Interpolatory refinement pairs with properties of symmetry and polynomial filling

by

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Thesis presented in partial fulfilment of the requirements for the degree of Masters in Mathematics at the University of Stellenbosch

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March 2008

Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

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Summary

Subdivision techniques have, over the last two decades, developed into a powerful tool in computer-aided geometric design (CAGD). In some applications it is required that data be preserved exactly; hence the need for interpolatory subdivision schemes. In this thesis, we consider the fundamentals of the mathematical analysis of symmetric interpolatory subdivision schemes for curves, also with the property of polynomial filling up to a given odd degree, in the sense that, if the initial control point sequence is situated on such a polynomial curve, all the subsequent subdivision iterates fills up this curve, for it to eventually also become also the limit curve.

A subdivision scheme is determined by its mask coefficients, which we find convenient to mathematically describe as a bi-infinite sequnce a with finite support. This sequence is in one-to-one correspondence with a corresponding Laurent polynomial A with coefficients given by the mask sequence a. After an introductory Chapter 1 on notation, basic definitions, and an overview of the thesis, we proceed in Chapter 2 to separately consider the issues of interpolation, symmetry and polynomial filling with respect to a subdivision scheme, eventually leading to a definition of the class $\mathscr{A}_{m,n}$ of mask symbols in which all of the above desired properties are combined.

We proceed in Chapter 3 to deduce an explicit characterization formula for the class $\mathscr{A}_{m,n}$, in the process also showing that its optimally local member is the well-known Dubuc–Deslauriers (DD) mask symbol D_m of order m. In fact, an alternative explicit characterization result appears in recent work by De Villiers and Hunter, in which the authors characterized mask symbols $A \in \mathscr{A}_{m,n}$ as arbitrary convex combinations of DD mask symbols. It turns out that $\mathscr{A}_{m,m} = \{D_m\}$, whereas the class $\mathscr{A}_{m,m+1}$ has one degree of freedom, which we interpret here in the form of a shape parameter $t \in \mathbb{R}$ for the resulting subdivision scheme.

In order to investigate the convergence of subdivision schemes associated with mask symbols in $\mathscr{A}_{m,n}$, we first introduce in Chapter 4 the concept of a refinement pair (a,ϕ), consisting of a finitely-supported sequence a and a finitely-supported function ϕ, where ϕ is a refinable function in the sense that it can be expressed as a finite linear combination, as determined by a, of the integer shifts of its own dilation by factor 2. After presenting proofs of a variety of properties satisfied by a given refinement pair (a,ϕ), we next introduce the concept of an interpolatory refinement pair as one for which the refinable function ϕ interpolates the delta sequence at the integers. A fundamental result is then that the existence of an interpolatory refinement pair (a,ϕ) guarantees the convergence of the interpolatory subdivision scheme with subdivision mask a, with limit function Φ expressible as a linear combination of the integer shifts of ϕ, and with all the subdivision iterates lying on Φ.

In Chapter 5, we first present a fundamental result by Micchelli, according to which interpolatory refinable function existence is obtained for mask symbols in $\mathscr{A}_{m,n}$ if the mask symbol A is strictly positive on the unit circle in complex plane. After showing that the DD mask symbol D_m satisfies this sufficient property, we proceed to compute the precise t-interval for such positivity on the unit circle to occur for the mask symbols $A = A_m(t|\cdot) \in \mathscr{A}_{m,m+1}$. Also, we compare our numerical results with analogous ones in the literature.

Finally, in Chapter 6, we investigate the regularity of refinable functions $\phi = \phi_m(t|\cdot)$ corresponding to mask symbols $A_m(t|\cdot)$. Using a standard result from the literature in which a lower bound on the Hölder continuity exponent of a refinable function ϕ is given explicitly in terms of the spectral radius of a matrix obtained from the corresponding mask sequence a, we compute this lower bound for selected values of m.

Opsomming

Tegnieke gebaseer op subdivisie het oor die laaste twee dekades ontwikkel in kragtige gereedskap in rekenaargesteunde geometriese ontwerp (CAGD). In sommige toepassings word dit vereis dat data presies behoue bly; en dus die nodigheid vir interpolerende subdivisieskemas. In hierdie tesis beskou ons die grondliggende beginsels van die wiskundige analise van simmetriese interpolerende subdivisieskemas vir krommes, met ook die eienskap van polinoomvulling tot by ‘n gegewe onewe graad, in die sin dat, indien die beginkontrolepuntry geleë is op so ‘n polinoomkromme, dan vul al die subdivisie-iterate hierdie kromme, sodat dit dan uiteindelik ook die limietkromme word.

‘n Subdivisieskema word bepaal deur die maskerkoëffisiënte daarvan, wat ons gerieflik vind om wiskunding te beskryf as ‘n dubbel-oneindige ry a met eindige steungebied. Hierdie ry is in een-tot-een korrespondensie met ‘n ooreenkomstige Laurent polinoom A waarvan die koëffisiënte gegee word deur die maskerry a. Na ‘n inleidende Hoofstuk 1 oor notasie, basiese definisies, en ‘n oorsig van die tesis, gaan ons in Hoofstuk 2 voort om afsonderlik te beskou die konsepte van interpolasie, simmetrie en polinoomvulling met betrekking tot ‘n subdivisieskema, en wat uiteindelik lei tot ‘n definisie van die klas $\mathscr{A}_{m,n}$ van maskersimbole waarin al die bogenoemde gunstige eienskappe gekombineer word.

In Hoofstuk 3 gaan ons voort om ‘n eksplisiete karakteriseringsformule vir die klas $\mathscr{A}_{m,n}$ af te lei, en in die proses wys ons dat die optimale lokale lid daarvan die bekende Dubuc–Deslauriers (DD) maskersimbool D_m van orde m is. ‘n Alternatiewe eksplisiete karakterisering verskyn in onlangse werk deur De Villiers en Hunter, waarin die outeurs maskersimbole $A \in \mathscr{A}_{m,n}$ as arbitrêre konvekse kombinasies van DD maskersimbole karakteriseer. Dit blyk dat $\mathscr{A}_{m,m} = \{D_m\}$, terwyl, daarteenoor, die klas $\mathscr{A}_{m,m+1}$ een vryheidsgraad het, wat ons hier interpreteer as ‘n vormparameter $t \in \mathbb{R}$ vir die betrokke subdivisieskema.

Om die konvergensie van die subdivisieskemas ge-assosieer met maskersim-

bole in $\mathscr{A}_{m,n}$ te ondersoek, stel ons in Hoofstuk 4 die konsep van 'n verfyningspaar (a,ϕ) bekend, wat bestaan uit 'n eindig-ondersteunde ry a en 'n eindig-ondersteunde funksie ϕ, waar ϕ 'n verfynbare funksie is in die sin dat dit uitgedruk kan word as 'n eindige lineêre kombinasie, soos bepaal deur a, van die heelgetalskuiwe van sy eie dilasie met faktor 2. Nadat bewyse van 'n verskeidenheid van eienskappe bevredig deur 'n gegewe verfyningspaar (a,ϕ) gegee is, gaan ons voort om die konsep van 'n interpolerende verfyningspaar te definieer as een waarin die verfynbare funksie ϕ die delta-ry by die heelgetalle interpoleer. 'n Fundamentele resultaat is dan dat die bestaan van 'n interpolerende verfyningspaar (a,ϕ) die konvergensie van die interpolerende subdivisieskema met subdivisiemasker a waarborg, met limietfunksie Φ uitdrukbaar as 'n lineêre kombinasie van die heelgetalskuiwe van ϕ, en met al die subdivisie-iterate op Φ geleë.

In Hoofstuk 5 gee ons 'n fundamentele resultaat deur Micchelli, waarvolgens interpolerende verfynbare funksie bestaan verkry word vir maskersimbole in $\mathscr{A}_{m,n}$ indien die maskersimbool A streng positief is op die eenheidsirkel in die komplekse vlak. Nadat getoon word dat die DD maskersimbool voldoen aan hierdie voldoende voorwaarde, gaan ons voort om die presiese t-interval vir sulke positiwiteit op die eenheidsirkel te bereken vir maskersimbole $A = A_m(t|\cdot) \in \mathscr{A}_{m,m+1}$. Ons verskaf ook 'n vergelyking tussen ons numeriese resultate en analoë resultate in die literatuur.

Laastens, in Hoofstuk 6, ondersoek ons die regulariteit van die verfynbare funksies $\phi = \phi_m(t|\cdot)$ wat ooreenstem met maskersimbole $A_m(t|\cdot)$. Deur gebruik te maak van 'n standaardresultaat uit die literatuur waarin die ondergrens vir die Hölder kontinuïteitsindeks van 'n verfynbare funksie eksplisiet gegee word in terme van die spektraalradius van 'n matriks verkry uit die ooreenkomstige maskerry a, bereken ons dan hierdie ondergrens vir geselekteerde waardes van m.

Acknowledgements

The author would like to thank the following people for their contribution towards this thesis:

- Prof J.M. de Villiers, my supervisor who introduced me to the very interesting field of wavelets and subdivision, and for his important support and guidance throughout my graduate studies at Stellenbosch University. His expertise in mathematics improved my research skills and prepared me for future challenges.
- Prof Mike Neamtu, for his valuable comments on the main results of this thesis during his visit to South Africa.
- Prof Ben Herbst and Dr Karin Hunter, for the Matlab subdivision code. D.N.J. Els for the US-Thesis Package.
- The Wavelet Research Group and to everyone at Department of Mathematical Sciences, Mathematics Division, University of Stellenbosch, thank you for the friendliness and help.
- The National Research Foundation (NRF), Ernst & Ethel Eriksen Trust and the University of Stellenbosch for financial support.
- Rembu, for sharing his experience of the thesis writing endeavor with me, for listening to complaints and frustrations, and for encouragement to follow my dreams and work hard.
- My stepfather Vho-Tshivhase, my sister, Mukovhe, my brothers Kenny and Simon, my career adviser Mr Mutepe and my grandmother Vho-Muofhe for their support and encouragement. My mother Vho-Avhurengwi has been an inspiration throughout my life and this thesis is dedicated to her.

Contents

List of Symbols

Symbol	Definition
$\mathbb{N}$	the set of natural numbers
$\mathbb{N}_k$	the set of natural numbers $\leq k$
$\mathbb{Z}$	the set of integers
$\mathbb{Z}_+$	the set of nonnegative integers
$\mathbb{Z}_k$	the set of nonnegative integers $\leq k$
$\mathbb{J}_k$	the set of integers $\{-k+1,\ldots,k\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\sum\limits_j$	the sum $\sum\limits_{j\in\mathbb{Z}}$
$\lfloor x \rfloor$	the largest integer $\leq x$
$\lceil x \rceil$	the smallest integer $\geq x$
$\mathcal{M}(\mathbb{Z})$	the linear space of bi-infinite real-valued sequences, i.e. $c \in \mathcal{M}(\mathbb{Z}) \Longleftrightarrow c = \{c_j : j \in \mathbb{Z}\} \subset \mathbb{R}$
$\mathcal{M}_0(\mathbb{Z})$	the subspace of $\mathcal{M}(\mathbb{Z})$ consisting of those sequences in $\mathcal{M}(\mathbb{Z})$ with finite support. A sequence $c \in \mathcal{M}(\mathbb{Z})$ is called *finitely-supported* if the set $\{j : c_j \neq 0,\ j \in \mathbb{Z}\}$ has a finite number of elements
$\mathcal{M}(\mathbb{R})$	the linear space of real-valued functions on $\mathbb{R}$, i.e. the set $\{f : \mathbb{R} \to \mathbb{R}\}$
$\mathcal{M}_0(\mathbb{R})$	the subspace of $\mathcal{M}(\mathbb{R})$ consisting of finitely supported functions in $\mathcal{M}(\mathbb{R})$. A function $f \in \mathcal{M}(\mathbb{R})$ is called *finitely-supported* if there exists a finite interval $[\alpha,\beta] \subset \mathbb{R}$ such that $f(x) = 0, \quad x \notin [\alpha,\beta]$
a	refinement mask, or subdision mask, in $\mathcal{M}_0(\mathbb{Z})$
$\text{supp}(a)$	support of the mask a, i.e. $\text{supp}(a) = \{j : a_j \neq 0\}$
A	the mask symbol, defined by $\sum\limits_j a_j(\cdot)^j$, a Laurent polynomial or a polynomial

	if $a_j = 0, j < 0$, corresponding to the mask $a \in \mathscr{M}_0(\mathbb{Z})$
$A^{(j)}$	for $j \in \mathbb{Z}_+$, the jth derivative of the Laurent polynomial A, where $A^{(0)} = A$
S_a	subdivision operator with mask $a \in \mathscr{M}_0(\mathbb{Z})$
S_a^r	subdivision operator, with mask a, applied r times
$c^{(r)}$	the resulting sequence after applying S_a^r to a sequence $c \in \mathscr{M}(\mathbb{Z})$
$\sup\limits_j$	the supremum over all $j \in \mathbb{Z}$
$\sup\limits_x$	the supremum over all $x \in \mathbb{R}$
$\ell^\infty(\mathbb{Z})$	the subspace of bounded sequences in $\mathscr{M}(\mathbb{Z})$, i.e. if $c \in \mathscr{M}(\mathbb{Z})$, and $\sup\limits_j \lvert c_j \rvert$
$\lVert . \rVert_\infty$	the sup norm $\sup\limits_j \lvert c_j \rvert$ for the linear space $\ell^\infty(\mathbb{Z})$
$\mathscr{C}(\mathbb{R})$	the linear space of continuous functions in $\mathscr{M}(\mathbb{R})$
$\mathscr{C}_0(\mathbb{R})$	$\{f \in \mathscr{C}(\mathbb{R})$: the function f is finitely supported$\}$
$\mathscr{C}^k(\mathbb{R})$	for $k \in \mathbb{Z}_+$, $\mathscr{C}^k(\mathbb{R}) := \{f \in \mathscr{M}(\mathbb{R}) : f^{(j)} \in \mathscr{C}(\mathbb{R}),\ \ j \in \mathbb{Z}_k\}$, with the convention $f^{(0)} = f$
$\mathscr{C}^{-1}(\mathbb{R})$	the subspace of $\mathscr{M}(\mathbb{R})$ consisting of piecewise continuous functions
$\mathscr{A}_{m,n}$	the class of symmetric interpolatory mask symbol Laurent polynomials A, with A possessing a zero of order m at -1, and where $a_j = 0, \quad j \notin \{-2n+1, \ldots, 2n-1\}, \quad a_{-2n+1} \neq 0, \quad a_{2n-1} \neq 0$
Φ	limit function of the subdivision scheme (S_a, c)
(a, ϕ)	refinement pair
ϕ	refinable function
π	the space of all polynomials
π_n	the linear space of polynomials of degree $\leq n$, where $n \in \mathbb{Z}_+$
δ_j	the Kronecker delta, equal to zero for all $j \in \mathbb{Z}$, except for $\delta_0 = 1$
$\delta_{j,k}$	the Kronecker delta, equal to zero for all $j, k \in \mathbb{Z}$, except for $\delta_{j,j} = 1$
δ	the sequence $\{\delta_{j,0} : j \in \mathbb{Z}\}$
$\binom{j}{k}$	$= \begin{cases} \dfrac{j!}{k!(j-k)!}, & k \in \{0, 1, \ldots, j\}, \\ 0, & k \notin \{0, 1, \ldots, j\}, \end{cases}$ are the binomial coefficients $\left\{\binom{j}{k} : j,\ \ k \in \mathbb{Z}_+\right\}$, with the convention that $0! = 1$
$L_{m,k}$	for $k \in \mathbb{J}_m$, the Lagrange fundamental polynomials of degree $(2m-1)$, with respect to the interpolation points $\mathbb{J}_m$
d_m	Dubuc–Deslauriers mask of order m
D_m	Dubuc–Deslauriers mask symbol of order m
(d_m, ϕ_m^D)	Dubuc–Deslauriers refinement pair

ϕ_m^D	Dubuc–Deslauriers refinable function
$p^{(e)}$	the even part $p = \sum_j p_{2j}(\cdot)^{2j}$ of a (Laurent) polynomial $p = \sum_j p_j(\cdot)^j$
$p^{(o)}$	the odd part $p = \sum_j p_{2j+1}(\cdot)^{2j+1}$ of a (Laurent) polynomial $p = \sum_j p_j(\cdot)^j$

List of Figures

Chapter 1

Introduction

1.1 A brief overview

In recent decades, subdivision schemes have become important and efficient ways to generate smooth curves and surfaces. See [14] and [23] for their application to computer-aided geometric design, and e.g., [2], [3], [5] and [6] for their applications to wavelet decomposition. In this thesis, we consider subdivision schemes for curves. The basic idea in subdivision is: if, given a sequence of data points, or control points, in the plane or in space, we compute a denser sequence of new control points by means of a rule which calculates each new set of control points as a linear combination of its neighbouring initial control points, the resulting rule is known as a *subdivision scheme*. For an appropriately chosen subdivision scheme, the increasingly dense control point sequences, as obtained by repeatedly using the same rule, approach a smooth curve in the limit. Each subdivision scheme is associated with a mask and is called *stationary subdivision scheme* if the same mask is used in each step of iteration. A general discussion of stationary subdivision schemes can be found in [1] and [14].

It is sometimes useful, when applying subdivision schemes, if the limit curve actually contains the initial control points. This situation can be achieved if, at each step of the subdivision algorithm, the new (denser) data set contains all of the previous data set. Such a subdivision scheme is called an *interpolatory subdivision scheme*. The corresponding refinable function obtained from such a scheme is called an *interpolatory refinable function*.

In [12] and [13], Deslauriers and Dubuc introduced an interpolatory subdivision scheme which was also symmetric. Their idea was if the orginal control points fall on a polynomial of a given odd degree, then the newly generated control points must also lie on the same polynomial. The corresponding interpolatory refinable functions are called the Dubuc–Deslauriers (DD) refinable functions.

The behaviour of the DD refinable functions has drawn the attention of several mathematicians. Following Meyer's suggestion, Daubechies noticed that there is a similarity between the techniques used in [7], and those in [12]. In [21], Micchelli discussed the connection between the DD refinable functions and the Daubechies orthogonal wavelets [6]. The smoothness analysis of these DD refinable functions was studied by several authors, e.g., [4], [6], [16], [20] and [24]. One of their approaches was to use the spectral radius of certain matrices.

In [8], [9], (see also [19] and [22]), De Villiers and Hunter introduced a general class $\mathscr{A}_{m,n}$ of symmetric interpolatory subdivision schemes with the property of polynomial filling up to a degree $2m-1$. It was shown there that the property of polynomial filling is equivalent to the fact that the corresponding Laurent polynomial mask symbol A has a zero of order at least $2m$ at -1. They also characterized mask symbols in the above class as arbitrary convex combinations of DD mask symbols.

In this thesis, we give an alternative explicit characterization for the class of symmetric interpolatory subdivision schemes with the property of polynomial filling up to a degree $2m-1$. This characterization includes DD mask symbols as a special case. The thesis is organized as follows:

- In Chapter 2, we present the concept of interpolatory subdivision schemes and investigate these with respect to their properties and explicit construction methods.

- Next, in Chapter 3, we explicitly characterize interpolatory masks in the class $\mathscr{A}_{m,n}$.

- In Chapter 4, we present the concept of interpolatory refinement pairs and investigate these with respect to their properties.

- In Chapter 5, we present a set of sufficient conditions on an interpolatory mask which guarantees the existence of the corresponding interpolatory refinable function and also give numerical examples to graphically illustrate the convergence and existence results.

- Finally, in Chapter 6, we investigate and give numerical examples to graphically illustrate the regularity (or smoothness) of the interpolatory refinable function of Chapter 5.

Before we proceed futher, we introduce some notation and results.

1.2 Notation

The following notation is used in this thesis.

By $\mathbb{N}$ we denote the set of natural numbers, by $\mathbb{Z}$ the set of integers, by $\mathbb{R}$ the set of real numbers, and by $\mathbb{C}$ the set of complex numbers. For the set of nonnegative integers we write $\mathbb{Z}_+$, and for any $k \in \mathbb{Z}_+$, we use the symbol $\mathbb{Z}_k$ to denote the set of nonnegative integers $\leq k$, i.e. $\mathbb{Z}_k = \{0, 1, \ldots, k\}$, whereas the symbol $\mathbb{N}_k$ is used to denote the set of positive integers $\leq k$, i.e. $\mathbb{N}_k = \{1, 2 \ldots, k\}$.

We write $\mathscr{M}(\mathbb{Z})$ for the linear space of bi-infinite real-valued sequences, i.e. $c \in \mathscr{M}(\mathbb{Z})$ if and only if $c = \{c_j \in \mathbb{R} : j \in \mathbb{Z}\}$, and we use the notation $\text{supp}(c) = \{j : c_j \neq 0\}$ to denote the support of a sequence $c \in \mathscr{M}(\mathbb{Z})$. The subspace of $\mathscr{M}(\mathbb{Z})$ consisting of those sequences in $\mathscr{M}(\mathbb{Z})$ with finite support will be denoted by $\mathscr{M}_0(\mathbb{Z})$, i.e. $c = \{c_j : j \in \mathbb{Z}\} \in \mathscr{M}_0(\mathbb{Z})$ if and only if $c \in \mathscr{M}(\mathbb{Z})$, and $\text{supp}(c)$ is a finite set.

Similarly, we write $\mathscr{M}(\mathbb{R})$ for the linear space of real-valued functions on $\mathbb{R}$, and we use the notation $\mathscr{M}_0(\mathbb{R})$ for the subspace of $\mathscr{M}(\mathbb{R})$ consisting of finitely supported functions in $\mathscr{M}(\mathbb{R})$, i.e. $f \in \mathscr{M}_0(\mathbb{R})$ if and only if $f \in \mathscr{M}(\mathbb{R})$, and there exists a bounded interval $[\alpha, \beta]$ such that $f(x) = 0, \quad x \notin [\alpha, \beta]$. The subspace of continuous functions in $\mathscr{M}_0(\mathbb{R})$ will be denoted $\mathscr{C}_0(\mathbb{R})$.

Moreover, we write $\mathscr{C}(\mathbb{R})$ for the linear space of continuous functions on $\mathbb{R}$, and, for $k \in \mathbb{Z}_+$, we define $\mathscr{C}^k(\mathbb{R}) := \{f \in \mathscr{M}(\mathbb{R}) : f^{(j)} \in \mathscr{C}(\mathbb{R}), j \in \mathbb{Z}_k\}$, with the convention $f^{(0)} = f$. Observe that $\mathscr{C}^0(\mathbb{R}) = \mathscr{C}(\mathbb{R})$. We shall use the symbol $\mathscr{C}^{-1}(\mathbb{R})$ to denote the space of piecewise continuous functions on $\mathbb{R}$.

1.3 Preliminaries

Definition 1.1. For a given sequence $a \in \mathscr{M}_0(\mathbb{Z})$, we define the *subdivision operator* $S_a : \mathscr{M}(\mathbb{Z}) \to \mathscr{M}(\mathbb{Z})$ by

$$(S_a c)_j = \sum_k a_{j-2k} c_k, \quad j \in \mathbb{Z}. \tag{1.1}$$

The resulting *subdivision scheme* then generates, for a given initial sequence $c \in \mathcal{M}(\mathbb{Z})$, the sequence $\{c^{(r)} : r \in \mathbb{Z}_+\} \subset \mathcal{M}(\mathbb{Z})$ by means of

$$c^{(0)} = c, \qquad c^{(r+1)} = S_a c^{(r)}, \quad r \in \mathbb{Z}_+, \tag{1.2}$$

or, equivalently,

$$c^{(0)} = c, \qquad c^{(r)} = S_a^r c, \quad r \in \mathbb{Z}_+.$$

The sequence $a = \{a_j : j \in \mathbb{Z}\}$ in (1.1) is called the *subdivision mask* of the subdivision scheme.

Henceforth, for a given mask $a \in \mathcal{M}_0(\mathbb{Z})$ and initial control point sequence $c \in \mathcal{M}(\mathbb{Z})$, whenever we refer to the subdivision scheme (S_a, c), we shall mean the subdivision scheme (1.1), (1.2). Note that the repeated application of S_a to a given set of control points c, yielding the sequence $\{S_a^r c : r \in \mathbb{Z}_+\}$, is called a *stationary subdivision scheme*, which means that the same subdivision rule is applied at every iteration level r.

Also note that, whereas the definitions and results on subdivision throughout this thesis are stated and proved for initial sequences $c \in \mathcal{M}(\mathbb{Z})$, they can easily be extended, componentwise, to the case of vector-valued initial sequences c. However, for simplicity of presentation, we restrict ourselves to the case where $c \in \mathcal{M}(\mathbb{Z})$, except possibly in the graphical examples, were we choose $c = \{c_j : j \in \mathbb{Z}\}$, with $c_j \in \mathbb{R}^2,\ j \in \mathbb{Z}$.

The concept of the convergence for a subdivision scheme is now defined as follows.

Definition 1.2. For a given subdivision mask $a \in \mathcal{M}_0(\mathbb{Z})$ and initial control point sequence $c \in \mathcal{M}(\mathbb{Z})$, we call the subdivision scheme (S_a, c) *convergent on a subset* $\mathcal{M}$ of $\mathcal{M}(\mathbb{Z})$ if, for every $c \in \mathcal{M}$, there exists a function $\Phi \in \mathscr{C}(\mathbb{R})$, with $\Phi \neq 0$, such that

$$\sup_j \left| \Phi\left(\frac{j}{2^r}\right) - c_j^{(r)} \right| \longrightarrow 0, \quad r \longrightarrow \infty. \tag{1.3}$$

We call Φ the *limit function* of the subdivision scheme (S_a, c).

Note the following implication of Definition 1.2. For any given $x \in \mathbb{R}$, since the dyadic set $\left\{\frac{j}{2^r} : j \in \mathbb{Z},\ r \in \mathbb{Z}_+\right\}$ is dense in $\mathbb{R}$, there exists a sequence $\{j_r : r \in \mathbb{Z}_+\}$ such that $\frac{j_r}{2^r} \longrightarrow x,\ r \longrightarrow \infty$, and thus

$$\left|\Phi(x) - c_{j_r}^{(r)}\right| \leq \left|\Phi\left(\frac{j_r}{2^r}\right) - \Phi(x)\right| + \left|\Phi\left(\frac{j_r}{2^r}\right) - c_{j_r}^{(r)}\right| \longrightarrow 0 + 0 = 0,\ r \longrightarrow \infty,$$

from (1.3) and the fact that Φ is continuous at x; hence $c_{j_r}^{(r)} \longrightarrow \Phi(x)$, $r \longrightarrow \infty$.

Another useful notation here is the following definition.

Definition 1.3. For a given mask $a \in \mathscr{M}_0(\mathbb{Z})$, the Laurent polynomial A defined by

$$A(z) = \sum_j a_j z^j, \quad z \in \mathbb{C}\backslash\{0\}, \tag{1.4}$$

is called the corresponding *mask symbol.*

The concept of sum rules plays an important role in our study of interpolatory masks; hence, throughout this thesis, we shall rely on the following relationship between a sequence $a \in \mathscr{M}_0(\mathbb{Z})$ and its corresponding Laurent polynomial A.

Definition 1.4. We say that $a = \{a_j : j \in \mathbb{Z}\} \in \mathscr{M}_0(\mathbb{Z})$ satisfies the *sum rules* if and only if

$$\sum_j a_{2j} = \sum_j a_{2j+1} = 1. \tag{1.5}$$

Note that, since, according to (1.5), the corresponding mask symbol A satisfies

$$A(1) = \sum_j a_{2j} + \sum_j a_{2j+1},$$

and

$$A(-1) = \sum_j a_{2j} - \sum_j a_{2j+1},$$

a mask $a \in \mathscr{M}_0(\mathbb{Z})$ satisfies the sum rules (1.5) if and only if the corresponding mask symbol A satisfies the conditions

$$A(1) = 2, \qquad A(-1) = 0. \tag{1.6}$$

We assume throughout that the sum rules (1.5) are satisfied, and therefore that the two conditions (1.6) are met, i.e. there exist an integer $\ell \in \mathbb{N}$ and a Laurent polynomial B such that

$$A(z) = \frac{1}{2^{\ell-1}}(1+z)^\ell B(z), \quad z \in \mathbb{C}\backslash\{0\}, \tag{1.7}$$

where

$$B(1) = 1, \quad B(-1) \neq 0. \tag{1.8}$$

Out of many subdivision schemes, we shall consider in this thesis particularly interpolatory subdivision schemes, which have the property that the limit curve interpolates the original control points. We will proceed to introduce this concept in Chapter 2.

Chapter 2

A class of symmetric interpolatory subdivision schemes

2.1 The interpolatory condition

Definition 2.1. For a given initial sequence $c = \{c_j : j \in \mathbb{Z}\} \in \mathcal{M}(\mathbb{Z})$, we say that (S_a, c) is an *interpolatory subdivision scheme* if and only if it holds in (1.2) that

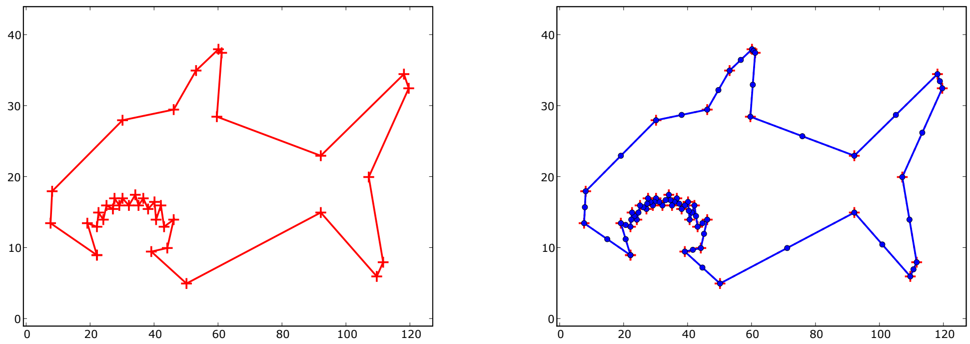
$$c_{2j}^{(r+1)} = c_j^{(r)}, \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+. \tag{2.1}$$

Note that (2.1) means that, at each step of the subdivision scheme, the even-indexed elements of the updated sequence $c^{(r+1)}$ correspond to the sequence $c^{(r)}$, whereas the odd-indexed elements of the updated sequence $c^{(r+1)}$ are calculated as a weighted average of a finite number of neighbouring elements of $c^{(r)}$.

As an example: if we are given the control points $c^{(0)} = \{c_j^{(0)} : j \in \mathbb{Z}\} \in \mathbb{R}^2$, and if we generate the new sequence using

$$\left.\begin{array}{rcl} c_{2j}^{(1)} & = & c_j^{(0)}, \\[2ex] c_{2j+1}^{(1)} & = & \dfrac{1}{2}\Big(c_j^{(0)} + c_{j+1}^{(0)}\Big), \end{array}\right\} \quad j \in \mathbb{Z}, \tag{2.2}$$

then the odd-indexed new points are generated halfway between the old ones while the even-indexed new points are simply the original points, indicated by $+'$s in Figure 2.1. This step can of course be repeated indefinitely, "roughly doubling" the number of points at each step. In this case the points fill in or converge to the straight lines connecting the initial control points (see Figure 2.1). This rule provides us with the familiar piecewise linear curve. Note that this is usually not smooth enough for practical applications. The scheme (2.2) is known as the *two-point scheme* and is the scheme discussed in [8, Section 1].



(a) (b)

Figure 2.1: (a) The original control points (+), (b) original control points (+) and one step of subdivision algorithm (o).

The interpolatory condition (2.1) has the following three equivalent formulations

$$(S_a c)_{2j} = c_j, \quad j \in \mathbb{Z}, \quad c \in \mathcal{M}(\mathbb{Z}), \tag{2.3}$$

$$a_{2j} = \delta_j, \quad j \in \mathbb{Z}, \tag{2.4}$$

and

$$A(z) + A(-z) = 2, \quad z \in \mathbb{C}\setminus\{0\}, \tag{2.5}$$

which we proceed to prove in Proposition 2.2 below. If a mask $a \in \mathcal{M}_0(\mathbb{Z})$ satisfies (2.3)–(2.5), we say that a is an *interpolatory mask.*

Proposition 2.2. *For a subdivision mask $a \in \mathcal{M}_0(\mathbb{Z})$, the conditions* (2.1), (2.3), (2.4) *and* (2.5) *are equivalent.*

Proof. (i) Let $c \in \mathcal{M}(\mathbb{Z})$ and suppose that (2.1) holds. Then, setting $r = 0$ in (2.1), and using (1.2), we see that (2.3) holds. Conversely, assume that (2.3) holds. Then, using (1.2), we obtain, for $c \in \mathcal{M}(\mathbb{Z})$, that

$$c_{2j}^{(r+1)} = \left(S_a c^{(r)}\right)_{2j} = c_j^{(r)},$$

i.e. (2.1) is satisfied. Hence we have now shown the equivalence of (2.1) and (2.3).

(ii) Suppose the sequence $a \in \mathscr{M}_0(\mathbb{Z})$ satisfies (2.4), and let $c \in \mathscr{M}(\mathbb{Z})$. Then, from (1.1), we have for $j \in \mathbb{Z}$ that

$$(S_a c)_{2j} = \sum_k a_{2j-2k} c_k = \sum_k \delta_{j-k} c_k = c_j,$$

thereby yielding (2.3). If (2.3) holds, we can choose $c = \delta$ in (2.3) to deduce from (1.2) that

$$\delta_j = \sum_k a_{2j-2k} \delta_k = a_{2j}, \quad j \in \mathbb{Z},$$

so that (2.4) holds. Hence (2.3) and (2.4) are equivalent.

(iii) Our proof will be complete if we can prove the equivalence of (2.4) and (2.5). First, use (1.4) to rewrite the left-hand side of (2.5), for $z \in \mathbb{C}\setminus\{0\}$, as

$$\begin{aligned} A(z) + A(-z) &= \sum_j a_j z^j + \sum_j a_j (-z)^j \\ &= \left[\sum_j a_{2j} z^{2j} + \sum_j a_{2j+1} z^{2j+1}\right] + \left[\sum_j a_{2j} z^{2j} - \sum_j a_{2j+1} z^{2j+1}\right], \end{aligned}$$

and thus

$$A(z) + A(-z) = 2\sum_j a_{2j} z^{2j}, \quad z \in \mathbb{C}\setminus\{0\}. \tag{2.6}$$

Now suppose that (2.4) holds. Then (2.6) gives

$$A(z) + A(-z) = 2\sum_j \delta_j z^{2j} = 2, \quad z \in \mathbb{C}\setminus\{0\},$$

so that (2.5) holds. If (2.5) holds, then (2.6) implies

$$2 = A(z) + A(-z) = 2\sum_j a_{2j} z^{2j}, \quad z \in \mathbb{C}\setminus\{0\},$$

and thus

$$\sum_j a_{2j} z^{2j} = 1, \quad z \in \mathbb{C}\setminus\{0\},$$

giving (2.4). ■

2.2 The symmetry condition

Definition 2.3. We say that the subdivision operator S_a yields a *symmetric subdivision scheme* if and only if it holds for the corresponding subdivision mask $a \in \mathscr{M}_0(\mathbb{Z})$ that

$$a_j = a_{-j}, \quad j \in \mathbb{Z}, \tag{2.7}$$

or, equivalently, as follows from (1.4), if the corresponding mask symbol A satisfies

$$A(z) = A(z^{-1}), \quad z \in \mathbb{C}\backslash\{0\}, \tag{2.8}$$

i.e., A is a *symmetric Laurent polynomial.*

The property of symmetry in an interpolatory subdivision scheme has the following consequences.

Proposition 2.4. *Suppose $a \in \mathscr{M}_0(\mathbb{Z})$ is the mask corresponding to a symmetric interpolatory subdivision scheme. Then*

(i) there is an integer $n \in \mathbb{N}$ such that

$$a_j = 0, \;\; j \notin \{-2n+1, \ldots, 2n-1\}, \;\; \textit{with } a_{-2n+1} \neq 0, \;\; a_{2n-1} \neq 0, \tag{2.9}$$

(ii) $A(e^{ix}) \in \mathbb{R}, \quad x \in \mathbb{R}.$

Proof. (i) Since a is the mask corresponding to an interpolatory subdivision scheme, we know from Proposition 2.2 that (2.4) holds. It follows from (2.4) and (2.7) that there exist an integer $n \in \mathbb{N}$ such that (2.9) holds.

(ii) Since (1.4) and (2.9) give

$$A(z) = \sum_{j=-2n+1}^{2n-1} a_j z^j, \quad z \in \mathbb{C}\backslash\{0\}, \tag{2.10}$$

we can use (2.4) and (2.7) in (2.10) to deduce that, for $x \in \mathbb{R}$, we have

$$\begin{aligned}
A(e^{ix}) = \sum_{j=-2n+1}^{2n-1} a_j (e^{ix})^j &= \sum_{j=-2n+1}^{2n-1} a_j e^{ijx} \\
&= \sum_{j=-n+1}^{n-1} a_{2j} e^{i(2j)x} + \sum_{j=-n}^{n-1} a_{2j+1} e^{i(2j+1)x} \\
&= 1 + \sum_{j=-n}^{n-1} a_{2j+1} e^{i(2j+1)x} \\
&= 1 + \sum_{j=-n}^{-1} a_{-2j-1} e^{i(2j+1)x} + \sum_{j=0}^{n-1} a_{2j+1} e^{i(2j+1)x} \\
&= 1 + \sum_{j=0}^{n-1} a_{2j+1} e^{-i(2j+1)x} + \sum_{j=0}^{n-1} a_{2j+1} e^{i(2j+1)x} \\
&= 1 + 2\sum_{j=0}^{n-1} a_{2j+1} \left[\frac{e^{-(2j+1)ix} + e^{(2j+1)ix}}{2} \right]
\end{aligned}$$

$$= \quad 1+2\sum_{j=0}^{n-1} a_{2j+1}\cos(2j+1)x,$$

from which it then follows that $A(e^{ix})$ is real for $x \in \mathbb{R}$. ■

2.3 The polynomial filling condition

Definition 2.5. An interpolatory subdivision scheme is said to possess, for an integer $\mu \in \mathbb{Z}_+$, the π_μ *polynomial filling property* if and only if the corresponding mask $a \in \mathscr{M}_0(\mathbb{Z})$ satisfies the property

$$\sum_k a_{2j+1-2k}p(k) = p\left(j+\frac{1}{2}\right), \quad j \in \mathbb{Z}, \quad p \in \pi_\mu, \tag{2.11}$$

and where μ is the largest integer for which (2.11) holds.

Observe from (2.4) that (2.11) has the equivalent formulation

$$\sum_k a_{j-2k}p(k) = p\left(\frac{j}{2}\right), \quad j \in \mathbb{Z}, \quad p \in \pi_\mu. \tag{2.12}$$

Our next result gives the relationship between the polynomial filling property in Definition 2.5 and the sum rules in Chapter 1.

Proposition 2.6. *Suppose an interpolatory subdivision scheme possesses, for an integer $\mu \in \mathbb{Z}_+$, the π_μ polynomial filling property. Then the sum rules* (1.5)*, as well as the equivalent mask symbol condition* (1.6)*, are satisfied.*

Proof. Suppose that for an integer $\mu \in \mathbb{Z}_+$, (2.11) holds. Since (2.11) and (2.12) are equivalent, we can choose the polynomial $p \in \pi_\mu$ in (2.12) as the constant polynomial $p(x) = 1,\ x \in \mathbb{R}$, to obtain

$$\sum_k a_{j-2k} = 1, \quad j \in \mathbb{Z}, \tag{2.13}$$

which is equivalent to the sum rules (1.5). In fact, if we set $j = 0$ and $j = 1$ in (2.13) we obtain (1.5), whereas, if (1.5) holds, then, for $\mu \in \mathbb{Z}$, if we successively set $j = 2\mu,\ j = 2\mu+1$ in (2.13), we obtain

$$\sum_k a_{j-2k} = \sum_k a_{2\mu-2k} = \sum_k a_{2k} = 1,$$

and

$$\sum_k a_{j-2k} = \sum_k a_{2\mu+1-2k} = \sum_k a_{2k+1} = 1,$$

thereby showing that (1.5) implies (2.13). ∎

The condition (2.13) was in fact shown in [1, Proposition 2.1] to be a neccessary condition for the convergence of the corresponding subdivision scheme (S_a, c).

We proceed to present and prove a sequence of equivalent formulations of the polynomial filling property in Definition 2.5 for integer μ, as follows. The proof below of the equivalence of (2.14a) and (2.17a) is from [9, Proposition 1], (see also [19, Proposition 2.5]).

Proposition 2.7. *For an interpolatory subdivision scheme with a mask $a \in \mathscr{M}_0(\mathbb{Z})$ and corresponding mask symbol A given by* (1.4)*, the following conditions are equivalent:*

(i)
$$\sum_k a_{2j+1-2k} p(k) = p\left(j + \frac{1}{2}\right), \quad j \in \mathbb{Z}, \quad p \in \pi_\mu, \tag{2.14a}$$
where the condition (2.14a) *does not hold if we replace*
$$\pi_\mu \text{ by } \pi_\ell, \text{ with } \ell \geq \mu + 1; \tag{2.14b}$$

(ii)
$$\sum_k a_{2j+1-2k} k^\ell = \left(j + \frac{1}{2}\right)^\ell, \quad j \in \mathbb{Z}, \quad \ell \in \mathbb{Z}_\mu, \tag{2.15a}$$
$$\sum_k a_{1-2k} k^{\mu+1} \neq \left(\frac{1}{2}\right)^{\mu+1}; \tag{2.15b}$$

(iii)
$$\sum_k (2k+1)^\ell a_{2k+1} = \delta_\ell, \quad \ell \in \mathbb{Z}_\mu, \tag{2.16a}$$
$$\sum_k (2k+1)^{\mu+1} a_{2k+1} \neq 0; \tag{2.16b}$$

(iv)
$$A^{(j)}(-1) = 0, \quad j \in \mathbb{Z}_\mu, \tag{2.17a}$$
$$A^{(\mu+1)}(-1) \neq 0; \tag{2.17b}$$

(v) *there exists a Laurent polynomial B such that*
$$A(z) = \frac{1}{2^\mu}(1+z)^{\mu+1} B(z), \quad z \in \mathbb{C}\setminus\{0\}, \tag{2.18a}$$
where
$$B(1) = 1, \quad B(-1) \neq 0. \tag{2.18b}$$

Proof. (a) First we show that (i) is equivalent to (ii). Suppose therefore that (i) holds. For $\ell \in \mathbb{Z}_\mu$ we then define the polynomial p by $p(x) = x^\ell$, $x \in \mathbb{R}$, so that $p \in \pi_\mu$. The condition (2.15a) then follows immediately from (2.14a).

Next, suppose that (2.15b) does not hold, i.e.

$$\sum_k a_{1-2k} k^{\mu+1} = \left(\frac{1}{2}\right)^{\mu+1}. \tag{2.19}$$

Let $p \in \pi_{\mu+1}$, with $\deg(p) = \mu + 1$ so that

$$p(x) = \sum_{\ell \in \mathbb{Z}_{\mu+1}} c_\ell x^\ell, \quad x \in \mathbb{R}, \quad \text{with } c_{\mu+1} \neq 0.$$

Then, for $j \in \mathbb{Z}$, we use (2.15a) with $j = 0$, and (2.19), to obtain

$$\begin{aligned}
\sum_k a_{2j+1-2k} p(k) &= \sum_k a_{2j+1-2k} \sum_{\ell \in \mathbb{Z}_{\mu+1}} c_\ell k^\ell \\
&= \sum_{\ell \in \mathbb{Z}_{\mu+1}} c_\ell \Big[\sum_k a_{2j+1-2k} k^\ell \Big] \\
&= \sum_{\ell \in \mathbb{Z}_{\mu+1}} c_\ell \Big[\sum_k a_{1-2k} (j+k)^\ell \Big] \\
&= \sum_{\ell \in \mathbb{Z}_{\mu+1}} c_\ell \sum_k a_{1-2k} \sum_{r \in \mathbb{Z}_\ell} \binom{\ell}{r} k^r j^{\ell-r} \\
&= \sum_{\ell \in \mathbb{Z}_{\mu+1}} c_\ell \sum_{r \in \mathbb{Z}_\ell} \binom{\ell}{r} j^{\ell-r} \Big[\sum_k a_{1-2k} k^r \Big] \\
&= \sum_{\ell \in \mathbb{Z}_{\mu+1}} c_\ell \sum_{r \in \mathbb{Z}_\ell} \binom{\ell}{r} j^{\ell-r} \Big(\frac{1}{2}\Big)^r \\
&= \sum_{\ell \in \mathbb{Z}_{\mu+1}} c_\ell \Big(j + \frac{1}{2}\Big)^\ell \\
&= p\Big(j + \frac{1}{2}\Big).
\end{aligned}$$

Thus (2.14a) holds with π_μ replaced by $\pi_{\mu+1}$, which contradicts (2.14b). Hence (2.15b) holds. We have therefore shown that (i) implies (ii).

To prove the converse, we suppose that (ii) holds. Let $p \in \pi_\mu$, i.e. $p(x) = \sum_{\ell \in \mathbb{Z}_\mu} c_\ell x^\ell$. Then, using (2.15a), we obtain, for $j \in \mathbb{Z}$,

$$\begin{aligned}
\sum_k a_{2j+1-2k} p(k) &= \sum_k a_{2j+1-2k} \sum_{\ell \in \mathbb{Z}_\mu} c_\ell k^\ell \\
&= \sum_{\ell \in \mathbb{Z}_\mu} c_\ell \Big[\sum_k a_{2j+1-2k} k^\ell \Big]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell\in\mathbb{Z}_\mu} c_\ell\Big(j+\frac{1}{2}\Big)^\ell \\
&= p\Big(j+\frac{1}{2}\Big).
\end{aligned}
$$

Hence (2.14a) holds.

Since (2.15b) holds, it is clear that (2.14a) does not hold if π_μ is replaced by π_ℓ with $\ell \geq \mu+1$. Thus (ii) implies (i) for $j=0$, i.e. (2.14a) does not hold, and thereby completing the proof of the equivalence of (i) and (ii).

(b) Next, we show the equivalence of (ii) and (iii). Suppose first that (ii) holds. For $\ell = 0$, we see that (2.16a) and (2.15a) are equivalent. Next, let $\ell \in \mathbb{N}_\mu$. Then, since (ii) implies (i), we can use (2.14a) with $j=0$ to deduce that

$$
\begin{aligned}
\sum_k (2k+1)^\ell a_{2k+1} &= \sum_k a_{1-2k}(1-2k)^\ell \\
&= \left(1-2\left(\frac{1}{2}\right)\right)^\ell = 0.
\end{aligned}
$$

Hence (2.16a) is true.

To prove that (2.16b) also holds, we use (2.15a) with $j=0$, together with (2.15b), to obtain

$$
\begin{aligned}
\sum_k (2k+1)^{\mu+1} a_{2k+1} &= \sum_k \sum_{\ell\in\mathbb{Z}_{\mu+1}} \binom{\mu+1}{\ell} 2^\ell k^\ell a_{2k+1} \\
&= \sum_{\ell\in\mathbb{Z}_{\mu+1}} \binom{\mu+1}{\ell} 2^\ell (-1)^\ell \left[\sum_k a_{1-2k} k^\ell\right] \\
&= \sum_{\ell\in\mathbb{Z}_{\mu}} \binom{\mu+1}{\ell} (-2)^\ell \left(\frac{1}{2}\right)^\ell + 2^{\mu+1}\sum_k a_{1-2k} k^{\mu+1} \\
&= \sum_{\ell\in\mathbb{Z}_{\mu}} \binom{\mu+1}{\ell} (-1)^\ell + 2^{\mu+1}\sum_k a_{1-2k} k^{\mu+1} \\
&= \sum_{\ell\in\mathbb{Z}_{\mu+1}} \binom{\mu+1}{\ell} (-1)^\ell - 1 + 2^{\mu+1}\sum_k a_{1-2k} k^{\mu+1} \\
&= (1-1)^\ell - 1 + 2^{\mu+1}\sum_k a_{1-2k} k^{\mu+1} \\
&= 2^{\mu+1}\sum_k a_{1-2k} k^{\mu+1} - 1 \\
&\neq 2^{\mu+1}\left(\frac{1}{2}\right)^{\mu+1} - 1 = 0,
\end{aligned}
$$

i.e. (2.16b) holds. Thus (ii) implies (iii).

Conversely, suppose that (iii) holds. After recalling the equivalence of (2.16a) and (2.15a) for $\ell = 0$, we let $\ell \in \mathbb{N}_\mu$ and choose $j \in \mathbb{Z}$. Then, it follows from (2.16a) that

$$\begin{aligned}
\sum_k a_{2j+1-2k} k^\ell &= \sum_k a_{2k+1}(j-k)^\ell \\
&= \frac{1}{2^\ell}\sum_k a_{2k+1}\Big[(2j+1)-(2k+1)\Big]^\ell \\
&= \frac{1}{2^\ell}\sum_k a_{2k+1}\left[\sum_{r\in\mathbb{Z}_\ell}\binom{\ell}{r}(-1)^r(2k+1)^r(2j+1)^{\ell-r}\right] \\
&= \frac{1}{2^\ell}\sum_{r\in\mathbb{Z}_\ell}\binom{\ell}{r}(-1)^r(2j+1)^{\ell-r}\left[\sum_k(2k+1)^r a_{2k+1}\right] \qquad (2.20) \\
&= \frac{1}{2^\ell}\sum_{r\in\mathbb{Z}_\ell}\binom{\ell}{r}(-1)^r(2j+1)^{\ell-r}\delta_r \\
&= \frac{1}{2^\ell}(2j+1)^\ell \\
&= \left(\frac{2j+1}{2}\right)^\ell = \left(j+\frac{1}{2}\right)^\ell .
\end{aligned}$$

Hence (2.15a) holds.

To prove that (2.15b) also holds, we use the fact that (2.20) also holds for $\ell = \mu+1$ and $j = 0$, together with (2.16a), to deduce that

$$\begin{aligned}
\sum_k a_{1-2k} k^{\mu+1} &= \frac{1}{2^{\mu+1}}\sum_{r\in\mathbb{Z}_{\mu+1}}\binom{\mu+1}{r}(-1)^r\left[\sum_k(2k+1)^r a_{2k+1}\right] \\
&= \frac{1}{2^{\mu+1}}\sum_{r\in\mathbb{Z}_\mu}\binom{\mu+1}{r}(-1)^r\left[\sum_k(2k+1)^r a_{2k+1}\right] \\
&\qquad\qquad + \frac{1}{2^{\mu+1}}\left[\sum_k(2k+1)^{\mu+1} a_{2k+1}\right] \\
&= \frac{1}{2^{\mu+1}}\sum_{r\in\mathbb{Z}_\mu}\binom{\mu+1}{r}(-1)^r\delta_r + \frac{1}{2^{\mu+1}}\sum_k(2k+1)^{\mu+1}a_{2k+1} \\
&= \frac{1}{2^{\mu+1}} + \frac{1}{2^{\mu+1}}\sum_k(2k+1)^{\mu+1}a_{2k+1} \\
&= \left(\frac{1}{2}\right)^{\mu+1} + \frac{1}{2^{\mu+1}}\sum_k(2k+1)^{\mu+1}a_{2k+1} \\
&\neq \left(\frac{1}{2}\right)^{\mu+1},
\end{aligned}$$

from (2.16b). Hence (2.15b) holds. Thus (iii) implies (i), and thereby completing the proof of the equivalence between (ii) and (iii).

(c) Our next step is to prove the equivalence of (i) and (iv). Suppose therefore that (i) holds. We show first that (2.17a) is true. To this end, we note first that (2.14a) has the equivalent formulation

$$\sum_k a_{2k+1} p(j-k) = p\left(j+\frac{1}{2}\right), \quad j \in \mathbb{Z}, \quad p \in \pi_\mu. \tag{2.21}$$

It will therefore suffice to prove that (2.21) implies (2.17a). Since (1.4) and (2.4) yield

$$A(z) = 1 + \sum_k a_{2k+1} z^{2k+1}, \quad z \in \mathbb{C}\setminus\{0\}, \tag{2.22}$$

repeated differentiation of (2.22) give the formula

$$A^{(j)}(-1) = \delta_j + (-1)^{j+1} \sum_k q_j(2k+1) a_{2k+1}, \quad j \in \mathbb{Z}_+, \tag{2.23}$$

where, for $x \in \mathbb{R}$,

$$q_0(x) = 1, \qquad q_j(x) = \prod_{\ell \in \mathbb{Z}_{j-1}} (x-\ell), \quad j \in \mathbb{N}. \tag{2.24}$$

Observe that $q_j \in \pi_j$, $j \in \mathbb{Z}_+$. Hence, if we define

$$p_j = q_j(-2\cdot+1+2j), \quad j \in \mathbb{Z}_+, \tag{2.25}$$

then also $p_j \in \pi_j$, $j \in \mathbb{Z}_+$. Moreover, (2.23) and (2.25) give

$$A^{(j)}(-1) = \delta_j + (-1)^{j+1} \sum_k p_j(j-k) a_{2k+1}, \quad j \in \mathbb{Z}_+, \tag{2.26}$$

whereas (2.25) and (2.24) yield

$$p_j\left(j+\frac{1}{2}\right) = q_j(0) = \delta_j, \quad j \in \mathbb{Z}_+. \tag{2.27}$$

Since, moreover, $p_j \in \pi_j \subset \pi_\mu$, $j \in \mathbb{Z}_+$, we can now use (2.26), (2.21) and (2.27) to deduce that, for any integer $j \in \mathbb{Z}_+$,

$$\begin{aligned} A^{(j)}(-1) &= \delta_j + (-1)^{j+1} \sum_k p_j(j-k) a_{2k+1} \\ &= \delta_j + (-1)^{j+1} p_j\left(j+\frac{1}{2}\right) \\ &= \delta_j [1+(-1)^{j+1}] = 0. \end{aligned}$$

Hence (2.17a) holds.

Next, to prove (2.17b), we note first from (2.24) that $q_{\mu+1}(x) = x^{\mu+1} + \tilde{q}(x),\ \ x \in \mathbb{R}$, where $\tilde{q} \in \pi_\mu$, and so that also $\tilde{q}(0) = q_{\mu+1}(0) = 0$, from (2.27). Then, with q the polynomial $\hat{q}$ defined by $\hat{q}(x) = \tilde{q}(1-2x),\ \ x \in \mathbb{R}$, so that also $\hat{q} \in \pi_\mu$, we use (2.23) and (2.14a) with $j = 0$ to deduce that

$$
\begin{aligned}
A^{(\mu+1)}(-1) &= -\sum_k q_{\mu+1}(2k+1)a_{2k+1} \\
&= -\sum_k (2k+1)\Big[(2k+1)-1\Big]\ldots\Big[(2k+1)-\mu\Big]a_{2k+1} \\
&= -\sum_k (2k+1)^{\mu+1}a_{2k+1} + \sum_k \tilde{q}(2k+1)a_{2k+1} \\
&= -\sum_k (2k+1)^{\mu+1}a_{2k+1} + \sum_k a_{1-2k}\tilde{q}(1-2k) \\
&= -\sum_k (2k+1)^{\mu+1}a_{2k+1} + \sum_k a_{1-2k}\hat{q}(k) \\
&= -\sum_k (2k+1)^{\mu+1}a_{2k+1} + \hat{q}\left(\frac{1}{2}\right) \\
&= -\sum_k (2k+1)^{\mu+1}a_{2k+1} + \tilde{q}(0) \\
&= -\sum_k (2k+1)^{\mu+1}a_{2k+1}. \qquad (2.28)
\end{aligned}
$$

Since (i) implies (iii), and therefore (2.16b) holds, we deduce from (2.28) that (2.17b) holds, and thus (i) implies (iv).

To prove the converse, we suppose that (iv) holds. It then follows from (2.17a) and (2.23) that

$$\sum_k q_j(2k+1)a_{2k+1} = (-1)^j\delta_j, \quad j \in \mathbb{Z}_\mu. \qquad (2.29)$$

Suppose now $p \in \pi_\mu$, and fix $j \in \mathbb{Z}$. From (2.24), we see that $\{q_\ell : \ell \in \mathbb{Z}_\mu\}$ is a basis for π_μ. Hence, from (2.25), we deduce that $\{q_\ell(-2\cdot+2j+1) : \ell \in \mathbb{Z}_\mu\}$ is a basis for π_μ. Thus there exists a unique coefficient sequence $\{\alpha_{j,\ell} : \ell \in \mathbb{Z}_\mu\} \subset \mathbb{R}$ such that $p = \sum_{\ell\in\mathbb{Z}_\mu} \alpha_{j,\ell}q_\ell(-2\cdot+2j+1)$, and thus, using also (2.29), we get

$$
\begin{aligned}
\sum_k a_{2k+1}p(j-k) &= \sum_k a_{2k+1}\sum_{\ell\in\mathbb{Z}_\mu}\alpha_{j,\ell}q_\ell\Big(-2(j-k)+2j+1\Big) \\
&= \sum_{\ell\in\mathbb{Z}_\mu}\alpha_{j,\ell}\sum_k a_{2k+1}q_\ell(2k+1) \\
&= \sum_{\ell\in\mathbb{Z}_\mu}\alpha_{j,\ell}(-1)^\ell\delta_\ell = \alpha_{j,0}. \qquad (2.30)
\end{aligned}
$$

Also, from (2.27),

$$p\left(j+\frac{1}{2}\right) = \sum_{\ell\in\mathbb{Z}_\mu}\alpha_{j,\ell}q_\ell\left(-2\Big(j+\frac{1}{2}\Big)+2j+1\right)$$

$$= \sum_{\ell \in \mathbb{Z}_\mu} \alpha_{j,\ell} q_\ell(0) = \sum_{\ell \in \mathbb{Z}_\mu} \alpha_{j,\ell} \delta_\ell = \alpha_{j,0}. \tag{2.31}$$

It follows from (2.30) and (2.31) that (2.15a) indeed holds.

Next, we observe from (2.28) that (2.17b) implies (2.16b). Also, from the proofs of the equivalences between (i), (ii) and (iii), we see that (2.14a), (2.15a) and (2.16a) are equivalent statements. Hence (2.14a) implies (2.16a), and we deduce that (iii) holds. But then also (i) holds. Hence (iv) implies (i), and thus (i) and (iv) are equivalent.

(d) Our final step is to prove the equivalence of (iv) and (v), thereby completing our proof. The fact that (v) implies (iv) is an immediate consequence of (2.18a) and (2.18b). Suppose next that (iv) holds. Then (2.17a) and (2.18b) show the existence of a Laurent polynomial B for which (2.18a) and the condition $B(-1) \neq 0$ hold. It remains to prove that we must have $B(1) = 1$. But (2.4) gives

$$A(1) = \sum_k a_{2k} + \sum_k a_{2k+1} = 1 + \sum_k a_{2k+1} = 1 + 1 = 2,$$

from (2.16a) with $\ell = 0$, together with the fact that (iv) and (iii) are equivalent. It follows from (2.18a) that

$$2 = A(1) = 2B(1),$$

and thus $B(1) = 1$. ■

2.4 The class $\mathscr{A}_{m,n}$

In this section we define a class of symmetric interpolatory subdivision schemes with the π_μ polynomial filling property for an appropriately chosen integer μ.

We shall rely on the following result.

Proposition 2.8. *An interpolatory subdivision scheme is symmetric, and has the π_μ polynomial filling property for an integer $\mu \in \mathbb{Z}_+$ if and only if $\mu = 2m-1$ for an integer $m \in \mathbb{N}$, and the corresponding mask symbol A is given by*

$$A(z) = \frac{1}{2^{m-1}} \left(1 + \frac{z + z^{-1}}{2}\right)^m C(z), \quad z \in \mathbb{C} \setminus \{0\}, \tag{2.32}$$

with C denoting a symmetric Laurent polynomial such that

$$C(1) = 1, \quad C(-1) \neq 0. \tag{2.33}$$

Proof. Suppose the mask A of an interpolatory subdivision scheme is given by (1.4) in terms of the corresponding subdivision mask $a \in \mathscr{M}_0(\mathbb{Z})$. Then, according to the equivalence of statements (i) and (v) in Proposition 2.7, we know that there exists a Laurent polynomial B such that (2.18a) and (2.18b) are satisfied. Since A is also a symmetric Laurent polynomial, we can now use (2.18a) and (2.8) to deduce that, for $z \in \mathbb{C}\setminus\{0\}$,

$$\frac{1}{2^\mu}(1+z)^{\mu+1}B(z) \quad = \quad \frac{1}{2^\mu}(1+z^{-1})^{\mu+1}B(z^{-1}) = \frac{1}{2^\mu}\frac{(1+z)^{\mu+1}}{z^{\mu+1}}B(z^{-1}),$$

and thus

$$z^{\mu+1}B(z) = B(z^{-1}), \quad z \in \mathbb{C}\setminus\{0\},$$

in which we now set $z = -1$ to obtain

$$B(-1)[(-1)^{\mu+1} - 1] = 0. \tag{2.34}$$

But, according to (2.18b), we have $B(-1) \neq 0$, which together with (2.34), implies that

$$(-1)^{\mu+1} - 1 = 0,$$

which holds if and only if μ is an odd integer, i.e. $\mu = 2m-1$ for an integer $m \in \mathbb{N}$, in terms of which (2.18a) becomes

$$A(z) = \frac{1}{2^{2m-1}}(1+z)^{2m}B(z), \quad z \in \mathbb{C}\setminus\{0\},$$

which is equivalent to the desired form (2.32), where the Laurent polynomial $C(z)$ is defined by

$$C(z) = z^m B(z), \quad z \in \mathbb{C}\setminus\{0\}. \tag{2.35}$$

It follows from (2.32) that

$$A(z^{-1}) = \frac{1}{2^{m-1}}\left(1+\frac{z+z^{-1}}{2}\right)^m C(z^{-1}), \quad z \in \mathbb{C}\setminus\{0\}. \tag{2.36}$$

Together, (2.32), (2.36) and (2.8) yields

$$C(z) = C(z^{-1}), \quad z \in \mathbb{C}\setminus\{0\},$$

i.e. C is a symmetric Laurent polynomial. The fact that C satisfies the condition (2.33), is a direct consequence of (2.35) and (2.18b). We have therefore proved that if an interpolatory subdivision scheme is symmetric and has the π_μ polynomial filling property, then $\mu = 2m-1$ for an integer $m \in \mathbb{N}$, and there exists a symmetric Laurent polynomial C such that its corresponding mask symbol A satisfies (2.32) and (2.33) hold.

Conversely, for an interpolatory subdivision scheme with corresponding mask symbol A, suppose that there exists a symmetric Laurent polynomial C such that (2.32) and (2.33) are satisfied. Then (2.8) holds, i.e. the interpolatory subdivision scheme is also symmetric. Also, define the Laurent polynomial B by

$$B(z) = z^{-m}C(z), \quad z \in \mathbb{C}\setminus\{0\}, \tag{2.37}$$

it follows from (2.32) that A is given by (2.18a), also (2.18b) follows from (2.37) and (2.33). Using the fact that, in Proposition 2.7, (v) implies (i), we deduce that the π_{2m-1} polynomial filling property is indeed satisfied. ∎

Following [9, Definition 1], (see also [19, Definition 2.4] as well as [22]), and based on our results in Proposition 2.7, 2.4 and 2.8, we proceed to introduce the general class $\mathscr{A}_{m,n}$ of subdivision mask symbols as follows:

Definition 2.9. For $m, n \in \mathbb{N}$, with $n \geq \lceil \frac{m+1}{2} \rceil$, we define the class $\mathscr{A}_{m,n}$ of Laurent polynomials as follows:

$$A \in \mathscr{A}_{m,n} \tag{2.38}$$

if and only if the following conditions are satisfied:

$$A(z) = \sum_j a_j z^j = \sum_{j=-2n+1}^{2n-1} a_j z^j, \quad z \in \mathbb{C}\setminus\{0\}, \quad \text{with} \quad a_{-2n+1} \neq 0, \quad a_{2n-1} \neq 0, \tag{2.39}$$

$$A(z) + A(-z) = 2, \quad z \in \mathbb{C}\setminus\{0\}, \tag{2.40}$$

$$A(z^{-1}) = A(z), \quad z \in \mathbb{C}\setminus\{0\}, \tag{2.41}$$

there exists a symmetric Laurent polynomial C such that

$$A(z) = \frac{1}{2^{m-1}}\left(1 + \frac{z+z^{-1}}{2}\right)^m C(z), \quad z \in \mathbb{C}\setminus\{0\}, \tag{2.42}$$

where

$$C(1) = 1, \quad C(-1) \neq 0. \tag{2.43}$$

Note in particular that the symmetric interpolatory subdivision scheme corresponding to a mask symbol $A \in \mathscr{A}_{m,n}$ has the π_{2m-1} polynomial filling property as defined by setting $\mu = 2m-1$ in Definition 2.5.

It should be pointed out that our Definition 2.9 is slightly more restrictive than Definition 1 in [9], in the sense that we include the condition $C(-1) \neq 0$ in (2.43), which was not implied by Definition 1 in [9].

Chapter 3

An explicit characterization of the class $\mathscr{A}_{m,n}$

The result of Proposition 2.7 enables us to explicitly characterize interpolatory masks in $\mathscr{A}_{m,n}$. Our main result of this chapter is as follows:

Theorem 3.1. For $m, n \in \mathbb{N}$, the class $\mathscr{A}_{m,n}$ is non-empty if and only if $n \geq m$. Moreover, a Laurent polynomial A belongs to the class $\mathscr{A}_{m,n}$ if and only if

$$A(z) = \frac{1}{2^{m-1}}(1+\zeta)^m \left[\sum_{j \in \mathbb{Z}_{m-1}} \binom{m+j-1}{j} \left[\frac{1}{2}(1-\zeta)\right]^j + \zeta(1-\zeta)^m P(\zeta^2) \right], \quad z \in \mathbb{C}\setminus\{0\}, \tag{3.1}$$

where

$$\zeta = \frac{z+z^{-1}}{2}, \quad z \in \mathbb{C}\setminus\{0\}, \tag{3.2}$$

and where either $P = 0$, in which case $n = m$, or P denotes an arbitrary polynomial, with $\deg(P) = n - m - 1$ if $n \geq m+1$, satisfying the condition

$$P(1) \neq \frac{1}{2^m}\binom{2m-1}{m-1}. \tag{3.3}$$

In addition,

$$\mathscr{A}_{m,m} = \{A_m\}, \tag{3.4}$$

where the Laurent polynomial A_m is obtained by choosing $P = 0$ in (3.1), i.e.

$$A_m(z) = \frac{1}{2^{m-1}}(1+\zeta)^m \sum_{j \in \mathbb{Z}_{m-1}} \binom{m+j-1}{j} \left[\frac{1}{2}(1-\zeta)\right]^j, \quad z \in \mathbb{C}\setminus\{0\}, \tag{3.5}$$

and with ζ given by (3.2).

In order to prove Theorem 3.1, we shall rely on the following sequence of propositions. For Proposition 3.2, we rely on techniques from [17, Lemma 2.3 and 2.4], (see also [18, Theorem 4.3]), whereas the results of Proposition 3.3 to 3.5 was first introduced in [7], (see also [11, Proposition 11.4]).

3.1 The fundamental Bezout identity

Proposition 3.2. *For $m,n \in \mathbb{N}$, with $m \leq 2n-1$, a Laurent polynomial $A \in \mathscr{A}_{m,n}$ if and only if the symmetric Laurent polynomial C in Proposition 2.8 is given by*

$$C(z) = p\left(\frac{1}{2}\left[1 - \frac{z+z^{-1}}{2}\right]\right), \quad z \in \mathbb{C}\setminus\{0\}, \tag{3.6}$$

where p is a polynomial of degree $(2n-1-m)$ satisfying the Bezout identity

$$(1-z)^m p(z) + z^m p(1-z) = 1, \quad z \in \mathbb{C}, \tag{3.7}$$

with

$$p(0) = 1, \quad p(1) \neq 0. \tag{3.8}$$

Proof. We see from (2.40) in Definition 2.9 that a Laurent polynomial A belongs to the class $\mathscr{A}_{m,n}$ if and only if the symmetric Laurent polynomial C in (2.42) satisfies the Bezout identity

$$\left(1 + \frac{z+z^{-1}}{2}\right)^m C(z) + \left(1 - \frac{z+z^{-1}}{2}\right)^m C(-z) = 2^m, \quad z \in \mathbb{C}\setminus\{0\}, \tag{3.9}$$

as obtained by substituting (2.42) into (2.40).

Observe that (3.9) holds if and only if it holds on the unit circle $\{z \in \mathbb{C} : |z| = 1\} = \{z \in \mathbb{C} : z = e^{ix}, \quad x \in \mathbb{R}\}$, i.e. (3.9) holds if and only if, for $x \in \mathbb{R}$, we have

$$\left(1 + \frac{e^{ix}+e^{-ix}}{2}\right)^m C(e^{ix}) + \left(1 - \frac{e^{ix}+e^{-ix}}{2}\right)^m C(-e^{ix}) = 2^m, \tag{3.10}$$

which is the same as

$$(1+\cos x)^m C(e^{ix}) + (1-\cos x)^m C(e^{i(x+\pi)}) = 2^m, \quad x \in \mathbb{R}, \tag{3.11}$$

or, equivalently, since $\cos x = 2\cos^2\frac{x}{2} - 1 = 1 - 2\sin^2\frac{x}{2}$, $x \in \mathbb{R}$,

$$\left(1 - \sin^2\frac{x}{2}\right)^m C(e^{ix}) + \left(\sin^2\frac{x}{2}\right)^m C(e^{i(x+\pi)}) = 1, \quad x \in \mathbb{R}. \tag{3.12}$$

Denote by $N \in \mathbb{Z}_+$ the non-negative integer, and by $\{c_j : j \in \mathbb{Z}\}$ the sequence in $\mathscr{M}_0(\mathbb{Z})$ for which it holds that

$$C(z) = \sum_j c_j z^j = \sum_{j=-N}^{N} c_j z^j, \quad z \in \mathbb{C}\setminus\{0\}, \tag{3.13}$$

where, according to (2.39) and (2.42), we have $N = 2n - 1 - m \geq 0$.

Since C is a symmetric Laurent polynomial, we have $c_{-j} = c_j, \;\; j \in \mathbb{Z}$, and thus

$$\begin{aligned} C(e^{ix}) &= \sum_{j=-N}^{N} c_j e^{ijx} \\ &= \sum_{j=-N}^{-1} c_j e^{ijx} + c_0 + \sum_{j=1}^{N} c_j e^{ijx} \\ &= \sum_{j=1}^{N} c_{-j} e^{-ijx} + c_0 + \sum_{j=1}^{N} c_j e^{ijx} \\ &= 2\sum_{j=1}^{N} c_j \left[\frac{e^{ijx} + e^{-ijx}}{2}\right] + c_0 \\ &= 2\sum_{j=1}^{N} c_j \cos(jx) + c_0, \quad x \in \mathbb{R}. \end{aligned} \tag{3.14}$$

From de Moivre's Theorem, we have for $j \in \mathbb{N}, \;\; x \in \mathbb{R}$, that

$$\begin{aligned} \cos(jx) &= Re[\cos(jx) + i\sin(jx)] \\ &= Re\left[(\cos x + i\sin x)^j\right] \\ &= Re\left[\sum_k \binom{j}{k} (\cos x)^{j-k} (i)^k (\sin x)^k\right] \\ &= Re\left[\sum_k \binom{j}{2k} (\cos x)^{j-2k} (-1)^k (\sin^2 x)^k \right. \\ &\quad \left. + i\sum_k \binom{j}{2k+1} (\cos x)^{j-2k-1} (-1)^k (\sin^2 x)^{2k+1}\right] \\ &= \sum_k \binom{j}{2k} (\cos x)^{j-2k} (\cos^2 x - 1)^k \\ &= \sum_k \binom{j}{2k} (\cos x)^{j-2k} \sum_\ell \binom{k}{\ell} (\cos x)^{2\ell} (-1)^{k-\ell} \\ &= \sum_k \binom{j}{2k} \sum_\ell (-1)^{k-\ell} \binom{k}{\ell} (\cos x)^{j-2(k-\ell)} \\ &= \sum_k \binom{j}{2k} \sum_\ell (-1)^{\ell} \binom{k}{k-\ell} (\cos x)^{j-2\ell} \end{aligned}$$

$$
\begin{aligned}
&= \sum_k \binom{j}{2k} \sum_\ell (-1)^\ell \binom{k}{\ell} (\cos x)^{j-2\ell} \\
&= \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j}{2k} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} (\cos x)^{j-2\ell} \\
&= \sum_{\ell=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^\ell \Big[\sum_{k=\ell}^{\lfloor \frac{j}{2} \rfloor} \binom{j}{2k} \binom{k}{\ell} \Big] (\cos x)^{j-2\ell},
\end{aligned}
$$

from which we see that there exists a sequence $\{\beta_{j,k},\ k = 0,1,\ldots,j;\ j \in \mathbb{N}\} \subset \mathbb{R}$, with $\beta_{j,j} = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j}{2k} = 2^{j-1},\ \ j \in \mathbb{N}$, such that

$$
\cos(jx) = \sum_{k=0}^{j} \beta_{j,k} (\cos x)^k, \quad x \in \mathbb{R}, \quad j \in \mathbb{Z}. \tag{3.15}
$$

Combining (3.14) and (3.15), we obtain, for $x \in \mathbb{R}$,

$$
\begin{aligned}
C(e^{ix}) &= c_0 + 2 \sum_{j=1}^{N} c_j \sum_{k=0}^{j} \beta_{j,k} (\cos x)^k \\
&= c_0 + 2 \sum_{j=1}^{N} c_j \sum_{k=0}^{j} \beta_{j,k} \left(1 - 2\sin^2 \frac{x}{2}\right)^k. \qquad (3.16)
\end{aligned}
$$

It follows from (3.16) that if we define the polynomial p by

$$
p(z) = c_0 + 2 \sum_{j=1}^{N} c_j \sum_{k=0}^{j} \beta_{j,k} (1-2z)^k, \quad z \in \mathbb{C}, \tag{3.17}
$$

then

$$
p \in \pi_N = \pi_{2n-1-m}, \tag{3.18}
$$

where, since $\beta_{j,j} = 2^{j-1} \neq 0,\ \ j \in \mathbb{N}$, we have $\deg(p) = 2n-1-m$, with also

$$
C(e^{ix}) = p\left(\sin^2 \frac{x}{2}\right), \quad x \in \mathbb{R}. \tag{3.19}
$$

Moreover, (3.19) gives

$$
C(e^{ix}) = p\left(\frac{1-\cos x}{2}\right) = p\left(\frac{1}{2}\left[1 - \frac{e^{ix}+e^{-ix}}{2}\right]\right),
$$

and thus

$$
C(z) = p\left(\frac{1}{2}\left[1 - \frac{z+z^{-1}}{2}\right]\right), \quad z \in \mathbb{C}, \quad |z| = 1, \tag{3.20}
$$

which holds if and only if C and p are related by (3.6).

Moreover, (3.19) gives

$$C(e^{i(x+\pi)}) = p\left(\sin^2\left(\frac{x+\pi}{2}\right)\right) = p\left(\cos^2\frac{x}{2}\right) = p\left(1-\sin^2\frac{x}{2}\right), \quad x \in \mathbb{R}. \tag{3.21}$$

It follows from (3.19) and (3.21) that the Bezout identity (3.10) is satisfied by a symmetric Laurent polynomial C if and only if the Bezout identity

$$\left(1-\sin^2\frac{x}{2}\right)^m p\left(\sin^2\frac{x}{2}\right) + \left(\sin^2\frac{x}{2}\right)^m p\left(1-\sin^2\frac{x}{2}\right) = 1, \quad x \in \mathbb{R}, \tag{3.22}$$

holds, and where C and p are related by (3.16), (3.17) and (3.6). Now observe that the two Bezout identities (3.22) and (3.7) are equivalent.

Finally, we note that, if C and p are related by (3.6), then $C(1) = p(0)$, and $C(-1) = p(1)$, from which we deduce that the conditions (2.43) and (3.8) are equivalent. ■

3.2 A polynomial solution of least possible degree

Our next step is to explicitly obtain the polynomial p of least possible degree that solves the Bezout identity (3.7).

The following result is of fundamental importance in this regard.

Proposition 3.3. *For $m \in \mathbb{N}$, there exists a unique polynomial $p = p_m \in \pi_{m-1}$ that satisfies the Bezout identity* (3.7).

Proof. Since the two polynomials f and g defined by $f(z) = (1-z)^m$, $z \in \mathbb{C}$, and $g(z) = z^m$, $z \in \mathbb{C}$, have no common factors, a standard result in polynomial algebra states that there exist two polynomials u and v such that

$$(1-z)^m u(z) + z^m v(z) = 1, \quad z \in \mathbb{C}. \tag{3.23}$$

According to the polynomial division theorem, there exist (unique) polynomials q and r, with $r \in \pi_{m-1}$, such that

$$v(z) = q(z)(1-z)^m + r(z), \quad z \in \mathbb{C}. \tag{3.24}$$

Inserting (3.24) into (3.23) then yields

$$(1-z)^m \rho(z) + z^m r(z) = 1, \quad z \in \mathbb{C}, \tag{3.25}$$

where the polynomial ρ is given by

$$\rho(z) = u(z) + z^m q(z), \quad z \in \mathbb{C}. \tag{3.26}$$

Since (3.25) gives

$$(1-z)^m\rho(z) = 1 - z^m r(z), \quad z \in \mathbb{C},$$

and thus

$$m + \deg(\rho) = m + \deg(r) \le m + (m-1) = 2m - 1,$$

so that $\deg(\rho) \le m-1$, we find that $\rho \in \pi_{m-1}$.

We claim that ρ and r are the unique solutions in π_{m-1} of (3.25). To prove this, suppose that $\tilde{\rho}$ and $\tilde{r}$ are both in π_{m-1}, and are such that

$$(1-z)^m\tilde{\rho}(z) + z^m\tilde{r}(z) = 1, \quad z \in \mathbb{C}. \tag{3.27}$$

By subtracting (3.27) from (3.25), we get

$$(1-z)^m[\rho(z) - \tilde{\rho}(z)] = z^m[\tilde{r}(z) - r(z)], \quad z \in \mathbb{C}. \tag{3.28}$$

If $\rho - \tilde{\rho} \neq 0$, it follows from (3.28) that the polynomial $\rho - \tilde{\rho}$ must contain the factor z^m, which is impossible, since $\rho - \tilde{\rho} \in \pi_{m-1}$. Hence $\rho = \tilde{\rho}$, and therefore also $r = \tilde{r}$, thereby establishing our uniqueness claim above.

Since (3.25) holds for all $z \in \mathbb{C}$, it also holds with z replaced by $1-z$, i.e.

$$(1-z)^m r(1-z) + z^m\rho(1-z) = 1, \quad z \in \mathbb{C}. \tag{3.29}$$

But the polynomials $r(1-z)$ and $\rho(1-z)$ also belong to π_{m-1}, so that, from the uniqueness result above, and by comparing (3.25) and (3.29), we deduce that $\rho(z) = r(1-z)$, $z \in \mathbb{C}$, and $r(z) = \rho(1-z)$, $z \in \mathbb{C}$. It follows that the polynomial $p = p_m \in \pi_{m-1}$ defined by $p_m(z) = r(1-z)$, $z \in \mathbb{C}$, is the unique polynomial in π_{m-1} which satisfies the Bezout identity (3.7). ■

We can now solve for the polynomial $p = p_m$ explicitly from the Bezout identity (3.7). We shall rely on the following power series result.

Proposition 3.4. *For $m \in \mathbb{N}$, we have*

$$\frac{1}{(1-z)^m} = \sum_{j=0}^{\infty} \binom{m+j-1}{j} z^j, \quad |z| < 1. \tag{3.30}$$

Proof. Differentiating the convergent geometric power series

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j, \quad |z| < 1,$$

$m-1$ times, yields the convergent power series

$$\frac{(m-1)!}{(1-z)^m} = \sum_{j=0}^{\infty}(j+(m-1))((j+(m-2))\ldots(j+1)z^j, \quad |z|<1,$$

which then immediately gives the desired result (3.30). ∎

Using Proposition 3.4, we can now prove the following explicit formulation of the polynomial p_m in Proposition 3.3.

Proposition 3.5. *In Proposition 3.3, the unique polynomial $p = p_m \in \pi_{m-1}$ which solves the Bezout identity* (3.7) *is given by*

$$p_m(z) = \sum_{j\in\mathbb{Z}_{m-1}} \binom{m+j-1}{j} z^j, \quad z\in\mathbb{C}. \tag{3.31}$$

Here $\deg(p_m) = m-1$*, and p_m is the polynomial of least possible degree satisfying* (3.7).

Proof. Observe from (3.7) that the polynomial $p = p_m$ satisfies the identity

$$p_m(z) = \frac{1}{(1-z)^m}[1-z^m p_m(1-z)], \quad |z|<1. \tag{3.32}$$

Hence, from (3.32) and (3.30), we have

$$p_m(z) = \sum_{j=0}^{\infty}\binom{m+j-1}{j} z^j [1-z^m p_m(1-z)], \quad |z|<1.$$

Hence there exists a sequence $\{\alpha_j : j = m, m+1,\ldots\} \subset \mathbb{R}$ such that

$$p_m(z) = \sum_{j=0}^{m-1}\binom{m+j-1}{j} z^j + \sum_{j=m}^{\infty}\alpha_j z^j, \quad |z|<1. \tag{3.33}$$

Since $p_m \in \pi_{m-1}$, we know that p has a unique Taylor series expansion of the form

$$p_m = \sum_{j=0}^{\infty} c_j z^j, \quad z\in\mathbb{C}, \tag{3.34}$$

with $c_j = 0,\ \ j \geq m$. It follows from (3.33) and (3.34) that $\alpha_j = 0,\ \ j\geq m$, which, together with (3.33), then yields the formula (3.31). Hence $\deg(p_m) = m-1$, and the uniqueness of p_m as a solution of (3.7) implies that p_m is indeed the polynomial of least possible degree such that the Bezout identity (3.7) is satisfied. ∎

3.3 The general polynomial solution

Using (3.31), we can now explicitly find the general polynomial solution of the Bezout identity (3.7). Our result is as follows:

Proposition 3.6. *For $m \in \mathbb{N}$, the general polynomial solution p of the Bezout identity* (3.7) *is given by*

$$p(z) = p_m(z) + z^m \tilde{p}(z), \quad z \in \mathbb{C}, \tag{3.35}$$

with p_m as in Proposition 3.5, and where $\tilde{p}$ is any polynomial such that the symmetry property

$$\tilde{p}\Big(\frac{1}{2} - z\Big) = -\tilde{p}\Big(\frac{1}{2} + z\Big), \quad z \in \mathbb{C}, \tag{3.36}$$

is satisfied. Moreover, in (3.35)*, we have $p(0) = 1$ for every polynomial $\tilde{p}$, whereas $p(1) \neq 0$ for every polynomial $\tilde{p}$ such that*

$$\tilde{p}(0) \neq \binom{2m-1}{m-1}. \tag{3.37}$$

Proof. For a general polynomial solution p of (3.7), and from the fact, from Proposition 3.5, that $p = p_m$ is a specific polynomial solution of (3.7), we deduce that

$$(1-z)^m [p(z) - p_m(z)] = -z^m [p(1-z) - p_m(1-z)], \quad z \in \mathbb{C}. \tag{3.38}$$

Since the two polynomials $(1-z)^m$ and z^m have no common factors, we deduce from (3.38) that there is a polynomial $\tilde{p}$ such that

$$p(z) - p_m(z) = z^m \tilde{p}(z), \quad z \in \mathbb{C}. \tag{3.39}$$

Substituting (3.39) into (3.38) yields

$$(1-z)^m z^m \tilde{p}(z) = -z^m (1-z)^m \tilde{p}(1-z), \quad z \in \mathbb{C},$$

and thus

$$\tilde{p}(z) = -\tilde{p}(1-z), \quad z \in \mathbb{C}, \tag{3.40}$$

i.e.

$$\tilde{p}\Big(\frac{1}{2} - z\Big) = -\tilde{p}\Big(\frac{1}{2} + z\Big), \quad z \in \mathbb{C}. \tag{3.41}$$

It follows from (3.39) and (3.41) that the general polynomial solution p of (3.7) is given by (3.35), with $\tilde{p}$ denoting any polynomial satisfying the symmetry condition (3.36).

Finally, let p be a general polynomial solution of (3.7). Then (3.35) and (3.31) yield $p(0) = p_m(0) = 1$, whereas

$$p(1) = \sum_{j\in\mathbb{Z}_{m-1}} \binom{m+j-1}{j} + \tilde{p}(1) = \sum_{j\in\mathbb{Z}_{m-1}} \binom{m+j-1}{j} - \tilde{p}(0), \tag{3.42}$$

from (3.40).

We claim that

$$\sum_{j\in\mathbb{Z}_{m-1}} \binom{m+j-1}{j} = \binom{2m-1}{m-1}, \tag{3.43}$$

which, together with (3.42), then yields the desired result that $p(1) \neq 0$ for every polynomial $\tilde{p}$ such that (3.37) holds. Our proof will therefore be complete if we can show that (3.43) holds. To this end, note that

$$\begin{aligned}
\sum_{j\in\mathbb{Z}_{m-1}} \binom{m+j-1}{j} &= \sum_{j\in\mathbb{Z}_{m-1}} \left[\binom{m+j}{j} - \binom{m+j-1}{j-1}\right] \\
&= \left[\binom{2m-1}{m-1} - \binom{2m-2}{m-2}\right] + \left[\binom{2m-2}{m-2} - \binom{2m-3}{m-3}\right] \\
&\quad + \cdots + \left[\binom{m+1}{1} - \binom{m}{0}\right] + \left[\binom{m}{0} - 0\right] \\
&= \binom{2m-1}{m-1},
\end{aligned}$$

and thereby concluding the proof of (3.43). ■

We are now in the position to prove our main result Theorem 3.1.

3.4 Proof of Theorem 3.1

Proof. We can now combine the results of Propositions 2.8, 3.2 and 3.5 to deduce that the class $\mathscr{A}_{m,n}$ is non-empty if and only if $n \geq m$, and that $A \in \mathscr{A}_{m,n}$ if and only if, with the variable ζ defined in (3.2), we have

$$A(z) = \frac{1}{2^{m-1}}(1+\zeta)^m \left[\sum_{j\in\mathbb{Z}_{m-1}} \binom{m+j-1}{j} \left[\frac{1}{2}(1-\zeta)\right]^j + \frac{1}{2^m}(1-\zeta)^m \tilde{p}\left(\frac{1}{2}(1-\zeta)\right)\right],$$
$$z \in \mathbb{C}\setminus\{0\}, \tag{3.44}$$

with $\tilde{p}$ denoting any polynomial with either $\tilde{p} = 0$, or $\deg(\tilde{p}) = 2(n-m-1)$ if $n \geq m+1$, and satisfying the symmetry condition (3.41), as well as the condition (3.37).

For any given polynomial $\tilde{p}$ as in (3.44) above, we now define the polynomial $\tilde{P}$ by

$$\tilde{P}(\zeta) = \frac{1}{2^m}\tilde{p}\left(\frac{1}{2}(1-\zeta)\right), \quad \zeta \in \mathbb{C}. \tag{3.45}$$

But then $\tilde{P} = 0$, or $\deg(\tilde{P}) = 2(n-m)-1$, whereas (3.37) and (3.45) yield

$$\tilde{P}(1) = \frac{1}{2^m}\tilde{p}(0) \neq \frac{1}{2^m}\binom{2m-1}{m-1},$$

i.e. the condition (3.3) is satisfied.

Moreover, (3.36) and (3.45) show that, for $\zeta \in \mathbb{C}$, we have

$$\tilde{P}(-\zeta) = \frac{1}{2^m}\tilde{p}\left(\frac{1}{2}+\frac{1}{2}\zeta\right) = -\frac{1}{2^m}\tilde{p}\left(\frac{1}{2}-\frac{1}{2}\zeta\right) = -\tilde{P}(\zeta),$$

i.e. $\tilde{P}$ is an odd polynomial if $\tilde{P} \neq 0$.

Our proof of the explicit characterization result of Theorem 3.1 is now complete by defining the polynomial P by means of $\tilde{P}(\zeta) = \zeta P(\zeta^2), \quad \zeta \in \mathbb{C}$.

The result in (3.4) and (3.5) is an immediate consequence of (3.1). ■

Observe that the formula (3.1) in Theorem 3.1 has the following alternative representation.

Corollary 3.7. *In Theorem 3.1, the representation formula* (3.1) *can alternatively be expressed as*

$$A(z) = A_m(z) + \zeta\left(1-\zeta^2\right)^m Q(\zeta^2), \quad z \in \mathbb{C}\backslash\{0\}, \tag{3.46}$$

where the symmetric Laurent polynomial A_m is defined by (3.5)*, with ζ defined by* (3.2)*, and where either $Q = 0$, or Q denotes an arbitrary polynomial, with* $\deg(Q) = n-m-1$ *if $n \geq m+1$, satisfying the condition*

$$Q(1) \neq \frac{1}{2^{2m-1}}\binom{2m-1}{m-1}. \tag{3.47}$$

Proof. Define the polynomial $Q = \frac{1}{2^{m-1}}P$, with P as in Theorem 3.1, then (3.46) follows immediately from (3.1). ■

Since $A_m \in \mathscr{A}_{m,m}$, we know from Definition 2.9 that there exists a sequence $\{a_{m,j},\ j \in \mathbb{Z}\} \in \mathscr{M}_0(\mathbb{Z})$, with

$$a_{m,j} = 0, \quad j \notin \{-2m+1,\ldots,2m-1\}, \quad a_{m,-2m+1} \neq 0, \qquad a_{m,2m-1} \neq 0, \tag{3.48}$$

$$a_{m,2j} = \delta_j, \quad j \in \mathbb{Z}, \tag{3.49}$$

and

$$a_{m,-j} = a_{m,j}, \quad j \in \mathbb{Z}, \tag{3.50}$$

such that

$$A_m(z) = \sum_{j=-2m+1}^{2m-1} a_{m,j} z^j, \quad z \in \mathbb{C}\setminus\{0\}. \tag{3.51}$$

Inserting (3.49) into (3.51) then yields the formula

$$A_m(z) = 1 + \sum_{j=-m}^{m-1} a_{m,2j+1} z^{2j+1}, \quad z \in \mathbb{C}\setminus\{0\}. \tag{3.52}$$

We proceed in the next section to develop a theory which will yield an explicit closed formula for the mask coefficients $\{a_{m,2j+1} : j = -m, \dots, m-1\}$ in (3.52).

3.5 Dubuc–Deslauriers subdivision

Following [8, Section 2.1], we consider for $m \in \mathbb{N}$ the problem of finding a minimally supported mask $a \in \mathscr{M}_0(\mathbb{Z})$ such that the polynomial filling property

$$\sum_k a_{j-2k} p(k) = p\left(\frac{j}{2}\right), \quad j \in \mathbb{Z}, \quad p \in \pi_{2m-1}, \tag{3.53}$$

as obtained by setting $\mu = 2m-1$ in (2.12), holds. To achieve this goal, we begin by introducing the Lagrange fundamental polynomials $L_{m,k} \in \Pi_{2m-1}$, $k \in \mathbb{J}_m := \{-m+1, \dots, m\}$, as defined by

$$L_{m,k} = \prod_{k \neq j \in \mathbb{J}_m} \frac{\cdot - j}{k-j}, \quad k \in \mathbb{J}_m, \tag{3.54}$$

for which it holds

$$L_{m,k}(j) = \delta_{k,j} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \quad k, j \in \mathbb{J}_m, \tag{3.55}$$

and

$$\sum_{k \in \mathbb{J}_m} p(k) L_{m,k} = p, \quad p \in \pi_{2m-1}. \tag{3.56}$$

Since $\{L_{m,k} : k = -m+1, \dots, m\}$ is a basis for the polynomial space π_{2m-1}, we see that the condition (3.53) has the equivalent formulation

$$\sum_k a_{j-2k} L_{m,\ell}(k) = L_{m,\ell}\left(\frac{j}{2}\right), \quad j \in \mathbb{Z}, \quad \ell \in \mathbb{J}_m. \tag{3.57}$$

A necessary condition for (3.57) to hold is obtained by setting $j = 0$ and $j = 1$ in (3.57), giving

$$\sum_k a_{-2k} L_{m,\ell}(k) = L_{m,\ell}(0), \quad \ell \in \mathbb{J}_m, \tag{3.58a}$$

$$\sum_k a_{1-2k} L_{m,\ell}(k) = L_{m,\ell}\left(\frac{1}{2}\right), \quad \ell \in \mathbb{J}_m. \tag{3.58b}$$

Using (3.55) in (3.58) yields

$$a_{-2\ell} + \sum_{k \notin \{-m+1,\ldots,m\}} a_{-2\ell} L_{m,\ell}(k) = L_{m,\ell}(0), \quad \ell \in \mathbb{J}_m, \tag{3.59a}$$

$$a_{1-2\ell} + \sum_{k \notin \{-m+1,\ldots,m\}} a_{1-2\ell} L_{m,\ell}(k) = L_{m,\ell}\left(\frac{1}{2}\right), \quad \ell \in \mathbb{J}_m, \tag{3.59b}$$

or, equivalently,

$$a_{2j} + \sum_{k \notin \{-m,\ldots,m-1\}} a_{2k} L_{m,-j}(-k) = \delta_j, \quad j = -m,\ldots,m-1, \tag{3.60a}$$

$$a_{2j+1} + \sum_{k \notin \{-m,\ldots,m-1\}} a_{2k+1} L_{m,-j}(-k) = L_{m,-j}\left(\frac{1}{2}\right), j = -m,\ldots,m-1. \tag{3.60b}$$

A minimally supported sequence $a = d_m = \{d_{m,j} : j \in \mathbb{Z}\} \in \mathscr{M}_0(\mathbb{Z})$ satisfying (3.60) is given by

$$\left.\begin{aligned} &d_{m,2j} = \delta_j, \quad j = -m,\ldots,m-1, \\ &d_{m,2j+1} = L_{m,-j}\left(\tfrac{1}{2}\right), \quad j = -m,\ldots,m-1, \\ &d_{m,j} = 0, \quad j \notin \{-2m+1,\ldots,2m-1\}. \end{aligned}\right\} \tag{3.61}$$

Observe in particular from the first line of (3.61) that d_m is an interpolatory mask in the sense that the condition (2.4) is satisfied by the sequence $a = d_m$.

Our next result proves that the choice (3.61) does indeed satisfy the condition (3.53).

Proposition 3.8. *The mask $a = d_m = \{d_{m,j} : j \in \mathbb{Z}\} \in \mathscr{M}_0(\mathbb{Z})$ defined by* (3.61) *is a minimally supported mask satisfying the polynomial filling condition* (3.53).

Proof. Let $j \in \mathbb{Z}$ and $p \in \pi_{2m-1}$, and note that if the polynomial q is defined by $q(x) = p(j + x)$, $x \in \mathbb{R}$, then q also belongs to π_{2m-1}. Now use the second line of (3.61), and (3.56), to obtain

$$\sum_k d_{m,2j+1-2k}\, p(k) \quad = \quad \sum_k d_{m,2k+1}\, p(j-k)$$

$$
\begin{aligned}
&= \sum_k d_{m,-2k+1} p(j+k) \\
&= \sum_{k\in\mathbb{J}_m} p(j+k) d_{m,-2k+1} \\
&= \sum_{k\in\mathbb{J}_m} p(j+k) L_{m,k}\left(\frac{1}{2}\right) \\
&= \sum_{k\in\mathbb{J}_m} q(k) L_{m,k}\left(\frac{1}{2}\right) \\
&= q\left(\frac{1}{2}\right) = p\left(j+\frac{1}{2}\right) = p\left(\frac{2j+1}{2}\right),
\end{aligned}
$$

thereby proving that (3.53) holds if j is odd. Next, we use the first line of (3.61) to deduce that

$$\sum_k d_{m,2j-2k} p(k) = \sum_k d_{m,2k} p(j-k) = \sum_k \delta_k p(j-k) = p(j) = p\left(\frac{2j}{2}\right),$$

i.e. (3.53) also holds if j is even. Hence (3.53) is satisfied.

The minimal support property of the mask $d_m = \{d_{m,j} : j \in \mathbb{Z}\}$ given by (3.61) follows from the fact that d_m is a minimally supported mask for which (3.53) holds for $j \in \{0,1\}$. ■

The subdivision scheme based on the mask $d_m = \{d_{m,j} : j \in \mathbb{Z}\}$ was first introduced by Dubuc and Deslauriers in [12], [13], and shall henceforth be referred to as Dubuc–Deslauriers (DD) subdivision.

We proceed to derive an explicit expression for the DD mask $d_m = \{d_{m,j} : j \in \mathbb{Z}\} \in \mathscr{M}_0(\mathbb{Z})$, as given by (3.61). Setting $x = \frac{1}{2}$ in (3.54), we have

$$L_{m,j}\left(\frac{1}{2}\right) = \prod_{j\neq k\in\mathbb{J}_m} \frac{\left(\frac{1}{2}-k\right)}{(j-k)} = \frac{\prod_{j\neq k\in\mathbb{J}_m}\left(\frac{1}{2}-k\right)}{\prod_{j\neq k\in\mathbb{J}_m}(j-k)}, \quad k \in \mathbb{J}_m.$$

But

$$
\begin{aligned}
\prod_{j\neq k\in\mathbb{J}_m}\left(\frac{1}{2}-k\right) &= \prod_{j\neq k\in\mathbb{J}_m}\left(\frac{1-2k}{2}\right) \\
&= \frac{2}{1-2j}\prod_{k\in\mathbb{J}_m}\left(\frac{1-2k}{2}\right) \\
&= \frac{1}{2^{2m-1}}\frac{1}{2j-1}\prod_{k=-m}^{m-1}(2k+1).
\end{aligned}
$$

Here

$$\prod_{k=-m}^{m-1}(2k+1) = (-2m+1)(-2m+3)(-2m+5)\ldots(-1)(1)(3)(5)\cdots(2m-1)$$

$$\begin{aligned}
&= (-1)^{m-1}(2m-1)(2m-3)(2m-5)\cdots(1)(1)(3)(5)\cdots(2m-1) \\
&= (-1)^{m-1}\frac{(2m-1)!}{2^{m-1}(m-1)!}\frac{(2m-1)!}{2^{m-1}(m-1)!} \\
&= \frac{(-1)^{m-1}}{2^{2(m-1)}}\left[\frac{(2m-1)!}{(m-1)!}\right]^2,
\end{aligned}$$

so that

$$\prod_{j\neq k\in\mathbb{J}_m}\left(\frac{1}{2}-k\right) = \frac{(-1)^{m-1}}{2^{4m-3}}\frac{1}{2j-1}\left[\frac{(2m-1)!}{(m-1)!}\right]^2, \tag{3.62}$$

whereas

$$\begin{aligned}
\prod_{j\neq k\in\mathbb{J}_m}(j-k) &= (j+m-1)(j+m-2)\cdots(1)(-1)(-2)\cdots(j-m) \\
&= (j+m-1)!(-1)^{m+j}(m-j)!.
\end{aligned} \tag{3.63}$$

Combining (3.62) and (3.63), we obtain

$$\begin{aligned}
L_{m,j}\left(\frac{1}{2}\right) &= \frac{\frac{(-1)^{m-1}}{2^{4m-3}}\frac{1}{1-2j}\left[\frac{(2m-1)!}{(m-1)!}\right]^2}{(j+m-1)!(-1)^{m+j}(m-j)!} \\
&= \frac{m}{2^{4m-3}}\binom{2m-1}{m}\frac{(-1)^{j+1}}{2j-1}\binom{2m-1}{m-j}, \quad j\in\mathbb{J}_m.
\end{aligned} \tag{3.64}$$

Hence the DD mask $d_m = \{d_{m,j} : j \in \mathbb{Z}\} \in \mathscr{M}_0(\mathbb{Z})$ in (3.61) has the following explicit expression:

$$\left.\begin{aligned}
&d_{m,2j} = \delta_j, \quad j\in\mathbb{Z}, \\
&d_{m,2j+1} = \frac{m}{2^{4m-3}}\binom{2m-1}{m}\frac{(-1)^j}{1+2j}\binom{2m-1}{m+j}, \\
&\qquad\qquad j=-m,\ldots,m-1, \\
&d_{m,2j+1} = 0, \quad j\notin\{-m,\ldots,m-1\}.
\end{aligned}\right\} \tag{3.65}$$

Observe from the second line of (3.61) and (3.64) that, for $j\in\{-m,\ldots,m-1\}$, we have

$$\begin{aligned}
d_{m,-2j-1} &= d_{m,2(-j-1)+1} \\
&= L_{m,j+1}\left(\frac{1}{2}\right) \\
&= \frac{m}{2^{4m-3}}\binom{2m-1}{m}\frac{(-1)^{-j+1}}{2j+1}\binom{2m-1}{m-j-1}
\end{aligned}$$

$$\begin{aligned} &= \frac{m+1}{2^{4m-3}}\binom{2m-1}{m}\frac{(-1)^j}{2(-j)-1}\binom{2m-1}{m+j} \\ &= L_{m,-j}\left(\frac{1}{2}\right) = d_{m,2j+1}. \end{aligned}$$

Hence the DD mask $d_m = \{d_{m,j} : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$, as given by (3.61), satisfies the symmetry condition (2.7).

Since the DD mask $d_m = \{d_{m,j} : j \in \mathbb{Z}\}$ is, moreover, an optimally local interpolatory mask with the polynomial filling property (3.53), we deduce that $a = d_m$ is the unique symmetric interpolatory mask such that the polynomial filling property (2.11) holds with $\mu = 2m-1$. Introducing the DD mask symbol definition

$$D_m(z) = \sum_j d_{m,j} z^j, \quad z \in \mathbb{C}\setminus\{0\},$$

and noting also from (3.65) that

$$d_{m,-2m+1} \neq 0, \quad d_{m,2m-1} \neq 0,$$

$$d_{m,j} = 0, \quad j \notin \{-2m+1,\ldots,2m-1\},$$

we deduce that $D_m \in \mathscr{A}_{m,m}$. It follows from (3.4) in Theorem 3.1 that $A_m = D_m$, and thus

$$a_{m,j} = d_{m,j}, \quad j \in \mathbb{Z}.$$

It immediately follows that Corollary 3.7 has the following alternative explicit formulation.

Corollary 3.9. *In Corollary 3.7, the representation formula* (3.46) *has the alternative explicit form*

$$A(z) = D_m(z) + \zeta\left(1-\zeta^2\right)^m Q(\zeta^2), \quad z \in \mathbb{C}\setminus\{0\}, \tag{3.66}$$

with the polynomial Q as in Corollary 3.7, where ζ is given by (3.2)*, and where D_m is the DD mask symbol as given by the explicit formula*

$$D_m(z) = 1 + \frac{m}{2^{4m-3}}\binom{2m-1}{m}\sum_{j=-m}^{m-1}\frac{(-1)^j}{1+2j}\binom{2m-1}{m+j}z^{2j+1}, \quad z \in \mathbb{C}\setminus\{0\}. \tag{3.67}$$

In [9, Theorem 1], (see also [19, Theorem 2.7] and [22, Theorem 2]) it was proved that the representation formula (3.66) in Corollary 3.9 has the alternative explicit form

$$A = \sum_{j=0}^{n-m} t_j D_{m+j}, \tag{3.68}$$

where the coefficient sequence $\{t_j : j = 0, \ldots, n-m\} \subset \mathbb{R}$ satisfies the condition

$$\sum_{j=0}^{n-m} t_j = 1. \tag{3.69}$$

We intend to pursue the issue of the relative advantages/disadvantages of the two alternative explicit characterization formulas (3.66) and (3.68), in particular with respect to their uses in existence and convergence analysis, in future research.

3.6 Special cases of $\mathscr{A}_{m,n}$

We close this chapter with the following examples, which will be used in Chapter 5.

3.6.1 The case $n = m$

For the case $n = m$, we choose $Q = 0$ in Corollary 3.9, to obtain $A = A_m = D_m$, as is also clear from the case $n = m$ in (3.68) and (3.69), where, as calculated by means of (3.65), we have, for $z \in \mathbb{C}\setminus\{0\}$,

$$D_1(z) = \frac{1}{2}\Big(z^{-1} + 2 + z\Big), \tag{3.70}$$

$$D_2(z) = \frac{1}{16}\Big(-z^{-3} + 9z^{-1} + 16 + 9z - z^3\Big), \tag{3.71}$$

$$D_3(z) = \frac{1}{256}\Big(3z^{-5} - 25z^{-3} + 150z^{-1} + 256 + 150z - 25z^3 + 3z^5\Big), \tag{3.72}$$

$$\begin{aligned} D_4(z) = \frac{1}{2048}\Big(&-5z^{-7} + 49z^{-5} - 245z^{-3} + 1225z^{-1} + 2048 + 1225z \\ &-245z^3 + 49z^5 - 5z^7\Big), \end{aligned} \tag{3.73}$$

$$\begin{aligned} D_5(z) = \frac{1}{65536}\Big(&35z^{-9} - 405z^{-7} + 2268z^{-5} - 8820z^{-3} + 39690z^{-1} \\ &+65536 + 39690z - 8820z^3 + 2268z^5 - 405z^7 + 35z^9\Big). \end{aligned} \tag{3.74}$$

3.6.2 The case $n = m+1$

If $n = m+1$, then $Q(z) = t$, $z \in \mathbb{C}$, with t denoting an arbitrary non–zero constant (independent of z) such that, according to (3.47), we have $t \neq \frac{1}{2^{2m-1}}\binom{2m-1}{m-1}$. Then Corollary 3.9 yields $A = A_m(t|\cdot)$, where for $z \in \mathbb{C}\setminus\{0\}$, we obtain, from (3.66), (3.2), and the formulas in Section 3.6.1 above,

$$A_1(t|z) = \frac{1}{8}\Big(-tz^{-3} + (4+t)z^{-1} + 8 + (4+t)z - tz^3\Big), \tag{3.75}$$

$$\begin{aligned} A_2(t|z) &= \frac{1}{32}\Big(tz^{-5} + (-2-3t)z^{-3} + (18+2t)z^{-1} + 32 + (18+2t)z \\ &\qquad + (-2-3t)z^3 + tz^5\Big), \end{aligned} \tag{3.76}$$

$$\begin{aligned} A_3(t|z) &= \frac{1}{256}\Big(-2tz^{-7} + (3+10t)z^{-5} + (-25-18t)z^{-3} + (150+10t)z^{-1} \\ &\qquad +256 + (150+10t)z + (-25-18t)z^3 + (3+10t)z^5 - 2tz^7\Big), \end{aligned} \tag{3.77}$$

$$\begin{aligned} A_4(t|z) &= \frac{1}{2048}\Big(4tz^{-9} + (-5-28t)z^{-7} + (49+80t)z^{-5} + (-245-112t)z^{-3} \\ &\qquad +(1225+56t)z^{-1} + 2048 + (1225+56t)z + (-245-112t)z^3 \\ &\qquad +(49+80t)z^5 + (-5-28t)z^7 + 4tz^9\Big). \end{aligned} \tag{3.78}$$

Note that $A = A_m(0|\cdot) = D_m \in \mathscr{A}_{m,m}$, as can be verified directly for the cases $m = 1,\ldots,4$ by setting $t = 0$ in (3.75),…,(3.78) to obtain (3.70),…,(3.73). The choice $t = \frac{1}{2^{2m-1}}\binom{2m-1}{m-1}$ in the formulas for $A_m(t|\cdot)$ also has an interesting interpretation in terms of DD mask symbols. Our result is as follows.

Theorem 3.10. For $m \in \mathbb{N}$, the mask $A = A_m\left(\frac{1}{2^{2m-1}}\binom{2m-1}{m-1}\Big|\cdot\right)$, as obtained by choosing Q in (3.46) as the constant polynomial $Q(z) = \frac{1}{2^{2m-1}}\binom{2m-1}{m-1}$, $z \in \mathbb{C}$, satisfies

$$A = A_m\left(\frac{1}{2^{2m-1}}\binom{2m-1}{m-1}\Big|\cdot\right) = A_{m+1} = D_{m+1}. \tag{3.79}$$

Proof. Recalling from the proof of Corollary 3.7 the relationship $Q = \dfrac{1}{2^{m-1}}P$, we deduce that here the mask symbol A is given by the formula

$$A(z) = \frac{1}{2^{m-1}}(1+\zeta)^m \left[\sum_{j\in\mathbb{Z}_{m-1}} \binom{m+j-1}{j} \left[\frac{1}{2}(1-\zeta)\right]^j + \frac{1}{2^m}\binom{2m-1}{m-1}\zeta(1-\zeta)^m \right], \quad z \in \mathbb{C}\backslash\{0\}, \tag{3.80}$$

with ζ defined by (3.2), i.e. (2.42) holds with

$$C(z) = \sum_{j\in\mathbb{Z}_{m-1}} \binom{m+j-1}{j} \left[\frac{1}{2}(1-\zeta)\right]^j + \frac{1}{2^m}\binom{2m-1}{m-1}\zeta(1-\zeta)^m, \quad z \in \mathbb{C}\backslash\{0\}, \tag{3.81}$$

according to which $C(1) = 1$.

Now note from (3.2) and (3.43) that

$$C(-1) = 0. \tag{3.82}$$

Since C is a symmetric Laurent polynomial, we know that

$$C(z^{-1}) = C(z), \quad z \in \mathbb{C}\backslash\{0\}, \tag{3.83}$$

which we now differentiate to obtain

$$C'(z^{-1})(-z^{-2}) = C'(z), \quad z \in \mathbb{C}\backslash\{0\}. \tag{3.84}$$

Setting $z = -1$ in (3.84), we obtain

$$C'(-1)(-1) = C'(-1),$$

i.e.

$$2C'(-1) = 0,$$

and thus

$$C'(-1) = 0. \tag{3.85}$$

It follows from (3.82) and (3.85) that there is a Laurent polynomial $\tilde{C}$ such that

$$C(z) = \frac{1}{4z}\left(1+z\right)^2 \tilde{C}(z), \quad z \in \mathbb{C}\backslash\{0\}. \tag{3.86}$$

Using (3.86) and (3.83), we obtain, for $z \in \mathbb{C}\backslash\{0\}$,

$$\frac{z}{4}\left(1+z^{-1}\right)^2 \tilde{C}(z^{-1}) = \frac{1}{4z}\left(1+z\right)^2 \tilde{C}(z),$$

i.e.

$$\frac{z(1+z)^2}{4z^2}\tilde{C}(z^{-1}) = \frac{(1+z)^2}{4z}\tilde{C}(z),$$

and thus

$$\tilde{C}(z^{-1}) = \tilde{C}(z), \quad z \in \mathbb{C}\setminus\{0\},$$

showing that $\tilde{C}$ is a symmetric Laurent polynomial.

Observe now, from (3.2), that (3.86) can be rewritten in the form

$$C(z) = \frac{1}{2}\Big(1+\zeta\Big)\tilde{C}(z), \quad z \in \mathbb{C}\setminus\{0\}. \tag{3.87}$$

Now substitute (3.87) into (2.42) to obtain

$$A(z) = \frac{1}{2^m}(1+\zeta)^{m+1}\tilde{C}(z), \quad z \in \mathbb{C}\setminus\{0\}. \tag{3.88}$$

Next, we note from (3.81) and (3.87) and (3.2) that there is a sequence $\{\tilde{c}_j : j \in \mathbb{Z}\} \in \mathscr{M}_0(\mathbb{Z})$ such that

$$\tilde{C}(z) = \sum_j \tilde{c}_j z^j = \sum_{j=-m}^{m} \tilde{c}_j z^j, \quad z \in \mathbb{C}\setminus\{0\}. \tag{3.89}$$

We can now deduce from (3.4) in Theorem 3.1, with m replaced by $m+1$, together with Propositions 3.2 and 3.5, as well as (3.86), and (3.87), that the result (3.79) of our theorem does indeed hold, after recalling the fact that $A_{m+1} = D_{m+1}$, as noted before Corollary 3.9. ∎

Chapter 4

Refinable functions and subdivision

To investigate the convergence of the interpolatory subdivision schemes constructed in Chapter 3, we rely on the concept of refinement pairs, which we now proceed to introduce and analyze in Section 4.1 below, before specialising to the case of an interpolatory refinable function, and its relationship with interpolatory subdivision convergence, in the subsequent Section 4.2.

4.1 Refinement pairs

Definition 4.1. If a sequence $a \in \mathscr{M}_0(\mathbb{Z})$ and a function $\phi \in \mathscr{M}_0(\mathbb{R})$, with $\phi \neq 0$, satisfy the equation

$$\phi = \sum_j a_j \phi(2\cdot - j), \tag{4.1}$$

we say that (a, ϕ) is a *refinement pair*, the sequence $a = \{a_j : j \in \mathbb{Z}\}$ is the *refinement mask*, the function ϕ is the *refinable function* corresponding to the mask a, and the equation (4.1) is called the *refinement equation* or *two-scale difference equation*. The Laurent polynomial A defined by (1.4) is the corresponding *refinement mask symbol*.

Our results and proofs of Theorems 4.2 to 4.6 below are based on those in [11, Chapter 2].

For a refinement pair (a, ϕ), since $a \in \mathscr{M}_0(\mathbb{Z})$, there exist $M, N \in \mathbb{Z}$, with $N > M$, such that

$$a_j = 0, \quad j \notin \{M, \ldots, N\}, \quad \text{with} \quad a_M \neq 0, \quad a_N \neq 0. \tag{4.2}$$

The condition (4.2) has the following implication.

Theorem 4.2. For any $M, N \in \mathbb{Z}$, with $N > M$, suppose that (a, ϕ) is a refinement pair such that (4.2) holds. Then

$$\inf\{x : \phi(x) \neq 0\} = M, \tag{4.3}$$

and

$$\sup\{x : \phi(x) \neq 0\} = N. \tag{4.4}$$

Proof. First, to prove (4.3), we note that if we set

$$\alpha = \inf\{x : \phi(x) \neq 0\}, \tag{4.5}$$

$$\beta = \sup\{x : \phi(x) \neq 0\}, \tag{4.6}$$

then $\phi \in \mathscr{M}_0(\mathbb{R})$ implies $\alpha, \beta \in \mathbb{R}$. Also, it follows from (4.5) that

$$\inf\{x : \phi(2x - M) \neq 0\} = \frac{\alpha + M}{2}, \tag{4.7}$$

whereas

$$\sup\{x : \phi(2x - N) \neq 0\} = \frac{\beta + N}{2}. \tag{4.8}$$

Using (4.1) and (4.2), we see that

$$\phi = \sum_{j=M}^{N} a_j \phi(2 \cdot - j). \tag{4.9}$$

Now we use (4.5), (4.7) and (4.9) to deduce that, if $\alpha < M$, then there exists a number $x_0 \in \left[\alpha, \frac{\alpha+M}{2}\right)$ such that

$$0 \neq \phi(x_0) = \sum_{j=M}^{N} a_j \phi(2x_0 - j) = 0,$$

whereas, if $\alpha > M$, there exists a number $\left[\frac{\alpha+M}{2}, \tilde{\alpha}\right)$, where $\tilde{\alpha} = \min\left\{\frac{\alpha+M+1}{2}, \alpha\right\}$, such that

$$0 = \phi(x_1) = a_M \phi(2x_1 - M) \neq 0,$$

since $a_M \neq 0$ from (4.2). Hence the assumption $\alpha \neq M$ yields a contradiction, and thus $\alpha = M$, thereby proving (4.3).

Similarly, we use (4.6), (4.8) and (4.9) to deduce that, if $\beta > N$, there exists a number $\tilde{x}_0 \in \left(\frac{N+\beta}{2}, \beta\right]$ such that

$$0 \neq \phi(\tilde{x}_0) = \sum_{j=M}^{N} a_j \phi(2\tilde{x}_0 - j) = 0,$$

whereas, if $\beta < N$, there exists a number $\tilde{x}_1 \in \left(\tilde{\beta}, \frac{N+\beta}{2}\right]$, where

$$\tilde{\beta} = \max\left\{\frac{N+\beta-1}{2}, \beta\right\},$$

such that

$$0 = \phi(\tilde{x}_1) = a_N \phi(2\tilde{x}_1 - N) \neq 0,$$

since also $a_N \neq 0$ from (4.2). Hence the assumption $\beta \neq N$ yields contradiction, and thus $\beta = N$, thereby proving (4.4). ∎

The following result immediately follows from Theorem 4.2.

Corollary 4.3. *For any $M, N \in \mathbb{Z}$, with $N > M$, suppose that (a, ϕ) is a refinement pair such that* (4.2) *holds. Then*

$$\phi(x) = 0, \quad x \notin [M, N]. \tag{4.10}$$

Next, we prove an important implication of the sum rules (1.5).

Theorem 4.4. For a refinement pair (a, ϕ), with $\phi \in \mathscr{C}_0(\mathbb{R})$, suppose that the refinement mask a satisfies the sum rules (1.5). Then

$$\sum_j \phi(x-j) = \sum_j \phi(j) = \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x, \quad x \in \mathbb{R}. \tag{4.11}$$

Proof. Defining the function $F \in \mathscr{M}(\mathbb{R})$ by

$$F = \sum_j \phi(\cdot - j), \tag{4.12}$$

it follows that $F \in \mathscr{C}(\mathbb{R})$, since $\phi \in \mathscr{C}_0(\mathbb{R})$. Moreover, since the dyadic set $\{\frac{j}{2^r} : j \in \mathbb{Z},\ \ r \in \mathbb{Z}_+\}$ is dense in $\mathbb{R}$, and F is continuous on $\mathbb{R}$, the first equality in (4.11) will follow if we can prove that

$$F\left(\frac{j}{2^r}\right) = \sum_{\ell} \phi(\ell), \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+. \tag{4.13}$$

To prove (4.13), we use (4.1) and (2.13) to obtain, for $j \in \mathbb{Z}, \;\; r \in \mathbb{Z}_+$,

$$
\begin{aligned}
F\left(\frac{j}{2^r}\right) &= \sum_k \phi\left(\frac{j}{2^r} - k\right) \\
&= \sum_k \sum_\ell a_\ell \phi\left(\frac{j}{2^{r-1}} - 2k - \ell\right) \\
&= \sum_k \sum_\ell a_{\ell-2k} \phi\left(\frac{j}{2^{r-1}} - \ell\right) \\
&= \sum_\ell \left[\sum_k a_{\ell-2k}\right] \phi\left(\frac{j}{2^{r-1}} - \ell\right) \\
&= \sum_\ell \phi\left(\frac{j}{2^{r-1}} - \ell\right) \\
&= \cdots \\
&= \sum_\ell \phi(j-\ell) = \sum_\ell \phi(\ell),
\end{aligned}
$$

thereby completing our proof of (4.13).

It remains to prove the second inequality in (4.11). To this end, we fix $r \in \mathbb{N}$, and use (4.1) repeatedly to obtain

$$
\begin{aligned}
\sum_j \phi\left(\frac{j}{2^r}\right) &= \sum_j \sum_{k_1} a_{k_1} \phi\left(\frac{j}{2^{r-1}} - k_1\right) \\
&= \sum_j \sum_{k_1} a_{k_1} \sum_{k_2} a_{k_2} \phi\left(\frac{j}{2^{r-2}} - 2k_1 - k_2\right) \\
&= \sum_{k_1} a_{k_1} \sum_{k_2} a_{k_2-2k_1} \sum_j \phi\left(\frac{j}{2^{r-2}} - k_2\right) \\
&= \cdots \\
&= \sum_{k_1} a_{k_1} \sum_{k_2} a_{k_2-2k_1} \cdots \sum_{k_r} a_{k_r-2k_{r-1}} \sum_j \phi(j-k_r) \\
&= \left[\sum_{k_1} a_{k_1} \sum_{k_2} a_{k_2} - 2k_1 \cdots \sum_{k_r} a_{k_r-2k_{r-1}}\right] \left[\sum_j \phi(j)\right] \\
&= \left[\sum_{k_1} a_{k_1} \sum_{k_2} a_{k_2} \cdots \sum_{k_r} a_{k_r}\right] \left[\sum_j \phi(j)\right] \\
&= \left(\sum_k a_k\right)^r \sum_j \phi(j) = 2^r \sum_j \phi(j),
\end{aligned}
\tag{4.14}
$$

since $\sum_j a_j = 2$, by virtue of (1.5).

Since (4.10) in Corollary 4.3 holds, and ϕ is continuous on $\mathbb{R}$, we know that $\phi(M) = \phi(N) = 0$, and thus, using the fact that a difinite integral is by definition the limit of Riemann sum, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\phi(x)\mathrm{d}x=\int_{M}^{N}\phi(x)\mathrm{d}x &= \lim_{r\to\infty}\left[\frac{1}{2^r}\sum_{j=2^rM}^{2^rN-1}\phi\left(\frac{j}{2^r}\right)\right]\\ &= \lim_{r\to\infty}\left[\frac{1}{2^r}\sum_{j}\phi\left(\frac{j}{2^r}\right)\right],\end{aligned}\tag{4.15}$$

which, together with (4.14), the yields the desired second equality in (4.11). ∎

Our uniqueness result in Theorem 4.5 below depends on the following results from polynomial approximation.

Let $[a,b]$ denote a bounded interval in $\mathbb{R}$, and let $f\in\mathscr{C}[a,b]$. Then the Bernstein polynomial approximation sequence $\{q_j=q_{f,j}: j\in\mathbb{N}\}$, where $q_j\in\pi_j,\ \ j\in\mathbb{N}$, as defined by

$$q_j=\sum_{k=0}^{j} f\left(a+k\left(\frac{b-a}{j}\right)\right)\binom{j}{k}\left(\frac{\cdot-a}{b-a}\right)^k\left(\frac{b-\cdot}{b-a}\right)^{j-k},\quad j\in\mathbb{N},\quad f\in\mathscr{C}[a,b],\tag{4.16}$$

satisfies the uniform convergence property

$$\max_{a\le x\le b}\left|f(x)-q_j(x)\right|\to 0,\quad j\to\infty,\quad f\in\mathscr{C}(a,b).\tag{4.17}$$

Also, the Legendre polynomial approximation sequence $\{p_j=p_{f,j}: j\in\mathbb{N}\}$, where $p_j\in\pi_j,\ \ j\in\mathbb{N}$, as defined by

$$p_j=\sum_{k=0}^{j}\left[\frac{2k+1}{(b-a)^{2k+1}}\binom{2k}{k}^2\int_a^b f(t)\tilde{p}_j(t)\mathrm{d}t\right]\tilde{p}_k,\quad j\in\mathbb{N},\quad f\in\mathscr{C}[a,b],\tag{4.18}$$

with $\{\tilde{p}_k: k\in\mathbb{N}\}$ denoting the Legendre polynomials on $[a,b]$, as given recursively by

$$\left.\begin{aligned}\tilde{p}_0(x) &= 1,\quad x\in\mathbb{R},\qquad \tilde{p}_1=\frac{2(\cdot)-(a+b)}{2},\\ \tilde{p}_{k+1} &= \frac{2(\cdot)-(a+b)}{2}\tilde{p}_k-\left(\frac{b-a}{2}\right)^2\frac{k^2}{(2k-1)(2k+1)}\tilde{p}_{k-1},\quad k\in\mathbb{N},\end{aligned}\right\}\tag{4.19}$$

and satisfying the orthogonality condition

$$\int_a^b \tilde{p}_j(x)\tilde{p}_k(x)\mathrm{d}x=0,\quad j,k\in\mathbb{N},\quad \text{with}\quad j\neq k,\tag{4.20}$$

satisfies the best approximation property

$$\int_a^b [f(x)-\tilde{p}_j(x)]^2\mathrm{d}x\le\int_a^b [f(x)-p(x)]^2\mathrm{d}x,\quad p\in\pi_j,\quad j\in\mathbb{N}.\tag{4.21}$$

Theorem 4.5. Suppose the sequence $a \in \mathscr{M}_0(\mathbb{Z})$ satisfies the sum rules (1.5). Then there exists at most one function $\phi \in \mathscr{C}_0(\mathbb{R})$, with

$$\sum_j \phi(j) = 1, \tag{4.22}$$

such that (a,ϕ) is a refinement pair.

Proof. We use a proof by contradiction. Suppose therefore that ϕ_1 and ϕ_2 are two fuctions in $\mathscr{C}_0(\mathbb{R})$, with $\phi_1 \neq \phi_2$, such that both (a,ϕ_1) and (a,ϕ_2) are refinement pairs, and such that

$$\sum_j \phi_1(j) = 1, \quad \sum_j \phi_2(j) = 1.$$

Since $a \in \mathscr{M}_0(\mathbb{Z})$, we know that there exist unique integers M and N such that (4.2) holds. Now define $\phi_0 = \phi_1 - \phi_2$, so that $\phi_0 \neq 0$, and

$$\begin{aligned}
\sum_j a_j \phi_0(2\cdot - j) &= \sum_j a_j \left[\phi_1(2\cdot - j) - \phi_2(2\cdot - j)\right] \\
&= \sum_j a_j \phi_1(2\cdot - j) - \sum_j a_j \phi_2(2\cdot - j) \\
&= \phi_1 - \phi_2 = \phi_0, \qquad (4.23)
\end{aligned}$$

i.e. (a,ϕ_0) is also a refinement pair. But then Theorem 4.4 gives

$$\int_{-\infty}^{\infty} \phi_0(x)\mathrm{d}x = \sum_j \phi_0(j) = \sum_j \phi_1(j) - \sum_j \phi_2(j) = 1 - 1 = 0. \tag{4.24}$$

Observe from Corollary 4.3 that $\phi_0(x) = 0, \quad x \in (M,N)$. Our next step is to prove that if (4.24) holds, then

$$\int_{-\infty}^{\infty} p(x)\phi_0(x)\mathrm{d}x = \int_M^N p(x)\phi_0(x)\mathrm{d}x = 0, \quad p \in \pi, \tag{4.25}$$

with π denoting the space of all polynomials.

To this end, we define the sequence $\{\mu_j : j \in \mathbb{Z}_+\}$ by

$$\mu_j = \int_{-\infty}^{\infty} x^j \phi_0(x)\mathrm{d}x = \int_M^N x^j \phi_0(x)\mathrm{d}x = 0, \quad j \in \mathbb{Z}_+. \tag{4.26}$$

Using (4.23), (4.10) and (1.5), it follows from (4.26) that, for $j \in \mathbb{N}$,

$$\mu_j = \int_M^N x^j \left[\sum_{k=M}^N a_k \phi_0(2x - k)\right] \mathrm{d}x$$

$$
\begin{aligned}
&= \sum_{k=M}^{N} a_k \int_M^N x^j \phi_0(2x-k)\mathrm{d}x \\
&= \sum_{k=M}^{N} a_k \left[\frac{1}{2}\int_{2M-k}^{2N-k} \left(\frac{x+k}{2}\right)^j \phi_0(x)\mathrm{d}x\right] \\
&= \sum_{k=M}^{N} a_k \left[\frac{1}{2^{j+1}}\int_{2M-k}^{2N-k} (x+k)^j \phi_0(x)\mathrm{d}x\right] \\
&= \frac{1}{2^{j+1}}\sum_{k=M}^{N} a_k \left[\int_M^N (x+k)^j \phi_0(x)\mathrm{d}x\right] \\
&= \frac{1}{2^{j+1}}\sum_{k=M}^{N} a_k \left[\int_M^N \left\{\sum_{\ell=0}^{j}\binom{j}{\ell} x^\ell k^{j-\ell}\right\} \phi_0(x)\mathrm{d}x\right] \\
&= \frac{1}{2^{j+1}}\sum_{k=M}^{N} a_k \left[\sum_{\ell=0}^{j}\binom{j}{\ell} k^{j-\ell}\left\{\int_M^N x^\ell \phi_0(x)\mathrm{d}x\right\}\right] \\
&= \frac{1}{2^{j+1}}\sum_{k=M}^{N} a_k \sum_{\ell=0}^{j}\binom{j}{\ell} k^{j-\ell}\,\mu_\ell \\
&= \frac{1}{2^{j+1}}\sum_{\ell=0}^{j}\binom{j}{\ell}\left[\sum_{k=M}^{N} a_k k^{j-\ell}\right]\mu_\ell \\
&= \frac{1}{2^{j+1}}\left[\sum_{k=M}^{N} a_k\right]\mu_j + \frac{1}{2^{j+1}}\sum_{\ell=0}^{j-1}\binom{j}{\ell}\left[\sum_{k=M}^{N} a_k k^{j-\ell}\right]\mu_\ell \\
&= \frac{1}{2^j}\mu_j + \frac{1}{2^{j+1}}\sum_{\ell=0}^{j-1}\binom{j}{\ell}\left[\sum_{k=M}^{N} a_k k^{j-\ell}\right]\mu_\ell,
\end{aligned}
$$

thereby yielding the recursive formula

$$
\mu_j = \frac{1}{2(2^j-1)}\sum_{\ell=0}^{j-1}\binom{j}{\ell}\left[\sum_{k=M}^{N} a_k k^{j-\ell}\right]\mu_\ell, \quad j \in \mathbb{N}. \tag{4.27}
$$

Noting from (4.26) and (4.24) that $\mu_0 = 0$, we deduce recursively from (4.27) that $\mu_j = 0, \;\; j \in \mathbb{Z}$, and thus, from (4.26),

$$
\int_{-\infty}^{\infty} x^j \phi_0(x)\mathrm{d}x = 0, \quad j \in \mathbb{Z}_+, \tag{4.28}
$$

which is equivalent to (4.25).

With the choices $[a,b] = [M,N]$ and $f = \phi_0$ in the definition (4.18) of the Legendre polynomial approximation sequence $\{p_j = p_{\phi_0,j} : j \in \mathbb{Z}_+\}$, we now observe that, since $\int_M^N \phi_0(t)\tilde{p}_j(t) = 0, \;\; j \in \mathbb{Z}_+$, by virtue of (4.25), we have, in (4.18), $p_j = 0, \;\; j \in \mathbb{Z}_+$, and thus, from (4.21),

$$
\int_{-\infty}^{\infty} [\phi_0(x)]^2\mathrm{d}x = \int_M^N [\phi_0(x)]^2\mathrm{d}x \le \int_M^N [\phi_0(x) - p(x)]^2\mathrm{d}x, \quad p \in \pi_j, \quad j \in \mathbb{Z}_+. \tag{4.29}
$$

Now choose in (4.29), for $j \in \mathbb{Z}_+$, $p = q_j \in \pi_j$, where $\{q_j = q_{\phi_0,j} : j \in \mathbb{Z}_+\}$ denotes the Bernstein polynomial approximation sequence as defined by (4.16), with $f = \phi_0$ and $[a,b] = [M,N]$, to obtain, for $j \in \mathbb{Z}_+$,

$$\begin{aligned}\int_{-\infty}^{\infty} [\phi_0(x)]^2 \mathrm{d}x &\le \int_M^N [\phi_0(x) - q_j(x)]^2 \mathrm{d}x \\ &\le M - N\left[\max_{M\le x\le N}\left|\phi_0(x) - q_j(x)\right|\right]^2 \to 0, \quad j \to \infty, \quad (4.30)\end{aligned}$$

from (4.17), and the fact that ϕ_0 is a continuous function on $[M,N]$. But (4.30) implies $\phi_0 = 0$, and thus $\phi_1 = \phi_2$, thereby contradicting our assumption that $\phi_1 \neq \phi_2$. Our proof is now complete. ■

The uniqueness result of Theorem 4.5 now enables us to prove the following preservation of symmetry result.

Theorem 4.6. For a sequence $a \in \mathscr{M}_0(\mathbb{Z})$ satisfying the sum rules (1.5), as well as the symmetry condition (2.7), suppose (a,ϕ) is a refinement pair, with $\phi \in \mathscr{C}_0(\mathbb{R})$, and such that the condition (4.22) is satisfied. Then the symmetry condition

$$\phi(-\cdot) = \phi \quad (4.31)$$

holds.

Proof. With the function $\tilde{\phi} \in \mathscr{C}_0(\mathbb{R})$ defined by $\tilde{\phi} = \phi(-\cdot)$, we use (2.7) and (4.1), to obtain

$$\begin{aligned}\sum_j a_j \tilde{\phi}(2\cdot - j) &= \sum_j a_j \phi(-2\cdot + j) \\ &= \sum_j a_{-j} \phi\Big(-2\cdot - j\Big) \\ &= \sum_j a_j \phi\Big(2(-\cdot) - j\Big) = \phi(-\cdot) = \tilde{\phi}.\end{aligned}$$

Moreover,

$$\sum_j \tilde{\phi}(j) = \sum_j \phi(-j) = \sum_j \phi(j) = 1, \quad (4.32)$$

from (4.22). Hence $(a,\tilde{\phi})$ is a refinement pair, with $\tilde{\phi}$ satisfying the condition (4.32). It follows from Theorem 4.5 that $\tilde{\phi} = \phi$, which is equivalent to the desired symmetry result (4.31). ■

4.2 The interpolatory case

Definition 4.7. If (a,ϕ) is a refinement pair with the property that ϕ interpolates the Kronecker delta sequence at the integers, i.e.

$$\phi(j) = \delta_j, \quad j \in \mathbb{Z}, \tag{4.33}$$

we say that (a,ϕ) is an *interpolatory refinement pair.*

The following result now follows from Definition 4.7.

Proposition 4.8. *Suppose* (a,ϕ) *is an interpolatory refinement pair. Then a is an interpolatory mask in the sense of* (2.4).

Proof. Using (4.1) and (4.33), we obtain, for $j \in \mathbb{Z}$,

$$\delta_j \quad = \quad \phi(j) = \sum_k a_k \phi(2j-k) = \sum_k a_{2j-k}\phi(k) = \sum_k a_{2j-k}\delta_k = a_{2j},$$

thereby proving (2.4). ■

The fundamental significance of the existence of an interpolatory refinable function in the context of the convergence of the corresponding interpolatory subdivision scheme is shown by the following result, as was first proved in [8, Theorem 2.2].

Theorem 4.9. Suppose (a,ϕ) is an interpolatory refinement pair. Then the corresponding interpolatory subdivision scheme (S_a, c) is convergent on $\mathscr{M}(\mathbb{Z})$, in the sense that if, in the definition (1.2), we choose $c \in \mathscr{M}(\mathbb{Z})$, then the function $\Phi \in \mathscr{C}(\mathbb{R})$ defined by

$$\Phi = \sum_j c_j \phi(\cdot - j), \tag{4.34}$$

satisfies

$$c_j^{(r)} = \Phi\left(\frac{j}{2^r}\right), \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+. \tag{4.35}$$

Proof. It will suffice to prove that the function $\Phi \in \mathscr{C}(\mathbb{R})$ satisfies (4.35) for every initial sequence $c \in \mathscr{M}(\mathbb{Z})$ in (1.2), since (1.3) then holds trivially for every $c \in \mathscr{M}(\mathbb{Z})$. If r = 0, then (4.35) follows from (4.34) and the interpolatory property

(4.33) of ϕ. For $r \geq 1$, we use the refinability (4.1) of ϕ, together with (1.1), (1.2) and (4.33) to deduce, for $j \in \mathbb{Z}$, that

$$\begin{aligned}
\Phi\Big(\frac{j}{2^r}\Big) &= \sum_k c_k \phi\Big(\frac{j}{2^r} - k\Big) \\
&= \sum_k c_k \Big[\sum_\ell a_\ell \phi\Big(\frac{j}{2^{r-1}} - 2k - \ell\Big)\Big] \\
&= \sum_k c_k \sum_\ell a_{\ell-2k} \phi\Big(\frac{j}{2^{r-1}} - \ell\Big) \\
&= \sum_\ell \Big[\sum_k a_{\ell-2k} c_k\Big] \phi\Big(\frac{j}{2^{r-1}} - \ell\Big) \\
&= \sum_\ell (S_a c)_\ell \phi\Big(\frac{j}{2^{r-1}} - \ell\Big) \\
&= \sum_\ell c_\ell^{(1)} \phi\Big(\frac{j}{2^{r-1}} - \ell\Big) = \cdots = \sum_\ell c_\ell^{(r)} \phi(j - \ell) = c_j^{(r)},
\end{aligned}$$

thereby proving (4.35). ■

For symmetric interpolatory mask symbols $A \in \mathscr{A}_{m,n}$, we can now deduce the following result, which first appeared in [8, Theorem 2.1].

Theorem 4.10. For integers $m, n \in \mathbb{N}$, with $n \geq m$, suppose the Laurent polynomial A belongs to the class $\mathscr{A}_{m,n}$, and suppose $a = \{a_j : j \in \mathbb{Z}\}$ is the corresponding sequence in $\mathscr{M}_0(\mathbb{Z})$, with A and a related by (1.4). If there exists a function $\phi \in \mathscr{C}_0(\mathbb{R})$ such that (a, ϕ) is an interpolatory refinement pair, then the refinable function ϕ is the unique solution in $\mathscr{C}_0(\mathbb{R})$ of the refinement equation (4.1), and ϕ satisfies the following properties:

$$\phi(x) = 0, \quad x \notin (-2n+1, 2n-1); \tag{4.36}$$

$$\sum_j \phi(x - j) = 1, \quad x \in \mathbb{R}; \tag{4.37}$$

$$\phi(-\,\cdot\,) = \phi; \tag{4.38}$$

$$\sum_j p(j)\phi(\,\cdot - j) = p, \quad p \in \pi_{2m-1}; \tag{4.39}$$

$$\phi\Big(j + \frac{1}{2}\Big) = a_{2j+1}, \quad j \in \mathbb{Z}; \tag{4.40}$$

$$\phi\left(2n - 1 - 2^{-j}\left(n - \frac{3}{2} - k\right)\right) = 0, \quad k, j \in \mathbb{Z}_+. \tag{4.41}$$

Moreover, the corresponding symmetric interpolatory scheme (1.2) is convergent on $\mathscr{M}(\mathbb{Z})$, in the sense of Theorem 4.9.

Proof. First, note that property (4.36) is an immediate consequence of (2.39) and Corollary 4.3.

Next, since, from Proposition 2.6, the sum rules (1.5) are satisfied by the mask a, we deduce from the first equality in (4.11) of Theorem 4.4, together with (4.33), that, for $x \in \mathbb{R}$, we have

$$\sum_j \phi(x-j) = \sum_j \phi(j) = \sum_j \delta_j = 1. \tag{4.42}$$

The property (4.38) of ϕ follows from the fact that the mask a satisfies the symmetry condition (2.7), together with Theorem 4.6.

To prove (4.39), it will suffice to show that

$$\sum_k p(k)\phi\left(\frac{j}{2^r} - k\right) = p\left(\frac{j}{2^r}\right), \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+, \quad p \in \pi_{2m-1},$$

or, equivalently,

$$\sum_k k^l \phi\left(\frac{j}{2^r} - k\right) = \left(\frac{j}{2^r}\right)^\ell, \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+, \quad \ell \in \mathbb{Z}_{2m-1}, \tag{4.43}$$

which then implies (4.39), since the dyadic set $\left\{\frac{j}{2^r},\ j \in \mathbb{Z},\ r \in \mathbb{Z}_+\right\}$ is dense in $\mathbb{R}$, and since $\phi \in \mathscr{C}_0(\mathbb{R})$.

If $r = 0$, then (4.43) is an immediate consequence of (4.33). We assume next that $r \geq 1$. Then, using consecutively the refinability (4.1) of ϕ, the polynomial filling property (2.11), and interpolatory property (4.33), we get

$$\begin{aligned}
\sum_k k^\ell \phi\left(\frac{j}{2^r} - k\right) &= \sum_k k^\ell \sum_n a_n \phi\left(\frac{j}{2^{r-1}} - 2k - n\right) \\
&= \sum_k k^\ell \sum_n a_{n-2k}\left(\frac{j}{2^{r-1}} - n\right) \\
&= \sum_n \Big[\sum_k a_{n-2k} k^\ell\Big] \phi\left(\frac{j}{2^{r-1}} - n\right) \\
&= \frac{1}{2^l}\sum_n n^\ell \phi\left(\frac{j}{2^{r-1}} - n\right) \\
&= \frac{1}{2^l}\sum_k k^\ell \phi\left(\frac{j}{2^{r-1}} - k\right) \\
&= \cdots \\
&= \left(\frac{1}{2^l}\right)^r \sum_k k^\ell \phi(j-k) \\
&= \frac{j^\ell}{2^{\ell r}} = \frac{j^\ell}{(2^r)^\ell} = \left(\frac{j}{2^\ell}\right)^r,
\end{aligned}$$

thereby completing the proof of (4.43).

The property (4.40) is an immediate consequence of the refinability (4.1) and the interpolatory property (4.33) of ϕ, i.e.

$$\phi\Big(j+\frac{1}{2}\Big)=\sum_k a_k\phi(2j+1-k)=\sum_k a_k\delta_{2j+1-k}=a_{2j+1}, \quad j\in\mathbb{Z}.$$

To prove (4.41), we proceed by induction on j. When $j=0$, property (4.41) follows from (4.40) and (4.36). To advance the inductive hypothesis from j to $j+1$, we we use the refinability of ϕ, and the finite support property, as implied by (2.39), of a, to deduce that, for $k\in\mathbb{Z}$,

$$\begin{aligned}\phi\left(2n-1-2^{-(j+1)}\left(n-\frac{3}{2}-k\right)\right) &= \sum_{\ell=-2n+1}^{2n-1} a_\ell\phi\left(4n-2-2^{-j}\left(n-\frac{3}{2}-k\right)-\ell\right)\\ &= \sum_{\ell=0}^{4n-2} a_{2n-1-\ell}\phi\left(2n-1-2^{-j}\left(n-\frac{3}{2}-[k+2^j l]\right)\right),\end{aligned}$$

thereby completing our inductive proof.

Finally, note that subdivision convergence has already been established in Theorem 4.9. ∎

Chapter 5

Interpolatory refinable function existence

According to Theorem 4.10, we know that, if for a given interpolatory refinement mask $a \in \mathscr{M}_0(\mathbb{Z})$ there exists an associated refinable function $\phi \in \mathscr{C}_0(\mathbb{R})$ such that (a, ϕ) is an interpolatory refinement pair, then the corresponding interpolatory subdivision scheme (S_a, c) converges for every initial control point sequence $c \in \mathscr{M}(\mathbb{Z})$, as described in Theorem 4.10.

5.1 A fundamental existence result

We proceed to state a fundamental existence result for interpolatory refinable function, which is due to Micchelli [21], (see also [18, Theorem 4.2]).

Theorem 5.1. For $m, n \in \mathbb{N}$, with $n \geq m$, suppose the Laurent polynomial A belongs to the class $\mathscr{A}_{m,n}$. If, moreover, A satisfies the positivity condition

$$A(e^{ix}) > 0, \quad x \in (-\pi, \pi), \tag{5.1}$$

then there exists a refinable function $\phi \in \mathscr{C}_0(\mathbb{R})$ such that (a, ϕ) is an interpolatory refinement pair.

We can now use Theorem 5.1, together with our representation formula (3.46), with $Q = 0$, to prove that Dubuc–Deslauriers subdivision is convergent.

Theorem 5.2. For $m \in \mathbb{N}$, the Dubuc–Deslauriers mask symbol D_m, as defined by (3.4), (3.5), satisfies

$$D_m(e^{ix}) > 0, \quad x \in (-\pi, \pi). \tag{5.2}$$

Proof. Since $D_m = A_m$, as noted before Corollary 3.9, if follows from (3.5) and (3.2) that

$$D_m(z) = \frac{1}{2^{m-1}} \left(1 + \frac{z + z^{-1}}{2}\right)^m \sum_{j \in \mathbb{Z}_{m-1}} \binom{m+j-1}{j} \left[\frac{1}{2}\left(1 - \frac{z+z^{-1}}{2}\right)\right]^j, \quad z \in \mathbb{C}\backslash\{0\}, \tag{5.3}$$

and thus, by restricting z in (5.3) to the unit circle in $\mathbb{C}$,

$$D_m(e^{ix}) = \frac{1}{2^{m-1}}(1+\cos x)^m \sum_{j \in \mathbb{Z}_{m-1}} \binom{m+j-1}{j} \left[\frac{1}{2}(1-\cos x)\right]^j, \quad x \in \mathbb{R}, \tag{5.4}$$

from which (5.2) then immediately follows, since, in (5.4), we have

$$\sum_{j \in \mathbb{Z}_{m-1}} \binom{m+j-1}{j} \left(\frac{1}{2}(1-\cos x)\right)^j \geq \binom{m-1}{0} = 1, \quad x \in \mathbb{R}.$$

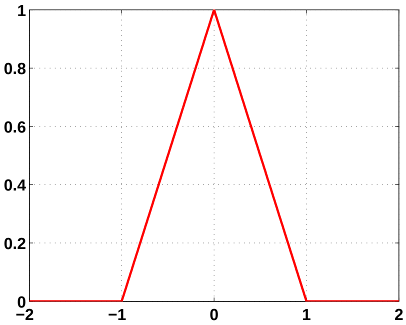
■

Combining the results of Theorems 4.9, 5.1 and 5.2, we thus have the following result.

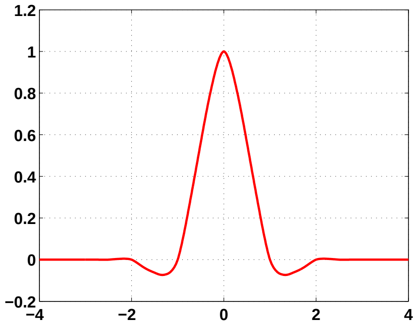
Corollary 5.3. *For $m \in \mathbb{N}$, there exists a unique refinable function ϕ_m^D such that (d_m, ϕ_m^D) is an interpolatory refinement pair, where $d_m = \{d_{m,j} : j \in \mathbb{Z}\} \in \mathcal{M}_0(\mathbb{Z})$ is the DD mask symbol defined by* (3.65)*, and with $\phi = \phi_m^D$ satisfying all the properties of Theorem 4.10, with $n = m$ and $a = d_m$. Moreover, the DD subdivision scheme (S_{d_m}, c) converges on $\mathcal{M}(\mathbb{Z})$ in the sense of Theorem 4.9.*

We shall call (d_m, ϕ_m^D) the DD refinement pair of order m, whereas ϕ_m^D shall be called the DD refinable function of order m.

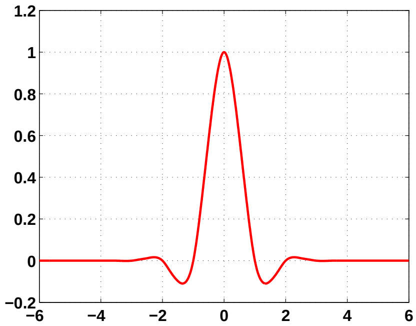
Using the explicit formulas in Section 3.6.1, together with the fact from Theorem 4.9 that the DD subdivision scheme (S_{d_m}, δ) converges to the DD refinable function ϕ_m^D in the sense of (4.35), with $\Phi = \phi_m^D$, we have drawn, in Figure 5.1, the function ϕ_m^D for $m = 1, 2, 3$ and 5. Observe in particular that Figure 5.1, graphically illustrate (4.36), (4.38) and (4.40) of Theorem 4.10 with $n = m$, $a = d_m$ and $\phi = \phi_m^D$. Also, the clustered zeros of ϕ_m^D, as implied by (4.41) are graphically illustrated for the case $m = 5$ by the zoom-in technique used in Figure 5.2.



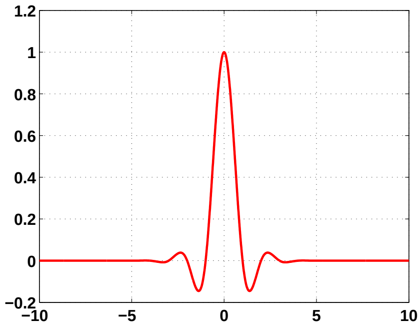
(a) $m = 1$



(b) $m = 2$

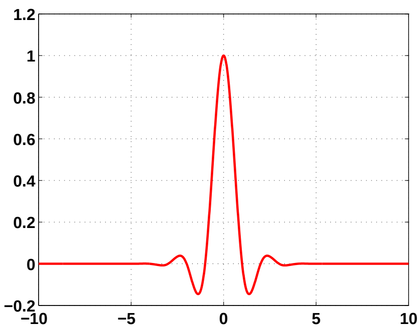


(c) $m = 3$

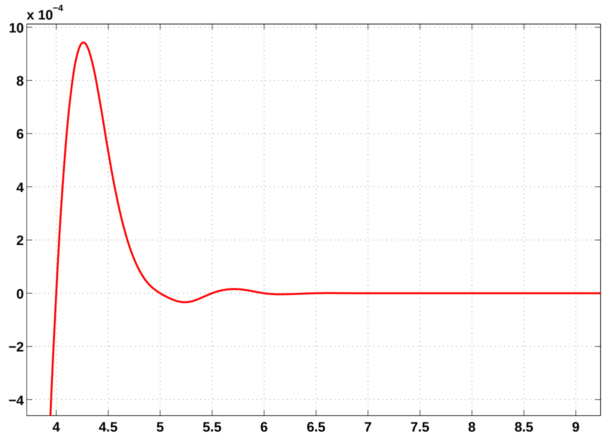


(d) $m = 5$

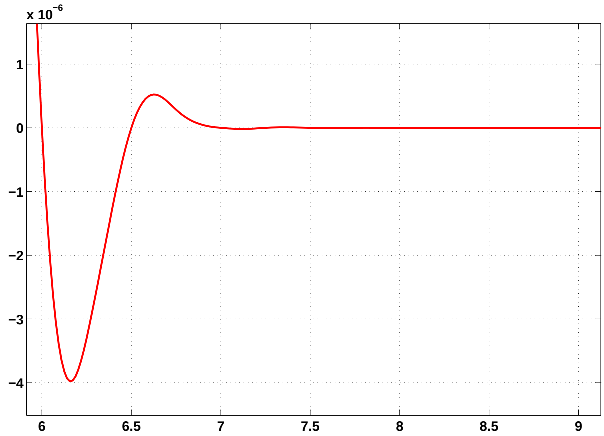
Figure 5.1: The DD refinable function ϕ_m^D for $m = 1, 2, 3$ and 5.



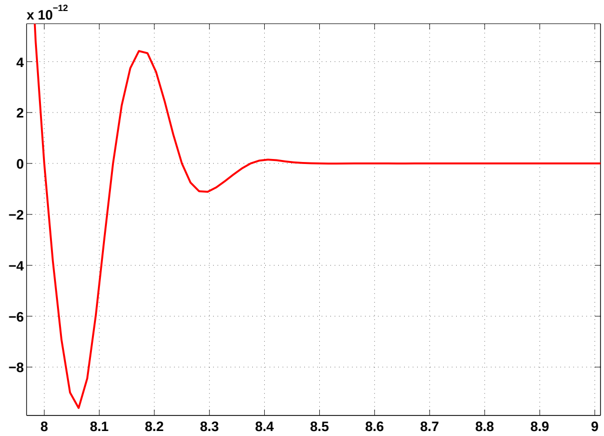
(a) ϕ_5^D with $\phi_5^D(x) = 0, \;\; x \notin (-9, 9)$



(b) ϕ_5^D on [4,9]



(c) ϕ_5^D on [6,9]



(d) ϕ_5^D on [8,9]

Figure 5.2: Graphical illustration of the clustered zeros property (4.41) of ϕ_5^D.

5.2 Positivity on the unit circle in $\mathbb{C}$ for $\mathscr{A}_{m,m+1}$

Next, we investigate the positivity condition (5.1) of Theorem 5.1 as it applies, for $m \in \mathbb{N}$, to the mask symbol class

$$\mathscr{A}_{m,m+1} = \left\{ A_m(t|\cdot) : \; t \in \mathbb{R} \backslash \left\{ 0, \frac{1}{2^{2m-1}} \binom{2m-1}{m-1} \right\} \right\}, \tag{5.5}$$

as was explicitly calculated in Section 3.6.2 by means of the representation formula (3.66) in Corollary 3.9.

Recalling also the relationship $Q = \frac{1}{2^{m-1}} P$ from the proof of Corollary 3.7, we deduce from (3.1) and (3.2) in Theorem 3.1, and with the parameter $t \in \mathbb{R} \backslash \left\{ 0, \frac{1}{2^{2m-1}} \binom{2m-1}{m-1} \right\}$ chosen as in Section 3.6.2, that

$$A_m(t|e^{ix}) = \frac{1}{2^{m-1}} \left(1 + \cos x\right)^m \left[\sum_{j \in \mathbb{Z}_{m-1}} \binom{m+j-1}{j} \left[\frac{1}{2} \left(1 - \cos x\right) \right]^j \right.$$
$$\left. + 2^{m-1} t \cos x \left(1 - \cos x\right)^m \right], \quad x \in \mathbb{R}, \tag{5.6}$$

or, equivalently,

$$A_m(t|e^{ix}) = \frac{1}{2^{m-1}} \left(1 + \cos x\right)^m \left[\sum_{j \in \mathbb{Z}_{m-1}} \binom{m+j-1}{j} \left(\sin^2 \frac{x}{2}\right)^j \right.$$
$$\left. + 2^{2m-1} t \left(1 - 2\sin^2 \frac{x}{2}\right) \left(\sin^2 \frac{x}{2}\right)^m \right], \quad x \in \mathbb{R}, \tag{5.7}$$

from which the following result is then immediately evident.

Proposition 5.4. *For $m \in \mathbb{N}$ and $t \in \mathbb{R} \backslash \left\{ 0, \frac{1}{2^{2m-1}} \binom{2m-1}{m-1} \right\}$, the mask symbol $A = A_m(t|\cdot) \in \mathscr{A}_{m,m+1}$ satisfies the positivity condition (5.1) of Theorem 5.1 if and only if the polynomial $p_m(t|\cdot)$ of degree $(m+1)$, as defined by*

$$p_m(t|x) = p_m(x) + 2^{2m-1} t x^m (1 - 2x), \quad x \in \mathbb{R}, \tag{5.8}$$

with p_m denoting the polynomial of degree $(m-1)$ of Proposition 3.5, satisfies the positivity condition

$$p_m(t|x) > 0, \quad x \in [0, 1). \tag{5.9}$$

We proceed to find the parameter interval for t such that the condition (5.9) holds.

We shall rely on the following properties of $p_m(t|\cdot)$. Statements on the number of zeros of a polynomial shall be given on the basis of also counting multiplicities. Our proof of Proposition 5.5 (d) below relies on a technique introduced in [17, Lemmas 2.4 and 2.5], (see also [18, Theorem 4.3]).

Proposition 5.5. *For $m \in \mathbb{N}$ and $t \in \mathbb{R}$, the polynomial $p_m(t|\cdot)$ defined by* (5.8) *satisfies the following properties:*

$$(a) \qquad p_m(t|0) = 1; \tag{5.10}$$

$$(b) \qquad p_m(t|1/2) = 2^{m-1}; \tag{5.11}$$

$$(c) \qquad p_m(t|1) = \binom{2m-1}{m-1} - 2^{2m-1}t; \tag{5.12}$$

(d) (i) $p_m(t|\cdot)$ *has at most two zeros in the interval* $(0,1/2)$;

(ii) $p_m(t|\cdot)$ *has at most one zero in the interval* $(1/2,1]$.

Proof. (a) The property (5.10) follows immediately from (5.8) and (3.31).

(b) Since, according to Proposition 3.5, we have

$$(1-x)^m p_m(x) + x^m p_m(1-x) = 1, \qquad x \in \mathbb{R}, \tag{5.13}$$

we can set $x = \frac{1}{2}$ in (5.13) to obtain

$$\left(\frac{1}{2}\right)^m p_m\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^m p_m\left(\frac{1}{2}\right) = 1,$$

which, together with (5.8), then yields (5.11).

(c) The property (5.12) is an immediate consequence of (5.8), (3.31) and (3.43).

(d) As can be verified directly from (5.8) and (5.13), we have that $p_m(t|\cdot)$ satisfies the Bezout identity

$$(1-x)^m p_m(t|x) + x^m p_m(t|1-x) = 1, \qquad x \in \mathbb{R}. \tag{5.14}$$

Hence, if we define the function $w_m(t|\cdot)$ by means of

$$w_m(t|x) = (1-x)^m p_m(t|x), \qquad x \in \mathbb{R}, \tag{5.15}$$

it follows that $w_m(t|\cdot)$ is a polynomial of degree $2m+1$ satisfying, according to (5.14) and (5.15), the Bezout identity

$$w_m(t|x) + w_m(t|1-x) = 1, \qquad x \in \mathbb{R}. \tag{5.16}$$

Differentiation of (5.16) with respect to x then shows that the polynomial $w'_m(t|\cdot)$ of degree $2m$ satisfies the Bezout identity

$$w'_m(t|x) = w'_m(t|1-x), \qquad x \in \mathbb{R}. \tag{5.17}$$

We see from (5.15), (5.10), (5.11) and (5.12) that

$$w_m(t|0) = 1, \tag{5.18}$$

$$w_m(t|1/2) = 1/2, \tag{5.19}$$

$$w_m(t|1) = 0, \tag{5.20}$$

whereas (5.15) and (5.17) show that the derivative $w'_m(t|\cdot)$ possesses a zero of multiplicity $(m-1)$ at both $x=1$ and $x=0$.

(i) Suppose that $p_m(t|\cdot)$ has at least three zeros in $(0,1/2)$, according to which (5.15) then shows that $w_m(t|\cdot)$ has at least three zeros in $(0,1/2)$. It follows from Rolle's theorem, together with (5.18) and (5.19), that the derivative $w'_m(t|\cdot)$ has at least two zeros in $(0,1/2)$, and thus $w'_m(t|\cdot)$ has at least $(m-1)+2 = m+1$ zeros in $[0,1/2)$.

But then (5.17) implies that $w'_m(t|\cdot)$ has at least $(m+1)$ zeros in $(1/2,1]$. In total therefore, $w'_m(t|\cdot)$ has at least $(2m+2)$ zeros in $[0,1]$, which is not possible, since $w'_m(t|\cdot)$ is a polynomial of degree $2m$. Hence $p_m(t|\cdot)$ has at most two zeros in $(0,1/2)$.

(ii) Finally, suppose $p_m(t|\cdot)$ has at least two zeros in $(1/2,1]$. It follows from (5.15) that then $w_m(t|\cdot)$ has at least $(m+2)$ zeros in $(1/2,1]$, including a zero of multiplicity at least m at $x=1$. Keeping in mind also (5.19), and using Rolle's theorem if $w_m(t|\cdot)$ has at least one zero in $(1/2,1)$, we deduce that the derivative $w'_m(t|\cdot)$ has at least $(m+1)$ zeros in $(1/2,1]$, at least $(m-1)$ of which are at $x=1$.

But then (5.17) shows that the $2m$–th degree polynomial $w'_m(t|\cdot)$ possesses at least $(2m+2)$ zeros in $[0,1]$, yielding a contradiction as before. Hence $p_m(t|\cdot)$ has at most one zero in $(1/2,1]$. ■

A graphical illustration of Proposition 5.5 is provided in Figure 5.3, where the polynomial $p_m(t|\cdot)$ is drawn on the interval $[0,1]$ for $m=1$, 2, 3, 4, and for the indicated values of the parameter t.

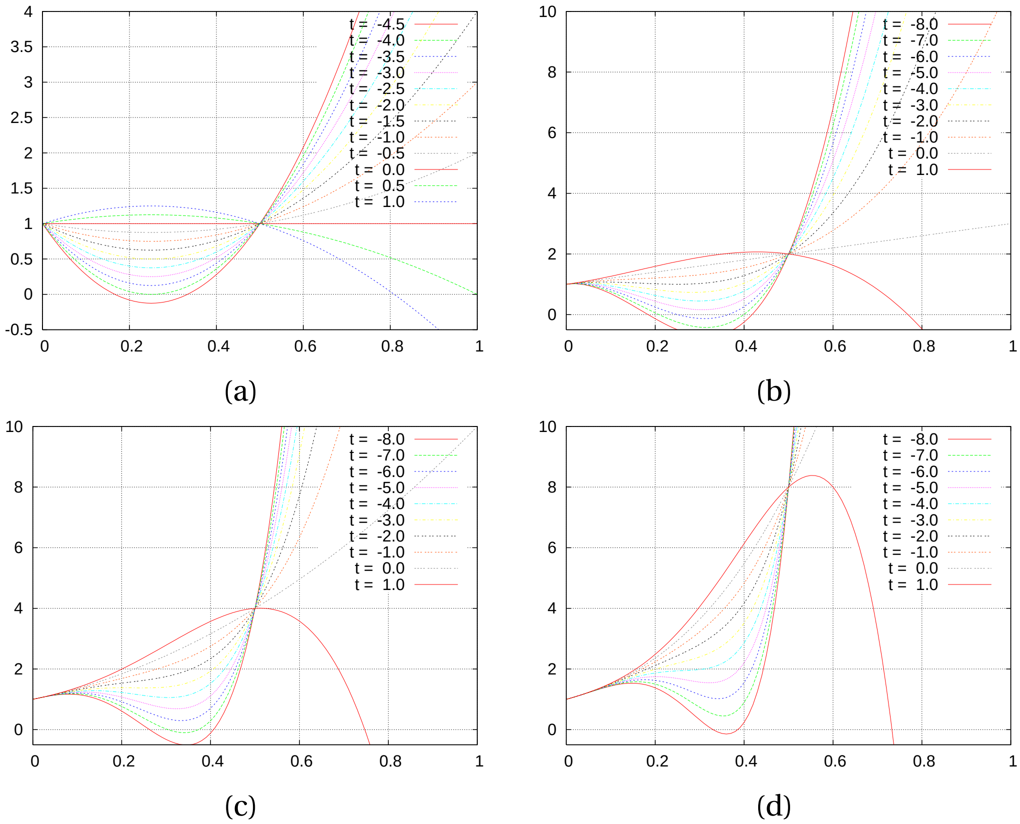


Figure 5.3: The polynomials (a) $p_1(t|\cdot)$, $t \in (-4.5, 1)$, (b) $p_2(t|\cdot)$, $t \in (-8, 1)$, (c) $p_3(t|\cdot)$, $t \in (-8, 1)$, and (d) $p_4(t|\cdot)$, $t \in (-8, 1)$.

We can now use Proposition 5.5 to prove the following result.

Theorem 5.6. In Proposition 5.4, the positivity condition (5.9) holds if and only if

$$t \in \big(-t_m, \tilde{t}_m\big], \tag{5.21}$$

where t_m is the positive number defined by

$$t_m = \frac{p_m(x_m)}{2^{2m-1} x_m^m (1-2x_m)}, \tag{5.22}$$

with x_m denoting the unique zero in $(0, 1/2)$ of the polynomial q_m defined by

$$q_m(x) = [m - 2(m+1)x]\, p_m(x) - x(1-2x)\, p_m'(x), \qquad x \in \mathbb{R}, \tag{5.23}$$

and where $\tilde{t}_m$ is the positive number defined by

$$\tilde{t}_m = \frac{1}{2^{2m-1}} \binom{2m-1}{m-1}. \tag{5.24}$$

Proof. First observe from (5.8) that, for a fixed $x \in [0,1]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} p_m(t|x) = 2^{2m-1} x^m (1-2x) \begin{cases} > 0, & \text{for } x \in (0,1/2), \qquad (5.25a) \\ \\ < 0, & \text{for } x \in (1/2,1]. \qquad (5.25b) \end{cases}$$

Combining the results (5.11), (5.12), (5.25b) and Proposition 5.5 (d)(ii), we conclude that

$$p_m(t|x) > 0, \quad x \in [1/2,1), \tag{5.26}$$

if and only if $t \in (-\infty, \tilde{t}_m]$, with $\tilde{t}_m$ defined by (5.24). Our proof will therefore be complete if we can show that

$$p_m(t|x) > 0, \quad x \in [0,1/2), \tag{5.27}$$

if and only if

$$t \in (-t_m, \infty), \tag{5.28}$$

with t_m denoting the positive number as in (5.22).

To this end, we first observe from (5.8), (5.10), (5.11), Proposition 5.5 (d)(i), as well as (5.25a), together with the fact that, according to (3.31), we have

$$p_m(0|x) = p_m(x) > 0, \quad x \in [0,1/2], \tag{5.29}$$

that there exists a unique solution pair $(x,t) = (x_m, t_m^*)$ in $(0,1/2) \times (-\infty, 0)$ of the system of equations given by

$$p_m(t|x) = 0, \qquad p_m'(t|x) = 0, \tag{5.30}$$

or, equivalently, the system of equations

$$\begin{cases} p_m(x) + 2^{2m-1} t x^m (1-2x) = 0, & (5.31a) \\ \\ p_m'(x) + 2^{2m-1} t x^{m-1} \Big[m - 2(m+1)x \Big] = 0, & (5.31b) \end{cases}$$

and where (5.27) holds if and only if the condition (5.28) is satisfied with the positive number t_m defined by $t_m = -t_m^*$.

It remains to prove that t_m is then given by the formula (5.22). To this end, we first use the fact that the pair (x_m, t_m) satisfies the system of equations

$$\begin{cases} p_m(x_m) + 2^{2m-1} t_m x_m^m (2x_m - 1) = 0, & (5.32a) \\ \\ p_m'(x_m) + 2^{2m-1} t_m x_m^{m-1} \Big[2(m+1)x_m - m \Big] = 0, & (5.32b) \end{cases}$$

from which we now eliminate t_m to deduce that the polynomial q_m defined by (5.23) indeed possesses a zero $x_m \in (0,1/2)$.

To prove our claim that x_m is the unique zero of q_m in $(0,1/2)$, suppose $\hat{x}_m \in (0,1/2)$ satisfies $q_m(\hat{x}_m) = 0$, so that (5.23) gives

$$\Big[m-2(m+1)\hat{x}_m\Big]p_m(\hat{x}_m) = \hat{x}_m(1-2\hat{x}_m)p'_m(\hat{x}_m). \tag{5.33}$$

If we now define the number $\hat{t}_m$ by

$$\hat{t}_m = \frac{p_m(\hat{x}_m)}{2^{2m-1}\hat{x}_m^m(2\hat{x}_m-1)}, \tag{5.34}$$

then $\hat{t}_m < 0$, and

$$p_m(\hat{x}_m) + 2^{2m-1}\hat{t}_m\hat{x}_m^m(1-2\hat{x}_m) = 0,$$

whereas

$$p'_m(\hat{x}_m) + 2^{2m-1}\hat{t}_m\hat{x}_m^{m-1}[m-2(m+1)\hat{x}_m] = p'_m(\hat{x}_m) + \frac{[m-2(m+1)\hat{x}_m]p_m(\hat{x}_m)}{\hat{x}_m(2\hat{x}_m-1)}. \tag{5.35}$$

But, since $q_m(\hat{x}_m) = 0$, it follows from (5.23) that

$$p'_m(\hat{x}_m) = \frac{[m-2(m+1)\hat{x}_m]p_m(\hat{x}_m)}{\hat{x}_m(1-2\hat{x}_m)}. \tag{5.36}$$

Substituting (5.36) into (5.35) then yields

$$p'_m(\hat{x}_m) = 2^{2m-1}\hat{t}_m\hat{x}_m^{m-1}[m-2(m+1)\hat{x}_m] = 0. \tag{5.37}$$

Hence $(x,t) + (\hat{x}_m, \hat{t}_m) \in (0,1/2)\times(-\infty,0)$ is a solution pair of the system of equations (5.31a), (5.31b). But we have already established that $(x,t) = (x_m, t_m^*)$ is the unique solution in $(0,1/2)\times(-\infty,0)$ of the system of equations (5.31a), (5.31b). Hence $\hat{x}_m = x_m$, and $\hat{t}_m = t_m^*$, and thus x_m is the unique zero in $(0,1/2)$ of the polynomial q_m. ■

A graphical illustration of the fact that the polynomial q_m possesses a unique zero in $(0,1/2)$ is provided in Figure 5.4, where we have drawn q_m on the interval $[0,1/2]$ for $m = 1, 2, \ldots, 7$.

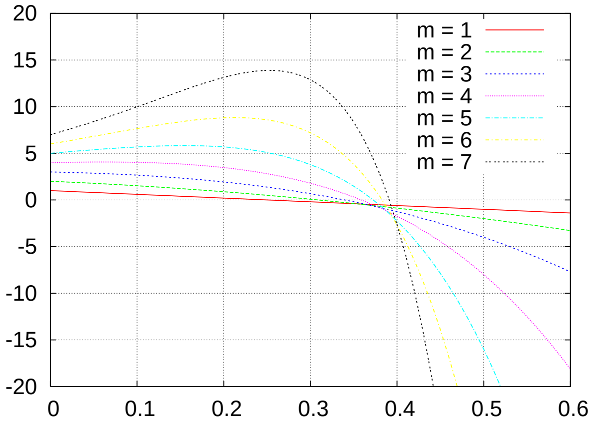


Figure 5.4: The polynomial $q_m(\cdot)$ for $m = 1, 2, \ldots, 7$.

Combining the results of Proposition 5.4 and Theorem 5.6, we immediately obtain the following result.

Corollary 5.7. *For $m \in \mathbb{N}$ and $t \in \mathbb{R}\backslash\{0, \tilde{t}_m\}$, the mask symbol $A = A_m(t|\cdot) \in \mathscr{A}_{m,m+1}$ satisfies the positivity condition* (5.1) *of Theorem 5.1 if and only if the parameter t satisfies the condition* (5.21) *of Theorem 5.6.*

We can now combine the results of Corollary 3.9, Theorem 5.1, Corollary 5.7 and Theorem 4.10, to deduce the following theorem.

Theorem 5.8. For $m \in \mathbb{N}$, and with the parametrisation

$$A_m(t|z) = D_m(z) + t\left(\frac{z+z^{-1}}{2}\right)\left[1-\left(\frac{z+z^{-1}}{2}\right)^2\right]^m, \quad z \in \mathbb{C}\backslash\{0\}, \quad t \in \mathbb{R}, \quad (5.38)$$

with D_m denoting the DD mask symbol of order m, of the mask symbols $A = A_m(t|\cdot) \in \mathscr{A}_{m,m+1}$ if $t \in \mathbb{R}\backslash\{0, \tilde{t}_m\}$, $A = A_m(0|\cdot) = D_m \in \mathscr{A}_{m,m}$, and $A = A_m(\tilde{t}_m|\cdot) = D_{m+1} \in \mathscr{A}_{m+1,m+1}$, there exists a function $\phi_m(t|\cdot) \in \mathscr{C}_0(\mathbb{R})$ such that $\left(a^{(m)}(t), \phi_m(t|\cdot)\right)$ is an interpolatory refinement pair with respect to the mask symbol $A_m(t|\cdot)$ if $t \in (-t_m, \tilde{t}_m]$, with the positive numbers t_m and $\tilde{t}_m$ defined as in Theorem 5.6, and where $\phi_m(0|\cdot) = \phi_m^D$, $\phi_m(\tilde{t}_m|\cdot) = \phi_{m+1}^D$, the DD refinable function of respective orders m and $m+1$. Moreover, all the results of Theorem 4.10 hold if we choose there $n = m+1$ and $\phi = \phi_m(t|\cdot)$.

In the context of interpolatory subdivision, the parameter t in Theorem 5.8 can be interpreted as a shape parameter with respect to the limit curve Φ in (4.34) of Theorem 4.9.

Observe also that the results $\phi_m(0|\cdot) = \phi_m^D$ and $\phi_m(\tilde{t}_m|\cdot) = \phi_{m+1}^D$ in Theorem 5.8 above are consistent with the alternative representation formulation of (3.68) and (3.69) if we set there $n = m+1$, and choose, respectively, $(t_0, t_1) = (1, 0)$ and $(t_0, t_1) = (0, 1)$.

5.3 Examples

We proceed to explicitly calculate the parameter interval (5.21) in Theorem 5.6.

5.3.1 The case $m = 1$

Using the notation of Section 5.2, we have here, from (3.31), that

$$p_1(x) = 1, \quad x \in \mathbb{R}, \tag{5.39}$$

so that (5.23) and (5.39) give

$$q_1(x) = 1 - 4x, \quad x \in \mathbb{R}.$$

Hence $x_1 = 1/4$, and thus, from (5.22), we get

$$t_1 = 4.$$

Also, (5.24) gives

$$\tilde{t}_1 = 1/2.$$

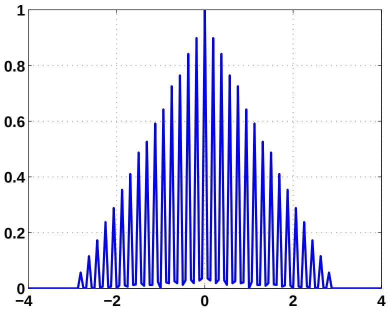
We deduce from Corollary 5.7 that the class of masks $\{A_1(t|\cdot) : \ t \in \mathbb{R}\}$, as given by (3.75), yields refinable function existence and resulting subdivision convergence if

$$t \in (-4, 1/2].$$

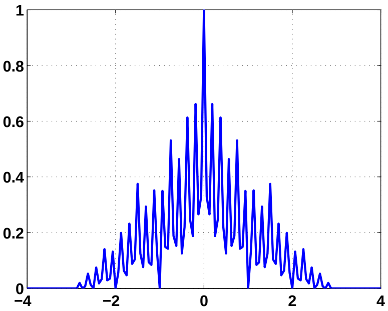
The corresponding interpolatory refinable function $\phi_1(t|\cdot)$ of Theorem 5.8, as calculated by use of the fact from Theorem 4.9 that the interpolatory subdivision scheme based on the mask symbol $A_1(t|\cdot)$ converges in the sense of (4.35) to $\phi_1(t|\cdot)$ if we choose $c = \delta$ in (1.2), is shown graphically in Figure 5.5 for the values $t = -3.9, \ -3, \ -2, \ -1, \ 0, \ 0.25, \ 0.5$, where the equivalences $\phi_1(0|\cdot) = \phi_1^D$

and $\phi_1(1/2|\cdot) = \phi_2^D$ can be seen by comparing Figure 5.7(e), (g) with respectively, Figure 5.1(a), (b). In addition, we graphically illustrate in Figures 5.6, 5.7 and 5.8 convergence of the corresponding interpolatory subdivision scheme for the above values of t.

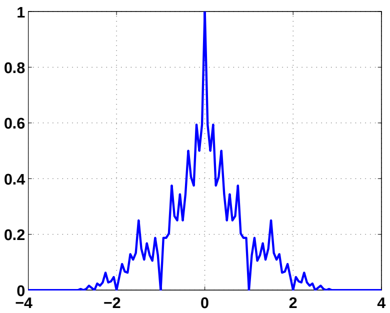
Observe in particular from Figure 5.5(a) and Figure 5.6(d) that it is impossible to conjecture about refinable function existence and subdivision convergence on the basis of only numerical/graphical information. This fact emphasizes the importance of the theoretical existence and convergence result Theorem 5.8, according to which we can see that $t = -3.9 \in (-4, 1/2]$, the existence and convergence t-interval for $m = 1$.



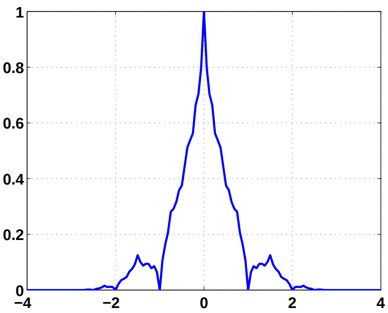
(a) $t = -3.9$



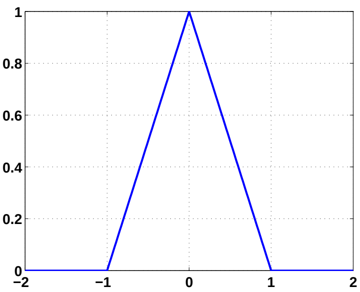
(b) $t = -3$



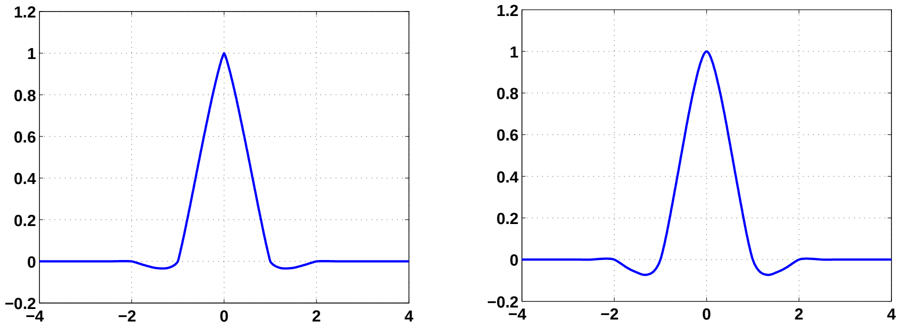
(c) $t = -2$



(d) $t = -1$

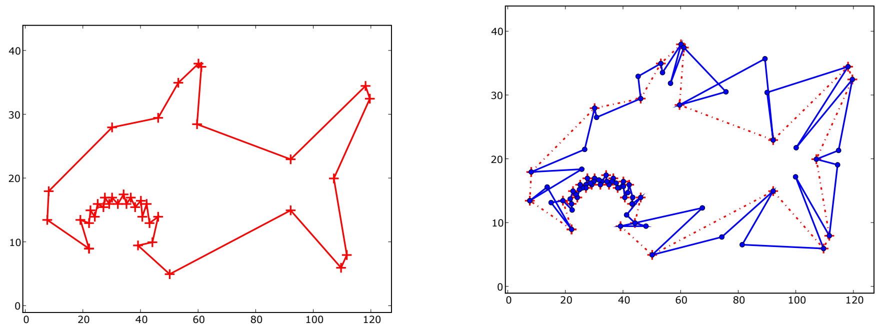


(e) $t = 0$



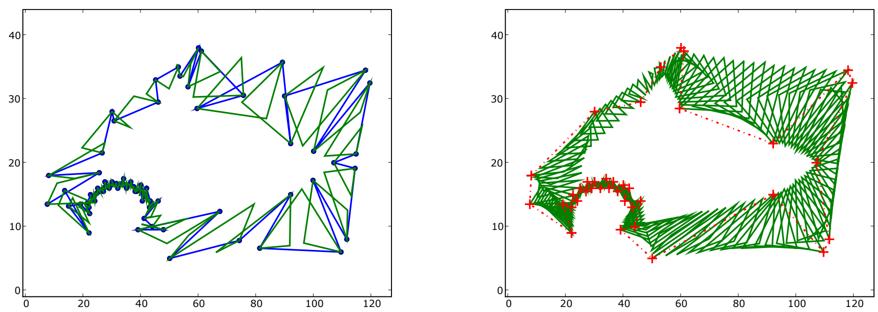
(f) $t = 0.25$ (g) $t = 0.5$

Figure 5.5: The refinable functions (a) $\phi_1(-3.9|\cdot)$, (b) $\phi_1(-3|\cdot)$, (c) $\phi_1(-2|\cdot)$, (d) $\phi_1(-1|\cdot)$, (e) $\phi_1(0|\cdot)$, (f) $\phi_1(0.25|\cdot)$ and (g) $\phi_1(0.5|\cdot)$.



(a) $r = 0$: $c = c^{(0)} = (+)$

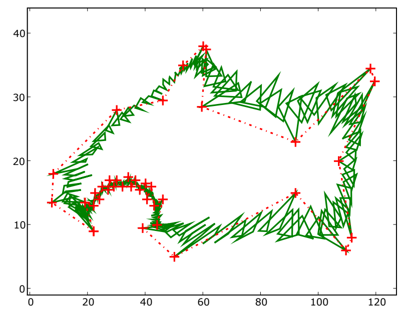
(b) $r = 0$ and $r = 1$:$c^{(0)} = (+)$, $c^{(1)} = (o)$



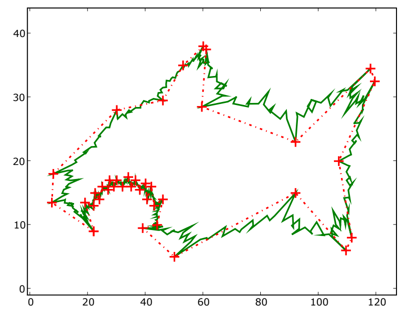
(c) $r = 1$ and $r = 2$: $c^{(1)} = (o)$, $c^{(2)} = (-)$

(d) $r = 0$ and $r = 4$: $c^{(0)} = (+)$, $c^{(4)} = (-)$

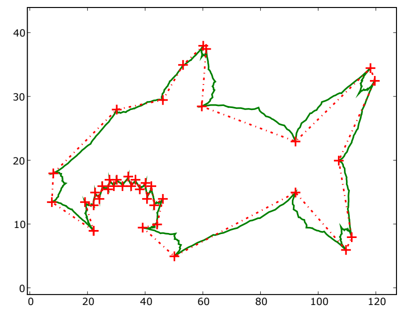
Figure 5.6: Illustration of the convergence of the interpolatory subdivision scheme with mask symbol $A_1(-3.9|\cdot)$, where r denotes the iteration level of the subdivision scheme.



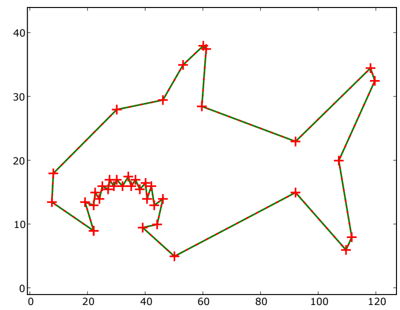
(a) $t = -3$



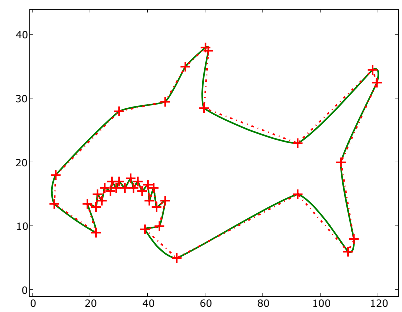
(b) $t = -2$



(c) $t = -1$

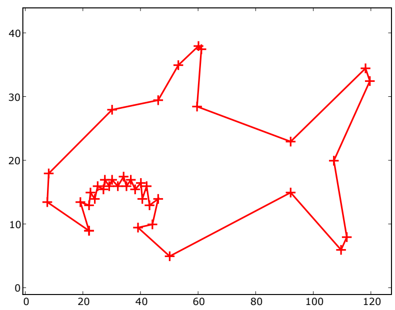


(d) $t = 0$

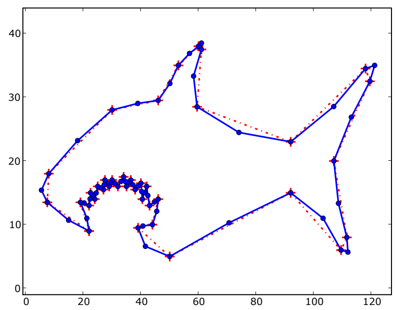


(e) $t = 0.25$

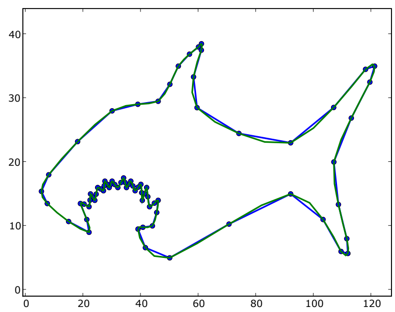
Figure 5.7: The limit curves produced by the convergent interpolatory subdivision scheme with mask symbol $A_1(t|\cdot)$ for $t = -3, -2, -1, 0, 0.25$.



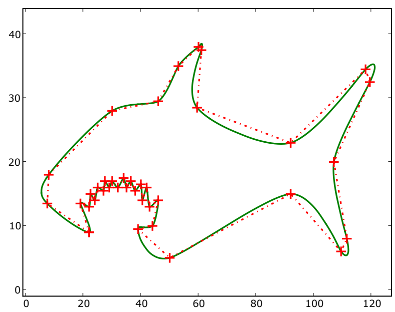
(a) $r = 0$: $c = c^{(0)} = (+)$



(b) $r = 0$ and $r = 1$:$c^{(0)} = (+)$, $c^{(1)} = (o)$



(c) $r = 1$ and $r = 2$: $c^{(1)} = (o)$, $c^{(2)} = (-)$



(d) $r = 0$ and $r = 4$: $c^{(0)} = (+)$, $c^{(4)} = (-)$

Figure 5.8: Illustration of the convergence of the interpolatory subdivision scheme with mask symbol $A_1(0.5|\cdot)$, where r denotes the iteration level of the subdivision scheme.

5.3.2 The case $m = 2$

From (3.31), we get here

$$p_2(x) = 1 + 2x, \quad x \in \mathbb{R}, \tag{5.40}$$

so that (5.23) and (5.40) yield

$$q_2(x) = -2(4x^2 + 2x - 1), \quad x \in \mathbb{R},$$

and thus

$$x_2 = \frac{\sqrt{5}-1}{4} \approx 0.3090,$$

so that, from (5.22),

$$\tilde{t}_2 = \frac{p_2\left(\frac{\sqrt{5}-1}{4}\right)}{8\left(\frac{\sqrt{5}-1}{4}\right)^2\left[1-2\left(\frac{\sqrt{5}-1}{4}\right)\right]} = \frac{1}{4}\left(11+5\sqrt{5}\right) \approx 5.5451.$$

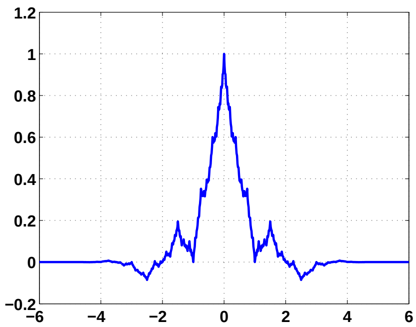
Morover, (5.24) gives

$$\tilde{t}_2 = 3/8.$$

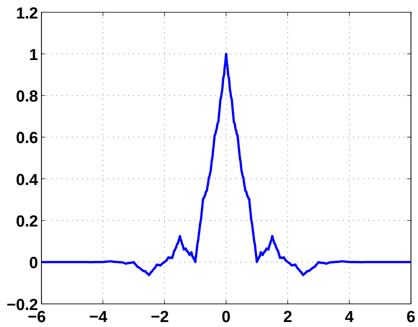
Hence the class of masks $\{A_2(t|\cdot): \ t \in \mathbb{R}\}$, as given by (3.76), yields refinable function existence and subdivision convergence if

$$t \in \left(-\frac{11+5\sqrt{5}}{4}, \frac{3}{8}\right] \approx (-5.5451, 0.3750].$$

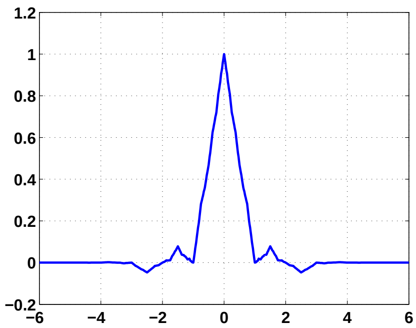
Similarly to the graphs in Section 5.3.1 above, we show in Figure 5.9–5.12 the graphs of the refinable function $\phi_2(t|\cdot)$ of Theorem 5.8 for $t = -5.5, \ -4, \ -3, -2, -1, \ 0, \ 0.2$ and 0.375 as well as graphical illustration of the corresponding subdivision convergence.



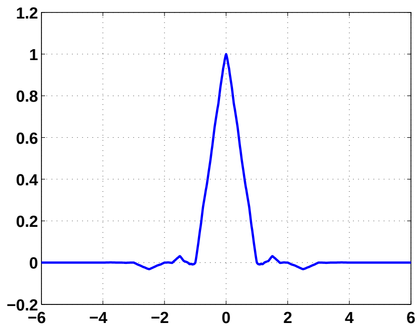
(a) $t = -5.5$



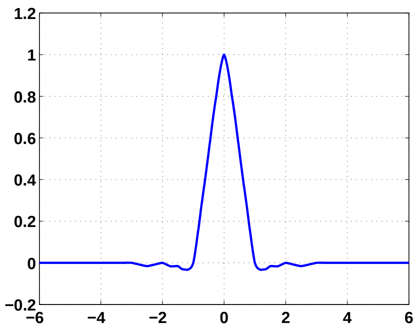
(b) $t = -4$



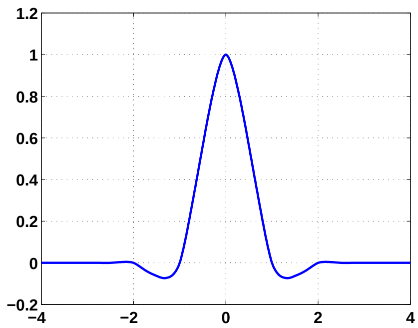
(c) $t = -3$



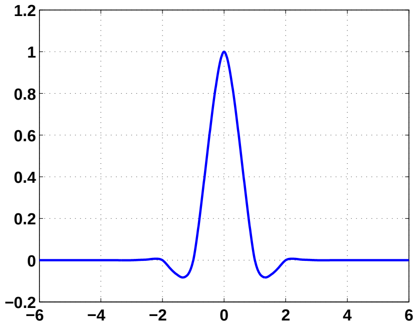
(d) $t = -2$



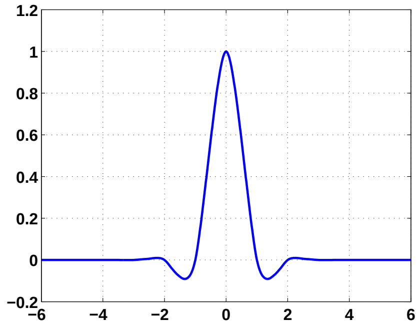
(e) $t = -1$



(f) $t = 0$

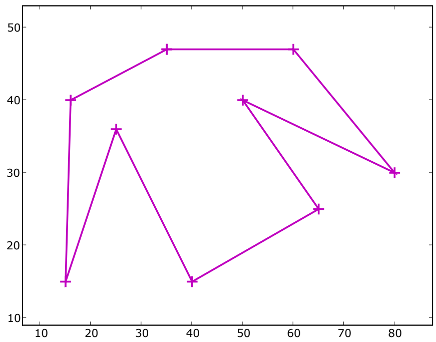


(g) $t = 0.2$

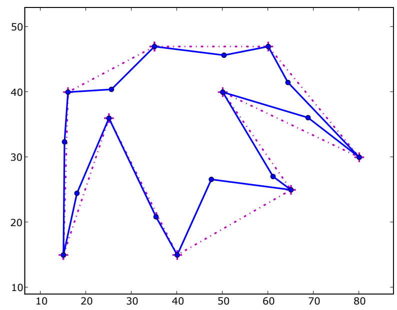


(h) $t = 0.375$

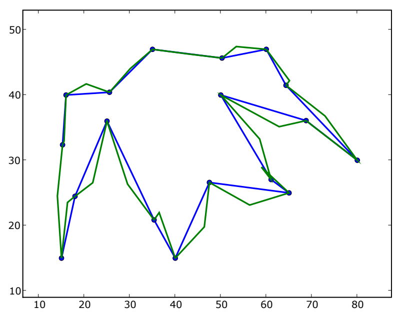
Figure 5.9: The refinable functions (a) $\phi_2(-5.5|\cdot)$, (b) $\phi_2(-4|\cdot)$, (c) $\phi_2(-3|\cdot)$, (d) $\phi_2(-2|\cdot)$, (e) $\phi_2(-1|\cdot)$, (f) $\phi_2(0|\cdot)$, (g) $\phi_2(0.2|\cdot)$ and (f) $\phi_2(0.375|\cdot)$.



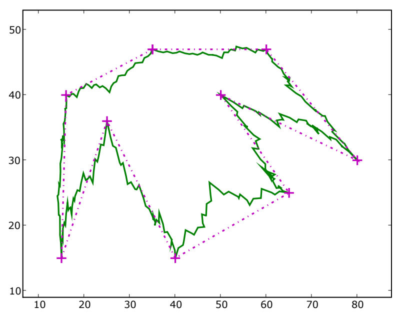
(a) $r = 0 : c = c^{(0)} = (+)$



(b) $r = 0$ and $r = 1 : c^{(0)} = (+)$, $c^{(1)} = (o)$

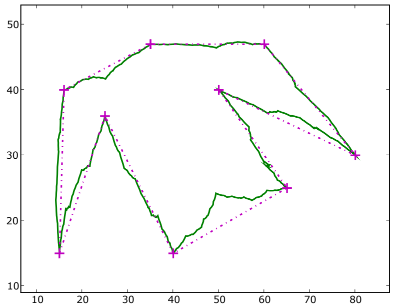


(c) $r = 1$ and $r = 2 : c^{(1)} = (o)$, $c^{(2)} = (-)$

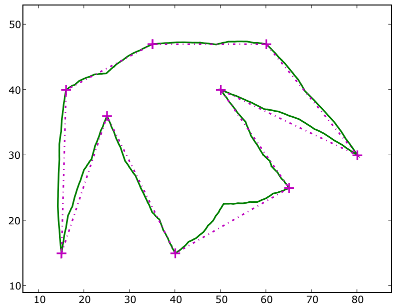


(d) $r = 0$ and $r = 4 : c^{(0)} = (+)$, $c^{(6)} = (-)$

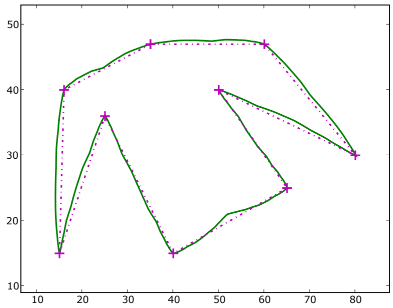
Figure 5.10: Illustration of the convergence of the interpolatory subdivision scheme with mask symbol $A_2(-5.5|\cdot)$, where r denotes the iteration level of the subdivision scheme.



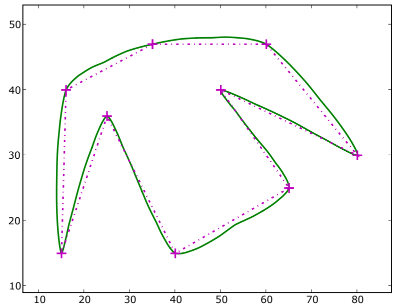
(a) $t = -4$



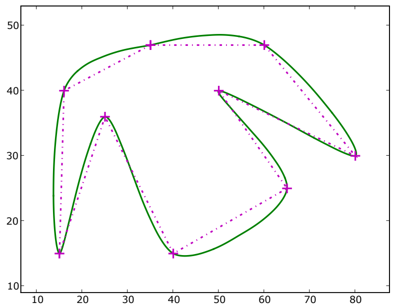
(b) $t = -3$



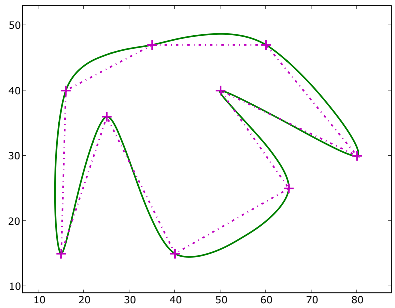
(c) $t = -2$



(d) $t = -1$

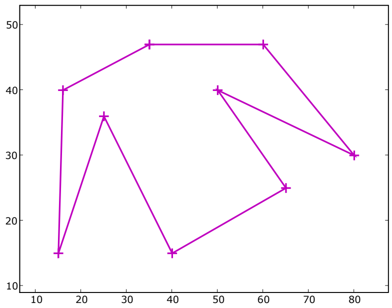


(e) $t = 0$

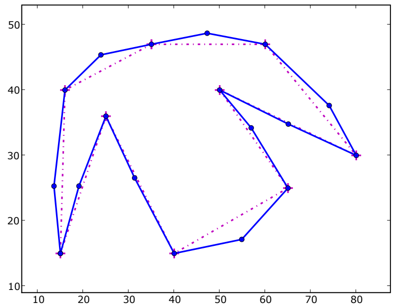


(d) $t = 0.2$

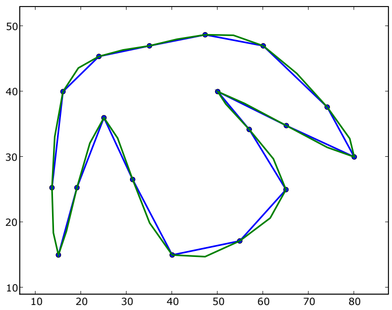
Figure 5.11: The limit curves produced by the convergent interpolatory subdivision scheme with mask symbol $A_2(t|\cdot)$ for $t = -4,\ -3,\ -2,\ -1,\ 0,\ 0.2$.



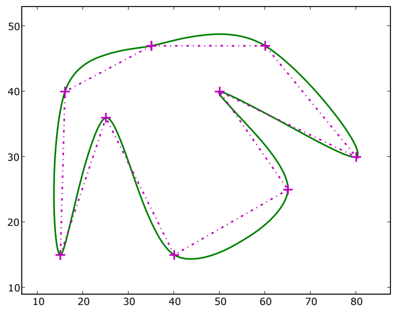
(a) $r = 0 : c = c^{(0)} = (+)$



(b) $r = 0$ and $r = 1 : c^{(0)} = (+)$, $c^{(1)} = (o)$



(c) $r = 1$ and $r = 2 : c^{(1)} = (o)$, $c^{(2)} = (-)$



(d) $r = 0$ and $r = 4 : c^{(0)} = (+)$, $c^{(6)} = (-)$

Figure 5.12: Illustration of the convergence of the interpolatory subdivision scheme with mask symbol $A_2(0.375|\cdot)$, where r denotes the iteration level of the subdivision scheme.

5.3.3 A table for x_m, t_m and $\tilde{t}_m$ for $m \in \{1, 2, \ldots, 10\}$

Continuing as above, we obtain, from the formulas in Theorem 5.6, the following table of values, for x_m, t_m and $\tilde{t}_m$, in which the numerical values are given to 4 decimal accuracy:

Table 5.1: The values of x_m, $\tilde{t}_m$ and $\tilde{t}_m$

m	x_m	t_m	$\tilde{t}_m$
1	0.2500	4.0000	0.5000
2	0.3090	5.5451	0.3750
3	0.3394	6.7423	0.3125
4	0.3588	7.7559	0.2734
5	0.3725	8.6512	0.2461
6	0.3828	9.4620	0.2256
7	0.3910	10.2885	0.2090
8	0.3977	10.9039	0.1964
9	0.4033	11.5575	0.1855
10	0.4080	12.1761	0.1762

5.4 A comparison with a result from the literature

In [9, Proposition 3], (see also [19, Lemma 4.1], and [22]), De Villiers and Hunter parametrized, for $m \in \mathbb{N}$, and by using also the representation formula (3.68), (3.69) with $n = m + 1$, the mask symbol $A_m(\tau|\cdot)$ on the unit circle by means of

$$A_m(\tau|e^{ix}) = D_m(e^{ix}) + \frac{1}{2^{2m}}\binom{2m}{m}\tau\cos x(\sin x)^{2m}, \quad x,\ \tau \in \mathbb{R}. \tag{5.41}$$

Now observe that our formula (5.6) can be rewritten, according to (3.66) in Corollary 3.9, together with (3.2), as

$$A_m(\tau|e^{ix}) = D_m(e^{ix}) + t\cos x(\sin x)^{2m}, \quad x,\ \tau \in \mathbb{R}. \tag{5.42}$$

Comparing (5.41) and (5.42), we deduce that the parameter t and τ in the two representations satisfies the relationship

$$t = \frac{1}{2^{2m}}\binom{2m}{m}\tau, \tag{5.43}$$

or, equivalently,

$$\tau = \frac{2^{2m}}{\binom{2m}{m}} t.$$

Note in particular from (5.24) that

$$\frac{2^{2m}}{\binom{2m}{m}} \tilde{t}_m = \frac{2^{2m}}{\binom{2m}{m} 2^{2m-1}} \binom{2m-1}{m-1} = 1.$$

It therefore follows from our Theorem 5.6, together with (5.43), that, with $A_m(\tau|\cdot)$ parametrised as in (5.41), the positivity condition (5.1) of Theorem 5.1 is satisfied if and only if

$$\tau \in (-\tau_m, 1],$$

where

$$\tau_m = \frac{1}{\binom{2m-1}{m-1}} \frac{p_m(x_m)}{x_m^m(1-2x_m)}.$$

The parameter interval for τ proved in [9, Proposition 3] for the positivity condition (5.1) in Theorem 5.1 to hold for $A = A_m(\tau|\cdot)$ was

$$\tau \in (-\hat{\tau}_m, 1],$$

where the positive number $\hat{\tau}_m$ is given by

$$\hat{\tau}_m = 2m.$$

Using the numerical results of Sections 5.3.1 and 5.3.2, together with (5.43), we find that

$$8 = \tau_1 > 2 = \hat{\tau}_1,$$

and

$$14.7869 \approx \frac{2}{3}\left(11 + 5\sqrt{5}\right) = \tau_2 > 4 = \hat{\tau}_2.$$

Since our estimates $-t_m < t < \tilde{t}_m$ are sharp with respect to the bounds on the parameter t for which the positivity condition (5.1) in Theorem 5.1 holds, whereas the left hand inequality in the estimates $-2m \leq \tau \leq 1$ of [9] and [22] is not sharp, it is not suprising to find, in Table 5.2, that $\tau_m > \hat{\tau}_m$, $m \in \mathbb{N}$, i.e. our precise method based on Theorem 5.6 considerably improves the left hand endpoint of the admissible parameter interval for t.

Table 5.2: The numerical values for τ_m and $\hat{t}_m$

m	τ_m	$\hat{t}_m$
1	8.0000	2
2	14.7869	4
3	21.5752	6
4	28.3644	8
5	35.1541	10
6	41.9440	12
7	48.7341	14
8	55.5243	16

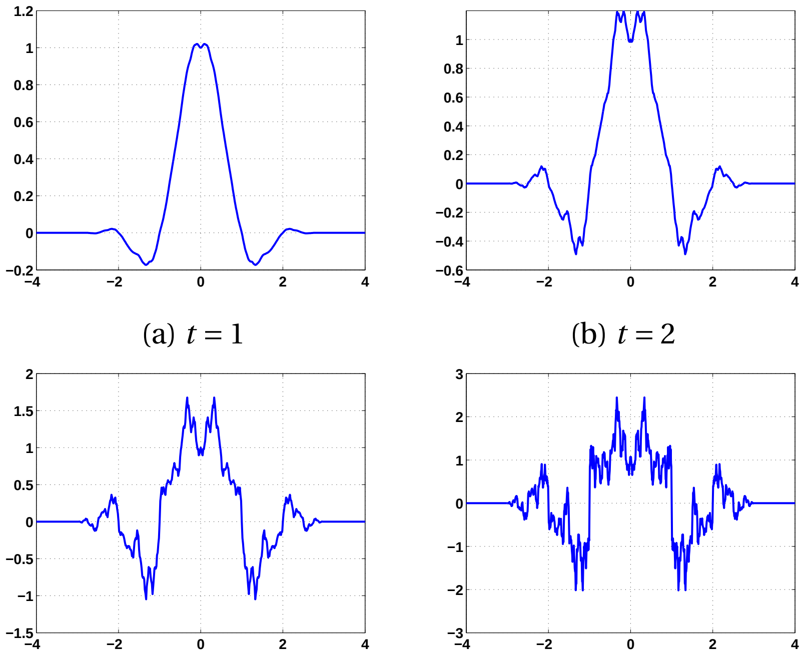
5.5 Results from the literature for the case where $A_m(t|\cdot)$ has zeros on the unit circle in $\mathbb{C}$

As shown in [9, p 167], (see also [19, Proposition 4.2]) the mask symbol $A_m(\tau|\cdot)$, as in (5.41), has exactly two zeros on the unit circle in $\mathbb{C}$ for $\tau > 1$, i.e. the positivity condition (5.1) in Theorem 5.1 does not hold for $A = A_m(\tau|\cdot)$ if $\tau > 1$, or, equivalently, for $A = A_m(t|\cdot)$ if $t > \tilde{t}_m$. It has been shown, however, in existing literature, that a corresponding refinable function $\phi_m(\tau|\cdot)$ does indeed exist for certain τ-intervals in which the right hand end point is larger than one.

For example, as noted [19, p 84], it has been shown in [1, Example 3.1] and [15] that the mask symbol $A_1(\tau|\cdot)$ has a corresponding refinable function $\phi_1(\tau|\cdot)$ for $\tau \in (-8,8)$, or, equivalently, from (5.43), $\phi_1(t|\cdot)$ for $t \in (-4,4)$. Observe from our Section 5.4 that $\tau_1 = 8$, i.e. our method, as based on Theorem 5.6, yields the same left hand endpoint $(= -8)$ as did the above mentioned work of [1] and [15].

Regarding the case $m = 2$, which, to our knowledge, has only be studied in [19], an existence result in [19, Theorem 4.8], the proof of which is based on the cascade algorithm, yields, according to [19, p 85], the result that the mask symbol $A_2(\tau|\cdot)$ has a corresponding refinable function $\phi_2(\tau|\cdot)$ for $\tau \in (-32/3, 16/3) \approx (-10.6667, 5.3333)$, or, equivalently, from (5.43), $\phi_2(t|\cdot)$ for $t \in (-4,2)$. Observe from our Section 5.4 that $\tau_2 = \frac{2}{3}\left(11 + 5\sqrt{5}\right) \approx 14.7869$, i.e. our left hand endpoint ≈ -14.7869 of the τ-interval for convergence improves on its counterpart $(= -32/3)$, as obtained by the methods of [19].

In Figures 5.13–5.16, we show the graph of the refinable function $\phi_1(t|\cdot)$ for $t = 1,\ 2,\ 3,\ 3.9$ and the refinable function $\phi_2(t|\cdot)$ for $t = 1,\ 1.5,\ 1.9$, as well as the limit curves for the corresponding subdivision schemes.



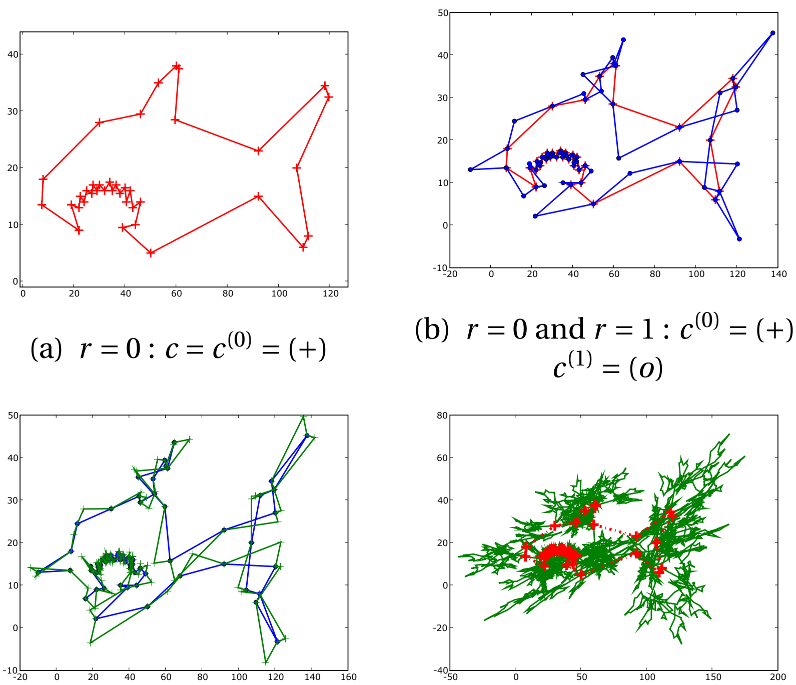
(a) $t = 1$

(b) $t = 2$

(c) $t = 3$

(d) $t = 3.9$

Figure 5.13: The refinable functions (a) $\phi_1(1|\cdot)$, (b) $\phi_1(2|\cdot)$, (c) $\phi_1(3|\cdot)$ and (d) $\phi_1(3.9|\cdot)$.



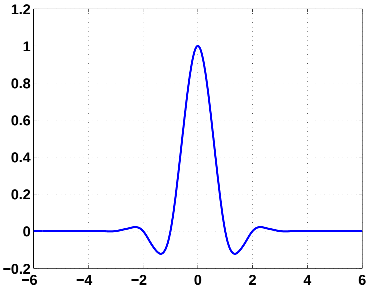
(a) $r = 0 : c = c^{(0)} = (+)$

(b) $r = 0$ and $r = 1 : c^{(0)} = (+)$, $c^{(1)} = (o)$

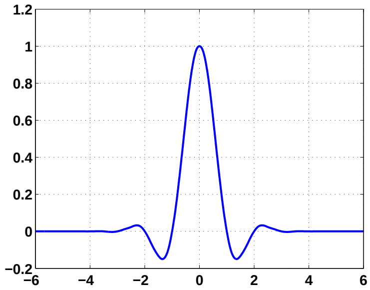
(c) $r = 1$ and $r = 2 : c^{(1)} = (o)$, $c^{(2)} = (-)$

(d) $r = 0$ and $r = 4 : c^{(0)} = (+)$, $c^{(8)} = (-)$

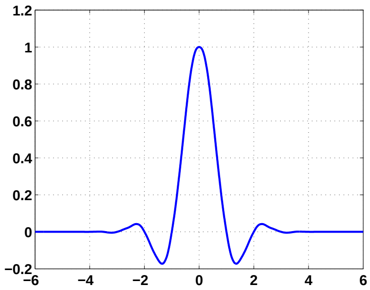
Figure 5.14: Illustration of the convergence of the interpolatory subdivision scheme with mask symbol $A_1(3.9|\cdot)$, where r denotes the iteration level of the subdivision scheme.



(a) $t = 1$

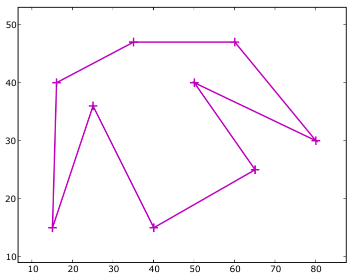


(b) $t = 1.5$

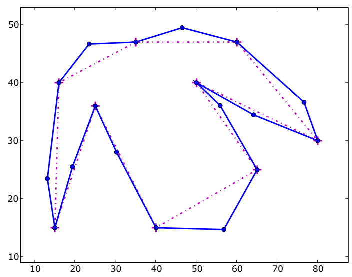


(c) $t = 1.9$

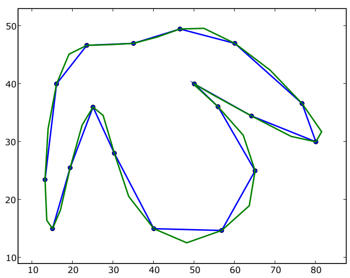
Figure 5.15: The refinable functions (a) $\phi_2(1|\cdot)$, (b) $\phi_1(1.5|\cdot)$ and (c) $\phi_1(1.9|\cdot)$.



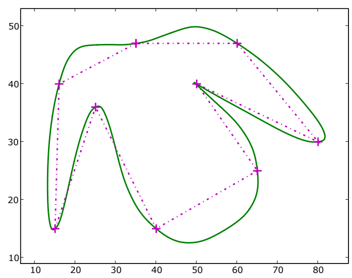
(a) $r = 0 : c = c^{(0)} = (+)$



(b) $r = 0$ and $r = 1 : c^{(0)} = (+)$, $c^{(1)} = (o)$



(c) $r = 1$ and $r = 2 : c^{(1)} = (o)$, $c^{(2)} = (-)$



(d) $r = 0$ and $r = 4 : c^{(0)} = (+)$, $c^{(6)} = (-)$

Figure 5.16: Illustration of the convergence of the interpolatory subdivision scheme with mask symbol $A_2(1.9|\cdot)$, where r denotes the iteration level of the subdivision scheme.

Chapter 6

On the regularity of the refinable function $\phi_m(t|\cdot)$

In this chapter we investigate the regularity (or smoothness) of the interpolatory refinable functions $\phi = \phi_m(t|\cdot)$ of Theorem 5.8 and Section 5.5.

6.1 A regularity result based on spectral radius

We shall base our results of this chapter on the general regularity result of Theorem 6.4 below, as obtained in [4], [6], [16], [20], (see also [24]), and which is based on the spectral radius of a certain matrix obtained from the refinement mask. First, however, we introduce the following concepts.

Definition 6.1. For $\beta \in (0,1]$, we say that a function $f \in \mathscr{M}(\mathbb{R})$ is *Lipschitz continuous* (or *Lipschitz regular*) *of order* β if and only if there exists a non-negative constant c such that

$$|f(x+h) - f(x)| \leq c|h|^{\beta}, \quad x, h \in \mathbb{R}, \tag{6.1}$$

where c is independent of x and h. The linear space of all such functions is denoted by $\mathscr{L}^{\beta}(\mathbb{R})$.

With the definition

$$\mathscr{L}_0^{\beta}(\mathbb{R}) = \mathscr{L}^{\beta}(\mathbb{R}) \cap \mathscr{M}_0(\mathbb{R}), \tag{6.2}$$

we then have the inclusions

$$\mathscr{C}_0^1(\mathbb{R}) \subset \mathscr{L}_0^{\alpha}(\mathbb{R}) \subset \mathscr{L}_0^{\beta}(\mathbb{R}) \subset \mathscr{C}_0(\mathbb{R}), \quad \alpha \geq \beta. \tag{6.3}$$

Definition 6.2. For $\gamma \in (0,\infty)$, and with the definitions $k := \lfloor \gamma \rfloor, \quad \beta := \gamma - k$, we define the linear space

$$\mathscr{C}_0^\gamma(\mathbb{R}) = \begin{cases} \mathscr{C}^k(\mathbb{R}), & \gamma \in \mathbb{Z}_+, \\ \{f : f \in \mathscr{C}_0^k(\mathbb{R}), \quad f^{(k)} \in \mathscr{L}_0^\beta(\mathbb{R})\}, & \gamma \in (0,\infty)\backslash\mathbb{N}. \end{cases}$$

If $f \in \mathscr{C}_0^\gamma(\mathbb{R})$, $\gamma \in (0,\infty)\backslash\mathbb{N}$, we say that f is *Hölder continuous* with *Hölder exponent* γ.

Observe that $\mathscr{C}_0^\gamma(\mathbb{R}) = \mathscr{L}_0^\alpha(\mathbb{R}), \quad \gamma \in (0,1)$. Moreover, we have, from (6.3), the inclusion

$$\mathscr{C}_0^\alpha(\mathbb{R}) \subset \mathscr{C}_0^\gamma(\mathbb{R}), \quad \alpha \geq \gamma. \tag{6.4}$$

If there exists a positive number γ^* such that

$$f \in \mathscr{C}_0^\gamma(\mathbb{R}), \quad \gamma \in (0,\gamma^*), \tag{6.5}$$

and γ^* is the largest possible value for which (6.5) holds, we shall say that γ^* is the *Hölder regularity index* of f.

Definition 6.3. For a square matrix M, we define the spectrum $\sigma(M)$ of M by

$$\sigma(M) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } M\}, \tag{6.6}$$

in terms of which the spectral radius $\rho(M)$ of M is defined by

$$\rho(M) = \max\Big\{|\lambda| : \lambda \in \sigma(M)\Big\}. \tag{6.7}$$

The following regularity result then holds.

Theorem 6.4. Suppose (a,ϕ) is a refinement pair, with corresponding mask symbol A satisfying (1.7), (1.8) for an integer $\ell \in \mathbb{N}$ and a polynomial B, and where $\phi \in \mathscr{C}_0(\mathbb{R})$. Also, with the coefficient sequence $\{b_j : j \in \mathbb{Z}\} \in \mathscr{M}_0(\mathbb{Z})$ defined by $B(z) = \sum_j b_j z^j, \quad z \in \mathbb{C}\backslash\{0\}$, let the $(2\nu+1)\times(2\nu+1)$ matrix $M_a = [M_{i,j} : -\nu \leq i,j \leq \nu]$ be given by

$$M_{i,j} = 2\sum_k b_k b_{k+i-2j}, \quad -\nu \leq i,j \leq \nu, \tag{6.8}$$

where

$$\nu = \lambda - \mu - \ell, \tag{6.9}$$

with λ and μ denoting the (unique) integers such that

$$a_j = 0, \quad j \notin \{\mu, \mu+1, \ldots, \lambda\}, \quad a_\mu \neq 0, \quad a_\lambda \neq 0. \tag{6.10}$$

Then, if the spectral radius $\rho(M_a)$ of M_a satisfies the inequality

$$1 < \rho(M_a) < 4^{\ell-1/2}, \tag{6.11}$$

we have that the Hölder regularity index γ^* of ϕ satisfies

$$\gamma^* \geq \ell - 1/2 - \log_4 \rho(M_a). \tag{6.12}$$

We proceed to apply Theorem 6.4 to the context of a refinement pair as in Theorem 5.8.

To this end, we first note from Definition 2.9 that (6.10) holds for the refinement mask sequence $a = a^{(m)}(t) \in \mathscr{M}_0(\mathbb{Z})$ corresponding to the mask symbol in $A = A_m(t|\cdot) \in \mathscr{A}_{m,m+1}$, as given by (5.38), with $\lambda = 2m+1$, $\mu = -2m-1$, whereas (2.37) and (2.42) yield $\ell = 2m$, and thus, from (6.9), we get

$$\nu = (2m+1) - (-2m-1) - 2m = 2m+2. \tag{6.13}$$

Moreover, (2.37) gives, for $z \in \mathbb{C}\setminus\{0\}$,

$$\sum_j b_j z^j = B(z) = z^{-m} C(z) = \sum_j c_j z^{j-m} = \sum_j c_{j+m} z^j,$$

so that

$$b_j = c_{j+m}, \quad j \in \mathbb{Z},$$

and thus

$$\sum_k b_k b_{k+i-2j} = \sum_k c_{k+m} c_{k+i-2j+m} = \sum_k c_k c_{k+i-2j}, \quad i, j \in \mathbb{Z}. \tag{6.14}$$

Next, we note from (5.38), (3.5), (3.6), and the fact, as noted before Corollary 3.9, that $A_m = D_m$, that, for $t \in \mathbb{R}$, we have

$$A_m(t|z) = \frac{1}{2^{m-1}} \left(1 + \frac{z+z^{-1}}{2}\right)^m C_m(t|z), \quad z \in \mathbb{C}\setminus\{0\}, \tag{6.15}$$

where $C_m(t|\cdot)$ is the symmetric Laurent polynomial defined by

$$C_m(t|z) = p_m\left(t \middle| \frac{1}{2}\left[1 - \frac{z+z^{-1}}{2}\right]\right), \quad z \in \mathbb{C}\setminus\{0\}, \tag{6.16}$$

in terms of the polynomial $p_m(t|\cdot)$ given, as implied by (5.8), by the formula

$$p_m(t|z) = p_m(z) + 2^{2m-1} t z^m (1-2z), \quad z \in \mathbb{C}, \tag{6.17}$$

with p_m denoting the polynomial of degree $(m-1)$ as defined by (3.31) in Proposition 3.5.

It follows from Theorem 6.4, together with (6.14) and all the other arguments above, that the following regularity result holds.

Theorem 6.5. For $m \in \mathbb{N}$, let the mask symbol $A_m(t|\cdot)$ be defined by (5.38), with the real parameter t chosen in such a way that there exists a function $\phi_m(t|\cdot) \in \mathscr{C}_0(\mathbb{R})$ such that $\left(a^{(m)}(t), \phi_m(t|\cdot)\right)$, with $a^{(m)}(t) \in \mathscr{M}_0(\mathbb{Z})$ denoting the refinement mask sequence corresponding to $A_m(t|\cdot)$, is a refinement pair. Futhermore, denote by $c^{(m)} = \left\{c_j^{(m)}(t) : j \in \mathbb{Z}\right\} \in \mathscr{M}_0(\mathbb{Z})$ the sequence defined by

$$C_m(t|z) = \sum_j c_j^{(m)}(t) z^j, \quad z \in \mathbb{C}\setminus\{0\}, \tag{6.18}$$

with the Laurent polynomial $C_m(t|\cdot)$ defined by (6.16), (6.17). Then, if the spectral radius $\rho\left(M_{a^{(m)}(t)}\right)$ of the $(4m+5) \times (4m+5)$ matrix

$$M_{a^{(m)}(t)} = \left[M_{i,j} : -2m-2 \le i, j \le 2m+2\right],$$

as defined by

$$M_{i,j} = 2\sum_k c_k^{(m)}(t) c_{k+i-2j}^{(m)}(t), \quad -2m-2 \le i, j \le 2m+2, \tag{6.19}$$

satisfies the inequality

$$1 < \rho\left(M_{a^{(m)}(t)}\right) < 4^{2m-1/2}, \tag{6.20}$$

then the Hölder regularity index $\gamma_m^*(t)$ of $\phi_m(t|\cdot)$ satisfies

$$\gamma_m^*(t) \ge 2m - 1/2 - \log_4 \rho\left(M_{a^{(m)}(t)}\right). \tag{6.21}$$

6.2 Examples

Using (6.16), (6.17) and (3.31), we obtain, for $z \in \mathbb{C}\backslash\{0\}$ and $t \in \mathbb{R}$, the explicit formulas

$$\begin{aligned}
C_1(t|z) &= \frac{1}{4}\Big(-tz^{-2}+2tz^{-1}+4-2t+2tz-tz^2\Big),\\
C_2(t|z) &= \frac{1}{4}\Big(tz^{-3}-4tz^{-2}+(-2+7t)z^{-1}+8-8t+(-2+7t)z+-4tz^{-2}+tz^{-3}\Big),\\
C_3(t|z) &= \frac{1}{8}\Big(-2tz^{-4}+12tz^{-3}-(3+32t)z^{-2}-(18-52t)z^{-1}+38-60t\\
&\qquad -(18-52t)z-(3+32t)z^2-2tz^4\Big),\\
C_4(t|z) &= \frac{1}{16}\Big(4tz^{-5}-32tz^{-4}+(-5+116t)z^{-3}+(40-256t)z^{-2}+(-131+392t)z^{-1}\\
&\qquad +208-448t+(-131+392t)z+(40-256t)z^2+(-5+116t)z^3-32tz^4+4tz^5\Big).
\end{aligned}$$

We next use (6.5) and (6.19), together with a MATLAB routine for calculating the eigenvalues of a matrix, to plot, in Figure 6.1, and for $m = 1, 2, 3, 4$, the function

$$U_m(t) = 2m - 1/2 - \log_4 \rho\left(M_{a^{(m)}}(t)\right)$$

for those values of $t > -t_m$ for which we know that the refinable function $\phi_m(t|\cdot)$ exists, and for which we know from (6.21) in Theorem 6.5 to be a lower bound for the Hölder regularity index $\gamma_m^*(t)$ of $\phi_m(t|\cdot)$, i.e.

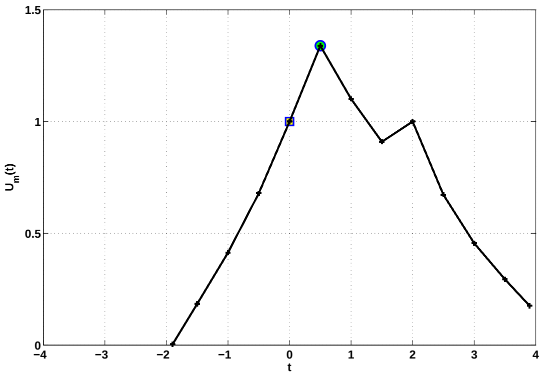
$$\gamma_m^*(t) \geq U_m(t),$$

for those real values of t for which there exists a function $\phi_m(t|\cdot) \in \mathscr{C}_0(\mathbb{R})$ such that $\left(a^{(m)}(t), \phi_m(t|\cdot)\right)$ is a refinement pair, and for which the inequality (6.20) holds.

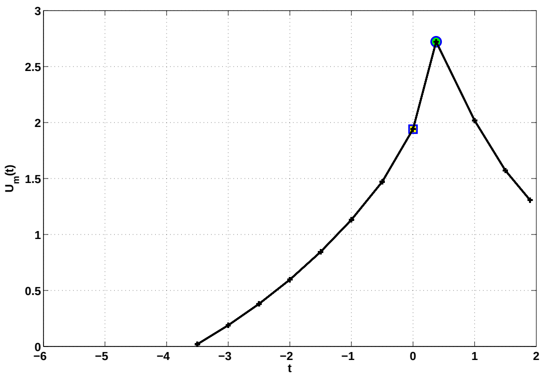
In each graph, the positions of $\phi_m^D = \phi_m(0|\cdot)$ and $\phi_{m+1}^D = \phi_m(\tilde{t}_m|\cdot)$ are indicated by the square and the circle respectively.

Based on numerical evidence, as displayed graphically in Figure 6.1, we obtain, after recalling also (6.4), the regularity results $\phi_1(t|\cdot) \in \mathscr{C}_0^1(\mathbb{R})$, $0 \leq t \leq 0.5$, $\phi_2(t|\cdot) \in \mathscr{C}_0^1(\mathbb{R})$, $-1 \leq t \leq 0$, $\phi_2(t|\cdot) \in \mathscr{C}_0^2(\mathbb{R})$, $t = 0.3750$, $\phi_3(t|\cdot) \in \mathscr{C}_0^1(\mathbb{R})$, $-2 \leq t \leq -1$, $\phi_3(t|\cdot) \in \mathscr{C}_0^2(\mathbb{R})$, $-0.5 \leq t \leq 0.3125$, $\phi_4(t|\cdot) \in \mathscr{C}_0^1(\mathbb{R})$, $-3.5 \leq t \leq -1.5$, $\phi_4(t|\cdot) \in \mathscr{C}_0^2(\mathbb{R})$, $-1 \leq t \leq -0.5$, and $\phi_4(t|\cdot) \in \mathscr{C}_0^3(\mathbb{R})$, $0 \leq t \leq 0.2734$.

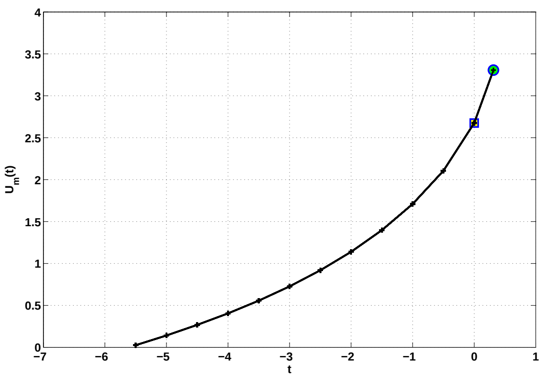
Since $\phi_m(0|\cdot) = D_m \in \mathscr{A}_{m,m}$ and $\phi_m(\tilde{t}_m|\cdot) = D_{m+1} \in \mathscr{A}_{m+1,m+1}$, we know that the lower bound indicated by the circle (o) in each of Figures 6.1(a), (b) and (c) below, can actually be improved to the larger value indicated by the square (□) in respectively, Figures 6.1(b), (c) and (d).



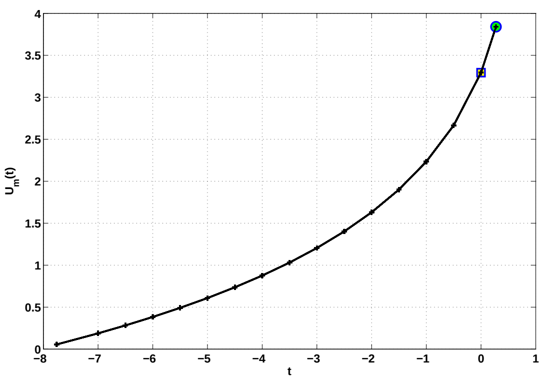
(a)



(b)



(c)



(d)

Figure 6.1: The Hölder regularity lower bounds (a) $U_1(t)$, $t \in (-4,4)$, (b) $U_2(t)$, $t \in (-5.5451,2)$, (c) $U_3(t)$, $t \in (-6.7423,0.3125)$, and (d) $U_4(t)$, $t \in (-7.7559,0.2734)$.

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