial differential equation for the price of an European call option for example, is given by the Black−Scholes formula

$$V(t,S_t) = S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-), \tag{1.4}$$

where

$$d_\pm = \frac{\ln\left(S_t/K\right) + \left(r \pm \frac{\sigma^2}{2}(T-t)\right)}{\sigma\sqrt{T-t}}$$

and $\Phi(\cdot)$ is the standard normal distribution function, given by

$$\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{y^2}{2}}dy.$$

An European call option on the underlying asset S and strike price K can be seen as an asset that pays to its holder a payoff $X = (S_T - K)^+$ at date T. European here meaning the option can only be exercised at exactly the maturity date T.

Under our framework, the asset price follows a geometric Brownian motion which implies that the asset price at any future time is described by the lognormal density function. If therefore, an Asian option is based on geometric average, the average is still lognormally distributed because the product of lognormal random variables remains lognormal. Kemna and Vorst [53] have shown that it is possible to derive explicit formulas for geometric average Asian options.

In contrast, if the Asian option is based on arithmetic average, there is no simple explicit representation for the distribution on the average of the underlying asset price because the sum of lognormal random variables is not lognormally distributed any more. Hence there is no explicit simple formula to price the arithmetic Asian option. We shall provide in the next section an overview of research related to the valuation of these options.

1.2 Overview of techniques for pricing Arithmetic Asian options

Arithmetic Asian options are very popular in the financial community for several reasons, one of which is the fact that they are based on an average price. This makes them attractive for thinly traded assets and commodities such as gold or crude oil, where price manipulations which could be done for example by putting through large buy orders to bid up the price near the option expiration date are possible. Notably, some options on domestic interest rates and options on interest rate swaps, exhibit this Asian feature when the base rate is an arithmetic average of spot rates [42].

The interest by academics and practitioners alike to learn more about the use of the arithmetic Asian option due to their wide variety of application has made the pricing of these options an important subject of intensive research. Several techniques have been proposed in the literature to tackle the difficulty in pricing these options and can generally be classified as follows:

1. **Monte Carlo Simulation.**
   Various methods using the Monte Carlo simulation have been offered to price Asian option (some of which we shall explore later). Kemna and Vorst [53] use Monte Carlo

simulations with the geometric based discrete Asian option as a control variate to reduce the variance of the Asian option price. On the other hand, Joy et al. [50] use quasi−Monte Carlo method, a method that uses deterministic numbers as opposed to random numbers, to price discrete arithmetic Asian options.

2. **Fourier and Laplace Transform Methods.**
   Though the price of the arithmetic Asian option does not have a closed form representation, Geman and Yor [42] computed for the first time its Laplace transform using Bessel processes. The option prices are then obtained by inverting numerically this transform. Carvehill and Clewlow [18] use the fast Fourier transform to calculate the density of the sum of random variables as the convolution of individual densities and then numerically integrated the payoff function against the density. Benhamou [5] improves this method by incorporating a re-centering step into the algorithm.

3. **Approximation of the Density of the Average.**
   Turnbull and Wakeman [72] approximate the true distribution with an alternative distribution by applying the Edgeworth series expansion up to the fourth term around the lognormal distribution function. By applying the work by Mitchell they assumed that this alternative distribution is lognormal, thereby obtaining the analytic approximation of the call price. Lévy [57] on the other hand, derived an approximating pricing formula for the discrete Asian option by matching the first two moments of the density of the average with that of the lognormal density. Milevsky and Posner [59] show that the density function for the infinite sum of correlated lognormal random variables is gamma distributed. The arithmetic Asian options are then valued by approximating the finite sum of correlated lognormal variables using the reciprocal gamma distribution as the state price density function.

4. **Binomial and Trinomial Trees.**
   Asian options can be priced using lattice/tree methods. At any point in time of the tree, the value of the option is dependent upon the average of the price that the path has taken. As the number of nodes on the tree grows, so does the number of the averages that must be taken, particularly in the central nodes. Hull and White [49] argument an additional state variable to each node in the tree to record the possible averages of the underlying asset price realized between time zero and the time of that node. Approximation is taken with interpolation technique in backward induction.

Chao and Lee [20] improved it by deriving the maximum and minimum averages for each node and Hsu and Lynu [75] further improved it by using an optimization technique which yields non-uniform allocation scheme of states in each node that is determined by the Lagrange multiplier. As a remark, the lattice methods require large amounts of computer memory since they have to keep track of every possible path throughout the tree, in fact, they are effectively unusable in practice.

5. **Lower and Upper Bounds.**

   Rogers and Shi [65] provided lower and upper bounds for both fixed and floating strike Asian options by computing the expectation of a process based on some non-zero mean Gaussian variable, in the view that it remains a Gaussian process. They restricted their derivation to options with maturity of one year. Chen and Lyuu [19] then extended their formulas to general maturities. Thompson [70] derived upper bounds that are computationally effective and more accurate[3] than those obtained by Rogers and Shi while the lower bound gives similar results.

   J. Dhaene et al. [32] use the concept of comonotonicity from actuarial science and finance to derive "comonotonic bounds" for the price of the discrete arithmetic Asian options (see [33, 32] for an extensive overview of this method and related application). The lower bound that they derived is closely related to that derived by Rogers and Shi.

6. **Partial Differential Equations and Finite Difference Methods.**

   Nieuwveldt [61] derived well known one−dimensional PDE for pricing arithmetic Asian options (see [73, 74, 65]) following the approach by Dubois and Lelievre [35]. By evaluating a partial differential equation with smooth coefficients and zero initial conditions, Zhang [79] presented a semi-analytical approximate formula for pricing continuous Asian options. Interestingly their prices report an absolute error of the order 10^{-7} which is quite remarkable. In general, when Thompson's lower and upper bounds are compared with these prices we observe that their accuracy is 10^{-4} and 10^{-3} respectively.

[3] As an indication for the accuracy of Thompson's bounds relative to Rogers and Shi by comparison for instance, when $\sigma = 0.3$, $r = 0.09$, $S = 100$, $K = 100$ and maturity time $T = 1$, the lower and upper bounds for Thompson are [8.8275 8.8333] and those of Rogers and Shi are [8.8275 9.039] and the "Exact" price from Zhang [79] is 8.8287588.

Moreover, Zvan et al. [80] produced stable numerical PDE techniques which are adapted from the field of computational fluid dynamics for pricing American style Asian options with continuously sampled prices.

7. **Other Methods**
   Other approaches not discussed above include conditioning on the geometric mean price by Curran [26], pricing bounds by Nielsen and Sandmann [60] and series expansion methods by Dufresne [36] and Ju [51].

All these methods involve some tradeoffs between accuracy and computational efficiency. Our work will be centered on the first two approaches namely, Monte Carlo simulation and Fourier and Laplace transform methods. The primary focus will be in valuation of continuous fixed strike arithmetic Asian call options, whose value at time $t \leq T$ is defined by

$$V_{t,T}(S_t, K) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{1}{T-t}\int_t^T S_u du - K\right)^+\right]. \tag{1.5}$$

Here S_t denotes the stock price at time t, T is the maturity date, K is the strike price and $\mathbb{E}_{\mathbb{Q}}$ is the expectation taken under the risk-neutral measure $\mathbb{Q}$ which we shall often write as $\mathbb{E}$ thereby suppressing the subscript $\mathbb{Q}$.

1.3 Organization of the Thesis

In Chapter 2 we demonstrate the use of the fast Fourier transform (FFT) method in pricing options. To highlight the effectiveness of the method we shall first price the European call option whose solution is known in closed form, hence allowing for the comparison of our results with those obtained by the Black−Scholes formula and the Monte Carlo method. We go on to compare the computation speed of the FFT method with the Monte Carlo method. This pricing approach was introduced by Carr and Madan [17]. Because our objective is to price continuous Asian options, we utilize the remarkable speed of the FFT algorithm in computing the Fourier transform by extending its application to the computation of Fourier convolutions, an approach pioneered in the field of finance by Carverhill and Clewlow [18]. In that respect, by extrapolating the discrete Asian options we obtain prices for the continuous Asian options.

In Chapter 3 we provide a hedging strategy for the Geman and Yor [42] pricing formula for the continuous fixed strike Asian call options. This indicates how to invest the price of the option so that the investment reproduces the value of the option at maturity. We look at this in two parts: the first part where the call price can be obtained explicitly hence the hedging strategy can be obtained by differentiating the price with respect to the stock price (Delta hedging). In the second part the hedging strategy is obtained in the form of a Laplace transform since the option price has no explicit pricing formula. In both cases we use proposition (1.1.1) to obtain our hedging strategies.

In Chapter 4 we review Monte Carlo methods for pricing options. The Monte Carlo methods have proved to be powerful and flexible tools available for valuing many types of derivatives and other financial securities. We first discuss the application of the Monte Carlo method to the European call option then we extend our discussions to Asian options. We note, from a series of examples, that the introduction of variance reduction techniques within the standard Monte Carlo method plays a very significant role in pricing options.

Chapter 5 deals with the application of the Laplace transform method for pricing Asian options. We review the Laplace transform method and discuss the Laplace inversion methods in [2, 3, 68]. We provide a concrete example on pricing European call options using the Laplace transform method extracted from [41] in Appendix D, on which we focus on the Laguerre inversion method. Following the approach by Fu et al. [39], we derive a double Laplace transform formula for pricing Asian options. The implementation of inversion methods to obtain the option price is carried out on the double Laplace formula and the results obtained are compared with other known methods in literature by different authors. Furthermore, we performed a series of experiments on the double transform formula for large values of stock and strike prices using the Talbot method and the results are compared with the Monte Carlo methods as discussed in Chapter 4 and we took the results by Zhang [79] as our benchmark values. Finally, Chapter 6 contains the discussions and conclusions.

Our numerical computations are performed on a Pentium 4, 1.8 Ghz processor equipped with 1GB of RAM, MATLAB version 7.1.0.246 R(14) Service pack 3, Mathematica 5.0 and Python software. Attached is a CD with all the computational codes used here.

Chapter 2

Option Pricing Using the Fast Fourier Transform (FFT) Method

Our objective in this chapter is to price continuous fixed strike Asian call options using the fast Fourier transform method. To provide insight into the application of this method in pricing options, we shall price the European call option following the algorithm described by Madan and Carr [17]. The later case allows us to justify the effectiveness in the application of the FFT algorithm and because we have a closed form solution for pricing European call options (the Black−Scholes formula) we can compare our results with absolute certainty.

We go on to discuss the pricing algorithm for discrete arithmetic Asian options based on the fast Fourier transform method by Benhamou [5]. Then, by means of the Richardson extrapolation method [67], we price the continuous arithmetic Asian option. The property of the fast Fourier transform used in this method is its efficiency to calculate convolutions. As we pointed out earlier, the distribution of the average is not known; therefore, we shall numerically compute the density function by means of Fourier convolution. Having obtained the approximated density of the average we then compute the expected payoff by numerical integration and upon discounting by the risk free interest rate we obtain the option price.

2.1 Pricing European Call Options

In this section we shall describe the numerical approach for pricing European options which utilizes the characteristic function of the underlying asset price process. The basic idea behind this approach is to develop an analytic expression for the Fourier transform of the option price and then get the price by Fourier inversion. This approach was introduced by Carr and Madan [17] and is based on the FFT algorithm to speed up the inversion process.

In a nutshell, the FFT algorithm[1] is an efficient algorithm for computing the discrete Fourier transform say $X(n)$ of the function which we denote $x(k)$ given as

$$X(n) := \sum_{k=0}^{N-1} e^{-\frac{2\pi i}{N}nk} x(k) \quad \text{for } n = 0, \ldots, N-1. \tag{2.1}$$

The application of the FFT algorithm in pricing options attains its motivation from the fact that it is fast in terms of computational speed and the calculation of the option prices is made possible for a whole range of strikes. To take advantage of this attractive feature of the algorithm, we shall ideally represent our option prices in terms of (2.1) and hence apply the FFT algorithm to obtain the prices.

First, we give the definition of a Fourier transform from which we shall go on to give the Fourier Transform of an option price.

Definition 2.1.1 (Fourier Integral)**.** *Let f be a function defined for all $x \in \mathbb{R}$ with values in $\mathbb{C}$. Then the Fourier Integral is defined by*

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx. \tag{2.2}$$

If the integral exist for every x, then equation (2.2) defines $F(\omega)$, the Fourier transform of $f(x)$. The inverse Fourier transform, which allows us to determine the function $f(x)$ from its Fourier transform, is defined as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega. \tag{2.3}$$

Remark 2.1.2. *If $f(x)$ is integrable in the sense*

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty, \tag{2.4}$$

[1]For a detailed discussion of the algorithm we refer to Chapters 10 and 11 of Brigham [12].

then its Fourier transform $F(\omega)$ *exists and satisfies the inverse Fourier transform in equation* (2.3).

2.1.1 Application of the Fourier Transform Method

The application of the FFT in the pricing of European options works well because the density of the underlying asset is known in closed form hence allowing the application of the FFT algorithm to compute the option price.

If we let $k = \ln(K)$, $s = \ln(S_T)$ where K is the strike price and S_T is the price of the underlying asset at maturity time T and let $V_{t,T}$ be the fair value of the European call option with maturity T. Then by the risk-neutral pricing formula (1.2)

$$\begin{aligned} V_{t,T} &= e^{-r(T-t)}\mathbb{E}\left[(S_T - K)^+ | \mathcal{F}_t\right] \\ &= e^{-r(T-t)}\mathbb{E}\left[(e^s - e^k)^+ | \mathcal{F}_t\right] \end{aligned} \tag{2.5}$$

If q_T is the risk-neutral density of the log price of the underlying asset, then the value of the option is given by

$$V_{t,T}(k) = e^{-r(T-t)} \int_k^{\infty} (e^s - e^k)\, q_T(s) ds.$$

This implies therefore that the initial call value of the option is given by (as is the case in [17])

$$V_T(k) = e^{-rT} \int_k^{\infty} (e^s - e^k)\, q_T(s) ds. \tag{2.6}$$

Defining the characteristic function of $s = \log(S_T)$ whose density function is given by $q_T(s)$ in equation (2.6) as

$$\Phi_T(u) = \int_{-\infty}^{\infty} e^{ius} q_T(s) ds,$$

we note by equation (2.2) that this characteristic function is the Fourier transform of the function $q_T(s)$. Letting

$$\begin{aligned} Y &= \log(S_T) \\ &= \log(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T - t) + \sigma(W_T - W_t), \end{aligned}$$

hence Y is normally distributed with mean $\log(S_t)+\left(r-\frac{\sigma^2}{2}\right)(T-t)$ and variance $\sigma^2(T-t)$. We can write the characteristic function as

$$\begin{aligned}\Phi_T(u) &= \mathbb{E}\left(e^{iuY}\right)\\ &= \exp\left[iu\left(\log(S_t)+\left(r-\frac{\sigma^2}{2}\right)(T-t)\right)-\frac{\sigma^2(T-t)u^2}{2}\right],\end{aligned}$$

which implies therefore that at time $t=0$

$$\Phi_T(u)=\exp\left[iu\left(\log(S_0)+\left(r-\frac{\sigma^2}{2}\right)T\right)-\frac{\sigma^2Tu^2}{2}\right]. \tag{2.7}$$

Since we seek to compute the Fourier transform of the call function we will require that it should be integrable as in remark (2.1.2). Unfortunately, V_T is not integrable [17]. We shall therefore consider the modified call price proposed by Carr and Madan where they propose we multiply V_T by $e^{\alpha k}$. Denoting the modified call price by $c_T(k)$, we have

$$c_T(k)=e^{\alpha k}V_T(k)$$

for $\alpha>0$. We expect that for a range of values of α, c_T would be square integrable in k. We denote the Fourier transform of the modified call price by $\Psi_T(v)$ defined as

$$\Psi_T(v)=\int_{-\infty}^{\infty}e^{ivk}c_T(k)dk. \tag{2.8}$$

We shall develop an analytic expression for the integral (2.8) in terms of the characteristic function of the density function q_T thus,

$$\begin{aligned}\Psi_T(v) &= \int_{-\infty}^{\infty}e^{ivk}e^{\alpha k}V_T(k)dk\\ &= \int_{-\infty}^{\infty}e^{ivk}\int_{k}^{\infty}e^{\alpha k}e^{-rT}(e^{s_T}-e^{k})q_T(s)dsdk\\ &= \int_{-\infty}^{\infty}e^{-rT}q_T(s)\int_{-\infty}^{s}\left(e^{s+\alpha k}-e^{(\alpha+1)k}\right)e^{ivk}dkds\\ &= \frac{e^{-rT}}{(iv+\alpha)(iv+\alpha+1)}\int_{-\infty}^{\infty}e^{(\alpha+1+iv)s}q_T(s)ds\\ &= \frac{e^{-rT}}{(iv+\alpha)(iv+\alpha+1)}\int_{-\infty}^{\infty}e^{(-i\alpha-i+v)is}q_T(s)ds\\ &= \frac{e^{-rT}\Phi_T(v-(\alpha+1)i)}{(iv+\alpha)(iv+\alpha+1)}\end{aligned}$$

Computing the inverse transform of $\Psi_T(v)$ we have

$$V_T(k) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-ivk}\Psi_T(v)dv \tag{2.9}$$

As V_T is real, we have $\overline{V_T(k)} = V_T(k)$. Thus changing the variable k into $-k$ in $\overline{V_T(k)}$ (conjugate of the call function), implies that the real part of Ψ_T is even and the imaginary part is odd. Therefore,

$$V_T(k) = \frac{e^{-\alpha k}}{\pi}\int_{0}^{\infty} e^{-ivk}\Psi_T(v)dv \tag{2.10}$$

2.1.2 Application of the FFT algorithm

We want to represent the option values in terms of a sum as in the representation (2.1) so that we can use the FFT algorithm to evaluate the price. Using the Trapezium rule for computing equation (2.10) numerically we have

$$V_T(k) \approx \frac{e^{-\alpha k}}{\pi}\sum_{j=1}^{N} e^{-iv_j k}\Psi(v_j)h, \tag{2.11}$$

where $v_j = h(j-1)$ and h is the step size of the numerical integration. The upper limit for the integration is

$$a = Nh.$$

We shall now consider a range of log strike price around the log initial price of the asset.

$$k_u = -\frac{1}{2}N\lambda + \lambda(u-1) \quad \text{for } u = 1, \dots, N \tag{2.12}$$

where $\lambda > 0$ is the distance between the log strike prices. Substituting equation (2.12) into equation (2.11) yields

$$V_T(k_u) \approx \frac{e^{-\alpha k_u}}{\pi}\sum_{j=1}^{N} e^{-iv_j(-\frac{1}{2}N\lambda+\lambda(u-1))}\Psi(v_j)h \quad \text{for } u = 1, \dots, N, \tag{2.13}$$

and noting that $v_j = h(j-1)$, we have

$$V_T(k_u) \approx \frac{e^{-\alpha k_u}}{\pi}\sum_{j=1}^{N} e^{-i\lambda h(j-1)(u-1)}e^{i\frac{1}{2}(j-1)N\lambda h}\Psi(v_j)h. \tag{2.14}$$

To apply the fast Fourier transform algorithm, we shall express equation (2.14) in the form of equation (2.1), a discrete Fourier transform. Thus, we let

$$\lambda h = \frac{2\pi}{N},$$

hence

$$\begin{aligned} V_T(k_u) &\approx \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)} e^{i(j-1)\pi} \Psi(v_j) h \\ &= \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} (-1)^{j-1} e^{-i\frac{2\pi}{N}(j-1)(u-1)} \Psi(v_j) h. \end{aligned} \tag{2.15}$$

We will choose the strike values near the initial stock price S_0. In that respect we should arrange the prices such that the initial price S_0 appears in the range of the strikes hence we should choose a small value of λ in order to have many strikes around it. This implies a large value of h which would give a large grid for the integration. To obtain an accurate integration with large values of h, Carr and Madan incorporated Simpson's rule weightings into equation (2.15) which gives

$$V_T(k_u) \approx \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} (-1)^{j-1} e^{-i\frac{2\pi}{N}(j-1)(u-1)} \Psi(v_j) \frac{h}{3} \left(3 + (-1)^j - \delta_{j-1}\right), \tag{2.16}$$

where δ_n is the Kronecker delta function given by

$$\delta_n = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The summation in equation (2.16) is the exact application for the FFT algorithm where

$$x_j = (-1)^{j-1} \Psi(v_j) \frac{h}{3} \left(3 + (-1)^j - \delta_{j-1}\right) \quad \text{for } j = 1, \ldots, N.$$

Now, for the appropriate choices of α and h, we can now apply the fast Fourier transform algorithm for pricing the European call option.

2.1.3 Numerical Computations

To conclude this section we shall compare the prices obtained from the Black−Scholes formula given in equation (1.4) and the standard Monte Carlo method[2] with those obtained

[2] A detailed discussion of the Monte Carlo pricing method is done in Chapter 4

from the FFT method as discussed above. We further show the effectiveness in terms of the computational speed of the FFT method by making comparisons with the standard Monte Carlo simulation which we take to be a very convenient way for comparison in this regard.

The computations are based on the strike prices which have been chosen to simplify our comparison and are given in the range $[0, 0.1, 5]$. The parameter settings have been adapted from those specified by Carr and Madan [17] so that a balance is struck between computational time and accuracy. We have however changed the value of h from 0.00613 to 0.146484375, which has proved to give better accuracy. The Monte Carlo prices are obtained through a sample of 1 000 evaluations and 10 000 repetitions (Monte Carlo paths).

In Figure 2.1 we show the prices of the European call options as a function of the strike price K, obtained by the three pricing methods: Black Scholes, Monte Carlo and the FFT.

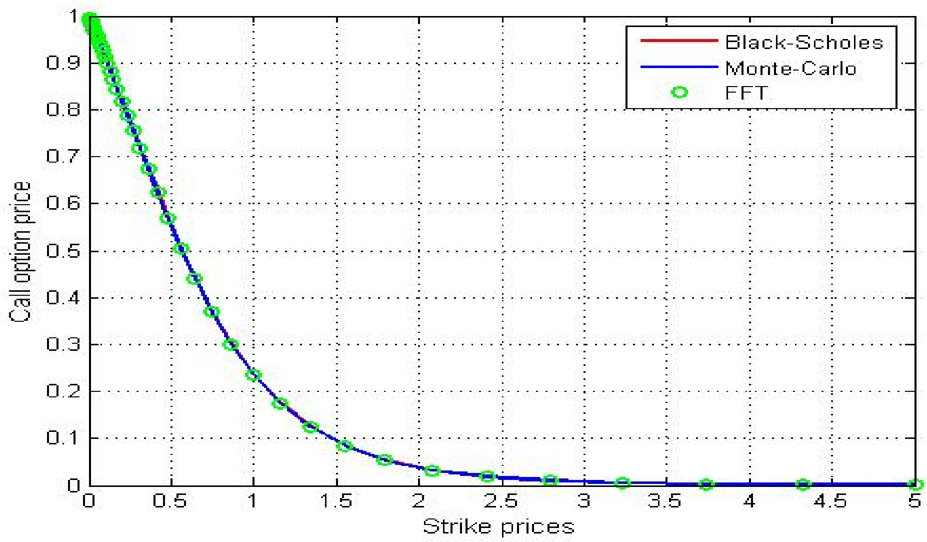

Figure 2.1. European call option prices for the three different pricing methods. The computation parameters are $S_0 = 1$, $r = 0.09$, $\sigma = 0.5$ and $T = 1$.

Notably, the FFT method seems to be more accurate for in-the money-calls. This is because in our computation, we have constructed the sum for the FFT such that the strikes are chosen around the initial price, hence the error increases when the strikes move away from the initial price. To support this assertion we illustrate clearly by plotting in Figure 2.2

the mean absolute percentage error (MAPE), which we shall define as

$$\text{MAPE} = \left| \frac{\text{Black-Scholes}-\text{FFT}}{\text{Black-Scholes}} \right|.$$

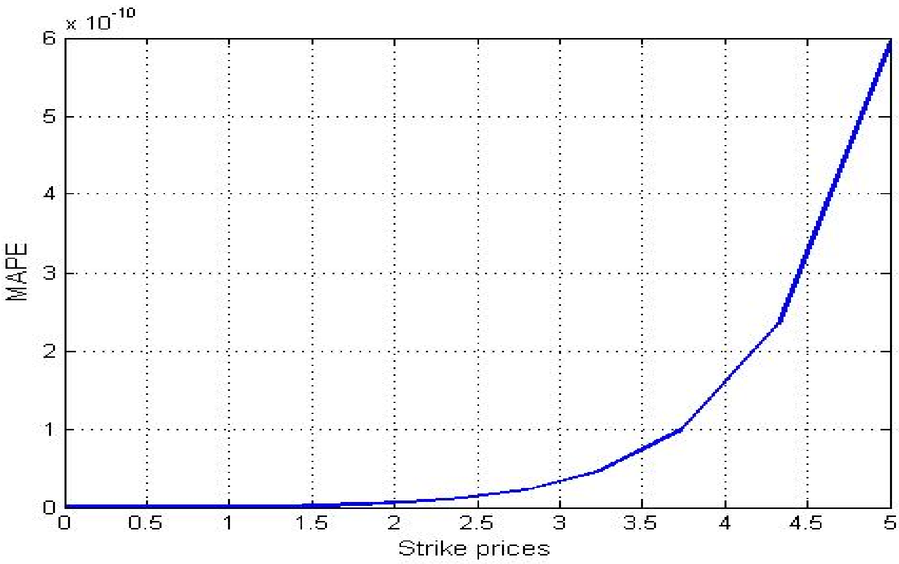

Figure 2.2. The percentage differences between the Black Scholes model and the FFT method.

Even though these methods yield similar results the computational time is very significantly different. We compared the speed for the two methods: standard Monte Carlo and the FFT and report the results in Table 2.1.

N	Monte Carlo	FFT
512	21.714	0.001
1024	43.733	0.021
4096	174.606	0.006

Table 2.1. A comparison of computational time in seconds for the FFT and the Monte Carlo method.

The speed of the FFT compared to the Monte Carlo is clearly visible as for example it took 174.61 seconds to compute 4096 prices using the Monte Carlo method but only 0.01 seconds using the FFT method. To support our findings, Borak et al. [8] compared using

C^{++} the Monte Carlo method on 20 different strikes, 500 evaluation and 5 000 Monte Carlo steps with the FFT methods on three different pricing models Merton, Heston and Bates. In their conclusion, they state that the speed superiority of the FFT-based method is more than 3000 times faster than the Monte Carlo methods which is quite remarkable.

2.2 Pricing Asian Call Options

Recall that the discretely sampled arithmetic Asian option written on the average of the underlying asset taken at predetermined dates denoted by $t_1, t_2, \ldots, t_N$ (which we shall consider to be equally spaced) has the average price given as

$$A_N = \frac{1}{N}\sum_{i=1}^{N} S_{t_i}. \tag{2.17}$$

We let the returns for each of these intervals be defined as

$$R_{t_i} = \ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) \tag{2.18}$$

which are independently distributed and have densities f_i. Within our framework, these densities are normally distributed with mean $(r-\sigma^2/2)(t_i - t_{i-1})$ and variance $\sigma^2(t_i - t_{i-1})$. Moreover, since we have taken the monitoring dates to be equally spaced it implies that all the returns follow the normal distribution with the same mean and variance.

From equation (2.18) we can alternatively write the underlying asset's price at any time t_i in terms of the initial price S_{t_0} as

$$S_{t_i} = S_{t_0}\exp(R_{t_1} + R_{t_2} + \cdots + R_{t_i}). \tag{2.19}$$

Substituting equation (2.19) into equation (2.17) (suppressing the subscript N), we have

$$A = \frac{1}{N}\sum_{i=1}^{N} S_{t_0}\exp(R_{t_1} + R_{t_2} + \cdots + R_{t_i}). \tag{2.20}$$

As we have seen, the value of any contingent claim using the risk neutral pricing formula is given by equation (1.2), which in this case is taken to be

$$V_T = e^{-rT}\mathbb{E}_{\mathbb{Q}}[(A-K)^+].$$

Since the distribution of the average is unknown, the price of this option can not be obtained in closed form. As a remedy, we shall show how the density function can be approximated numerically by means of Fourier convolution via the fast Fourier transform.

2.2.1 The Fourier Convolution Method via FFT

The Fourier convolution method first introduced in finance by Carverhill and Clewlow [18] represents the density function by discrete grid points within a fixed-width window of the density function's domain. The property of the Fourier transform which would be utilized here is its efficiency to calculate the convolution products, see Appendix B. To convolve we take the fast Fourier transform of two functions (which we shall develop), multiply the result and then take the inverse transform of the product. This procedure is made possible by the following propositions:

Proposition 2.2.1. *Suppose that X and Y are independent random variables with distribution functions f and g respectively and we let $Z = X+Y$. Then the distribution function of Z is the convolution of the distribution function of X and Y.*

Proposition 2.2.2. *The Fourier transform of the convolution of two functions f and g is given by the product of their Fourier transforms F and G. By taking the inverse Fourier transform of the product we obtain the convolution of the two functions.*

Unfortunately, for the Asian option the average is not a straightforward sum of independent variables. We show using the Hodges factorization [18] how we can transform the average in equation (2.20) as summands of independent variables.

Proposition 2.2.3 (Hodges Factorization)**.** *The average in equation* (2.20) *can be expressed as*

$$A = \frac{S_{t_0}}{N}\left[\exp\left(R_{t_1} + \ln\left(1 + \exp\left(R_{t_2} + \ln\left(1 + \cdots + \ln\left(1 + \exp\left(R_{t_N}\right)\right)\ldots\right)\right)\right)\right)\right] \tag{2.21}$$

Proof. From equation (2.20), we have

$$\begin{aligned} A &= \frac{S_{t_0}}{N}\sum_{i=1}^{N}\exp(R_{t_1}+R_{t_2}+\cdots+R_{t_i}) \\ &= \frac{S_{t_0}}{N}\left[e^{R_{t_1}}+e^{R_{t_1}+R_{t_2}}+e^{R_{t_1}+R_{t_2}+R_{t_3}}+\cdots+e^{R_{t_1}+R_{t_2}+\cdots+R_{t_N}}\right] \\ &= \frac{S_{t_0}}{N}\left[e^{R_{t_1}}\left(1+e^{R_{t_2}}\left(1+e^{R_{t_3}}\left(1+\cdots+e^{R_{t_{N-1}}}\left(1+e^{R_{t_N}}\right)\dots\right)\right)\right)\right] \\ &= \frac{S_{t_0}}{N}\left[\exp\left(R_{t_1}+\ln\left(1+\exp\left(R_{t_2}+\ln\left(1+\cdots+\ln\left(1+\exp\left(R_{t_N}\right)\right)\dots\right)\right)\right)\right)\right] \end{aligned}$$

□

Defining a recursive sequence B_i such that $B_1 = R_{t_N}$ as

$$B_i = R_{t_{N+1-i}} + \ln(1+\exp(B_{i-1})) \quad \text{for } i = 2, 3, \dots, N, \tag{2.22}$$

and noting that B_i is a sum of independent random variables, we have

$$A = \frac{S_{t_0}}{N}\exp(B_N). \tag{2.23}$$

We now seek to compute the density function of B_N from equation (2.22) recursively knowing that B_1 is normally distributed. This density function can be computed by applying propositions (2.2.1) and (2.2.2) hence obtaining the density of A (the average).

2.2.2 Recentering Intermediate Densities

The initial density function B_1 would be located at the center of the fixed-width widow which defines the domain of the density function. However, the term $\ln(1 + \exp(B_{i-1}))$ in equation (2.22) would cause the distribution of B_{i+1} to be shifted to the right of the distribution of B_i as shown in Figure 2.3 for $N = 4$. As a result the discretization grid would be expected to be large enough to contain these distributions.

This is not attractive in that the larger the number of averaging dates, then the larger the grid. To cope with a smaller grid and hence reduced computation time, we shall at each step recenter the densities to make the mean of the densities 'roughly' at the center of the window.

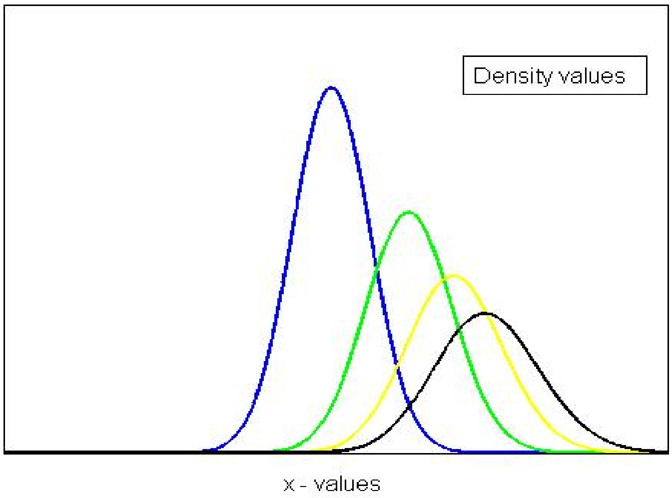

Figure 2.3. Evolution of the densities

Since we do not know the mean of B_i, we shall suppose that we know the mean of B_{i-1} which we denote as m_{i-1} from which we shall approximate the mean of $\ln(1+\exp(B_{i-1}))$ by $\ln(1+\exp(m_{i-1}))$. This implies therefore that the mean of the variable B_i for $i = 2, \ldots, N$, will be approximated by the sequence initialized with $m_1 = \mu_n$ defined as

$$m_i = \mu_{N+1-i} + \ln(1 + \exp(m_{i-1})) \tag{2.24}$$

where $\mu_N = \mathbb{E}(R_{t_N})$. By Jensen's inequality we know that we are underestimating this mean implying that we do not perfectly center the variable B_i hence the meaning of 'roughly' as used above. However, there is no new error implied by the recentering [5]. As a remark we point out that since all R_{t_i} are normally distributed with the same mean and variance it implies therefore that all the μ_i's would be the same and equal to $(r - \sigma^2/2)dt$ where $dt = t_i - t_{i-1}$.

Defining the recentered sequence for equation (2.22) as $A_i = B_i - m_i$ for $i = 1, \ldots, N$, the expression for the average defined in (2.23) can be expressed as

$$A = \frac{S_{t_0}}{N} \exp(A_N + m_N). \tag{2.25}$$

where A_N is given by the initial condition $A_1 = R_{t_N} - m_1$ and for $i = 2, \ldots, N$

$$\begin{aligned} A_i &= B_i - m_i \\ &= R_{t_{N+1-i}} + \ln(1 + \exp(B_{i-1})) - m_i \\ &= R_{t_{N+1-i}} + \ln(1 + \exp(A_{i-1}) \exp(m_{i-1})) - m_i. \end{aligned}$$

The initial random variable A_1 is normally distributed and its mean is centered in the window. The key to obtaining the densities A_i is hinged on the convolution method using the fast Fourier transform on the sum of the two independent variables $R_{t_{N+1-i}}$ and $\ln(1+\exp(A_{i-1}) \exp(m_{i-1}))-m_i$. In Figure 2.4 we show the nature of the densities resulting from the recentering method.

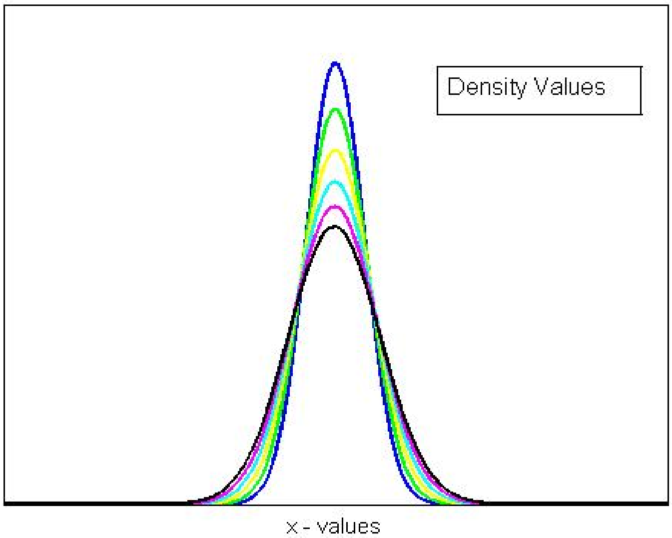

Figure 2.4. Evolution of the density with recentering at each step.

Since we are approximating the mean, Figure 2.4 is only showing the perfectly centered densities. However, for higher volatilities the approximation of the mean leads to a shift at each iteration to the right for the different densities [5].

2.2.3 Interpolation and Extrapolation Formula

The distribution function for $R_{t_{N+1-i}}$ is known and initially we know the distribution of A_1. Now, to get the density of A_i with respect to that of A_{i-1} we will need to compute the density function of $\ln(1+\exp(A_{i-1})\exp(m_{i-1})) - m_i$. For that we shall use the standard variable change theorem which states that given a random variable X with probability density f and another random variable Y related to X by the equation $y = \Phi(x)$, then the probability density of Y is given by

$$f_Y(y) = f_X(\Phi^{-1}(y)) \left| \frac{d\Phi^{-1}}{dy} \right|. \tag{2.26}$$

Taking $y = \ln(1+\exp(A_{i-1}+m_{i-1})) - m_i$ and applying the result in (2.26) we have the interpolation formula given as

$$f_Y(y) = \left| \frac{e^{y+m_i}}{e^{y+m_i}-1} \right| f_{A_{i-1}}(\ln(\exp(y+m_i)-1) - m_{i-1}) \mathbb{I}_{\{y>-m_i\}} \tag{2.27}$$

The errors incurred by this procedure emanates from the realization that if y is on the grid point in the domain of A_{i-1}, then $\ln(\exp(y+m_i)-1) - m_{i-1}$ would not be on the grid point in the domain A_{i-1}. Hence the more we apply the interpolation formula (2.27) the more errors accumulate in our algorithm.

To obtain the prices for the continuous Asian call options we shall use the Richardson extrapolation method on the pricing formula for the discrete case. To carry out the extrapolation we shall compute the call option using the formula

$$f = \frac{N_1 f(N_1) - N_2 f(N_2)}{N_1 - N_2} \tag{2.28}$$

where N_1 and N_2 are two different choices of the number of grid points and f is our method function. When $N_2 = 2N_1$ the extrapolation method is called the two-point Richardson extrapolation and equation (2.28) simplifies to

$$f = 2f(2N_1) - f(N_1).$$

In summary, the algorithm initially computes the values of $m_1 = \mu_N$ and $A_1 = R_{t_N} - m_1$ and then recursively computes the next approximated mean and centered density function up until we obtain m_N and A_N. Upon approximating the density function for the average, the expected payoff is then numerically computed using the trapezium rule to get the option prices.

2.2.4 Numerical Computations

To conclude this section, we report the results for one year continuous Asian call options obtained by applying the 2-point Richardson extrapolation formula on the discrete Asian option [67]. In both Table 2.2 and 2.3 the Zhang column represents the results from [79] and the width is taken to be $6\sigma\sqrt{T}$. Using the extrapolation formula with $N_2 = 2N_1$, in Table 2.2 we have $N_1 = 100$ and 180 for Table 2.3.

				Fourier Convolution method			
K	σ	Zhang	Hsu-Lyuu	2^{13}	2^{14}	2^{15}	2^{16}
95	0.05	8.8088392	8.808717	8.80827	8.80836	8.80838	8.80839
100	0.05	4.3082350	4.309247	4.30739	4.30770	4.30778	4.30780
105	0.05	0.9583841	0.960068	0.95689	0.95780	0.95802	0.95808
95	0.1	8.9118509	8.912238	8.91052	8.91120	8.91137	8.91142
100	0.1	4.9151167	4.914254	4.91283	4.91427	4.91463	4.91472
105	0.1	2.0700634	2.072473	2.06733	2.06919	2.06965	2.06977
95	0.2	9.9956567	9.995661	9.99093	9.99419	9.99501	9.99521
100	0.2	6.7773481	6.777748	6.77182	6.77572	6.77670	6.77695
105	0.2	4.2965626	4.297021	4.29073	4.29482	4.29585	4.29610
95	0.3	11.6558858	11.656062	11.64664	11.65263	11.65494	11.65533
100	0.3	8.8287588	8.829033	8.81950	8.82616	8.82782	8.82824
105	0.3	6.5177905	6.518063	6.50845	6.51518	6.51685	6.51726
95	0.4	13.5107083	13.510861	13.49733	13.50687	13.50925	13.50985
100	0.4	10.9237708	10.923943	10.91007	10.91988	10.92232	10.92292
105	0.4	8.7299362	8.730102	8.71629	8.72607	8.72849	8.72910

Table 2.2. Values of the continuous fixed strike Asian options for varied strike and volatility. The parameters used are $S = 100$, $r = 0.09$, $T = 1$ and $N_1 = 100$.

On comparison, for example the prices for column 2^{13} in Table 2.3 are less accurate than those in Table 2.2 since we have more interpolation approximation hence more accumulated errors due to an increased number of application of the interpolation formula (2.27). However, these accumulated errors can be lowered by increasing the number of the grid points hence increasing the accuracy of the results as demonstrated by the 2^{16} column in Table 2.3.

				Fourier Convolution method			
K	σ	Zhang	Hsu-Lyuu	2^{13}	2^{14}	2^{15}	2^{16}
95	0.05	8.8088392	8.808717	8.80957	8.80868	8.80865	8.80863
100	0.05	4.3082350	4.309247	4.30980	4.30841	4.30814	4.30807
105	0.05	0.9583841	0.960068	0.96295	0.95939	0.95854	0.95833
95	0.1	8.9118509	8.912238	8.91551	8.91248	8.91185	8.91169
100	0.1	4.9151167	4.914254	4.92241	4.91679	4.91544	4.91509
105	0.1	2.0700634	2.072473	2.07960	2.07229	2.07052	2.07008
95	0.2	9.9956567	9.995661	10.01228	9.99959	9.99651	9.99570
100	0.2	6.7773481	6.777748	6.79731	6.78213	6.77842	6.77747
105	0.2	4.2965626	4.297021	4.31761	4.30152	4.29761	4.29662
95	0.3	11.6558858	11.656062	11.68719	11.66350	11.65757	11.65605
100	0.3	8.8287588	8.829033	8.86252	8.83694	8.83059	8.82898
105	0.3	6.5177905	6.518063	6.55222	6.52606	6.51963	6.51801
95	0.4	13.5107083	13.510861	13.55918	13.52229	13.51311	13.51082
100	0.4	10.9237708	10.923943	10.97388	10.93571	10.92626	10.92391
105	0.4	8.7299362	8.730102	8.78023	8.74184	8.73242	8.73009

Table 2.3. Values of the continuous fixed strike Asian option for varied strike and volatility. The parameters used are $S = 100$, $r = 0.09$, $T = 1$ and $N_1 = 180$.

Chapter 3

Hedging Strategy for Asian Call Option

3.1 Introduction

Suppose that we have sold an Asian call option. Since the future values of the underlying asset are unknown, we have exposed ourselves to a certain amount of financial risk at the time of expiration of the option. The question then is, "how do we protect ('hedge') ourselves against this risk"? In that respect, the availability of a hedging strategy for a financial product becomes more important than the determination of its price. Our aim in this chapter is to determine hedging strategies for Asian call option based on the pricing result by Geman and Yor [42].

We shall consider a fixed strike Asian call option with payoff $(A_T - K)^+$, where

$$A_t := \frac{1}{t-t_0}\int_{t_0}^{t} S_u du\,; \quad (t \geq t_0).$$

The value of this option at time t is expressed by arbitrage arguments as

$$V_{t,T} = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[(A_T - K)^+ \,|\mathcal{F}_t\right].$$

Geman and Yor give the details of the different mathematical steps using Bessel processes which led to the following expression for the Asian call option

$$V_{t,T}(K) = \frac{e^{-r(T-t)}}{T-t_0}\left(\frac{4S_t}{\sigma^2}\right) C^{(v)}(h,q) \tag{3.1}$$

where,

$$v = \frac{2r}{\sigma^2} - 1\,; \quad h = \frac{\sigma^2}{4}(T-t)\,; \quad q = \frac{\sigma^2}{4S_t}\left(K(T-t_0) - \int_{t_0}^{t} S_u du\right)$$

and by defining

$$C^{(v)}(x,q) := \mathbb{E}_{\mathbb{Q}}\left[\left(\int_0^x \exp\left(2\left(W_u + vu\right)\right) du - q\right)^+\right]; \quad q \in \mathbb{R}.$$

Notably, at time t, q is no longer a random variable. Thus, if the observed values of the underlying asset S over the time interval $[t_0, T]$ are big enough then q can be negative. This would imply therefore that at time t we already know that the Asian option will be in the money at maturity time T. Therefore we have a closed form expression for the Asian option which we shall derive in Section 3.2, hence the hedging strategy can be obtained explicitly.

On the other hand, if $q > 0$, we do not have a closed form expression for the Asian option. We adapt in Section 3.3 the derivation of the Geman and Yor formula by Dewynne and Shaw [30]. They give a simple derivation of the formula without any knowledge of Bessel processes, by means of a transformed PDE. They went on to express the Laplace transform $C^{(v)}(h,q)$ with respect to h in terms of a hyper-geometric function. Geman and Yor originally gave their result as an integral and Shaw [66] shows by using Mathematica that the integral is in fact a hyper-geometric function.

3.2 Hedging strategy for the case $q \leq 0$

We begin by considering the price of the Asian option given by equation (3.1) when $q \leq 0$. Then

$$K \leq \frac{1}{T-t_0}\int_{t_0}^{t} S_u du \leq A_T,$$

such that our payoff simplifies to

$$(A_T - K)^+ = A_T - K.$$

By writing A_T as

$$A_T = \frac{1}{T-t_0}\int_{t_0}^{t} S_u du + \frac{1}{T-t_0}\int_{t}^{T} S_u du$$

and observing that the values of our underlying asset S are known between $[t_0, t]$, we can apply the martingale property satisfied by the discounted price S_t under $\mathbb{Q}$ hence computing explicitly our call price as follows

$$\begin{aligned} V_{t,T}(K) &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[(A_T - K)|\mathcal{F}_t] \\ &= e^{-r(T-t)}\left[\frac{1}{T-t_0}\int_{t_0}^{t} S_u du + \frac{1}{T-t_0}\int_{t}^{T}\mathbb{E}_{\mathbb{Q}}(S_u|\mathcal{F}_t)du - K\right] \\ &= e^{-r(T-t)}\left[\frac{1}{T-t_0}\int_{t_0}^{t} S_u du + \frac{S_t}{T-t_0}\int_{t}^{T} e^{r(T-u)}du - K\right] \\ &= e^{-r(T-t)}\left[\frac{t-t_0}{T-t_0}A_t - \frac{S_t}{r(T-t_0)}\left(1-e^{r(T-t)}\right) - K\right] \\ &= S_t\left(\frac{1-e^{-r(T-t)}}{r(T-t_0)}\right) - e^{-r(T-t)}\left(K - \frac{t-t_0}{T-t_0}A_t\right). \end{aligned} \tag{3.2}$$

This explicit pricing formula was also obtained by Geman and Yor [42]. Notably, it has some striking resemblance to the Black-Scholes formula in structure and we are interested in setting up a hedge for this formula.

We define

$$F_t(x,y) := x\left(\frac{1-e^{-r(T-t)}}{r(T-t_0)}\right) - e^{-r(T-t)}\left(K - \frac{t-t_0}{T-t_0}y\right) \tag{3.3}$$

so that $V_t = F_t(S_t, A_t)$. By setting the discounted value $\widetilde{V}_t = \widetilde{F}_t(\widetilde{S}_t, e^{-rt}A_t)$, we seek to use proposition (1.1.1) to find a predictable process $\{\phi_t\}_{0\le t\le T}$. From equation (3.3) we can write $\widetilde{F}_t(\widetilde{S}_t, e^{-rt}A_t)$ as

$$\widetilde{F}_t(\widetilde{S}_t, e^{-rt}A_t) = \frac{e^{-rt}S_t}{r(T-t_0)} - \frac{e^{-rT}S_t}{r(T-t_0)} - e^{-r(T-t)}K + \frac{t-t_0}{T-t_0}e^{-rT}A_t.$$

This implies that

$$\begin{aligned} d\widetilde{F}_t(\widetilde{S}_t, e^{-rt}A_t) &= \frac{1}{r(T-t_0)}\left(e^{-rt}dS_t - re^{-rt}S_t\right) - \frac{e^{-rT}}{r(T-t_0)}dS_t \\ &\quad + re^{-r(T-t)}Kdt + \frac{e^{-rT}}{T-t_0}\left(A_t dt + (t-t_0)dA_t\right). \end{aligned} \tag{3.4}$$

We recall that $A_t = \frac{1}{t-t_0}\int_{t_0}^{t} S_u du$ which implies that

$$dA_t = \frac{1}{t-t_0}S_t dt - \frac{1}{(t-t_0)^2}\int_{t_0}^{t} S_u du dt, \tag{3.5}$$

therefore from equation (3.4) we have

$$\begin{aligned} d\widetilde{F}_t(\widetilde{S}_t, e^{-rt}A_t) &= \frac{1-e^{-r(T-t)}}{r(T-t_0)}e^{-rt}dS_t + \left(re^{-r(T-t)}K + \frac{e^{-rT}}{T-t_0}S_t - \frac{e^{-rt}}{T-t_0}S_t\right)dt \\ &= \frac{1-e^{-r(T-t)}}{r(T-t_0)}d\widetilde{S}_t. \end{aligned}$$

Integrating both sides we have

$$\widetilde{F}_t(\widetilde{S}_t, e^{-rt}A_t) = F_0(S_0, A_0) + \int_0^t \frac{1-e^{-r(T-t)}}{r(T-t_0)}d\widetilde{S}_u.$$

From proposition (1.1.1) we can choose ϕ_t to be given by

$$\phi_t = \frac{1-e^{-r(T-t)}}{r(T-t_0)}, \tag{3.6}$$

which is the delta of the option and can also be obtained by taking the derivative of equation (3.2) with respect to S_t. We choose ψ_t such that $\psi_t = \widetilde{F}_t(\widetilde{S}_t, e^{-rt}A_t) - \phi_t\widetilde{S}_t$ thus,

$$\begin{aligned} \psi_t &= \frac{e^{-rt}S_t}{r(T-t_0)} - \frac{e^{-rT}S_t}{r(T-t_0)} - e^{-r(T-t)}K + \frac{t-t_0}{T-t_0}e^{-rT}A_t - e^{-rt}S_t\frac{1-e^{-r(T-t)}}{r(T-t_0)} \\ &= e^{-rT}\left(\frac{t-t_0}{T-t_0}A_t - e^{rt}K\right) \\ &= e^{-rT}\left(\frac{1}{T-t_0}\int_{t_0}^t S_u du - e^{rt}K\right). \end{aligned} \tag{3.7}$$

We can therefore, conclude by proposition (1.1.1) that the hedging strategy $\Phi = \{\psi_t, \phi_t\}$ given by equations (3.7) and (3.6) respectively, is a self-financing strategy for the Geman and Yor formula when $q \leq 0$.

3.3 Hedging strategy for the case $q > 0$

Now, we seek to find the hedging strategy $\Phi = \{\psi_t, \phi_t\}_{0\leq t\leq T}$ of the Asian option for $q > 0$. Unlike when $q \leq 0$, in this case we do not have a closed form expression of the Asian call price. However, Shaw [66] and Dewynne and Shaw [30] have shown that $C^{(v)}(h, q)$ in the Geman and Yor expression can be expressed in terms of a hyper-geometric function through its Laplace transform with respect to h. We first prove the Geman and Yor formula for this case.

Proof. Defining a new variable

$$I_t = \int_0^t S_u du,$$

the PDE for the continuous fixed strike Asian call option is given by (see [30])

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial I} - rV &= 0 \\ V(T, S, I_T) &= \left(\frac{I}{T} - K\right)^+. \end{aligned} \tag{3.8}$$

By making the transformation

$$\Phi = \frac{V}{S}, \qquad \eta = \frac{I - KT}{TS}$$

we have Φ satisfying the PDE

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\sigma^2\eta^2\frac{\partial^2\Phi}{\partial\eta^2} + \left(\frac{1}{T} - r\eta\right)\frac{\partial\Phi}{\partial\eta} = 0.$$

If we make the time reversal setting $\tau = T - t$, we obtain the PDE

$$\begin{aligned} \frac{\partial\Phi}{\partial\tau} &= \frac{1}{2}\sigma^2\eta^2\frac{\partial^2\Phi}{\partial\eta^2} + \left(\frac{1}{T} - r\eta\right)\frac{\partial\Phi}{\partial\eta} \\ \Phi(\eta, 0) &= \eta^+. \end{aligned}$$

Introducing the Geman and Yor variables

$$v = \frac{2r}{\sigma^2} - 1, \qquad h = \frac{1}{4}\sigma^2\tau, \qquad q = -\frac{1}{4}\sigma^2 T\eta, \qquad C = \frac{1}{4}\sigma^2 e^{r\tau}T\Phi,$$

the PDE in (3.8) becomes

$$\begin{aligned} \frac{\partial C}{\partial h} &= 2\alpha^2\frac{\partial^2 C}{\partial q^2} - (1 + 2(v+1)q)\frac{\partial C}{\partial q} + 2(1+v)C \\ C(h, 0) &= \frac{e^{2(1+v)h} - 1}{2(1+v)} \end{aligned}$$

on the domain $q > 0$, $h > 0$ [30] and subject to the initial conditions $C(0, q) = 0$. We now introduce the Laplace transform

$$\hat{C}(\lambda, q) = \int_0^\infty e^{-\lambda h}C(h, q)dh$$

which satisfies the transformed PDE

$$0 = 2q^2\frac{\partial^2\hat{C}}{\partial q^2} - (1 + 2(v+1)q)\frac{\partial\hat{C}}{\partial q} + (2(1+v) - \lambda)\hat{C} \tag{3.9}$$

$$\hat{C}(\lambda, 0) = \frac{1}{\lambda(\lambda - 2(1+v))}, \tag{3.10}$$

where λ in the boundary condition is chosen such that $\Re(\lambda) > 2(1+v)^+$ to ensure that the transform of the boundary condition exists. The solution of equation (3.9) can be expressed in terms of a pair of confluent hyper-geometric functions ${}_1F_1$ thus,

$$\hat{C}(\lambda, q) = C_1(p)A_1(\lambda, q) + C_2(p)A_2(\lambda, q),$$

where by defining $\mu = \sqrt{2\lambda + v^2}$, we have

$$\begin{aligned} A_1(\lambda, q) &= (2q)^{\frac{1}{2}(2+v+\mu)}\, {}_1F_1\left(-\frac{1}{2}(\mu + v + 2), 1 - \mu; \frac{-1}{2q}\right) \\ A_2(\lambda, q) &= (2q)^{\frac{1}{2}(2+v-\mu)}\, {}_1F_1\left(\frac{1}{2}(\mu - v - 2), 1 + \mu; \frac{-1}{2q}\right) \end{aligned}$$

For a valid Laplace transform, we need to choose the solution that is analytic in the right half-plane and hence this excludes A_1 [30]. By imposing the boundary conditions coupled with the identity that if $\Re(z) < 0$ and as $|z| \to \infty$, then

$${}_1F_1(a; b; z) \sim \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a},$$

we have

$$\frac{1}{\lambda(\lambda - 2(1+v))} = C_2(\lambda)\frac{\Gamma(1+\mu)}{\Gamma(2 + \frac{1}{2}(\mu + v))}.$$

Hence obtaining the Laplace transform of the term $C^{(v)}(h, q)$ in the German and Yor formula given as

$$\hat{C}(\lambda, q) = \frac{(2q)^{\frac{1}{2}(2+v-\mu)}\Gamma(2 + \frac{1}{2}(\mu + v))}{\lambda(\lambda - 2v - 2)\Gamma(\mu + 1)}\, {}_1F_1\left(\frac{1}{2}(\mu - v - 2), \mu + 1; \frac{-1}{2q}\right). \tag{3.11}$$

$\square$

We shall define

$$Z_t(S_t, A_t) := \frac{e^{-r(T-t)}}{T - t_0}\left(\frac{4S_t}{\sigma^2}\right) C^{(v)}(h, q) \tag{3.12}$$

so that $V_t = Z_t(S_t, A_t)$. By setting the discounted value $\widetilde{V}_t = \widetilde{Z}_t(\widetilde{S}_t, e^{-rt}A_t)$ and using Ito's formula we have

$$\begin{aligned} d\widetilde{Z}_t(\widetilde{S}_t, e^{-rt}A_t) &= \widetilde{Z}_1(t, \widetilde{S}_t, e^{-rt}A_t)dt + \widetilde{Z}_2(t, \widetilde{S}_t, e^{-rt}A_t)d\widetilde{S}_t + \widetilde{Z}_3(t, \widetilde{S}_t, e^{-rt}A_t)d(e^{-rt}A_t) \\ &\quad + \frac{1}{2}\widetilde{Z}_{22}(t, \widetilde{S}_t, e^{-rt}A_t)d\left\langle \widetilde{S}_t \right\rangle_t. \end{aligned} \tag{3.13}$$

Here we have used the following notation

$$\begin{aligned}\widetilde{Z}_i(t,x,y) &= \frac{\partial}{\partial j}\widetilde{Z}_t(x,y)\\ \widetilde{Z}_{22}(t,x,y) &= \frac{\partial^2}{\partial x^2}\widetilde{Z}_t(x,y)\end{aligned}$$

for $i = 1, 2, 3$ and $j = t, x, y$. Letting $x = A_t$ we have from equation (3.5),

$$\begin{aligned}df(t,A_t) &= -re^{-rt}A_t dt + e^{rt}dA_t\\ &= \frac{e^{-rt}}{t-t_0}S_t dt - e^{-rt}A_t\left(r - \frac{1}{t-t_0}\right)dt. \end{aligned} \tag{3.14}$$

Substituting equation (3.14) and $d\widetilde{S}_t = \sigma\widetilde{S}_t dW_t$ into equation (3.13) and integrating both sides noting that the du integrals are zero since our process $\widetilde{Z}_t(\widetilde{S}_t, e^{-rt}A_t)$ is a martingale under $\mathbb{Q}$. It implies therefore that,

$$\begin{aligned}\widetilde{Z}_t(\widetilde{S}_t, e^{-rt}A_t) &= \widetilde{Z}_0(\widetilde{S}_0, A_0) + \int_0^t \widetilde{Z}_2(u, \widetilde{S}_u, e^{-ru}A_u)\sigma\widetilde{S}_u dW_u\\ &= \widetilde{Z}_0(\widetilde{S}_0, A_0) + \int_0^t \widetilde{Z}_2(u, \widetilde{S}_u, e^{-ru}A_u)d\widetilde{S}_u.\end{aligned} \tag{3.15}$$

By construction,

$$\widetilde{Z}_2(t,x,y) = \frac{\partial}{\partial x}\widetilde{Z}_t(x,y).$$

Now from equation (3.12) and letting

$$h := \frac{\sigma^2}{4}(T-t), \quad X := \int_0^h \exp(2(W_u + vu))du, \quad Y := \frac{\sigma^2}{4x}\left[K(T-t_0) - (t-t_0)y\right] \tag{3.16}$$

we have

$$\frac{\partial}{\partial x}\widetilde{Z}_t(x,y) = \frac{e^{-r(T-t)}}{T-t_0}\frac{4}{\sigma^2}\left(\mathbb{E}_{\mathbb{Q}}\left[(X-Y)^+\right] + x\frac{\partial}{\partial x}\mathbb{E}_{\mathbb{Q}}\left[(X-Y)^+\right]\right).$$

Now,

$$\begin{aligned}\frac{\partial}{\partial x}\mathbb{E}_{\mathbb{Q}}\left[(X-Y)^+\right] &= \frac{\partial}{\partial x}\left[-\int_{+\infty}^{Y}(\gamma - Y)f_h(\gamma)d\gamma\right]\\ &= \frac{Y}{x}\int_{+\infty}^{Y} f_h(\gamma)d\gamma\\ &= \frac{Y}{x}\mathbb{Q}(X \geq Y).\end{aligned}$$

Therefore, from equation (3.15) we choose $\phi_t = Z_2(t, S_t, A_t)$ which implies that

$$\begin{aligned}\phi_t &= \frac{e^{-r(T-t)}}{T-t_0}\left[\frac{4}{\sigma^2}\mathbb{E}_{\mathbb{Q}}\left[(X-Y)^+\right] + \frac{K(T-t_0)-(t-t_0)A_t}{S_t}\mathbb{Q}(X \geq Y)\right] \\ &= \frac{e^{-r(T-t)}}{T-t_0}\left(\frac{4}{\sigma^2}\right)\left[C^{(v)}(h,q) + q\,\mathbb{Q}(X \geq Y)\right] \qquad (3.17)\end{aligned}$$

Recall $q = \frac{\sigma^2}{4S_t}(K(T-t_0)-(t-t_0)A_t)$. We choose ψ_t such that $\psi_t = \widetilde{Z}_t(\widetilde{S}_t, e^{-rt}A_t) - \phi_t\widetilde{S}_t$ from which we obtain an expression for ψ_t given as

$$\psi_t = -\frac{e^{-rT}}{T-t_0}\left(K(T-t_0) - \int_{t_0}^{t} S_u du\right)\mathbb{Q}(X \geq Y), \qquad (3.18)$$

where h, X and Y are given in (3.16). Eydeland and Geman [38], Fu et al. [39] and Nieuwveldt [61] applied different algorithms to invert the Laplace transform (3.11) which provides for the value of $C^{(v)}(h,q)$. Hence the hedging strategy $\{\phi_t, \psi_t\}$ given by equations (3.17) and (3.18) can then be computed. In addition, we can now use the off-the shelf software for inverting transforms, for example the statistical tools box in MATLAB 7.0.1 and in Mathematica 2.0 and 3.0, these may be useful for numerical solutions as has been shown recently by Shaw [66].

In conclusion, we have derived in this chapter the hedging strategies for the continuous fixed strike Asian call options using the option price formula obtained by Geman and Yor [42] for $q \leq 0$ and $q > 0$.

Chapter 4

Pricing Asian Options Using Monte Carlo Simulation

The Monte Carlo method have proved to be a powerful and flexible tool available for valuing many types of derivatives and other financial securities. In particular, the method has played an increasingly important role in handling complex[1] financial instruments in the field of financial mathematics. The literature on this subject dates back to Boyle [10] until the paper on quasi−Monte Carlo by Joy et al. [50].

In this chapter, we shall price the continuous fixed strike Asian call option whose payoff $X = (A_T - K)^+$, depends on the average of the price of a risky asset over a period of time where,

$$A_t := \frac{1}{t - t_0} \int_{t_0}^{t} S_u du \,; \quad (t \geq 0),$$

and the value of this option at time T is given by

$$V_{t,T}(S_t) = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left[(A_T - K)^+ \, | \mathcal{F}_t \right]. \tag{4.1}$$

To use the Monte Carlo simulation we will have to simulate the average of S_t, which in this case would require the approximation of the integral A_t. In that respect, Lapeyre and Temam [55] proposed time schemes for estimating the integrals of the form A_t (we choose without lose of generality $t_0 = 0$). We shall therefore discuss and adopt two of these schemes in order for use in the computation of the option value in equation (4.1). The

[1]Many of the new "exotic" options involve several underlying assets, different currencies and path dependency.

interval $[0, T]$ will be divided into N steps, where the step size will be given as $h = T/N$ and each time step will be given by $t_k = kT/N = kh$.

4.1 The Riemann Scheme

This is the standard and widely used scheme for estimating integrals of the form A_T. Since we can simulate S_t at any given t, A_T can be approximated by using the Riemann sums:

$$A_T^r = \frac{h}{T}\sum_{k=0}^{N-1} S_{t_k}.$$

Using this scheme, the approximate value of the option in equation (4.1) is given by

$$V_T(S) = \frac{1}{M}e^{-rT}\sum_{j=1}^{M}\left(\frac{h}{T}\sum_{k=0}^{N-1} S_{t_k} - K\right)^+, \tag{4.2}$$

where M is the number of Monte Carlo loops. As a remark, we point out that the time complexity of this algorithm is $O(1/NM)$ which involves the step error and the Monte Carlo error $(\sigma/\sqrt{M})$ see [55]. In general, this time complexity is true for every kind of Monte Carlo method.

4.2 The Trapezoidal Scheme

This scheme is equivalent to the trapezoidal method and it gives high accuracy for the integral approximation. Assume that

$$\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \middle| \mathcal{B}_h\right] \tag{4.3}$$

is the closest random variable to

$$\left(\frac{1}{T}\int_0^T S_u du - K\right)^+,$$

where $\mathcal{B}_h$ is the σ-field generated by the $(S_{t_k}, k = 0, \ldots, N)$. By using the conditional law of W_u with respect to $\mathcal{B}_h$ for $t_k \leq u \leq t_{k+1}$ which is given as

$$\mathcal{L}\left(W_u | W_{t_k} = x, W_{t_{k+1}} = y\right) = \mathcal{N}\left(\frac{t_{k+1} - u}{h}x + \frac{u - t_k}{h}y, \frac{(t_{k+1} - u)(u - t_k)}{h}\right) \tag{4.4}$$

where $h = t_{k+1} - t_k$, we take

$$\left[\mathbb{E}\left(\frac{1}{T}\int_0^T S_u du \middle| \mathcal{B}_h\right) - K\right]^+ = \left[\frac{1}{T}\int_0^T \mathbb{E}\left(S_u|\mathcal{B}_h\right) du - K\right]^+ \tag{4.5}$$

which by Jensen's inequality, we know to be less than expression (4.3). However, Lapeyre and Temam have shown (see proposition 3.3 in [55]) that this is a "really good approximation" of the integral A_T.

Using the conditional law of W_u, we write the integral as follows

$$\mathbb{E}\left(\frac{1}{T}\int_0^T S_u du \middle| \mathcal{B}_h\right) = \frac{1}{T}\int_0^T \mathbb{E}\left(S_u|\mathcal{B}_h\right) du \tag{4.6}$$

where we take $S_u = e^{(r-\frac{\sigma^2}{2})u+\sigma B_u}$ and we recall that, if $X \sim \mathcal{N}(\mu, \sigma^2)$ then $\mathbb{E}(e^{tX}) = e^{\mu t + \sigma^2 t^2/2}$. We can therefore compute the expectation in (4.6) (we shall represent it for convenience by $\mathbb{I}$) as follows

$$\begin{aligned}
\mathbb{I} &= \frac{1}{T}\int_0^T \mathbb{E}\left(e^{(r-\frac{\sigma^2}{2})u+\sigma B_u}|B_{t_k} = W_{t_k}, B_{t_{k+1}} = W_{t_{k+1}}\right) du \\
&= \frac{1}{T}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} e^{(r-\frac{\sigma^2}{2})u} e^{\sigma\left(\frac{t_{k+1}-u}{h}\right)W_{t_k}+\sigma\left(\frac{u-t_k}{h}\right)W_{t_{k+1}}+\frac{\sigma^2}{2}\frac{(t_{k+1}-u)(u-t_k)}{h}} du \\
&= \frac{1}{T}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} e^{ru-\frac{\sigma^2}{2}u+\sigma\left(\frac{t_{k+1}-u}{h}\right)W_{t_k}+\sigma\left(\frac{u-t_k}{h}\right)W_{t_{k+1}}+\frac{\sigma^2}{2h}\left(ut_{k+1}-t_{k+1}t_k-s^2+st_k\right)} du \\
&= \frac{1}{T}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} e^{\sigma\left(\frac{u-t_k}{h}\right)\left(W_{t_{k+1}}-W_{t_k}\right)-\frac{\sigma^2}{2}\left(\frac{u-t_k}{h}\right)^2+ru} e^{\sigma W_{t_k}-\frac{\sigma^2}{2}t_k} du.
\end{aligned} \tag{4.7}$$

Now, we seek to further simplify the expression in (4.7) using Taylors' expansion since when implementing the Monte Carlo simulation to attain our Asian call value, we shall have a double sum; hence the need to make (4.7) as simple as possible. By using Taylors' expansion, e^x is given as

$$e^x = 1 + x + \frac{x^2}{2} + \ldots.$$

Now, letting $\xi = u - t_k \in (0, h)$, $ru = r(u-t_k) + rt_k$, $h = t_{k+1} - t_k$ and $W_{t_{k+1}} - W_{t_k} = \Delta W_{t_k}$

we have

$$\begin{aligned}
\mathbb{I} &= \frac{1}{T}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} e^{\sigma\left(\frac{u-t_k}{h}\right)\left(W_{t_{k+1}}-W_{t_k}\right)-\frac{\sigma^2}{2}\left(\frac{u-t_k}{h}\right)^2+ru} e^{\sigma W_{t_k}-\frac{\sigma^2}{2}t_k} du \\
&= \frac{1}{T}\sum_{k=0}^{N-1}\int_0^h e^{\sigma\frac{\xi}{h}\Delta W_{t_k}-\frac{(\sigma\xi)^2}{2}+r\xi} S_{t_k} d\xi \\
&= \frac{1}{T}\sum_{k=0}^{N-1} S_{t_k}\int_0^h \left(1+\frac{\sigma\xi}{h}\Delta W_{t_k}-\frac{(\sigma\xi)^2}{2}+r\xi+O(h)\right) d\xi \\
&= \frac{1}{T}\sum_{k=0}^{N-1} hS_{t_k}\left(1+\frac{\sigma}{2}\Delta W_{t_k}+\frac{rh}{2}\right).
\end{aligned}$$

Hence, we have the scheme given as

$$A_T^* = \frac{h}{T}\sum_{k=0}^{N-1} S_{t_k}\left(1+\frac{rh}{2}+\sigma\frac{W_{t_{k+1}}-W_{t_k}}{2}\right).$$

Using this scheme, the approximate price of our Asian option[2] in equation (4.1) would be given by

$$V_T(S) = \frac{1}{M}e^{-rt}\sum_{j=1}^{M}\left(\frac{h}{T}\sum_{k=0}^{N-1} S_{t_k}\left(1+\frac{rh}{2}+\sigma\frac{W_{t_{k+1}}-W_{t_k}}{2}\right)-K\right)^+. \tag{4.8}$$

4.3 Variance Reduction and Efficiency Improvement Techniques

The main concern in Monte Carlo work is to obtain a respectably small standard error in the final result. Though it is possible to reduce the standard error by taking the average of say n independent values of an estimator this is rarely a rewarding procedure as usually the standard error is inversely proportional to the square root of the sample size n. Therefore, to reduce the standard error by a factor of k, the sample size needs to be increased by k^2-fold. This tends to be impractical when k is large, say 100. To escape this impracticable amount of experimental requirement, it is profitable to change or at least distort the original problem in such a way that the uncertainty in a result is reduced. Such procedures are

[2]'Our Asian option' here would refer to the continuous fixed strike Asian call option

known as variance reduction techniques (uncertainty can be measured in terms of variance). There are many such techniques [9] and they include control variates, antithetic variates, stratified sampling and importance sampling. These variance reduction techniques do not introduce bias into the estimation, and thus they make results more precise without sacrificing reliability. We shall discuss here the control and antithetic variates.

To motivate the need for variance reduction in the use of standard Monte Carlo method, we shall simulate our Asian option prices for five different strike (K) levels using the following parameter settings: $r = 0.09, \sigma = 0.30, S_0 = 100, T = 1, N = 1\,000$ and $M = 100\,000$. The error (E_M) and the 95% confidence interval (CI) are given by

$$\mathrm{E_M} = \frac{\bar{\sigma}}{\sqrt{M}}, \qquad \mathrm{CI} = \left[\bar{X}_M - 1.96\frac{\bar{\sigma}}{\sqrt{M}}, \bar{X}_M + 1.96\frac{\bar{\sigma}}{\sqrt{M}}\right]$$

and $\bar{X}_M$ is the price of the option computed using the scheme in equation (4.2).

Strike (K)	SMC price	$\mathrm{E_M}$	Variance	CI	Zhang
90	15.0630	0.09210	225.1326	[14.9700,15.1560]	14.9840
95	11.7287	0.0856	192.3991	[11.6428,11.8147]	11.6559
100	8.89394	0.0777	157.0163	[8.81627,8.97161]	8.82876
105	6.57539	0.0686	122.5812	[6.50677,6.64401]	6.51780
110	4.74735	0.0594	91.89543	[4.68793,4.80677]	4.69671

Table 4.1. Simulation results for pricing Asian call option using standard Monte Carlo (SMC) method and Zhang prices from [79].

From the table, if we take for instance when $K = 100$, the simulated price is 8.89394 and the price from Zhang is 8.82876 of which both belong to our confidence interval $[8.81627, 8.97161]$. The interval is very large such that we can not, with great confidence accept this simulated price as our option price, hence the need to reduce the variance, which is computed to be 157.0163.

However, Boyle et al. [9] argue that if we have a choice between two Monte Carlo estimates, smaller variance should not be a sufficient justification for preferring one estimator over another, instead efficiency in terms of computational effort should be the basis of preference. In that respect, we shall use the ratio of the variances as the ratio of computational effort required for a given predetermined accuracy, thus

$$\text{Efficiency} = \frac{\text{Variance of standard Monte Carlo}}{\text{Variance of new estimator}},$$

where we shall take the variance of new estimator to be the variance obtained by using antithetic or control variate method. What this efficiency means is that if the value of 100 is obtained, the standard Monte Carlo estimator would require 100 times more evaluations to achieve the same variance or standard error as the new method. Variance reduction methods, therefore, have the added advantage of improving computational efficiency and reducing variance. We shall discuss the specific techniques in the next subsections.

4.3.1 Antithetic Variates Technique

The method of antithetic variates attempts to reduce variance by introducing negative dependence between pairs of replications. Generally, the method is based on the observation that if U is uniformly distributed over $[0, 1]$, then $1-U$ is too. Hence if we generate a path using as inputs $U_1, \ldots, U_n$, then we can generate a second path using $1-U_1, \ldots, 1-U_n$ without changing the law of the simulated process. The variable U_i and $1-U_i$ form an antithetic pair in the sense that a large value of one is accompanied by a small value of the other. This suggests that an unusually large or small output computed from the first path may be balanced by the value computed from the antithetic path, resulting in reduction in variance.

In the same way, in a simulation driven by independent standard normal random variables, antithetic variates can be implemented by pairing a sequence of $Z_1, \ldots, Z_n$ with the sequence $-Z_1, \ldots, -Z_n$ of both i.i.d $\mathrm{N}(0, 1)$ variables. If the Z_i are used to simulate the increments of a Brownian motion, then $-Z_i$ simulate the increments of the reflection of the path about the origin. This suggests that running a pair of simulations using the original path and then use its reflection may result in lower variance. The success of the antithetic scheme hinges on whether

$$\mathrm{Var}\left(\frac{f(Z_i)+f(-Z_i)}{2}\right) \leq \frac{1}{2}\mathrm{Var}(f(Z_i)). \tag{4.9}$$

This would require that the negative dependence in Z_i's (i.e Z_i and $-Z_i$) would produce the negative correlation between $f(Z_i)$ and $f(-Z_i)$ and a simple sufficient condition ensuring this condition is when f is monotonic. The inequality (4.9) in general, is only a bound the actual improvement in the use of the antithetic variates can even be much better.

When applying the antithetic variates technique we shall take the average payoff from one path with samples $\{Z_0, \ldots, Z_{N-1}\}$ and another path with samples $\{-Z_0, \ldots, -Z_{N-1}\}$. If one path zig-zags, the other path zag-zigs. In Figure 4.1, we show such a sample path.

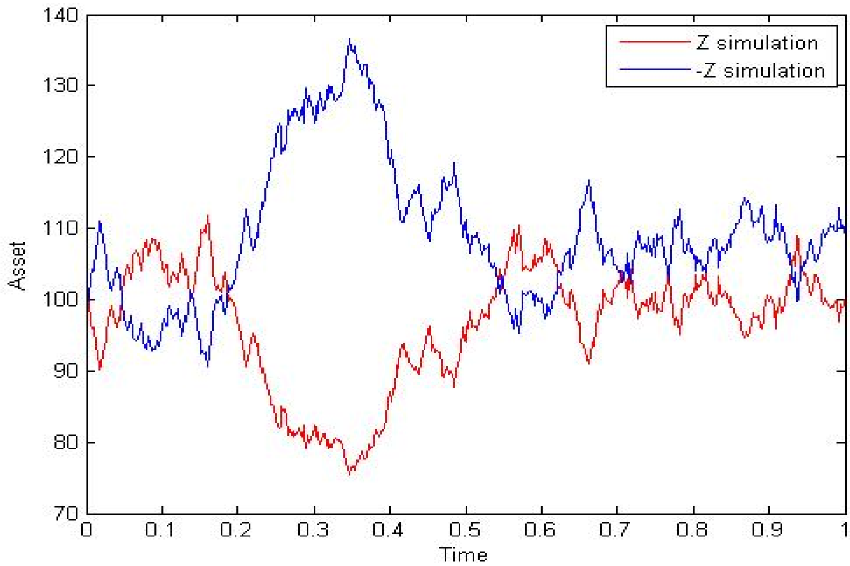

Figure 4.1. The sample path of an asset simulated using antithetic variate

We give an example where we use the antithetic variates to value our Asian option. The same parameters as in Section 4.3 are used and $M = 10\,000$ instead. We present in Table 4.2, the results for the confidence intervals obtained for the standard Monte Carlo and the antithetic variate method and give the efficiency gained due to the antithetic variates application.

Strike	Standard Monte Carlo		Antithetic Variate		Efficiency
	Variance	Confidence Interval	Variance	Confidence Interval	
90	226.75436	[14.780541, 15.370829]	113.903640	[14.716151, 15.134516]	1.9907560
95	193.98759	[11.469098, 12.015074]	97.1950150	[11.411663, 11.798126]	1.9958595
100	158.32112	[8.6704693, 9.1637058]	79.0967484	[8.6250124, 8.9736433]	2.0016136
105	123.67704	[6.3865304, 6.8224740]	61.5272789	[6.3594189, 6.6669006]	2.0101173
110	92.839576	[4.5867772, 4.9644823]	45.9577217	[4.5686474, 4.8343925]	2.0201082

Table 4.2. The table shows the variance, 95 percent confidence intervals and efficiency for the standard Monte Carlo and the antithetic variates method.

We note the significant reduction in the variance across all the strike levels. We can roughly say that the efficiency gained by the use of antithetic variates lies in that we would require approximately half of the standard Monte Carlo simulation to obtain the same precision as the standard Monte Carlo method.

4.3.2 Control Variate Technique

In contrast to the antithetic variate which relies on finding samples that are anti-correlated with the origin random variables, control variate techniques seek to find samples that have some general known correlation. This technique (when it works) is very powerful in reducing variance as we shall see herein.

The idea is, given that we wish to estimate $\mathbb{E}(f(X))$ from simulated random variables X_i's, we suppose that we can find an arbitrary function g which has a similar shape as f and whose expectation $\mathbb{E}(g(X))$ is known, so that we can write

$$\mathbb{E}(f(X)) = \mathbb{E}(f(X_i)) - \mathbb{E}\left(g(X_i) - \mathbb{E}(g(X))\right).$$

However, it can happen that the choice of g is not the best estimator for f, therefore we shall consider a linear function of g that is better by using regression analysis techniques. For a fixed $\beta \in \mathbb{R}$, we have

$$\begin{aligned} f(X_i) - \mathbb{E}(f(X)) &= \beta\left(g(X_i) - \mathbb{E}(g(X))\right) + \epsilon && (4.10)\\ \mathbb{E}(f(X)) - \epsilon &= f(X_i) - \beta\left(g(X_i) - \mathbb{E}(g(X))\right) && (4.11)\end{aligned}$$

where the error ϵ has expectation zero. By computing the sample mean, we have

$$\bar{Z}_{cv}(\beta) = \mathbb{E}\left[f(X_i) - \beta g(X_i)\right] + \beta\mathbb{E}(g(X)) \tag{4.12}$$

The standard Monte Carlo will be used to estimate $\mathbb{E}\left((f(X_i) - \beta g(X_i)\right)$ and the value $\mathbb{E}(g(X))$ is known, hence $\bar{Z}_{cv}(\beta)$ can then be approximated. The term $(g(X_i) - \mathbb{E}(g(X)))$ serves as the control in estimating $\mathbb{E}(f(X))$, hence the name control variate. The control variate $\bar{Z}_{cv}(\beta)$ is an unbiased estimator of $\mathbb{E}(f(X))$ and its variance is given by

$$\begin{aligned} \mathrm{Var}(\bar{Z}_{cv}(\beta)) &= \mathrm{Var}\left[f(X) - \beta g(X)\right]\\ &= \mathrm{Var}(f(X)) - 2\beta\mathrm{Cov}(f(X), g(X)) + \beta^2\mathrm{Var}(g(X)). && (4.13)\end{aligned}$$

The control variate will have a smaller variance estimate than the standard estimate if $\beta^2\mathrm{Var}(g(X)) < 2\beta\mathrm{Cov}(f(X), g(X))$ and therefore the value that minimizes equation (4.13) is given by

$$\beta := \frac{\mathrm{Cov}(f(X), g(X))}{\mathrm{Var}(g(X))}.$$

In practice however, we do not know the covariance of $f(X)$ and $g(X)$ and the variance of $f(X)$, we can use the standard least squares estimator to estimate β during the Monte Carlo simulation.

Kemna and Vorst [53] suggest that we take the closed form solution of the geometric Asian options as the control variate for pricing arithmetic Asian options. Paul Glasserman [43] has shown that there is an extremely strong correlation between the payoffs of the arithmetic and geometric Asian option, in fact, it is greater than 0.99. Therefore to demonstrate the application of this method to option pricing we use $\beta = 1$.

Taking the value of our Asian option given by the Riemann scheme (4.2), we seek to use the control variate technique to compute this price. Following Kemna and Vorst, we take the value of the geometric continuous Asian call option formula to be our control variate.

The value of the geometric Asian call option for the continuous case is given by

$$V_G^c = e^{-rT}\mathbb{E}\left[(G_T - K)^+\right],$$

where $\ln(G_t) \sim N\left((r - \sigma^2/2)\, T/2 + \ln(S_0), \sigma^2 T/3\right)$

$$\begin{aligned} G_T &= \exp\left(\frac{1}{T}\int_0^T \ln\left(S_t\right) dt\right) \\ &= \exp\left(\frac{1}{T}\int_0^T \left[\ln(S_0) + \left[r - \frac{\sigma^2}{2}\right] t + \sigma W_t\right] dt\right) \\ &= S_0 \exp\left(\left[r - \frac{\sigma^2}{2}\right]\frac{T}{2} + \frac{\sigma}{T}\int_0^T W_t dt\right) \end{aligned}$$

and the closed form expression for this option is given by (see [48])

$$V_G^c = e^{-rT}\left[S_0 \exp(C_M)\Phi\left(d_M + \frac{1}{\sqrt{3}}\sigma\sqrt{T}\right) - K\Phi(d_M)\right], \tag{4.14}$$

where $\Phi(\cdot)$ is the standard normal distribution function,

$$\begin{aligned} C_M &= \frac{1}{2}\left(r - \sigma^2/2\right) T + \sigma^2 T/6 \\ d_M &= \sqrt{3}\frac{(r - \sigma^2/2)\,\frac{T}{2} - \ln(K/S_0)}{\sigma\sqrt{T}}. \end{aligned}$$

We therefore choose our control variate to be

$$V_{cv} = e^{-rT}\left(S_0 \exp\left(\left[r - \frac{\sigma^2}{2}\right]\frac{T}{2} + \frac{\sigma}{T}\int_0^T W_t dt\right) - K\right)^+. \tag{4.15}$$

The integral of the Brownian motion $\int_0^T W_t dt$ shall be approximated using the schemes (4.2) and (4.8). Implementing the control variate technique, the value of our Asian option is given by

$$V_T^{cv}(S) \quad = \quad \mathbb{E}(V_T(S) - V_{cv}) + V_G^c$$

where $V_T(S)$, V_{cv} and V_G^c are defined by the equations (4.2), (4.15) and (4.14) respectively. The table (4.3) shows the computational results for standard Monte Carlo and the control variate method and the efficiency gained due to the use of control variate technique.

	Standard Monte Carlo		Control Variate		
Strike	Variance	Confidence Interval	Variance	Confidence Interval	Efficiency
90	223.68349	[14.841992, 15.428269]	0.6967472	[14.976906, 15.009626]	321.03964
95	191.03795	[11.520420, 12.062229]	0.7178910	[11.647822, 11.681036]	266.10999
100	155.58347	[8.7051115, 9.1940651]	0.7359374	[8.8207245, 8.8543530]	211.40855
105	121.18618	[6.3987365, 6.8302684]	0.7503113	[6.5113425, 6.5452976]	161.51456
110	90.698570	[4.5711225, 4.9444494]	0.7538181	[4.6864243, 4.7204590]	120.31891

Table 4.3. The table shows the variance, 95 percent confidence intervals and efficiency for the standard Monte Carlo and the control variate method.

We note the significant reduction in the confidence interval when the control variate method is used. The efficiency is very high which could be due to the strong correlation between the payoff of the arithmetic and the geometric options.

Chapter 5

Pricing Asian Options Using Laplace Transforms

The Laplace transform[1], named after the French Mathematician Pierre Simon Marquis de Laplace, is another form of integral transform which is used in many important applications for instance, mathematics, physics, engineering and signal processing. In short, by using Laplace transform methods we can transform a partial differential equation (PDE) into an ordinary differential equation (ODE) which is, in general, easy to solve. The solution of the PDE is then obtained by inverting numerically and/or analytically the Laplace transform expression.

The use of Laplace transforms in finance dates back to Buser [15] and it has recently been extensively used in the context of option pricing by for example Pelsser [63], Geman [47], Carr and Schröder [16] and Fusai [40], who used Laplace transform to price exotic options. Unfortunately, it has proved to be difficult to find the analytical expressions for the inverse of Laplace transform for instance when pricing Asian options. Hence the need for the numerical inversion methods. Davies and Martin [28] give a survey and comparison of different approaches used in literature for inverting Laplace transforms. On the other hand, Craddock et al. [25] investigate and compare different approaches for numerical inversion of Laplace transform in the context of computational finance.

We recall the basic definitions and properties of Laplace transform.

[1] For the historical introduction see the preface in [22].

Definition 5.0.1. *The Laplace transform of a function $f(t)$, defined for all $t \geq 0$, is the function $\hat{f}(s)$ defined by*

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t) dt, \tag{5.1}$$

where the parameter s belongs to the real line or in the complex plane. We use the notation $s = x+iy$ when s is complex. The integral converges in a right-plane $\Re(s) > s_0$ and diverges for $\Re(s) < s_0$. The number s_0 which may be $\pm\infty$, is called the abscissa of convergence.

As a remark, we note that not every function of t has a Laplace transform because the defining integral can fail to converge. For example, the functions $1/t$, $\exp(t^2)$, $\tan(t)$ do not possess Laplace transforms. A large class of functions that have Laplace transform are of exponential order. Thus, a function f is of exponential order if there exist some constants M and k, for which $|f(t)| \leq Me^{kt}$ for all $t \geq 0$. Then the Laplace transform surely exists if the real part of s is greater than k, hence k in this case coincides with the abscissa of convergence s_0.

If the Laplace transform is known, then the original function $f(t)$ can be recovered using the inversion formula (Bromwich inversion formula), that can be represented as an integral in the complex plane.

Proposition 5.0.2. ***Laplace Transform Inverse.*** *If the function $f(t)$ has a Laplace transform as given in equation (5.1), then for $t > 0$, we have the following relation*

$$f(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{st} \hat{f}(s) ds, \tag{5.2}$$

and the vertical line at $\Re(s) = x$ is known as the Bromwich line. This line is chosen such that the path of integration lies to the right of all the singularities of $\hat{f}(s)$. Equation (5.2) is called the Bromwich inversion integral.

The Laplace transform has a number of interesting properties some of which we present in the Table 5.1 below. Notably, Buser [15] presented an interesting summary for these properties in relation to finance noting that the present value $V(r)$ of a given cash flow $C(t)$ for a given rate of discount r is a Laplace transform, thus

$$V(r) = \int_0^\infty e^{-rt} C(t) dt.$$

Property	Function	Laplace transform
Linearity	$af(t) + bg(t)$	$a\hat{f}(s) + b\hat{g}(s)$
Translation	$f(t-a),\ a > 0$	$\exp(-as)\hat{f}(s)$
Shift	$\exp(at)f(t)$	$\hat{f}(s-a),\ s > a$
Scaling	$af(at)$	$\hat{f}(\frac{s}{a})$
Integral	$\int_0^t f(u)du$	$\dfrac{\hat{f}(s)}{s}$
Differentiation	$\dfrac{\partial^n f(t)}{\partial t^n}$	$s^n\hat{f}(s) - s^{n-1}f(0) + \cdots - f^{(n-1)}(0)$
Multiplication by polynomials	$t^n f(t)$	$(-1)^n\hat{f}^{(n)}(s)$

Table 5.1. Basic properties of Laplace transform.

Our objective is to apply the Laplace transform method to price continuous fixed strike Asian call options. In Appendix D we demonstrate the power of the method in pricing European call option. Upon applying the Laplace inversion algorithms, accurate option prices are obtained. In the same spirit, we seek to obtain a Laplace transform formula for the continuous fixed strike Asian call option and then apply the Laplace inverse to obtain the price of the option numerically.

To achieve our objective, we relate the Laplace transform of the call option with respect to the strike price to the Laplace transform of the average price of the underlying asset. Because the Laplace transform of the underlying asset (which is lognormal) does not admit any analytic expression, we shall obtain the Laplace transform of the resulting relation with respect to the maturity time thereby obtaining the double Laplace transform for the continuous fixed strike Asian call option. The option prices would then be obtained by numerical inversion of this double Laplace transform.

We remark that this approach was taken by Fu et al. [39]. However, they did not do any computation on the resulting transform. Our goal is to perform the numerical computation on the resulting pricing formula. Actually, because the transform involves a very complicated function (generalized hyper-geometric function, ${}_1F_2$) the inversion of the transform proves to be somehow difficult to perform. Nonetheless, we shall show here by using the multidimensional Laplace transform version of the Laguerre method how the inversion can be achieved. Furthermore, because of the limitations in our Laguerre method, we shall employ the iterated Talbot method to perform the inversion for large arguments of asset and strike prices since as stated earlier, the method is easy to program and implement.

5.1 Motivation

Suppose we have a call option maturing at time T, written on a general real variable A whose payoff at time T is $X = (A-K)^+$, where K is the fixed strike price. Assuming that the risk neutral density of A is well defined under the measure $\mathbb{Q}$ at time t and is $f_{t,T}(a)$, we shall interpret its moment generating function as the Laplace transform given by

$$\begin{aligned}\psi_{t,T}(\lambda) &= \mathbb{E}_{t,T}(e^{-\lambda A}) \\ &= \int_0^\infty e^{-\lambda a} f_{t,T}(a)da. \end{aligned} \tag{5.3}$$

Recall from equation (1.5) that by no arbitrage arguments, the option price is given by

$$\begin{aligned}C_{t,T}(S,K) &= e^{-r(T-t)}\mathbb{E}\left[(A-K)^+\right] \\ &= e^{-r(T-t)}\int_K^\infty (a-K) f_{t,T}(a)da. \end{aligned} \tag{5.4}$$

Defining the Laplace transform with respect to the strike price of the call option by

$$\phi_{t,T}(\lambda) := \int_0^\infty e^{-\lambda K} C_{t,T}(S,K)dK, \tag{5.5}$$

we shall relate this transform to the Laplace transform on the density on which the option is written, thus

$$\begin{aligned}\phi_{t,T}(\lambda) &= e^{-r(T-t)}\int_0^\infty e^{-\lambda K}\int_K^\infty (a-K) f_{t,T}(a)dadK \\ &= e^{-r(T-t)}\int_0^\infty \left[\int_0^a e^{-\lambda K}(a-K)dK\right] f_{t,T}(a)da \\ &= e^{-r(T-t)}\int_0^\infty \left[\frac{a}{\lambda}+\frac{e^{-\lambda a}}{\lambda^2}-\frac{1}{\lambda^2}\right] f_{t,T}(a)da \\ &= \frac{e^{-r(T-t)}}{\lambda^2}\left[\int_0^\infty e^{-\lambda a} f_{t,T}(a)da + \lambda\int_0^\infty a f_{t,T}(a)da - \int_0^\infty f_{t,T}(a)da\right] \\ &= \frac{e^{-r(T-t)}}{\lambda^2}\left[\mathbb{E}_{t,T}\left(e^{-\lambda A}\right)+\lambda\mathbb{E}_{t,T}(A)-1\right] \\ &= e^{-r(T-t)}\frac{\psi_{t,T}(\lambda)+\lambda\mathbb{E}_{t,T}(A)-1}{\lambda^2}, \end{aligned} \tag{5.6}$$

where $\mathbb{E}_{t,T}(A)$ is the mean of the density $f_{t,T}(a)$. Notably, equation (5.6) is equation (9) of Fu et al. [39], though derived differently. Having derived this relation, we shall go on to take the Laplace transform with respect to maturity time thereby obtaining the double Laplace transform.

5.2 Double Laplace Transform for Asian Call Options

For the continuous fixed strike Asian call option we shall take $A = A_T$ where

$$A_T = \frac{1}{T}\int_0^T S_u du.$$

We shall therefore derive the Laplace transform with respect to time of $\psi_{t,T}(\lambda)$ as follows

$$\begin{aligned}
\psi_{t,T}(\lambda) &= \mathbb{E}_{t,T}\left(e^{-\lambda A}\right) \\
&= \mathbb{E}_{t,T}\left[\exp\left(-\frac{\lambda}{T}\int_0^T S_u du\right)\right] \\
&= \mathbb{E}_{t,T}\left[\exp\left(-\frac{\lambda}{T}\int_0^t S_u du - \frac{\lambda}{T}\int_t^T S_u du\right)\right] \\
&= \exp\left(-\frac{\lambda t}{T}A_t\right)\mathbb{E}_{t,T}\left[\exp\left(-\frac{\lambda}{T}\int_t^T S_u du\right)\right]. \qquad (5.7)
\end{aligned}$$

If we define by Φ the Laplace transform of the uncertainty in equation (5.7), we have

$$\Phi(t,\lambda,T) := \mathbb{E}_{t,T}\left[\exp\left(-\lambda\int_t^T S_u du\right)\right], \qquad (5.8)$$

which therefore implies that

$$\psi_{t,T}(\lambda) = \exp\left(-\frac{\lambda t}{T}A_t\right)\Phi(t,\lambda/T,T).$$

In the derivation of the solution to this Laplace transform we let

$$\Phi(t,\lambda,T) = \Psi(t,S,T,\lambda),$$

and upon the realization that definition (5.8) is similar to the pricing of pure discount bonds with short rate r replaced by λS, it follows from Cox et al. [24] that Ψ satisfies the partial differential equation given as

$$\begin{aligned}
\Psi_t + rS\Psi_S + \frac{1}{2}\sigma^2 S^2\Psi_{SS} - \lambda S\Psi &= 0 \\
\Psi(T,S,T,\lambda) &= 1.
\end{aligned}$$

Proof. From Cox et al. [24] we have the term structure equation given as

$$\begin{aligned}
F_t + \bar{\mu}F_r + \frac{1}{2}\bar{\sigma}^2 - rF &= 0 \qquad (5.9)\\
F(T,r) &= 1
\end{aligned}$$

where F is the price of a pure discount bond and is given by

$$F(t,r) = \mathbb{E}_{t,r}\left[\exp\left(-\int_t^T r_s ds\right)\right] \tag{5.10}$$

and the short rate has the dynamics

$$\begin{aligned} dr_s &= \bar{\mu} ds + \bar{\sigma} dW_s \\ r_t &= r. \end{aligned}$$

Now, if we let $r_s = \lambda S_s$ in equation (5.10) we obtain equation (5.8) and from the dynamics of S given by equation (1.1), we have

$$\begin{aligned} dr_s &= \lambda r S_s ds + \lambda \sigma S_s dW_s \\ r_t &= \lambda S. \end{aligned}$$

Taking $\frac{\partial}{\partial r} = \frac{1}{\lambda}\frac{\partial}{\partial S}$ we have

$$\begin{aligned} \Psi_t + rS\Psi_S + \frac{1}{2}\sigma^2 S^2 \Psi_{SS} - \lambda S \Psi &= 0 \\ \Psi(T,S,T,\lambda) &= 1. \end{aligned}$$

$\square$

As the underlying process which is a homogeneous Markov process, the function Ψ depends only on the time $(T-t)$. We shall therefore consider a solution of the form

$$\Psi(t,S,T,\lambda) = U(S,\tau,\lambda).$$

where $\tau = T - t$. Hence the partial differential equation in U is given as

$$\begin{aligned} U_\tau - rSU_S - \frac{\sigma^2}{2} S^2 U_{SS} + \lambda S U &= 0 \\ U(S,0,\lambda) &= 1. \end{aligned}$$

Pre-multiplying by $e^{-v\tau}$ and applying the boundary conditions for U and integrating, we obtain the ordinary differential equation in W, which is the Laplace transform of U, given as

$$S^2 W_{SS} + \frac{2r}{\sigma^2} S W_S - \frac{S}{\sigma^2}(v + \lambda S) W = -\frac{2}{\sigma^2} \tag{5.11}$$

whose solution we obtain to be

$$W(S,v,\lambda) = \frac{1}{v}\left(1 - \frac{\sigma^2}{2(r-v)}\right)\left(1 + \frac{\lambda S}{r-v}\right) + \frac{\sigma^2}{2v(r-v)}\, {}_1F_2\left(1; 1-\alpha_1, 1-\alpha_2; \frac{2\lambda S}{\sigma^2}\right) \tag{5.12}$$

where ${}_1F_2$ is the generalized hyper-geometric function and

$$\begin{aligned} \alpha_1 &= 1/2 - r/\sigma^2 - \sqrt{(1/2 - r/\sigma^2)^2 + 2(r+v)/\sigma^2} \\ \alpha_2 &= 1/2 - r/\sigma^2 + \sqrt{(1/2 - r/\sigma^2)^2 + 2(r+v)/\sigma^2}. \end{aligned} \tag{5.13}$$

In Appendix (C) we provide the proof of this result so that our work is self contained, a similar proof can be found in [39]. In summary, the solution of U is obtained by taking the Laplace inverse of W and replacing λ by λ/T, hence obtaining Φ.

We use the above formulation in pricing continuous fixed strike Asian call options. To obtain the expression for $\mathbb{E}_{t,T}(A)$ in equation (5.6), we note that a continuous Asian call option written at time $t = 0$, with maturity and strike T and K respectively is equivalent to $1/T$ units of a call option on the integral over the interval $[0, T]$ with strike KT. We therefore consider, without loss of generality, the option written on the integral of the stock price over the interval $[0, T]$, thus

$$R = \int_0^T S_u du.$$

Using the relation in (5.6), we let $\psi_{0,T}(\lambda)$ be the Laplace transform of the integral of the stock price over the interval $[0, T]$ such that

$$\psi_{0,T}(\lambda) = \mathbb{E}_{0,T}\left[e^{-\lambda R}\right].$$

It follows therefore from equation (5.6) that the Laplace transform with respect to the strike price of the call option is given by

$$\phi_{0,T}(\lambda) = e^{-rT}\frac{\psi_{0,T}(\lambda) + \lambda\mathbb{E}_{0,T}(R) - 1}{\lambda^2},$$

where

$$\begin{aligned} \mathbb{E}_{0,T}(R) &= \mathbb{E}_{0,T}\left[\int_0^T S_u du\right] \\ &= \frac{S}{r}\left(e^{rT} - 1\right) \end{aligned}$$

which implies that

$$\phi_{0,T}(\lambda) = \frac{S}{r\lambda}\left(1 - e^{-rT}\right) + \frac{e^{-rT}\psi_{0,T}(\lambda)}{\lambda^2} - \frac{e^{-rT}}{\lambda^2}.$$

Taking the Laplace transform with respect to maturity time T, to obtain the double Laplace transform, we have

$$\begin{aligned}\hat{C}(\lambda, v) &= \int_0^\infty e^{-vT}\phi_{0,T}(\lambda)dT \\ &= \int_0^\infty \frac{S}{r\lambda}\left[e^{-vT} - e^{-(r+v)T}\right]dT + \frac{1}{\lambda^2}\int_0^\infty e^{-(r+v)T}\psi_{0,T}(\lambda)dT - \frac{1}{\lambda^2}\int_0^\infty e^{-(r+v)T}dT \\ &= \frac{S}{r\lambda}\left(\frac{1}{v} - \frac{1}{r+v}\right) - \frac{1}{\lambda^2(r+v)} + \frac{1}{\lambda^2}\int_0^\infty e^{-(r+v)T}\psi_{0,T}(\lambda)dT. \end{aligned} \tag{5.14}$$

Since we have computed the Laplace transform with respect to the maturity time (T) for $\psi_{0,T}$, we can now then substitute equation (5.12) for the Laplace transform of $\psi_{0,T}$, noting that v is replaced by $r+v$. Therefore the double transform is given by

$$\begin{aligned}\hat{C}(\lambda, v) = \frac{S}{r\lambda}\left(\frac{1}{v} - \frac{1}{r+v}\right) - \frac{1}{\lambda^2(r+v)} + \frac{1}{\lambda^2(r+v)}\Bigg[\left(1 + \frac{\sigma^2}{2v}\right)\left(1 - \frac{\lambda S}{v}\right) \\ - \frac{\sigma^2}{2v}\,{}_1F_2\left(1; 1-\alpha_1, 1-\alpha_2; \frac{2\lambda S}{\sigma^2}\right)\Bigg]. \end{aligned} \tag{5.15}$$

By applying Laplace inversion techniques to the double Laplace formula derived above we shall show how the option prices can be obtained. This, as stated earlier, was not done by Fu et al. [39].

5.3 Laplace Transform Inversion Methods

Our objective is to use the Laplace transform methods to calculate the values of a real-valued function $f(t)$ of a positive real variable t from the Laplace transform given in equation (5.1) at any desired complex s. This can be done if the transform is given explicitly. Unfortunately, often the Laplace transform is given implicitly by some complicated functional equations which are difficult to solve. However, accurate and efficient algorithms for the inversion of Laplace transforms have been developed in the literature, for instance, Weideman [76], Abate and Whitt [3], Petrella [64] to name just a few, give some references and some of their work on the inversion algorithms. See also [21, 22, 68].

For the problems considered in this thesis, we shall focus on the algorithms discussed by Abate and Whitt, Weideman and Abate et al. [1] and Talbot namely the Euler, Laguerre and the Talbot methods for inverting Laplace transforms respectively. Though the Euler method described herein was developed for inverting functions defined only on the positive real line, Petrella [64] showed that this method can be extended to functions defined on the entire real line. We discuss the Talbot method which we intend to use in Section 5.5 for inverting our Asian call option for large values of the stock and strike price because of its simplicity in implementation and programming.

5.3.1 Euler Inversion Method

The underlying idea of this method is to discretize equation (5.16) using the trapezoidal rule which result in the integral being given by a sum of infinite terms. We then employ the Euler algorithm which allows the computation of the integral with great accuracy using a limited number of terms [3, 41].

By letting the contour be any vertical line $\Re(s) = x$ such that $\hat{f}(t)$ has no singularities on or to the right of it, we obtain

$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{st}\hat{f}(s)ds$$

Now, setting $s = y + iu$ we have,

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(y+iu)t}\hat{f}(y+iu)du \\ &= \frac{e^{yt}}{2\pi}\int_{-\infty}^{\infty} \left(\cos(ut) + i\sin(ut)\right)\hat{f}(y+iu)du \\ &= \frac{e^{yt}}{2\pi}\int_{-\infty}^{\infty} \left[\Re\left(\hat{f}(y+iu)\right)\cos(ut) - \Im\left(\hat{f}(y+iu)\right)\sin(ut)\right] du \\ &= \frac{2e^{yt}}{\pi}\int_{0}^{\infty} \Re\left(\hat{f}(y+iu)\right)\cos(ut)du, \end{aligned} \tag{5.16}$$

where $\Re(z)$ and $\Im(z)$ are the real and the imaginary parts of z. We numerically evaluate this integral by means of the trapezoidal rule. Thus, choosing the step size to be h it implies that

$$\begin{aligned} f(t) &\approx f_h(t) \\ &= \frac{he^{yt}}{\pi}\Re\left(\hat{f}(y)\right) + \frac{2he^{yt}}{\pi}\sum_{k=1}^{\infty}\Re\left(\hat{f}(y+ikh)\right)\cos(kht). \end{aligned} \tag{5.17}$$

Letting $h = \pi/(2t)$ and $y = A/(2t)$, we obtain a nearly alternating series given by

$$f_h(t) = \frac{e^{A/2}}{2t}\Re\left[\hat{f}\left(\frac{A}{2t}\right)\right] + \frac{e^{A/2}}{t}\sum_{k=1}^{\infty}(-1)^k\Re\left[\hat{f}\left(\frac{A+2k\pi i}{2t}\right)\right]; \tag{5.18}$$

this is equation (21) of Dubner and Abate [34].

To numerically compute equation (5.18), which involves an infinite sum, we use the Euler summation which is one of the most elementary technique as described in Abate and Whitt [3] who noted that for practical purposes it seems to provide adequate computational efficiency. Briefly, the Euler summation is an algorithm consisting of summing explicitly the first n terms of the series and then taking a weighted average of the additional m partial sums of a binomial probability distribution with parameters m and $p = \frac{1}{2}$. This method is however not well known as pointed out by [3] though its detailed representation was given by Wimp [77].

The method works as follows: let $S_n(t)$ be the approximation $f_h(t)$ in equation (5.18) with the infinite series truncated to n terms such that

$$S_n(t) = \frac{e^{A/2}}{2t}\Re\left[\hat{f}\left(\frac{A}{2t}\right)\right] + \frac{e^{A/2}}{t}\sum_{k=1}^{n}(-1)^k a_k(t),$$

where

$$a_k(t) = \Re\left[\hat{f}\left(\frac{A+2k\pi i}{2t}\right)\right].$$

We now apply the Euler summation to m terms after an initial n, so that the Euler sum which would be our approximation to equation (5.18) would be given by

$$E(m,n,t) = \sum_{k=0}^{m}\begin{pmatrix} m \\ k \end{pmatrix} 2^{-m} S_{n+k}(t). \tag{5.19}$$

Although this method was originally developed to analyze queuing systems, it has earned much interest in solving computational finance problems and among other applications is Davydov and Linetsky [29] and Fu et al. [39].

5.3.2 Laguerre Inversion Method

The Laguerre[2] method has its basis for the inversion of the Laplace transform in equation (5.1) through the Laguerre series representation of f which is given as

$$f(t) = \sum_{n=0}^{\infty} q_n \ell_n(t) \tag{5.20}$$

where ℓ_n's are the Laguerre functions and are computed using the recursion

$$\ell_n(t) = \left(\frac{2n-1-t}{n}\right)\ell_{n-1}(t) - \left(\frac{n-1}{n}\right)\ell_{n-2}(t), \tag{5.21}$$

with $\ell_0(t) = e^{-t/2}$ and $\ell_1(t) = (1-t)e^{-t/2}$. The method works because the Laplace transform of the n^{th} Laguerre function has a special form that allows for the computation of the Laguerre generating function in terms of the Laplace transform $\hat{f}$, see [2, 1]. We shall interest our computation to the use of the scaled version of this method, since we have clearly developed method for computing the free parameters for computing the Laguerre coefficients q_n through the Laguerre generating function associated with $\hat{f}$.

The reason for scaling is essentially to ensure fast convergence of q_n, this is done by choosing two positive real free parameters σ' and b' such that

$$f(t) = e^{\sigma' t}\sum_{0}^{\infty} q_n e^{b't}\ell_n(2b't). \tag{5.22}$$

The Laguerre generating function given by

$$Q(z) = \frac{2b'}{1-z}\hat{f}\left(\sigma' + \frac{2b'}{1-z} - b'\right),$$

would be used to derive the Laguerre coefficients. For the derivation of these formulas we refer to Weideman [76]. Abate et al. [2] have also derived the same formulas but with b' replaced with $b'/2$ and another notable difference between these two approaches was the derivation of the coefficients q_n. We shall use here the formula obtained by Weideman which is given as

$$\widetilde{q}_n = \frac{e^{-inh/2}}{2N}\sum_{j=-N}^{N-1} Q(e^{iu_{j+1/2}})e^{-inu_j}, \quad n = 0, \ldots, N-1$$

[2]The method is often called the Weeks method [2] largely because of his early contribution back in the 1950's.

where $u_j = jh$, $h = \pi/N$ and we shall employ the FFT method to compute this sum, this result differs from that in Abate et al. [2] because of the different choices of the variable z.

Weideman went further to provide two algorithms for determining the free parameters σ' and b', making the algorithm more attractive. We shall use the second of these algorithms since it does not require any information about the singularities of the Laplace transform, hence suitable for a inversion for wide range of function [76].

5.3.3 Talbot Inversion Method

The Talbot method seeks to replace the Bromwich contour integral in equation (5.2) by the Talbot's [71] contour given as

$$z(\beta) = \frac{N}{t}\left[0.5017\beta\cot(0.6407\beta) + 0.2645i\beta - 0.6122\right] \tag{5.23}$$

where $-\pi \leq \beta \leq \pi$ such that equation (5.2) is then written as

$$f(t) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{z(\beta)t}\hat{f}(z(\beta))z'(\beta)d\beta. \tag{5.24}$$

The derivative of z is represented by z' above. By partitioning β into N uniformly distributed points of distance $h = 2\pi/N$ such that

$$\beta_k = -\pi + \left(k - \frac{1}{2}\right)h, \quad 1 \leq k \leq N$$

and applying the midpoint rule to the integral in (5.24), we have

$$f_N(t) = \frac{h}{2\pi i}\sum_{k=1}^{N} e^{z(\beta_k)t}\hat{f}(z(\beta_k))z'(\beta_k). \tag{5.25}$$

For the derivation of these formulas we refer to [68]. Clearly, this is a simple method for inverting Laplace transforms.

5.3.4 Multidimensional Laplace Inversion Method

Developed by Abate et al. [1], the method is an extension of the Laguerre method as presented in Section 5.3.2 for inverting the multidimensional Laplace transform. The algorithm is based on the construction of Laguerre polynomials which are associated with the Laguerre functions, the Laguerre coefficients and the Laguerre generating functions. We shall not attempt to prove their results here, instead for brevity and to do away with duplication of work, we shall present their main algorithm which we shall implement in Section 5.5.

By considering the bivariate case which has the Laplace transform given as

$$\hat{f}(s_1, s_2) = \int_0^\infty \int_0^\infty e^{-(s_1 t_1 + s_2 t_2)} f(t_1, t_2) dt_1 dt_2 \tag{5.26}$$

which we assume is well defined thus, it is convergent and analytic for $\Re(s_1) > 0$ and $\Re(s_2) > 0$. Our goal is to compute the bivariate function f from its two dimensional Laplace transform $\hat{f}$ as in equation (5.26). The Laguerre series representation takes the form

$$f(t_1, t_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} q_{n_1,n_2} \ell_{n_1}(t_1) \ell_{n_2}(t_2), \qquad t_1, t_2 \geq 0$$

where ℓ_n's are the Laguerre functions and are computed using the recursion as in equation (5.21) and

$$Q(z_1, z_2) = (1 - z_1)^{-1}(1 - z_2)^{-1} \hat{f}\left(\frac{1 + z_1}{2(1 - z_1)}, \frac{1 + z_2}{2(1 - z_2)}\right). \tag{5.27}$$

For the derivation of these functions we refer the reader to Abate et al. [1] and we shall however, state here the formulas for computing the above algorithm.

$$\begin{aligned}
q_{n_1,n_2} &= \frac{1}{r_1^{n_1} r_2^{n_2}} a_{n_1,n_2}, \qquad 0 \leq n_1 \leq N_1 - 1,\ 0 \leq n_2 \leq N_2 - 1 \\
a_{n_1,n_2} &= \frac{1}{m_1 m_2} \sum_{j=0}^{m_1-1} \sum_{k=0}^{m_2-1} \exp\left(-\frac{2\pi i j n_1}{m_1} - \frac{2\pi i k n_2}{m_2}\right) b_{j,k} \\
b_{j,k} &= Q\left(r_1 e^{2\pi i n_1/m_1}, r_1 e^{2\pi i n_2/m_2}\right)
\end{aligned}$$

where $Q(\cdot, \cdot)$ is as in equation (5.27), $r_1 = 10^{-A_1/m_1}$, $r_2 = 10^{-A_2/m_2}$. The constants A_1 and A_2 are chosen to control the error in the approximation of q_{n_1,n_2} and l_i are the roundoff error control parameters which have been suggested to be $l_1 = 1$ or 2 and $l_2 = 2$. Furthermore, N_i are chosen as powers of 2, $m_1 = 2l_1N_1$ and $m_2 = 2l_2N_2$. We shall apply the two dimensional FFT method for computing the Laguerre coefficients q_{n_1,n_2} since the form of a_{n_1,n_2} resembles the nature of the discrete Fourier transform in two dimensions hence the direct application of the FFT algorithm.

5.4 Inversion of the Double Laplace Transform

We have shown that by relating the Laplace transform in the strike of the Asian call option to the Laplace transform of the density of the average price on which the call is written and further taking the Laplace transform of the relation with respect to maturity time, we obtain an expression for the Asian call price i.e the double Laplace transform. Given this transform, which is denoted by $\hat{C}(\lambda, v)$, we shall in this section pursue its inversion to obtain the original function hence the Asian call price.

By introducing the notation $\mathcal{L}^{-1}$ for the Laplace inverse, the call price $C(K,T)$ is given by

$$\begin{aligned} C(K,T) &= \mathcal{L}^{-1}\left[\mathcal{L}^{-1}\left[\hat{C}(\lambda, v); \lambda \to K\right]; v \to T\right] \\ &:= \mathcal{L}^{-1}\left[\mathcal{L}^{-1}\left[\hat{C}(\lambda, v)\right]\right]. \end{aligned}$$

We can break the double Laplace transform in equation (5.15) into two parts; the first part which we can perform the inversion analytical as we shall show and then the second part which we shall invert numerically by using the Laguerre or Talbot methods as discussed in Section (5.3). We denote the two parts by $\hat{C}_1(\lambda, v)$ and $\hat{C}_2(\lambda, v)$ and are given as follows

$$\begin{aligned} \hat{C}_1(\lambda, v) &= \frac{S}{r\lambda}\left(\frac{1}{v} - \frac{1}{r+v}\right) - \frac{1}{\lambda^2(r+v)} + \frac{1}{\lambda^2(r+v)}\left(1 + \frac{\sigma^2}{2v}\right)\left(1 - \frac{\lambda S}{v}\right) \\ \hat{C}_2(\lambda, v) &= \frac{1}{2\lambda^2 v(r+v)}\, {}_1F_2\left(1; 1-\alpha_1, 1-\alpha_2; \frac{2\lambda}{\sigma^2}S\right), \end{aligned} \tag{5.28}$$

where α_1 and α_2 are as defined in (5.13) and hence we shall recover the option price by

$$\begin{aligned} C(K,T) &= \mathcal{L}^{-1}\left[\mathcal{L}^{-1}\left[\hat{C}_1(\lambda, v)\right]\right] - \mathcal{L}^{-1}\left[\mathcal{L}^{-1}\left[\hat{C}_2(\lambda, v)\right]\right] \\ &= C_1(K,T) - C_2(K,T). \end{aligned}$$

We obtain $C_1(K,T)$ by first computing the inverse with respect to v and then with respect to λ which we obtain to be

$$C_1(K,T) = \frac{S\sigma^2}{2r^2} - e^{-rT}\frac{S\sigma^2}{2r^2} + \frac{KT\sigma^2}{2r} - e^{-rT}\frac{KT\sigma^2}{2r} - \frac{ST\sigma^2}{2r}$$

On the other hand, $C_2(K,T)$ is obtained by numerically inverting equation (5.28) as follows

$$C_2(K,T) = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{\lambda^2}\mathcal{L}^{-1}\left[\frac{1}{v(r+v)}\, {}_1F_2\left(1; 1-\alpha_1, 1-\alpha_2; \frac{2\lambda}{\sigma^2}S\right)\right]\right].$$

Nieuwveldt [61] proposed a new method based on the Talbot contour for inversion of the Geman and Yor [42] formula which involves the hyper−geometric function, ${}_1F_1$. We however cannot extend this method to our work since our hyper−geometric function, ${}_1F_2$ does not poses a general Laplace transform. Before we proceed to do the numerical inversion of $\hat{C}_2$, we shall give a brief outlook of the generalized hyper−geometric function, ${}_1F_2$.

5.4.1 Generalized Hyper-geometric Function

The generalized hyper-geometric function denoted ${}_pF_q$ has the notation given as

$${}_pF_q := {}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z).$$

${}_pF_q$ converges for all z if $p \leq q$, converges for $|z| < 1$ if $p = q+1$ and diverges for all $z \neq 0$ if $p > q+1$. To our interest, is when $p = 1$ and $q = 2$ which we recognize to be written as

$${}_1F_2 := {}_1F_2(a_1; b_1, b_2; z).$$

The series representation of ${}_1F_2$ is given as

$${}_1F_2(a_1; b_1, b_2; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k z^k}{(b_1)_k (b_2)_k} k!$$

where

$$\begin{aligned} (x)_k &= x(1+x)(2+x)\ldots(k-1+x) \\ &= \frac{\Gamma(k+x)}{\Gamma(x)}. \end{aligned}$$

Figure (5.1) shows a plot for the generalized hyper-geometric function ${}_1F_2$ for $z \in (-300, 0)$ and we fixed $a = 1$, $b_1 = b_2 = 1.1$.

For the purpose of our inversion problem (see [39]), we shall consider the integral representation of the function ${}_1F_2$ given by Lee [56] which is

$${}_1F_2(a; b_1, b_2, z) = \int_0^1 \Gamma(b_2)(tz)^{-(b_2-1)/2} \mathcal{I}_{(b_2-1)}(2\sqrt{tz}) \frac{\Gamma(b_1)}{\Gamma(a)\Gamma(b_1-a)} t^{a-1}(1-t)^{b_1-a-1} dt,$$

where $\mathcal{I}_v(\omega)$ is the Bessel function of imaginary arguments. Considering the arguments for ${}_1F_2$ in $\hat{C}_2$, we have $a = 1$, $b_1 = 1-\alpha_1$ and $b_2 = 1-\alpha_2$, which then simplifies to give

$${}_1F_2(a; b_1, b_2, z) = \int_0^1 \Gamma(1-\alpha_2)(tz)^{\alpha_2/2} \mathcal{I}_{-\alpha_2}(2\sqrt{tz})(-\alpha_1)(1-t)^{-(1+\alpha_1)} dt \tag{5.29}$$

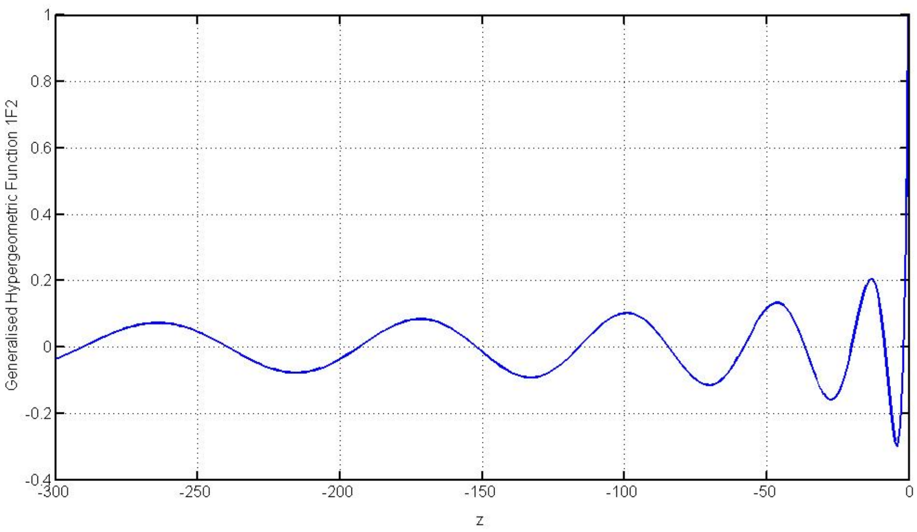

Figure 5.1. A plot for the generalized hyper-geometric function ${}_1F_2(a; b_1, b_2; z)$ for selected parameters which are $a = 1$, $b_1 = b_2 = 1.1$ and $z \in (-300, 0)$.

where α_1 and α_2 are as in equation (5.13). The statistical tools box in MATLAB and the built-in function in Mathematica for computing the generalized hyper-geometric function tends to be slow for large arguments of z, b_1 and b_2. To that effect we shall utilize the representation in equation (5.29) for our computation, and for that we employ the midpoint rule to approximate the integral thus,

$$\begin{aligned} {}_1F_2(1; 1-\alpha_1, 1-\alpha_2; z) &= \int_0^1 \Gamma(1-\alpha_2)(tz)^{\alpha_2/2}\mathcal{I}_{-\alpha_2}(2\sqrt{tz})(-\alpha_1)(1-t)^{-(1+\alpha_1)}dt \\ &= \int_0^1 f(t)dt \\ &:= \frac{1}{N}\sum_{i=0}^{N-1} f((i+1/2)/N). \end{aligned}$$

5.4.2 Discussions

As is evident from the example in Appendix D where we apply the Laplace transform method for pricing European call options, accurate numerical approximation for the option prices are obtained. Moreover, we note the accuracy of the inversion methods precisely the Euler method whose results (in all the tables) gives to six decimal places a 100% similarity with the analytic results obtained by the Black Scholes formula. On the other hand, we see that the use of the scaling parameters σ' and b' in the Laguerre method plays a very significant role in the accuracy of the method.

In presenting the multidimensional Laplace inversion algorithm we pointed out that it was an extension of the Laguerre method. However, as we have noted in the application of the Laguerre method the scaling parameters plays a significant role in the inversion algorithm. To that effect we need to incorporate these scaling parameters into the extended algorithm. Unfortunately in doing so the algorithm gets to be so complicated such that Abate et al. [1] had this to say, "... we have shown that it is possible to resolve many of the difficulties that arise with the Laguerre method, but the remedies make the algorithm more complicated." That brings us to the issue of parameter settings with many of the Laplace inversion method.

Though Weideman [76] has develop a method of computing the scaling parameters for the one dimensional Laguerre method, the extended version of the method has no defined way to determine these parameters. The same applies to the Euler algorithm. At the very least in most of the examples that are given in literature when it comes to how the parameters have been determined what is commonly said is, "by experimenting the parameters used are ..." [3, 21, 28]. This is not an attractive feature with many algorithms and hence poses a difficulty when faced with a problem that requires different parameter specification.

In our application of the multidimensional Laplace transform we shall not attempt to use the scaled version of the algorithm because of its complexity, thus if we introduce the scaling parameters we are faced with the problem of estimating six parameters in the algorithm which is clearly computationally expensive to do. Also to note is the fact that, with the multidimensional Laguerre method it can be difficult to calculate the desired function for large arguments because of round off errors, hence the Talbot method becomes our opted method to perform the inversion on such arguments.

5.5 Numerical Computations

In this section we shall perform the numerical computations for the pricing of continuous Asian option in Mathematica (Talbot) and Python (Laguerre). First we shall compare the results that we obtain from numerically inverting the double Laplace transform using the Laguerre and Talbot method with those reported in Table 4 of Fu et al. [39]. This table comprise of the numerical results of other known methods in literature, hence the choice as a benchmark for our comparisons.

We highlighted above that due to the slow computations of the built-in functions of the generalized hyper-geometric function (${}_1F_2$) in Mathematica, we employ the midpoint rule in our numerical computations to approximate the function. Figure 5.2 shows the comparison of the exact and approximated values of ${}_1F_2$ together with the mean absolute percentage error, for the parameter setting given as $a = 1$, $b_1 = 3$, $b_2 = 2$ and $z \in (-100, 0)$.

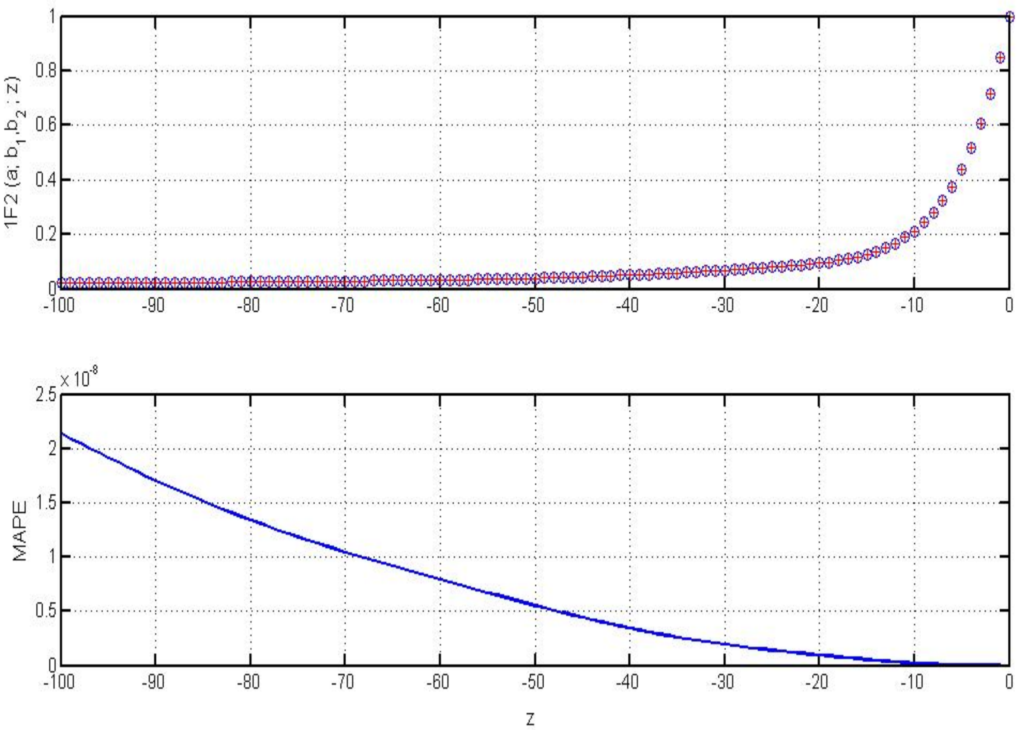

Figure 5.2. A comparison of the exact and approximate values of the function ${}_1F_2(1; 3, 2, z)$ and the mean absolute percentage error for the resulting computation.

Generally, as N increases the error obtained between the exact and the approximated result decreases significantly, we show in Figure 5.3 a plot of the N values against the mean absolute percentage error which clearly decreases as N increases.

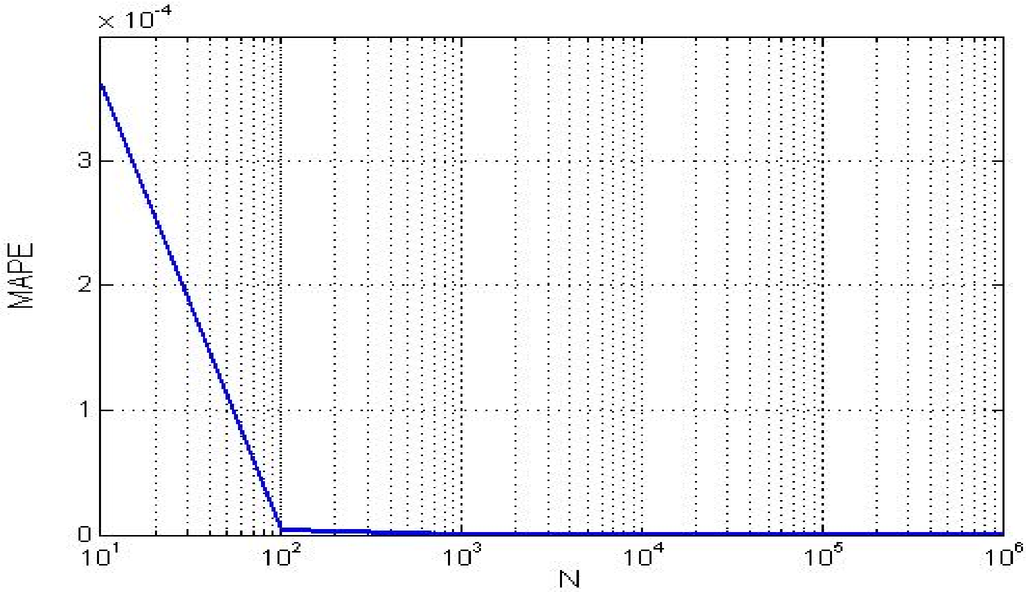

Figure 5.3. Mean absolute percentage error (MAPE) obtained by valuating ${}_1F_2(1; 2, 2; -2)$ as a function of N.

For our computation we used $N = 100$ which produces results that are correct to three decimal places and the computational time is relatively fast. On the other hand, when we use large values of N say, $N = 1\,000, 10\,000$, this only increases the number of decimal places on which the result is correct however increasing the computational time. For instance, in Table 5.3 the price of the option when $S = 1.9$ for $N = 100$ is computed in 1.016 seconds and when we increase N to 10 000 we obtain the price in 100.407 seconds which is correct to 6 significant figures.

Upon implementing the Laguerre and Talbot methods for inversion of our double Laplace transform for continuous Asian call options using the model parameters given in Table 5.2, we obtained the prices given in Table 5.3.

For the Laguerre method, the parameters used are $N_1 = 16$, $N_2 = 64$, $l_1 = 1$, $l_2 = 2$, $A_1 = 18$ and we varied A_2 according to the model parameter settings. For the Talbot method we took in all the cases $n = m = 10$, we scaled the contour by a factor inversely proportional

r	σ	T	S_0	K
0.05	0.5	1	1.9	2
0.05	0.5	1	2	2
0.05	0.5	1	2.1	2
0.02	0.1	1	2	2
0.18	0.3	1	2	2
.0125	0.25	2	2	2
0.05	0.5	2	2	2

Table 5.2. The model parameters used in the inversion of the double Laplace transform.

to sigma such that we control the z argument in the generalized hyper-geometric function. We present below a comparison of our results with those obtained in Table 4 of Fu et al. [39].

Laguerre	Talbot	GE	Shaw	Euler	PW	TW	MC100
0.196	0.193	0.195	0.193	0.194	0.194	0.195	0.196
0.248	0.246	0.248	0.246	0.247	0.247	0.250	0.249
0.306	0.309	0.308	0.306	0.307	0.307	0.311	0.309
0.055	0.057	0.058	0.056	0.056	.0624	.0568	.0565
0.219	0.217	0.227	0.217	0.219	0.219	0.220	0.220
0.173	0.173	0.172	0.172	0.172	0.172	0.173	0.172
0.352	0.350	0.351	0.350	0.352	0.352	0.359	0.348

Table 5.3. Values of the continuous fixed strike Asian option - comparison of results in (GE)-Geman Eydeland, Shaw, Euler, (PW)-Post Widder, (TW)-Turnbull Wakeman, (MC)-Monte Carlo approximation methods with Laguerre and Talbot inversion methods. Parameters used are those in Table 5.2.

We observe that all the calculated prices are close to each other despite being based on different methodologies. We shall now explore the inversion of our double transform formula for large values of S and K using the Talbot method. We compare our results with the Monte Carlo methods as discussed in Chapter 4 and take the results by Zhang [79] as our benchmark values.

In Table 5.4, the abbreviation MC is used to indicate that the method used is the standard Monte Carlo and on the other hand, MC1, MC2 is the Monte Carlo method using antithetic variates, the Riemann Scheme and the Trapezoidal Scheme discussed in Section 4.1 and 4.2 respectively. MC3 and MC4 use the control variates with the two schemes respectively.

K	σ	Zhang	Talbot	MC	MC1	MC2	MC3	MC4
95	0.5	15.4427	15.4438	15.6148	15.3977	15.4363	15.4583	15.4455
100	0.5	13.0282	13.0270	13.1958	13.0077	13.0096	13.0427	13.0286
105	0.5	10.9296	10.9299	11.0969	10.9304	10.9095	10.9491	10.9290
95	0.4	13.5107	13.5105	13.6394	13.4607	13.4995	13.5229	13.5123
100	0.4	10.9238	10.9253	11.0491	10.8970	10.9207	10.9365	10.9238
105	0.4	8.72993	8.72996	8.8541	8.7267	8.7364	8.7448	8.7205
95	0.3	11.6559	11.6522	11.7421	11.6049	11.6469	11.6644	11.6560
100	0.3	8.82876	8.82735	8.9171	8.7993	8.8301	8.8375	8.8190
105	0.3	6.51779	6.51579	6.6045	6.5132	6.5181	6.5283	6.5176
95	0.2	9.99566	9.99641	10.0528	9.9557	9.9941	10.0025	9.9938
100	0.2	6.77735	6.77677	6.8334	6.7493	6.7819	6.7838	6.7776
105	0.2	4.29646	4.2967	4.3504	4.2922	4.2994	4.3038	4.2985
95	0.1	8.91185	8.95457	8.9522	8.8948	8.9153	8.9164	8.9135
100	0.1	4.91512	4.91522	4.9427	4.8959	4.9186	4.9199	4.9134
105	0.1	2.07006	2.07010	2.0959	2.0685	2.0743	2.0745	2.0706

Table 5.4. Values of the continuous fixed strike Asian option for varied strike and volatility. The parameters used are $S = 100$, $r = 0.09$ and $T = 1$.

What we can observe from the results obtained above is the rapid improvements on the Monte Carlo method when the "variance reduction" techniques are incorporated within the MC method. Clearly the use of control variance technique is a good technique to improve the results of the MC approximation method. However, our Laplace transform pricing method gives better results as compared to the MC method.

Moreover, we consider the results obtained by varying the strike and the interest rate, the strike price, interest rate and the volatility. We present below the tables that result from this approach.

We observe that in almost all the cases considered above, our double transform formula provides values that are almost consistent with the results of Zhang whose prices are usually considered to be "exact" in literature [19, 27]. With the help of the variance reduction techniques, the Monte Carlo method can be a good candidate for the determination of option prices especially when we are dealing with complex problems.

In summary, we note that by addressing the pricing aspect of Asian options using the Laplace transform method its success is largely judged by the availability of the inversion method to obtain the option price. As a remark we point out that the inversion of our dou-

K	r	Zhang	Talbot	MC	MC1	MC2	MC3	MC4
90	0.05	13.9538	13.9564	14.0404	13.8923	13.9486	13.9619	13.9543
100	0.05	7.94563	7.94546	8.0324	7.9252	7.9384	7.9538	7.9456
110	0.05	4.07179	4.07179	4.1482	4.0815	4.0593	4.0785	4.0727
90	0.09	14.9840	14.9838	15.0757	14.9253	14.9720	14.9933	14.9817
100	0.09	8.82876	8.82735	8.9171	8.7993	8.8286	8.8375	8.8289
110	0.09	4.69671	4.69749	4.7756	4.7015	4.6835	4.7034	4.6962
90	0.15	16.51291	16.5128	16.6101	16.4591	16.5089	16.5233	16.5128
100	0.15	10.2098	10.2082	10.2999	10.1700	10.2074	10.2207	10.2062
110	0.15	5.73012	5.73074	5.8145	5.7311	5.7348	5.7423	5.7302

Table 5.5. Values of the continuous fixed strike Asian option for varied strike and interest rate. The parameters used are $S = 100$, $\sigma = 0.30$ and $T = 1$.

K	r	Zhang	Talbot	MC	MC1	MC2	MC3	MC4
90	0.05	12.5960	12.5973	12.6628	12.5585	12.5874	12.6007	12.5954
100	0.05	5.7631	5.76786	5.8178	5.7427	5.7594	5.7683	5.7634
110	0.05	1.9899	1.98954	2.0285	1.9973	1.9789	1.9944	1.9890
90	0.09	13.8315	13.8315	13.9069	13.7984	13.8271	13.8376	13.8318
100	0.09	6.7773	6.77677	6.8334	6.7493	6.7824	6.7838	6.7763
110	0.09	2.5462	2.54677	2.5931	2.5522	2.5406	2.5514	2.5466
90	0.15	15.6418	15.6415	15.7218	15.6095	15.6517	15.6496	15.6420
100	0.15	8.4088	8.40812	8.4618	8.3713	8.4068	8.4175	8.4082
110	0.15	3.5556	3.55524	3.6078	3.5570	3.5570	3.5644	3.5552

Table 5.6. Values of the continuous fixed strike Asian option for varied strike and interest rate. The parameters used are $S = 100$, $\sigma = 0.20$ and $T = 1$.

ble Laplace transform pricing formula is a nontrivial procedure mainly because the formula involves nonstandard function ${}_1F_2$ whose computation leads to numerical instability for low volatilities and its slow decaying oscillatory nature for given parameter settings. Though the formula is amenable to two inversion techniques as described above i.e Laguerre and Talbot methods, parameter settings on these methods is a very difficult step to undertake which is not an attractive feature. However, our formula for the continuous fixed strike Asian option approximates the true value very well in cases considered above.

Chapter 6

Discussions and Conclusions

The methodologies involving Monte Carlo and integral transforms have proved to be very efficient in the valuation of options especially when acceleration techniques are introduced within these methods. In our work, by first pricing using these methods the European call options which simplified very pleasingly resulting in accurate prices being obtained, we have motivated the use of these methods in valuing Asian call options.

Carr and Madan pioneered the use of characteristic functions in option pricing. We have shown in Chapter 2 how we can use the FFT algorithm to recover the option price from the Fourier transform of the call price function. By means of transforming the option price to ensure integrability hence applying the FFT algorithm to invert the Fourier transform, the call prices are obtained at many strike levels. Furthermore we have, by utilizing the effectiveness of the FFT method in computing convolutions, successfully managed to price the continuous Asian options by means of the Richardson extrapolating formula for the discrete Asian option. In both these approaches the effectiveness of the FFT algorithm turns out to be an attractive way for pricing options due to its speed and accuracy in pricing. As a result we incorporated the FFT algorithm as an accelerating technique for inverting the double Laplace transform for Asian options.

Following the results by Geman and Yor, we have successfully obtained the hedging strategy for the continuous Asian call option as presented in Chapter 3. Since the inversion of the Laplace transform term in Geman and Yor formula has been effectively performed by Fu et al. [39], Craddock et al. [25], Shaw [66] and Dewynne and Shaw [30] the numerical computation of our hedging strategy becomes an easy task to perform.

Pricing Asian options using Monte Carlo methods is very effective when variance reduction techniques are incorporated within the standard Monte Carlo as we have shown in Chapter 4. Apart from our findings as discussed herein, Fu et al. [39] distinguished the use of the control variates as variance reduction techniques between biased and unbiased control variates in the view that in actual fact simulating continuous geometric Asian call price is actually discrete time Asian option. With that in mind, it implies technically that the continuous geometric option price is a biased control variate in this respect. Paraphrasing their findings, since numerical computations in general are discretely performed by any selected numerical technique, therefore to curb for the discretization bias in the simulation of continuous problems appropriate biased control variates can be very effective which is evident from our pricing of the Asian call option using the Monte Carlo method where we incorporated the continuous geometric Asian call option as the control variate.

In our work herein, we have derived the double Laplace formula for continuous fixed strike Asian call options following the approach by Fu et al. [39]. By using the multi-Laguerre and the iterated Talbot inversion methods for Laplace transforms we have managed to obtain option prices with reasonable accuracy. This computational approach was not performed in [39].

As demonstrated first by our examples, one critical point for the Laplace method is the selection of adequate parameters for the inversion algorithms especially when computed prices are to give guaranteed numerical accuracy to a large number of decimal places. In other words, our predictions will be strengthened if we reduce uncertainty and get better estimates on specific parameters of the inversion method. This will enhance the performance of the algorithms and hence accurate prices would be obtained and the efficiency and effectiveness of the methods would be improved. Our example in one dimensional inversion (Appendix D) proved that with a good method for parameter selection Laplace transform method is very effective.

For instance, the multi−Laguerre inversion method requires the use of Laguerre; polynomials, functions, coefficients and generating functions. We noted that the parameter values that were suggested by Abate et al. [1] are not consistent in our double Laplace transform formula and to curb for that shortcoming we have through experimentation determined ideal parameters that proved to be sufficient for our respective model parameter specifications. The prices obtained by the use of these parameters are in agreement with those reported in literature despite being based on different methodologies.

In conclusion, there being a variety of methods for pricing Asian options we focused here on Monte Carlo and integral transform methods. A general finding from our work and true for many available methods, is that the trade off for any available pricing method is between accuracy, speed and simplicity. Monte Carlo simulation have proved to be the simplest for pricing Asian options and its accuracy can be improved by the use of reduction techniques in the computation of the option prices.

On the other hand, the use of integral transform methods precisely the Laplace transform, can be very effective and favorable provided that the inversion methods to be used to obtain the prices has a clearly defined method for parameter settings. This makes the transform methods less favorable since often the inversion algorithms are more complicated to implement and there is no single method that gives optimum results for all purposes and occasions [25, 28].

Appendix A

The Girsanov Theorem

Theorem A.0.1 (Girsanov theorem)**.** *Let $\mathbb{X}_t \in \mathbb{R}^n$ be a stochastic process given by*

$$d\mathbb{X}_t = \boldsymbol{a}(t,\omega)dt + \boldsymbol{b}(t,\omega)d\mathbb{B}_t\,; \quad t \leq T.$$

Where $\boldsymbol{a}(t,\omega) \in \mathbb{R}^n$, $\boldsymbol{b}(t,\omega) \in \mathbb{R}^{n\times m}$ and $\mathbb{B}_t$ is a m-dimensional Brownian motion (i.e $\mathbb{B}_t = B_t^1, B_t^2, \ldots, B_t^m$ where B_t^i's are Brownian motions) with respect to the filtered space $(\Omega, \mathbb{P}, \mathcal{F})$. Suppose there exist a predictable processes $\lambda(t,\omega) \in \mathbb{R}^m$ and $\alpha(t,\omega) \in \mathbb{R}^n$ such that

$$\boldsymbol{b}(t,\omega)\lambda(t,\omega) = \boldsymbol{a}(t,\omega) - \alpha(t,\omega)$$

and we assume that $\lambda(t,\omega)$ satisfies the Novikov's condition[1]

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \lambda^2(s,\omega)ds\right)\right] < \infty. \tag{A.1}$$

Let

$$M_t = \exp\left(-\int_0^t \lambda^2(s,\omega)d\mathbb{B}_s - \frac{1}{2}\int_0^t \lambda^2(s,\omega)ds\right)\,; \quad t \leq T.$$

Since M_t is a martingale, we let $\mathbb{Q}$ be another measure equivalent to $\mathbb{P}$(i.e $\mathbb{P}$ and $\mathbb{Q}$ have the same null-measurable sets) such that

$$\begin{aligned} d\mathbb{Q} &= M_t d\mathbb{P} \\ \frac{d\mathbb{Q}}{d\mathbb{P}} &= M_t. \end{aligned}$$

M_t is called the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ and $\mathbb{Q}$ is a probability measure on $\mathcal{F}$. The process

$$\widetilde{\mathbb{B}}_t := \int_0^t \lambda^2(s,\omega)ds + \mathbb{B}_t\,; \quad t \leq T$$

[1] The Novikov condition (A.1) guarantees that $\{M_t\}_{t\leq T}$ is a martingale

is a Brownian motion with respect to the probability space $(\Omega, \mathbb{Q}, \mathcal{F})$ *and in terms of* $\widetilde{\mathbb{B}}_t$ *the process* $\mathbb{X}(t)$ *has the following representation*

$$d\mathbb{X}_t = \alpha(t, \omega) + \boldsymbol{b}(t, \omega)d\widetilde{\mathbb{B}}_t.$$

Proof can be found in [62].

Basically the Girsanov theorem says that the properties of stochastic processes do not drastically change with the change of their drift. In fact, by changing the drift of the process we change also its probability law which however turns out to be the same as the original one and we can compute explicitly the Radon-Nikodym derivative.

In this thesis, the process λ_t has been taken as the market price of risk. Where $\mathbb{X}_t = S_t \in \mathbb{R}$ which is the Black-Scholes stock price process such that we have $\mathbf{a}(t, \omega) = \mu S_t$ and $\mathbf{b}(t, \omega) = \sigma S_t$ and $\mathbb{Q}$ is the risk-neutral measure or the equivalent martingale measure, then $\alpha_t = rS_t \in \mathbb{R}$ and $\lambda_t = \frac{\mu - r}{\sigma}$ which is the risk market price of stock. The existence of an risk-neutral measure is related to the absence of arbitrage while uniqueness of the measure is related to market completeness.

Appendix B

A Comparison of Convolution Computational Methods

The convolution of two functions f and g is defined as

$$C(z) = \sum_{x=-\infty}^{\infty} f(x)g(z-x).$$

To demonstrate the speed for using the FFT method to compute convolutions we choose two independent functions X and Y such that $X = \exp(-t)$, $Y = 1$; $\forall\, t$ and we choose $t = 0 : N-1$. Our objective is to compare the computational time taken to compute the density function of Z where $Z = X + Y$ by means of convolution method.

	Convolution Methods		Ratio
N	Direct	FFT	Direct/FFT
2^9	0.1720	0.0150	11.4667
2^{10}	1.9220	0.0160	120.1250
2^{11}	6.3120	0.0310	203.6129
2^{12}	24.046	0.0780	308.2821
2^{13}	76.938	0.1570	470.0510
2^{14}	267.67	0.3440	778.1160
2^{15}	2308.078	0.8910	2 590.44

Table B.1. A comparison of the computational time in seconds for the Direct and FFT method for computing convolution varying N and repeating the computation 100 times.

In Table B.1, direct convolution means that we computed the convolution of X and Y by using the direct method $\text{conv}(X, Y)$. On the other hand, FFT convolution means that we computed the Fourier transforms of X and Y, multiplied the resulting transforms and then we computed the inverse Fourier transform of the resulting multiplication.

For $N < 2^9$, we noted that the two methods take relatively the same time to compute the convolutions, hence the direct method can be opted instead. From our results above it is clear that the FFT method performs much faster than the direct convolution method especially for large N. In fact, the FFT algorithm reduces the number of computations needed for N points from $2N^2$ to $2N\log(N)$.

Appendix C

Solution of the O.D.E in Equation (5.11)

By construction, the boundary conditions of $U(0,\tau,\lambda) = 1$ hence $W(0,v,\lambda) = 1/v$ and $U_S(0,\tau,\lambda) = -\lambda/r(e^{r\tau-1})$ hence $W_S(0,v,\lambda) = \lambda/(vr - v^2)$. We shall consider a solution of the form

$$W(S,v,\lambda) = L\left(\frac{2\lambda}{\sigma^2}S, v\right).$$

Expressing equation (5.11) in terms of L we have

$$z^2L^{zz} + \alpha z L_z - (\beta + z)L = -\gamma \tag{C.1}$$

where $\alpha = 2r/\sigma^2, \beta = 2v/\sigma^2$ and $\gamma = 2/\sigma^2$ and from the boundary conditions of W we obtain those of L as $L(0,\lambda) = 1/v$ and $L_z(0,v,\lambda) = \sigma^2/(2v(r-v))$.

To solve for L we shall employ the method of analytic coefficients where we consider the solution of the form

$$L(z,v) = \sum_{m=0}^{\infty} a_m z^m. \tag{C.2}$$

Now, rewriting equation (C.1) in the form $z^2L^{zz} + \alpha z L_z - \beta L = -\gamma + zL$ and applying (C.2) and upon equating coefficients in powers of z we obtain

$$\begin{aligned} -\beta a_0 &= -\gamma \\ (\alpha - \beta)a_1 &= a_0 \\ (m(m-1) + \alpha m - \beta)a_m &= a_{m-1}. \end{aligned}$$

If we let α_1 and α_2 be the roots of equation $x^2 + (\alpha - 1)x - \beta = 0$, then it follows that

$$a_m = \frac{a_{m-1}}{(m - \alpha_1)(m - \alpha_2)}.$$

Substituting into equation (C.2) we have

$$L(z, v) = \frac{1}{v} + \frac{z\sigma^2}{2v(r - v)} + \sum_{m=2}^{\infty} \frac{\sigma^2}{2v(r - v)} \frac{\Gamma(1 - \alpha_1)\Gamma(1 - \alpha_2)\Gamma(m + 1)z^m}{\Gamma(m + 1 - \alpha_1)\Gamma(m + 1 - \alpha_2)m!}.$$

By adding and subtracting the terms of order $m = 0$ and $m = 1$ in the infinite sum yields a simplification of the result given as

$$L(z, v) = \frac{1}{v}\left(1 - \frac{\sigma^2}{2(r - v)}\right)\left(1 + \frac{z\sigma^2}{2(r - v)}\right) + \frac{\sigma^2}{2v(r - v)}\, {}_1F_2(1; 1 - \alpha_1, 1 - \alpha_2; z). \quad \text{(C.3)}$$

Upon substituting $2S\lambda/\sigma^2$ for z we obtain equation (5.12).

Appendix D

Numerical Application of Laplace Transform to Option Pricing

We shall demonstrate the application of Laplace transform method to option pricing and for that we consider the European call option on the standard Black−Scholes model. We compare our results with the exact solution obtained from the Black−Scholes formula. Fusai and Roncoroni [41] have provided the results from the inversion method using the Euler algorithm therefore, we shall focus on the Laguerre one dimensional algorithm which we shall extend to use its multi-dimensional method for inverting the double Laplace transform for the continuous Asian call option.

We consider the Black-Scholes PDE given as

$$\partial_t C + rS\partial_s C + \frac{1}{2}\sigma^2 S^2 \partial_{ss} C = rC \tag{D.1}$$

$$C(T,S) = (S_T - K)^+, \tag{D.2}$$

where S is the risky asset having the dynamics given by the geometric Brownian motion, K is the strike price and the option has $T-t$ time to mature. We let

$$C(t,S) = f(\tau, z) \qquad \text{where } \tau = \frac{\sigma^2}{2}(T-t) \quad \text{and } z = \ln(S).$$

We derive the PDE satisfied by f as follows

$$\partial_t C = \frac{-\sigma^2}{2}\partial_\tau f \qquad\qquad \partial_s C = \frac{1}{S}\partial_z f \qquad\qquad \partial_{ss} C = \frac{1}{S^2}\partial_{zz} f - \frac{1}{S^2}\partial_z f.$$

Substituting into equation (D.1) we have

$$\begin{aligned}
\frac{-\sigma^2}{2}\partial_\tau f + re^z\frac{1}{e^z}\partial_z f + \frac{1}{2}\sigma^2 e^{2z}\left(\frac{1}{e^{2z}}\partial_{zz}f - \frac{1}{e^{2z}}\partial_z f\right) - rf &= 0 \\
\frac{-\sigma^2}{2}\partial_\tau f + \left(r - \frac{1}{2}\sigma^2\right)\partial_z f + \frac{1}{2}\sigma^2\partial_{zz}f - rf &= 0 \\
-\partial_\tau f + \left(\frac{r}{\sigma^2/2} - 1\right)\partial_z f + \partial_{zz}f - \frac{r}{\sigma^2/2}f &= 0, \qquad \text{(D.3)}
\end{aligned}$$

and the initial condition is given by $f(0,z) = C(T, e^z)$ and we note that with our payoff function as in equation (D.2) $f(\tau, z) \to (e^z - e^k)$ as $z \to +\infty$ and $f(\tau, z) \to 0$ as $z \to -\infty$. Now to apply the Laplace transform method to the PDE in (D.3), we use the properties as illustrated in Table 5.1 to get

$$\begin{aligned}
\int_0^\infty e^{-s\tau} f(\tau, z)d\tau &= \mathcal{L}[f(\tau, z)] \\
&= \hat{f}(s, z) \qquad \text{(D.4)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\left[\frac{\partial f}{\partial \tau}\right] &= \int_0^\infty e^{-s\tau}\frac{\partial f}{\partial \tau}d\tau \\
&= \int_0^\infty e^{-s\tau}\partial f(\tau, z) \\
&= e^{-s\tau}f(\tau, z)\Big|_0^\infty + s\int_0^\infty e^{-s\tau}f(\tau, z)d\tau \\
&= -f(0, z) + s\hat{f}(s, z) \quad \text{and} \qquad \text{(D.5)} \\
\mathcal{L}\left[\frac{\partial f}{\partial z}\right] &= \int_0^\infty e^{s\tau}\frac{\partial df}{\partial \tau}d\tau \\
&= \frac{\partial}{\partial z}\int_0^\infty e^{-s\tau}f(\tau, z)d\tau \\
&= \partial_z\hat{f}(s, z) \quad \text{(in the same way,)} \qquad \text{(D.6)} \\
\mathcal{L}\left[\frac{\partial^2 f}{\partial z^2}\right] &= \partial_{zz}\hat{f}(s, z). \qquad \text{(D.7)}
\end{aligned}$$

By substituting equation (D.4), (D.5), (D.6) and (D.7) into equation (D.3) we obtain

$$\begin{aligned}
-(-f(0, z) + s\hat{f}(s, z)) + \left(\frac{r}{\sigma^2/2} - 1\right)\partial_z\hat{f}(s, z) + \partial_{zz}\hat{f}(s, z) - \frac{r}{\sigma^2/2}f &= 0 \\
\partial_{zz}\hat{f}(s, z) + (m-1)\partial_z\hat{f}(s, z) - (m+s)\hat{f}(s, z) + (e^z - e^k)^+ &= 0, \quad \text{(D.8)}
\end{aligned}$$

where $m = r/(\sigma^2/2)$. We obtain the boundary conditions for this ODE by using the Laplace transform on the boundary conditions of the PDE in equation (D.3), thus, when

$z \to +\infty$ we have

$$\begin{aligned}\hat{f}(s,z) &\to \mathcal{L}\left[e^z - e^{-m\tau}e^k\right] \\ &= \int_0^\infty e^{-s\tau}\left(e^z - e^{-m\tau}e^k\right)d\tau \\ &= \frac{e^z}{s} - \frac{e^k}{s+m} \qquad \text{(D.9)}\end{aligned}$$

and when $z \to -\infty$, we have $\hat{f}(s,z) \to \mathcal{L}(0) = 0$ and we note that the initial conditions for the PDE have been absorbed into the ODE which makes the problem manageable to solve. Since the ODE is in terms of the Laplace transform, we would have to invert the solution of this ODE to recover the solution of the original problem in equation (D.1).

As has been proposed by Madan and Carr [17], we shall consider the modified $\hat{f}$ which we shall define as

$$\hat{f}(s,z) = \exp(\alpha z)\hat{g}(s,z),$$

and we choose $\alpha = (1-m)/2$. The ODE satisfied by $\hat{g}$ is obtained as follows

$$\begin{aligned}\partial_z\hat{f} &= \alpha e^{\alpha z}\hat{g}(s,z) + e^{\alpha z}\partial_z\hat{g}(s,z) \quad \text{and} \\ \partial_{zz}\hat{f} &= \alpha\left(\alpha e^{\alpha z}\hat{g}(s,z) + e^{\alpha z}\partial_z\hat{g}(s,z)\right) + \alpha e^{\alpha z}\partial_z\hat{g}(s,z) + e^{\alpha z}\partial_{zz}\hat{g}(s,z) \\ &= \alpha^2 e^{\alpha z}\hat{g}(s,z) + 2\alpha e^{\alpha z}\partial_z\hat{g}(s,z) + e^{\alpha z}\partial_{zz}\hat{g}(s,z),\end{aligned}$$

substituting these in equation (D.8) we have

$$\begin{aligned}0 &= \alpha^2\hat{g} + 2\alpha\partial_z\hat{g} + \partial_{zz}\hat{g} + (m-1)(\alpha\hat{g} + \partial_z\hat{g}) - (m+s)\hat{g} + e^{-\alpha z}(e^z - e^k)^+ \\ &= \left(\alpha^2 + \alpha(m-1) - (m+s)\right)\hat{g} + (2\alpha + m - 1)\partial_z\hat{g} + \partial_{zz}\hat{g} + e^{-\alpha z}(e^z - e^k)^+ \\ &= \partial_{zz}\hat{g}(s,z) - (b+s)\hat{g}(s,z) + e^{-\alpha z}(e^z - e^k)^+ \qquad \text{(D.10)}\end{aligned}$$

where $b = \alpha^2 + m$. Before we begin to solve the ODE for $\hat{g}$ we shall look back at what we have done, and provide a detailed explanation on how the application of Laplace transform works. We begun with a PDE given in equation (D.3) together with the initial conditions and the boundary conditions. Upon application of the Laplace transform method we noted that the initial conditions where absorbed in the new transform which then resulted in the ODE given by equation (D.10). Fusai represented the general procedure for this process by means of a diagram as shown in Figure D.1

Original Function	$\left\{\begin{array}{c}\text{PDE}\\ +\text{IC}\\ +\text{BC's}\end{array}\right\}$		Solution to the Original function	
	$\downarrow$		$\nearrow\nwarrow$	
	$\mathcal{L}$-transform	Analytic		Numerical
	$\vert$		$\nwarrow\nearrow$	
	$\vert$		$\mathcal{L}^{-1}$-transform	
	$\downarrow$		$\uparrow$	
Image Space	$\left\{\begin{array}{c}\text{ODE}\\ +\text{BC's}\end{array}\right\}$	$\longrightarrow$	Solution to the $\mathcal{L}$ -transform	

Table D.1. A pictorial representation of the application of Laplace transform method in solving PDEs.

We shall now seek to recover the solution of the original problem by applying the Laplace inversion methods namely the Euler and Laguerre algorithm on the solution of $\hat{f}$ obtained from solving the ODE for $\hat{g}$.

D.1 Solution of the ODE in $(D.10)$

We shall employ the method of undetermined coefficients to solve our ODE and to find the particular solution of the non-homogeneous equation we shall use the superposition principle thus, if say g_1 is a particular solution of

$$\hat{g}'' - (b+s)\hat{g} = -e^{-(\alpha-1)z}$$

and g_2 is a particular solution of

$$\hat{g}'' - (b+s)\hat{g} = e^{-\alpha z+k},$$

then $g_p = g_1 + g_2$ is the solution of the non-homogeneous equation. First we shall consider the solution of (D.10) for $z > k$. Let $g_1 = Ae^{-(\alpha-1)z}$, then

$$g_1' = -(\alpha-1)Ae^{-(\alpha-1)z} \qquad\qquad g_1'' = (\alpha-1)^2Ae^{-(\alpha-1)z}.$$

By substituting into equation (D.10) we have

$$\begin{aligned}
(\alpha-1)^2 Ae^{-(\alpha-1)z} - (b+s)Ae^{-(\alpha-1)z} &= -e^{-(\alpha-1)z} \\
A &= \frac{1}{(\alpha^2-1)^2-(b+s)} \\
&= \frac{1}{s}.
\end{aligned}$$

Let $g_2 = Be^{-\alpha z}$, then

$$g_2' = -\alpha Be^{-\alpha z} \qquad\qquad g_2'' = \alpha^2 Be^{-\alpha z}.$$

And substituting into equation (D.10) we have

$$\begin{aligned}
\alpha^2 Be^{-\alpha z} - (b+s)Be^{-\alpha z} &= e^{-\alpha z}e^{k} \\
B &= \frac{e^k}{\alpha^2-b+s} \\
&= \frac{e^k}{s-m}.
\end{aligned}$$

The particular solution is therefore given by

$$g_p = \frac{e^{-(\alpha-1)z}}{s} - \frac{e^{-\alpha z+k}}{s+m}.$$

To get the general solution we would require the fundamental solution of the homogeneous equation which is

$$\hat{g}'' - (b+s)\hat{g} = 0.$$

hence, the characteristic equation is

$$r^2 - (b+s) = 0,$$

and it follows that the fundamental solutions are given as $e^{-z\sqrt{b+s}}$ and $e^{z\sqrt{b+s}}$, hence the general solution is given as

$$\hat{g}(s,z) = \frac{e^{-(\alpha-1)z}}{s} - \frac{e^{-\alpha z+k}}{s+m} + A_1 e^{-z\sqrt{b+s}} + A_2 e^{z\sqrt{b+s}}. \tag{D.11}$$

Now, for $z \leq k$ we have only the homogeneous equation since $(e^z - e^k)^+ = 0$, therefore the general solution in this case is given as

$$\hat{g}(s,z) = B_1 e^{-z\sqrt{b+s}} + B_2 e^{z\sqrt{b+s}}. \tag{D.12}$$

where A_1, A_2, B_1 and B_2 are constants. From the boundary conditions of $\hat{f}$ we note that since,

$$\lim_{z\to+\infty} e^{-z\left(\sqrt{b+s}-\alpha\right)} = 0 \qquad \lim_{z\to+\infty} e^{z\left(\alpha+\sqrt{b+s}\right)} = +\infty,$$

we should have $A_2 = 0$ and in the same way $B_1 = 0$. Moreover, to determine A_1 and B_2 we shall require that

$$\lim_{z\to k^+} \hat{f}(s,z) = \lim_{z\to k^-} \hat{f}(s,z)$$

which is equivalent to say

$$\lim_{z\to k^+} e^{\alpha z}\hat{g}(s,z) = \lim_{z\to k^-} e^{\alpha z}\hat{g}(s,z)$$

Now,

$$\begin{aligned}
\hat{g}(s,z) &= \frac{e^{-(\alpha-1)z}}{s} - \frac{e^{-\alpha z+k}}{s+m} + A_1 e^{-z\sqrt{b+s}} \quad \text{for } z>k \\
e^{\alpha z}\hat{g}(s,z) &= \frac{e^{z}}{s} - \frac{e^{k}}{s+m} + A_1 e^{-z\sqrt{b+s}+\alpha z} \\
\lim_{z\to k^+} \hat{f}(s,z) &= \frac{me^{k}}{s(s+m)} + A_1 e^{(\sqrt{b+s}-\alpha)k}
\end{aligned}$$

$$\begin{aligned}
e^{\alpha z}\hat{g}(s,z) &= e^{\alpha z} B_2 e^{z\sqrt{b+s}} \quad \text{for } z\le k \\
\lim_{z\to k^-} \hat{f}(s,z) &= B_2 e^{(\sqrt{b+s}+\alpha)k}
\end{aligned}$$

Therefore, we have

$$\frac{me^{k}}{s(s+m)} + A_1 e^{(\sqrt{b+s}-\alpha)k} = B_2 e^{(\sqrt{b+s}+\alpha)k}. \tag{D.13}$$

We would like also to satisfy the condition that

$$\lim_{z\to k^+} \partial_z \hat{f}(s,z) = \lim_{z\to k^-} \partial_z \hat{f}(s,z),$$

equivalently we write

$$\lim_{z\to k^+} \partial_z e^{\alpha z}\hat{g}(s,z) = \lim_{z\to k^-} \partial_z e^{\alpha z}\hat{g}(s,z).$$

Now, for $z>k$ we have

$$\begin{aligned}
\partial_z \hat{f}(s,z) &= \alpha e^{\alpha z}\hat{g}(s,z) + e^{\alpha z}\partial_z \hat{g}(s,z) \\
&= \alpha \hat{f}(s,z) + e^{\alpha z}\left(\frac{-(\alpha-1)e^{-(\alpha-1)z}}{s} + \frac{\alpha e^{-\alpha z+k}}{s+m} - \sqrt{b+s}A_1 e^{-z\sqrt{b+s}}\right) \\
&= \alpha \hat{f}(s,z) - \frac{-(\alpha-1)e^{z}}{s} + \frac{\alpha e^{k}}{s+m} - \sqrt{b+s}A_1 e^{-z\sqrt{b+s}+\alpha z} \\
\lim_{z\to k^+} \partial_z \hat{f}(s,z) &= \frac{e^{k}}{s} + A_1\left(\alpha - \sqrt{b+s}\right)e^{-\left(\sqrt{b+s}-\alpha\right)k},
\end{aligned}$$

and for $z \leq k$ we have

$$\begin{aligned}
\partial_z \hat{f}(s,z) &= \left(\alpha + \sqrt{b+s}\right) e^{(\sqrt{b+s}+\alpha)z} B_2 \\
\lim_{z \to k^-} \partial_z \hat{f}(s,z) &= \left(\alpha + \sqrt{b+s}\right) B_2 e^{(\sqrt{b+s}+\alpha)k}.
\end{aligned}$$

Therefore, we have

$$\frac{e^k}{s} + A_1 \left(\alpha - \sqrt{b+s}\right) e^{-\left(\sqrt{b+s}-\alpha\right)k} = \left(\alpha + \sqrt{b+s}\right) B_2 e^{(\sqrt{b+s}+\alpha)k}. \tag{D.14}$$

Now we have two simultaneous equation which we shall now solve for A_1 and B_2. Multiplying equation (D.13) by $(\alpha + \sqrt{b+s})$ and equation (D.14) by 1 and subtracting the two we have

$$\begin{aligned}
0 &= \frac{\left(\alpha + \sqrt{b+s}\right) m e^k}{s(s+m)} + A_1 \left(\left(\alpha + \sqrt{b+s}\right) e^{-\left(\sqrt{b+s}-\alpha\right)k} - \left(\alpha - \sqrt{b+s}\right) e^{-\left(\sqrt{b+s}-\alpha\right)k}\right) - \frac{e^k}{s} \\
A_1 &= \frac{\left(s - \left(\alpha - 1 + \sqrt{b+s}\right) m\right) e^{\left(1-\alpha+\sqrt{b+s}\right)k}}{2s(s+m)\sqrt{b+s}}.
\end{aligned}$$

By substituting A_1 in equation (D.13) and simplifying we obtain the expression for B_2 as

$$B_2 = \frac{\left(s - \left(\alpha - 1 - \sqrt{b+s}\right) m\right) e^{\left(1-\alpha-\sqrt{b+s}\right)k}}{2s(s+m)\sqrt{b+s}}.$$

We can now, write the expression for our option price as follows

$$\hat{f}(s,z) = e^{\alpha z} \left[\left(\frac{e^{-z(\alpha-1)}}{s} - \frac{e^{-\alpha z + k}}{s+m} \right) \mathbb{I}_{z>k} + e^{-|z-k|\sqrt{b+s}} \left(s - \left(\alpha - 1 + \sqrt{b+s}\ \mathrm{sgn}(z-k) \right) m \right) \right] \tag{D.15}$$

where $\mathbb{I}_{x\geq 0}$ is the indicator function and $\mathrm{sgn}(x) = \mathbb{I}_{x\geq 0} - \mathbb{I}_{x<0}$. This is equation (7.14) in Fusai and Roncoroni. Our main focus now is to perform the Laplace inversion of this expression which as we have highlighted before that we intend to use the Euler and Laguerre method.

D.2 Numerical results

We present the numerical results obtained by numerically inverting the Laplace transform in equation (D.15). To appreciate the accuracy of our inversion methods, we compare our results with the Black−Scholes closed form solution given in equation (1.4).

In our numerical computations, we conduct experiments to a wide scope of varied parameter settings as reported by the tables below.

Strike (K)	I. rate (r)	BS	Euler		Weeks	
			n=15, m=10	n=50, m=10	$\sigma' = 1$, $b' = 1$	$\sigma' = 0.394$, $b' = 29.745$
95	0.05	5.898922	5.898922	5.898922	5.897821	5.898921
100		2.512067	2.512067	2.512067	2.512376	2.511913
105		0.744020	0.744020	0.744020	0.742720	0.744019
95	0.09	6.160725	6.160725	6.160725	6.159699	6.160725
100		2.687294	2.687294	2.687294	2.687601	2.687144
105		0.820509	0.820509	0.820509	0.819118	0.820508
95	0.15	6.560016	6.560016	6.560016	6.559095	6.560015
100		2.963197	2.963197	2.963197	2.963500	2.963046
105		0.946187	0.946187	0.946187	0.944650	0.946186

Table D.2. Numerical results for the European call option with volatility, $\sigma = 0.05$, initial stock price $S_0 = 100$, expiry 1 month with varied interest rate r and strike price, K.

Strike (K)	Vol. (σ)	BS	Euler		Weeks	
			n=15, m=10	n=50, m=10	$\sigma' = 1$, $b' = 1$	$\sigma' = 0.394$, $b' = 29.605$
95	0.1	13.50830	13.50830	13.50830	13.50751	13.50830
100		9.566265	9.566265	9.566265	9.569359	9.566285
105		6.252705	6.252705	6.252705	6.250456	6.252705
95	0.2	15.80272	15.80272	15.80272	15.80247	15.80272
100		12.68209	12.68209	12.68209	12.68279	12.68209
105		9.987905	9.987905	9.987905	9.987536	9.987905
95	0.3	18.92766	18.92767	18.92767	18.92760	18.92766
100		16.21927	16.21927	16.21927	16.21942	16.21926
105		13.81067	13.81067	13.81067	13.81059	13.81067

Table D.3. Values for the European call option computed on interest rate, $r = 0.09$, initial stock price, $S_0 = 100$, expiry 1 year with varied volatility, σ and strike price, K.

Strike (K)	Expiry (T)	BS	Euler		Weeks	
			n=15, m=10	n=50, m=10	$\sigma' = 1$, $b' = 1$	$\sigma' = 0.394$, $b' = 29.605$
95	3	44.46298	44.46298	44.46298	44.46297	44.46298
100		42.69422	42.69422	42.69422	42.69424	42.69423
105		41.01276	41.01276	41.01276	41.01276	41.01275
95	5	56.69365	56.69365	56.69365	56.69358	56.69365
100		55.26921	55.26921	55.26921	55.26940	55.26921
105		53.90098	53.90098	53.90098	53.90091	53.90098
95	10	75.38894	75.38894	75.38894	75.38898	75.38894
100		74.53783	74.53783	74.53783	74.53770	74.53783
105		73.71175	73.71175	73.71175	73.71180	73.71175

Table D.4. Numerical results for long term European call option written on the initial stock price, $S_0 = 100$ interest rate, $r = 0.09$, volatility, $\sigma = 0.5$ and varied strike price, K.

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