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Cardinal Spline Wavelet Decomposition based on Quasi-Interpolation and Local Projection

by

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Declaration

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Veronica Sitsofe Ahiati Date

Summary

Wavelet decomposition techniques have grown over the last two decades into a powerful tool in signal analysis. Similarly, spline functions have enjoyed a sustained high popularity in the approximation of data.

In this thesis, we study the cardinal B-spline wavelet construction procedure based on quasi-interpolation and local linear projection, before specialising to the cubic B-spline on a bounded interval.

First, we present some fundamental results on cardinal B-splines, which are piecewise polynomials with uniformly spaced breakpoints at the dyadic points $\mathbb{Z}/2^r$, for $r \in \mathbb{Z}$. We start our wavelet decomposition method with a quasi-interpolation operator $\mathcal{Q}_{m,r}$ mapping, for every integer r, real-valued functions on $\mathbb{R}$ into $\mathcal{S}_m^r$ where $\mathcal{S}_m^r$ is the space of cardinal splines of order m, such that the polynomial reproduction property $\mathcal{Q}_{m,r}p = p$, $p \in \pi_{m-1}$, $r \in \mathbb{Z}$ is satisfied. We then give the explicit construction of $\mathcal{Q}_{m,r}$.

We next introduce, in Chapter 3, a local linear projection operator sequence $\{\mathcal{P}_{m,r} : r \in \mathbb{Z}\}$, with $\mathcal{P}_{m,r} : \mathcal{S}_m^{r+1} \to \mathcal{S}_m^r$, $r \in \mathbb{Z}$, in terms of a Laurent polynomial Λ_m solution of minimally length which satisfies a certain Bezout identity based on the refinement mask symbol A_m, which we give explicitly.

With such a linear projection operator sequence, we define, in Chapter 4, the error space sequence $W_m^r = \{f - \mathcal{P}_{m,r}f : f \in \mathcal{S}_m^{r+1}\}$. We then show by solving a certain Bezout identity that there exists a finitely supported function $\psi_m \in \mathcal{S}_m^1$ such that, for every $r \in \mathbb{Z}$, the integer shift sequence $\{\psi_m(2 \cdot -j)\}$ spans the linear space W_m^r. According to our definition, we then call ψ_m the m^{th} order cardinal B-spline wavelet. The wavelet decomposition algorithm based on the quasi-interpolation operator $\mathcal{Q}_{m,r}$, the local linear projection operator $\mathcal{P}_{m,r}$, and the wavelet ψ_m, is then based on finite sequences, and is shown to possess, for a given signal f, the essential property of yielding relatively small wavelet coefficients in regions where the support interval of $\psi_m(2^r \cdot -j)$ overlaps with a C^m-smooth region of f.

Finally, in Chapter 5, we explicitly construct minimally supported cubic B-spline wavelets on a bounded interval $[0, n]$. We also develop a corresponding explicit decomposition algorithm for a signal f on a bounded interval.

Throughout Chapters 2 to 5, numerical examples are provided to graphically illustrate the theoretical results.

Opsomming

Dekomposisietegnieke gebaseer op golfies ("wavelets") het oor die afgelope twee dekades ontwikkel in 'n kragtige stuk gereedskap in seinanalise. Soortgelyk geniet latfunksies 'n voortgesette hoë gewildheid in die benadering van data.

In hierdie tesis bestudeer ons kardinale B-latfunksies, wat stuksgewyse polinome is met uniform gespasieerde knooppunte by die diadiese punte $\mathbb{Z}/2^r$, vir $r \in \mathbb{Z}$. Ons begin ons golfie dekomposisiemetode met 'n kwasi-interpolasie operator $\mathcal{Q}_{m,r}$ wat, vir elke heelgetal r, reëlwaardige funksies op $\mathbb{R}$ in $\mathcal{S}_m^r$ afbeeld, waar $\mathcal{S}_m^r$ die ruimte van kardinale latfunksies van orde m aandui, sodat die polinoom reproduksie eienskap $\mathcal{Q}_{m,r}p = p,\ p \in \pi_{m-1}, r \in \mathbb{Z}$, bevredig word. Ons gee dan die eksplisiete konstruksie van $\mathcal{Q}_{m,r}$.

Vervolgens, in Hoofstuk 3, stel ons bekend 'n lokale lineêre projeksie operator ry $\{\mathcal{P}_{m,r} : r \in \mathbb{Z}\}$, met $\mathcal{P}_{m,r} : \mathcal{S}_m^{r+1} \to \mathcal{S}_m^r, r \in \mathbb{Z}$, in terme van 'n Laurent polinoom Λ_m oplossing van minimum lengte wat 'n sekere Bezout identiteit gebaseer op die verfyningsmaskersimbool A_m, wat eksplisiet gegee word.

Met so 'n lineêre projeksie operator ry definieer ons, in Hoofstuk 4, die foutruimte ry $W_m^r = \{f - \mathcal{P}_{m,r}f : \ f \in \mathcal{S}_m^{r+1}\}$. Ons toon dan aan, deur 'n sekere Bezout identiteit op te los, dat daar 'n eindig-ondersteunde funksie $\psi_m \in \mathcal{S}_m^1$ bestaan sodat, vir elke $r \in \mathbb{Z}$, die heelgetal skuif ry $\{\psi_m(2 \cdot -j)\}$ die lineêre ruimte W_m^r onderspan. Volgens ons definisie, noem ons ψ_m dan die m^{te} orde kardinale B-golfie. Die golfie dekomposisie algoritme gebaseer op die kwasi-interpolasie operator $\mathcal{Q}_{m,r}$, sowel as die lokale lineêre projeksie operator $\mathcal{P}_{m,r}$, asook die golfie ψ_m, is dan gebaseer op eindige rye, en word getoon om, vir 'n gegewe sein f, die essensiële eienskape te besit van om relatiewe klein golfie koëffisiënte te lewer in gebiede waar die steuninterval van $\psi_m(2^r \cdot -j)$ oorvleuel met 'n C^m-gladde gebied van f.

Ten slotte, in Hoofstuk 5, konstrueer ons minimum-ondersteunde kubiese B-latfunksie golfies op 'n begrensde interval $[0, n]$. Ons ontwikkel ook 'n ooreenstemmende eksplisiete dekomposisie algoritme vir 'n sein f op 'n begrensde interval.

Deurgaans in Hoofstukke 2 tot 5 word numeriese voorbeelde verskaf om die teoretiese resultate grafies te illustreer.

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Contents

List of Symbols

$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of non-negative real numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Z}_+$	the set of non-negative integers
$\mathbb{C}$	the set of complex numbers
π_k	the linear space of polynomials of degree less than or equal to k, $k \in \mathbb{Z}_+$
$M(\mathbb{Z})$	the space of bi-infinite real-valued sequences
$M(\mathbb{R})$	the space of real-valued functions on $\mathbb{R}$
$M_0(\mathbb{Z})$	the space of sequences $c \in M(\mathbb{Z})$ such that c is finitely supported, i.e. if $c = \{c_j : j \in \mathbb{Z}\}$, then there exits integers α and β, such that $c_j = 0$, $j \notin \{\alpha, \dots, \beta\}$
$C(\mathbb{R})$	the set of continuous functions on $\mathbb{R}$
$C[a,b]$	the set of continuous functions on $[a,b]$
$C^{-1}(\mathbb{R})$	the space of piecewise continuous functions in $M(\mathbb{R})$
$C^k(\mathbb{R})$	$f \in M(\mathbb{R}) : f^{(k)} \in C(\mathbb{R})$, $k \in \mathbb{N}$
$C_0^k(\mathbb{R})$	$C^k(\mathbb{R}) \cap C_0(\mathbb{R})$
deg	the degree of a polynomial
$\delta_{j,k}$	the Kronecker symbol, $\delta_{j,k} = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \quad j,k \in \mathbb{Z}$
δ_j	$\delta_{j,0}$, $j \in \mathbb{Z}$
$\binom{n}{j}$	$\begin{cases} \frac{n!}{j!(n-j)!}, & j = 0,1,\dots,n, \\ 0, & j \neq 0,1,\dots,n, \end{cases} \quad j \in \mathbb{Z}, \quad n \in \mathbb{Z}_+$ with $0! = 1$
$\lceil x \rceil$	the smallest integer greater than x
$\lfloor x \rfloor$	the largest integer less than x
x_+^k	$\begin{cases} x^k, & x \geq 0, \\ 0, & x < 0 \end{cases}$
$\sum_j$	the sum $\sum_{j \in \mathbb{Z}}$

$p^{(e)}$	the even part $p = \sum_j p_{2j}(\cdot)^{2j}$ of a (Laurent) polynomial $p = \sum_j p_j(\cdot)^j$
$p^{(o)}$	the odd part $p = \sum_j p_{2j+1}(\cdot)^{2j+1}$ of a (Laurent) polynomial $p = \sum_j p_j(\cdot)^j$
ψ	a wavelet

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1. Introduction

The term wavelet was itself coined in 1982, [see [10]]. Wavelets are functions that satisfy certain mathematical requirements and are used in representing data or functions. Approximation using superposition of functions has existed since the early 1800's, when Joseph Fourier discovered that he could superpose sines and cosines to represent other functions.

For many decades, scientists have wanted more appropriate functions than the sines and cosines which comprise the basis of Fourier analysis, to approximate choppy signals.

Wavelet analysis, in contrast to Fourier analysis, uses approximating functions that are localized in both time and frequency space. It is this unique characteristic that makes wavelets particularly useful, in approximating data with sharp discontinuities. The wavelet transform is a tool for carving up functions, or data into components of different frequency, as required in signal analysis.

Over the last two decades, wavelet decomposition and reconstruction algorithms have been playing an increasingly important role in signal analysis application areas such as image processing, statistics and theoretical physics [see [3] and [7]]. Wavelet construction method based on multi-resolutional analysis, in which Fourier transform methods are employed, [see [19], [2], [22], [21]], usually require conditions like orthogonality and Riesz-stability. In particular, the so-called Chui-Wang bi-orthogonal cardinal spline wavelets [see [4]] provide a Riesz-stable basis for the square-(Lebesgue)-inetegrable functions on the real line. The decomposition algorithms based on these wavelets are infinite, albeit with exponential decay, so that truncation is necessary in practical applications.

In recent work [see [11] [17]], a new class of cardinal spline wavelets was developed, where finite decomposition algorithms were obtained at the cost of giving up bi-orthogonality.

Our approach to wavelet decomposition in this thesis is based on the concept of quasi-interpolation and local linear projection. We shall concentrate on the construction of cardinal B-spline wavelets, which are special in the sense that they can be formulated explicitly, are smooth, and computationally easy to work with.

Motivated by the following statement made by Gilbert Strang in his forward to Charles Chui's book [2], " . . . when students come to ask advice about their thesis, the problem is always at the boundary" , and the fact that real life data always has to do with a finite amount of data, we study also in this thesis the construction of cubic spline wavelets on a bounded interval. These results seem likely to extend to hold for general m^{th} order cardinal B-splines, and we intend to pursue this generalization in our future study.

1.1 Overview

In Chapter 2, we introduce some fundamental results on m^{th} order cardinal B-splines which are piecewise polynomials with uniformly spaced breakpoints at the dyadic points $Z/2^r$, for $r \in \mathbb{Z}$, and from which other cardinal spline functions of the same degree are obtained by linear combination of integer-shifts, that is, their integer-shifts form a basis for the cardinal spline space denoted by $\mathcal{S}_m^r$. We next explicitly construct a quasi-interpolant $\mathcal{Q}_{m,r}$ which maps real valued functions on $\mathbb{R}$ into the space $\mathcal{S}_m^r$, for every $r \in \mathbb{Z}$ such that polynomials in π_{m-1} are reproduced. The fundamental property of polynomial reproduction of quasi-interpolants allows a large range of constructions in a space containing polynomials [see [5], [15] and [9]].

In Chapter 3, we characterise a local linear projection in terms of a Laurent polynomial Λ_m solution to a certain Bezout identity. It is then shown how Λ_m can in fact be explicitly found.

Next, in Chapter 4, we define the error space sequence $W_m^r = \{f - \mathcal{P}_{m,r} f : f \in \mathcal{S}_m^{r+1}\}$, and we show that there exists a finitely supported function $\psi_m \in \mathcal{S}_m^1$ such that the integer shift sequence $\{\psi_m(2^r \cdot -j)\}$ spans the linear space W_m^r, for every $r \in \mathbb{Z}$. Such a function is called a cardinal B-spline wavelet of order m. We proceed to develop a general theory of cardinal spline wavelet decomposition based on the quasi-interpolation operator, the projection operator and the wavelet, to decompose any signal f defined on $\mathbb{R}$.

Finally, in Chapter 5, we give an explicit formulation of the cubic spline wavelet construction and decomposition algorithm on a bounded interval based on the methods of Chapters 2 to 4. We show that a finite wavelet decomposition algorithm can in fact be obtained on a bounded interval without demanding any type of stability, e.g. Riesz-stability, or orthogonality, on our generating linear spaces. Our wavelet decomposition algorithm then uses only a finite data set. We therefore obtain a cardinal B-spline wavelet decomposition algorithm that is local, as opposed to previous boundary wavelet construction methods in work by Chui and De Villiers (see [6]) and Chui and

Quak (see [8]), obtained from bi-orthogonal wavelets, and in which the decomposition algorithm depends on all the full data set.

1.2 Notation

In this thesis, the symbol $\mathbb{R}$ represents the real line, the symbol $\mathbb{N}$ denotes the set of natural numbers, whereas the symbol $\mathbb{Z}$ denotes the set of integers, and $\mathbb{Z}_+ = \{x \in \mathbb{Z} : x \geq 0\}$. We write $\lceil x \rceil$ for the smallest integer greater than or equal to x, $\lfloor x \rfloor$ for the largest integer less than or equal to x and $C[a,b]$ for the set of continuous functions on a closed interval $[a,b]$. We write $M(\mathbb{R})$ for the set of real-valued functions on $\mathbb{R}$. For $k \in \mathbb{Z}_+$, we denote by $C^k(\mathbb{R})$ the subspace of $M(\mathbb{R})$ consisting of functions f such that the k^{th} derivative $f^{(k)}$ is continuous on $\mathbb{R}$, and where $f^{(0)} = f$. We write $C(\mathbb{R})$ for $C^{(0)}(\mathbb{R})$ and $C^{-1}(\mathbb{R})$ for the space of piecewise continuous functions in $M(\mathbb{R})$.

For $k \in \mathbb{Z}_+$, we write π_k for the space of polynomials of degree less than or equal to k. A function $f \in M(\mathbb{R})$ is called a finitely supported function if there exists a bounded interval $[a,b] \subset \mathbb{R}$ such that $f(x) = 0$, $x \notin [a,b]$. We write $M_0(\mathbb{R}) = \{f \in M(\mathbb{R}) : f \text{is finitely supported}\}$, $C_0(\mathbb{R}) = C(\mathbb{R}) \cap M_0(\mathbb{R})$, and, for $k \in \mathbb{N}$, $C_0^k(\mathbb{R}) = C^k(\mathbb{R}) \cap M_0(\mathbb{R})$.

We write

$$\chi_{[0,1)}(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & \text{elsewhere.} \end{cases} \tag{1.1}$$

The binomial coefficient is defined as

$$\binom{n}{j} = \begin{cases} \frac{n!}{j!(n-j)!}, & j = 0, 1, \ldots, n, \\ 0, & j \neq 0, 1, \ldots, n, \end{cases} \quad j \in \mathbb{Z}, \quad n \in \mathbb{Z}_+ \tag{1.2}$$

and with the convention $0! = 1$.

For $k \in \mathbb{Z}_+$, the truncated power function $(\cdot)_+^k \in C^{k-1}(\mathbb{R})$ is defined by

$$x_+^k = \begin{cases} x^k, & x \geq 0 \\ 0, & x < 0. \end{cases} \tag{1.3}$$

with $0^0 = 1$.

2. Cardinal Spline Quasi-Interpolation

2.1 Cardinal B-Splines

For $m \in \mathbb{N}$, $r \in \mathbb{Z}$, we consider the linear space S_m^r of cardinal splines of order m defined by

$$\mathcal{S}_m^r = \mathcal{S}_m\left(\mathbb{Z}/2^r\right) = \left\{ f \in C^{m-2}(\mathbb{R}) : f|_{\left[\frac{j}{2^r}, \frac{j+1}{2^r}\right)} \in \pi_{m-1}, \quad j \in \mathbb{Z} \right\}. \tag{2.1}$$

We observe that the relation $\mathcal{S}_m^r \subset \mathcal{S}_m^{r+1}$, $r \in \mathbb{Z}$ holds, i.e. $\{\mathcal{S}_m^r : r \in \mathbb{Z}\}$ is a nested sequence of a linear spaces. We write $\mathcal{S}_m(\mathbb{Z}) = \mathcal{S}_m^0\left(\mathbb{Z}/2^0\right)$.

Definition 2.1 *For $m \in \mathbb{N}$, we define the sequence $\{N_m : m \in \mathbb{N}\}$ of real-valued functions on $\mathbb{R}$ recursively by*

$$N_m(x) = \int_0^1 N_{m-1}(x-t)dt, \quad m = 2, 3, \cdots, \quad x \in \mathbb{R}, \tag{2.2}$$

where

$$N_1(x) = \chi_{[0,1)}, \quad x \in \mathbb{R}, \tag{2.3}$$

The function N_m is called the cardinal B-spline of order m.

The following properties are proved in [1, Chapter 4]; see also [11, Theorem 1.1].

Theorem 2.2 *For $m \in \mathbb{N}$, and $x \in \mathbb{R}$, the cardinal B-splines as defined by (2.2) and (2.3), have the following properties:*

$$N_m(x) = \frac{1}{(m-1)!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (x-j)_+^{m-1}; \tag{2.4}$$

$$N_m(\cdot - j) \in \mathcal{S}_m, \quad j \in \mathbb{Z}; \tag{2.5}$$

$$N_m(x) = 0, \quad x \notin \begin{cases} [0,1), \ m = 1, \\ (0,m), \ m \geq 2; \end{cases} \tag{2.6}$$

$$N_m(x) = \frac{1}{(m-1)} \left[x N_{m-1}(x) + (m-x) N_{m-1}(x-1) \right], \quad m \geq 2; \tag{2.7}$$

$$N_m(x) > 0, \quad x \in (0,m); \tag{2.8}$$

$$N_m'(x) = N_{m-1}(x) - N_{m-1}(x-1), \quad m = 2, 3, \cdots, \text{ if } \quad m \geq 3; \tag{2.9}$$

$$N_m(x) = N_m(m-x), \quad m \geq 2. \tag{2.10}$$

Graphs of the cardinal B-spline N_m for $m = 2, 3, 4$ are shown in Figures 2.1, 2.2 and 2.3 by means of (2.7) together with (2.3).

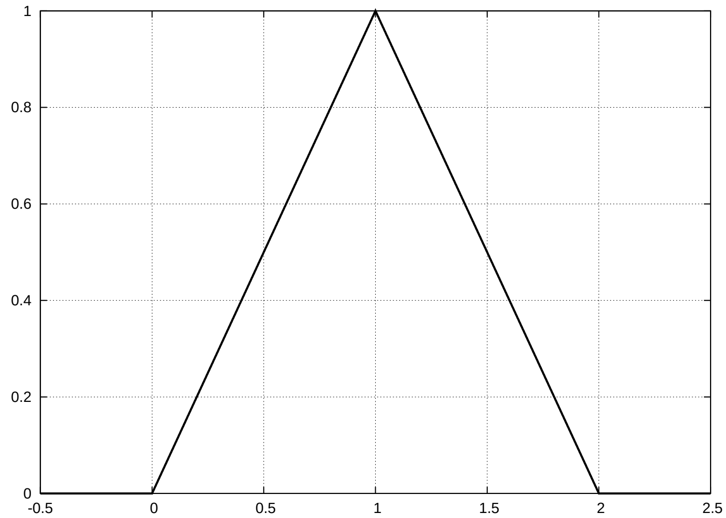


Figure 2.1: Graph of the function N_2

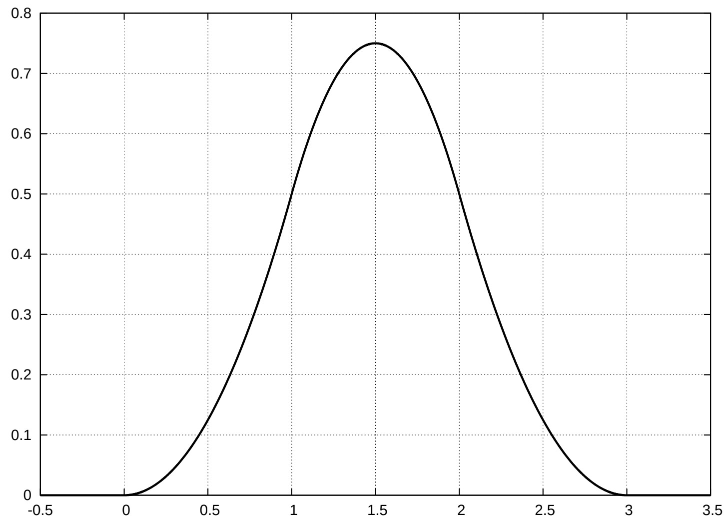


Figure 2.2: Graph of the function N_3

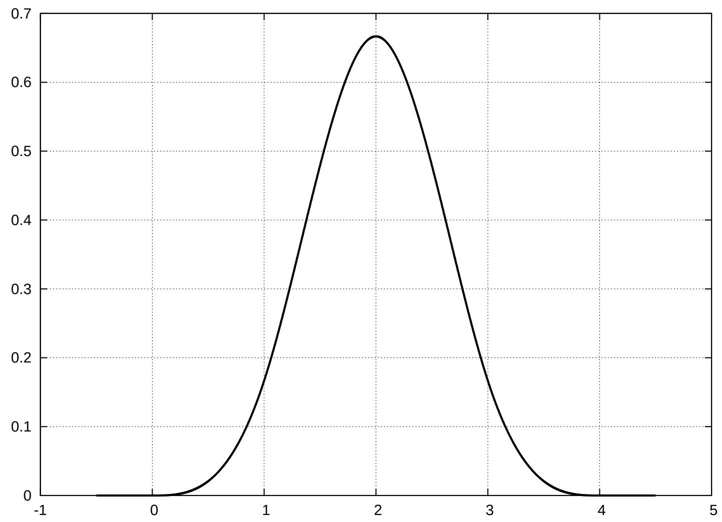


Figure 2.3: Graph of the function N_4

The following result is proved in [16, Theorem 2.1]; see also [11, Theorem 1.2].

Theorem 2.3 *For $m \in \mathbb{N}$, the integer shift sequence $\{N_m(\cdot - j) : j \in \mathbb{Z}\}$ of spline functions is a basis of $\mathcal{S}_m$ in the sense that, for each $f \in \mathcal{S}_m$, there exists a unique sequence $\{c_j\} \subset \mathbb{R}$ such that*

$$f = \sum_j c_j N(\cdot - j). \tag{2.11}$$

The following refinement equation was proved in [1, Chapter 4]; see also [11, Theorem 1.4].

Theorem 2.4 *For $m \in \mathbb{N}$,*

$$N_m = \frac{1}{2^{m-1}} \sum_{j=0}^{m} \binom{m}{j} N_m(2 \cdot -j). \tag{2.12}$$

If a sequence $a \in M_0(\mathbb{Z})$ and a function $\phi \in M_0(\mathbb{R})$ with $\phi \neq 0$ is such that

$$\phi = \sum_j a_j \phi(2 \cdot -j), \tag{2.13}$$

then (a, ϕ) is called a refinement pair, the function ϕ is called the refinable function,the sequence a is called the refinement mask and equation (2.13) is called the refinement equation.

From Theorem 2.4, we have that, for a given integer $m \in \mathbb{N}$, (a_m, N_m) is a refinement pair with the sequence $a_m = a_{m,j} \in M_0(\mathbb{Z})$ given by

$$a_{m,j} = \frac{1}{2^{m-1}} \binom{m}{j}, \quad j \in \mathbb{Z}. \tag{2.14}$$

The polynomial A_m defined by

$$A_m(z) = \sum_{j=0}^{m} a_{m,j} z^j = \frac{1}{2^{m-1}}(1+z)^m, \quad z \in \mathbb{C}, \tag{2.15}$$

is called the cardinal B-spline refinement mask symbol of order m.

2.2 Marsden's Identity

Since (2.1) gives the inclusion $\pi_{m-1} \subset \mathcal{S}_m$, we know from Theorem 2.3 that, if $f \in \pi_{m-1}$, there exists a unique sequence $\{c_j\}$ such that (2.11) holds. We proceed to establish a result by means of which this sequence $\{c_j\}$ can be explicitly calculated.

We shall rely on the following so called Marsden's identity, (see [16, p 65]), the proof of which we take from [11, Theorem 6.6].

Theorem 2.5 *For $m \in \mathbb{N}$, $m \geq 2$, we have*

$$(x+t)^{m-1} = \sum_j Q_m(j+t) N_m(x-j), \quad x, t \in \mathbb{R}, \tag{2.16}$$

where Q_m is the polynomial of degree $m-1$ defined by

$$Q_m(x) = \prod_{k=1}^{m-1}(x+k), \quad x \in \mathbb{R}. \tag{2.17}$$

Proof. Our proof is by induction on the cardinal spline order m. Using (2.4), we obtain

$$N_2(x) = \begin{cases} x, & x \in [0,1), \\ 2-x, & x \in [1,2), \\ 0, & \text{elsewhere}, \end{cases} \tag{2.18}$$

according to which

$$N_2(j+1) = \delta_j, \quad j \in \mathbb{Z}. \tag{2.19}$$

For a fixed $t \in \mathbb{R}$, we define the polynomial $p \in \pi_1$ by

$$p(x) = x + t. \tag{2.20}$$

Since $\pi_1 \in \mathcal{S}_2$, it follows from Theorem 2.3 that there exists a unique sequence $c \in M(\mathbb{Z})$ such that

$$p(x) = \sum_j c_j N_2(x-j), \quad x \in \mathbb{R}, \tag{2.21}$$

and thus

$$p(k+1) = \sum_j c_j N_2(k+1-j) = \sum_j c_j \delta_{k,j} = c_k, \quad k \in \mathbb{Z},$$

from equation (2.19). Hence

$$p(x) = \sum_j p(j+1) N_2(x-j), \quad x \in \mathbb{R},$$

and thus

$$(x+t) = \sum_j (j+1+t) N_2(x-j), \quad x,t \in \mathbb{R}.$$

Therefore, the theorem holds for $m = 2$.

Suppose now that the theorem holds for a fixed integer $m \geq 2$. Since (2.17) gives $Q_{m+1}(x) = (x+m)Q_m(x)$, and $Q_{m+1}(x-1) = xQ_m(x)$, we can use (2.7) and (2.17) to deduce that, for $x \in \mathbb{R}$,

$$\begin{aligned}
\sum_j Q_{m+1}(j+t) N_{m+1}(x-j) &= \frac{1}{m} \sum_j Q_{m+1}(j+t) \left[(x-j) N_m(x-j) \right. \\
&\qquad \left. + (m+1+j-x) N_m(x-j-1) \right] \\
&= \frac{1}{m} \left[\sum_j Q_{m+1}(j+t)(x-j) N_m(x-j) \right. \\
&\qquad \left. + \sum_j Q_{m+1}(j+t)(m+1+j-x) N_m(x-j-1) \right] \\
&= \frac{1}{m} \left[\sum_j Q_{m+1}(j+t)(x-j) N_m(x-j) \right. \\
&\qquad \left. + \sum_j Q_{m+1}(j+t-1)(m+j-x) N_m(x-j) \right] \\
&= \frac{1}{m} \sum_j \left[Q_{m+1}(j+t)(x-j) \right. \\
&\qquad \left. + Q_{m+1}(j+t-1)(m+j-x) \right] N_m(x-j)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{m}\sum_j Q_m(j+t)\left[(j+t+m)(x-j)\right. \\
&\quad \left. +(j+t)(m+j-x)\right] N_m(x-j) \\
&= \frac{1}{m}\sum_j m(x+t)Q_m(j+t)N_m(x-j) \\
&= (x+t)\sum_j Q_m(j+t)N_m(x-j) \\
&= (x+t)(x+t)^{m-1} \\
&= (x+t)^m,
\end{aligned}$$

from the inductive hypothesis, and thereby completing our proof. ■

Corollary 2.6 *For $m \in \mathbb{N}$ and $n \geq 2$, we have*

$$x^l = \frac{l!}{(m-1)!}\sum_j Q_m^{(m-1-l)}(j)N_m(x-j), \quad x \in \mathbb{R}, \quad l = 0,1,\ldots,m-1, \tag{2.22}$$

Proof. Taking the l^{th} derivative with respect to t of both sides of the identity (2.16) yields

$$\frac{(m-1)!}{(m-1-l)!}(x+t)^{m-1-l} = \sum_j Q_m^{(l)}(j+t)N_m(x-j), \quad x \in \mathbb{R}, \quad l = 0,1,\ldots,m-1,$$

in which we set $t = 0$ to obtain (2.22). ■

The identity (2.22) can be use to explicitly calculate the coefficients $\{c_j : j \in \mathbb{Z}\}$ in the cardinal B-spline series (2.11) for any polynomial $f \in \pi_{m-1}$.

2.3 Explicit Recursive Formulation of an Optimally Local Quasi-Interpolant

In our eventual wavelet decomposition algorithm, we shall require an approximation to map a given signal $f \in M(\mathbb{R})$ into the space $\mathcal{S}_m^r$ for an appropriate value of r. For this purpose, we define, for $m \geq 2$, an approximation operator $\mathcal{Q}_{m,r} : M(\mathbb{R}) \to \mathcal{S}_m^r$, such that the polynomial reproduction property

$$\mathcal{Q}_{m,r}p = p, \quad p \in \pi_{m-1}, \quad r \in \mathbb{Z}, \tag{2.23}$$

is satisfied. Such an operator is referred to as a quasi-interpolation operator. The construction method given below is from [17, Chapter 3].

Our first step towards the construction of $\mathcal{Q}_{m,r}$ is to seek, for a parameter $\tau \in \mathbb{R}$, a function $u = u^{m,\tau} \in \mathcal{S}_m$ with finite support such that

$$\sum_j p(j+\tau)u(\cdot - j) = p, \quad p \in \pi_{m-1}, \quad \tau \in \mathbb{R}, \tag{2.24}$$

where

$$u = \sum_j u_j N_m(\cdot - j), \tag{2.25}$$

and $\{u_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$.

To explicitly construct the quasi-interpolation operator $\mathcal{Q}_{m,r}$, we first observe that the condition (2.24) holds for a fixed $\tau \in \mathbb{R}$ if and only if

$$\sum_j (j+\tau)^l u(x-j) = x^l, \quad x \in \mathbb{R}, \quad l = 0, 1, \ldots, m-1. \tag{2.26}$$

Substituting (2.25) into the left-hand-side of (2.26), we obtain, for $l = 0, 1, \ldots, m-1$,

$$\begin{aligned} \sum_j (j+\tau)^l u(x-j) &= \sum_j (j+\tau)^l \sum_k u_k N_m(x-j-k) \\ &= \sum_j (j+\tau)^l \sum_k u_{k-j} N_m(x-k) \\ &= \sum_k \left[\sum_j (j+\tau)^l u_{k-j}\right] N_m(x-k). \end{aligned} \tag{2.27}$$

It follows from (2.27) and (2.22) that the condition (2.26) holds if and only if the sequence $\{u_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ in (2.25) is such that

$$\sum_k \left[\sum_j (j+\tau)^l u_{k-j} - \frac{l!}{(m-1)!} Q^{(m-1-l)}(k)\right] N_m(.-k) = 0, \quad l = 0, 1, \ldots, m-1. \tag{2.28}$$

It follows from Theorem 2.3 that a sequence $\{u_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ satisfies (2.28) if and only if

$$\sum_j (j+\tau)^l u_{k-j} = \frac{l!}{(m-1)!} Q^{(m-1-l)}(k), \quad l = 0, 1, \ldots, m-1. \tag{2.29}$$

A necessary condition for (2.29) to hold is obtained by setting $k = 0$ in (2.29) to yield

$$\sum_j (j+\tau)^l u_{-j} = \frac{l!}{(m-1)!} Q^{(m-1-l)}(0), \quad l = 0, 1, \ldots, m-1, \tag{2.30}$$

or, equivalently,

$$\sum_j (j-\tau)^l u_j = \frac{(-1)^l l!}{(m-1)!} Q^{(m-1-l)}(0), \quad l = 0, 1, \ldots, m-1. \tag{2.31}$$

To get a minimally supported solution $\{u_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ of (2.31), we set

$$u_j = 0, \quad j \notin \{0, 1, \ldots, m-1\}, \tag{2.32}$$

so that (2.31) becomes the $m \times m$ linear system

$$\sum_{j=0}^{m-1} (j-\tau)^l u_j = \frac{(-1)^l l!}{(m-1)!} Q^{(m-1-l)}(0), \quad l = 0, 1, \ldots, m-1. \tag{2.33}$$

By defining

$$A = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ x_0 & x_1 & \ldots & x_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_0^{m-1} & x_1^{m-1} & \ldots & x_{m-1}^{m-1} \end{bmatrix}, \tag{2.34}$$

where

$$x_j = x_{j,\tau} = j - \tau, \quad j = 0, 1, \ldots, m-1; \tag{2.35}$$

$$\mathbf{u} = [u_0, u_1, \ldots, u_{m-1}]^T; \tag{2.36}$$

and

$$\mathbf{b} = [b_0, b_1, \ldots, b_{m-1}]^T; \tag{2.37}$$

with

$$b_l = \frac{(-1)^l l!}{(m-1)!} Q_m^{(m-1-l)}(0), \quad l = 0, 1, \ldots, m-1; \tag{2.38}$$

we obtain the matrix-vector formulation

$$A\mathbf{u} = \mathbf{b} \tag{2.39}$$

of the $m \times m$ linear system (2.33).

To solve the linear system (2.33), we shall rely on the following result from [11, Proposition 7.1].

Proposition 2.7 *For $\mu \in \mathbb{N}$, suppose $\{x_j : j = 0, 1, \ldots, \mu\}$ are $\mu + 1$ distinct points in $\mathbb{R}$, and suppose $\{d_l : l = 0, 1, \ldots, \mu\} \subset \mathbb{R}$. Then the $(\mu+1) \times (\mu+1)$ linear system*

$$\sum_{j=0}^{\mu} x_j^l u_j = d_l, \quad l = 0, 1, \ldots, \mu, \tag{2.40}$$

has the unique solution

$$u_j = \sum_{k=0}^{\mu} \frac{1}{k!} L_j^{(k)}(0) d_k, \quad j = 0, 1, \ldots, \mu, \tag{2.41}$$

with

$$L_j(x) = \prod_{j \neq k = 0}^{\mu} \frac{x - x_k}{x_j - x_k}, \quad x \in \mathbb{R}, \quad j = 0, 1, \ldots, \mu, \tag{2.42}$$

denoting the fundamental Lagrange polynomials of degree μ with respect to the point set $\{x_0, x_1, \ldots, x_\mu\}$.

Proof. Since it holds that

$$L_j\left(x_{\tilde{j}}\right) = \delta_{j,\tilde{j}}, \quad j, \tilde{j} = 0, 1, \ldots, \mu,$$

we can appeal to a standard uniqueness result in polynomial interpolation to deduce that

$$\sum_{j=0}^{\mu} x_j^l L_j(x) = x^l, \quad x \in \mathbb{R}, \quad l = 0, 1, \ldots, \mu. \tag{2.43}$$

It follows from (2.41) and (2.43) that

$$\begin{aligned}
\sum_{j=0}^{\mu} x_j^l u_j &= \sum_{j=0}^{\mu} x_j^l \sum_{k=0}^{\mu} \frac{1}{k!} L_j^{(k)}(0) d_k \\
&= \sum_{k=0}^{\mu} \frac{1}{k!} \left[\sum_{j=0}^{\mu} x_j^l L_j^{(k)}(0) \right] d_k \\
&= \sum_{k=0}^{\mu} \frac{1}{k!} \left[\left(\frac{d}{dx} \right)^k (x^l) \right]_{x=0} d_k \\
&= \sum_{k=0}^{\mu} \frac{1}{k!} \frac{l!}{(l-k)!} \delta_{l,k} d_k \\
&= \sum_{k=0}^{\mu} \binom{l}{k} \delta_{l,k} d_k = d_l,
\end{aligned}$$

which shows that the formula (2.41) satisfies (2.40).

We see from (2.34) that A is the transpose of the (invertible) Vandermonde matrix with respect to the m distinct point set $\{x_0, x_1, \ldots, x_{m-1}\}$, so that A is an invertible matrix. Hence (2.41) does indeed give the unique solution of the linear system (2.40). ■

It follows from Proposition 2.7 that the unique solution $\{u_j = u_{m,j} : j = 0, 1, \ldots, m-1\}$ of the linear system (2.33) is given by

$$u_{m,j} = \frac{1}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k L_j^{(k)}(0) Q_m^{(m-1-k)}(0), \quad j = 0, 1, \ldots, m-1, \tag{2.44}$$

where $\{L_j : j = 0, \ldots, m-1\}$ is the Lagrange fundamental polynomial sequence given by (2.42), (2.35).

We now show that the sequence $\{u_{m,j} : j = 0, 1, \ldots, m-1\}$ defined by (2.44) and (2.32) satisfies the condition (2.29). To this end, we use (2.31) to obtain, for $l \in \{0, 1, \ldots, m-1\}$ and $k \in \mathbb{Z}$,

$$\begin{aligned}
\sum_j (j+\tau)^l u_{m,k-j} &= \sum_j \left[k - (j-\tau)\right]^l u_{m,j} \\
&= \sum_j \sum_{n=0}^{l} \binom{l}{n} k^n (-1)^{l-n} (j-\tau)^{l-n} u_{m,j} \\
&= \sum_{n=0}^{l} \binom{l}{n} k^n (-1)^{l-n} \sum_j (j-\tau)^{l-n} u_{m,j} \\
&= \sum_{n=0}^{l} \binom{l}{n} k^n (-1)^{l-n} \frac{(-1)^{l-n}(l-n)!}{(m-1)!} Q_m^{(m-1-l+n)}(0) \\
&= \sum_{n=0}^{l} \frac{l! k^n}{n!(l-n)!} \frac{(l-n)!}{(m-1)!} Q_m^{(m-1-l+n)}(0) \\
&= \frac{l!}{(m-1)!} \sum_{n=0}^{l} \frac{Q_m^{(m-1-l+n)}(0)}{n!} k^n \\
&= \frac{l!}{(m-1)!} \sum_{n=0}^{l} \frac{\left(Q_m^{(m-1-l)}\right)^{n}(0)}{n!} k^n \\
&= \frac{l!}{(m-1)!} Q_m^{(m-1-l)}(k),
\end{aligned}$$

since $\deg\,(Q_m) = m-1$ implies that $Q_m^{(m-1-l)} \in \pi_l$ and thereby showing that (2.29) is indeed satisfied.

Suppose now that the Lagrange fundamental polynomials given by (2.42) are given by

$$L_j(x) = \sum_{k=0}^{m-1} h_{j,k} x^k, \quad x \in \mathbb{R}, \quad j = 0, 1, \ldots, m-1, \tag{2.45}$$

according to which

$$L_j^{(k)}(0) = k! h_{j,k}, \quad j, k = 0, \ldots, m-1, \tag{2.46}$$

and, similarly, let

$$Q_m(x) = \sum_{k=0}^{m-1} q_{m,k} x^k, \quad x \in \mathbb{R}, \quad j = 0, 1, \ldots, m-1, \tag{2.47}$$

so that

$$Q_m^{(k)}(0) = k! q_{m,k}, \quad k = 0, 1, \ldots, m-1. \tag{2.48}$$

Substituting (2.46) and (2.48) into (2.44) yields

$$u_{m,j} = \sum_{k=0}^{m-1} \frac{(-1)^k}{\binom{m-1}{k}} h_{j,k} q_{m,m-1-k}. \tag{2.49}$$

Moreover, since $\{u_{m,j} : j = 0, 1, \ldots, m-1\}$ is the unique solution of the linear system (2.40), we deduce that the sequence $\{u_{m,j}\}$ defined by (2.49) and (2.32) is a sequence of shortest possible length for which the condition (2.29) holds.

We have therefore established the first part of the following result.

Theorem 2.8 *The function*

$$u^m(x) = \sum_{j=0}^{2m-1} u_{m,j} N_m(x-j), \tag{2.50}$$

where the sequence $\{u_{m,j} : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ is given, for a fixed $\tau \in \mathbb{R}$, by (2.32) and (2.49), is the function of shortest possible support in $\mathcal{S}_m$ such that the polynomial reproduction property (2.24) holds. Moreover,

$$u^m(x) = 0, \quad x \notin (0, 2m-1). \tag{2.51}$$

Proof. It remains to prove the finite support property (2.51). Combining (2.25), (2.32) and (2.6), we conclude that (2.51) is indeed satisfied. ■

We next construct an optimally local quasi-interpolation operator sequence $\{\mathcal{Q}_{m,r} : r \in \mathbb{Z}\}$ from the function u of Theorem 2.8 such that the polynomial reproduction property (2.23) is satisfied.

For $f \in M(\mathbb{R})$ and $r \in \mathbb{Z}$, using (2.25) we deduce that, for $x \in \mathbb{R}$,

$$\begin{aligned}
\sum_j f\left(\frac{j+\tau}{2^r}\right) u^m(2^r x - j) &= \sum_j f\left(\frac{j+\tau}{2^r}\right) \sum_k u_{m,k} N_m(2^r x - j - k) \\
&= \sum_j f\left(\frac{j+\tau}{2^r}\right) \sum_k u_{m,k-j} N_m(2^r x - k) \\
&= \sum_k \left[\sum_j u_{m,k-j} f\left(\frac{j+\tau}{2^r}\right) \right] N_m(2^r x - k). \quad (2.52)
\end{aligned}$$

Suppose $f \in \pi_{m-1}$, and let $g = f\left(\frac{\cdot}{2^r}\right)$, i.e. $f = g(2^r \cdot)$, according to which $g \in \pi_{m-1}$. But then (2.52) and Theorem 2.8 yield, for $x \in \mathbb{R}$,

$$\sum_k \left[\sum_j u_{m,k-j} f\left(\frac{j+\tau}{2^r}\right) \right] N_m(2^r x - k) = \sum_j g(j+\tau) u^m(2^r x - j) = g(2^r \cdot) = f. \quad (2.53)$$

The following result is an immediate consequence of (2.52) and (2.53).

Theorem 2.9 *For $\tau \in \mathbb{R}$, the operator sequence $\{\mathcal{Q}_{m,r} : r \in \mathbb{Z}\}$, where $\mathcal{Q}_{m,r} : M(\mathbb{R}) \to S_m^r$, $r \in \mathbb{Z}$, as defined by*

$$\mathcal{Q}_{m,r} f = \sum_j \left[\sum_k u_{j-k} f\left(\frac{j+\tau}{2^r}\right) \right] N_m(2^r \cdot - j), \quad r \in \mathbb{Z}, \quad f \in M(\mathbb{R}), \quad (2.54)$$

with the sequence $\{u_{m,j} : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ defined as in Theorem 2.8, is an optimally local quasi-interpolation operator sequence such that the polynomial reproduction property (2.23) is satisfied.

We have from (2.52) that the quasi-interpolation operator $\mathcal{Q}_{m,r}$, as defined by (2.54), has the equivalent formulation

$$\mathcal{Q}_{m,r} f = \sum_j f\left(\frac{j+\tau}{2^r}\right) u^m(2^r \cdot - j), \quad r \in \mathbb{Z}, \quad f \in M(\mathbb{R}), \quad (2.55)$$

with the function $u^m \in \mathcal{S}_m$ as given in Theorem 2.8.

From (2.51) and (2.55), it seems natural to choose the real number τ in the definition (2.54) of the operator $\mathcal{Q}_{m,r}$ as

$$\tau = \tau_0 = m - 1, \quad (2.56)$$

i.e. τ_0 is the integer closest from the left to the midpoint of the interval $(0, 2m-1)$ in (2.52).

By using the equations (2.17), (2.48), (2.56), (2.35), (2.42), (2.46), (2.47) and (2.51), we obtain the following table of values for the sequence $\{u_j^m : j = 0, 1, \ldots, 2m-1\}$ for the choice $\tau = m-1$, for $m = 2, 3, 4$.

m	$\{u_{m,j}\}$
2	$\{u_{2,0}, u_{2,1}\} = \{1, 0\}$
3	$\{u_{3,0}, u_{3,1}, u_{3,2}\} = \{\frac{1}{4}, 1, -\frac{1}{4}\}$
4	$\{u_{4,0}, u_{4,1}, u_{4,2}, u_{4,3}\} = \{-\frac{1}{6}, \frac{4}{3}, -\frac{1}{6}, 0\}$

Table 2.1: The sequence $\{u_{m,j} : j = 0, 1, \ldots, 2m-1\}$ for $m = 2, 3, 4$

We obtain the graphs of the function $u^m(x)$, for $m = 2, 3, 4$ in Figures 2.4, 2.5 and 2.6 by using the values of Table 2.1 in the definition (2.50).

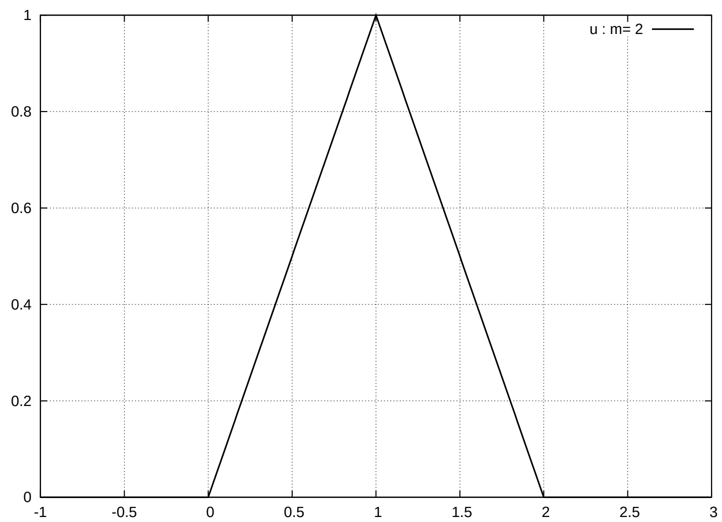


Figure 2.4: The function u^2

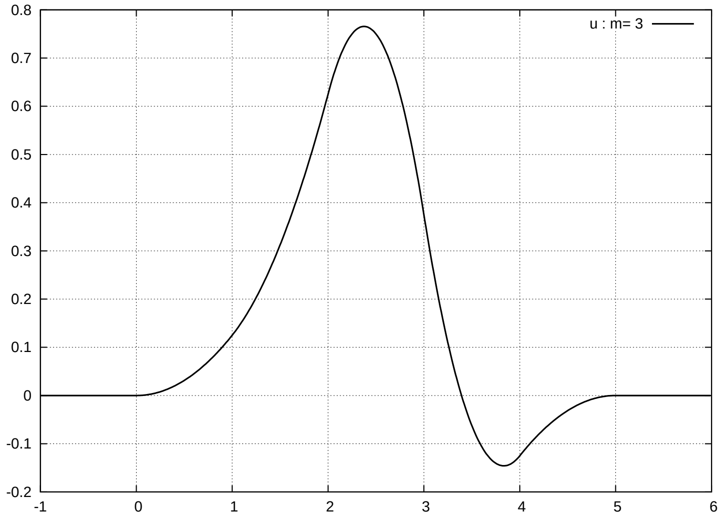


Figure 2.5: The function u^3

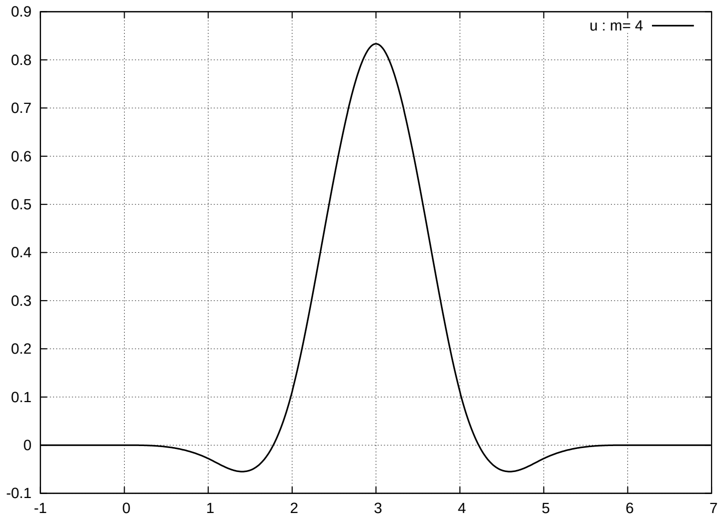


Figure 2.6: The function u^4

Next, we show that if a given signal $f \in M(\mathbb{R})$ is a polynomial in π_{m-1} on an interval, then this polynomial is locally preserved by $\mathcal{Q}_{m,r}$.

Theorem 2.10 *Suppose that, in Theorem 2.9, we choose $\tau = \tau_0 = m-1$ as in (2.56), and suppose that $f \in M(\mathbb{R})$ is such that there exists a bounded interval $[\alpha, \beta]$ and a polynomial $p \in \pi_{m-1}$ such that*

$$f(x) = p(x), \quad x \in [\alpha, \beta]. \tag{2.57}$$

Then

$$(\mathcal{Q}_{m,r} f)(x) = p(x), \quad x \in \left[\alpha + \frac{\tau_0 + 1}{2^r}, \beta - \frac{\tau_0}{2^r}\right], \tag{2.58}$$

for every integer r such that

$$r > \log_2 \frac{2\tau_0 + 1}{\beta - \alpha}. \tag{2.59}$$

Proof. Let r denote an integer such that the inequality (2.59) is satisfied. From (2.55) and the fact that $u(x) = 0$, $x \notin (0, 2\tau_0 + 1)$, we have that

$$(\mathcal{Q}_{m,r} f)(x) = \sum_{j=\lceil 2^r x - 2\tau - 1\rceil}^{\lfloor 2^r x \rfloor} f\left(\frac{j + \tau_0}{2^r}\right) u(2^r x - j), \quad x \in \mathbb{R}, \tag{2.60}$$

and thus

$$(\mathcal{Q}_{m,r} f)(x) = \sum_{j=\lceil 2^r \alpha - \tau_0\rceil}^{\lfloor 2^r \beta - \tau_0 \rfloor} f\left(\frac{j + \tau_0}{2^r}\right) u(2^r x - j), \quad x \in \left[\alpha + \frac{\tau_0 + 1}{2^r}, \beta - \frac{\tau_0}{2^r}\right]. \tag{2.61}$$

Observe that

$$\alpha \leq \frac{j + \tau_0}{2^r} \leq \beta \quad \text{for} \quad j = \lceil 2^r \alpha - \tau_0 \rceil, \ldots, \lfloor 2^r \beta - \tau_0 \rfloor. \tag{2.62}$$

The desired result (2.58) is then a consequence of (2.61), (2.62), (2.57) and (2.23). ■

2.4 Example

We illustrate Theorem 2.10 by choosing, for $m = 2, 3, 4$, the function $f_m \in M(\mathbb{R})$ as

$$f_m(x) = \begin{cases} \frac{1}{2} + \frac{1}{2}\sin\left[\pi\left(x - \frac{1}{2}\right)\right], & x \in [0,1), \\ 1 + (x-1)^{m-1}, & x \in [1,2), \\ 2, & x \in [2,3), \\ 1 + \cos\left[\pi(x-3)\right], & x \in [3,4), \\ 0, & x \notin [0,4), \end{cases} \tag{2.63}$$

which is shown for $m = 2, 3, 4$ in Figures 2.7 and 2.8.

The inequality (2.59) is here given by

$$r \geq \log_2(2m-1). \tag{2.64}$$

Observe from (2.63) that

$$\begin{aligned} f_m(x) &= p(x), \quad x \in [1,2], \\ f_m(x) &= \tilde{p}(x), \quad x \in [2,3], \end{aligned}$$

where $p \in \pi_{m-1}$ and $\tilde{p} \in \pi_0 \subset \pi_{m-1}$ are given by

$$p(x) = 1 + (x-1)^{m-1}$$

and

$$\tilde{p}(x) = 2.$$

It follows from (2.58) in Theorem 2.10 that

$$(\mathcal{Q}_{m,r} f_m)(x) = \begin{cases} 1 + (x-1)^{m-1}, & x \in \left[1 + \frac{m}{2^r}, 2 - \frac{m-1}{2^r}\right], \\ 2, & x \in \left[2 + \frac{m}{2^r}, 3 - \frac{m-1}{2^r}\right]. \end{cases} \tag{2.65}$$

Hence, if we choose

$$r = \begin{cases} 3, & m = 2, \\ 4, & m = 3, \\ 4, & m = 4, \end{cases} \tag{2.66}$$

then the inequality (2.64) is satisfied, and (2.65) gives,

$$(\mathcal{Q}_{2,3}f_2)(x) = \begin{cases} x, & x \in \left[\frac{5}{4}, \frac{16}{8}\right], \\ 2, & x \in \left[\frac{9}{4}, \frac{23}{8}\right], \end{cases} \tag{2.67}$$

$$(\mathcal{Q}_{3,4}f_3)(x) = \begin{cases} 1+(x-1)^2, & x \in \left[\frac{19}{16}, \frac{15}{8}\right], \\ 2, & x \in \left[\frac{35}{16}, \frac{23}{8}\right], \end{cases} \tag{2.68}$$

$$(\mathcal{Q}_{4,4}f_4)(x) = \begin{cases} 1+(x-1)^3, & x \in \left[\frac{5}{4}, \frac{29}{16}\right], \\ 2, & x \in \left[\frac{9}{4}, \frac{45}{16}\right]. \end{cases} \tag{2.69}$$

The results (2.67), (2.68) and (2.69) are illustrated in Figures 2.9 to 2.11, where we have used the values of Table 2.1, (2.55), (2.50), (2.7), (2.56), (2.63). We also give the graphs of the error functions $E_{m,r} = f - \mathcal{Q}_{m,r}f$.

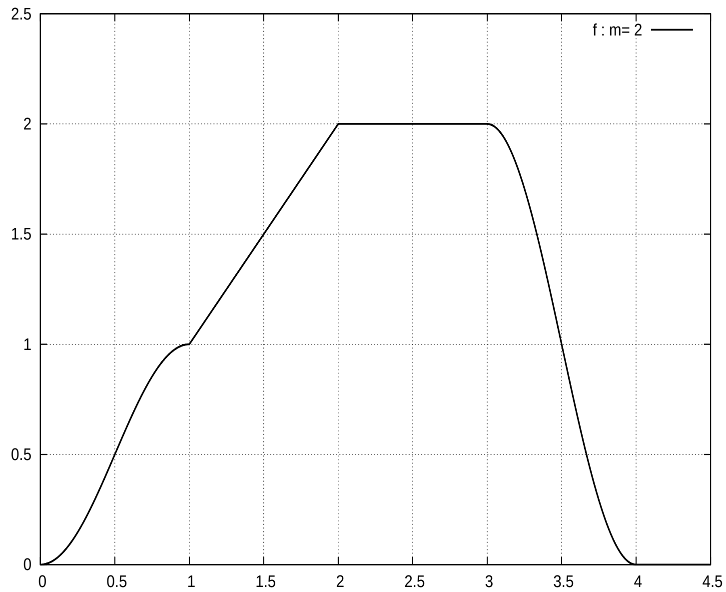


Figure 2.7: Graph of the function f_2

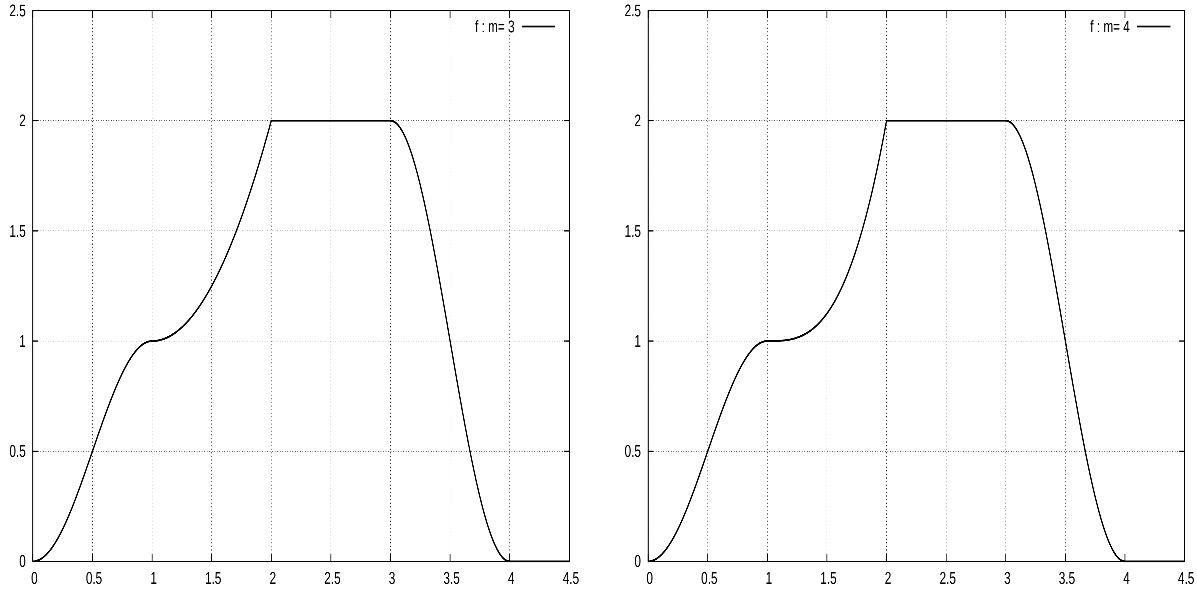


Figure 2.8: The functions f_3 and f_4

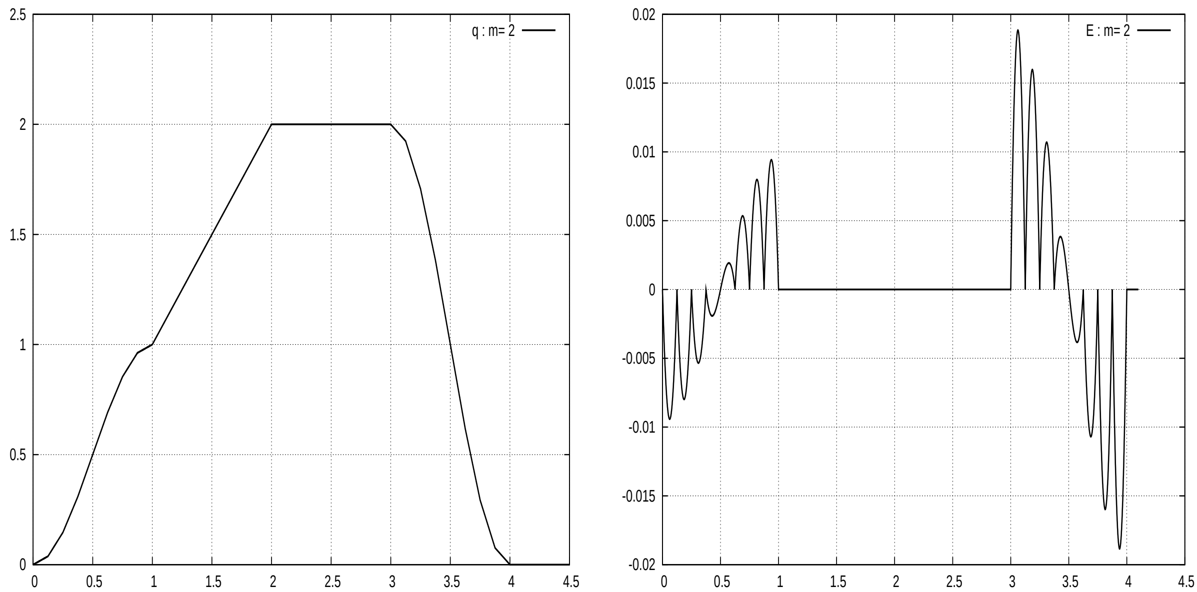


Figure 2.9: The function $\mathcal{Q}_{2,3}f$ and the error function $E_{2,3}f$

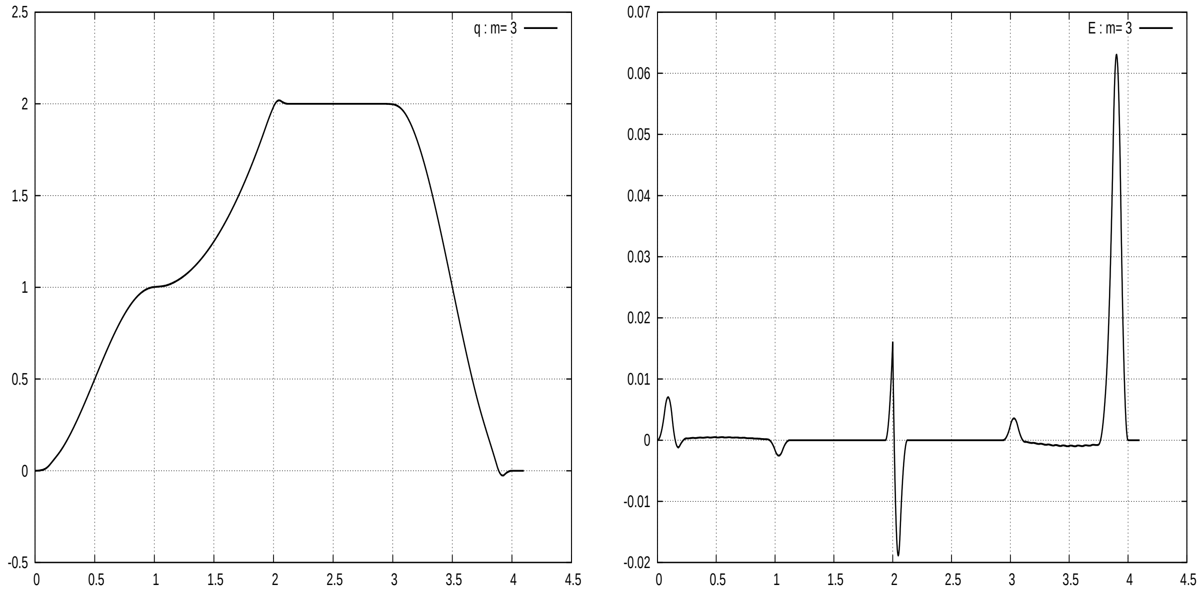


Figure 2.10: The function $\mathcal{Q}_{3,4}f$ and the error function $E_{3,4}f$

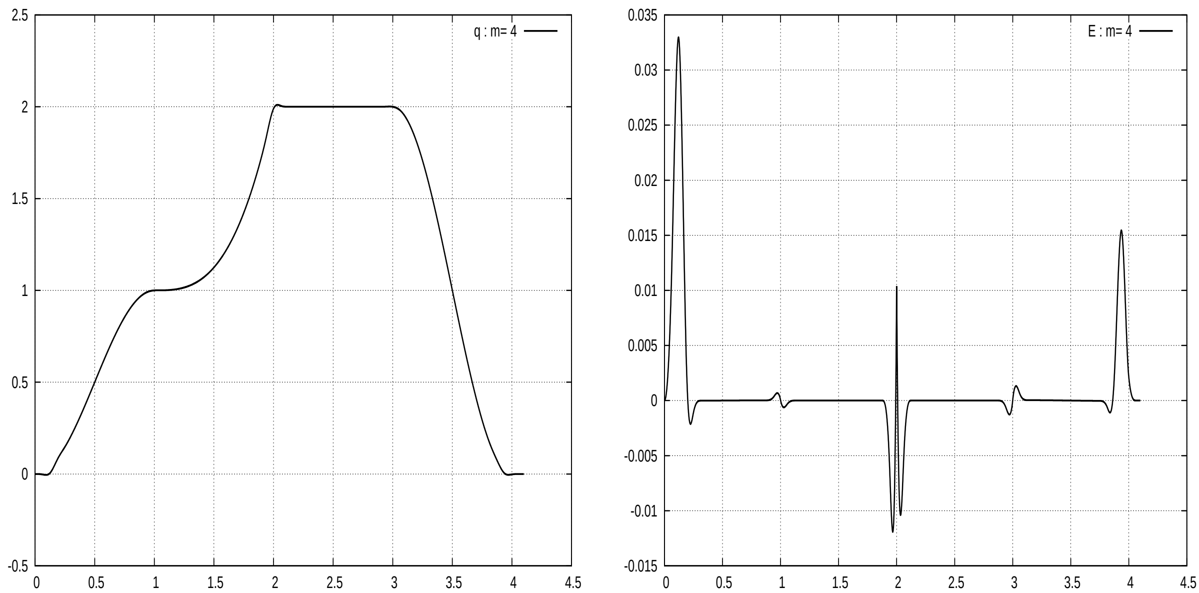


Figure 2.11: The function $\mathcal{Q}_{4,4}f$ and the error function $E_{4,4}f$

3. Local Linear Projection

For a given B-spline refinement pair $\left(a^{(m)}, N_m\right)$ and corresponding refinement space sequence $\{\mathcal{S}_m^r : r \in \mathbb{Z}, m \in \mathbb{N}\}$ as defined by (2.1), our wavelet construction method will depend on the existence of an operator sequence $\{\mathcal{P}_r : r \in \mathbb{Z}\}$ with $\mathcal{P}_r : \mathcal{S}_m^{r+1} \to S_m^r, \quad r \in \mathbb{Z}$, for which the reproduction property

$$\mathcal{P}_r f = f, \quad f \in \mathcal{S}_m^r, \quad r \in \mathbb{Z}, \tag{3.1}$$

holds, according to which $\mathcal{P}_r$ is then, for each $r \in \mathbb{Z}$, a local linear projection on $\mathcal{S}_m^r$.

The following result shows that such a projection operator sequence can be obtained by solving a Bezout identity.

3.1 The Fundamental Bezout Identity

For a Laurent polynomial P defined by

$$P(z) = \sum_j p_j z^j, \quad z \in \mathbb{C}\backslash\{0\}, \tag{3.2}$$

we define the even part $P^{(e)}$ and the odd part $P^{(o)}$ respectively by

$$P^{(e)}(z) = \sum_j p_{2j} z^{2j} \quad \text{and} \quad P^{(o)}(z) = \sum_j p_{2j+1} z^{2j+1}, \quad z \in \mathbb{C}\backslash\{0\}, \tag{3.3}$$

so that

$$\left.\begin{array}{rcl} P(z) & = & P^{(e)}(z) + P^{(o)}(z) \\ P(-z) & = & P^{(e)}(z) - P^{(o)}(z) \end{array}\right\}, \quad z \in \mathbb{C}\backslash\{0\}, \tag{3.4}$$

and thus

$$\left.\begin{array}{rcl} P^{(e)}(z) & = & \frac{P(z)+P(-z)}{2} \\ P^{(o)}(z) & = & \frac{P(z)-P(-z)}{2} \end{array}\right\}, \quad z \in \mathbb{C}\backslash\{0\}. \tag{3.5}$$

Theorem 3.1 *For an integer $m \geq 2$ and a sequence $\{\lambda_j : j \in M_0(\mathbb{Z})\}$, the local linear operator sequence $\{\mathcal{P}_r : r \in \mathbb{Z}\}$, where $\mathcal{P}_r : \mathcal{S}_m^{r+1} \to \mathcal{S}_m^r$, as defined by*

$$\mathcal{P}_r f = \sum_j \left[\sum_k \lambda_{2j-k} c_k\right] N_m(2^r \cdot -j) \quad \textit{for} \quad f = \sum_j c_j N_m(2^{r+1} \cdot -j), \tag{3.6}$$

satisfies the reproduction property (3.1) if and only if the Laurent polynomial Λ given by

$$\Lambda(z) = \sum_j \lambda_j z^j, \quad z \in \mathbb{C}\backslash\{0\}, \tag{3.7}$$

satisfies the Bezout identity

$$(1+z)^m \Lambda(z) + (1-z)^m \Lambda(-z) = 2^m, \quad z \in \mathbb{C}\backslash\{0\}. \tag{3.8}$$

Proof. Let $r \in \mathbb{Z}$ be fixed, and suppose $f \in \mathcal{S}_m^r$, i.e. there exists a sequence $c \in M(\mathbb{Z})$ such that $f = \sum_j c_j N_m(2^r \cdot -j)$. From (2.12) and (2.14), we deduce that

$$\begin{aligned} f(x) &= \sum_j c_j \sum_k a_{m,k} N_m(2^{r+1}x - 2j - k) \\ &= \sum_j c_j \sum_k a_{m,k-2j} N_m(2^{r+1}x - k) \\ &= \sum_k \left[\sum_j a_{m,k-2j} c_j\right] N_m(2^{r+1}x - k). \end{aligned} \tag{3.9}$$

For a sequence $\{\lambda_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$, it follows from (3.6) and (3.9) that

$$\begin{aligned} (\mathcal{P}_r f)(x) &= \sum_j \left[\sum_k \lambda_{2j-k} \left(\sum_l a_{m,k-2l} c_l\right)\right] N_m(2^r x - j) \\ &= \sum_j \left[\sum_l \left(\sum_k a_{m,k-2l} \lambda_{2j-k}\right) c_k\right] N_m(2^r x - j). \end{aligned} \tag{3.10}$$

It follows from (3.9) and (3.10) that, for $x \in \mathbb{R}$,

$$\begin{aligned} f(x) - (\mathcal{P}_r f)(x) &= \sum_j \left[c_j - \sum_l \left(\sum_k a_{m,k-2l} \lambda_{2j-k}\right) c_l\right] N_m(2^r x - j) \\ &= \sum_j \left[\sum_l \left(\delta_{j,l} - \sum_k a_{m,k-2l} \lambda_{2j-k}\right) c_k\right] N_m(2^r x - j). \end{aligned} \tag{3.11}$$

Since also Theorem 2.3 implies $\sum_j c_j N(\cdot - j) = 0$ if and only if $c_j = 0,\ j \in \mathbb{Z}$, we deduce from (3.11) that the reproduction property (3.1) holds if and only if the sequence $\{\lambda_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ satisfies the condition

$$\sum_k a_{m,k-2l} \lambda_{2j-k} = \delta_{j,l}, \quad j, l \in \mathbb{Z}. \tag{3.12}$$

It remains to show that the two conditions (3.12) and (3.8) are equivalent. To this end, we use (2.15) and (3.7) to deduce that, for $j \in \mathbb{Z}$ and $z \in \mathbb{C}\backslash\{0\}$,

$$\begin{aligned}
\sum_l \sum_k a_{m,k-2l}\lambda_{2j-k}z^{2l} &= \sum_l \sum_k a_{m,2k-2l}\lambda_{2j-2k}z^{2l-2k}z^{2k} + \sum_l \sum_k a_{m,2k-2l+1}\lambda_{2j-2k-1}z^{2l-2k-1}z^{2k+1} \\
&= \sum_k \lambda_{2j-2k}\left[\sum_l a_{m,2k-2l}z^{2l-2k}\right]z^{2k} \\
&\quad + \sum_k \lambda_{2j-2k-1}\left[\sum_l a_{m,2k-2l+1}z^{2l-2k-1}\right]z^{2k+1} \\
&= \sum_k \lambda_{2j-2k}\left[\sum_l a_{m,2l}z^{-2l}\right]z^{2k} \\
&\quad + \sum_k \lambda_{2j-2k-1}\left[\sum_l a_{m,2l+1}z^{-(2l+1)}\right]z^{2k+1} \\
&= A_m^{(e)}(z^{-1})\left[\sum_k \lambda_{2j-2k}z^{2k-2j}\right]z^{2j} + A_m^{(o)}(z^{-1})\left[\sum_k \lambda_{2j-2k-1}z^{2k-2j+1}\right]z^{2j} \\
&= A_m^{(e)}(z^{-1})\left[\sum_k \lambda_{2k}z^{-2k}\right]z^{2j} + A_m^{(o)}(z_{-1})\left[\sum_k \lambda_{2k+1}z^{-(2k+1)}\right]z^{2j} \\
&= z^{2j}\left[A_m^{(e)}(z^{-1})\Lambda^{(e)}(z^{-1}) + A_m^{(o)}(z^{-1})\Lambda^{(o)}(z^{-1})\right] \\
&= z^{2j}\left[\frac{A_m(z^{-1})+A_m(-z^{-1})}{2}\frac{\Lambda(z^{-1})+\Lambda(-z^{-1})}{2}\right. \\
&\quad \left.+\frac{A_m(z^{-1})-A_m(-z^{-1})}{2}\frac{\Lambda(z^{-1})-\Lambda(-z^{-1})}{2}\right] \\
&= \frac{1}{2}z^{2j}\left[A_m(z^{-1})\Lambda(z^{-1}) + A_m(-z^{-1})\Lambda(-z^{-1})\right], \qquad (3.13)
\end{aligned}$$

whereas

$$\sum_l \delta_{j,l}z^{2l} = z^{2j}, \quad j \in \mathbb{Z}, \quad z \in \mathbb{C}. \qquad (3.14)$$

It follows from (3.13) and (3.14) that, for $j \in \mathbb{Z}$ and $z \in \mathbb{C}\backslash\{0\}$, we have

$$\sum_l \left[\sum_k a_{m,k-2l}\lambda_{2j-k} - \delta_{j,l}\right]z^{2l} = z^{2j}\left[\frac{A_m(z^{-1})\Lambda(z^{-1}) + A_m(-z^{-1})\Lambda(-z^{-1})}{2} - 1\right]. \quad (3.15)$$

According to (3.15), $\{\lambda_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ is a sequence satisfying (3.12) if and only if the corresponding Laurent polynomial Λ defined by (3.7) satisfies the Bezout identity

$$A_m(z^{-1})\Lambda(z^{-1}) + A_m(-z^{-1})\Lambda(-z^{-1}) = 2, \quad z \in \mathbb{C}\backslash\{0\},$$

or, equivalently,

$$A_m(z)\Lambda(z) + A_m(-z)\Lambda(-z) = 2, \quad z \in \mathbb{C}\backslash\{0\}. \tag{3.16}$$

By substituting (2.15) into (3.16), we conclude that (3.16) is equivalent to (3.8), which then completes our proof. ■

We proceed to find the Laurent polynomial $\Lambda = \Lambda_m$ of shortest possible length satisfying the Bezout identity (3.8).

3.2 The Generating Polynomial H_m

Based on results in [11] and [17], we first prove the following result with respect to a polynomial solution H of the Bezout identity

$$(1+z)^m H(z) - (1-z)^m H(-z) = 2^m z^{2\lfloor \frac{1}{2}m \rfloor -1}, \quad z \in \mathbb{C}, \tag{3.17}$$

with $\lfloor x \rfloor$ denoting the largest integer less than or equal to x.

Theorem 3.2 *The recursion formulation*

$$\left.\begin{array}{rcl} H_2(z) & = & 1, \\ & & \\ H_{2k+1}(z) & = & \dfrac{2H_{2k}(z) - 2^{1-2k}H_{2k}(-1)(1-z)^{2k}}{1+z}, \\ & & \\ H_{2k+2}(z) & = & \dfrac{2z^2H_{2k+1}(z) - 2^{-2k}H_{2k+1}(-1)(1-z)^{2k+1}}{1+z}, \end{array}\right\}, \quad z \in \mathbb{C}, \tag{3.18}$$

yields a sequence $\{H_m : m = 2, 3, \ldots, \}$ *of polynomials such that*

$$\deg\,(H_m) = m-2, \quad m = 2, 3, \ldots \tag{3.19}$$

and where $H = H_m$ *is the only polynomial in* π_{m-1} *satisfying the Bezout identity (3.17).*

Proof. Since the numerator in the second and third lines of (3.18) both vanish at $z = -1$, it follows inductively from (3.18) that $\{H_m : m = 2, 3, \ldots\}$ is indeed a sequence of polynomials such that (3.19) is satisfied.

Next, we prove by induction on m that the polynomial H_m satisfies the Bezout identity (3.17).

For $m = 2$, we see from first line of (3.18) that, for $z \in \mathbb{C}$, we have

$$(1+z)^2 H_2(z) - (1-z)^2 H_2(-z) = (1+z)^2 - (1-z)^2 = 4z,$$

i.e. the polynomial H_2 satisfies (3.17) for $m = 2$. Suppose next that, for a fixed integer $k \in \mathbb{N}$, it holds that

$$(1+z)^{2k} H_{2k}(z) - (1-z)^{2k} H_{2k}(-z) = 2^{2k} z^{2k-1}, \quad z \in \mathbb{C}. \tag{3.20}$$

Using the second line of (3.18), together with the inductive hypothesis (3.20), we obtain, for $z \in \mathbb{C}$,

$$\begin{aligned}
&(1+z)^{2k+1} H_{2k+1}(z) - (1-z)^{2k+1} H_{2k+1}(-z) \\
&\quad = (1+z)^{2k} \left[2H_{2k}(z) - 2^{1-2k} H_{2k}(-1)(1-z)^{2k}\right] \\
&\quad -(1-z)^{2k} \left[2H_{2k}(-z) - 2^{1-2k} H_{2k}(-1)(1+z)^{2k}\right] \\
&\quad = 2\left[(1+z)^{2k} H_{2k}(z) - (1-z)^{2k} H_{2k}(-z)\right] \\
&\quad = 2^{2k+1} z^{2k-1}
\end{aligned} \tag{3.21}$$

i.e. the polynomial $H = H_{2k+1}$ satisfies the Bezout identity (3.17) for $m = 2k+1$. Now we use the third line of (3.18), together with (3.21), to deduce, for $z \in \mathbb{C}$, that

$$\begin{aligned}
&(1+z)^{2k+2} H_{2k+2}(z) - (1-z)^{2k+2} H_{2k+2}(-z) \\
&\quad = (1+z)^{2k+1} \left[2z^2 H_{2k+1}(z) - 2^{-2k} H_{2k+1}(-1)(1-z)^{2k+1}\right] \\
&\quad -(1-z)^{2k+1} \left[2z^2 H_{2k+1}(-z) - 2^{-2k} H_{2k+1}(-1)(1+z)^{2k+1}\right] \\
&\quad = 2z^2 \left[(1+z)^{2k+1} H_{2k+1}(z) - (1-z)^{2k+1} H_{2k+1}(-z)\right] \\
&\quad = 2z^2 (2^{2k+1} z^{2k-1}) \\
&\quad = 2^{2k+2} z^{2k+1},
\end{aligned}$$

i.e. the polynomial $H = H_{2k+2}$ satisfies the Bezout identity (3.17) for $m = 2k+2$, and thereby concluding our inductive proof of the fact that the polynomial $H = H_m$ satisfies the Bezout identity (3.17) for every integer $m \geq 2$.

Finally, we prove that H_m is the only solution in π_{m-1} of the Bezout identity (3.17).

Suppose $\tilde{H}_m \in \pi_{m-1}$ satisfies

$$(1+z)^m \tilde{H}_m(z) - (1-z)^m \tilde{H}_m(-z) = 2^m z^{2\lfloor \frac{1}{2}m \rfloor - 1}, \quad z \in \mathbb{C}. \tag{3.22}$$

Since also

$$(1+z)^m H_m(z) - (1-z)^m H_m(-z) = 2^m z^{2\lfloor \frac{1}{2}m \rfloor - 1}, \quad z \in \mathbb{C}, \tag{3.23}$$

it follows by subtracting (3.22) from (3.23) that

$$(1+z)^m \left[H_m(z) - \tilde{H}_m(z) \right] = (1-z)^m \left[H_m(-z) - \tilde{H}_m(-z) \right], \quad z \in \mathbb{C}. \tag{3.24}$$

Since the two polynomials $(1+z)^m$ and $(1-z)^m$ have no common factors, we deduce from (3.24) that there exists a polynomial $J(z)$ such that

$$H_m(z) - \tilde{H}_m(z) = J(z)(1-z)^m, \quad z \in \mathbb{C}. \tag{3.25}$$

Suppose $J \neq 0$. Then (3.25) gives

$$m - 1 \geq \deg\, (H_m - \tilde{H}_m) = \deg\, (J) + m \geq m,$$

a contradiction. Hence $J = 0$, so that (3.25) yields $H_m = \tilde{H}_m$, i.e. H_m is the only solution in π_{m-1} of the Bezout identity (3.17). ■

Using (3.18), we calculate in Table 3.1 the sequence $\{h_{m,j} : j \in \mathbb{Z}\}$, where we denote by $\{h_{m,j} : j = 0, 1, \ldots, m-2\}$ the coefficients of the polynomial H_m, i.e.

$$H_m(z) = \sum_{j=0}^{m-2} h_{m,j} z^j, \quad z \in \mathbb{C}. \tag{3.26}$$

m	$h_{m,j}$
2	$\{h_{2,0}\} = \{1\}$
3	$\{h_{3,0}, h_{3,1}\} = \{\frac{3}{2}, -\frac{1}{2}\}$
4	$\{h_{4,0}, h_{4,1}, h_{4,2}\} = \{-\frac{1}{2}, 2, -\frac{1}{2}\}$

Table 3.1: The sequence $h_{m,j}$ for $m = 2, 3, 4$

Now let the polynomial Λ_m be defined by

$$\Lambda_m(z) = z^{-2\lfloor \frac{1}{2}m \rfloor + 1} H_m(z), \quad z \in \mathbb{C}\backslash\{0\}, \tag{3.27}$$

which, together with (3.23), then yields, for $z \in \mathbb{C}\backslash\{0\}$,

$$\begin{aligned} (1+z)^m \Lambda_m(z) - (1-z)^m \Lambda_m(-z) &= z^{-2\lfloor \frac{1}{2}m \rfloor + 1} \left[(1+z)^m H_m(z) - (1-z)^m H_m(-z) \right] \\ &= z^{-2\lfloor \frac{1}{2}m \rfloor + 1} \left[2^m z^{2\lfloor \frac{1}{2}m \rfloor - 1} \right] = 2^m, \end{aligned}$$

so that the following consequence of Theorem 3.2 can now be stated.

Corollary 3.3 *For an integer $m \geq 2$, the Laurent polynomial $\Lambda = \Lambda_m$, as defined by (3.27), with the polynomial H_m obtained recursively from (3.18), is a Laurent polynomial of shortest possible length satisfying the Bezout identity (3.8).*

Combining the results of Theorem 3.1 and Corollary 3.3, we immediately get the following result.

Theorem 3.4 *For an integer $m \geq 2$, the sequence $\{\lambda_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ defined by*

$$\lambda_j = \lambda_{m,j}, \quad j \in \mathbb{Z}, \tag{3.28}$$

where

$$\sum_j \lambda_{m,j} z^j = z^{-2\lfloor \frac{1}{2}m \rfloor + 1} H_m(z), \quad z \in \mathbb{C}\setminus\{0\}, \tag{3.29}$$

with H_m denoting the polynomial of degree $m-2$ obtained recursively from (3.18), is a sequence of shortest possible length such that the operator sequence $\{\mathcal{P}_r : \mathcal{S}_m^{r+1} \to \mathcal{S}_m^r,\ r \in \mathbb{Z}\}$, as defined by (3.6), satisfies the reproduction property (3.1).

Accordingly, we define, for any $r \in \mathbb{Z}$, the projection operator $\mathcal{P}_{m,r} : \mathcal{S}_m^{r+1} \to \mathcal{S}_m^r$ by

$$\mathcal{P}_{m,r} f = \sum_j \left[\sum_k \lambda_{m,2j-k} c_k \right] N_m(2^r \cdot -j) \text{ for } f = \sum_j c_j N_m(2^{r+1} \cdot -j), \tag{3.30}$$

with the sequence $\{\lambda_{m,j} : j \in \mathbb{Z}\}$ defined by (3.29), and for which it then holds that

$$\mathcal{P}_{m,r} f = f, \quad f \in \mathcal{S}_m^r. \tag{3.31}$$

We shall also rely on the following properties of the polynomial H_m of Theorem 3.2.

A polynomial P of degree n, as given by

$$P(z) = \sum_{j=0}^{n} p_j z^j, \quad \text{with} \quad p_n \neq 0, \tag{3.32}$$

is said to be symmetric polynomial if it holds that

$$p_{n-j} = p_j, \quad j = 0, 1, \ldots, n. \tag{3.33}$$

Since it holds for $z \in \mathbb{C}$ that, from (3.32) and (3.33),

$$z^n p\left(\frac{1}{z}\right) = z^n \sum_{j=0}^{n} p_j z^{-j} = \sum_{j=0}^{n} p_j z^{n-j} = \sum_{j=0}^{n} p_{n-j} z^j,$$

we see that the symmetry condition (3.33) has the equivalent formulation,

$$z^n p\left(\frac{1}{z}\right) = p(z), \quad z \in \mathbb{C}\backslash\{0\}. \tag{3.34}$$

The following result holds.

Proposition 3.5 *For $m \geq 2$, the polynomial H_m of Theorem 3.2 satisfies the following properties;*

(a) *If m is even, then H_m is a symmetric polynomial;*

(b) *$H_m(0) \neq 0$.*

Proof.

(a) Suppose $m = 2n$ for an integer $n \in \mathbb{N}$, according to which it follows from (3.23) that

$$(1+z)^{2n} H_{2n}(z) - (1-z)^{2n} H_{2n}(-z) = 2^{2n} z^{2n-1}, \quad z \in \mathbb{C}. \tag{3.35}$$

By replacing z by $\frac{1}{z}$ in (3.35), we obtain

$$\left(1+\frac{1}{z}\right)^{2n} H_{2n}\left(\frac{1}{z}\right) - \left(1-\frac{1}{z}\right)^{2n} H_{2n}\left(-\frac{1}{z}\right) = 2^{2n} z^{-2n+1}, \quad z \in \mathbb{C}$$

and thus

$$(1+z)^{2n} \tilde{H}_{2n}(z) - (1-z)^{2n} \tilde{H}_{2n}(-z) = 2^{2n} z^{2n-1}, \quad z \in \mathbb{C}, \tag{3.36}$$

where

$$\tilde{H}_{2n}(z) = z^{2n-2} H_{2n}\left(\frac{1}{z}\right), \quad z \in \mathbb{C}. \tag{3.37}$$

Since (3.19) gives $\deg\,(H_{2n}) = 2n-2$, it follows from (3.37) that $\tilde{H}_{2n}$ is a polynomial, with $\deg\,(\tilde{H}_{2n}) = 2n-2$. But, as shown in the proof of Theorem 3.2, $H = H_{2n}$ is the only polynomial in π_{2n-1} satisfying the Bezout identity (3.17), and thus, using also (3.37), we get

$$H_{2n}(z) = \tilde{H}_{2n}(z) = z^{2n-2} H_{2n}\left(\frac{1}{z}\right), \quad z \in \mathbb{C}\backslash\{0\},$$

and it follows that H_{2n} is a symmetric polynomial.

(b) According to (3.19), the polynomial H_m can be formulated as

$$H_m(z) = \sum_{j=0}^{m-2} h_{m,j} z^j, \quad z \in \mathbb{C}, \quad \text{with} \quad h_{m,m-2} \neq 0. \tag{3.38}$$

If $m = 2n$ for $n \in \mathbb{N}$, then the symmetry result in (a) gives

$$h_{2n,0} = h_{2n,2n-2} \neq 0,$$

from (3.38), and thus property (b) holds for even integers m.

If $m = 3$, we see from Table 3.1 that $H_3(0) = \dfrac{3}{2} \neq 0$. Next, for $m = 2n+1$, with $n \geq 2$, suppose $H_{2n+1}(0) = 0$. But then there exists a polynomial $\tilde{H}_{2n+1}$, with $\deg\,(\tilde{H}_{2n+1}) = 2n-2$, such that

$$H_{2n+1}(z) = z\tilde{H}_{2n+1}(z), \quad z \in \mathbb{C}. \tag{3.39}$$

Since also (3.23) yields

$$(1+z)^{2n+1} H_{2n+1}(z) - (1-z)^{2n+1} H_{2n+1}(-z) = 2^{2n+1} z^{2n-1}, \quad z \in \mathbb{C}, \tag{3.40}$$

we now substitute (3.39) into (3.40) to deduce that

$$(1+z)^{2n+1} \tilde{H}_{2n+1}(z) + (1-z)^{2n+1} \tilde{H}_{2n+1}(-z) = 2^{2n+1} z^{2n-2}, \quad z \in \mathbb{C}. \tag{3.41}$$

By setting $z = 0$ in (3.41), and recalling that $n \geq 2$, we obtain $2\tilde{H}_{2n+1}(0) = 0$, and thus $\tilde{H}_{2n+1}(0) = 0$, i.e. there exists a polynomial $\tilde{\tilde{H}}_{2n+1}$, with $\deg\,(\tilde{\tilde{H}}_{2n+1}) = 2n-3$, such that

$$\tilde{H}_{2n+1}(z) = z\tilde{\tilde{H}}_{2n+1}(z), \quad z \in \mathbb{C}. \tag{3.42}$$

By substituting (3.42) into (3.41), we obtain

$$(1+z)^{2n-1} \left[(1+z)^2 \tilde{\tilde{H}}_{2n+1}(z)\right] - (1-z)^{2n-1} \left[(1-z)^2 \tilde{\tilde{H}}_{2n+1}(-z)\right] = 2^{2n+1} z^{2n-3}, \quad z \in \mathbb{C}. \tag{3.43}$$

Since also (3.23) gives

$$(1+z)^{2n-1} H_{2n-1}(z) - (1-z)^{2n-1} H_{2n-1}(z) = 2^{2n-1} z^{2n-3}, \quad z \in \mathbb{C}, \tag{3.44}$$

we can now subtract (3.43) from (3.44) to obtain

$$\begin{aligned} &(1+z)^{2n-1} \left[4H_{2n-1}(z) - (1+z)^2 \tilde{\tilde{H}}_{2n+1}(z)\right] \\ &\qquad = (1-z)^{2n-1} \left[4H_{2n-1}(-z) - (1-z)^2 \tilde{\tilde{H}}_{2n+1}(-z)\right], \quad z \in \mathbb{C}. \end{aligned} \tag{3.45}$$

Since the two polynomials $(1+z)^{2n-1}$ and $(1-z)^{2n-1}$ have no common factors, and since $\deg\,(H_{2n-1}) = \deg\,(\tilde{\tilde{H}}_{2n+1}) = 2n-3$, it follows from (3.45) that there exists a constant c such that

$$4H_{2n-1}(z) - (1+z)^2\tilde{\tilde{H}}_{2n+1}(z) = c(1-z)^{2n-1}, \quad z \in \mathbb{C}. \tag{3.46}$$

By setting $z = -1$ in (3.46), it follows that $c = 2^{3-2n}H_{2n-1}(-1)$, which we can now substitute into (3.46) to obtain

$$4H_{2n-1}(z) = (1+z)^2\tilde{\tilde{H}}_{2n+1}(z) + 2^{3-2n}H_{2n-1}(-1)(1-z)^{2n-1}, \quad z \in \mathbb{C}. \tag{3.47}$$

Now substitute (3.47) into the third line of (3.18) with $k = n-1$ to deduce that

$$4H_{2n}(z) = \frac{1}{2}z^2(1+z)\tilde{\tilde{H}}_{2n+1}(z) + \frac{2^{-2n}(8z^2-1)H_{2n-1}(-1)(1-z)^{2n-1}}{1+z}, \quad z \in \mathbb{C}\backslash\{0\}. \tag{3.48}$$

Since both H_{2n} and $\tilde{\tilde{H}}_{2n+1}$ are polynomials, we deduce from (3.48) that we must have

$$H_{2n-1}(-1) = 0. \tag{3.49}$$

Substituting (3.49) into (3.48) yield

$$H_{2n}(z) = 2z^2(1+z)\tilde{\tilde{H}}_{2n+1}(z), \quad z \in \mathbb{C},$$

and thus

$$2n-2 = \deg\,(H_{2n}) = 3 + \deg\,(\tilde{\tilde{H}}_{2n+1}) = 3 + (2n-3) = 2n,$$

a contradiction. Hence $H_{2n+1}(0) \neq 0$, and thereby completing our proof of (b). ■

3.3 Example

Using Theorem 3.2, Corollary 3.3 and equations (3.28) and (3.29), we explicitly calculate the sequence $\lambda = \lambda_m \in M_0(\mathbb{Z})$ for $m = 2, 3, 4, 5$ in Table 3.2.

By choosing, respectively, $c_j = \delta_j$ and $c_j = \delta_{j-1}$, $j \in \mathbb{Z}$ in (3.30), together with Table 3.2, we compute the functions $\mathcal{P}_{m,0}N_m(2\cdot) = \sum_j \lambda_{m,2j}N_m(\cdot - j)$ and $\mathcal{P}_{m,0}N_m(2\cdot -1) = \sum_j \lambda_{m,2j-1}N_m(\cdot - j)$ for $m = 2, 3, 4$. The resulting graphs are shown in Figures 3.1 to 3.3.

m	$\lambda_{m,j}$
2	$\{\lambda_{2,-1}\} = \{1\}$
3	$\{\lambda_{3,0}, \lambda_{3,-1}\} = \{-\frac{1}{2}, \frac{3}{2}\}$
4	$\{\lambda_{4,-1}, \lambda_{4,-2}, \lambda_{4,-3}\} = \{-\frac{1}{2}, 2, -\frac{1}{2}\}$

Table 3.2: The sequences $\{\lambda_{m,j} : j \in \mathbb{Z}, m = 2, 3, 4\}$.

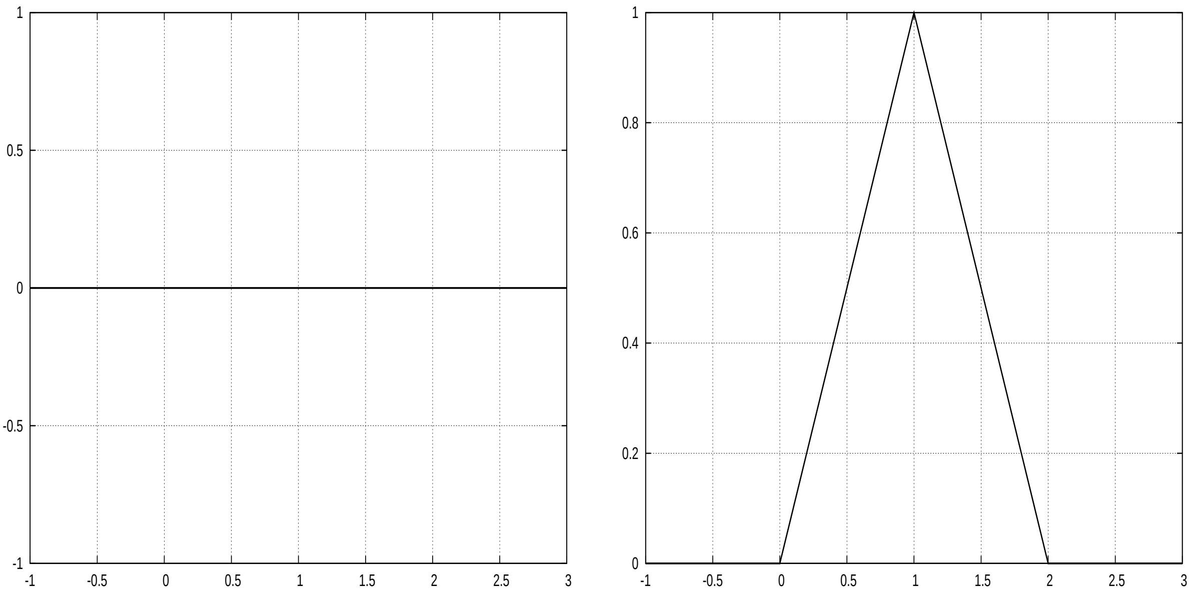


Figure 3.1: The functions $\mathcal{P}_{2,0}N_2(2\cdot)$ and $\mathcal{P}_{2,0}N_2(2\cdot -1)$

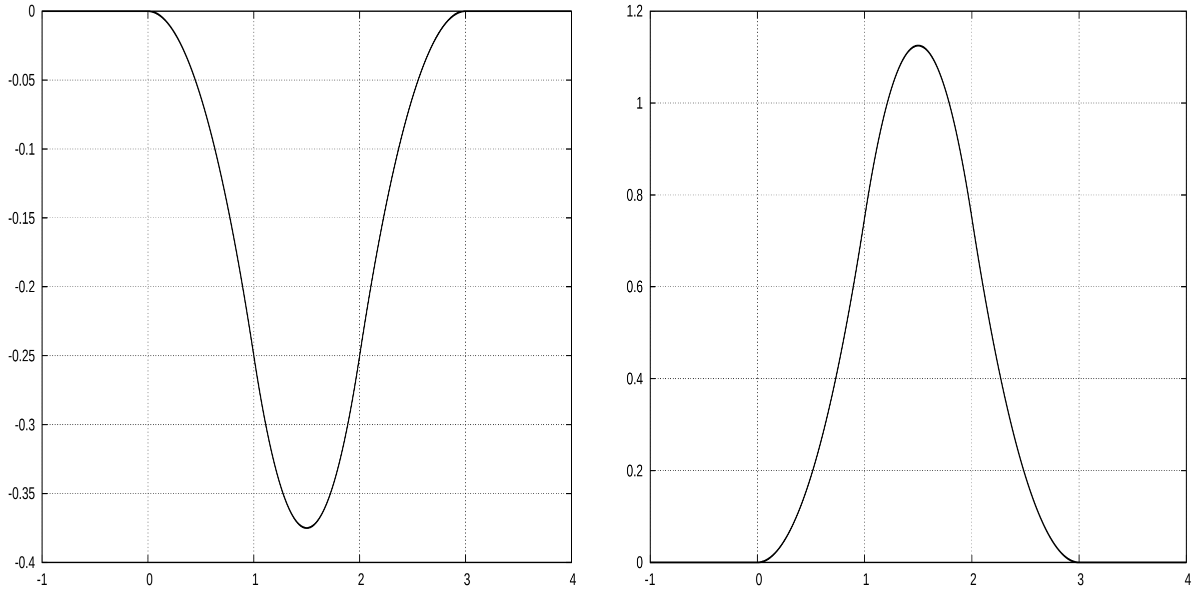


Figure 3.2: The functions $\mathcal{P}_{3,0}N_3(2\cdot)$ and $\mathcal{P}_{3,0}N_3(2\cdot -1)$

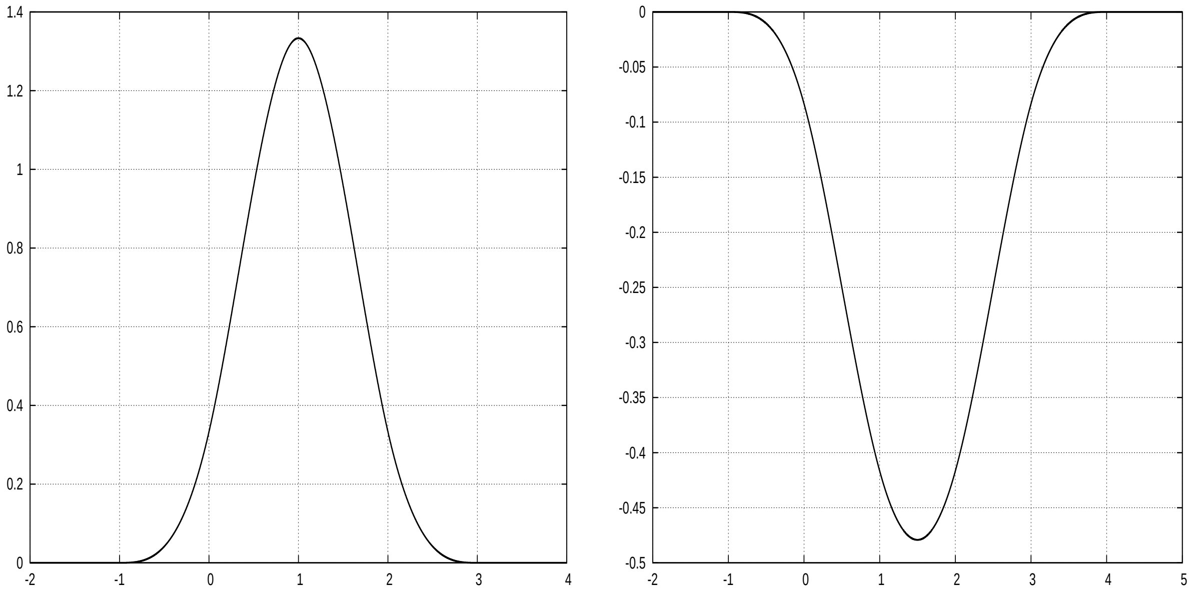


Figure 3.3: The functions $\mathcal{P}_{4,0}N_4(2\cdot)$ and $\mathcal{P}_{4,0}N_4(2\cdot -1)$

4. Cardinal Spline Wavelets

With the local linear projection sequence operator $\{\mathcal{P}_{m,r} : r \in \mathbb{Z}\}$ as in Theorem 3.4, we now define the linear space sequence $\{W_m^r : r \in \mathbb{Z}\}$ by

$$W_m^r = \left\{ f - \mathcal{P}_{m,r} f : f \in \mathcal{S}_m^{r+1} \right\}, \quad r \in \mathbb{Z}. \tag{4.1}$$

Since $\mathcal{P}_{m,r} : \mathcal{S}_m^{r+1} \to \mathcal{S}_m^r$, and $\mathcal{S}_m^r \subset \mathcal{S}_m^{r+1}$, we observe from (4.1) that

$$W_m^r \subset \mathcal{S}_m^{r+1}, \quad r \in \mathbb{Z}. \tag{4.2}$$

Hence, if for a given $r \in \mathbb{Z}$ we have $f \in \mathcal{S}_m^{r+1}$, then $f = g + h$, where $g = \mathcal{P}_{m,r} f \in \mathcal{S}_m^r$ and $h = f - \mathcal{P}_{m,r} f \in W_m^r$.

A function $\psi_m \in \mathcal{S}_m^1$ which is such that

$$W_m^r = \left\{ \sum_j c_j \psi_m(2^r \cdot - j) : c \in M(\mathbb{Z}) \right\}, \quad r \in \mathbb{Z}, \tag{4.3}$$

is called the m^{th} order cardinal B-spline wavelet generated by the local linear projection sequence $\{\mathcal{P}_{m,r} : r \in \mathbb{Z}\}$.

4.1 The Wavelet Bezout Identity

To find a wavelet ψ_m, we first prove the following result.

Proposition 4.1 *The linear space sequence $\{W_m^r : r \in \mathbb{Z}\}$ defined by (4.1) satisfies*

$$W_m^r = \left\{ f \in \mathcal{S}_m^{r+1} : \mathcal{P}_{m,r} f = 0 \right\}, \quad r \in \mathbb{Z}. \tag{4.4}$$

Proof. Let $r \in \mathbb{Z}$ be fixed and suppose that $f \in W_m^r$. Then, according to (4.1), there exists a function $g \in \mathcal{S}_m^{r+1}$ such that $f = g - \mathcal{P}_{m,r} g$. From the reproduction property (3.31), the linearity of $\mathcal{P}_{m,r}$, and the fact that $\mathcal{P}_{m,r} g \in \mathcal{S}_m^r$, we have

$$\mathcal{P}_{m,r} f = \mathcal{P}_{m,r} g - \mathcal{P}_{m,r}^2 g = \mathcal{P}_{m,r} g - \mathcal{P}_{m,r} g = 0, \tag{4.5}$$

i.e. $f \in \{g \in \mathcal{S}_m^{r+1} : \mathcal{P}_{m,r} g = 0\}$.

Suppose $f \in \mathcal{S}_m^{r+1}$ is such that $\mathcal{P}_{m,r} f = 0$. Then

$$f = f - 0 = f - \mathcal{P}_{m,r} f, \tag{4.6}$$

so that, from (4.4), we have the inclusion $W_m^r \subset \{f \in \mathcal{S}_m^{r+1} : \mathcal{P}_{m,r} = 0\}$, and thereby completing our proof. ■

Using Proposition 4.1, we can now obtain the following characterization of all finitely supported functions $\psi \in W_m^0$. Note first from (4.2) that, if $\psi \in W_m^0 \cap M_0(\mathbb{R})$, there exists a sequence $\{\gamma_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ such that

$$\psi = \sum_j \gamma_j N_m(2 \cdot -j), \tag{4.7}$$

and for which we define its corresponding Laurent polynomial symbol by

$$\Gamma(z) = \sum_j \gamma_j z^j, \quad z \in \mathbb{C}\backslash\{0\}. \tag{4.8}$$

Our result is as follows.

Theorem 4.2 *For a sequence $\{\gamma_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$, let the cardinal spline $\psi \in \mathcal{S}_m^1$ be defined by (4.7). Then ψ belongs to the space*

$$W_m^0 = \{f - \mathcal{P}_{m,0} f : f \in \mathcal{S}_m^1\} \tag{4.9}$$

if and only if the symbol Γ defined by (4.8) is given by

$$\Gamma(z) = K(z) H_m(-z), \quad z \in \mathbb{C}\backslash\{0\}, \tag{4.10}$$

where H_m is the polynomial of degree $m-2$ defined in Theorem 3.2, and with K denoting an arbitrary even Laurent polynomial, i.e.

$$K(-z) = K(z), \quad z \in \mathbb{C}\backslash\{0\}. \tag{4.11}$$

Proof. From (3.30) and (4.7), we have

$$\mathcal{P}_{m,0}\psi = \sum_j \left[\sum_k \lambda_{m,2j-k} \gamma_k \right] N_m(\cdot - j), \tag{4.12}$$

from which, together with Proposition 4.1 and Theorem 2.3, we see that the cardinal spline ψ in (4.7) belongs to W_m^0 if and only if the sequence $\{\gamma_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ is chosen such that

$$\sum_k \lambda_{m,2j-k}\gamma_k = 0. \tag{4.13}$$

Using the fact that $\Lambda_m(z) = \sum_j \lambda_{m,j}z^j,\ z \in \mathbb{C}\backslash\{0\}$, together with (4.8), we obtain, for $z \in \mathbb{C}\backslash\{0\}$,

$$\begin{aligned}
\sum_j \left[\sum_k \lambda_{m,2j-k}\gamma_k\right] z^{2j} &= \sum_j \left[\sum_k \lambda_{m,2j-2k}\gamma_{2k}\right] z^{2j} + \sum_j \left[\sum_k \lambda_{m,2j-2k-1}\gamma_{2k+1}\right] z^{2j} \\
&= \sum_j \left[\sum_k \lambda_{m,2j-2k} z^{2j-2k}\right] \gamma_{2k} z^{2k} \\
&\quad + \sum_j \left[\sum_k \lambda_{m,2j-2k-1} z^{2j-2k-1}\right] \gamma_{2k+1} z^{2k+1} \\
&= \sum_j \left[\sum_k \lambda_{m,2j} z^{2j}\right] \gamma_{2k} z^{2k} + \sum_j \left[\sum_k \lambda_{m,2j+1} z^{2j+1}\right] \gamma_{2k+1} z^{2k+1} \\
&= \Lambda_m^{(e)}(z)\Gamma^{(e)}(z) + \Lambda_m^{(o)}(z)\Gamma^{(o)}(z) \\
&= \frac{\Lambda_m(z)+\Lambda_m(-z)}{2}\frac{\Gamma(z)+\Gamma(-z)}{2} + \frac{\Lambda_m(z)-\Lambda_m(-z)}{2}\frac{\Gamma(z)-\Gamma(-z)}{2} \\
&= \frac{1}{2}\left[\Lambda_m(z)\Gamma(z) + \Lambda_m(-z)\Gamma(-z)\right],
\end{aligned}$$

from which it follows that a sequence $\{\gamma_j : j \in \mathbb{Z}\}$ satisfies the condition (4.13) if and only if the corresponding Laurent polynomial Γ, as given by (4.8), satisfies the Bezout identity

$$\Lambda_m(z)\Gamma(z) = -\Lambda_m(-z)\Gamma(-z), \quad z \in \mathbb{C}\backslash\{0\} \tag{4.14}$$

Since also Λ_m is given by the formula (3.27), it follows that (4.14) is equivalent to the Bezout identity

$$H_m(z)\Gamma(z) = H_m(-z)\Gamma(-z), \quad z \in \mathbb{C}\backslash\{0\}. \tag{4.15}$$

Now observe from (3.23), together with Proposition (3.5), that the polynomials $H_m(z)$ and $H_m(-z)$ have no common factors, so that we can deduce from (4.15) that there exists a Laurent polynomial $K(z)$ such that (4.10) holds. Also, by substituting (4.10) into (4.15), we obtain

$$H_m(z)K(z)H_m(-z) = H_m(-z)K(-z)H_m(z), \quad z \in \mathbb{C}\backslash\{0\},$$

from which it follows that (4.11) holds, and thereby completing our proof. ∎

The following result can now be deduced from Theorem 4.2.

Theorem 4.3 *The function $\psi_m \in \mathcal{S}_m^1$ defined by*

$$\psi_m(x) = \sum_{j=0}^{m-2} (-1)^j h_{m,j} N_m(2x - j), \tag{4.16}$$

where

$$\sum_{j=0}^{m-2} h_{m,j} z^j = H_m(z), \quad z \in \mathbb{C}, \tag{4.17}$$

with H_m denoting the polynomial of Theorem 3.2, is a minimally-supported non-trivial function in the space W_m^0 defined by (4.9), with

$$\psi_m(x) = 0, \quad x \notin (0, m-1). \tag{4.18}$$

Proof. Since a polynomial $K(z)$ of minimally degree satisfying (4.11) is given by $K(z) = 1$, $z \in \mathbb{C}$, we deduce from (4.10) that

$$\Gamma(z) = \Gamma_m(z) = H_m(-z), \quad z \in \mathbb{C}, \tag{4.19}$$

is a polynomial of least possible degree as described by (4.10) and (4.11) in Theorem 4.2. Writing

$$\Gamma_m(z) = \sum_{j=0}^{m-2} \gamma_{m,j} z^j, \quad z \in \mathbb{C}, \tag{4.20}$$

it follows from (4.19), (4.17) and (4.20) that

$$\gamma_{m,j} = (-1)^j h_{m,j}, \quad j \in \mathbb{Z}. \tag{4.21}$$

It follows from Theorem 4.2, together with (4.7), (4.20) and (4.21) that the function ψ_m defined by (4.16) is indeed a minimally supported non-trivial function in W_m.

The finite support property of (4.18) is a direct consequence of (4.16) and (2.6). ∎

Using (4.16) and Table 3.1, we obtain the spline-wavelets

$$\begin{aligned}
\psi_2(x) &= N_2(2x), \\
\psi_3(x) &= \frac{3}{2} N_3(2x) + \frac{1}{2} N_3(2x-1), \\
\psi_4(x) &= -\frac{1}{2} N_4(2x) - 2N_4(2x-1) - \frac{1}{2} N_4(2x-2),
\end{aligned}$$

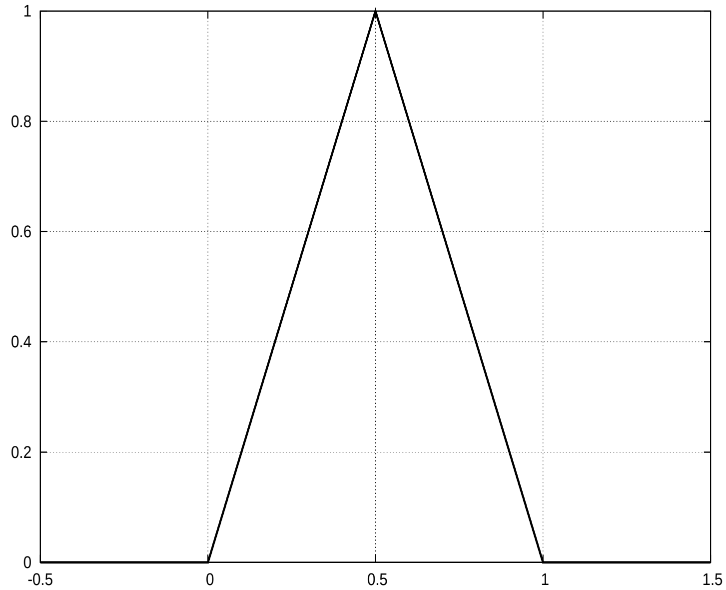


Figure 4.1: Graph of the function ψ_2

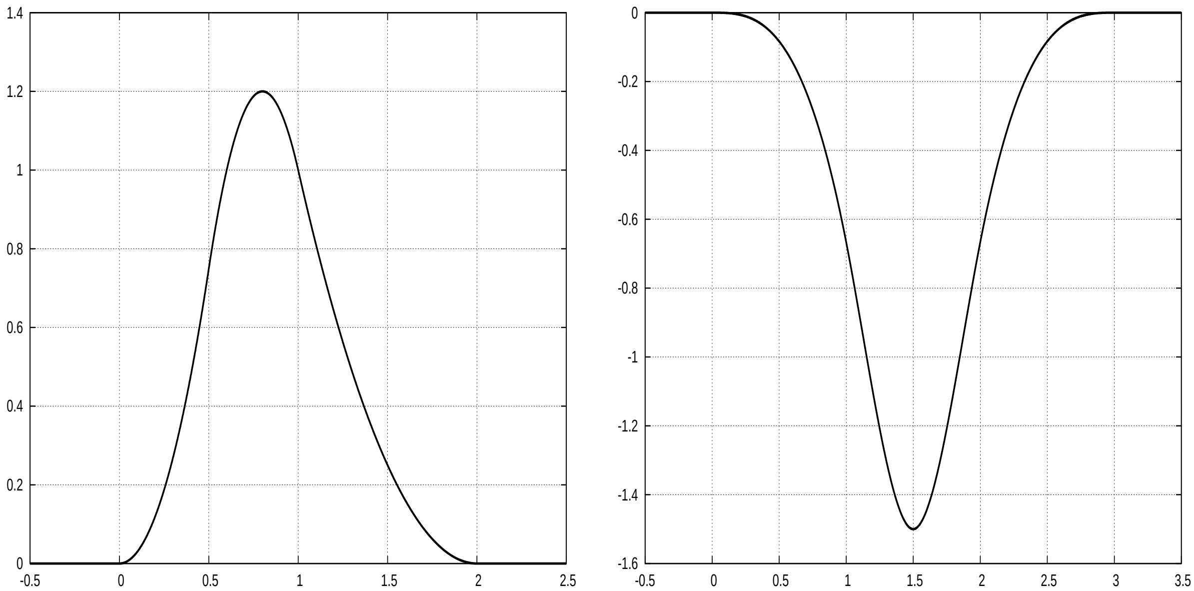


Figure 4.2: Graph of the functions ψ_3 and ψ_4

of which the graphs are shown in Figures 4.1 and 4.2.

We proceed to show that the function ψ_m of Theorem 4.3 satisfies the property (4.3), according to which we will then have shown that ψ_m is indeed a wavelet.

4.2 The Fundamental Decomposition Result

Our fundamental decomposition result is as follows.

Theorem 4.4 *For an integer $m \geq 2$, let the function $\psi_m \in W_m^0$ be defined as in Theorem 4.3. Then, for $r \in \mathbb{Z}$, it holds for any function $f = \sum_j c_j N_m(2^{r+1} \cdot -j) \in \mathcal{S}_m^{r+1}$ that*

$$\begin{aligned} f &= \sum_j \left[\sum_k h_{m,2j-k+2\lfloor\frac{1}{2}m\rfloor-1} c_k \right] N_m(2^r \cdot -j) \\ &+ \frac{1}{2^{m-1}} \sum_j \left[\sum_k (-1)^k \binom{m}{2j-k+2\lfloor\frac{1}{2}m\rfloor-1} c_k \right] \psi_m(2^r \cdot -j), \end{aligned} \tag{4.22}$$

where

$$\sum_j h_{m,j} z^j = \sum_{j=0}^{m-2} h_{m,j} z^j = H_m(z), \quad z \in \mathbb{C}, \tag{4.23}$$

with H_m denoting the polynomial of Theorem 3.2.

Remark. Observe from (4.22) that, for $k \in \mathbb{Z}$, by choosing $f = N_m(2^{r+1} \cdot -k)$, i.e. $f = \sum_l c_l N_m(2^{r+1} \cdot -l)$ with $c_l = \delta_{l,k}$, we obtain the decomposition result

$$\begin{aligned} N_m(2^{r+1} \cdot -k) &= \sum_j h_{m,2j-k+2\lfloor\frac{1}{2}m\rfloor-1} N_m(2^r \cdot -j) \\ &+ \frac{(-1)^k}{2^{m-1}} \sum_j \binom{m}{2j-k+2\lfloor\frac{1}{2}m\rfloor-1} \psi_m(2^r \cdot -j), \\ & k \in \mathbb{Z}. \end{aligned} \tag{4.24}$$

Proof of Theorem 4.4. Let $r \in \mathbb{Z}$ be fixed and suppose $f = \sum_j c_j N_m(2^{r+1} \cdot -j)$ for a sequence $c \in M(\mathbb{Z})$. Using (3.30) and the refinement equation (2.12) for cardinal B-splines, we obtain, with the sequence $\{a_{m,j} : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ defined by (2.14),

$$\begin{aligned}
(f - \mathcal{P}_{m,r}f)(x) &= \sum_j c_j N_m(2^{r+1}x - j) - \sum_j \sum_k \lambda_{m,2j-k} c_k \left[\sum_l a_{m,l} N_m(2^{r+1}x - 2j - l)\right] \\
&= \sum_j c_j N_m(2^{r+1}x - j) - \sum_j \sum_k \lambda_{m,2j-k} c_k \sum_l a_{m,l-2j} N_m(2^{r+1}x - l) \\
&= \sum_j c_j N_m(2^{r+1}x - j) - \sum_l \left[\sum_k \lambda_{m,2l-k} c_k\right] \sum_j a_{m,j-2l} N_m(2^{r+1}x - j) \\
&= \sum_j c_j N_m(2^{r+1}x - j) - \sum_j \sum_l \left[\sum_k \lambda_{m,2l-k} c_k\right] a_{m,j-2l} N_m(2^{r+1}x - j) \\
&= \sum_j \left[\sum_k \left(\delta_{j,k} - \sum_l \lambda_{m,2l-k} a_{m,j-2l}\right) c_k\right] N_m(2^{r+1}x - j). \qquad (4.25)
\end{aligned}$$

Let $\{\omega_j : j \in \mathbb{Z}\}$ denote a sequence in $M_0(\mathbb{Z})$. Then, using (4.16), we have

$$\begin{aligned}
\sum_j \left[\sum_k \omega_{2j-k} c_k\right] \psi_m(2^r x - j) &= \sum_j \sum_k \omega_{2j-k} c_k \left[\sum_l \gamma_{m,l} N_m(2^{r+1}x - 2j - l)\right] \\
&= \sum_j \sum_k \omega_{2j-k} c_k \left[\sum_l \gamma_{m,l-2j} N_m(2^{r+1}x - l)\right] \\
&= \sum_l \sum_k \omega_{2l-k} c_k \sum_j \gamma_{m,j-2l} N_m(2^{r+1}x - j) \\
&= \sum_j \left[\sum_k \left(\sum_l \gamma_{m,j-2l} \omega_{2l-k}\right) c_k\right] N_m(2^{r+1}x - j). \quad (4.26)
\end{aligned}$$

It follows from (4.25) and (4.26) that $\{\omega_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ is such that

$$(f - \mathcal{P}_{m,r}f)(x) = \sum_j \left[\sum_k \omega_{2j-k} c_k\right] \psi_m(2^r x - j), \quad x \in \mathbb{R}, \qquad (4.27)$$

if and only if

$$\sum_j \left[\sum_k \left(\delta_{j,k} - \sum_l a_{m,j-2l} \lambda_{m,2l-k} - \sum_l \gamma_{m,j-2l} \omega_{2l-k}\right) c_k\right] N_m(2^{r+1}x - j) = 0, \quad x \in \mathbb{R}, \qquad (4.28)$$

which holds if and only if the sequence $\{\omega_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ satisfies the condition

$$\sum_l a_{m,j-2l} \lambda_{m,2l-k} + \sum_l \gamma_{m,j-2l} \omega_{2l-k} = \delta_{j,k}, \quad j,k \in \mathbb{Z}. \qquad (4.29)$$

We proceed to show that the condition (4.29) is equivalent to a pair of Bezout identities.

With the Laurent polynomial Ω defined by

$$\Omega(z) = \sum_j \omega_j z^j, \quad z \in \mathbb{C}\backslash\{0\}, \tag{4.30}$$

we use (3.7) and (4.30) to deduce that, for $j \in \mathbb{Z}$ and $z \in \mathbb{C}\backslash\{0\}$, we have

$$\begin{aligned}
&\sum_k \left[\sum_l a_{m,j-2l}\lambda_{m,2l-k} + \sum_l \gamma_{m,j-2l}\omega_{2l-k}\right] z^k \\
&\quad = \sum_l \left[\sum_k \lambda_{m,2l-k}(z^{-1})^{2l-k}\right] a_{m,j-2l}z^{2l} + \sum_l \left[\sum_k \omega_{2l-k}(z^{-1})^{2l-k}\right]\gamma_{m,j-2l}z^{2l} \\
&\quad = \sum_l \left(\sum_k \lambda_{m,k}(z^{-1})^k\right) a_{m,j-2l}z^{2l} + \sum_l \left(\sum_k \omega_k(z^{-1})^k\right)\gamma_{m,j-2l}z^l \\
&\quad = z^j\left[\left(\sum_l a_{m,j-2l}z^{2l-j}\right)\Lambda_m(z^{-1}) + \left(\sum_l \gamma_{m,j-2l}z^{2l-j}\right)\Omega(z^{-1})\right].
\end{aligned} \tag{4.31}$$

Since also

$$\sum_k \delta_{j,k}z^k = z^j, \quad z \in \mathbb{C},$$

we deduce from (4.31) that the condition (4.29) is satisfied if and only if

$$\left[\sum_l a_{m,j-2l}z^{2l-j}\right]\Lambda_m(z^{-1}) + \left[\sum_l \gamma_{m,j-2l}z^{2l-j}\right]\Omega(z^{-1}) = 1, \quad z \in \mathbb{C}\backslash\{0\}, \quad j \in \mathbb{Z}, \tag{4.32}$$

or equivalently,

$$\left[\sum_l a^m_{j-2l}z^{j-2l}\right]\Lambda_m(z) + \left[\sum_l \gamma_{m,j-2l}z^{j-2l}\right]\Omega(z) = 1, \quad z \in \mathbb{C}\backslash\{0\}, \quad j \in \mathbb{Z}. \tag{4.33}$$

But (4.33) holds for every $j \in \mathbb{Z}$ if and only if it holds for all even j and for all odd j. Hence, using (2.15), we find that (4.33) is equivalent to the pair of Bezout identities

$$\left.\begin{array}{rcl} A_m^{(e)}(z)\Lambda_m(z) + \Gamma_m^{(e)}(z)\Omega(z) & = & 1 \\ A_m^{(o)}(z)\Lambda_m(z) + \Gamma_m^{(o)}(z)\Omega(z) & = & 1 \end{array}\right\} z \in \mathbb{C}\backslash\{0\}, \tag{4.34}$$

i.e.,

$$\left.\begin{array}{rcl} \left[A_m(z) + A_m(-z)\right]\Lambda_m(z) + \left[\Gamma_m(z) + \Gamma_m(-z)\right]\Omega(z) & = & 2 \\ \left[A_m(z) - A_m(-z)\right]\Lambda_m(z) + \left[\Gamma_m(z) - \Gamma_m(-z)\right]\Omega(z) & = & 2 \end{array}\right\} z \in \mathbb{C}\backslash\{0\}, \tag{4.35}$$

which holds if and only if

$$\left.\begin{array}{rcl} A_m(z)\Lambda_m(z)+\Gamma_m(z)\Omega(z) & = & 2 \\ A_m(-z)\Lambda_m(z)+\Gamma_m(-z)\Omega(z) & = & 0 \end{array}\right\} z\in\mathbb{C}\backslash\{0\}. \tag{4.36}$$

By using (2.15), (3.27) and (4.19), we find that Ω is a Laurent polynomial satisfying (4.36) if and only if the pair of Bezout identities

$$\left.\begin{array}{rcl} z^{-2\lfloor\frac{1}{2}m\rfloor+1}(1+z)^m H_m(z)+2^{m-1}H_m(-z)\Omega(z) & = & 2^m \\ \left[\frac{1}{2^{m-1}}z^{-2\lfloor\frac{1}{2}m\rfloor+1}(1-z)^m+\Omega(z)\right]H_m(z) & = & 0 \end{array}\right\} z\in\mathbb{C}\backslash\{0\} \tag{4.37}$$

are satisfied.

The second line in (4.37) is satisfied if and only if the Laurent polynomial Ω is chosen as

$$\Omega(z)=\Omega_m(z)=-\frac{1}{2^{m-1}}z^{-2\lfloor\frac{1}{2}m\rfloor+1}(1-z)^m,\quad z\in\mathbb{C}\backslash\{0\}. \tag{4.38}$$

Substituting (4.38) into the left hand side of the first equation in (4.37) yields the expression

$$z^{-2\lfloor\frac{1}{2}m\rfloor+1}\left[(1+z)^m H_m(z)-(1-z)^m H_m(-z)\right],\quad z\in\mathbb{C}\backslash\{0\}, \tag{4.39}$$

which together with the Bezout identity (3.17), shows that the choice (4.38) of Ω also satisfies the first equation in (4.37). Therefore, the pair of Bezout identities (4.37) are satisfied by a Laurent polynomial Ω if and only if Ω is given by (4.38), according to which also, from (4.30), we have for $z\in\mathbb{C}\backslash\{0\}$ that

$$\begin{aligned} \sum_j \omega_j z^j &= -\frac{1}{2^{m-1}}z^{-2\lfloor\frac{1}{2}m\rfloor+1}\sum_j(-1)^j\binom{m}{j}z^j \\ &= -\frac{1}{2^{m-1}}\sum_j(-1)^j\binom{m}{j}z^{j-2\lfloor\frac{1}{2}m\rfloor+1} \\ &= \frac{1}{2^{m-1}}\sum_j(-1)^j\binom{m}{j+2\lfloor\frac{1}{2}m\rfloor-1}z^j, \end{aligned}$$

and thus the sequence $\{\omega_j : j\in\mathbb{Z}\}\in M_0(\mathbb{Z})$ satisfies the condition (4.29) if and only if

$$\omega_j=\omega_{m,j}=\frac{(-1)^j}{2^{m-1}}\binom{m}{j+2\lfloor\frac{1}{2}m\rfloor-1},\quad j\in\mathbb{Z}. \tag{4.40}$$

By combining (3.39), (3.30), (4.27) and (4.40), we see that (4.22) does indeed hold. ■

4.3 Decomposition and Reconstruction Algorithms

Given a signal $f \in M(\mathbb{R})$, we proceed to describe the cardinal B-spline wavelet decomposition of f into detail components.

Algorithm 4.5 *1. For a sufficiently large positive integer N, we set*

$$f_N = \mathcal{Q}_{m,N} f \tag{4.41}$$

where $\mathcal{Q}_{m,N}$ is the quasi-interpolation operator as in Theorem 2.9 and with the choice $\tau = m-1$. Then

$$f_N = \sum_j c_j^{(N)} N_m(2^N. - j) \tag{4.42}$$

with the sequence $c^N \in M(\mathbb{Z})$ given by

$$c_j^{(N)} = \sum_k u_{j-k} f\left(\frac{k+m-1}{2^N}\right). \tag{4.43}$$

2. For a sufficiently small positive integer $M \in \mathbb{N}$ and using Theorem 4.4, if we define the sequences $\{f_r : r = N-1, \ldots, N-M\}$ and $\{g_r : r = N-1, \ldots, N-M\}$ by

$$\left.\begin{aligned} f_r &= \textstyle\sum_j c_j^{(r)} N_m(2^r. - j) \\ g_r &= \textstyle\sum_j d_j^{(r)} \psi_m(2^r. - j) \end{aligned}\right\} r = N-1, \ldots, N-M, \tag{4.44}$$

where the sequences $c^{(r)} \in M(\mathbb{Z})$ and $d^{(r)} \in M(\mathbb{Z})$ are obtained recursively from (4.43) and the equations

$$\left.\begin{aligned} c_j^{(r)} &= \textstyle\sum_k h_{m,2j-k+2\left\lfloor \frac{1}{2}m \right\rfloor -1} c_k^{(r+1)} \\ d_j^{(r)} &= \textstyle\sum_k \frac{(-1)^k}{2^{m-1}} \binom{m}{2j-k+2\left\lfloor \frac{1}{2}m \right\rfloor -1} c_k^{(r+1)} \end{aligned}\right\} j \in \mathbb{Z}, \quad r = N-1, \ldots, N-M \tag{4.45}$$

so that $f_r \in \mathcal{S}_m^r$, $r = N-1, \ldots, N-M$ and $g_r \in W_m^r$, $r = N-1, \ldots, N-M$, then the decomposition result

$$f_{r+1} = f_r + g_r, \quad r = N-1, \ldots, N-M \tag{4.46}$$

holds.

It follows from (4.46) that

$$f_N = f_{N-M} + \sum_{r=N-M}^{N-1} g_r. \tag{4.47}$$

The equations (4.43) and (4.45) are known as the wavelet decomposition algorithm associated with the cardinal spline projection operator sequence $\{\mathcal{P}_{m,r} : r \in \mathbb{Z}\}$ and the corresponding wavelet ψ_m.

Once a signal has been decomposed using the wavelet decomposition algorithm (4.43) and (4.45), and the data has been processed, we describe next how to reconstruct the signal according to the right hand side of (4.47).

Algorithm 4.6 *1. By using (4.44) and the refinement equation (2.12), we obtain, for $r = N-1,\ldots,N-M$,*

$$\begin{aligned} f_r &= \sum_j c_j^{(r)} \sum_k a_{m,k} N_m(2^{r+1}. - 2j - k) \\ &= \sum_j c_j^{(r)} \sum_k a_{m,k-2j} N_m(2^{r+1}. - k) \\ &= \sum_k \left[\sum_j a_{m,k-2j} c_j^{(r)} \right] N_m(2^{r+1}. - k) \end{aligned} \tag{4.48}$$

whereas, from (4.16),

$$\begin{aligned} g_r &= \sum_j d_j^{(r)} \sum_k (-1)^k h_{m,k} N_m(2^{r+1}. - 2j - k) \\ &= \sum_j d_j^{(r)} \sum_k (-1)^k h_{m,k-2j} N_m(2^{r+1}. - k) \\ &= \sum_k \left[\sum_j (-1)^k h_{m,k-2j} d_j^{(r)} \right] N_m(2^{r+1}. - k) \end{aligned} \tag{4.49}$$

2. *Combining (4.48) and (4.49) and the first line of (4.44), we deduce that the desired equation (4.46) holds if and only if the condition*

$$\sum_j \left[c_j^{(r+1)} - \left(\sum_k a_{m,j-2k} c_k^{(r)} + \sum_k (-1)^k h_{m,j-2k} d_k^{(r)} \right) \right] N_m(2^{r+1} \cdot -k) = 0 \tag{4.50}$$

is satisfied. Hence, if we set

$$c_j^{(r+1)} = \sum_k a_{m,j-2k} c_j^{(r)} + \sum_k (-1)^k h_{m,j-2k} d_k^{(r)}, \quad j \in \mathbb{Z}, \quad r = N-1,\ldots,N-M, \tag{4.51}$$

then (4.50), and therefore also (4.49) are satisfied.

The equation (4.51) is called the cardinal spline wavelet reconstruction algorithm.

4.4 Singularity Detection Property

We proceed to prove that our decomposition algorithm defined by Algorithm 4.5 possesses the essential property to locally detect singularities in a given signal f.

We shall rely on the following result.

Proposition 4.7 *The sequence $a_m \in M_0(\mathbb{Z})$ defined by (2.14) satisfies the condition*

$$\sum_k (-1)^k a_{m,2j+2\left\lfloor \frac{1}{2}m \right\rfloor -1-k} p(k) = 0, \quad p \in \pi_{m-1}, \quad j \in \mathbb{Z}. \tag{4.52}$$

Proof. We shall prove that

$$\sum_k (-1)^k a_{m,2j+2\left\lfloor \frac{1}{2}m \right\rfloor -1-k} k^l = 0, \quad j \in \mathbb{Z}, \quad l = 0, 1, \ldots, m-1 \tag{4.53}$$

which is equivalent to the condition (4.52)

Now note that, for $j \in \mathbb{Z}$, by using also the fact that $2\left\lfloor \frac{1}{2}m \right\rfloor - 1$ is an odd integer, as well as (2.13), we have

$$\begin{aligned}
&\sum_k (-1)^k a_{m,2j+2\left\lfloor \frac{1}{2}m \right\rfloor -1-k} k^l \\
&= \sum_k (-1)^{2j+2\left\lfloor \frac{1}{2}m \right\rfloor -1-k} a_{m,k} \left(2j + 2\left\lfloor \frac{1}{2}m \right\rfloor - 1 - k\right)^l \\
&= \sum_k (-1)^{2j+2\left\lfloor \frac{1}{2}m \right\rfloor -1-k} \sum_{q=0}^{l} \binom{l}{q} (-1)^q k^q \left(2j + 2\left\lfloor \frac{1}{2}m \right\rfloor - 1\right)^{l-q} \\
&= -\sum_k (-1)^k a_{m,k} \sum_{q=0}^{l} \binom{l}{q} (-1)^q k^q \left(2j + 2\left\lfloor \frac{1}{2}m \right\rfloor - 1\right)^{l-q} \\
&= -\frac{1}{2^{m-1}} \sum_k (-1)^k \binom{m}{k} \sum_{q=0}^{l} \binom{l}{q} (-1)^q k^q \left(2j + 2\left\lfloor \frac{1}{2}m \right\rfloor - 1\right)^{l-q} \\
&= -\frac{1}{2^{m-1}} \sum_{q=0}^{l} (-1)^q \binom{l}{q} \left(2j + 2\left\lfloor \frac{1}{2}m \right\rfloor - 1\right)^{l-q} \sum_k (-1)^k \binom{m}{k} k^q.
\end{aligned} \tag{4.54}$$

Our proof will be complete if we can show that

$$\sum_k (-1)^k \binom{m}{k} k^q = 0, \quad q = 0, 1, \ldots, m-1, \tag{4.55}$$

which, together with (4.54), then yield the desired result, (4.53).

We prove (4.55) by induction on the integer m.

Note that (4.55) holds for $m = 1$. Next suppose that (4.55) holds for a fixed $m \in \mathbb{N}$. We define

$$\mathcal{A}_{m,q} = \sum_k (-1)^k \binom{m}{k} k^q, \quad q \in \mathbb{Z}_+.$$

First, note that

$$\mathcal{A}_{m+1,0} = \sum_k (-1)^k \binom{m+1}{k} = (1-1)^{m+1} = 0.$$

Next, for $q \in \{1, 2, \ldots, m\}$, we have

$$\begin{aligned}
\mathcal{A}_{m+1,q} &= \sum_k (-1)^k \binom{m+1}{k} k^q \\
&= \sum_k (-1)^k \frac{(m+1)!}{k!(m+1-k)!} k^q \\
&= (m+1) \sum_k (-1)^k \frac{m!}{(k-1)!(m+1-k)!} k^{q-1} \\
&= (m+1) \sum_k (-1)^{k+1} \frac{m!}{k!(m-k)!} (k+1)^{q-1} \\
&= -(m+1) \sum_k (-1)^k \binom{m}{k} \sum_{r=0}^{q-1} \binom{q-1}{r} k^r \\
&= -(m+1) \sum_{r=0}^{q-1} \binom{q-1}{r} \sum_k (-1)^k \binom{m}{k} k^r \\
&= -(m+1) \sum_{r=0}^{q-1} \binom{q-1}{r} \mathcal{A}_{m,r} = 0
\end{aligned}$$

from the inductive hypothesis, and thereby completing our proof of (4.55). ■

The result of Proposition 4.7 enables us to prove the following fundamental property of the decomposition algorithm given by Algorithm 4.5.

Theorem 4.8 *For the wavelet ψ_m of Theorem 4.3, if we choose $f \in \pi_{m-1}$ in the decomposition algorithm given by Algorithm 4.5, then the wavelet coefficient sequences $\{d_j^{(r)} : r = N -$*

$1, \dots, N-M\} \in M(\mathbb{Z})$ *satisfy*

$$d_j^{(r)} = 0, \quad j \in \mathbb{Z}, \quad r = N-1, \dots, N-M. \tag{4.56}$$

Proof. Let $\sigma \in \{0, 1, \dots, m-1\}$ be fixed. It will suffice to prove our theorem for the choice

$$f(x) = x^\sigma, \quad x \in \mathbb{R}. \tag{4.57}$$

First, for $r = N-1$, we have from (2.30), (4.43), (4.45), (4.57) and Proposition 4.7, that, for $j \in \mathbb{Z}$,

$$\begin{aligned} d_j^{N-1} &= \sum_k (-1)^k a_{m,2j+2\lfloor\frac{1}{2}m\rfloor-1-k} \left[\sum_k u_{k-l} \left(\frac{l+\tau_0}{2^N} \right)^\sigma \right] \\ &= \frac{\sigma!}{2^{N\sigma}(m-1)!} \sum_k (-1)^k a_{m,2j+2\lfloor\frac{1}{2}m\rfloor-1-k} Q^{(m-1-\sigma)}(k) = 0, \end{aligned} \tag{4.58}$$

since $Q^{(m-1-\sigma)} \in \pi_{m-1}$, $\sigma \in \{0, 1, \dots, m-1\}$.

Similarly, with the polynomial $\tilde{Q} \in \pi_{m-1}$ defined by

$$\tilde{Q} = \frac{\sigma!}{2^{N\sigma}(m-1)!} Q,$$

we deduce that, for $j \in \mathbb{Z}$, we have

$$\begin{aligned} d_j^{(N-2)} &= \sum_k (-1)^k a_{m,2j+2\lfloor\frac{1}{2}m\rfloor-1-k} c_k^{(N-1)} \\ &= \sum_k (-1)^k a_{m,2j+2\lfloor\frac{1}{2}m\rfloor-1-k} \sum_l \lambda_{m,2k-l} c_l^{(N)} \\ &= \sum_k (-1)^k a_{m,2j+2\lfloor\frac{1}{2}m\rfloor-1-k} \sum_l \lambda_{m,2k-l} \tilde{Q}^{(m-1-\sigma)}(l) \\ &= \sum_k (-1) a_{m,2j+2\lfloor\frac{1}{2}m\rfloor-1-k} \left[\sum_l \lambda_{m,2k-2l} \tilde{Q}^{(m-1-\sigma)}(2l) \right. \\ &\qquad \left. + \sum_l \lambda_{m,2k-2l-1} \tilde{Q}^{(m-1-\sigma)}(2l+1) \right] \\ &= \sum_k (-1) a_{m,2j+2\lfloor\frac{1}{2}m\rfloor-1-k} \left[\sum_l \lambda_{m,2l} \tilde{Q}^{(m-1-\sigma)}(2k-2l) \right. \\ &\qquad \left. + \sum_l \lambda_{m,2l+1} \tilde{Q}^{(m-1-\sigma)}(2k-2l-1) \right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_l \lambda_{m,2l}\left[\sum_k (-1)^k a_{m,2j+2\lfloor\frac{1}{2}m\rfloor-1-k}\tilde{Q}^{(m-1-\sigma)}(2k-2l)\right] \\
&\quad + \sum_l \lambda_{m,2l+1}\left[\sum_k (-1)^k a_{m,2j+2\lfloor\frac{1}{2}m\rfloor-1-k}\tilde{Q}^{(m-1-\sigma)}(2k-2l-1)\right] \\
&= \left[\sum_l \lambda_{m,2l}\right](0) + \left[\sum_l \lambda_{m,2l+1}\right](0) = 0, \qquad (4.59)
\end{aligned}
$$

from Proposition 4.7, after having used also the fact that, since $\tilde{Q}^{(m-1-\sigma)} \in \pi_{m-1}$, $\sigma \in \{0,1,\ldots,m-1\}$, we also have $\tilde{Q}^{(m-1-\sigma)}(2\cdot -2l) \in \pi_{m-1}$, and $\tilde{Q}^{(m-1-\sigma)}(2\cdot -2l-1) \in \pi_{m-1}$.

Repeated use of this procedure yields

$$
d_j^{(r)} = 0, \quad j \in \mathbb{Z}, \quad r = N-3,\ldots,N-M. \qquad (4.60)
$$

Our result (4.56) then follows from (4.58), (4.59) and (4.60). ■

The result of Theorem 4.8 has the following important implication with respect to the cardinal B-spline wavelet decomposition algorithm given by Algorithm 4.5. If the signal f is locally C^m-smooth in a certain region, so that, according to Taylors theorem, f is locally well approximated by a polynomial in π_{m-1} in that region, it follows from Theorem 4.8 that, for a given $r \in \mathbb{Z}$, the wavelet coefficients $d_j^{(r)}$ can be expected to be relatively small whenever the support interval $\left[\frac{j}{2^r}, \frac{j+m-1}{2^r}\right]$, as implied by (4.18), of the wavelet $\psi_m(2^r\cdot -j)$, overlaps with this C^m-smooth region of f, thereby providing localised information, at each resolution level r, on the smoothness of f.

4.5 Example

We consider the signal $f \in C(\mathbb{R})$ given by $f = N_3$, so that, from (2.4), we have

$$
f(x) = \begin{cases} \frac{1}{2}x^2, & 0 \le x \le 1, \\ \frac{1}{2}(-2x^2+6x-3), & 1 \le x \le 2, \\ \frac{1}{2}(3-x)^2, & 2 \le x \le 3, \\ 0, & x \notin [0,3), \end{cases} \qquad (4.61)
$$

as shown in Figure 4.3. Then $f \in C^1(\mathbb{R})\backslash C^2(\mathbb{R})$, with discontinuities in the second derivatives

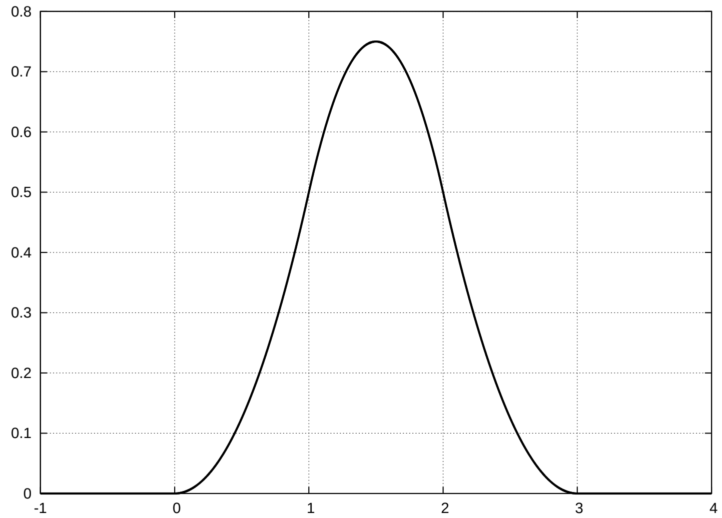


Figure 4.3: The signal f

f'' at $x \in \{0, 1, 2, 3, \}$.

We use the wavelet decomposition Algorithm 4.5 based on the cardinal B-spline of order $m = 4$, the local linear projection operator sequence $\{\mathcal{P}_{m,r} : r \in \mathbb{Z}\}$, as given by (3.30), and where the corresponding wavelet ψ_4 according to (4.16) and Table 3.1, is given by

$$\psi_4(x) = -\frac{1}{2}N_4(2x) - 2N_4(2x-1) - \frac{1}{2}N_4(2x-2), \tag{4.62}$$

as shown in Figure 4.2. The quasi-interpolant approximation $f_{10} = \mathcal{Q}_{4,10}f$ is shown in Figure 4.4.

Next, we graph, for $r = N, \ldots, N-M$, where $N = 10$ and $M = 5$, the functions f_r and g_r as shown in Figures 4.5 to 4.9 by using (4.44), (4.45), (4.43), (4.16), (2.7) and (2.3). We see that the singularities in the second derivatives f'' at $x \in \{0, 1, 2, 3\}$ are efficiently detected by our decomposition, with sharply defined localisation.

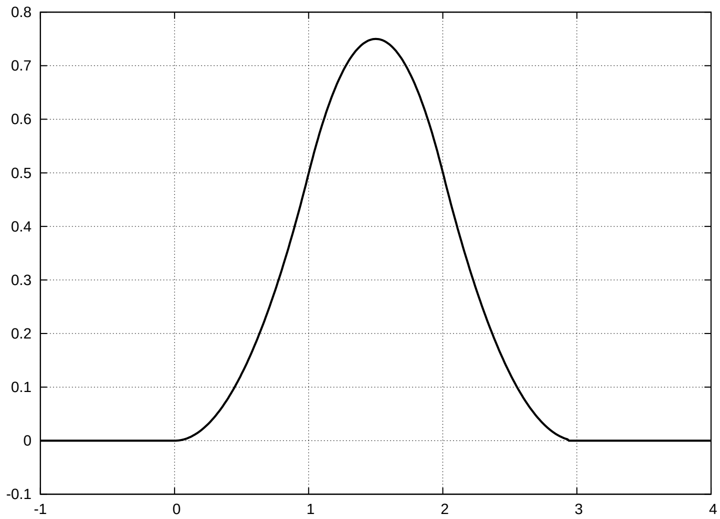


Figure 4.4: The function $\mathcal{Q}_{4,10}f$

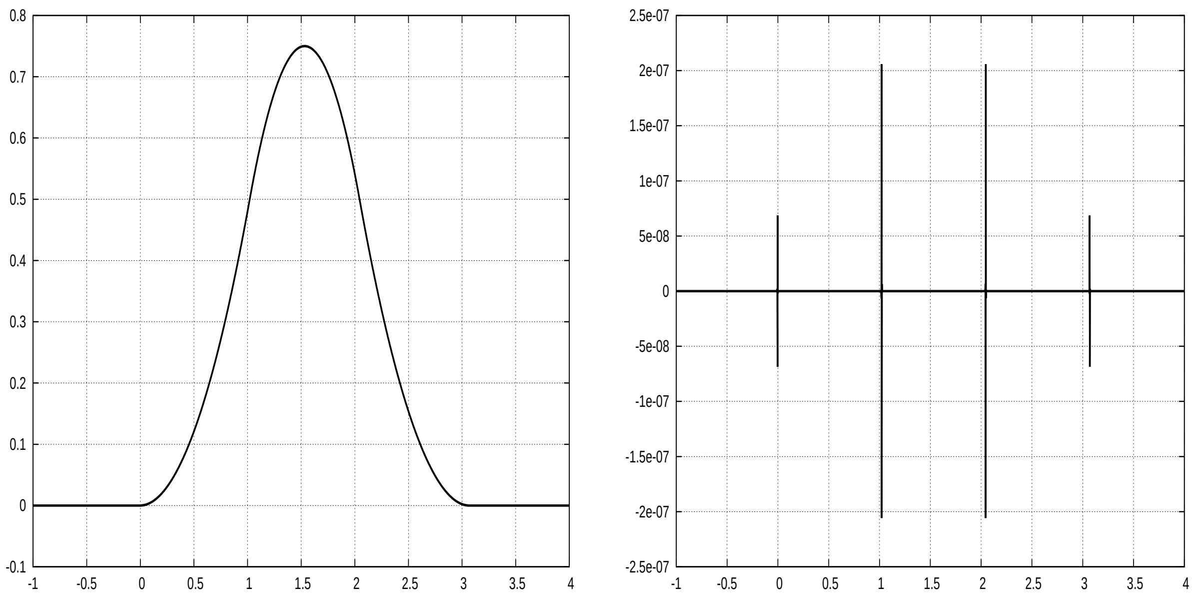


Figure 4.5: The functions f_9 and g_9

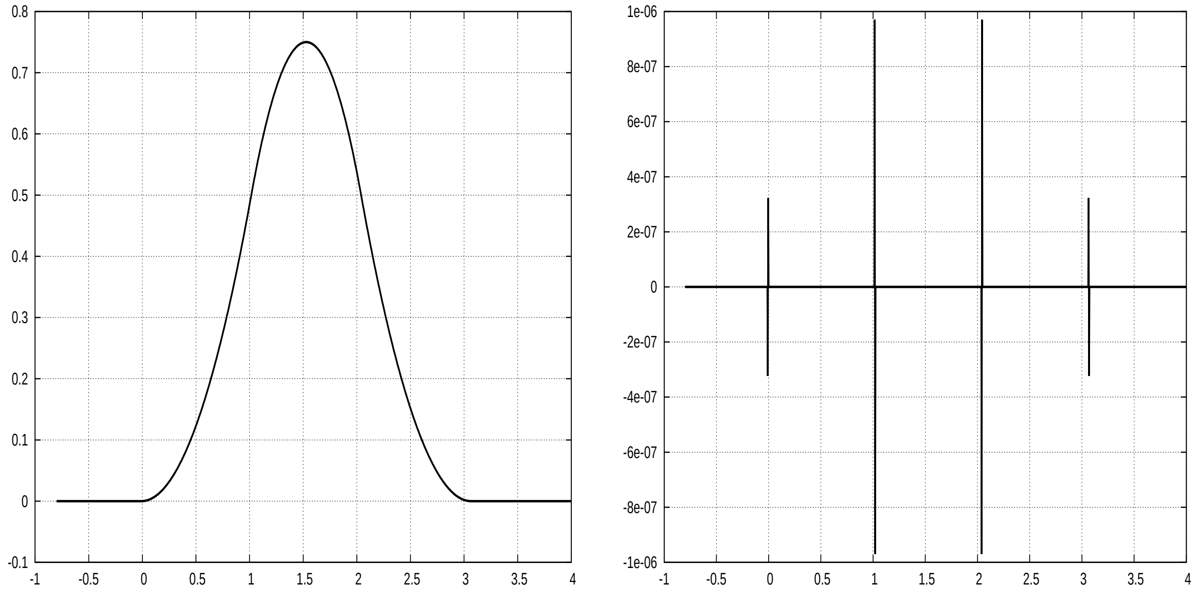


Figure 4.6: The functions f_8 and g_8

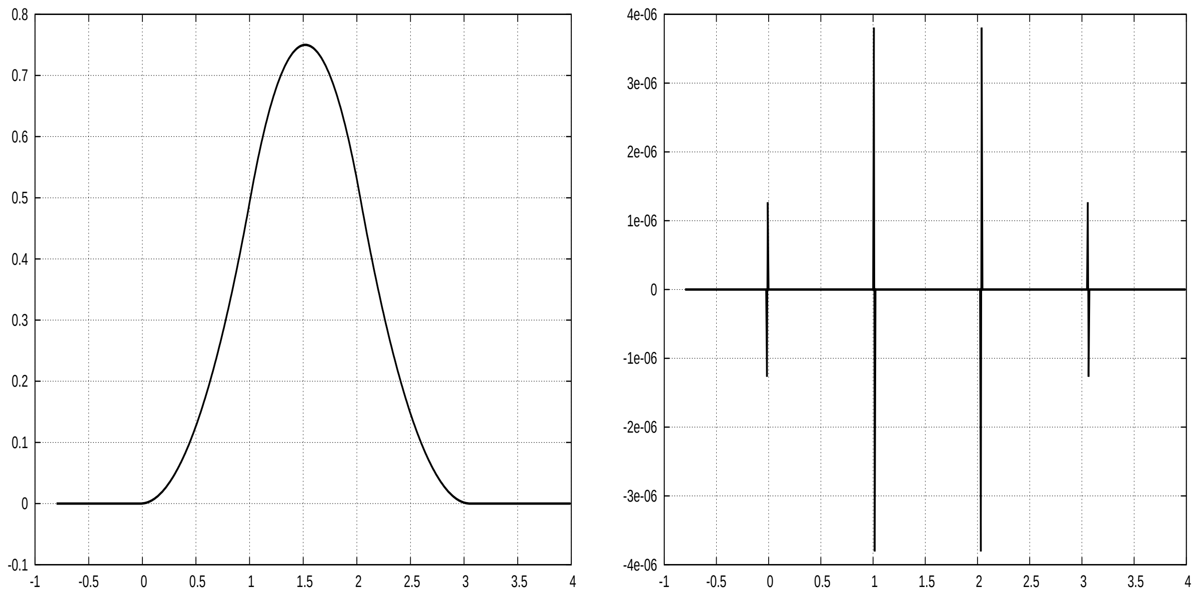


Figure 4.7: The functions f_7 and g_7

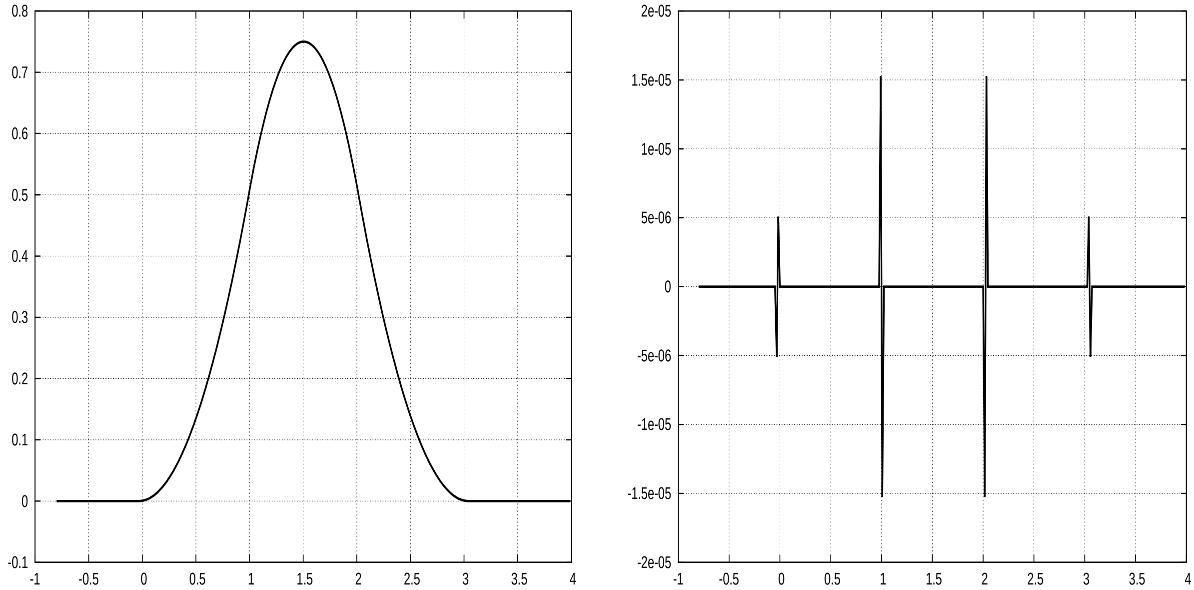


Figure 4.8: The functions f_6 and g_6

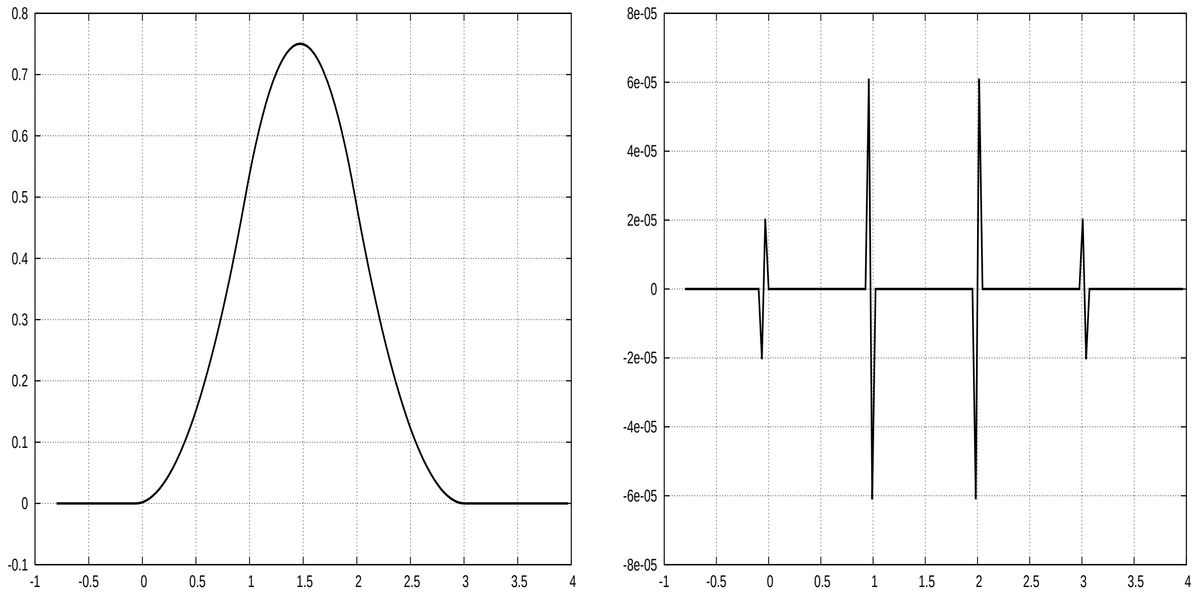


Figure 4.9: The functions f_5 and g_5

5. Cubic Spline Wavelet Decomposition on a Bounded Interval

In Chapter 4, we constructed a minimally supported cardinal B-spline wavelet ψ_m as a finitely supported continuous function in $\mathcal{S}_m^1$. In this chapter, we show how our wavelet construction method of Chapter 4 can be adapted to yield a cubic B-spline wavelet on a bounded interval.

5.1 Finite-Dimensional Cubic Spline Refinement Spaces

For an integer $n \geq 1$ and $r \in \mathbb{Z}_+$, define the cubic spline spaces

$$\mathcal{S}^r[0,n] = \left\{ s \in C^2[0,n] : s|_{\left[\frac{j}{2^r}, \frac{j+1}{2^r}\right)} \in \pi_3, \quad j = 0, 1, \ldots, 2^r n - 1 \right\}, \tag{5.1}$$

from which it immediately follows that the nesting property

$$\mathcal{S}^r[0,n] \subset \mathcal{S}^{r+1}[0,n], \quad r \in \mathbb{Z}_+, \tag{5.2}$$

is satisfied.

According to a standard result in spline theory [18, Theorem 2.6], it holds for any $r \in \mathbb{Z}_+$ that the sequence

$$\mathcal{N}_n^r = \{N_4(2^r x - j), \quad x \in [0,n]; \quad j = -3, -2, \ldots, 2^r n - 1\} \tag{5.3}$$

is a basis for $\mathcal{S}^r[0,n]$. Hence

$$\dim(\mathcal{S}^r[0,n]) = 2^r n + 3. \tag{5.4}$$

The following refinement equation holds on $[0,n]$.

Proposition 5.1 *For $r \in \mathbb{Z}_+$, it holds that*

$$N_4(2^r x - j) = \frac{1}{8} \sum_{k=-3}^{2^{r+1}n-1} \binom{4}{k-2j} N_4(2^{r+1}x - k), \quad x \in [0,n], \quad j = -3, -2, \ldots, 2^r n - 1. \tag{5.5}$$

Proof. Let $r \in \mathbb{Z}_+$ be fixed. It follows from (2.12) that, for $x \in \mathbb{R}$ and $j \in \mathbb{Z}$,

$$N_4(x-j) = \frac{1}{8}\sum_k \binom{4}{k} N_4(2x-2j-k) = \frac{1}{8}\sum_k \binom{4}{k-2j} N_4(2x-k). \tag{5.6}$$

Since $N_4 \in C_0(\mathbb{R})$, and (2.6) holds, we have

$$N_4(x) = 0, \quad x \notin (0,4). \tag{5.7}$$

Moreover, since $0 < 2x-k < 4$ if and only if $2x-4 < k < 2x$, we deduce from (5.6) and (5.7) that (5.5) does indeed hold. ■

5.2 Local Linear Projection

We proceed to obtain a local linear projection operator between cubic spline refinement spaces at successive resolution levels.

For $r \in \mathbb{Z}_+$, we seek a sequence $\{\lambda^r_{j,k} : k = -3,-2,\ldots,2^{r+1}n-1; j = -3,-2,\ldots,2^r n-1\}$ such that the linear operator $\mathcal{P}^r_n : \mathcal{S}^{r+1}[0,n] \to \mathcal{S}^r[0,n]$ defined for $x \in [0,n]$ by

$$(\mathcal{P}^r_n f)(x) = \sum_{j=-3}^{2^r n-1}\left[\sum_{k=-3}^{2^{r+1}n-1} \lambda^r_{j,k} c_k\right] N_4(2^r x - j), \quad \text{for} \quad f(x) = \sum_{j=-3}^{2^{r+1}n-1} c_j N_4(2^{r+1}x-j), \tag{5.8}$$

satisfies the reproduction property

$$(\mathcal{P}^r_n f)(x) = f(x), \quad x \in [0,n], \quad f \in \mathcal{S}^r[0,n], \tag{5.9}$$

so that $\mathcal{P}^r_n$ is then a linear projection on $\mathcal{S}^r[0,n]$.

To this end, let $r \in \mathbb{Z}_+$, suppose $f \in \mathcal{S}^r[0,n]$, and denote by $\{c_j : j = -3,-2,\ldots,2^r n-1\}$ the sequence such that

$$f(x) = \sum_{k=-3}^{2^r n-1} c_k N_4(2^r x - k), \quad 0 \le x \le n. \tag{5.10}$$

By substituting (5.5) into (5.10) we obtain, for $x \in [0,n]$,

$$\begin{aligned} f(x) &= \frac{1}{8}\sum_{j=-3}^{2^r n-1} c_j \sum_{k=-3}^{2^{r+1}n-1} \binom{4}{k-2j} N_4(2^{r+1}x-k) \\ &= \frac{1}{8}\sum_{k=-3}^{2^{r+1}n-1}\left[\sum_{j=-3}^{2^r n-1}\binom{4}{k-2j} c_j\right] N_4(2^{r+1}x-k). \end{aligned} \tag{5.11}$$

It follows from (5.11) that, if the linear operator $\mathcal{P}_n^r : \mathcal{S}^{r+1}[0,n] \to \mathcal{S}^r[0,n]$ is defined by (5.8), with $\{\lambda_{j,k}^r : k = -3,-2,\ldots,2^{r+1}n-1; j = -3,-2,\ldots,2^r n-1\}$ denoting an arbitrary sequence, then, for $x \in [0,n]$,

$$\begin{aligned}(\mathcal{P}_n^r f)(x) &= \frac{1}{8}\sum_{j=-3}^{2^r n-1}\left[\sum_{k=-3}^{2^{r+1}n-1}\lambda_{j,k}^r \sum_{l=-3}^{2^r n-1}\binom{4}{k-2l}c_l\right]N_4(2^r x - j)\\ &= \frac{1}{8}\sum_{j=-3}^{2^r n-1}\left[\sum_{l=-3}^{2^r n-1}\left\{\sum_{k=-3}^{2^{r+1}n-1}\binom{4}{k-2l}\lambda_{j,k}^r\right\}c_l\right]N_4(2^r x - j). \qquad (5.12)\end{aligned}$$

By combining (5.10) and (5.12), we obtain the formula

$$(f - \mathcal{P}_n^r f)(x) = \sum_{j=-3}^{2^r n-1}\left[\sum_{l=-3}^{2^r n-1}\left\{\delta_{j,l} - \frac{1}{8}\sum_{k=-3}^{2^{r+1}n-1}\binom{4}{k-2l}\lambda_{j,k}^r\right\}c_l\right]N_4(2^r x - j), \quad x \in [0,n]. \qquad (5.13)$$

Since $\mathcal{N}_n^r$ is a basis of $\mathcal{S}^r[0,n]$, we immediately deduce from (5.13) the following result.

Proposition 5.2 *For $r \in \mathbb{Z}_+$, the linear operator $\mathcal{P}_n^r : \mathcal{S}^{r+1}[0,n] \to \mathcal{S}^r[0,n]$, as defined by (5.8), satisfies the reproduction property (5.9), and is therefore a linear projection on $\mathcal{S}^r[0,n]$, if and only if the sequence $\{\lambda_{j,k}^r : k = -3,-2,\ldots,2^{r+1}n-1;\ j = -3,-2,\ldots,2^r n-1\}$ satisfies the condition*

$$\sum_{k=-3}^{2^{r+1}n-1}\binom{4}{k-2l}\lambda_{j,k}^r = 8\delta_{j,l}, \quad l = -3,-2,\ldots,2^{r+1}n-1;\ j = -3,-2,\ldots,2^r n-1. \qquad (5.14)$$

We proceed to explicitly solve for $\{\lambda_{j,k}^r\}$ from the linear equation (5.14).

Based on the case $m = 4$ of Theorem 3.4 and Table 3.2, we first prove the following result.

Proposition 5.3 *For $r \in \mathbb{Z}_+$ and $j = -1,0,\ldots,2^r n-3$, the sequence $\{\lambda_{j,k}^r\}$ defined by*

$$\lambda_{j,k}^r = \lambda_{4,2j-k}, \quad k = -3,-2,\ldots,2^{r+1}n-1, \qquad (5.15)$$

where

$$\left.\begin{aligned}\lambda_{4,-3} &= -\tfrac{1}{2}\\ \lambda_{4,-2} &= 2\\ \lambda_{4,-1} &= -\tfrac{1}{2}\\ \lambda_{4,j} &= 0, \quad j \notin \{-3,-2,-1\},\end{aligned}\right\} \qquad (5.16)$$

satisfies the condition (5.14).

Proof. Let $r \in \mathbb{Z}_+$ and $j \in \{-1, 0, \ldots, 2^r n - 3\}$. Then, if $l = -3$, it follows from (5.15) that

$$\sum_{k=-3}^{2^{r+1}n-1} \binom{4}{k-2l} \lambda_{j,k}^r = \sum_{k=-3}^{-2} \binom{4}{k+6} \lambda_{j,k}^r = \sum_{k=-3}^{-2} \binom{4}{k+6} \lambda_{4,2j-k}.$$

Since $2j - k \geq (-2) - (-2) = 0$ for $j \geq -1$ and $k \in \{-3, -2\}$, we see from (5.16) that $\lambda_{4,2j-k} = 0$ for $j \geq -1$ and $k \in \{-3, -2\}$. Hence

$$\sum_{k=-3}^{2^{r+1}n-1} \binom{4}{k-2l} \lambda_{j,k}^r = 0 = 8\delta_{j,l}, \quad l = -3, \quad j = -1, 0, \ldots, 2^r n - 3,$$

which shows that (5.14) holds for $l = -3$.

If $l = -2$, we argue as above from (5.15) to deduce that

$$\begin{aligned}
\sum_{k=-3}^{2^{r+1}n-1} \binom{4}{k-2l} \lambda_{j,k}^r &= \sum_{k=-3}^{2^{r+1}n-1} \binom{4}{k+4} \lambda_{4,2j-k} \\
&= \sum_{k=-3}^{0} \binom{4}{k+4} \lambda_{4,2j-k} \\
&= \sum_{k=-1}^{0} \binom{4}{k+4} \lambda_{4,2j-k} \\
&= 4\lambda_{4,2j+1} + \lambda_{4,2j}. \qquad (5.17)
\end{aligned}$$

If $j = -1$, we see from (5.16) that

$$4\lambda_{4,2j+1} + \lambda_{4,2j} = 4\lambda_{4,-1} + \lambda_{4,-2} = 4\left(-\frac{1}{2}\right) + 2 = 0, \qquad (5.18)$$

whereas if $j \in \{0, 1, \ldots, 2^r n - 3\}$, the last line of (5.16) gives

$$4\lambda_{4,2j+1} + \lambda_{4,2j} = 4(0) + 0 = 0. \qquad (5.19)$$

It follows from (5.17), (5.18) and (5.19) that

$$\sum_{k=-3}^{2^{r+1}n-1} \binom{4}{k-2l} \lambda_{j,k} = 0 = 8\delta_{j,l}, \quad l = -2, \quad j = -1, \ldots, 2^r n - 3,$$

which shows that (5.14) also holds for $l = -2$.

We have therefore now shown that the condition (5.14) is satisfied by the choice (5.15), (5.16) for $l \in \{-3, -2\}$ and $j = -1, 0, \ldots, 2^r n - 3$.

The proof that the choice (5.15), (5.16) satisfies (5.14) for $l \in \{2^r n-2, 2^r n-1\}$ and $j \in \{-1, 0, \ldots, 2^r n-3\}$ is similar to the above.

Next, suppose $l \in \{-1, 0, \ldots, 2^r n-3\}$. But then (5.15) and (5.16) give

$$\begin{aligned}
\sum_{k=-3}^{2^{r+1}n-1} \binom{4}{k-2l} \lambda_{j,k}^r &= \sum_{k=2l}^{2l+4} \binom{4}{k-2l} \lambda_{j,k}^r \\
&= \sum_{k=0}^{4} \binom{4}{k} \lambda_{j,2l+k}^r \\
&= \sum_{k=0}^{4} \binom{4}{k} \lambda_{4,2j-2l-k} \\
&= \sum_{k=2j-2l+1}^{2j-2l+3} \binom{4}{k} \lambda_{4,2j-2l-k} \\
&= -\frac{1}{2}\binom{4}{2j-2l+1} + 2\binom{4}{2j-2l+2} - \frac{1}{2}\binom{4}{2j-2l+3}
\end{aligned} \tag{5.20}$$

For $l \in \{-1, 0, \ldots, 2^r n-3\}$ and $j \in \{-1, 0, \ldots, 2^r n-3\}$, we have, if $j = l$,

$$-\frac{1}{2}\binom{4}{2j-2l+1} + 2\binom{4}{2j-2l+2} - \frac{1}{2}\binom{4}{2j-2l+3} = -\frac{1}{2}(4) + 2(6) - \frac{1}{2}(4) = 8 = 8\delta_{j,l},$$

which shows from (5.20) that (5.14) holds if $j = l$. If $l = j-1$, then

$$-\frac{1}{2}\binom{4}{2j-2l+1} + 2\binom{4}{2j-2l+2} - \frac{1}{2}\binom{4}{2j-2l+3} = -\frac{1}{2}(4) + 2(1) - \frac{1}{2}(0) = 0,$$

whereas if $l = j+1$, then

$$-\frac{1}{2}\binom{4}{2j-2l+1} + 2\binom{4}{2j-2l+2} - \frac{1}{2}\binom{4}{2j-2l+3} = -\frac{1}{2}(0) + 2(1) - \frac{1}{2}(4) = 0,$$

which shows from (5.20) that (5.14) also holds for $l = j-1$ and $l = j+1$.

If $l \leq j-2$ or $l \geq j+2$, we have

$$-\frac{1}{2}\binom{4}{2j-2l+1} + 2\binom{4}{2j-2l+2} - \frac{1}{2}\binom{4}{2j-2l+3} = -\frac{1}{2}(0) + 2(0) - \frac{1}{2}(0) = 0,$$

which shows that (5.14) also holds for these values of l, and thereby completing our proof. ■

Our next step is to solve the linear system (5.14) for $j \in \{-3, -2\}$ and $j \in \{2^r n-2, 2^r n-1\}$.

Suppose first $j = -3$. Then (5.14) is given by

$$\sum_{k=-3}^{2^{r+1}n-1} \binom{4}{k-2l} \lambda_{-3,k}^r = 8\delta_{-3,l}, \quad l = -3, -2, \ldots, 2^r n-1, \tag{5.21}$$

or equivalently, in matrix-vector notation,

$$\begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 6 & 4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 6 & 4 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 4 & 6 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 4 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} \lambda_{-3,-3} \\ \lambda_{-3,-2} \\ \lambda_{-3,-1} \\ \lambda_{-3,0} \\ \vdots \\ \lambda_{-3,2^{r+1}n-4} \\ \lambda_{-3,2^{r+1}n-3} \\ \lambda_{-3,2^{r+1}n-2} \\ \lambda_{-3,2^{r+1}n-1} \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{5.22}$$

We now observe that the minimally supported sequence $\{\lambda^r_{j,k}\}$ satisfying (5.22) is the one satisfying

$$\begin{bmatrix} 4 & 1 & 0 \\ 4 & 6 & 4 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} \lambda^r_{-3,-3} \\ \lambda^r_{-3,-2} \\ \lambda^r_{-3,-1} \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix}, \tag{5.23}$$

with also

$$\lambda^r_{-3,k} = 0, \quad k = 0, 1, \cdots, 2^{r+1}n - 1. \tag{5.24}$$

The unique solution of the 3×3 linear system (5.23) is

$$\{\lambda^r_{-3,-3}, \lambda^r_{-3,-2}, \lambda^r_{-3,-1}\} = \frac{1}{2}\{5, -4, 1\}. \tag{5.25}$$

Next, suppose $j = -2$. Then (5.14) is given by

$$\sum_{k=-3}^{2^{r+1}n-1} \binom{4}{k-2l} \lambda^r_{-2,k} = 8\delta_{-2,l}, \quad l = -3, -2, \cdots, 2^r n - 1, \tag{5.26}$$

or equivalently, in matrix-vector notation,

$$\begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 6 & 4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 6 & 4 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 4 & 6 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 4 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} \lambda_{-2,-3} \\ \lambda_{-2,-2} \\ \lambda_{-2,-1} \\ \lambda_{-2,0} \\ \vdots \\ \lambda_{-2,2^{r+1}n-4} \\ \lambda_{-2,2^{r+1}n-3} \\ \lambda_{-2,2^{r+1}n-2} \\ \lambda_{-2,2^{r+1}n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{5.27}$$

Analogously to the case $j = -3$, we see that the minimally supported sequence $\{\lambda^r_{j,k}\}$ satisfying (5.27) is the one that solves

$$\begin{bmatrix} 4 & 1 & 0 \\ 4 & 6 & 4 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} \lambda^r_{-2,-3} \\ \lambda^r_{-2,-2} \\ \lambda^r_{-2,-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix}, \tag{5.28}$$

with also

$$\lambda^r_{-2,k} = 0, \quad k = 0, 1, \cdots, 2^{r+1}n - 1. \tag{5.29}$$

The unique solution of the 3×3 linear system (5.28) is given by

$$\{\lambda^r_{-2,-3}, \lambda^r_{-2,-2}, \lambda^r_{-2,-1}\} = \frac{1}{2}\{-1, 4, -1\}. \tag{5.30}$$

Similarly, if $j = 2^r n - 2$, the minimally supported sequence $\{\lambda_{2^r n-2,k}\}$ satisfying the condition (5.14) for $j = 2^r n - 2$ is given by

$$\{\lambda^r_{2^r n-2,2^{r+1}n-3}, \lambda^r_{2^r n-2,2^{r+1}n-2}, \lambda^r_{2^r n-2,2^{r+1}n-1}\} = \frac{1}{2}\{-1, 4, -1\}, \tag{5.31}$$

with

$$\lambda^r_{2^r n-2,k} = 0, \quad k = -3, -2, \cdots, 2^{r+1}n - 4, \tag{5.32}$$

whereas the minimally supported sequence $\{\lambda_{2^r n-1,k}\}$ satisfying the condition (5.14) for $j = 2^r n - 1$ is given by

$$\{\lambda^r_{2^r n-1,2^{r+1}n-3}, \lambda^r_{2^r n-1,2^{r+1}n-2}, \lambda^r_{2^r n-1,2^{r+1}n-1}\} = \frac{1}{2}\{1, -4, 5\}, \tag{5.33}$$

with

$$\lambda^r_{2^r n-1,k} = 0, \quad k = -3, -2, \cdots, 2^{r+1}n - 4. \tag{5.34}$$

We have therefore proved the following result.

Proposition 5.4 *For $r \in \mathbb{Z}_+$ and $j \in \{-3,-2,-1\} \cup \{2^r n - 2, 2^r n - 1\}$, the sequence $\{\lambda^r_{j,k}\}$ defined by (5.24), (5.25), (5.29), (5.30), (5.31), (5.32), (5.33), (5.34) satisfies the condition (5.14).*

Combining the results of Proposition 5.2, 5.3 and 5.4, we have the following result.

Theorem 5.5 *For $r \in \mathbb{Z}_+$, let the linear operator $\mathcal{P}^r_n : \mathcal{S}^{r+1}[0,n] \to \mathcal{S}^r[0,n]$ be defined for $x \in [0,n]$ by*

$$\begin{aligned}
(\mathcal{P}^r_n f)(x) \quad = \quad & \left(\frac{5}{2}c_{-3} - 2c_{-2} + \frac{1}{2}c_{-1}\right) N_4(2^r x + 3) \\
& + \sum_{j=-2}^{2^r n-2} \left(-\frac{1}{2}c_{2j+1} + 2c_{2j+2} - \frac{1}{2}c_{2j+3}\right) N_4(2^r x - j) \\
& + \left(\frac{1}{2}c_{2^{r+1}n-3} - 2c_{2^{r+1}n-2} + \frac{5}{2}c_{2^{r+1}n-1}\right) N_4(2^r x - 2^r n + 1), \\
& \textit{for} \quad f(x) = \sum_{j=-3}^{2^{r+1}n-1} c_j N_4(2^{r+1}x - j).
\end{aligned} \tag{5.35}$$

Then the reproduction property (5.9) is satisfied, i.e. $\mathcal{P}^r_n$ is a projection on $\mathcal{S}^r_n[0,n]$.

5.3 The Fundamental Space Decomposition Result

For $r \in \mathbb{Z}_+$, we now define the linear subspace $W^r[0,n]$ of $\mathcal{S}^{r+1}[0,n]$ by

$$W^r[0,n] = \{f - \mathcal{P}^r_n f : f \in \mathcal{S}^{r+1}_n\}, \tag{5.36}$$

with the operator $\mathcal{P}^r_n : \mathcal{S}^{r+1}[0,n] \to \mathcal{S}^r[0,n]$ defined by (5.35). We proceed to show that $W^r[0,n]$ is the kernel of the operator $\mathcal{P}^r_n$, as follows.

Proposition 5.6 *For $r \in \mathbb{Z}_+$, it holds that*

$$W^r[0,n] = \{f \in \mathcal{S}^{r+1}[0,n] : \mathcal{P}^r_n f = 0\}. \tag{5.37}$$

Proof. Let $r \in \mathbb{Z}_+$ and suppose $f \in W^r[0,n]$. Then, according to the definition (5.36), there exists a function $g \in \mathcal{S}^{r+1}[0,n]$ such that

$$f = g - \mathcal{P}_n^r g.$$

Hence, since $\mathcal{P}_n^r$ is a linear operator, we have that

$$\mathcal{P}_n^r f = \mathcal{P}_n^r g - \mathcal{P}_n^r(\mathcal{P}_n^r g) = \mathcal{P}_n^r g - \mathcal{P}_n^r g = 0,$$

after having used also the fact that $\mathcal{P}_n^r g \in \mathcal{S}^r[0,n]$, together with the reproduction property (5.9) of $\mathcal{P}_n^r$. It follows that

$$W^r[0,n] \subset \{f \in \mathcal{S}^{r+1}[0,n] : \mathcal{P}_n^r f = 0\}.$$

Next, suppose $f \in \mathcal{S}^{r+1}[0,n]$ is such that $\mathcal{P}_n^r f = 0$. But then $f = f - \mathcal{P}_n^r f$, and thus $f \in \{g - \mathcal{P}_n^r g : g \in \mathcal{S}^{r+1}[0,n]\} = W^r[0,n]$, so that $\{f \in \mathcal{S}^{r+1}[0,n] : \mathcal{P}_n^r f = 0\} \subset W^r[0,n]$, and thereby concluding our proof of (5.37). ∎

By using Proposition 5.6, we can now prove the following fundamental property of the linear space $W^r[0,n]$.

Proposition 5.7 *For $r \in \mathbb{Z}_+$, it holds that*

$$\mathcal{S}^r[0,n] \cap W^r[0,n] = \{0\}. \tag{5.38}$$

Proof. Since $\mathcal{S}^r[0,n]$ and $W^r[0,n]$ are both linear spaces, we have

$$\{0\} \subset \mathcal{S}^r[0,n] \cap W^r[0,n].$$

Suppose next $f \in \mathcal{S}^r[0,n] \cap W^r[0,n]$. But then $f \in \mathcal{S}^r[0,n]$, so that (5.9) gives

$$f = \mathcal{P}_n^r f. \tag{5.39}$$

Also, $f \in W^r[0,n]$, and it follows from (5.37) that

$$\mathcal{P}_n^r f = 0. \tag{5.40}$$

Together, (5.39) and (5.40) yield $f = 0$. It follows that $\mathcal{S}^r[0,n] \cap W^r[0,n] \subset \{0\}$, and thereby completing our proof of (5.38). ∎

Recalling from the definition (5.36) that $W^r[0,n]$ is a subspace of the linear space $\mathcal{S}^{r+1}[0,n]$, our fundamental space decomposition result is now as follows.

Theorem 5.8 *For $r \in \mathbb{Z}_+$,*

$$\mathcal{S}^{r+1}[0,n] = \mathcal{S}^r[0,n] \dot{+} W^r[0,n], \tag{5.41}$$

in the sense that, for any $f \in \mathcal{S}^{r+1}[0,n]$, there exist functions $g \in \mathcal{S}^r[0,n]$ and $h \in W^r[0,n]$ such that $f = g + h$, and with g and h uniquely determined by f. Moreover, $g = \mathcal{P}_n^r f$ and $h = f - \mathcal{P}_n^r f$.

Proof. Let $r \in \mathbb{Z}_+$ and suppose $f \in \mathcal{S}^{r+1}[0,n]$. Since $\mathcal{P}_n^r : \mathcal{S}^{r+1}[0,n] \to \mathcal{S}^r[0,n]$, and recalling also the definition (5.36) of $W^r[0,n]$, it follows that, if we define $g = \mathcal{P}_n^r f$ and $h = f - \mathcal{P}_n^r f$, then $f = g + h$, with $g \in \mathcal{S}^r[0,n]$ and $h \in W^r[0,n]$. Suppose $\tilde{g} \in \mathcal{S}^r[0,n]$ and $\tilde{h} \in W^r[0,n]$ are such that $f = \tilde{g} + \tilde{h}$. Hence $g + h = \tilde{g} + \tilde{h}$, and thus $g - \tilde{g} = \tilde{h} - h$. Since $\mathcal{S}^r[0,n]$ and $W^r[0,n]$ are both linear spaces, we know that $g - \tilde{g} \in \mathcal{S}^r[0,n]$ and $h - \tilde{h} \in W^r[0,n]$. But $g - \tilde{g} = \tilde{h} - h$, and thus $g - \tilde{g} \in \mathcal{S}^r[0,n] \cap W^r[0,n]$ and $\tilde{h} - h \in \mathcal{S}^r[0,n] \cap W^r[0,n]$. It then follows from (5.38) that $g - \tilde{g} = 0$ and $\tilde{h} - h = 0$, i.e. $\tilde{g} = g$ and $\tilde{h} = h$, which completes our proof. ■

The following dimension result can now immediately be deduced from Theorem 5.8.

Corollary 5.9 *For $r \in \mathbb{Z}_+$, it holds that*

$$\dim\,(W_n^r) = 2^r n. \tag{5.42}$$

Proof. According to a standard result from linear algrebra (see [20, Theorem 8.4.2]), it follows from (5.41) that

$$\dim\,(\mathcal{S}_n^{r+1}) = \dim\,(\mathcal{S}_n^r) + \dim\,(W_n^r),$$

and thus, from (5.4),

$$\dim\,(W_n^r) = (2^{r+1}n + 3) - (2^r n + 3) = 2^r n.$$

■

5.4 Construction of a Wavelet Basis

In this section, we explicitly construct, for $r \in \mathbb{Z}_+$, a sequence

$$\mathcal{W}_n^r = \{\psi_j^r : j = -1, 0, \ldots, 2^r n - 2\} \tag{5.43}$$

such that:

$$\text{(a)} \qquad \mathcal{W}_n^r \subset W^r[0,n]; \tag{5.44}$$

$$\text{(b)} \qquad \mathcal{W}_n^r \text{ is a linearly independent set.} \tag{5.45}$$

Since the set (5.43) contains precisely $2^r n$ functions, it will then follow from (5.42) that $\mathcal{W}_n^r$ is a basis for $W^r[0,n]$. Such a sequence $\mathcal{W}_n^r$ is called a wavelet basis for $W^r[0,n]$, and the functions $\{\psi_j^r : j = -1, \ldots, 2^r n - 2\}$, are called wavelets.

First, recall that a function ψ belongs to $\mathcal{S}^{r+1}[0,n]$ if and only if there exists a sequence $\{\gamma_j : j = -3, -2, \ldots, 2^{r+1}n - 1\}$ such that

$$\psi(x) = \sum_{j=-3}^{2^{r+1}n-1} \gamma_j N_4(2^{r+1}n - j), \quad x \in [0,n]. \tag{5.46}$$

Moreover, we see from (5.37) that $\psi \in W^r[0,n]$ if and only if $\mathcal{P}_n^r \psi = 0$, or, equivalently, from (5.35) and (5.46), and the fact that $\mathcal{N}_n^r$ is a basis for $\mathcal{S}^r[0,n]$, if and only if the sequence $\{\gamma_j\}$ in (5.46) satisfies

$$\left.\begin{array}{lcll} 5\gamma_{-3} - 4\gamma_{-2} + \gamma_{-1} & = & 0, & \\ -\gamma_{2j+1} + 4\gamma_{2j+2} - \gamma_{2j+3} & = & 0, & j = -2, \ldots, 2^r n - 2, \\ \gamma_{2^{r+1}n-3} - 4\gamma_{2^{r+1}n-2} + 5\gamma_{2^{r+1}n-1} & = & 0, & \end{array}\right\} \tag{5.47}$$

or, in equivalent matrix-vector formulation,

$$\begin{bmatrix} 5 & -4 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 4 & -1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 4 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} \gamma_{-3} \\ \gamma_{-2} \\ \gamma_{-1} \\ \gamma_0 \\ \vdots \\ \gamma_{2^{r+1}n-4} \\ \gamma_{2^{r+1}n-3} \\ \gamma_{2^{r+1}n-2} \\ \gamma_{2^{r+1}n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{5.48}$$

Now consider first the 3×4 homogeneous linear system

$$\left.\begin{array}{rrrrcl} 5\gamma_{-3} & -4\gamma_{-2} & +\gamma_{-1} & & = & 0 \\ -\gamma_{-3} & +4\gamma_{-2} & -\gamma_{-1} & & = & 0 \\ & & -\gamma_{-1} & +4\gamma_0 & = & 0 \end{array}\right\}, \tag{5.49}$$

for which the general solution is given by

$$\{\gamma_{-3}, \gamma_{-2}, \gamma_{-1}, \gamma_0\} = \{0, t, 4t, t\}, \quad t \in \mathbb{R}. \tag{5.50}$$

By choosing $t = -\frac{1}{2}$ in (5.50), we deduce from (5.49) and (5.48) that, for $r \in \mathbb{Z}_+$, the sequence $\{\gamma_j\} = \{\gamma^r_{-1,j}\}$ defined by

$$\left.\begin{array}{c} \gamma^r_{-1,-2} = -\frac{1}{2}, \quad \gamma^r_{-1,-1} = -2, \quad \gamma^r_{-1,0} = -\frac{1}{2}, \\ \gamma^r_{-1,j} = 0, \quad j \in \{-3\} \cup \{1, 2, \ldots, 2^{r+1}n - 1\}, \end{array}\right\} \tag{5.51}$$

solves the linear system (5.47).

Next, we note that the 4×6 homogeneous linear system

$$\left.\begin{array}{rrrrrrcl} 5\gamma_{-3} & -4\gamma_{-2} & +\gamma_{-1} & & & & = & 0 \\ -\gamma_{-3} & +4\gamma_{-2} & -\gamma_{-1} & & & & = & 0 \\ & & -\gamma_{-1} & +4\gamma_0 & -\gamma_1 & & = & 0 \\ & & & & -\gamma_1 & +4\gamma_2 & = & 0 \end{array}\right\} \tag{5.52}$$

has the general solution

$$\{\gamma_{-3}, \gamma_{-2}, \gamma_{-1}, \gamma_0, \gamma_1, \gamma_2\} = \{0, t, 4t, t+s, 4s, s\}, \quad t, s \in \mathbb{R}. \tag{5.53}$$

By choosing $t = 0$ and $s = -\frac{1}{2}$ in (5.53), we deduce from (5.52) and (5.48) that, for $r \in \mathbb{Z}_+$, the sequence $\{\gamma_j\} = \{\gamma^r_{0,j}\}$ defined by

$$\left.\begin{array}{c} \gamma^r_{0,0} = -\frac{1}{2}, \quad \gamma^r_{0,1} = -2, \quad \gamma^r_{0,2} = -\frac{1}{2}, \\ \gamma^r_{0,j} = 0, \quad j \in \{-3, -2, -1\} \cup \{3, 4, \ldots, 2^{r+1}n - 1\}, \end{array}\right\} \tag{5.54}$$

solves the linear system (5.47).

Repeated application of the above procedure yields, for $r \in \mathbb{Z}_+$, the sequence $\{\gamma^r_{j,k} : k = -3, -2, \ldots, 2^{r+1}n - 1; j = -1, 0, \ldots, 2^r n - 2\}$, where

$$\left.\begin{array}{rcl} \gamma^r_{j,2j} & = & -\frac{1}{2}, \quad \gamma^r_{j,2j+1} = -2, \quad \gamma^r_{j,2j+2} = -\frac{1}{2}, \\ \gamma^r_{j,k} & = & 0, \ k \in \{-3, -2, \ldots, 2j-1\} \cup \{2j+3, , \ldots, 2^{r+1}n - 1\}, \quad j = -1, 0, \ldots, 2^r n - 2 \end{array}\right\}, \tag{5.55}$$

and with $\{\gamma_k\} = \{\gamma^r_{j,k}\}$ satisfying the homogeneous linear system (5.47) for $j = -1, 0, \ldots, 2^r n - 2$.

Substituting (5.55) into (5.46) yield, for $r \in \mathbb{Z}_+$, the function sequence $\{\psi_j : j = -1, 0, \ldots, 2^r n - 2\}$ defined by

$$\psi_j^r(x) = -\frac{1}{2}N_4(2^{r+1}x - 2j) - 2N_4(2^{r+1}x - 2j - 1) - \frac{1}{2}N_4(2^{r+1}x - 2j - 2), \quad x \in [0, n], \quad j = -1, 0, \ldots, 2^r n - 2. \tag{5.56}$$

Now observe from (4.16) and Table 3.2 that

$$\psi_4(x) = -\frac{1}{2}N_4(2x) - 2N_4(2x - 1) - \frac{1}{2}N_4(2x - 2), \quad x \in \mathbb{R}, \tag{5.57}$$

and thus

$$\psi_4(2^r x - j) = -\frac{1}{2}N_4(2^{r+1}x - 2j) - 2N_4(2^{r+1}x - 2j - 1) - \frac{1}{2}N_4(2^{r+1}x - 2j - 2), \quad x \in \mathbb{R}, \quad j \in \mathbb{R}. \tag{5.58}$$

It follows from (5.56) and (5.58) that, for $r \in \mathbb{Z}_+$,

$$\psi_j^r(x) = \psi_4(2^r x - j), \quad x \in [0, n], \quad j = -1, 0, \ldots, 2^r n - 2. \tag{5.59}$$

Observe from (4.18) that, for $r \in \mathbb{Z}_+$,

$$\psi_4(2^r x - j) = 0, \quad x \notin \left(\frac{j}{2^r}, \frac{j+3}{2^r}\right), \quad j \in \mathbb{R},$$

which, together with (5.59), gives, for $r \in \mathbb{Z}_+$,

$$\left.\begin{array}{lll} \psi_{-1}^r(x) = 0, & x \notin [0, 2^{r-1}), & \\ \psi_j^r(x) = 0, & x \notin \left(\frac{j}{2^r}, \frac{j+3}{2^r}\right), & j = 0, 1, \ldots, 2^r n - 3, \\ \psi_{2^r n - 2}^r(x) = 0, & x \notin (n - 2^{r-1}, n], & \end{array}\right\} \quad x \in [0, n]. \tag{5.60}$$

We have therefore now proved the following result.

Proposition 5.10 *For $r \in \mathbb{Z}_+$, the sequence $\mathcal{W}_n^r$ defined by (5.43) and (5.56) satisfies the inclusion (5.44).*

We proceed to prove that $\mathcal{W}_n^r$ is a linearly independent set.

To this end, for a fixed $r \in \mathbb{Z}_+$, we let $\{d_{-1}, d_0, \ldots, d_{2^r n - 2}\}$ denote a sequence such that

$$\sum_{j=-1}^{2^r n - 2} d_j \psi_j^r(x) = 0, \quad x \in [0, n]. \tag{5.61}$$

Substituting (5.56) into (5.61) yields, for $x \in [0, n]$,

$$
\begin{aligned}
0 &= \sum_{j=-1}^{2^r n-2} d_j \left[-\frac{1}{2} N_4(2^{r+1}x - 2j) - 2N_4(2^{r+1}x - 2j - 1) - \frac{1}{2} N_4(2^{r+1}x - 2j - 2) \right] \\
&= -\frac{1}{2} \sum_{j=-1}^{2^r n-2} d_j N_4(2^{r+1}x - 2j) - 2 \sum_{j=-1}^{2^r n-2} d_j N_4(2^{r+1}x - 2j - 1) - \frac{1}{2} \sum_{j=0}^{2^r n-1} d_{j-1} N_4(2^{r+1}x - 2j),
\end{aligned}
$$

according to which, since $\mathcal{N}_n^{r+1}$ is a basis for $\mathcal{S}^{r+1}[0, n]$, it must hold that

$$d_j = 0, \quad j = -1, 0, \ldots, 2^r n - 2.$$

We have therefore now proved the following result.

Proposition 5.11 *For $r \in \mathbb{Z}_+$, the set $\mathcal{W}_n^r$ defined by (5.43) and (5.56) is linearly independent.*

Combining the results of Propositions 5.10 and 5.11, as well as Corollary 5.9, we have the following.

Theorem 5.12 *The sequence $\mathcal{W}_n^r$ defined by (5.43) and (5.56) is a wavelet basis for the set $W^r[0, n]$.*

According to (5.59), we have therefore shown that the restriction to $[0, n]$ of the cubic spline-wavelet sequence $\{\psi_4(2^r \cdot -j) : j = -1, 0, \ldots, 2^r n - 2\}$, as previously obtained in Theorem 4.3, constitutes a wavelet basis for the space $W^r[0, n]$.

In Figures 5.1, 5.2 and 5.3, we show, for $n = 6$, the sequences $\mathcal{N}_6^0, \mathcal{N}_6^1$ and $\mathcal{W}_6^0$.

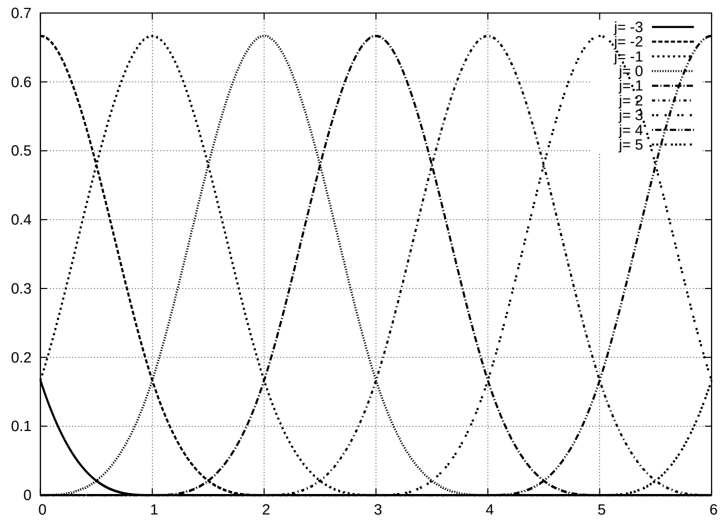


Figure 5.1: The sequence $\mathcal{N}_6^0$

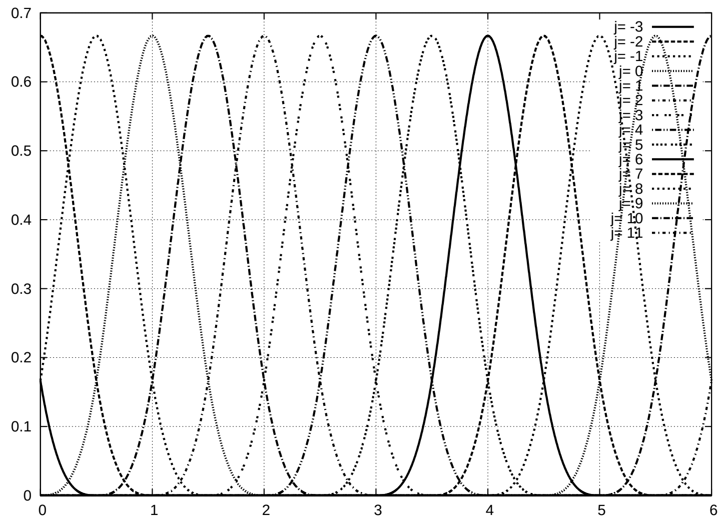


Figure 5.2: The sequence $\mathcal{N}_6^1$

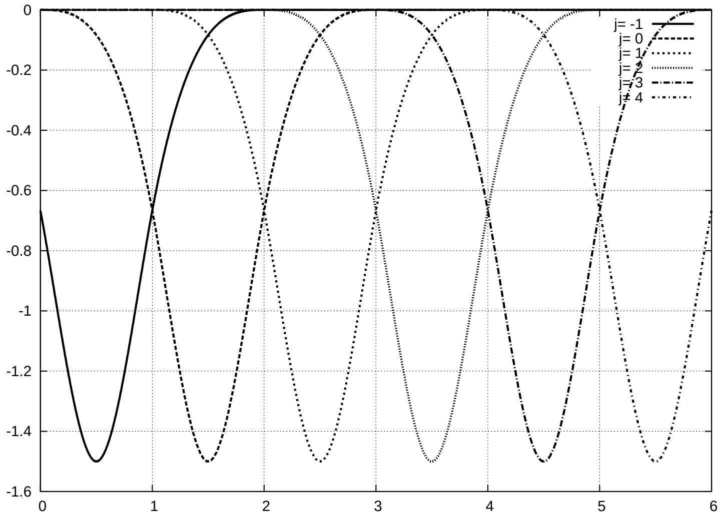


Figure 5.3: The sequence $\mathcal{W}_6^0$

5.5 The Decomposition Algorithm

According to (5.36) and Theorem 5.12, there exists, for $r \in \mathbb{Z}_+$, a unique sequence $\{\omega_{j,k}^r : k = -1, 0, \ldots, 2^r n - 1,\ j = -3, -2, \ldots, 2^{r+1}n - 1\}$ such that, for $x \in [0, n]$, it holds that

$$\begin{aligned}\left(N_4(2^{r+1} \cdot - j) - \mathcal{P}_n^r N_4(2^{r+1} \cdot - j)\right)(x) &= \sum_{k=-1}^{2^r n - 2} \omega_{j,k}^r \psi_k^r(x), \quad x \in [0, n], \\ j &= -3, -2, \ldots, 2^{r+1}n - 1. \end{aligned} \tag{5.62}$$

We proceed to explicitly calculate the sequence $\{\omega_{j,k}^r\}$ in (5.62).

Let $r \in \mathbb{Z}_+$ and $j \in \{-3, -2, \ldots, 2^{r+1}n - 1\}$ be fixed. Then we have for $x \in [0, n]$ that

$$\begin{aligned} N_4(2^{r+1}x - j) &= \sum_{k=-3}^{2^{r+1}n-1} \delta_{j,k} N_4(2^{r+1}x - k) \\ &= \sum_{k=-1}^{2^r n - 1} \delta_{j,2k} N_4(2^{r+1}x - 2k) + \sum_{k=-2}^{2^r n - 1} \delta_{j,2k+1} N_4(2^{r+1}x - 2k - 1), \end{aligned} \tag{5.63}$$

and thus, from (5.35) and (5.5), it holds for $x \in [0,n]$ that

$$\begin{aligned}
& (\mathcal{P}_n^r N_4(2^{r+1}\cdot -j))(x) \\
= & \left(\frac{5}{2}\delta_{j,-3} - 2\delta_{j,-2} + \frac{1}{2}\delta_{j,-1}\right) N_4(2^r x + 3) \\
& + \sum_{k=-2}^{2^r n-1} \left(-\frac{1}{2}\delta_{j,2k+1} + 2\delta_{j,2k+2} - \frac{1}{2}\delta_{j,2k+3}\right) N_4(2^r x - k) \\
& + \left(\frac{1}{2}\delta_{j,2^{r+1}n-3} - 2\delta_{j,2^{r+1}n-2} + \frac{5}{2}\delta_{j,2^{r+1}n-1}\right) N_4(2^r x - 2^r n + 1) \\
= & \frac{1}{8}\left(\frac{5}{2}\delta_{j,-3} - 2\delta_{j,-2} + \frac{1}{2}\delta_{j,-1}\right) \sum_{k=-3}^{2^{r+1}n-1} \binom{4}{k+6} N_4(2^{r+1}x - k) \\
& + \frac{1}{8}\sum_{k=-2}^{2^r n-1} \left(-\frac{1}{2}\delta_{j,2k+1} + 2\delta_{j,2k+2} - \frac{1}{2}\delta_{j,2k+3}\right) \sum_{l=-3}^{2^{r+1}n-1} \binom{4}{l-2k} N_4(2^{r+1}x - l) \\
& + \frac{1}{8}\left(\frac{1}{2}\delta_{j,2^{r+1}n-3} - 2\delta_{j,2^{r+1}n-2} + \frac{5}{2}\delta_{j,2^{r+1}n-1}\right) \sum_{k=-3}^{2^{r+1}n-1} \binom{4}{k-2^{r+1}n+2} N_4(2^{r+1}x - k) \\
= & \frac{1}{8}\left(\frac{5}{2}\delta_{j,-3} - 2\delta_{j,-2} + \frac{1}{2}\delta_{j,-1}\right) \left[4N_4(2^{r+1}x + 3) + N_4(2^{r+1}x + 2)\right] \\
& + \frac{1}{8}\sum_{k=-2}^{2^r n-1} \left(-\frac{1}{2}\delta_{j,2k+1} + 2\delta_{j,2k+2} - \frac{1}{2}\delta_{j,2k+3}\right) \sum_{l=-1}^{2^r n-1} \binom{4}{2l-2k} N_4(2^{r+1}x - 2l) \\
& + \frac{1}{8}\sum_{k=-2}^{2^r n-1} \left(-\frac{1}{2}\delta_{j,2k+1} + 2\delta_{j,2k+2} - \frac{1}{2}\delta_{j,2k+3}\right) \sum_{l=-2}^{2^r n-1} \binom{4}{2l+1-2k} N_4(2^{r+1}x - 2l - 1) \\
& + \frac{1}{8}\left(\frac{1}{2}\delta_{j,2^{r+1}n-3} - 2\delta_{j,2^{r+1}n-2} + \frac{5}{2}\delta_{j,2^{r+1}n-1}\right) \\
& \qquad \left[N_4(2^{r+1}x - 2^{r+1}n + 2) + 4N_4(2^{r+1}x - 2^{r+1}n + 1)\right]. \qquad (5.64)
\end{aligned}$$

Next, we use (5.56) to deduce that, for $j \in \{-3,-2,\ldots,2^{r+1}n-1\}$ and $x \in [0,n]$, we have

$$\begin{aligned}
& \sum_{k=-1}^{2^r n-2} \omega_{j,k}^r \psi_k^r(x) \\
& = \sum_{k=-1}^{2^r n-2} \omega_{j,k}^r \left[-\frac{1}{2}N_4(2^{r+1}x - 2k) - 2N_4(2^{r+1}x - 2k - 1) - \frac{1}{2}N_4(2^{r+1}x - 2k - 2)\right] \\
& = -\frac{1}{2}\sum_{k=-1}^{2^r n-2} \omega_{j,k}^r N_4(2^{r+1}x - 2k) - 2\sum_{k=-1}^{2^r n-2} \omega_{j,k}^r N_4(2^{r+1}x - 2k - 1) \\
& -\frac{1}{2}\sum_{k=0}^{2^r n-1} \omega_{j,k-1}^r N_4(2^{r+1}x - 2k)
\end{aligned}$$

$$= -\frac{1}{2}\sum_{k=-1}^{2^r n-1}\left(\omega_{j,k}^r + \omega_{j,k-1}^r\right) N_4(2^{r+1}x - 2k) - 2\sum_{k=-1}^{2^r n-2}\omega_{j,k}^r N_4(2^{r+1}x - 2k - 1), \quad (5.65)$$

where we have defined

$$\omega_{j,-2}^r = \omega_{j,2^r n-1}^r = 0, \quad j = -3, -2, \ldots, 2^{r+1}n - 1. \quad (5.66)$$

Using also the fact that $\mathcal{N}_n^{r+1}$ is a basis for $\mathcal{S}^{r+1}[0,n]$, we can now derive from (5.62), (5.64), (5.65) and (5.66), as in (a), ..., (g) below, the following for $x \in [0,n]$ and $j \in \{-3, -2, \ldots, 2^{r+1}n - 1\}$.

(a) $j = -3$:

By equating coefficients of $N_4(2^{r+1}x + 2)$, we obtain

$$0 - \frac{5}{16} - \frac{1}{8}\left[\left(-\frac{1}{2}\right)\binom{4}{2(-1) - 2(-2)}\right] = -\frac{1}{2}(\omega_{-3,-1}^r + 0),$$

and thus

$$\omega_{-3,-1}^r = -\frac{1}{8}. \quad (5.67)$$

By equating coefficients of $N_4(2^{r+1}x)$, we obtain

$$0 - \frac{1}{8}\left[\left(-\frac{1}{2}\right)\binom{4}{2(0) - 2(-2)}\right] = \frac{1}{2}(\omega_{-3,0}^r + \omega_{-3,-1}^r),$$

and thus

$$\omega_{-3,-1}^r + \omega_{-3,0}^r = -\frac{1}{8}, \quad (5.68)$$

which, together with (5.67), yield

$$\omega_{-3,0}^r = 0. \quad (5.69)$$

By equating coefficients of $N_4(2^{r+1}x - 2k)$ with $k \geq 1$, we obtain

$$0 - \frac{1}{8}\left[\left(-\frac{1}{2}\right)\binom{4}{2k+4}\right] = -\frac{1}{2}(\omega_{-3,k}^r + \omega_{-3,k-1}^r),$$

i.e.

$$\omega_{-3,k-1}^r + \omega_{-3,k}^r = 0, \quad k \geq 1,$$

which, together with (5.69), yields

$$\omega_{-3,k}^r = 0, \quad k = 0, 1, \ldots, 2^r n - 2. \quad (5.70)$$

Hence we have, according to (5.67), (5.68) and (5.70), shown that

$$\omega^r_{-3,k} = \begin{cases} -\frac{1}{8}, & k=-1, \\ 0, & k=0,1,\ldots,2^r n-2. \end{cases} \tag{5.71}$$

(b) $j=-2$:

By equating coefficients of $N_4(2^{r+1}x+2)$, we obtain

$$1-\frac{1}{8}(-2)(1)-\frac{1}{8}(2)\binom{4}{2(-1)-2(-2)} = -\frac{1}{2}(\omega^r_{-2,-1}+0),$$

and thus

$$\omega^r_{-2,-1} = \frac{1}{2}. \tag{5.72}$$

By equating coefficients of $N_4(2^{r+1}x)$, we obtain

$$0-\frac{1}{8}(2)\binom{4}{2(0)-2(-2)} = -\frac{1}{2}(\omega^r_{-2,0}+\omega^r_{-2,-1}),$$

which, together with (5.72), give

$$\omega^r_{-2,0} = 0. \tag{5.73}$$

Since, moreover,

$$\binom{4}{2l-2k} = \binom{4}{2l+1-2k} = 0 \quad \text{for} \quad k=-2 \text{ and } l \geq 1,$$

it follows from (5.73), (5.63), (5.64) and (5.65) that

$$\omega^r_{-2,k} = 0, \quad k=-1,\ldots,2^r n-2. \tag{5.74}$$

Together, (5.72) and (5.74) yield

$$\omega^r_{-2,k} = \begin{cases} \frac{1}{2}, & k=-1, \\ 0, & k=0,1,\ldots,2^r n-2. \end{cases} \tag{5.75}$$

(c) $j=2^{r+1}n-1$:

We obtain, as in (a) above,

$$\omega^r_{2^{r+1}n-1,k} = \begin{cases} -\frac{1}{8}, & k=2^r n-2, \\ 0, & k=-1,\ldots,2^r n-3. \end{cases} \tag{5.76}$$

(d) $j = 2^{n+1}n - 2$:

We obtain, as in (b) above,

$$\omega^r_{2^{r+1}n-2,k} = \begin{cases} \frac{1}{2}, & k = 2^r n - 2, \\ 0, & k = -1, \ldots, 2^r n - 3. \end{cases} \tag{5.77}$$

(e) $j = -1$:

By equating coefficients of $N_4(2^{r+1}x + 2)$, we obtain

$$0 - \frac{1}{8}\left(\frac{1}{2}\right)(1) - \frac{1}{8}\left[\left(-\frac{1}{2}\right)\binom{4}{2(-1)-2(-1)} + \left(-\frac{1}{2}\right)\binom{4}{2(-1)-2(-2)}\right] = -\frac{1}{2}(\omega^r_{-1,-1} + 0),$$

and thus

$$\omega^r_{-1,-1} = -\frac{3}{4}. \tag{5.78}$$

By equating coefficients of $N_4(2^{r+1}x)$, we obtain

$$0 - \frac{1}{8}\left[\left(-\frac{1}{2}\right)\binom{4}{2(0)-2(-1)} + \left(-\frac{1}{2}\right)\binom{4}{2(0)-2(-2)}\right] = -\frac{1}{2}(\omega^r_{-1,0} + \omega^r_{-1,-1})$$

and thus

$$\omega^r_{-1,-1} + \omega^r_{-1,0} = -\frac{7}{8}, \tag{5.79}$$

which, together with (5.78), gives

$$\omega^r_{-1,0} = -\frac{1}{8}. \tag{5.80}$$

By equating coefficients of $N_4(2^{r+1}x - 2)$, we obtain

$$0 - \frac{1}{8}\left[\left(-\frac{1}{2}\right)\binom{4}{2(1)-2(-1)} + \left(-\frac{1}{2}\right)\binom{4}{2(1)-2(-2)}\right] = -\frac{1}{2}(\omega^r_{-1,1} + \omega^r_{-1,0}),$$

and thus

$$\omega^r_{-1,0} + \omega^r_{-1,1} = -\frac{1}{8}, \tag{5.81}$$

which, together with (5.80), yield

$$\omega^r_{-1,1} = 0. \tag{5.82}$$

Now observe that

$$\binom{4}{2l+1-2k} = 0 \quad \text{for} \quad l \geq 1 \quad \text{and} \quad k \in \{-2, -1\},$$

whereas

$$\binom{4}{2l-2k} = 0 \quad \text{for} \quad l \geq 2 \quad \text{and} \quad k \in \{-2,-1\},$$

so that we can deduce from (5.62), (5.63), (5.64), (5.65) and (5.80) that

$$\omega^r_{-1,k} = 0, \quad k = 0,1,\ldots,2^r n - 1. \tag{5.83}$$

Together, (5.78), (5.80) and (5.83) yield

$$\omega^r_{-1,k} = \begin{cases} -\frac{3}{4}, & k = -1, \\ -\frac{1}{8}, & k = 0, \\ 0, & k = 1, \ldots 2^r n - 2. \end{cases} \tag{5.84}$$

(f) $j = 2^{r+1}n - 3$:

Similarly, we obtain, as in (e) above,

$$\omega_{2^{r+1}n-3,k} = \begin{cases} -\frac{3}{4}, & k = 2^r n - 2, \\ -\frac{1}{8}, & k = 2^r n - 3, \\ 0, & k = -1, \ldots, 2^r n - 4. \end{cases} \tag{5.85}$$

(g) $j = 2m$, for $m \in \{0,1,\ldots,2^r n - 2\}$:

By equating coefficients of $N_4(2^{r+1}x - 2l - 1)$ for $l \in \{-1,\ldots,2^r n - 2\}$, we obtain

$$\delta_{2m,2l+1} - \frac{1}{8}(2)\binom{4}{2l+1-2(m-1)} = -2\omega^r_{2m,l},$$

and thus

$$\omega^r_{2m,l} = \frac{1}{8}\binom{4}{2l-2m+3}, \quad l = -1,\ldots,2^r n - 2. \tag{5.86}$$

By equating coefficients of $N_4(2^{r+1}x - 2l - 1)$ for $l \in \{-1,-2,\ldots,2^r n - 2\}$, we obtain

$$\delta_{2m+1,2l+1} - \frac{1}{2}\left[\left(-\frac{1}{2}\right)\binom{4}{2l+1-2(m)} + \left(-\frac{1}{2}\right)\binom{4}{2l+1-2(m-1)}\right] = -2\omega^r_{2m+1,l},$$

and thus

$$\omega^r_{2m+1,l} = -\frac{1}{2}\delta_{2m+1,2l+1} - \frac{1}{32}\left[\binom{4}{2l+1-2m} + \binom{4}{2l+3-2m}\right], l = -1,\ldots,2^r n - 2. \tag{5.87}$$

Now observe that

$$-\frac{1}{2}\delta_{2m+1,2l+1} \quad - \quad \frac{1}{32}\left[\binom{4}{2l+1-2m}+\binom{4}{2l+3-2m}\right] = -\frac{1}{8}\binom{4}{2l-2m+2},$$
$$l = -1,\ldots,2^r n-2. \tag{5.88}$$

It follows from (5.87) and (5.88) that

$$\omega^r_{2m+1,l} = -\frac{1}{8}\binom{4}{2l-2m+2}, \quad l = -1,\ldots,2^r n-2. \tag{5.89}$$

We can now combine (5.86) and (5.89) to obtain the formula

$$\omega^r_{j,k} = (-1)^j \frac{1}{8}\binom{4}{2k-j+3}, \quad k = -1,\ldots,2^r n-2; \quad j = 0,\ldots,2^{r+1}n-4. \tag{5.90}$$

Furthermore, since it holds that

$$(-1)^{-1}\frac{1}{8}\binom{4}{2k-(-1)+3} = \begin{cases} -\frac{3}{4}, & k=-1, \\ -\frac{1}{8}, & k=0, \\ 0, & k=1,\ldots,2^r n-2, \end{cases}$$

we deduce from (5.90), (5.84) and (5.85) that,

$$\omega^r_{j,k} = (-1)^j \frac{1}{8}\binom{4}{2k-j+3}, \quad k = -1,\ldots,2^r n-2, \quad j = -1,\ldots,2^{r+1}n-3. \tag{5.91}$$

Moreover, since also

$$(-1)^{-2}\frac{1}{8}\binom{4}{2k-(-2)+3} = \begin{cases} \frac{1}{2}, & k=-1, \\ 0, & k=0,\ldots,2^r n-1, \end{cases}$$

we deduce from (5.91), (5.75) and (5.76) that

$$\omega^r_{j,k} = (-1)^j \frac{1}{8}\binom{4}{2k-j+3}, \quad k = -1,\ldots,2^r n-2, \quad j = -2,\ldots,2^{r+1}n-2. \tag{5.92}$$

Finally, since also

$$(-1)^{-3}\frac{1}{8}\binom{4}{2k-(-3)+3} = \begin{cases} -\frac{1}{8}, & k=-1, \\ 0, & k=0,\ldots,2^r n-1, \end{cases}$$

we deduce from (5.92), (5.71) and (5.76) that

$$\omega^r_{j,k} = (-1)^j \frac{1}{8}\binom{4}{2k-j+3}, \quad k = -1,\ldots 2^r n-2, \quad j = -3,\ldots,2^{r+1}n-1. \tag{5.93}$$

It follows from (5.93) and (5.62), together with Theorems 5.8 and 5.12, as well as (5.59), that the following decomposition result holds.

Theorem 5.13 *For $r \in \mathbb{Z}_+$, it holds that, if $f(x) = \sum_{j=-3}^{2^{r+1}n-1} c_j N_4(2^{r+1}x - j)$, $x \in [0, n]$, then*

$$f(x) = (\mathcal{P}_n^r f)(x) + \frac{1}{8} \sum_{j=-1}^{2^r n-2} \left[\sum_{k=-3}^{2^r n-1} (-1)^k \binom{4}{2j-k+3} c_k \right] \psi_4(2^r x - j), \quad x \in [0, n], \quad (5.94)$$

with $(\mathcal{P}_n^r f)(x)$ defined for $x \in [0, n]$ by (5.35), and with the cubic cardinal spline-wavelet ψ_4 defined by (5.57).

5.6 Cubic Spline Quasi-Interpolation on a Bounded Interval

By choosing $m = 4$ in Theorem 2.9 and Table 2.1, we deduce that the approximation operator $\mathcal{Q}_{4,r} : M(\mathbb{R}) \to \mathcal{S}_4^r$ defined by

$$(\mathcal{Q}_{4,r} f)(x) = \sum_j \left[-\frac{1}{6} f\left(\frac{j}{2^r}\right) + \frac{4}{3} f\left(\frac{j+1}{2^r}\right) - \frac{1}{6} f\left(\frac{j+2}{2^r}\right) \right] N_4(2^r x - j), \quad x \in \mathbb{R}, \quad (5.95)$$

satisfies the polynomial reproduction property

$$(\mathcal{Q}_{4,r} p)(x) = p(x), \quad x \in \mathbb{R}, \quad p \in \pi_3, \quad (5.96)$$

i.e. $\mathcal{Q}_{4,r}$ is a quasi-interpolation operator.

For $n \in \mathbb{N}$, with $n \geq 1$, and $r \in \mathbb{Z}_+$, we seek to construct a bounded interval quasi-interpolation operator $\mathcal{Q}_n^r : C[0, n] \to \mathcal{S}^r[0, n]$, where

$$(\mathcal{Q}_n^r f)(x) = \sum_{j=-3}^{2^r n-1} \sum_{k=-3}^{2^r n-3} \left[u_{j,k}^n f\left(\frac{k+3}{2^r}\right) \right] N_4(2^r x - j), \quad x \in [0, n], \quad (5.97)$$

and where the sequence $\{u_{j,k}^n : k = -3, \ldots, 2^r n - 3; \quad j = -3, \ldots, 2^r n - 1\}$ is to be chosen in such a way that

$$(\mathcal{Q}_n^r p)(x) = p(x), \quad p \in \pi_3, \quad x \in [0, n]. \quad (5.98)$$

To this end, following the polynomial extrapolation idea introduced in De Villiers and Rohwer, [13], (see also [12] and [14]), we first define the Lagrange interpolation polynomials $\{\tilde{L}_j : j = 0, 1, 2, 3\}$ by

$$\tilde{L}_j(x) = \prod_{j \neq k=0}^{3} \frac{x-k}{j-k}, \quad x \in \mathbb{R}, \quad j = 0, 1, 2, 3. \tag{5.99}$$

Since

$$\prod_{j \neq k=0}^{3} (j-k) = \frac{(-1)^{j+1} 6}{\binom{3}{j}}, \quad j = 0, 1, 2, 3,$$

it follows from (5.99) that

$$\tilde{L}_j(x) = \frac{(-1)^{j+1}}{6} \binom{3}{j} \prod_{j \neq k=0}^{3} (x-k), \quad x \in \mathbb{R}, \quad j = 0, 1, 2, 3. \tag{5.100}$$

For $f \in C[0, n]$ and $r \in \mathbb{Z}_+$, let the polynomial $p_L^r = p_{f,L}^r \in \pi_3$ be defined by

$$p_L^r(x) = \sum_{j=0}^{3} f\left(\frac{j}{2^r}\right) \tilde{L}_j(2^r x), \quad x \in \mathbb{R}. \tag{5.101}$$

Since (5.99) gives

$$\tilde{L}_j(k) = \delta_{j,k}, \quad j, k = 0, 1, 2, 3, \tag{5.102}$$

we see from (5.101) that, for $j \in \{0, 1, 2, 3\}$,

$$p_L^r\left(\frac{j}{2^r}\right) = \sum_{k=0}^{3} f\left(\frac{k}{2^r}\right) \tilde{L}_k(j) = \sum_{k=0}^{3} f\left(\frac{k}{2^r}\right) \delta_{j,k} = f\left(\frac{j}{2^r}\right). \tag{5.103}$$

Next, we define, for $r \in \mathbb{Z}_+$, the polynomial $p_R^r = p_{f,R}^r \in \pi_3$ by

$$p_R^r(x) = \sum_{j=2^r n-3}^{2^r n} f\left(\frac{j}{2^r}\right) \tilde{L}_{2^r n-j}\left(2^r(n-x)\right), \quad x \in \mathbb{R}. \tag{5.104}$$

It follows from (5.104) and (5.102) that, for $j \in \{2^r n-3,\ 2^r n-2,\ 2^r n-1,\ 2^r n\}$, we have

$$p_R^r\left(\frac{j}{2^r}\right) = \sum_{k=2^r n-3}^{2^r n} f\left(\frac{k}{2^r}\right) \tilde{L}_{2^r n-k}(2^r n-j) = \sum_{k=0}^{3} f\left(\frac{2^r n-k}{2^r}\right) \tilde{L}_k(2^r n-j) = f\left(\frac{j}{2^r}\right). \tag{5.105}$$

The following construction is analogous to the one introduced in [13].

Theorem 5.14 *For $r \in \mathbb{Z}_+$, the approximation operator $\mathcal{Q}_n^r : C[0,n] \to \mathcal{S}^r[0,n]$ defined for $x \in [0,n]$ by*

$$\begin{aligned}
(\mathcal{Q}_n^r f)(x) &= \left[-\frac{1}{6}p_L^r\left(\frac{-3}{2^r}\right) + \frac{4}{3}p_L^r\left(\frac{-2}{2^r}\right) - \frac{1}{6}p_L^r\left(\frac{-1}{2^r}\right)\right] N_4(2^r x + 3) \\
&\quad + \left[-\frac{1}{6}p_L^r\left(\frac{-2}{2^r}\right) + \frac{4}{3}p_L^r\left(\frac{-1}{2^r}\right) - \frac{1}{6}f(0)\right] N_4(2^r x + 2) \\
&\quad + \left[-\frac{1}{6}p_L^r\left(\frac{-1}{2^r}\right) + \frac{4}{3}f(0) - \frac{1}{6}f\left(\frac{1}{2^r}\right)\right] N_4(2^r x + 1) \\
&\quad + \sum_{j=0}^{2^r n-2} \left[-\frac{1}{6}f\left(\frac{j}{2^r}\right) + \frac{4}{3}f\left(\frac{j+1}{2^r}\right) - \frac{1}{6}f\left(\frac{j+2}{2^r}\right)\right] N_4(2^r x - j) \\
&\quad + \left[-\frac{1}{6}f\left(\frac{2^r n-1}{2^r}\right) + \frac{4}{3}f(n) - \frac{1}{6}p_R^r\left(\frac{2^r n+1}{2^r}\right)\right] N_4(2^r x - 2^r n - 1), \qquad (5.106)
\end{aligned}$$

with the polynomials $p_L^r = p_{f,L}^r$ and $p_R^r = p_{f,R}^r$ defined by (5.101) and (5.104), satisfies the polynomial reproduction property (5.98).

Proof. Let $p \in \pi_3$. Then, for $r \in \mathbb{Z}_+$ and $x \in [0,n]$, it follows from (5.106), (5.103), (5.105), (5.7), (5.95) and (5.96) that

$$\begin{aligned}
(\mathcal{Q}_n^r p)(x) &= \sum_{j=-3}^{2^r n-1} \left[-\frac{1}{6}p\left(\frac{j}{2^r}\right) + \frac{4}{3}p\left(\frac{j+1}{2^r}\right) - \frac{1}{6}p\left(\frac{j+2}{2^r}\right)\right] N_4(2^r x - j) \\
&= \sum_{j} \left[-\frac{1}{6}p\left(\frac{j}{2^r}\right) + \frac{4}{3}p\left(\frac{j+1}{2^r}\right) - \frac{1}{6}p\left(\frac{j+2}{2^r}\right)\right] N_4(2^r x - j) \\
&= p(x).
\end{aligned}$$

■

5.7 Decomposition Algorithm on a Bounded Interval

Based on our work of this chapter, we now formulate the following cubic spline wavelet decomposition algorithm.

Algorithm 5.15 *Let $f \in C[0,n]$ be given.*

1. *For a sufficiently large value of $N \in \mathbb{N}$, define*

$$f(x) = \sum_{j=-3}^{2^N n-1} c_j^N N_4(2^N x - j) = (\mathcal{Q}_n^N f)(x), \quad x \in [0,n], \qquad (5.107)$$

by means of (5.106).

2. *Define the sequences* $\{f^r : r = N-1,\ldots,M\}$ *and* $\{g^r : r = N-1,\ldots,M\}$, *with* $M < N$ *and* M *a sufficiently small integer in* $\mathbb{Z}_+$, *by*

$$f^r(x) = \sum_{j=-3}^{2^r n-1} c_j^r N_4(2^r x - j) = (\mathcal{P}_n^r f^{r+1})(x), \quad x \in [0,n], \tag{5.108}$$

with $\mathcal{P}_n^r$ *as in (5.35), and*

$$g^r(x) = \sum_{j=-1}^{2^r n-2} d_j^r \psi_4(2^r x - j), \quad x \in [0,n], \tag{5.109}$$

with

$$d_j^r = \frac{1}{8}\sum_{k=-3}^{2^r n-1} (-1)^k \binom{4}{2j-k+3} c_k^{r+1}, \quad j = -1,\ldots,2^r n-2; \tag{5.110}$$

according to which Theorem 5.13 gives

$$f^{r+1}(x) = f^r(x) + g^r(x), \quad x \in [0,n], \quad r = N-1,\ldots,N-M. \tag{5.111}$$

Since, as implied by the definition (5.1), $\pi_3 \subset \mathcal{S}^r[0,n]$, $r \in \mathbb{Z}_+$, and (5.60) holds, we deduce from Theorem 5.13 that local smooth polynomial-like behaviour of f in an interval I will be detected by means of relatively small wavelet coefficients for these wavelets $\psi(2^r x - j)$ with support intervals overlapping the interval I, as illustrated by our following example.

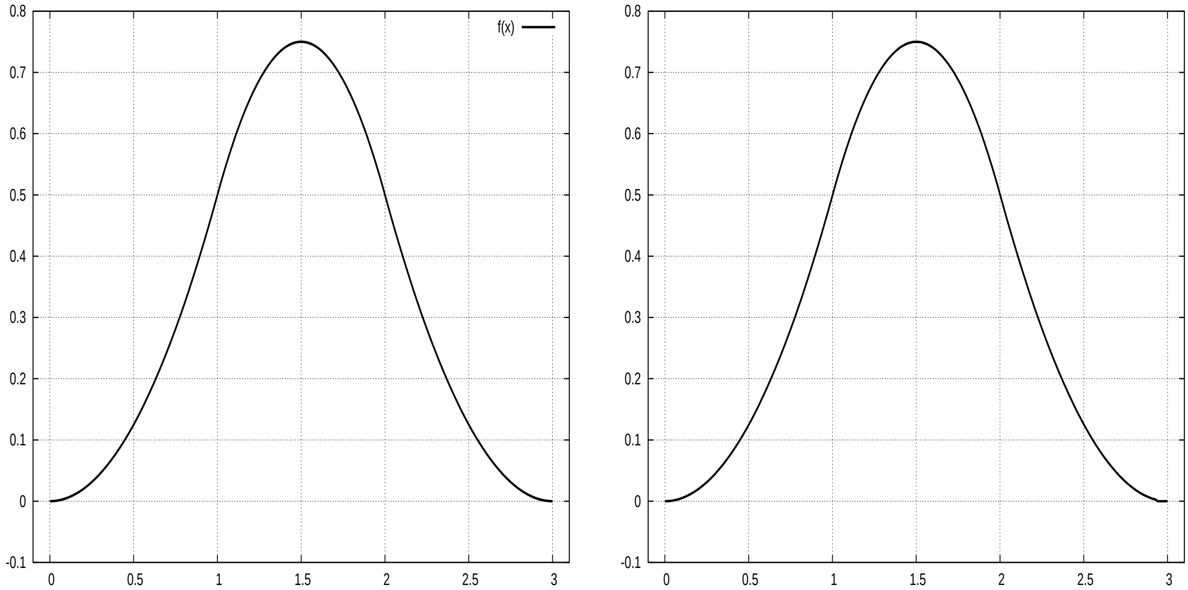
5.8 Example

We consider the signal $f \in C[0,3]$ defined by $f(x) = N_3(x)$, $x \in [0,3]$, i.e., as in Example 4.5,

$$f(x) = \begin{cases} \frac{1}{2}x^2, & 0 \le x \le 1, \\ \frac{1}{2}(-2x^2 + 6x - 3), & 1 \le x \le 2, \\ \frac{1}{2}(3-x)^2, & 2 \le x \le 3, \end{cases} \tag{5.112}$$

the graph of which is given in Figure 5.4a. Note that we have here $n = 3$. Then $f \in C^1[0,3]\backslash C^2[0,3]$, with discontinuities in the second derivatives f'' at $x \in \{1,2\}$.

We use the wavelet decomposition algorithm given by Algorithm 5.15 with $N = 10$ and $M = 5$. The quasi-interpolant approximation $f_{10} = \mathcal{Q}_4^{10} f$, as defined in (5.106), is shown in Figure 5.4b.



(a) The signal f

(b) The function $\mathcal{Q}_4^{10} f$

Figure 5.4: The signal f and approximation $f_{10} = \mathcal{Q}_4^{10} f$

We plot, for $r = N-1, \ldots, M$, the functions f^r and g^r in Figures 5.5 to 5.9 by means of (5.107), (5.108), (5.35), (5.109), (5.110), (5.56), (2.7) and (2.3).

Observe in particular from our graphs below of the wavelet component g^r that, in contrast to the analogous graphs of Example 4.5, which locally detected the discontinuities in f'' at $x \in \{0, 1, 2, 3\}$, Figures 5.5 to 5.9 below only detect the discontinuities in f'' at the interior points $x = 1$ and $x = 2$. Our decomposition algorithm therefore succeeds, as expected from our theoretical results in this chapter, in eliminating any edge artefacts close to the endpoints $x = 0$ and $x = 3$.

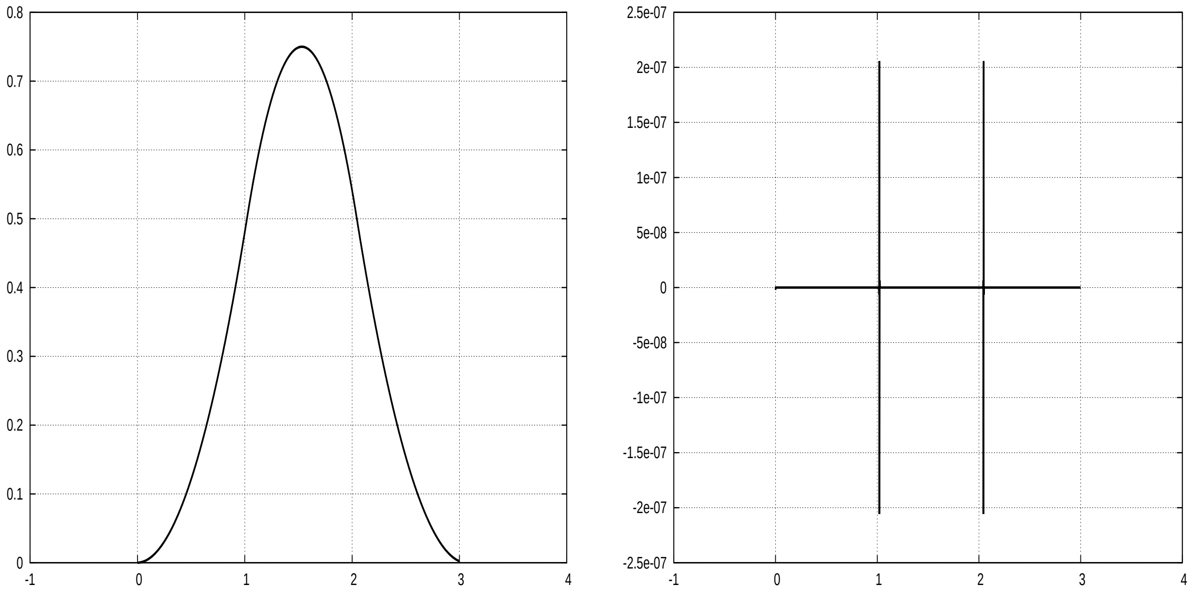


Figure 5.5: The functions f^9 and g^9

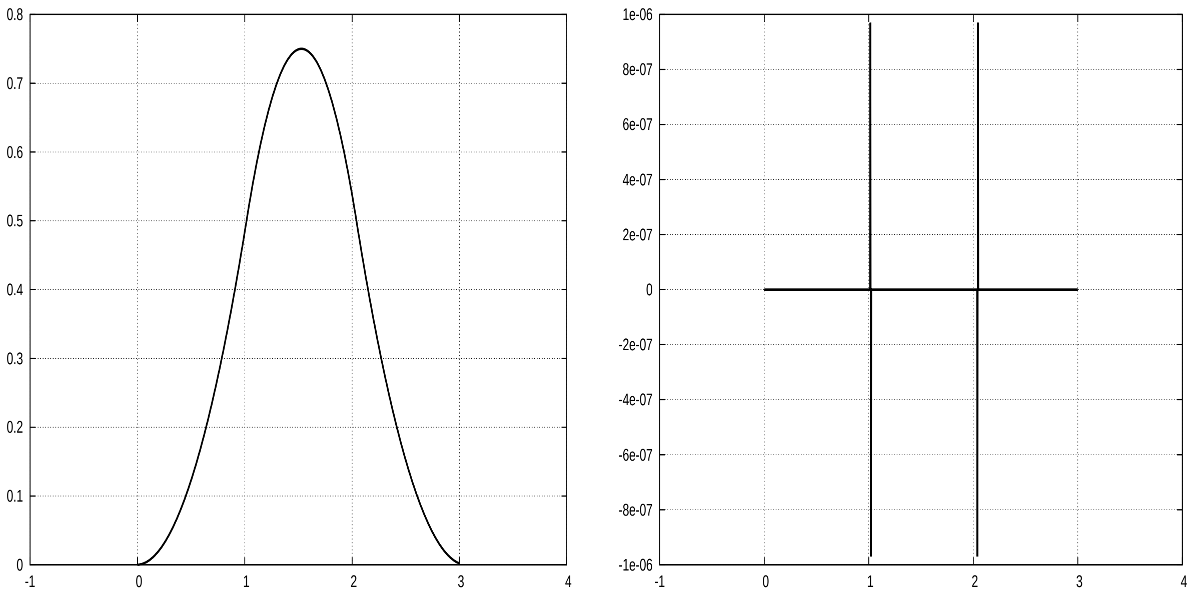


Figure 5.6: The functions f^8 and g^8

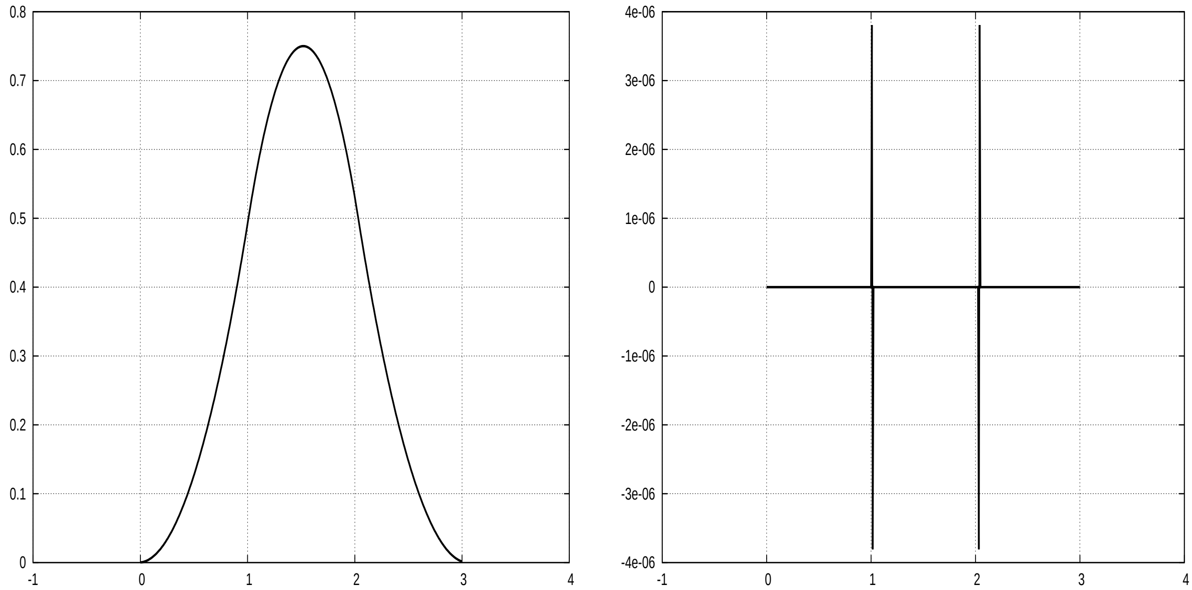


Figure 5.7: The functions f^7 and g^7

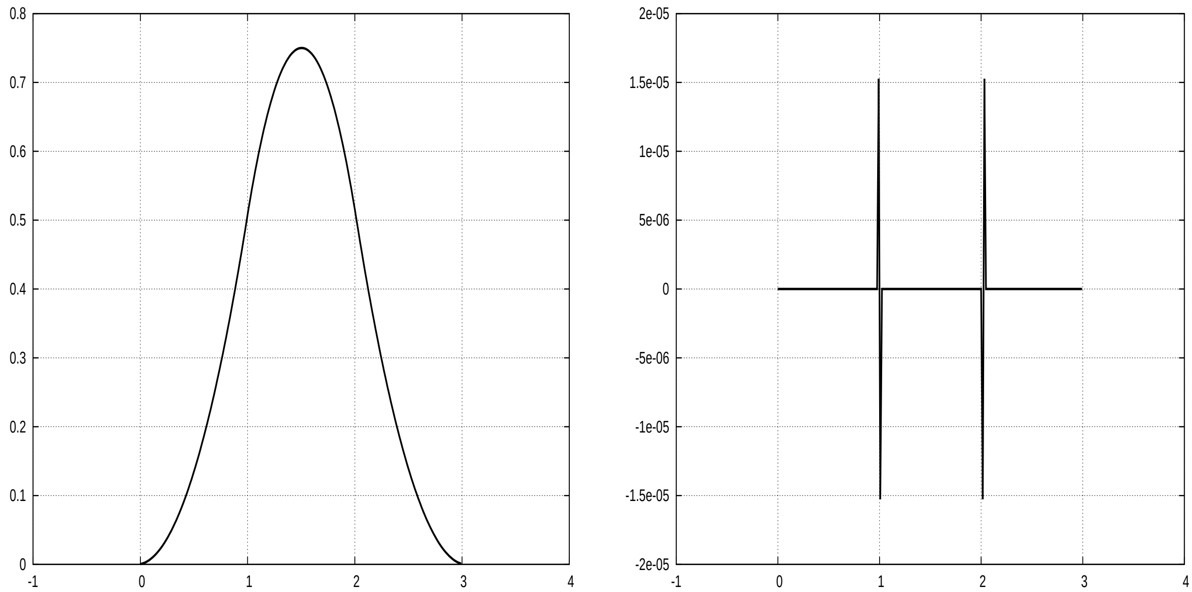


Figure 5.8: The functions f^6 and g^6

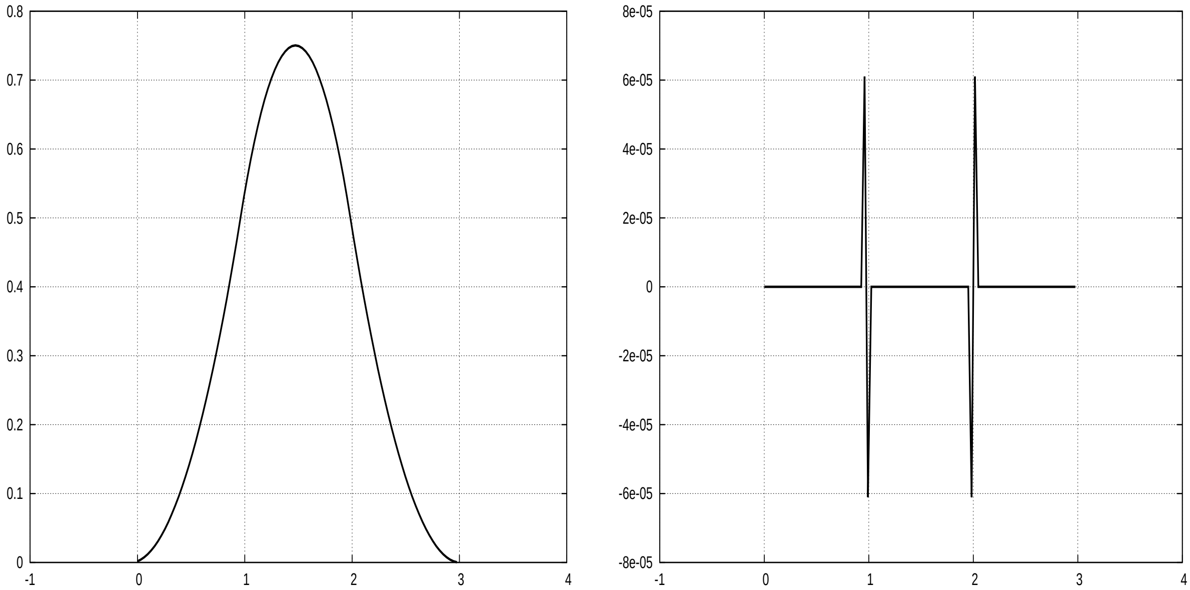


Figure 5.9: The functions f^5 and g^5

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