INTEREST RATE MODEL THEORY WITH REFERENCE TO THE SOUTH AFRICAN MARKET

by

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Declaration

I, the undersigned hereby declare that the work contained in this dissertation is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

Signed

Date

ABSTRACT

An overview of modern and historical interest rate model theory is given with the specific aim of derivative pricing. A variety of stochastic interest rate models are discussed within a South African market context. The various models are compared with respect to characteristics such as mean reversion, positivity of interest rates, the volatility structures they can represent, the yield curve shapes they can represent and weather analytical bond and derivative prices can be found. The distribution of the interest rates implied by some of these models is also found under various measures. The calibration of these models also receives attention with respect to instruments available in the South African market. Problems associated with the calibration of the modern models are also discussed.

OPSOMMING

'n Oorsig van moderne en histories belangrike rentekoersmodelle word gegee met die oog op die prysing van afgeleide instrumente. Die modelle word bespreek in 'n Suid-Afrikaanse konteks en die onderskeie modelle word vergelyk met betrekking tot, rentekoers strukture, volatiliteits strukture en terugkeer eienskappe. Formules vir effekte en rentekoers-afgeleides onder die verskillende modelle word ook ondersoek. Die statistiese verdelings van rentekoerse onder verskillende mate en of die verskillende modelle positiewe rentekoerse toelaat geniet ook aandag. Dit word geillustreer hoe die modelle gekalibreer kan word met behulp van instrumente wat beskikbaar is in die Suid-Afrikaanse mark. Probleme met die kalibrasie van die moderne modelle word ook bespreek.

To my father

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NOTATION

Probability Measures

$\mathbb{P}$	Real world probability measure.
$\widetilde{\mathbb{P}}$	Traditional risk neutral measure (when M(t) is the numéraire).
$\widetilde{\mathbb{P}}^{(N)}$	Risk neutral measure when the asset N(t) is the numéraire.
$\widetilde{\mathbb{P}}^{T}$	T-forward risk neutral measure when the bond B(t,T) is the numéraire.

Expectations

$E[\cdot]$	Expectation with respect to $\mathbb{P}$.
$\widetilde{E}[\cdot]$	Expectation with respect to $\widetilde{\mathbb{P}}$.
$\widetilde{E}^{(N)}[\cdot]$	Expectation with respect to $\widetilde{\mathbb{P}}^{(N)}$.
$\widetilde{E}^{T}[\cdot]$	Expectation with respect to $\widetilde{\mathbb{P}}^{T}$.

Partial Derivatives are denoted by a subscript

$f_t(t,x)$	$\frac{\partial}{\partial t}f(t,x)$
$f_x(t,x)$	$\frac{\partial}{\partial x}f(t,x)$
$f_{xx}(t,x)$	$\frac{\partial^2}{\partial x^2}f(t,x)$

Interest Rates

$R(t)$	The short rate.
$R(t,T)$	The continuously compounded spot rate.
$R(t,S,T)$	The continuously compounded forward rate.
$F(t,T)$	The instantaneously compounded forward rate.
$\widehat{F}(t,T)$	The market instantaneously compounded forward rate.
$L(t,T)$	The simply compounded spot rate.
$L(t,S,T)$	The simply compounded forward rate.

GLOSSARY

ALBI. All Bond Index. An index consisting of the 20 largest S.A. government and corporate bonds. The ALBI is also subdivided into two other indices, the GOVI and the OTHI. Also see GOVI, OTHI.

BESA. Bond Exchange of South Africa

Cap. An interest rate derivative that is made up of a sequence of caplets. A cap can be seen as a European call option on the forward rate over a specified period. It is primarily used to hedge the risk of a surge in interest rates. Also see **Floor**, **Caplet**.

Caplet. An interest rate derivative that forms the basic building block of a cap. A caplet can be seen as a European call option on a simply compounded forward rate over a single tenor period. Also see **Cap, Floor, Floorlet**.

CIR. Cox-Ingersoll-Ross

Day Count Convention. The convention according to which the distance between two dates are measured.

DFB. Default-Free Bond

Floor. An interest rate derivative that is made up of a sequence of floorlets. A floor can be seen as a European put option on the forward rate over a specified period. It is primarily used to hedge the risk of a decline in interest rates. Also see **Cap, Floorlet**.

Floorlet. An interest rate derivative that forms the basic building block of a floor. A floorlet can be seen as a European put option on a simply compounded forward rate over a single tenor period. Also see **Floor, Cap, Caplet**.

Forward Curve. The yield curve of instantaneous forward rates. Also see Yield Curve.

Forward Rate Agreement. A bilateral agreement that fixes the interest rate over a future time period on a fixed size loan.

FRA. Forward Rate Agreement

GOVI. An index consisting of all government bonds used in the construction of the ALBI index. Also see ALBI, OTHI.

IRS. Interest Rate Swap

Interest Rate Swap. See Swap.

HJM. Heath-Jarrow-Morton

JIBAR. Johannesburg Inter Bank Acceptance Rate

Market Makers. Those institutions that are always willing to buy and sell and earn their income from a bid-offer spread. These institutions provide liquidity to the market..

MtM. Mark to market

Numéraire. The numéraire can be any strictly positive non-dividend paying asset price process.

LIBOR. London Inter Bank Offer Rate.

OTHI. An index consisting of all corporate bonds used in the construction of the ALBI index. Also see ALBI, GOVI.

PDE. Partial Differential Equation

SAONIA. South African Over Night Interbank Average

SDE. Stochastic Differential Equation

Spot Curve. The yield curve of instantaneous spot rates. Also see Yield Curve.

SVD. Singular Value Decomposition

Swap. A bilateral agreement between two parties to exchange a fixed rate of interest for a floating rate on some nominal amount N.

Swaption. An option on a swap contract. The strike price of the option is the swap rate at which the swap contract will be initiated. The tenor of the swap is the period over which the swap contract will be in effect.

Term Structure of Interest Rates. See, Yield Curve.

Volatility Structure. A function associating some aspect of a derivative (usually the time to maturity) with a volatility. In the case of swaptions it is the two dimensional function associating the time to maturity and the tenor length with a volatility

Yield Curve. A real valued function associating interest rates with time. Can also be used to refer to a family of equivalent definitions of the same function, e.g. a function for the instantaneous forward rate and the equivalent function for the instantaneous spot rate and the equivalent function for the simple compounded spot rate etc.

Zero Curve. See Spot Curve.

ZCB. Zero Coupon Bond

Chapter 1

INTRODUCTION

1.1 Overview

Though the literature on interest rate modelling is vast, this dissertation attempts to highlight fundamental academic and practical concepts surrounds a few important models to serve as a general but informative introduction to **interest rate modelling theory** with reference to derivative pricing.

In the pricing of equity and exchange rate derivatives, there are two main factors that affect the derivative price,

1) volatility of equity prices and
2) volatility of interest rates.

For these derivatives, the second factor is often assumed to be deterministic because the effect of interest rate fluctuations on the derivative price is overshadowed by the effect of fluctuations of equity prices on the derivative price.

When dealing with interest rate derivatives however, the volatility of interest rates is the only factor affecting the price of the derivative and the assumption that the yield curve follows a known function of time will prove inaccurate. Therefore, from the need for accurate interest rate derivative pricing stems the need for stochastic interest rate modelling.

Chapter 3 introduces descriptive yield curve models that do not incorporate randomness but reflect the current state of the yield curve implied by market instruments. This will be used to calibrate the models in chapters 4 and 5. Considerations for models selection and calibration are discussed in section 3.3.

Chapter 4 introduces endogenous yield curve models and compares the characteristics of three historically important yield curve models, the Vasicek model, the Dothan model and the Cox-Ingersoll-Ross model.

Chapter 5 introduces exogenous yield curve models and elaborates on the current "state of the market" interest rate models. The Hull-White model, the Heath-Jarrow-Morton framework and the LIBOR market models are inspected and their place in the market is discussed.

Finally Chapter 6 concludes with some foresight into the direction of current research and briefly discusses the state of interest rate modelling in the South-African market as well as the applicability of the models from chapters 4 and 5 to the South African environment.

C h a p t e r   2

DEFINITIONS

2.1 Introduction

The chapter starts with a list of general definitions that are important throughout this dissertation. Section 2.3 gives defines the various incarnations of interest rates and yield curves. The chapter ends with an overview of the important explanations for the shape of the yield curve observed in markets.

2.2 General Definitions

The following is a list of fundamental definitions which will prove essential for a thorough understanding of the later chapters.

Definition 2.2.1 Short Rate

> *The short rate, R(t), is an $\mathscr{F}(t)$-measurable quantity representing the annualised, instantaneous rate of interest that will be earned over an infinitesimally small time increment following t.*

Remark 2.1: This will be the primary quantity of interest for an entire class of interest rate models, some of which will be discussed in chapters 4 and 5. The short rate will be modelled with an Ito process.

Definition 2.2.2 Money Market Account

> *The money market account, $M(t)$, is an $\mathscr{F}(t)$-measurable quantity representing the value at time t, of an investment of 1 at time 0, in a bank account which accumulates interest at the short rate. An expression for $M(t)$, in terms of the short rate, is given by (2.1).*

$$M(t) = e^{\int_0^t R(u)du} \tag{2.1}$$

Remark 2.2: $M(t)$ is the numéraire in the traditional risk neutral world. All general asset price processes expressed with $M(t)$ as the denominator will be martingales in the traditional risk neutral world.

Definition 2.2.3 Stochastic Discount Process

The stochastic discount process, $D(t)$ *, is the value at time 0 of an investment of 1 at time t, in a bank account which accumulates interest at the short rate. An expression for* $D(t)$ *in terms of the short rate is given by (2.2).*

$$D(t) = e^{-\int_0^t R(u)du} \tag{2.2}$$

Multiplying a time t cashflow by $D(t)$ gives the equivalent cashflow at time 0.

Remark 2.3: Note that $D(t) = \frac{1}{M(t)}$.

Definition 2.2.4 Default-Free Zero Coupon Bond

A zero coupon bond, $B(t,T)$ *, is an* $\mathscr{F}(t)$ *-measurable quantity representing the no-arbitrage value at time t, of a payoff of 1 at time T. An expression for* $B(t,T)$ *in terms of the short rate is given by (2.3).*

$$B(t,T) = \widetilde{E}\left[\exp\left(-\int_t^T R(u)du\right) \mid \mathscr{F}(t)\right] \tag{2.3}$$

Remark 2.4: In accordance with the fundamental theorem of asset pricing, the expectation is taken under the traditional risk neutral measure, i.e. the measure under which discounted stock prices are martingales. All general asset price processes expressed with $B(t,T)$ as the denominator will be martingales under the T-Forward measure, $\widetilde{\mathbb{P}}^T$.

Default-Free zero coupon bonds are the fundamental building blocks of default-free bonds which trade in the market. The values of these bonds are dependent on unknown future interest rates and since these bonds trade in the market today they must reflect investors' expectation of future interest rates.

Definition 2.2.5 Day Count Conventions

The method by which the distance between two dates, t and T, is expressed as a fraction of a year, is referred to as the day count convention.

The number of years between two dates t and T will be defined by $\tau(t,T)$ where t < T. Exactly how $\tau(t,T)$ should be calculated depends on the specific day count convention. This text does not commit to a specific day count convention and merely refers to $\tau(t,T)$ whenever a day count convention is appropriate. An important application of day count conventions is in the determination of the accrued interest between two dates for coupon bearing bonds.

Remark 2.5: When T and t are very close together $\tau(t,T) = T - t$.

Example 2.1 (Day Count Conventions). All South African bonds use the day count convention of $\frac{Actual}{365}$. This convention assumes that there are 365 days in a year and interest is accrued for each day in the compounding period. The R153 bond issued by the South African government has a coupon rate of 13% per annum and coupons are paid biannually on the 28th of February and the 31st of August. Suppose we are interested in finding the amount of interest accrued on the 15th of June 2005 since the last coupon date. The actual number of days between the 28th of February 2005 and the 15th of June 2005 is 107. Therefore the total amount of interest accrued on the 15th of June is $\frac{107}{365} \times 13 = 3.81$.

Definition 2.2.6 Default-Free Fixed Coupon Bond

The value of default-free coupon bond, $DFB(t,T,\tau(t,T),C,N)$, is an $\mathcal{F}(t)$-measurable quantity representing the value at time t of a notional N at time T and coupon payments at rate C with coupon intervals given by the day count convention $\tau(t,T)$.

The present value at time t of a default-free bond can be constructed using default-free zero coupon bonds as follows,

$$DFB(t,T,\tau(t,T),C,N) = N\sum_{i=1}^{n} C\tau(T_{i-1},T_i)B(t,T_i) + N.B(t,T). \qquad (2.4)$$

Remark 2.6: In this dissertation it is assumed that all South African government bonds are default-free and can be constructed from default-free zero coupon bonds. The allowance for default risk on these bonds is beyond the scope of this text. The terms default-free bonds, government bonds and bonds will be used synonymously throughout this dissertation.

Definition 2.2.7 Stock Price Process

In a market with d sources of uncertainty represented by a d-dimensional Brownian motion vector $\underline{W}(t)$, let the short rate process be given by R(t), the filtration be given by $\mathcal{F}(t)$ and let $\mu(t)$ and $\underline{\sigma}(t)$ be measurable processes. The stock price process is given by (2.5) and its differential form is given by (2.6).

$$S(t) = S(0)\exp\left(\int_0^t \mu(u) - \tfrac{1}{2}\|\underline{\sigma}(u)\|^2\, du + \int_0^t \underline{\sigma}(u)\cdot d\underline{W}(u)\right) \tag{2.5}$$

$$dS(t) = \mu(t)S(t)dt + S(t)\sum_{j=1}^{d}\sigma_j(t)dW_j(t) = \mu(t)S(t)dt + S(t)\underline{\sigma}(t)\cdot d\underline{W}(t) \tag{2.6}$$

2.3 Interest Rates Definitions

There are two features which define an interest rate quotation,

(1) The Day Count Convention and
(2) The Compounding Frequency.

Every interest rate quotation must specify both these quantities to be uniquely identifiable. Any interest rate quotation can be transformed into an equivalent quotation for a different day count convention or compounding frequency. These compounding frequencies fall into two broad categories,

(1) Continuously compounded and
(2) Simply Compounded.

These two categories are described below.

Definition 2.3.1 Continuously Compounded Spot Rate

The continuously compounded spot rate of interest between t and T, $R(t,T)$, is an $\mathscr{F}(t)$-measurable quantity representing the constant implied rate of interest associated with a zero coupon bond $B(t,T)$, expressed as the rate at which interest accrues continuously over the life of the contract. An expression for $R(t,T)$ in terms of $B(t,T)$ is given by (2.7).

$$R(t,T) = -\tfrac{1}{\tau(t,T)}\ln B(t,T) \tag{2.7}$$

Remark 2.7: Note that (2.7) can be rewritten in the form,

$$B(t,T) = \exp(-\tau(t,T)R(t,T)).$$

Definition 2.3.2 Simply Compounded Spot Rate

The simply compounded spot rate of interest between t and T, $L(t,T)$, is an $\mathscr{F}(t)$-measurable quantity representing the constant implied rate of interest associated with a *zero coupon bond B(t,T), expressed as the rate proportional to the length of the contract. An expression for $L(t,T)$ in terms of $B(t,T)$ is given by (2.8).*

$$L(t,T) = \frac{1-B(t,T)}{\tau(t,T)B(t,T)} \tag{2.8}$$

Remark 2.8: Note that (2.8) can be rewritten in the form,

$$B(t,T) = (1+\tau(t,T)L(t,T))^{-1}.$$

Definition 2.3.3 Forward Rate Agreement

An FRA is a bilateral agreement at time t between parties A and B, where A agrees to borrow N from B over [S,T] at an interest rate K with day count convention of $\tau(S,T)$. The value of the FRA at time t is given by 2.9.

$$FRA(t,\ S,\ T,\ \tau(S,T),\ K,\ N) \tag{2.9}$$

The payoff of the contract for counterparty B at maturity T will be,

$$N\tau(S,T)[K - L(S,T)].$$

A logical question can now be raised. What will constitute a fair value for K? The fair value for K will be that value for which the contract value is zero at the outset. Therefore K = $L(S,T)$ will be fair. Unfortunately $L(S,T)$ is unknown at the outset (i.e. time t) since it is a future spot rate. The expected value of $L(S,T)$ will be determined using arbitrage arguments.

Suppose large institutions can always borrow and lend at the risk free rate, then any large institution can effectively lock in the rate of interest, under the current term structure for a future period. Suppose they wish to lock in the rate at which they can borrow N over the future period [S,T]. In order to receive N at S,

a) they borrow N. $B(t,T)$ up until time T and
b) lend out N. $B(t,T)$ up until time S.

Table 2.1 shows the institution's capital flows over the life of the contract.

Table 2.1: Cashflows in an FRA contract

Time	Capital Inflow	Capital Outflow	Net
t	N.B(t,S)	-N.B(t,S)	0
S	N	0	N
T	0	-N.(1+$\tau(S,T)$ K)	-N.(1+$\tau(S,T)$ K)

K can be solved using the fundamental theorem of asset pricing. The fair value of K will be that value which makes the expected value of the net cashflows equal to zero under the traditional risk neutral measure. The net present value of the cashflows is,

$$Ne^{-\int_t^S R(u)du} - N(1+\tau(S,T)K)e^{-\int_t^T R(u)du}$$

Each of the net cashflows in Table 2.1 is discounted back to time t. The expectation of this quantity has must be taken to infer the value of K, since it depends on unknown future short rates. Taking the expectation yields,

$$\begin{aligned}0 &= \widetilde{E}[Ne^{-\int_t^S R(u)du} - N(1+\tau(S,T)K)e^{-\int_t^T R(u)du} \mid \mathscr{F}(t)] \\ &= \widetilde{E}[e^{-\int_t^S R(u)du} \mid \mathscr{F}(t)] - (1+\tau(S,T)K)\widetilde{E}[e^{-\int_t^T R(u)du} \mid \mathscr{F}(t)].\end{aligned}$$

The remaining expectations have the form as in Definition 2.2.4 of the zero coupon bond and the above expression can therefore be written as,

$$0 = B(t,S) - (1+\tau(S,T)K)B(t,T)\,.$$

From this the fair value of K can be solved,

$$K = \tfrac{1}{\tau(S,T)}\tfrac{B(t,S)-B(t,T)}{B(t,T)}\,.$$

Remark 2.9: An FRA is an interest rate derivative and an important building block of many other interest rate derivatives such as swaps, caps and floors.

Definition 2.3.4 Simply Compounded Forward Rate

The simply compounded forward rate of interest, $L(t,S,T)$*, represents the fair borrowing rate implied by an FRA(t, S, T,* $\tau(S,T)$*, K, 1) contract. An expression for* $L(t,S,T)$ *is given by (2.10).*

$$L(t,S,T) = \frac{1}{\tau(S,T)}\frac{B(t,S)-B(t,T)}{B(t,T)} \tag{2.10}$$

Remark 2.10: From the arbitrage arguments leading to the fair value of an FRA we can see that $L(t,S,T)$ can be intuitively interpreted as our best guess of $L(S,T)$ given the information available at time t. This is the prime modelling quantity underlying a whole class of interest rate models and will receive further attention in chapter 5 when the LIBOR market models are discussed.

Definition 2.3.5 Continuously Compounded Forward Rate

The continuously compounded forward rate, $R(t,S,T)$*, is the simply compounded forward rate* $L(t,S,T)$*, expressed with continuous compounding. An expression for* $R(t,S,T)$*, in terms of the corresponding simply compounded rate* $L(t,S,T)$*, is given by (2.11).*

$$R(t,S,T) = \frac{1}{\tau(S,T)}\ln(1+\tau(S,T)L(t,S,T)) \tag{2.11}$$

Remark 2.11: Note that expression (2.11) can be rewritten as,

$$1+\tau(S,T)L(t,S,T) = \exp(\tau(S,T)R(t,S,T)).$$

Definition 2.3.6 Instantaneous Forward Rate of Interest

The instantaneous forward rate, $F(t,T)$*, is the instantaneous continuously compounded forward rate at time T implied by the term structure of interest rates at time t. An expression for the instantaneous forward rate in terms of the simply compounded forward rate is given by (2.12).*

$$F(t,T) = \lim_{S\uparrow T} L(t,S,T) \tag{2.12}$$

Remark 2.12: The instantaneous forward rate is the prime modelling quantity for an entire class of interest rate models. These models will receive further attention in chapter 5 when the Heath-Jarrow-Morton framework is discussed.

Remark 2.13: If the definition of the simply compounded forward is substituted into the definition of the instantaneous forward rate, it follows that,

$$\begin{aligned} F(t,T) &= \lim_{\delta t \to 0} L(t,T,T+\delta t) \\ &= \lim_{\delta t \to 0} \tfrac{1}{\delta t} \tfrac{B(t,T)-B(t,T+\delta t)}{B(t,T+\delta t)} \\ &= \tfrac{1}{B(t,T)} \lim_{\delta t \to 0} \tfrac{B(t,T)-B(t,T+\delta t)}{\delta t} \\ &= -\tfrac{1}{B(t,T)} \tfrac{\partial}{\partial T} B(t,T) \end{aligned}$$

finally, recognising that this is the derivative of the natural logarithm, leads to

$$F(t,T) = -\tfrac{\partial}{\partial T} \ln B(t,T). \tag{2.13}$$

Remark 2.14: By integrating the above expression the value of a zero coupon bond can be written as,

$$B(t,T) = e^{-\int_t^T F(t,u)du} \tag{2.14}$$

But the definition of a zero coupon bond states that,

$$B(t,T) = \widetilde{E}[e^{-\int_t^T R(u)du} \mid \mathscr{F}(t)]$$

Therefore, just like $R(t,S,T)$ is our best guess of $L(S,T)$ so also is $F(t,T)$ our best guess of $R(T)$ given the information up to time t.

Remark 2.15: Also note that, from (2.14) it follows that,

$$\tfrac{B(t,S)}{B(t,T)} = e^{\int_S^T F(t,u)du} = e^{\tau(t,T)R(t,T)-\tau(t,S)R(t,S)} = e^{\tau(S,T)R(t,S,T)} \tag{2.15}$$

so that $R(t,S,T)$ can also be rewritten in the following form,

$$R(t,S,T) = \tfrac{\tau(t,T)R(t,T)-\tau(t,S)R(t,S)}{\tau(S,T)}. \tag{2.16}$$

Remark 2.16: It has been mentioned that for small intervals [t,T] it will be assumed that $\tau(t,T) = T - t$. From this an alternative expression can be derived for $F(t,T)$. Starting at (2.13),

$$\begin{aligned} F(t,T) &= -\tfrac{\partial}{\partial T} \ln B(t,T) \\ &= -\tfrac{\partial}{\partial T} (-R(t,T)\tau(t,T)) \\ &= -\tfrac{\partial}{\partial T} (-R(t,T)(T-t)). \end{aligned}$$

Finally, using the chain rule, expression (2.17) is obtained.

$$F(t,T) = R(t,T) + (T-t)\tfrac{\partial}{\partial T} R(t,T) \tag{2.17}$$

This is why a forward curve always intersects the equivalent spot curve at the turning points in the spot curve. Figure 2.1 illustrates this phenomenon. It can also clearly be seen how the instantaneous forward curve magnifies movements in the corresponding spot curve and is effectively more volatile. It is due to this characteristic that the market forward curve is often used for model-fitting rather than the corresponding spot curve.

Table 2.2 summarises the interest rate relations derived in this section.

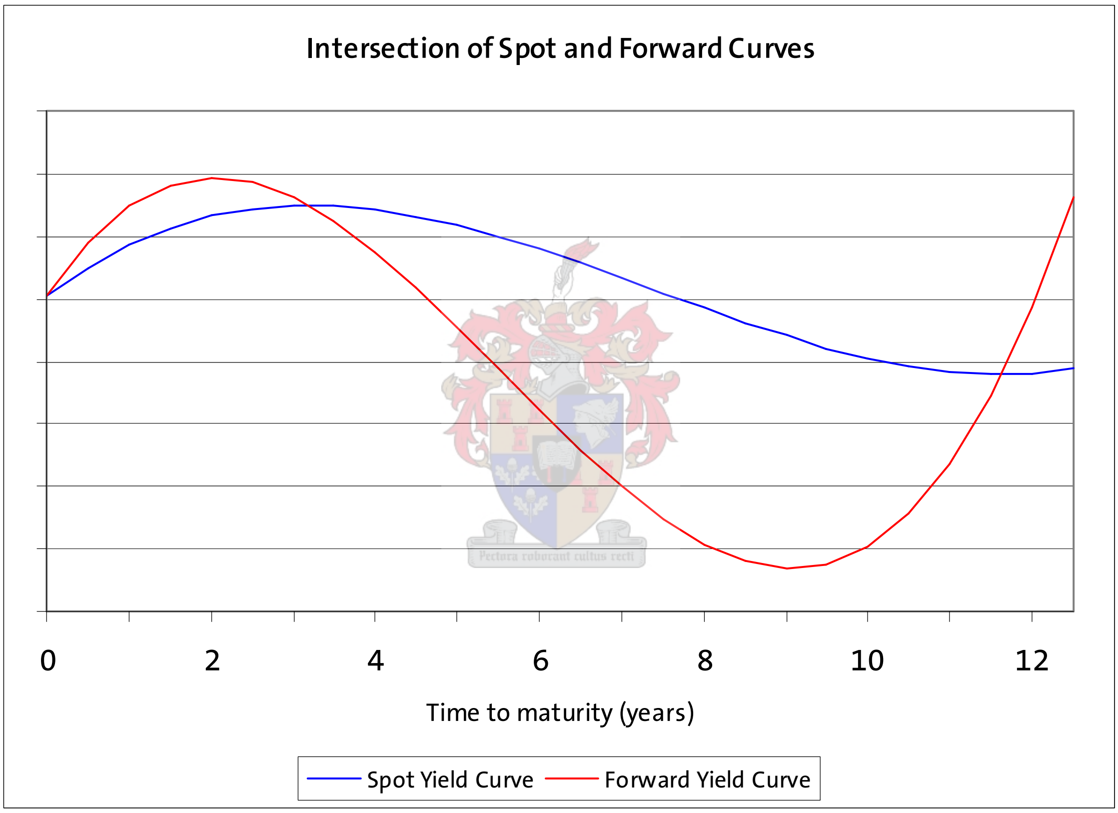

Figure 2.1: An illustration of how the continuously compounded forward curve always intersects the continuously compounded spot curve at the turning points.

Table 2.2: Relations Between Interest Rate Definitions

	Zero Coupon Bond	**Continuously Compounded Forward Rate**	**Instantaneous Forward Rate**	**Spot Rate**	**Short Rate**
	$B(t,T)$	$R(t,S,T)$	$F(t,T)$	$R(t,T)$	$R(t)$
$B(t,T)$	-	$e^{-\tau(t,S)R(t,S)-\tau(S,T)R(t,S,T)}$	$e^{-\int_t^T F(t,u)du}$	$e^{-\tau(t,T)R(t,T)}$	$\widetilde{E}\left[\exp\left(\int_t^T R(u)du\right) \mid \mathcal{F}(t)\right]$
$R(t,S,T)$	$\frac{\ln(B(t,T))-\ln(B(t,S))}{T-S}$	-	$\frac{1}{\tau(S,T)}\int_S^T F(t,u)du$	$\frac{\tau(t,T)R(t,T)-\tau(t,S)R(t,S)}{\tau(S,T)}$	$-\frac{1}{\tau(S,T)}\ln \widetilde{E}[\mathrm{e}^{-\int_S^T R(u)du} \mid \mathcal{F}(t)]$
$F(t,T)$	$-\frac{\partial}{\partial T}\ln B(t,T)$	$\lim_{\delta t\to 0} R(t,T,T+\delta t)$	-	$-\frac{\partial}{\partial T}\tau(t,T)R(t,T)$	$-\frac{\partial}{\partial T}\ln \widetilde{E}[e^{-\int_t^T R(u)du} \mid \mathcal{F}(t)]$
$R(t,T)$	$-\frac{1}{\tau(t,T)}\ln B(t,T)$	$R(t,t,T)$	$\frac{1}{\tau(t,T)}\int_t^T F(t,u)du$	-	$-\frac{1}{\tau(t,T)}\ln \widetilde{E}[e^{-\int_t^T R(u)du} \mid \mathcal{F}(t)]$

Definition 2.3.7 Floating Rate Note

A floating rate note provides a series of interest payments on the dates $\{T_1, T_2, \ldots, T_n\}$. *The interest rate that will be paid on the dates* T_i *are reset on the dates* T_{i-1} *for* $i = 1, \ldots, n$. *At each reset date,* T_{i-1}, *the rate that will be paid at* T_i *is set equal to the prevailing LIBOR rate,* $L(T_{i-1}, T_i)$. *At maturity the notional is repaid along with the floating interest rate.*

The period, $\tau(T_{i-1}, T_i)$ over which the rates are fixed is called the tenor of the note and is usually equal to 3 or 6 months.

Example: A floating rate note that has a tenor of 3 months and maturity of 12 months will pay $L(0, \frac{1}{4})$ at the end of month 3, $L(\frac{1}{4}, \frac{1}{2})$ at the end of month 6, $L(\frac{1}{2}, \frac{3}{4})$ at the end of month 9 and $L(\frac{3}{4}, 1)$ at maturity in one year's time.

Definition 2.3.8 Caplet

A caplet $c(t, L_K, T_{i-1}, T_i)$ *is the value at time t of a European call option on the simply compounded forward rate* $L(T_{i-1}, T_i)$ *over one tenor period,* $\tau(T_{i-1}, T_i)$, *with a strike price of* L_K *and maturity* T_i.

For the strike price L_K the payoff of a caplet at time T_i is,

$$\tau(T_{i-1}, T_i) \max\left(L(T_{i-1}, T_i) - L_K, 0\right).$$

It can be shown that, when $L(T_{i-1}, T_i)$ can be assumed to be log-normally distributed at maturity, by taking expectations and discounting the above expression, will yield the well know Black-Scholes stock pricing formula. Though this assumption is seldom realistic, it is the market convention to price caplets using the Black-Scholes formula.

The purpose of a caplet is to provide insurance against a rise in the interest rate above the level L_K, that is paid by a floating rate note.

Definition 2.3.9 Cap

A cap $C(t, L_k, T_{i-1}, T_i)$ *is the value of an insurance contract that provides protection against the interest rate of a floating rate note rising above* L_K.

The value of a cap is merely the sum of the corresponding sequence of caplets.

Definition 2.3.10 Yield Curve

A real valued function that associates interest rates with a future time T, is referred to as a yield curve.

If the function associates the time T with the spot rate $R(t,T)$, it is called a spot yield curve or a zero coupon yield curve.

$$f:[t,T] \rightarrow R(t,T) \ ,0 \leq t \leq T \tag{2.18}$$

If the function associates the time T with the forward rate $F(t,T)$, it is called a forward yield curve.

$$f:[t,T] \rightarrow F(t,T) \ ,0 \leq t \leq T \tag{2.19}$$

Remark 2.17: The yield curve at time t will be the primary modelling quantity of chapter 3. Figure 2.1 shows the yield curve on the 5th of December 2005 as quoted by the Bond Exchange of South Africa.

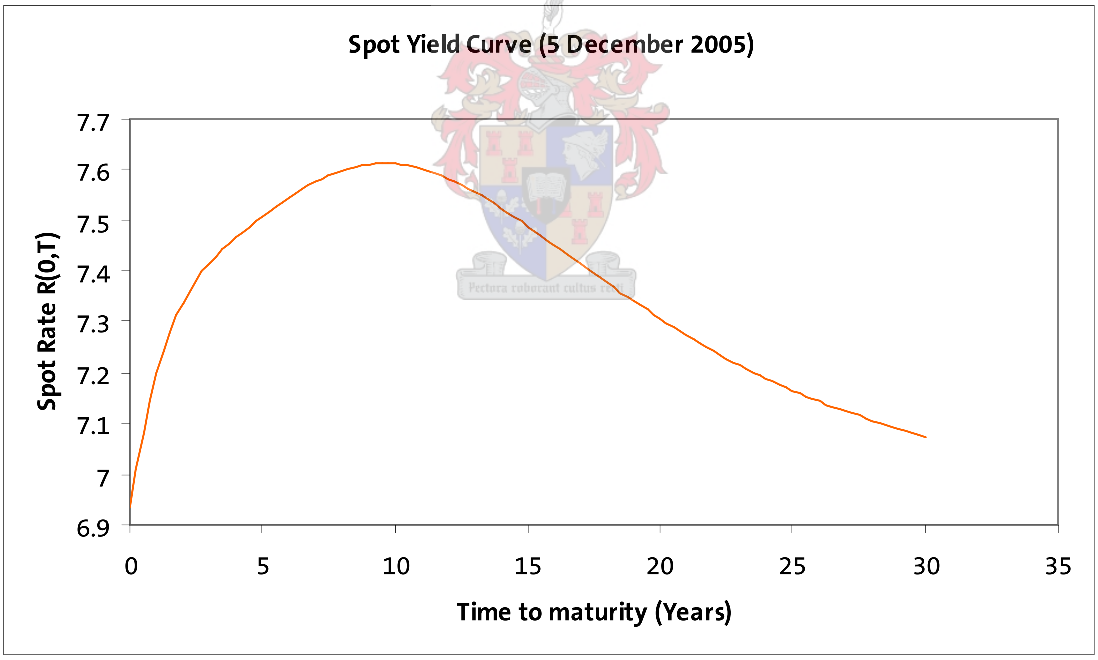

Figure 2.2: Government Bond Yield Curve on 5 December 2005.

The South African bond market often represents a humped yield curve. The three most common shapes for yield curves are,

(1) Increasing,
(2) Decreasing and
(3) Humped

When modelling interest rates it is important that the proposed model is capable of representing all possible shapes which can be expected from a given market. When the various models are discussed in future chapters, specific attention will be paid to this feature.

2.4 Term Structure Theories

Numerous theories exist which explain the different shapes of yield curves observed in financial markets. The four most prevalent theories and the possible shapes that they allow will be discussed in turn.

2.4.1 The Expectations Theory

Under the expectations theory, the expectations of investors with regards to future interest rate movements are the primary driving factor of the term structure. If short term interest rates are expected to fall, there will be an increased demand for investments with a longer maturity. Consequently the demand for long term bonds will rise and the term structure will be decreasing. The inverse will be true if investors expect a decrease in the short term interest rates.

The most common approach to making this theory mathematically tractable is by setting,

$$e^{R(0,S,T)} = E[e^{R(S,T)}] \; \forall \; S < T \tag{2.20}$$

This has the following two implications,

Due to the fact that e^x is a convex function, we know from Jensen's inequality that $R(0,S,T) > E[R(S,T)]$. Therefore the current forward rate is greater than the expected future spot rate.

From the relation between the continuously compounded forward rate and the instantaneous forward rate in table 2.2, it can be seen that,

$$\begin{aligned} R(0,S,S+2) &= \tfrac{1}{2}\int_{S}^{S+2} F(0,t)dt \\ &= \tfrac{1}{2}\int_{S}^{S+1} F(0,t)dt + \tfrac{1}{2}\int_{S+1}^{S+2} F(0,t)dt \\ &= \tfrac{1}{2}R(0,S,S+1) + \tfrac{1}{2}R(0,S+1,S+2) \end{aligned}$$

Multiplying both sides of the above equation by two and using relation (2.20) it can be seen that,

$$\begin{aligned} e^{2R(0,S,S+2)} &= e^{R(0,S,S+1)+R(0,S+1,S+2)} \\ &= E[e^{R(S,S+1)}]E[e^{R(S+1,S+2)}] \end{aligned} \tag{2.21}$$

However it is also follows from (2.20) that,

$$e^{2R(0,S,S+2)} = E[e^{2R(S,S+2)}]. \tag{2.22}$$

Now, from the relation between the continuously compounded spot rate $R(S,T)$ and the instantaneous forward rate $F(t,T)$ in table 2.2, it is clear that,

$$\begin{aligned} R(S,S+2) &= \tfrac{1}{2}\int_S^{S+2} F(S,t)dt \\ &= \tfrac{1}{2}\int_S^{S+1} F(S,t)dt + \tfrac{1}{2}\int_{S+1}^{S+2} F(S,t)dt \\ &= \tfrac{1}{2}R(S,S+1) + \tfrac{1}{2}R(S+1,S+2) \end{aligned}$$

Substituting this into the right hand side of (2.22) yields,

$$e^{2R(0,S,S+2)} = E[e^{R(S,S+1)+R(S+1,S+2)}] \tag{2.23}$$

Therefore equating (2.21) and (2.23) it can be seen that,

$$E[e^{R(S,S+1)+R(S+1,S+2)}] = E[e^{R(S,S+1)}]E[e^{R(S+1,S+2)}]$$

That implies that $e^{R(S,S+1)} = \frac{1}{B(S,S+1)}$ and $e^{R(S+1,S+2)} = \frac{1}{B(S+1,S+2)}$ are uncorrelated, which is unlikely. This deficiency is addressed by the arbitrage-free pricing theory.

2.4.2 The Liquidity Preference Theory

Under the liquidity preference theory, investors prefer more liquid investments and require additional compensation for holding investments with a longer term to maturity. Consequently, there will be a risk premium associated with long term bonds because they are more sensitive to interest rate movements. The liquidity preference theory gives rise to an increasing term structure curve.

2.4.3 The Market Segmentations Theory

The demand for bonds of various maturities is the primary driving factor of the term structure of interest rates. Some investors (such as banks) have a high demand for short termed instruments while other investors (such as pension funds) have a high demand for longer termed instruments.

The relative demands for the various types of instruments ultimately determine the term structure. The market segmentation theory can give rise to a humped yield curve.

2.4.4 Arbitrage-Free Pricing Theory

This extensive theory is based on the law of one price and assumes that no arbitrage opportunities must persist in efficient markets. Arbitrage-free pricing theory can be used to pull together the expectations, liquidity preference and market segmentations theories. All three yield curve shapes can be attained under the arbitrage free pricing theory given a sufficiently complex model is used. This text will focus on this theory and its application with respect to interest rate models.

2.5 Summary

This chapter introduced the various incarnations of interest rates and yield curves and also presented some basic interest rate derivatives which will be referenced in the forthcoming chapters.

The next chapter will introduce the concept of a non-stochastic, descriptive yield curve model which will be used to calibrate the stochastic interest rate models of chapters 4 and 5.

Chapter 3

DESCRIPTIVE YIELD CURVE MODELS

3.1 Introduction

Descriptive yield curve models reflect a static view of the yield curve based solely on the latest market data. No historical data is used in the construction of a descriptive mode and the primary goal of these models is to give an accurate reflection of the yield curve implied by current prices. Descriptive yield curve models have many uses but cannot explain the evolution of the yield curve into the future. In order to forecast this evolution, more sophisticated stochastic models like those in chapters 4 and 5 are required. If interest rates are thought of as a movie, then descriptive models will try to reflect a single frame of that film whereas a stochastic yield curve model will try to reflect the entire flow of interest rates. Practitioners that price interest rate derivatives prefer stochastic models which return prices that are consistent with today's market prices. Therefore the more sophisticated models of chapter 5 have to be calibrated using, amongst others, a descriptive yield curve model.

This chapter represents the first step in the calibration process of some of the more advanced stochastic models. It starts with a brief overview of the application areas of descriptive models and a discussion of how market data for model calibration should be selected. This follows with an in depth look at the yield curve methodology currently used by all major South African investors, the Bond Exchange of South Africa's (BESA) yield curve methodology. The chapter ends with a worked example of the BESA yield curve methodology.

3.2 Applications of Descriptive Yield Curve Models

The most important applications of descriptive models are:

(1) They can be used to identify arbitrage opportunities between over- and under-priced bonds.
(2) They give an idea of which term structure of interest rates is implied by market data e.g. swap rates, bond prices or Johannesburg Inter Bank Average Rates (JIBAR) rates.
(3) They can be used to value forward contracts on bonds.
(4) They can be used as input for the calibration of exogenous interest rate models.

(5) They can be used to inspect monetary policy.
(6) They can be used to construct yield indices.

3.3 Considerations for Calibration

Before constructing a descriptive yield curve model, it is important to understand the objective of the model. If the primary aim of the model is to assist with bond pricing, then bonds should be used as calibration securities. If on the other hand the goal is to price swap contracts then market swap rates should be used in its construction.

Three broad classes of rates are used when constructing descriptive yield curve models:

(1) Money Market Rates,
(2) Forward/Swap Rates and
(3) Default Free Interest Rates.

The literature has proven it difficult to build a single model that produces market consistent prices across all three classes of underlying instruments. Consequently, separate models must be constructed and the derivatives that are to be priced will determine which instruments must be included in the calibration procedure. Table 3.1 shows which descriptive yield curve models can be constructed and which South African securities are used to extract the required data.

Table 3.1: Calibration Securities for descriptive yield curve models

	(1) Money Market Rates	**(2) Forward / Swap Rates**	**(3) Default Free Rates**
Calibration Securities	Overnight and Inter-bank rates.	Swap contracts, FRA's and futures[1]	Government Bonds
Securities used in the South-African Market	SAONIA, JIBAR rates	2, 3, 4, 5, 10 year[2] swap rates. FRA rates for all 3-month periods up to 1 year.	R194, R153, R157, R186, R201, R203, R204

[1] A convexity adjustment is required for converting futures rates into an equivalent FRA rates.
[2] 6-9 year swap rates are also available but are insufficiently liquid to be used for calibration purposes.

These are not the only considerations when selecting calibration instruments, risk premiums and liquidity preferences should also be taken into account. Suppose a model of the default free rates is required, then defaultable bonds cannot be used for calibration since they include a risk premium, using illiquid riskless instruments will also pose a problem since they include a liquidity premium. Using either of these securities will lead to a skewed view of the risk free yield curve.

Due to the small size and liquidity constraints of the South African market, some of these curves may be joined after taking account of some liquidity or risk premium adjustment. For example, it is often found that the short end of the descriptive yield curve constructed from government bonds has a poor resolution. This is because there are usually few government bonds with imminent maturity. This can be remedied by joining the short end of the swap curve with the bond curve.

There are two approaches to descriptive yield curve modelling,

a) Parametric Modelling and
b) Spline Based Modelling.

Parametric modelling is primarily used in econometric and actuarial applications where parsimony is of greater concern. Spline based models on the other hand is more concerned with goodness-of-fit. For derivative pricing it is important that the calibrated model reflects current market prices and not admit arbitrage opportunities. Consequently, spline based models are preferred for derivative pricing.

There is a vast amount of literature surrounding spline based interpolation methods. Unfortunately a detailed explanation and comparison of the various techniques is beyond the scope of this document. The interested reader can consult Hagan and West (2006) for comparison of various interpolation methods applicable to the South African market.

In the South African market there is no consensus regarding construction of descriptive yield curves based on forward and swap rates. There is however a method for the construction of bond yield curves which enjoys widespread appeal. This technique is described by Quant Financial Research (2003) and can also be used to construct swap curves. It is used by BESA as well as by various other market makers.

3.4 BESA Yield Curve Methodology

In 2001 BESA commissioned Quant Financial Research to develop a methodology for constructing zero coupon yield curves, specifically suited to the South African market. These curves had to provide suitable accuracy in the relatively small and illiquid South African market. The development process took nearly three years to complete but the methodology, though quite technical, is very robust and flexible. Not only can this yield curve methodology be applied to extract yield curves from bonds but also from swaps, futures or any number of different instruments for which the cashflows and present values are known. In fact, since it is usually the case that there are no bonds that expire within the next day, week or month, the bond yield curve is often augmented by the short end of the swap curve. The BESA methodology provides a combined approach to constructing such an augmented curve. The swap and bond prices are simultaneously used by the procedure and a curve is extracted from all inputs provided. It is not necessary to construct two separate curves and then join them.

This section gives an overview of the BESA methodology which is followed in the next section by a worked example using bond data from the 12th of December 2005. This approach to finding the yield curve is formulated as a multidimensional optimisation problem. Such problems are usually characterised by:

(1) The solution space to be searched.
(2) The objective function, measuring the quality of a candidate solution.
(3) An optimisation algorithm used for searching the solution space.
(4) A convergence criterion, signalling the end of the procedure.

3.4.1 The Solution Space

The set of financial calibration instruments from which the yield curve must be extracted is denoted by,

$$\underline{I} = \{I_1, I_2, ..., I_{\mathrm{m}}\}. \tag{3.1}$$

The set $\underline{I}$ is not limited to any specific type of instrument. The set of all cashflow dates from set $\underline{I}$ will be denoted by,

$$\underline{dt} = \{dt_0, dt_1, ..., dt_n\}. \tag{3.2}$$

Also define,

$$\underline{T} = \{t_0, \dots, t_n\},\ \ t_i = \tfrac{dt_i - dt_0}{365}. \tag{3.3}$$

From the set $\underline{I}$, a cashflow matrix is constructed with a row for each instrument and a column for each cashflow date. The cashflow matrix is denoted by,

$$C = \left[c_{ij}\right],\ \text{where } c_{ij} \text{ is the cashflow of } I_{\mathrm{i}} \text{ on date } t_{\mathrm{j}}. \tag{3.4}$$

The methodology prescribes that cashflows be entered in such a fashion that the present value of each row should be zero when discounting under the desired yield curve. Consequently the first entry is equal to minus the market value of the instrument. The reason for this will become clear once the multi-dimensional optimisation problem is stated below.

Denote the (n+1) dimensional spot curve corresponding to the cashflow dates of $\underline{dt}$ by,

$$\underline{r}^T = \{r_0, r_1, \dots, r_n\}, \tag{3.5}$$

where $r_i = R(0, t_i)$ from Definition 2.3.1.

Also define,

$$\underline{f}^T = \{f_1, f_2, \dots, f_n\} \tag{3.6}$$

where $f_i = F(0, t_i)$ and

$$\underline{q}^T = \{q_1, q_2, \dots, q_n\}, \tag{3.7}$$

where q_i is a discreet approximation to $F(0, t_i)$ (the instantaneous forward rate from Definition 2.3.6). The construction of $\underline{q}$ using $\underline{r}$, is discussed in appendix A.

Denote the (n+1) dimensional discount vector as a function of the spot yield curve $\underline{r}$ by,

$$\underline{d}(\underline{r})^T = \left\{e^{-r_0 t_0}, e^{-r_1 t_1}, \dots, e^{-r_n t_n}\right\}. \tag{3.8}$$

The optimisation problem can now be formulated as follows.

> ***Multidimensional Optimisation Problem:*** *Let $e(\cdot)$ be an objective function and take $\underline{\varepsilon} = \{\varepsilon_1, \dots, \varepsilon_m\}$, where $\varepsilon_i > 0$. Find an n+1 dimensional vector $\underline{r}$ that optimises $e(\cdot)$ such that equation (3.9) holds within some tolerance level $\underline{\varepsilon}$.*

$$\|C\underline{d}(\underline{r}) - \underline{\varepsilon}\| = 0 \tag{3.9}$$

Remark 3.1: From this formulation it is clear why the cashflow matrix C had to be constructed such that the present values of the rows are zero. The problem reduces to one of finding the rates $\underline{r}$ which brings $C\underline{d}(\underline{r})$ close enough to zero.

Note that this is an underdetermined system with more unknowns than equations. As such BESA provides two different zero curves on a daily basis. These curve are calculated from the daily close prices of bonds in the GOVI index. The first is a "Perfect Fit" curve for which $\underline{\varepsilon} = 0$ and the second is a "Best Decency" curve for which $\underline{\varepsilon} \neq 0$. Finding solutions to each problem follows a similar course and thus only the "Perfect Fit" case will be discussed since practitioners who wish to price derivatives are only interested in those solutions that give a "perfect fit", in other words, which price the calibration securities perfectly. The "perfect fit" restriction imposes a constraint on the solution space for each calibration security and will consequently decrease the degrees of freedom of the solution space by the number of securities, m.

Since the system is underdetermined i.e. C has strictly more columns than rows, no unique solution exists for the discount vector $\underline{d}(\underline{r})$. Singular value decomposition (SVD) can however be used to find the null-space of C, i.e. the space consisting of all vectors $\underline{d}$ such that $C\underline{d} = 0$. The SVD form of C is given in (3.10).

$$\underset{m \times n+1}{C} = \underset{m \times n+1}{U} \cdot \underset{n+1 \times n+1}{W} \cdot \underset{n+1 \times n+1}{V^T} \tag{3.10}$$

where $U^T U = V^T V = I$ and W is a diagonal matrix of the singular values of C.

Let B contain those columns of V corresponding to the diagonal entries of W that are zero. It can be shown that the columns of B form a basis for the null-space of C. The multidimensional optimisation problem can now be reformulated for the "Perfect Fit" case,

> ***Multidimensional Optimisation Problem:*** *Let* $e(\cdot)$ *be an objective function and take* $\underline{\varepsilon} = \{\varepsilon_1, \ldots, \varepsilon_m\}$, where $\varepsilon_i > 0$. *Since* $CB\underline{x} = \underline{0}$ *holds for all n-m+1 dimensional vectors* $\underline{x}$, *find that* $\underline{x}$ *that optimises* $e(\cdot)$.

The dimensionality of the problem has now been reduced from n+1 to n-m+1.

Remark 3.2: Note that any solution $\underline{x}$, can be transformed back to a zero curve $\underline{r}$, by calculating the discount vector,

$$\underline{d}(\underline{r}) = B\underline{x} = (d_0,\ldots,d_n). \tag{3.11}$$

Hence, the zero rates can be computed using:

$$r_i = -\tfrac{1}{t_i}\ln(d_i),\ \mathrm{i} = 0,\ldots,\mathrm{n}. \tag{3.12}$$

3.4.2 The Objective Function

This section defines a measure called Decency, which will be used as the objective function, $e(\cdot)$, specified in the formulation of the multi-dimensional optimisation problem.

Each candidate solution is characterised by the n-m+1 points. The candidate solution space is therefore an n-m+1 dimensional vector space that has to be searched for the "best" candidate. The BESA methodology defines three criteria which characterise a solution,

(1) Global Smoothness,
(2) Local Smoothness and
(3) Goodness of Fit.

These three criteria are combined into a single expression called the Decency. This Decency is defined to be the objective function of a multidimensional minimisation problem. The "best" candidate will then be the one with the "best" Decency.

3.4.2.1 Global Smoothness

As the name suggests, the global smoothness criterion is a measure of the overall roughness of a candidate yield curve. The global smoothness is defined as the quadratic variation of the quadratic forward curve and can be calculated as:

$$QVar(\underline{q}) = \sum\nolimits_{i=2}^{n}\left(q_i - q_{i-1}\right)^2. \tag{3.13}$$

It has been shown that forward curves are more volatile than their implied spot curves. It is due to this fact that forward curves actually magnify deficiencies in the spot curves, that the global smoothness is defined on $\underline{q}$ rather than the $\underline{r}$.

Remark 3.3: In general a candidate with a lower Global Smoothness will be better.

3.4.2.2 Local Smoothness

In contrast, the local smoothness is a measure of roughness between the individual points of the quadratic forward. The local smoothness is defined as the quadratic variation of the first derivative of the quadratic forward curve given by:

$$QVar(\tfrac{\Delta q}{\Delta t}) = \sum\nolimits_{i=2}^{n} \left(\tfrac{\Delta q_i}{\Delta t} - \tfrac{\Delta q_{i-1}}{\Delta t}\right)^2 . \tag{3.14}$$

Remark 3.4: In general a candidate with a lower Local Smoothness will be better.

3.4.2.3 Goodness of Fit

The goodness of fit is a measure of how accurately a candidate solution prices the calibration securities. The goodness of fit is defined as:

$$g = \sum\nolimits_{i=1}^{m} u_i g_i^2 \ , \tag{3.16}$$

where $\underline{g} = (g_1, ..., g_m)$ is the solution of:

$$C\underline{d}(\underline{r} + \underline{g}) = \underline{0} . \tag{3.15}$$

Remark 3.5: Note that g_i is the required parallel shift in the yield curve in order to price I_i precisely. For the BESA "Perfect Fit" curve $g = 0$ and for the "Best Decency" curve $g \neq 0$. In general a lower g is associated with a higher goodness-of-fit and therefore a candidate with a lower g generally will be better.

3.4.2.4 Decency

The Decency combines the above three criteria into a single objective function to be minimised. The Decency is given by:

$$e = (w_1 QVar(Q) + w_2 QVar(\tfrac{\Delta Q}{\Delta t}) + w_3 g) \,/ \sum\nolimits_{i=1}^{3} w_i \ . \tag{3.17}$$

For the "perfect fit" case this reduces to:

$$e = (w_1 QVar(Q) + w_2 QVar(\tfrac{\Delta Q}{\Delta t})) \,/ \sum\nolimits_{i=1}^{2} w_i \ . \tag{3.18}$$

The choice of weights will not be subjective and is discussed in section (3.4.5).

Remark 3.6: A candidate with a lower Decency value will be better.

3.4.3 The Multidimensional Optimisation Algorithm

Our multidimensional minimisation problem for the "Perfect Fit" case is now as follows,

> ***Multidimensional Minimisation Problem:*** *Since $CB\underline{x} = \underline{0}$ holds for all n-m+1 dimensional vectors $\underline{x}$, find the $\underline{x}$ that minimises the Decency when $w_3 = 0$.*

The BESA methodology assumes that the solution space contains only one global minimum and no local minima. The BESA methodology further recommends the use of the Conjugate Gradient search method of Fletcher and Reeves, as amended by Polak and Ribiere as the optimisation algorithm, see Numerical Recipes in C (1992). A detailed explanation of this technique is beyond the scope of this document but the basic principle is fairly intuitive. The algorithm essentially starts out at an initial point in the solution space and then uses an intelligent scheme for traversing this space in search of the global minimum of some objective function (in this case, Decency).

Algorithm: Polak-Ribiere Conjugate Gradient Minimisation Algorithm

Inputs: An objective function $e = f(\underline{x})$ and a starting point $\underline{x}_0 = (x_{0,0}, \ldots, x_{0,n-m})$.
Outputs: That $\underline{x}_m$ which minimises the objective function.

(1) Determine the direction of movement for the next step. All conjugate gradient algorithms start out by moving in the steepest decent direction on the first iteration i.e. the direction in which the objective function decreases the fastest. The steepest decent direction is given by the vector of partial derivatives of the objective function i.e. Decency, at $\underline{x}_0$,

$$\underline{k}_0^T = \frac{\partial e_{X_0}}{\partial X_0} = \left(\frac{\partial e_{X_0}}{\partial x_{0,0}}, \ldots, \frac{\partial e_{X_0}}{\partial x_{0,n-m}}\right). \tag{3.19}$$

Since the optimisation procedure is a minimisation problem, it moves down this slope. In the case of a maximisation problem one would move up the slope. The direction of movement on the first iteration is therefore,

$$\underline{p}_0 = -\underline{k}_0. \tag{3.20}$$

(2) Calculate the next candidate $\underline{x}_i$ using 3.21.

$$\underline{x}_i = \underline{x}_{i-1} + \alpha_{i-1}\underline{p}_{i-1} \tag{3.21}$$

α_i is the optimal distance to be travelled as given by some line search algorithm. A discussion of the various line search algorithms is beyond the scope of this text but the interested reader can refer to Numerical Recipes in C (1992) for further information.

(3) Determine the direction of movement for the next step. It can be shown that moving in the steepest decent direction on each iteration is not optimal. A direction is chosen which is conjugate to the previous one. Set the direction of movement for the next step to,

$$\underline{p}_i = -\underline{k}_i + \beta_i \underline{p}_{i-1} \,. \tag{3.22}$$

Various choices of β_i exist and some can be shown to be optimal for certain types of problems. The Polak-Ribiere form for β_i is,

$$\beta_i = \frac{(\underline{k}_i - \underline{k}_{i-1})^T \cdot \underline{k}_i}{\underline{k}_{i-1}^T \cdot \underline{k}_{i-1}} \,. \tag{3.23}$$

Steps 2 & 3 are iterated until the differences between the decencies of successive candidates decrease to below a certain threshold for a sufficient number of iterations.

3.4.4 The Convergence Criteria

The BESA methodology defines the concept of "Blur" as convergence criterion. Blur is measured on the path along which each point on the zero curve has moved over past iterations. The algorithm will only converge if each point of the latest candidate is "close enough" to its historical exponentially weighted mean.

Let $\underline{x}_i$ be the ith candidate solution in a Polak-Ribiere execution cycle. For each $\underline{x}_i$ the corresponding spot yield curve must be calculated. Let

$$\underline{d}(\underline{r}_i) = B\underline{x}_i = \left(d_{i,0}, \ldots, d_{i,n}\right),$$

where $\underline{r}_i$ is the spot yield curve corresponding to candidate $\underline{x}_i$ and $d_{i,j}$ is the discount factor corresponding to candidate $\underline{x}_i$ with maturity t_j. Also define,

$$r_{i,j} = -\tfrac{1}{t_j}\ln(d_{i,j}),\ \mathrm{j} = 0,\ldots,\mathrm{n},$$

where $r_{i,j}$ is the continuously compounded spot rate corresponding to candidate $\underline{x}_i$ with maturity t_j.

The search path $\{\underline{x}_i\}$ of the Polak-Ribiere algorithm has now been transformed into an equivalent set of yield curves $\{\underline{r}_i\}$. The exponentially weighted mean and standard deviation of each point on the yield curve is sought. The weighting parameter is α.

Define,

$$A = \sum_{j=0}^{k} \alpha^j \text{ ,} \tag{3.25}$$

$$S_i = \sum_{j=0}^{k} \alpha^j r_{i,k-j}, \ \mathrm{i} = 0,\ldots,\mathrm{n} \text{ and} \tag{3.26}$$

$$SS_i = \sum_{j=0}^{k} \alpha^j r_{i,k-j}^2, \ \mathrm{i} = 0,\ldots,\mathrm{n}\,. \tag{3.27}$$

Next, calculate the exponentially weighted mean and standard deviation of each point on the yield curve, along the search path of the Polak-Ribiere algorithm by using (3.28) and (3.29).

$$\mu_i = \frac{S_i}{A}, \ \mathrm{i} = 0,\ldots,\mathrm{n} \tag{3.28}$$

$$\sigma_i = \sqrt{\frac{SS_i - u_i S_i}{A-1}}, \ \mathrm{i} = 0,\ldots,\mathrm{n} \tag{3.29}$$

The distance of each point of the current candidate from the far tail of its distribution is calculated using,

$$\Delta_i = \kappa\sigma_i + |\, z_i - \mu_i \,|, \ \mathrm{i} = 0,\ldots,\mathrm{n}\,, \tag{3.30}$$

where BESA currently recommends $\kappa = 2.5$. This distance, at each point, is then converted into a mispricing per R1m. The blur is defined as the maximum mispricing over all points on the candidate curve and is defined by:

$$blur = \max\left\{10^6 t_i \Delta_i : i = 0,\ldots n\right\} \tag{3.31}$$

Convergence occurs once the blur becomes less than some predetermined tolerance.

Blur can be understood as follows. The final solution will be found only once the value of blur is small enough. This value will only be small if each point in the final solution is fairly close to it's long run average. The "weight of past candidates" or the "influence of past candidates" in the long run mean and standard deviation decays exponentially fast according to the choice of α. This means that initial candidates will have little impact on the shape of the final solution after a sufficient number of iterations. This in turn allows the candidate solutions to settle into the true average/best solution before the final candidate gets accepted.

3.4.5 The Minimisation Procedure

Once the objective function, the optimisation algorithm and the convergence criteria have been defined, it is possible to formulate a minimisation routine.

Algorithm: BESA Minimisation Procedure

Inputs: A starting point $\underline{x}_0$, from which to begin the optimisation process. Other inputs are defined below.

Outputs: A vector $\underline{x}_m$ which minimises Decency and for which the blur is within tolerance.

(1) Initialise the weights for the objective function and reset the blur value. Currently BESA prescribes an equal initial weighting.

(2) Perform a fixed number of conjugate gradient iterations.

(3) Recalculate the weights for the objective function. If the change in the weights remain unchanged up to within two decimal places, continue to Step 4 else return to Step 1 and initialise the procedure with the new weights. The reweighing procedure is not difficult but quite tedious, refer to the original article Quant Financial Research, (2005) for full details.

(4) Calculate the blur on the current history since the last time new weights were established and the procedure was reset. If the blur value is not within tolerance, return to Step 2. This tolerance is another input variable to the algorithm and currently equal to a maximum mispricing at any point of R5 per R1m. If this blur value is within tolerance, tighten the tolerance, return to Step 2 and reset the conjugate gradient algorithm until the tolerance has been tightened n times. The value of n is another input to the procedure and presently, BESA recommends a value of 5. If the tolerance has been tightened n times, the minimisation is complete.

Once the minimisation routine had finished, the final solution $\underline{x}_m$ can be converted into a set of points on the continuously compounded spot yield curve, using expression (3.13). This spot curve will be denoted by $\underline{r}_m$. Finally, since a continuous curve is required and not merely a set of points on the curve, the points have to be interpolated.

Remark 3.7: It is recommended that $\underline{x}_0$ be calculated using (3.32), where $\underline{r}_0$ is calculated using the bootstrap procedure discussed in Appendix B.

$$\underline{x}_0 = B^T \underline{d}(\underline{r}_0) \tag{3.32}$$

3.4.6 Interpolation

It has been shown, see Adams and Van Deventer (1994), that the interpolation function which yields the smoothest continuously compounded forward curve is a quartic spline. This is in contrast with a cubic spline which can be shown to yield the smoothest continuously compounded spot curve. Yet a smooth forward is more desirable due to the aforementioned reasons that the forward curve magnifies blemishes in the spot curve. A quartic spline will be fit to $\underline{r}_m$, the final solution of the minimisation procedure of the previous section.

A quartic spline is a series of fourth order polynomials fitted between each two dates.

$$\widehat{F}(0,t) = a_j + b_j t + c_j t^2 + d_j t^3 + e_j t^4,\ t_{j\text{-}1} \le t \le t_j,\ j = 1,...,n \tag{3.33}$$

From the relation between the continuously compounded spot and forward in table 2.2, the spot curve can be calculated as,

$$\begin{aligned}
\widehat{R}(0,t) &= \tfrac{1}{t}\int_0^t \widehat{F}(0,u)du \\
&= \tfrac{1}{t}\left[t_{j-1}\widehat{R}(0,t_{j-1}) + \int_{t_{j-1}}^{t_j} \widehat{F}(0,u)du\right] \\
\widehat{R}(0,t) &= \tfrac{1}{t}\begin{bmatrix} t_{j-1}\widehat{R}(0,t_{j-1}) + a_j(t-t_{j-1}) + \frac{b_j}{2}(t-t_{j-1})^2 \\ +\frac{c_j}{3}(t-t_{j-1})^3 + \frac{d_j}{4}(t-t_{j-1})^4 + \frac{e_j}{5}(t-t_{j-1})^5 \end{bmatrix} \\
&,\ t_{j\text{-}1} \le t \le t_j,\ j = 1,...,n.
\end{aligned} \tag{3.34}$$

Since there are n+1 points that must be interpolated, n polynomials must be fitted, one between each two points. Each polynomial has 5 unknown parameters a, b, c, d and e. Consequently, 5 restrictions must be placed on each polynomial in order to solve the system.

Firstly, it is required that each polynomial go through one of the points in $\underline{r}_m$. In other words it is required that the quartic forward of (3.33), integrate to a zero curve that goes through the points given by the minimisation algorithm. Equation (3.35) represents these restrictions

$$\begin{aligned}
r_j &= \widehat{R}(0,t_j) \\
&= \tfrac{1}{t_j}\left[t_{j-1}r_{j-1} + \int_{t_{j-1}}^{t_j} \widehat{F}(0,t)dt\right] \\
&= \tfrac{1}{t_j}\begin{bmatrix} t_{j-1}r_{j-1} + a_j(t-t_{j-1}) + \frac{b_j}{2}(t-t_{j-1})^2 \\ +\frac{c_j}{3}(t-t_{j-1})^3 + \frac{d_j}{4}(t-t_{j-1})^4 + \frac{e_j}{5}(t-t_{j-1})^5 \end{bmatrix} \quad j = 1,...,n
\end{aligned} \tag{3.35}$$

This accounts for n equations.

Secondly, it is required that the polynomials agree at the knot points for the spline to be continuous. Implicit in the second requirement is that each polynomial pass through both knot points between which they are defined. (3.36) accounts for n-1 equations.

$$\begin{gathered}\widehat{F}(0,t_j)=\widehat{F}(0,t_{j-1})\\ a_j+b_jt_j+c_jt_j^2+d_jt_j^3+e_jt_j^4=a_{j-1}+b_jt_{j-1}+c_jt_{j-1}^2+d_jt_{j-1}^3+e_jt_{j-1}^4 \quad \text{j=1,..,n-1}\end{gathered} \tag{3.36}$$

Thirdly, it is required that the spline be continuous in the first, second and third derivatives. This implies that the first, second and third derivatives of neighbouring polynomials must agree at their joint. Each of (3.37), (3.38) and (3.39) account for n-1 equations.

$$\begin{gathered}\left.\frac{\partial}{\partial t}\widehat{F}(0,t)\right|_{t_j}=\left.\frac{\partial}{\partial t}\widehat{F}(0,t)\right|_{t_{j-1}}\\ b_j+c_jt_j+d_jt_j^2+e_jt_j^3=b_{j-1}+c_jt_{j-1}+d_jt_{j-1}^2+e_jt_{j-1}^3 \quad \text{j=1,..,n-1}\end{gathered} \tag{3.37}$$

$$\begin{gathered}\left.\frac{\partial^2}{\partial t^2}\widehat{F}(0,t)\right|_{t_j}=\left.\frac{\partial^2}{\partial t^2}\widehat{F}(0,t)\right|_{t_{j-1}}\\ c_j+d_jt_j+e_jt_j^2=c_{j-1}+d_jt_{j-1}+e_jt_{j-1}^2 \quad \text{j=1,..,n-1}\end{gathered} \tag{3.38}$$

$$\begin{gathered}\left.\frac{\partial^3}{\partial t^3}\widehat{F}(0,t)\right|_{t_j}=\left.\frac{\partial^3}{\partial t^3}\widehat{F}(0,t)\right|_{t_{j-1}}\\ d_j+e_jt_j=d_{j-1}+e_jt_{j-1} \quad \text{j=1,..,n-1}\end{gathered} \tag{3.39}$$

All polynomials now have 5 restrictions each, except the first and last which only have 3. The first derivative of the forward at t_0 will be set equal to the first derivative at t_1 (3.40) and the second derivative of the forward at t_0 will be set equal to zero (3.41). This implies a constant slope for the spline between t_0 and t_1. The change in the quartic forward between t_n and t_{n-1} will be set equal to the change in the quadratic forward between t_n and t_{n-1} (3.42) and the second derivative at t_n will be set equal to zero (3.43). This implies a constant slope for the spline between t_{n-1} and t_n.

$$c_1=0 \tag{3.40}$$

$$3d_1+4e_1t_1=0 \tag{3.41}$$

$$c_n+3d_nt_n+6e_nt_n^2=0 \tag{3.42}$$

$$\widehat{F}(0,t_n)-\widehat{F}(0,t_{n-1})=q_n-q_{n-1} \tag{3.43}$$

These equations can be written in the form $A\underline{x}=\underline{b}$ and solved using LU decomposition. Firstly A is decomposed into its LU form $A=LU$ using a statistical program such as the Matlab function lu(). Now L is a lower triangular and U is an upper triangular matrix. The original problem has now been reduced to two problems involving triangular matrices. The following two systems have to be solved,

$$\begin{aligned} L\underline{z} &= \underline{b} \\ U\underline{x} &= \underline{z}, \end{aligned} \tag{3.43}$$

such that $A\underline{x} = LU\underline{x} = L\underline{z} = \underline{b}$. Linear systems with triangular matrices can easily be solved via Gauss-Jordan elimination.

Remark 3.8: Even though it would be more accurate to used the quartic forward instead of the quadratic forward in the multidimensional minimisation problem, it can be seen that calculating the quartic forward is much more intensive. Since the quadratic forward has to be fit on each iteration of the minimisation algorithm, substituting this with the quartic forward is infeasible.

3.5 Example

The zero curve implied by the government bonds in the ALBI index are calculated daily and is available from the BESA website. This section will give an example of the construction of the "Perfect Fit" zero curve using bond data on 12 December 2005.

The set of instruments that is applicable to our example are listed in table 3.2 and comprise those government bonds in the ALSI index on the 12th of December 2005.

Table 3.2: GOVI bonds on 12 December 2005

Bond Code	Coupon Rate	Maturity	Interest Payable 1	Interest Payable 2	Books Closed 1	Books Closed 2	Issue Date	MtM on 12/12/2005	Price on 12/12/2005
R194	10.000	28-Feb-08	28-Feb	31-Aug	18-Feb	21-Aug	01-Apr-01	7.26%	R 108.34
R153	13.000	31-Aug-10	28-Feb	31-Aug	18-Feb	21-Aug	22-Jun-89	7.39%	R 125.67
R201	8.750	21-Dec-14	21-Jun	21-Dec	11-Jun	11-Dec	27-May-03	7.56%	R 107.47
R157	13.500	15-Sep-15	15-Mar	15-Sep	05-Mar	05-Sep	18-Jan-91	7.59%	R 143.47
R203	8.250	15-Sep-17	15-Sep	15-Mar	05-Sep	05-Mar	07-May-04	7.58%	R 107.04
R204	8.000	21-Dec-18	21-Dec	21-Jun	11-Dec	11-Jun	11-Aug-04	7.55%	R 100.09
R186	10.500	21-Dec-26	21-Jun	21-Dec	11-Jun	11-Dec	01-Apr-98	7.25%	R 134.54

The above prices were calculated from the market quoted MtM yields using the BESA bond pricing formula.

For this example m = 7. When a bond is bought today, the price that must be paid will be today's quoted price but settlement will only occur within three business days. Therefore dt_0 will be Thursday the 15th of December 2005. The set $\underline{I}$ as given by table 3.2 has 77 different cashflow dates, consequently n = 77.

The cashflows for fixed coupon government bonds will therefore take the from given in table 3.3. The Matlab program "createBond", given in Appendix D section D.1, was written to construct a matrix, with contents defined by table 3.3, for each bond.

Table 3.3: Cashflow structure of a single bond

dt_0	dt_i	$dt_{maturity}$
-(All In Bond Price on dt_0)	Coupon/2	100 + Coupon/2

The first 10 columns of the cashflow matrix C: 7x77, given in expression 3.4, is listed in table 3.4. The Matlab program "buildCashflowMatrix", given in Appendix D section D.1, was written to construct the cashflow matrix for a given set of cashflows as defined by table 3.3.

Table 3.4: An extract from the cashflow matrix C

	15-Dec-05	28-Feb-06	15-Mar-06	21-Jun-06	31-Aug-06	15-Sep-06	21-Dec-06	28-Feb-07	15-Mar-07	21-Jun-07
R194	-108.34	5	0	0	5	0	0	5	0	0
R153	-125.67	6.5	0	0	6.5	0	0	6.5	0	0
R201	-107.47	0	0	4.375	0	0	4.375	0	0	4.375
R157	-143.47	0	6.75	0	0	6.75	0	0	6.75	0
R203	-107.04	0	4.125	0	0	4.125	0	0	4.125	0
R204	-100.09	0	0	4	0	0	4	0	0	4
R186	-134.54	0	0	5.25	0	0	5.25	0	0	5.25

The reason why there are no coupon payments for the R201, R204 and R186 on 21 December 2005 is because these bonds were ex-dividend on 12 December 2005.

The bootstrapped zero, forward and quadratic curves are given in figure 3.1.

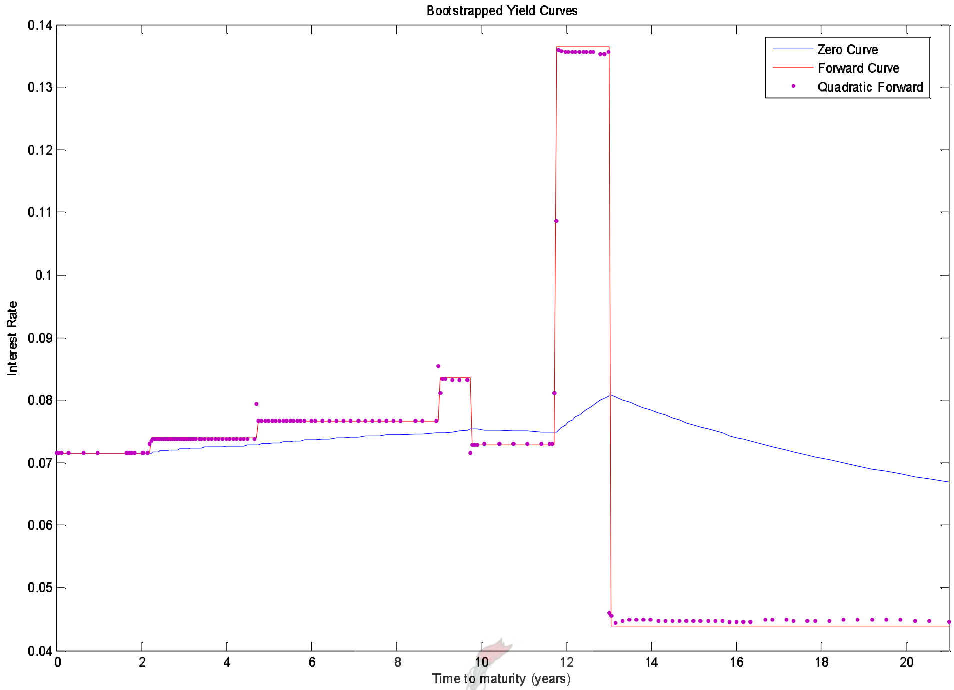

Figure 3.1: Bootstrapped Yield Curves

At first glance this graph may look somewhat disproportional with a large spike at the 12 year mark. This has however been a characteristic of the South African interest rate market for the years leading up to the period under consideration. Had there been more liquid securities for the bootstrapping procedure, a gentler slope would have prevailed. The large spike is an anomaly of the bootstrapping algorithm. Consider what happens when the bootstrapping is performed.

- For the bond with the earliest maturity, the interest rate is sought which makes its discounted cashflows equal to its current market value. This interest rate is then fixed for the period up to the maturity date of this bond.

- For the bond with the second earliest maturity date, the interest rate remains fixed for the period between its maturity and the maturity of the first bond. The rate at which the interest rate is fixed, is that rate for which the second bond's discounted cashflows are equal to its current market value.

- This process continues for each subsequent bond, each time calculating a new interest rate for the period spanning the maturities of two bonds.

Note that in the example above the maturities of the R203 and the R204 are very close together. Furthermore, the discounted present value of the cashflows of the R204 up to the maturity for the R203 is nearly equal to the market value of the R204. The discounted value of the remaining cashflows for the R204, between their maturities, has to make up this difference. Consequently, since this difference is small, the interest rate over this period has to be high.

Using expression (3.32) $\underline{x}_0$ can be calculated from $\underline{r}_0$ and is given in table 3.5.

Table 3.5: Starting candidate for multidimensional optimisation algorithm

$\underline{x}_0$	0.9085	0.9802	0.8471	0.8759	...	0.2227	0.217	0.2114	-0.541

A program was written in Matlab to perform the optimisation procedure and is given in Appendix D, section D.1 as "optDecency". An alternative to the Polak-Ribiere Conjugate Gradient Algorithm was used. A Levenberg-Marquardt line-search algorithm was used instead, since this is already implemented in Matlab. Figure 3.2 shows the convergence of Decency using this line-search algorithm.

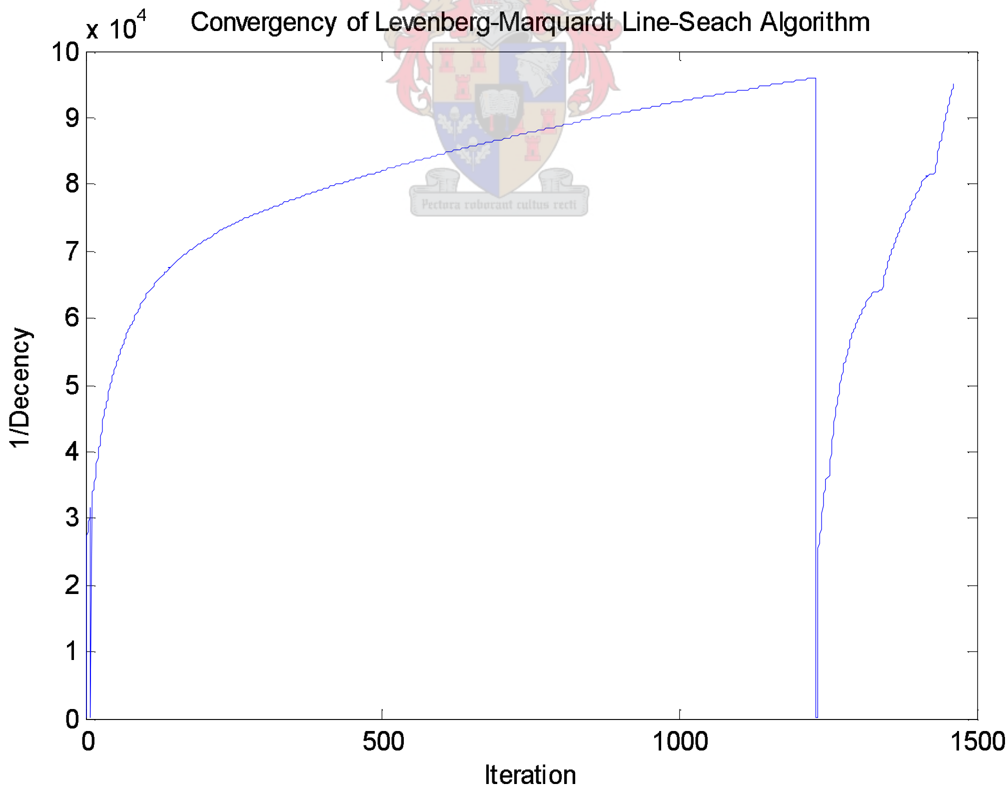

Figure 3.2: Convergence of Objective Function

The y-axis is defined as 1/Decency which implies that an increase in the graph is associated with a better candidate solution.

Each discontinuity in figure 3.2 represents a change of weights for local and global smoothness in the Decency calculation. Finally the optimised points on the zero curve is given in figure 3.3. The Decency weights of the final solution, for local and global smoothness respectively, are:

$$w_1 = 3.64 \text{ and } w_2 = 1.$$

The Decency of the final solution is:

$$e = 1.0516e\text{-}5.$$

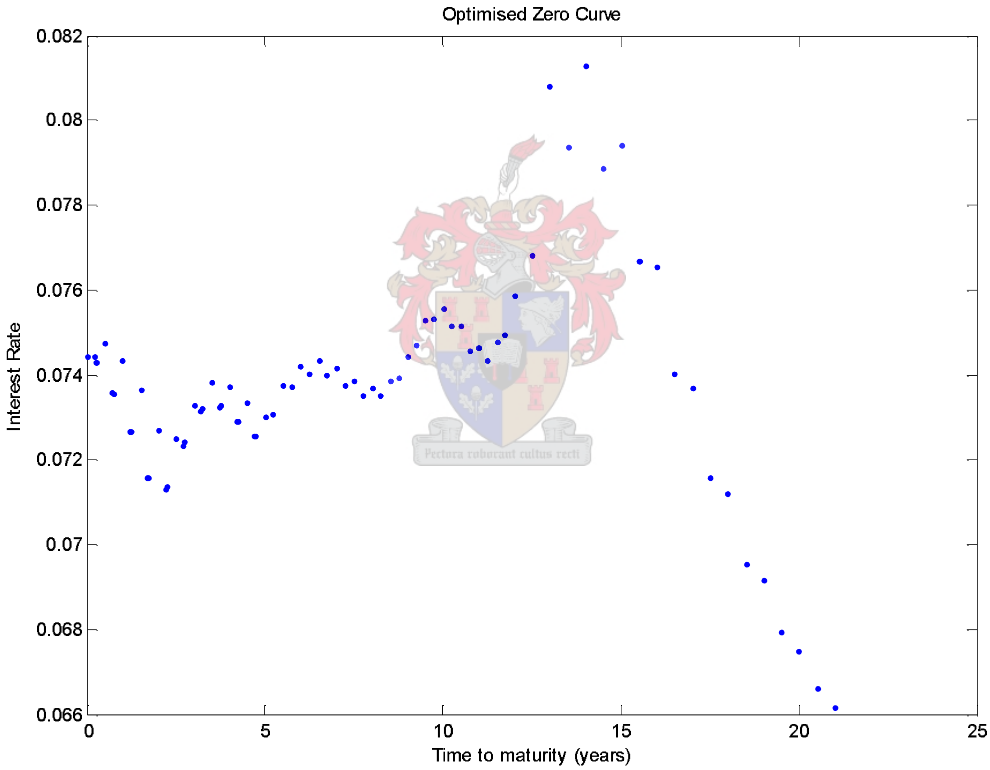

Figure 3.3: Optimised Zero Curve on 12 December 2005

Finally the quartic forward curve is constructed which is given in figure 3.4. A Matlab program was written to construct the linear system which must be solved to find the quartic forward parameters. This program is given in Appendix D, section D.2.

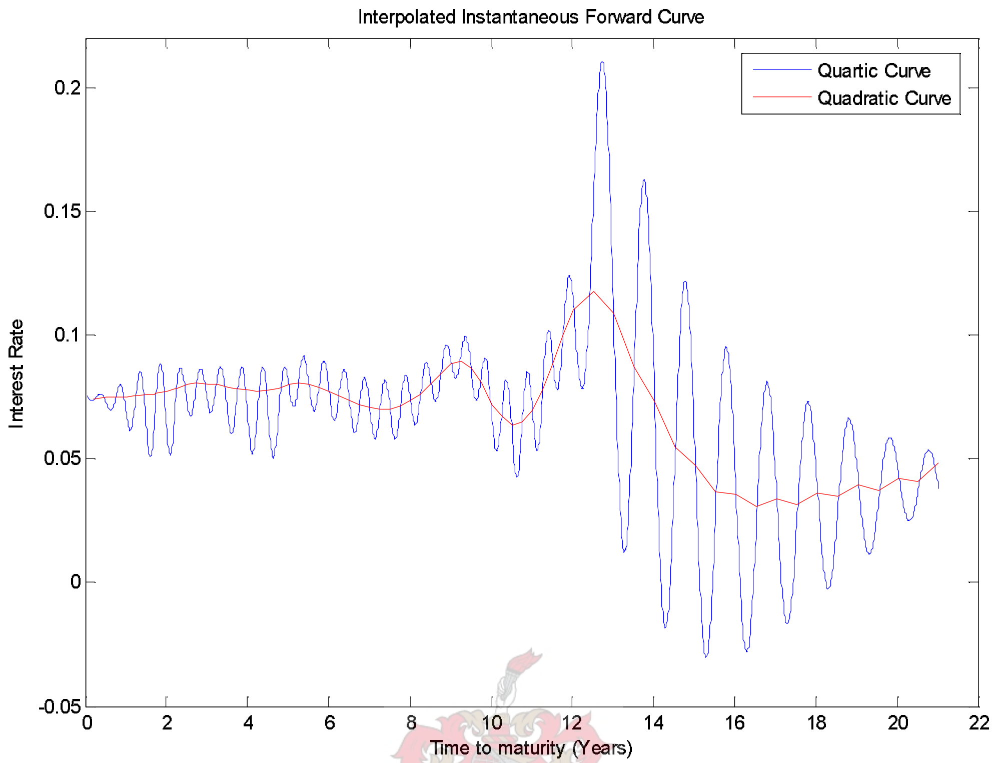

Figure 3.4: Interpolated Instantaneous Forward Curve

It can be seen that the quartic forward curve is unnaturally volatile for the perfect fit case. This is due to the jagged sequence of points in figure 3.3. A "best Decency" fit would have smoothed out this roughness because a perfect goodness of fit would not be required. Judging by the shape of the quartic forward compared to the quadratic forward, it might be prudent to use the quadratic forward for model fitting in subsequent chapters, rather than the quartic forward.

3.6 Summary

This chapter explained the concept of a descriptive yield curve and discussed some of its applications. Some considerations pertaining to its construction were also reviewed. The primary focus, however, was the explanation of the BEASSA yield curve methodology which is widely implemented throughout the South African market.

This descriptive yield curve will be a fundamental building block for the stochastic models of chapters 4 and 5.

Chapter 4

ENDOGENOUS INTEREST RATE MODELS

4.1 Introduction

In the previous chapter a descriptive model of the yield curve was built based on the current market price of instruments. This descriptive model was static and represented a snapshot of the yield curve, which is an evolving process through time. This chapter introduces stochastic yield curve models which are used to model this evolutionary process. Knowledge of this process is crucial for pricing interest rate derivatives which depend on the level of future interest rates.

Three classic endogenous models will be presented. Even though they are quite naive, they are useful for developing an intuition for the theory while not being overly complex. The models of this chapter are called endogenous because the current term structure is not supplied as an input but can be calculated as a function of the model parameters. For endogenous models this implied term structure cannot exactly reflect the current market term structure, no matter how the parameters are chosen. This is in contrast with exogenous models which perfectly reproduce the current term structure of interest rates. Exogenous models achieve this by accepting the entire current term structure as an input. In this respect exogenous models can be considered as "infinite parameter" models. Endogenous models on the other hand typically only have three or four parameters and therefore can only approximate the current term structure.

Endogenous models are calibrated by finding those parameters that minimise the discrepancy between market prices or interest rates and model implied prices or interest rates. For derivative pricing purposes these models often provide analytical expressions due to their relatively uncomplicated nature but admit arbitrage opportunities due to their inconsistency with market interest rates. These models were popular during the 1980's but the growth in the derivatives market combined with the emergence of more sophisticated models have restricted these models to the realm of benchmarks and academic history.

Table 4.1 gives a list of the most famous classical endogenous short rate models, i.e. the Dothan model (D), the Vasicek model (V) and the Cox-Ingersoll-Ross model (CIR).

Table 4.1: Endogenous Short Rate Models

Model	Dynamics	Strictly Positive Rates	Short Rate Distribution	Analytical Bond Prices	Analytical Option Prices
V	$dR(t) = k(\theta - R(t))dt + \sigma d\widetilde{W}(t)$	No	Normal	Yes	Yes
D	$dR(t) = \alpha R(t)dt + \sigma R(t)d\widetilde{W}(t)$	Yes	Log-Normal	Yes	No
CIR	$dR(t) = k(\theta - R(t))dt + \sigma\sqrt{R(t)}d\widetilde{W}(t)$	Yes	Non-Central χ^2	Yes	Yes

Each of the entries in the table will be discussed in detail in the subsequent sub-section.

All of the models in this chapter are also so called "single-factor" models. In other words, there is only one source of uncertainty, the single Brownian motion in the dynamics of the model. This implies that the short rate is modelled directly and not considered a consequence of various influences. Multi-factor models, though very important, are not yet the standard for derivative pricing and therefore fall beyond the scope of this dissertation.

This chapter will discuss each of the above models based on the characteristics outlined below. The Vasicek case serves as an example to illustrate how most of the calculations can be done.

4.2 Characteristics of a Stochastic Interest Rate Model

It has already been mentioned in Chapter 2 that stochastic interest rates will be modelled using an Ito process for the short rate dynamics. The primary motivation for the stochastic modelling of interest rates is the pricing of interest rate derivatives. With this goal in mind, the following features should be considered when selecting a short rate model.

4.2.1 Mean Reversion

Interest rates often reflect the property that the risk free rate varies around some long term average, never straying too farafield before it is "pulled back" by market forces. There are some very compelling economic arguments in favour of this phenomenon. Low interest rates are generally coupled with high investment spending by consumers which leads to an increased demand for money and an increase in the interest rate. In contrast, investors tend to save money in a high interest rate environment and this slows down the economy and decreases the demand for money which eventually leads to lower interest rates.

4.2.2 Distribution

It will be a desirable feature of the model if the distribution of the short rate can be found under both the traditional risk neutral measure as well as the T-Forward measure. When pricing interest rate derivatives, expectations are often simplified under these measures which make knowledge of the distribution of the short rate under these measures extremely useful. If the distribution can indeed be found, two questions regarding the distribution are important.

(1) Does the distribution allow negative rates (e.g. Normal Distribution)?

(2) Does the distribution have fat-tails?

It is generally accepted that interest rate changes have heavier tails than that of the normal distribution.

4.2.3 Positivity

Interest rates are generally considered to be strictly positive. Some models imply a normal distribution of the short rate which is often tolerated because of the analytical tractability of these models.

4.2.4 Bond Prices

It will be desirable if analytical expressions can be found for bond prices and hence (from the relations derived in chapter 2) for spot and forward rates.

4.2.5 Derivative Prices

It will be desirable if analytical expressions can be found for options on zero coupon bonds, caps, floors and swaptions. The bulk of interest rate derivative markets consist of caps and swaptions, therefore practitioners will often prefer a model that produces such analytical expressions.

4.2.6 Pricing Methods

When the goal is the pricing of derivatives that requires the use of a recombining lattice, it is vital that the chosen model be suited to building these structures efficiently. If Monte Carlo simulation is required, it is vital that these simulations are not too time consuming.

4.2.7 Realistic Shapes

Another crucial requirement is that the model be able to reflect yield curve structures observed in the market. The South African market has a humped yield curve which is a shape that some models cannot reflect. In other markets an inverted yield curve has been observed which none of the models in this chapter can reflect. An inverted yield curve is illustrated in figure 4.1. Instead of one turning point like a humped yield curve, an inverted yield curve has two turning points.

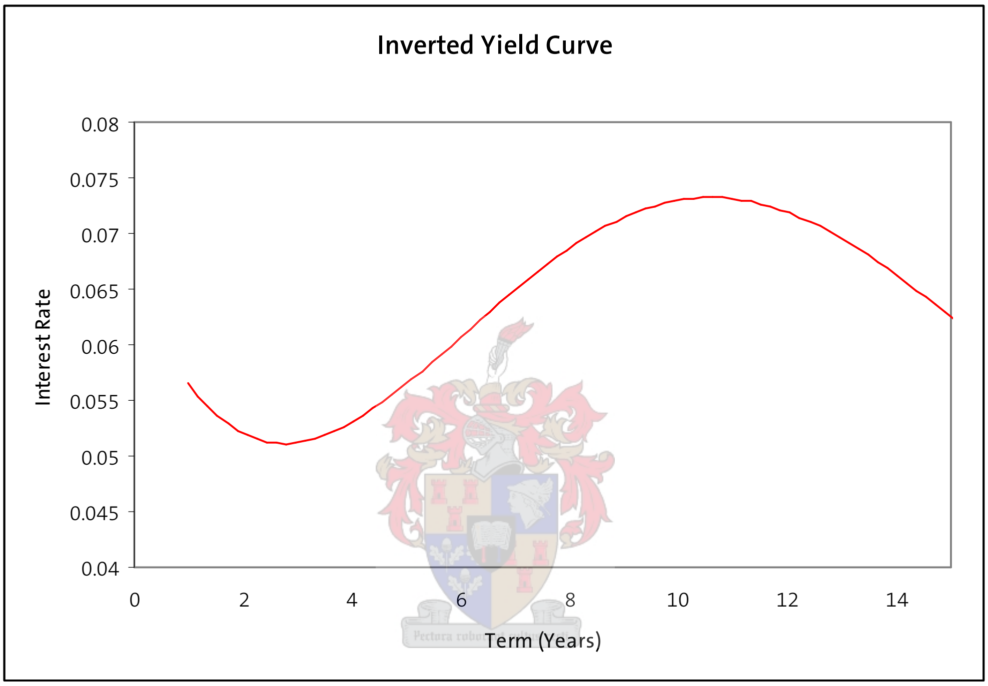

Figure 4.1: Inverted yield curve

4.2.8 Realistic Volatility Structures

Suppose a model is built with the specific aim of pricing swaptions. It is important that a model be chosen which is capable of reflecting the volatility structure of the swaptions market. In other words the volatilities of swaptions, priced using the model, should match the implied volatilities of swaptions in the market. More attention will be paid to this feature in the next chapter, since the endogenous models of this chapter cannot match the current term structure, calibrating them to volatility structures is futile.

4.3 The Vasicek Model (1977)

The Vasicek model dynamics are given by expression (4.1).

$$dR(t) = k(\theta - R(t))dt + \sigma d\widetilde{W}(t) \text{ under } \widetilde{\mathbb{P}} \tag{4.1}$$

The Vasicek model (Vasicek, 1997) was developed and was the first continuous time interest rate model to gain widespread acceptance. The different features of the model will be outlined below with proofs where appropriate.

4.3.1 Bond Prices

As was stated in table 4.1, analytical formulas exist for the bond prices. The proof however is quite long and will not be given here those interested can refer to Cairns (2004). The bond prices are given by expression (4.2).

$$B(t,T) = A(t,T)e^{-P(t,T)R(t)} \tag{4.2}$$

where,

$$A(t,T) = \exp\left\{\left(\theta - \tfrac{\sigma^2}{2k^2}\right)\left(P(t,T) - T + t\right) - \tfrac{\sigma^2}{4k}P(t,T)^2\right\}$$
$$P(t,T) = \tfrac{1}{k}\left[1 - e^{-k(T-t)}\right].$$

This analytical tractability is a strong point of the Vasicek model.

4.3.2 Distribution

It will be shown that the Vasicek interest rate R(t) has a normal distribution under both the traditional risk neutral measure, $\widetilde{\mathbb{P}}$, and the T-Forward measure $\widetilde{\mathbb{P}}^T$.

4.3.2.1 Distribution under $\widetilde{\mathbb{P}}$

Set

$$f(t,R(t)) = e^{kt}R(t).$$

Let the partial derivatives of $f(t,x)$ be defined by,

$$
\begin{aligned}
f_t(t,x) &= \frac{\partial}{\partial t} f(t,x) = ke^{kt}x, \\
f_x(t,x) &= \frac{\partial}{\partial x} f(t,x) = e^{kt} \text{ and} \\
f_{xx}(t,x) &= \frac{\partial^2}{\partial x^2} f(t,x) = 0.
\end{aligned}
$$

Applying now the Ito formula to $f(t, R(t))$, leads to,

$$
\begin{aligned}
df(t,R(t)) &= f_t(t,R(t))dt + f_x(t,R(t))dR(t) + \tfrac{1}{2} f_{xx}(t,R(t))d[R,R](t) \\
&= ke^{kt}R(t)dt + e^{kt}k\left(\theta - R(t)\right)dt + e^{kt}\sigma d\widetilde{W}(t) \\
&= e^{kt}\theta k dt + e^{kt}\sigma d\widetilde{W}(t).
\end{aligned}
$$

Integrating both sides of the above equation yields,

$$
f(t,R(t)) = e^{kt}R(t) = e^{ks}R(s) + \theta k \int_s^t e^{ku} du + \int_s^t e^{ku}\sigma d\widetilde{W}(u).
$$

Finally, after dividing by e^{kt}, expression (4.3) is obtained for the short rate.

$$
R(t) = e^{-k(t-s)}R(s) + \theta\left(1 - e^{-k(t-s)}\right) + \sigma \int_s^t e^{k(t-u)} d\widetilde{W}(u) \tag{4.3}
$$

Given the information up to time s, the only random component of (4.3) is the Ito integral which has a normal distribution with mean zero and variance given by (4.4).

$$
\begin{aligned}
VAR\left(\sigma \int_s^t e^{k(t-u)} d\widetilde{W}(u)\right) &= \sigma^2 \int_s^t e^{2k(t-u)} du \\
&= \tfrac{\sigma^2}{2k}\left(1 - e^{2k(t-s)}\right)
\end{aligned} \tag{4.4}
$$

Therefore, given the information $\mathscr{F}(s)$, the distribution of the short rate R(t) under the Vasicek model is represented by (4.5).

$$
R(t) \sim N\left[e^{-k(t-s)}R(s) + \theta\left(1 - e^{-k(t-s)}\right),\ \tfrac{\sigma^2}{2k}\left(1 - e^{2k(t-s)}\right)\right] \text{ under } \widetilde{\mathbb{P}} \tag{4.5}
$$

4.3.2.2 Distribution under $\widetilde{\mathbb{P}}^T$

It is known that a change of numéraire changes the drift but not the volatility of a process. When a change of measure is performed on the Vasicek process of (4.1) from the traditional risk neutral measure to the T-Forward measure, the new drift is found by applying Theorem C.7.

Theorem C.7 is employed with the substitutions in (4.6).

$$\begin{aligned} N(t) &= M(t) \Rightarrow \sigma_N(t) = 0 \\ U(t) &= B(t,T) \Rightarrow \sigma_U(t) = -P(t,T)B(t,T)\sigma \\ X(t) &= R(t) \Rightarrow \sigma_X(t) = \sigma \end{aligned} \tag{4.6}$$

The first and the last expressions are obvious but the middle one requires some thought. Set

$$f(t, R(t)) = B(t,T)$$

where $B(t,T)$ is the Vasicek bond price expression (4.2). Applying Ito's lemma to the above function yields expression (4.7),

$$\begin{aligned} f(t,x) &= A(t,T)e^{-P(t,T)x} \\ df(t,x) &= f_t(t,x)dt + f_x(t,x)dx + \tfrac{1}{2} f_{xx}(t,x)dxdx\,. \end{aligned} \tag{4.7}$$

Note that the partial derivative with respect to x will be the only one containing a Brownian motion. Consequently, since it must be shown that the coefficient of the Brownian motion is given by (4.6), the other terms need not be calculated and can be grouped into the coefficient of dt. This partial derivative w.r.t. x can now be calculated as,

$$\begin{aligned} f_x(t,x) &= \frac{\partial}{\partial x} A(t,T)e^{-P(t,T)x} \\ &= -P(t,T)f(t,x). \end{aligned}$$

This can be substituted into the Ito expansion of (4.7) to get,

$$\begin{aligned} df(t,R(t)) &= (\ldots)dt + f_x(t,R(t))dR(t) \\ &= (\ldots)dt + f_x(t,R(t))d(k(\theta - R(t))dt + \sigma d\widetilde{W}(t)) \\ &= (\ldots)dt - P(t,T)B(t,T)\sigma d\widetilde{W}(t) \\ dB(t,T) &= (\ldots)dt - P(t,T)B(t,T)\sigma d\widetilde{W}(t) \end{aligned}$$

Therefore the quadratic volatility coefficient of $B(t,T)$ is,

$$-P(t,T)B(t,T)\sigma\,. \tag{4.8}$$

The expressions in (4.6) can be entered into Theorem C.7 to obtain the drift rate of $R(t)$ under numéraire $B(t,T)$ i.e.

$$\mu^U(t,R(t)) = k(\theta - R(t)) - \sigma.1.\left(\tfrac{0}{S(t)} - \tfrac{-P(t,T)B(t,T)\sigma}{B(t,T)}\right)$$
$$= k\theta - P(t,T)\sigma^2 - kR(t).$$

Finally, since volatility does not change with a change in measure, the process of the short rate under the T-Forward measure will be,

$$dR(t) = k\left(\theta - P(t,T)\tfrac{\sigma^2}{k} - R(t)\right)dt + \sigma d\widetilde{W}^T(t). \tag{4.9}$$

This expression has a similar form as the original Vasicek dynamics of (4.1) but with an adjustment to the original theta in the drift. Therefore the distribution of the Vasicek short rate under the T-Forward measure can be found in the same way as under the traditional risk neutral measure. This distribution is given by expression (4.10).

$$R(t) \sim N\left[R(s)e^{-k(t-s)} + M^T(s,t), \tfrac{\sigma^2}{2k}\left(1 - e^{2k(t-s)}\right)\right], \text{ under } \widetilde{\mathbb{P}}^T \tag{4.10}$$

where,

$$M^T(s,t) = \left(\theta - \tfrac{\sigma^2}{k^2}\right)\left(1 - e^{-k(t-s)}\right) + \tfrac{\sigma^2}{2k^2}\left[e^{-k(T-s)} - e^{-k(T+t-2s)}\right].$$

An explicit knowledge of the distribution of $R(t)$ under $\widetilde{\mathbb{P}}^T$ will be valuable when expectations need to be performed under this measure.

4.3.3 Mean Reversion

The distribution of $R(t)$ under the traditional and T-forward measures behaves in a similar way with respect to the mean reversion of the process. Consider specifically the drift coefficient of the Vasicek model under the risk neutral measure,

$$\mu_R(t) = k\left(\theta - R(t)\right).$$

It is clear that when the current rate $R(s)$ is below θ, rates are expected to rise and when the current rate is above θ, they are expected to fall.

Next, consider the expectation of (4.5),

$$E[R(t)] = e^{-k(t-s)}R(s) + \theta\left(1 - e^{-k(t-s)}\right), \text{ under } \widetilde{\mathbb{P}}. \tag{4.11}$$

In the long run, this rate is expected to equal θ as t goes to infinity since,

$$\lim_{t\to\infty} E[R(t)] = \lim_{t\to\infty} e^{-k(t-s)}R(s) + \theta\left(1 - e^{-k(t-s)}\right) = \theta\,, \tag{4.12}$$

From this it can be seen that the process is mean reverting with a long-run average of θ. The result under the T-Forward measure is exactly the same except that $\lim_{t\to\infty} E[R(t)] = \theta - \frac{\sigma^2}{k^2}$.

4.3.4 Positivity

Unfortunately a normal distribution allows negative rates to occur which is undesirable for an interest rate model. This is the major drawback of the Vasicek model as an endogenous model.

4.3.5 Option Prices

Analytical expressions for a European call and put options on a zero coupon bond is also available. The results have been derived by Jamshidian (1989) but a more intuitive derivation will be given. In order to derive the option prices the following well known result will be used.

Proposition 4.1

Given a log-normal variable V where VAR(ln(V))=s^2,

$$E[\max(V-K,0)] = E[V]\Phi(d_1) - K\Phi(d_2), \tag{4.13}$$

where

$$d_1 = \frac{\ln {}^{E[V]}\!/\!{}_K + {}^{s^2}\!/\!{}_2}{s},$$

$d_2 = d_1 - s$ and

$\Phi(\cdot)$ is the standard normal cumulative distribution function.

Proof: See Hull, J.C., (2003) pp. 262-263.

Consider a European call option on a zero coupon bond, with strike price K, maturity date S and bond maturity T. The pricing formula under the T-forward measure is given in appendix C by expression C.10 where the price process V(t) is taken as a European call process, c(t), with payoff at maturity equal to $C(T) = \left(B(S,T) - K\right)^+$.

$$\begin{aligned} c(t) &= B(t,S)\widetilde{E}^T\left[\left(B(S,T) - K\right)^+ \mid \mathcal{F}(t)\right] \\ &= B(t,S)\widetilde{E}^T\left[\left(e^{\ln A(S,T) - P(S,T)R(S)} - K\right)^+ \mid \mathcal{F}(t)\right] \end{aligned} \tag{4.14}$$

Relating this to proposition 4.1 it can be seen that

$$V = e^{\ln A(S,T) - P(S,T)R(S)} \tag{4.15}$$

Because R(S) has the normal distribution, V will be log-normal with s^2 equal to,

$$\begin{aligned} s^2 &= VAR(\ln B(S,T)) \\ &= P(S,T)^2 VAR(R(S)) \\ &= P(S,T)^2 \tfrac{\sigma^2}{2k}\left(1 - e^{2k(S-t)}\right) \end{aligned} \tag{4.16}$$

Finally note that,

$$\widetilde{E}^T \left[B(S,T) \mid \mathscr{F}(t) \right] = \tfrac{B(t,T)}{B(t,S)} \tag{4.17}$$

Substituting (4.14), (4.15) and (4.16) into (4.13) the option prices are given by,

$$\begin{aligned} & c(t,S,T,K) = B(t,T)\Phi\left(d_1\right) - KB(t,S)\Phi(d_2) \\ & p(t,S,T,K) = KB(t,S)\Phi\left(-d_2\right) - B(t,T)\Phi(-d_1) \\ & s = P(S,T)\sigma\sqrt{\tfrac{1-e^{2k(S-t)}}{2k}} \\ & d_1 = \frac{\ln B(t,T) - \ln B(t,S) - \ln K + {}^{s^2}\!/_{2}}{s} \\ & d_2 = d_1 - s \end{aligned} \tag{4.18}$$

Because bond options are so popular in the market, processes that yield analytic formulas for these prices are preferred by practitioners. This is another strength of the Vasicek model.

4.3.6 Realistic Shapes

Depending on the choice of the parameters θ, k and σ the Vasicek model can have an increasing, decreasing or humped shape.

Figure 4.2 shows the shape of the spot yield curve at time t=0 for σ ranging over (0.07-0.12) and keeping the other parameters constant at r_0=0.05, k = 0.06 and σ = 0.07. The Matlab program that was written to draw figures 4.2 and 4.3 is given in Appendix D, section D.4. The top most yield curve in figure 4.2 corresponds to σ = 0.07 and the bottom yield curve corresponds to σ = 0.12. It can be seen that a greater volatility forces the yield curve away from the long run average.

Figure 4.3 shows the shape of the spot yield curve at time t=0 for k ranging over (0.2-1) and keeping the other parameters constant at r_0=0.05, θ = 0.08 and σ = 0.07. It is clear that parameter k controls the speed of mean reversion, where a larger k speeds up and a smaller k will slow down the rate of convergence. When k is small the volatility seems to dominate the shape of the yield curve. The top graph in figure 4.3 corresponds to a yield curve where k = 1 and the bottom graph corresponds to a yield curve where k = 0.2.

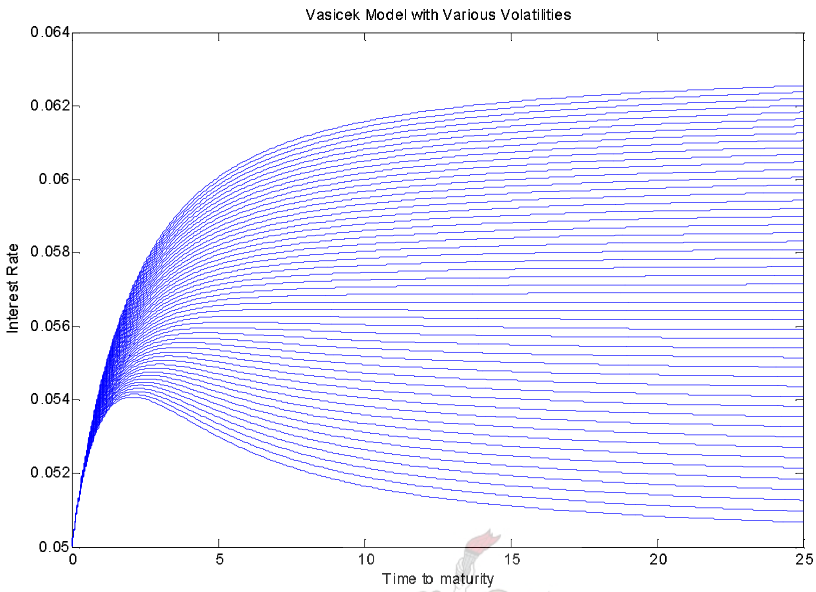

Figure 4.2: Vasicek spot yield curve with $r_0 = 0.05$ $k = 0.6$, $\theta = 0.07$, $\sigma = (0.07,\ldots,0.12)$

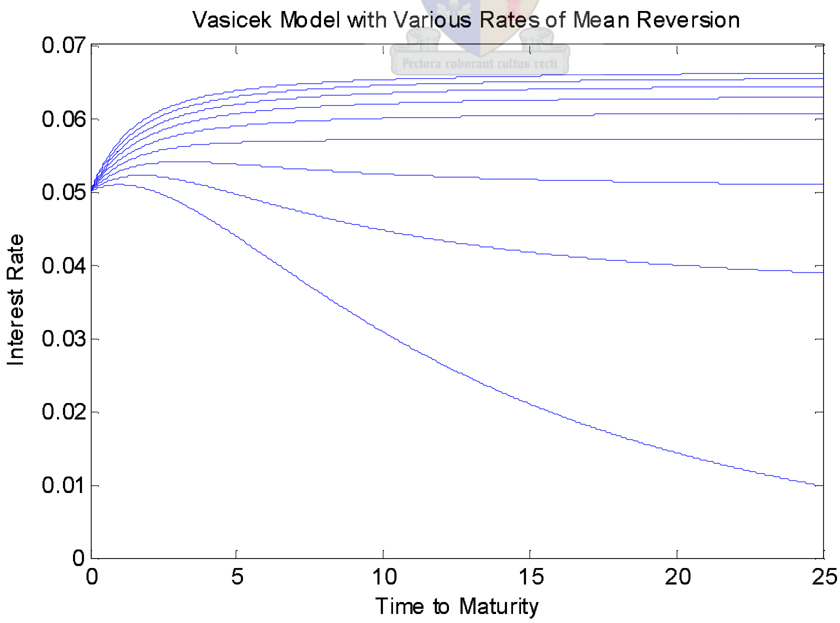

Figure 4.3: Vasicek spot yield curve with $r_0 = 0.05$ $\theta = 0.08$, $\sigma = 0.07$, $k = (0.2,\ldots,1)$

4.3.7 Parameter Estimation

There are various approaches to calibrating an interest rate model. Two different approaches will be outlined below.

4.3.7.1 Calibration to Historical Data

It is important to realise that the statistical properties of historical data, characterise the distribution of the short rate process R(t) under the real world measure $\mathbb{P}$. Therefore calibrating a model to historical data requires knowledge of its dynamics under the real world measure. Furthermore, for purposes of derivative pricing, knowledge of the dynamics of the short rate R(t) under the real world measure is of little use, since all pricing is done under the risk neutral measure. It should also be kept in mind that a model that has been calibrated to historical data will reflect the characteristics of the data into the future. This is not always realistic and practitioners prefer to calibrate models based on current prices alone, since current prices reflect the market's expectations of the future. Nevertheless, historical data can be employed to find an initial estimate of the volatility term of a market process, which can prove especially useful when data is scarce and few calibration securities are available. The reason why the volatility term can be estimated is because it does not change with a change in measure, it is the same under both the real world and traditional risk neutral measures. Also, it can only serve as an initial estimate since it is a reflection of the past, though in the case of the Vasicek model, it is a constant. A procedure will therefore be presented through which this volatility can be estimated based on historical data.

Firstly, the dynamics of R(t) has to be found under the real world measure $\mathbb{P}$. The Vasicek dynamics in (4.1) was defined under the traditional risk neutral measure $\tilde{\mathbb{P}}$ and in section 4.3.2.2 an equivalent model (4.9) was found under the T-Forward measure $\mathbb{P}^T$. Now in order to find the equivalent dynamics under the real world measure another change of measure has to be performed. Under the assumption that the market is complete and arbitrage free, there exists a square integrable function $\lambda(t)$ such that the Girsanov theorem (C.1) can be employed to make the change from the risk neutral world to the real world. The Girsanov change of measure is illustrated by (4.19).

$$
\begin{aligned}
Z(t) &= \exp\Big(\int_0^t \underline{\lambda}(u)\cdot d\underline{W}(u) - \tfrac{1}{2}\int_0^t \|\underline{\lambda}(u)\|^2\,du\Big)\\
Z &= Z(T)\\
\mathbb{P}(A) &= \int_A Z(\omega)d\widetilde{\mathbb{P}}(\omega) \;\; \forall\; A \in \mathscr{F}(T)\\
dW_i(t) &= d\widetilde{W}_i(t) - \lambda_i(t)dt \text{ for } i = 1,\dots,d
\end{aligned}
\tag{4.19}
$$

The Vasicek model (4.1) can now be restated under the real world measure by substituting (4.19) into the Vasicek dynamics to obtain expression (4.20).

$$
\begin{aligned}
dR(t) &= k(\theta - R(t))dt + \sigma d\widetilde{W}(t)\\
&= k(\theta - R(t))dt + \sigma(dW(t) + \lambda(t)dt)\\
dR(t) &= (k\theta - kR(t) - \sigma\lambda(t))dt + \sigma dW(t)
\end{aligned}
\tag{4.20}
$$

Again it can be seen that only the drift changes and the volatility of the process stays intact. An assumption on $\lambda(t)$ is required to make this model tractable. The following assumption will be placed on the shape of $\lambda(t)$,

$$\lambda(t) = \lambda R(t). \tag{4.21}$$

This assumption is not necessarily realistic but it is required to make the model tractable. This restricted model is given in (4.22).

$$dR(t) = (k\theta - (k + \sigma\lambda)R(t))dt + \sigma dW(t) \text{ under } \mathbb{P} \tag{4.22}$$

The parameters in the drift of (4.22) are of little importance and, as long as the volatility remains intact, the model can be reparameterised. Since there is no way to separate the parameters in the drift anyway, (4.22) will be rewritten in the form of (4.23), where $b = k\theta$ and $a = k + \sigma\lambda$.

$$dR(t) = (b - aR(t))dt + \sigma dW(t) \tag{4.23}$$

The market data that will be used to calibrate the model are the South African Overnight Interbank Acceptance (SAONIA) rate which is an average of the over-nightly borrowing rates between SA banks. The SAONIA rate is equivalent to the simply compounded spot rate L(1,1+1/252) from definition 2.3.2, which is a fair approximation of R(t). Since the model (4.23) is in a similar form to the original model, the earlier results from section 4.3.2 regarding the distribution of R(t) will still hold. Therefore, the short rate R(t) will have the normal distribution under the real world measure as presented in (4.24) below, where δ = 1/252.

$$
\begin{aligned}
R(t) &= R(s)e^{-a\delta} + \tfrac{b}{a}\left(1-e^{-a\delta}\right) + \sigma\int_s^t e^{a(t-u)}dW(u) \\
R(t) &\sim N\left[R(s)e^{-a\delta} + \tfrac{b}{a}\left(1-e^{-a\delta}\right), \tfrac{\sigma^2}{2a}(1-e^{-2a\delta})\right]
\end{aligned}
\tag{4.24}
$$

Let the SAONIA rates be represented by $\{l_0,\ldots,l_n\}$. Assuming the SAONIA rates have approximately the same distribution as the short rate R(t) above, maximum likelihood estimates of the parameter values can be found. However, the model is first rewritten in a form that will simplify the estimation procedure. Substituting (4.25), (4.26) and (4.27) into (4.24) leads to (4.28).

$$\alpha = e^{-a\delta} \tag{4.25}$$

$$\beta = \frac{b}{a} \tag{4.26}$$

$$V^2 = \tfrac{\sigma^2}{2a}(1-e^{-2a\delta}) \tag{4.27}$$

$$l_i \sim N\left[l_{i-1}\alpha + \beta(1-\alpha), V^2\right],\ \mathrm{i} = 1,\ldots,\mathrm{n} \tag{4.28}$$

The log-likelihood function and its derivative with respect to each parameter is given by (4.29).

$$
\begin{aligned}
\log L &= -n\log V - \tfrac{n}{2}\log(2\pi) - \tfrac{1}{2V^2}\sum_{i=1}^n\left(l_i - l_{i-1}\alpha - \beta - \alpha\beta\right)^2 \\
\frac{\partial \log L}{\partial \beta} &= -\tfrac{1}{V^2}\sum_{i=1}^n\left(l_i - l_{i-1}\alpha - \beta - \alpha\beta\right)(\alpha-1) \\
\frac{\partial \log L}{\partial \alpha} &= -\tfrac{1}{V^2}\sum_{i=1}^n\left(l_i - l_{i-1}\alpha - \beta - \alpha\beta\right)(\beta - l_{i-1}) \\
\frac{\partial \log L}{\partial V} &= -\tfrac{n}{V} + \tfrac{1}{V^3}\sum_{i=1}^n\left(l_i - l_{i-1}\alpha - \beta - \alpha\beta\right)^2
\end{aligned}
\tag{4.29}
$$

In each case the derivative is set equal to zero and the parameter estimates follow as,

$$\hat{\alpha} = \frac{n\sum_{i=1}^n l_i l_{i-1} - \sum_{i=1}^n l_i \sum_{i=1}^n l_{i-1}}{n\sum_{i=1}^n l_i^2 - \left(\sum_{i=1}^n l_{i-1}\right)^2}, \tag{4.30}$$

$$\hat{\beta} = \frac{\sum_{i=1}^n\left(l_i - l_{i-1}\hat{\alpha}\right)}{n(1-\hat{\alpha})} \quad \text{and} \tag{4.31}$$

$$\hat{V} = \tfrac{1}{n}\sum_{i=1}^n\left(l_i - l_{i-1}\hat{\alpha} - \hat{\beta} - \hat{\alpha}\beta\right)^2. \tag{4.32}$$

Using these expressions it follows directly that σ can be estimated as:

$$\hat{\sigma} = \sqrt{\frac{2\widehat{V}^2 \ln \hat{\alpha}}{\delta(\hat{\alpha}^2 - 1)}}. \tag{4.33}$$

The 260 SAONIA values between 6 December 2004 and 5 December 2005 were used to calculate this volatility estimate. The calculated value was **0.187**.

4.3.7.2 Calibration to Current Data

A more widely accepted method of model calibration is by using only current market data. The yield curve from chapter 3 can be used since it was constructed using only current bond data. As mentioned earlier, the single factor models of this chapter do not have enough degrees of freedom to allow calibration of the volatility structure; therefore the best case scenario is a rough representation of the primary characteristics of the current term structure of interest rates. The model will be calibrated to market data using a conventional least squares approach.

Procedure: General Least Squares Interest Rate Model Calibration

Inputs: Market values $\{v_1, v_2, \ldots, v_n\}$. Initial parameter values k_0, θ_0 and σ_0.
Outputs: Calibrated parameter values k_m, θ_m and σ_m.

(1) Calculate the fair values of the market instruments under the model $\{\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n\}$. These fair values can be calculated using the following expectations,

$$\begin{aligned} \hat{v}_i(t, R(t)) &= M(t)\widetilde{E}\left[\tfrac{v_i(T,R(T))}{M(T)} \mid \mathscr{F}(t)\right] \text{ under } \widetilde{\mathbb{P}} \text{ or} \\ \hat{v}_i(t, R(t)) &= B(t,T)\widetilde{E}^T\left[v_i(T, R(T)) \mid \mathscr{F}(t)\right] \text{ under } \widetilde{\mathbb{P}}^T. \end{aligned} \tag{4.34}$$

(2) Calculate the sum of the squares of the errors in the prices. A weighting can be applied if certain instruments are considered more important than others. It is common practice to place a heavier weighting on instruments with a longer maturity since they are more sensitive to changes in the interest rate. A mispricing of a longer dated security is more dangerous since it can lead to a greater mispricing of derivative securities. The greater weight will force the least squares procedure to provide a better fit to the longer dated securities.

$$Q = \sum_{i=1}^{n} w_i (v_i - \hat{v}_i)^2$$

(3) If

- the error is within tolerance or
- the maximum allowed number of iterations has expired or
- the decrease in Q is below a certain threshold,

the procedure stops, otherwise the parameter values are adjusted and the procedure returns to step 1.

The choice of instruments $\{v_1, v_2, \ldots, v_n\}$ will determine the quality of the fit. It is preferable that instruments be chosen that

(1) are liquid and consequently do not contain any illiquidity premia,
(2) can easily be priced using 4.18. It is preferable if an analytic formula can be found.

For calibration purposes a range of choices of the calibration securities is available.

(1) ***Bond Options***: An analytical formula for ZCB options was presented by (4.18). Unfortunately bond options are not actively traded in the South African market which makes the use of bond options as the choice of calibration securities impractical.

(2) ***Zero Coupon Bonds***: Zero coupon bonds are also not actively traded in the South African market but their implied market values can be inferred from the descriptive yield curve of chapter 3. Unfortunately a problem might arise during the optimisation procedure. The bond price expression (4.2) is very non-linear in the parameter values. This might make it difficult for an optimisation routine to find the best-fit values.

(3) ***Instantaneous Forward Rates***: Using instantaneous forward rates is the most direct approach to calibrating a short rate model.

(4) ***Forward Rates***: These are the rates quoted for FRAs in the market. Forward Rates are defined as the integral of the instantaneous forward rate which makes them less variable than the instantaneous forwards rates. Using forward rates is a viable alternative to instantaneous forward rates, but a less direct approach.

(5) ***Swap Rates***: These are the rates quoted in swap contracts. It can be shown that swap rates can be expressed as a weighted average of forward rates. Therefore using swap rates will be a viable alternative to instantaneous forward rates, but a less direct approach.

Instantaneous forward rates will be used to calibrate the model. Proposition 4.2 will be employed which is a well known result in the interest rate literature.

Proposition 4.2

The expected value of the future instantaneous short rate, under the T-forward measure is equal to the related instantaneous forward rate.

$$F(t,T) = \widetilde{E}^T\left[R(T) \mid \mathcal{F}(t)\right] \tag{4.35}$$

Proof: *Utilising the two pricing formulas of (4.34)*

$$\begin{aligned}
B(t,T)\widetilde{E}^T\left[R(T) \mid \mathcal{F}(t)\right] &= \widetilde{E}[R(T)\tfrac{M(t)}{M(T)} \mid \mathcal{F}(t)] \\
\widetilde{E}^T\left[R(T) \mid \mathcal{F}(t)\right] &= \tfrac{1}{B(t,T)}\widetilde{E}[R(T)e^{-\int_t^T R(s)ds} \mid \mathcal{F}(t)] \\
\widetilde{E}^T\left[R(T) \mid \mathcal{F}(t)\right] &= -\tfrac{1}{B(t,T)}\widetilde{E}[\tfrac{\partial}{\partial T}e^{-\int_t^T R(s)ds} \mid \mathcal{F}(t)] \\
\widetilde{E}^T\left[R(T) \mid \mathcal{F}(t)\right] &= -\tfrac{1}{B(t,T)}\tfrac{\partial B(t,T)}{\partial T} \\
\widetilde{E}^T\left[R(T) \mid \mathcal{F}(t)\right] &= F(t,T)
\end{aligned} \tag{4.36}$$

Another motivation for using the instantaneous forward rates is that it is much more sensitive to changes in the short rate than zero coupon bond prices. From the earlier calculation of the expectation of R(t) under the T-Forward measure it can be seen that the instantaneous forward rate at time t under the Vasicek model is,

$$F(t,T) = R(t)e^{-k(T-t)} + \left(\theta - \tfrac{\sigma^2}{k^2}\right)\left(1 - e^{-k(T-t)}\right) + \tfrac{\sigma^2}{2k^2}e^{-k(T-t)}\left[1 - e^{-k(T-t)}\right] \tag{4.37}$$

A change of parameters was applied to the above expression to obtain an expression that is more linear in the parameters. Expressions (4.38) is substituted into (4.37) to obtain (4.39).

$$\begin{aligned}
\alpha &= e^k \\
\beta &= \tfrac{\sigma^2}{2k^2}
\end{aligned} \tag{4.38}$$

$$\begin{aligned}
\widetilde{E}^T\left[R(T) \mid \mathcal{F}(t)\right] &= R(t)\alpha^{-(T-t)} + (\theta - 2\beta)\left(1 - \alpha^{-(T-t)}\right) + \beta\alpha^{-(T-t)}\left[1 - \alpha^{-(T-t)}\right] \\
F(t,T) &= R(t)\alpha^{-(T-t)} + \left(\theta - 2\beta + \beta\alpha^{-(T-t)}\right)\left(1 - \alpha^{-(T-t)}\right)
\end{aligned} \tag{4.39}$$

Upon completion of the minimisation procedure these substitutions were inverted to obtain the parameter values.

The set of market rates that were used in the minimisation procedure was the quadratic forward rates obtained after the BESA optimisation procedure was performed. This set is denoted by $\{q_1,\ldots,q_n\}$. The model rates that was used were calculated from the above expression for the corresponding maturities and is denoted by $\{\widehat{F}(0,t_1),...,\widehat{F}(0,t_n)\}$. The starting values for the minimisation procedure were as follows,

- the starting value for r_0 was q_1,
- the starting value for σ was 0.187, which was calculated from historical data, and
- the starting value for θ was $\overline{q} = \frac{1}{n}\sum_{i=1}^{n} q_i$, the average over the quadratic forward rates.

The procedure was performed multiple times for various starting values of k. The general least squares procedure as it was performed in the Vasicek case is outlined below.

Procedure: Vasicek Least Squares Interest Rate Model Calibration

Inputs: Market values $\{q_1,\dots,q_n\}$. Initial parameter values $r_0 = q_1$, k_0=[0.01,...,1], σ_0 = 0.187 and $\theta_0 = \overline{q}$.
Outputs: Calibrated parameter values r_m, k_m, σ_m and θ_m.

(1) Calculate the fair values of the forward rates under the model $\{\widehat{F}(0,t_0),...,\widehat{F}(0,t_n)\}$. These fair values can be calculated using expectation under $\widetilde{\mathbb{P}}^T$ where $T = t_i$ as,

$$\widehat{F}(0,t_i) = \widetilde{E}^{t_i}\left[R(t_i)\right] \tag{4.40}$$

(2) Calculate the sum of the squares of the errors in the prices. A subjective weighting equal to the square root of the "maturity" of the forward rate was used to emphasise the importance of a decent fit at later maturities. This weighted sum of squares can be calculated using,

$$Q = \sum_{i=1}^{n} w_i(q_i - \widehat{F}(0,t_i))^2, \text{ where } w_i = \frac{\sqrt{t_i}}{\sum_{i=1}^{n}\sqrt{t_i}}.$$

(3) The multi-dimensional minimisation algorithm in Matlab, "fminunc" was used to find the parameter values.

This minimisation procedure was performed twice. The first time the σ parameter was kept fixed at 0.187, this is the broken line in figure 4.4. On the second execution the σ parameter was allowed to vary and the resulting model can be seen as the solid line in figure 4.4. It is clear that the Vasicek model is not nearly flexible enough to reflect the quadratic forward curve. Further problems arise for the model where σ was kept constant, since the yield curve drops to zero at short maturities. A model with more parameters will be required to reflect the subtleties of the quadratic forward curve.

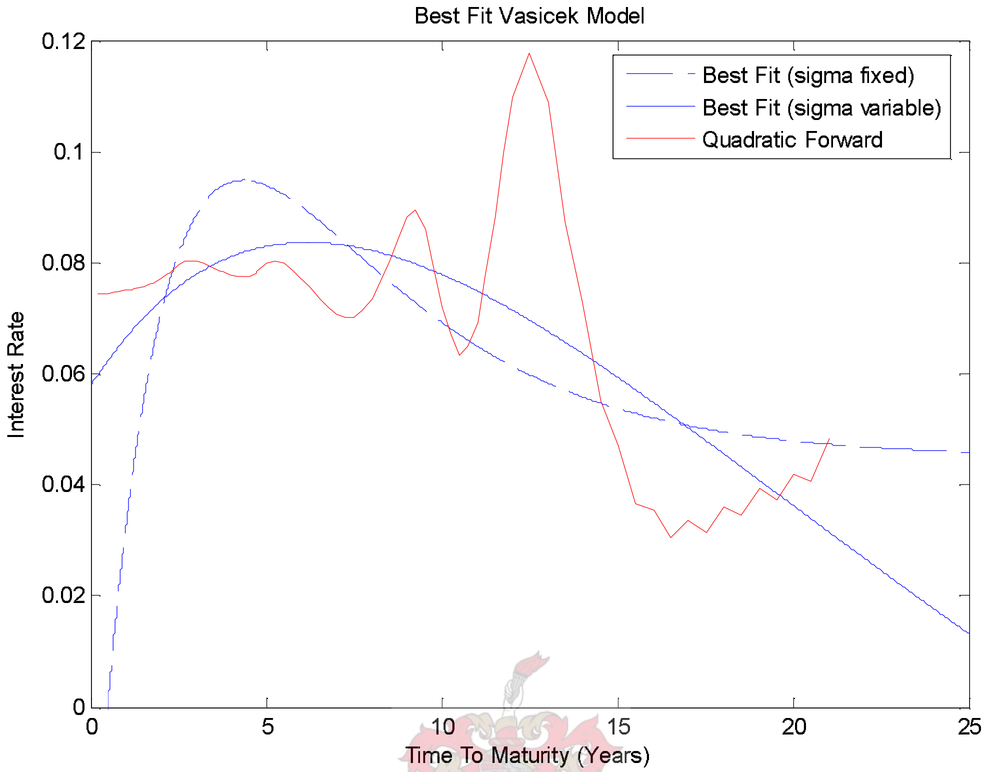

Figure 4.4: Best fit 1 - Vasicek Model with: r_0 = 0.0598 k= 0.0389, = 0.7406, = 0.0378, Best fit 2 - Vasicek Model with: r_0 = 0.0545 k= 0.0474, = 0.6815, = 0.0428,

4.4 The Dothan Model (1978)

The Dothan model is given below by (4.41).

$$dR(t) = \alpha R(t)dt + \sigma R(t)d\widetilde{W}(t) \tag{4.41}$$

This is the continuous time version of the Rendleman and Bartter model which was developed in Dothan (1978). This model is immediately familiar since it has exactly the same form as a stock price process with constant drift and volatility.

4.4.1 Bond Prices

The Dothan model is the only model in the literature that has log-normal short rates and analytical formulas for the zero coupon bond prices. The ZCB price is given by,

$$B(t,T) = \tfrac{\bar{r}^p}{\pi^2} \int_0^\infty \sin(2\sqrt{\bar{r}}\sinh(y)) \int_0^\infty f(z)\sin(yz)dzdy + \tfrac{2}{\Gamma(2p)}\bar{r}^p K_{2p}(2\sqrt{\bar{r}}) \tag{4.42}$$

where K_{2p} is the modified Bessel function of the second kind of order 2p and

$$\begin{aligned} f(z) &= z \exp\left(\frac{-\sigma^2(4p^2+z^2)(T-t)}{8}\right)\left|\Gamma(-p+i\tfrac{z}{2})\right|^2 \cosh\frac{\pi z}{2} \\ \bar{r} &= \frac{2r(t)}{\sigma^2} \\ p &= \tfrac{1}{2} - a. \end{aligned}$$

See Brigo and Mercurio (2001). Even though this expression is analytical, it is computationally intensive to calculate.

4.4.2 Distribution

The model can be integrated by applying the Ito formula to the natural logarithm of R(t), to obtain the familiar expression.

$$R(t) = R(s)\exp\left(\left(\alpha - \tfrac{1}{2}\sigma^2\right)(t-s) + \sigma\left(\widetilde{W}(t) - \widetilde{W}(s)\right)\right) \tag{4.43}$$

The Dothan short rate therefore has a log-normal distribution given by (4.44).

$$R(t) \sim \log N\left[R(s)e^{-\alpha(t-s)}, R(s)^2 e^{2\alpha(t-s)}\left(e^{\sigma^2(t-s)} - 1\right)\right] \text{ under } \widetilde{\mathbb{P}} \tag{4.44}$$

Calculating the distribution under the T-Forward measure can be done in a similar fashion to the example in the Vasicek case.

4.4.3 Positivity

Due to the log-normality of the Dothan model, the short rate will always be positive. This addresses one of the flaws of the Vasicek model.

4.4.4 Mean Reversion

The process will be mean reverting if the expectation tends towards a fixed value as t tends to infinity. This will only be the case if $a < 0$ and then the long run average of the Dothan model will be zero. This is one of the drawbacks of the Dothan model.

4.4.5 Option Prices

No analytical formula for the prices of options on zero coupon bonds has been found for this model or log-normal models in general. This is another drawback of the Dothan model which makes it unpopular amongst practitioners.

4.5 The Cox-Ingersoll-Ross Model (1985)

The CIR dynamics are given by

$$dR(t) = k(\theta - R(t))dt + \sigma\sqrt{R(t)}d\widetilde{W}(t) \tag{4.45}$$

The CIR model was developed by Cox et al (1985), and at first glace it seems very similar to the Vasicek model but, with a square root term in the volatility. It will be shown that this has a dramatic impact on the characteristics of the process. The CIR process combined the strengths of the Vasicek and Dothan models.

4.5.1 Bond Prices

The price at time t of a zero coupon bond maturing at time T under the CIR model is

$$B(t,T) = A(t,T)e^{-P(t,T)R(t)} \tag{4.46}$$

where

$$\begin{aligned} A(t,T) &= \left[\frac{2h\exp(\frac{1}{2}(k+h)(T-t))}{2h+(k+h)(\exp((T-t)h)-1)}\right]^{2k\theta/\sigma^2}, \\ P(t,T) &= \frac{2(\exp((T-t)h)-1)}{2h+(k+h)(\exp((T-t)h)-1)} \quad \text{and} \\ h &= \sqrt{k^2+2\sigma^2}. \end{aligned}$$

The proof however is quite long, see Cairns (2004), and will not be given here. This has the same analytical form as that of the Vasicek model. It is also much less arduous than the Dothan formula.

4.5.2 Distribution

The distribution of the CIR can be shown to be non-central chi-squared under both the traditional risk neutral and T-Forward measures. The interested reader can refer to Brigo and Mercurio (2001) for further details.

4.5.3 Positivity

The square root term in the volatility is responsible for keeping the interest rate positive. Consider what happens when the interest rate approaches zero. The volatility becomes negligible compared to the drift which will force the process back towards its long run average. This is the major advantage this model has over the Vasicek model.

4.5.4 Mean Reversion

Similar to the Vasicek model, the CIR model is also mean reverting with long run average equal to θ. This is an improvement on the Dothan model.

4.5.5 Option Prices

Analytic expression for ZCB options are also available, see Brigo and Mercurio (2001) for further details.

All of these features combine to make the CIR model the most realistic of the single-factor endogenous interest rate models in this chapter.

4.6 Summary

The three most prominent classical interest rate models were reviewed in this chapter. It was seen that the Vasick model is quite tractible but has the drawback of allowing negative interest rates. The CIR model solves this problem, yet in practice it is only a slight improvement over the Vasicek model and judging by the results of section 4.3, these models are not flexible enough for the pricing of exotic interest rate derivatives. Many of the calculations that were presented for the Vasicek model can be repeated for the Dothan and CIR models, with only slight adjustments in each case. For this reason they have been left out since neither of the subsequent models show a dramatic improvement or departure from the Vasicek case.

In the next chapter exogenous models will be presented which can be calibrated to fit the current term structure of interest rates exactly and will prove much more useful for valuing interest rate derivatives.

Chapter 5

EXOGENOUS INTEREST RATE MODELS

5.1 Introduction

The previous chapter introduced three endogenous interest rate models and it was shown that these models have insufficient parameters to accurately reflect the current term structure of interest rates. This deficiency results in bond and derivative prices which are inconsistent with market prices and consequently allow arbitrage opportunities. Though such models can be useful for speculative purposes to determine which securities are cheap or dear (under the assumptions of the model), they are not used by market makers. This chapter discusses exogenous interest rate models that can be calibrated to exactly fit the current term structure and to approximate the volatility structure of certain securities. Section 5.2 introduces the different approaches to stochastic yield curve modelling. Sections 5.3-5.5 discusses three exogenous models, the Hull-White model (HW), the Heath-Jarrow-Morton modelling framework (HJM) and the log-normal LIBOR forward model (LFM). The dynamics of these models are presented in table 5.1

Table 5.1: Dynamics of three exogenous interest rate models

Model	Dynamics	Strictly Positive Shot Rate	Distribution of the Short Rate	Analytical Bond Prices	Analytical Option Prices
HW	$dR(t) = k(\theta(t) - R(t))dt + \sigma d\widetilde{W}(t)$	No	Normal	Yes	Yes
HJM	$dF(t,T) = \sigma(t,T)\sigma_T(t,T)dt + \sigma_T(t,T)d\widetilde{W}(t)$	-	-	-	-
LFM	$dL(t,S,T) = L(t,S,T)\sigma(t)d\widetilde{W}^T(t)$ for $t < S$	Yes	Log-Normal	Yes	Yes

5.2 Three Approaches to Stochastic Interest Rate Modelling

In the interest rate literature three approaches have emerged for defining the dynamics of stochastic interest rates.

(1) The dynamics are defined on the short rate R(t).
(2) The dynamics are defined on the instantaneous forward rate F(t,T).
(3) The dynamics are defined on the zero coupon bond prices B(t,T).

The first case was inspected in the previous chapter and will be further analysed under the Hull-White model. The second case is the departure point of the Heath-Jarrow-Morton framework and the LIBOR market model which will be discussed in sections 5.4. and 5.5. The relation between these approaches will be discussed in section 5.5.

5.3 The Hull-White Model (1990)

Hull and White (1990) proposed an extension to the Vasicek model which allows the current term structure to be fit exactly. The model dynamics are given below,

$$dR(t) = (\theta(t) - kR(t))dt + \sigma d\widetilde{W}(t) \tag{5.1}$$

The principal difference between this model and the Vasicek model is that the theta parameter is now function of time. The functional form is assumed to be known.

5.3.1 Bond Prices

Hull and White (1990) have shown that ZCB prices can be calculated using the explicit formula,

$$\begin{aligned} B(t,T) &= A(t,T)e^{-P(t,T)R(t)}, \text{ where} \\ A(t,T) &= \frac{B^M(0,T)}{B^M(0,t)} \exp\left\{P(t,T)F^M(0,t) - \frac{\sigma^2}{4k}(1-e^{-2kt})P(t,T)^2\right\} \\ P(t,T) &= \frac{1}{k}\left[1-e^{-k(T-t)}\right] \end{aligned} \tag{4.47}$$

Here the ZCB prices $B^M(0,t)$, can be calculated directly from the descriptive yield curve $F^M(0,t)$ which was found in chapter 3.

5.3.2 Distribution

The distribution of the Hull-White model is calculated by analogy to section 4.3.2.1 where the distribution of the Vasicek model was found.

5.3.2.1 Distribution under $\widetilde{\mathbb{P}}$

Set

$$f(t,R(t)) = e^{kt}R(t)$$

Applying the Ito formula to $f(t,R(t))$ yields,

$$
\begin{aligned}
df(t,R(t)) &= f_t(t,R(t))dt + f_x(t,R(t))dR(t) + \tfrac{1}{2} f_{xx}(t,R(t))d[R,R](t) \\
&= ke^{kt}R(t)dt + e^{kt}\left(\theta(t) - kR(t)\right)dt + e^{kt}\sigma d\widetilde{W}(t) \\
&= e^{kt}\theta(t)dt + e^{kt}\sigma d\widetilde{W}(t).
\end{aligned}
$$

Integrating both sides of the above expression yields,

$$
e^{kt}R(t) = e^{ks}R(s) + \int_s^t e^{ku}\theta(u)du + \sigma\int_s^t e^{ku}d\widetilde{W}(u)\,.
$$

Finally dividing both sides by e^{kt} it follows that,

$$
R(t) = e^{-k(t-s)}R(s) + e^{-kt}\int_s^t e^{ku}\theta(u)du + \sigma e^{-kt}\int_s^t e^{ku}d\widetilde{W}(u) \tag{5.2}
$$

Just like the Vasicek model, the Hull-White short rate has a normal distribution with mean and variance given by (5.3) and (5.4) respectively.

$$
\widetilde{E}[R(t)] = e^{-k(t-s)}R(s) + e^{-kt}\int_s^t e^{ku}\theta(u)du \tag{5.3}
$$

$$
VAR[R(t)] = \tfrac{\sigma^2}{2k}\left(1 - e^{-2k(t-s)}\right) \tag{5.4}
$$

5.3.2.2 Distribution under $\widetilde{\mathbb{P}}^T$

Following exactly the same arguments as in the Vasicek case, the drift rate of the Hull-White dynamics is amended under the T-Forward measure to yield expression (5.5).

$$
dR(t) = (\theta(t) - \sigma^2 P(t,T) - kR(t))dt + \sigma d\widetilde{W}^T(t) \tag{5.5}
$$

Applying the Ito formula again to $d\left(e^{kt}R(t)\right)$ to solve the above stochastic differential equation yields,

$$
R(t) = e^{-k(t-s)}R(s) + e^{-kt}\int_s^t e^{ku}\theta(u)du - \sigma^2 e^{-kt}\int_s^t e^{ku}P(u,T)du + \sigma e^{-kt}\int_s^t e^{ku}d\widetilde{W}^T(u)\,. \tag{5.6}
$$

Under the T-Forward measure the distribution is again normal but with an adjusted drift rate. Under the T-Forward measure the mean and variance of the Hull-White short rate is given by (5.7) and (5.8) respectively.

$$
\widetilde{E}^T[R(t)] = e^{-k(t-s)}R(s) + e^{-kt}\int_s^t e^{ku}\theta(u)du - \sigma^2 e^{-kt}\int_s^t e^{ku}P(u,T)du \tag{5.7}
$$

$$
VAR[R(t)] = \tfrac{\sigma^2}{2k}(1 - e^{-2k(t-s)}) \tag{5.8}
$$

5.3.3 Mean Reversion

Similar to the Vasicek model, the Hull-White model is also mean reverting with a long term average dictated by $\theta(t)$. With careful choice of the $\theta(t)$ function, the model will fit the current term structure perfectly. The correct choice of $\theta(t)$, is the one that satisfies (5.9) according to proposition 4.2. Here the instantaneous forward rate is taken as the quadratic or quartic forward curves $\widehat{F}(t,T)$,from chapter 3.

$$\widehat{F}(t,T) = \widetilde{E}^{T}[R(T) \mid \mathscr{F}(t)] \text{ or equivalently } \widehat{F}(0,t) = \widetilde{E}^{t}[R(t)] \tag{5.9}$$

This expectation under the T-Forward measure has already been calculated in (5.7). By substituting equation (5.7) into (5.9) it follows that,

$$\widehat{F}(0,t) = e^{-k(t-s)}R(s) + e^{-kt}\int_s^t e^{ku}\theta(u)du - \sigma^2 e^{-kt}\int_s^t e^{ku}P(u,t)du\,. \tag{5.10}$$

The last term on the right hand side can be integrated to yield,

$$\begin{aligned}
\sigma^2 e^{-kt}\int_0^t e^{ku}P(u,t)du &= \sigma^2 e^{-kt}\int_0^t e^{ku}\tfrac{1}{k}(1-e^{-k(t-u)})du \\
&= \tfrac{\sigma^2}{k}e^{-kt}\int_0^t e^{ku} - e^{-k(t-2u)}du \\
&= \tfrac{\sigma^2}{k}e^{-kt}\left(\tfrac{1}{k}e^{ku}\Big|_0^t - \tfrac{1}{2k}e^{-k(t-2u)}\Big|_0^t\right) \\
&= \tfrac{\sigma^2}{k^2}e^{-kt}\left(e^{kt} - 1 - \tfrac{1}{2}e^{kt} + \tfrac{1}{2}e^{-kt}\right) \\
&= \tfrac{\sigma^2}{2k^2} - \tfrac{\sigma^2}{k^2}e^{-kt} + \tfrac{\sigma^2}{2k^2}e^{-2kt}.
\end{aligned}$$

Substituting this into (5.10) yields,

$$\widehat{F}(0,t) = e^{-k(t-s)}R(0) + e^{-kt}\int_0^t e^{ku}\theta(u)du - \tfrac{\sigma^2}{2k^2} + \tfrac{\sigma^2}{k^2}e^{-kt} - \tfrac{\sigma^2}{2k^2}e^{-2kt}\,. \tag{5.11}$$

Remember that an expression is sought for $\theta(t)$ in terms of the market instantaneous forward curve $\widehat{F}(0,t)$. Differentiate the above expression with respect to t to obtain,

$$\widehat{F}_t(0,t) = -ke^{-k(t-s)}R(0) - ke^{-kt}\int_0^t e^{ku}\theta(u)du + e^{-kt}e^{kt}\theta(t) - \tfrac{\sigma^2}{k}e^{-kt} + \tfrac{\sigma^2}{k}e^{-2kt}\,. \tag{5.12}$$

Using expression (5.11) and (5.12) it can be seen that,

$$\widehat{F}_t(0,t) + k\widehat{F}(0,t) = \theta(t) - \tfrac{\sigma^2}{2k}(1 - e^{-2kt}).$$

Finally, making $\theta(t)$ the object of the equation, expression (5.13) is obtained.

$$\theta(t) = \widehat{F}_t(0,t) + k\widehat{F}(0,t) + \tfrac{\sigma^2}{2k}(1 - e^{-2kt}) \tag{5.13}$$

For this choice of $\theta(t)$, the term structure implied by the ZCBs under this model will be exactly the current term structure defined by the descriptive yield curve $\widehat{F}(0,t)$. It is interesting to contemplate the dynamics under this choice of $\theta(t)$. The final term is generally quite small and will be ignored for now. The dynamics can be written as,

$$dR(t) = \left(\widehat{F}_t(0,t) + k\left(\widehat{F}(0,t) - R(t)\right)\right)dt + \sigma d\widetilde{W}(t) \tag{5.14}$$

On average the drift will be the slope of the market instantaneous forward curve, adjusted upward if the model short rate is below the market forward rate and adjusted downward if the model short rate is above the market forward rate. Therefore when it strays from the market curve, it reverts at a speed determined by k.

5.3.4 Positivity

Due to the fact that the short rate has a normal distribution, negative rates are attainable. In fact the probability of attaining a negative interest rate at any future time t is given by,

$$\widetilde{P}\left\{R(t) < 0\right\} = \Phi\left(-\frac{\widetilde{E}[R(t)]}{\sqrt{VAR[R(t)]}}\right) = \Phi\left(-\frac{e^{-k(t-s)}R(0) + e^{-kt}\displaystyle\int_s^t e^{ku}\theta(u)du}{\sqrt{\frac{\sigma^2}{2k}\left(1 - e^{-2k(t-s)}\right)}}\right) \tag{5.15}$$

5.3.5 Option Prices

Explicit formulas are also available for European put and call options on ZCBs, see Hull (2003). There are also explicit formulas available for the pricing of caps, floors and swaptions, see Brigo and Mercurio (2001).

5.3.6 Realistic Shapes

By definition, any exogenous model gives a perfect fit to any shape of the current yield curve. The problem of realistic shapes therefore becomes naught and it is a realistic volatility structure that becomes the focus of attention. There is a vast amount of literature covering the topic of volatility structure of yield curve models and interest rate derivatives. Unfortunately such detail is beyond the scope of this dissertation and the interested reader can refer to Amin. and Morton (1994), Andersen and Andreasen (2000) and Brace et al (1997).

5.3.7 Parameter Estimation

Compared to the Vasicek model the calibration of the Hull-White model poses somewhat of a problem. In the Vasicek case the model was calibrated to the instantaneous forward rates given by the descriptive yield curve of chapter 3. This cannot be done with the Hull-White model since the descriptive yield curve is entirely incorporated into the model, that is, the instantaneous forward rates under the model will always match those of the market, independent of the choice of k and σ. Consequently a different set of calibration securities has to be employed. The choice of calibration securities will depend on the purpose of the model for instance, if the goal is swaption pricing, the model should be calibrated to the prices of swaptions in the market. Again it is important that the model be calibrated to prices that are liquid and do not contain any risk premia. Unfortunately, since there are no liquidly traded interest rate derivatives in the South African market that can serve as calibration securities, none of the conventional calibration techniques can be used. This will be a good area for future research and would be of specific interest to institutions that wish to start trading interest rate derivatives.

5.4 The Heath-Jarrow-Morton Framework (1992)

In Heath, Jarrow and Morton (1992) a framework was presented that takes a different approach to the modelling of interest rates than the previous models. The HJM-framework models the evolution of the instantaneous forward curve instead of the spot curve. The HJM dynamics are given by the following,

$$\begin{aligned} dF(t,T) &= \alpha(t,T)dt + v(t,T)d\widetilde{W}(t) \\ F(0,t) &= \widehat{F}(0,t) \end{aligned} \tag{5.16}$$

where $\widehat{F}(0,t)$ is a boundary condition at time zero to the stochastic dynamics and represents the descriptive yield curve at time zero. An advantage of this formulation is that the current market term structure is incorporated into the model at construction. There is a restriction that must be placed on the drift parameter, $\alpha(t,T)$, for the dynamics to be arbitrage free, this is the so called "Heath-Jarrow-Morton No Arbitrage Condition". Armed with this condition, the HJM-framework provides a completely general specification of all arbitrage-free interest rate models.

5.4.1 The Heath-Jarrow-Morton No-Arbitrage Condition

To see why the dynamics of (5.16) are not arbitrage free, the dynamics are deduced in an arbitrage-free setting in two steps,

(1) Define a completely general ZCB price process in an arbitrage-free risk neutral world,

(2) Find the general form of the instantaneous forward rate process in this setting.

Firstly the zero coupon bond dynamics will be constructed in an arbitrage free setting. Under the assumption that the market is arbitrage free, Theorem C.4 shows that there exists an $\mathscr{F}(t)$-measurable volatility process $\sigma(t,T)$ such that,

$$dB(t,T) = R(t)B(t,T)dt + \sigma(t,T)B(t,T)d\widetilde{W}(t). \tag{5.17}$$

In (5.17) the term R(t) represents the $\mathscr{F}(t)$-measurable risk free, short rate process. This expression of the ZCB dynamics is equivalent to that of the general stock price process. By applying Ito's formula the log-dynamics can be written as,

$$d\ln B(t,T) = \left(R(t) - \tfrac{1}{2}\sigma(t,T)^2\right)dt + \sigma(t,T)d\widetilde{W}(t). \tag{5.18}$$

Note that, since all default-free zero coupon bonds have a value of 1 at maturity, their volatility must be zero at maturity, therefore,

$$\sigma(t,t) = 0\,. \tag{5.19}$$

Now that a completely general expression of the ZCB dynamics was found in an arbitrage free world, a contingent expression for the instantaneous forward rate process is sought. An expression will first be found for the continuously compounded forward rate $R(t,T,T+\delta t)$ of Definition 2.3.5 and then the dynamics of the instantaneous forward rate $F(t,T)$ of Definition 2.3.6 will be found using the limiting argument $F(t,T) = \lim_{\delta t \to 0} L(t,T,T+\delta t) = \lim_{\delta t \to 0} R(t,T,T+\delta t)$.

From table 2.2 the following relation between the forward rate and the ZCB prices can be seen,

$$e^{(\delta t)R(t,T,T+\delta t)} = \frac{B(t,T)}{B(t,T+\delta t)}\,. \tag{5.20}$$

By writing the continuously compounded forward rate on the right hand side it is clear that,

$$R(t,T,T+\delta t)=\frac{\ln(B(t,T))-\ln(B(t,T+\delta t))}{\delta t}. \tag{5.21}$$

Therefore the differential form of the continuously compounded forward rate process will be,

$$dR(t,T,T+\delta t)=\frac{d\ln(B(t,T))-d\ln(B(t,T+\delta t))}{\delta t}. \tag{5.22}$$

After substituting these log-dynamics of the bond price dynamics, (5.18), into (5.22), the dynamics of the forward rate can be written as,

$$\begin{aligned}
dR(t,T,T+\delta t) &= \frac{d\ln(B(t,T))-d\ln(B(t,T+\delta t))}{\delta t}\\
&= \frac{\sigma(t,T+\delta t)^2-\sigma(t,T)^2}{2\delta t}dt+\frac{\sigma(t,T)-\sigma(t,T+\delta t)}{\delta t}d\widetilde{W}(t)\\
&= \left(\frac{\sigma(t,T+\delta t)+\sigma(t,T)}{2}\right)\left(\frac{\sigma(t,T)-\sigma(t,T+\delta t)}{\delta t}\right)dt+\frac{\sigma(t,T)-\sigma(t,T+\delta t)}{\delta t}d\widetilde{W}(t)
\end{aligned} \tag{5.23}$$

The instantaneous forward rate, $F(t,T)$, is obtained by letting $\delta t\to 0$. Each of the coefficients of expression (5.23) has the following limiting values,

$$\begin{aligned}
&\lim_{\delta t\to 0}\frac{\sigma(t,T+\delta t)+\sigma(t,T)}{2}=\sigma(t,T),\\
&\lim_{\delta t\to 0}\frac{\sigma(t,T)-\sigma(t,T+\delta t)}{\delta t}=-\frac{\partial}{\partial T}\sigma(t,T)=-\sigma_T(t,T),
\end{aligned}$$

where the subscript denotes the corresponding partial derivative. Therefore the dynamics of the instantaneous forward curve can finally be written as,

$$dF(t,T)=\sigma(t,T)\sigma_T(t,T)dt+\sigma_T(t,T)d\widetilde{W}(t). \tag{5.24}$$

Writing the volatility process as an integral and remembering that $\sigma(t,t)=0$ it can be seen that,

$$\begin{aligned}
\sigma(t,T)-\sigma(t,t) &= \int_t^T \sigma_T(t,u)du \text{ and}\\
\sigma(t,T) &= \int_t^T \sigma_T(t,u)du.
\end{aligned} \tag{5.25}$$

Therefore the instantaneous dynamics can be written as,

$$dF(t,T)=\sigma_T(t,T)\int_t^T \sigma_T(t,u)du\,dt+\sigma_T(t,T)d\widetilde{W}(t). \tag{5.26}$$

Finally, setting $v(t,T) = \sigma_T(t,T)$, the no-arbitrage condition on the drift becomes clear. If zero coupon bond prices are represented by a no-arbitrage process (5.17), the instantaneous forward rate dynamics must be,

$$dF(t,T) = \left(v(t,T) \int_t^T v(t,u)du \right) dt + v(t,T) d\widetilde{W}(t)\,. \tag{5.27}$$

This no-arbitrage condition is formally stated below.

The HJM No-Arbitrage Condition

Let F(t,T) be a term structure model for the instantaneous forward rate of all maturities in (0,T] and driven by a single Brownian motion with dynamics given by,

$$\begin{aligned} dF(t,T) &= \alpha(t,T)dt + v(t,T)d\widetilde{W}(t) \\ F(0,t) &= \widehat{F}(0,t)\,. \end{aligned} \tag{5.28}$$

This model does not admit arbitrage if $\alpha(t,T) = v(t,T) \int_t^T v(t,u)du\,.$

This is the fundamental result of Heath, Jarrow and Morton which can be extended to a market with multiple sources of uncertainty. It implies that, given knowledge of the volatility process, the drift is completely determined in an arbitrage free setting. In other words, the volatility is the only quantity that needs to be calibrated for the entire process to be known. Furthermore, since the volatility does not change with a change in measure, it can be approximated from historical data.

It has been shown that all arbitrage free models can be represented by the HJM-framework, therefore a few questions are now of immediate interest,

(1) For an arbitrary volatility structure $v(t,T)$, what are the dynamics on the short rate?

(2) What choices of $v(t,T)$ correspond to each of the earlier short rate models?

(3) What choices of $v(t,T)$ result in a Markov process for the short rate?

Each of these questions will be answered in turn.

5.4.1.1 For an arbitrary volatility structure $v(t,T)$, what are the dynamics of the short rate?

The first question can be answered by noting that,

$$\begin{aligned} R(t) &= F(t,t) \\ &= F(0,t) + \int_o^t dF(\tau,t). \end{aligned}$$

Substitute expression (5.26) in the above to obtain,

$$R(t) = F(0,t) + \int_o^t \sigma(\tau,t)\sigma_T(\tau,t)d\tau + \int_o^t \sigma_T(\tau,t)d\widetilde{W}(\tau)\,. \tag{5.29}$$

The dynamics of the short rate can be found by differentiating the above expression with respect to t to obtain the following expression,

$$dR(t) = d\big(F(0,t)\big) + d\left(\int_o^t \sigma(\tau,t)\sigma_T(\tau,t)d\tau\right) + d\left(\int_o^t \sigma_T(\tau,t)d\widetilde{W}(\tau)\right). \tag{5.30}$$

Each of the three terms on the right hand side of (5.30) is differentiated separately and listed as (5.31), (5,32) and (5.33) respectively.

$$dF(0,t) = F_t(0,t)dt \tag{5.31}$$

$$\begin{aligned} d\left(\int_o^t \sigma(\tau,t)\sigma_T(\tau,t)d\tau\right) &= \left(\sigma(t,t)\sigma_T(t,t) + \int_o^t \tfrac{\partial}{\partial\tau}\big(\sigma(\tau,t)\sigma_T(\tau,t)\big)\,d\tau\right)dt \\ &= \left(\sigma(t,t)\sigma_T(t,t) + \int_o^t \sigma(\tau,t)\sigma_{TT}(\tau,t) + \sigma_T(\tau,t)^2\,d\tau\right)dt \\ &= \left(\int_o^t \sigma(\tau,t)\sigma_{TT}(\tau,t) + \sigma_T(\tau,t)^2\,d\tau\right)dt \text{ since } \sigma(t,t) = 0 \end{aligned} \tag{5.32}$$

$$d\left(\int_o^t \sigma_T(\tau,t)d\widetilde{W}(\tau)\right) = \left(\int_o^t \sigma_{TT}(\tau,t)d\widetilde{W}(\tau)\right)dt + \sigma_T(\tau,t)\big|_{\tau=t}\,d\widetilde{W}(t) \tag{5.33}$$

Substituting (5.31), (5,32) and (5.33) back into (5.30) yields the dynamics of the short rate,

$$\begin{aligned} dR(t) = &\left[F_t(0,t) + \int_o^t \sigma(\tau,t)\sigma_{TT}(\tau,t) + \sigma_T(\tau,t)^2\,d\tau + \int_o^t \sigma_{TT}(\tau,t)d\widetilde{W}(\tau)\right]dt \\ &+ \sigma_T(\tau,t)\big|_{\tau=t}\,d\widetilde{W}(t). \end{aligned}$$

These are the dynamics of an general short rate process in an arbitrage free market. Each of the drift parameters can be interpreted to some degree.

- $F_t(0,t)$ is the slope of the time zero descriptive forward curve.

- $\int_o^t \sigma(\tau,t)\sigma_{TT}(\tau,t)+\sigma_T(\tau,t)^2 d\tau$ though a specific interpretation is difficult this quantity depends on the history of the volatility process.

- $\int_o^t \sigma_{TT}(\tau,t)d\widetilde{W}(\tau)$ this process depends on the history of both the volatility and the Brownian motion processes.

From this it is clear that, in general, the drift is dependent on the entire history of R(t) and not just on the value of R(t) in the previous instant, in other words, this process is non-Markovian for most choices of the volatility.

5.4.1.2 What choices of $\upsilon(t,T)$ correspond to each of the earlier short rate models?

The answer to the second question requires a more comprehensive overview of the topic an will not be given here, but a discussion on this subject can be found in Brigo and Mercurio (2001).

5.4.1.3 What choices of $\upsilon(t,T)$ result in a Markov process of the short rate?

The third question is of practical importance. When pricing derivatives, the interest rates are often modelled using a binomial tree. In the case where the interest rate process is Markov, a recombining binomial tree can be built which has a size of O(N), where N is the number of time steps to maturity. When the interest rate process is not Markovian, the tree usually grows at a rate of O(2^N) and accurate evaluation becomes infeasible for long dated instruments. Hull and White (1993) has shown that the short rate process will be Markovian if and only if $\upsilon(t,T)$ has the functional form,

$$\upsilon(t,T) = x(t)\left[y(T) - y(t)\right]$$

5.4.2 Other Characteristics

The distribution, mean reversion, positivity, along with all the other characteristics that define an interest rate model, will be determined by the choice of the volatility function $\upsilon(t,T)$. Certain choices of $\upsilon(t,T)$ will yield models like that of Hull and White that have a normal distribution and are mean reverting while other choices might yield models that have a fat-tailed distribution of the short rate. The fact that the HJM-framework and the no-arbitrage condition gives structure to the form of all no-arbitrage interest rate models make this approach especially powerful and of great academic value. The question that needs to be answered, and which is currently the source of much research, is, what choice of $\upsilon(t,T)$ constitutes a very realistic model whilst remaining computationally feasible.

5.4.3 Model Calibration

All conventional methods of calibrating a model based on the HJM framework requires a range of prices for interest rate derivatives that trade liquidly in the market. The volatility function $\upsilon(t,T)$, has to be found which most accurately reflects the prices observed in the market. Suppose caps, $C(t,R_k,T_1,T_2)$ are trading in the market at time t. The cap $C(t,R_k,T_1,T_2)$, fixes the interest rate at R_k for the period following T_1 up to time T_2. Suppose these prices are available for various values of $T_1 < T_2$ up to a maturity of T_n. The prices of these instruments can be calculated under the model as $\widehat{C}(t,R_k,T_1,T_2)$, given some initial choice of the volatility function. It might be the case that analytical expressions for the cap prices are not available, then tree based methods, numerical techniques or simulation have to be used to value the derivatives. The volatility function can be parameterised and the parameters can then be found by following the general least squares minimisation procedure outlined in chapter 4. Let $\underline{\theta}$ be the parameter that defines some volatility function $\upsilon(t,T,\underline{\theta})$. Then the optimisation problem is that of finding the minimum for (5.34).

$$\min_{\underline{\theta}\in\Omega} \sum_{T_1<T_2\le T_n} \left[\widehat{C}(t,R_k,T_1,T_2) - C(t,R_k,T_1,T_2)\right]^2 \tag{5.34}$$

5.5 The LIBOR Market Models (1997)

The LIBOR market models will not be discussed in depth since a detailed overview will require a thorough review of stochastic calculus and interest rate theory which is beyond the scope of this study but a discussion of the reasons for their popularity will be given.

In order to address the problem of finding a practically appealing choice of the HJM volatility, Brace et al (1997) proposed what has become known as the LIBOR market models. In the same way that the HJM-model has become very popular amongst academics due to its attractive theoretical properties, the market models have become very popular amongst practitioners due to their analytical tractability. The problem with the HJM-framework is that it can be difficult to calibrate because the instantaneous forward curve is not directly observable in the market. Quantities that are observable are the LIBOR (and JIBAR) rates. These are not instantaneous rates but rather the simply compounded forward rates, L(t, S, T) from Definition 2.3.4. Furthermore, the most prolific interest rate derivatives, namely caps, floors and swaptions, are defined on these forward rates. It is also the convention of the market to value these derivatives using Black's formula which assumes that these forward and swap rates are log-normally distributed. With the introduction of market models they were the only interest rate models which were compatible with either Black's formula for caps or Black's formula for swaptions. Traditionally, when these formulas were used by practitioners they were based on inexact assumptions of the interest rate distribution but under the market models Black's formula is based on rigorous interest rate dynamics. These market models are not without problems however. In the formulation of these models a critical inconsistency has arisen. When the model is formulated such that simply compounded forward rates are log-normal, the corresponding swap rates cannot be log-normal as well. Yet Black's formula is used to value both swaptions, which implies log-normal swap rates, and caps which require log-normal forward rates. This sometimes leads practitioners to build two separate incarnations of the market model,

- The log-normal forward-LIBOR model (LFM) that prices caps with Black's formula, and
- The log-normal forward-swap model (LSM) that prices swaption with Black's formula.

Unfortunately this double standard leads to arbitrage opportunities between the models.

5.5.1 Formulation of the Log-Normal Forward-LIBOR Model (LFM)

Consider the expression for the simply compounded forward rate from Definition 2.3.4,

$$L(t,S,T) = \tfrac{1}{\tau(S,T)} \tfrac{B(t,S)-B(t,T)}{B(t,T)}\,.$$

Rewrite this as,

$$B(t,T)L(t,S,T) = \tfrac{1}{\tau(S,T)}\big(B(t,S)-B(t,T)\big)\,.$$

The right hand side is clearly a tradable security, it is the value of a portfolio consisting of a short position in $\frac{1}{\tau(S,T)}$ units of $B(t,T)$ and a long position in $\frac{1}{\tau(S,T)}$ units of $B(t,S)$. Under the assumption that the market is complete and arbitrage free, the right hand side has to be a martingale under the traditional risk neutral measure. Equivalently, if the T-maturity ZCB price is taken as the numéraire, this asset must be a martingale under the T-Forward measure. Therefore,

$$\frac{B(t,T)L(t,S,T)}{B(t,T)} = L(t,S,T)\,.$$

The simply compounded forward rate must be a martingale under the T-forward measure and its dynamics can be written as,

$$dL(t,S,T) = L(t,S,T)\sigma(t)d\widetilde{W}^{T}(t) \quad \text{for} \quad t < S\,.$$

This is also a geometric Brownian motion process which implies that the simply compounded forward rate has a log-normal distribution under the T-Forward measure.

Caps provide insurance against the risk that the interest rate will rise above a certain level for a floating rate note. Let $\{T_1, T_2, ..., T_n\}$ be the reset dates on the underlying floating rate note and let T_0 be the settlement date. Let $\tau(T_{i-1}, T_i)$ be the corresponding tenor. The simply compounded forward rates can now be written as,

$$dL(t,T_{i-1},T_i) = L(t,T_{i-1},T_i)\sigma(t)d\widetilde{W}^{T_i}(t) \quad \text{for} \quad t < T_{i-1}, i = 1,...,n\,.$$

A cap consists of a sequence of caplets where each caplet fixes the simply compounded forward rate over one tenor period. A caplet can therefore be seen as a European call option on the simply compounded forward rate with some strike rate L_k. Therefore according to Definition 2.3.8, the payoff of a caplet with maturity T_{i+1}, can be written as,

$$\tau(T_{i-1},T_i)\max\left(L(T_{i-1},T_{i-1},T_i) - L_k, 0\right).$$

Therefore according to (C.8) in Definition C.3, the present value of this payoff can be calculated from,

$$P(0,T_{i+1})\widetilde{E}^{T_i}\left[\tau(T_{i-1},T_i)\max\left(L(T_{i-1},T_{i-1},T_i) - L_k, 0\right)\right].$$

Since the simply compounded forward has the log-normal distribution, proposition 4.1 can again be employed which results in the standard Black option pricing formula given below.

$$\tau(T_{i-1},T_i)P(0,T_{i+1})\left[L(0,T_{i-1},T_i)\Phi(d_1) - L_k\Phi(d_2)\right],$$

where $\sigma_{L(T_{i-1},T_{i-1},T_i)}$ is the volatility of $\ln L(T_{i-1},T_{i-1},T_i)$ and

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{L(0,T_{i-1},T_i)}{L_k}\right) + \frac{1}{2}T_i\sigma^2_{L(T_{i-1},T_{i-1},T_i)}}{\sqrt{T_i}\sigma_{L(T_{i-1},T_{i-1},T_i)}}, \\ d_2 &= d_1 - \sqrt{T_i}\sigma_{L(T_{i-1},T_{i-1},T_i)}. \end{aligned}$$

A cap is priced by valuing the corresponding series of caplets. From the above formulation the practical appeal of the model is obvious since it prices the most common interest rate derivatives using the same formula as the market.

The corresponding formulation of the LSM will not be given since it follows a similar approach.

5.6 Summary

This chapter introduced the most prominent exogenous models that are currently used for derivative pricing. The models address the inadequacies of the models in chapter 4 by allowing an entire descriptive yield curve to be used as a calibration input. It was shown that the HJM framework provides a general way of presenting all no-arbitrage interest rate models and it is therefore of great academic value. It was also shown how market conventions are reflected by the LIBOR market models, which make them of great practical value.

The next chapter will reference some researchers that are attempting to amalgamate these approaches.

C h a p t e r   6

CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

This dissertation set out to give an overview of the current state of interest rate model theory with specific attention to its applicability to the South African market and derivative pricing.

Chapter 3 introduced descriptive yield curve models. Throughout the following chapters the importance of a proper descriptive model was witnessed, specifically for the purpose of calibrating stochastic yield curve models. It was shown that the BESA methodology are quite general and very flexible since any fixed income market instruments can be used together to extract a yield curve.

Chapter 4 gave an overview of three famous single factor endogenous short rate models. It was shown that these models are analytically tractable but of little practical value for derivative pricing since they provide a poor fit to the current yield curve and therefore allow arbitrage opportunities.

Chapter 5 reviewed three exogenous models that are currently used by practitioners for derivative pricing. It was shown that the Hull-White model extended the Vasicek model in order to perfectly fit the current term structure. The drawback of the Hull-White model was that it allowed negative interest rate and only had two free parameters for the calibration of the volatility structure of market instruments. The HJM-framework was shown to be the primary focus of current academic activity due to its characterisation of the entire family of arbitrage free models. The LIBOR market models were shown to be primarily used by practitioners due to the fact that they reflect market standards for derivative pricing such as Black's formula for caps and swaptions. It was also shown that the LIBOR market models have the drawback that they are not compatible and cannot hold simultaneously.

The current state of the interest rate derivatives market in South Africa is such that stochastic interest rate models are implemented by very few institutions. Through discussions with SANLAM and CADIZ it was learned that such models can only be found in large banking institutions since they are the main market markets in South Africa. Even so it will be unlikely that they employ any of the sophisticated models in it will be extremely difficult, if not impossible to calibrate the sophisticated models in any meaningful way. However as soon as these derivative products become more popular so also will their calibration become more viable.

6.2 Current and Future Research

The models that have been discussed are the most popular and widely implemented amongst both academics and practitioners but scores of variants have been proposed.

For researchers interested in the development of the short rate under the real world measure a three-factor model has been developed by Balduzzi et al (1996).

Another model which is gaining much attention is the Positive Interest model by Flesaker and Hughston (1996). The Positive Interest model develops a process for the bank account numéraire M(t). The result is a model that always provides positive interest rates, has an elegant formula for the ZCB prices and prices caps and swaptions analytically using formulas that are similar but different to Black's formula. The model works especially well in cases where yield curves of different currencies are involved.

Research continues, in search of a model that marries the theoretical qualities of the HJM-framework with the practical requirements of market participants.

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QUADRATIC FORWARD

The quadratic forward curve gives a discrete approximation of the instantaneous forward curve implied by a set of discrete points on the zero curve. The quadratic forward will be denoted by,

$$\underline{q}^T = \{q_1, q_2, \ldots, q_n\}.$$

Define the zero rates at times $\{t_0, t_1, \ldots, t_n\}$ to which the forward curve must be fit as,

$$\underline{r}^T = \{r_0, r_1, \ldots, r_n\}.$$

Define the following spline,

$$\widehat{R}(0,t) = a_j + b_j t + c_j t^2 \text{ for } t_{j-1} \leq t \leq t_j$$

The spline consists of a series of parabolas between each two points t_{j-1} and t_j on the zero curve. The $\{a_j\}, \{b_j\}$ and $\{c_j\}$ parameters of the spline has to be solved by fitting the spline to the zero rates $\underline{r}$. Each of these parabolas will be fitted to three successive point on $\underline{r}$. The following three simultaneous equations are obtained for each parabola,

$$r_{j-1} = a_j + b_j t_{j-1} + c_j t_{j-1}^2 \text{ for } j = 1, \ldots, n,$$

$$r_j = a_j + b_j t_j + c_j t_j^2 \text{ for } j = 1, \ldots, n \text{ and}$$

$$r_{j+1} = a_j + b_j t_{j+1} + c_j t_{j+1}^2 \text{ for } j = 1, \ldots, n.$$

From these the values of c_j and b_j can be found for j=1…n as,

$$c_j = \frac{\Delta r_{j+1}\Delta t_j - \Delta r_j \Delta t_{j+1}}{\Delta t_{j+1}\Delta t_j(\Delta t_j + \Delta t_{j+1})} \text{ and}$$

$$b_j = \frac{\Delta r_j}{\Delta t_j} - c_j(t_j + t_{j-1})$$

where $\Delta r_j = r_j - r_{j-1}$.

Remark: Note that the instantaneous forward rate is expressed in terms of $\frac{\partial}{\partial t} R(0,t)\Big|_{t_j} = b_j - 2c_j t_j$ and it is therfore unnecessary to calculate a_j.

From remark (2.13). under the definition of the instantaneous forward curve. it can be seen that,

$$F(0,t_j) = R(0,t_j) + t_j \frac{\partial}{\partial t} R(0,t)\Big|_{t_j}.$$

From this relation the quadratic forward curve can be found using,

$$\begin{aligned} q_j &= r_j + t_j \frac{\partial}{\partial t} \widehat{R}(0,t)\Big|_{t_j} \\ &= r_j + t_j(b_j - 2c_j t_j). \end{aligned}$$

A p p e n d i x B

BOOTSTRAPPING

Bootstrapping is the most rudimentary method of finding the yield curve implied by the market prices of a set of fixed income securities. Bootstrapped curves are often discontinuous and if continuous, they are discontinuous in their derivatives. The bootstrapping procedure recommended by the BESA methodology follows the series of steps outlined below.

Step 0: Sort the instrument set I in order of increasing time to maturity. No instruments with equal time to maturity are allowed. Let their times to maturity be,

$$t_1, \ldots, t_m \,. \tag{B.1}$$

Step 1: Calculate the required, constant rate of interest such that the future value of the first instrument at its maturity is zero via (B.2).

$$FV_0(t_1) = \sum_{j=1}^{n_1} c_{0j} e^{f_1(t_1 - t_j)} = 0 \tag{B.2}$$

where $FV_i(t_j)$ is the future value of all cashflows of security i at time t_j.

Remark: Each instrument has a cashflow at t_0 equal to minus its market value.

Step 2: For each subsequent instrument, calculate their future value on the maturity date of the previous instrument and find the value of f_j that will make it zero.

$$FV_i(t_{i-1}) = \sum_{j:t_{i-1} > t_j} c_{ij} e^{f_j(t_{i-1} - t_j)} + \sum_{j:t_j > t_{i-1}} c_{ij} e^{-f_j(t_j - t_{i-1})} = 0 \tag{B.3}$$

The interest rates in the first term of B.3 are known. There are two alternatives for modelling the forward rates in the second term. Let the interest rate in the second term be constant over the period $[t_{i-1}, t_i]$ and find the rate that solves B.3. This gives a discontinuous forward curve. Let the forward rate over the interval $[t_{i-1}, t_i]$ be,

$$f_t = f_{i-1} + s_j(t - t_{i-1}) \,. \tag{B.4}$$

Then solve B.3 for the slope s_j. This gives a continuous forward curve with discontinuous derivatives.

Appendix C

STOCHASTIC CALCULUS PRIMER

Some fundamental theorems and results are listed below which are used throughout chapters 4-7.

Finding the Traditional Risk Neutral Measure

Consider an arbitrage free market with d sources of uncertainty represented by a d-dimensional Brownian motion vector $\underline{W}(t)$ and the m stocks,

$$dS_i(t) = \mu_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t) \text{ for } i = 1,\ldots,m\,.$$

The corresponding discounted stock price processes are,

$$dD(t)S_i(t) = (\,\mu_i(t) - R(t)\,)\,D(t)S_i(t)dt + D(t)S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t) \text{ for } i = 1,\ldots,m\,.$$

It can be shown that the "market price of risk equations" can always be solved for $\underline{\theta}(t)$, in a market that does not admit arbitrage. These equations are given by,

$$\mu_i(t) - R(t) = \sum_{j=1}^{d}\sigma_{ij}(t)\theta_j(t)dt \text{ for } i = 1,\ldots,m\,. \tag{C.1}$$

The discounted stock price processes can then be rewritten as,

$$dD(t)S_i(t) = D(t)S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)\left(\theta_j(t)dt + dW_j(t)\right) \text{ for } i = 1,\ldots,m\,. \tag{C.2}$$

A change of measure is now performed using Theorem C.2 below and the traditional risk neutral measure is given by,

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega) \;\; \forall \; A \in \mathscr{F}(T), \tag{C.3}$$

where,

$$Z = \exp\!\left(-\int_0^T \underline{\theta}(u)\cdot d\underline{W}(u) - \tfrac{1}{2}\int_0^T \left\|\underline{\theta}(u)\right\|^2 du\right).$$

The discounted stock price processes are now martingales under $\widetilde{\mathbb{P}}$,

$$
\begin{aligned}
dD(t)S_i(t) &= D(t)S_i(t)\sum\nolimits_{j=1}^{d}\sigma_{ij}(t)d\widetilde{W}_j(t) \text{ for } i = 1,\ldots,m \text{ where} \\
\widetilde{W}_i(t) &= W_i(t) + \int_0^t \theta_i(u)du \text{ for } i = 1,\ldots,d
\end{aligned} \tag{C.4}
$$

Theorem C.1 : Fundamental Theorem of Asset Pricing

Let V(t) be the price process of an asset in a complete market. Then the discounted asset price process is a martingale under the risk neutral measure.

$$
\frac{V(t)}{M(t)} = \widetilde{E}\left[\frac{V(T)}{M(T)} \mid \mathscr{F}(t)\right] \tag{C.5}
$$

Remark C.1: Note that the differential form of the discounted stock price process is,

$$
dD(t)S(t) = D(t)S(t)\underline{\sigma}(t) \cdot d\underline{\widetilde{W}}(t) \tag{C.6}
$$

Proof: See Shreve (2004) chapter 5.

Theorem C.2 : Multi-dimensional Girsanov Theorem

Let $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space and* $\underline{W}(t)$ *be a d-dimensional standard Brownian motion over* $0 \le t \le T$ *under* $\mathbb{P}$. *Also let* $\{\mathscr{F}(t)\}_{0 \le t \le T}$ *be the filtration generated by this process and* $\{\underline{\theta}(t)\}_{0 \le t \le T}$ *be an adapted d-dimensional process. Define,*

$$
\begin{aligned}
Z(t) &= \prod\nolimits_{j=1}^{d} \exp\left(-\int_0^t \theta_j(u)dW_j(u) - \tfrac{1}{2}\int_0^t \theta_j^2(u)du\right) \\
&= \exp\left(-\int_0^t \underline{\theta}(u)\cdot d\underline{W}(u) - \tfrac{1}{2}\int_0^t \left\|\underline{\theta}(u)\right\|^2 du\right), \\
Z &= Z(T), \\
\widetilde{\mathbb{P}}(A) &= \int_A Z(\omega)d\mathbb{P}(\omega) \ \ \forall\ A \in \mathscr{F}(T) \text{ and} \\
\widetilde{W}_i(t) &= W_i(t) + \int_0^t \theta_i(u)du \text{ for } i = 1,\ldots,d.
\end{aligned}
$$

Subject to the restriction that

$$
E\left[\int_0^T \left\|\underline{\theta}(u)\right\|^2 Z^2(u)du\right] < \infty
$$

$\underline{\widetilde{W}}(t)$ *will be a d-dimensional standard Brownian motion over* $0 \le t \le T$ *under* $\widetilde{\mathbb{P}}$.

Proof: See Shreve (2004) chapter 5.

Theorem C.3 : Multi-dimensional Martingale Representation Theorem

Let $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space and* $\underline{W}(t)$ *be a d-dimensional standard Brownian motion over* $0 \leq t \leq T$ *under* $\mathbb{P}$*. Also let* $\{\mathscr{F}(t)\}_{0 \leq t \leq T}$ *be the filtration generated by this process,* $\{\underline{\theta}(t)\}_{0 \leq t \leq T}$ *be an adapted d-dimensional process,* $M(t)$ *be a martingale over* $0 \leq t \leq T$ *under* $\mathbb{P}$ *and* $\widetilde{M}(t)$ *be a martingale over* $0 \leq t \leq T$ *under* $\widetilde{\mathbb{P}}$*. Then there exists a d-dimensional adapted process* $\underline{\Gamma}(t)$ *such that*

$$M(t) = M(0) + \int_0^t \underline{\Gamma}(u) \cdot d\underline{W}(u)$$

Similarly, there exist a d-dimensional adapted process $\underline{\widetilde{\Gamma}}(t)$ *such that*

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \underline{\widetilde{\Gamma}}(u) \cdot d\underline{\widetilde{W}}(u)\,.$$

Proof: See Shreve (2004) chapter 5.

Theorem C.4 : Stochastic Representation Theorem

Let $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space and N(t) be a strictly positive non-dividend paying asset price process (either a stock or derivative). In a market with d sources of uncertainty represented by a d-dimensional Brownian motion vector* $\underline{\widetilde{W}}(t)$*, there exists an adapted volatility vector* $\underline{\upsilon}(t)$*, such that*

$$\begin{aligned}
dD(t)N(t) &= D(t)N(t)\underline{\upsilon}(t) \cdot d\underline{\widetilde{W}}(t) \quad \Leftrightarrow \\
dN(t) &= R(t)N(t)dt + N(t)\underline{\upsilon}(t) \cdot d\underline{\widetilde{W}}(t) \quad \Leftrightarrow \\
D(t)N(t) &= N(0)\exp\Big(\int_0^t \underline{\upsilon}(u) \cdot d\underline{\widetilde{W}}(u) - \tfrac{1}{2}\int_0^t \|\underline{\upsilon}(u)\|^2\, du\Big) \quad \Leftrightarrow \\
D(t)N(t) &= N(0)\exp\Big(\int_0^t \underline{\upsilon}(u) \cdot d\underline{\widetilde{W}}(u) + \int_0^t R(u) - \tfrac{1}{2}\|\underline{\upsilon}(u)\|^2\, du\Big)
\end{aligned}$$

Proof: See Shreve (2004) chapter 6.

Theorem C.5 : Change of Measure

Let S(t) and N(t) be asset price processes with respective volatility vectors $\underline{\sigma}(t)$ *and* $\underline{\upsilon}(t)$*. Such that,*

$$\begin{aligned}
dD(t)S(t) &= D(t)S(t)\underline{\sigma}(t) \cdot d\underline{\widetilde{W}}(t) \equiv d\Big(\tfrac{S(t)}{M(t)}\Big) = \Big(\tfrac{S(t)}{M(t)}\Big)\underline{\sigma}(t) \cdot d\underline{\widetilde{W}}(t) \\
dD(t)N(t) &= D(t)N(t)\underline{\upsilon}(t) \cdot d\underline{\widetilde{W}}(t) \equiv d\Big(\tfrac{N(t)}{M(t)}\Big) = \Big(\tfrac{N(t)}{M(t)}\Big)\underline{\sigma}(t) \cdot d\underline{\widetilde{W}}(t)
\end{aligned}$$

A change of measure such that N(t) is the numéraire instead of M(t). Define,

$$\begin{aligned} Z(t) &= \tfrac{D(t)N(t)}{N(0)} = \exp\Big(\int_0^t \underline{\upsilon}(u)\cdot d\underline{\widetilde{W}}(u) - \tfrac{1}{2}\int_0^t \|\underline{\upsilon}(u)\|^2\,du\Big) \\ Z &= Z(T) \\ \widetilde{\mathbb{P}}^{(N)}(A) &= \int_A Z(\omega)d\widetilde{\mathbb{P}}(\omega) \quad \forall\ A \in \mathcal{F}(T) \\ \widetilde{W}_i^{(N)}(t) &= \widetilde{W}_i(t) - \int_0^t \upsilon_i(u)du \text{ for } i = 1,..,d \end{aligned}$$

Then

$$d\Big(\tfrac{S(t)}{N(t)}\Big) = \Big(\tfrac{S(t)}{N(t)}\Big)\big(\underline{\sigma}(t) - \underline{\upsilon}(t)\big)\cdot d\underline{\widetilde{W}}^N(t) \tag{C.7}$$

Proof: See Shreve (2004) chapter 9.

Remark C.2: Note that this theorem applies the multi-dimensional Girsanov theorem with $\underline{\theta}(t) = -\underline{\upsilon}(t)$.

Also note that, when discounting is done by the money market process M(t), the discounted stock price process $\frac{S(t)}{M(t)}$, is a martingale under the traditional risk neutral measure $\widetilde{\mathbb{P}}$. But when "discounting" is done by the asset N(t), the discounted stock price process $\frac{S(t)}{N(t)}$, is a martingale under measure $\widetilde{\mathbb{P}}^{(N)}$.

Definition C.3 : T-Forward Measure

Apply the change of measure result with N(t) = B(t, T). Then,

$$\begin{aligned} Z(t) &= \tfrac{D(t)B(t,T)}{B(0,T)} \\ Z &= Z(T) = \tfrac{D(T)}{B(0,T)}, \text{ since } B(T,T) = 1 \\ \widetilde{\mathbb{P}}^T(A) &= \tfrac{1}{B(0,T)}\int_A D(T)(\omega)d\widetilde{\mathbb{P}}(\omega) \quad \forall\ A \in \mathcal{F}(T) \end{aligned}$$

Remark C.3: The T-forward measure often simplifies calculations when pricing derivatives in a stochastic interest rate environment. Consider the asset price under the traditional risk neutral measure (when the numéraire N(t) = M(t)),

$$V(t) = M(t)\widetilde{E}\Big[\tfrac{V(T)}{M(T)} \mid \mathcal{F}(t)\Big]$$

In order to calculate this expectation, the interaction between the asset price process and the interest rate process must be modelled. Compare this with asset price under the T-forward measure (when the numéraire N(t) = B(t,T)),

$$V(t) = B(t,T)\widetilde{E}^{T}\left[V(T) \mid \mathscr{F}(t)\right] \tag{C.8}$$

Discounting now occurs outside the expectation and the interaction between the processes does not need to be modelled.

Theorem C.6 : Feynman-Kac

Consider the stochastic differential equation,

$$dX(t) = \beta(t, X(t))dt + \gamma(t, X(t))dW(t)$$

Define,

$g(t,x) = E[h(X(T) \mid X(t) = x],$ for $t \in [0,T],$ and h(·) borrel-measurable

then g(t,x) satisfies the partial differential equation,

$$g_t(t,x) + \beta(t,x)g_x(t,x) + \tfrac{1}{2}\gamma^2(t,x)g_{xx}(t,x) = 0\,,$$

with terminal condition,

$$g(T,x) = h(x),\ \forall\ x \in \mathbb{R}\,.$$

Proof: See Shreve (2004) chapter 6.

Theorem C.7 : The Change of Drift implied by a change of Numéraire

Assume that the two numéraire N and U evolve under $\widetilde{\mathbb{P}}^{(U)}$ *according to,*

$$dN(t) = (\ldots)dt + \underline{\sigma}_N(t)^T C d\underline{\widetilde{W}}^{(U)}(t)$$
$$dU(t) = (\ldots)dt + \underline{\sigma}_U(t)^T C d\underline{\widetilde{W}}^{(U)}(t)$$

where $\underline{\sigma}^N(t)$ *and* $\underline{\sigma}^U(t)$ *are n x 1 volatility vectors and* $\underline{\widetilde{W}}^{(U)}(t)$ *is an n-dimensional standard Brownian motion and* C *is the covariance matrix for the Brownian motions such that* $CC' = \Sigma$. *Then the drift of a process* $X(t)$ *under numéraire U is,*

$$\mu_X^U(t) = \mu_X^N(t) - \underline{\sigma}_X(t)^T \Sigma\left(\tfrac{\sigma_N(t)}{N(t)} - \tfrac{\sigma_U(t)}{U(t)}\right)$$

Proof: See Brigo and Mercurio (2001) chapter 2.

Appendix D

MATLAB SOURCE CODE

D.1 Source code for the BEASSA Yield Curve minimisation procedure

LoadBonds is a helper function that was written to return a bond variable "I" which contains the bond information for the bonds in the GOVI index on 5 December 2005. The input parameters are a valuation date and the marketValues on that date. If no parameters are specified, the default values for 5 December is used.

```
function result = loadBonds(marketValue, valuationDate)

    if (marketValue == 0)
        marketValue = [108.34, 125.67, 107.47, 143.47, 107.04, 100.09, 134.54];
        valuationDate = '12-Dec-2005';
    end

    I(1) = createBond('R194', '29-Feb-2008', 2, 10,   3, 1, '01-Apr-2001',
         marketValue(1), valuationDate);
    I(2) = createBond('R153', '31-Aug-2010', 2, 13,   3, 1, '22-Jun-1989',
         marketValue(2), valuationDate);
    I(3) = createBond('R201', '21-Dec-2014', 2, 8.75, 3, 1, '27-May-2003',
         marketValue(3), valuationDate);
    I(4) = createBond('R157', '15-Sep-2015', 2, 13.5, 3, 1, '18-Jan-1991',
         marketValue(4), valuationDate);
    I(5) = createBond('R203', '15-Sep-2017', 2, 8.25, 3, 1, '07-May-2004',
         marketValue(5), valuationDate);
    I(6) = createBond('R204', '21-Dec-2018', 2, 8,    3, 1, '11-Aug-2004',
         marketValue(6), valuationDate);
    I(7) = createBond('R186', '21-Dec-2026', 2, 10.5, 3, 1, '01-Apr-1998',
         marketValue(7), valuationDate);

    result = I;
end
```

Createbond accepts a list of parameters which define a bond then calculates coupon payment dates taking into account day count conventions and book closed dates. A cashflow vector is also generated to be used in the construction of the BEASSA cashflow matrix. Finally createBond returns a single variable which defines the bond.

```
function result = createBond(name, maturity, period, coupon, ...
    basis, endMonthRule, issueDate, marketValue, valuationDate)
    % Examples,
    % createBond('R153', '31-Aug-2010', 2, 13, 3, 1,
    % '22-Jun-1989', 'Book Close Day 1', 'Book Close Month 1',
    % 'Book Close Day 2', 'Book Close Month 2');

    I.name = name;
    I.type = 'bond';
    I.settle = datenum(valuationDate) + 3;
    I.valuationDate = datenum(valuationDate);
    I.maturity = datenum(maturity);
    I.period = period;
    I.coupon = coupon;
    I.basis = basis;
    I.endMonthRule = endMonthRule;
    I.issueDate = datenum(issueDate);
    I.t = cfdates(I.settle, maturity, period, basis, ...
        endMonthRule, I.settle, 'LastCouponDate', maturity);

    %Check if first cashflow should be removed because bond is
    %evaluated after its book close date
    if I.t(1) < I.valuationDate + 10
        if length(I.t) == 1
            error('ERROR: Bond has no cashflows!');
        end
        I.t(1) = [];
    end

    if (I.t(1) ~= I.settle)
       I.t = [I.settle I.t];
    end
    n = length(I.t);
    I.c = [-marketValue coupon/2*ones(1,n-2) coupon/2+100];

    result = I;
end
```

```
The following sequence of commands initialises the variable "in" which will be used as a
parameter for the BEASSA minimisation routine.  This variable contains all the
parameters values that are currently recommended by BEASSA for the minimisation routine.

in.bootstrapType = 'FF';
in.runType       = 'PF';
in.verose        = 1;
in.iterLimit     = 100000;
in.eta           = 10;
in.firstStep     = 250;
in.laterStep     = 250;
in.alpha         = 100;
in.kappa         = 2.5;
in.gamma0        = 250;
in.gamman        = 5;
in.K             = [1 1];
in.lambda        = 1000;
in.omega         = 0.1;
in.nu            = 2;
in.I             = I;



optDecency i.e. Optimise Decency is the function which executes the BEASSA minimisation
procedure.  The all parameters are contained in the variable "in".  The procedure
returns a set of zero coupon interest rates which corresponds to the cashflow dates for
the cashflow matrix defined by the bond set "in.I".

% input.bootstrapType = 'FF' / 'SL'              : Flat Forward / Straight Line Forward
% input.runType       = 'BD' / 'PF' / 'BDw'      : Best Decency / Perfect Fit
% input.verose        = 0 / 1                    : Display optimisation information
% input.iterLimit     = [0..500000] (50000)      : Maximum number of iterations
% input.eta           = [0..20]     (10)         : Maximum sub-path number
% input.firstStep     = [1..5000]   (250)        : Initial # steps down a path
% input.laterStep     = [1..5000]   (250)        : Consequent # steps down a path
% input.alpha         = [0..inf]    (100)        : Blur Weighting Number
% input.kappa         = [1..5]      (2.5)        : Blur Tail SDs
% input.gamma0        = [1..50000]  (250)        : Rand Tolerance for Sub-Path 0
% input.gamman        = [1..gamm0]  (5)          : Rand Tolerance for Sub-Path n
% input.K(1)          = [0..inf]    (1)          : Global Smoothness Importance Factor
% input.K(2)          = [0..inf]    (1)          : Local  Smoothness Importance Factor
% input.K(3)          = [0..inf]    (1)          : Goodness Importance Factor
% input.lambda        = [100..inf]  (1000)       : Weight Limit
% input.omega         = [0..1]      (0.1)        : Weight Threshold Trigger
% input.nu            = [1..15]     (2)          : Weights: Significant Digits
% input.I             = Instruments

function [z w fvalNew decencies] = optDecency(in)
    global path S SS A blur alpha kappa;

    [C t B]    = buildCashflowMatrix(in.I);
    [t z f q] = bootstrap(in.I, 0);

    blur      = inf;
    A         = 1;
    S         = z;
    SS        = z.*z;
    alpha     = in.alpha;
    kappa     = in.kappa;

    x0        = exp(-t.*z)*B;
    x         = x0;
    path      = x0;
    w         = pfWeights(B, t, x0, in.K, in.lambda, in.nu);
    fvalOld   = inf;
    flag      = 0;
    iterCount = 0;
    iters     = in.firstStep;
    wCycle    = 0;
    wOld      = w*2;
```

```
    tol       = in.gamma0*(in.gamman/in.gamma0).^((0:in.eta)./in.eta);
    subPath   = sum(blur < tol);

    decencies = zeros(1,4000);
    loops     = 1;

    while (subPath <= in.eta)
        disp(['Starting sub-path # ',int2str(subPath),'...']);
        options = optimset('Display','final','LargeScale','off','MaxIter',iters, ...
            'TolFun',10e-10,'TolX',10e-10,'MaxFunEvals', inf,'OutputFcn',@calcBlur);
        [x, fvalNew] = lsqnonlin('minPFDecency', x, [], [], options, B, t, w);
        decencies(loops) = fvalNew;
        loops = loops + 1;
        iterCount = iterCount + iters;
        if (iterCount > in.iterLimit)
            break;
        end

        if sum(abs(wOld - wNew)) < 0.01
            wCycle = 1;
        end

        if ~(in.omega == 1)
           dw    = max(abs((wNew./sum(wNew))./(w./sum(w))-1)) > in.omega;
        else
            wOld = w; wNew = w;
        end

        if (fvalOld == fvalNew)
            if (w == wNew)
                flag = 1;
                break;
            end
            dw = 1;
        end

        if dw && ~wCycle
            wOld = w; w = wNew; subPath = 0; disp(['Starting a new path with weights '
                ,num2str(w,3),'...']);
            blur = inf; x = x0; path = x0; fvalOld = inf; iters = in.firstStep;
            subPath = sum(blur < tol);
        elseif isequal(w,wNew)
            disp('Performing additional iterations...');
            in.firstStep = ceil(in.firstStep*1.1);
            in.laterStep = ceil(in.laterStep*1.1);
            iters = ceil(in.laterStep/10);
        else
            %blur = calcBlur(B, t, path, in.alpha, in.kappa);
            disp(['Blur = ',num2str(blur,7)]);
            fvalOld = fvalNew; iters = in.laterStep;
            subPath = sum(blur < tol);
        end
    end

    if flag
        disp('Path has converged to machine precition');
    elseif (iterCount > in.iterLimit)
        disp('No convergence after maximum allowed iterations');
    end

    z = x2z(B, t, x);

end
```

This function generates the BEASSA cashflow matrix from the bond set parameter "I".

```
function [C t B] = buildCashflowMatrix(I)

    m = length(I);

    t = I(1).t;
    for i = 2:m
        t  = union(t, I(i).t);
    end

    n = length(t);
    C = zeros(m,n);
    for i = 1:m
        if strcmp(I(i).type,'bond')
            temp = nonzeros((1:n).*ismember(t, I(i).t));
            C(i,temp) = I(i).c;
        else
            error('ERROR: Unknown instrument type');
        end
    end

    [U W V] = svd(C);
    cols = nnz(W)+1;
    B = V(:,cols:end);

end
```

This procedure follows the process described in Appendix B to infer the interest rates from the bond set parameter "I".

```
% Bootstrap interest rates from instrument set I according to the BEASSA
% methodology.
% Method = 0 selects the Flat Forward method
% Method = 1 selects the Straight-Line Forward method
function [dates zrate frate qrate] = bootstrap(I, method)

    m = length(I);
    f = zeros(1,m);
    t = zeros(1,m);
    s = zeros(1,m);

    for i = 1:m
        t(i) = (I(i).maturity-I(i).settle)/365;
    end

    if ( length(unique(t)) < m )
        error('ERROR: All instruments in I must have different maturities');
    end

    % Bootstrap the forward rates
    for i = 1:m
        init = 0.08;
        cfdates = (I(i).t - I(i).settle)./365;
        cashflows = I(i).c;
        options = optimset(optimset('fzero'),'Display','off');
        s(i) = fzero('futureValue', init, options, cashflows, cfdates, t, f, method);
        f(i) = s(i);
        if (i > 1) && (method == 1) %Straight line forward method selected
            f(i) = f(i-1) + s(i)*(t(i)-t(i-1));
        end
    end

    [C dates] = buildCashflowMatrix(I);
    dates = (dates - I(1).settle)/365;
    n = length(dates);
    frate = zeros(1,n);
    zrate = zeros(1,n);

    % Now we must find the zero rates implied by the bootstrapped forwards
    frate(1) = f(1);
    zrate(1) = f(1);
    i = 2;
    for j = 1:m
        while (i <= n) && (dates(i) <= t(j))
            frate(i) = f(j);
            if (j > 1) && (method == 1)
                frate(i) = f(j-1) + s(j)*(dates(i)-t(j-1));
            end
            % The calculation below is inaccurate if method = 1
            zrate(i) = frate(i) + (zrate(i-1) - frate(i)) * dates(i-1)/dates(i);
            i = i+1;
        end
    end

    qrate = quadratic(dates,frate);
end
```

This function returns the quadratic yield curve values corresponding to a zero coupon yield curve using the procedure defined in Appendix A. The zero coupon yield curve is defined by the input vectors
z - containing the zero coupon values and
t - containing the corresponding times of the z - rates.

```
% t = [t0 t1 t2 ... tn] & z = [z0 z1 z2 ... zn]
function result = quadratic(t, z)

    % Note that z(1) is z0 & z(end) is zn
    dt = t(2:end)-t(1:(end-1));
    dz = z(2:end)-z(1:(end-1));
    % Now dz(1) = z1-z0 = z(2)-z(1)
    n = length(dz);

    c = zeros(1,n-2);
    b = zeros(1,n-2);
    q = zeros(1,n);
    q(1) = z(1) + t(1)*dz(2)/dt(2);
    for j = 2:(n-1)
        c(j) = (dz(j+1)*dt(j)-dz(j)*dt(j+1))/(dt(j)*dt(j+1)*(dt(j)+dt(j+1)));
        b(j) = dz(j)/dt(j)-c(j)*(t(j)+t(j-1));
        q(j) = z(j) + t(j)*(b(j)+2*c(j)*t(j));
    end
    q(n) = z(n) + t(n)*dz(n)/dt(n);

    result = q;
end
```

This function is used by "optDecency" to update the blur value after each iteration of the minimisation procedure.

```
function stop = calcBlur(x, optimValues, state, B, t, w)
    global S SS A blur alpha kappa;

    if (alpha > 0)
        a = 1-1/alpha;
    else
        a = 1;
    end

    z  = x2z(B,t,x);
    A  = a*A  + 1;
    S  = a*S  + z;
    SS = a*SS + z.*z;

    mu  = S./A;
    sig = ((SS-mu.*S)./(A-1)).^0.5;

    dz = kappa*sig + abs(z - mu);
    dR = 1000000.*t.*dz;
    blur = max(dR);

    stop = 0;

end
```

minPFDecency i.e. Minimise Perfect Fit Decency is helper function used by "optDecency" during the minimisation process. "minPFDecency" returns the Decency corresponding to the parameter values. optDecency traverses the solution space defined by the vector x by repeatedly calling "minPFDecency", each time with an intelligent adjustment of x in order to find the x which minimises the Decency.

```
function d = minPFDecency(x, B, t, w)

    z = x2z(B, t, x);
    q = quadratic(t, z);
    [d ex] = pfDecency(B, t, x, w);

end
```

pfDecency i.e. Perfect Fit Decency calculates the Decency defined by its parameters.

```
function [in ex] = pfDecency(B, t, x, w)

    z = x2z(B, t, x);
    q = quadratic(t, z);
    [ST s2 S2 s1 S1] = smoothness(t(2:end), q);
    in = w*[s1 s2]'/sum(w);
    ex = ST;

end
```

smoothness calculates the smoothness of the quadratic yield curve given by the parameter vectors t and q. Where
t - time vector
q - interest rates corresponding to the times in t

```
function [total locIn locEx gloIn gloEx] = smoothness(t, q)

    [locIn locEx] = localSmoothness(t, q);
    [gloIn gloEx] = globalSmoothness(q);
    total  = ((locEx^2+gloEx^2)/2)^0.5;

end
```

localSmoothness calculates the local smoothness associated with a quadratic yield curve defined by the parameters vectors t and q. Where
t - time vector
q - interest rates corresponding to the times in t

```
function [in ex] = localSmoothness(t, q)

    n = length(q);
    dt = t(2:end)-t(1:(end-1));
    dq = q(2:end)-q(1:(end-1));
    dqdt = dq./dt;
    ddqdt = dqdt(2:end)-dqdt(1:(end-1));
    in = sum(ddqdt.^2);
    A = (t(end)-t(2))/(n-1);
    ex = 10000*A*(in/(n-2))^0.5;

end
```

globalSmoothness calculates the global smoothness associated with the quatratic yield rates given by q.

```
function [in ex] = globalSmoothness(q)

    n = length(q);
    dq = q(2:end)-q(1:(end-1));
    in = sum(dq.^2);
    ex = 10000*(in/(n-1))^0.5;

end
```

pfWeights i.e. Perfect Fit Weights calculates new weights for the global and local smoothness for "optDecency". The new set of weights are defined by the parameters.

```
function w = pfWeights(B, t, x, K, lambda, nu)

    z = x2z(B, t, x);
    q = quadratic(t, z);
    [ST s2 S2 s1 S1] = smoothness(t(2:end), q);

    if (K(1) == 0) && (K(2) == 0)
        error('ERROR: At least one of the importance factors for smoothness must be non-
              zero');
    end

    if (s1 == 0)
        if (K(1) == 0)
            w(1) = 0;
            w(2) = K(2);
        else
            w(1) = 1;
            w(2) = K(2)/K(1);
        end
    elseif (s2 == 0)
        if (K(1) == 0)
            w(1) = 0;
            w(2) = K(2);
        else
            w(1) = 1;
            w(2) = lambda*K(2)/K(1);
        end
    elseif (K(1) == 0)
        w(1) = 0;
        w(2) = K(2)/s2;
    elseif (K(2) == 0)
        w(1) = K(1)/s1;
        w(2) = 0;
    else
        w(1) = K(1)/s1;
        w(2) = min(K(2)/s2,lambda*K(2)/s1); %MAKE SURE THIS IS RIGHT
    end

    w = round(10^nu*w./min(w))/10^nu;
end
```

x2z is a helper function which transforms a vector of x values into the corresponding set of zero coupon interest rates as defined by expressions (3.11) - (3.13).

```
function z = x2z(B, t, x)

    D = -log(B*x')';
    z = D(2:end)./t(2:end); % Note that t(0) = 0 : Divition by zero not allowed
    z = [z(1) z];           % z(0) should not affect smoothness or goodness

end
```

D.2 Source code for the construction of the Quartic yield curve

buildQuarticMat i.e. Build Quartic Matrix contructs the linear system that must be solved to generate the quartic yield curve from the parameters
t - time vector
z - spot interest rates corresponding to the times in t
q - quartic interest rates corresponding to the times in t

```
function [A b] = buildQuarticMat(t, z, q)

    n = length(t)-1;
    m1 = zeros(n,5*n);
    m2 = zeros(n-1,5*n);
    m3 = zeros(n-1,5*n);
    m4 = zeros(n-1,5*n);
    m5 = zeros(n-1,5*n);
    m6 = zeros(4,5*n);
    t2 = t.^2;
    t3 = t.^3;
    t4 = t.^4;
    t5 = t.^5;

    % Quartic spline must fit zero rates z
    k = [t' t2'./2 t3'./3 t4'./4 t5'./5];
    k = k(2:end,:)-k(1:(end-1),:);
    for i = 1:n
       m1(i,(1+5*(i-1)):(5*i)) = k(i,:);
    end

    % Quartic spline must be continuous
    k = [ones(n+1,1) t' t2' t3' t4'];
    for i = 1:(n-1)
        m2(i,(1+5*(i-1)):(5+(5*i))) = [-k(i+1,:) k(i+1,:)];
    end

    % Quartic spline must have continuous first derivatives
    k = [ones(n+1,1) 2*t' 3*t2' 4*t3'];
    for i = 1:(n-1)
        m3(i,(1+5*(i-1)):(5+(5*i))) = [0 -k(i+1,:) 0 k(i+1,:)];
    end

    % Quartic spline must have continuous second derivatives
    k = [ones(n+1,1) 3*t' 6*t2'];
    for i = 1:(n-1)
        m4(i,(1+5*(i-1)):(5+(5*i))) = [0 0 -k(i+1,:) 0 0 k(i+1,:)];
    end

    % Quartic spline must have continuous third derivatives
    k = [ones(n+1,1) 4*t'];
    for i = 1:(n-1)
        m5(i,(1+5*(i-1)):(5+(5*i))) = [0 0 0 -k(i+1,:) 0 0 0 k(i+1,:)];
    end

    m6(1,3)            = 1;
    m6(2,3:4)          = [3 4*t(2)];
    m6(3,(end-2):end) = [1 3*t(end) 6*t2(end)];
    m6(4,(end-9):end) = [-1 -t(end-1) -t2(end-1) -t3(end-1) -t4(end-1) 1 t(end) t2(end)
                          t3(end) t4(end)];

    A = [m1;m2;m3;m4;m5;m6];

    r = (z.*t)';
    b = zeros(n*5,1);
    b(1:n,1) = r(2:end)-r(1:(end-1));
    b(end,1) = q(end)-q(end-1);

end
```

D.3 Source code for fitting of a Vasicek yield curve to input data

minVasicek i.e. Minimise Vasicek is used to fit a Vasicek model to market data, given as set of initial parameter values. The initial parameter values are given by r0, a, b and sig. The market data that was used were the quadratic yield curve calculates using the procedure of chapter 3. The quadratic yield curve is defined by the parameters t and data where,
t - time vector
data - quartic interest rates corresponding to the times in t

```
function [r0 a b sig SSE] = minVasicek(r0, a, b, sig, t, data)

    x = [r0 a b];
    options = optimset('MaxFunEvals',100000,'MaxIter',100000,'Display','final'
                      ,'LargeScale','off', 'TolFun',10e-10,'TolX',10e-10);
    [x, SSE] = fminunc('LSVasicek', x, options, sig, t, data);

    r0  = x(1);
    a   = x(2);
    b   = x(3);

end
```

LSVasicek i.e. Least Squares Vasicek returns the sum of squares of the differences of the market data given by "data" and the Vasicek yield curve defined by the parameter vector "params". "LSVasicek" is called repeatedly by minVasicek, each time with an intelligent adjustment to the parameter vector "params" in order to find those parameters for which LSVasicek returns the smallest sum of squares.

```
function Q = LSVasicek(params, sig, T, data)

    r0  = params(1);
    a   = params(2);
    b   = params(3);

    et = exp(-a*T);
    sa = sig^2/a^2;
    modelData = r0*et + (b - sa)*(1-et) + 0.5*sa*(et-et.^2);

    T = T.^(0.5);
    T = T./sum(T);
    Q = 1000000*sum((T.*(data - modelData)).^2);

end
```

D.4 Source code for functions plotting spot and forward Vasicek yield curves

```
plotVasicek plots the spot and/or forward yield curves of the Vasicek model defined by
the user defined parameters.  The code below will draw various spot curves for different
sigma parameters.  With slight adjustments either spot or forward curves can be drawn
for various values of any of the model parameters.
function plotVasicek(r0, a, b, sig, n, T);

    sig = 0.07:0.005:0.12;

    for i = 1:length(sig)
        temp = spot_Vasicek(r0, a, b, sig(i), n, T);
        plot(temp.T,temp.spot,'b');
        %temp = forward_Vasicek(r0, a, b, sig(i), n, T);
        %plot(temp.T,temp.forward,'r');
        hold on;
    end

end



spot_Vasicek returns the spot yield curve values associates with the times in "T" for
the Vasicek model defined by the parameters r0, a, b and sig.
function result = spot_Vasicek(r0, a, b, sig, n, T)
    result.type = 'Vasicek';
    result.r0 = r0;
    result.a = a;
    result.b = b;
    result.sig = sig;
    result.dt = 1/n;
    result.n = n;
    result.T = (1/n):(1/n):T;

    T = (1/n):(1/n):T;
    k = a;
    t = b;
    B = 1/k*(1-exp(-k*T));
    X = (t-sig^2/2/k/k).*(B-T)-sig^2/4/k.*B.^2;
    A = exp(X);
    result.bonds = A.*exp(-B*r0);
    result.spot  = -log(result.bonds)./T;

    plot(T,result.spot);

end



forward_Vasicek returns the forward yield curve values associates with the times in "T"
for the Vasicek model defined by the parameters r0, a, b and sig.
function result = forward_Vasicek(r0, a, b, sig, n, T)
    result.type = 'Vasicek';
    result.r0 = r0;
    result.a = a;
    result.b = b;
    result.sig = sig;
    result.dt = 1/n;
    result.n = n;
    result.T = (1/n):(1/n):T;

    T = (1/n):(1/n):T;
    et = exp(-a*T);
    sa = sig^2/a^2;
    result.forward = r0*et + (b - sa)*(1-et) + 0.5*sa*(et-et.^2);

    plot(T,result.forward);

end
```

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