Fredholm Theory in General Banach Algebras

by

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Declaration

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Abstract

This thesis is a study of a generalisation, due to R. Harte (see [9]), of Fredholm theory in the context of bounded linear operators on Banach spaces to a theory in a Banach algebra setting. A bounded linear operator T on a Banach space X is Fredholm if it has closed range and the dimension of its kernel as well as the dimension of the quotient space $X/T(X)$ are finite. The index of a Fredholm operator is the integer $\dim T^{-1}(0) - \dim X/T(X)$. Weyl operators are those Fredholm operators of which the index is zero. Browder operators are Fredholm operators with finite ascent and descent. Harte's generalisation is motivated by Atkinson's theorem, according to which a bounded linear operator on a Banach space is Fredholm if and only if its coset is invertible in the Banach algebra $\mathcal{L}(X)/\mathcal{K}(X)$, where $\mathcal{L}(X)$ is the Banach algebra of bounded linear operators on X and $\mathcal{K}(X)$ the two-sided ideal of compact linear operators in $\mathcal{L}(X)$. By Harte's definition, an element a of a Banach algebra A is Fredholm relative to a Banach algebra homomorphism $T : A \to B$ if Ta is invertible in B. Furthermore, an element of the form $a + b$ where a is invertible in A and b is in the kernel of T is called Weyl relative to T and if $ab = ba$ as well, the element is called Browder. Harte consequently introduced spectra corresponding to the sets of Fredholm, Weyl and Browder elements, respectively. He obtained several interesting inclusion results of these sets and their spectra as well as some spectral mapping and inclusion results. We also convey a related result due to Harte which was obtained by using the exponential spectrum. We show what H. du T. Mouton and H. Raubenheimer found when they considered two homomorphisms. They also introduced Ruston and almost Ruston elements which led to an interesting result related to work by B. Aupetit. Finally, we introduce the notions of upper and lower semi-regularities – concepts due to V. Müller. Müller obtained spectral inclusion results for spectra corresponding to upper and lower semi-regularities. We could use them to recover certain spectral mapping and inclusion results obtained earlier in the thesis, and some could even be improved.

Opsomming

Hierdie tesis is ‘n studie van ’n veralgemening deur R. Harte (sien [9]) van Fredholm-teorie in die konteks van begrensde lineêre operatore op Banach-ruimtes tot ’n teorie in die konteks van Banach-algebras. ’n Begrensde lineêre operator T op ’n Banach-ruimte X is Fredholm as sy waardeversameling geslote is en die dimensie van sy kern, sowel as dié van die kwosiëntruimte $X/T(X)$, eindig is. Die indeks van ’n Fredholm-operator is die heelgetal $\dim T^{-1}(0) - \dim X/T(X)$. Weyl-operatore is daardie Fredholm-operatore waarvan die indeks gelyk is aan nul. Fredholm-operatore met eindige styging en daling word Browder-operatore genoem. Harte se veralgemening is gemotiveer deur Atkinson se stelling, waarvolgens ’n begrensde lineêre operator op ’n Banach-ruimte Fredholm is as en slegs as sy neweklas inverteerbaar is in die Banach-algebra $\mathcal{L}(X)/\mathcal{K}(X)$, waar $\mathcal{L}(X)$ die Banach-algebra van begrensde lineêre operatore op X is en $\mathcal{K}(X)$ die twee-sydige ideaal van kompakte lineêre operatore in $\mathcal{L}(X)$ is. Volgens Harte se definisie is ’n element a van ’n Banach-algebra A Fredholm relatief tot ’n Banach-algebrahomomorfisme $T : A \to B$ as Ta inverteerbaar is in B. Verder word ’n Weyl-element relatief tot ’n Banach-algebrahomomorfisme $T : A \to B$ gedefinieer as ’n element met die vorm $a + b$, waar a inverteerbaar in A is en b in die kern van T is. As $ab = ba$ met a en b soos in die definisie van ’n Weyl-element, dan word die element Browder relatief tot T genoem. Harte het vervolgens spektra gedefinieer in ooreenstemming met die versamelings van Fredholm-, Weyl- en Browder-elemente, onderskeidelik. Hy het heelparty interessante resultate met betrekking tot insluitings van die verskillende versamelings en hulle spektra verkry, asook ’n paar spektrale-afbeeldingsresultate en spektrale-insluitingsresultate. Ons dra ook ’n verwante resultaat te danke aan Harte oor, wat verkry is deur van die eksponensiële-spektrum gebruik te maak. Ons wys wat H. du T. Mouton en H. Raubenheimer verkry het deur twee homomorfismes gelyktydig te beskou. Hulle het ook Ruston- en byna Ruston-elemente gedefinieer, wat tot ’n interessante resultaat, verwant aan werk van B. Aupetit, gelei het. Ten slotte stel ons nog twee begrippe bekend, naamlik ’n onder-semi-regulariteit en ’n bo-semi-regulariteit – konsepte te danke aan V. Müller. Müller het spektrale-insluitingsresultate verkry vir spektra wat ooreenstem met bo- en onder-semi-regulariteite. Ons kon dit gebruik om sekere spektrale-afbeeldingsresultate en spektrale-insluitingsresultate wat vroeër in hierdie tesis verkry is, te herwin, en sommige kon selfs verbeter word.

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Contents

Notation

A^{-1}	The set of invertible elements of the Banach algebra A . . . 3
a^{-1}	The inverse of the element a . . . 2
$\mathcal{W}, \mathcal{B}, \mathcal{F}$	The sets Weyl, Browder and Fredholm elements, respectively . . . 8
$\mathcal{L}(X)$	The algebra of bounded linear operators on the Banach space X . . . 8
$\mathcal{K}(X)$	The set of compact linear operators on the Banach space X . . . 8
$\mathcal{F}(X)$	The set of linear operators on the Banach space X with finite dimensional range . . . v
$\operatorname{Exp} A$	The set of generalised exponentials of a Banach algebra A . . . 42
$\sigma(a)$	The spectrum of the element a in the relevant Banach algebra . . . 3
$\rho(a)$	The spectral radius of the element a in the relevant Banach algebra. 3
$\sigma_B(Ta)$	The Fredholm spectrum of the element a relative to some fixed Banach algebra homomorphism $T : A \to B$ . . . 17
$\omega_T(a)$	The Weyl spectrum of the element a relative to some fixed Banach algebra homomorphism $T : A \to B$ . . . 17
$\omega_T^{\mathrm{comm}}(a)$	The Browder spectrum of the element a relative to some fixed Banach algebra homomorphism $T : A \to B$ . . . 17
I	The identity operator on any space . . . 4
$\operatorname{acc}(X)$	The set of accumulation points of the set X . . . 5
$p(a, \alpha)$	The spectral idempotent of a associated with α . . . 5
$\mathcal{B}(\zeta, \delta)$	The open ball with center ζ and radius δ . . . 15
∂X	The boundary of the set X . . . 39
ηX	The convex hull of the set $X \subseteq \mathbb{C}$ . . . 6
$\tau(a)$	The singular spectrum of the element a in the relevant Banach algebra . . . 39
$\varepsilon(a)$	The exponential spectrum of the element a in the relevant Banach algebra . . . 42
$\sigma^R(a)$	The spectrum corresponding to the upper or lower semi-regularity R of a Banach algebra . . . 66
$\operatorname{ins} \Gamma$	The inside of the contour Γ . . . 35

Introduction

This thesis is about Fredholm theory in a Banach algebra relative to a fixed Banach algebra homomorphism – a generalisation, due to R. Harte, of Fredholm theory in the context of bounded linear operators on a Banach space. Only complex Banach algebras are considered in this thesis.

As described in [5] (p. 2), Fredholm theory of bounded linear operators on a Banach space originated from Fredholm integral equations. These integral equations led to the study of operators of the form $\lambda I - T$, where $T : X \to X$ is a compact operator on a Banach space X, the symbol I denotes the identity operator on X and λ is a non-zero complex number. An operator of this form has the properties that $(\lambda I - T)^{-1}(0)$ is finite dimensional, $(\lambda I - T)(X)$ is closed and the quotient space $X/\big((\lambda I - T)(X)\big)$ is finite dimensional. Denote the set of bounded linear operators on X by $\mathcal{L}(X)$ and the set of compact linear operators on X by $\mathcal{K}(X)$. Note that, if $T \in \mathcal{K}(X)$, then $\lambda I - T + \mathcal{K}(X) = \lambda I + \mathcal{K}(X)$, so that it is clear that, for $\lambda \neq 0$, the element $\lambda I - T + \mathcal{K}(X)$ is invertible in the Banach algebra $\mathcal{L}(X)/\mathcal{K}(X)$. A Fredholm operator has later been defined as a linear operator $T : X \to X$ such that $T^{-1}(0)$ is finite dimensional, $T(X)$ is closed and the quotient space $X/T(X)$ is finite dimensional (see [3], p. 3, for instance). We mention that the index of a Fredholm operator is defined as the integer $\dim T^{-1}(0) - \dim X/T(X)$.

There has been a few generalisations of Fredholm Theory. To name a some examples, it has been generalised to a theory of elements of a Banach algebra (e.g. in [9]), a semi-simple algebra (e.g. in [3], p. 31), a ring with identity (e.g. in [5], p. 107) and even to a theory of morphisms of an additive category (e.g. in [13]). All of the generalisations that I have come across were motivated directly or indirectly by Atkinson's Theorem (sometimes called Atkinson's characterisation). Denote the set of bounded linear operators on a Banach space X with finite range by $\mathcal{F}(X)$. Then Atkinson's theorem states that, if X is a Banach space, then $T \in \mathcal{L}(X)$ is Fredholm if and only if $T + \mathcal{K}(X)$

is invertible in the quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$, which is the case if and only if $T+\mathcal{F}(X)$ is invertible in $\mathcal{L}(X)/\mathcal{F}(X)$ (see, for instance, [3], p. 4).

An example of a generalisation to a semi-simple algebra setting using a specific two-sided ideal of a semi-simple algebra, namely the socle, can be found in [3] (p. 31). The socle of a semi-prime ring is the sum of the minimal left ideals (which coincides with the sum of the minimal right ideals) (see [5], p. 105). A semi-simple algebra without scalar multiplication is a semi-prime ring. Denote the socle of a semi-simple algebra A by $\operatorname{soc}(A)$. In this case, an element a of a semi-simple algebra A is defined to be Fredholm if the coset $x+\operatorname{soc}(A)$ is invertible in the quotient algebra $A/\operatorname{soc}(A)$. This is motivated by Atkinson's theorem and the fact that, if X is a Banach space, then $\mathcal{F}(X)$ is the socle of $\mathcal{L}(X)$ (see, for instance, [5], p. 106). In exactly the same way, a generalisation can be obtained in the more general setting of rings with identity elements with respect to any two-sided ideal, as shown in [5] (p. 107). A few of these cases are discussed in ([5], Chapter 6).

Harte went a step further in his article [9] and it is his generalisation that forms the central theme of this thesis. His generalisation was motivated by the fact that, if X is a Banach space, then $\mathcal{L}(X)$ and $\mathcal{L}(X)/\mathcal{K}(X)$ are both Banach algebras and the canonical mapping from $\mathcal{L}(X)$ onto $\mathcal{L}(X)/\mathcal{K}(X)$ is a bounded Banach algebra homomorphism. Furthermore, by Atkinson's theorem, an operator $T\in\mathcal{L}(X)$ is Fredholm if and only if its image under the canonical Banach algebra homomorphism, which maps each element of $\mathcal{L}(X)$ to its coset, is invertible in $\mathcal{L}(X)/\mathcal{K}(X)$. Harte then defined an element a of a Banach algebra A to be Fredholm relative to a fixed bounded Banach algebra homomorphism $T:A\to B$ if Ta is invertible in B. The Fredholm spectrum of an element $a\in A$ with respect to T is defined as the set $\{\lambda\in\mathbb{C}: a-\lambda\mathbf{1}$ is not Fredholm with respect to $T\}$.

Harte's definition of Fredholm elements relative to a Banach algebra homomorphism would still be valid if A and B were only rings with identity, by formulating the definitions relative to a ring homomorphism. However, in this setting one would not be able to define a spectrum, because a ring does not have scalar multiplication. If A and B were only algebras, one would be able to define a spectrum, but holomorphic functional calculus and hence the spectral mapping theorem is only applicable in a Banach algebra setting. One could therefore only study spectral mapping and/or inclusion theorems in a Banach algebra setting. The greatest part of this thesis is related to these types of results and hence, almost the entire thesis is in a Banach algebra setting.

Related to Fredholm theory is Weyl and Riesz-Schauder Theory. In [4] (p. 530) the Weyl spectrum of an operator T on a Hilbert space is defined as the set of complex numbers λ such that $T - \lambda I$ is not a Fredholm operator with index 0. Let H be a Hilbert space. By ([4], Corollary 2.8, p. 531) a bounded linear operator $T : H \to H$ is Fredholm of index 0 if and only if it can be written as a sum $S + K$, where S is invertible in $\mathcal{L}(H)$ and $K \in \mathcal{K}(H)$. Analogous to such operators, Harte introduced Weyl elements of a Banach algebra relative to a fixed bounded Banach algebra homomorphism. Let A be a Banach algebra and $T : A \to B$ a bounded Banach algebra homomorphism. Harte defined Weyl elements as elements of the form $s + k$ where s is invertible in A and k is in the kernel of T. He also introduced the Weyl spectrum of an element $a \in A$ as the set $\omega_T(a) = \{\lambda \in \mathbb{C} : a - \lambda\mathbf{1}$ is not Weyl with respect to $T\}$.

Let X be a Banach space and $T \in \mathcal{L}(X)$. If there exists an integer n such that $T^n(X) = T^{n+1}(X)$, then it is said that T has finite descent. Also, if there exists an integer n such that the kernel of T^n coincides with the kernel of T^{n+1}, then we say that T has finite ascent (see [5], p. 10). An operator $T \in \mathcal{L}(X)$ is defined to be Riesz-Schauder if T is Fredholm and has finite ascent and descent (see [5], Definition 1.4.4, p. 12). Such operators are sometimes also called Browder (see [15], for instance). Furthermore, by ([5], Theorem 1.4.5, p. 12), an operator $T \in \mathcal{L}(X)$ is Riesz-Schauder if and only if $T = U + V$, where $U : X \to X$ is an isomorphism, $V \in \mathcal{K}(X)$ (i.e. V is in the kernel of the canonical mapping from $\mathcal{L}(X)$ onto $\mathcal{L}(X)/\mathcal{K}(X)$) and $UV = VU$. Analogous to Riesz-Schauder operators, Harte also introduced Browder elements relative to a bounded Banach algebra homomorphism in [9]. He called an element of a Banach algebra A Browder relative to a bounded Banach algebra homomorphism T, if it had the form $u + v$, where u is invertible in A, v is in the kernel of T and $uv = vu$. He consequently introduced the Browder spectrum of an element a of A as the set $\omega_T^{\text{comm}}(a) = \{\lambda \in \mathbb{C} : a - \lambda\mathbf{1}$ is not Browder with respect to $T\}$. Hence, the set of Browder elements of a Banach algebra is a subset of the set of Weyl elements of the same Banach algebra.

We mention that Harte later introduced the notions of Fredholm, Weyl and Browder morphisms of an additive category (see [13]), but in this setting, spectra cannot be defined.

We now briefly explain the composition of this thesis.

The purpose of Chapter 1 is to state the preliminary results and definitions that are used in the rest of the thesis.

Chapter 2 is an exposition of the greatest part of the article of Harte ([9]) in which he introduced Fredholm, Weyl and Browder elements as well as their respective spectra. The motivation has been explained above. However, we define Fredholm, Weyl and Browder elements relative to any fixed Banach algebra homomorphism and do not require the homomorphism to be bounded, as was also done, for instance, by H. du T. Mouton and H. Raubenheimer in [19] – a paper in which the concepts introduced by Harte in [9] were used and studied further. We then investigate in which results Harte used the boundedness in his proofs and are consequently able to state some results slightly stronger than Harte did in [9]. (In Chapter 3 we show that, due to a result of J.J. Grobler and H. Raubenheimer, more of Harte's results hold without requiring the homomorphism to be bounded.)

In the first section of Chapter 2, we convey some inclusion results (see Corollary 2.1.20, for instance) of Harte concerning the sets of Fredholm, Browder and Weyl elements of a Banach algebra A relative to a Banach algebra homomorphism T. The Banach algebra homomorphism T has the Riesz property if $\operatorname{acc} \sigma(a) \subseteq \{0\}$ for all $a \in T^{-1}(0)$. Almost invertible elements are introduced which leads to an interesting result of Harte (see Theorem 2.1.18) stating that the set of almost invertible Fredholm elements and the set of Browder elements coincide if T is bounded and Riesz.

The second section of Chapter 2 is about the Fredholm, Weyl and Browder spectra and some related results of Harte. These include that, like the ordinary spectrum of an element, the Fredholm, Weyl and Browder spectra are always non-empty and compact. The almost invertible Fredholm spectrum is also introduced and a chain of inclusions (Proposition 2.2.11) is obtained.

The third section of Chapter 2 is about some spectral mapping and spectral inclusion results of Harte related to the newly introduced spectra. If T is bounded, a spectral mapping theorem for the Fredholm spectrum (Proposition 2.3.1) is easily obtained. In most other cases, only spectral inclusion results can be obtained, but without requiring T to be bounded (Theorem 2.3.3, for instance). We saw that a similar spectral inclusion result could be obtained for the Fredholm spectrum when T is not required to be bounded (Proposition 2.3.13). After reading a remark of Harte in [11], we also noticed that, for each $a \in A$, a spectral mapping theorem could be obtained for Browder and Weyl spectra, if the function f is invertible and holomorphic on a neighbourhood of $\sigma(a)$ such that f^{-1} is holomorphic on a neighbourhood of $\sigma(f(a))$ (Corollary 2.3.9). Examples of such functions are mobius functions (see [11]). This holds for any spectrum for which one has a similar spectral inclusion result.

In Chapter 3 we show that, using a result (Proposition 3.3) of Grobler and Raubenheimer, many results of Harte stated in Chapter 2 in which we still require T to be bounded can be stated more generally by dropping the boundedness requirement of T. We then obtain, for instance, that if T is Riesz (not necessarily bounded) and $a \in A$, then a spectral mapping theorem for the Browder spectrum holds for all functions which are holomorphic on a neighbourhood of $\sigma(a)$ and non-constant on every component of this neighbourhood (Corollary 3.12).

In Chapter 4 we consider the results of a paper of Harte on the exponential spectrum in a Banach algebra ([10]). The main result (Theorem 4.3.2) is that, if $T : A \to B$ is a bounded Banach algebra homomorphism which has closed range, then the inclusion $\sigma_B(Ta) \subseteq \omega_T(a) \subseteq \eta\sigma_B(Ta)$ holds for all $a \in A$. The exponential spectrum is used to obtain this. This result is interesting in relation to inclusion results obtained in the first section of Chapter 2.

Chapter 5 is about results obtained when two Banach algebra homomorphisms are considered. Most of this work is due to Mouton and Raubenheimer ([18]), although one of the most interesting results is also found in a paper by Harte ([12]) in a slightly more general form. This result is that, under certain circumstances, the ordinary spectrum of an element is the union of the two respective Fredholm spectra (see Theorem 5.6 and Corollary 5.15).

In Chapter 6 we consider some results from the paper [19] by Mouton and Raubenheimer. We introduce Ruston and almost Ruston elements as well as Ruston and almost Ruston spectra relative to a bounded Banach algebra homomorphism as defined by Mouton and Raubenheimer. Let $T : A \to B$ be a bounded Banach algebra homomorphism. An element is called Riesz relative to the closed two-sided ideal $T^{-1}(0)$ if its coset has a spectrum consisting of zero in the Banach algebra $A/T^{-1}(0)$. The sum of two commuting elements, one of which is invertible and the other one Riesz, is called a Ruston element. An almost Ruston element is a sum of an invertible element b and a Riesz element c such that $TbTc = TcTb$ in B. This leads to more interesting inclusion results and equaltities (see, for instance, Corollary 6.1.11, Corollary 6.1.14, Corollary 6.1.15, Corollory 6.2.5 and Proposition 6.2.7) when related to the sets of Fredholm, Weyl and Browder elements as well as the respective spectra which we have introduced already. It also leads to results related to perturbation by Riesz elements. The most interesting result of Mouton and Raubenheimer in this paper is a perturbation result (Theorem 6.3.8) which improves Theorem 2 in [8] by Groenewald, Harte and Raubenheimer and gives a sharper form of Theorem 6.3.4 by Aupetit.

In Chapter 7 we give definitions of upper and lower semi-regularities as introduced by V. Müller in [20]. Actually, the notion of a regularity had been introduced earlier by Müller and V. Kordula in [14]. In [20] we see that a subset of a Banach algebra is a regularity if and only if it is both an upper and a lower semi-regularity. Furthermore, Müller shows that interesting results can be obtained for both upper and lower semi-regularities and that the results obtained for regularities follow by combining the results of upper and lower semi-regularities. Corresponding spectra are defined in the usual way and spectral inclusion results (Theorem 7.1.6, Theorem 7.1.7 and Theorem 7.2.16) are obtained by Müller.

Under certain circumstances, we then have the opposite spectral inclusion property for a spectrum corresponding to a lower semi-regularity than the spectral inclusion property obtained earlier for both the Weyl and the Browder spectrum. We show that some sets that we have encountered earlier in the thesis are lower semi-regularities, the most interesting ones perhaps being the set of almost invertible Fredholm elements and the set of Fredholm elements of a Banach algebra. (Müller showed in [20] that the set of Fredholm operators in $\mathcal{L}(X)$ is a regularity.) This leads to a spectral mapping theorem for the almost invertible Fredholm spectrum (equality (7.1.21)) and one for the Fredholm spectrum (equality (7.1.20)) that we have not obtained earlier in the thesis. We recover some spectral mapping theorems that have been obtained earlier in this thesis.

We show that some sets that have been introduced earlier in the thesis are upper semi-regularities, including the set of generalised exponentials $\operatorname{Exp} A$ of a Banach algebra A (which has also been shown by Müller in [20]). We obtain a spectral inclusion result for the corresponding spectrum (Proposition 7.2.18 (1)) which has not been obtained earlier in the thesis, but there has been no attempt earlier in the thesis to obtain spectral inclusion/mapping results for the exponential spectrum. Furthermore, we recover a few spectral inclusion results that have been obtained earlier in this thesis.

Finally, there is a short section on regularities where the results of upper and lower semi-regularities are combined and we recover the spectral mapping theorems obtained in the section on lower semi-regularities, without using results of other chapters of this thesis. In this way, we can prove, for instance, the spectral mapping theorem for the Fredholm spectrum (equality (7.1.20)) obtained at the end of the section on lower semi-regularities.

Chapter 1

Preliminaries

In this chapter some definitions of concepts used throughout this thesis are given. Some theorems which are necessary are stated without proof.

Definition 1.1 (Ring)**.** ([1], Definition 3.2.2, p. 85) Let R be a non-empty set equipped with two binary operators $\cdot$ and $+$. Suppose that the binary operators satisfy the following properties:

$$\begin{gathered} a+b=b+a; \\ (a+b)+c=a+(b+c); \\ (a\cdot b)\cdot c=a\cdot(b\cdot c); \\ a\cdot(b+c)=a\cdot b+a\cdot c; \\ (a+b)\cdot c=a\cdot c+b\cdot c \end{gathered}$$

for all $a, b, c \in R$. Furthermore, suppose that there exists an element $z \in R$ such that

$$a+z=z+a=a$$

for all elements $a \in R$ and that, for each element $a \in R$, there exists an element $a^* \in R$ such that

$$a+a^*=a^*+a=0.$$

Then R is called a *ring*.

Definition 1.2 (Ideal)**.** ([1], Definition 3.4.6, p. 98) Let R be a ring and I a non-empty subset of R, satisfying the following two properties:

- for any two elements $x, y \in I$, the element $x + y \in I$;
- if $x \in I$ then $-x \in I$.

If it holds for all $a \in I$ and $b \in R$ that $ba \in I$, then I is called a *left ideal* of R. If it holds for all $a \in I$ and $b \in R$ that $ab \in I$, then I is called a *right ideal* of R. If I is both a left and a right ideal of R, then I is called a two-sided ideal of R.

Definition 1.3 (Algebra)**.** ([16], p. 394). An algebra A is a ring R which is equipped with scalar multiplication over a field K such that the set of elements of the ring together with the operator $+$ and the scalar multiplication forms a vector space.

If the field K in the above definition is $\mathbb{C}$ or $\mathbb{R}$, then we speak of a *complex* or *real* algebra, respectively.

Definition 1.4 (Banach algebra)**.** ([16], p. 395). A *Banach algebra* A is a complete normed space which is also an algebra and for which

$$\|xy\| \leq \|x\|\|y\|,$$

for all $x, y \in A$. Also, if A has an identity $\mathbf{1}$, then

$$\|\mathbf{1}\| = 1.$$

If the algebra in the above definition is real or complex, we speak of a *real* or *complex Banach algebra*, respectively. In this thesis, all Banach algebras will be assumed to have an identity element $\mathbf{1}$ and to be complex to ensure that the spectrum (as defined in Definition 1.6) of each element of the Banach algebra is non-empty (see Theorem 1.7). The identity element of the Banach algebra A could be written as $\mathbf{1}_A$, but in most cases it is clear from the context to which Banach algebra $\mathbf{1}$ belongs and it suffices to write $\mathbf{1}$.

Theorem 1.5 ([2], p. 35, Theorem 3.2.1)**.** *Let A be a Banach algebra and $a \in A$. If $\|a\| < 1$, then $\mathbf{1} - a$ is invertible and*

$$(\mathbf{1} - a)^{-1} = \sum_{k=0}^{\infty} a^k.$$

Definition 1.6 (The spectrum of an element of a Banach algebra)**.** ([16]). The *spectrum* $\sigma(a)$ of an element a of a Banach algebra A is the set

$$\{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin A^{-1}\},$$

where A^{-1} denotes the set of invertible elements of A. If needed we write $\sigma_A(a)$ for the spectrum of a.

Theorem 1.7. ([2], Theorem 3.2.8 (ii), p. 38). *Let a be an element of a Banach algebra A. Then $\sigma(a)$ is a non-empty compact set.*

Definition 1.8 (Spectral radius)**.** ([2], p. 36) The *spectral radius* of an element a of a Banach algebra A denoted by $\rho(a)$ is the real number

$$\rho(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}$$

Theorem 1.9. *Let A be a Banach algebra. If the elements $a, b \in A$ satisfy $ab = ba$, then*

$$\sigma(a+b) \subseteq \sigma(a) + \sigma(b)$$

and

$$\sigma(ab) \subseteq \sigma(a)\,\sigma(b)$$

and hence

$$\rho(a+b) \leq \rho(a) + \rho(b)$$

and

$$\rho(ab) \leq \rho(a)\rho(b).$$

Theorem 1.10 ([2], Theorem 4.1.2, p. 70)**.** *Let A be a commutative Banach algebra. Denote the set of non-zero mutiplicative linear functionals which are not identical to 0 on A by $\mathcal{M}(A)$. Then the following properties hold:*

1. *the map $f \mapsto f^{-1}(0)$ defines a bijection from $\mathcal{M}(A)$ onto the set of maximal ideals of A;*

2. *for every $a \in A$ we have $\sigma(a) = \{f(a) : f \in \mathcal{M}(A)\}$.*

Definition 1.11 (Inessential ideal)**.** ([2], p. 106). Let A be a Banach algebra and I an ideal of A. Then I is *inessential* if, for each $a \in I$, the spectrum of a has at most 0 as a limit point.

Definition 1.12 (topological zero-divisor)**.** ([2], p. 41). Let A be a Banach algebra and $a \in A$. If there exists a sequence (a_n) in A such that $\|a_n\| = 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} aa_n = \lim_{n\to\infty} a_n a = 0$, then a is called a *topological divisor of zero.*

Definition 1.13 (Smooth contour)**.** A *smooth contour* in $\mathbb{C}$ surrounding a compact set K is a set of disjoint directed paths, each of which is piece-wise smooth and such that the union of the paths surrounds K once.

Theorem 1.14 (Holomorphic Functional Calculus)**.** ([2], p. 45, Theorem 3.3.3) *Let A be a Banach algebra and $a \in A$. If Ω is an open set containing $\sigma(a)$ and Γ is a smooth contour in Ω surrounding $\sigma(a)$, then the mapping $f \mapsto f(a) := \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda\mathbf{1} - a)^{-1}\mathrm{d}\lambda$ from $H(\Omega)$ (the set of all holomorphic functions on Ω) into A has the following properties:*

1. $(f_1 + f_2)(a) = f_1(a) + f_2(a)$;
2. $(f_1 f_2)(a) = f_1(a)f_2(a) = f_2(a)f_1(a)$;
3. $1(a) = \mathbf{1}$ *and* $I(a) = a$, *where* $1(\zeta) := 1$ *for all* $\zeta \in \mathbb{C}$ *and* $I(\zeta) := \zeta$ *for all* $\zeta \in \mathbb{C}$;
4. *if* (g_n) *converges uniformly to* f *on compact subsets of* Ω, *then* $f(a) = \lim g_n(a)$;
5. $\sigma(f(a)) = f(\sigma(a))$

for all $f_1, f_2, g_n, f \in H(\Omega)$.

In the above theorem, property no. 5 is referred to as *the spectral mapping theorem.*

Theorem 1.15. ([2], Theorem 3.3.4, p. 45). *Let A be a Banach algebra. Suppose $a \in A$ has a disconnected spectrum. Take any two disjoint open sets U_0 and U_1 such that $\sigma(a) \subseteq U_0 \cup U_1$. Let $f \in H(U_0 \cup U_1)$ be defined by*

$$f(\zeta) := \begin{cases} 0 & \text{if } \zeta \in U_0, \\ 1 & \text{if } \zeta \in U_1. \end{cases}$$

Then $p := f(a)$ is a projection commuting with a. If $\sigma(a) \cap U_0 \neq \emptyset$ and $\sigma(a) \cap U_1 \neq \emptyset$, then p is a non-trivial projection and

$$\sigma(pa) = (\sigma(a) \cap U_1) \cup \{0\}$$

and

$$\sigma(a - pa) = (\sigma(a) \cap U_0) \cup \{0\}.$$

Definition 1.16 (Spectral idempotent)**.** ([2], p. 106). Let $a \in A$. If $\alpha \notin \operatorname{acc}\sigma(a)$, let U_0 and U_1 be any disjoint sets such that $\alpha \in U_1$ and $\sigma(a) \setminus \{\alpha\} \subseteq U_0$. If $f \in H(U_0 \cup U_1)$ is defined as in the above theorem, then the *spectral idempotent* of a associated with α, denoted by $p(a,\alpha)$, is defined as $f(a)$.

Theorem 1.17 (Spectral idempotent properties)**.** ([2], p. 106 and proof of Theorem 3.3.4, p. 45). *Let $a \in A$. If $\alpha \notin \operatorname{acc}\sigma(a)$, then $p(a,\alpha)$ is an idempotent of A which commutes with a as well as with all elements commuting with a. If and only if $\alpha \notin \sigma(a)$, we have $p(a,\alpha) = 0$ and if $\{\alpha\} = \sigma(a)$, then $p(a,\alpha) = \mathbf{1}$.*

Lemma 1.18. *Let A be a Banach algebra and $p \in A$ an idempotent. Then $\sigma(p) \subseteq \{0,1\}$. If $\sigma(p) = \{0\}$ then $p = 0$ and if $\sigma(p) = \{1\}$, then $p = \mathbf{1}$.*

Proof. Because p is an idempotent, the equality $p(p-1) = 0$ holds. It follows from the spectral mapping theorem that, if $\lambda \in \sigma(p)$, then

$$\lambda(\lambda - 1) \in \sigma(0) = \{0\}.$$

Therefore, $\lambda \in \{0,1\}$. If $p \notin \{0,\mathbf{1}\}$, then $\mathbf{1} - p \notin \{0,\mathbf{1}\}$ and $0 \in \sigma(p)$. Furthermore $(\mathbf{1} - p)^2 = \mathbf{1} - p$ and $0 \in \sigma(\mathbf{1} - p)$. But then

$$1 = 1 - 0 \in \sigma(\mathbf{1} - (\mathbf{1} - p)) = \sigma(p)$$

and similarly $1 \in \sigma(\mathbf{1} - p)$. Therefore, if $\sigma(p) = \{0\}$, then $p = 0$, if $\sigma(p) = \{1\}$ then $p = \mathbf{1}$ and $\sigma(p) = \{0,1\}$ whenever p is a nontrivial idempotent. $\square$

Theorem 1.19 ([2], Corollary 5.7.5, p. 110)**.** *Let I be a two-sided inessential ideal of a Banach algebra A. If $x \in A$ and $\sigma_{A/\overline{I}}\left(x + \overline{I}\right) = \{0\}$, then $\operatorname{acc}\sigma_A(x) \subseteq \{0\}$. Furthermore, for every non-zero spectral value α of x, the spectral idempotent of x associated with α is in I.*

Definition 1.20 (Banach algebra homomorphism)**.** Let A and B be Banach algebras. A *Banach algebra homomorphism* is a linear operator $T : A \to B$ such that $T\mathbf{1}_A = \mathbf{1}_B$ and $Tab = TaTb$ for all $a, b \in A$.

Definition 1.21 (Bounded below operator)**.** Let A and B be Banach algebras. A linear operator $T : A \to B$ is *bounded below* if there exists $c > 0$ such that $\|Ta\| \geq c\|a\|$ for all $a \in A$.

Definition 1.22 (Retraction)**.** ([21], Problem 7J, p. 49) A continuous function r with domain the space X and range A a subspace of X is a *retraction* of X onto A if the restriction of r to A is the identity map on A.

Theorem 1.23 ([21], Theorem 34.5, p. 236)**.** *Let* $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ *and* $S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$*. Then there is no retraction from* D *onto* S^1*.*

Definition 1.24 (Convex hull)**.** Let X be a subset of $\mathbb{C}$ and G the union of the bounded components of $\mathbb{C} \backslash X$. Then the *convex hull* of X denoted by ηX is defined as $X \cup G$.

Chapter 2

General Fredholm Theory

In Harte's article "Fredholm Theory relative to Banach algebra homomorphisms" ([9]) Atkinson's theorem is used to generalise the concept of Fredholm operators in the algebra of bounded linear operators on a Banach space to Fredholm elements in an arbitrary Banach algebra, relative to some fixed Banach algebra homomorphism. He also defined Weyl and Browder elements in a Banach algebra setting. In [9] only bounded homomorphisms are considered. However, Definition 2.1.3 in this chapter of Fredholm, Weyl and Browder elements is more general in the sense that it does not require the homomorphism to be bounded.

2.1 Inclusion properties of Fredholm, Weyl and Browder sets

We state Atkinson's theorem which is the motivation of Definition 2.1.3.

Theorem 2.1.1 (Atkinson)**.** ([3], p. 4). *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then the following are equivalent:*

1. *$T^{-1}(0)$ is finite dimensional, $T(X)$ is closed and the quotient space $X/T(X)$ is finite dimensional;*
2. *$T + \mathcal{K}(X)$ is invertible in the quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$;*
3. *$T + \mathcal{F}(X)$ is invertible in the quotient algebra $\mathcal{L}(X)/\mathcal{F}(X)$.*

Remark 2.1.2. Property (1) in the theorem above is the Fredholm property of an operator which has been generalised by Harte to a Banach algebra setting as in the next definition.

Definition 2.1.3 (Fredholm, Weyl and Browder elements of a Banach algebra)**.** ([9], p. 431). Let $T : A \to B$ be a homomorphism where A and B are Banach algebras. An element $a \in A$ is called:

- *Fredholm* if Ta is invertible in B;
- *Weyl* if there exist elements $b, c \in A$, where b is invertible in A and c is in the kernel of T, such that $a = b + c$;
- *Browder* if there exist commuting elements $b, c \in A$, where b is invertible in A and c is in the kernel of T, such that $a = b + c$.

Remark 2.1.4. Clearly, the properties of being Fredholm, Weyl or Browder is dependent upon a specific Banach algebra homomorphism. In this thesis, we will write Fredholm, Weyl or Browder *with respect to* a certain Banach algebra homomorphism in the cases where it is not clear which homomorphism is relevant. The symbols $\mathcal{F}$, $\mathcal{W}$ and $\mathcal{B}$ will denote the sets of Fredholm, Weyl and Browder elements respectively. If there could be confusion, the symbols $\mathcal{F}_T$, $\mathcal{W}_T$ and $\mathcal{B}_T$ will be used to show that T is the relevant Banach algebra homomorphism.

Example 2.1.5. Let X be a Banach space. Then the set of bounded linear operators $\mathcal{L}(X)$, with multiplication defined by composition, is a Banach algebra. The quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$, where $\mathcal{K}(X)$ is the set of compact linear operators on X is also a Banach algebra. Let $\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{K}(X)$ be the homomorphism which maps an operator $T \in \mathcal{L}(X)$ to its coset $T + \mathcal{K}(X)$ in $\mathcal{L}(X)/\mathcal{K}(X)$. Then $T \in \mathcal{L}(X)$ is Fredholm in the sense of Definition 2.1.3 if $\pi(T) = T + \mathcal{K}(X)$ is invertible in $\mathcal{L}(X)/\mathcal{K}(X)$. By Atkinson's theorem, T is invertible in $\mathcal{L}(X)/\mathcal{K}(X)$ if and only if $\dim T^{-1}(0)$ is finite, the range of T is closed and $\dim X/T(X)$ is finite.

Example 2.1.6. Let $A = C(X)$, the continuous complex-valued functions on X, and $B = C(Y)$, the continuous complex-valued functions on Y, where X and Y are compact Hausdorff spaces. With the supremum norm and pointwise multiplication, A and B are Banach algebras. Define $T : A \to B$ as the Banach algebra homomorphism induced by composition with any fixed continuous map $\theta : Y \to X$, that is $Tf := f \circ \theta$ for all $f \in C(X)$. Then T is a Banach algebra homomorphism. Take any $f \in A$. Then $f \in \mathcal{F}$ if and only if $Tf \in B^{-1}$, i.e. $0 \notin (f \circ \theta)(Y)$. This means that $\theta(Y) \cap f^{-1}(0) = \emptyset$.

Furthermore, if $f \in \mathcal{W}$ then $f = g + h$ for some $g \in A^{-1}$ and $h \in T^{-1}(0)$. Observe that $h \in T^{-1}(0)$ if and only if $h(\theta(Y)) = \{0\}$. So, we have that if $f \in \mathcal{W}$, then

$$\begin{aligned} f|_{\theta(Y)} &= g|_{\theta(Y)} + h|_{\theta(Y)} \\ &= g|_{\theta(Y)} + 0 \\ &= g|_{\theta(Y)} \end{aligned}$$

for some $g \in A^{-1}$ and $h \in T^{-1}(0)$. Then f restricted to $\theta(Y)$ has an invertible extension to X, namely g. Conversely, if the restriction of $f \in A$ to $\theta(Y)$ has an invertible extension to X, say g, then $f = g + (f - g)$. Then $g \in A^{-1}$ and $T(f - g) = (f - g) \circ \theta = 0$, so that $f \in \mathcal{W}$.

Therefore, $f \in A$ is Fredholm if and only if $\theta(Y) \cap f^{-1}(0) = \emptyset$ and f is Weyl if and only if its restriction to $\theta(Y)$ has an invertible extension to X. Because A is a commutative algebra, the Weyl and Browder elements are the same.

The following proposition is stated without proof in [9].

Proposition 2.1.7. ([9], p. 431, (1.4)) *Let $T : A \to B$ be a Banach algebra homomorphism and $a \in A$. Consider the properties defined in Definition 2.1.3 relative to T. Then the following inclusions hold:*

$$A^{-1} \subseteq \mathcal{B} \subseteq \mathcal{W} \subseteq \mathcal{F}.$$

Proof. It is obvious from Definition 2.1.3 that $\mathcal{B} \subseteq \mathcal{W}$, because for an element to be in $\mathcal{B}$ it has to satisfy one more property (namely that b and c in the definition must commute) than to be in $\mathcal{W}$.

We will now prove the first inclusion. Suppose that $a \in A^{-1}$. Because T is a linear operator, $T0 = 0$. We also know that $0a = a0 = 0$. So, because $a = a + 0$, it follows from Definition 2.1.3 that $a \in \mathcal{B}$.

It remains to prove that $\mathcal{W} \subseteq \mathcal{F}$. Suppose that $a \in \mathcal{W}$. Then there exist elements $b \in A^{-1}$ and c in the kernel of T such that $a = b + c$. Take note that $TbTb^{-1} = T(bb^{-1}) = T\mathbf{1} = \mathbf{1}$ which means that Tb is invertible in B. We then see that

$$\begin{aligned} Ta &= Tb + Tc \\ &= Tb + 0 \\ &= Tb, \end{aligned}$$

which is invertible, as we have shown above. It therefore follows from Definition 2.1.3 that $a \in \mathcal{F}$. $\square$

To illustrate that the above inclusions can be strict, the following example elaborates on Example 2.1.6.

Example 2.1.8. Using the notation of Example 2.1.6, let $X := \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $Y := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Set $\theta = \theta_1$ where $\theta_1 : Y \to X$ is defined by

$$\theta_1(\lambda) := \lambda,$$

for all $\lambda \in Y$. The Banach algebra homomorphism T is induced by composition with θ_1. Consider the function f on X which maps z to z for all $z \in X$. We see that $\{0\} = f^{-1}(0)$ and $0 \notin \theta(Y) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Therefore $f \in \mathcal{F}_T$. Now, suppose that $f|_{\theta_1(Y)}$ can be extended continuously to an invertible function on X and that $g : X \to \mathbb{C}$ is such an extension. Then $0 \notin g(X)$. Define the function $h : g(X) \to Y$ by

$$h(z) := \frac{z}{|z|}$$

for all $z \in g(X)$. Then h is a well-defined, continuous function which maps each point in Y to itself. That means that $h \circ g$ is a continuous function with domain X and range Y which maps each point in Y to itself. Hence, it is a retraction of X onto Y, but, by Theorem 1.23, no such retraction exists. Therefore, there is no invertible continuous extension of $f|_{\theta_1(Y)}$ to X, which means that the image of any continuous extension of $f|_{\theta_1(Y)}$ to X must contain 0. Hence $f \notin \mathcal{W}_T$.

Now, set $\theta = \theta_2$ where $\theta_2 : Y \to X$ is defined as the constant function which maps all values of Y to 1 and denote the Banach algebra homomorphism induced by composition with θ_2 by S. Once again, consider the function f on X which maps z to z for all $z \in X$. Because $\theta_2(Y) = \{1\}$, we see that $f|_{\theta_2(Y)}$ can be extended continuously to the constant function 1 on X which is invertible. Therefore f is Weyl with respect to S, but f is not invertible, because $0 \in f(X)$.

Remark 2.1.9. The inclusion $\mathcal{B} \subseteq \mathcal{W}$ can also be strict. See [13] for an example of a Weyl element which is not Browder in an operator algebra ([13], Example 4.3, p. 173).

Proposition 2.1.10 ([9], p. 434, (2.3))**.** *Let $a \in A$ and $T : A \to B$ a Banach algebra homomorphism. Then $\sigma_B(Ta) \subseteq \sigma_A(a)$.*

Proof. Suppose $\lambda \in \sigma_B(Ta)$. Then $Ta - \lambda\mathbf{1} = T(a - \lambda\mathbf{1})$ is not invertible in B and hence $a - \lambda\mathbf{1} \notin \mathcal{F}$. It follows from Proposition 2.1.7 that $a - \lambda\mathbf{1} \notin A^{-1}$, which means that $\lambda \in \sigma_A(a)$. □

Definition 2.1.11 (Almost invertible element)**.** ([9], p. 432). An element $a \in A$, where A is a Banach algebra, is called *almost invertible* if 0 is not an accumulation point of the spectrum of a.

Proposition 2.1.12. ([9], p. 432, (1.5)) *If $T : A \to B$ is a Banach algebra homomorphism, then the following inclusions hold:*

$$A^{-1} \subseteq \mathcal{F} \cap \{x \in A : x \textit{ is almost invertible in } A\} \subseteq \mathcal{F}.$$

Proof. We only need to prove the first inclusion. Let a be an invertible element of A. Then 0 is not an element of the spectrum of a and therefore 0 is not an accumulation point of the spectrum of a. Furthermore, it follows from Proposition 2.1.7 that a is Fredholm. □

Before we give the proof of the next theorem of Harte (Theorem 2.1.18), which extends the inclusion properties already observed in Proposition 2.1.7 and 2.1.12, we will prove two lemmas.

Lemma 2.1.13 states a fact that is mentioned without proof in [9].

Lemma 2.1.13. *Let A be a Banach algebra and $T : A \to B$ a bounded Banach algebra homomorphism. If $a \in A$ is an almost invertible Fredholm element and p is the spectral idempotent of a associated with 0, then $Tp = 0$.*

Proof. Suppose that $a \in A$ is almost invertible and Fredholm. Let U_1 and U_0 be disjoint open sets such that U_1 contains $\{0\}$ and U_0 contains $\sigma(a)\backslash\{0\}$. This is possible, because 0 is not an accumulation point of $\sigma(a)$, i.e. 0 is either an isolated point of $\sigma(a)$ or 0 is not in $\sigma(a)$. Let Γ_1 be a small circle in U_1 surrounding $\{0\}$ and let Γ_2 be a smooth contour in U_0 surrounding $\sigma(a)\backslash\{0\}$. Define the function $f : U_1 \cup U_0 \to \mathbb{C}$ as

$$f(\zeta) := \begin{cases} 0 & \text{if } \zeta \in U_0, \\ 1 & \text{if } \zeta \in U_1. \end{cases}$$

Then, by Definition 1.16,

$$\begin{aligned} f(a) &= \frac{1}{2\pi i}\int_{\Gamma} f(\zeta)\,(\zeta\mathbf{1}-a)^{-1}\ \mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\int_{\Gamma_1} 1\,(\zeta\mathbf{1}-a)^{-1}\ \mathrm{d}\zeta + \frac{1}{2\pi i}\int_{\Gamma_2} 0\,(\zeta\mathbf{1}-a)^{-1}\ \mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\int_{\Gamma_1} (\zeta\mathbf{1}-a)^{-1}\ \mathrm{d}\zeta \\ &= p, \end{aligned}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$. Because T is a bounded homomorphism, $\sigma_B(Ta) \subseteq \sigma_A(a)$ and $a \in \mathcal{F}$, we have

$$\begin{aligned} Tp &= T\left(\frac{1}{2\pi i}\int_{\Gamma_1} (\zeta\mathbf{1}-a)^{-1}\mathrm{d}\zeta\right) \\ &= \frac{1}{2\pi i}\int_{\Gamma_1} T\left((\zeta\mathbf{1}-a)^{-1}\right)\ \mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\int_{\Gamma_1} (\zeta\mathbf{1}-Ta)^{-1}\ \mathrm{d}\zeta \\ &= 0, \end{aligned}$$

by Cauchy's theorem. □

The following theorem states a fact that is mentioned in the proof of ([9], Theorem 1, p. 432). The result ([19], Theorem 2.4, p. 19) is a slightly more general form (boundedness of the homomorphism is not required here) of this theorem and has a similar but more detailed proof.

Theorem 2.1.14. *Let $T : A \to B$ be a bounded Banach algebra homomorphism. Take any $a \in \mathcal{F}$ such that a is almost invertible and let p be the spectral idempotent of a associated with 0. Define*

$$b := p + a(\mathbf{1}-p)$$

and

$$c := (a-\mathbf{1})p,$$

so that

$$a = b + c.$$

Then $b \in A^{-1}$, $c \in T^{-1}(0)$ *and* $bc = cb$ *and hence* $a \in \mathcal{B}$.

Proof. We first show that, if a is almost invertible in A, then there exist elements b and c such that $a = b + c$, $b \in A^{-1}$ and $bc = cb$. Let $a \in A$ be almost invertible. Let p be the spectral idempotent of a associated with 0. Then $p = p^2$ and $pa = ap$, by Theorem 1.17. Note that, because p commutes with a, it also commutes with $a - \mathbf{1}$ and a commutes with $\mathbf{1} - p$. It now follows that

$$\begin{aligned} bc &= \big(p + a(\mathbf{1} - p)\big)\big((a - \mathbf{1})p\big) \\ &= p(a - \mathbf{1})p + a(\mathbf{1} - p)(a - \mathbf{1})p \\ &= (a - \mathbf{1})p^2 + (a - \mathbf{1})pa(\mathbf{1} - p) \\ &= \big(a - \mathbf{1}\big)p\big(p + a(\mathbf{1} - p)\big) \\ &= cb. \end{aligned}$$

We will now show that b is invertible.

Choose U_1 and U_0 to be disjoint open sets such that U_1 contains $\{0\}$ and U_0 contains $\sigma(a)\backslash\{0\}$, exactly as in the proof of Lemma 2.1.13. Let Γ_1 be a small circle in U_1 surrounding $\{0\}$ and let Γ_2 be a smooth contour in U_0 surrounding $\sigma(a)\backslash\{0\}$. Define the function $f : U_1 \cup U_0 \to \mathbb{C}$ as

$$f(\zeta) := \begin{cases} 0 & \text{if } \zeta \in U_0, \\ 1 & \text{if } \zeta \in U_1. \end{cases}$$

Then, by Definition 1.16,

$$f(a) = p.$$

Define the function $g : U_1 \cup U_0 \to \mathbb{C}$ as

$$g(\zeta) := f(\zeta) + \zeta(1 - f(\zeta))$$

for all $\zeta \in U_1 \cup U_0$. Because f is analytic on $U_1 \cup U_0$, g is also analytic on $U_1 \cup U_0$. Therefore, by Theorem 1.14, $g(a)$ is an element of A and

$$\begin{aligned} g(a) &= f(a) + a(\mathbf{1} - f(a)) \\ &= p + a(\mathbf{1} - p) \\ &= b. \end{aligned}$$

It also follows from Theorem 1.14 that

$$\sigma(g(a)) = g(\sigma(a)).$$

To determine whether $\sigma(g(a))$ contains 0, we see that it follows from the definition of f that, for $\zeta \in U_1 \supseteq \{0\}$,

$$g(\zeta) = f(0) + 0(1 - f(0)) = f(0) = 1,$$

and for $\zeta \in U_0 \supseteq \sigma(a) \setminus \{0\}$,

$$g(\zeta) = f(\zeta) + \zeta(1 - f(\zeta)) = 0 + \zeta(1 - 0) = \zeta.$$

Therefore, $\sigma(b) = \sigma(g(a)) = g(\sigma(a)) \subseteq (\sigma(a)\setminus\{0\}) \cup \{1\}$. So $0 \notin \sigma(b)$ and hence b is invertible in A.

Note that a will be Browder if c is in the kernel of T. We will show that, if we assume that a is Fredholm as well, then c is indeed in the kernel of T and thus a is Browder. Suppose that a is Fredholm. It then follows from Lemma 2.1.13 that $Tp = 0$ and therefore

$$\begin{aligned} Tc &= T\big((a - \mathbf{1})p\big) \\ &= T(a - \mathbf{1})Tp \\ &= T(a - \mathbf{1})0 \\ &= 0. \end{aligned}$$

Hence, c is in the kernel of T and a is Browder. □

The following lemma is a more general version of a statement in [19] (p. 18).

Lemma 2.1.15. *Let A be a Banach algebra such that $b \in A$ and $c \in A$ are commuting elements where b and c are almost invertible. Then bc is almost invertible in A.*

Proof. Let $a := bc$. We know that

$$\sigma(a) = \sigma(bc) \subseteq \sigma(b)\,\sigma(c),$$

as was observed by Mouton and Raubenheimer in [19]. The remainder of this proof consists of our own arguments.

Because b is almost invertible, we have that either $0 \notin \sigma(b)$ or 0 is an isolated point of $\sigma(b)$. It follows from this and the fact that $\sigma(b)$ is closed, that there exists $\delta_1 > 0$ such that $\mathcal{B}(0, \delta_1)$ is disjoint from $\sigma(b) \backslash \{0\}$. Similarly, there exists $\delta_2 > 0$ such that $\mathcal{B}(0, \delta_2)$ is disjoint from $\sigma(c) \backslash \{0\}$. Now, choose arbitrary elements $\lambda_1 \in \sigma(b)$ and $\lambda_2 \in \sigma(c)$. If $\lambda_1 = 0$ or $\lambda_2 = 0$, we have that $|\lambda_1 \lambda_2| = 0$. If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, then

$$\begin{aligned} |\lambda_1 \lambda_2| &= |\lambda_1||\lambda_2| \\ &\geq |\delta_1||\delta_2| \\ &= \delta_1 \delta_2 \\ &> 0. \end{aligned}$$

Therefore, for every element ζ in $\sigma(b)\sigma(c)$, we have that either $|\zeta| \geq \delta_1 \delta_2 > 0$, or $|\zeta| = 0$. Therefore, 0 is not an accumulation point of $\sigma(b)\sigma(c)$. Since $\sigma(a) \subseteq \sigma(b)\sigma(c)$, it follows that 0 is not an accumulation point of $\sigma(a)$ and we see that a is almost invertible.

□

Definition 2.1.16 (Riesz property of a homomorphism)**.** ([9], p. 432). A Banach algebra homomorphism $T : A \to B$ has the *Riesz property* if the kernel of T is an inessential ideal.

Lemma 2.1.17. *Let I a two-sided inessential ideal of the Banach algebra A and $T : A \to A/\overline{I}$ the quotient map. Then T is Riesz.*

Proof. Suppose that $a \in T^{-1}(0)$. Then $a + \overline{I} = 0 + \overline{I}$ and hence $\sigma_{A/\overline{I}}\left(a + \overline{I}\right) = \{0\}$. It follows from Theorem 1.19 that $\operatorname{acc} \sigma_A(a) \subseteq \{0\}$ and therefore T is Riesz. □

In the proof of the next theorem, a result of Harte, we use an approach similar to Harte's in [9] and also to the approach of Mouton and Raubenheimer in [19] (Theorem 2.4, p. 19) to prove the first part (Theorem 2.1.18 (1)). We isolated the proof of the first part by including Theorem 2.1.14. To prove the second part (Theorem 2.1.18 (2)), we use Lemma 2.1.15, rather than using joint spectra as Harte did in [9].

Theorem 2.1.18. ([9], Theorem 1, p. 432)*. Let $T : A \to B$ be a Banach algebra homomorphism. Then we have the following two properties.*

1. *If T is bounded, then the inclusion*

$$\{a \in A : a \text{ is almost invertible in } A\} \cap \mathcal{F} \subseteq \mathcal{B}$$

holds.

2. *The homomorphism T (not necessarily bounded) has the Riesz property if and only if the converse inclusion*

$$\{a \in A : a \text{ is almost invertible in } A\} \cap \mathcal{F} \supseteq \mathcal{B}$$

holds.

Proof. 1. The inclusion $\{a \in A : a \text{ is almost invertible in } A\} \cap \mathcal{F} \subseteq \mathcal{B}$ is given by Theorem 2.1.14.

2. Suppose that T has the Riesz property and let a be a Weyl element of A. Then $a = b + c$ for some $b \in A^{-1}$ and $c \in T^{-1}(0)$. So, $b^{-1}a = \mathbf{1} + b^{-1}c$ and $b^{-1}c \in T^{-1}(0)$. Because T has the Riesz property, we have that $\sigma(b^{-1}c)$ has no non-zero accumulation points. So, because $\sigma(b^{-1}a) = \sigma(\mathbf{1}+b^{-1}c) = 1+\sigma(b^{-1}c)$, it follows that 1 is the only possible accumulation point of $\sigma(b^{-1}a)$. Hence, 0 is not an accumulation point of $\sigma(b^{-1}a)$, i.e. $b^{-1}a$ is almost invertible.

   Now, suppose that $bc = cb$, i.e. that $a \in \mathcal{B}$. Then a commutes with b and hence b commutes with $b^{-1}a$. Now, because $a = b(b^{-1}a)$ and the elements b and $b^{-1}a$ are almost invertible, it follows from Lemma 2.1.15 that a is almost invertible.

   It is clear from Proposition 2.1.7 that a is Fredholm, because a is Browder.

   Conversely, suppose that every Browder element is almost invertible Fredholm. For any non-zero $\lambda \in \mathbb{C}$ and any c in the kernel of T, $c - \lambda\mathbf{1}$ is Browder, because $-\lambda\mathbf{1}$ is invertible and $-\lambda\mathbf{1}$ commutes with all elements of A. It therefore follows from the assumption that $c - \lambda\mathbf{1}$ is almost invertible. So, $0 \notin \operatorname{acc}\sigma\,(c - \lambda\mathbf{1})$ and it follows from the spectral mapping theorem that $\lambda \notin \operatorname{acc}\sigma\,(c)$. Hence, $\sigma\,(c)$ has no non-zero accumulation points and therefore $T^{-1}(0)$ is an inessential ideal of A. Therefore, T has the Riesz property. $\square$

Example 2.1.19. Let I be a two-sided, inessential ideal of a Banach algebra A, X the set of almost invertible elements of A and $T : A \to A/\overline{I}$ the Banach algebra homomorphism which maps each element of A to its coset in $A/\overline{I}$. Then T has the Riesz property, by Lemma 2.1.17, and is bounded. Hence, by Theorem 2.1.18, $\mathcal{B} = X \cap \mathcal{F}$. That means that, if $a = b + c$ where $b \in A^{-1}$, $c \in \overline{I}$ and $bc = cb$, then $b + \overline{I}$ is invertible in $A/\overline{I}$ and $0 \notin \operatorname{acc} \sigma(a)$. Furthermore, if $a + \overline{I}$ is invertible in $A/\overline{I}$ and $0 \notin \operatorname{acc} \sigma(a)$, then $a = b + c$ where $b \in A^{-1}$, $c \in \overline{I}$ and $bc = cb$.

The following corollary is not stated explicitly in [9], but follows directly from Proposition 2.1.7, Proposition 2.1.12 and Theorem 2.1.18 (1).

Corollary 2.1.20. *If $T : A \to B$ is a bounded Banach algebra homomorphism, then we have the following inclusions:*

$$A^{-1} \subseteq \{a \in A : a \textit{ is almost invertible in } A\} \cap \mathcal{F} \subseteq \mathcal{B} \subseteq \mathcal{W} \subseteq \mathcal{F}.$$

2.2 Inclusion properties of spectra

In the next part of Harte's article ([9]) the concept of the spectrum of an element in a Banach algebra is extended. In the same way that the group of invertible elements A^{-1} of a Banach algebra A is used to define the ordinary spectrum (Definition 2.1.3), the sets of Fredholm, Weyl and Browder elements are used to define three new types of spectra.

Definition 2.2.1 (Fredholm, Weyl and Browder spectra of an element of a Banach algebra)**.** ([9], pp. 433–434). Let $T : A \to B$ be a Banach algebra homomorphism.

1. The *Fredholm spectrum* $\sigma_B(Ta)$ of an element a of a Banach algebra A is the set
$$\{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathcal{F}\}.$$
2. The *Weyl spectrum* $\omega_T(a)$ of an element a of a Banach algebra A is the set
$$\{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathcal{W}\}.$$
3. The *Browder spectrum* $\omega_T^{\mathrm{comm}}(a)$ of an element a of a Banach algebra A is the set
$$\{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathcal{B}\}.$$

Remark 2.2.2. Strictly speaking, one should call $\sigma_B(Ta)$ the Fredholm spectrum of the element a *relative to* T and likewise for $\omega_T(a)$ and $\omega_T^{\mathrm{comm}}(a)$, but in this thesis the relevant Banach algebra homomorphism will only be specified, should it be necessary to avoid confusion.

It makes sense to use the notation $\sigma_B(Ta)$ for the Fredholm spectrum of the element a because the Fredholm spectrum of an element $a \in A$ and the spectrum of $Ta \in B$ are the same. This follows immediately from the fact that $T(\lambda\mathbf{1} - a) = \lambda\mathbf{1} - Ta$, which means that $T(\lambda\mathbf{1} - a) \in B^{-1}$ if and only if $\lambda\mathbf{1} - Ta \in B^{-1}$, i.e. $\lambda\mathbf{1} - a \in \mathcal{F}$ if and only if $\lambda \notin \sigma_B(Ta)$.

Note, also, that $\sigma_B(Ta) \subseteq \sigma(a)$, as shown in Proposition 2.1.10.

Corollary 2.2.3. ([9], p. 434) *The Fredholm spectrum of an element relative to any Banach algebra homomorphism is non-empty and compact.*

Proposition 2.2.4. ([9], p. 434, (2.3)) *Let $T : A \to B$ be a Banach algebra homomorphism. For each element $a \in A$, the following inclusions hold:*

$$\sigma_B(Ta) \subseteq \omega_T(a) \subseteq \omega_T^{\mathrm{comm}}(a) \subseteq \sigma(a).$$

Proof. This follows directly from Proposition 2.1.7. □

Corollary 2.2.5. ([9], p. 434) *Let $T : A \to B$ be a Banach algebra homomorphism. Then the sets $\omega_T(a)$ and $\omega_T^{\mathrm{comm}}(a)$ are both non-empty.*

The following example illustrates that some of the above inclusions can be strict.

Example 2.2.6. Let T and S be the Banach algebra homomorphisms and f the function defined in example 2.1.8. Then $\sigma_B(Tf) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $\sigma_A(f) = \omega_T(f) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. So we have that

$$\sigma_B(Tf) \subsetneqq \omega_T(f) = \sigma_A(f).$$

Furthermore, $\sigma_B(Sf) = \omega_S(f) = \{1\}$. Therefore

$$\sigma_B(Sf) = \omega_S(f) \subsetneqq \sigma_A(f).$$

Proof. Let $\lambda \in \mathbb{C}$. The function $f - \lambda\mathbf{1}$ is invertible if and only if $0 \notin (f - \lambda\mathbf{1})(X) = X - \lambda$, which is the case if and only if $\lambda \notin X$. Hence $\sigma_A(f) = X$.

From Example 2.1.6, $f - \lambda\mathbf{1} \in \mathcal{F}_T$ if and only if $\theta_1(Y) \cap (f - \lambda\mathbf{1})^{-1}(0) = Y \cap (f - \lambda\mathbf{1})^{-1}(0) = \emptyset$. If $\lambda \in X$, then $(f - \lambda\mathbf{1})^{-1}(0) = \lambda$, otherwise

$(f-\lambda\mathbf{1})^{-1}(0)=\emptyset$. It follows that $\lambda \in \sigma_B(Tf)$ if and only if $\lambda \in Y$. Hence $\sigma_B(Tf)=Y$.

From Example 2.1.6, we have that $f-\lambda\mathbf{1}$ is Weyl with respect to T if and only if its restriction to $\theta_1(Y)=Y$ has an invertible extension to X. Note that $(f-\lambda\mathbf{1})|_Y(Y)=Y-\lambda$. If $\lambda \notin X$, then $0 \notin Y-\lambda$ and then the function which maps z to $z-\lambda$ for all $z \in X$ is an invertible extension of $(f-\lambda\mathbf{1})|_Y$ to X and hence $f-\lambda\mathbf{1} \in \mathcal{W}_T$. If $|\lambda|=1$, then $0 \in Y-\lambda$ and hence $(f-\lambda\mathbf{1})|_Y$ is not invertible and cannot have an invertible extension to X. If $\lambda \in X$, suppose that $(f-\lambda\mathbf{1})|_Y$ has an invertible extension g to X. Then $0 \notin g(X)$. Therefore $\lambda \notin (g+\lambda\mathbf{1})(X)$. Define the function $h:(g+\lambda\mathbf{1})(X)\to\mathbb{C}$ as follows: For $z \in (g+\lambda\mathbf{1})(X)$ where $|z|\geq 1$ define $h(z):=\frac{z}{|z|}$. For $z\in(g+\lambda\mathbf{1})(X)$ where $|z|<1$, construct a line starting from λ and passing through z and map z to the point where this line intersects Y. Then h is a continuous function from $(g+\lambda\mathbf{1})(X)$ onto Y which restricts to the identity function on Y. Furthermore $g+\lambda\mathbf{1}$ is a continuous function from X onto $(g+\lambda\mathbf{1})(X)$ which restricts to the identity function on Y. Hence $h\circ(g+\lambda\mathbf{1})$ is a retraction of X onto Y. This is a contradiction, by Theorem 1.23, and therefore there exists no invertible extension of $(f-\lambda\mathbf{1})|_Y$ to X. It follows that $\omega_T(f)=\{\lambda\in\mathbb{C}:|\lambda|\leq 1\}=X$.

As above, $f-\lambda\mathbf{1}\in\mathcal{F}_S$ if and only if $\theta_2(Y)\cap(f-\lambda\mathbf{1})^{-1}(0)=\{1\}\cap(f-\lambda\mathbf{1})^{-1}(0)=\emptyset$. If $\lambda=1$, then $(f-\lambda\mathbf{1})^{-1}(0)=\lambda$, otherwise $\{1\}\cap(f-\lambda\mathbf{1})^{-1}(0)=\emptyset$. It follows that $\lambda\in\sigma_B(Sf)$ if and only if $\lambda=1$. Hence $\sigma_B(Sf)=\{1\}$.

As before, we have that $f-\lambda\mathbf{1}$ is Weyl with respect to S if and only if its restriction to $\theta_2(Y)=\{1\}$ has an invertible extension to X. If $\lambda=1$, then $(f-\lambda\mathbf{1})(1)=1-1=0$ and hence $(f-\lambda\mathbf{1})|_{\theta_2(Y)}$ is not invertible and cannot have an invertible extension to X. If $\lambda\neq 1$, then $(f-\lambda\mathbf{1})(1)=1-\lambda\neq 0$. Hence, $(f-\lambda\mathbf{1})|_{\theta_2(Y)}$ can be extended to the function which maps every $z\in X$ to $\lambda-1$, which is invertible in A. It follows that $\omega_S(f)=\{1\}$. $\square$

The following proposition provides characterisations of the Weyl and Browder spectra which are stated without proof in [9].

Proposition 2.2.7. ([9], pp. 433–434, (2.1) and (2.2)) *Let $T:A\to B$ be a Banach algebra homomorphism. For any element $a\in A$ the following two* *equalities hold:*

$$\omega_T(a) = \bigcap_{c \in T^{-1}(0)} \sigma(a+c), \tag{2.2.8}$$

$$\omega_T^{\mathrm{comm}}(a) = \bigcap_{\substack{c \in T^{-1}(0) \\ ac=ca}} \sigma(a+c). \tag{2.2.9}$$

Proof. Let $a \in A$.

We first prove equality (2.2.8). Suppose that $\lambda \notin \bigcap_{c \in T^{-1}(0)} \sigma(a+c)$. Then there exists an element $d \in T^{-1}(0)$ such that $z := \lambda \mathbf{1} - (a+d)$ is invertible in A. So, $\lambda \mathbf{1} - a = z + d$ with $z \in A^{-1}$ and $d \in T^{-1}(0)$, which means that $\lambda \mathbf{1} - a$ is Weyl and therefore $\lambda \notin \omega_T(a)$. It follows that

$$\omega_T(a) \subseteq \bigcap_{c \in T^{-1}(0)} \sigma(a+c).$$

Conversely, suppose that $\lambda \notin \omega_T(a)$. Then there exist elements $b \in A^{-1}$ and $c \in T^{-1}(0)$ such that $\lambda \mathbf{1} - a = b + c$. Therefore $\lambda \mathbf{1} - (a+c) = b$, which is invertible. We see that $\lambda \notin \sigma(a+c)$ with $c \in T^{-1}(0)$. Therefore $\lambda \notin \bigcap_{c \in T^{-1}(0)} \sigma(a+c)$. Therefore

$$\omega_T(a) \supseteq \bigcap_{c \in T^{-1}(0)} \sigma(a+c),$$

and equality (2.2.8) is proved.

We now prove equation (2.2.9) in a similar way. Suppose that $\lambda \notin \bigcap_{\substack{c \in T^{-1}(0) \\ ac=ca}} \sigma(a+c)$. Then there exists an element $d \in T^{-1}(0)$, commuting with a, such that $z := \lambda \mathbf{1} - (a+d)$ is invertible in A. Note that d commutes with z as well. So $\lambda \mathbf{1} - a = z + d$, with $z \in A^{-1}$, $d \in T^{-1}(0)$ and $zd = dz$. This means that $\lambda \mathbf{1} - a$ is Browder and therefore $\lambda \notin \omega_T^{\mathrm{comm}}(a)$. Hence

$$\omega_T^{\mathrm{comm}}(a) \subseteq \bigcap_{\substack{c \in T^{-1}(0) \\ ac=ca}} \sigma(a+c).$$

Conversely, suppose that $\lambda \notin \omega_T^{\mathrm{comm}}(a)$. Then there exist commuting elements $b \in A^{-1}$ and $c \in T^{-1}(0)$ such that $\lambda \mathbf{1} - a = b + c$. Therefore $\lambda \mathbf{1} - (a+c) = b$ which is invertible. Note that c commutes with a as well. We see that $\lambda \notin \sigma(a+c)$ with $c \in T^{-1}(0)$ and $ac = ca$. Therefore $\lambda \notin \bigcap_{\substack{c \in T^{-1}(0) \\ ac=ca}} \sigma(a+c)$ and hence

$$\omega_T^{\mathrm{comm}}(a) \supseteq \bigcap_{\substack{c \in T^{-1}(0) \\ ac=za}} \sigma(a+c).$$

This proves equality (2.2.9). $\square$

Because an arbitrary intersection of compact sets in a Hausdorff space is compact, the following corollary is obtained from equations (2.2.8) and (2.2.9).

Corollary 2.2.10 ([9], p. 434)**.** *Let $T : A \to B$ be a Banach algebra homomorphism. For every $a \in A$, the sets $\omega_T(a)$ and $\omega_T^{\mathrm{comm}}(a)$ are compact.*

In the following proposition the set $\sigma_B(Ta) \cup \operatorname{acc}\sigma(a)$, where $T : A \to B$ is a Banach algebra homomorphism and $a \in A$, is regarded in relation to the four types of spectra we have introduced up until this point. The set $\sigma_B(Ta) \cup \operatorname{acc}\sigma(a)$ is another type of spectrum, called the *almost invertible Fredholm spectrum*. We find a way of writing $\sigma_B(Ta) \cup \operatorname{acc}\sigma(a)$ which is analogous to the definitions of the other spectra. Let X be the set of almost invertible elements of A. Then we call $\mathcal{F} \cap X$ *the set of almost invertible Fredholm elements* and

$$\sigma_B(Ta) \cup \operatorname{acc}\sigma(a) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathcal{F} \cap X\}.$$

This becomes clear in the proof of the proposition below.

Proposition 2.2.11. ([9], p. 434, (2.3)) *Let $T : A \to B$ be a bounded Banach algebra homomorphism. For any element $a \in A$, the following inclusions hold:*

$$\sigma_B(Ta) \subseteq \omega_T(a) \subseteq \omega_T^{\mathrm{comm}}(a) \subseteq \sigma_B(Ta) \cup \operatorname{acc}\sigma(a) \subseteq \sigma(a).$$

Proof. From Proposition 2.2.4, we have that

$$\sigma_B(Ta) \subseteq \omega_T(a) \subseteq \omega_T^{\mathrm{comm}}(a) \subseteq \sigma(a),$$

and it is clear that $\sigma_B(Ta) \cup \operatorname{acc}\sigma(a) \subseteq \sigma(a)$, because $\operatorname{acc}\sigma(a) \subseteq \sigma(a)$ and $\sigma_B(Ta) \subseteq \sigma(a)$ (from Remark 2.2.2).

It remains to prove the inclusion $\omega_T^{\mathrm{comm}}(a) \subseteq \sigma_B(Ta) \cup \operatorname{acc}\sigma(a)$. Suppose that $\lambda \in \omega_T^{\mathrm{comm}}(a)$. Then $\lambda\mathbf{1} - a$ is not Browder. It follows from Theorem 2.1.18 (1) that $\lambda\mathbf{1} - a$ is not almost invertible Fredholm. This means that at least one of the following two conditions holds:

$$T(\lambda\mathbf{1} - a) \notin B^{-1}; \tag{2.2.12}$$

$$0 \in \operatorname{acc}\sigma(\lambda\mathbf{1} - a). \tag{2.2.13}$$

Condition 2.2.12 means that $\lambda \in \sigma_B(Ta)$. Furthermore, $\lambda \in \operatorname{acc}\sigma(a)$ is the same as condition 2.2.13 by the spectral mapping theorem (Theorem 1.14 (5)). We conclude that

$$\lambda \in \sigma_B(Ta) \cup \operatorname{acc}\sigma(a),$$

which completes this proof. $\square$

The following corollary is not stated explicitly in Harte's article ([9]), but follows immediately from the non-emptiness and compactness of the Fredholm spectrum of an element.

Corollary 2.2.14. *If $T : A \to B$ is a Banach algebra homomorphism, then the almost invertible Fredholm spectrum $\sigma_B(Ta) \cup \operatorname{acc}\sigma(a)$ of an element $a \in A$ is non-empty and compact.*

Proof. It follows from the fact that $\sigma_B(Ta)$ is non-empty that $\sigma_B(Ta) \cup \operatorname{acc}\sigma(a)$ is non-empty.

We know that $\sigma_B(Ta)$ is compact, so we only need to show that $\operatorname{acc}\sigma(a)$ is compact. The set $\operatorname{acc}\sigma(a)$ is bounded, because it is a subset of $\sigma(a)$ which is bounded. We now show that the set $\operatorname{acc}\sigma(a)$ is closed. For each $\lambda \notin \operatorname{acc}\sigma(a)$ there exists a $\delta > 0$ such that $\mathcal{B}(\lambda, \delta)$ is disjoint from $\sigma(a) \setminus \{\lambda\} \supseteq \operatorname{acc}\sigma(a)$. Therefore the complement of the set $\operatorname{acc}\sigma(a)$ is open and the set $\operatorname{acc}\sigma(a)$ is closed. Therefore, from the Heine-Borel theorem, $\operatorname{acc}\sigma(a)$ is compact. $\square$

In the second of the following two results, boundedness of the Banach algebra homomorphism is required and it is a result that can be found in [9]. The first of the two is what we obtain when we consider the same situation, but without requiring the Banach algebra homomorphism to be bounded and this result is not stated in [9].

Proposition 2.2.15. *If $T : A \to B$ is a Banach algebra homomorphism with the Riesz property, then $\omega_T^{\mathrm{comm}}(a) \supseteq \sigma_B(Ta) \cup \operatorname{acc}\sigma(a)$.*

Proof. Suppose that T has the Riesz property and that $\lambda \in \sigma_B(Ta) \cup \operatorname{acc}\sigma(a)$. Then $\lambda\mathbf{1} - a \notin \mathcal{F} \cap X$, where X is the set of almost invertible elements of A. It now follows from Theorem 2.1.18 (2) that $\lambda\mathbf{1} - a \notin \mathcal{B}$, which means that $\lambda \in \omega_T^{\mathrm{comm}}(a)$. □

Corollary 2.2.16. ([9], p. 434) *If $T : A \to B$ is a bounded Banach algebra homomorphism with the Riesz property, then $\omega_T^{\mathrm{comm}}(a) = \sigma_B(Ta) \cup \operatorname{acc}\sigma(a)$.*

Proof. This follows directly from the third inclusion in Proposition 2.2.11 and from Proposition 2.2.15. □

Example 2.2.17. Let I be a two-sided, inessential ideal of a Banach algebra A and $T : A \to A/\overline{I}$ the Banach algebra homomorphism which maps each element of A to its coset in $A/\overline{I}$. Then T has the Riesz property, by Lemma 2.1.17, and is bounded. It then follows from Corollary 2.2.16 or directly from Example 2.1.19 that

$$\omega_T^{\mathrm{comm}}(a) = \sigma_{A/\overline{I}}\left(a + \overline{I}\right) \cup \operatorname{acc}\sigma_A(a)$$

for all $a \in A$.

2.3 The spectral mapping theorem

We have already established that the four new types of spectra are always non-empty compact sets, like the ordinary spectrum. The next question is which of them and to what extent they obey a spectral mapping theorem. If the relevant Banach algebra homomorphism is bounded, it is easy to find an answer for the Fredholm spectrum as we see in the following proposition which is stated in [9] without proof.

Proposition 2.3.1. ([9], p. 434, (2.4)) *Let $T : A \to B$ be a bounded Banach algebra homomorphism and $a \in A$. If $f : U \to \mathbb{C}$ is holomorphic, with U a neighbourhood of $\sigma(a)$, then $f(\sigma_B(Ta)) = \sigma_B(T(f(a)))$.*

Proof. By Remark 2.2.2, $\sigma_B(Ta) \subseteq \sigma(a)$. Let $f : U \to \mathbb{C}$ be holomorphic, with U a neighbourhood of $\sigma(a) \supseteq \sigma_B(Ta)$. Note that, by choosing a smooth contour Γ surrounding $\sigma(a) \supseteq \sigma_B(Ta)$, we see that, since T is a bounded homomorphism,

$$\begin{aligned} T(f(a)) &= T\left(\frac{1}{2\pi i}\int_\Gamma f(\zeta)(\zeta\mathbf{1}-a)^{-1}\,\mathrm{d}\zeta\right) \\ &= \frac{1}{2\pi i}\int_\Gamma T\left(f(\zeta)(\zeta\mathbf{1}-a)^{-1}\right)\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\int_\Gamma f(\zeta)(\zeta\mathbf{1}-Ta)^{-1}\,\mathrm{d}\zeta \\ &= f(Ta). \end{aligned}$$

It now follows from the spectral mapping theorem (Theorem 1.14 (5)) that

$$f(\sigma_B(Ta)) = \sigma_B(f(Ta)) = \sigma_B(T(f(a))). \qquad \square$$

The next theorem gives us a weaker form of the spectral mapping theorem for the Weyl and Browder spectra of an element, as well as for the almost invertible Fredholm spectrum. Although we only have one way inclusions in this theorem, it is interesting to note that boundedness of the relevant operator is not required, while the boundedness was necessary in our proof of 2.3.1. Our method of proof is similar to that of Harte in [9]. We first prove a lemma. The first part of this lemma is part of the proof of ([9], Theorem 2, p. 434) by Harte. The second part is our own observation.

Lemma 2.3.2. *Let A be a Banach algebra and $a \in A$. Then, if U is a neighbourhood of $\sigma(a)$ and $f : U \to \mathbb{C}$ an analytic function on U, then* $\operatorname{acc}\sigma(f(a)) \subseteq f(\operatorname{acc}\sigma(a))$. *If f is non-constant on every component of U, then* $\operatorname{acc}\sigma(f(a)) = f(\operatorname{acc}\sigma(a))$.

Proof. Suppose that $t \in \operatorname{acc}\sigma(f(a)) = \operatorname{acc} f(\sigma(a))$. Then there is a sequence $(f(s_n))$ in $f(\sigma(a))$ not containing t, but converging to t. Because (s_n) is a sequence in the compact set $\sigma(a)$, it contains a convergent subsequence (s_{n_k}), with limit s, say. Then $s \in \operatorname{acc}\sigma(a)$. But then $f(s_{n_k}) \to f(s)$ as $k \to \infty$, so $f(s) = t$ and we see that $t \in f(\operatorname{acc}\sigma(a))$. Therefore $\operatorname{acc}\sigma(f(a)) \subseteq f(\operatorname{acc}\sigma(a))$.

Suppose that f is non-constant on every component of U and that $t \in f(\operatorname{acc}\sigma(a))$. Then there exists an element $s \in \operatorname{acc}\sigma(a)$ such that $t = f(s)$ and there is a sequence (s_n) in $\sigma(a)$ not containing s, but converging to s. We can assume that each element of the sequence (s_n) is in the same component of U, say U_1. Because $\sigma(a)$ is closed, $s \in \sigma(a) \cap U_1$. Because f is continuous, the sequence $(f(s_n))$ converges to $f(s)$. By the spectral mapping theorem, $f(s) \in \sigma(f(a))$ and $f(s_n) \in \sigma(f(a))$ for all $n \in \mathbb{N}$. Furthermore, if $f(s_n) = f(s)$ for an infinite number of elements s_n in the sequence (s_n), then the function which maps each $z \in U_1$ to $f(z) - f(s)$ would have a limit point of zeros in U_1 and, by complex analysis, f would be constant on U_1, a contradiction. Hence, we can only have $f(s_n) = f(s)$ for finitely many elements s_n of the sequence (s_n). It follows that there is a subsequence of $(f(s_n))$ not containing $f(s)$, but converging to $f(s)$. Therefore, $t = f(s) \in \operatorname{acc} \sigma(f(a))$. □

Theorem 2.3.3. ([9], p. 434, Theorem 2) *Let $T : A \to B$ be a Banach algebra homomorphism, $a \in A$ and $f : U \to \mathbb{C}$ a holomorphic function on U, a neighbourhood of $\sigma(a)$. Then we have the following two sets of conditional inclusions.*

1. *If f is non-constant on every component of U, then*

$$\omega_T(f(a)) \subseteq f(\omega_T(a)), \tag{2.3.4}$$

and

$$\omega_T^{\text{comm}}(f(a)) \subseteq f(\omega_T^{\text{comm}}(a)). \tag{2.3.5}$$

2. *If T is bounded, we have*

$$\begin{aligned} f(\omega_T^{\text{comm}}(a)) &\subseteq \sigma_B\big(T(f(a))\big) \cup \operatorname{acc} \sigma(f(a)) \\ &\subseteq f\big(\sigma_B(Ta) \cup \operatorname{acc} \sigma(a)\big). \end{aligned} \tag{2.3.6}$$

Proof. If U is not bounded, redefine it, so that it is bounded, but remains a neighbourhood of $\sigma(a)$.

1. We first prove inclusion (2.3.4). Suppose that $t \notin f(\omega_T(a))$. We know, from Proposition 2.2.11 and the spectral mapping theorem, that

$$\omega_T(f(a)) \subseteq \sigma(f(a)) = f(\sigma(a)).$$

Therefore, if $t \notin f(\sigma(a))$, then $t \notin \omega_T(f(a))$. We therefore suppose that $t \in f(\sigma(a)) \backslash f(\omega_T(a))$. Now, define the function $h : U \to \mathbb{C}$ by

$$h(z) := f(z) - t$$

for all $z \in U$. Then h has at least one zero in $\sigma(a)$. Also, h is holomorphic on U and not identically zero on each component of U. We can deduce from Theorem 3.7 in [6] that, in each component C of U, the set of zeros of h in C does not have a limit point in C. Because $\sigma(a) \cap C$ is closed and bounded, this means that h has only a finite number of zeros in $\sigma(a) \cap C$. Because $\sigma(a) \cap C_1$ and $\sigma(a) \cap C_2$ are disjoint, closed sets for every pair of distinct components C_1 and C_2 of U, it now follows that the zeros of h cannot have a limit point in $\sigma(a)$, so that h has only a finite number of zeros in $\sigma(a)$. Let

$$\begin{aligned}\{s_1, s_2, \ldots, s_n\} &= \{z \in \sigma(a) : h(z) = 0\}\\ &= \{z \in \sigma(a) : f(z) = t\}.\end{aligned}$$

Because $t \notin f(\omega_T(a))$, $\{s_1, s_2, \ldots, s_n\} \cap \omega_T(a) = \emptyset$. It also follows from Corollary 3.9 in [6] that we can write

$$f(z) - t = h(z) = (z - s_1)^{m_1}(z - s_2)^{m_2} \cdots (z - s_n)^{m_n} g(z),$$

where $m_1, m_2, \ldots, m_n \in \mathbb{N}$ and $g : U \to \mathbb{C}$ is holomorphic and has no zeros in $\sigma(a)$. So, by holomorphic functional calculus (Theorem 1.14),

$$h(a) = (a - s_1\mathbf{1})^{m_1}(a - s_2\mathbf{1})^{m_2} \cdots (a - s_n\mathbf{1})^{m_n} g(a).$$

For $j \in \{1, 2, \ldots, n\}$, $s_j \notin \omega_T(a)$ and therefore $a - s_j\mathbf{1} = a_j + b_j$ for some $a_j \in A^{-1}$ and $b_j \in T^{-1}(0)$. Because g has no zeros in $\sigma(a)$, $\frac{1}{g}$ is a holomorphic function on a neighbourhood of $\sigma(a)$ and $g(a)$ is invertible with $g(a)^{-1} = \left(\frac{1}{g}\right)(a)$. So,

$$\begin{aligned}f(a) - t\mathbf{1} &= (a_1 + b_1)^{m_1}(a_2 + b_2)^{m_2} \cdots (a_n + b_n)^{m_n} g(a)\\ &= a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} g(a) + d_2,\end{aligned}$$

where d_2 is the sum of terms of which each contains at least one b_j. Now, $a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} g(a) \in A^{-1}$, while d_2 must be in the kernel of T. So we can write

$$f(a) - t\mathbf{1} = d_1 + d_2$$

for some $d_1 \in A^{-1}$ and $d_2 \in T^{-1}(0)$, and therefore $t \notin \omega_T(f(a))$. We have shown that

$$\omega_T(f(a)) \subseteq f(\omega_T(a)),$$

This proves inclusion (2.3.4).

We will now prove (2.3.5). Suppose that $t \notin f(\omega_T^{\mathrm{comm}}(a))$. We know from Proposition 2.2.11 and the spectral mapping theorem that

$$\omega_T^{\mathrm{comm}}(f(a)) \subseteq \sigma(f(a)) = f(\sigma(a)).$$

Hence, if $t \notin f(\sigma(a))$, then $t \notin \omega_T^{\mathrm{comm}}(f(a))$. We therefore suppose that $t \in f(\sigma(a)) \backslash f(\omega_T^{\mathrm{comm}}(a))$. Now, define the function $h : U \to \mathbb{C}$ by

$$h(z) := f(z) - t$$

for all $z \in U$. As in the proof of inclusion (2.3.4), we can write

$$f(z) - t = h(z) = (z - s_1)^{m_1}(z - s_2)^{m_2} \cdots (z - s_n)^{m_n} g(z),$$

where

$$\begin{aligned} \{s_1, s_2, \ldots, s_n\} &= \{z \in \sigma(a) : h(z) = 0\} \\ &= \{z \in \sigma(a) : f(z) = t\}, \end{aligned}$$

$m_1, m_2, \ldots, m_n \in \mathbb{N}$ and $g : U \to \mathbb{C}$ is holomorphic and has no zeros in $\sigma(a)$. So we have

$$h(a) = (a - s_1\mathbf{1})^{m_1}(a - s_2\mathbf{1})^{m_2} \cdots (a - s_n\mathbf{1})^{m_n} g(a),$$

with $g(a)$ invertible. For $j \in \{1, 2, \ldots, n\}$, $s_j \notin \omega_T^{\mathrm{comm}}(a)$ and therefore $a - s_j\mathbf{1} = a_j + b_j$ for some $a_j \in A^{-1}$ and $b_j \in T^{-1}(0)$ such that $a_j b_j = b_j a_j$. We know that a commutes with each $s_j\mathbf{1}$, and therefore a also commutes with each a_j and b_j, because

$$\begin{aligned} a a_j &= (s_j\mathbf{1} + a_j + b_j)a_j \\ &= s_j a_j + a_j^2 + b_j a_j \\ &= s_j a_j + a_j^2 + a_j b_j \\ &= a_j(s_j\mathbf{1} + a_j + b_j) \\ &= a_j a, \end{aligned}$$

and similarly $ab_j = b_j a$. Now,

$$f(a) - t\mathbf{1} = a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} g(a) + d,$$

where d is the sum of terms each containing at least one b_j and therefore $d \in T^{-1}(0)$. So,

$$f(a) - d - t\mathbf{1} = a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} g(a)$$

which is invertible. Also, d commutes with a, because a commutes with each a_j as well as with $g(a)$ and $f(a)$. So, $t \notin \sigma\left(f(a) - d\right)$ and $Td = 0$ and $ad = da$. Therefore

$$\begin{aligned} t &\notin \bigcap \{\sigma\left(f(a) - c\right) : Tc = 0, ca = ac\} \\ &\supseteq \bigcap \{\sigma\left(f(a) - c\right) : Tc = 0, cf(a) = f(a)c\} \\ &= \omega_T^{\mathrm{comm}}\left(f(a)\right). \end{aligned}$$

Therefore, $\omega_T^{\mathrm{comm}}\left(f(a)\right) \subseteq f(\omega_T^{\mathrm{comm}}\left(a\right))$.

2. We now show that, if T is bounded, then

$$\sigma_B\left(T(f(a))\right) \cup \operatorname{acc}\sigma\left(f(a)\right) \subseteq f(\sigma_B\left(Ta\right) \cup \operatorname{acc}\sigma\left(a\right)),$$

which is the second inclusion of (2.3.6). From Proposition 2.3.1, we know that $\sigma_B\left(T(f(a))\right) = f(\sigma_B\left(Ta\right))$. Furthermore, it follows from Lemma 2.3.2 that $\operatorname{acc}\sigma\left(f(a)\right) \subseteq f(\operatorname{acc}\sigma\left(a\right))$. We therefore obtain the inclusion $\sigma_B\left(T(f(a))\right) \cup \operatorname{acc}\sigma\left(f(a)\right) \subseteq f(\sigma_B\left(Ta\right)) \cup f(\operatorname{acc}\sigma\left(a\right)) = f(\sigma_B\left(Ta\right) \cup \operatorname{acc}\sigma\left(a\right))$.

It remains to prove the first inclusion of (2.3.6) for the case where T is bounded. Suppose that t is not in the almost invertible Fredholm spectrum of $f(a)$. Let q be the spectral idempotent of $f(a) - t\mathbf{1}$ associated with 0. By the definition of the spectral idempotent (1.16), $q = h(f(a) - t\mathbf{1})$, where $h : U \cup V \to \mathbb{C}$, with U and V disjoint open sets, $0 \in U$, $\sigma\left(f(a) - t\mathbf{1}\right) \setminus \{0\} \subseteq V$ and

$$h(\zeta) := \begin{cases} 0 & \text{if } \zeta \in V, \\ 1 & \text{if } \zeta \in U. \end{cases}$$

Now, suppose there exists $s \in \mathbb{C}$ such that $t = f(s)$. We will show that $a - s\mathbf{1}$ is Browder, i.e. that $s \notin \omega_T^{\mathrm{comm}}\left(a\right)$. Note that

$$a - s\mathbf{1} = b + c$$

where

$$b = q + (a - s\mathbf{1})(\mathbf{1} - q)$$

and

$$c = (a - s\mathbf{1} - \mathbf{1})q.$$

Observe that

$$\begin{aligned} b &= q + (a - s\mathbf{1})(\mathbf{1} - q) \\ &= h(f(a) - t\mathbf{1}) + (a - s\mathbf{1})(\mathbf{1} - h(f(a) - t\mathbf{1})). \end{aligned}$$

Now, define the function $g : U \cup V \to \mathbb{C}$ as

$$g(\lambda) := h(f(\lambda) - t) + (\lambda - s)(1 - h(f(\lambda) - t))$$

for all $\lambda \in U \cup V$. Then $g(a) = q + (a - s\mathbf{1})(1 - q)$, so, by the spectral mapping theorem,

$$\begin{aligned} \sigma\left(q + (a - s\mathbf{1})(\mathbf{1} - q)\right) &= \sigma\left(g(a)\right) \\ &= g(\sigma\left(a\right)). \end{aligned}$$

If $z \in \sigma(a)$ and $z = s$, then

$$g(z) = h(0) + 0 = 1.$$

If $z \in \sigma(a)\,, z \neq s$ and $f(z) = t$, then

$$\begin{aligned} g(z) &= h(0) + (z - s)(1 - h(0)) \\ &= h(0) + (z - s)(1 - 1) \\ &= h(0) + 0 \\ &= 1. \end{aligned}$$

If $z \in \sigma(a)\,, z \neq s$ and $f(z) \neq t$, then

$$\begin{aligned} g(z) &= h(f(z) - t) + (z - s)(1 - h(f(z) - t)) \\ &= 0 + (z - s)(1 - 0) \\ &= z - s \\ &\neq 0. \end{aligned}$$

Therefore, $0 \notin \sigma\,(q + (a - s\mathbf{1})(\mathbf{1} - q))$ and hence $b = q + (a - s\mathbf{1})(\mathbf{1} - q)$ is invertible. Note that, because T is bounded, Lemma 2.1.13 states that $q \in T^{-1}(0)$ and therefore $c = (a - s\mathbf{1} - \mathbf{1})q \in T^{-1}(0)$. We now have

$$a - s\mathbf{1} = b + c$$

where $b \in A^{-1}$ and $c \in T^{-1}(0)$. Also, these two elements commute, because a and q commute. Therefore, $a - s\mathbf{1}$ is Browder and $s \notin \omega_T^{\mathrm{comm}}\,(a)$. It follows that $t \notin f(\omega_T^{\mathrm{comm}}\,(a))$. □

Corollary 2.3.7. *Let $T : A \to B$ be a Banach algebra homomorphism, $a \in A$ and $f : U \to \mathbb{C}$ a holomorphic function on U, a neighbourhood of $\sigma\,(a)$. If f is non-constant on every component of U and T is bounded, then*

$$\sigma_B\,(T(f(a))) \cup \operatorname{acc} \sigma\,(f(a)) = f(\sigma_B\,(Ta) \cup \operatorname{acc} \sigma\,(a)).$$

Proof. It follows from Proposition 2.3.1 that $\sigma_B\,(T(f(a))) = f(\sigma_B\,(Ta))$. Furthermore, by Lemma 2.3.2, we have $\operatorname{acc} \sigma\,(f(a)) = f(\operatorname{acc} \sigma\,(a))$. The corollary follows. □

Corollary 2.3.8. ([9], Theorem 2 (second part), p. 434) *Let $T : A \to B$ be a bounded Banach algebra homomorphism with the Riesz property. Let $f : U \to \mathbb{C}$ be a holomorphic function on U, a neighbourhood of $\sigma\,(a)$. If f is non-constant on every component of U, then*

$$\omega_T^{\mathrm{comm}}\,(f(a)) = f(\omega_T^{\mathrm{comm}}\,(a)).$$

Proof. This corollary follows immediately from Corollary 2.3.7 and Corollary 2.2.16. It also follows directly from Theorem 2.3.3 together with Corollary 2.2.16 as we show now. From inclusion (2.3.5) and the first inclusion of (2.3.6), we have that

$$\omega_T^{\mathrm{comm}}\,(f(a)) \subseteq f(\omega_T^{\mathrm{comm}}\,(a)) \subseteq \sigma_B\,(T(f(a))) \cup \operatorname{acc} \sigma\,(f(a)).$$

It follows from this and Corollary 2.2.16 that

$$\omega_T^{\mathrm{comm}}\,(f(a)) = f(\omega_T^{\mathrm{comm}}\,(a)).$$ □

We obtained the following corollary after reading a remark of Harte on page 204 of [11] with regard to mobius spectra.

Corollary 2.3.9. *Let $T : A \to B$ be a Banach algebra homomorphism, $a \in A$ and $f : U \to \mathbb{C}$ a holomorphic function on U, a neighbourhood of $\sigma(a)$. If f is non-constant on every component of U and f is an invertible function and f^{-1} is holomorphic on a neighbourhood of $\sigma(f(a))$, then we have the equalities*

$$f(\omega_T(a)) = \omega_T(f(a)); \tag{2.3.10}$$

$$f(\omega_T^{\mathrm{comm}}(a)) = \omega_T^{\mathrm{comm}}(f(a)). \tag{2.3.11}$$

Proof. We prove (2.3.10). Suppose that f is an invertible function and that f^{-1} is holomorphic on a neighbourhood of $\sigma(f(a))$. It follows from inclusion (2.3.4) of Theorem 2.3.3 that

$$\omega_T(f(a)) \subseteq f(\omega_T(a)) \tag{2.3.12}$$

and

$$\omega_T\left(f^{-1}(f(a))\right) \subseteq f^{-1}(\omega_T(f(a))).$$

This implies that

$$f\left(\omega_T\left(f^{-1}(f(a))\right)\right) \subseteq f\left(f^{-1}\left(\omega_T(f(a))\right)\right)$$

which is equivalent to

$$f(\omega_T(a)) \subseteq \omega_T(f(a)).$$

It now follows from the above inclusion and (2.3.12) that

$$f(\omega_T(a)) = \omega_T(f(a)).$$

Equation (2.3.11) follows in exactly the same way from inclusion (2.3.5) of Theorem 2.3.3. ∎

It is natural to investigate whether we could obtain the same one way inclusion that we have for the Weyl and Browder spectra in Theorem 2.3.3 for the Fredholm spectrum when the relevant Banach algebra homomorphism is not required to be bounded. By using the same approach as in the proofs of (2.3.4) and (2.3.5) of Theorem 2.3.3, we found that it is indeed the case. This proposition is weaker than Proposition 2.3.1 in the sense that we only have an inclusion and not equality and it is proved here only for holomorphic functions which are non-constant on each component of a neighbourhood of the spectrum of an element. It is, however, stronger in the sense that the homomorphism is not required to be bounded.

Proposition 2.3.13. *Let $T : A \to B$ be a Banach algebra homomorphism, $a \in A$ and U a neighbourhood of $\sigma(a)$ and $f : U \to \mathbb{C}$ a holomorphic function on U which is non-constant on each component of U. Then we have the inclusion*

$$\sigma_B(T(f(a))) \subseteq f(\sigma_B(Ta)).$$

Proof. If U is not bounded, redefine it so that it is bounded. Suppose that $t \notin f(\sigma_B(Ta))$. We know, from Proposition 2.2.11 and the spectral mapping theorem, that $\sigma_B(T(f(a))) \subseteq \sigma(f(a)) = f(\sigma(a))$. Therefore, if $t \notin f(\sigma(a))$, then $t \notin \sigma_B(T(f(a)))$. We therefore suppose that $t \in f(\sigma(a)) \backslash f(\sigma_B(Ta))$. Now, define the function $h : U \to \mathbb{C}$ by

$$h(z) := f(z) - t$$

for all $z \in U$. As in the proof of inclusion (2.3.4), we can write

$$f(z) - t = h(z) = (z - s_1)^{m_1}(z - s_2)^{m_2} \cdots (z - s_n)^{m_n} g(z),$$

where

$$\begin{aligned}\{s_1, s_2, \ldots, s_n\} &= \{z \in \sigma(a) : h(z) = 0\} \\ &= \{z \in \sigma(a) : f(z) = t\},\end{aligned}$$

$m_1, m_2, \ldots, m_n \in \mathbb{N}$ and $g : U \to \mathbb{C}$ is holomorphic and has no zeros in $\sigma(a)$. So we have

$$h(a) = (a - s_1\mathbf{1})^{m_1}(a - s_2\mathbf{1})^{m_2} \cdots (a - s_n\mathbf{1})^{m_n} g(a)$$

with $g(a)$ invertible. But then $T(g(a)) \in B^{-1}$ and for $j \in \{1, 2, \ldots, n\}$, $s_j \notin \sigma_B(Ta)$ and hence $T(a - s_j\mathbf{1}) \in B^{-1}$. Therefore

$$\begin{aligned}T(f(a) - t\mathbf{1}) &= T(h(a)) \\ &= T((a - s_1\mathbf{1})^{m_1}(a - s_2\mathbf{1})^{m_2} \cdots (a - s_n\mathbf{1})^{m_n} g(a)) \\ &= T(a - s_1\mathbf{1})^{m_1} T(a - s_2\mathbf{1})^{m_2} \cdots T(a - s_n\mathbf{1})^{m_n} T(g(a)) \\ &\in B^{-1},\end{aligned}$$

and hence $t \notin \sigma_B(T(f(a)))$. □

Because we now have Proposition 2.3.13, we show in the next proposition that the second inclusion of (2.3.6) holds when the Banach algebra homomorphism T is not required to be bounded.

Proposition 2.3.14. *Let $T : A \to B$ be a Banach algebra homomorphism, $a \in A$ and $f : U \to \mathbb{C}$ a holomorphic function on U which is non-constant on every component of U, where U is a neighbourhood of $\sigma(a)$. Then*

$$\sigma_B(T(f(a))) \cup \operatorname{acc}\sigma(f(a)) \subseteq f(\sigma_B(Ta) \cup \operatorname{acc}\sigma(a)).$$

Proof. We have $\sigma_B(T(f(a))) \subseteq f(\sigma_B(Ta))$ from Proposition 2.3.13 and the equality $\operatorname{acc}\sigma(f(a)) = f(\operatorname{acc}\sigma(a))$ follows from Lemma 2.3.2. We obtain $\sigma_B(T(f(a))) \cup \operatorname{acc}\sigma(f(a)) \subseteq f(\sigma_B(Ta)) \cup f(\operatorname{acc}\sigma(a))$ which completes the proof. $\square$

Chapter 3

Some Generalisations with Respect to Boundedness

In Chapter 2 we used a bounded Banach algebra homomorphism T in a number of results. In some of these results boundedness of T was only required because Lemma 2.1.13 was used in the proofs. Because of Proposition 2.1 in [7], a result of Grobler and Raubenheimer which was published in 1991, we can now prove Lemma 2.1.13 in the more general case where T is not required to be bounded. In this chapter we show which results can now be stated more generally.

Definition 3.1 (Clopen set)**.** A set $D \subseteq \mathbb{C}$ is *clopen* in a set $K \subseteq \mathbb{C}$ if D is compact and there exists an open set U in $\mathbb{C}$ such that $D \subseteq U$ and $K \cap U \subseteq D$.

We now prove a lemma which is necessary to prove Proposition 3.3 which is Grobler and Raubenheimer's result from [7]. This lemma states a fact which is used, but not proved, in [7].

Lemma 3.2. *Let A be a Banach algebra and $a \in A$. Let D be a clopen set in $\sigma_A(a)$ and define*

$$e := \frac{1}{2\pi i}\int_\Gamma (\lambda\mathbf{1} - a)^{-1}\mathrm{d}\lambda$$

where Γ is a smooth contour surrounding D in $\mathbb{C}\backslash\sigma_A(a)$. Then e is idempotent.

Proof. By the definition of clopen sets, we can find an open set U in $\mathbb{C}$ such that $D \subseteq U$ and $U \cap \sigma_A(a) \subseteq D$. We can choose U in such a way that ∂U is disjoint from $\sigma_A(a)$. Then $\sigma(a) \subseteq U \cup \mathbb{C}\backslash\overline{U}$. Furthermore, we can choose Γ in such a way that $\Gamma \cup \operatorname{ins}\Gamma \subseteq U$. Define the function $f : U \cup \mathbb{C}\backslash\overline{U} \to \mathbb{C}$ by

$$f(\lambda) := \begin{cases} 1 & \text{if } \lambda \in U \\ 0 & \text{if } \lambda \in \mathbb{C}\backslash\overline{U}. \end{cases}$$

Then $f(a)$ exists and $f(a) = e$. We observe that $f(a)$ is idempotent and this completes the proof. □

Proposition 3.3 ([7], Proposition 2.1, p. 11)**.** *Let $T : A \to B$ be a Banach algebra homomorphism and $a \in A$. Let D be a clopen set in both $\sigma_A(a)$ and $\sigma_B(Ta)$ and define*

$$e_A := \frac{1}{2\pi i}\int_\Gamma (\lambda\mathbf{1} - a)^{-1}\mathrm{d}\lambda$$

and

$$e_B := \frac{1}{2\pi i}\int_\Gamma (\lambda\mathbf{1} - Ta)^{-1}\mathrm{d}\lambda,$$

where Γ is a smooth contour surrounding D in $\mathbb{C}\backslash\sigma_A(a)$. Then $Te_A = e_B$.

Proof. Let C_A and C_B be maximal commutative subalgebras of A and B respectively, such that $a \in C_A$ and $Ta \in C_B$ and $T(C_A) \subseteq C_B$. Let f be a multiplicative linear functional on C_B. Then $f \circ T$ is a multiplicative linear functional on C_A. It follows from Theorem 1.10 (2) that $f \circ T$ is bounded. Therefore,

$$\begin{aligned}(f \circ T)(e_A) &= \frac{1}{2\pi i}\int_\Gamma f\Big(T\left((\lambda\mathbf{1} - a)^{-1}\right)\Big)\mathrm{d}\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma f\left((\lambda\mathbf{1} - Ta)^{-1}\right)\mathrm{d}\lambda \\ &= f\left(\frac{1}{2\pi i}\int_\Gamma (\lambda\mathbf{1} - Ta)^{-1}\mathrm{d}\lambda\right) \\ &= f(e_B).\end{aligned} \tag{3.4}$$

Now, by Lemma 3.2, the element e_A is an idempotent in A and therefore Te_A is an idempotent in C_B. It follows from this and equation (3.4) that

$$\begin{aligned}f\big(Te_A - (Te_A)e_B\big) &= f\big(Te_A(Te_A - e_B)\big) \\ &= (f \circ T)(e_A)f(Te_A - e_B) \\ &= (f \circ T)(e_A)\big((f \circ T)(e_A) - f(e_B)\big) \\ &= (f \circ T)(e_A)0 \\ &= 0.\end{aligned}$$

Because f is an arbitrary multiplicative linear functional on C_B, it follows from Theorem 1.10 (2) that $\sigma_{C_B}(Te_A - (Te_A)e_B) = \{0\}$. Also, by Lemma 3.2, we have that e_B is idempotent. Similarly, $\sigma_{C_B}(e_B - (Te_A)e_B) = \{0\}$. We also see that $Te_A - (Te_A)e_B$ and $e_B - (Te_A)e_B$ are idempotents in C_B. Therefore, by Lemma 1.18, they are both 0. It follows that

$$\begin{aligned} Te_A &= (Te_A)e_B \\ &= e_B. \end{aligned}$$

□

Corollary 3.5. *Let A be a Banach algebra, $T : A \to B$ a Banach algebra homomorphism and $a \in A$. If $\lambda \notin \operatorname{acc} \sigma(a)$, then $T(p(a, \lambda)) = p(Ta, \lambda)$.*

Proof. Take any $\lambda \notin \operatorname{acc} \sigma_A(a)$. Since, by Proposition 2.1.10, the inclusion $\sigma_B(Ta) \subseteq \sigma_A(a)$ holds, we have that $\lambda \notin \operatorname{acc} \sigma_B(Ta)$. Let $D := \{\lambda\}$. then D is clopen in $\sigma_A(a)$ as well as in $\sigma_B(Ta)$. It now follows from Proposition 3.3 that $T(p(a, \lambda)) = \frac{1}{2\pi i}\int_{\Gamma_1} (\zeta\mathbf{1} - Ta)^{-1}\, \mathrm{d}\zeta$ where Γ_1 is a smooth contour surrounding $\{\lambda\}$ in $\mathbb{C}\backslash\sigma_A(a)$. By Definition 1.16, we have that $T(p(a, \lambda)) = p(Ta, \lambda)$. □

We now have the following corollary, which is the same as Lemma 2.1.13, except that T is not required to be bounded.

Corollary 3.6. *Let A be a Banach algebra and $T : A \to B$ a Banach algebra homomorphism. If $a \in A$ is an almost invertible Fredholm element and p is the spectral idempotent of a associated with 0, then $Tp = 0$.*

Proof. It follows from Corollary 3.5 that $Tp = p(Ta, 0)$. Since $a \in \mathcal{F}$, we have that $0 \notin \sigma_B(Ta)$ and it follows from Theorem 1.17 that $p(Ta, 0) = 0$. □

In the proof of Theorem 2.1.18 (1), the boundedness of T was only necessary because Lemma 2.1.13 was used in the proof. Using Corollary 3.6 we can now state a more general version of Theorem 2.1.18 as well as of Corollary 2.1.20. This result was obtained by Mouton and Raubenheimer.

Theorem 3.7 ([19], Theorem 2.4, p. 19)**.** *Let $T : A \to B$ be a Banach algebra homomorphism. Then we have the following two properties.*

1. *The inclusion*

$$\{a \in A : a \textit{ is almost invertible in } A\} \cap \mathcal{F} \subseteq \mathcal{B}$$

holds in general.

2. *The homomorphism T has the Riesz property if and only if the equality*

$$\{a \in A : a \text{ is almost invertible in } A\} \cap \mathcal{F} = \mathcal{B}$$

holds.

Corollary 3.8 ([19], Corollary 2.5, p. 19)**.** *If $T : A \to B$ is a Banach algebra homomorphism, then we have that following inclusions:*

$$A^{-1} \subseteq \{a \in A : a \text{ is almost invertible in } A\} \cap \mathcal{F} \subseteq \mathcal{B} \subseteq \mathcal{W} \subseteq \mathcal{F}.$$

In Proposition 2.2.11 boundedness of the Banach algebra homomorphism T is required because Theorem 2.1.18 (1) is used in the proof. Because we now have Theorem 3.7, we state a more general version of this proposition.

Proposition 3.9. *Let $T : A \to B$ be a Banach algebra homomorphism. For any element $a \in A$, the following inclusions hold:*

$$\sigma_B (Ta) \subseteq \omega_T (a) \subseteq \omega_T^{\text{comm}} (a) \subseteq \sigma_B (Ta) \cup \operatorname{acc} \sigma (a) \subseteq \sigma (a) .$$

In Corollary 2.2.16 the Banach algebra homomorphism T is required to be bounded, because Proposition 2.2.11 is used in the proof. We now have Proposition 3.9 and can therefore state the following corollary.

Corollary 3.10. *If $T : A \to B$ is a Banach algebra homomorphism with the Riesz property, then $\omega_T^{\text{comm}} (a) = \sigma_B (Ta) \cup \operatorname{acc} \sigma (a)$.*

We now also have the first inclusion of (2.3.6) without requiring T to be bounded. This is because Lemma 2.1.13 was used in the proof, but we now have Corollary 3.6. This is stated in the following proposition.

Proposition 3.11. *Let $T : A \to B$ be a Banach algebra homomorphism, $a \in A$ and $f : U \to \mathbb{C}$ a holomorphic function on U, a neighbourhood of $\sigma (a)$. Then*

$$f(\omega_T^{\text{comm}} (a)) \subseteq \sigma_B (T(f(a))) \cup \operatorname{acc} \sigma (f(a)).$$

In Corollary 2.3.8 the homomorphism was required to be bounded, because Corollary 2.2.16 and the first inclusion of (2.3.6) were used. Because we now have Corollary 3.10 and Proposition 3.11, we now have the following proposition.

Corollary 3.12. *Let $T : A \to B$ be a Banach algebra homomorphism with the Riesz property, $a \in A$ and $f : U \to \mathbb{C}$ a holomorphic function U, a neighbourhood of $\sigma (a)$. If f is non-constant on every component of U, then*

$$f(\omega_T^{\text{comm}} (a)) = \omega_T^{\text{comm}} (f(a)) .$$

Chapter 4

The Exponential Spectrum of an Element of a Banach Algebra

In this chapter we consider the results in Harte's paper on the exponential spectrum in a Banach algebra setting ([10]). In this article, the exponential spectrum is used to prove an inclusion result which is of interest if the inclusion properties discussed in Chapter 2, especially those given by Proposition 2.2.11, are considered. The main result is that if $T : A \to B$ is a bounded Banach algebra homomorphism with closed range and $a \in A$, then the inclusions $\sigma_B(Ta) \subseteq \bigcap_{Tc=0} \sigma(a+c) \subseteq \eta\sigma_B(Ta)$ hold. This article was published in 1976, while the article in which Harte introduced the concept of Fredholm, Weyl and Browder elements and spectra was only published in 1982. We now know that $\bigcap_{Tc=0} \sigma(a+c) = \omega_T(a)$, so Harte's result is equivalent to saying that the inclusions $\sigma_B(Ta) \subseteq \omega_T(a) \subseteq \eta\sigma_B(Ta)$ hold under the conditions specified above. We also know from Proposition 2.2.4 in Chapter 2 that the inclusion $\sigma_B(Ta) \subseteq \omega_T(a)$ holds in general (neither boundedness of T nor T having a closed range is required). Finding conditions under which the inclusion $\omega_T(a) \subseteq \eta\sigma_B(Ta)$ holds, is therefore what most of Harte's article ([10]) is about.

4.1 The case where T is bounded as well as bounded below

First of all, Harte showed that the inclusion $\omega_T(a) \subseteq \eta\sigma_B(Ta)$ holds when the relevant Banach algebra homomorphism T is bounded as well as bounded below. In order to show that, the singular spectrum of an element of a Banach algebra is introduced. This is similar to what Harte did in [10].

Definition 4.1.1 (Singular spectrum of an element of a Banach algebra)**.** ([10], p. 114). Let a be an element of a Banach algebra A. Then the *singular spectrum* of a, denoted by $\tau(a)$, is defined as

$$\tau(a) := \{\lambda \in \mathbb{C} : a - \lambda\mathbf{1} \text{ is a topological zero-divisor in } A\}.$$

Lemma 4.1.2. ([2], p. 41, Theorem 3.2.11) *Let A be a Banach algebra containing a sequence (a_n). If the sequence (a_n) consists of invertible elements and it converges to a non-invertible element, then $\|a_n^{-1}\| \to \infty$ as $n \to \infty$.*

Lemma 4.1.3. ([2], p. 41, Corollary 3.2.12) *Let x be an element of a Banach algebra A and $\alpha \in \partial\sigma(x)$. Then $x - \alpha\mathbf{1}$ is a topological zero-divisor in A.*

Proof. Because $\alpha \in \partial\sigma(x)$, there exists a sequence (α_n) in $\mathbb{C}\backslash\sigma(x)$ which converges to α. For each $n \in \mathbb{N}$ we define $x_n := \frac{(x-\alpha_n\mathbf{1})^{-1}}{\|(x-\alpha_n\mathbf{1})^{-1}\|}$, so that $\|x_n\| = 1$. We then have

$$\begin{aligned}(x - \alpha\mathbf{1})x_n &= (x - \alpha_n\mathbf{1})x_n - (\alpha - \alpha_n)x_n \\ &= (x - \alpha_n\mathbf{1})\frac{(x - \alpha_n\mathbf{1})^{-1}}{\|(x - \alpha_n\mathbf{1})^{-1}\|} - (\alpha - \alpha_n)x_n \\ &= \frac{1}{\|(x - \alpha_n\mathbf{1})^{-1}\|} - (\alpha - \alpha_n)x_n.\end{aligned}$$

It follows that

$$\begin{aligned}\|(x - \alpha\mathbf{1})x_n\| &\leq \frac{1}{\|(x - \alpha_n\mathbf{1})^{-1}\|} + |\alpha - \alpha_n|\|x_n\| \\ &= \frac{1}{\|(x - \alpha_n\mathbf{1})^{-1}\|} + |\alpha - \alpha_n|.\end{aligned}$$

We now show that the right-hand side of the above inequality converges to 0. The sequence $(x - \alpha_n\mathbf{1})$ consists of invertible elements and it converges to

$x-\alpha\mathbf{1}$ which is not invertible. Therefore, by Lemma 4.1.2, $\frac{1}{\|(x-\alpha_n\mathbf{1})^{-1}\|} \to 0$ as $n \to \infty$. Clearly $|\alpha-\alpha_n| \to 0$ as $n \to \infty$. Hence, the right-hand side of the above inequality and also $\|(x-\alpha\mathbf{1})x_n\|$ converges to 0 as $n \to \infty$.

Because we also have that $x_n(x-\alpha\mathbf{1}) = (\alpha_n-\alpha)x_n + x_n(x-\alpha_n\mathbf{1}) = (\alpha_n-\alpha)x_n + \frac{1}{\|(x-\alpha\mathbf{1})^{-1}\|}$, it follows that $\|x_n(x-\alpha\mathbf{1})\| \to 0$ as $n \to \infty$. □

Lemma 4.1.4. ([10], p. 114) *Let a be an element of a Banach algebra A. Then the following inclusions hold:*

$$\partial\sigma(a) \subseteq \tau(a) \subseteq \sigma(a).$$

Proof. The first inclusion follows directly from Lemma 4.1.3.

The second inclusion follows immediately from the fact that a topological zero-divisor cannot be invertible. □

Lemma 4.1.5. ([10], p. 114–115) *Let $T : A \to B$ be a Banach algebra homomorphism which is bounded as well as bounded below. Then, for any $a \in A$,*

$$\tau_A(a) \subseteq \tau_B(Ta).$$

Proof. Take any $\lambda \in \tau_A(a)$. Then we have by definition that $a-\lambda\mathbf{1}$ is a topological zero-divisor. Hence, there exists a sequence (y_n) in A such that $\|y_n\| = 1$ for each y_n and $y_n(a-\lambda\mathbf{1}) \to 0$ as $n \to \infty$. Because T is bounded, there exists $0 \le c_1 \in \mathbb{R}$ such that $\|Tx\| \le c_1\|x\|$ for all $x \in A$. Because T is bounded below, there exists $0 < c_2 \in \mathbb{R}$ such that $\|Tx\| \ge c_2\|x\|$ for all $x \in A$. Note that $\|Ty_n\| \ge c_2\|y_n\| = c_2 > 0$ for all $n \in \mathbb{N}$. We now consider the sequence $\left(\frac{Ty_n}{\|Ty_n\|}\right)$. It follows from the above and the fact that T is a homomorphism that

$$\begin{aligned}
\left\|\left(\frac{Ty_n}{\|Ty_n\|}\right)(Ta-\lambda\mathbf{1})\right\| &= \left\|\frac{1}{\|Ty_n\|}T(y_n(a-\lambda\mathbf{1}))\right\| \\
&= \frac{\|T(y_n(a-\lambda\mathbf{1}))\|}{\|Ty_n\|} \\
&\le \frac{c_1\|y_n(a-\lambda\mathbf{1}\|)}{c_2\|y_n\|} \\
&= \frac{c_1}{c_2}\|y_n(a-\lambda\mathbf{1}\|) \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Clearly $\left\| \frac{Ty_n}{\|Ty_n\|} \right\| = 1$ for all $n \in \mathbb{N}$.

Now, there also exists a sequence $(x_n) \in A$ such that $\|x_n\| = 1$ for each x_n and $(a - \lambda\mathbf{1})x_n \to 0$ as $n \to \infty$. In exactly the same way as above, it follows that $\left\| (Ta - \lambda\mathbf{1}) \left(\frac{Tx_n}{\|Tx_n\|} \right) \right\| \to 0$ as $n \to \infty$. Also, $\left\| \frac{Tx_n}{\|Tx_n\|} \right\| = 1$ for all $n \in \mathbb{N}$.

Therefore $Ta - \lambda\mathbf{1}$ is a topological zero-divisor and we have that $\lambda \in \tau_B(Ta)$. $\square$

Lemma 4.1.6. ([11], Theorem 1.2, p. 202) *If X and Y are compact sets in $\mathbb{C}$, then the implication*

$$\partial X \subseteq Y \Rightarrow X \subseteq \eta Y$$

holds.

Proof. Suppose that $\partial X \subseteq Y$. Let H be any connected subset of $\mathbb{C} \backslash Y$. Then $H \cap \partial X$ is empty. Therefore, either $X \cap H = \emptyset$ or $X \supseteq H$. If we have the latter inclusion, then H is bounded, because it is contained in a compact set. It follows that $X \cap G = \emptyset$ for the unbounded component G of $\mathbb{C} \backslash Y$. Hence $X \subseteq \eta Y$. $\square$

Theorem 4.1.7. ([10], p. 115) *Let $T : A \to B$ be a Banach algebra homomorphism which is bounded as well as bounded below. Then, for any $a \in A$,*

$$\omega_T(a) \subseteq \eta\sigma_B(Ta).$$

Proof. Let $a \in A$. It follows from Lemma 4.1.4 and Lemma 4.1.5 that

$$\partial\sigma_A(a) \subseteq \tau_A(a) \subseteq \tau_B(Ta) \subseteq \sigma_B(Ta).$$

By applying Lemma 4.1.6 with $X = \sigma_A(a)$ and $Y = \sigma_B(Ta)$, we obtain the inclusion

$$\sigma_A(a) \subseteq \eta\sigma_B(Ta).$$

The result now follows from Proposition 2.2.4. $\square$

4.2 The case where T is bounded and onto

Harte then showed that the result holds when the Banach algebra homomorphism $T : A \to B$ is bounded and onto. In order to do this the exponential spectrum of an element $a \in A$ is introduced.

Definition 4.2.1 (Generalized exponentials of a Banach algebra)**.** ([11], p. 115). Let A be a Banach algebra. The set of generalized exponentials $\operatorname{Exp} A$ is defined by

$$\operatorname{Exp} A = \{e^{c_1}e^{c_2}\cdots e^{c_k} : k \in \mathbb{N} \text{ and } c_1, c_2, \ldots, c_k \in A\}.$$

The following theorem by E.R. Lorch will be accepted without proof.

Theorem 4.2.2. ([2], Theorem 3.3.7, p. 47) *If A is a Banach algebra, then the set* $\operatorname{Exp} A$ *forms the connected component of A^{-1} containing the identity* $\mathbf{1}$.

Corollary 4.2.3. ([11], p. 115) *If A is a Banach algebra, then* $\operatorname{Exp} A$ *is an open set.*

Definition 4.2.4 (Exponential spectrum of an element of a Banach algebra)**.** ([11], Definition 1, p. 115) Let A be a Banach algebra. For every element $a \in A$, the *exponential spectrum* $\varepsilon(a)$ is defined by

$$\varepsilon(a) := \{\lambda \in \mathbb{C} : a - \lambda\mathbf{1} \notin \operatorname{Exp} A\}.$$

Theorem 4.2.5. ([11], Theorem 1, p. 115) *Let A be a Banach algebra. For every $a \in A$ the exponential spectrum $\varepsilon(a)$ is non-empty and compact and the following inclusions hold:*

$$\partial\varepsilon(a) \subseteq \tau(a) \subseteq \sigma(a) \subseteq \varepsilon(a) \subseteq \eta\sigma(a).$$

Proof. Let $a \in A$. Because $\operatorname{Exp} A$ is open, it follows that $\varepsilon(a)$ is closed. The non-emptiness of $\varepsilon(a)$ will follow from the third inclusion of this theorem.

We will now prove that $\varepsilon(a)$ is bounded. Let $\lambda \in \varepsilon(a)$. Then $a - \lambda\mathbf{1} \notin \operatorname{Exp} A$ and $a - \lambda\mathbf{1} = -\lambda(\mathbf{1} - \frac{1}{\lambda}a)$ if $\lambda \neq 0$. Therefore, if $\|a\| < |\lambda|$, then $\mathbf{1} - \frac{1}{\lambda}a \in \mathcal{B}(\mathbf{1}, 1) \subseteq A^{-1}$, by Theorem 1.5. It now follows from Theorem 4.2.2 that $\mathbf{1} - \frac{1}{\lambda}a \in \operatorname{Exp} A$. Also, $-\lambda\mathbf{1}$ is an invertible element in the connected component of A^{-1} containing $\mathbf{1}$ and hence $-\lambda\mathbf{1} \in \operatorname{Exp} A$. This means that $a - \lambda\mathbf{1} \in \operatorname{Exp} A$, which is a contradiction. Therefore, $|\lambda| \leq \|a\|$ and it follows that $\varepsilon(a)$ is bounded.

We have the second inclusion from Lemma 4.1.4. The third inclusion follows directly from the fact that $\operatorname{Exp} A \subseteq A^{-1}$.

We will now prove the first inclusion. Suppose that $\lambda \in \partial\varepsilon(a)$. Then, because $\varepsilon(a)$ is compact, $a - \lambda\mathbf{1} \notin \operatorname{Exp} A$ and there exists a sequence (λ_n) in

$\mathbb{C}\backslash\varepsilon\,(a)$ converging to λ. For each λ_n, we have that $a-\lambda_n\mathbf{1}\in\operatorname{Exp}A\subseteq A^{-1}$. Define $x_n:=(\lambda-\lambda_n)(a-\lambda_n\mathbf{1})^{-1}$ for each $n\in\mathbb{N}$. Now, suppose that

$$\|x_k\|=|\lambda-\lambda_k|\|(a-\lambda_k\mathbf{1})^{-1}\|<1$$

for some $k\in\mathbb{N}$. Then $1-x_k\in\operatorname{Exp}A$, because $\mathcal{B}(\mathbf{1},1)\subseteq A^{-1}$, by Theorem 1.5, and $\mathcal{B}(\mathbf{1},1)$ is contained in the connected component of A^{-1} containing $\mathbf{1}$. But

$$\begin{aligned}(a-\lambda\mathbf{1})(a-\lambda_k\mathbf{1})^{-1}&=\big((a-\lambda_k\mathbf{1})-(\lambda\mathbf{1}-\lambda_k\mathbf{1})\big)(a-\lambda_k\mathbf{1})^{-1}\\&=(a-\lambda_k\mathbf{1})(a-\lambda_k\mathbf{1})^{-1}-(\lambda-\lambda_k)(a-\lambda_k\mathbf{1})^{-1}\\&=1-(\lambda-\lambda_k)(a-\lambda_k\mathbf{1})^{-1}\\&=1-x_k\\&\in\operatorname{Exp}A\end{aligned}$$

and since $(a-\lambda_k\mathbf{1})^{-1}\in\operatorname{Exp}A$, it follows that $a-\lambda\mathbf{1}\in\operatorname{Exp}A$, which is a contradiction. Therefore,

$$\|x_n\|=|\lambda-\lambda_n|\|(a-\lambda_n\mathbf{1})^{-1}\|\geq 1$$

for all $n\in\mathbb{N}$. We now have that

$$\|(a-\lambda_n\mathbf{1})^{-1}\|\geq\frac{1}{|\lambda-\lambda_n|}$$

and therefore $\|(a-\lambda_n\mathbf{1})^{-1}\|\to\infty$ as $n\to\infty$. Now, define

$$b_n:=\frac{(a-\lambda_n\mathbf{1})^{-1}}{\|(a-\lambda_n\mathbf{1})^{-1}\|}$$

for all $n\in\mathbb{N}$. Then $\|b_n\|=1$ for all $n\in\mathbb{N}$ and

$$\begin{aligned}(a-\lambda\mathbf{1})b_n&=(a-\lambda_n\mathbf{1})b_n-(\lambda\mathbf{1}-\lambda_n\mathbf{1})b_n\\&=\frac{1}{\|(a-\lambda_n\mathbf{1})^{-1}\|}-(\lambda\mathbf{1}-\lambda_n\mathbf{1})b_n\end{aligned}$$

and therefore

$$\begin{aligned}\|(a-\lambda\mathbf{1})b_n\|&<\frac{1}{\|(a-\lambda_n\mathbf{1})^{-1}\|}+|\lambda\ -\lambda_n|\\&\to 0\end{aligned}$$

as $n \to \infty$. Similarly $\|b_n(a - \lambda\mathbf{1})\| \to 0$ as $n \to \infty$. Therefore, $(a - \lambda\mathbf{1})$ is a topological divisor of zero, so that $\lambda \in \tau(a)$ and this completes the proof of the first inclusion.

We will now prove the last inclusion. We know that $\varepsilon(a)$ and $\sigma(a)$ are compact sets and it follows from the first and the second inclusions that $\partial\varepsilon(a) \subseteq \sigma(a)$. Therefore, by Lemma 4.1.6, we have that $\varepsilon(A) \subseteq \eta\sigma(a)$. $\square$

Lemma 4.2.6. ([10], p. 115) *If $T : A \to B$ is a bounded Banach algebra homomorphism, then we have the inclusion*

$$T(\operatorname{Exp} A) \subseteq \operatorname{Exp} B. \tag{4.2.7}$$

It follows that, if $a \in A$, then

$$\varepsilon_B(Ta) \subseteq \bigcap_{Tc=0} \varepsilon_A(a + c).$$

Proof. If $a \in \operatorname{Exp} A$, then there exist elements $c_1, c_2, \ldots, c_n$ in A such that $a = e^{c_1}e^{c_2}\cdots e^{c_n}$. Since T is a continuous homomorphism, it follows that $Ta = T(e^{c_1}e^{c_2}\cdots e^{c_n}) = e^{Tc_1}e^{Tc_2}\cdots e^{Tc_n} \in \operatorname{Exp} B$.

Suppose that $\lambda \notin \bigcap_{Tc=0} \varepsilon_A(a + c)$. Then there exists an element $c \in T^{-1}(0)$ such that $a + c - \lambda\mathbf{1} \in \operatorname{Exp} A$. It follows from equation (4.2.7) that $T(a + c - \lambda\mathbf{1}) = Ta - \lambda\mathbf{1} \in \operatorname{Exp} B$ and hence $\lambda \notin \varepsilon_B(Ta)$ $\square$

Theorem 4.2.8. ([10], Theorem 2, p. 115) *If $T : A \to B$ is a bounded Banach algebra homomorphism which is onto, then*

$$T(\operatorname{Exp} A) = \operatorname{Exp} B \tag{4.2.9}$$

and hence

$$\varepsilon_B(Ta) = \bigcap_{Tc=0} \varepsilon_A(a + c). \tag{4.2.10}$$

Proof. Let $T : A \to B$ be a bounded Banach algebra homomorphism which is onto. From Lemma 4.2.6 we have that $T(\operatorname{Exp} A) \subseteq \operatorname{Exp} B$. We will now show that the reverse inclusion $T(\operatorname{Exp} A) \supseteq \operatorname{Exp} B$ follows from the fact that T is onto. Suppose that $b \in \operatorname{Exp} B$. Then $b = e^{d_1}e^{d_2}\cdots e^{d_n}$, where $d_1, d_2, \ldots, d_n \in B$ and $n \in \mathbb{N}$. Because T is onto, there exist $c_1, c_2, \ldots, c_n \in A$ such that $Tc_j = d_j$ for each $j \in \{1, 2, \ldots, n\}$. It follows from this and the fact that T is a bounded homomorphism, that

$$\begin{aligned} b &= e^{d_1}e^{d_2}\cdots e^{d_n} \\ &= e^{Tc_1}e^{Tc_2}\cdots e^{Tc_n} \\ &= Te^{c_1}Te^{c_2}\cdots Te^{c_n} \\ &= T(e^{c_1}e^{c_2}\cdots e^{c_n}) \\ &= Ta, \end{aligned}$$

where $a = e^{c_1}e^{c_2}\cdots e^{c_n} \in \operatorname{Exp} A$. Therefore, $b \in T(\operatorname{Exp} A)$ and this completes the proof of equality (4.2.9).

We know that $\varepsilon_B(Ta) \subseteq \bigcap_{Tc=0} \varepsilon_A(a+c)$ from Lemma 4.2.6. It therefore remains to prove the converse inclusion. Suppose that $\lambda \notin \varepsilon_B(Ta)$. Then $Ta - \lambda\mathbf{1} \in \operatorname{Exp} B = T(\operatorname{Exp} A)$. Therefore, there exists an element $x \in \operatorname{Exp} A$ such that $Ta - \lambda\mathbf{1} = Tx$. But then $x' := x - a + \lambda\mathbf{1} \in T^{-1}(0)$ and $a + x' - \lambda\mathbf{1} = x \in \operatorname{Exp} A$. Hence $\lambda \notin \varepsilon(a + x') \supseteq \bigcap_{Tc=0} \varepsilon(a+c)$ and we have equality 4.2.10. □

Theorem 4.2.11. ([10], p. 116) *If $T : A \to B$ is a bounded homomorphism which is onto, then*

$$\omega_T(a) \subseteq \eta\sigma_B(Ta)$$

for all $a \in A$.

Proof. For all $c \in T^{-1}(0)$, we have from Theorem 4.2.5 that $\sigma(a+c) \subseteq \varepsilon(a+c)$. It follows that

$$\begin{aligned} \bigcap_{Tc=0} \sigma(a+c) &\subseteq \bigcap_{Tc=0} \varepsilon(a+c) \\ &= \varepsilon_B(Ta) \\ &\subseteq \eta\sigma_B(Ta), \end{aligned}$$

where the equality follows from Theorem 4.2.8 and the last inclusion from Theorem 4.2.5. We know from (2.2.8) that $\omega_T(a) = \bigcap_{Tc=0} \sigma(a+c)$ and therefore the proof is complete. □

Example 4.2.12. Let I be a two-sided, inessential ideal of a Banach algebra A and $T : A \to A/\overline{I}$ the Banach algebra homomorphism which maps each element of A to its coset in $A/\overline{I}$. Then T is bounded and onto. It follows from Theorem 4.2.11

$$\omega_T(a) \subseteq \eta\sigma_{A/\overline{I}}\left(a + \overline{I}\right)$$

for all $a \in A$.

4.3 The case where T is bounded and has closed range

In this section Theorem 4.1.7 and Theorem 4.2.11 are used to prove the main result.

Lemma 4.3.1. *If $T : A \to B$ is a bounded homomorphism which is injective and has closed range, then T is bounded below.*

Proof. Because T has closed range, $T(A)$ is a complete subspace of B and therefore a Banach algebra in itself. If we consider the mapping of T onto its range and call it T^*, it follows from the open mapping theorem that $T^* : A \to T(A)$ is an open mapping. We also know that T^* is bijective and therefore invertible. Hence, the inverse mapping of T^* is bounded. Therefore, there exists a constant $k > 0$ such that $\|T^{*-1}b\| \leq k\|b\|$ for all $b \in B$. We now have that

$$\|a\| = \|T^{*-1}T^*a\| \leq k\|T^*a\| = k\|Ta\|,$$

for all $a \in A$, which means that

$$\|Ta\| \geq \frac{1}{k}\|a\|$$

for all $a \in A$. $\square$

Theorem 4.3.2. ([10], Theorem 3, p. 116) *If $T : A \to B$ is a bounded homomorphism which has closed range and $a \in A$, then the inclusions*

$$\sigma_B(Ta) \subseteq \omega_T(a) \subseteq \eta\sigma_B(Ta)$$

hold.

Proof. We only need to prove the last inclusion. We factorise T as

$$A \xrightarrow{T_1} A/T^{-1}(0) \xrightarrow{T_2} B,$$

where T_1 is the natural mapping $a \mapsto a + T^{-1}(0)$ and T_2 is the mapping $a + T^{-1}(0) \mapsto Ta$. Then T_1 is onto, and T_2 is one-to-one. Let $a \in A$. Since

$$\begin{aligned}
\omega_{T_1}(a) &= \bigcap_{T_1c=T^{-1}(0)} \sigma(a+c) \\
&= \bigcap_{Tc=0} \sigma(a+c) \\
&= \omega_T(a),
\end{aligned}$$

it follows from Theorem 4.2.11 that

$$\omega_T(a) \subseteq \eta\sigma_{A/T^{-1}(0)}\left(a + T^{-1}(0)\right). \tag{4.3.3}$$

Because T is bounded and has closed range, T_2 is also bounded with closed range. We therefore have from Lemma 4.3.1 that T_2 is bounded below. We also have

$$\begin{aligned} \omega_{T_2}\left(a + T^{-1}(0)\right) &= \bigcap_{T_2(c+T^{-1}(0))=0} \sigma_{A/T^{-1}(0)}\left(a + c + T^{-1}(0)\right) \\ &= \sigma_{A/T^{-1}(0)}\left(a + T^{-1}(0)\right), \end{aligned}$$

for all $a \in A$, where the second equality follows from the fact that T_2 is one-to-one. Hence, by Theorem 4.1.7,

$$\sigma_{A/T^{-1}(0)}\left(a + T^{-1}(0)\right) \subseteq \eta\sigma_B(Ta). \tag{4.3.4}$$

We now obtain from the inclusions (4.3.3) and (4.3.4) that

$$\begin{aligned} \omega_T(a) &\subseteq \eta\sigma_{A/T^{-1}(0)}\left(a + T^{-1}(0)\right) \\ &\subseteq \eta\eta\sigma_B(Ta) \\ &= \eta\sigma_B(Ta). \end{aligned}$$

□

Chapter 5

Fredholm Theory Relative to two Banach Algebra Homomorphisms

In this chapter we consider properties of different types of spectra relative to two Banach algebra homomorphisms, as was done in [18] by Mouton and Raubenheimer as well as in [12] by Harte.

Theorem 5.1 ([18], Theorem 3.1, p. 377)**.** *Let $T : A \to B$ and $S : A \to D$ be Banach algebra homomorphisms. If S has the Riesz property and $a \in \mathcal{F}_T$, then we have the following two implications:*

1. *if $a \in \mathcal{B}_S$, then $a \in \mathcal{B}_T$;*
2. *if $a \in \mathcal{W}_S$, then $a \in \mathcal{W}_T$.*

Proof. 1. Suppose that $a \in \mathcal{B}_S$. Because S has the Riesz property, it follows from Theorem 2.1.18 (2) that a is almost invertible in A. So, by our assumption, a is almost invertible Fredholm with respect to T. It follows from Theorem 3.7 (1) that $a \in \mathcal{B}_T$. This proves the first implication.

2. Suppose that $a \in \mathcal{W}_S$. Then $a = b + c$ for some $b \in A^{-1}$ and $c \in S^{-1}(0)$. Then we have that

$$ab^{-1} = \mathbf{1} + cb^{-1}.$$

Hence, $ab^{-1} \in \mathcal{B}_S$. It now follows from inclusion (1) that $ab^{-1} \in \mathcal{B}_T$. Therefore, $ab^{-1} = e + f$ for some $e \in A^{-1}$ and $f \in T^{-1}(0)$. It follows that $a = eb + fb$, but clearly $eb \in A^{-1}$ and $fb \in T^{-1}(0)$ and hence $a \in \mathcal{W}_T$. □

Corollary 5.2 ([18], Corollary 3.2, p. 378)**.** *Let $T : A \to B$ and $S : A \to D$ be Banach algebra homomorphisms. If S has the Riesz property, $a \in A$ and $\sigma_B(Ta) \subseteq \sigma_D(Sa)$, then the following two inclusions hold:*

$$\omega_T^{\mathrm{comm}}(a) \subseteq \omega_S^{\mathrm{comm}}(a); \tag{5.3}$$

$$\omega_T(a) \subseteq \omega_S(a). \tag{5.4}$$

Proof. Take any $\lambda \notin \omega_S^{\mathrm{comm}}(a)$. Then $a - \lambda\mathbf{1} \in \mathcal{B}_S$. It follows from Proposition 2.2.4 that $\lambda \notin \sigma_D(Sa)$. Therefore, because $\sigma_B(Ta) \subseteq \sigma_D(Sa)$, we have that $\lambda \notin \sigma_B(Ta)$ and hence $a - \lambda\mathbf{1} \in \mathcal{F}_T$. It now follows from Theorem 5.1 (1) that $a - \lambda\mathbf{1} \in \mathcal{B}_T$ which means that $\lambda \notin \omega_T^{\mathrm{comm}}(a)$. This proves the first inclusion.

We now prove the second inclusion. Take any $\lambda \notin \omega_S(a)$. Then $a - \lambda\mathbf{1} \in \mathcal{W}_S$. It follows from Proposition 2.2.4 that $\lambda \notin \sigma_D(Sa)$. Therefore, $\lambda \notin \sigma_B(Ta)$ and we have that $a - \lambda\mathbf{1} \in \mathcal{F}_T$. From Theorem 5.1 (2), we see that $a - \lambda\mathbf{1} \in \mathcal{W}_T$ and we obtain that $\lambda \notin \omega_T(a)$. □

The following corollary follows directly from the previous one. It states the interesting fact that, when two operators both have the Riesz property and the respective Fredholm spectra of an element are equal, then so are the Weyl and Browder spectra.

Corollary 5.5 ([18], Corollary 3.3, p. 378)**.** *Let $T : A \to B$ and $S : A \to D$ be Banach algebra homomorphisms. If T and S have the Riesz property, $a \in A$ and $\sigma_B(Ta) = \sigma_D(Sa)$ then*

$$\omega_T^{\mathrm{comm}}(a) = \omega_S^{\mathrm{comm}}(a)$$

and

$$\omega_T(a) = \omega_S(a).$$

The following theorem is an interesting result of Mouton and Raubenheimer. It shows that, when we have two Banach algebra homomorphisms with the same domain, then, under certain conditions, the spectrum of an element in the domain can be written as the union of the two respective Fredholm spectra.

Theorem 5.6 ([18], Theorem 3.4, p. 378)**.** *Let $T : A \to B$ and $S : A \to D$ be Banach algebra homomorphisms such that T is injective and S is surjective. If $T(S^{-1}(0))$ is a two-sided ideal in B, then*

$$\sigma_A(a) = \sigma_D(Sa) \cup \sigma_B(Ta)$$

for all $a \in A$.

Proof. It is only necessary to prove that

$$A^{-1} = T^{-1}(B^{-1}) \cap S^{-1}(D^{-1}).$$

The inclusion $A^{-1} \subseteq T^{-1}(B^{-1}) \cap S^{-1}(D^{-1})$ is trivial, because, by Proposition 2.1.7, we have that $A^{-1} \subseteq \mathcal{F}_T$ and $A^{-1} \subseteq \mathcal{F}_S$.

We now prove the converse inclusion. Take any $a \in T^{-1}(B^{-1}) \cap S^{-1}(D^{-1})$. Because $a \in S^{-1}(D^{-1})$, and S is a surjective homomorphism, there exists an element $b \in A$ such that $SaSb = SbSa = S(ab) = \mathbf{1}$. Therefore, $S(ab - \mathbf{1}) = 0$. Define $c := ab - \mathbf{1} \in S^{-1}(0)$. We then have that

$$ab = \mathbf{1} + c.$$

Now, because T is a homomorphism, it follows that

$$TaTb = \mathbf{1} + Tc. \tag{5.7}$$

Also, because $a \in T^{-1}(B^{-1})$, there exists an element $y \in B$ such that $yTa = (Ta)y = \mathbf{1}$. By left multiplication by y in equation (5.7), we obtain

$$Tb = y + yTc$$

and it follows that

$$y = Tb - yTc. \tag{5.8}$$

Because $Tc \in T(S^{-1}(0))$ which is a two-sided ideal in B, we have that $yTc \in T(S^{-1}(0)) \subseteq T(A)$. It follows from this and equation (5.8) that $y \in T(A)$. We can therefore find an element $x \in A$ such that $y = Tx$ and then we have that $TxTa = TaTx = T(ax) = T(xa) = \mathbf{1}$. Since T is injective, $ax = xa = \mathbf{1}$ and hence $a \in A^{-1}$. □

The following theorem together with its corollary is a more general result of Harte which was published in the same year.

Theorem 5.9 ([12], Theorem 5, p. 85)**.** *Let $T : A \to B$ and $S : A \to D$ be ring homomorphisms which are respectively injective and surjective. If there is a two-sided ideal $J \subseteq A$ satisfying the inclusions*

$$BT(J) + T(J)B \subseteq T(J) \tag{5.10}$$

and

$$\mathbf{1} + S^{-1}(0) \subseteq A^{-1} + J, \tag{5.11}$$

then

$$A^{-1} = T^{-1}(B^{-1}) \cap S^{-1}(D^{-1}).$$

Proof. The inclusion $A^{-1} \subseteq T^{-1}(B^{-1}) \cap S^{-1}(D^{-1})$ is trivial, as we saw in the proof of Theorem 5.6.

We now prove the inclusion

$$T^{-1}(B^{-1}_{\text{left}}) \cap S^{-1}(D^{-1}_{\text{right}}) \subseteq A^{-1}_{\text{left}}, \tag{5.12}$$

where X^{-1}_{left} and X^{-1}_{right} denote the respective sets of left and right invertible elements of the ring X. Suppose that $a \in T^{-1}(B^{-1}_{\text{left}}) \cap S^{-1}(D^{-1}_{\text{right}})$. Then there exist elements $b \in B$ and $d \in D$ such that

$$bTa = \mathbf{1} = T\mathbf{1} \tag{5.13}$$

and

$$(Sa)d = \mathbf{1} = S\mathbf{1}.$$

Because S is surjective, there is an element $y \in A$ such that $d = Sy$. Therefore, $S(ay - \mathbf{1}) = 0$, so that $ay - \mathbf{1} \in S^{-1}(0)$. Thus $ay \in \mathbf{1} + S^{-1}(0)$. It now follows from inclusion (5.11) that $ay = x + z$ for some $x \in A^{-1}$ and $z \in J$. By inclusion (5.10), the inclusion $BT(J) \subseteq T(J)$ also holds. Therefore, we can find an element $u \in J$ such that $bTz = Tu$. It follows that

$$\begin{aligned} Ty &= \mathbf{1}Ty \\ &= bTaTy \\ &= bT(ay) \\ &= bT(x+z) \\ &= bTx + Tu \end{aligned}$$

and therefore

$$bTx = Ty - Tu.$$

Now, because $x \in A^{-1}$, we have that

$$b = T((y-u)x^{-1}).$$

By this and equation (5.13) we have that

$$T((y-u)x^{-1}a) = bTa = T\mathbf{1}.$$

Hence, because T is injective, $(y-u)x^{-1}a = \mathbf{1}$ which means that a is left invertible. This proves inclusion (5.12).

We now prove the inclusion

$$T^{-1}(B^{-1}_{\text{right}}) \cap S^{-1}(D^{-1}_{\text{left}}) \subseteq A^{-1}_{\text{right}}. \tag{5.14}$$

We define new multiplication operators on the rings A, B and D, such that the new product $a \times b$ is equal to the old product ba for any two elements b and a of A, B or D respectively. Then inclusion (5.12) with respect to the new multiplication operators is equivalent to inclusion (5.14) with respect to the old multiplication operators. Since inclusions (5.10) and (5.11) still hold with respect to the new multiplication operators, inclusion (5.12) is also still satisfied. Therefore, inclusion (5.14) holds.

Now, take any $a \in T^{-1}(B^{-1}) \cap S^{-1}(D^{-1})$. Then a is in the left-hand sides of both inclusion (5.12) and inclusion (5.14). Therefore a is left and right invertible in A and hence $a \in A^{-1}$. □

Corollary 5.15 ([12], p. 86)**.** *If the homomorphisms in the above theorem are between Banach algebras, then*

$$\sigma_A(a) = \sigma_B(Ta) \cup \sigma_D(Sa)$$

for all $a \in A$.

Remark 5.16. Harte required the homomorphism S and T to be bounded in the above corollary, but the boundedness is of no relevance here and therefore not necessary.

Remark 5.17. If T and S are Banach algebra homomorphisms which are respectively injective and surjective and $T(S^{-1}(0))$ is a two-sided ideal in B, we see that if we define $J := S^{-1}(0)$, then the inclusions (5.10) and (5.11) are satisfied. Therefore, by the above corollary, $\sigma_A(a) = \sigma_B(Ta) \cup \sigma_D(Sa)$ and thus Theorem 5.6 is a special case of this above result of Harte. However, we find that the statement of Theorem 5.6 is simpler and more elegant.

Chapter 6

Ruston and Almost Ruston Elements

In this chapter we show how Mouton and Raubenheimer introduced Ruston and almost Ruston elements relative to a bounded Banach algebra homomorphism in [19]. We illustrate some relations to Weyl and Browder elements as well as to Riesz elements as was done by Mouton and Raubenheimer in [19].

In this chapter we only consider bounded Banach algebra homomorphisms. That is because we consider the properties of elements in the Banach algebra $A/T^{-1}(0)$ where A is a Banach algebra. It is necessary that $T^{-1}(0)$ is closed for $A/T^{-1}(0)$ to be a Banach algebra (otherwise the norm is not well defined) and the kernel of a bounded homomorphism is always closed. Of course $A/\overline{T^{-1}(0)}$ is always a Banach algebra, but we will see that our proofs are often only sufficient for the cases where $A/T^{-1}(0)$ is a Banach algebra.

We will use the notation $\overline{x} := x + T^{-1}(0)$.

6.1 Introducing Ruston and almost Ruston elements

Definition 6.1.1 (Riesz elements)**.** Let $T : A \to B$ be a bounded Banach algebra homomorphism. An element $a \in A$ is called Riesz if the spectrum of the element $a + T^{-1}(0)$ in the quotient algebra $A/T^{-1}(0)$ is $\{0\}$. We denote the set of Riesz elements relative to the ideal $T^{-1}(0)$ by $\mathscr{R}_T$.

Definition 6.1.2 (Ruston and almost Ruston elements)**.** ([19], Definition

3.1). Let $T : A \to B$ be a bounded Banach algebra homomorphism. We define an element $a \in A$ as

- *Ruston* with respect to T if $a = b + c$ for some $b \in A^{-1}$ and $c \in \mathscr{R}_T$ such that $bc = cb$;
- *almost Ruston* with respect to T if $a = b + c$ for some $b \in A^{-1}$ and $c \in \mathscr{R}_T$ such that $TbTc = TcTb$.

We denote the set of Ruston elements by $\mathcal{R}$ and the set of almost Ruston elements by $\mathcal{R}_{\text{alm}}$. We will only state the relevant homomorphism explicitly if there could otherwise be confusion.

The following proposition illustrates inclusion properties of Ruston and almost Ruston elements in relation to Weyl and Browder elements.

Proposition 6.1.3 ([19], p. 20)**.** *Let $T : A \to B$ be a bounded Banach algebra homomorphism and $a \in A$. Then the following inclusions hold:*

$$\begin{array}{ccccc} & \rotatebox{45}{$\subseteq$} & \mathcal{R} & \rotatebox{-45}{$\subseteq$} & \\ \mathcal{B} & & & & \mathcal{R}_{\text{alm}}. \\ & \rotatebox{-45}{$\subseteq$} & \mathcal{W} & \rotatebox{45}{$\subseteq$} & \end{array}$$

Proof. We first prove the inclusion $\mathcal{B} \subseteq \mathcal{R}$. Let $a \in \mathcal{B}$. Then $a = b + c$ for some $b \in A^{-1}$ and $c \in T^{-1}(0)$ such that $bc = cb$. So $c + T^{-1}(0) = 0 + T^{-1}(0)$ and hence $\sigma_{A/T^{-1}(0)}\left(c + T^{-1}(0)\right) = \{0\}$ which means that $c \in \mathscr{R}_T$. Therefore $a \in \mathcal{R}$.

It is clear from Definition 6.1.2 that $\mathcal{R} \subseteq \mathcal{R}_{\text{alm}}$, since $bc = cb$ implies that $TbTc = TcTb$ for all $b, c \in A$.

We have $\mathcal{B} \subseteq \mathcal{W}$ from Proposition 2.1.7 and it remains to prove the inclusion $\mathcal{W} \subseteq \mathcal{R}_{\text{alm}}$. Let $a \in \mathcal{W}$. This means that $a = b + c$ for some $b \in A^{-1}$ and $c \in T^{-1}(0)$. We have, as in the first part of this proof, that $c \in \mathscr{R}_T$. Furthermore, $TbTc = (Tb)0 = 0(Tb) = TcTb$ and therefore $a \in \mathcal{R}_{\text{alm}}$. $\square$

Proposition 6.1.4 ([19], p. 20)**.** *Let $T : A \to B$ be a bounded Banach algebra homomorphism. If the quotient algebra $A/T^{-1}(0)$ is commutative, then the set $\mathcal{R}_{\text{alm}}$ is closed under finite products.*

In particular, if $a_1 = b_1 + c_1$ and $a_2 = b_2 + c_2$ for some elements $b_1, b_2 \in A^{-1}$ and $c_1, c_2 \in \mathscr{R}_T$ such that $Tb_1Tc_1 = Tc_1Tb_1$ and $Tb_2Tc_2 = Tc_2Tb_2$, then $a_1a_2 = b_1b_2 + b_1c_2 + c_1b_2 + c_1c_2$ with $b_1b_2 \in A^{-1}$ and $b_1c_2 + c_1b_2 + c_1c_2 \in \mathscr{R}_T$ and $T(b_1b_2)T(b_1c_2 + c_1b_2 + c_1c_2) = T(b_1c_2 + c_1b_2 + c_1c_2)T(b_1b_2)$.

Proof. It suffices to prove that the proposition holds for binary products. Suppose that the quotient algebra $A/T^{-1}(0)$ is commutative. Take any two elements $a_1, a_2 \in \mathcal{R}_{\text{alm}}$. Then $a_1 = b_1 + c_1$ and $a_2 = b_2 + c_2$ for some elements $b_1, b_2 \in A^{-1}$ and $c_1, c_2 \in \mathscr{R}_T$ such that $Tb_1Tc_1 = Tc_1Tb_1$ and $Tb_2Tc_2 = Tc_2Tb_2$. So, we have

$$a_1a_2 = b_1b_2 + b_1c_2 + c_1b_2 + c_1c_2.$$

We see that $b_1b_2 \in A^{-1}$ and we will show that $b_1c_2 + c_1b_2 + c_1c_2 \in \mathscr{R}_T$. Then, since $A/T^{-1}(0)$ is commutative, it follows from Theorem 1.9 that

$$\begin{aligned} \rho(\overline{b_1c_2 + c_1b_2 + c_1c_2}) &\leq \rho(\overline{b_1})\rho(\overline{c_2}) + \rho(\overline{c_1})\rho(\overline{b_2}) + \rho(\overline{c_1})\rho(\overline{c_2}) \\ &= 0, \end{aligned}$$

because $\rho(\overline{c_1}) = \rho(\overline{c_2}) = 0$. Therefore,

$$\sigma_{A/T^{-1}(0)}\left(b_1c_2 + c_1b_2 + c_1c_2 + T^{-1}(0)\right) = \{0\}$$

and hence $b_1c_2 + c_1b_2 + c_1c_2 \in \mathscr{R}_T$. Furthermore, since $A/T^{-1}(0)$ is commutative,

$$T(b_1b_2)T(b_1c_2 + c_1b_2 + c_1c_2) = T(b_1c_2 + c_1b_2 + c_1c_2)T(b_1b_2). \tag{6.1.5}$$

It follows that $a_1a_2 \in \mathcal{R}_{\text{alm}}$. □

Remark 6.1.6. If we do not require T to be bounded and consider the Banach algebra $A/\overline{T^{-1}(0)}$, then the above proof will not be valid any more. We will not obtain (6.1.5), since, at that step in the proof, we will only have that $x := (b_1b_2)(b_1c_2 + c_1b_2 + c_1c_2) - (b_1c_2 + c_1b_2 + c_1c_2)(b_1b_2) \in \overline{T^{-1}(0)}$. So, there will be a sequence in $T^{-1}(0)$ converging to x, but since we cannot assume that T is continuous, we will not be able to conclude that $x \in T^{-1}(0)$.

Theorem 6.1.7 ([19], Theorem 3.2, p. 20)**.** *Let A be a Banach algebra and $T : A \to B$ a bounded Banach algebra homomorphism. If $a \in A$ is almost Ruston, then $a + T^{-1}(0)$ is invertible in the algebra $A/T^{-1}(0)$.*

Proof. Take any $a \in \mathcal{R}_{\text{alm}}$. Then $a = b + c$ for some $b \in A^{-1}$ and $c \in \mathscr{R}_T$ such that $TbTc = TcTb$. The elements $b + T^{-1}(0)$ and $c + T^{-1}(0)$ commute in the Banach algebra $A/T^{-1}(0)$, because $T(bc - cb) = TbTc - TcTb = 0$.

Furthermore, $\sigma_{A/T^{-1}(0)}\left(c + T^{-1}(0)\right) = \{0\}$, since $c \in \mathscr{R}_T$. It follows from this and Theorem 1.9 that

$$\begin{aligned}
\sigma_{A/T^{-1}(0)}\big(a + T^{-1}(0)\big) &= \sigma_{A/T^{-1}(0)}\big(b + c + T^{-1}(0)\big) \\
&\subseteq \sigma_{A/T^{-1}(0)}\big(b + T^{-1}(0)\big) + \sigma_{A/T^{-1}(0)}\big(c + T^{-1}(0)\big) \\
&= \sigma_{A/T^{-1}(0)}\big(b + T^{-1}(0)\big),
\end{aligned}$$

but $0 \notin \sigma_{A/T^{-1}(0)}\left(b + T^{-1}(0)\right)$, since $b + T^{-1}(0)$ is invertible in $A/T^{-1}(0)$. Hence $0 \notin \sigma_{A/T^{-1}(0)}\left(a + T^{-1}(0)\right)$, and therefore $a + T^{-1}(0)$ is invertible in the Banach algebra $A/T^{-1}(0)$. □

Corollary 6.1.8 ([19], Corollary 3.3, p. 20)**.** *Let $T : A \to B$ be a bounded Banach algebra homomorphism. Then we have the inclusion*

$$\mathcal{R}_{\text{alm}} \subseteq \mathcal{F}.$$

Proof. Suppose that $a \in \mathcal{R}_{\text{alm}}$. Then, by Theorem 6.1.7, we have that $a + T^{-1}(0)$ is invertible in $A/T^{-1}(0)$. Therefore there exists an element $b \in A$ such that $ab + T^{-1}(0) = ba + T^{-1}(0) = \mathbf{1} + T^{-1}(0)$. Hence

$$T(ab - \mathbf{1}) = T(ba - \mathbf{1}) = 0. \tag{6.1.9}$$

It follows that $TaTb = TbTa = \mathbf{1}$, so that Ta is invertible in B. We see that $a \in \mathcal{F}$, which completes the proof. □

Remark 6.1.10. The above proof would not hold if T were not bounded. We would not obtain (6.1.9), since we would only have that $ab - \mathbf{1} \in \overline{T^{-1}(0)}$. Similar to what we stated in Remark 6.1.6, there would be a sequence in $T^{-1}(0)$ converging to $ab - \mathbf{1}$, but that would not necessarily imply that $ab - \mathbf{1} \in T^{-1}(0)$.

The following corollary is obtained directly from Corollary 2.1.20, Proposition 6.1.3 and Corollary 6.1.8.

Corollary 6.1.11 ([19], p. 21)**.** *Let $T : A \to B$ be a bounded Banach algebra homomorphism. Then we have the inclusions*

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & \mathcal{R} \arrow[dr, phantom, "\subsetneq"] & \\
A^{-1} \subseteq X \cap \mathcal{F} \subseteq \mathcal{B} \arrow[urr, phantom, "\subsetneq"] \arrow[drr, phantom, "\subsetneq"] & & & & \mathcal{R}_{\text{alm}} \subseteq \mathcal{F}, \\
 & & & \mathcal{W} \arrow[ur, phantom, "\subsetneq"] &
\end{tikzcd}$$

where $X := \{a \in A : a \text{ is almost invertible in } A\}$.

The following theorem is an interesting result of Mouton and Raubenheimer in [19] which leads to a corollary which expands Theorem 2.1.18 for the case where a bounded Banach algebra homomorphism with the Riesz property is considered. In the theorem below, we added the reverse implication and we give a proof which is analogous to the last part of the proof of Theorem 2.1.18 (2).

Theorem 6.1.12 ([19], Theorem 3.4, p. 21). *Let $T : A \to B$ be a bounded Banach algebra homomorphism. Then T has the Riesz property if and only if every Ruston element is almost invertible.*

Proof. Suppose that T has the Riesz property and take any $a \in \mathcal{R}$. Then $a = b + c$ for some elements $b \in A^{-1}$ and $c \in \mathscr{R}_T$ such that $bc = cb$. By multiplying a from the right by the inverse b^{-1} of b, we obtain

$$ab^{-1} = \mathbf{1} + cb^{-1}. \tag{6.1.13}$$

Since T has the Riesz property, $T^{-1}(0)$ is an inessential ideal of A. Since $c \in \mathscr{R}_T$, we have $\sigma_{A/T^{-1}(0)}\left(c + T^{-1}(0)\right) = \{0\}$. Because $c + T^{-1}(0)$ and $b^{-1} + T^{-1}(0)$ commute in the Banach algebra $A/T^{-1}(0)$, we have by Theorem 1.9 that

$$\begin{aligned}\sigma_{A/T^{-1}(0)}\left(cb^{-1} + T^{-1}(0)\right) &\subseteq \sigma_{A/T^{-1}(0)}\left(c + T^{-1}(0)\right)\sigma_{A/T^{-1}(0)}\left(b^{-1} + T^{-1}(0)\right)\\ &= \{0\}.\end{aligned}$$

It therefore follows from Theorem 1.19 that $\operatorname{acc}\sigma\left(cb^{-1}\right) \subseteq \{0\}$. By applying the spectral mapping theorem to the right hand side of equation (6.1.13), we obtain the inclusion $\operatorname{acc}\sigma\left(ab^{-1}\right) \subseteq \{1\}$ and hence $0 \notin \operatorname{acc}\sigma\left(ab^{-1}\right)$, so that ab^{-1} is almost invertible in A. Therefore a is the product of an almost invertible element and an invertible element. Furthermore, ab^{-1} and b commute, because b and c commute. Therefore, by Lemma 2.1.15, we have that a is almost invertible in A.

Conversely, suppose that every Ruston element is almost invertible. It can now be seen that T is Riesz from Theorem 2.1.18 (2) and the fact that, by Corollary 6.1.11, every Browder element is almost invertible Fredholm. We also prove it directly from the definitions. Take any $\lambda \neq 0$ in $\mathbb{C}$ and any $c \in T^{-1}(0)$. Consider the element $c - \lambda\mathbf{1}$. We see that $-\lambda\mathbf{1} \in A^{-1}$ and $-\lambda\mathbf{1}$ commutes with all elements of A. Furthermore, $c \in \mathscr{R}_T$, because $c + T^{-1}(0) = 0 + T^{-1}(0)$. Therefore, $c - \lambda\mathbf{1}$ is Ruston and hence, by assumption, $c - \lambda\mathbf{1}$ is almost invertible. This means that $0 \notin \operatorname{acc}\sigma(c-\lambda\mathbf{1})$ and it then follows from the spectral mapping theorem that $\lambda \notin \operatorname{acc}\sigma(c)$. Therefore, $\operatorname{acc}\sigma(c) \subseteq \{0\}$. We conclude that $T^{-1}(0)$ is an inessential ideal of A and therefore T has the Riesz property. □

The following corollary is found in [19], but only states that the condition of T being Riesz implies the equalities $X \cap \mathcal{F} = \mathcal{B} = \mathcal{R}$. We see that we also have the converse implication.

Corollary 6.1.14 ([19], Corollary 3.6, p. 21)**.** *Let $T : A \to B$ be a bounded Banach algebra homomorphism. Then T has the Riesz property if and only if*

$$X \cap \mathcal{F} = \mathcal{B} = \mathcal{R},$$

where $X := \{a \in A : a \text{ is almost invertible in } A\}$.

Proof. If T has the Riesz property, the equalities $X \cap \mathcal{F} = \mathcal{B} = \mathcal{R}$ follow directly from Corollary 6.1.11 and Theorem 6.1.12.

Conversely, if the equalities $X \cap \mathcal{F} = \mathcal{B} = \mathcal{R}$ hold, then it follows from Theorem 2.1.18 (2) that T is Riesz. □

Corollary 6.1.15 ([19], Corollary 3.7, p. 21)**.** *Let $T : A \to B$ be a bounded Banach algebra homomorphism. Then, if T has the Riesz property, the equality $\mathcal{R}_{\mathrm{alm}} = \mathcal{W}$ holds.*

Proof. We already know, by Proposition 6.1.3, that $\mathcal{W} \subseteq \mathcal{R}_{\mathrm{alm}}$.

We now prove the converse inclusion. Suppose that T has the Riesz property. Take any element $a \in \mathcal{R}_{\mathrm{alm}}$. Then $a = b + c$ for some $b \in A^{-1}$ and $c \in \mathscr{R}_T$ such that $TbTc = TcTb$. By multiplying a from the right by b^{-1}, we obtain

$$ab^{-1} = \mathbf{1} + cb^{-1}.$$

Since Tc and Tb^{-1} commute, $cb^{-1} - b^{-1}c \in T^{-1}(0)$, so $c + T^{-1}(0)$ and $b^{-1} + T^{-1}(0)$ commute in the Banach algebra $A/T^{-1}(0)$. Therefore, by Theorem 1.9, we have $\rho(\overline{cb^{-1}}) \leq \rho(\overline{c})\rho(\overline{b^{-1}}) = 0$, so that $cb^{-1} \in \mathscr{R}_T$. Hence $ab^{-1} = \mathbf{1} + cb^{-1} \in \mathcal{R}$. It follows from Corollary 6.1.14 and Proposition 2.1.7 that $ab^{-1} \in \mathcal{B} \subseteq \mathcal{W}$. Therefore, $a = ab^{-1}b$ is the product of a Weyl element and an invertible element and it follows that a itself is Weyl. □

6.2 Ruston and almost Ruston spectra

In this section we introduce the Ruston and almost Ruston spectra of an element in a Banach algebra with respect to a bounded Banach algebra homomorphism as Mouton and Raubenheimer did in [19]. We also look at a spectral inclusion theorem for Ruston and almost Ruston spectra which is also a result of Mouton and Raubenheimer in [19].

Definition 6.2.1 (Ruston and almost Ruston spectrum)**.** ([19], Definition 4.1, p. 22) Let $T : A \to B$ be a bounded Banach algebra homomorphism and $a \in A$. Then we define the following:

- The *Ruston spectrum* of a is the set $\{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathcal{R}\}$ and is denoted by $\vartheta_T^{comm}(a)$;
- The *almost Ruston spectrum* of a is the set $\{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathcal{R}_{\mathrm{alm}}\}$ and is denoted by $\vartheta_T(a)$.

The following proposition provides characterisations of the Ruston and almost Ruston spectra which are stated without proof in [19].

Proposition 6.2.2 ([19], p. 22)**.** *Let $T : A \to B$ be a bounded Banach algebra homomorphism and $a \in A$. Then*

$$\vartheta_T(a) = \bigcap_{\substack{c \in \mathscr{R}_T \\ TaTc = TcTa}} \sigma(a - c) \tag{6.2.3}$$

and

$$\vartheta_T^{comm}(a) = \bigcap_{\substack{c \in \mathscr{R}_T \\ ac = ca}} \sigma(a - c). \tag{6.2.4}$$

Proof. We first prove equality (6.2.3). Suppose that $\lambda \notin \vartheta_T(a)$. Then $\lambda\mathbf{1} - a = b + c$ for some $b \in A^{-1}$ and $c \in \mathscr{R}_T$ such that $TbTc = TcTb$. It follows that $\lambda\mathbf{1} - (a + c) \in A^{-1}$ and therefore $\lambda \notin \sigma(a + c)$. Furthermore, $TaTc = TcTa$ and hence $\lambda \notin \bigcap\{\sigma(a - c) : c \in \mathscr{R}_T; TaTc = TcTa\}$. Therefore, $\vartheta_T(a) \supseteq \bigcap\{\sigma(a - c) : c \in \mathscr{R}_T; TaTc = TcTa\}$.

Conversely, suppose that $\lambda \notin \bigcap\{\sigma(a - c) : c \in \mathscr{R}_T; TaTc = TcTa\}$. Then there exists an element $c \in \mathscr{R}_T$ such that $TcTa = TaTc$ and $\lambda \notin \sigma(a - c)$. Therefore, the element $b := \lambda\mathbf{1} - (a - c)$ is invertible in A,. It follows that

$\lambda\mathbf{1} - a = b - c$, where $b \in A^{-1}$ and $-c \in \mathscr{R}_T$. Furthermore, $TbTc = TcTb$ and hence $\lambda\mathbf{1} - a \in \mathcal{R}_{\mathrm{alm}}$ which means that $\lambda \notin \vartheta_T(a)$. Therefore, $\vartheta_T(a) \subseteq \bigcap\{\sigma(a-c) : c \in \mathscr{R}_T; TaTc = TcTa\}$.

Similarly, equality (6.2.4) holds. □

Proposition 6.2.5 ([19], p. 22)**.** *Let $T : A \to B$ be a bounded Banach algebra homomorphism and $a \in A$. Then we have the inclusions*

$$\begin{tikzcd}[column sep=small] & \vartheta_T^{comm}(a) \arrow[dr, phantom, sloped, "\subsetneq"] & \\ \sigma_B(Ta) \subseteq \vartheta_T(a) \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \omega_T^{\mathrm{comm}}(a) \subseteq \sigma_B(Ta) \cup \operatorname{acc}\sigma(a) \subseteq \sigma(a). \\ & \omega_T(a) \arrow[ur, phantom, sloped, "\subsetneq"] & \end{tikzcd}$$

Proof. This proposition follows directly from Proposition 6.1.11. □

Corollary 6.2.6 ([19], p. 22)**.** *Let $T : A \to B$ be a bounded Banach algebra homomorphism and $a \in A$. Then the Ruston and almost Ruston spectra of a are both non-empty and compact.*

Proof. This proposition follows directly from Proposition 6.2.2 and Proposition 6.2.5. □

Proposition 6.2.7 ([19], Remark on p. 22)**.** *Let $T : A \to B$ be a bounded Banach algebra homomorphism and $a \in A$. If T has the Riesz property, then the following equalities hold:*

$$\sigma_B(Ta) \cup \operatorname{acc}\sigma(a) = \omega_T^{\mathrm{comm}}(a) = \vartheta_T^{comm}(a) \tag{6.2.8}$$

$$\vartheta_T(a) = \omega_T(a) \tag{6.2.9}$$

Proof. The equations (6.2.8) follows directly from Definition 6.2.1 and Corollary 6.1.14.

Equation (6.2.9) follows directly from Corollary 6.1.15. □

Theorem 6.2.10 ([19], Theorem 4.2, p. 23)**.** *Let $T : A \to B$ be a bounded Banach algebra homomorphism such that the quotient algebra $A/T^{-1}(0)$ is commutative. Take any $a \in A$ and a neighbourhood U of $\sigma(a)$. If the function $f : U \to \mathbb{C}$ is analytic and non-constant on every component of U, then*

$$\vartheta_T(f(a)) \subseteq f(\vartheta_T(a)) \tag{6.2.11}$$

and

$$\vartheta_T^{comm}(f(a)) \subseteq f(\vartheta_T^{comm}(a)). \tag{6.2.12}$$

Proof. We first prove 6.2.11. Observe that, by Proposition 6.2.5 and the spectral mapping theorem, $\vartheta_T\left(f(a)\right) \subseteq \sigma\left(f(a)\right) \subseteq f(\sigma\left(a\right))$. Therefore, if $\lambda \notin f(\sigma\left(a\right))$, then $\lambda \notin \vartheta_T\left(f(a)\right)$. Take any $\lambda \in f(\sigma\left(a\right))\backslash f(\vartheta_T\left(a\right))$. Now, define the function $h : U \to \mathbb{C}$ by

$$h(z) := f(z) - \lambda$$

for all $z \in U$. As in the proof of inclusion (2.3.4), we can write

$$f(z) - \lambda = h(z) = (z - s_1)^{m_1}(z - s_2)^{m_2} \cdots (z - s_n)^{m_n} g(z),$$

where

$$\begin{aligned} \{s_1, s_2, \ldots, s_n\} &= \{z \in \sigma\left(a\right) : h(z) = 0\} \\ &= \{z \in \sigma\left(a\right) : f(z) = \lambda\}, \end{aligned}$$

$m_1, m_2, \ldots, m_n \in \mathbb{N}$ and $g : U \to \mathbb{C}$ is holomorphic and has no zeros in $\sigma\left(a\right)$. So we have

$$h(a) = (a - s_1\mathbf{1})^{m_1}(a - s_2\mathbf{1})^{m_2} \cdots (a - s_n\mathbf{1})^{m_n} g(a) \tag{6.2.13}$$

with $g(a) \in A^{-1} \subseteq \mathcal{R}_{\text{alm}}$. For $1 \leq j \leq n$, we have that $s_j \in f^{-1}(\lambda)$ and therefore $s_j \notin \vartheta_T\left(a\right)$ so that $a - s_j \in \mathcal{R}_{\text{alm}}$. Since $A/T^{-1}(0)$ is commutative, it follows from Proposition 6.1.4 and equation (6.2.13) that $h(a) = f(a) - \lambda\mathbf{1} \in \mathcal{R}_{\text{alm}}$ and hence $\lambda \notin \vartheta_T\left(f(a)\right)$. Therefore, $\vartheta_T\left(f(a)\right) \subseteq f(\vartheta_T\left(a\right))$.

We now prove inclusion (6.2.12). Observe that, by Proposition 6.2.5 and the spectral mapping theorem, $\vartheta_T^{comm}\left(f(a)\right) \subseteq \sigma\left(f(a)\right) \subseteq f(\sigma\left(a\right))$. It follows that, if $\lambda \notin f(\sigma\left(a\right))$, then $\lambda \notin \vartheta_T^{comm}\left(f(a)\right)$. We therefore take any $\lambda \in f(\sigma\left(a\right))\backslash f(\vartheta_T^{comm}\left(a\right))$. Now, define the function $h : U \to \mathbb{C}$ by

$$h(z) := f(z) - \lambda$$

for all $z \in U$. As in the first part of the proof, we can write

$$f(z) - \lambda = h(z) = (z - s_1)^{m_1}(z - s_2)^{m_2} \cdots (z - s_n)^{m_n} g(z),$$

where

$$\begin{aligned} \{s_1, s_2, \ldots, s_n\} &= \{z \in \sigma\left(a\right) : h(z) = 0\} \\ &= \{z \in \sigma\left(a\right) : f(z) = \lambda\}, \end{aligned}$$

$m_1, m_2, \ldots, m_n \in \mathbb{N}$ and $g : U \to \mathbb{C}$ is holomorphic and has no zeros in $\sigma(a)$. So we have

$$h(a) = (a - s_1\mathbf{1})^{m_1}(a - s_2\mathbf{1})^{m_2} \cdots (a - s_n\mathbf{1})^{m_n} g(a) \tag{6.2.14}$$

with $g(a) \in A^{-1} \subseteq \mathcal{R} \subseteq \mathcal{R}_{\mathrm{alm}}$. For $1 \leq j \leq n$, we have that $s_j \in f^{-1}(\lambda)$ and therefore $s_j \notin \vartheta_T^{comm}(a)$ so that $a - s_j\mathbf{1} \in \mathcal{R} \subseteq \mathcal{R}_{\mathrm{alm}}$. Therefore, for each j, there exist elements $b_j \in A^{-1}$ and $c_j \in \mathscr{R}_T$ such that $a - s_j\mathbf{1} = b_j + c_j$ and $b_j c_j = c_j b_j$ and hence also $ab_j = b_j a$ and $ac_j = c_j a$. Since $A/T^{-1}(0)$ is commutative, it follows from Proposition 6.1.4 and equation (6.2.14) that $h(a) = f(a) - \lambda\mathbf{1} \in \mathcal{R}_{\mathrm{alm}}$ and $f(a) - \lambda\mathbf{1} = b + c$ with $b = b_1^{m_1} b_2^{m_2} \cdots b_n^{m_n} g(a) \in A^{-1}$ and $c \in \mathscr{R}_T$ such that $TbTc = TcTb$. Also, c is the sum of terms containing only b_j's and c_j's and $g(a)$. It follows that b and c both commute with a and therefore also with $f(a)$, but since $f(a) - \lambda\mathbf{1} = b + c$ we see that $bc = cb$. This means that $f(a) - \lambda\mathbf{1} \in \mathcal{R}$. Therefore, $\lambda \notin \vartheta_T^{comm}(f(a))$ and hence $\vartheta_T^{comm}(f(a)) \subseteq f(\vartheta_T^{comm}(a))$. $\square$

Proposition 6.2.15. *Let $T : A \to B$ be a bounded Banach algebra homomorphism with the Riesz property, $a \in A$ and $f : U \to \mathbb{C}$ a holomorphic function U, a neighbourhood of $\sigma(a)$. If f is non-constant on every component of U, then*

$$f(\vartheta_T^{comm}(a)) = \vartheta_T^{comm}(f(a)).$$

Proof. This follows immediately from Proposition 6.2.7 and Corollary 3.12. $\square$

Proposition 6.2.16. *Let $T : A \to B$ be a bounded Banach algebra homomorphism with the Riesz property, $a \in A$ and $f : U \to \mathbb{C}$ a holomorphic function U, a neighbourhood of $\sigma(a)$. If f is non-constant on every component of U, then*

$$\vartheta_T(f(a)) \subseteq f(\vartheta_T(a)).$$

Proof. This follows immediately from Proposition 6.2.7 and Theorem 2.3.3. $\square$

6.3 Perturbation by Riesz elements

In this section the Ruston elements are used to prove an interesting perturbation result of Mouton and Raubenheimer. As we see later, this result is an improvement of a theorem of Groenewald, Harte and Raubenheimer and it is of importance when related to a similar result of Aupetit.

Theorem 6.3.1 ([19], Theorem 5.1, p. 24)**.** *Let $T : A \to B$ be a bounded homomorphism which satisfies the Riesz property. If $a \in A$ and $f \in \mathscr{R}_T$ such that $af = fa$, then $a \in \mathcal{B}$ if and only if $a + f \in \mathcal{B}$.*

Proof. Take any elements $a \in A$ and $f \in \mathscr{R}_T$ such that $af = fa$. Suppose that $a \in \mathcal{B}$. Then, because T has the Riesz property, it follows from Theorem 2.1.18 (2) that a is almost invertible Fredholm. If $p = p(a, 0)$, we have by Theorem 2.1.14 that $a = b+c$ where $b = p+a(\mathbf{1}-p) \in A^{-1}$ and $c = (a-\mathbf{1})p \in T^{-1}(0)$ and by Theorem 1.17 that p commutes with a as well as with all elements of A commuting with a. Therefore, $pf = fp$ and hence also $fb = bf$ and $fc = cf$. It follows from Theorem 1.9 that $\rho(\overline{f + c}) \leq \rho(\overline{f}) + \rho(\overline{c}) = 0$ and we obtain that $f + c \in \mathscr{R}_T$. We now have that $a + f = b + (c + f) \in \mathcal{R}$ and therefore, by Corollary 6.1.14, we have $a + f \in \mathcal{B}$.

Conversely, suppose that $a + f \in \mathcal{B}$. We use the part of this theorem which has already been proved. Observe that $a = (a + f) - f$. Since $a + f$ and $-f$ commute and $-f \in \mathscr{R}_T$, it follows that $a \in \mathcal{B}$. $\square$

Corollary 6.3.2 ([19], Corollary 5.2, p. 24)**.** *Let $T : A \to B$ be a bounded homomorphism which satisfies the Riesz property. If $a \in A$ and $f \in \mathscr{R}_T$ such that $af = fa$, then $\omega_T^{\mathrm{comm}}(a) = \omega_T^{\mathrm{comm}}(a + f)$.*

Proof. Take any $a \in A$ and $f \in \mathscr{R}_T$ such that $af = fa$. Then, for any $\lambda \in \mathbb{C}$, we have $(a - \lambda\mathbf{1})f = f(a - \lambda\mathbf{1})$ and therefore, by Theorem 6.3.1, we have $a-\lambda\mathbf{1} \in \mathcal{B}$ if and only if $a+f-\lambda\mathbf{1} \in \mathcal{B}$. Hence $\omega_T^{\mathrm{comm}}(a) = \omega_T^{\mathrm{comm}}(a + f)$. $\square$

Definition 6.3.3 ([2], p. 108)**.** Let A be a Banach algebra, I a two-sided inessential ideal of A and $a \in A$. The set $\mathscr{D}_I(a)$ in $\mathbb{C}$ is defined by

$$\mathscr{D}_I(a) := \sigma(a) \setminus \{\lambda \in \operatorname{iso}\sigma(a) : p(a, \lambda) \in I\}.$$

We now state a theorem of Aupetit which is related to the theorems of Mouton and Raubenheimer which also follow.

Theorem 6.3.4 ([2], Theorem 5.7.4, p. 108)**.** *Let I be a two-sided inessential ideal of a Banach algebra A. For $x \in A$ and $y \in I$ the following properties hold:*

1. *if G is a connected component of $\mathbb{C}\setminus\mathscr{D}_I(x)$ intersecting $\mathbb{C}\setminus\sigma(x + y)$ then it is a component of $\mathbb{C}\setminus\mathscr{D}_I(x + y)$;*
2. *the unbounded connected components of $\mathbb{C}\setminus\mathscr{D}_I(x)$ and $\mathbb{C}\setminus\mathscr{D}_I(x + y)$ coincide, in particular $\mathscr{D}_I(x)$ and $\mathscr{D}_I(x+y)$ have the same external boundaries;*

3. *if we consider the element $\overline{x}$ in the quotient algebra $A/\overline{I}$, then we have $\sigma_{A/\overline{I}}(\overline{x}) \subseteq \mathscr{D}_I(x)$ and $\eta\mathscr{D}_I(x) = \eta\sigma_{A/\overline{I}}(\overline{x})$.*

Lemma 6.3.5 ([2], part of proof of Lemma 5.7.1, p. 107)**.** *Let I be a two-sided ideal of a Banach algebra A. Then I and $\overline{I}$ have the same idempotents.*

Proof. It is obvious that the set of idempotents of I is contained in $\overline{I}$. To prove the converse inclusion, suppose that $p \in \overline{I}$ is idempotent. The set $p\overline{I}p$ can be regarded as a Banach algebra with identity element p. The set pIp is a two-sided ideal of $p\overline{I}p$ which is dense in the Banach algebra $p\overline{I}p$. Since the open ball with centre p and radius 1 in $p\overline{I}p$ is a subset of $\left(p\overline{I}p\right)^{-1}$ and an ideal cannot contain any invertible elements, we conclude that $pIp = p\overline{I}p$. Therefore $p = ppp \in p\overline{I}p = pIp \subseteq I$. $\square$

Theorem 6.3.6 ([19], Theorem 5.3, p. 24)**.** *Let A be a Banach algebra with I a two-sided inessential ideal of A and $T : A \to A/\overline{I}$ the quotient map from A onto $A/\overline{I}$. Take any $a \in A$. Then $\mathscr{D}_I(a) = \sigma_{A/\overline{I}}(Ta) \cup \operatorname{acc}\sigma_A(a)$, which is the almost invertible Fredholm spectrum of a.*

Proof. Let $a \in A$. Take any $\lambda \notin \mathscr{D}_I(a)$. Then $\lambda \in (\mathbb{C}\backslash\sigma_A(a)) \cup \{\lambda \in \operatorname{iso}\sigma(a) : p(a,\lambda) \in I\}$. In the case where $\lambda \in \mathbb{C}\backslash\sigma_A(a)$, we have that $\lambda \notin \sigma_A(a) \supseteq \sigma_{A/\overline{I}}(Ta)$ and $\lambda \notin \sigma_A(a) \supseteq \operatorname{acc}\sigma_A(a)$, so we see that $\lambda \notin \sigma_{A/\overline{I}}(Ta) \cup \operatorname{acc}\sigma_A(a)$. In the case where $\lambda \in \{\lambda \in \operatorname{iso}\sigma(a) : p(a,\lambda) \in I\}$ then $T(p(a,\lambda)) = 0 + \overline{I}$. Furthermore, by Corollary 3.5, we have that $T(p(a,\lambda)) = p(Ta,\lambda)$, so we obtain $p(Ta,\lambda) = 0 + \overline{I}$. Therefore, by Theorem 1.17, we have that $\lambda \notin \sigma_{A/\overline{I}}(Ta)$. Furthermore, $\lambda \notin \operatorname{acc}\sigma_A(a)$ and hence $\lambda \notin \sigma_{A/\overline{I}}(Ta) \cup \operatorname{acc}\sigma_A(a)$.

Conversely, take any $\lambda \in \mathscr{D}_I(a)$. Then either $\lambda \in \operatorname{acc}\sigma_A(a)$ or $\lambda \in \{\operatorname{iso}\sigma_A(a) : p(a,\lambda) \notin I\}$. Suppose that $\lambda \in \{\operatorname{iso}\sigma_A(a) : p(a,\lambda) \notin I\}$. Then, by Lemma 6.3.5, $p(a,\lambda) \notin \overline{I}$ so that $T(p(a,\lambda)) \neq 0 + \overline{I}$ and it follows from Theorem 1.17 that $\lambda \in \sigma_{A/\overline{I}}(Ta)$. $\square$

Corollary 6.3.7 ([19], Corollary 5.4, p. 25)**.** *Let I be a two-sided inessential ideal of the Banach algebra A and $a \in A$. If $T : A \to A/\overline{I}$ is the quotient map, then $\omega_T^{\mathrm{comm}}(a) = \mathscr{D}_I(a)$.*

Proof. We have from Lemma 2.1.17 that T has the Riesz property. It now follows from Corollary 3.10 and Theorem 6.3.6 that $\omega_T^{\mathrm{comm}}(a) = \mathscr{D}_I(a)$. $\square$

The next theorem is the result of Mouton and Raubenheimer which is the main result of this section. In Theorem 6.3.4 of Aupetit, two elements a and b of a Banach algebra A with inessential ideal I are considered. One, say b, is required to be in I and we then obtain that the external boundaries of $\mathscr{D}_I(a)$ and $\mathscr{D}_I(a+b)$ coincide. In this result of Mouton and Raubenheimer, it is only required that $\sigma_{A/\overline{I}}\left(\overline{b}\right) = \{0\}$, but in addition, that $ab = ba$. Then the equality $\mathscr{D}_I(a) = \mathscr{D}_I(a+b)$ is obtained.

Theorem 6.3.8 ([19], Theorem 5.5, p. 25)**.** *Let I be a two-sided inessential ideal of the Banach algebra A and $T : A \to A/\overline{I}$ the quotient map from A onto $A/\overline{I}$. Take any $a \in A$ and $b \in \mathscr{R}_T$ such that $ab = ba$. Then $\mathscr{D}_I(a) = \mathscr{D}_I(a+b)$.*

Proof. By Lemma 2.1.17, we see that T has the Riesz property. Because T is also bounded, it follows from Corollary 6.3.2 and Corollary 6.3.7 that $\mathscr{D}_I(a) = \mathscr{D}_I(a+b)$. □

The above theorem of Mouton and Raubenheimer is also an improvement of a theorem of Groenewald, Harte and Raubenheimer which we state below. The equality in Theorem 6.3.8 implies both (1) and (2) of Theorem 6.3.9.

Theorem 6.3.9 ([8], Theorem 2)**.** *Let I be a two-sided inessential ideal of the Banach algebra A and $T : A \to A/\overline{I}$ the quotient map from A onto $A/\overline{I}$. Take any $a \in A$ and $b \in \mathscr{R}_T$ such that $ab = ba$. Then*

1. *any component of $\mathbb{C}\backslash\mathscr{D}_I(a)$ intersecting $\mathbb{C}\backslash\sigma(a+b)$ is also a component of $\mathbb{C}\backslash\mathscr{D}_I(a+b)$;*
2. *the unbounded components of $\mathbb{C}\backslash\mathscr{D}_I(a)$ and $\mathbb{C}\backslash\mathscr{D}_I(a+b)$ coincide.*

Chapter 7

Regularities in Relation to Fredholm Theory

In this chapter we introduce the notions of upper and lower semi-regularities in Banach algebras which are concepts due to V. Müller (see [20]). This leads to the introduction of spectra and spectral inclusion theorems as obtained by Müller. We can then recover and even improve some spectral inclusion theorems related to Fredholm theory which we have already obtained.

This article of Müller was published four years later than a joint article of Kordula and Müller ([14]) in which they introduced the notion of regularities in Banach algebras. We will see that a set is a regularity if and only if it is both an upper and a lower semi-regularity.

7.1 Lower semi-regularities

Definition 7.1.1 (Lower semi-regularity)**.** ([20], Definition 1, p. 160) Let R be a non-empty subset of a Banach algebra A. Then R is called a *lower semi-regularity* if the following properties are satisfied:

1. if $a \in A$ and $a^n \in R$ for some $n \in \mathbb{N}$, then $a \in R$;
2. if $a, b, c, d \in A$ are mutually commutative elements of A such that $ac + bd = \mathbf{1}$, then $ab \in R \Rightarrow a, b \in R$.

Definition 7.1.2 ([20], p. 160)**.** Let R be a lower semi-regularity of a Banach algebra A and $a \in A$. The corresponding spectrum $\sigma^R(a)$ is defined by

$$\sigma^R(a) := \{\lambda \in \mathbb{C} : a - \lambda\mathbf{1} \notin R\}.$$

Remark 7.1.3 ([20], Remark 3, p. 160). Let R be a subset of A and suppose that we have the following implication: if a and b are commuting elements such that $ab \in R$, then $a \in R$ and $b \in R$. Then R is a lower semi-regularity.

Remark 7.1.4 ([20], p. 160). It follows from the Definition 7.1.1 that the intersection of any system of lower semi-regularities is a lower semi-regularity.

Lemma 7.1.5 ([20], Lemma 2, p. 160)**.** *Let R be a lower semi-regularity of the Banach algebra A. Then the following properties hold:*

1. $\mathbf{1} \in R$;
2. $A^{-1} \subseteq R$;
3. *if* $a \in R$, $b \in A^{-1}$ *and* $ab = ba$, *then* $ab \in R$;
4. $\sigma^R(a) \subseteq \sigma(a)$, *for all* $a \in A$;
5. $\sigma^R(a + \lambda\mathbf{1}) = \sigma^R(a) + \lambda$, *for all* $a \in A$ *and* $\lambda \in \mathbb{C}$ *(the translation property).*

Proof.

1. Take any b in the non-empty set R. We can write $\mathbf{1} = b \cdot 0 + \mathbf{1} \cdot \mathbf{1}$. Clearly the elements $\mathbf{1}$, b and 0 are mutually commutative and, by assumption, $b\mathbf{1} = b \in R$. Therefore, by (2) of Definition 7.1.1, we have that $\mathbf{1} \in R$.
2. Let $a \in A^{-1}$. Then we write $\mathbf{1} = a^{-1}a + a \cdot 0$. Since $aa^{-1} = \mathbf{1}$ and we have shown that $\mathbf{1} \in R$, it follows from (2) in Definition 7.1.1 that $a \in R$ and $a^{-1} \in R$.
3. Suppose that $a \in R$, $b \in A^{-1}$ and $ab = ba$. Then we have $\mathbf{1} = ab \cdot 0 + b^{-1}b$. Since the elements $ab, 0, b^{-1}, b$ are mutually commutative and $abb^{-1} = a \in R$, it follows from Definition 7.1.1 (2) that $ab \in R$.
4. Take any $a \in A$. It follows directly from (2) of this lemma that $\sigma^R(a) \subseteq \sigma(a)$.
5. Take any $a \in A$ and $\lambda \in \mathbb{C}$. Let $\alpha \in \mathbb{C}$. By definition, $\alpha \in \sigma^R(a)$ if and only if $a - \alpha\mathbf{1} = a + \lambda\mathbf{1} - (\alpha + \lambda)\mathbf{1} \notin R$, which holds if and only if $\alpha + \lambda \in \sigma^R(a + \lambda\mathbf{1})$. □

Theorem 7.1.6 ([20], Theorem 4, p 160)**.** *Let R be a lower semi-regularity of the Banach algebra A and $a \in A$ with U a neighbourhood of $\sigma(a)$. If $f : U \to \mathbb{C}$ is an analytic function which is non-constant on each component of U, then*

$$f\left(\sigma^R(a)\right) \subseteq \sigma^R(f(a)).$$

Proof. Suppose that there is $\lambda \in f(\sigma^R(a)) \backslash \sigma^R(f(a))$. Since $f(\sigma^R(a)) \subseteq f(\sigma(a))$ by Lemma 7.1.5 (4), we know that the function $f(z) - \lambda$ has at least one zero in $\sigma(a)$. As in the proof of inclusion (2.3.4), we can now write

$$f(z) - \lambda = (z - s_1)^{m_1}(z - s_2)^{m_2} \cdots (z - s_n)^{m_n} g(z)$$

for some $n \in \mathbb{N}$, where $\{s_1, s_2, \ldots, s_n\} = \{z \in \sigma(a) : f(z) = \lambda\}$ and g is analytic on U with no zeros in $\sigma(a)$. By the spectral mapping theorem, $0 \notin \sigma(g(a))$. We can assume that $s_j \neq s_k$ if $j \neq k$. By holomorphic functional calculus, we have that

$$f(a) - \lambda\mathbf{1} = (a - s_1\mathbf{1})^{m_1}(a - s_2\mathbf{1})^{m_2} \cdots (a - s_n\mathbf{1})^{m_n} g(a)$$

with $g(a) \in A^{-1}$. We therefore have that

$$(a - s_1\mathbf{1})^{m_1}(a - s_2\mathbf{1})^{m_2} \cdots (a - s_n\mathbf{1})^{m_n} = (f(a) - \lambda\mathbf{1})\, g(a)^{-1}$$

with $(f(a) - \lambda\mathbf{1}) \in R$, $g(a)^{-1} \in A^{-1}$ and $(f(a) - \lambda\mathbf{1})$ commutes with $g(a)^{-1}$. It now follows from Lemma 7.1.5 (3) that $(a - s_1\mathbf{1})^{m_1}(a - s_2\mathbf{1})^{m_2} \cdots (a - s_n\mathbf{1})^{m_n} \in R$. Now, for each $k \in \{1, 2, \ldots, m\}$, because the greatest common divisor of the polynomials $(z - s_k)^{m_k}$ and $\prod_{j \neq k}(z - s_j)^{m_j}$ in the polynomial ring $\mathbb{C}[z]$ is 1, there exist polynomials p_k and q_k in $\mathbb{C}[z]$ such that

$$(z - s_k)^{m_k} p_k(z) + \prod_{j \neq k}(z - s_j)^{m_j} q_k(z) = 1.$$

By holomorphic functional calculus, we now have the following equality:

$$(a - s_k\mathbf{1})^{m_k} p_k(a) + \prod_{j \neq k}(a - s_j\mathbf{1})^{m_j} q_k(a) = \mathbf{1}.$$

Because the elements $(a - s_k\mathbf{1})^{m_k}$, $p_k(a)$, $\prod_{j \neq k}(a - s_j\mathbf{1})^{m_j}$ and $q_k(a)$ are mutually commutative and $(a - s_k\mathbf{1})^{m_k} \prod_{j \neq k}(a - s_j\mathbf{1})^{m_j} = (a - s_1\mathbf{1})^{m_1}(a - s_2\mathbf{1})^{m_2} \cdots (a - s_n\mathbf{1})^{m_n} \in R$, it follows from Definition 7.1.1 (2) that $(a - s_k\mathbf{1})^{m_k} \in R$ and hence, by Definition 7.1.1 (1), also $a - s_k\mathbf{1} \in R$. Therefore $s_k \notin \sigma^R(a)$ for each $k \in \{1, 2, \ldots, m\}$. Hence $\lambda \notin f(\sigma^R(a))$, which is a contradiction. $\square$

Theorem 7.1.7 ([20], Theorem 6, p. 161)**.** *Let $R \subseteq A$ be a lower semi-regularity satisfying the following condition: if $c = c^2 \in R$, $a \in A$ and $ac = ca$ then $c + (1 - c)a \in R$. Then*

$$f(\sigma^R(a)) \subseteq \sigma^R(f(a))$$

for all $a \in A$ and f analytic on a neighbourhood U of $\sigma(a)$.

Proof. Let $a \in A$, U a bounded neighbourhood of $\sigma(a)$ and $f : U \to \mathbb{C}$ an analytic function. If the distance between any two distinct components of U is 0, redefine U so that this is not the case. Suppose that there exists a complex number $\lambda \in f(\sigma^R(a)) \backslash \sigma^R(f(a))$. Define U_1 as the union of all components of U on which f is identically equal to λ and define $U_2 := U \backslash U_1$. Define the function

$$h(z) := \begin{cases} 0 & \text{if } z \in U_1, \\ 1 & \text{if } z \in U_2. \end{cases}$$

The function $f|_{U_2}$ is analytic and not identical to λ on each component of U_2.

We now consider two cases. In each case we show that $h(a) \in R$ and that there exists $\beta \in U_1 \cap \sigma^R(a)$ such that $f(\beta) = \lambda$.

1. In the case where $\lambda \notin f(U_2)$, define

$$g(z) := \begin{cases} 1 & \text{if } z \in U_1; \\ f(z) - \lambda & \text{if } z \in U_2. \end{cases}$$

Then g has no zeros in $\sigma(a)$ and hence, by the spectral mapping theorem, $g(a)$ is invertible in A. We can now write

$$f(z) - \lambda = h(z)g(z)$$

for all $z \in U$. Therefore, by holomorphic functional calculus,

$$f(a) - \lambda \mathbf{1} = h(a)g(a),$$

so that $h(a) = (f(a) - \lambda \mathbf{1})g(a)^{-1}$. Because $\lambda \notin \sigma^R(f(a))$, we have $f(a) - \lambda \mathbf{1} \in R$. The elements $f(a) - \lambda \mathbf{1}$ and $g(a)^{-1}$ commute and hence, by Lemma 7.1.5 (3), $h(a) \in R$. Now, because $\lambda \in f\left(\sigma^R(a)\right)$ and $f(z) \neq \lambda$ for all $z \in U_2$, there exist $\beta \in U_1 \cap \sigma^R(a)$ such that $f(\beta) = \lambda$.

2. Consider the case where $\lambda \in f(U_2)$. As in the proof of inclusion (2.3.4), we can write

$$f(z) - \lambda = (z - s_1)^{m_1}(z - s_2)^{m_2} \cdots (z - s_n)^{m_n} g_1(z)$$

for all $z \in U_2$, where $\{s_1, s_2, \ldots s_n\} = \{s \in \sigma(a) \cap U_2 : f(s) = \lambda\}$ and g_1 is analytic on U_2 with no zeros in $\sigma(a) \cap U_2$. We can assume that $s_j \neq s_k$ if $j \neq k$. Extend g_1 to the function $g : U \to \mathbb{C}$ as follows

$$g(z) := \begin{cases} 1 & \text{if } z \in U_1, \\ g_1(z) & \text{if } z \in U_2. \end{cases}$$

Then g is analytic on U and has no zeros in $\sigma(a)$. Furthermore, we now have that

$$f(z) - \lambda = h(z)(z - s_1)^{m_1}(z - s_2)^{m_2} \cdots (z - s_n)^{m_n} g(z) \tag{7.1.8}$$

for all $z \in U$. Define $q(z) := (z - s_1)^{m_1}(z - s_2)^{m_2} \cdots (z - s_n)^{m_n}$ for all $z \in U$. Then equation (7.1.8) is equivalent to

$$f(z) - \lambda = h(z)q(z)g(z)$$

for all $z \in U$. By holomorphic functional calculus, we have that

$$f(a) - \lambda\mathbf{1} = h(a)q(a)g(a).$$

Since g has no zeros in $\sigma(a)$, it follows from the spectral mapping theorem that $g(a)$ is invertible in A. Therefore

$$h(a)q(a) = (f(a) - \lambda\mathbf{1})g(a)^{-1}$$

and $f(a) - \lambda\mathbf{1}$ and $g(a)^{-1}$ commute. Since $\lambda \notin \sigma^R(f(a))$ by assumption, $f(a) - \lambda\mathbf{1} \in R$. It now follows from Lemma 7.1.5 that $h(a)q(a) \in R$. Define the function

$$r(z) := \begin{cases} \frac{1}{q(z)} & \text{if } z \in U_1; \\ 0 & \text{if } z \in U_2. \end{cases}$$

Then, if $z \in U_1$, we have that

$$\begin{aligned} q(z) \cdot (1 - h(z))r(z) + h(z) &= q(z)(1 - 0)\frac{1}{q(z)} + 0 \\ &= 1 \end{aligned}$$

and if $z \in U_2$, then

$$\begin{aligned} q(z) \cdot (1 - h(z))r(z) + h(z) &= q(z)(1 - 1)0 + 1 \\ &= 1. \end{aligned}$$

Therefore

$$q(a) \cdot (\mathbf{1} - h(a))r(a) + h(a) \cdot \mathbf{1} = \mathbf{1}.$$

Since the elements $q(a)$, $(\mathbf{1} - h(a))r(a)$, $h(a)$ and $\mathbf{1}$ are mutually commutative and $q(a)h(a) \in R$, it follows from Definition 7.1.1 (2) that

$q(a) \in R$ and $h(a) \in R$. It now follows in exactly the same way as in the proof of Theorem 7.1.6 that $a - s_k\mathbf{1} \in R$ and hence $s_k \notin \sigma^R(a)$ for each $k \in \{1, 2, \ldots, m\}$. This implies that $f(z) \neq \lambda$ for all $z \in \sigma^R(a) \cap U_2$. So, because $\lambda \in f(\sigma^R(a))$, there exists an element $\beta \in \sigma^R(a) \cap U_1$ such that $f(\beta) = \lambda$.

Because $h(a)$ is an idempotent in R commuting with $a - \beta\mathbf{1}$, it follows by assumption that

$$(a - \beta\mathbf{1})(\mathbf{1} - h(a)) + h(a) \in R. \tag{7.1.9}$$

Furthermore, if $z \in U_1$, then $(1-h(z))+(z-\beta)h(z) = (1-0)+(z-\beta)0 = 1 \neq 0$ and if $z \in U_2$, then $(1-h(z))+(z-\beta)h(z) = (1-1)+(z-\beta)1 = z-\beta \neq 0$, since $\beta \in U_1$ and U_1 is disjoint from U_2. Therefore, by the spectral mapping theorem, $0 \notin \sigma((\mathbf{1} - h(a)) + (a - \beta\mathbf{1})h(a))$ and hence $(\mathbf{1} - h(a)) + (a - \beta\mathbf{1})h(a)$ is invertible. It follows from this, (7.1.9) and Lemma 7.1.5 (3) that

$$a - \beta\mathbf{1} = \big((a - \beta\mathbf{1})(\mathbf{1} - h(a)) + h(a)\big) \cdot \big((\mathbf{1} - h(a)) + (a - \beta\mathbf{1})h(a)\big) \in R$$

which contradicts the fact that $\beta \in \sigma^R(a)$. Therefore $f(\sigma^R(a)) \subseteq \sigma^R(f(a))$. $\square$

The following proposition shows that some of the sets that we have encountered are lower semi-regularities. We remind the reader that, relative to a fixed Banach algebra homomorphism, $\mathcal{F}$ is the set of Fredholm elements, $\mathcal{B}$ the set of Browder elements and $\mathcal{R}$ the set of Ruston elements of a Banach algebra. We mention that it follows from a remark by Müller in [20] on p. 164 that the set of Fredholm operators on a Banach space X is a lower semi-regularity of $\mathcal{L}(X)$. Furthermore, it follows from ([17], Proposition 7.1, p. 572) by L. Lindeboom and H. Raubenheimer that, in the case where J is a two-sided closed ideal of an algebra A and $T : A \to A/J$ is the canonical algebra homomorphism from A onto A/J, the set of almost invertible Fredholm elements of A relative to T is a lower semi-regularity.

Proposition 7.1.10. *Let A be a Banach algebra, $T : A \to B$ a Banach algebra homomorphism and X the set of almost invertible elements of A. Then the following holds.*

1. *The sets $\mathcal{F}$, X and $\mathcal{F} \cap X$ are lower semi-regularities.*
2. *If T is Riesz, then $\mathcal{B}$ is a lower semi-regularity.*
3. *If T is bounded and has the Riesz property, then $\mathcal{R}$ is a lower semi-regularity.*

Proof. 1. We use the fact stated in Remark 7.1.3. Take any two commuting elements $a, b \in A$ such that $ab \in \mathcal{F}$. Then $T(ab)$ is invertible in B. Since $T(ab) = TaTb = TbTa$, it follows that Ta and Tb are left and right invertible and hence invertible in B. Therefore $a, b \in \mathcal{F}$ so that $\mathcal{F}$ is a lower semi-regularity.

Take any four mutually commutative elements a, b, c, d in A such that $ac+bd = \mathbf{1}$ and $ab \in X$. Then $0 \notin \operatorname{acc}\sigma(ab)$. Suppose that $0 \in \operatorname{acc}\sigma(a)$. Let C be the maximal commutative sub-algebra of A containing a, b, c and d. If $x \in C$, $\lambda \in \mathbb{C}$ and $x - \lambda\mathbf{1}$ is invertible in A, then $x - \lambda\mathbf{1}$ is invertible in C and hence $\sigma_A(x) = \sigma_C(x)$. We can therefore write $\sigma(x)$ unambiguously. It follows from Theorem 1.10 that

$$\sigma(a) = \{f(a) : f \in \mathcal{M}(C)\}; \tag{7.1.11}$$

$$\sigma(b) = \{f(b) : f \in \mathcal{M}(C)\}; \tag{7.1.12}$$

$$\begin{aligned} \sigma(ab) &= \{f(ab) : f \in \mathcal{M}(C)\} \\ &= \{f(a)f(b) : f \in \mathcal{M}(C)\}; \end{aligned} \tag{7.1.13}$$

and

$$\begin{aligned} \{1\} &= \sigma(\mathbf{1}) \\ &= \sigma(ac+bd) \\ &= \{f(ac+bd) : f \in \mathcal{M}(C)\} \\ &= \{f(a)f(c) + f(b)f(d) : f \in \mathcal{M}(C)\}. \end{aligned} \tag{7.1.14}$$

From equation (7.1.11) we see that there exists a sequence (f_n) of non-zero multiplicative linear functionals on C such that the sequence of non-zero elements $(f_n(a))$ in $\mathbb{C}$ converges to 0. Furthermore, from equation (7.1.12), we have that $|f_n(b)| \leq \rho(b) \leq \|b\|$ for all $n \in \mathbb{N}$. From equation (7.1.13), $(f_n(a)f_n(b))$ is a sequence in $\sigma(ab)$. We now have that

$$\begin{aligned} |f_n(a)f_n(b)| &= |f_n(a)||f_n(b)| \\ &\leq |f_n(a)|\|b\| \\ &\to 0 \end{aligned} \tag{7.1.15}$$

if $n \to \infty$. Furthermore, from equation (7.1.14),

$$\begin{aligned} 1 &= |f_n(a)f_n(c) + f_n(b)f_n(d)| \\ &\leq |f_n(a)||f_n(c)| + |f_n(b)||f_n(d)| \\ &\leq |f_n(a)|\|c\| + |f_n(b)||f_n(d)|. \end{aligned} \tag{7.1.16}$$

For n large enough $|f_n(a)|\|c\| < 1$, since $f_n(a) \to 0$ if $n \to \infty$. Hence, from equation (7.1.16), $|f_n(b)| > 0$ for n large enough. From this and equation (7.1.15) we see that $(f_n(a)f_n(b))$ is a sequence of non-zero elements in the spectrum of ab converging to 0. Therefore 0 is an accumulation point of $\sigma(ab)$, which is a contradiction. Hence $0 \notin \operatorname{acc}\sigma(a)$. In exactly the same way, we have that $0 \notin \operatorname{acc}\sigma(b)$. We now have that a and b are both almost invertible elements of A. Therefore property (2) of Definition 7.1.1 is satisfied by X.

We now prove that property (1) of Definition 7.1.1 is satisfied as well. Take any $a \in A$ such that $a^n \in X$ for some $n \in \mathbb{N}$. Suppose that $0 \in \operatorname{acc}\sigma(a)$. Then there exists a sequence of non-zero elements (s_k) in $\sigma(a)$ converging to 0. It then follows by the spectral mapping theorem that $((s_k)^n)$ is a sequence of non-zero elements in $\sigma(a^n)$ converging to 0. Hence $0 \in \operatorname{acc}\sigma(a^n)$ which is a contradiction. It follows that X is a lower semi-regularity and hence, by Remark 7.1.4, $\mathcal{F} \cap X$ is a lower semi-regularity as well.

2. If T has the Riesz property, we have from Theorem 3.7 that $\mathcal{B} = \mathcal{F} \cap X$ and hence $\mathcal{B}$ is a lower semi-regularity.
3. If T is bounded and has the Riesz property, it follows from Corollary 6.1.14 that $\mathcal{R} = \mathcal{F} \cap X$ and hence $\mathcal{R}$ is a lower semi-regularity. □

We will now apply Theorem 7.1.6 and Theorem 7.1.7 to the sets that we have shown to be lower semi-regularities. We can then compare the spectral inclusions that we obtain with the spectral inclusions already obtained in this thesis.

Remark 7.1.17. If X is the set of almost invertible elements of a Banach algebra A and $a \in A$, then X is a lower semi-regularity and the spectrum of a corresponding to X is $\operatorname{acc}\sigma(a)$. That is because

$$\begin{aligned}\{\lambda \in \mathbb{C} : a - \lambda\mathbf{1} \notin X\} &= \{\lambda \in \mathbb{C} : 0 \in \operatorname{acc}\sigma(a - \lambda\mathbf{1})\}\\ &= \{\lambda \in \mathbb{C} : \lambda \in \operatorname{acc}\sigma(a)\}\\ &= \operatorname{acc}\sigma(a).\end{aligned}$$

Proposition 7.1.18. *Let A be a Banach algebra, $T : A \to B$ a Banach algebra homomorphism and $a \in A$ with U a bounded neighbourhood of a. Then, for a function $f : U \to \mathbb{C}$ which is analytic on U, the following holds:*

1. $f(\sigma_B(Ta)) \subseteq \sigma_B(T(f(a)))$.

Furthermore, if f is non-constant on every component of U, then the following properties hold:

2. $f(\operatorname{acc}\sigma(a)) \subseteq \operatorname{acc}\sigma(f(a))$*;*
3. $f(\sigma_B(Ta) \cup \operatorname{acc}\sigma(a)) \subseteq \sigma_B(T(f(a))) \cup \operatorname{acc}\sigma(f(a))$*;*
4. *if T is Riesz, then* $f(\omega_T^{\mathrm{comm}}(a)) \subseteq \omega_T^{\mathrm{comm}}(f(a))$*;*
5. *if T is bounded and has the Riesz property, then we have* $f(\vartheta_T^{comm}(a)) \subseteq \vartheta_T^{comm}(f(a))$.

Proof. 1. We have from Proposition 7.1.10 that $\mathcal{F}$ is a lower semi-regularity. We must show that the condition of Theorem 7.1.7 is satisfied. Suppose that $c = c^2 \in \mathcal{F}$. Then $Tc = (Tc)^2$ and Tc is invertible in B and hence $Tc = \mathbf{1}$. If $ac = ca$, then $T(c + (\mathbf{1} - c)a) = \mathbf{1} + 0 = \mathbf{1} \in B^{-1}$. Therefore $c + (\mathbf{1} - c)a \in \mathcal{F}$. It now follows from Theorem 7.1.7 that $f(\sigma_B(Ta)) \subseteq \sigma_B(T(f(a)))$.

2. It follows from Propostion 7.1.10 and Theorem 7.1.6 that $f(\operatorname{acc}\sigma(a)) \subseteq \operatorname{acc}\sigma(f(a))$.
3. It follows from Propostion 7.1.10 and Theorem 7.1.6 or from (1) and (2) of this proposition that $f(\sigma_B(Ta) \cup \operatorname{acc}\sigma(a)) \subseteq \sigma_B(T(f(a))) \cup \operatorname{acc}\sigma(f(a))$.
4. Suppose that T is Riesz. It follows from Theorem 3.7 that $\mathcal{B} = \mathcal{F} \cap X$, so we have from (3) of this proposition that $f(\omega_T^{\mathrm{comm}}(a)) \subseteq \omega_T^{\mathrm{comm}}(f(a))$.
5. Suppose that T is bounded and has the Riesz property. By Corollary 6.1.14, $\mathcal{R} = \mathcal{F} \cap X$. Therefore, by (3) of this proposition, $f(\vartheta_T^{comm}(a)) \subseteq \vartheta_T^{comm}(f(a))$. ☐

Corollary 7.1.19. *Let A be a Banach algebra, $T : A \to B$ a Banach algebra homomorphism and $a \in A$ with U a bounded neighbourhood of $\sigma(a)$. Then, for any function $f : U \to \mathbb{C}$ which is analytic on U and non-constant on every component of U, the following holds:*

$$f(\sigma_B(Ta)) = \sigma_B(T(f(a)))\,; \tag{7.1.20}$$

$$f(\sigma_B(Ta) \cup \operatorname{acc}\sigma(a)) = \sigma_B(T(f(a))) \cup \operatorname{acc}\sigma(f(a)). \tag{7.1.21}$$

If T is Riesz then

$$f(\omega_T^{\mathrm{comm}}(a)) = \omega_T^{\mathrm{comm}}(f(a))\,. \tag{7.1.22}$$

If T is bounded and Riesz then

$$f(\vartheta_T^{comm}(a)) = \vartheta_T^{comm}(f(a)). \tag{7.1.23}$$

Proof. This follows directly from the above proposition together with Proposition 2.3.13, Proposition 2.3.14, Theorem 3.7 and Corollary 6.1.14. □

Equality (7.1.20) has not been obtained in earlier chapters of this thesis. We had an inclusion in Proposition 2.3.13. We also had an equality for the case where T is bounded (Proposition 2.3.1), but here boundedness of T is not required.

Equality (7.1.21) has also not been obtained in other parts of this thesis. The strongest we had was the inclusion result of Proposition 2.3.14.

Equality (7.1.22) is the same as the equality obtained in Corollary 3.12.

Equality (7.1.23) is exactly the same as the equality obtained in Proposition 6.2.15.

We mention that a spectral mapping theorem for the spectrum corresponding to the set of almost invertible elements of a Banach algebra has also been obtained in Lemma 2.3.2.

7.2 Upper semi-regularities

Definition 7.2.1 (Upper semi-regularity)**.** ([20], p. 164) Let R be a subset of a Banach algebra A. Then R is called an upper semi-regularity if it satisfies the following properties:

1. if $a \in R$, then $a^n \in R$ for all $n \in \mathbb{N}$;
2. if a, b, c, d are mutually commutative elements of A satisfying $ac+bd = \mathbf{1}$ and $a, b \in R$ then $ab \in R$;
3. R contains a neighbourhood of the identity element $\mathbf{1}$.

Remark 7.2.2. To show that properties (1) and (2) of Definition 7.2.1 hold for a subset R of A, it suffices to show that if a and b are commutative elements of R, then $ab \in R$.

Definition 7.2.3 ([20], p. 164)**.** Let R be an upper semi-regularity of a Banach algebra A. The spectrum $\sigma^R(a)$ corresponding to R is defined by

$$\sigma^R(a) := \{\lambda \in \mathbb{C} : a - \lambda\mathbf{1} \notin R\}.$$

Proposition 7.2.4 ([20], p. 164)**.** *The intersection of any finite family of upper semi-regularities is an upper semi-regularities.*

Proof. Suppose that $\{R_i : i \in I\}$ is a family of upper semi-regularities where I is some finite index set. Let $R := \bigcap_{i \in I} R_i$. Take any $a \in R$. Then $a \in R_i$ for all $i \in I$ and hence $a^n \in R_i$ for all $i \in I$, by (1) of Definition 7.2.1 and the fact that each R_i is an upper semi-regularity. If $a, b, c, d \in A$ are mutually commutative elements such that $ac + bd = \mathbf{1}$ and $a, b \in R$, then $a, b \in R_i$ for all $i \in I$ and hence, by (2) of Definition 7.2.1, $ab \in R_i$ for all $i \in I$. Therefore $ab \in R$. Furthermore, the finite intersection of neighbourhoods of $\mathbf{1}$ must be a neighbourhood of $\mathbf{1}$ and hence R contains a neighbourhood of $\mathbf{1}$. $\square$

Lemma 7.2.5 ([20], Lemma 12, p. 164)**.** *Let R be an upper semi-regularity of the Banach algebra A and $a \in A^{-1} \cap R$. Then there exists $\epsilon > 0$ such that*

$$\{b \in A : ab = ba, \|b - a\| < \epsilon\} \subseteq R.$$

Proof. Since R contains a neighbourhood of $\mathbf{1}$, there exists $\delta > 0$ such that $\{c \in A : \|c - \mathbf{1}\| < \delta\} \subseteq R$. Let $\epsilon := \frac{\delta}{\|a^{-1}\|}$. Take any $b \in A$ such that $ab = ba$ $\|b - a\| < \epsilon$. Then

$$\begin{aligned}
\|a^{-1}b - \mathbf{1}\| &= \|a^{-1}(b - a)\| \\
&\leq \|a^{-1}\|\|b - a\| \\
&< \|a^{-1}\|\epsilon \\
&= \|a^{-1}\|\frac{\delta}{\|a^{-1}\|} \\
&= \delta
\end{aligned}$$

and hence $a^{-1}b \in R$. We also have that $aa^{-1} + (a^{-1}b)0 = \mathbf{1}$ and therefore, since $a, a^{-1}b \in R$, it follows from Definition 7.2.1 (2) that $a^{-1}ab = b \in R$. $\square$

Lemma 7.2.6 ([20], Lemma 13, p. 165)**.** *Let R be an upper semi-regularity of A and (a_n) a sequence in $R \cap A^{-1}$ converging to $a \in A^{-1}$ such that $a_n a = a a_n$ for all $n \in \mathbb{N}$. Then $a \in R$.*

Proof. The function $\Phi : A^{-1} \to A^{-1}$ which maps each element in A^{-1} to its inverse is continuous. Therefore, the sequence (a_n^{-1}) converges to a^{-1}. Hence the sequence $a_n^{-1}a$ converges to $\mathbf{1}$ and it follows from the fact that R contains a neighbourhood of $\mathbf{1}$ that $a_n^{-1}a \in R$ for n large enough. Furthermore, for each $n \in \mathbb{N}$ we have $a_n a_n^{-1} + (a_n^{-1}a)0 = \mathbf{1}$. Therefore, by Definition 7.2.1 (2), $a_n(a_n^{-1}a) = a \in R$. $\square$

Theorem 7.2.7 ([20], Theorem 14, p. 165)**.** *Let R be an upper semi-regularity of a Banach algebra A. Let C be a component of $\mathbb{C}\backslash\sigma(a)$. Then either $C \subseteq \sigma^R(a)$ or $C \cap \sigma^R(a) = \emptyset$.*

Proof. Define the set $L := \{\lambda \in \mathbb{C} : \lambda \in C, a - \lambda\mathbf{1} \in R\}$. We first show that L is open in C. Take any $\lambda \in L$. Then $\lambda \in C \subseteq \mathbb{C}\backslash\sigma(a)$ and hence $a - \lambda\mathbf{1} \in A^{-1}$. By Lemma 7.2.5, there exists $\epsilon_1 > 0$ such that if $|\alpha - \lambda| = \|(a - \lambda\mathbf{1}) - (a - \alpha\mathbf{1})\| < \epsilon_1$, then $a - \alpha\mathbf{1} \in R$. Since A^{-1} is open in $\mathbb{C}$, there exists $\epsilon_2 > 0$ such that if $|\alpha - \lambda| = \|(a - \lambda\mathbf{1}) - (a - \alpha\mathbf{1})\| < \epsilon_2$, then $a - \alpha\mathbf{1} \in A^{-1}$. Let $\epsilon := \min\{\epsilon_1, \epsilon_2\}$. Then we have that $\alpha \in C$, if $|\alpha - \lambda| < \epsilon$. It follows that L is open in $\mathbb{C}$ and therefore also in C.

We now show that L is closed in C. Let (λ_n) be a convergent sequence in L with limit λ in C. Then $(a - \lambda_n\mathbf{1})$ converges to $(a - \lambda\mathbf{1})$ in A and we have that $a - \lambda_n\mathbf{1} \in A^{-1} \cap R$ and $(a - \lambda_n\mathbf{1})(a - \lambda\mathbf{1}) = (a - \lambda\mathbf{1})(a - \lambda_n\mathbf{1})$ for each $n \in \mathbb{N}$. Furthermore, $a - \lambda\mathbf{1} \in A^{-1}$. It now follows from Lemma 7.2.6 that $a - \lambda\mathbf{1} \in R$. Therefore $\lambda \in L$, so that L is closed in C.

Since L is both open and closed in C, it is either the empty set or a component of C. Because C is a component of $\mathbb{C}\backslash\sigma(a)$ and therefore connected, we have that either $L = \emptyset$ or $L = C$. By definition, $L = C \cap \mathbb{C}\backslash\sigma^R(a)$ and it follows that either $C \subseteq \sigma^R(a)$ or $C \cap \sigma^R(a) = \emptyset$. $\square$

Corollary 7.2.8 ([20], Corollary 15, p. 165)**.** *Let R be an upper semi-regularity of a Banach algebra A. Then $\lambda\mathbf{1} \in R$ for each non-zero $\lambda \in \mathbb{C}$.*

Proof. The set $C := \{\lambda \in \mathbb{C} : \lambda \neq 0\}$ is a component of $\mathbb{C}\backslash\sigma(0)$. Since $0-(-\mathbf{1}) \in R$ we have that $-1 \notin \sigma^R(0)$ and hence $C \nsubseteq \sigma^R(0)$. It follows from Theorem 7.2.7 that $C \cap \sigma^R(0) = \emptyset$ and therefore $\lambda\mathbf{1} \in R$ for all $\lambda \neq 0$. $\square$

Lemma 7.2.9 ([20], Lemma 16, p. 165)**.** *Let R be an upper semi-regularity of a Banach algebra A. If $a \in R$, $b \in R \cap A^{-1}$ and $ab = ba$, then $ab \in R$.*

Proof. It follows from the fact that $a \cdot 0 + bb^{-1} = \mathbf{1}$ and Definition 7.2.1 (2) that $ab \in R$. $\square$

Corollary 7.2.10. *Let R be an upper semi-regularity of a Banach algebra A. If $a \in R$ and $0 \neq \lambda \in \mathbb{C}$, then $\lambda a \in R$.*

Proof. This follows directly from Corollary 7.2.8 and Lemma 7.2.9 by writing $\lambda a = \lambda\mathbf{1}a$. $\square$

Theorem 7.2.11 ([20], Theorem 17, p. 165)**.** *Let R be an upper semi-regularity of a Banach algebra A and $a \in A$. Then $\sigma^R(a) \subseteq \eta\sigma(a)$. Furthermore, $\sigma^R(a) \backslash \sigma(a)$ is the union of some bounded components of $\mathbb{C} \backslash \sigma(a)$.*

Proof. Because R contains a neighbourhood of $\mathbf{1}$, we have that $\mathbf{1} - \frac{1}{\lambda}a \in R$ for $\lambda \in \mathbb{C}$ such that $|\lambda|$ is large enough. It follows from Corollary 7.2.10 that $a - \lambda\mathbf{1} = -\lambda(\mathbf{1} - \frac{1}{\lambda}a) \in R$. Therefore $\sigma^R(a)$ is bounded and cannot be equal to the unbounded component of $\mathbb{C} \backslash \sigma(a)$. Hence, by Theorem 7.2.7, $\sigma^R(a)$ and the unbounded component of $\mathbb{C} \backslash \sigma(a)$ are disjoint, so that $\sigma^R(a) \subseteq \eta\sigma(a)$.

It follows from this and Theorem 7.2.7 that $\sigma^R(a) \backslash \sigma(a)$ is the union of some bounded components of $\mathbb{C} \backslash \sigma(a)$. □

Corollary 7.2.12 ([20], Corollary 18, p. 165)**.** *The set $\sigma^R(a) \cup \sigma(a)$ is a compact subset of $\mathbb{C}$ for all $a \in A$.*

Proof. It follows from Theorem 7.2.11 that $\sigma^R(a) \cup \sigma(a)$ is the union of the compact set $\sigma(a)$ with some of its holes and hence $\sigma^R(a) \cup \sigma(a)$ is compact. □

Lemma 7.2.13. *Let A be a Banach algebra and $R \subseteq A$ an upper semi-regularity. Let $\{s_1, s_2, \ldots, s_n\}$ be a subset of $\mathbb{C}$ and $\{m_1, m_2, \ldots, m_n\}$ a subset of $\mathbb{N}$. If $a - s_j\mathbf{1} \in R$ for each $j = 1, 2, \ldots n$, then $\prod_{k=1}^{n}(a - s_k\mathbf{1})^{m_k} \in R$.*

Proof. Suppose that $a - s_j\mathbf{1} \in R$ for each $j = 1, 2, \ldots n$. It follows from Definition 7.2.1 (1) that $(a - s_j\mathbf{1})^{m_j} \in R$ for $j = 1, 2, \ldots n$. If $n = 1$ then we are done. If $n > 1$, we have that, because the greatest common divisor of the polynomials $(z - s_1)^{m_1}$ and $(z - s_2)^{m_2}$ in the polynomial ring $\mathbb{C}[z]$ is 1, there are polynomials p_{11} and p_{12} in $\mathbb{C}[z]$ such that

$$(z - s_1)^{m_1} p_{11}(z) + (z - s_2)^{m_2} p_{12}(z) = 1$$

and hence, by holomorphic functional calculus

$$(a - s_1\mathbf{1})^{m_1} p_{11}(a) + (a - s_2\mathbf{1})^{m_2} p_{12}(a) = \mathbf{1}.$$

Since $(a - s_1\mathbf{1})^{m_1}, p_{11}(a), (a - s_2\mathbf{1})^{m_2}$ and $p_{12}(a)$ are mutually commutative elements of A and $(a - s_1\mathbf{1})^{m_1}, (a - s_2\mathbf{1})^{m_2} \in R$, it follows from Definition 7.2.1 (2) that $(a - s_1\mathbf{1})^{m_1}(a - s_2\mathbf{1})^{m_2} \in R$. If $n = 2$ then we are done. If $n > 2$, because the greatest common divisor of the polynomials $(z - s_1)^{m_1}(z - s_2)^{m_2}$ and $(z - s_3)^{m_3}$ is 1, we obtain that $(a - s_1\mathbf{1})^{m_1}(a - s_2\mathbf{1})^{m_2}(a - s_3\mathbf{1})^{m_3} \in R$. If $n > 3$ and we proceed similarly, it follows after a finite number of steps that $\prod_{j=1}^{n}(a - s_j\mathbf{1})^{m_j} \in R$. □

Theorem 7.2.14 ([20], Theorem 19, p. 165)**.** *Let A be a Banach algebra and $R \subseteq A$ an upper semi-regularity. If $a \in A$, then $\sigma^R(p(a)) \subseteq p\left(\sigma^R(a)\right)$ for all non-constant polynomials p.*

Moreover, if $\sigma^R(b) \neq \emptyset$ for all $b \in A$, then $\sigma^R(p(a)) \subseteq p\left(\sigma^R(a)\right)$ for all polynomials p.

Proof. Let $a \in A$ and p a non-constant polynomial. Take any $\lambda \notin p\left(\sigma^R(a)\right)$. Define the polynomial h by

$$h(z) := p(z) - \lambda$$

for all $z \in \mathbb{C}$. Since h has at least one and at most finitely many roots, we can write

$$h(z) = \beta(z - s_1)^{n_1}(z - s_2)^{n_2} \cdots (z - s_m)^{n_m}$$

where $\beta, s_1, s_2, \ldots, s_m \in \mathbb{C}$, $\beta \neq 0$ and $n_1, n_2, \ldots, n_m \in \mathbb{N}$ with $n \geq 1$. We can assume that $s_j \neq s_k$ if $j \neq k$. It follows from holomorphic functional calculus that

$$p(a) - \lambda\mathbf{1} = \beta(a - s_1\mathbf{1})^{n_1}(a - s_2\mathbf{1})^{n_2} \cdots (a - s_m\mathbf{1})^{n_m}.$$

Since $p(s_j) = \lambda$ for $j = 1, 2, \ldots, m$ and $\lambda \notin p\left(\sigma^R(a)\right)$, we have that $s_j \notin \sigma^R(a)$. Therefore $a - s_j\mathbf{1} \in R$ for $j = 1, 2, \ldots m$. It follows from Lemma 7.2.13 that $\prod_{j=1}^{m}(a - s_j\mathbf{1})^{n_j} \in R$ and hence, by Corollary 7.2.10, we have that $\beta\prod_{j=1}^{m}(a - s_j\mathbf{1})^{n_j} = p(a) - \lambda\mathbf{1} \in R$. Therefore $\lambda \notin \sigma^R(p(a))$. Hence $\sigma^R(p(a)) \subseteq p\left(\sigma^R(a)\right)$ for all non-constant polynomials p.

Suppose that $\sigma^R(a) \neq \emptyset$. We must show that $\sigma^R(p(a)) \subseteq p\left(\sigma^R(a)\right)$ for all constant polynomials p. Take any $\lambda \in \mathbb{C}$ and let $p(z) := \lambda$ for all $z \in \mathbb{C}$. Note that, by Corollary 7.2.8, $\lambda\mathbf{1} - \alpha\mathbf{1} = (\lambda - \alpha)\mathbf{1} \in R$ for all $\alpha \in \mathbb{C}$ such that $\alpha \neq \lambda$. Therefore

$$\sigma^R(p(a)) = \sigma^R(\lambda\mathbf{1}) \subseteq \{\lambda\} = p\left(\sigma^R(a)\right)$$

and this completes the proof. □

The proof of the following lemma is part of the proof of ([20], Theorem 20, p. 166).

Lemma 7.2.15. *Let A be a Banach algebra and $R \subseteq A$ an upper semi-regularity satisfying the condition*

$$b \in R \cap A^{-1} \Rightarrow b^{-1} \in R.$$

Let $a \in A$ and U a neighbourhood of $\sigma(a) \cup \sigma^R(A)$. If $g : U \to \mathbb{C}$ is a holomorphic function on U with no zeros in $\sigma(a) \cup \sigma^R(a)$. Then $g(a) \in R$.

Proof. By Corollary 7.2.12, the set $\sigma(a) \cup \sigma^R(a)$ is a compact subset of $\mathbb{C}$. There exists a bounded neighbourhood U_0 of $\sigma(a) \cup \sigma^R(a)$ such that g has no zeros in the compact set $\overline{U}_0$ and $\overline{U}_0 \subseteq U$. By Runge's theorem, there exists a sequence of rational functions $\frac{p_n}{q_n}$ with poles outside $\overline{U}_0$, converging uniformly to g on $\overline{U}_0$. We can assume that p_n and q_n are non-constant polynomials for all $n \in \mathbb{N}$ and that $p_n(z) \neq 0$ for all $z \in \overline{U}_0$. We also have that $q_n(z) \neq 0$ for all $z \in \overline{U}_0$ and $n \in \mathbb{N}$. It follows from Theorem 7.2.14 that $0 \notin \sigma^R(p_n(a))$ and $0 \notin \sigma^R(q_n(a))$ and hence $p_n(a), q_n(a) \in R$ for all $n \in \mathbb{N}$. Because $q_n(a) \in A^{-1} \cap R$, it follows by assumption that $q_n(a)^{-1} \in R$ for all $n \in \mathbb{N}$. Therefore, by Lemma 7.2.9, $p_n(a)q_n(a)^{-1} \in R$ for all $n \in \mathbb{N}$. By holomorphic functional calculus, $(p_n(a)q_n(a)^{-1})$ converges to $g(a)$, so we have by Lemma 7.2.6 that $g(a) \in R$. □

Theorem 7.2.16 ([20], Theorem 20, p. 166). *Let A be a Banach algebra and $R \subseteq A$ an upper semi-regularity satisfying the condition*

$$b \in R \cap A^{-1} \Rightarrow b^{-1} \in R.$$

Then $\sigma^R(f(a)) \subseteq f\left(\sigma^R(a)\right)$ for all $a \in A$ and functions f which are analytic on a neighbourhood U of $\sigma(a) \cup \sigma^R(a)$ and non-constant on every component of U.

Furthermore, $\sigma^R(f(a)) \subseteq f\left(\sigma^R(a) \cup \sigma(a)\right)$ for all functions f which are analytic on a neighbourhood of $\sigma^R(a) \cup \sigma(a)$.

Proof. Let $a \in A$ with U a neighbourhood of $\sigma(a) \cup \sigma^R(a)$ and $f : U \to \mathbb{C}$ an analytic function which is non-constant on every component of U. Suppose that there is $\lambda \in \sigma^R(f(a)) \setminus f\left(\sigma^R(a)\right)$. Define $h : U \to \mathbb{C}$ by

$$h(z) := f(z) - \lambda$$

for all $z \in U$. The function h is analytic on $\sigma(a) \cup \sigma^R(a)$. We now consider two cases and obtain a contradiction in both.

1. Consider the case in which h has no zeros in $\sigma(a) \cup \sigma^R(a)$. Then, by the spectral mapping theorem, $h(a)$ is invertible in A. Note that, by Lemma 7.2.12, $\sigma(a) \cup \sigma^R(a)$ is compact. It follows from Lemma 7.2.15 that $h(a) \in R$. That means that $f(a) - \lambda\mathbf{1} \in R$ which is a contradiction, because $\lambda \in \sigma^R(f(a))$.

2. Consider the case in which h has at least one zero in $\sigma(a) \cup \sigma^R(a)$. As in the proof of inclusion (2.3.4), we can write

$$h(z) = (z - s_1)^{n_1}(z - s_2)^{n_2} \cdots (z - s_m)^{n_m} g(z)$$

where $\{s_1, s_2, \ldots, s_m\} = \{\alpha \in \sigma(a) \cup \sigma^R(a) : f(\alpha) = \lambda\}$ and g is analytic on U with no zeros in $\sigma(a) \cup \sigma^R(a)$. So, if we define $q(z) := \prod_{k=1}^{m}(z - s_k)^{n_k}$, then we have that

$$h(z) = q(z)g(z).$$

Therefore, by holomorphic functional calculus, we have

$$\begin{aligned} h(a) &= (a - s_1\mathbf{1})^{n_1}(a - s_2\mathbf{1})^{n_2} \cdots (a - s_m\mathbf{1})^{n_m} g(a) \\ &= q(a)g(a) \end{aligned}$$

where $g(a) \in A^{-1}$. It follows from Lemma 7.2.15 that $g(a) \in R$. Since $\lambda \notin f(\sigma^R(a))$ by assumption, $\{s_1, s_2, \ldots, s_m\} \cap \sigma^R(a) = \emptyset$. Therefore $a - s_j\mathbf{1} \in R$ for $j = 1, 2, \ldots, m$. It follows from Lemma 7.2.13 that $q(a) = (a - s_1\mathbf{1})^{n_1}(a - s_2\mathbf{1})^{n_2} \cdots (a - s_m\mathbf{1})^{n_m} \in R$. Because $q(a)$ and $g(a)$ commute, it now follows from Lemma 7.2.9 that $q(a)g(a) = h(a) = f(a) - \lambda\mathbf{1} \in R$ which is a contradiction.

Therefore $\sigma^R(f(a)) \subseteq f\left(\sigma^R(a)\right)$.

To prove the second part of the theorem, let f be any analytic function on U. Suppose there is $\lambda \in \sigma^R(f(a)) \backslash f\left(\sigma(a) \cup \sigma^R(a)\right)$. Let U_1 be the union of all the components of U on which f is identically equal to λ and $U_2 := U \backslash U_1$. Since $\lambda \notin f\left(\sigma(a) \cup \sigma^R(a)\right)$, we have that $U_1 \cap \sigma(a) \cap \sigma^R(A) = \emptyset$. Therefore f is analytical on U_2 which is a neighbourhood of $\sigma(a) \cup \sigma^R(a)$ and f is not identically equal to λ on every component of U_2. We now obtain, exactly as in the proof of the first part of this theorem, that $f(a) - \lambda\mathbf{1} \in R$ which is a contradiction. Therefore the inclusion $\sigma^R(f(a)) \subseteq f\left(\sigma^R(a) \cup \sigma(a)\right)$ holds. □

In the proposition that follows we see that some sets that we have introduced earlier are examples of upper semi-regularities. We mention that it follows from a remark by Müller in [20] on p. 164 that the set of Fredholm operators on a Banach space X is an upper semi-regularity of $\mathcal{L}(X)$. Furthermore, it follows from ([17], Proposition 7.1, p. 572) by L. Lindeboom and H. Raubenheimer that, in the case where J is a two-sided closed ideal of an algebra A and $T : A \to A/J$ is the canonical algebra homomorphism from A onto A/J, the set of almost invertible Fredholm elements of A relative to T is an upper semi-regularity. Also, Müller showed in [20] (Examples (1) and (3)) that Exp and $\mathcal{W}$ are an upper semi-regularities.

Proposition 7.2.17. *Let A be a Banach algebra, $T : A \to B$ a Banach algebra homomorphism and X the set of almost invertible elements of A. Then we have the following.*

1. *The sets $A^{-1}, \mathcal{F}, \mathcal{W}, \operatorname{Exp} A$, X and $\mathcal{F} \cap X$ are upper semi-regularities.*
2. *If T is Riesz, then $\mathcal{B}$ is an upper semi-regularity.*
3. *If T is bounded and $A/T^{-1}(0)$ is commutative, then $\mathcal{R}_{\mathrm{alm}}$ is an upper semi-regularity.*
4. *If T is Riesz and bounded, then the sets $\mathcal{R}$ and $\mathcal{R}_{\mathrm{alm}}$ are upper semi-regularities.*
5. *If A is commutative, then $\mathcal{B}$ is an upper semi-regularity.*
6. *If A is commutative and T is bounded, then the sets $\mathcal{R}$ and $\mathcal{R}_{\mathrm{alm}}$ are upper semi-regularities.*

Proof. 1. It is easy to see that A^{-1} is an upper semi-regularity, because $\mathcal{B}(\mathbf{1}, 1) \subseteq A^{-1}$ and products of invertible elements are invertible.

Note that, since $A^{-1} \subseteq \mathcal{W} \subseteq \mathcal{F}$ and $A^{-1} \subseteq \mathcal{F} \cap X$, the sets $\mathcal{W}, \mathcal{F}, X$ and $\mathcal{F} \cap X$ each contains a neighbourhood of $\mathbf{1}$. Also, $\operatorname{Exp} A$ is the component of A^{-1} which contains $\mathbf{1}$ and therefore contains a neighbourhood of $\mathbf{1}$.

We will now use the fact stated in Remark 7.2.2 to show that properties (1) and (2) of Definition 7.2.1 hold for the sets $\mathcal{F}, \mathcal{W}, X$ and $\operatorname{Exp} A$.

If $a, b \in \mathcal{F}$, then $Ta, Tb \in B^{-1}$, so that $T(ab) = TaTb \in B^{-1}$ and thus $ab \in \mathcal{F}$.

Take any $a, b \in \mathcal{W}$. Then $a = c_1 + d_1$ and $b = c_2 + d_2$ for some elements $c_1, c_2 \in A^{-1}$ and $d_1, d_2 \in T^{-1}(0)$. Therefore $ab = c_1c_2 + (c_1d_2 + d_1c_2 + d_1d_2)$. We see that $c_1c_2 \in A^{-1}$ and $c_1d_2 + d_1c_2 + d_1d_2 \in T^{-1}(0)$ and hence $ab \in \mathcal{W}$.

Take any $a, b \in X$ such that $ab = ba$. It follows from Lemma 2.1.15 that $ab \in X$. Hence $ab \in X$.

Take any two commuting elements $a, b \in \operatorname{Exp} A$. Then $a = e^{x_1}e^{x_2} \cdots e^{x_n}$ and $b = e^{y_1}e^{y_2} \cdots e^{y_m}$ where $x_i, y_j \in A$ for all i and j. The product ab is clearly in the set $\operatorname{Exp} A$.

It follows that properties (1), (2) and (3) of Definition 7.2.1 are satisfied for the sets $\mathcal{F}, \mathcal{W}$, $\mathcal{F} \cap X$ and $\operatorname{Exp} A$ respectively. Therefore the sets

$\mathcal{F}, \mathcal{W}, X$ and $\operatorname{Exp} A$ are upper semi-regularities.
Because $X \cap \mathcal{F}$ is the intersection of two upper semi-regularities, $X \cap \mathcal{F}$ is an upper semi-regularity as well.

2. Suppose that T is Riesz. Then, by Theorem 3.7, $\mathcal{B} = \mathcal{F} \cap X$ and hence $\mathcal{B}$ is an upper semi-regularity.
3. Suppose that T is bounded and that $A/T^{-1}(0)$ is commutative. It follows from Corollary 6.1.11 that $A^{-1} \subseteq \mathcal{R}_{\text{alm}}$. Therefore $\mathcal{R}_{\text{alm}}$ contains a neighbourhood of $\mathbf{1}$. By Proposition 6.1.4, the set $\mathcal{R}_{\text{alm}}$ is closed under finite products. Hence $\mathcal{R}_{\text{alm}}$ is an upper semi-regularity.
4. Suppose that T is bounded and has the Riesz property. It follows from Corollary 6.1.11 that $A^{-1} \subseteq \mathcal{R}$ and hence $\mathcal{R}$ contains a neighbourhood of $\mathbf{1}$. By Corollary 6.1.14, we have the equality $\mathcal{R} = \mathcal{F} \cap X$ and, by Corollary 6.1.15, we have $\mathcal{R}_{\text{alm}} = \mathcal{W}$. Since we have shown that $\mathcal{F} \cap X$ and $\mathcal{W}$ are upper semi-regularities, we conclude that the sets $\mathcal{R}_{\text{alm}}$ and $\mathcal{R}$ are upper semi-regularities as well.
5. Suppose that A is commutative. Then $\mathcal{B} = \mathcal{W}$ and hence $\mathcal{B}$ is an upper semi-regularity.
6. Suppose that A is commutative and T is bounded. Note that $A/T^{-1}(0)$ is also commutative. Since A is commutative $\mathcal{R} = \mathcal{R}_{\text{alm}}$. It follows from (3) of this proposition that $\mathcal{R} = \mathcal{R}_{\text{alm}}$ is an upper semi-regularity. $\square$

We apply Theorem 7.2.16 to the sets that we have shown to be upper semi-regularities to obtain and/or recover some spectral inclusion theorems. Most of these results have already been obtained in other chapters, some even in stronger forms. We indicate this after the proposition.

Proposition 7.2.18. *Let A be a Banach algebra and $T : A \to B$ a Banach algebra homomorphism with $a \in A$. We then have the following.*

1. *If U is a neighbourhood of $\varepsilon(a)$ and $f : U \to \mathbb{C}$ is an analytic function on U, then $\varepsilon(f(a)) \subseteq f(\varepsilon(a))$;*

If U is a neighbourhood of $\sigma(a)$, the function $f : U \to \mathbb{C}$ is analytic on U and non-constant on every component of U, then the following inclusion properties hold:

2. $\sigma(f(a)) \subseteq f(\sigma(a))$;
3. $\sigma_B(T(f(a))) \subseteq f(\sigma_B(Ta))$;
4. $\omega_T(f(a)) \subseteq f(\omega_T(a))$;
5. $\operatorname{acc}\sigma(f(a)) \subseteq f(\operatorname{acc}\sigma(a))$;

6. *$\sigma_B\left(T\left(f(a)\right)\right) \cup \operatorname{acc}\sigma\left(f(a)\right) \subseteq f(\sigma_B\left(T\left(a\right)\right) \cup \operatorname{acc}\sigma\left(a\right))$;*
7. *If T is Riesz, then $\omega_T^{\text{comm}}\left(f(a)\right) \subseteq f(\omega_T^{\text{comm}}\left(a\right))$;*
8. *If T is bounded and $A/T^{-1}(0)$ is commutative, then we have $\vartheta_T\left(f(a)\right) \subseteq f(\vartheta_T\left(a\right))$;*
9. *If T is bounded and has the Riesz property, then we obtain $\vartheta_T^{comm}\left(f(a)\right) \subseteq f(\vartheta_T^{comm}\left(a\right))$;*
10. *If T is bounded and has the Riesz property, then $\vartheta_T\left(f(a)\right) \subseteq f(\vartheta_T\left(a\right))$;*
11. *If A is commutative, then $\omega_T^{\text{comm}}\left(f(a)\right) \subseteq f(\omega_T^{\text{comm}}\left(a\right))$;*
12. *If A is commutative and T is bounded, then $\vartheta_T^{comm}\left(f(a)\right) \subseteq f(\vartheta_T^{comm}\left(a\right))$;*
13. *If A is commutative and T is bounded, then $\vartheta_T\left(f(a)\right) \subseteq f(\vartheta_T\left(a\right))$.*

Proof. We first prove (1) of this proposition. Let U be a neighbourhood of $\varepsilon\left(a\right)$ and $f : U \to \mathbb{C}$ an analytic function on U. The condition in Theorem 7.2.16 is satisfied by $\operatorname{Exp} A$, since, if $a \in \operatorname{Exp} A$, then $a^{-1} \in \operatorname{Exp} A$. Furthermore, $\sigma\left(a\right) \subseteq \varepsilon\left(a\right)$, because $\operatorname{Exp} A \subseteq A^{-1}$ and hence $\varepsilon\left(a\right) \cup \sigma\left(a\right) = \varepsilon\left(a\right)$. It now follows from the second part of Theorem 7.2.16 that $\varepsilon\left(f(a)\right) \subseteq f(\varepsilon\left(a\right))$.

We now prove the remainder of this proposition. Note that, except for $\operatorname{Exp} A$, each of the upper semi-regularities obtained in Proposition 7.2.17 contains the set of invertible elements of A. Therefore, the condition in Theorem 7.2.16 is satisfied by all of these upper semi-regularities.

Furthermore, $\sigma^R\left(a\right) \subseteq \sigma\left(a\right)$ for each upper semi-regularity R obtained in Proposition 7.2.17 except for $\varepsilon\left(a\right)$. Hence U is a neighbourhood of $\sigma\left(a\right) \cup \sigma^R\left(a\right) = \sigma\left(a\right)$ for each R except for $R = \operatorname{Exp} A$.

The remainder of this proposition now follows directly from Theorem 7.2.16 and Proposition 7.2.17. □

We compare these results of the above corollary with results obtained in previous chapters.

We have not previously looked at spectral inclusion results for the exponential spectrum of an element and hence (1) of the above proposition has not been obtained in any form earlier in this thesis.

Number (2) is of course a weaker form of the spectral mapping theorem.

Numbers (3), (4) and (6) have already been obtained in Theorem 2.3.3 and Proposition 2.3.14.

Number (5) is a one-way inclusion of the equality obtained in Lemma 2.3.2.

Number (7) is weaker than Corollary 3.12.

Number (8) is part of Theorem 6.2.10.

Number (9) is a weaker form of Proposition 6.2.15.

Number (10) is the same as Proposition 6.2.16.

Number (11) is not stronger than Theorem 2.3.3, since $\mathcal{B} = \mathcal{W}$ if A is commutative.

Numbers (12) and (13) are weaker than Theorem 6.2.10, since commutativity of A is a stronger condition than commutativity of $A/T^{-1}(0)$ and $\mathcal{R} = \mathcal{R}_{\text{alm}}$ if A is commutative.

7.3 Regularities

Definition 7.3.1 (Regularity)**.** ([14], Definition 1.2, p. 110) Let R be a subset of a Banach algebra A. Then R is called a regularity if it satisfies the following properties:

1. if $a \in R$ and $n \in \mathbb{N}$, then $a \in R$ if and only if $a^n \in R$;
2. if a, b, c, d are mutually commutative elements of A satisfying $ac+bd = \mathbf{1}$, then $a, b \in R$ if and only if $ab \in R$.

Definition 7.3.2 ([14], p. 110)**.** Let R be a regularity of a Banach algebra A and $a \in A$. The corresponding spectrum $\sigma^R(a)$ is defined by

$$\sigma^R(a) := \{\lambda \in \mathbb{C} : a - \lambda\mathbf{1} \notin R\}.$$

It is clear now that a subset of a Banach algebra is a regularity if and only if it is both an upper and a lower semi-regularity.

The following theorem is now obtained directly from Theorem 7.1.6 and Theorem 7.2.16. This theorem is found in [14], a paper which was published when the notions of upper and lower semi-regularities had not yet been defined. The proof in [14] is therefore done from the definition of a regularity and not by combining results for upper and lower semi-regularities as we do.

Theorem 7.3.3 ([14], Theorem 1.4, p. 111)**.** *Let $R \subseteq A$ be a regularity. Then*

$$f(\sigma^R(a)) = \sigma^R(f(a))$$

for all $a \in A$ and f analytic on a neighbourhood U of $\sigma(a)$ and non-constant on every component of U.

Proof. Since R is a lower semi-regularity, it contains A^{-1} and hence the condition of Theorem 7.2.16 is satisfied. The corollary follows immediately. □

By using Theorem 7.3.3 we can now recover Corollary 7.1.19 and the second part of Lemma 2.3.2 without using spectral inclusion results of earlier chapters. Below is an alternative proof of Corollary 7.1.19 and the second part of Lemma 2.3.2.

Proof. Let X be the set of almost invertible elements if A. We have shown that the sets $\mathcal{F}, X$ and $X \cap \mathcal{F}$ are all upper and lower semi-regularities; if T is Riesz then $\mathcal{B}$ is both an upper and a lower semi-regularity and if T is bounded and Riesz then $\mathcal{R}$ is both an upper and a lower semi-regularity. Corollary 7.1.19 and the second part of Lemma 2.3.2 follow from Theorem 7.3.3. □

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