Powered Addition as Modelling Technique for Flow Processes

by

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Thesis presented in partial fulfilment of the requirements for the degree of Master of Sciences

at Stellenbosch University

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March 2010
Declaration

By submitting this thesis electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the owner of the copyright thereof (unless to the extent explicitly otherwise stated) and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

Signature: ..........................  Date: ....../....../.............
Abstract

The interpretation – and compilation of predictive equations to represent the general trend – of collected data is aided immensely by its graphical representation. Whilst, by and large, predictive equations are more accurate and convenient for use in applications than graphs, the latter is often preferable since it visually illustrates deviations in the data, thereby giving an indication of reliability and the range of validity of the equation. Combination of these two tools – a graph for demonstration and an equation for use – is desirable to ensure optimal understanding. Often, however, the functional dependencies of the dependent variable are only known for large and small values of the independent variable; solutions for intermediate quantities being obscure for various reasons (e.g. narrow band within which the transition from one regime to the other occurs, inadequate knowledge of the physics in this area, etc.). The limiting solutions may be regarded as asymptotic and the powered addition to a power, $s$, of such asymptotes, $f_0$ and $f_\infty$, leads to a single correlating equation that is applicable over the entire domain of the dependent variable. This procedure circumvents the introduction of ad hoc curve fitting measures for the different regions and subsequent, unwanted jumps in piecewise fitted correlative equations for the dependent variable(s). Approaches to successfully implement the technique for different combinations of asymptotic conditions are discussed. The aforementioned method of powered addition is applied to experimental data and the semblances and discrepancies with literature and analytical models are discussed; the underlying motivation being the aspiration towards establishing a sound modelling framework for analytical and computational predictive measures. The purported procedure is revealed to be highly useful in the summarising and interpretation of experimental data in an elegant and simplistic manner.
Opsomming

Die interpretasie – en samestelling van vergelykings om die algemene tendens voor te stel – van versamelde data word onoorsienbaar bygestaan deur die grafiese voorstelling daarvan. Ten spyte daarvan dat vergelykings meer akkuraat en geskik is vir die gebruik in toepassings as grafieke, is laasgenoemde dikwels verskieslik aangesien dit afwykings in die data visueel illustreer en sodoende ‘n aanduiding van die betroubaarheid en omvang van geldigheid van die vergelyking bied. ‘n Kombinasie van hierdie twee instrumente – ‘n grafie vir demonstrasie en ‘n vergelyking vir aanwending – is wenslik om optimale begrip te verseker. Die funksionele afhanklikheid van die afhanklike veranderlike is egter dikwels slegs bekend vir groot en klein waardes van die onafhanklike veranderlike; die oplossings by intermediêre hoeveelhede onduidelik as gevolg van verskeie redes (waaronder, bv. ‘n smal band van waardes waarbinne die oorgang tussen prosesse plaasvind, onvoldoende kennis van die fisika in hierdie area, ens.). Beperkende oplossings / vergelykings kan as asimptote beskou word en magsaddisie tot ‘n mag, $s$, van sodanige asimptote, $f_0$ en $f_\infty$, lei tot ‘n enkel, saamgestelde oplossing wat toepaslik is oor die algehele domein van die onafhanklike veranderlike. Dié prosedure voorkom die instelling van ad hoc passingstegnieke vir die verschillende gebiede en die gevolglike ongewensde spronge in stuksgewys-passende vergelykings van die afhanklike veranderlike(s). Na aanleiding van die moontlike kombinasies van asimptotiese toestande word verschillende benaderings vir die suksesvolle toepassing van hierdie tegniek bespreek. Die bogemelde metode van magsaddisie word toegepas op eksperimentele data en die ooreenkoms van verskillende modelle met literatuur en analitiesise modelle bespreek; die onderliggend motivering ‘n strewe na die daarstelling van ‘n modellering-voorspellingsraamwerk vir analitiese- en rekenaarvoorspellingsmaatreëls. Die voorgestelde prosedure word aangetoon om, op ‘n elegante en eenvoudige wyse, hoogs bruikbaar te wees vir die lees en interpretasie van eksperimentele data.
Acknowledgements

I would like to express my sincere gratitude to the following people for their contribution:

• My supervisor, Prof. Jean Prieur du Plessis, for patient academic guidance, long- ganimity and an open-door policy throughout the course of both my undergraduate and postgraduate studies at Stellenbosch University;

• Prof. Britt Halvorsen at Høgskolen i Telemark (Telemark University College), Porsgrunn, Norway, for being my host and generous financial assistance during my visit to their institution during the months of August – October 2008; for encouraging me to write a paper for submission to The 5th International Conference on Computational & Experimental Methods in Multiphase and Complex Flow (Multiphase Flow V); and for making it financially possible to attend the conference in New Forest, UK, from 15 - 17 June 2009;

• Dr. Finn Haugen for writing the experiment-specific software in LABVIEW that we required to perform the experiments at Høgskolen i Telemark (Telemark University College), Porsgrunn, Norway.

• The South African National Research Foundation (NRF) for financial support.
vir die Twee Outes
# Contents

Declaration i

Abstract ii

Acknowledgements iv

Dedication v

Nomenclature ix

Introduction xiii

1 Powered addition as curve fitting technique 1

1.1 Asymptotic behaviour of transfer processes 1

1.2 Shifting of the matching curve 4

1.2.1 Increasing dependence 4

1.2.2 Decreasing dependence 6

1.2.2.1 Bounded from below 6

1.2.2.2 Bounded from above 7

1.2.3 Only limiting values known 9

1.2.4 Crossing of one limiting solution 15
CONTENTS

1.3 Normalisation to obtain one horizontal asymptote ........................................ 17
1.4 Critical point and shifting-exponent ............................................................ 18

2 Flow in straight-through diaphragm valves ...................................................... 21
  2.1 Definitions of pressures and heads ............................................................. 22
  2.2 Choice of Reynolds number ..................................................................... 25
  2.3 Mbiya’s empirical correlation ................................................................. 26
  2.4 Powered addition applied to Mbiya’s work ............................................. 28

3 Flow through a packed bed .............................................................................. 34
  3.1 Ergun equation ......................................................................................... 35
  3.2 RUC model ............................................................................................. 38
    3.2.1 Granular porous media .................................................................... 39
    3.2.2 Spongelike porous media ............................................................... 40
    3.2.3 Unidirectional two-dimensional fibre-bed porous media ............. 42

4 Fluidised bed ................................................................................................ 44
  4.1 Newtonian fluid ......................................................................................... 45
    4.1.1 Experimental procedure .................................................................... 45
      4.1.1.1 Superficial velocity of the traversing fluid ............................... 47
      4.1.1.2 Porosity of the packed bed .................................................... 47
    4.1.2 Asymptotic dependencies ................................................................ 49
      4.1.2.1 The lower asymptote .............................................................. 49
      4.1.2.2 The upper asymptote .............................................................. 51
    4.1.3 Powered addition of the asymptotes ............................................. 52
      4.1.3.1 Critical point and shifting-exponent ..................................... 52
## CONTENTS

4.1.3.2 Crossing of the upper asymptote ............................................. 53
4.1.4 Correlation of experimental results .............................................. 54
4.2 Non-Newtonian fluid ........................................................................ 58
  4.2.1 Asymptotic dependencies ................................................................ 58
    4.2.1.1 The lower asymptote .............................................................. 58
    4.2.1.2 The upper asymptote .............................................................. 60
  4.2.2 Powered addition of the asymptotes ............................................ 60
    4.2.2.1 Critical point and shifting-exponent ...................................... 61
  4.2.3 Correlation of experimental results ............................................ 62

5 Conclusion / Closure ............................................................................. 64

A Fluid classification ................................................................................ 66
  A.1 Newtonian flow ............................................................................... 66
  A.2 Non-Newtonian flow ....................................................................... 68

B Derivation of Slatter Reynolds number ................................................ 70

C Plots with Mbiya’s data sets ................................................................. 77

D Høgskolen i Telemark (Telemark University College) data sets .......... 98
# Nomenclature

## Constants

<table>
<thead>
<tr>
<th>Symbol</th>
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## Variables

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<td>([m^{-1}])</td>
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<tr>
<td>( A )</td>
<td>arbitrary coefficient</td>
<td>[-]</td>
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<tr>
<td>( A_{\text{ann}} )</td>
<td>area of annulus</td>
<td>([m^2])</td>
</tr>
<tr>
<td>( A_c )</td>
<td>cross-sectional area of bed</td>
<td>([m^2])</td>
</tr>
<tr>
<td>( A_p )</td>
<td>surface area of single, non-spherical particle</td>
<td>([m^2])</td>
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<tr>
<td>( A_{\text{plug}} )</td>
<td>area of plug</td>
<td>([m^2])</td>
</tr>
<tr>
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<tr>
<td>( B )</td>
<td>arbitrary coefficient</td>
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</tr>
<tr>
<td>( c )</td>
<td>constant</td>
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</tr>
<tr>
<td>( c_d )</td>
<td>form drag coefficient</td>
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</tr>
<tr>
<td>( C_\Omega )</td>
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<td>( d )</td>
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<td>$k_{v,c}$</td>
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<td>$K$</td>
<td>fluid consistency index</td>
<td>$[Pa.s^n]$</td>
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<tr>
<td>$L$</td>
<td>length of straight channel / bed height</td>
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<td>$m_0$</td>
<td>total or bulk mass</td>
<td>[kg]</td>
</tr>
<tr>
<td>$m_f$</td>
<td>mass of traversing fluid</td>
<td>[kg]</td>
</tr>
<tr>
<td>$m_s$</td>
<td>solid mass</td>
<td>[kg]</td>
</tr>
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<td>$M$</td>
<td>empirically determined, constant coefficient</td>
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<td>$n$</td>
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<td>$N$</td>
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</tr>
<tr>
<td>$p$</td>
<td>pressure</td>
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<td>$p_H$</td>
<td>total pressure head</td>
<td>$[N/m^2]$</td>
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<tr>
<td>$q$</td>
<td>superficial velocity / specific discharge</td>
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<td>$q_{mf}$</td>
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<td>$Q_{ann}$</td>
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<td>flux through plug</td>
<td>$[m^3/s]$</td>
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<td>$r$</td>
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<td>$R$</td>
<td>pipe radius</td>
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<td>$Re$</td>
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<td>$Re_s$</td>
<td>Slatter Reynolds number</td>
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<tr>
<td>$Re_{3,c}$</td>
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</tr>
<tr>
<td>$Re_p$</td>
<td>particle Reynolds number</td>
<td>[-]</td>
</tr>
<tr>
<td>$Re_s$</td>
<td>Reynolds number for packed bed of spheres</td>
<td>[-]</td>
</tr>
<tr>
<td>$s$</td>
<td>arbitrary shifting exponent</td>
<td>[-]</td>
</tr>
<tr>
<td>$t$</td>
<td>arbitrary shifting exponent</td>
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<td>$U_0$</td>
<td>total or bulk volume</td>
<td>$[m^3]$</td>
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<tr>
<td>$U_f$</td>
<td>volume of fluid phase</td>
<td>$[m^3]$</td>
</tr>
<tr>
<td>$U_s$</td>
<td>volume of solid phase</td>
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</tr>
<tr>
<td>$\nu$</td>
<td>velocity</td>
<td>$[m/s]$</td>
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<td>Symbol</td>
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<td>Unit</td>
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<td>--------</td>
<td>---------------------------------------------------------------------------</td>
<td>--------</td>
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<td>$\overline{u}_{\text{ann}}$</td>
<td>corrected mean velocity in the annulus</td>
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<td>velocity of plug</td>
<td>$[m/s]$</td>
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<td>$v_x$</td>
<td>velocity component in $x$-direction</td>
<td>$[m/s]$</td>
</tr>
<tr>
<td>$v_y$</td>
<td>velocity component in $y$-direction</td>
<td>$[m/s]$</td>
</tr>
<tr>
<td>$v_z$</td>
<td>velocity component in $z$-direction</td>
<td>$[m/s]$</td>
</tr>
<tr>
<td>$V_p$</td>
<td>volume of single, non-spherical particle</td>
<td>$[m^3]$</td>
</tr>
<tr>
<td>$w$</td>
<td>specific weight</td>
<td>$[kg/m^2s^2]$</td>
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<tr>
<td>$x$</td>
<td>independent variable</td>
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</tr>
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<td>$x^*$</td>
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<td>[-]</td>
</tr>
<tr>
<td>$x_B$</td>
<td>arbitrary constant</td>
<td>[-]</td>
</tr>
<tr>
<td>$x_c$</td>
<td>independent variable at central or critical point</td>
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</tr>
<tr>
<td>$Y$</td>
<td>normalised dependent variable</td>
<td>[-]</td>
</tr>
<tr>
<td>$z$</td>
<td>height above arbitrary reference point</td>
<td>[m]</td>
</tr>
<tr>
<td>$Z$</td>
<td>normalised independent variable</td>
<td>[-]</td>
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**Greek letters**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Unit</th>
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<tbody>
<tr>
<td>$\alpha$</td>
<td>exponent in asymptotic solution for $x \to 0$ / arbitrary exponent</td>
<td>[-]</td>
</tr>
<tr>
<td>$\beta$</td>
<td>exponent in asymptotic solution for $x \to \infty$ / arbitrary exponent</td>
<td>[-]</td>
</tr>
<tr>
<td>$\dot{\gamma}$</td>
<td>shear rate</td>
<td>$[s^{-1}]$</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>change in stream-wise property</td>
<td>[-]</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>bed porosity / void fraction</td>
<td>[-]</td>
</tr>
<tr>
<td>$\epsilon_0$</td>
<td>porosity at incipient fluidisation</td>
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<tr>
<td>$\eta$</td>
<td>apparent viscosity</td>
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<tr>
<td>$\theta$</td>
<td>valve opening coefficient</td>
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<td>$\kappa$</td>
<td>hydrodynamic permeability</td>
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<td>$\mu$</td>
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<td>$\nu$</td>
<td>kinematic viscosity</td>
<td>$[s^{-1}]$</td>
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<td>functional dependence of pressure drop for $q \to 0$</td>
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<td>functional dependence of pressure drop for $q \to \infty$</td>
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<td>$\rho$</td>
<td>mass density</td>
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<td>total or bulk mass density</td>
<td>$[kg/m^3]$</td>
</tr>
<tr>
<td>$\rho_f$</td>
<td>mass density of traversing fluid</td>
<td>$[kg/m^3]$</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>mass density of solids</td>
<td>$[kg/m^3]$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>shear stress</td>
<td>$[N/m^2]$</td>
</tr>
<tr>
<td>$\tau_0$</td>
<td>shear stress at the pipe wall</td>
<td>$[N/m^2]$</td>
</tr>
</tbody>
</table>
\( \tau_y \) yield stress \([N/m^2]\)

\( \phi_p \) particle shape factor (sphericity) \([-\])

\( \Phi \) variable defined for simplicity in Ergun equation \([-\])

\( \psi \) geometric factor \([-\])

\( \Psi \) Waddell sphericity factor \([-\])

**Vectors and Tensors**

\( f_b \) body forces \([N/kg]\)

\( \hat{n} \) unit vector in stream-wise direction \([-\])

\( \nabla \) del operator \([m^{-1}]\)

\( \sigma \) stress tensor \([N/m^2]\)

\( \tau \) viscous stress dyadic \([N/m^2]\)

**Acronyms**

- CHE: Churchill-Usagi Equation
- RUC: Representative Unit Cell
Introduction

The dependence of modern engineering research on precise, credible experimental and computational practices is undeniable. These procedures provide the requisite predictive information needed for design purposes and the in-depth understanding of complex processes. It is common practice to represent the general trend in a set of such collected data by drawing a line through the individual datum points on the plot. Correlation between the drawn predictive curve and the data is then evaluated against some norm, e.g. least squares fit to a straight line or polynomial function passing through the data, visual inspection, etc. Theoretical knowledge of the functional behaviour is helpful but not a prerequisite for the construction of graphical correlation and thus a line best suited to the particular problem is chosen – the better the predictive line on the graphical presentation corresponds to the physical reality, especially in the limits of the independent variable, the greater the trustworthiness of obtained results.

If it is possible to accurately determine or predict the asymptotic behaviour – traits at extreme values of the independent variable – of the dependent variable under consideration, the results can usually be presented in a neat and elegant format. The basic procedure of asymptotic matching by straightforward addition of the expressions for the asymptotic conditions is a method that has been been in use for some time, especially in engineering practice. However, the article by Churchill & Usagi [1], which appeared in 1972, for the first time really formalised the use and accentuated the wide application possibilities of the method and variations thereof. Their method yields an equation of simple form with one arbitrary constant that interpolates between the limiting solutions; the value of which may be determined by either experimental or theoretical procedure. The routine is applicable to any phenomenon which varies uniformly between known, limiting solutions and is especially useful for the evaluation and summarising of experimental and computational data. Furthermore it is particularly convenient for design purposes as it yields an expression that is relevant over the entire domain of the dependent variable and has the same form for all correlations. Whether it presents an exact representation of the transfer process cannot be proven scientifically, yet the method is widely applicable and accepted.
In the first chapter an outline is given of powered addition. The articles by Churchill & Usagi [1; 2] form the backbone of this chapter. Different scenarios of the limiting functions and / or values are investigated and simple examples provided. Curve adjustment and the importance of the point of intersection of the asymptotes are discussed. Chapter 2 and 4 sees the application of the method to the collected experimental data for two diverse processes. The results of the former chapter were presented at The 2nd Southern African Conference on Rheology (SASOR), Cape Peninsula University of Technology, Cape Town, 6 - 8 October 2008 [3]; those of the latter were published in the proceedings of The 5th International Conference on Computational & Experimental Methods in Multiphase and Complex Flow (Multiphase Flow V), New Forest, UK, 15 - 17 June 2009 [4]. Chapter 3 serves as précis of two approaches used in predicting pressure drop over a packed bed or porous medium – the one (Ergun equation) being itself an example of powered addition with an exponent of unity, the other (RUC model) forming a keystone to the work of the following chapter. Supplementary material and the original data sets of the experimental investigations conducted at Høgskolen i Telemark (Telemark University College), Porsgrunn, Norway are collected in the appendices.
Chapter 1

Powered addition as curve fitting technique

Powered addition of expressions valid for two opposing ranges as described by Churchill and Usagi [1; 2; 5; 6] is a procedure used to produce a combined result which is valid for both of these ranges. Since each of the limiting expressions predominates in their respective regions of applicability, a unified model can be obtained using such an ‘asymptote matching’ technique.

1.1 Asymptotic behaviour of transfer processes

In many continuum processes, such as momentum and thermal transfer processes, the value of a sought after parameter is expressible as a function of certain known parameter(s) at low and high values. These limiting solutions for large and small values of the independent variable(s) may be regarded as asymptotic conditions of the dependent variable.

By stating that
\[
f\{x\} \to g\{x\} \quad \text{as} \quad x \to a
\]

it is meant that
\[
\left( \frac{f\{x\}}{g\{x\}} \right) \to 1 \quad \text{as} \quad x \to a.
\]

In other words, it is said that \( f \) is asymptotic to \( g \) as \( x \to a \) [7]. Very often the functional expression of the dependent variable is in the form of a power dependency upon some independent variable, \( x \).
1.1 Asymptotic behaviour of transfer processes

Let the functional dependence, \( f \), of such a process be described by

\[
\begin{align*}
 f & \to f_0 \{x\} = A x^\alpha \quad \text{as} \quad x \to 0, \\
 f & \to f_\infty \{x\} = B x^\beta \quad \text{as} \quad x \to \infty.
\end{align*}
\]  

(1.1) (1.2)

Here equations (1.1) and (1.2) denote the functional expressions at the lower and upper extremal values of \( x \) respectively. However, solutions for intermediate cases are seldom as simply expressed. (It is important to take note that, for the discussion to follow, the explicit expression of the asymptotes in terms of a power dependency is not permutable, i.e. the lower asymptote is always associated with coefficient \( A \) and exponent \( \alpha \); the upper with \( B \) and \( \beta \)).

The direct summation of two such asymptotic solutions or approximations is often effected to obtain a single solution that holds over the entire range of the independent variable, i.e.

\[
f \{x\} = f_0 \{x\} + f_\infty \{x\} = Ax^\alpha + Bx^\beta.
\]

(1.3)

Equation (1.3) may now be considered as a matching or coupled curve connecting the two dependencies as it satisfies the asymptotic conditions and also provides values for \( f \) at intermediate values of the independent variable, \( x \).

![Figure 1.1: Linear plot of the function \( f \{x\} = 1 + x \).](http://scholar.sun.ac.za)
As an example, consider the very simple case of a function \( f(x) = 1 + x \); i.e. \( A = 1, B = 1, \alpha = 0 \) and \( \beta = 1 \) in equation (1.3). In the above formulation this corresponds to the case,

\[
\begin{align*}
  f(x) &\to 1 \quad \text{as} \quad x \to 0, \\
  f(x) &\to x \quad \text{as} \quad x \to \infty.
\end{align*}
\]  

(1.4) (1.5)

Hence the asymptotes governing the behaviour of the coupled function will be given by

\[
\begin{align*}
  f_0(x) &= 1 \\
  f_\infty(x) &= x.
\end{align*}
\]

Plotting this relation on a linear-linear Cartesian scale, generates a straight line as shown in Figure 1.1. It is only once the function is drawn on a log-log graph that more insight is gained; the asymptotic behaviour that results from addition of the functional expressions at the extremal values now becomes apparent. This is illustrated in Figure 1.2.

![Log-log plot of the function \( f(x) = 1 + x \).](http://scholar.sun.ac.za)

The advantage of using logarithmic coordinates when plotting data is that equal percentage changes yield equal displacements over the entire range, where-as with
1.2 Shifting of the matching curve

arithmetic coordinates (Cartesian axes) on the other hand, the displacement increases in accordance with the magnitude of the variable. In other words, logarithmic coordinates display percentage deviations and perceptually suppress these deviations compared to arithmetic plots – the former may thus obscure the magnitude of scatter in the data, the latter distort such scatter unduly by displaying absolute differences.

It is important to note that, as seen in Figure 1.1, the matched curve merely approaches, yet never reaches, the upper limiting functional value. An increase in the independent variable leads to the diminishing influence of the lower asymptotic function on the overall solution, which only becomes visually apparent once the solution is plotted on log-log axes as in Figure 1.2. The method is therefore best suited to approximate the general trend in a process, rather than predict the exact values of the constituent limiting functions.

1.2 Shifting of the matching curve

Frequently the values of the dependent variable at the transition between the asymptotic extremities do not lie exactly on this matching solution. Churchill & Usagi [1; 2; 5; 6] demonstrated that the use of powered addition, the most general form of which is shown in equations (1.6) and (1.7) below, may lead to dramatic improvement in correlation with experimental data

\[
 f^s\{x\} = f^s_0\{x\} + f^s_\infty\{x\},
 \]

whence

\[
 f\{x\} = [f^s_0\{x\} + f^s_\infty\{x\}]^{1/s}.
 \]  

By adjusting the value of the shifting exponent, \(s\), the level of the solution may be modified so as to more closely trace the expected or empirical values, yielding better correspondence between predictive equation and experimental results. The right hand side of equation (1.7) may be considered as the \(s^{th}\) order sum of the asymptotic solutions [1; 2; 5].

1.2.1 Increasing dependence

When the dependent variable is an increasing power of the independent variable, in other words if the power of \(x\) in equation (1.3) is greater at the higher limit, that is

\[
 \alpha < \beta,
 \]  

(1.8)
1.2 Shifting of the matching curve

Figure 1.3: Linear plot of the function \( f\{x\} = [f_0^s\{x\} + f_\infty^s\{x\}]^{1/s} = [1 + x^s]^{1/s} \) for varying values of the shifting exponent, \( s \).

the expression

\[
f\{x\} = [(A x^a)^s + (B x^\beta)^s]^{1/s},
\]

(1.9)
is usually desirable for interpolation between the extremal values.

Theoretically the matched function in equation (1.9) will have no upper bound and will only be bounded from below by the the functional expression for small values of \( x \); i.e the term \( A x^a \) in equation (1.3) will form a lower bound on the values that the independent variable may take on. The arbitrary exponent, \( s \) will now have a positive value. The shifting effect obtained is illustrated in Figures 1.3 and 1.4; the same conditions were used as in equations (1.4) and (1.5) to obtain

\[
f\{x\} = [f_0^s\{x\} + f_\infty^s\{x\}]^{1/s} = [1 + x^s]^{1/s}.
\]

(1.10)

For the sake of simplicity, the function-notation \((f_0^s\) and \(f_\infty^s\)) will henceforth be favoured over the explicit expression in terms of power dependencies.
1.2 Shifting of the matching curve

In some instances the dependence of \( f\{x\} \) decreases with an increase in the independent variable, i.e.

\[
\alpha > \beta, \tag{1.11}
\]

in equation (1.3). Two possibilities now exist – the asymptotes may either form the lower bound or the upper bound of the resulting matched curve; knowledge of the process being modelled and/or experimental data will dictate the specific case.

1.2.2 Decreasing dependence

In some instances the dependence of \( f\{x\} \) decreases with an increase in the independent variable, i.e.

\[
\alpha > \beta, \tag{1.11}
\]

in equation (1.3). Two possibilities now exist – the asymptotes may either form the lower bound or the upper bound of the resulting matched curve; knowledge of the process being modelled and/or experimental data will dictate the specific case.

1.2.2.1 Bounded from below

Decreasing dependence upon the independent variable is such that the solutions for extremal values – i.e. the functional expressions for the asymptotes – bind all possible solutions to the process from below. Suppose, for the sake of an illustrative example,
1.2 Shifting of the matching curve

Figure 1.5: Linear plot of the function \( f\{x\} = \left( \frac{1}{x} \right)^s + 1 \right)^{1/s} \) for varying values of the shifting exponent, \( s \).

that the limiting solutions to such a process are given by the simple relations

\[
\begin{align*}
    f\{x\} &\to \frac{1}{x} \text{ as } x \to 0, \quad (1.12) \\
    f\{x\} &\to 1 \text{ as } x \to \infty. \quad (1.13)
\end{align*}
\]

This corresponds to equation (1.3) with coefficients \( A = 1, B = 1 \) and exponents, \( \alpha = -1 \) and \( \beta = 0 \). The matched solution, raised to the shifting exponent will thus be

\[
    f\{x\} = \left( \frac{1}{x} \right)^s + 1 \right)^{1/s}. \quad (1.14)
\]

The result of varying the values of the shifter, \( s \), is graphically represented on Cartesian and log-log axes in Figures 1.5 and 1.6 respectively.

1.2.2.2 Bounded from above

The asymptotes, \( f_0\{x\} \) and \( f_\infty\{x\} \), of the process being modelled form an upper bound on the possible values that the function can assume. Using the formulation of
equation (1.3) consider, as an example, the very simple case in which the asymptotes constituting the matched equation are given by

\[ f\{x\} \to x \quad \text{as} \quad x \to 0, \quad (1.15) \]
\[ f\{x\} \to 1 \quad \text{as} \quad x \to \infty. \quad (1.16) \]

To ensure that the matched solution approaches the limiting functions from below, the shifting exponent now needs to take on a negative value. However, the obtained curve will still approach the asymptotes as \(|s|\) increases; illustrated in Figures 1.7 and 1.8.

The introduction of a negative value for \(s\) may be circumvented by taking the reciprocal of the original dependent variable, i.e. by defining

\[ \frac{1}{g\{x\}} = \frac{1}{Ax^p} + \frac{1}{Bx^q} = \frac{1}{g_0\{x\}} + \frac{1}{g_\infty\{x\}}, \quad (1.17) \]

before it is raised to \(s\), ensures that \(s > 0\). Applying this to the above example, outlined in equations (1.15) and (1.16), yields the function

\[ f\{x\} = \frac{x}{1 + x}, \quad (1.18) \]
1.2 Shifting of the matching curve

and in so doing Figures 1.7 and 1.8 in \( g\{x\} \) are converted to Figures 1.5 and 1.6 in \( f\{x\} = 1/g\{x\} \).

1.2.3 Only limiting values known

In many cases the functional dependence of the independent variable is known at the extremal values. Often, however, only the limiting values in both limits, i.e. \( f\{0\} \) and \( f\{\infty\} \), are known beforehand. In cases such as these the straight-forward application of equation (1.6) is not possible and an alternative approach is to be followed.

To commence, a functional dependence of the dependent variable upon the independent variable is postulated for either \( x \to 0 \) or \( x \to \infty \). Any convenient function which approximates the behaviour of the data may be chosen. Churchill & Usagi [2] recommend the use of a power function since its use is widely applicable and the simplicity of such a function ties in with that of equation (1.6) and the philosophy behind the method in general. Once an applicable function for either of the limiting values has been chosen it is, as per the discussion in Section 1.1, matched to the constant value.
that binds the process in the other limit (it is important to choose the approximating function such that no singularities are introduced once the functions are combined).

As an example, the power function

\[ f_0\{x\} = f\{0\} + (f\{\infty\} - f\{0\}) \left( \frac{x}{x_A} \right)^\alpha, \]  

may be suggested to represent the functional dependence at the lower limiting value \([2]\). Here \(x_A\) is an arbitrary constant and \(\alpha\) an arbitrary exponent; the influence of these values on the obtained curves will be discussed shortly. The function in equation (1.19) is chosen such that

\[ f_0\{x\} \rightarrow f\{0\} \quad \text{as} \quad x \rightarrow 0, \]

i.e.

\[ (f\{\infty\} - f\{0\}) \left( \frac{x}{x_A} \right)^\alpha \rightarrow 0 \quad \text{as} \quad x \rightarrow 0. \]  

In equation (1.20) the coefficient \((f\{\infty\} - f\{0\})\) is a constant value and therefore it should hold that

\[ \left( \frac{x}{x_A} \right)^\alpha \rightarrow 0 \quad \text{as} \quad x \rightarrow 0, \]  

\[ f = \frac{f_0}{10^0} \]
1.2 Shifting of the matching curve

Figure 1.9: Postulated function $f_0(x) = f\{0\} + (f\{\infty\} - f\{0\}) (x/x_A)^\alpha$ for constant value of arbitrary exponent, $\alpha = 1.5$, and varying values of the arbitrary constant, $x_A$.

which will only be the case if both the arbitrary constant and exponent is such that $x_A > 0$ and $\alpha \geq 0$; a restriction that should be kept in mind when choosing these values.

The postulated function and upper limiting asymptote will now intersect where

$$f_0(x) = f\{\infty\}$$

that is

$$f\{0\} + (f\{\infty\} - f\{0\}) \left(\frac{x}{x_A}\right)^\alpha = f\{\infty\},$$

whence, after rearrangement and division,

$$\left(\frac{x}{x_A}\right)^\alpha = 1.$$  (1.24)

It thus follows from equation (1.24) that $x = x_A$ at the intersection of these two functions; by changing the value of $x_A$ the point of intersection may be altered. In Section 1.4 the importance of this value, the so-called critical point, will be discussed. Plotting of the postulated function in equation (1.19) for different values of the arbitrary constant $x_A$ – illustrated in Figure 1.9 – graphically clarifies its influence.
1.2 Shifting of the matching curve

The influence of the arbitrary exponent, \( \alpha \), becomes clear when equation (1.19) is rearranged as

\[
\left( \frac{x}{x_A} \right)^\alpha = \frac{f_0(x) - f(0)}{f^{\infty} - f(0)},
\]

and the logarithm taken on either side to yield

\[
\log \left( \frac{x}{x_A} \right)^\alpha = \log \left( \frac{f_0(x) - f(0)}{f^{\infty} - f(0)} \right),
\]

i.e.

\[
\alpha (\log x - \log x_A) = \log (f_0(x) - f(0)) - \log (f^{\infty} - f(0)).
\]

Defining a new variable, \( x^* \), equation (1.26) thus takes the form of a linear depen-
1.2 Shifting of the matching curve

Figure 1.11: Application of powered addition with only limiting values known. A function of the form $f_0(x) = f(0) + (f(\infty) - f(0))(x/x_A)^s$ was postulated for the lower limiting dependency. The effect of varying the value of the shifting exponent, $s$, on the solution is shown ($x_A = 1$ and $\alpha = 2$ were kept constant).

dency in $x^*$ (straight line graph in Cartesian coordinates), such that

$$h(x^*) = \alpha x^* + c,$$

where

$$h(x^*) = \log(f_0(x) - f(0)),$$

$$x^* = \log x,$$

and $c$ is a constant value

$$c = \log(f(\infty) - f(0)) - \alpha \log x_A = \log \left( \frac{f(\infty) - f(0)}{x_A^\alpha} \right).$$

As can be seen from equation (1.27), altering the value of $\alpha$ thus influences the 'curvature' of the postulated function; this is illustrated graphically in Figure 1.10.

The postulated function of equation (1.19) will thus form an upper bound on the possible values that the dependent variable may take in the lower limit. Furthermore
1.2 Shifting of the matching curve

Figure 1.12: Application of powered addition with only limiting values known. A function of the form $f_\infty \{x\} = f\{\infty\} - (f\{\infty\} - f\{0\}) (x_B / x)^\beta$ was postulated for the upper limit. The effect on the matched solution for selected values of the shifting exponent, $s$, is demonstrated ($x_B = 1$ and $\beta = 1.5$ were kept constant).

its contribution to the final solution should diminish as the value of the independent variable increases, in other words, once matched, the solution should show a decreasing dependence upon this function: a decreasing dependence, bounded from above as outlined in Section 1.2.2. By setting $g\{x\} = 1 / f\{x\}$, cf. equation (1.17), the expressions $g_0 \{x\} = 1 / f_0 \{x\}$ and $g_\infty \{x\} = 1 / f_\infty \{x\}$ are obtained for the respective dependencies; the latter being a constant value. Inserting the aforementioned together with the proposed dependency of equation (1.19) into equation (1.6), yields

$$\frac{1}{f^s \{x\}} = \frac{1}{f_0^s \{x\}} + \frac{1}{f_\infty^s \{x\}} = \frac{1}{[f\{0\} + (f\{\infty\} - f\{0\}) (x / x_A)^s]^{s} + \frac{1}{f^s \{\infty\}}}.$$ (1.31)

A plot of equation (1.31) for different values of the shifting exponent, $s$, is shown in Figure 1.11.

Instead of postulating a function for the lower limit, the behaviour of the process in
1.2 Shifting of the matching curve

the upper limit may be considered. The function is now required to act such that

\[ f_\infty(x) \rightarrow f(\infty) \quad \text{as} \quad x \rightarrow \infty. \]  \hspace{1cm} (1.32)

Once again a power function, now of the form

\[ f_\infty(x) = f(\infty) - (f(\infty) - f(0)) \left( \frac{X_B}{x} \right)^\beta, \]  \hspace{1cm} (1.33)

may be utilised as arbitrary function, approximating the dependency at upper extremal values. Applying the same reasoning as above to equation (1.32) imposes the restrictions \( x_B > 0 \) and \( \beta \geq 0 \) (the function now forming a lower bound). A power-added function, similar to that of equation (1.31), covering the entire range of the independent variable, may now be constructed by choosing \( g(x) = 1/f(x) \), \( g_0(x) = 1/f_0(x) = 1/f(0) \) and \( g_\infty(x) = 1/f_\infty(x) \), where \( 1/f_0(x) \) is now given by equation (1.33). Figure 1.12 illustrates the use of the function proposed in equation (1.33) for approximation of the behaviour at upper extremal values; altering the value of the shifting exponent, \( s \), having the desired effect.

1.2.4 Crossing of one limiting solution

In some phenomena the data is not bound completely by the limiting solutions; one of the limiting functions may be crossed as the solution approaches it. Although it is presumed that both the lower functional dependency, \( f_0(x) \), and the upper limiting value, \( f(\infty) \), is known, equation (1.6) is not directly applicable, since for any positive values of the shifting exponent, equation (1.6) gives values that fall above \( f_0(x) \) and \( f(\infty) \) (see Sections 1.2.1 and 1.2.2.1). As in the preceding section, a function is postulated viz.,

\[ f_\infty(x) \rightarrow f(\infty) \quad \text{as} \quad x \rightarrow \infty, \]  \hspace{1cm} (1.34)

but it should now differ from equation (1.33) in that it not only forms an upper bound on attainable values of the dependent variable, but also approaches the limiting value from above. Using a function of the form \([1; 2]\),

\[ f_\infty(x) = f(\infty) \left[ 1 + \left( \frac{X_A}{x} \right)^\alpha \right], \]  \hspace{1cm} (1.35)

in stead of \( f(\infty) \), solves this problem, since as \( x \rightarrow \infty \) the second term in square brackets on the left hand side of equation (1.35) approaches zero (once again, provided that \( \alpha \geq 0 \)).

Constructing a new dependency of the form suggested by equation (1.6), with \( g(x) = 1/f(x) \), \( g_0(x) = 1/f_0(x) \) and \( g_\infty(x) = 1/f_\infty(x) \), with \( f_\infty(x) \) the newly defined re-
1.2 Shifting of the matching curve

Figure 1.13: Application of powered addition when the solution crosses one of the limiting functions; a function of the form $f_{\infty}\{x\} = [1 + (x_A/x)^{\alpha}]$ utilized to approximate the upper limit. The effect on the matched solution for selected values of the shifting exponent, $s$, is demonstrated ($x_A = 5$ and $\alpha = 2$ were kept constant).

The application of equation (1.35), yields

$$\left(\frac{f_0\{x\}}{f\{x\}}\right)^s = 1 + \left(\frac{f_0\{x\}}{f_{\infty} \left[1 + \left(\frac{x_A}{x}\right)^\alpha\right]}\right)^s$$

(1.36)

after simplification. In Figure 1.13 the family of curves found for selected values of the shifting exponent, $s$ is illustrated; the arbitrary variables, $x_A = 5$ and $\alpha = 2$, were kept constant, their allocated values having been selected purely for demonstrative purposes. Investigation of the influence of the arbitrary constant, $x_A$, and arbitrary exponent, $\alpha$, in equation (1.36) can be done in a fashion similar to the procedures followed to obtain equations (1.26) and (1.27) – the graphical representation of a change in the values assigned to these constants are illustrated by Figures 1.14 and 1.15. The results are, as was to be expected, akin to those of Figures 1.9 and 1.10.
1.3 Normalisation to obtain one horizontal asymptote

Frequently neither of the expressions for the limiting solutions, (1.1) and (1.2), are linear in form or of a constant value. To aide visual interpretation it is often beneficial to divide equation (1.7) by one of the asymptotic expressions, namely

\[
\frac{f\{x\}}{f_0\{x\}} = \left[1 + \left(\frac{f_\infty\{x\}}{f_0\{x\}}\right)^s\right]^{1/s}, \tag{1.37}
\]

or

\[
\frac{f\{x\}}{f_\infty\{x\}} = \left[\left(\frac{f_0\{x\}}{f_\infty\{x\}}\right)^s + 1\right]^{1/s}, \tag{1.38}
\]

to obtain non-dimensional, normalised forms of the original function. Both equation (1.37) and (1.38) can now be written in generic form as

\[
Y = (1 + Z^s)^{1/s}, \tag{1.39}
\]
yielding a horizontal asymptote at \(Y = 1\) (\(Z \to 0\)); the exact functional definition of the newly defined variables, \(Y\) and \(Z\), will be case specific. Plotting of the expression
1.4 Critical point and shifting-exponent

The central or critical point, $x_c$, of the matching curve is the value of the independent variable at which the asymptotes meet. Since the asymptotes intersect here, the numerical value of their respective functional expressions must be equal, that is

$$f_0\{x_c\} = f_\infty\{x_c\}. \quad (1.40)$$

As both functions, $f_0$ and $f_\infty$, contribute equally to the added solution at this point, the resultant curve is most sensitive to variations in the value of the shifter, $s$, in the vicinity of $x_c$. Furthermore, looking at equations (1.37), (1.38) and (1.39), it becomes apparent that the maximal fractional deviation of the matched solution from either of
1.4 Critical point and shifting-exponent

The limiting solutions or asymptotic values will occur at precisely this point and take on the value

$$Y\{1\} - 1 = 2^{1/s} - 1.$$  \hspace{1cm} (1.41)

That is

$$\left( \frac{f\{x_c\}}{f_0\{x_c\}} \right) - 1 = \left( \frac{f\{x_c\}}{f_\infty\{x_c\}} \right) - 1 = 2^{1/s} - 1,$$  \hspace{1cm} (1.42)

if written in terms of the original equations for the extremal values.

Determining the value of the shifting-exponent, $s$, we use the same argument as above in equation (1.40). Thus,

$$f^s\{x_c\} = f_0^s\{x_c\} + f_\infty^s\{x_c\} = 2f_0^s\{x_c\} = 2f_\infty^s\{x_c\}$$  \hspace{1cm} (1.43)

whence it follows that

$$\left( \frac{f\{x_c\}}{f_\infty\{x_c\}} \right)^s = \left( \frac{f\{x_c\}}{f_0\{x_c\}} \right)^s = Y\{1\}^s = 2.$$  \hspace{1cm} (1.44)

The value of $s$ may now be determined straightforwardly from equation (1.44) as

$$s = \frac{\ln 2}{\ln f\{x_c\} - \ln f_\infty\{x_c\}} = \frac{\ln 2}{\ln f\{x_c\} - \ln f_0\{x_c\}} = \frac{\ln 2}{\ln Y\{1\}}.$$  \hspace{1cm} (1.45)

Figure 1.16: Linear plot of the function $f\{x\} = (f_0^s\{x\} + f_\infty^s\{x\})^{1/s} = (1 + x^s)^{1/s}$ to indicate the location of the critical point and equivalent function value.
1.4 Critical point and shifting-exponent

In performing an experiment, it is therefore advantageous to arrange the physical conditions in such a manner that the independent variable is in close vicinity of $x_c$. Whenever the experimental value of $f\{x_c\}$ is known, we proceed to determine the value of the shifter by equation (1.45). In Figures 1.16 and 1.17 the critical point and the corresponding function value at this point is indicated for the illustrative example, $f\{x\} = 1 + x$, that has been used thus far. Note that in this, most simple form, the value of the shifter, $s = 1$.

Alternatively, visual inspection by trial and error adjustment of the correlation between the predictive curve and data points may lead to an assignment of a value to $s$. As noted by Churchill & Usagi [1] the matched curve is relatively insensitive to variations in $s$; the required acuity being determined by considerations such as the process involved, tunability of other parameters and allowable error-margin.
Chapter 2

Flow in straight-through diaphragm valves

Diaphragm valves possess several advantages that lead to their extensive use in diverse industrial applications. There are two types of diaphragm valves: the "weir" type used in piping systems that carry less viscous fluids; and the "straight-through" type - a schematic representation of which is shown in Figure 2.1 - suited to slurries and suspensions [8]. The data sets of Mbiya [9; 10], on which this chapter is based, is concerned with the latter type of valve.

Figure 2.1: Schematic representation showing the cross-section of a straight-through diaphragm valve (http://www.engvalves.com)

Despite the broad scope of their use, Mbiya [9; 11] notes that few studies dealing
with valve openings of aperture less than unity is available in the literature. This limits
the use of the information contained therein, rendering it inapplicable to cases where
valves are used as flow impeding devices. Furthermore the studies that had been done
were restricted to specific intervals of the Reynolds number. The aim of Mbiya’s [9; 11]
experimental investigation was to more accurately predict the additional pressure
loss incurred, for four different opening positions, once a pipe had been fitted with a
straight-through diaphragm valve.

A brief outline will be given on the definition of the pressure loss coefficient, the
experimental determination of which is one of the main focuses of Mbiya’s work
[9; 10; 11]. Hereafter the specific Reynolds number used in his study will be discussed
shortly (for a complete derivation of the Slatter Reynolds number, refer to Appendix
B). In the second half of this chapter Mbiya’s [9; 11] results are investigated for pos-
sible asymptotic bounds whereafter powered addition is applied to these functional
dependencies. The outcomes of powered addition is compared to those of Mbiya’s
model for a few selected cases (a complete set of comparative graphs are available in
Appendix C) and the results discussed.

2.1 Definitions of pressures and heads

The fitting of a valve into a pipe section causes a change in shape of the plane perpen-
dicular to the direction of flow and hence also in that of the flow path, thereby leading
to an increase in the pressure drop as the fluid traverses the constriction caused by the
valve. In Figure 2.2 this resulting additional pressure loss is graphically illustrated.

A wide variety of parameters are used to express the pressure drop characteristics of
the different components in a piping system [12]. The data on pressure losses may be
arrived at by either experiment or by theoretical solution of the equations governing
flow. Pressure loss data obtained by the former method usually concern measurements
taken at two stations, one upstream and one downstream of the component. Reference
is seldom made to the details of the change in pressure within the component itself.

The well-known Bernoulli equation for steady, incompressible flow states

$$\frac{1}{2}v^2 + \frac{p}{\rho} + gz = \text{constant}, \quad (2.1)$$

where $v$ is the velocity, $p$ the pressure, $\rho$ the fluid density (which is presumed constant),
$z$ is the height above some arbitrary reference point and $g$ is the acceleration due to
gravity. Division of equation (2.1) by $g$ yields

$$\frac{v^2}{2g} + \frac{p}{\rho}w_1 + z_1 = \frac{v^2}{2g} + \frac{p}{\rho}w_2 + z_2 = \text{constant} = h_t, \quad (2.2)$$
2.1 Definitions of pressures and heads

Pressure, \( p \), Pipe length, \( \Delta p \), pressure drop over valve, additional pressure drop due to valve.

Figure 2.2: Increased pressure drop due to the presence of a valve.

where \( w \), the specific weight, has been written in stead of \( \rho g \). The terms in equation (2.2) all have the dimensions of length and are referred to as *heads* in hydraulics. Hence, \( v^2/2g \) is known as the velocity head \( (h_v) \), \( p/w \) the pressure head, \( z \) is the position head and their sum, \( h_t \) – constant along the stream tube – is called the total head [12]. Frequently the pressure head and position head together are referred to as the static head [13]; the choice of this grouping becomes clear upon regarding Figure 2.3, where the flow in a horizontal pipe is schematically represented.

The height to which a fluid rises in a tube connected to a tapping in the pipe wall is called the static head. The difference between the static head and the head yielded upon placing a forward facing tube (pitot) into the fluid stream, is called the velocity head. The total head is simply the sum of the static and velocity heads. Comparison with equation (2.2) yields an expression for the static head

\[
h_s = \frac{p}{w} + z. \tag{2.3}
\]

Since it is the same fluid being regarded, and assuming incompressibility (i.e. constant density), that is \( \rho = \rho_1 = \rho_2 \), equation (2.1) may be multiplied by \( \rho \), resulting in

\[
\frac{1}{2} \rho v_1^2 + p + \rho g z_1 = \frac{1}{2} \rho v_2^2 + p + \rho g z_2 = \text{constant} = p_H. \tag{2.4}
\]

In equation (2.4) the contribution to the energy balance (the principle on which the
2.1 Definitions of pressures and heads

Figure 2.3: Graphical representation of the static, velocity and total heads.

Bernoulli equation is based) of the term $\rho gz$ is negligible [12] and hence it simplifies to

$$\frac{1}{2} \rho v_1^2 + p_1 = \frac{1}{2} \rho v_2^2 + p_2 = \text{constant} = p_H. \quad (2.5)$$

In the form of equation (2.5) all terms have the dimensions of pressure; the first term, $\frac{1}{2} \rho v^2$, is known as the dynamic pressure, the second term, $p$, as the static pressure and their sum, $p_H$, (once again a constant along the stream tube) as the total pressure [12].

Considering equation (2.5), it follows that any change in the total or static pressure within the flow will be proportional to the local dynamic pressure. This leads to the definition of the total pressure loss coefficient

$$k \equiv \frac{p_{H_2} - p_{H_1}}{\frac{1}{2} \rho v^2} = \frac{\Delta p_H}{\frac{1}{2} \rho v^2}. \quad (2.6)$$

Hence, using the change of total head, the head loss, $\Delta h_t$, in a straight pipe section is approximately proportional to the square of the velocity, $v^2$, of the fluid. The relation in equation (2.6) may thus be expressed as

$$k = \frac{h_{t_1} - h_{t_2}}{\frac{v^2}{2g}} = \frac{2g \Delta h_t}{v^2} = \frac{2g H_{\text{loss}}}{v^2}. \quad (2.7)$$

whereby division of the head loss by the mean velocity head, $\frac{v^2}{2g}$, yields a non-dimensional loss coefficient [12; 13]. The presence of a component, such as a valve, in a piping system will, via an increase in the loss of dynamic pressure, lead to a
2.2 Choice of Reynolds number

According to Newton’s law of viscosity the shear stress, \( \tau \), and viscosity, \( \mu \), for one-dimensional flow of a fluid are related by

\[
\tau = -\mu \frac{dv_x}{dy},
\]

where \( \frac{dv_x}{dy} \) is the shear rate or velocity gradient as a function of position [14]. Fluids that obey this criterion are referred to as Newtonian fluids. This is however an idealisation as many fluids exhibit a more complicated relationship than the mere linearity described by equation (2.8). Often the relation between the velocity gradient and shear stress of a fluid is best described by the power dependency,

\[
\tau = K \left( -\frac{dv_x}{dy} \right)^n,
\]

where \( K \) is the fluid consistency index and \( n \) the flow behaviour index; fluids exhibiting such behaviour are called power-law fluids (for a brief outline on the classification of fluids, refer to Appendix A). Depending on the value of the flow behaviour index, power-law fluids are classified into three broad groups: pseudo-plastic fluids if \( n < 1 \); Newtonian fluids for \( n = 1 \), since equation (2.9) reverts to equation (2.8) in this case; and dilatant fluids for \( n > 1 \). In pseudo-plastic substances shear thinning is observed, in other words the viscosity decreases with an increase in rate of the shear stress.

A true plastic substance has an initial yield stress that needs to be overcome before it assumes fluid-like properties, i.e. continuous deformation when subjected to a (further) shear stress [14]. The constitutive equation for the yield pseudo-plastic model can thus be formulated as

\[
\tau = \tau_y + K \left( -\frac{dv_x}{dy} \right)^n,
\]

with \( \tau_y \) denoting the yield stress. Setting \( n = 1 \) in equation (2.10) yields the so-called Bingham-plastic model, while \( \tau_y = 0 \) results in it reverting back to that for power-law fluids, equation (2.9).

The Slatter Reynolds number, \( Re_3 \), is based on the yield pseudo-plastic model and starts from the assumption that, in the presence of a yield stress, the core of the fluid
2.3 Mbiya’s empirical correlation

moves as a solid, unsheared plug [10; 15] resulting in annular flow (see derivation in Appendix B). It can be expressed as

\[ Re_3 = \frac{8\rho \bar{v}_{ann}^2}{\tau_y + K \left( \frac{8\bar{v}_{ann}}{D_{shear}} \right)^n} . \]  

(2.11)

In equation (2.11) \( \bar{v}_{ann} \) denotes the corrected mean velocity in the annulus and \( D_{shear} \) the sheared diameter.

2.3 Mbiya’s empirical correlation

The addition of a component, such as a valve, to a piping system leads to a local constriction (or dilation) of the cross-sectional area and consequently also to a change in the flow path. Initially, at low Reynolds numbers – the region of laminar flow –

\[ Re \to 0 \]

\[ 0 < Re < 10 \]

\[ Re \to \infty \]

Figure 2.4: Schematic representation of recirculation within the valve section due to an increase in the Reynolds number.

streamlines will trace out the irregular geometry caused by the valve’s presence. As the Reynolds number increases however, localised areas of recirculation will gradually develop within the indentations of the diaphragm until, at turbulent flow conditions,
the streamlines bypass these areas altogether; Figure 2.4 represents this schematically. This is an intuitive explanation accounting for the constant resistance coefficients (pressure loss) obtained at high Reynolds numbers, i.e. turbulent flow, and reflected in the data of Mbiya [9; 10; 11].

Mbiya’s [9; 10; 11] proposed two-constant model is based on a large set of accrued experimental data. A test rig, constructed at the Cape Peninsula University of Technology, consisted of pipes of different diameters (40mm, 50mm, 65mm, 80mm and 100mm), each of which was fitted with the appropriately sized diagram valve. The fluids carboxymethyl cellulose [CMC] (at 5% and 8% concentration), glycerine or glycerol (concentrations of 75% and 100%), kaolin, a claylike mineral (10% and 13% concentrations) and water were pumped through the pipes for four different valve opening positions (25%, 50%, 75% and 100% open) and the pressure drop in the pipe was recorded. The aim was to predict the pressure loss coefficient, \( k_v(R_e^3) \), as defined in equation (2.7) – the \( v \)-subscript denoting valve – for straight-through diaphragm valves. Mbiya [9; 10; 11] concludes by summarizing his model, as being applicable to all sizes of valves tested, by straight-forward addition of

\[
 k_v = \begin{cases} 
 1000 \frac{1}{R_e^3}, & R_e^3 < 10 \\
 \frac{C_{\Omega}}{\sqrt{R_e^3} \theta^2} + \frac{\lambda_{\Omega}}{\theta^2}, & R_e^3 \geq 10 
\end{cases}
\]

(2.12)

Here \( C_{\Omega} \) is a new constant (model parameter) introduced by Mbiya [9; 11], \( \lambda_{\Omega} \) is the nominal turbulent loss coefficient, and \( \theta \) is the partial valve opening coefficient as ratio of the fully opened position, i.e. \( \theta = 0 \) for a closed valve and \( \theta = 1 \) for a fully opened valve. Note that an open valve does not correspond to an open tube flow condition; the diaphragm still protrudes into the lumen as can be seen in Figure 2.1 (right).

Unfortunately, to obtain good agreement with experimental results, an ‘if’-condition had to be introduced at a Slatter Reynolds number of 10. The two different curve fitted solutions on either side of this value lead to an unwanted jump in the values of the dependent variable, i.e. the predicted crossover at this Reynolds number is not smooth; Figure 2.5 shows a typical correlation for such a case – cf equation (2.12) – the ‘jump’ in the value of the dependent variable evident. This contradicts the expected, intuitive-orderly behaviour of such a continuum transfer process. (The constant \( C_{\Omega} \) is an unfortunate fudge factor introduced for proper agreement, in the transitional region, between the experimental data and correlative equations (2.12). It is also this factor that leads to the unwanted jump in the proposed model).
2.4 Powered addition applied to Mbiya’s work

Regarding equation (2.12) in the limit where \( Re_3 \to \infty \), it is clear that \( k_v \to \lambda_\Omega / \theta^2 \). Hence, \( \lambda_\Omega / \theta^2 \) may be regarded as an asymptotic lower bound on \( k_v \). The direct addition of this result to the dependency of \( k_v \) on \( Re_3 \) for \( Re_3 < 10 \), is then considered as a matching between the two asymptotic conditions, yielding a single solution that covers the entire range of the Reynolds numbers, namely

\[
k_v = \frac{1000}{Re_3} + \frac{\lambda_\Omega}{\theta^2}.
\] (2.13)

Inspection of Mbiya’s proposal thus evidently leads to the following definitions

\[
k_0 \equiv \frac{1000}{Re_3} \quad \text{for} \quad Re_3 \to 0,
\] (2.14)

and

\[
k_\infty \equiv \frac{\lambda_\Omega}{\theta^2} \quad \text{for} \quad Re_3 \to \infty,
\] (2.15)
whence equation (2.13) becomes

\[ k_v = k_0 + k_\infty. \]  

(2.16)

Figure 2.6: Application of powered addition to the data sets generated by Mbiya [9] for a pipe with internal diameter of 40mm and a valve opening of 25%; s-values of 0.4 (solid line) and 1.4 (dashed) are shown for comparison.

Instead of the direct addition of the two asymptotes as in equation (2.13), powered addition, as discussed in Chapter 1, may now be applied to the asymptotic expressions, yielding

\[ k^*_v = k^*_0 + k^*_\infty, \]  

(2.17)

which may, analogous to equation (1.38), be re-written as

\[ \frac{k_v}{k_\infty} = \left[ \left( \frac{k_0}{k_\infty} \right)^s + 1 \right]^{1/s}. \]  

(2.18)

If two new variables, \( Y \) and \( Z \), are defined as

\[ Y \equiv \frac{k_v}{k_\infty}, \]  

(2.19)

\[ Z \equiv \frac{k_0}{k_\infty}, \]  

(2.20)
2.4 Powered addition applied to Mbiya’s work

equation (2.18) simplifies to
\[ Y = [Z^s + 1]^{1/s}, \]  
(2.21)
cf equation (1.39).

Instead of dividing equation (2.17) by \( k_\infty \), one may alternatively have chosen to designate \( k_0 \) as denominator, followed by the corresponding redefinition of variables \( Y \) and \( Z \) in equations (2.19) and (2.20). One extremely useful consequence of this type of modelling is the direct possibility of non-dimensionalisation into either of the following forms

\[ \frac{k_v \lambda \Omega \theta^2}{\lambda \Omega} = \left[ \frac{1000 \theta^2}{\lambda \Omega \lambda \Omega \theta^2} \right]^{1/s} + 1 \]  
(2.23)

Furthermore, in so doing equations (2.22) and (2.23) have been normalized with regards to different values of the nominal turbulent loss coefficient, \( \lambda \Omega \), and valve flow ratio (or valve opening), \( \theta \), and a single horizontal asymptote obtained.

Determination of the critical point and the value of the shifting exponent may now be done in the manner outlined in Section 1.4. The critical point will thus be where

\[ k_0 = k_\infty, \]  
(2.24)

whence

\[ \frac{1000}{\lambda \Omega \theta^2} \Rightarrow \lambda \Omega \theta^2 = \lambda \Omega \theta^2 \]  
(2.25)

Since both \( \lambda \Omega \) and \( \theta \) are constants for a given pipe diameter and valve opening, the Slatter Reynolds number at which the critical point is to be found may easily be determined; these values are listed in Table 2.1. The pressure loss coefficient at the critical point is thus given by \( k_{v,c} = k_v(Re_{3,c}) \), the corresponding functional value obtained by equation (2.25).

The discussion in Section 1.4, equation (1.45), now yields a value for the shifting exponent, i.e.

\[ s = \frac{\ln 2}{\ln k_{v,c} - \ln k_{\infty,c}} = \frac{\ln 2}{\ln k_{v,c} - \ln k_{0,c}}, \]  
(2.26)

or in explicit form as

\[ s = \frac{\ln 2}{\ln k_v(Re_{3,c}) - \ln \left( \frac{\lambda \Omega}{\theta^2} \right)} = \frac{\ln 2}{\ln k_v(Re_{3,c}) - \ln \left( \frac{1000}{Re_{3,c}} \right)}, \]  
(2.27)
Table 2.1: Calculated values $Re_{3,c} = (1000 \theta^2)/\lambda_\Omega$ for all possible combinations of pipe diameter and valve opening. The $\lambda_\Omega$-values in this table were obtained from Mbiya [9].

<table>
<thead>
<tr>
<th>Valve opening, $\theta$</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>40mm, $\lambda_\Omega = 8.0$</td>
<td>7.8594</td>
<td>31.438</td>
<td>70.734</td>
<td>125.75</td>
</tr>
<tr>
<td>50mm, $\lambda_\Omega = 3.4$</td>
<td>18.493</td>
<td>73.971</td>
<td>166.43</td>
<td>295.88</td>
</tr>
<tr>
<td>65mm, $\lambda_\Omega = 1.5$</td>
<td>41.917</td>
<td>167.67</td>
<td>377.25</td>
<td>670.67</td>
</tr>
<tr>
<td>80mm, $\lambda_\Omega = 2.9$</td>
<td>21.681</td>
<td>86.724</td>
<td>195.13</td>
<td>346.90</td>
</tr>
<tr>
<td>100mm, $\lambda_\Omega = 4.1$</td>
<td>15.335</td>
<td>61.341</td>
<td>138.02</td>
<td>245.37</td>
</tr>
</tbody>
</table>

Traversal of the data sets in search of the $Re_{3,c}$-value closest to those listed in Table 2.1 may now be effected, the objective being to find the corresponding value of the dependent variable, $k_{v,c}$, at this point. Plugging these values into equation (2.27) will then yield a possible value for the shifting exponent. However, the datum point chosen may be a poor choice (an outlier, the result of a poor reading, etc.) and grounding the $s$-value solely on this one, single reading may lead to erroneous results. It is therefore recommended that the value of the intersection of the asymptotes be determined beforehand and the bulk of experimentation conducted in the area of the yielded independent variable, i.e. $Re_{3,c}$. In so doing a more accurate prediction will be obtained (from averaging numerous data points) and the fractional deviation of the matched solution from either of the limiting solutions or asymptotic values minimized (see Section 1.4). It is nevertheless important to note that the method is still an empirical one, based on experimental results; the wish being for an analytical expression in which this shifting exponent is linked to some quantifiable parameter in the process under consideration.

Since Mbiya’s experimental readings were not arranged in such a manner as to focus on the transitional area between the asymptotes, the aforementioned methodical approach was not used. In lieu, to circumvent the shifting exponent being based on an incorrect or inaccurate reading, a trial-and-error graphical approach was used. The results of two such curve fittings for different pipe diameters and valve openings are shown in Figures 2.6 and 2.7, with $s$-values of 0.4 (solid line) and 1.4 (dashed) plotted for comparison.

Ideally, the normalised, non-dimensional expressions of equations (2.22) and (2.23) would also allow for the experimental data of all valve sizes to be plotted on a single plot, irrespective of the valve flow ratio, $\theta$. However, as can be seen in Table 2.1, there is no discernable relation between the valve diameter and the $\lambda_\Omega$-values. Mbiya
2.4 Powered addition applied to Mbiya’s work

Figure 2.7: Powered addition applied to Mbiya’s [9] data sets for a pipe with internal diameter of 50mm and a valve opening of 50%. $s$-values of 0.4 (solid line) and 1.4 (dashed) are shown for comparison.

[9; 10; 11] notes that $\lambda$ is obtained by the minimisation of the overall logarithmic difference between his calculated and experimental $k_v$-values and cites the lack of dynamic similarity between valves of different sizes as rationalisation for these discrepancies.

An attempt at plotting all the data for a specific valve diameter, regardless of the valve opening, on a single plot afforded no clear visual results. To prevent clutter, data sets were plotted on separate axes according to valve diameter and valve opening (see Appendix C). The plots show a gradual shift towards and beyond (below) the asymptotes with an increase in the valve opening, that is to lower values of both the dependent and independent variables. This observation suggests a dependence upon some parameter that is yet to be considered or identified. To arrive at an accurate prediction of the shifter it is recommended that experiments be tailored so as to specifically investigate the flow parameters in the transitional regime; this was, however, not the focus of Mbiya’s [9; 11] study.

Although the graphs obtained through the use of powered addition still leave a lot
to be desired, overall they exhibit, for our particular goal, a qualitatively improved prediction of the process than the model of Mbiya.
Chapter 3

Flow through a packed bed

Packed beds are of significant interest to chemical engineers since they are widely used for mass transfer in industrial plants. A crucial piece of information required for effective design is the difference of piezometric head necessary to ensure steady fluid flow, be it through a pipe, porous medium, etc. See e.g. [16] (piezo is derived from the Greek piezein, which means to squeeze or press). In the study of porous media this translates into a relation being sought between the pressure gradient and the superficial or discharge velocity.

The two predominant approaches used in the theoretical study of pressure drop through a packed bed differ from one-another in the manner in which the solid and fluid parts of the bed are regarded. The first method regards the pores between the solid phase as a bundle of tangled tubes of irregular and inconsistent cross-section and proceeds to develop a model based on applying, modifying and expanding the well-established results for the flow in single, straight tube. The second method involves the solid phase being seen as an aggregation of individual objects submersed in the fluid phase; the pressure drop being calculated by addition of the resistance of each of the particles, e.g. [17]. The former approximation to reality has enjoyed more publicity and attention in the literature.

In this chapter two methods used in predicting pressure drop over a packed bed or porous medium are discussed: the first, the Ergun equation, is semi-empirical in nature and, surprisingly, one of the very few known example of powered addition with an exponent of unity; the other method, the RUC model, is an analytical approach based on pore-scale modelling of the microstructure of the porous medium and forms a key concept in the discussion following in Chapter 4.
One of the pioneers in experimental fluid mechanics was the French engineer, Henry Darcy (1803 - 1858) of Dijon. Involvement in the design of municipal water supply systems led to his research into the flow of water through sand bed filters [18; 19] (i.e. flow through porous media) and the eventual publication in 1856 of Les Fontaines Publiques de la Ville de Dijon. The empirical law that he proposed, and that consequently carries his name, states that the rate of flow through a bed of packed particles is proportional to the pressure drop (hydraulic gradient) over the bed [18; 20] – specific discharge is proportional to the fluid pressure gradient in the direction of flow. In one dimension this linear relation is expressed as

\[- \frac{dp}{dx} = \frac{\mu}{\kappa} q, \]  

(3.1)

where \(-dp/dx\) is the streamwise pressure drop, \(\mu\) the viscosity of the traversing fluid and \(q\) the superficial velocity or specific discharge. The coefficient of proportionality, \(\kappa\), is referred to as the hydrodynamic permeability and is generally determined experimentally (a great deal of work has gone into determining an analytical expression and will be discussed in Section 3.2).

In practice, however, results obtained from experiments tend to deviate from the linear relation predicted by equation (3.1) and become nonlinear at higher velocities, despite the fact that the Reynolds number, \(Re\), may still be fairly small [17]. Hence, wide application of Darcy’s law is impeded by its limitation to a fairly narrow range of Reynolds numbers.

Throughout history various attempts have been made to more accurately relate the pressure gradient and superficial velocity. In 1901 Forchheimer proposed [21] (referenced in [22]) the introduction of a quadratic term to take this nonlinear behaviour into account, leading to an empirical generalized form of the Darcy equation, namely

\[- \frac{dp}{dx} = Mq + Nq^2, \]  

(3.2)

with the coefficients \(M\) and \(N\) empirical constants that depend on the structural and geometric properties of the porous medium and fluid viscosity [23; 24]. In the case of a superficial velocity, \(q\), less than unity the second term in equation (3.2) becomes negligible and it effectively reduces to the Darcy law, equation (3.1). Conversely, for a high superficial velocity the quadratic term in equation (3.2) predominates. Between these two flow regimes a transitional area exists.

The Ergun equation – a capillary tube model widely used for the prediction of pressure drop of flow through a packed bed – is in essence based upon a combination of
3.1 Ergun equation

specific asymptotic solutions for the Darcy and Forchheimer regimes and is therefore semi-empirical in nature. In it’s derivation the assumption is made that the bed consists of smooth, uniformly sized, spherical particles; that the particles are packed in a statistically uniform random manner; and, that the diameter of the containing column is orders of magnitude larger than that of the particles. Ergun simply added the equation proposed by Blake and Kozeny for laminar flow (Darcy regime)

\[ \Phi = 150 \frac{Re_s}{D} \]  

(3.3)

to the Burke-Plummer equation for turbulent flow (Forchheimer regime) [1; 17; 25],

\[ \Phi = 1.75 \left( \frac{Re_s}{D} \right)^2 \]  

(3.4)

to obtain

\[ \Phi = 150 \frac{Re_s}{D} + 1.75 \left( \frac{Re_s}{D} \right)^2 \]  

(3.5)

an expression for the pressure drop across a packed bed for Reynolds numbers ranging from the laminar to the turbulent flow regimes. It should be noted that the non-linear effects, represented by the second term on the righthand side of equation (3.5),

![Figure 3.1: Logarithmic correlation for the pressure drop through a packed bed of spheres – the Ergun equation is an instance of powered addition with \( s = 1 \) (the equation of Carman and Kozeny was added to the Burke-Plummer equation).](http://scholar.sun.ac.za)
3.1 Ergun equation

are already noticeable at Reynolds numbers well below that of the region which is
normally associated with turbulence.

His result mostly yields satisfactory agreement with experimental data for \( \varepsilon < 0.5 \).
In equations (3.3) to (3.5) the variable \( \Phi \) has been used for simplicity and is defined as

\[
\Phi = \frac{\rho_f \varepsilon^3 D_n^3}{\mu^2 (1 - \varepsilon)^3} \left( -\frac{dp}{dL} \right),
\]

(3.6)

with the Reynolds number for a packed bed of spheres, \( Re_s \), given by [1]

\[
Re_s = \frac{\rho_f q D_s}{\mu (1 - \varepsilon)}.
\]

(3.7)

In equations (3.6) and (3.7), \( \rho_f \) denotes the density of the traversing fluid, \( \varepsilon \) the bed
porosity or void fraction, \( D_s \) the diameter of a perfectly spherical particle and \( L \) the
length of the straight channel (or height of bed).

Ergun had thus, in effect, applied the matching technique proposed by Churchill &
Usagi [1; 2; 5; 6; 25] as discussed in Chapter 1 to produce a combined result,

\[
\Phi = \left[ (150 Re_s)^s + (1.75 Re_s^2)^s \right]^{1/s},
\]

(3.8)

with equation (3.8) reverting to equation (3.5) by choosing the shifting exponent, \( s = 1 \);
his result is shown in Figure 3.1. The value of the exponent was however not known a
priori. Utilizing the generic form suggested by equation (1.39) with

\[
Y = \frac{\Phi}{150 Re_s},
\]

(3.9)

and

\[
Z = \frac{1.75 Re_s}{150} = \frac{Re_s}{85.7},
\]

(3.10)

the value of the critical point, as considered in Section 1.4, is obtained at \( Z = 1 \), that is
at \( Re_s = 85.7 \). Churchill [1] notes that this process happens to be the only one thus far
examined in which the best choice of exponent turned out to be unity.

The Ergun equation, expressed in its more well-known form

\[
-\frac{\Delta p}{L} = 150 (1 - \varepsilon)^2 \mu q^2 \frac{\phi_p^2}{\phi_p d_p^3} + 1.75 (1 - \varepsilon) \frac{\rho q^2}{\phi_p d_p},
\]

(3.11)

takes into account cases in which the bed consists of non-spherical particles. In equation (3.11) \( d_p \) denotes the mean particle diameter and \( \phi_p \) the particle shape factor
(sphericity). The mean particle diameter, \( d_p \), is defined in terms of the specific sur-
face, \( a_v \) of the particle,

\[
d_p = \frac{6}{a_v},
\]

(3.12)
with the specific surface being defined as

$$a_v = \frac{\text{total particle surface}}{\text{volume of the particles}}.$$  \hfill (3.13)

The reason for the above definitions becomes apparent when regarding a bed consisting of $N$ perfectly spherical particles. Since the surface area of a sphere is given by $4\pi r^2$ and its volume by $\frac{4}{3}\pi r^3$, substitution into equation (3.13) will yield

$$a_v = \frac{4N\pi r^2}{\frac{4}{3}N\pi r^3} = \frac{3}{r} = \frac{6}{D_s},$$  \hfill (3.14)

and plugging this result into equation (3.12) sets $d_p = D_s$ (here $r$ and $D_s$ respectively denote the radius and diameter of the sphere) [17].

More often than not the bed does not consist of perfectly spherical particles, in which case it is customary to construct a hypothetical equivalent-volume sphere, $\phi_p d_p$, with

$$d_p = \frac{6V_p}{A_p \phi_p} = \frac{6V_p}{A_{sp}},$$  \hfill (3.15)

as used in equation (3.11). Here $V_p$ is the volume of a single, non-spherical particle, $A_p$ its surface area and $A_{sp}$ the surface area of an equivalent volume sphere. Equation (3.15) thus relates the surface area of the particle to the surface area of a sphere of equal volume [24]; cf equation (3.13). Thus, $\phi_p = 1$ and $\phi_s < 1$ for spherical and non-spherical particles respectively.

According to Geldart [26] the most appropriate parameter for use in flow through packed beds is the external surface area of the powder per unit particle volume. Hence, the diameter of a sphere with the same external surface area to volume ratio as that of the particle, $d_{sv}$, is the most relevant diameter. The Waddell sphericity factor, $\Psi$, defined as

$$\Psi = \frac{\text{surface area of equivalent volume sphere}}{\text{surface area of particle}},$$  \hfill (3.16)

can be shown to link $d_{sv}$ and $d_v$, by

$$\Psi = \frac{d_{sv}}{d_v},$$  \hfill (3.17)

with $d_v$ the volume diameter, i.e. the diameter of a sphere having the same volume as the particle.

### 3.2 RUC model

The analytical approach to flow through porous media is based on modelling the microstructure of the porous media on pore-scale level and aims to provide a theoretical
derivation of the empirical Ergun equation. In dealing with porous media that is assumed to have a regular pattern, the idea of constructing a repetitive unit or cell, holds the advantage that flow around each and every solid particle in the medium need not be analyzed. It would thus be possible to ‘reconstruct’ the particular porous medium under regard from such a building block or unit cell, provided that these cells are both volume-conserving and space-filling.

The actual interstitial pore space, especially in cases where the media under regard is isotropic, and the solid material is thus randomly positioned within the volume, is very seldom open to precise analysis. To circumvent this obstacle, a hypothetical representative unit cell (RUC) was introduced [27; 28] which theoretically approximates the porous material in that it contains the average geometric properties of the material. The RUC model is thus a generalised geometric approximation of the actual geometric properties of the medium in such a manner as to preserve the physical attributes of flow through such a medium; an abstraction of the porous medium rendering it amenable to physical analysis.

The pore-scale geometric relation between the solid and fluid phases of porous media allows for classification into three fundamental classes: granular, spongelike and unidirectional two-dimensional fibre-bed. Analytical expressions for the hydrodynamic permeability, \( k \), and stream-wise pressure drop, \( -dp/dx \), of the different RUCs are known [27; 28; 29; 30; 31; 32].

For the remainder of this section the particle Reynolds number, \( Re_p \) – an indicator of the ratio between the inertial and viscous forces of the fluid with \( d_p \) used as characteristic length – has been defined as

\[
Re_p = \frac{\rho q d_p}{\mu},
\]

(3.18)

where \( \rho \) and \( \mu \) denote the density and viscosity of the traversing fluid respectively, \( q \) the magnitude of the superficial velocity or specific discharge and \( d \) the linear RUC dimension. The practical use of the Reynolds number is to distinguish among flow regimes, for example laminar or turbulent flow in pipes or around immersed objects, its value depending upon the situation.

### 3.2.1 Granular porous media

In media that fall into this class, the solid phase is present in the form of isolated to semi-connected units that are submerged in the surrounding fluid phase. Some examples of granular porous media would be the glass spheres used in packed bed chemical reactors and a pipe filled with sand or marbles. A schematic representation
3.2 RUC model

of the isotropic granular RUC model [29; 31; 32] is shown in Figure 3.2. $U_s$ denotes the volume of the solid phase, $U_f$ the volume of the fluid phase and $U_o$ the total volume of the RUC.

The relation between the linear dimensions $d$ and $d_s$ of the granular RUC, with $\varepsilon$ denoting the porosity, is given by

$$d_s = (1 - \varepsilon)^{1/3}d, \quad (3.19)$$

and the stream-wise pressure drop, $-dp/dx$, by

$$-\frac{dp}{dx} = \frac{25.4(1 - \varepsilon)^{4/3}}{(1 - (1 - \varepsilon)^{1/3})(1 - (1 - \varepsilon)^{2/3})^2} \cdot \frac{\mu q}{d_s^2} + \frac{c_d(1 - \varepsilon)}{2\varepsilon(1 - (1 - \varepsilon)^{2/3})^2} \cdot \frac{\rho q^2}{d_s}, \quad (3.20)$$

where $c_d$ denotes the drag coefficient and $Re$ is the Reynolds number as defined in equation (3.18).

3.2.2 Spongelike porous media

Spongelike or consolidated porous media are distinguished from granular porous media by the fact that the solid and fluid phases are interconnected. In other words, one would be able to move from any point in a given phase to any other point in the same phase, without ever having to leave it. Examples would be plastic and metallic foams, such as those used in electronics, or any substance containing interlinking ducts.
Foamlike media is further differentiated into three sub-models depending on the number of stagnant pore-sections assumed to be present within the foam model \cite{29, 33, 34}. For porous media of this type, the stream-wise pressure drop, $-dp/dx$ is given by

$$\frac{-dp}{dx} = \frac{36\psi^2(\psi - 1)}{\epsilon^2} \frac{\mu q}{d^2} + \frac{c_d(\psi - 1)(\psi - 3)^2}{4\epsilon^3} \frac{\rho q^2}{d},$$  \hspace{1cm} \text{(3.21)}

with $\psi$ a geometric factor, defined in this case as

$$\psi = 2 + 2 \cos\left(\frac{4\pi}{3} + \frac{1}{3} \cos^{-1}(2\epsilon - 1)\right),$$  \hspace{1cm} \text{(3.22)}

subject to the condition that $\psi = 1$ when $\epsilon = 1$.

![Figure 3.3](http://scholar.sun.ac.za)

Figure 3.3: Cubic geometry of the RUC model for modelling fluid flow through foamlike media. The streamwise direction is indicated by $\hat{n}$. Note that here the shaded volume is representative of the fluid phase.

In Figure 3.3 a schematic representation is given of the isotropic foamlike RUC model. It is important to note that in this figure, as opposed to the other figures, the shaded volume is representative of the fluid phase, $U_f$. Fluid traversing such a theoretically constructed medium would thus enter through the façade facing the reader, move along the side of the cube and then turn Eastwards to move along the middle channel. It will travel a distance $d_s$ along the side of the cube, before turning in a Northerly direction, to finally exit the model at the backward facing plane.

Expressing the relation between the linear dimensions $d$ and $d_s$ of the foamlike RUC in terms of the porosity, $\epsilon$, yields a polynomial of the third order,

$$\epsilon = 1 - 3 \left(\frac{d_s}{d}\right)^2 + 2 \left(\frac{d_s}{d}\right)^3,$$  \hspace{1cm} \text{(3.23)}
which may be rewritten in terms of the geometric factor, $\psi$, as [29]

$$\frac{\psi}{\varepsilon} = \frac{4}{(3 - \psi)^2}.$$  \hfill (3.24)

### 3.2.3 Unidirectional two-dimensional fibre-bed porous media

Should the solid phase of the medium consist of aligned, disjoint fibre strands surrounded by an interconnected fluid phase, it is classified as a two-dimensional prismatic or unidirectional fibre-bed porous medium. Air flowing through a stack in a timber-drying kiln would be an example of porous media of this kind. Figure 3.4 shows a schematic representation of porous media of this type. It should be noted that the dimension $L$ should be of sufficient length so as to clearly distinguish it from the granular RUC model illustrated in Figure 3.2. The relation between $d$ and $d_s$ for the fibre-bed RUC may be expressed as

$$d_s = d\sqrt{1 - \varepsilon},$$  \hfill (3.25)

while the stream-wise pressure drop, $-dp/dx$, for the isotropic fibre-bed model is given by [29; 30]

$$-\frac{dp}{dx} = \frac{13.5\sqrt{1 - \varepsilon}}{(1 - \sqrt{1 - \varepsilon})^3} \cdot \frac{\mu q}{d^2} + \frac{c_d \sqrt{1 - \varepsilon}}{2\varepsilon(1 - \sqrt{1 - \varepsilon})} \cdot \rho q^2 \cdot \frac{d}{d}. $$  \hfill (3.26)
Whereas the Ergun equation was limited to a fairly narrow range of porosities, the fact that the RUC model is based on physical principles (and doesn’t contain empirical coefficients) ensures that the latter is applicable over the entire range of porosities from zero through unity. The geometries of the different fundamental classes have also been designed as such that it is physically adaptable to predict the pressure drop in beds consisting of non-spherical particles.
Chapter 4

Fluidised bed

A fluidised bed is formed when a fluid, usually a gas, is forced upwards through a bed of particles. The packed bed of particles, supported on some kind of distributor, is converted into an expanded, suspended bed and in the process takes on many liquid-like properties: the bed has zero angle of repose, pressure increases linearly with distance below the surface, wave motion is observed, denser objects sink and lighter objects may be floated on its surface, their movement almost unhindered [35]. This only happens once the pressure drop across the bed, created by the flow, becomes sufficient enough to support the weight of the constituent particles and is referred to as incipient fluidisation. The superficial velocity at which the aforementioned phenomenon occurs is known as the minimum fluidisation velocity, \( q_{mf} \). Additional increase in the fluid flow rate leads to a further expansion of the bed and eventually the formation of bubbles. This is commonly known as aggregative fluidisation and leads to agitation of the solid constituents and a consequent fluctuation in the height of the bed [35].

A fluidised system has a number of highly useful properties that may be utilised in industrial applications. Foremost among them is the agitation caused by the bubbles – the already large specific area of a fine powder is further exploited as the stirring action continually moves the particles around, exposing it to the gas. The good mixing of the solids is responsible for a high heat transfer rate from the gas to the particles, causing dispersion of localised temperature regions and leading to radial and axial isothermal conditions. The large contact area between the solid and fluid phases also enhances chemical reaction rate, making fluidisation a technique well suited for catalytic reactions. Thus, although the mechanism may be both physical and chemical in nature, the dominating attribute utilised in a specific industrial process, will determine its application. A broad classification of fluid bed applications is given by [36], from which it is evident that fluidisation is an interdisciplinary field of inquiry.

Despite many attractive features, fluidised beds are not suited to all fluid-solid pro-
cesses and some of the disadvantages may overshadow the potential benefits. Both chemical and mechanical difficulties arise with scale-up in size from laboratory equipment to industrial plant – some of the gas in the bubbles may by-pass particles completely, lowering efficiency; erosion of the containing vessel will inevitably occur in areas of high gas velocity. To optimally reap the benefits of this promising and still incompletely understood process there thus exists a particular need for the development of predictive models; holding challenges for both fundamental and applied research.

Upon analysis of sets of collected data of flow through a porous medium, it is evident that asymptotes exist for some variable dependencies in the transition from packed- to fluidised bed. The transition between such asymptotes is governed, amongst others, by parameters such as particle size, particle size distribution, superficial gas velocity and bed height. Powered addition to a power, $s$, of such asymptotes $f_0$ and $f_\infty$, leads to a single correlating equation that is applicable over the whole range of flow rates.

4.1 Newtonian fluid

Experimental data was obtained from measurements performed by the author on laboratory-scale fluidized beds at Telemark University College, Porsgrunn, Norway. The aforementioned method of powered addition is applied to the experimental data and the outcomes discussed; the underlying motivation being an aspiration towards establishing a sound modelling framework for analytical and computational predictive measures. The results of this section were presented at the 5th International Conference on Computational & Experimental Methods in Multiphase and Complex Flow (Multiphase Flow V), New Forest, UK, 15 - 17 June 2009 [4].

4.1.1 Experimental procedure

The bed was contained in a cylindrical perspex tube with an inner diameter of 72mm. Since the investigation was only concerned with the pressure drop across a specified section of the bed (i.e. between two pressure sensors), the porosity of the plate on which the bed was supported was irrelevant. It was only required not to allow particles to drop through into the antechamber and to function as a uniform gas distributor. To prevent particle entrainment the uppermost end of the containing perspex tube was covered with a porous paper cap and the bed illuminated from behind to allow easy visual detection of the onset of fluidisation.
The beds comprised of glass powders, consisting of spherical glass particles, with a density of 2485 kg/m$^3$ and available in three different diameter-ranges: 100 $\mu$m – 200 $\mu$m; 400 $\mu$m – 600 $\mu$m; and 750 $\mu$m – 1000 $\mu$m. These particles all fall into Group B and D according to the Geldart powder classification [37]. Since the particles are manufactured they were assumed, for the sake of simplicity, to be perfectly spherical in shape and thus have a Waddell sphericity factor – see equation (3.16) – of $\Psi = 1$ [26].

In all of the experiments performed the fluid used to fluidise the bed was air at ambient conditions, with a density of 1.2 kg/m$^3$ and viscosity of $1.78 \times 10^{-5} N \cdot s/m^2$. Air was pumped into the bed by a compressor and flow into the antechamber was controlled via a manual valve. A digital flow meter, situated between the valve and the antechamber, registered the fluid velocity. By using different gas velocities and only one particle diameter-range or a mixture of the particle diameter-ranges, the parameters of the experiment could be varied. Eight equidistantly spaced pressure tubes were placed along the height of the bed and were connected to digital pressure sensors (Honeywell, model 142PC02G) that measured the pressure (in mbar) at these fixed...
4.1 Newtonian fluid

intervals within the bed. An experiment-specific software program was written with the aid of the LABVIEW package with which the data from each sensor were recorded and written to file; the program was calibrated and allowed for the interval between consecutive readings to be varied. Figure 4.1 shows the bed, antechamber, fluid inlet from below and some of the silicone pressure tubes (leading off to the digital sensors).

A minimum of three runs were performed for each of the three different diameter-ranges mentioned above. A run consisted of a gradual increase until and beyond fluidisation, followed by a gradual and controlled decrease to zero fluid flow. It was noted that hysteresis only became apparent if the bed was allowed a period of rest between consecutive fluidisations. Consequently, runs for a particular diameter-range were conducted in succession allowing the obtained pressure-values to be averaged. It is to this averaged data that the curve-fitting technique was applied.

4.1.1.1 Determining the superficial velocity of the traversing fluid

A digital flow meter, connected to a compressor, measured the flow rate of the air entering the bed. The meter registered rates in liters per minute, \( [\ell/min] \), and these readings were recorded with the mentioned software program. To convert \( \ell/min \), the inner diameter (72 mm) of the cylindrical perspex tube containing the bed was used, i.e. readings from the meter were multiplied by a factor,

\[
\frac{\ell}{min} = \frac{(0.1 m)^3}{60 s} = 4.09 \times 10^{-3} m/s,
\]

(4.1)

to obtain the superficial velocity. Here \( A_c \) is the cross-sectional area of bed and \( r \) the radius of the containing cylinder.

4.1.1.2 Determining the porosity of the packed bed

For each of the diameter-ranges an amount of powder was weighed out and poured into the cylindrical perspex tube comprising the fluidised bed. Once inside the tube, the powders were briefly allowed to settle before commencing experimentation. By measuring the distance between the supporting porous plate at the base of the bed and the bed surface, the bulk volume, \( U_0 \), occupied by the powder (and hence that of the bed) could be calculated by

\[
U_0 = \pi r^2 H,
\]

(4.2)

where \( r \) is the radius of the containing cylinder and \( H \) the height of the packed bed.

Porosity or bed voidage, \( \varepsilon \), is defined as the ratio between the fluid volume, \( U_f \) and
4.1 Newtonian fluid

the bulk volume, \( U_0 \), i.e.

\[
\varepsilon \equiv \frac{U_f}{U_0},
\]

and density, \( \rho \), as the mass, \( m \), per unit volume, \( U \)

\[
\rho_i \equiv \frac{m_i}{U_i},
\]

with the subscript denoting the phase under consideration.

It follows naturally that the total or bulk volume, \( U_0 \), is merely the sum of the volumes occupied by the solids, \( U_s \), and the fluid, \( U_f \), i.e.

\[
U_0 = U_s + U_f,
\]

with the same being true of the total or bulk mass, \( m_0 \)

\[
m_0 = m_s + m_f,
\]

where \( m_s \) and \( m_f \) are the masses of the solids and fluid respectively.

From definition (4.4) and equations (4.5) and (4.6) it follows that

\[
U_0 = \frac{m_s}{\rho_s} + \frac{m_f}{\rho_f} = \frac{m_0 - m_f}{\rho_s} + \frac{m_f}{\rho_f}.
\]

Some further simple algebraic manipulation, utilisation of definition (4.4), substitution with equation (4.6) and rearrangement to make \( m_f \) the subject of the formula, yields

\[
m_f = U_0 \rho_f \left( \frac{\rho_s - \rho_0}{\rho_s - \rho_f} \right).
\]

Division of equation (4.8) by \( \rho_f \) yields

\[
U_f = U_0 \left( \frac{\rho_s - \rho_0}{\rho_s - \rho_f} \right),
\]

whence follows, by definition (4.3)

\[
\varepsilon = \frac{\rho_s - \rho_0}{\rho_s - \rho_f},
\]

an expression for the porosity in terms of known or easily measurable quantities.

In an idealised situation of infinite size the diameter of spherical particles would have no influence on the porosity of the bed. However, in the constrained bed found in practical applications, the cross-sectional area of the bed will influence the packing of the particles. Taking this into account and regarding equation (4.10) it now becomes evident that the diameter of particles impacts indirectly on the porosity of the bed by having an influence on the bulk volume, and thus the bulk density thereof. The porosity for each of the diameter ranges were calculated with the aid of equation (4.10) and are listed in Table 4.1.
4.1 Newtonian fluid

<table>
<thead>
<tr>
<th>Particle diameter</th>
<th>Porosity, $\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 – 200µm</td>
<td>0.391</td>
</tr>
<tr>
<td>400 – 600µm</td>
<td>0.368</td>
</tr>
<tr>
<td>750 – 1000µm</td>
<td>0.362</td>
</tr>
</tbody>
</table>

Table 4.1: Porosity or bed voidage for the different diameter-ranges.

4.1.2 Asymptotic dependencies

Three regimes are to be identified, namely the two regimes corresponding respectively to the physical conditions related to the two asymptotes $f_0$ and $f_\infty$ and the change-over regime connecting the two. This latter regime surrounds the critical point, $x_c$, where the asymptotes meet. Of particular interest is the relation between the extent of the change-over regime surrounding the critical point and the shifting exponent $s$; in the present case of onset of fluidization the change-over between a packed bed condition and a fluidized state is fairly abrupt leading to a relatively high value of $s$. These effects will be discussed in the following subsections.

4.1.2.1 The lower asymptote

Before the onset of fluidisation, the bed may be regarded as a packed bed or porous medium consisting of spherical particles. To describe the pressure drop of Newtonian flow through such a structure, the Ergun equation has proven to be satisfactory in most applications as is evident from its extensive utilisation in chemical engineering. In their paper [32], Du Plessis & Woudberg compare the RUC (Representative Unit Cell) model to the Ergun equation for the description of Newtonian flow through a packed bed of uniformly sized spherical granules and find the agreement to be satisfactory. The choice, in this paper, of the RUC model to describe the asymptotic relation at small values of the independent variable is due to the fact that it is adaptable to different physical situations, whereas the Ergun equation is empirically based and will thus vary according to the situation to which it is applied. Furthermore the RUC model allows the usage of the average bed porosity and is applicable over both the entire porosity and laminar Reynolds number ranges.

In the original Ergun equation, which is already a special case of powered addition with shifter, $s = 1$ as discussed in Section 3.1,

$$\frac{\Delta p}{H} = M \frac{(1 - \varepsilon)^2 \mu q}{\varepsilon^3 \frac{D_h^2}{D_n}} + N \frac{1 - \varepsilon \rho f \theta^2}{\varepsilon^3 \frac{D_h^2}{D_n}},$$

(4.11)

the values of coefficients $M$ and $N$ were acquired experimentally and are given as 150...
Figure 4.2: Particle diameter-range $100 \mu m - 200 \mu m$. Influence of chosen particle diameter on the positioning of the lower asymptote, $f_0$, and the position of the critical point, $f_c$. The RUC model is used to describe the lower asymptote.

and 1.75 respectively. Here $\Delta p$ denotes the finite pressure difference (measured in the streamwise direction of fluid flow), $H$ the bed height, $\varepsilon$ the porosity or bed voidage, $\mu$ the fluid viscosity, $\rho_f$ the fluid density, $q$ the superficial velocity of the traversing fluid and $D_h$ the hydraulic diameter (which is equal to the diameter of the spherical particles).

The work of Du Plessis & Woudberg, as briefly discussed in Section 3.2, allows one to purge equation (4.11) of its empirical elements. Pore-scale analysis of interstitial flow conditions lead to the following expression of coefficients

$$M = \frac{25.4 \varepsilon^3}{(1 - \varepsilon)^{2/3}(1 - (1 - \varepsilon)^{1/3})(1 - (1 - \varepsilon)^{2/3})^2}, \quad (4.12)$$

and

$$N = \frac{\varepsilon^2 c_d}{2(1 - (1 - \varepsilon)^{2/3})^2}. \quad (4.13)$$

They thus succeed in rewriting (4.11) such that it is independent of, and not limited
4.1 Newtonian fluid

Figure 4.3: Particle diameter-range 750µm – 1000µm. The choice of particle diameter determines orientation of the lower asymptote, $f_0$, and critical point, $f_c$. Note that the asymptotes need not be straight lines – slight curvature of $f_0$ due to the quadratic nature of equation (4.11) is evident.

by, the range of porosities used. Here the particle Reynolds number, $Re_p$, is defined as

$$Re_p \equiv \frac{\rho_f q D_h}{\mu},$$

(4.14)

and as in [32] the value of the form drag coefficient, $c_d$ in (4.13), was taken to be 1.9, presenting the most empirical aspect of the procedure.

4.1.2.2 The upper asymptote

As noted by Geldart [26], the pressure drop across a fluidised bed, given by

$$\Delta p = \frac{m_0 g}{A_c} = \frac{\rho_0 U_0 g}{A_c} = \rho_0 g H,$$

(4.15)

is the only parameter that can be predicted with accuracy, since at all times during fluidisation the downward force, i.e. the weight of the bed, $m_0 g$, is balanced by the
upward force, $\Delta p A_c$. Division of equation (4.15) by the bed height, yields

$$
\frac{\Delta p}{H} = \frac{m_0 g}{A_c H} = \frac{\rho_0 U_0 g}{A_c H} = \rho_0 g,
$$

which forms the upper limiting asymptote. Here $m_0$ denotes the bulk mass, $\rho_0$ the bulk density, $U_0$ the bulk volume, $g$ acceleration due to gravity (taken as $9.81\text{ m/s}^2$) and $A_c$ the cross-sectional area of the bed.

### 4.1.3 Powered addition of the asymptotes

The original Ergun equation, (4.11), was obtained through simple addition of the Blake-Kozeny and Burke-Plummer equations, the former being a Darcy-type equation predominating in the regime where $Re_p \to 0$ and the latter dominating in the Forchheimer regime (refer to Section 3.1). If powered addition, as discussed in Chapter 1, is used to match the asymptotic conditions – i.e. equations (4.11) and (4.16) are combined – a single correlative measure,

$$
\frac{\Delta p}{H} = \left[ M \frac{\left( 1 - \varepsilon \right)^2 \mu q}{D_h^5} + N \frac{\left( 1 - \varepsilon \right) \rho_f q^2}{\varepsilon^3 D_h^5} \right]^s + \left( \rho_0 g \right)^s,
$$

is obtained for the pressure drop over the bed. Here coefficients $M$ and $N$ are as expressed in equations (4.12) and (4.13).

#### 4.1.3.1 Critical point and shifting-exponent

As discussed in Section 1.4 the critical point of the matching curve is the value of the independent variable at which the asymptotes meet. To determine this value for the case of the fluidised bed, we set

$$
M \frac{\left( 1 - \varepsilon \right)^2 \mu q}{D_h^5} + N \frac{\left( 1 - \varepsilon \right) \rho_f q^2}{\varepsilon^3 D_h^5} = \rho_0 g,
$$

which yields a quadratic equation in $q$. Let $q_c$ be the value of the independent variable at which the asymptotes meet, i.e. the critical point. Solving $q = q_c$ in equation (4.18) yields

$$
q_c = \frac{M}{N} \frac{\mu}{2 \rho_f D_h} (1 - \varepsilon) \left[ -1 \pm \sqrt{1 + \frac{N \varepsilon^3}{M^2 (1 - \varepsilon)^3} \frac{4 \rho_0 \rho_f g D_h^3}{\mu^2}} \right].
$$

Since $\frac{N \varepsilon^3}{M^2 (1 - \varepsilon)^3} \frac{4 \rho_0 \rho_f g D_h^3}{\mu^2} \geq 0$ in equation (4.19) and $q_c \geq 0$, it follows that we may disregard the negative root. Substitution of this $q_c$ value into equation (4.17) will yield the
function value at the intersection of the asymptotes. It is also worth noting that once the pressure drop created by the fluid flow becomes sufficient to support the weight of the bed, fluidisation will take an onset. In the general introductory discussion on fluidised beds at the beginning of this chapter it was remarked that this is referred to as the point of incipient fluidisation and the corresponding superficial velocity of the fluid as the minimum fluidisation velocity, $q_{mf}$. Seeing as the upper limiting asymptote predominates in the composition of the matched solution beyond the critical point, this point of intersection is the threshold value of the superficial velocity at which fluidisation will occur. The value found by equation (4.19) will thus be a theoretical prediction of the minimum velocity required to fluidise the bed, i.e. $q_{mf} = q_c$.

Once the experimental value of $f_c = f\{q_c\}$ for each of the diameter ranges has been determined by the method discussed above, we may proceed to calculate the value of the shifter by equation (1.45). As an alternative we may manually adjust the predictive curve until satisfactory visual correlation with the data is achieved and then assign a value to $s$.

### 4.1.3.2 Crossing of the upper asymptote

In Section 1.2.4 a method was outlined to construct a (postulated) dependence should the collected data cross the upper limiting asymptote. Referring to Figure 4.2 it is evident that this is the case for particles within the diameter range $100\mu m - 200\mu m$. This phenomenon is, according to Geldard [26] and Davidson & Harrison [35], caused by the wedging action within the bed and cohesion between the particles and is prevalent in beds composed of Group B particles (into which this diameter range resorts).

Letting $\zeta\{\infty\} = \rho_0g$, and making use of the functional form suggested by equation (1.35), the dependence

$$\frac{\Delta p}{H} = \rho_0 g \left[1 + \left(\frac{q_A}{q}\right)^\alpha\right],$$

(4.20)

was postulated to represent behaviour of the bed in the upper limits of the superficial velocity. Here $q_A$ is equivalent to the arbitrary constant of equation (1.35) and $\alpha$ an arbitrary exponent (see Section 1.2.4). The values of both these variables were chosen by trial and error; selecting the former to be equal to the value of the critical point, calculated in equation (4.19), that is $q_A = q_c$, and the latter as $\alpha = 12$, yielded encouraging results. The RUC model, as per equations (4.11), (4.12) and (4.13) was kept as representative of the lower asymptote. After applying the powered addition method used when an asymptote is crossed (outlined in Section 1.2.4) to the constituent terms in the functional expression and some rearrangement, the pressure drop across the
Figure 4.4: Curve-fitting by powered addition with a postulated function for the upper limiting dependency to enable crossing of the upper extremal value. Particles in diameter-range $100\mu m - 200\mu m$. The functional relation is given by equation (4.21), with $x_A = 0.026$, $\alpha = 12$ and $s = 4$ having been used.

The pressure drop may now be expressed as

$$\frac{\Delta p}{H} = \frac{\zeta_0\{q\}\zeta_\infty\{q\}}{(\zeta_0\{q\}^s + \zeta_\infty\{q\}^s)^{1/s}}.$$  \hspace{1cm} (4.21)

Here $\zeta_0\{q\}$ is the functional dependence in the lower limit of the pressure drop on the superficial velocity as given by equation (4.11) and $\zeta_\infty\{q\}$ as per equation (4.20). In Figure 4.4 the matched curve yielded after plotting equation (4.21) against the collected data is shown; the degree of agreement between the two being a satisfactory result. Here, and in subsequent figures, $f_{\text{CHE}}$ refers to the Churchill-Usagi-equation, or matched solution, as per equation (1.6).

### 4.1.4 Correlation of experimental results

The pressure drop was plotted against the superficial velocity for each of the diameter ranges. Figures 4.2 and 4.3 serve to illustrate the influence of the use of the mini-
4.1 Newtonian fluid

mum ($d_{\text{min}}$), average ($d_{\text{mean}}$) and maximum ($d_{\text{max}}$) particle diameters on the orientation of the lower asymptote. A similar plot may be drawn for particle diameter-range

![Graph](http://scholar.sun.ac.za)

Figure 4.5: Curve-fitting by powered addition to experimental readings for particles in diameter-range $100\mu m - 200\mu m$. The functional relation is given by equation (4.17). An $s$-value of -2 was used for $d_{\text{mean}}$; an $s$-value of -14 for $d_{\text{max}}$ (as per Table 4.2).

400$\mu m$ – 600$\mu m$. It is important to note that the asymptotes need not be straight lines; this is apparent in Figure 4.3 where the quadratic nature of the RUC model – equation (4.11) with coefficients $M$ and $N$ as expressed in equations (4.12) and (4.13) – starts to dominate due to an increase in the superficial velocity before the onset of fluidisation.

<table>
<thead>
<tr>
<th>Diameter</th>
<th>100 – 200$\mu m$</th>
<th>400 – 600$\mu m$</th>
<th>750 – 1000$\mu m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{\text{min}}$</td>
<td>-0.593</td>
<td>-0.825</td>
<td>-1.878</td>
</tr>
<tr>
<td>$d_{\text{mean}}$</td>
<td>-2.248</td>
<td>-3.155</td>
<td>-8.193</td>
</tr>
<tr>
<td>$d_{\text{max}}$</td>
<td>-14.280</td>
<td>-9.567</td>
<td>-11.786</td>
</tr>
</tbody>
</table>

Table 4.2: Calculated values of shifter-exponent, $s$, for the different particle diameter possibilities

From these graphical results the particle-diameter yielding the best correlation with
Figure 4.6: Powered addition-curve fitted to data for particles in diameter-range 400$\mu$m – 600$\mu$m. The functional relation is once again given by equation (4.17); $s$-value of -10 was used.

the experimental data was chosen to be used in the fitting of a predictive curve. In the case where the data points were not noticeably favouring a specific particle diameter, the experimental data was traversed in search of the $q_c$-value matching the theoretically predicted value of the critical point, as expressed by equation (4.19), most closely. This value was then used to determine an $s$-value. Calculated values of $s$ are shown in Table 4.2. The corresponding curves were plotted and the best match was chosen by visual inspection. From equation (4.19) it is clear that the diameter of the particle impacts on the value of $q_c$, and thus on $f_c = f\{q_c\}$. The latter in turn has a direct influence on the value of $s$, as calculated by equation (1.45). In the format of equation (4.17), with $\Delta p/H$ the dependent and $q$ the independent variable, the relation shows a decreasing dependence as the superficial velocity grows. Furthermore, both the functions at the extremal values are presumed to form upper bounds on the value that the pressure drop may take. Hence, it is expected that the plot should be qualitatively analogous to the case outlined in Section 1.2.2.2 and a negative value of the shifting exponent is to be expected. Indeed, this turns out to be the case.

It should however be noted that the datum point chosen upon traversal of the ex-
experimental data may be a poor choice (an outlier, the result of a poor reading, etc.). Establishing the \( s \)-value purely based on this principle may lead to erroneous results; the value of visual inspection should never be underestimated (besides, the method is empirical). The false ‘accuracy’ portrayed by the values listed in Table 4.2 is merely the result of the algorithm used to traverse the original data sets; as Churchill and Usagi [1; 2; 5; 6] notes the solution is relatively insensitive to changes in \( s \) and hence these values were rounded in each of the corresponding figures.

For diameter-range 100\( \mu \text{m} \) – 200\( \mu \text{m} \), shown in Figure 4.2, the experimental values lie between the average, \( d_{\text{mean}} \), and maximum, \( d_{\text{max}} \), particle diameters. Curves for the matched solution were plotted for a particle diameter of both 153\( \mu \text{m} \) and 200\( \mu \text{m} \) – the former yielding results that closer match the trend of the data; illustrated by Figure 4.5.

Figure 4.7: Curve-fitting by the method of powered addition of asymptotic solutions as applied to particles in diameter-range 750\( \mu \text{m} \) – 1000\( \mu \text{m} \). Equation (4.17) represents the functional relation with an \( s \)-value of -8 having been used.

In the case of diameter-range 400\( \mu \text{m} \) – 600\( \mu \text{m} \), the data points were distributed around the asymptote predicted by the upper limit of the range, \( d_{\text{max}} \). Use of a particle diameter of 600\( \mu \text{m} \) yields the best graphical results, as shown in Figure 4.6. Examining diameter-range 750\( \mu \text{m} \) – 1000\( \mu \text{m} \), it was once again difficult to discern the particle di-
ameter to be used by merely regarding data point distribution about the asymptotes. Figure 4.3 shows experimental points crossing the asymptotes for both $d_{\text{max}}$ and $d_{\text{mean}}$. Curves were plotted for both solutions; the average diameter of 960$\mu$m more closely followed the trend exhibited by the experimental data. The graphical result of a plot with this diameter is shown in Figure 4.7.

### 4.2 Non-Newtonian fluid

Digitised data was obtained from the work of Sabiri [38; 39] and Ciceron [40] for the flow of the non-Newtonian fluids glycerol (glycerine) and carboxymethyl cellulose (CMC), a cellulose derivative, through a bed of spherical particles. As in the case of Newtonian flow discussed above in Section 4.1, the bed was contained in a PMMA (poly methyl methacrylate) or perspex cylinder, the interior diameter of which was 90mm. Two particle sizes were used, the diameters being 2.92mm and 5.00mm respectively, with the corresponding bed porosities at incipient fluidisation taken as 0.42 and 0.437. The CMC solution of 6g/ℓ was said to have a consistency index, $K = 0.624\text{Pa.s}^n$, behaviour index, $n = 0.7662$, and density of $\rho = 1020.3\text{kg/m}^3$. The corresponding parameters for the 75% glycerine solution was taken as $K = 0.025\text{Pa.s}^n$, $n = 1$ and $\rho = 1192.5\text{kg/m}^3$. These authors consider the latter solution to be an example of non-Newtonian fluid flow, though strictly speaking the use of a behaviour index value, $n = 1$ reverts the power law model for non-Newtonian fluids back to the model for Newtonian fluids.

#### 4.2.1 Asymptotic dependencies

From plots of the digitised data an abrupt transition between the packed- and fluidised bed regimes is evident and hence a shifting exponent larger than unity is expected.

##### 4.2.1.1 The lower asymptote

The assumption is made, as in Section 4.1.2.1, that the unfluidised or packed bed may be viewed as a porous medium consisting of spherical particles. The RUC model of Du Plessis & Woudberg [29; 22; 32] was extended [31] to include non-Newtonian fluids, with the stream-wise pressure gradient expressed as

$$-\frac{dp}{dx} = KF_nq^n,$$

(4.22)
where

$$F_n = \frac{2^n (4.47) (1 - \epsilon)^{2/3}}{d_n^{n+1} \left( 1 - (1 - \epsilon)^{1/3} \right)^n \left( 1 - (1 - \epsilon)^{2/3} \right)^{n+1}} \left( \frac{2n + 1}{n} \right)^n \times \left[ 1 + \frac{(1 - (1 - \epsilon)^{2/3})}{12 (4.47) \epsilon} Re_{p_n} \right],$$  \hspace{1cm} (4.23)

and

$$Re_{p_n} = \frac{12 \rho d_n^{n} q^{2-n}}{K} \frac{(1 - (1 - \epsilon)^{1/3})}{(1 - (1 - \epsilon)^{2/3})^{2-n}} \left( \frac{n}{2(2n+1)} \right)^n.$$  \hspace{1cm} (4.24)

Figure 4.8: Fitting of the matched solution to Sabiri [38; 39] and Ciceron [40]'s data. The bed consisted of particles with diameter 2.92 mm and was fluidised with glycerine. An s-value of -10 was used to obtain the fitted curve.

The rational behind the choice of this model to represent the functional dependence of the lower asymptote is similar to that outlined in Section 4.1.2.1.
4.2 Non-Newtonian fluid

4.2.1.2 The upper asymptote

As mentioned in Section 4.1.2.2, the pressure drop at incipient fluidisation is sufficient to support the weight of the particles \[35; 41\], and hence

\[
\frac{\Delta p}{H} = (1 - \varepsilon_0)(\rho_s - \rho_f)g.
\]

(4.25)

with \(\varepsilon_0\) the porosity at incipient fluidisation. The relation given by equation (4.25) above will be taken to represent functional dependence at the upper extreme.

![Figure 4.9: Curve-fitting by the method of powered addition of asymptotic solutions as applied to non-Newtonian flow. Equation (4.26) represents the functional relation with an s-value of -10 having been used. The fluid used to fluidise the bed was glycerine; particle diameter 4.99mm.](http://scholar.sun.ac.za)

4.2.2 Powered addition of the asymptotes

Application of Churchill & Usagi’s method of powered addition, outlined in Chapter 1, to the results obtained for the respective dependencies at the extremal values – that
is, combing equations (4.22) and (4.25) and taking the $s^{th}$ order sum of the solutions – produces the desired single measure,

$$\frac{\Delta p}{H} = \left[ [KF_n q^n]^s + [(1 - \varepsilon_0)(\rho_s - \rho_f)g]^s \right]^{1/s}, \quad (4.26)$$

applicable over the entire range of $q$-values.

![Graph](http://scholar.sun.ac.za)

Figure 4.10: Curve-fitting by the method of powered addition of asymptotic solutions as applied to non-Newtonian flow through a bed consisting of spherical particles, diameter 2.92 mm. Equation (4.26) represents the functional relation with an $s$-value of -2 having been used.

### 4.2.2.1 Critical point and shifting-exponent

Analogous to Section 4.1.3.1, by equating

$$KF_n q^n = (1 - \varepsilon_0)(\rho_s - \rho_f)g, \quad (4.27)$$

and isolating the independent variable as,

$$q_c = \left[ \frac{(1 - \varepsilon_0)(\rho_s - \rho_f)g}{KF_n} \right]^{1/n}, \quad (4.28)$$
the critical point, \( f(q_c) \), of the matching curve where the asymptotic solutions intersect, is obtained. The dependency of this value is thus closely linked to the behaviour index of the particular non-Newtonian fluid under regard.

### 4.2.3 Correlation of experimental results

For the digitised data of Sabiri [38] and Ciceron [40] the pressure loss over the bed versus the superficial velocity of the traversing non-Newtonian fluid was plotted. The value of the shifting exponent, \( s \), was adjusted by trial-and-error visual evaluation to obtain the graphical representations shown in Figures 4.8 to and 4.11. Figures 4.8 and 4.9 show the results obtained for a bed fluidised with a 75% glycerine solution. The expansion of the RUC model to encompass non-Newtonian flow [31] was based on the power law model (or Ostwald de Waele model). The power law model for time-independent, non-Newtonian flow (see Appendix A), given by

\[
\tau_{xy} = -K|\dot{\gamma}|^n,
\]  

\[(4.29)\]
reverts to the mathematical model for Newtonian flow when \( n = 1 \). Hence, since a value of unity is assigned to the behaviour index of glycerine, this fluid is strictly speaking not an example of a non-Newtonian substance.

In the work of Sabiri [38] no mention was made of the bed height in the cases where CMC was used as fluidising liquid. Consequently, the uppermost limiting value had to be adjusted accordingly to fit the experimental data. Since the pressure drop in equation (4.25) is linearly related to the height, an incorrect choice of value would only cause a parallel offset and not influence the curvature of the resultant match. In both the cases of glycerine and CMC the data show an abrupt changeover in the transitional areas, which corresponds to the relatively high values obtained for the shifter. As with the Newtonian fluidised bed, studied in Section 4.1, equation (4.26) exhibits a decrease in the dependence of the pressure drop as the superficial velocity increases (for constant \( n \)). As anticipated from the discussion in Section 1.2, a negative shifting exponent ensures a satisfactory match.
Chapter 5

Conclusion / Closure

The straightforward addition of the solutions at extremal values to obtain a single relation that holds over the domain of the independent variable, is a practice that has been in use for some time, especially in engineering applications. The method of powered addition, formalised by Churchill & Usagi [1], succeeds, through simple exponentiation of the constituent terms, in adjusting the solution so as to more closely trace the experimental or computational data. The procedure should not be regarded as a trick; rather linear addition of asymptotic models should be regarded as a special case of powered addition when \( s = 1 \). Prior knowledge of the functional behaviour in at least one of the limits makes the direct application of the proposed method possible. Should, however, only the limiting values be known, but not the functional dependence at either of the extremities, an empirical expression for one or the other limit may be postulated. In cases such as these, a power function is pertinent; its uncomplicated form making it a satisfactory choice in tune with the underlying philosophy of simplicity.

Although the routine has not yet been proven to accurately describe the relative behaviour of the different parameters during a transfer process, it may be argued to be more appropriate, since the rate of change between the two asymptotes can be adjusted to fit experimental readings. For instance, the higher the (absolute) value of the shifter-exponent, \( s \), the more abrupt the changeover between the two asymptotic processes. Furthermore, irrespective of the rate at which the changeover occurs, it will be a smooth transition as is expected for the crossover from one continuum process to another. In the examined cases in this study, and the various examples listed in the literature by Churchill et al. [1; 2; 5; 6; 25; 42; 43], powered addition appears to be favoured over the introduction of \textit{ad hoc} curve-fitting or bridging functions which might introduce unwanted jumps. Thus, although primarily a curve-fitting exercise, this procedure leads to better physical modelling since the only ‘tuneable’ parameter is
the shifter-exponent, $s$. Adjusting its value does not change the value of the asymptotic conditions and leaves the double-asymptote character of the transfer process intact.

The main advantages of the method is that a singular functional dependence of the independent variable upon the dependent variable is established over the entire range of the latter. The inherent simplicity of the method suggests that in such a curve-fitting exercise the greater deal of effort should be exerted in determination of the asymptotes, the value of the exponent, $s$, and possible relation of the latter to some quantifiable parameter.

The sensitivity of the matched curve to changes in the shifter, $s$, in the vicinity of intersection of the asymptotes, suggests that experiments should be designed so as to focus on this area. More specifically, in performing an experiment it is advantageous to arrange the physical conditions in such a manner that the independent variable is in close proximity of the critical point, $x_c$. Should the experimental values of the dependent variable at the critical point, $f\{x_c\}$, be known, a good indication of the value of the shifting exponent may be obtained by the logarithmic expression of equation (1.45). It is however important to keep in mind that, since the method is empirical in nature, an exact solution is not obtained and therefore the potential benefit of visual inspection and evaluation on the result should not be disregarded (visual inspection by trial and error adjustment of the correlation between the predictive curve and data points often lead to a better assignment of a value to $s$). As noted by Churchill & Usagi [1] the matched curve is relatively insensitive to variations in $s$; the required accuracy being determined by considerations such as the process involved, tunability of other parameters and allowable error-margin.
Appendix A

Fluid classification

In the classification of fluids two different approaches are followed. Hence, the category into which a fluid will fall depends on the criterion being used: either the fluid’s reaction to an externally applied pressure, or the response of the fluid due to the application of a shearing force or shear stress, being used as distinguishing factor. The former approach differentiates between compressible and incompressible fluids, depending on the relation between the pressure and volume of the fluid; the latter is contingent upon the relationship between shear stress and shear rate and leads to the distinction between Newtonian (linear relation) and non-Newtonian (non-linear relation) fluids [44; 45]. In other words classification is either based on the compressibility or viscosity. Most liquids may be regarded as incompressible [44] (whereas this is not the case for gasses) and accordingly the second approach will be discussed in this section.

Viscosity can be regarded as a fluid’s resistance to a shearing force when the fluid is in motion; the nature of the relation between the shear stress and the velocity gradient of the fluid determining into which category the fluid is classified.

A.1 Newtonian flow

For Newtonian fluids the velocity gradient in the direction of flow, $dv_x/dy$ is linearly related to the shear stress, $\tau_{yx}$, by

$$\tau_{yx} = \mu \left(-\frac{dv_x}{dy}\right) = \mu \dot{\gamma}_{yx},$$  

(A.1)

where $\mu$ denotes the viscosity, a proportionality constant linking the former two factors. In rheology the symbol $\dot{\gamma}$ is frequently used to represent the shear rate of the fluid, with the first subscript referring to the surface on which the stress is acting and
the second to the direction in which it acts, i.e. \( \dot{\gamma}_{yx} \) is the shear rate of the stress acting on a surface defined by a normal vector in the \( y \)-direction and with component in the \( x \)-direction (the direction of flow) [44; 45]. The negative sign in equation (A.1) serves to indicate that \( \tau_{yx} \) opposes the motion of the fluid.

Alternatively, for the case of an incompressible fluid (i.e. \( \rho \) constant), equation (A.1) may be written as

\[
\tau_{yx} = -\frac{\mu}{\rho} \frac{d}{dy} (\rho v_x) = -v \frac{d}{dy} (\rho v_x),
\]

(A.2)

where \( \rho \) represents the fluid’s mass density. Now the quantity \( \rho v_x \) may be thought of as the linear momentum in the \( x \)-direction per unit volume of fluid, with \( \tau_{yx} \) the flux of momentum in the direction of flow, the \( x \)-direction [44]. The ratio between viscosity, \( \mu \), and mass density, \( \rho \) is referred to as the kinematic viscosity and represented by the Greek letter \( \nu \), in equation (A.2), i.e

\[
\nu = \frac{\mu}{\rho},
\]

(A.3)

By definition the Newtonian viscosity, \( \mu \), is independent of both the shear rate, \( \dot{\gamma}_{yx} \), and the shear stress, \( \tau_{yx} \), and is dependent only upon the properties of the material, its temperature and pressure. This single constant therefore serves to completely characterize the flow behaviour of the fluid at a given pressure and temperature.

It is important to note that equation (A.1) is for the simple case of unidirectional flow, the velocity vector having only one component directed in the \( x \)-direction and being dependent only on the \( y \)-coordinate for its magnitude. In the case of three-dimensional flow, the appropriate partial differential equations for each of the nine entries in the shear stress tensor need to be evaluated. This more general case, for the \( x \)-plane, of an incompressible fluid is expressed as [17; 44]

\[
\tau_{xx} = -2\mu \frac{\partial v_x}{\partial x} + \frac{2}{3} \mu \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right),
\]

(A.4)

\[
\tau_{xy} = -\mu \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right),
\]

(A.5)

\[
\tau_{xz} = -\mu \left( \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right).
\]

(A.6)

Similar expressions to those of equations (A.4) to (A.6) may be set up for each of the \( y \)- and \( z \)-planes; as above there will be two shearing components (in the plane being regarded) and one component normal to the plane in each of the cases. Furthermore, it holds for a Newtonian fluid that [17; 44]

\[
\tau_{xx} = \tau_{yy} = \tau_{zz} = 0.
\]

(A.7)

Therefore, the complete definition of a fluid that falls into this category is not only that it has a constant viscosity, but also that the fluid obeys equation (A.7).
A.2 Non-Newtonian flow

A fairly large category of fluids does not obey the Newtonian law of viscosity, in other word the viscosity is not independent of the shear rate and the relation between the shear stress and shear rate is not a linear one. These fluids are classified as non-Newtonian and require, besides viscosity, the measurement of additional parameters (normal stresses, etc.). Broadly these fluids are classified into three general classes as (a) time independent or purely viscous; (b) time dependent; and (c) viscoelastic, though most real materials exhibit a combination of features. Each of these categories may be further subdivided [44; 45]: time independent fluids into shear-thinning or pseudo-plastic fluids, viscoplastic fluids and shear-thickening or dilatant fluids; and time dependent fluids into thixotropic and rheopectic fluids. The classification is by no means sharply defined and merely serves to aid in the mathematical modelling of the fluids. Indeed, rheology – from the Greek word *rheos* meaning stream – is the branch of physics concerned exclusively with the study of the flow of matter and the quantitative and qualitative relationships between deformations and stresses and their derivatives.

The extension of the RUC model to incorporate non-Newtonian fluids [31], used in Section 4.2.1.1, is based on the power law (or Ostwald de Waele) model and will thus be the only model outlined here. In this model, which is effectively a generalised expression used to model Newtonian as well as non-Newtonian fluids, the non-linear relation between the shear stress and shear rate is given by

\[ \tau_{yx} = K(\dot{\gamma}_{yx})^n, \]  

(A.8)

where \( K \), the fluid consistency index, and \( n \), the flow behaviour index, are empirical curve fitting parameters used to characterise the fluid and are constant at a fixed temperature and pressure. The exponent, \( n \), is a measure of how much the fluid deviates from a Newtonian fluid and hence for \( n = 1 \) equation (A.1) will revert to equation (A.8) with \( K = \mu \). Division of the shear stress by the shear rate,

\[ \eta = \frac{\tau_{yx}}{\dot{\gamma}_{yx}}, \]  

(A.9)

gives the so-called apparent viscosity, \( \eta \), a parameter frequently used in the literature. For values of \( n < 1 \) the fluid exhibits shear-thinning properties, meaning that the apparent viscosity of the fluid decreases as with an increase in shear rate. The opposite behaviour is observed when \( n > 1 \) and hence these fluid are known as shear-thickening.

Criticism against the power law is that for zero shear rate the apparent viscosity in equation takes on an infinite value and that for real fluids the exponent, \( n \), is not
A.2 Non-Newtonian flow

constant over the entire range of flow. Another objection is that the consistency index, \( K \), is dependent upon the behaviour index, \( n \), and may thus not be compared unless the latter values are similar. Despite all the critique, the power law model has the advantage of simplicity and is therefore, according to Chhabra [44], one of the models used most widely in literature dealing with process engineering.
Appendix B

Derivation of Slatter Reynolds number

In its most general form, without any simplifications, the Navier-Stokes equation is a complicated, nonlinear partial differential equation used to describe the motion of a fluid substance (as yet, no general solution has been obtained) [14]. During the last part of the 19th century Osborne Reynolds studied these equations governing flow in an attempt to determine when two different flow situations may be deemed similar.

Two instances of flow are said to be dynamically similar if (a) their corresponding linear dimensions have a constant ratio (that is, they are geometrically similar); and (b) pressures at corresponding points have a constant ratio (geometrically similar streamlines)[14]. Reynolds considered geometrically similar flow situations and came to the conclusion that these flows would be dynamically similar, providing the differential equations describing them were identical. He found that if the dimensionless group \( \frac{\rho vl}{\mu} \) was the same, this criterion was met (here, \( \rho \) is the mass density, \( v \) is a characteristic velocity, \( l \) a characteristic length, and \( \mu \) the viscosity). Thus, the parameter

\[
Re = \frac{\rho vl}{\mu},
\]

has taken his name, Reynolds number, and may be viewed as the ratio of the inertial forces to viscous forces [14; 46; 16], i.e.

\[
Re \propto \frac{\text{inertial forces}}{\text{viscous forces}}.
\]

Many variations on the above definition of the Reynolds number for straight, round tubes exist; the characteristic quantities being chosen so as to be specific to the case where it is being applied. The Slatter Reynolds number, \( Re_3 \), is one such variant based on the yield pseudo-plastic model. It starts from the assumption that, in the presence of a yield stress, the core of the fluid moves as a solid, unsheared plug [47; 15; 10] resulting in annular flow.
The fact that velocity components perpendicular to a surface cannot exist in the region of the surface, accounts for the existence of a laminar sub-layer immediately adjacent to the solid surface. It also implies that solid boundaries suppress turbulence. Slatter [47; 15] notes that the definition of yield stress states by implication that if the shear stresses are insufficient to overcome the former, the material will behave as a solid. Such is the case in the unsheared core of plug flow – a coaxial solid plug existing in the centre of the pipe with flow in an annular region – a representation of which is shown in Figure B.1. He argues that this plug may be regarded as a solid boundary and that it will affect the stability of the laminar flow in the surrounding annulus [47; 15; 10]. Hence, the flow represented by that of the plug must be subtracted since it is no longer being treated as part of the fluid flow. The velocity profile of annular flow also differs from that of plane Poiseuille flow (the flow of an incompressible Newtonian fluid in a tube), as illustrated in Figure B.2.

The constitutive rheological equation for the yield pseudo-plastic model is formulated as [47; 15]

\[ \tau = \tau_y + K \left( \frac{-dv_z}{dr} \right)^n = \tau_y + K(\dot{\gamma}_{rz})^n, \tag{B.3} \]

with \( \tau_y \) denoting the yield stress, \( K \) the fluid consistency index and \( n \) the flow behaviour index. The shear rate, \((-dv_z/dr) = \dot{\gamma}_{rz}\), may be expressed as the velocity gradient in the direction perpendicular to that of the shear force (cf. viscous stress dyadic in equation (B.6)). Setting \( n = 1 \) in equation (B.3) yields the so-called Bingham-plastic model, while \( \tau_y = 0 \) results in it reverting back to that for power-law fluids.

Cauchy’s differential equation for the movement of any continuum, which holds at
any arbitrary point within the continuum, states

\[ \rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{f}_b + \nabla \cdot \mathbf{g} \]  (B.4)

where \( \rho \) denotes the density, \( \mathbf{v} \) the velocity, \( \mathbf{f}_b \) the body forces and \( \mathbf{g} \) the stress tensor. Since the pressure in a continuum with fluid properties is linked to the internal surface forces, there should be a relation between the pressure and the stress tensor. This is indeed the case and the stress tensor may be written as

\[ \mathbf{g} = -p I + \mathbf{\tau} (\mathbf{v}) \]  (B.5)

with \( p \) the pressure and \( \mathbf{\tau} \) the so-called viscous stress dyadic, represented in its matrix form for cylindrical coordinates as

\[
\mathbf{\tau} = \begin{bmatrix}
\tau_{rr} & \tau_{r\theta} & \tau_{rz} \\
\tau_{\theta r} & \tau_{\theta \theta} & \tau_{\theta z} \\
\tau_{z r} & \tau_{z \theta} & \tau_{zz}
\end{bmatrix} .
\]  (B.6)

According to convention the first subscript refers to the face upon which the stress is acting and the second to the direction in which it acts (first face, second stress). In other words the entry \( \tau_{rz} \) corresponds to the stress acting on a surface defined by a normal vector in the \( r \)-direction and with component in the \( z \)-direction (the direction of flow as per Figure B.3).

Equation (B.4) may now, with the aid of equation (B.5), be rewritten as

\[ \rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{f}_b - \nabla p + \nabla \cdot \mathbf{\tau} . \]  (B.7)
Once the flow has developed fully, there is no more acceleration of fluid particles. Hence the velocity profile is constant, no longer time dependent and \( \mathbf{v} = v(r) \mathbf{k} \). This being the case, one may then set the left-hand side of equation (B.7) equal to zero, since

\[
\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{u} = \frac{\partial v}{\partial t} + v_r \frac{\partial v}{\partial r} + \frac{v_\theta}{r} \frac{\partial v}{\partial \theta} + v_z \frac{\partial v}{\partial z},
\]

(B.8)

where

\[
v_r = \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial z} = 0,
\]

(B.9)

and \( \frac{\partial v}{\partial t} = 0 \) (no acceleration). If it is further assumed that the influence of gravity upon the flow is negligible (or has reached an equilibrium at fully developed flow) it implies that equation (B.7) reduces to

\[
\nabla p = \nabla \cdot \tau.
\]

(B.10)

However, a pressure gradient only exists in the direction of flow. Using a cylindrical coordinate system with the z-direction orientated along the axis of the pipe, this means that

\[
-\frac{\partial p}{\partial z} \mathbf{k} = \frac{dp}{dz} \mathbf{k} = \nabla \cdot \tau,
\]

(B.11)

but since \( \frac{\partial p}{\partial z} \) is a constant for fully developed flow, equation (B.11) may be written as

\[
c \mathbf{k} = \nabla \cdot \tau,
\]

(B.12)

where \( c \) is a constant value.

![Figure B.3: Schematic representation of flow in a tube to show influence of shear stress. The fluid flow is from left to right.](http://scholar.sun.ac.za)

Flow is caused by a pressure difference across the ends of the pipe. Once the flow has fully developed the radial velocity is zero and the fluid is flowing at a steady rate, in other words a fixed pressure gradient exists between any two reference points of the pipe. (See Figure B.3 illustrating the conditions present during fully developed, steady flow of an incompressible fluid in a pipe of radius \( R \)).
Regarding an arbitrary fluid element, \(ABCD\), as per Figure B.3, the linear momentum balance (which is in the direction of flow, i.e. the \(z\)-direction) may be written as
\[
p(\pi r^2) - (p + \Delta p)(\pi r^2) = \tau_{rz} (2\pi r L), \tag{B.13}
\]
with \(r\) and \(L\) as indicated. Rearranging this yields,
\[
\tau_{rz}(r) = \left( -\frac{\Delta p}{L} \right) \frac{r}{2} = \left( -\frac{dp}{dz} \right) \frac{r}{2} = c \frac{r}{2}. \tag{B.14}
\]
Thus the shear stress varies linearly over the pipe or tube’s cross-section, with the maximal value obtained at the wall and no shear stress at the axis \[44\]. What is important to note is that equation (B.13) is based only on the balancing of forces and that no assumptions have been made regarding fluid type or flow pattern. Hence it is applicable to laminar as well as turbulent flow, providing that the fluid is incompressible and the flow is steady and fully developed \[44\]. Even in the presence of a yield stress, this statement would still be true since, once the flow has developed fully, the linear momentum balance should still hold. The plug or unsheared core, although regarded as a solid boundary, moves along with the fluid and thus the flow in the annular region surrounding it may be considered as a special instance of plane Couette-Poiseuille flow (the resulting velocity profile, as previously mentioned, is shown in Figure B.2).

Letting \(\tau_0\) denote the shear stress at the pipe wall, i.e.
\[
\tau_{rz}(R) = \tau_0, \tag{B.15}
\]
equation (B.14) may be re-written as
\[
\tau_{rz}(r) = \frac{\tau_0}{R} r, \tag{B.16}
\]
Rearranging equation (B.3) as
\[
-(\frac{\tau_{rz}(r) - \tau_y}{K})^{1/n} = \frac{dv_z}{dr}, \tag{B.17}
\]
it may now be integrated as follows
\[
\int_{v_z(r)}^{v_z(R)} dv_z = -\left( \frac{1}{K} \right)^{1/n} \int_r^R \left[ \tau_{rz}(r) - \tau_y \right]^{1/n} dr
\]
\[
\Rightarrow v_z(R) - v_z(r) = -\left( \frac{1}{K} \right)^{1/n} \int_r^R \left[ \tau_0 \frac{r}{R} - \tau_y \right]^{1/n} dr
\]
\[
0 - v_z(r) = -\left( \frac{1}{K} \right)^{1/n} \left[ \left( \frac{\tau_0}{K} r - \tau_y \right)^{1+1/n} R \right]_r^{R} 
\]
\[
v_z(r) = \frac{n}{n + 1} \frac{R}{\tau_0} \left( \frac{1}{K} \right)^{1/n} \left[ (\tau_0 - \tau_y)^{(n+1)/n} - \left( \frac{\tau_0}{K} r - \tau_y \right)^{(n+1)/n} \right]
\]
\[
= \frac{n}{n + 1} \frac{R}{\tau_0} \left( \frac{1}{K} \right)^{1/n} \left[ (\tau_0 - \tau_y)^{(n+1)/n} - (\tau_{rz} - \tau_y)^{(n+1)/n} \right] \tag{B.18}
\]
Since the plug moves along with the fluid, yet does not shear (i.e. behaves as a solid), it implies that by setting $\tau_{rz} = \tau_y$ in equation (B.18) the velocity of the plug may be obtained as

$$v_{plug} = \frac{n}{n + 1} \frac{R}{\tau_0} \left( \frac{1}{R} \right)^{1/n} (\tau_0 - \tau_y)^{(n+1)/n}.$$  \hspace{1cm} (B.19)

By the same reasoning as above, it follows from equation (B.16) that for $\tau_{rz} = \tau_y$, the radius of the plug is determined as

$$r_{plug} = \frac{\tau_y}{\tau_0} R,$$  \hspace{1cm} (B.20)

and the area of the annulus by

$$A_{ann} = \pi (R^2 - r_{plug}^2).$$  \hspace{1cm} (B.21)

The sheared diameter, $D_{shear}$, as indicated in Figure B.1, may be calculated from

$$D_{shear} = D - D_{plug},$$  \hspace{1cm} (B.22)

and

$$D_{plug} = 2r_{plug}.$$  \hspace{1cm} (B.23)

As mentioned earlier the flow represented by the plug needs to be subtracted since it is treated as a solid body and doesn’t form part of the fluid flow. The flux through the annular part of the pipe is given by

$$Q_{ann} = Q - Q_{plug},$$  \hspace{1cm} (B.24)

with

$$Q_{plug} = v_{plug} A_{plug},$$  \hspace{1cm} (B.25)

where $Q$ denotes the flux indicated by the respective subscripts and $A$ the cross-sectional area. The corrected mean velocity in the annulus is thus obtained,

$$\bar{v}_{ann} = \frac{Q_{ann}}{A_{ann}}.$$  \hspace{1cm} (B.26)

Returning to the fundamental assumption of equation (B.2) where, according to Massey [16],

$$\text{inertial force } \propto \rho l^2 \bar{v}^2,$$  \hspace{1cm} (B.27)

and

$$\text{viscous force } \propto l^2 t_{visc},$$  \hspace{1cm} (B.28)

with $t_{visc}$ the representative viscous shear stress. The constitutive rheological equation – here, the yield pseudo-plastic model – for the fluid being considered relates the viscous stress to the shear rate. Slatter [47; 15; 48] notes that the so-called bulk shear
rate, \( \overline{\tau}/D \), has successfully been used in other rheological formulations and proposes its use as representative shear rate. Its use yields

\[
\tau_{\text{visc}} = \tau_y + K \left( \frac{8\overline{\tau}}{D} \right)^n,
\]

as representative shear rate. Since the shearing of the fluid only takes place in the annular region surrounding the plug, taking the sheared diameter, \( D_{\text{shear}} \), as per equation (B.22), as the characteristic length is preferable. The same reasoning leads to the choice of the velocity in the annulus, \( \overline{v}_{\text{ann}} \), as characteristic velocity. Substitution of these parameters into equation (B.29) together with relations (B.27) and (B.28) produces

\[
Re \propto \frac{\rho \overline{v}^2_{\text{ann}}}{\tau_y + K \left( \frac{8\overline{\tau}_{\text{ann}}}{D_{\text{shear}}} \right)^n}.
\]

Only the choice of a proportionality constant remains. The choice should result in the newly defined relation reverting back to standard form under Newtonian conditions. In other words setting \( \tau_y = 0, K = \mu \) and \( n = 1 \) in equation (B.30) should reduce it to equation (B.1). This is achieved by selecting the value 8, whence

\[
Re_3 = \frac{8\rho \overline{\tau}^2_{\text{ann}}}{\tau_y + K \left( \frac{8\overline{\tau}_{\text{ann}}}{D_{\text{shear}}} \right)^n},
\]

is obtained; the so-called Slatter Reynolds number.
Appendix C

Plots with Mbiya’s data sets

The following plots are based on the data sets generated by Mbiya [9] for pipes with varying diameters and at pre-selected opening values – Figures C.1 to C.4 are for a pipe with internal diameter of $40\text{mm}$; Figures C.5 to C.8 for $50\text{mm}$; Figures C.9 to C.12 for $65\text{mm}$; Figures C.13 to C.16 for $80\text{mm}$; and Figures C.17 to C.20 for $100\text{mm}$. In each of the representations (bottom) two matched solutions have been plotted: $s = 0.4$ and $s = 1.4$ in a solid and dashed line respectively.
Figure C.1: Internal pipe diameter: 40mm. Valve opening: 25%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.2: Internal pipe diameter: 40mm. Valve opening: 50%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.3: Internal pipe diameter: 40mm. Valve opening: 75%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.4: Internal pipe diameter: 40mm. Valve opening: 100%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.5: Internal pipe diameter: 50 mm. Valve opening: 25%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.6: Internal pipe diameter: 50mm. Valve opening: 50%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.7: Internal pipe diameter: 50mm. Valve opening: 75%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.8: Internal pipe diameter: 50mm. Valve opening: 100%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.9: Internal pipe diameter: 65mm. Valve opening: 25%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.10: Internal pipe diameter: 65mm. Valve opening: 50%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.11: Internal pipe diameter: 65mm. Valve opening: 75%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.12: Internal pipe diameter: 65mm. Valve opening: 100%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.13: Internal pipe diameter: 80mm. Valve opening: 25%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.14: Internal pipe diameter: 80 mm. Valve opening: 50%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.15: Internal pipe diameter: 80mm. Valve opening: 75%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.16: Internal pipe diameter: 80mm. Valve opening: 100%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.17: Internal pipe diameter: 100mm. Valve opening: 25%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.18: Internal pipe diameter: 100mm. Valve opening: 50%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.19: Internal pipe diameter: 100mm. Valve opening: 75%. Mbiya’s two-constant model (top), powered addition (bottom).
Figure C.20: Internal pipe diameter: 100mm. Valve opening: 100%. Mbiya’s two-constant model (top), powered addition (bottom).
Appendix D

Høgskolen i Telemark (Telemark University College) data sets

Measurements were performed on a laboratory-scale fluidized beds at Høgskolen i Telemark (Telemark University College), Porsgrunn, Norway between 13 – 20 October 2008. The beds, contained in a cylindrical perspex tube with inner diameter of 72 mm, comprised of glass powders, consisting of spherical glass particles, with a density of 2485 kg/m³ and available in three different diameter-ranges: 100 µm – 200 µm; 400 µm – 600 µm; and 750 µm – 1000 µm. In all of the experiments performed the fluid used to fluidise the bed was air at ambient conditions, with a density of 1.22 kg/m³ and viscosity of $1.78 \times 10^{-5} N \cdot s/m^2$. Below are the tables containing the experimental data.

Eight pressure tubes were placed along the height of the bed and were connected to digital pressure sensors (Honeywell, model 142PC02G) that measured the pressure at these fixed intervals within the bed. In Tables D.1 to D.3 these distances are indicated in the heading of each column under the sensor number. The unprocessed pressure data was plotted against distance of the sensor from the porous plate on which the bed was supported; the result of this plot is shown in Figure D.1. Since the pressure drop is a linear function of the height, this plot served to demonstrate possible anomalies in individual sensors (e.g. sensor 3 may be disregarded); for the results discussed in Section 4.1.4 sensors number 2 and 8 were used.
Figure D.1: Plot of raw pressure data, \( p \) [\text{mbar}], against vertical sensor distance from porous plate supporting the bed, \( z \) [\text{mm}].
Table D.1: **EXPERIMENTAL DATA: 100\(\mu\)m – 200\(\mu\)m**

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