Spectral Difference Methods for solving Equations of the KdV Hierarchy

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Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

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Abstract

The Korteweg-de Vries (KdV) hierarchy is an important class of nonlinear evolution equations with various applications in the physical sciences and in engineering.

In this thesis analytical solution methods were used to find exact solutions of the third and fifth order KdV equations, and numerical methods were used to compute numerical solutions of these equations.

Analytical methods used include the Fan sub-equation method for constructing exact traveling wave solutions, and the simplified Hirota method for constructing exact N-soliton solutions. Some well known cases were considered.

The Fourier spectral method and the finite difference method with Runge-Kutta time discretisation were employed to solve the third and the fifth order KdV equations with periodic boundary conditions. The one soliton and the two soliton solutions were used as initial conditions. The numerical solutions are obtained and compared with the exact solutions. The propagation of a single soliton as well as the interaction of double soliton solutions is modeled well by both numerical methods, although the Fourier spectral method performs better.

The stability, consistency and convergence of these numerical methods were investigated. Error propagation is studied. The theoretically predicted quadratic convergence of the finite difference method as well as the exponential convergence of the Fourier spectral method is confirmed in numerical experiments.

Opsomming

Die Korteweg-de Vries (KdV) hierargie is 'n belangrike klas van nie-lineere evolusie-vergelykings met verskeie toepassings in die fisiese wetenskappe en ingenieurswese.

In hierdie skripsie is analitiese oplossingsmetodes gebruik om eksakte oplossings vir die derde orde en vyfde orde KdV vergelykings te vind, en numeriese metodes is gebruik om numeriese oplossings van hierdie vergelykings te bereken.

Analitiese metodes wat gebruik is sluit die Fan sub-vergelyking metode in, vir die vind van bestendige golf-oplossings sowel as die vereenvoudigde Hirota metode vir die vind van N-soliton oplossings.

Die Fourier spektraalmetode en die eindige-verskilmetode met Runge-Kutta tydsdiskretisasie is gebruik om die derde en vyfde orde KdV vergelykings op te los met periodiese randvoorwaardes. Die een-soliton en twee-soliton oplossings is gebruik as aanvangsvoorwaardes. Die verkreë numeriese oplossings is vergelyk met die eksakte oplossings. Die beweging van 'n enkel-soliton sowel as die interaksie van twee-soliton oplossings word goed gemodelleer deur beide numeriese metodes alhoewel die Fourier spektraal-metode beter resultate lewer.

Die stabiliteit, konsistensie en konvergensie van hierdie numeriese metodes is ondersoek. Foutvoortplanting is ook bestudeer. Die teoreties voorspelde kwadratiese konvergensie van die eindige verskil-metode sowel as die eksponensiële kovergensie van die Fourier spektraal-metode is bevestig in die numeriese eksperimente.

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List of Acronyms and Symbols

- NLPDE: nonlinear partial differential equation
- NLODE: nonlinear ordinary differential equation
- ODE: ordinary differential equation.
- PDE : partial differential equation.
- IST : inverse scattering transform
- KdV : Korteweg-de Vries
- KdV3 : the third order Korteweg-de Vries
- KdV5 : the fifth order Korteweg-de Vries
- KK : Kaup-Kuperschmidt
- FD : finite difference
- PS : Fourier pseudospectral
- RK : Runge-Kutta
- RK4 : fourth order Runge-Kutta
- FFT : Fast Fourier Transform
- $u_{nx} = \frac{\partial^n}{\partial x^n}$: the n-partial derivative of the function $u(x,t)$
- $O((g(x)))$: a function $f(x)$ for which there exist a constant $x_0 > 0$ and $C > 0$ where $|f(x)| \leqslant C|g(x)|$ for all $x \geqslant x_0$
- $*$: element by element matrix multiplication
- δ_j^k : the Kronecker delta symbol

Chapter 1

Introduction

Nonlinear partial differential equations (NLPDEs) are widely used to describe many important phenomena in various fields of science such as physics, biology, chemistry [5] and classical mechanics [6]. The investigation of exact solutions of the NLPDEs plays an important role in the study of those problems, facilitates the verification of the performance of numerical solvers and can help to enhance the understanding the mechanisms of complicated phenomena.

Among those exact solutions there are some that do not change form during their propagation, called *travelling waves*. Often, they result from a balance between dispersion and nonlinearity. A localised travelling is called a *solitary wave*. Solitary waves that preserve their shape and speed after colliding with each other are called *solitons*.

In recent decades, many mathematicians and physicists have understood the importance of solitary waves and solitons. As a result, they decided to pay attention to the development of sophisticated methods for constructing exact solutions to the NLPDEs. Therefore, a number of powerful methods have been developed recently.

Among them is a class of sub-equation methods such as the sinh-cosh-tanh method [15], the Jacobi elliptic functions method [14] and the Fan-sub-equation method [2, 3, 4]. The sinh-cosh-tanh method seeks solutions of the NLPDEs as a polynomial of hyperbolic functions. The Jacobi elliptic functions method enables us to write solutions of NLPDEs as a polynomial of Jacobi elliptic functions. A more general method, the Fan-sub-equation provides solutions of the NLPDEs as a polynomial of solutions of the general elliptic equation.

Another class of methods deals with the integrability of the NLPDEs. We shall use integrability here only in the sense meaning that the NLPDE admits soliton solutions. Painlevé analysis [16] provides an algorithm for testing whether or not a given NLPDE is a good candidate for integrability. The inverse scattering transform (IST) [9, 12] is a powerful method for obtaining exact solutions for integrable NLPDEs given arbitrary initial conditions. The Hirota bilinear method [13, 17] allows us to find N-soliton solutions of large classes of integrable NLPDEs. More recently, a simplified version of the Hirota method that does not require any knowledge of the Hirota operator, has been introduced by Hereman and Nusier [25].

Moreover, several numerical methods, such as the finite difference [52], spectral [36] and pseudospectral methods [39, 40] are employed for numerical solutions of NLPDEs.

1.1 Historical Perspective

The discovery of solitary waves and solitons begins with a remarkable scientific discovery made by a young Scottish engineer named John Scott Russell (1808-1882) while conducting experiments to determine the most efficient design for canal boats. Nothing can describes better this new phenomena than his own words: "*I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.*"

Russell replicated the observation of solitary waves in a tank in his garden and he was convinced of the importance of them but unfortunately his work was hotly debated and rejected by the nineteenth and the early twentieth century scientists [54] .

The issue on solitary wave was solved only in 1895 when the Dutch professor Diederik Korteweg and his doctoral student Gustav de Vries first proposed a model of shallow water wave but scientists did not pay attention into it.

It was only in the sixties that new applications of the KdV equation were discovered. In 1965, Norman Zabusky and Martin Kruskal while conducting numerical simulations for the particle behavior observed in the Fermi-Pasta-Ulam experiment discovered that during the interaction, two solitary pulses behaved in a nonlinear way. They emerged from their collision with their former heights, widths and velocities. They called them 'solitons'. Surprisingly enough, the mathematical model for this apparently unrelated problem led to the KdV equation again.

In 1967, Gardner *et al.* [55] discovered an exact method, the Inverse Scattering Transform, for solving the initial value problem of the KdV equation. Since then, the KdV equation and its fifth order family that admit solitary wave and soliton solutions have been subject to intense study. In 1971, Hirota developed an ingenious method for obtaining the exact multi-soliton solutions of the KdV equation and derived an explicit expression for its N-soliton solutions.

Today, soliton theory has numerous applications in physics, in fields such as plasma physics and nonlinear optics. Solitons can be observed in the ocean, in the atmosphere, etc. In general solitons may propagate in many media, ranging from infinitesimal to meteorological and astrophysical.

1.2 Aims of the Thesis

The main objective of this thesis is to study and implement the Fourier Pseudospectral method for solving the initial value problem of the Korteweg-de Vries hierarchy of PDEs. However, the availability of the exact solutions to this problem can be employed well in testing the performance of this numerical method. Therefore an additional objective is to study and elaborate on the origin and derivation of simple exact solutions such as travelling waves and N-soliton solutions. More general exact solutions obtainable via the Inverse Scattering Transform (apart from pure soliton solutions) are not covered at all. In order to achieve this objective, the following points are covered:

1. The conditions for which the KdV family of equations admits travelling waves solutions are investigated and then analytical solutions are presented.

2. The conditions for which the KdV family of equations admits soliton solutions are investigated.

3. Some analytical N-soliton solutions of the third and the fifth order KdV equations are derived.

4. Efficient numerical schemes for finding approximate solutions to the KdV family equations are reviewed.

   - To this end, finite difference (used for comparison) and Fourier pseudospectral methods are employed in space discretisation, and the fourth order Runge-Kutta method is used for time discretisation.

5. A linear convergence analysis of the KdV family equations using the above numerical schemes is performed.

   - Consistency and stability conditions to the above schemes are established.

6. Numerical solutions of the KdV family equations are computed with the above schemes.

   - Periodic boundary conditions are used, one soliton and two solitons are used as initial conditions.
   - The accuracy of the two numerical schemes is investigated by experimentation.

1.3 Thesis Outline

In Chapter 2, we investigate conditions for which the third and the fifth order KdV admit travelling wave solutions. Once the condition is satisfied, the Fan method is used in order to compute analytical solutions of these equations. The computer software MATHEMATICA was used in the realisation of these analytical solutions.

The aim of Chapter 3 is to derive soliton solutions of the third and the fifth order KdV equation using the simplified Hirota method. But first a fundamental property, integrability, has to be satisfied. To this end, the Painlevé method is the appropriate method to verified this property. MATHEMATICA is needed in this chapter.

Chapter 4 reviews three different numerical methods. The finite difference and Fourier pseudospectral methods are used for space discretisation and the RK4 method is employed for time discretisation. Several issues such as linear stability, consistency and convergence of these methods for solving the third and the fifth order KdV are investigated.

In Chapter 5, numerical solutions of the third and the fifth order KdV equations are computed and presented. Periodic boundary conditions are used and initial conditions are taken to be one and two soliton solutions. The maximum norm and the L^2-norm error are used in order to investigate the accuracy of the numerical methods. All experimental results are obtained by using MATLAB.

Chapter 6 contains the summary of the results of the thesis and future perspectives.

Chapter 2

Travelling waves

A wave, in other words, a disturbance that propagates and transfers energy from one point to another, is one of the key concept in physics and applied mathematics. For many years, many important discoveries in physics, including classical mechanics [6], have involved wave phenomena.

In this chapter we will study two important effects, dispersion and nonlinearity. When these two effects are combined, they may cancel, allowing the possibility of travelling waves. We will next focus our attention on a family of evolution equations, in particular the Korteweg-de Vries (KdV) family. We are interested in finding travelling wave solutions of the generalized third and fifth order KdV equations. This means that using the travelling wave transformation, the study of nonlinear partial differential equations (NLPDEs) will be simplified into the study of nonlinear ordinary differential equations (NLODEs). The Fan sub-equation method will be of paramount importance in obtaining exact travelling wave solutions of the KdV equation family.

2.1 Dispersive and non-dispersive waves

A wave which spreads as it travels is defined to be a dispersive wave. This is usually the behaviour of a localised water wave, and the study of this helps us to understand complex natural disasters, such as the tsunami described by Shigihara [8]. In the following subsection we will discuss the following cases, linear dispersive and nonlinear waves, and we will observe that the two effects combined are candidates for travelling waves.

2.1.1 Linear non-dispersive waves

Consider the well-known one dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0, \tag{2.1}$$

where $u(x,t)$ is some property associated with the wave and c^2 is a constant wave speed (*phase velocity* of each wave). The solution of (2.1) is

$$u(x,t) = f(x - ct) + g(x + ct),$$

where f and g are arbitrary functions.

Assuming that u is periodic with angular frequency ω and wave number k, the most fundamental solution of (2.1) can be written as

$$u(x,t) = e^{i(kx-\omega t)}. \tag{2.2}$$

Equation (2.2) is chosen because each of its superposed plane wave solutions is physically possible, since it remains bounded along both boundaries at $x \to \pm\infty$. We note that the exponential solution

$$u(x,t) = e^{\pm(Kx-\Omega t)}, \tag{2.3}$$

also satisfies equation (2.1), but diverges at one of the boundaries. We therefore reject this solution in the linear theory of PDEs. In order to verify whether or not the wave is non-dispersive, one way to do is to determine the relationship between the angular frequency ω and the wave number k which satisfies the PDE, defined as the dispersion relation.

Definition 2.1.1 *The phase velocity is the ratio of the frequency to the wave number*

$$c_p(k) = \frac{\omega}{k}.$$

Definition 2.1.2 *The group velocity is the rate of change of the frequency with respect to the wave number*

$$c_g(k) = \frac{\partial \omega}{\partial k}.$$

Definition 2.1.3 *A wave is dispersive if*

$$\frac{\partial^2 \omega}{\partial k^2} \neq 0. \tag{2.4}$$

The dispersion relation of (2.1) is obtained by substituting (2.2) into (2.1)

$$\omega = \pm kc, \tag{2.5}$$

where k is the wave number, ω is the frequency and c is the velocity of the wave crests. It is easy to verify from (2.5) that $\frac{\partial^2 \omega}{\partial k^2} = 0$. This means that each superposed wave travels at the same speed and the wave solution is non-dispersive. This is the behaviour of a travelling wave.

2.1.2 Linear dispersive waves

Consider the following linear third order equation

$$u_t + \kappa u_{3x} = 0. \tag{2.6}$$

We would like to know the nature of the wave governed by (2.2). Substituting (2.2) into (2.6) yields the dispersion relation

$$\omega(k) = \kappa k^3. \tag{2.7}$$

Therefore, the group velocity is given by

$$c_g(k) = 3\kappa k^2, \tag{2.8}$$

and the phase velocity can be expressed as

$$c_p(k) = \kappa k^2. \tag{2.9}$$

Clearly, the group velocity differs from the phase velocity and the wave is said to be dispersive, i.e. the wave changes its shape as it travels. This is illustrated in figure 2.1.

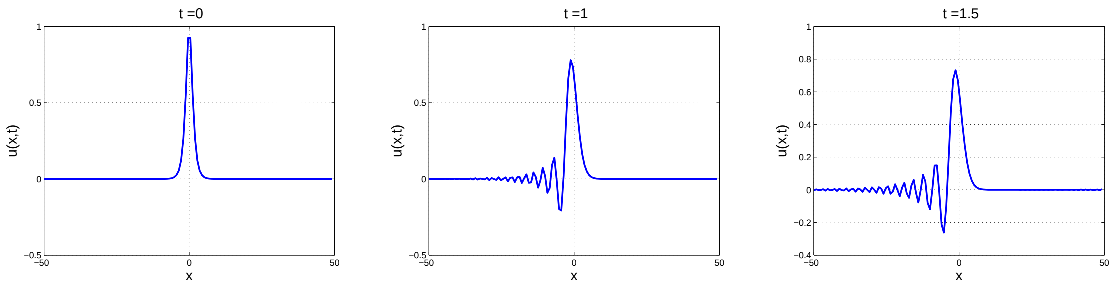


Figure 2.1: Graphical representation of the model (Eq. (2.6)), with $\kappa = 1$ at respective time $t = 0$, $t = 1$ and $t = 1.5$, with initial conditions $u(x, 0) = \text{sech}(x)$

2.1.3 Non-linear non-dispersive waves

Consider the inviscid Burgers' equation

$$u_t + \gamma u u_x = 0, \tag{2.10}$$

where γ is an arbitrary constant. The solution of (2.10) is obtained by Lagrange's method. The characteristic equations of (2.10) are

$$\frac{dt}{ds} = 1, \tag{2.11}$$

$$\frac{dx}{ds} = \gamma u, \tag{2.12}$$

$$\frac{du}{ds} = 0. \tag{2.13}$$

If the independent functions $\phi = \phi(t, x, u)$ and $\psi = \psi(t, x, u)$ can be found such that

$$\begin{aligned} \phi_t + \gamma u \phi_x &= 0 \\ \psi_t + \gamma u \psi_x &= 0, \end{aligned} \tag{2.14}$$

then the relation

$$F(\phi, \psi) = 0 \quad \text{(where } F \text{ is an arbitrary function)} \tag{2.15}$$

or equivalently

$$\phi = f(\psi) \quad \text{(where } f \text{ is an arbitrary function)} \tag{2.16}$$

would provide the solution of 2.10. Solving (2.13) yields

$$u = \phi = \text{const.} \tag{2.17}$$

By eliminating ds from (2.11) and (2.12) the following equation is obtained

$$\frac{dx}{dt} = \gamma u. \tag{2.18}$$

Integrating (2.18) gives

$$\psi = x - \gamma \phi t = \text{const.} \tag{2.19}$$

One can easily check that (2.17) and (2.19) solves (2.14). Therefore the general solution of (2.10) is

$$u(x, t) = f(x - \gamma u t), \tag{2.20}$$

where f is an arbitrary function.

We would like to show that the wave profile (2.20) undergoes a progressive deformation as t increases. One way to do it is to investigate the change of the slope of $u(x, t)$ as time t increases. We follow a similar analysis employed by Bhatnagar [1]. The first derivative of (2.20) with respect to x is given by

$$u_x(x, t) = (1 - \gamma u_x(x, t) t) f_\xi, \text{ with } \xi = x - \gamma u t. \tag{2.21}$$

After algebraic simplifications, (2.21) yields

$$u_x(x, t) = \frac{f_\xi}{1 + t\gamma f_\xi}, \tag{2.22}$$

which expresses the slope of the u-profile at the point (x, t) in terms of the slope of the initial profile at ξ, where $\xi = x$ at $t = 0$. If $f_\xi < 0$, $u_x(x, t)$ is infinite at $t = \dfrac{-1}{\gamma f_\xi}$. Therefore, if the initial profile has a negative slope at some point ξ, then for $t > T = \left(\dfrac{-1}{\gamma f_\xi}\right)_{min}$, the solution ceases to be single valued in the neighbourhood of a point $x_0 = \xi_0 + \gamma T f(\xi_0)$ where ξ_0 is the point at which $\left(\frac{-1}{\gamma f_\xi}\right)$ is the minimum value. We would like to find the changes in the slope of u at $\xi = \xi_0$ when $t > T$. Let $X_0(t)$ be the position of ξ at any time t. Let

$$t = T + \epsilon = \left(\frac{-1}{\gamma f_\xi}\right)_{min} + \epsilon, \tag{2.23}$$

where $|\epsilon| \ll 1$. Therefore

$$u_x(X_0(t), T+\epsilon) = \left(\frac{f_\xi}{1 + t\gamma f_\xi}\right)_{\xi=\xi_0, t=T+\epsilon}. \tag{2.24}$$

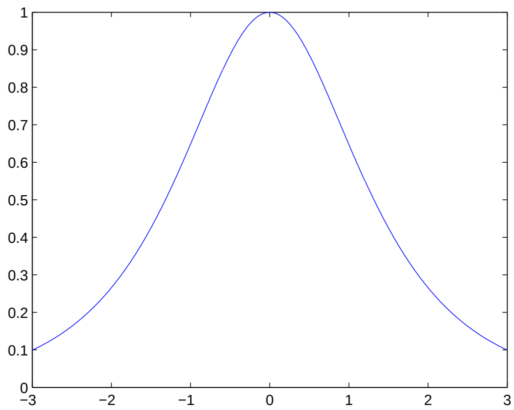
After algebraic simplifications, we obtain

$$u_x(X_0(t), T+\epsilon) = \frac{1}{\gamma\epsilon}. \tag{2.25}$$

Consequently, we have

$$u_x(X_0(T-0), T-0) = -\infty \quad \text{and} \quad u_x(X_0(T+0), T+0) = +\infty. \tag{2.26}$$

We conclude that as t increases, the wave profile $u(x,t)$ undergoes a progressive deformation as shown in Figure 2.2.



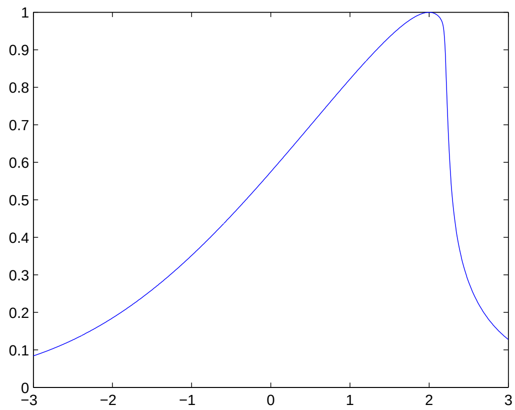


Figure 2.2: Graphical representation of the model (2.10), with $\gamma = 1$ at respective time $t = 0$ and $t = 1.5$, with initial conditions $u(x, 0) = \text{sech}(x)$

2.1.4 Non-linear dispersive waves

In the sections above we have seen that non-linearity alone or dispersion alone does not allow travelling waves. We would like to investigate under which conditions non-linearity together with dispersion can be candidate for a travelling wave.

If we add both a non-linear and a dispersion term to our governing wave equation, we obtain the following KdV equation

$$u_t + \gamma u u_x + \kappa u_{xxx} = 0. \tag{2.27}$$

Although it is well known that the KdV equation has travelling wave solutions, we shall investigate this question from a different angle. Consider a localised travelling wave moving at constant velocity.

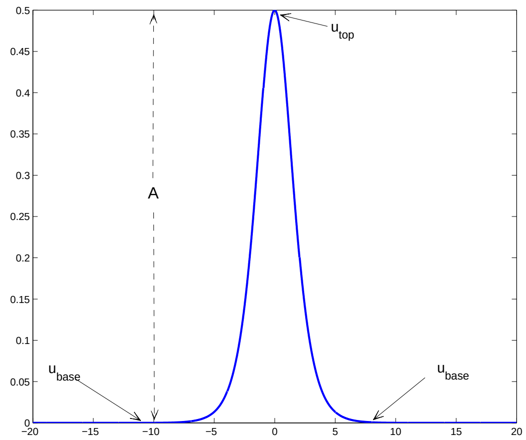


Figure 2.3: A localised travelling wave

Assume that the localised travelling wave solution in is symmetrical around the point of maximum amplitude. We can approximate u by a function u_{top} in the neighbourhood of the maximum amplitude, and the dispersion term u_{xxx} will be zero. Therefore the function u_{top} satisfies (2.10). Whereas, close to the base of the wave where the non-linear term can be neglected, since u is small, the approximation of the solution is denoted u_{base}. In the vicinity of the base of the localised travelling wave u_{base} on each side satisfies (2.6), where the group velocity and the phase velocity are given respectively by

$$c_g(k) = \frac{\partial \omega}{\partial k} = 3\kappa k^2 \quad \text{and} \quad c_p(k) = \frac{\omega}{k} = \kappa k^2. \tag{2.28}$$

Consequently, the top and the bottom of the wave above do not move at the same speed. But we claim that the wave above is a travelling wave, which is a contradiction. This happens because the equation is non-linear and the superposition principle of waves is no longer valid. As a result, we need to choose

$$u(x,t) = e^{-\Psi} \;\text{ as }\; \Psi \to +\infty \quad \text{and} \quad u(x,t) = e^{\Psi} \;\text{ as }\; \Psi \to -\infty, \tag{2.29}$$

where $\Psi = Kx - \Omega t$. Consider the wave solution at the base of the wave as

$$u_{base}(x,t) = e^{\pm(Kx - \Omega t)}. \tag{2.30}$$

We substitute (2.30) into (2.6) and we obtain a single non-linear dispersion relation from the linear equation (2.6)

$$\Omega = \kappa K^3 \tag{2.31}$$

The phase velocities at the top and base of the wave are respectively

$$c_{top} = \gamma A \quad \text{and} \quad c_{base} = \kappa K^2 \tag{2.32}$$

where A is the maximum amplitude of the wave. Equating the top and the base volocity of the wave, yields the following relation

$$\kappa K^2 = \gamma A. \tag{2.33}$$

Since the top and the base of the wave travel at the same velocity, travelling wave solutions may be possible. This is illustrated in Figure 2.3.

2.2 Travelling wave solutions

In recent years, several methods have been developed to find analytic solutions of NLPDEs. Among these methods we mainly cite, for example, the inverse scattering method [11-14], Hirota's bilinear method [13], the Painlevé expansion method [16], the sinh-cosh-tanh method [15] and the Jacobi elliptic functions method [14].

In this section, the Fan sub-equation method [2, 3, 4] is used to generate travelling wave solutions of two types of equations of the Korteweg-de-Vries equation family:

- The generalized third order KdV equation of the form

$$u_t + \gamma u u_x + \kappa u_{3x} = 0, \tag{2.34}$$

where γ and κ are arbitrary constants.

- The generalized fifth order KdV equation of the form

$$u_t + \alpha u^2 u_x + \beta u u_{3x} + \gamma u_x u_{2x} + \kappa u_{5x} = 0, \tag{2.35}$$

where α, β, γ and κ are arbitrary constants.

This method consists of looking for the travelling wave solutions of the general elliptic equation. We therefore first discuss the solution of the elliptic ODE.

2.2.1 Elliptic ordinary differential equation

Consider a function of ξ, $\phi(\xi)$, and consider the general elliptic equation

$$\phi'^2 = \left(\frac{d\phi}{d\xi}\right)^2 = c_0 + c_1\phi + c_2\phi^2 + c_3\phi^3 + c_4\phi^4. \tag{2.36}$$

Following Yomba [3], this equation admits five groups of travelling wave solutions. However, we will be interested in the fourth group whose solutions are single and combined Jacobi elliptic functions. This group of travelling wave solutions occurs when $c_1 = c_3 = 0$. Consequently the general elliptic equation becomes

$$\phi'^2 = \left(\frac{d\phi}{d\xi}\right)^2 = c_0 + c_2\phi^2 + c_4\phi^4 \tag{2.37}$$

Solutions of (2.37) can be expressed in terms of Jacobi elliptic functions (see Appendix), as follows

$$\phi = \pm\sqrt{-\frac{c_2 m^2}{c_4(2m^2-1)}}\mathrm{cn}\left(\sqrt{\frac{c_2}{(2m^2-1)}}\xi\right), \quad c_4 < 0, \ c_2 > 0, \quad c_0 = -\frac{c_2^2 m^2(1-m^2)}{c_4(2m^2-1)^2} \tag{2.38}$$

$$\phi = \pm\sqrt{-\frac{c_2}{c_4(2-m^2)}}\mathrm{dn}\left(\sqrt{\frac{c_2}{(2-m^2)}}\xi\right), \quad c_4 < 0, \quad c_2 > 0, \quad c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2} \tag{2.39}$$

$$\phi = \pm\sqrt{-\frac{c_2 m^2}{c_4(m^2+1)}}\mathrm{sn}\left(\sqrt{-\frac{c_2}{(m^2+1)}}\xi\right), \quad c_4 > 0, \ c_2 < 0, \quad c_0 = \frac{c_2^2 m^2}{c_4(m^2+1)^2} \tag{2.40}$$

As $m \to 1$, the Jacobi elliptic function solutions (2.38) and (2.39) degenerate into single hump solutions

$$\phi = \pm\sqrt{-\frac{c_2}{c_4}}\mathrm{sech}\left(\sqrt{c_2}\xi\right), \quad c_4 < 0, \quad c_2 > 0, \quad c_0 = 0, \tag{2.41}$$

and the Jacobi elliptic function solution (2.40) degenerates into the following

$$\phi = \pm\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right), \quad c_4 > 0, \quad c_2 < 0, \quad c_0 = \frac{c_2^2}{4c_4}. \tag{2.42}$$

In the two next subsections, the Fan sub-equation method will be introduced first, and finally used in order to find travelling waves of the third and fifth order KdV equation, taking into account only the cases above.

2.2.2 Fan sub-equation method: general description

Consider the non-linear partial differential equation with $\ell + 1$ independent variables $x_0 = t, x_1, x_2, ..., x_\ell$ and dependent variable u

$$F(u, u_t, u_{x_i}, u_{x_i x_j}, u_{x_i x_j x_k}, ...) = 0, \tag{2.43}$$

where F is a polynomial function of its arguments, and the subscripts denote partial derivatives. Suppose that the solution of (2.43) is a travelling wave, which means

$$u = u(\xi), \quad \xi = \sum_{i=0}^{\ell} k_i x_i, \tag{2.44}$$

where ξ is a phase variable and k_i with $i = 0, 1, ..., m$, are all arbitrary constants. Substituting (2.44) into (2.43) yields a non-linear ordinary differential equation

$$F(u, u', u'', ...) = 0. \tag{2.45}$$

We try to find solutions $u(\xi)$ that can be expanded into a polynomial in $\phi(\xi)$ of degree n

$$u(\xi) = \sum_{i=0}^{n} a_j \phi^j(\xi), \tag{2.46}$$

where ϕ satisfies the following general elliptic equation

$$\phi'^2 = \left(\frac{d\phi}{d\xi}\right)^2 = c_0 + c_1\phi + c_2\phi^2 + c_3\phi^3 + c_4\phi^4, \tag{2.47}$$

where c_k and a_j are constants with $k = 0, \ldots, 4$ and $j = 0, \ldots, m$. The solution u is determined explicitly using the following algorithm.

Step 1. The integer n is determined by balancing the linear term of highest order with the non-linear terms in (2.45). This means, we substitute (2.46) into (2.45) taking into account (2.47). Then (2.45) is expressed as a polynomial in ϕ. We equate two possible highest powers of ϕ in order to determine the value n.

Example 2.2.1 *For example, consider the generalized KdV equation*

$$u_t + u^p u_x + u_{xxx} = 0, \tag{2.48}$$

where $p \geq 0$. By considering the travelling wave transformation $u(\xi)$, where $\xi = kx + \omega t$, equation (2.48) yields an ordinary differential equation

$$\omega u + ku^p u' + k^3 u''' = 0. \tag{2.49}$$

Then balancing the non-linear term $k(p+1)u^p u'$ and the linear term of highest order $k^3 u'''$ yields

$$pn + n + 1 = n + 3. \tag{2.50}$$

By looking only for positive integer p, then two cases are $p = 1,\ n = 2$ and $p = 2,\ n = 1$. As a result, only the third KdV and the modified KdV admits travelling wave solutions.

Step 2. Then (2.46) is substituted along with (2.47) into (2.45). All the coefficients of ϕ^k with $(k = 0, 1, 2...)$ are collected and their coefficients are set to zero. A system of algebraic equations with respect to a_j and k_i with $j = 0, 1 \cdots n$ and $i = 0, 1, 2..., \ell$, is obtained.

Step 3. The system of algebraic equations is solved with MATHEMATICA in order to obtain the explicit values of a_j and k_i. These results are inserted into (2.46) in order to obtain the general form of the travelling wave solutions.

<u>*Step 4*</u>. Obtain exact solutions depending on the special conditions chosen for the c_0, c_1, c_2, c_3 and c_4. Travelling wave solutions are classified in five groups: ϕ_ℓ^I, ϕ_ℓ^{II}, ϕ_ℓ^{III}, ϕ_ℓ^{IV} and ϕ_ℓ^V. The superscripts $I-V$ determine the group of solutions and the subscript ℓ determines the rank of the solution. All groups of solutions have been successfully classified by Yomba [3]. In order to test the validity of the method, we find travelling wave solutions of the third and the fifth order KdV equations.

2.2.3 Applications to the generalized third order KdV equation

Consider the generalized third order KdV equation

$$u_t + \gamma u u_x + \kappa u_{3x} = 0, \tag{2.51}$$

where γ and κ are real constants. By applying the travelling wave substitution

$$u = u(\xi), \quad \xi = kx + \omega t, \tag{2.52}$$

this method will give a series of travelling wave solutions for (2.51). We substitute (2.52) into (2.51) to obtain the following ordinary differential equation

$$\omega u' + \gamma k u u' + \kappa k^3 u''' = 0. \tag{2.53}$$

Equation (2.53) is integrated once to yield

$$\omega u' + \frac{\gamma k}{2} u^2 + \kappa k^3 u'' = C_1, \tag{2.54}$$

where C_1 is a constant of integration. Therefore (2.54) can be written as

$$k_1 u'' + k_2 u^2 + k_3 u' + k_4 = 0, \tag{2.55}$$

where

$$k_1 = \kappa k^3, \quad k_2 = \frac{\gamma k}{2}, \quad k_3 = \omega, \quad k_4 = -C_1. \tag{2.56}$$

Balancing $\kappa k^3 u'''$(the linear term of highest order) and the $\gamma k u u'$(non-linear term) in (2.53) yields the following equation $n+3 = 2n+1$ from which $n = 2$. Hence, we obtain the travelling wave solutions of (2.51) by assuming that the solution is of the form

$$u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2. \tag{2.57}$$

Substituting (2.57) along with (2.37) into (2.55) yields a polynomial equation in ϕ. By equating the coefficients to zero we obtain the following system,

$$\begin{cases} k_2 a_0^2 + k_3 a_0 + 2a_2 c_0 k_1 + k_4 = 0 \\ a_1 c_2 k_1 + 2a_0 a_1 k_2 + a_1 k_3 = 0 \\ k_2 a_1^2 + 4a_2 c_2 k_1 + 2a_0 a_2 k_2 + a_2 k_3 = 0 \\ 2a_1 c_4 k_1 + 2a_1 a_2 k_2 = 0 \\ 6a_2 c_4 k_1 + a_2^2 k_2 = 0. \end{cases} \tag{2.58}$$

Since k_1 and k_2 are given, the only unknowns are a_0, a_1, a_2, k_3 and k_4 that are found in terms of c_i, with $i = 0, 2, 4$. So if k_1 and k_2 are known, then k_3 and k_4 will be known also. Solving the system (2.58) with the aid of MATHEMATICA, yields the following solutions

$$\begin{cases} a_0 = \text{const} \\ a_1 = 0 \\ a_2 = -\dfrac{6c_4k_1}{k_2} \\ k_3 = -4c_2k_1 - 2a_0k_2 \\ k_4 = k_2a_0^2 + 4c_2k_1a_0 + \dfrac{12c_0c_4k_1^2}{k_2}. \end{cases} \tag{2.59}$$

Substituting (2.56) into (2.59), the travelling wave solution of the third order KdV, is

$$u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2 \tag{2.60}$$

with

$$\begin{cases} a_0 = \text{constant} \\ a_1 = 0 \\ a_2 = -\dfrac{12k^2\kappa c_4}{\gamma} \\ \omega = -a_0k\gamma - 4k^3\kappa c_2 \\ C_1 = -\dfrac{24\kappa^2c_0c_4k^3}{\gamma} - 4\kappa a_0c_2k^2 - \dfrac{1}{2}\gamma a_0^2k. \end{cases} \tag{2.61}$$

Using the results obtained in subsection 2.2.1, the travelling wave solutions of (2.51) can be obtained as follows

$$u(x,t) = a_0 - a_2\frac{c_2m^2}{c_4(2m^2-1)}\text{cn}^2\left(\sqrt{\frac{c_2}{(2m^2-1)}}\xi\right), \quad c_0 = \frac{c_2^2m^2(1-m^2)}{c_4(2m^2-1)^2} \tag{2.62}$$

$$u(x,t) = a_0 - a_2\frac{c_2}{c_4(2-m^2)}\text{dn}^2\left(\sqrt{\frac{c_2}{(2-m^2)}}\xi\right), \quad c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2} \tag{2.63}$$

$$u(x,t) = a_0 - a_2\frac{c_2m^2}{c_4(m^2+1)}\text{sn}^2\left(\sqrt{-\frac{c_2}{(m^2+1)}}\xi\right), \quad c_0 = \frac{c_2^2m^2}{c_4(m^2+1)^2} \tag{2.64}$$

As $m \to 1$, the Jacobi elliptic function solutions (2.84) and (2.85) degenerate to the following solitary wave

$$u(x,t) = a_0 - a_2\frac{c_2}{c_4}\text{sech}^2\left(\sqrt{c_2}\xi\right), \quad c_0 = 0, \tag{2.65}$$

and the Jacobi elliptic function solution (2.86) degenerates to the following

$$u(x,t) = a_0 - a_2\frac{c_2}{2c_4}\tanh^2\left(\sqrt{-\frac{c_2}{2}}\xi\right), \quad c_0 = \frac{c_2^2}{4c_4}. \tag{2.66}$$

Substituting some numerical values for the constants some well known solutions are obtained. Consider $\gamma = 6$, $\kappa = 1$ and $u(x,0) = 2\text{sech}^2(x)$ the initial conditions function of (2.51). Using (2.88), one obtains

$$u(x,0) = a_0 - a_2 \frac{c_2}{c_4} \text{sech}^2\left(\sqrt{c_2} kx\right) \tag{2.67}$$

Equating (2.90) and (2.92) yields the following relations

$$\begin{aligned} &k\sqrt{c_2} = 1, \quad \text{therefore} \quad k = 1 \quad \text{and} \quad c_2 = 1, \\ &a_0 = 0 \\ &a_2 = -2c_4 \\ &\omega = -4. \end{aligned} \tag{2.68}$$

Consequently, the solution of (2.51) with (2.92) as initial condition can be written as

$$u(x,t) = 2\ \text{sech}^2(x - 4t), \tag{2.69}$$

which has the behaviour of a travelling wave.

2.2.4 Applications to the generalized fifth order KdV equation

Consider the generalized fifth order KdV equation

$$u_t + \alpha u^2 u_x + \beta u u_{3x} + \gamma u_x u_{2x} + \kappa u_{5x} = 0, \tag{2.70}$$

where α, β, γ and κ are real constants. By applying the travelling wave solution

$$u = u(\xi), \quad \xi = kx + \omega t, \tag{2.71}$$

this method will give a series of travelling wave solutions for (2.71). We substitute (2.71) into (2.70) to obtain the following ordinary differential equation

$$\omega u' + \alpha k u^2 u' + \beta k^3 u u^{(3)} + \gamma k^3 u' u^{(2)} + \kappa k^5 u^{(5)} = 0. \tag{2.72}$$

Equation (2.72) can be written as

$$k_1 u' + k_2 u^2 u' + k_3 u u^{(3)} + k_4 u' u^{(2)} + k_5 u^{(5)} = 0. \tag{2.73}$$

where

$$k_1 = \omega, \quad k_2 = \alpha k, \quad k_3 = \beta k^3, \quad k_4 = \gamma k^3 \quad \text{and} \quad k_5 = \kappa k^5 \tag{2.74}$$

Balancing $k_2 u^2 u' + k_3 u u^{(3)} + k_4 u' u^{(2)}$ (non-linear term) and the $k_5 u^{(5)}$(the linear term of highest order) in (2.73) yields the following equation

$$3n + 1 = n + 5, \tag{2.75}$$

with solution $n = 2$. Hence, we obtain the travelling wave solutions of (2.70) by assuming that the solution is of the form

$$u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2 \tag{2.76}$$

Substituting (2.76) along with (2.37) into (2.73) yields a polynomial equation in ϕ. By equating their coefficients to zero we obtain the following system,

$$a_1k_2a_0^2 + a_1c_2k_3a_0 + a_1k_1 + 2a_1a_2c_0k_4 + a_1c_2^2k_5 + 12a_1c_0c_4k_5 = 0 \tag{2.77}$$

$$\begin{aligned}2a_2k_2a_0^2 + 2a_1^2k_2a_0 + 8a_2c_2k_3a_0 + 2a_2k_1 + a_1^2c_2k_3 + 4a_2^2c_0k_4 \\ +a_1^2c_2k_4 + 32a_2c_2^2k_5 + 144a_2c_0c_4k_5 = 0\end{aligned} \tag{2.78}$$

$$\begin{aligned}k_2a_1^3 + 6a_0a_2k_2a_1 + 9a_2c_2k_3a_1 + 6a_0c_4k_3a_1 + 6a_2c_2k_4a_1 \\ +60c_2c_4k_5a_1 = 0\end{aligned} \tag{2.79}$$

$$\begin{aligned}4a_2k_2a_1^2 + 6c_4k_3a_1^2 + 2c_4k_4a_1^2 + 4a_0a_2^2k_2 + 8a_2^2c_2k_3 + 24a_0a_2c_4k_3 \\ +8a_2^2c_2k_4 + 480a_2c_2c_4k_5 = 0\end{aligned} \tag{2.80}$$

$$5a_1k_2a_2^2 + 30a_1c_4k_3a_2 + 10a_1c_4k_4a_2 + 120a_1c_4^2k_5 = 0 \tag{2.81}$$

$$2k_2a_2^3 + 24c_4k_3a_2^2 + 12c_4k_4a_2^2 + 720c_4^2k_5a_2 = 0. \tag{2.82}$$

We choose only variables to have the maximum of all the c_i's. For this purpose, we will choose a_0, a_1, a_2 and k_1 as variables. So if k_2, k_3, k_4 and k_5 are known, then k_1 will be known also. Solving the system above with the aid of MATHEMATICA, yields the following solutions,

$$\begin{cases} a_1 = 0, \\ a_2 = -\dfrac{3c_4}{k_2}\left(2k_3 + k_4 \pm \sqrt{(2k_3+k_4)^2 - 40k_2k_5}\right), \\ k_1 = -k_2a_0^2 - 4c_2k_3a_0 - 2a_2c_0k_4 - 16c_2^2k_5 - 72c_0c_4k_5. \end{cases} \tag{2.83}$$

It is obvious that the parameters k_2, k_3, k_4 and k_5 need to satisfy $(2k_3 + k_4)^2 - 40k_2k_5 > 0$, in order to obtain real solutions.

From (2.78) and (2.80), we have the following cases

<u>*Case 1.*</u> If $a_2k_2 + 6c_4k_3 \neq 0$ and $(k_3 + k_4) + 60c_4k_5 = 0$ then $a_0 = 0$. Therefore, (2.70) has the following solutions

$$u(x,t) = -a_2\frac{c_2m^2}{c_4(2m^2-1)}\mathrm{cn}^2\left(\sqrt{\frac{c_2}{(2m^2-1)}}\xi\right), \quad c_0 = \frac{c_2^2m^2(1-m^2)}{c_4(2m^2-1)^2}, \tag{2.84}$$

$$u(x,t) = -a_2\frac{c_2}{c_4(2-m^2)}\mathrm{dn}^2\left(\sqrt{\frac{c_2}{(2-m^2)}}\xi\right), \quad c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2}, \tag{2.85}$$

$$u(x,t) = -a_2\frac{c_2m^2}{c_4(m^2+1)}\mathrm{sn}^2\left(\sqrt{-\frac{c_2}{(m^2+1)}}\xi\right), \quad c_0 = \frac{c_2^2m^2}{c_4(m^2+1)^2}, \tag{2.86}$$

with

$$\begin{cases} a_0 = 0, \\ a_1 = 0, \\ a_2 = -\dfrac{3k^2c_4}{\alpha}\left(2\beta+\gamma \pm \sqrt{(\gamma+2\beta)^2 - 40\alpha\kappa}\right), \\ \omega = -16\kappa c_2^2k^5 - 72\kappa c_0c_4k^5 - 2\gamma a_2c_0k^3 - 4\beta a_0c_2k^3 - k\alpha a_0^2. \end{cases} \tag{2.87}$$

Case 2. If $a_2k_2 + 6c_4k_3 = 0$ and $(k_3+k_4) + 60c_4k_5 = 0$, then a_0 is an arbitrary constant. In this case, (2.70) admits the following solutions

$$u(x,t) = a_0 - a_2\frac{c_2m^2}{c_4(2m^2-1)}\mathrm{cn}^2\left(\sqrt{\frac{c_2}{(2m^2-1)}}\xi\right), \quad c_0 = \frac{c_2^2m^2(1-m^2)}{c_4(2m^2-1)^2}, \tag{2.88}$$

$$u(x,t) = a_0 - a_2\frac{c_2}{c_4(2-m^2)}\mathrm{dn}^2\left(\sqrt{\frac{c_2}{(2-m^2)}}\xi\right), \quad c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2}, \tag{2.89}$$

$$u(x,t) = a_0 - a_2\frac{c_2m^2}{c_4(m^2+1)}\mathrm{sn}^2\left(\sqrt{-\frac{c_2}{(m^2+1)}}\xi\right), \quad c_0 = \frac{c_2^2m^2}{c_4(m^2+1)^2}, \tag{2.90}$$

with

$$\begin{cases} a_0 = \text{constant}, \\ a_1 = 0, \\ a_2 = -\dfrac{3k^2c_4}{\alpha}\left(2\beta+\gamma \pm \sqrt{(\gamma+2\beta)^2 - 40\alpha\kappa}\right), \\ \omega = -16\kappa c_2^2k^5 - 72\kappa c_0c_4k^5 - 2\gamma a_2c_0k^3 - 4\beta a_0c_2k^3 - k\alpha a_0^2. \end{cases} \tag{2.91}$$

Case 3. If $a_2k_2 + 6c_4k_3 \neq 0$ and $(k_3+k_4) + 60c_4k_5 \neq 0$, then $a_0 = -\dfrac{a_2\,(k_3+k_4)\,60c_4k_5}{a_2\,(a_2k_2+6c_4k_3)}$. Therefore, (2.70) has the following solutions

$$u(x,t) = a_0 - a_2\frac{c_2m^2}{c_4(2m^2-1)}\mathrm{cn}^2\left(\sqrt{\frac{c_2}{(2m^2-1)}}\xi\right), \quad c_0 = \frac{c_2^2m^2(1-m^2)}{c_4(2m^2-1)^2} \tag{2.92}$$

$$u(x,t) = a_0 - a_2\frac{c_2}{c_4(2-m^2)}\mathrm{dn}^2\left(\sqrt{\frac{c_2}{(2-m^2)}}\xi\right), \quad c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2}, \tag{2.93}$$

$$u(x,t) = a_0 - a_2 \frac{c_2 m^2}{c_4(m^2+1)} \mathrm{sn}^2 \left(\sqrt{-\frac{c_2}{(m^2+1)}} \xi \right), \quad c_0 = \frac{c_2^2 m^2}{c_4(m^2+1)^2}, \tag{2.94}$$

with

$$\begin{cases} a_0 = \dfrac{60\kappa c_4 k^4 + k^2 a_2 (\beta+\gamma)}{4a_2 (6\beta c_4 k^2 + \alpha a_2)}, \\ a_1 = 0, \\ a_2 = -\dfrac{3k^2 c_4}{\alpha} \left(2\beta + \gamma \pm \sqrt{(\gamma+2\beta)^2 - 40\alpha\kappa} \right), \\ \omega = -16\kappa c_2^2 k^5 - 72\kappa c_0 c_4 k^5 - 2\gamma a_2 c_0 k^3 - 4\beta a_0 c_2 k^3 - k\alpha a_0^2. \end{cases} \tag{2.95}$$

When $m \to 1$, all the solutions of the three cases degenerate to solitary wave solutions.

2.3 Summary of the chapter

In this chapter, we have applied the Fan sub-equation method [2, 3, 4] for constructing exact travelling wave solutions of the third and the fifth order generalized KdV equations. Some special cases, whose solutions are expressed in terms of Jacobi elliptic functions, have been found. It is shown that these solutions degenerate into solitary waves solutions when $m \to 1$. We successfully recover for the third order KdV equation solitary wave solutions that have been found using the inverse scattering transform [1]. This type of solutions expressed in term of Jacobi elliptic solutions are called 'cnoidal' wave solutions, and are a generalisation of solitary wave solutions. In the next chapter, we will first define what a soliton is, and finally using various methods, the Painlevé and the simplified Hirota's metod will construct soliton solutions of the third and fifth order of the KdV family.

Chapter 3

Soliton solutions

In [1, 9, 12], the inverse scattering transform method has been used to find soliton solutions of NLPDEs. Hirota [17] developed a technique for finding soliton solutions of NLPDEs which avoid the heavy machinery of the inverse scattering transform. However, the Hirota's method requires the knowledge of the Hirota's bilinear operator D. In order to overcome this difficulty, Hereman and Nusier have developed a more simplified version of the Hirota's method [25] which does not require any knowledge of the Hirota's operator. Both the original Hirota's method and the simplified Hirota's method are based on a change of variables. This change of variable is obtained using the Painlevé test. The Painlevé test reveals whether or not a given PDE soliton solutions.

3.1 The Painlevé test

Nonlinear evolution equations, in particular the KdV family, are widely used as models in various fields to describe physical phenomena. In order to test the integrability of nonlinear evolution equations, the Painlevé analysis or Painlevé test has been widely used [18, 19]. The Painlevé analysis is the study of singularity structure of differential equations. A singularity can be defined as a point or a set of points at which a differential equation fails to be well-behaved. There exist different types of singularities for solutions of ODEs. This is illustrated here by four examples:

Simple fixed pole

$$zu' + u = 0 \qquad \Rightarrow \qquad u(z) = \frac{u_0}{z} \tag{3.1}$$

Simple movable pole

$$u' + u^2 = 0 \qquad \Rightarrow \qquad u(z) = \frac{1}{z - u_0} \tag{3.2}$$

Movable algebraic branch point

$$2uu' - 1 = 0 \qquad \Rightarrow \qquad u(z) = \sqrt{z - u_0} \tag{3.3}$$

Movable logarithmic branch point

$$u'' - u'^2 = 0 \qquad \Rightarrow \qquad u(z) = \log(z - u_0) + c. \tag{3.4}$$

More details about singularities can be found in [34].

Definition 3.1.1 *An ODE satisfies the Painlevé property if all movable singularities of all solutions are poles.*

Definition 3.1.2 *A PDE is said to pass the Painlevé test if its solutions are "single-valued" about arbitrary noncharacteristic movable singularity manifold.*

Definition 3.1.3 *A noncharacteristic manifold for a given PDE is a surface on which there are enough free Cauchy data functions.*

The following example is an illustration of the Definition 3.1.2.

Example 3.1.4

$$u_{tt} - c^2 u_{xx} = 0, \tag{3.5}$$

has the general solution

$$u(x,t) = f(x - ct) + g(x + ct), \tag{3.6}$$

where f and g are arbitrary functions of their arguments. We can construct a solution u with any type of singularity along the characteristic manifolds $k_1 = t - cx$ and $k_2 = t + cx$ by an appropriate choice of f and g. For this particular problem, the Cauchy data are defined as the prescription of the functions on the line $t = 0$,

$$u(x,0) = \phi(x) \quad \text{and} \quad u_x(x,0) = \psi(x). \tag{3.7}$$

Inserting (3.7) into (3.6), gives

$$\begin{aligned} f(x) + g(x) &= \phi(x) && (3.8)\\ -cf'(x) + cg'(x) &= \psi(x). && (3.9) \end{aligned}$$

After integration of (3.8) and (3.9), the fundamental theorem of calculus yields

$$\begin{aligned} f(x) &= \frac{\phi(x)}{2} - \frac{1}{2c}\int_{x_0}^{x} \psi(w)dw && (3.10)\\ g(x) &= \frac{\phi(x)}{2} + \frac{1}{2c}\int_{x_0}^{x} \psi(w)dw. && (3.11) \end{aligned}$$

Inserting (3.10) and (3.11) into (3.6) yields

$$u(x,t) = \frac{1}{2}\left(\phi(x - ct) + \phi(x + ct)\right) + \frac{1}{2}\int_{x-ct}^{x+ct} \psi(w)dw, \tag{3.12}$$

which is known as D'Alembert's solution to the Cauchy problem of (3.5). The solution u is constructed with any type of singularity on the curves $k_1 = t - cx$ and $k_2 = t + cx$. Therefore (3.5) passes the Painlevé test.

3.1.1 Algorithm

Consider the following one dimensional partial differential equation

$$F(u, u_x, u_t, u_{xt}, ...) = 0. \tag{3.13}$$

Here $u = u(x,t)$ and F is a polynomial in u and its derivatives. The movable, noncharacteristic singularity manifold is given by $g(x,t) = 0$, where u is an analytic function of (x,t). If u is a solution of (3.13), then it can be expressed as a Laurent series in g

$$u(x,t) = g^{\eta}(x,t) \sum_{k=0}^{\infty} u_k(x,t) g^k(x,t) \tag{3.14}$$

where η is a negative integer, and g and u_k are analytic functions.

In this section, the Wies, Tabor, Carnevale (WTC) [20] method is employed in order to test the Painlevé property of the third and the fifth order KdV equations. However, in order to reduce the cumbersome computations of the WTC method, one can use the simplified Kruskal method [21] to test the Painlevé property of a PDE. Following the WTC method, the algorithm for the Painlevé test contains three steps:

Step 1 (Determine the dominant behaviour)

In order to determine the value of the leading order η in (3.14), it is sufficient to consider

$$u(x,t) = \chi g^{\eta}(x,t) \tag{3.15}$$

where χ is a constant. Substitution of (3.15) into the PDE (3.13) yields a polynomial equation in g. In the resulting polynomial equation in g, one equates two possible lowest exponents in the polynomial equation. This yields a linear equation in η that needs to be solved. Once η is known, the following equation

$$u(x,t) = u_0(x,t) g^{\eta}(x,t) \tag{3.16}$$

can be substituted into (3.13) to determine $u_0(x,t)$ when balancing the leading terms. If η is non-integer then the algorithm terminates, and the PDE does not pass the Painlevé test .

Step2 (Determine the resonances)

For (η, u_0), we calculate the resonance r for which u_r defined in (3.17), is an arbitrary function in (3.14). This is done by substituting the following equation

$$u(x,t) = u_0(x,t) g^{\eta}(x,t) + u_r(x,t) g^{\eta+r} \tag{3.17}$$

into (3.13). Then, keeping only the most singular terms of $g(x,t)$, the coefficients of $u_r(x,t)$ are equated to zero. If any of the resonances are non-integer, then the solutions of (3.13) have a movable algebraic branch, the algorithm terminates and the PDE does not have a Painlevé property. The universal resonance always occurs at $r = -1$ corresponding to the arbitrary choice of $g(x,t)$ [34]. The existence of negative resonances other than $r = -1$ compels us to do further analysis based on the pertubative Painlevé approach [35] since the Laurent series solution (3.14) is not a general solution.

Step 3 (Verify the compatibility conditions)

Equation (3.13) passes the Painlevé test if the Laurent expansion of the general solution has as many arbitrary functions u_r as the order of the PDE. This is done by substituting the following truncated expansion

$$u(x,t) = g^{\eta}(x,t)\sum_{k=0}^{r_{max}} u_k(x,t)g^k(x,t) \tag{3.18}$$

into (3.13), where r_{max} is the largest resonance.

Kruskal's simplification has been widely used in proving the Painlevé property [22, 23]. It is a way of making the computation of the WTC method drastically shorter by separating a variable in the singular manifold, which means that the singular manifold can be defined as

$$g(x,t) = x - h(t) \tag{3.19}$$

where $h(t)$ is an arbitrary function. In this case $u_k = u_k(t)$, thus Kruskal's simplification considerably reduces the length of the involved calculations. We shall not look at Kruskal's simplification. In the next section, we use this method to determine when the third and fifth order KdV equations satisfy the Painlevé property.

3.1.2 Applications to the generalised third order KdV equation

Consider the generalised third order KdV equation

$$u_t + \gamma u u_x + \kappa u_{3x} = 0, \tag{3.20}$$

where γ and κ are real constants. Equation (3.15) is substituted into (3.20) to yield

$$\begin{aligned} &\eta^3\kappa\chi g_x^3 g^{\eta-3} - 3\eta^2\kappa\chi g_x^3 g^{\eta-3} + 2\eta\kappa\chi g_x^3 g^{\eta-3} + 3\eta^2\kappa\chi g_x g_{2x} g^{\eta-2} \\ &-3\eta\kappa\chi g_x g_{2x} g^{\eta-2} + \eta\chi g_t g^{\eta-1} + \eta\kappa\chi g_{3x} g^{\eta-1} + \gamma\eta\chi^2 g_x g^{2\eta-1} = 0. \end{aligned} \tag{3.21}$$

The exponents of g are $\eta-1,\ \eta-2,\ \eta-3$ and $2\eta-1$. Therefore, equating the most negative exponents of g yields a linear equation in η

$$\eta - 3 = 2\eta - 1 \tag{3.22}$$

whose root is $\eta = -2$. Substituting (3.16), $u(x,t) = u_0(x,t)g^{-2}(x,t)$, into (3.20) and requiring that the leading terms balance, we obtain

$$u_0(x,t) = \frac{-12\kappa}{\gamma} g_x^2(x,t). \tag{3.23}$$

We substitute (3.17), $u(x,t) = \dfrac{-12\kappa}{\gamma} g_x^2(x,t)g^{-2}(x,t) + u_r(x,t)g^{r-2}(x,t)$, into (3.20) and we equate the leading terms of $u_r(x,t)$ (in this case, terms with $g^{r-5}(x,t)$) to zero, and then we obtain

$$(r+1)(r-4)(r-6)g_x^3(x,t) = 0. \tag{3.24}$$

Therefore, if $g_x(x,t) \neq 0$, the resonances of (3.20) are $r=-1,\ r=4$ and $r=6$. The universal resonance at $r=-1$ corresponds to the arbitrariness of singular manifold $g(x,t)=0$. In order to check the existence of a sufficient number of arbitrary functions at the resonance values, we substitute

$$u(x,t) = g^{-2}(x,t)\sum_{k=0}^{6} u_k(x,t)g^k(x,t), \tag{3.25}$$

into (3.20) and by grouping the terms in like powers of $g(x,t)$. From the coefficient of $g^{-5}(x,t)$, the explicit value of $u_0(x,t)$ is obtained as given in (3.23).
Equating the coefficients of $g^{-4}(x,t)$ and after algebraic simplifications, we obtain

$$u_1(x,t) = \frac{12\kappa g_{2x}}{\gamma}. \tag{3.26}$$

Collecting the coefficient of $g^{-3}(x,t)$, the explicit value of $u_2(x,t)$ is obtained as

$$u_2(x,t) = \frac{3\kappa g_{2x}^2 - g_t g_x - 4\kappa g_x g_{3x}}{\gamma g_x^2}. \tag{3.27}$$

The explicit value of $u_3(x,t)$ is obtained by equating coefficients of $g^{-2}(x,t)$

$$u_3(x,t) = \frac{3\kappa g_{2x}^3 - g_t g_x g_{2x} - 4\kappa g_x g_{3x} g_{2x} + g_{tx} g_x^2 + \kappa g_x^2 g_{4x}}{\gamma g_x^4}. \tag{3.28}$$

Collecting the coefficients in $g^{-1}(x,t)$ yields

$$(u_1)_t + \kappa\,(u_1)_{3x} + \gamma\,(u_3)_x\,u_0 + \gamma\,(u_2)_x\,u_1 + \gamma\,(u_1)_x\,u_2 + \gamma\,(u_0)_x\,u_3 = 0. \tag{3.29}$$

The absence of $u_4(x,t)$ in (3.29), proves the arbitrariness of $u_4(x,t)$. Equation (3.27) is satisfied when the explicit values of $u_0(x,t),\ldots,u_3(x,t)$ are substituted. Proceeding further to the coefficient of $g^0(x,t)$, the value of $u_5(x,t)$ is obtained

$$\begin{aligned} u_5(x,t) = \frac{1}{6\gamma\kappa g_x^8}[&-6\gamma\kappa\,(u_4)_x\,g_x^7 - 6\gamma\kappa g_{2x}u_4 g_x^6 - g_{2t}g_x^4 - 2\kappa g_{3xt}g_x^4 - \kappa^2 g_{6x}g_x^4 + 2g_t g_{tx}g_x^3 \\ &+9\kappa g_{2xt}g_{2x}g_x^3 + 8\kappa g_{tx}g_{3x}g_x^3 + 2\kappa g_t g_{4x}g_x^3 + 17\kappa^2 g_{3x}g_{4x}g_x^3 + 9\kappa^2 g_{2x}g_{5x}g_x^3 \\ &-21\kappa g_{tx}g_{2x}^2 g_x^2 - 70\kappa^2 g_{2x}g_{3x}^2 g_x^2 - g_t^2 g_{2x}g_x^2 - 17\kappa g_t g_{2x}g_{3x}g_x^2 - 48\kappa^2 g_{2x}^2 g_{4x}g_x^2 \\ &+21\kappa g_t g_{2x}^3 g_x + 174\kappa^2 g_{2x}^3 g_{3x}g_x - 81\kappa^2 g_{2x}^5]. \end{aligned} \tag{3.30}$$

Similarly, collecting the coefficient of $g^1(x,t)$, the result is obtained as zero. The absence of $u_6(x,t)$ proves that $u_6(x,t)$ is arbitrary. This corresponds to the resonance at $r=6$. Therefore, (3.20) admits a sufficient number of arbitrary functions which are $g(x,t)$, $u_4(x,t)$ and $u_6(x,t)$; it is said to pass the Painlevé test and is expected to be integrable [24].

Bäcklund transformation

In order to construct the Bäcklund transformation of (3.20), the Laurent series (3.14) is truncated at a constant level term

$$u(x,t) = u_0(x,t)g^{-2}(x,t) + u_1(x,t)g^{-1}(x,t) + u_2(x,t) \tag{3.31}$$

where either u and u_2 satisfy (3.20). Therefore, we take the trivial solution $u_2(x,t) = 0$ in (3.31) and we obtain the following transformation

$$u(x,t) = \frac{12\kappa}{\gamma} (\ln g)_{2x} . \tag{3.32}$$

The transfomation (3.32) will be useful in obtaining polysoliton solutions using the Simplified Hirota's method in the next section.

3.1.3 Applications to the generalised fifth order KdV equation

We are interested in verifying under which conditions the fifth order KdV equation passes the Painlevé test. Consider the generalised fifth order KdV equation

$$u_t + \alpha u^2 u_x + \beta u u_{3x} + \gamma u_x u_{2x} + \kappa u_{5x} = 0, \tag{3.33}$$

where α, β, γ and κ are real constants. Equation (3.15) is substituted into (3.33) to yield a polynomial in powers of $g(x,t)$ which is not displayed here because of its length. The exponents of $g(x,t)$ in that polynomial are $\eta - 1$, $\eta - 2$, $\eta - 3$, $\eta - 4$, $\eta - 5$, $2\eta - 1$, $2\eta - 2$, $2\eta - 3$ and $3\eta - 1$. Hence, equating the lowest exponents of $g(x,t)$ yields $3\eta - 1 = \eta - 5$, i.e., $\eta = -2$. Substituting (3.16), $u(x,t) = u_0(x,t)g^{-2}(x,t)$, into (3.33) and requiring that the leading terms balance, gives

$$-720\kappa g_x^5 + (12\gamma + 24\beta) g_x^3 u_0 - 2\alpha g_x u_0^2 = 0. \tag{3.34}$$

This is a quadratic equation in $u_0(x,t)$. Therefore, the solution of (3.34) is given explicitly by

$$u_0(x,t) = -\frac{3\left(2\beta + \gamma \pm \sqrt{(2\beta+\gamma)^2 - 40\alpha\kappa}\right)}{\alpha} g_x^2. \tag{3.35}$$

Substituting (3.17) into (3.34) and equating the leading terms of $u_r(x,t)$ (in this case terms with $g^{r-7}(x,t)$) to zero gives

$$\begin{aligned}\frac{1}{\alpha}(r-6)(r+1)[-6(r-4)\beta^2 - 3(r-8)\gamma\beta + 6\gamma^2 + \left(r^3 - 15r^2 + 86r - 240\right)\alpha\kappa \\ +(6\gamma - 3(r-4)\beta)\sqrt{(2\beta+\gamma)^2 - 40\alpha\kappa}\ ]\ g_x^5 = 0.\end{aligned} \tag{3.36}$$

The only trivial integer roots of (3.36) are $r = -1$ and $r = 6$. For arbitrary values of α, β, γ and κ, other integer roots r of (3.36) seem to be difficult to find. One way to overcome this difficulty is to find relationships between constants α, β, γ and κ. In order to fulfil this aim, we investigate the scaling properties of (3.33). Substituting $u = kv$, where u and v are arbitrary, we obtain

$$u_t + \alpha k^2 u^2 u_x + \beta k u u_{3x} + \gamma k u_x u_{2x} + \kappa u_{5x} = 0. \tag{3.37}$$

Let a, b and c be arbitrary constants such that $a = \alpha k^2$, $b = \beta k$ and $c = \gamma k$. After simplification, we obtain

$$\gamma = \frac{c}{b}\beta \quad \text{and} \quad \alpha = \frac{a}{b^2}\beta^2. \tag{3.38}$$

Without loss of generality, we investigate integer roots of (3.36) for three cases.

The Kaup-Kuperschmidt equation

If we choose $a = 20$, $b = 10$ and $c = 25$ in (3.38), we obtain

$$\gamma = \frac{5}{2}\beta, \text{ and } \alpha = \frac{1}{5}\beta^2. \tag{3.39}$$

Substituting (3.39) into (3.33) yields a more general form of the KK equation

$$u_t + \frac{1}{5}\beta^2 u^2 u_x + \beta u u_{3x} + \frac{5}{2}\beta u_x u_{2x} + \kappa u_{5x} = 0. \tag{3.40}$$

Clearly, when $\beta = 10$ and $\kappa = 1$ the classic KK equation [16, 25] is obtained. If $g_x^5 \neq 0$ and $\kappa = 1$ in (3.36), we obtain two sets of resonances

$$r = \begin{cases} -7, -1, 6, 10, 12 \\ -1, 3, 5, 6, 7. \end{cases} \tag{3.41}$$

The existence of sufficient arbitrary functions at the first set of resonances are verified by substituting

$$u(x,t) = g^{-2}(x,t) \sum_{k=0}^{12} u_k(x,t) g^k(x,t), \tag{3.42}$$

into (3.40) and by grouping the terms in like powers of $g(x,t)$. From the coefficient of $g^{-7}(x,t)$, the explicit form of $u_0(x,t)$ is obtained as given in (3.35)

$$u_0(x,t) = -\frac{120}{\beta} g_x^2(x,t) \quad \text{or} \quad u_0(x,t) = -\frac{15}{\beta} g_x^2(x,t). \tag{3.43}$$

We first the show compatibility condition for the first value of $u_0(x,t) = -\dfrac{120}{\beta} g_x^2(x,t)$. Equating the coefficients of $g^{-6}(x,t)$ and after algebraic simplifications, the explicit value of $u_1(x,t)$ is obtained as

$$u_1(x,t) = \frac{120}{\beta} g_{2x}(x,t). \tag{3.44}$$

Collecting coefficients of $g^{-5}(x,t)$ yields the explicit value of $u_2(x,t)$

$$u_2(x,t) = -\frac{10\,(4 g_x g_{3x} - 3 g_{2x}^2)}{\beta g_x^2}. \tag{3.45}$$

Proceeding in the same way, from the coefficients of $g^{-4}(x,t)$ is obtained the explicit value of $u_3(x,t)$

$$u_3(x,t) = \frac{10\,(3 g_{2x}^3 - 4 g_x g_{3x} g_{2x} + g_x^2 g_{4x})}{\beta g_x^4}. \tag{3.46}$$

Similarly, from the coefficients of $g^{-3}(x,t)$, the explicit value of u_4 is found

$$u_4(x,t) = \frac{825 g_{2x}^4 - 1320 g_x g_{3x} g_{2x}^2 + 330 g_x^2 g_{4x} g_{2x} + g_t g_x^3 + 220 g_x^2 g_{3x}^2 - 44 g_x^3 g_{5x}}{22\beta g_x^6}. \tag{3.47}$$

The explicit value of $u_5(x,t)$ is obtained from the coefficient of $g^{-2}(x,t)$ as

$$u_5(x,t) = \frac{1}{66\beta g_x^8}[3465g_{2x}^5 - 6600g_x g_{3x} g_{2x}^3 + 1650g_x^2 g_{4x} g_{2x}^2 + 6g_t g_x^3 g_{2x} + 2200g_x^2 g_{3x}^2 g_{2x} \\ -264g_x^3 g_{5x} g_{2x} - 3g_{tx} g_x^4 - 440g_x^3 g_{3x} g_{4x} + 22g_x^4 g_{6x}]. \quad (3.48)$$

Collecting the coefficients of $g^{-1}(x,t)$, the result is obtained as zero. The absence of $u_6(x,t)$ proves that $u_6(x,t)$ is arbitrary. We proceed in the same manner to obtain the explicit values of the remaining functions $u_7(x,t),\ldots,u_{12}(x,t)$. The explicit form of these functions will not be displayed here because of their length. One keeps in mind that collecting the coefficients of $g^3(x,t)$ and $g^5(x,t)$ the result is obtained as zero. This prove that $u_{10}(x,t)$ and $u_{12}(x,t)$ are arbitrary. By convention, the resonance $r = -7$ is ignored since it violates the hypothesis that $g^{-2}(x,t)$ is the dominant term in the expansion near $g(x,t) = 0$. The term corresponding to the resonance $r = -7$ does not contribute to the expansion. Therefore, this leads to a particular solution and the general solution may still be multi-valued. This set of resonances does not allow the KK equation to pass the Painlevé test.

Now we verify the compatibility condition for the second value of $u_0(x,t) = -\frac{15}{\beta} g_x^2(x,t)$ and the second set of resonances. Substituting

$$u(x,t) = g^{-2}(x,t)\sum_{k=0}^{12} u_k(x,t) g^k(x,t), \quad (3.49)$$

into (3.40) yields a function in power of $g(x,t)$. Equating the coefficients of $g^{-6}(x,t)$ and after algebraic simplifications, the explicit value of $u_1(x,t)$ is obtained as

$$u_1(x,t) = \frac{15}{\beta} g_{2x}(x,t). \quad (3.50)$$

Collecting coefficients of $g^{-5}(x,t)$ yields the explicit value of $u_2(x,t)$

$$u_2(x,t) = -\frac{5\,(4g_x g_{2x} - 3g_{2x}^2)}{4\beta g_x^2}. \quad (3.51)$$

Proceeding in the same way, from the coefficients of $g_{-4}(x,t)$ is obtained the explicit value zero. This means $u_3(x,t)$ is an arbitrary function. Similarly, from the coefficients of $g^{-3}(x,t)$, the explicit value of u_4 is found

$$u_4(x,t) = \frac{1}{16\beta g_x^6}[-16\beta\,(u_3)_x\, g_x^5 - 8\beta g_{2x} u_3(x,t) g_x^4 + 16g_t g_x^3 + 16g_{5x} g_x^3 \\ -60g_{3x}^2 g_x^2 - 80g_{2x} g_{4x} g_x^2 + 260g_{2x}^2 g_{3x} g_x - 135g_{2x}^4]. \quad (3.52)$$

Collecting the coefficients of $g^{-2}(x,t)$, the result is obtained as zero. The absence of $u_5(x,t)$ proves that $u_5(x,t)$ is arbitrary. Similarly, collecting the coefficients of $g^1(x,t)$ and $g^0(x,t)$

the result is obtained as zero. This proves that $u_6(x,t)$ and $u_7(x,t)$ are arbitrary. Therefore, this set of resonances allows the KK equation to pass the Painlevé test. One of the two sets of resonances allows the KK equation to pass the Painlevé test. This is a sufficient condition for the KK equation (3.40) to pass the Painlevé test. The KK equation is expected to be integrable.

The Lax equation

From (3.38), if one chooses $a = 30$, $b = 10$ and $c = 20$ one obtains

$$\gamma = 2\beta, \quad \alpha = \frac{3}{10}\beta^2 \text{ and } \kappa = 1. \tag{3.53}$$

Substituting (3.53) into (3.33) yields a more general form of the Lax equation

$$u_t + \frac{3}{10}\beta^2 u^2 u_x + \beta u u_{3x} + 2\beta u_x u_{2x} + \kappa u_{5x} = 0. \tag{3.54}$$

One can see that when $\beta = 10$ and $\kappa = 1$ the classic Lax equation [26] is obtained. If $g_x^5 \neq 0$ and $\kappa = 1$ in (3.36) one obtains two sets of resonance values

$$r = \begin{cases} -3, -1, 6, 8, 10 \\ -1, 2, 5, 6, 8. \end{cases} \tag{3.55}$$

The Sawada-Kotera equation

Similarly, if one chooses $a = 5$, $b = 5$ and $c = 5$ in (3.38), one obtains

$$\gamma = \beta, \quad \text{and} \quad \alpha = \frac{1}{5}\beta^2. \tag{3.56}$$

Substituting (3.56) into (3.33) yields a more general form of the Lax equation

$$u_t + \frac{1}{5}\beta^2 u^2 u_x + \beta u u_{3x} + \beta u_x u_{2x} + \kappa u_{5x} = 0. \tag{3.57}$$

Clearly, when $\beta = 5$ and $\kappa = 1$ the classic SK equation is obtained. If $g_x^5 \neq 0$ and $\kappa = 1$ in (3.36), two sets of resonance values are obtained resonances of (3.36) are at

$$r = \begin{cases} -2, -1, 5, 6, 12 \\ -1, 2, 3, 6, 10. \end{cases} \tag{3.58}$$

One can verify as we did it for the KK equation that the general form of the Lax (3.54) and the Sawada-Kotera (3.57) equations both pass the Painlevé test since one of the two sets of resonances passes the Painlevé test. Therefore they are expected to be integrable.

The Bäcklund transformation

To construct the Bäcklund transformation of (3.33), we truncate the Laurent series (3.14) at a constant level term to obtain

$$u(x,t) = u_0(x,t)g^{-2}(x,t) + u_1(x,t)g^{-1}(x,t) + u_2(x,t), \tag{3.59}$$

where u and u_2 satisfy (3.33). If we take the trivial solution $u_2(x,t) = 0$ in (3.59) we get

$$u(x,t) = u_0(x,t)g^{-2}(x,t) + u_1(x,t)g^{-1}(x,t). \tag{3.60}$$

Substituting (3.60) into (3.33), and after algebraic simplification, the explicit form of $u_1(x,t)$ is given by

$$u_1(x,t) = \frac{3\left((\gamma+2\beta) \pm \sqrt{(\gamma-2\beta)^2 - 40\alpha\kappa}\right)}{\alpha} g_{xx}. \tag{3.61}$$

Finally if (3.35) and (3.61) are substituted into (3.60), the following transformation is obtained

$$u(x,t) = \frac{3\left((\gamma+2\beta) \pm \sqrt{(\gamma-2\beta)^2 - 40\alpha\kappa}\right)}{\alpha} (\ln g)_{xx}. \tag{3.62}$$

Equation (3.62) is of paramount importance in finding multisoliton solutions of the fifth oder integrable KdV equations.

3.2 Simplified Hirota method

In order to test the validity of the simplified Hirota method introduced by Hereman and Nusier [16, 25], we derive N-soliton solutions of (3.20) and (3.40).

3.2.1 Generalised third order equation

Consider the generalised third order equation

$$u_t + \gamma u u_x + \kappa u_{3x} = 0. \tag{3.63}$$

If we substitute the following change of variables

$$u(x,t) = \frac{12\kappa}{\gamma} (\ln f)_{xx} \tag{3.64}$$

into (3.63), the following quadratic equation in f and its derivatives is obtained

$$f\,(f_{xt} + \kappa f_{4x}) - f_t f_x + 3\kappa f_{2x}^2 - 4\kappa f_x f_{3x} = 0. \tag{3.65}$$

The transformation (3.64) is the Bäcklund transformation of (3.63) obtained in Section 3.1.2. Equation (3.65) is a polynomial in f of degree one where the coefficients are polynomials of derivatives of f.

Let $\mathcal{L}$ be a linear differential operator and $\mathcal{N}$ be a nonlinear differential operator defined as follows

$$\mathcal{L}\bullet = \frac{\partial\bullet}{\partial x\partial t} + \kappa\frac{\partial^4\bullet}{\partial x^4} \quad \text{and} \quad \mathcal{N}(f,g) = -f_t g_x + 3\kappa f_{2x}g_{2x} - 4\kappa f_x g_{3x}. \tag{3.66}$$

Note the coefficient of the highest power of the polynomial is defined as the linear differential operator, while other coefficients are defined as nonlinear differential operators. The substitution of (3.66) into (3.65) allows one to write (3.65) as

$$f\mathcal{L}(f) + \mathcal{N}(f,f) = 0. \tag{3.67}$$

Suppose that we want to find f such that

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x,t), \tag{3.68}$$

where $f^{(n)}(n = 1,2,3\ldots)$ are to be determined later and for simplicity we take $\epsilon = 1$. Substituting (3.116) into (3.67) yields a polynomial in power of ϵ. We equate to zero terms in like powers of ϵ to obtain a hierarchy of equations for $f^{(n)}$,

$$O(\epsilon^1) \;:\; \mathcal{L}(f^{(1)}) = 0 \tag{3.69}$$

$$O(\epsilon^2) \;:\; \mathcal{L}(f^{(2)}) = -\mathcal{N}\left(f^{(1)}, f^{(1)}\right) \tag{3.70}$$

$$O(\epsilon^3) \;:\; \mathcal{L}(f^{(3)}) = -f^{(1)}\mathcal{L}f^{(2)} - \mathcal{N}\left(f^{(1)}, f^{(2)}\right) - \mathcal{N}\left(f^{(1)}, f^{(2)}\right) \tag{3.71}$$

$$\vdots$$

$$O(\epsilon^n) \;:\; \mathcal{L}(f^{(n)}) = -\sum_{j=1}^{n-1}\left[\mathcal{N}\left(f^{(j)}, f^{(n-j)}\right) + f^{(j)}\mathcal{L}(f^{(n-j)})\right] \tag{3.72}$$

In order to determine the $f^{(n)}(n = 1,2,3\ldots)$ functions, we need to solve the family of the equations above simultaneously . If there exists an integer $m \leq n$ such that $f^n = 0$, the series (3.116) truncates at order $m-1$ to become

$$f(x,t) = 1 + \sum_{n=1}^{m-1} \epsilon^n f^{(n)}(x,t). \tag{3.73}$$

Note that (3.69) is linear, and therefore admits exponential solutions.

One soliton solution

In order to obtain one-soliton solutions, we let

$$f^{(1)} = \mathrm{e}^{\xi_1}, \tag{3.74}$$

with $\xi_1 = k_1 x - \omega_1 t + \delta_1$. The computation of (3.69) yields the following dispersion term $\omega_1 = \kappa k_1^3$. Computing the right hand side of (3.70) gives

$$-\mathcal{N}\left(f^{(1)}, f^{(1)}\right) = 0. \tag{3.75}$$

This means that $f^{(2)}$ will be equal to zero. It is easy to check that for $n \geq 2$ $f^{(n)} = 0$. The series (3.116) therefore truncates and yields

$$f(x,t) = 1 + \mathrm{e}^{\xi_1}. \tag{3.76}$$

The one-soliton solutions of (3.63) is generated when we substitute (3.77) into (3.64). After some algebraic manipulation we get

$$u(x,t) = \frac{3\kappa k_1^2}{\gamma}\mathrm{sech}^2\left(\frac{\xi_1}{2}\right). \tag{3.77}$$

Two soliton solution

In order to obtain the two-soliton solution, one has to use

$$f^{(1)} = \mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2}, \tag{3.78}$$

with $\xi_1 = k_1 x - \omega_1 = \delta_1$ and $\xi_2 = k_2 x - \omega_2 = \delta_2$. The dispersion terms are obtained from the computation of (3.69) as $\omega_1 = \kappa k_1^3$ and $\omega_2 = \kappa k_2^3$. We next compute the right-hand side of (3.70) as

$$-\mathcal{N}\left(f^{(1)}, f^{(1)}\right) = 3\kappa k_1 k_2 \left(k_1 - k_2\right)^2 e^{\xi_1+\xi_2}. \tag{3.79}$$

This means that $f^{(2)}$ will take the following form

$$f^{(2)} = a\mathrm{e}^{\xi_1+\xi_2}. \tag{3.80}$$

Substituting (3.80) into (3.70), one can compute the left hand side of (3.70)

$$\mathcal{L}f^{(2)} = 3a\kappa k_1 k_2 \left(k_1 + k_2\right)^2 \mathrm{e}^{\xi_1+\xi_2}. \tag{3.81}$$

Balancing (3.79) and (3.80) gives the explicit value of $a = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$. Consequently, we obtain the explicit value of $f^{(2)}$ as

$$f^{(2)} = \frac{\left(k_1 - k_2\right)^2}{\left(k_1 + k_2\right)^2}\mathrm{e}^{\xi_1+\xi_2}. \tag{3.82}$$

It is easy to check that $f^{(n)} = 0$ for $n \geq 3$. Therefore, using (3.64), the two-soliton solution, generated by

$$f = 1 + \mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2} + \frac{\left(k_1 - k_2\right)^2}{\left(k_1 + k_2\right)^2}\mathrm{e}^{\xi_1+\xi_2}, \tag{3.83}$$

is given as follows

$$u(x,t) = \frac{12\kappa\left(k_1^2 e^{\xi_1}\left(1 + e^{\xi_2}\right)\left(e^{\xi_2}a + 1\right) + 2k_2 k_1 (a-1)e^{\xi_1+\xi_2} + k_2^2 e^{\xi_2}\left(1 + e^{\xi_1}\right)\left(e^{\xi_1}a + 1\right)\right)}{\gamma\left(e^{\xi_1+\xi_2}a + e^{\xi_1} + e^{\xi_2} + 1\right)^2}. \tag{3.84}$$

Three soliton solution

For the three soliton solution, one may start with

$$f^{(1)} = \mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2} + \mathrm{e}^{\xi_3}, \tag{3.85}$$

with $\xi_i = k_i x + \omega_i t + \delta_i$ for $i = 1, 2, 3$. From the computation of (3.69), one gets the dispersion terms $\omega_i = \kappa k_i^3$ for $i = 1, 2, 3$. In order to determine $f^{(2)}$, we first compute the right-hand side of (3.70)

$$\begin{aligned} -\mathcal{N}\left(f^{(1)}, f^{(1)}\right) &= 3\kappa k_1 k_2 \left(k_1 - k_2\right)^2 e^{\xi_1+\xi_2} \\ &\quad +3\kappa k_1 k_3 \left(k_1 - k_3\right)^2 e^{\xi_1+\xi_3} \\ &\quad +3\kappa k_2 k_3 \left(k_2 - k_3\right)^2 e^{\xi_2+\xi_3}. \end{aligned} \tag{3.86}$$

From equation (3.86), we conclude that $f^{(2)}$ is of the form

$$f^{(2)} = a_{12} e^{\xi_1+\xi_2} + a_{13} e^{\xi_1+\xi_3} + a_{23} e^{\xi_2+\xi_3}. \tag{3.87}$$

We next compute the left-hand side of (3.69)

$$\begin{aligned} \mathcal{L} f^{(2)} &= 3\kappa k_1 k_2 a_{12} \left(k_1 + k_2\right)^2 e^{\xi_1+\xi_2} \\ &\quad +3\xi k_1 k_3 a_{13} \left(k_1 + k_3\right)^2 e^{\xi_1+\xi_3} \\ &\quad +3\kappa k_2 k_3 a_{23} \left(k_2 + k_3\right)^2 e^{\xi_2+\xi_3}. \end{aligned} \tag{3.88}$$

Equating (3.86) and (3.88) yields the explicit values of a_{12}, a_{13} and a_{23}

$$a_{12} = \frac{\left(k_1 - k_2\right)^2}{\left(k_1 + k_2\right)^2}, \quad a_{13} = \frac{\left(k_1 - k_3\right)^2}{\left(k_1 + k_3\right)^2} \quad \text{and} \quad a_{23} = \frac{\left(k_2 - k_3\right)^2}{\left(k_2 + k_3\right)^2}. \tag{3.89}$$

Similarly, one computes the right-hand side of (3.71)

$$\begin{aligned} -\mathcal{N}\left(f^{(2)}, f^{(2)}\right) &= (-3\kappa a_{12} \left(k_1 + k_2\right) \left(k_1 - k_3\right) \left(k_2 - k_3\right) \left(k_1 + k_2 - k_3\right) \\ &\quad +3\kappa a_{13} \left(k_1 - k_2\right) \left(k_2 - k_3\right) \left(k_1 + k_3\right) \left(k_1 - k_2 + k_3\right) \\ &\quad +3\kappa a_{23} \left(k_1 - k_2\right) \left(k_2 + k_3\right) \left(k_1 - k_3\right) \left(k_1 - k_2 - k_3\right)) e^{\xi_1+\xi_2+\xi_3}. \end{aligned} \tag{3.90}$$

Therefore $f^{(3)}$ is of the form

$$f^{(3)} = b_{123} e^{\xi_1+\xi_2+\xi_3}. \tag{3.91}$$

The computation of the left-hand side of (3.71) yields

$$\mathcal{L} f^{(3)} = 3\kappa b_{123} \left(k_1 + k_2\right) \left(k_1 + k_3\right) \left(k_2 + k_3\right) \left(k_1 + k_2 + k_3\right) e^{\xi_1+\xi_2+\xi_3}. \tag{3.92}$$

Equating (3.90) and (3.90), one can determine the explicit value of b_{123} as follows

$$b_{123} = a_{12} a_{13} a_{23} = \frac{\left(k_1 - k_2\right)^2 \left(k_1 - k_3\right)^2 \left(k_2 - k_3\right)^2}{\left(k_1 + k_2\right)^2 \left(k_1 + k_3\right)^2 \left(k_2 + k_3\right)^2}. \tag{3.93}$$

One can check that for $n \geq 4$, $f^{(n)} = 0$. The series (3.116) truncates into

$$f(x,t) = 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + a_{12}e^{\xi_1+\xi_2} + a_{13}e^{\xi_1+\xi_3} + a_{23}e^{\xi_2\xi_3} + b_{123}e^{\xi_1+\xi_2+\xi_3}, \tag{3.94}$$

from which the three-soliton solution is obtained

$$u(x,t) = \frac{12\kappa}{\gamma}(\ln(f))_{xx}. \tag{3.95}$$

The results above can be generalised to seek the N-soliton solution of the (3.63) by taking

$$f^{(1)} = \sum_{i=1}^{N} e^{\xi_i} = \sum_{i=1}^{N} e^{k_i x - \omega_i t + \delta_i}. \tag{3.96}$$

We substitute (3.96) into (3.69) to obtain the following dispersion term

$$\omega_i = \kappa k_i^3. \tag{3.97}$$

N-soliton solution

One can obtain the right hand side of (3.70) by substituting (3.96) and (3.97) into (3.70)

$$\begin{aligned} & 3\kappa\left[\left(\sum_{i=1}^{N} k_i^2 e^{\xi_i}\right)\left(\sum k_j^2 e^{\xi_j}\right) - \left(\sum_{i=1}^{N} e^{\xi_i}\right)\left(\sum_{j=1}^{N} k_j^3 e^{\xi_j}\right)\right] \\ = & -\sum_{i,j}^{N} 3\kappa k_i k_j^2 \left(k_i - k_j\right) e^{\xi_i - \xi_j} \\ = & \sum_{1\leq i<j\leq N} 3\kappa k_i k_j \left(k_i - k_j\right)^2 e^{\xi_i - \xi_j}. \end{aligned} \tag{3.98}$$

Therefore $f^{(2)}$ will be of the form

$$f^{(2)} = \sum_{1\leq i<j\leq N} a_{ij} e^{\xi_i + \xi_j}. \tag{3.99}$$

Substituting (3.99) into the left hand side of (3.70) yields

$$\mathcal{L} f^{(2)} = \sum_{1\leq i<j\leq N} 3k_i k_j \left(k_i + k_j\right)^2 a_{ij} e^{\xi_i + \xi_j}. \tag{3.100}$$

Equating (3.98) and (3.100) gives the explicit value of a_{ij}

$$a_{ij} = \frac{\left(k_i - k_j\right)^2}{\left(k_i + k_j\right)^2}, \quad 1 \leq i < j \leq N. \tag{3.101}$$

We proceed in the same way in order to determine the explicit values of $f^{(3)}, f^{(4)} \ldots f^{(n)}$.

3.2.2 Generalised Kaup-Kupershmidt equation

In this subsection, we follow the work of Hereman and Nuseir [16, 25] in order to generate soliton solutions of the generalised Kaup-Kupershmidt equation

$$u_t + \frac{1}{5}\beta^2 u^2 u_x + \beta u u_{3x} + \frac{5}{2}\beta u_x u_{2x} + u_{5x} = 0. \tag{3.102}$$

Two transformations have been obtained from the Painlevé analysis

$$u(x,t) = \frac{120}{\beta} \ln\left(f(x,t)\right)_{xx} \quad \text{or} \quad u(x,t) = \frac{15}{\beta} \ln\left(f(x,t)\right)_{xx}. \tag{3.103}$$

Substituting the second relation of (3.103) into (3.102) yields a third degree polynomial in f whose coefficients are polynomials of its derivatives

$$\begin{aligned} &4f^3\left(f_{tx} + f_{6x}\right) - f^2\left(4f_t f_x - 5f_{3x}^2 + 24 f_x f_{5x}\right) \\ &-30f\left(f_x f_{2x} f_{3x} - 2f_x^2 f_{4x}\right) + 15\left(3f_x^2 f_{2x}^2 - 4f_x^3 f_{3x}\right) = 0. \end{aligned} \tag{3.104}$$

The coefficient of f^3 is defined as the linear differential operator $\mathcal{L}$. The coefficients of f^2, f^1 and f^0 are defined as the nonlinear operators of $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$ respectively such that

$$\mathcal{L}\bullet = 4\left(\frac{\partial^2\bullet}{\partial x \partial t} + \frac{\partial^6 \bullet}{\partial x^6}\right), \tag{3.105}$$

$$\mathcal{N}_1(f,g) = -4f_t g_x + 5f_{3x}g_{3x} - 24 f_x g_{5x}, \tag{3.106}$$

$$\mathcal{N}_2(f,g,h) = -30 f_x g_{2x} h_{3x} + 60 f_x g_x h_{4x}, \tag{3.107}$$

$$\mathcal{N}_3(f,g,h,j) = 54 f_x g_x h_{2x} j_{2x} - 60 f_x g_x h_x j_{3x}, \tag{3.108}$$

where $f(x,t), g(x,t), h(x,t)$ and $j(x,t)$ are auxiliary functions. Therefore (3.104) can be written in an operator form

$$f^3\mathcal{L}(f) + f^2\mathcal{N}_1(f,f) + f\mathcal{N}_2(f,f,f) + \mathcal{N}_3(f,f,f,f). \tag{3.109}$$

We are interested only in solutions of the form

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}. \tag{3.110}$$

Substituting (3.128) into (3.109) gives a polynomial in powers of ϵ. We equate the coefficients in like powers of ϵ to obtain the following hierarchy equation in $f^{(n)}$.

$$
\begin{aligned}
O(\epsilon^1): \quad \mathcal{L}(f^{(1)}) &= 0 && (3.111)\\
O(\epsilon^2): \quad \mathcal{L}(f^{(2)}) &= -\mathcal{N}_1\left(f^{(1)}, f^{(1)}\right) && (3.112)\\
O(\epsilon^3): \quad \mathcal{L}(f^{(3)}) &= -3f^{(1)}\mathcal{L}f^{(2)} - 2\mathcal{N}_1\left(f^{(1)}, f^{(1)}\right) - \mathcal{N}_1\left(f^{(2)}, f^{(1)}\right)\\
&\quad -\mathcal{N}_1\left(f^{(1)}, f^{(2)}\right) - \mathcal{N}_2\left(f^{(1)}, f^{(2), f^{(1)}}\right) && (3.113)\\
O(\epsilon^4): \quad \mathcal{L}(f^{(4)}) &= -3f^{(1}\mathcal{L}f^{(3)} - 3f^{(1)^2}\mathcal{L}f^{(2)} - 3f^{(2)}\mathcal{L}f^{(2)} - \mathcal{N}_1\left(f^{(2)}, f^{(2)}\right) - \mathcal{N}_1\left(f^{(1)}, f^{(3)}\right)\\
&\quad -\mathcal{N}_1\left(f^{(3)}, f^{(1)}\right) - 2f^{(1)}\mathcal{N}_1\left(f^{(1)}, f^{(2)}\right) - 2f^{(1)}\mathcal{N}_1\left(f^{(2)}, f^{(1)}\right)\\
&\quad -f^{(1)^2}\mathcal{N}_1\left(f^{(1)}, f^{(1)}\right) - 2f^{(2)}\mathcal{N}_1\left(f^{(1)}, f^{(1)}\right) - f^{(1)}\mathcal{N}_2\left(f^{(1)}, f^{(1)}, f^{(1)}\right)\\
&\quad -\mathcal{N}_2\left(f^{(1)}, f^{(1)}, f^{(2)}\right) - \mathcal{N}_2\left(f^{(1)}, f^{(2)}, f^{(1)}\right)\\
&\quad -\mathcal{N}_2\left(f^{(2)}, f^{(1)}, f^{(1)}\right) - \mathcal{N}_3\left(f^{(1)}, f^{(1)}, f^{(1)}, f^{(1)}\right) && (3.114)\\
O(\epsilon^5): \quad \mathcal{L}(f^{(5)}) &= -3f^{(1}\mathcal{L}f^{(4)} - 3f^{(1)^2}\mathcal{L}f^{(3)} - f^{(1)^3}\mathcal{L}f^{(2)} - 3f^{(2)}\mathcal{L}f^{(3)} - 6f^{(1)}f^{(2)}\mathcal{L}f^{(2)}\\
&\quad -f^3\mathcal{L}f^{(3)} - \mathcal{N}_1\left(f^{(3)}, f^{(2)}\right) - \mathcal{N}_1\left(f^{(2)}, f^{(3)}\right) - \mathcal{N}_1\left(f^{(1)}, f^{(4)}\right)\\
&\quad -\mathcal{N}_1\left(f^{(4)}, f^{(1)}\right) - 2f^{(1)}\mathcal{N}_1\left(f^{(2)}, f^{(2)}\right) - 2f^{(1)}\mathcal{N}_1\left(f^{(3)}, f^{(1)}\right)\\
&\quad -2f^{(1)}\mathcal{N}_1\left(f^{(1)}, f^{(3)}\right) - f^{(1)^2}\mathcal{N}_1\left(f^{(2)}, f^{(1)}\right) - f^{(1)^2}\mathcal{N}_1\left(f^{(1)}, f^{(2)}\right)\\
&\quad -2f^{(2)}\mathcal{N}_1\left(f^{(2)}, f^{(1)}\right) - 2f^{(2)}\mathcal{N}_1\left(f^{(1)}, f^{(2)}\right) - 2f^{(1)}f^{(2)}\mathcal{N}_1\left(f^{(1)}, f^{(1)}\right)\\
&\quad -2f^{(3)}\mathcal{N}_1\left(f^{(1)}, f^{(1)}\right) - \mathcal{N}_2\left(f^{(1)}, f^{(2)}, f^{(2)}\right) - \mathcal{N}_2\left(f^{(2)}, f^{(1)}, f^{(2)}\right)\\
&\quad -\mathcal{N}_2\left(f^{(2)}, f^{(2)}, f^{(1)}\right) - \mathcal{N}_2\left(f^{(1)}, f^{(1)}, f^{(3)}\right) - \mathcal{N}_2\left(f^{(1)}, f^{(3)}, f^{(1)}\right)\\
&\quad -\mathcal{N}_2\left(f^{(3)}, f^{(1)}, f^{(1)}\right) - f^{(1)}\mathcal{N}_2\left(f^{(1)}, f^{(1)}, f^{(2)}\right) - f^{(1)}\mathcal{N}_2\left(f^{(1)}, f^{(2)}, f^{(1)}\right)\\
&\quad -f^{(1)}\mathcal{N}_2\left(f^{(2)}, f^{(1)}, f^{(2)}\right) - f^{(2)}\mathcal{N}_2\left(f^{(1)}, f^{(1)}, f^{(1)}\right)\\
&\quad -2\mathcal{N}_3\left(f^{(1)}, f^{(2)}, f^{(1)}, f^{(1)}\right) - 2\mathcal{N}_3\left(f^{(1)}, f^{(1)}, f^{(2)}, f^{(1)}\right) && (3.115)
\end{aligned}
$$

One can notice that the number of terms of the right hand side of the equations above increases rapidly as the power of ϵ increases. Therefore, we do not show all the equations needed in finding the $f^{(n)}$'s. In order to construct soliton solutions, we solve simultaneously the hierarchy equation in $f^{(n)}$ from right hand side to left hand side. The right hand side of each equation reveals the implicit form of $f^{(n)}$, while the left hand side of each equation allows us to determine the explicit form of $f^{(n)}$. If there exists an integer $m \leq n$ such that $f^n = 0$, the series (3.116) truncates at order $m-1$. We obtain

$$f(x,t) = 1 + \sum_{n=1}^{m-1} \epsilon^n f^{(n)}(x,t). \tag{3.116}$$

Equation (3.111) is linear, hence it can admit exponential solutions.

One soliton solution

The one soliton solution is constructed by considering $f^{(1)} = \mathrm{e}^{\xi}$, with $\xi = kx + \omega t + \delta$. The computation of (3.111) reveals the dispersion term $\omega = -k^5$. To find $f^{(2)}$, we first compute the right hand side of (3.112)

$$-\mathcal{N}_1\left(f^{(1)}, f^{(1)}\right) = -15k^6 \mathrm{e}^{2\xi}. \tag{3.117}$$

Therefore, $f^{(2)}$ will be of the form

$$f^{(2)} = a\mathrm{e}^{2\xi}. \tag{3.118}$$

In order to determine the value of a, we substitute (3.117) and (3.118) into (3.112), and we solve a linear equation in a. One obtains $a = \dfrac{1}{16}$. The computation of the right hand side of (3.113) yields zero, which means $f^{(3)} = 0$. It can be easily checked that for $n \geq 3$, $f^{(n)} = 0$. As as result,

$$f = 1 + \mathrm{e}^{\xi} + \frac{1}{16}\mathrm{e}^{2\xi}. \tag{3.119}$$

The one soliton soliton solution is obtained by substituting (3.119) into the second expression of (3.105).

Two soliton solution

In order to construct the two soliton solution, we choose

$$f^{(1)} = \mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2} \text{ with } \xi_1 = k_1 x + \omega_1 t + \delta_1 \text{ and } \xi_2 = k_2 x + \omega_2 t. \tag{3.120}$$

The computation of (3.112) reveals the dispersion terms $\omega_1 = -k_1^5$ and $\omega_2 = -k_2^5$. In order to find $f^{(2)}$, we first compute the right hand side of (3.111)

$$-\mathcal{N}_1\left(f^{(1)}, f^{(1)}\right) = 15k_1^6\mathrm{e}^{\xi_1} + 10k_1k_2\left(2k_1^4 - k_1^2k_2^2 + 2k_2^4\right).\mathrm{e}^{\xi_1+\xi_2} + 15k_2^6\mathrm{e}^{\xi_2}. \tag{3.121}$$

Therefore $f^{(2)}$ is of the form

$$f^{(2)} = a\mathrm{e}^{\xi_1} + b\mathrm{e}^{\xi_1+\xi_2} + c\mathrm{e}^{\xi_2}. \tag{3.122}$$

The computation of the left-hand side of (3.112) together with the expression of the right-hand side of (3.112) reveals the explicit values of the constants

$$a = c = \frac{1}{16}, \quad b = \frac{2k_1^4 - k_1^2k_2^2 + 2k_2^4}{2\left(k_1+k_2\right)^2\left(k_1^2 + k_1k_2 + k_2^2\right)}. \tag{3.123}$$

In the same manner we solve (3.113) in order to find $f^{(3)}$. Equation (3.113) shows that the function $f^{(3)}$ is of the form

$$f^{(3)} = d\mathrm{e}^{2\xi_1+\xi_2} + d_1\mathrm{e}^{\xi_1+2\xi_2} + d_3\mathrm{e}^{3\xi_1} + d_4\mathrm{e}^{3\xi_2} \tag{3.124}$$

Solving (3.113) gives $d_1 = d$ and $d_2 = d_3 = 0$. Therefore $f^{(3)}$ becomes

$$f^{(3)} = d\left(\mathrm{e}^{2\xi_1+\xi_2} + \mathrm{e}^{\xi_1+2\xi_2}\right) \text{ with } d = \frac{\left(k_1-k_2\right)^2\left(k_1^2 - k_1k_2 + k_2^2\right)}{16\left(k_1+k_2\right)^2\left(k_1^2 + k_1k_2 + k_2^2\right)}. \tag{3.125}$$

The computation of the right hand side of (3.114) reveals that $f^{(4)}$ is of the form

$$f^{(4)} = e_1 \mathrm{e}^{2\xi_1+2\xi_2} + e_2 \mathrm{e}^{3\xi_1+\xi_2} + e_3 \mathrm{e}^{\xi_1+3\xi_2}. \tag{3.126}$$

Substituting (3.126) into the left hand side of (3.114), the explicit expression of $f^{(4)}$ is obtained

$$f^{(4)} = e_1 \left(\mathrm{e}^{2\xi_1+2\xi_2}\right) \quad \text{with} \quad e_1 = d^2 = \frac{(k_1-k_2)^4 \left(k_1^2 - k_1k_2 + k_2^2\right)^2}{256\,(k_1+k_2)^4 \left(k_1^2 + k_1k_2 + k_2^2\right)^2}. \tag{3.127}$$

The computation of (3.115) reveals that $f^{(5)} = 0$. One can easily verify that for $n \geq 5$, $f^{(n)} = 0$. One concludes that in the case of the two soliton solution the KK equation, the series (3.128) truncates at order four and becomes

$$f(x,t) = 1 + f^{(1)} + f^{(2)} + f^{(3)} + f^{(4)}, \tag{3.128}$$

where $f^{(1)}$, $f^{(2)}$, $f^{(3)}$ and $f^{(4)}$ are given above. The two soliton solution of (3.102) are obtained by substituting (3.128) into (3.103).

3.3 Chapter Summary

In this chapter, we construct soliton solutions of the third and the fifth order KdV equation by the simplified Hirota method. This construction is only possible if these equations are integrable. In order to verify the integrability of these equation the Painlevé test was used. As a result, the third order KdV is integrable for any choice of its constants, where the fifth order KdV equation required a suitable choice of its constants. We successfully determined these suitable constants for three cases. These three cases give arise to the general version of the KK equation, SK equation and the Lax equation. For the third order KdV equation there exists only one set of resonance which allows it to pass the Painlevé test. For the KK, SK and the Lax equations, there exist two sets of resonances. Only one of the two sets of resonances allow them to pass the Painlevé which is a sufficient condition. The one and the two soliton solutions of the third order KdV equation and the KK equation have been successfully found with help of MATHEMATICA. We notice that only one of the two Bäcklund transformations of the KK equation satisfies this equation, strangely enough, only one of the two sets of resonances allow the KK to be integrable.

Chapter 4

Numerical Methods

This chapter begins our study of time-dependent partial differential equations, the third and the fifth KdV equations from a numerical point of view. The key for solving these equations is discretisation. The process of discretisation is divided into two steps. First, we discretise the problem with respect to the space, resulting in a system of ordinary differential equations in time. Next, we solve this system using some discrete method in time.

As far as the space discretisation is concerned, two methods are considered in this chapter, the finite difference method and spectral method. In the former method, the discrete approximation to derivatives will be converted into Toeplitz matrices, in the last method we consider the pseudospectral method using Fourier matrix differentiation. The discretisation in time uses the fourth order Runge-Kutta method. Since computational experiments are conducted, important issues such as consistency, stability and convergence are investigated. Once these issues are addressed, we solve numerically the third and the fifth order KdV with different methods mentioned above, we investigate numerically the accuracy and the convergence of these methods.

4.1 Finite difference methods

Our goal is to approximate solutions of differential equations, i.e., to find a function (or some discrete approximation to this function) which satisfies a given relationship between various of its derivatives on some given region of space, along with some boundary conditions along the edges of this domain. In general this is a difficult problem and only rarely this problem can be solved analytically. A finite difference method proceeds by replacing the derivatives in the differential equations by finite difference approximations. The most common way of deriving finite difference approximations is done by means of Taylor series expansions.

Let $h > 0$ and $k > 0$ be a fixed space step and time step, respectively, set $x_j = jh$ and $t_n = nk$ for each integer $j = 1, 2, \cdots N$ and $n = 1, 2, \cdots M$. Denote by $v_j = u(x_j, t)$ the numerical approximation of $u(x, t)$ at (x_j, t). Also denote $\dfrac{\partial^m v_j}{\partial x^m}$ the m-th numerical derivative of $u(x_j, t)$. The Taylor series expansions for respectively v_{j+1} and v_{j-1} about the point (x_j, t) are given by

$$v_{j+1} = v_j + h\frac{\partial v_j}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 v_j}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 v_j}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 v_j}{\partial x^4} + O(h^5). \tag{4.1}$$

$$v_{j-1} = v_j - h\frac{\partial v_j}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 v_j}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 v_j}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 v_j}{\partial x^4} + O(h^5). \tag{4.2}$$

$$\tag{4.3}$$

The idea is now to use these expressions to give good approximations to a derivative. For example, the substraction of (4.2) and (4.1) yields the approximation of the first derivative

$$\frac{\partial v_j}{\partial x} = \frac{v_{j+1} - v_{j-1}}{2h} + O(h^2). \tag{4.4}$$

Adding (4.1) and (4.2) together gives an approximation to the second derivative

$$\frac{\partial^2 v_j}{\partial x^2} = \frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} + O(h^2). \tag{4.5}$$

In the same manner, one can construct higher derivatives. The equations (4.4) and (4.5) are called *central difference approximation*. There are other ways to derive finite difference approximation, for example by using polynomial interpolation, however, this is not covered in this thesis.

4.2 Spectral methods

Finite difference methods, based on local representation of functions, have become classical methods in approximating time-dependent problems. These methods require a large number of nodal points in order to obtain satisfactory results.

In the last decades, spectral methods have become popular [36]. These methods distinguish themselves from finite difference methods by the fact that they use global representations by high order polynomials or Fourier series. Spectral methods can yield greater accuracy, in fact exponential for a smooth solution, with far fewer nodes and therefore less computational time than the finite-difference schemes [37].

The main idea behind spectral methods was introduced by Lanczos late in the 1930's for solving ordinary differential equations. Since then, spectral methods have been intensively used in various fields of science. In fluid mechanics, Azaiez *et al.* [38] performed very accurate 3D numerical experiments modeling the cooling of electronic components for the computer industry and in geophysics, Talbot and Crampton [39], obtained very accurate solutions of Navier's equations.

4.2.1 An overview of Spectral methods

A PDE boundary value problem may be posed generally by employing two linear differential operators $\mathcal{L}$ and $\mathcal{B}$. The operator $\mathcal{L}$ acts on a function u to produce a function f on a domain U and $\mathcal{B}$ acts on the function u to produce zero on the boundary of the domain ∂U:

$$\begin{aligned} \mathcal{L}u(x,t) - f(x,t) &= 0 \quad x \in U \subset \mathbb{R} && (4.6) \\ \mathcal{B}u(y,t) &= 0 \quad y \in \partial U. && (4.7) \end{aligned}$$

We wish to approximate the function u by a function u_N that can be expressed as a truncated series of smooth basis functions of the form

$$u_N(x,t) = \sum_{j=1}^{N} \widetilde{u}_j(t)\phi_j(x), \tag{4.8}$$

where the functions $\phi_j(x)$ satisfy the orthogonality relation,

$$\int_{\partial U} \phi_j(x)\phi_m(x)\mathrm{d}x \begin{cases} = 0 & \text{if} \quad m \neq j \\ \neq 0 & \text{if} \quad m = j \end{cases} \tag{4.9}$$

and the coefficients $\widetilde{u}_j(t)$ are to be determined by solving a system of ordinary differential equations. Clearly, when we substitute the numerical approximation (4.8) into (4.6), the result will not be zero, but will produce a residual R defined by

$$R(x,t) = \mathcal{L}u_N(x,t) - f(x,t). \tag{4.10}$$

The residual R is a continuous function of x and t. If N is large enough, then the coefficients $\widetilde{u}_j(t)$ can be chosen so that R is very small over domain U. In integral form, this can be achieved by

$$\int_{\partial U} W_m(x)R(x,t)\mathrm{d}x = 0, \quad m = 1, 2 \ldots N \;\text{ for all } t, \tag{4.11}$$

where W_m is a set of weight functions. Spectral methods are classified in three main groups, the Galerkin, the Tau and the pseudospectral (collocation) method, and their main difference lies in the choice of the weight functions. Although the Galerkin and the Tau methods will not be used here, we shall briefly describe them.

The Galerkin method consists of choosing the weight function such that $W_j(x) = \phi_j(x)$ where each $\phi_j(x)$ satisfies the boundary condition, i.e. $\mathcal{B}\phi_j(y) = 0$. Consequently, (4.11) becomes

$$\int_{\partial U} \phi_j(x)R(x,t)\mathrm{d}x = \int_{\partial U} \phi_j(x)\left(Lu_N(x,t) - f(x)\right)\mathrm{d}x = 0. \tag{4.12}$$

We substitute (4.8) into (4.12), and after algebraic simplifications we obtain

$$\sum_{k=0}^{N} L_{jk}\widetilde{u}_k(t) = \int_{\partial U} \phi_j(x)f(x)\mathrm{d}x \tag{4.13}$$

where $L_{jk} = \displaystyle\int_{\partial U} \phi_j(x)\mathcal{L}\phi_k(x)\mathrm{d}x$ is an element of an $N \times N$ matrix. The coefficients $\widetilde{u}_k(t)$ are obtained by solving a system of ordinary differential equations. The Tau method consists of choosing the weight function such that $W_j(x) = \phi_j(x)$ but the ϕ_j's do not satisfy the boundary condition, i.e. $\mathcal{B}\phi_j(y) \neq 0$. Let $\{e_i,\ i = 1, 2 \dots M\}$ be an orthonormal basis, then we write

$$B\phi_j(y) = \sum_{i=1}^{M} b_{ij} e_i(y). \tag{4.14}$$

We substitute (4.8) and (4.14) into (4.7), the boundary conditions becomes

$$Bu(y,t) = \sum_{k=1}^{N} \sum_{i=1}^{M} \widetilde{u}_k(t) b_{ik} e_i(y) = 0, \tag{4.15}$$

it follows that

$$\sum_{k=1}^{N} b_{ik} \widetilde{u}_k(t) = 0, \quad 1 \leq i \leq M. \tag{4.16}$$

Therefore, the system of ordinary differential equations for the N coefficients $\widetilde{u}_j(t)$ is taken to be the first $N - M$ rows of the Galerkin system (4.13) plus the M equations above

$$\sum_{k=0}^{N} L_{jk} \widetilde{u}_k(t) = \int_{\partial U} \phi_j(x) f \mathrm{d}x, \qquad 1 \leq j \leq N - M, \tag{4.17}$$

$$\sum_{k=1}^{N} b_{ik} \widetilde{u}_k(t) = 0, \qquad 1 \leq i \leq M. \tag{4.18}$$

In the collocation method the weight functions are chosen from a family of the Dirac δ functions,

$$W_j = \delta(x - x_j) = \begin{cases} 1 & \text{if} \quad x = x_j, \\ 0 & \text{otherwise}, \end{cases} \tag{4.19}$$

where the x_j's are the collocation points. Therefore the condition (4.11) is equivalent to $R(x_j, t) = 0$, i.e. $Lu(x_j, t) = f(x_j, t)$ and the boundary conditions are imposed as in the Tau method. Consequently, we obtain

$$\sum_{k=0}^{N} L\phi_k(x_j) \widetilde{u}_k(t) = f(x_j, t), \qquad 1 \leq j \leq N - M, \tag{4.20}$$

$$\sum_{k=1}^{N} b_{ik} \widetilde{u}_k(t) = 0, \qquad 1 \leq i \leq M. \tag{4.21}$$

4.2.2 Fourier pseudospectral discretisation

In this section we will consider the collocation method using Fourier differentiation matrices to solve the third an the fifth order KdV equations. In the literature, several authors have paid attention to solving the KdV family of equations using spectral methods. Darvishi *et al.* [40, 41] reduced the round off error and obtained very accurate solutions of the third and seventh order KdV equations, Rashid [58] demonstrated the high accuracy of the Fourier pseudospectral method by applying the artificial viscosity on the nonlinear term of the KdV equation.

We discuss the main idea behind the Fourier pseudospectral method for a one-dimensional problem. Two main steps are involved in the Fourier spectral method. Firstly, we approximate the solution by the trigonometric interpolation at collocation points. Secondly, we differentiate the approximate solution at the collocation points. More details on spectral methods can be found in [31, 36, 42, 43].

Let $u(x,t)$ be a $2L$-periodic function, whose values are $u_j(t) = u(x_j,t)$ at the collocation points

$$x_j = \frac{2L}{N} j, \qquad j = 0, 1, \dots, N-1, \tag{4.22}$$

where N is even. Gottlieb et al. [44] show that the numerical approximation $u_N(x,t)$ of the function $u(x,t)$ can be expressed as a truncated series of smooth basis functions of the form

$$u_N(x,t) = \sum_{j=1}^{N} u_j(t)\phi_j(x), \quad j = 1, \dots N, \tag{4.23}$$

where the basis functions are given as in [42] by

$$\phi_j(x) = \frac{1}{N} \sin\left(\frac{\mu N(x-x_j)}{2}\right) \cot\left(\frac{\mu(x-x_j)}{2}\right), \tag{4.24}$$

with $\mu = \dfrac{\pi}{L}$ and $\phi_j(x_k) = \delta_j^k$. One can also define the basis functions by a complex approach

$$\phi_j(x) = \frac{1}{N} \sum_{\ell=-N/2}^{N/2-1} \frac{1}{c_\ell} e^{i\ell\mu(x-x_j)}, \quad i^2 = -1, \tag{4.25}$$

where $c_\ell = \begin{cases} 1 & \text{if} \quad |\ell| \neq \frac{N}{2} \\ 2 & \text{if} \quad |\ell| = \frac{N}{2} \end{cases}$. The substitution of (4.25) into (4.23) yields

$$u_N(x,t) = \sum_{\ell=-N/2}^{N/2-1} \widehat{u}_\ell(t) e^{i\ell\mu x} \quad \text{where} \quad \widehat{u}_\ell(t) = \frac{1}{N c_\ell} \sum_{j=1}^{N} u_j(t) e^{-i\ell\mu x_j}. \tag{4.26}$$

This definition of the basis function allows us to express the spectral Fourier differentiation by means of the discrete Fourier transform. Without loss of generality, we will consider the case where the basis functions ϕ_j's are expressed by (4.24). Therefore the (j,k)-th entry of the differentiation matrix D denoted by $d_{kj}^{(1)}$ is

$$d_{kj}^{(1)} = \left.\frac{\mathrm{d}\phi_j}{\mathrm{d}x}\right|_{x_k} = \begin{cases} 0, & k = j \\ \\ \dfrac{1}{2}(-1)^{k+j} \mu \cot\left(\dfrac{x_k - x_j}{2}\right), & k \neq j \end{cases}$$

where $\left\{d_{kj}^{(1)}\right\} = D = D^{(1)}$. The differentiation matrix D is not only a Toeplitz matrix, but is also circulant, i.e., $d_{kj}^{(1)} = d_{k-j}^{(1)} = d_{k-j+N}^{(1)}$. In order to derive the second order differentiation matrix $D^{(2)}$, we differentiate the weight function ϕ_j twice at the grid points x_k

$$d_{kj}^{(2)} = \left.\frac{\mathrm{d}^2\phi_j}{\mathrm{d}x^2}\right|_{x_k} = \begin{cases} -\dfrac{\pi^2}{3h^2} - \dfrac{1}{6}, & k = j \\ -\dfrac{1}{2}(-1)^{k+j}\mu^2 \csc^2\left(\dfrac{x_k - x_j}{2}\right), & k \neq j \end{cases}$$

where $\left\{d_{kj}^{(2)}\right\} = D^{(2)}$. Note that $D^{(2)} \neq D^2$. In general, it can be shown that $D^{(m)} = D^m$ if m is odd and $D^{(m)} = (D^{(2)})^{\frac{m}{2}}$ if m is even [31].

If N is odd, the weight functions ϕ_j's are explicitly expressed by

$$\phi_j(x) = \frac{1}{N} \sin\left(\frac{\mu N(x - x_j)}{2}\right) \csc\left(\frac{\mu(x - x_j)}{2}\right). \tag{4.27}$$

The Fourier collocation methods can be also implemented by using the Fast Fourier Transform (FFT). This algorithm requires $O(N\log(N))$ operations rather than $O(N^2)$ operations when using the matrix multiplication approach. However, in some cases it is more convenient to use the matrix multiplication approach than the FFT. For instance for small values of N, the differentiation matrix approach is faster than the FFT approach, also it is more convenient to investigate the Fourier pseudospectral discretisation using the differentiation matrix approach than the FFT approach [42].

4.3 Runge-Kutta methods

ODEs are not often analytically solvable, numerical methods must then be used. We shall discuss only the following two families of methods:

- linear multistep methods,
- Runge-Kutta methods.

On one hand, linear multistep method raises difficulties to be implemented at each level of time. For example, consider the following ODE

$$u' = f(t, u), \quad u(0) = u_0, \tag{4.28}$$

Numerical solutions of (4.28) by the leap frog method gives

$$v^{n+1} = v^{n-1} + 2kf^n, \tag{4.29}$$

where $t_n = nk$ and v^n is the approximation of $u(t_n) = u^n$. Let $v^0 = u_0$ be the initial condition. If we desire to compute v^1 with the formula (4.29), where shall we obtain v^{-1}?

Similarly, if we desire to begin the computation by considering v^3, where shall we obtain v^2? This difficulty becomes greater as the number of steps increases.

On the other hand, the RK family consists of one-step but multistage methods, i.e in order to obtain v^{n+1} from v^n, we evaluate $f(u,t)$ a number of times during each time step. As a result, Runge-Kutta methods seem to be easier to implement than linear multistep methods. We therefore choose to look at Runge-Kutta methods in order to investigate linear stability of the linearised KdV equations and henceforth to solve them. We will not discuss Runge-Kutta methods in full detail, since our aim is to use them only to solve our systems of ODEs. More details can be found in the book by Butcher [32].

The simplest Runge-Kutta method is the Euler formula defined by

$$v^{n+1} = v^n + kf^n, \tag{4.30}$$

and it is first order accurate. In the following sections, we shall describe some higher order RK methods.

4.3.1 Order of accuracy

The 2-stage Runge-Kutta method is written as

$$\begin{aligned} \Delta_1 &= f\left(t_n, v^n\right), & (4.31)\\ \Delta_2 &= f\left(t_n + k\alpha_2, v^n + k\beta_{21}\Delta_1\right), & (4.32)\\ v^{n+1} &= v^n + k\left(\omega_1\Delta_1 + \omega_2\Delta_2\right), & (4.33) \end{aligned}$$

where α_2, β_{21}, ω_1 and ω_2 are parameters to be determined . One way to find these parameters is to look a the truncation error analysis. Consider

$$\begin{aligned} \frac{du}{dt} &= f\left(u\left(t\right), t\right), & (4.34)\\ \frac{d^2u}{dt^2} &= \frac{d}{dt} f\left(u\left(t\right), t\right) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u}\frac{du}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u} f, & (4.35)\\ \frac{d^3u}{dt^3} &= \frac{d}{dt}\left(\frac{d^2u}{dt^2}\right) = \left(\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial t \partial u} f\right) + \left(\frac{\partial^2 f}{\partial u \partial t} + \frac{\partial^2 f}{\partial u^2}\right) f + \frac{\partial f}{\partial u}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial u} f\right) & (4.36) \end{aligned}$$

and the Taylor series expansion

$$u(t+k) = u(t) + k\frac{du}{dt} + \frac{k^2}{2!}\frac{d^2u}{dt^2} + \frac{k^3}{3!}\frac{d^3u}{dt^3} + \cdots + \frac{k^p}{p!}\frac{d^p}{dt^p} + O\left(k^{p+1}\right). \tag{4.37}$$

First of all, if we expand $\varphi\left(t^n, v^n; k\right)$ in terms of powers of k using the bivariate Taylor expansion and assuming that $v^n = u^n$, we obtain

$$\begin{aligned} \varphi\left(t_n, u^n; k\right) &= \omega_1 f(t_n, u^n) + \omega_2 f(t_n + k\alpha_2, u^n + \beta_{21}k_1)\\ &= \omega_2\left[f + k\alpha_2 f_t + k\beta_{21} f f_u + \frac{(k\alpha_2)^2}{2} f_{2t} + (k\alpha_2)(k\beta_{12}) f_{tu} + \frac{(k\beta_{21}f)^2}{2} f_{2u}\right]\\ &\quad + \omega_1 f + O(k^3), & (4.38) \end{aligned}$$

where $f = f(t_n, u^n)$, $f_{\ell x} = \dfrac{\partial^\ell f}{\partial x^\ell}$ and $f_{xy} = \dfrac{\partial^2 f}{\partial x \partial y}$. The local truncation error can be constructed as follows

$$\begin{aligned} T_n &= u^{n+1} - u^n - k\varphi(t_n, u^n; k) \\ &= -k\left[\omega_1 f + \omega_2\left(f + (k\alpha_2 f_t + k\beta_{21} f f_u) + \frac{1}{2!}\left((k\alpha_2)^2 f_{2t} + 2(k\alpha_2)(k\beta_{12}) f_{tu} + (k\beta_{21} f)^2 f_{2u}\right)\right)\right] \\ &\quad + k\left(f + \frac{k}{2}(f_t + f f_u) + \frac{k^2}{3!}(f_{2t} + 2f f_{tu} + f^2 f_{2u} + f_u(f_t + f f_u))\right) + O(k^4). \end{aligned} \tag{4.39}$$

The parameters $\alpha_2,\ \beta_{21},\ \omega 1$ and ω_2 will be chosen to minimise the truncation error, i.e. we equate to zero all coefficients of powers of k. Under this assertion, we obtain the following equations

$$\omega_1 + \omega_2 = 1 \tag{4.40}$$
$$\omega_2 \alpha_2 = \frac{1}{2} \tag{4.41}$$
$$\omega_2 \beta_{21} = \frac{1}{2}. \tag{4.42}$$

Under these conditions, we can see that it is not possible to further eliminate the terms in k^3 by adjusting the parameters. Consequently, the 2-stage Runge-Kutta method can not be of order accuracy 3, its order of accuracy is 2. Similarly, it can be shown that the 3-stage and the 4-stage Runge-Kutta method have maximal orders 3 and 4 respectively. We shall not prove this here. More details can be found in [32, 33].

Equations (4.40), (4.41) and (4.42) together form a system of three equations and four unknowns. Solving this system in terms of ω_2 gives the general form of the 2-stage Runge-Kutta method

$$\Delta_1 = f\left(t_n, v^n\right), \tag{4.43}$$
$$\Delta_2 = f\left(t_n + \frac{k}{\omega_2}, v^n + \frac{k}{\omega_2}\Delta_1\right), \tag{4.44}$$
$$v^{n+1} = v^n + k\left((1-\omega_2)\Delta_1 + \omega_2\Delta_2\right). \tag{4.45}$$

This shows that the 2-stage Runge-Kutta method is not uniquely determined. Similarly the 3-stage and the 4-stage Runge-Kutta methods will not be uniquely determined. Therefore, only best known examples are considered :

Improved Euler or Heun formula ($s = p = 2$)

$$\Delta_1 = f\left(t_n, v^n\right), \tag{4.46}$$
$$\Delta_2 = f\left(t_n + k, v^n + k\Delta_1\right), \tag{4.47}$$
$$v^{n+1} = v^n + \frac{k}{2}\left(\Delta_1 + \Delta_2\right). \tag{4.48}$$

Heun's third-order formula ($s = p = 3$)

$$\begin{aligned}
\Delta_1 &= f\left(t_n, v^n\right), && (4.49)\\
\Delta_2 &= f\left(t_n + k/3, v^n + k\Delta_1/3\right), && (4.50)\\
\Delta_3 &= f\left(t_n + 2k/3, v^n + 2k\Delta_2/3\right), && (4.51)\\
v^{n+1} &= v^n + \frac{k}{4}\left(\Delta_1 + 3\Delta_3\right). && (4.52)
\end{aligned}$$

Classical fourth-order Runge-Kutta formula ($s = p = 4$)

$$\begin{aligned}
\Delta_1 &= f\left(t_n, v^n\right), && (4.53)\\
\Delta_2 &= f\left(t_n + k/2, v^n + k\Delta_1/2\right), && (4.54)\\
\Delta_3 &= f\left(t_n + k/2, v^n + k\Delta_2/2\right), && (4.55)\\
\Delta_4 &= f\left(t_n + k, v^n + k\Delta_3\right), && (4.56)\\
v^{n+1} &= v^n + \frac{k}{6}\left(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4\right). && (4.57)
\end{aligned}$$

4.3.2 Linear Stability Analysis

The linear stability analysis of a numerical scheme, in particular the Runge-Kutta scheme is conducted via the linear model problem

$$\begin{aligned}
u' &= \lambda u, && (4.58)\\
u(0) &= u_0, \quad 0 \le t, && (4.59)
\end{aligned}$$

where λ is complex. The analytical solution of (4.58) is

$$u(t) = u_0 e^{\lambda t}. \tag{4.60}$$

The problem admits a stable fixed point at $u = 0$ with λ in the left half of the complex plane.

Second order RK method

To begin with, we consider the Heun's method evaluated at a fixed point and we assume that f is linear as in (4.58). The scheme becomes

$$\begin{aligned}
\Delta_1 &= \lambda v^n,\\
\Delta_2 &= v^n + k\Delta_1,\\
v^{n+1} &= v^n + \frac{k}{2}\left(\Delta_1 + \Delta_2\right). && (4.61)
\end{aligned}$$

After algebraic manipulation, we obtain the following scheme

$$v^{n+1} = \left(1 + \lambda k + \frac{(\lambda k)^2}{2!}\right) v^n. \tag{4.62}$$

If we substitute the complex λk by z, (4.62) becomes

$$v^{n+1} = \left(1 + z + \frac{z^2}{2!}\right) v^n. \tag{4.63}$$

The numerical solution is bounded and the scheme is stable if

$$\left|1 + z + \frac{z^2}{2!}\right| \leq 1. \tag{4.64}$$

In order to determine the boundary of the stability region, we calculate the curve of values z corresponding to the following quadratic equation

$$1 + z + \frac{z^2}{2} = e^{i\theta}, \quad \theta \in [0,\ 2\pi], \tag{4.65}$$

which has the following roots

$$z = -1 \pm \sqrt{1 - 2\left(1 - e^{i\theta}\right)}. \tag{4.66}$$

The graphical representation of (4.66) is illustrated in Figure 4.1. We wish to check whether

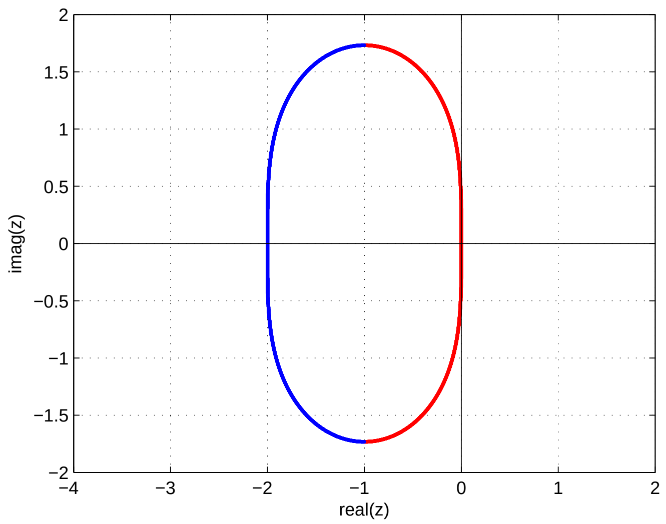


Figure 4.1: Stability region of the Runge-Kutta 2 method

or not the imaginary axis is included in the stability region, since the eigenvalues of the linearised KdV family lie on the imaginary axis. If $z = i\alpha$, with α real, then

$$\left|1 + z + \frac{z^2}{2}\right|^2 = \left|1 + i\alpha - \frac{\alpha^2}{2}\right|^2 = 1 + \frac{\alpha^4}{4}. \tag{4.67}$$

Therefore, the equation

$$1 + \frac{\alpha^4}{4} \leq 1 \tag{4.68}$$

admits only the origin as a solution. As a result, Heun's method is not a suitable method for solving ODEs that have imaginary eigenvalues.

Third order RK method

We next would like to determine the stability region of the Heun's third order method at a fixed point. For the model problem (4.58), the scheme (4.49)-(4.52) becomes

$$\begin{aligned} \Delta_1 &= \lambda v^n, &(4.69)\\ \Delta_2 &= \lambda(v^n + \frac{k\Delta_1}{3}), &(4.70)\\ \Delta_3 &= \lambda(v^n + \frac{2k\Delta_2}{3}), &(4.71)\\ v^{n+1} &= v^n + \frac{k}{4}\left(\Delta_1 + 3\Delta_3\right). &(4.72) \end{aligned}$$

Inserting (4.69), (4.70) and (4.71) into (4.72) yields the following

$$v^{n+1} = \left(1 + \lambda k + \frac{(\lambda k)^2}{2!} + \frac{(\lambda k)^3}{3!}\right) v^n. \quad (4.73)$$

Hence the third order Runge-Kutta method is stable if

$$\left|1 + z + \frac{z^2}{2!} + \frac{z^3}{3!}\right| \leq 1, \quad (4.74)$$

where $z = \lambda k$. The stability boundary condition will be determined by calculating the set of all z given by the following cubic equation

$$1 + z + \frac{z^2}{2} + \frac{z^3}{6} = e^{i\theta}. \quad (4.75)$$

In order to determine the parameterized representation of the stability region, we substitute

$$z = y - 1, \quad (4.76)$$

into (4.75) to obtain the cubic equation with no quadratic terms

$$y^3 + 3y + 2 - 6e^{i\theta} = 0. \quad (4.77)$$

In order to write (4.77) as an equation of degree six, we use the the following transformation

$$y = x - \frac{1}{x}. \quad (4.78)$$

After algebraic simplifications we obtain

$$x^6 + \left(2 - 6e^{i\theta}\right) x^3 - 1 = 0. \quad (4.79)$$

Clearly one can see that by using the transformation

$$x^3 = u, \quad (4.80)$$

a quadratic equation is obtained

$$u^2 + \left(2 - 6e^{i\theta}\right) u - 1 = 0. \tag{4.81}$$

Equation (4.81) admits two roots, which are

$$u_{1,2} = -1 + 3e^{i\theta} \pm \sqrt{2 - 6e^{i\theta} + 9e^{2i\theta}}. \tag{4.82}$$

Each root of (4.81) yields three values of x. Therefore, six values of x are obtained

$$\begin{aligned} x_1 &= \sqrt[3]{u_1}e^{2i\pi/3}, & x_4 &= \sqrt[3]{u_2}e^{2i\pi/3}, \\ x_2 &= \sqrt[3]{u_1}e^{-2i\pi/3}, & x_5 &= \sqrt[3]{u_2}e^{-2i\pi/3}, \\ x_3 &= \sqrt[3]{u_1}, & x_6 &= \sqrt[3]{u_2}. \end{aligned} \tag{4.83}$$

The substitution of all values of (4.83) into (4.78) gives three values of y

$$y_1 = y_6, \quad y_2 = y_5 \quad \text{and} \quad y_3 = y_4. \tag{4.84}$$

Finally, by inserting all values of (4.84) into (4.76), the exact roots of (4.76) are

$$z_1 = -1 + e^{\frac{2i\pi}{3}} A_1 - e^{\frac{-2i\pi}{3}} A_2 \tag{4.85}$$

$$z_2 = -1 - e^{\frac{2i\pi}{3}} A_2 + e^{\frac{-2i\pi}{3}} A_1 \tag{4.86}$$

$$z_3 = -1 + A_1 - A_2. \tag{4.87}$$

where $A_1 = \left(-1 + 3e^{i\theta} - \sqrt{2 - 6e^{i\theta} + 9e^{2i\theta}}\right)^{1/3}$ and $A_2 = \left(-1 + 3e^{i\theta} - \sqrt{2 - 6e^{i\theta} + 9e^{2i\theta}}\right)^{-1/3}$.

Figure 4.2 shows the stability region of the RK3 method.

The eigenvalues of the linearised KdV3 and KdV5 equations are purely imaginary. Therefore, it is necessary to verify which part of the imaginary axis satisfies the condition (4.74). If $z = i\alpha$, then (4.74) becomes

$$\left|1 + i\alpha - \frac{\alpha^2}{2} - i\frac{\alpha^3}{6}\right|^2 = \left(1 - \frac{\alpha^2}{2}\right)^2 + \left(\alpha - \frac{\alpha^3}{6}\right)^2 = 1 - \frac{\alpha^4}{12} + \frac{\alpha^6}{36} \leq 1 \tag{4.88}$$

which is equivalent to $|\alpha| \leq \sqrt{3}$.

Fourth order RK method

We evaluate the Runge-Kutta of order 4 at a fixed point and assume that f is linear as defined by (4.58). We obtain

$$\Delta_1 = \lambda v^n, \tag{4.89}$$

$$\Delta_2 = \lambda(v^n + \frac{k\Delta_1}{2}), \tag{4.90}$$

$$\Delta_3 = \lambda(v^n + \frac{k\Delta_2}{2}), \tag{4.91}$$

$$\Delta_4 = \lambda(v^n + k\Delta_3), \tag{4.92}$$

$$v^{n+1} = v^n + \frac{k}{6}\left(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4\right). \tag{4.93}$$

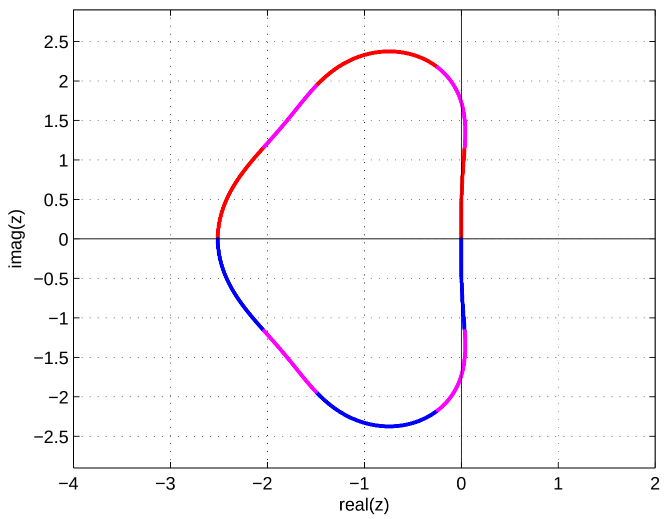


Figure 4.2: Stability region of the Runge-Kutta 3 method

Inserting (4.89), (4.90), (4.91) and (4.92) into (4.93) gives the following

$$v^{n+1} = \left(1 + \lambda k + \frac{(\lambda k)^2}{2!} + \frac{(\lambda k)^3}{3!} + \frac{(\lambda k)^4}{4!}\right) v^n. \tag{4.94}$$

The stability is defined by the region

$$\left|1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!}\right| \leq 1 \tag{4.95}$$

where $z = \lambda k$. The boundary of the stability region is found by computing the roots of the following quartic polynomial

$$1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} = e^{i\theta}. \tag{4.96}$$

Using the change of variables

$$z = y - 1, \tag{4.97}$$

the quartic equation (4.96) is reduced to the following equation

$$y^4 + 6y^2 + 8y + 9 - 24e^{i\theta} = 0. \tag{4.98}$$

The roots of the incomplete equation (4.98) are given by

$$y_1 = -\sqrt{x_1} - \sqrt{x_2} - \sqrt{x_3}, \qquad y_2 = -\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}, \tag{4.99}$$

$$y_3 = \sqrt{x_1} - \sqrt{x_2} + \sqrt{x_3}, \qquad y_4 = \sqrt{x_1} + \sqrt{x_2} - \sqrt{x_3}, \tag{4.100}$$

where x_1, x_2 and x_3 are roots of the cubic equation

$$x^3 + 3x^2 + 6e^{i\theta}x - 1 = 0. \tag{4.101}$$

In order to solve (4.101), we first insert into (4.101) the change of variable

$$x = w - 1. \tag{4.102}$$

We obtain a cubic equation with no second order term

$$w^3 + \left(-3 + 6e^{i\theta}\right) w + \left(1 - 6e^{i\theta}\right) = 0. \tag{4.103}$$

We next substitute the transformation

$$w = v + \frac{1 - 2e^{i\theta}}{v}, \tag{4.104}$$

into (4.103) and sixth order equation is obtained

$$v^6 + \left(1 - 6e^{i\theta}\right) v^3 - \left(1 - 2e^{i\theta}\right)^3, \tag{4.105}$$

which is reducible into a quadratic equation

$$u^2 + \left(1 - 6e^{i\theta}\right) u - \left(1 - 2e^{i\theta}\right)^3, \tag{4.106}$$

where $v^3 = u$ is substituted into (4.106). The roots of (4.106) are given by

$$u_{1,2} = \frac{1}{2}\left(-1 + 6e^{i\theta} \mp \sqrt{-3 + 12e^{i\theta} - 12e^{2i\theta} + 32e^{3i\theta}}\right). \tag{4.107}$$

Each root of (4.107) yields three values of v. Therefore, six values of v are obtained

$$\begin{array}{rclr} v_1 &=& \sqrt[3]{u_1}e^{2i\pi/3} & v_4 = \sqrt[3]{u_2}e^{2i\pi/3} \\ v_2 &=& \sqrt[3]{u_1}e^{-2i\pi/3} & v_5 = \sqrt[3]{u_2}e^{-2i\pi/3} \\ v_3 &=& \sqrt[3]{u_1} & v_6 = \sqrt[3]{u_2}. \end{array} \tag{4.108}$$

The substitution of all values of (4.108) into (4.104) gives three values of w

$$w_1 = w_6, \quad w_2 = w_5 \ \text{ and } \ w_3 = w_4. \tag{4.109}$$

Consequently, the roots of (4.101) are given by

$$x_1 = w_1 - 1; \quad x_2 = w_2 - 1 \quad x_3 = w_3 - 1. \tag{4.110}$$

Finally the roots of (4.96) will be found using (4.99) and (4.100). Figure 4.3 illustrates the stability region of the Runge-Kutta 4 method. In order to seek points on the imaginary axis, we consider the value $z = i\alpha$. Therefore (4.95) becomes

$$\left|1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}\right|^2 = \left|1 + i\alpha - \frac{\alpha^2}{2} + \frac{-i\alpha^3}{6} + \frac{\alpha^4}{24}\right|^2 = 1 - \frac{\alpha^6}{72} + \frac{\alpha^8}{576} \leq 1, \tag{4.111}$$

which is equivalent to $|\alpha| \leq 2\sqrt{2}$.

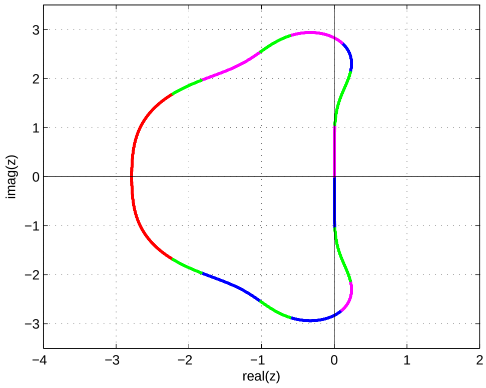


Figure 4.3: Stability region of the Runge-Kutta 4

The stability region of the 2, 3 and 4 stage Runge-Kutta methods of respective order 2, 3, and 4 are plotted in the $z = \lambda k$-space. The ordinate and abscissa are the image and the real part of z respectively. Note that roots of parametric equations (4.65), (4.75) and (4.96) are plotted in different colors, and also that the size of the region increases as the order of the method increases. But this does not mean that the stability region of the Runge-Kutta of order 3 (RK3) is included in the stability region of the Runge-Kutta of order 4 (RK4) since there is a region where the stability region of RK3 is outside of the stability region of RK4. Does it mean that there are regions where the RK3 is more stable than the RK4? The answer to this question is no, because these regions are located in the right half complex plane, while the stability region is concerned with left half complex plane. A graphical illustration is presented in Figure 4.4.

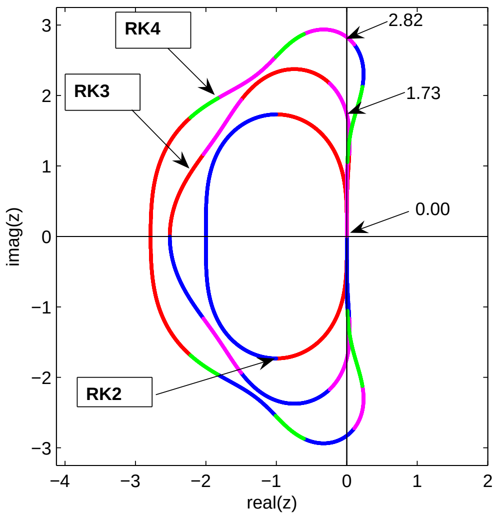


Figure 4.4 : Stability regions of the RK2, RK3 and RK4 methods.

4.4 Consistency, Stability and Convergence

In 1928, Richard Courant, Kurt Friedrichs, and Hans Lewy, of the University of Göttingen Germany [57], introduced necessary conditions for convergence of finite difference schemes known as the CFL condition. This condition states that the *mathematical domain of dependence* is contained in the *numerical domain of dependence*. If $u(x,t)$ is the exact solution of an initial boundary value problem and U_j^n is the solution of finite difference equations approximating the solution of initial boundary value problem, the mathematical or numerical domain of respective $u(x,t)$ or U_j^n is the set of all points in the space where the initial data may have some effect on the respective solution $u(x,t)$ or U_j^n as $h,k \to 0$. The approximation solution is useful if and only if $\left|u(t_n,x_j) - U_j^n\right| \to 0$ as $h,k \to 0$. The finite difference scheme is then said to be *convergent*.

In this section, we investigate convergence of the finite difference and the Fourier spectral method in space discretisation and the fourth order Runge-Kutta method in space discretisation for the third and fifth order KdV equations. These equations are nonlinear and in general the stronger notion of *nonlinear convergence* is very difficult to verify in these particular problems, therefore, one seeks *weaker convergence* known as *linear convergence*. In order to analyse the convergence of the above schemes for the linearised third order and fifth order KdV equation, and understand the difference between the two numerical schemes, we need to introduce two important notions. The first notion is *consistency* and the second notion is *stability*.

4.4.1 Consistency of the KdV3

A numerical method is said to be consistent if, when the exact solution is substituted into the numerical scheme, the error (*local truncation error*) tend to zero when the discretisation parameters tend to zero.

Consider the third order KdV equation

$$u_t + \gamma u u_x + \kappa u_{3x} = 0. \tag{4.112}$$

In order to analyse the linear consistency of (4.112) around a constant solution u_e, the linearised version of (4.112) has to be considered. This linearised version is obtained by substituting $u = u_e + \epsilon\widetilde{u}$ into (4.112) and by neglecting terms of order ϵ^2, namely

$$\widetilde{u}_t = -\gamma u_e \widetilde{u}_x - \kappa \widetilde{u}_{3x}. \tag{4.113}$$

Consistency of KdV3 by the composite FD-RK4

We start our analysis of numerical methods for PDEs with finite difference methods of (4.113). The third order KdV equation defined on a periodic grid only requires initial conditions. Typically we solve the following initial value problem

$$\begin{aligned} u_t &= -\gamma u_e u_x - \kappa u_{3x}, && (4.114)\\ u(x,0) &= f(x), \quad \text{for} \quad x \in [-L,\ L] && (4.115)\\ u(x+2L,t) &= u(x). && (4.116)\end{aligned}$$

First we define a computational grid $x_j = jh$; $h = 2L/N$ and $t_n = nk$ with h, k the step size in space and time respectively. We define $v_j(t)$ to be the numerical approximation to $u(x,t)$ at $x = x_j$. We write

$$v_j(t) \approx u(x_j,t), \quad j = 1, \cdots N. \tag{4.117}$$

There are many possible finite difference approximations of (4.114). We consider the central difference approximation which is second order accurate. The equation (4.114) with spacial derivatives replaced by central difference approximations becomes

$$\frac{dv_j(t)}{dt} = -\frac{\gamma u_e}{2h}\left(v_{j+1}(t) - v_{j-1}(t)\right) - \frac{\kappa}{2h^3}\left(v_{j+2}(t) - 2v_{j+1}(t) + 2v_{j-1}(t) - v_{j-2}(t)\right) \tag{4.118}$$

If we consider all interior points we obtain a system of ODEs which can be written in matrix form

$$\frac{d\mathbf{v}}{dt} = \left(-\gamma u_e D_1 - \kappa D_3\right)\mathbf{v}, \tag{4.119}$$

where

$$\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix}^T, \tag{4.120}$$

and the matrices D_1 and D_3 are defined by

$$D_1 = \frac{h^{-1}}{2}\begin{pmatrix} 0 & 1 & \dots & \dots & -1 \\ -1 & \ddots & & & \\ \vdots & & & & \\ \vdots & & & & \\ 1 & & & & \end{pmatrix}, \tag{4.121}$$

$$D_3 = \frac{h^{-3}}{2}\begin{pmatrix} 0 & -2 & 1 & \dots & -1 & 2 \\ 2 & 0 & -2 & \dots & & -1 \\ -1 & & & & & \\ \vdots & & & & & \\ 1 & & & & & \\ -2 & & & & & \end{pmatrix}. \tag{4.122}$$

Let

$$D = -\gamma u_e D_1 - \kappa D_3, \tag{4.123}$$

then the equation (4.171) becomes

$$\frac{d\mathbf{v}}{dt} = D\mathbf{v}, \tag{4.124}$$

where

$$D = \begin{pmatrix} 0 & a_1 & a_2 & & \\ a_{-1} & \ddots & \ddots & \ddots & \\ a_{-2} & \ddots & \ddots & \ddots & a_2 \\ & \ddots & \ddots & \ddots & a_1 \\ & & a_{-2} & a_{-1} & 0 \end{pmatrix}, \quad a_1 = -a_{-1} = \frac{-\gamma u_e h^2 + 2\kappa}{2h^3}, \quad \text{and} \quad a_2 = -a_{-2} = \frac{-\kappa}{2h^2}. \tag{4.125}$$

We have reduced the initial PDE (4.114) to a system of ODEs. This method is also known as the method of lines, since we are solving the initial PDE on the $x = x_j$ lines. Of course there is an approximation involved. We have used a finite spatial step size h to obtain a finite sized ODE system whereas the initial PDE is equivalent to an infinite system of ODEs. After applying the fourth order Runge-Kutta method defined in the section 4.3, the following fully discrete scheme is obtained,

$$\begin{aligned} \Delta_1 &= f\left(t_n, \mathbf{v}^n\right), & (4.126)\\ \Delta_2 &= f\left(t_n + k/2, \mathbf{v}^n + h\Delta_1/2\right), & (4.127)\\ \Delta_3 &= f\left(t_n + k/2, \mathbf{v}^n + k\Delta_2/2\right), & (4.128)\\ \Delta_4 &= f\left(t_n + k, \mathbf{v}^n + k\Delta_3\right), & (4.129)\\ \mathbf{v}^{n+1} &= \mathbf{v}^n + \frac{k}{6}\left(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4\right). & (4.130) \end{aligned}$$

By using the Taylor expansion, one can check that the truncation error of the fully discrete scheme is

$$T_j^n \leq \left| \frac{1}{2880}\left(\frac{75u_{2t}^4}{u_t^3} - \frac{65u_{2t}^2 u_{3t}}{u_t^2} + \frac{10u_{3t}^2}{u_t} + \frac{5u_{2t}u_{4t}}{u_t} - u_{5t} \right) k^4 + \left(\frac{\gamma u_e}{6} u_{3x} + \frac{\kappa}{4} u_{5x} \right) h^2 \right|, \tag{4.131}$$

where $T_j^n = u(x_j, t_n) - v_j^n$. Therefore

$$\left|T_j^n\right| \leq C_1 k^4 + C_2 h^2, \tag{4.132}$$

where C_1 and C_2 denote two constant that depend on some derivatives of u. As a result, the scheme (4.144) is consistent of order of accuracy 2 in space and 4 in time.

Consistency of the KdV3 the composite PS-RK4

The accuracy of our numerical scheme for the problem (4.114) is $O(k^4)$ for the time variable t due to the fourth order Runge-Kutta scheme (see previous subsection), $O(e^{-qN})$ for the space variable x, where q is a constant [36, 37].

4.4.2 Consistency of the KdV5 equation

Consider the fifth order KdV equation

$$u_t + \alpha u^2 u_x + \beta u u_{3x} + \gamma u_x u_{2x} + \kappa u_{5x} = 0. \tag{4.133}$$

The linearised version of (4.112) is obtained by substituting $u = u_e + \epsilon \widetilde{u}$ into (4.112) and by neglecting terms of order ϵ^2

$$\widetilde{u}_t = -\alpha u_e^2 \widetilde{u}_x - \beta \widetilde{u}_e u_{3x} - \kappa u_{5x}. \tag{4.134}$$

Consistency of the KdV5 by the composite FD-RK4

The discrete approximation of (4.134) by the central difference method is given by

$$\begin{aligned} \frac{dv_j(t)}{dt} &= -\frac{\alpha u_e^2}{2h}\left(v_{j+1}(t) - v_{j-1}(t)\right) - \frac{\beta u_e}{2h^3}\left(v_{j+2}(t) - 2v_{j+1}(t) + 2v_{j-1}(t) - v_{j-2}(t)\right) \\ &\quad -\frac{\kappa}{2h^5}\left(v_{j+3}(t) - 4v_{j+2}(t) + 5v_{j+1}(t) - 5v_{j-1}(t) + 4v_{j-2}(t) + v_{j-3}(t)\right) \end{aligned} \tag{4.135}$$

Considering all interior points we obtain a system of ODEs that can be written in matrix form as

$$\frac{d\mathbf{v}}{dt} = \left(-\alpha u_e^2 D_1 - \beta u_e D_3 - \kappa D_5\right)\mathbf{v}, \tag{4.136}$$

where D_1 and D_3 are defined in (4.121) and (4.122),

$$\mathbf{v} = [\ v_1 \quad v_2 \quad \cdots \quad v_N\ ]^T, \quad \text{and} \tag{4.137}$$

$$D_5 = \frac{1}{2h^5}\begin{bmatrix} 0 & 5 & -4 & 1 & \ldots\ldots & -1 & 4 & -5 \\ -5 & 0 & 5 & -4 & \ldots\ldots & & -1 & 4 \\ 4 & & \ddots & & & & & -1 \\ -1 & & & & & & & \\ \vdots & & & & & & & \\ \vdots & & & & & & & \\ 1 & & & & & & & \\ -4 & & & & & & & \\ 5 & & & & & & & \end{bmatrix}.$$

Let us write

$$D = -\alpha u_e^2 D_1 - \beta u_e D_3 - \kappa D_5, \tag{4.138}$$

consequently, (4.136) becomes

$$\frac{d\mathbf{v}}{dt} = D\mathbf{v}, \tag{4.139}$$

One obtain a system of ODEs (4.136) which is to be solved by the RK4.

$$\begin{aligned}
\Delta_1 &= f\left(t_n, \mathbf{v}^n\right), && (4.140)\\
\Delta_2 &= f\left(t_n + k/2, \mathbf{v}^n + k\Delta_1/2\right), && (4.141)\\
\Delta_3 &= f\left(t_n + k/2, \mathbf{v}^n + k\Delta_2/2\right), && (4.142)\\
\Delta_4 &= f\left(t_n + k, \mathbf{v}^n + k\Delta_3\right), && (4.143)\\
\mathbf{v}^{n+1} &= \mathbf{v}^n + \frac{k}{6}\left(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4\right). && (4.144)
\end{aligned}$$

The truncation error of the fully discrete scheme is obtained by Taylor expansion

$$T_j^n \leq \left| \frac{1}{2880}\left(\frac{75u_{2t}^4}{u_t^3} - \frac{65u_{2t}^2 u_{3t}}{u_t^2} + \frac{10u_{3t}^2}{u_t} + \frac{5u_{2t}u_{4t}}{u_t} - u_{5t}\right)k^4 + \left(-\frac{1}{6}\alpha u_e^2 u_{3x} + \frac{1}{4}\beta u_e u_{5x} - \frac{\kappa}{3}u_{7x}\right)h^2 \right| \tag{4.145}$$

One can see that the truncation error is bounded by the $C_1k^4 + C_2h^2$, where C_1 and C_2 denote two constants which depend on some norms of the derivatives of u. This means that the scheme (4.135) is consistent with order of accuracy 2 in space and 4 in time.

Consistency of the KdV5 by the composite PS-RK4

The accuracy of our numerical scheme for the problem (4.134) is $O(k^4)$ for the time variable t due to the fourth order Runge-Kutta scheme (see previous subsection), $O(e^{-qN})$ for the space variable x, where q is a constant [36, 37].

4.4.3 Stability

In order investigate the stability conditions of the combined FD-RK4 and PS-RK4 for the third and the fifth order KdV equations, we shall introduce some definitions.

Definition 4.4.1 *A Toeplitz matrix, is a matrix in which each ascending diagonal from left to right has constant entries, i.e, a matrix of the form*

$$\boldsymbol{A} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{N-1} \\ a_{-1} & a_0 & a_1 & \ddots & \vdots \\ a_{-2} & \ddots & \ddots & \ddots & a_2 \\ \vdots & \ddots & \ddots & \ddots & a_1 \\ a_{-N+1} & \cdots & a_{-2} & a_{-1} & a_0 \end{bmatrix}$$

where the entries of $\boldsymbol{A}$ *has the following property*

$$a_{ij} = a_{j-i}. \tag{4.146}$$

If in addition, the entries of $\boldsymbol{A}$ *are such that*

$$a_i = a_{i+N}, \tag{4.147}$$

then $\boldsymbol{A}$ *is said to be a circulant matrix.*

Definition 4.4.2 *Consider the following sequence*

$$s = \{a_k : \quad k = -p, \cdots, -1, 0, 1, \cdots, q\} \tag{4.148}$$

where p and q are positive integers. A square Toeplitz matrix $\boldsymbol{A}$ of order N is of bandwidth $(p+q+1) \leq N$ if its entry a_{ij} is member of the sequence s and zero otherwise. For instance, if $p = q = 2$, $\boldsymbol{A}$ is a pentadiagonal Toeplitz matrix.

Our interest will be in the eigenvalues of Toeplitz matrices obtained from finite difference schemes. This will be of paramount importance in determining the linear stability of our linear KdV equations. The eigenvalues of a tridiagonal Toeplitz matrix of arbitrary order N are well known [28]. For Toeplitz matrix with higher bandwidth, the eigenvalue problems become intractable [29], although some algorithm have been derived for pentadiagonal matrices [29, 30]. In order to avoid the difficulty of determining eigenvalues of higher bandwidth Toeplitz matrices, our main attention will be focussed on the eigenvalues of circulant Toeplitz matrices, which is related to our assumption of periodic boundary conditions.

The eigenvalue problem

Let $\mathbf{A}$ represent the Toeplitz banded matrix of Definition 4.4.2, a complex number λ, is called an *eigenvalue* of the matrix $\mathbf{A}$ if there exists a complex vector $\mathbf{v} \neq 0$ such that

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}. \tag{4.149}$$

The vector $\mathbf{v}$ is called the eigenvector for the eigenvalue λ. If we write

$$\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix}^T, \tag{4.150}$$

then the eigenvalue problem (4.149) is equivalent to

$$\sum_{k=-p}^{q} a_k v_{j+k} = \lambda v_j, \quad j = 1, 2, \cdots, N. \tag{4.151}$$

We would like to look for solutions of v_j of the form

$$v_j = r^j, \tag{4.152}$$

where r is a complex number. The substitution of (4.152) into (4.151) leads to the following polynomial equation of degree $p+q$ with unknown r

$$\lambda = \sum_{k=-p}^{q} a_k r^k. \tag{4.153}$$

The solution of (4.151) will be of the form

$$v_j = \sum_{k=1}^{p+q} \beta_k r_k^j \tag{4.154}$$

where r_k's are the roots of (4.153) and β's are arbitrary constants determined by using the boundary conditions.

Consider the problem (4.149) with periodic boundary conditions:

$$v_j = v_{j+N}, \tag{4.155}$$

the equation (4.152) becomes a cyclotomic equation

$$r^N = 1, \tag{4.156}$$

whose Nth roots of unity are ,

$$r_\ell = e^{i\theta_\ell}, \quad \text{with} \quad \theta_\ell = 2\ell\pi/N \quad \text{and} \quad \ell = 1, 2, \cdots, N. \tag{4.157}$$

Inserting (4.157) into (4.153) yields the following

$$\lambda = \sum_{k=-p}^{q} a_k e^{ik\theta_\ell}. \tag{4.158}$$

This result will be used in the next subsection to find the eigenvalues of the linearised KdV equations.

4.4.4 Linear stability of the KdV3

The stability conditions of the third order KdV equation is investigated for the composite FD-RK4 on one hand and for the PS-RK4 on the other hand.

Linear stability of the KdV3 by the composite FD-RK4

We seek eigenvalues of the circulant Toeplitz matrix D. Using the result (4.158), one obtains

$$\lambda = a_{-2}e^{-2i\theta_\ell} + a_{-1}e^{-i\theta_\ell} + a_1 e^{i\theta_\ell} + a_2 e^{2i\theta_\ell}, \tag{4.159}$$

which is reduced to the form

$$\lambda = \frac{i}{h^3} \sin\theta_\ell \left(-\gamma u_e h^2 + 2\kappa - 2\kappa\cos\theta_\ell\right). \tag{4.160}$$

The modulus of the eigenvalue λ is maximal only when its derivative with respect to θ_ℓ is zero. It is sufficient to check this condition via the function f defined as follows

$$f(\theta_\ell) = \sin\theta_\ell \left(-\gamma u_e h^2 + 2\kappa - 2\kappa\cos\theta_\ell\right). \tag{4.161}$$

The derivative of the function can be written as a quadratic equation in $\cos\theta_\ell$

$$f'(\theta_\ell) = -4\kappa\cos^2\theta_\ell + \left(2\kappa - h^2\gamma u_e\right)\cos\theta_\ell + 2\kappa. \tag{4.162}$$

For the sake of graphical illustration, we consider a special case where

$$[\ \kappa,\ \gamma,\ u_e\ ] = [\ 1,\ 6,\ 1\ ]. \tag{4.163}$$

Inserting (4.163) into (4.164), the following equation

$$-4\cos^2\theta_\ell + \left(2 - 6h^2\right)\cos\theta_\ell + 2 = 0 \tag{4.164}$$

admits two roots

$$(\cos\theta_\ell)_{1,2} = \frac{1}{4}\left(A_1 \mp A_2\right) \tag{4.165}$$

where $A_1 = 1 - 3h^2$ and $A_2 = \sqrt{9h^4 - 6h^2 + 9}$. Substituting (4.165) and (4.163) into (4.161) one obtains the following

$$f_1 = \frac{1}{2}\left(3A_1 + A_2\right)\sqrt{1 - \frac{1}{16}\left(-A_1 + A_2\right)^2} \tag{4.166}$$

$$f_2 = \frac{1}{2}\left(3A_1 - A_2\right)\sqrt{1 - \frac{1}{16}\left(A_1 + A_2\right)^2}. \tag{4.167}$$

Clearly $f_1 > f_2$ since $A_2 > 0$. Therefore the maximal modulus of eigenvalue λ can be expressed as

$$|\lambda_{max}| = \frac{1}{2h^3}\left(3A_1 + A_2\right)\sqrt{1 - \frac{1}{16}\left(-A_1 + A_2\right)^2}. \tag{4.168}$$

We note from (4.160) that the eigenvalue of this problem is imaginary, hence using the result (4.67), we conclude that Heun's method is inappropriate to solve the KdV equation of order 3. Consequently, we only seek the stability region of the third and fourth order Runge-Kutta method. To this end, we use the result (4.88) and (4.111) where $\alpha = -i\lambda k$. Therefore the combined FD-RK4 method is stable for

$$k \leq \frac{2\sqrt{2}}{|\lambda_{max}|} \tag{4.169}$$

Linear stability of the KdV3 by the composite PS-RK4

In this section, we investigate linear stability of the combined Fourier pseudospectral and fourth order Runge-Kutta methods for the problem (4.114). The equation (4.114) can be written at the k-th collocation point as

$$\frac{dv_k(t)}{dt} = -\gamma u_e \sum_{j=1}^{N} v_j d_{kj}^{(1)} - \kappa \sum_{j=1}^{N} v_j d_{kj}^{(3)} \tag{4.170}$$

If we consider all interior points we obtain a system of ODEs which is of the form

$$\frac{d\mathbf{v}}{dt} = \left(-\gamma u_e D - \kappa D^3\right)\mathbf{v}, \tag{4.171}$$

where $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix}^T$, $D^3 = D^{(3)}$ and D is the matrix whose entries are defined by (4.27). Let us write

$$M = -\gamma u_e D - \kappa D^3, \tag{4.172}$$

so that the equation (4.171) becomes

$$\frac{d\mathbf{v}}{dt} = M\mathbf{v}, \tag{4.173}$$

where M is a circulant Toeplitz matrix with entries $m_{kj} = -\gamma u_e d_{kj}^{(1)} - \kappa d_{kj}^{(3)}$. From Theorem 7.2 [31], the maximum absolute value of the eigenvalue is

$$|\lambda|_{max} = \gamma u_e \left(\frac{\pi}{h} - 1\right) + \kappa \left(\frac{\pi}{h} - 1\right)^3 \text{ with } h = \frac{2L}{N}. \tag{4.174}$$

As a result, the stability region of the combined Fourier pseudospectral method and the fourth order Runge-Kutta method satisfies the following relation and

$$k \leq \frac{2\sqrt{2}}{|\lambda|_{max}}. \tag{4.175}$$

This is illustrated graphically by Figure 4.4.

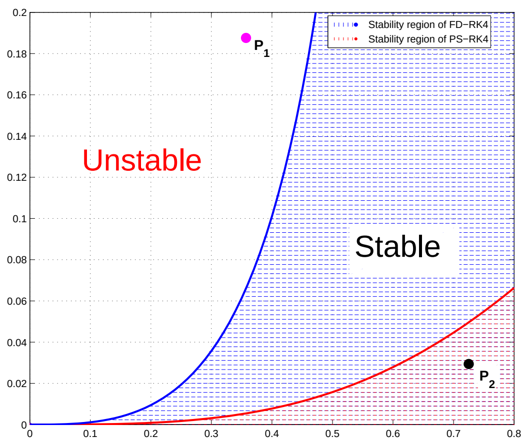
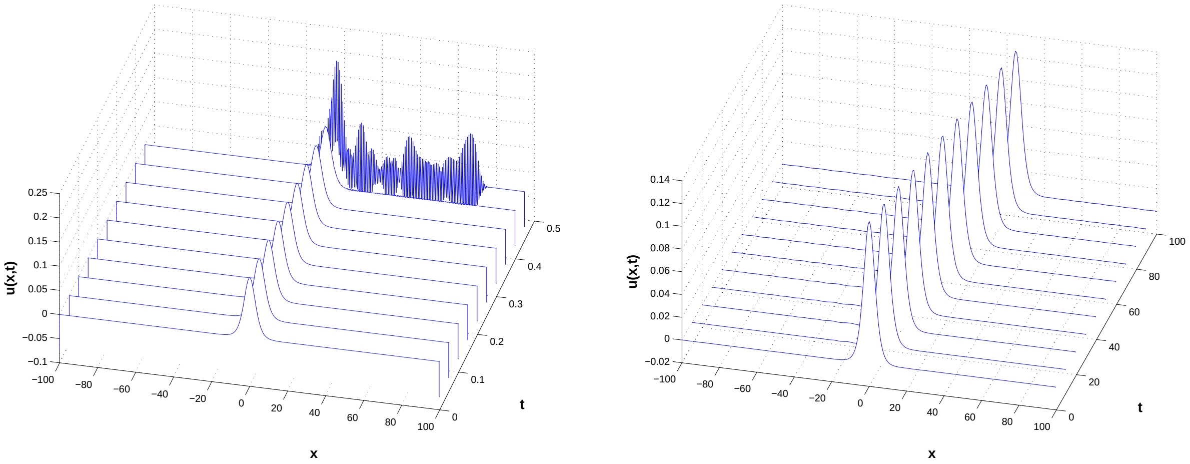


Figure 4.4: Graphical representation of (4.169) (blue) and (4.175)(red).The $x-$axes represents the space step h and the $y-$axes represents the time step k

4.4.5 Linear stability of the KdV5

We investigate the stability conditions of the fifth order KdV for the composite FD-RK4 on one hand and for the PS-RK4 on the other hand.



(a) Unstable computation

(b) Stable computation

Figure 4.5: Numerical approximation of the KdV3 by FD-RK4 at two different points $P_1(0.37,\ 0.185)$ and $P_2(0.73,\ 0.027)$; $\gamma = 6$, $\kappa = 1$ with periodic boundary conditions. The initial condition is the function $2\text{sech}^2(x)$.

Linear stability of the KdV5 by the composite FD-RK4

The matrix D defined by (4.138) is a circulant Toeplitz matrix. Therefore, the eigenvalue problem of (4.139) can be expressed by the relation (4.158). Algebraic simplifications give

$$\lambda = \frac{i}{h^5}\left(a \sin\theta_\ell + b\sin 2\theta_\ell + c\sin 3\theta_\ell\right) \tag{4.176}$$

where

$$\begin{cases} c = -\kappa \\ b = -\beta u_e h^2 + 4\kappa \\ a = -\alpha u_e^2 h^4 + 2\beta u_e - 5\kappa. \end{cases} \tag{4.177}$$

The modulus of the eigenvalue λ is maximal when its derivative with respect to θ_ℓ is zero. It is sufficient to check this condition via the function f defined as follows

$$f(\theta_\ell) = a\sin\theta_\ell + b\sin 2\theta_\ell + c\sin 3\theta_\ell. \tag{4.178}$$

The derivative of the function can be written as a cubic equation in $\cos\theta_\ell$

$$f'(\theta_\ell) = 12c\cos^3\theta_\ell + 4d\cos^2\theta_\ell + (a-9c)\cos\theta_\ell - 2b. \tag{4.179}$$

For the case $\alpha = 20, \quad \beta = 10, \quad \kappa = 1 \quad \text{and} \quad u_e = 1$, the solution of $f'(\theta_\ell) = 0$ is given by

$$\cos\theta_{\ell 1} = z_1 - \frac{1}{3z_1}C_1 - C_2, \tag{4.180}$$

$$\cos\theta_{\ell 2} = z_2 - \frac{1}{3z_2}C_1 - C_2, \tag{4.181}$$

$$\cos\theta_{\ell 3} = z_3 - \frac{1}{3z_3}C_1 - C_2, \tag{4.182}$$

where

$$C_1 = \left(-\frac{25}{27} + \frac{35}{27}h^2 - \frac{55}{27}h^4\right), \quad C_2 = \frac{1}{3}\left(-\frac{4}{3} + \frac{10}{3}h^2\right), \tag{4.183}$$

$$x_1 = \sqrt[3]{u_1}e^{2i\pi/3}, \quad x_2 = \sqrt[3]{u_1}e^{-2i\pi/3}, \quad x_3 = \sqrt[3]{u_1}, \tag{4.184}$$

$$u_1 = \frac{5}{1458}\left(B_1 - B_2\right), \tag{4.185}$$

$$B_1 = -50 + 105h^2 + 102h^4 - 130h^6, \tag{4.186}$$

$$B_2 = 9\sqrt{3}\left(-125h^4 + 360h^6 - 325h^8 + 100h^{10} - 40h^{12}\right)^{\frac{1}{2}}. \tag{4.187}$$

Clearly, Figure 4.6 shows that the function f is maximal when θ_ℓ achieves the value $\theta_{\ell 1}$.

Therefore, the maximal absolute value of the eigenvalue can be written as

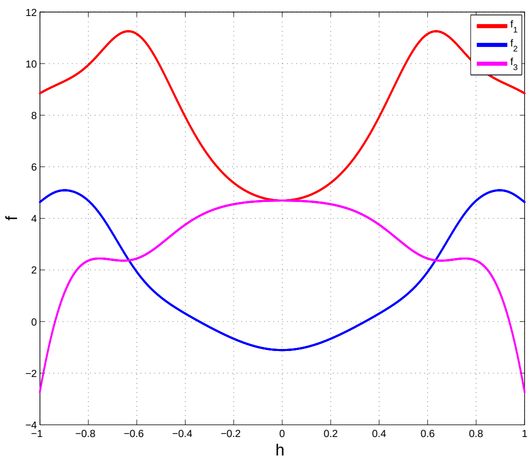


Figure 4.6: Graphs of the functions f_1, f_2 and f_3 obtained when θ_ℓ takes the values $\theta_{\ell 1}$, $\theta_{\ell 2}$ and $\theta_{\ell 3}$.

$$|\lambda|_{max} = \frac{1}{h^5}\left(a\sin\theta_{\ell 1} + b\sin 2\theta_{\ell 1} + c\sin 3\theta_{\ell 1}\right) \tag{4.188}$$

where a, b, c, $\theta_{\ell 1}$ are defined above. Therefore, the stability region of the composite FD-RK4 for the linearised fifth order KdV equation can be written respectively as

$$k \leq \frac{2\sqrt{2}}{|\lambda|_{max}}. \tag{4.189}$$

Graphical illustration can be seen on Figure 4.7.

Linear stability of the KdV5 by the composite PS-RK4

In this subsection, the stability conditions of the linearised fifth order KdV equation is established using the PS in space discretisation and RK4 in time discretisation,

$$u_t = -\alpha u_e^2 u_x - \beta u_e u_{3x} - \kappa u_{5x}, \tag{4.190}$$

$$u(x,t) = u(x+L,t). \tag{4.191}$$

First we define a computational grid $x_j = jh$; $h = 2L/N$ and $t^n = nk$ with h, k the step size in space and time respectively. Let $v_j(t)$ the numerical approximation of $u(x,t)$ at $x = x_j$. We write

$$v_j(t) \approx u(x_j,t), \quad j = 1, \cdots, N. \tag{4.192}$$

The equation (4.190) can be written at the k-th collocation point as

$$\frac{dv_k(t)}{dt} = -\alpha u_e^2 \sum_{j=1}^{N} v_j d_{kj}^{(1)} - \beta u_e \sum_{j=1}^{N} d_{kj}^{(3)} - \kappa \sum_{j=1}^{N} d_{kj}^{(5)}. \tag{4.193}$$

If we consider all interior points we obtain a system of ODEs of the form

$$\frac{d\mathbf{v}}{dt} = \left(-\alpha u_e^2 D - \beta u_e D^3 - \kappa D^5\right)\mathbf{v}, \tag{4.194}$$

where $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix}^T$, $D^3 = D^{(3)}$, $D^5 = D^{(5)}$ and D is the matrix whose entries are defined by (4.27). Let us write

$$M = -\alpha u_e^2 D - \beta u_e D^3 - \kappa D^5, \tag{4.195}$$

then the equation (4.194) becomes

$$\frac{d\mathbf{v}}{dt} = M\mathbf{v}, \tag{4.196}$$

where M is a circulant Toeplitz matrix with entries

$$m_{kj} = -\alpha u_e^2 d_{kj}^{(1)} - \beta u_e d_{kj}^{(3)} - \kappa d_{kj}^{(5)}. \tag{4.197}$$

From theorem 4.2 [31], the maximum absolute value of the eigenvalue is

$$|\lambda|_{max} = \alpha u_e^2 \left(\frac{\pi}{h} - 1\right) + \beta u_e \left(\frac{\pi}{h} - 1\right)^3 + \kappa \left(\frac{\pi}{h} - 1\right)^5 \text{ with } h = \frac{2L}{N}. \tag{4.198}$$

As a result, the stability region of the composite PS-RK4 is represented by the inequality

$$k \leq \frac{2\sqrt{2}}{|\lambda|_{max}}. \tag{4.199}$$

This is illustrated graphically by Figure4.7.

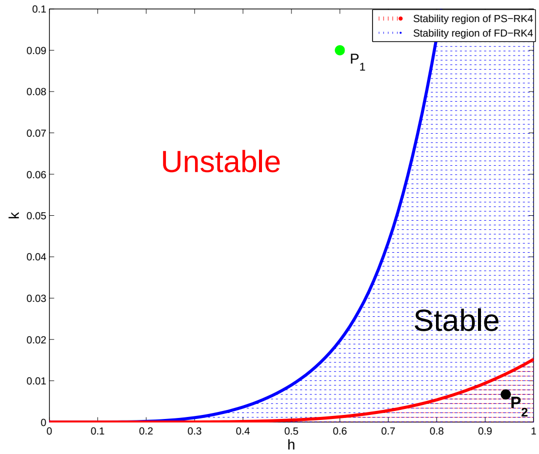
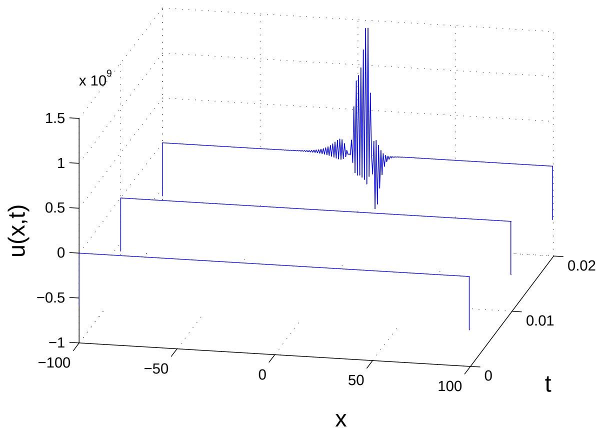
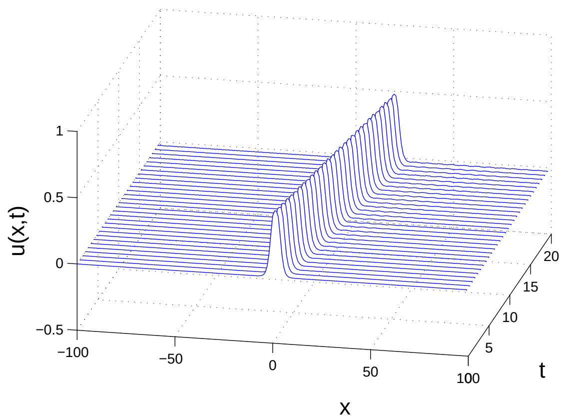


Figure 4.7: Graphical representation of (4.189)(blue) and (4.199)(red).



(a) Unstable computation



(b) Stable computation

Figure 4.8: Numerical approximation of the KdV3 by FD-RK4 at two two different points $P_1(0.5,\ 0.09)$ and $P_2(0.95,\ 0.005)$ with periodic boundary conditions.The initial condition is a $2\text{sech}^2(x)$ function.

4.5 Chapter Summary

In this chapter, we investigate the validity of the composite FD-RK and the composite PS-RK in order to solve the third and the fifth order KdV equations. By validity of the numerical schemes we mean, how do we know that by solving an appropriate, nearby numerical model, we obtain a solution which is nearby the solution of the exact problem? This is the problem of *convergence*. Since the KdV family of equations are nonlinear PDEs and since nonlinear convergence is difficult, we then seek convergence of the linearised problem. In order to investigate the convergence of these equations, we address the issues of consistency and stability. As a result, we show that the composite FD-RK4 is consistent of order 2 in space and 4 in time, while the composite PS-RK4 has exponential accuracy in space. The stability conditions of these method have been successfully determined and verified by testing the numerical solution in the region of stability and in the region of instability. The proposed numerical schemes are then linearly convergent. We also see that the RK of order one and two are always instable for the proposed numerical method in space for the KdV family of equations.

Chapter 5

Numerical Solutions and Results

In this chapter, we present numerical solutions and results for the third and fifth order KdV equations using the central difference and the Fourier pseudospectral methods in space and the fourth order Runge-Kutta method in time. We use a single soliton and a double soliton solution in order to test the accuracy of our methods. We first solve the third order KdV equation using the central difference method with periodic boundary conditions in order to avoid boundary induced errors [36]. We then solve the fifth order KdV equation using the same schemes. Finally we compare the numerical solutions and accuracy obtained by both schemes.

5.1 Finite Difference Discretisation

Consider the KdV equation with periodic boundary conditions

$$\begin{aligned} u_t + \gamma u u_x + \kappa u_{3x} &= 0, \quad x \in [-L, \quad L]\,, \; t \in [0, \quad T]\,, && (5.1)\\ u(x+2L,t) &= u(x), && (5.2)\\ u(x,0) &= f(x). && (5.3) \end{aligned}$$

The computational grid is defined by $x_j = jh$ with $h = 2L/N$ and $t_n = kn$. Δx and k are the step size in space and time respectively. The numerical approximation of $u(x,t)$ at $x = x_j$ is denoted by $v_j(t)$. Therefore the semi-discretisation by the central difference scheme yields

$$\frac{dv_j(t)}{dt} = -\frac{\gamma}{2\Delta x}\left(v_{j+1}(t) - v_{j-1}(t)\right) v_j(t) - \frac{\kappa}{2h^3}\left(v_{j+2}(t) - 2v_{j+1}(t) + 2v_{j-1}(t) - v_{j-2}(t)\right). \tag{5.4}$$

If we consider all interior points we obtain a system of ODEs which can be written in the vector form

$$\frac{d\mathbf{v}}{dt} = F(\mathbf{v}, t) \tag{5.5}$$

where $F(\mathbf{v}, t) = -\gamma \mathbf{v} * D_1 \mathbf{v} - \kappa D_3 \mathbf{v}$, D_1 and D_3 are defined by (4.121) and (4.122) respectively. The asterisk, $*$, denotes element by element multiplication of two vectors.

Using the Runge-Kutta method of order four, we obtain a fully discrete scheme

$$\begin{aligned}
\Delta_1 &= F\left(t_n, \mathbf{v}^n\right), && (5.6)\\
\Delta_2 &= F\left(t_n + k/2, \mathbf{v}^n + k\Delta_1/2\right), && (5.7)\\
\Delta_3 &= F\left(t_n + k/2, \mathbf{v}^n + k\Delta_2/2\right), && (5.8)\\
\Delta_4 &= F\left(t_n + k, \mathbf{v}^n + k\Delta_3\right), && (5.9)\\
\mathbf{v}^{n+1} &= \mathbf{v}^n + \frac{k}{6}\left(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4\right). && (5.10)
\end{aligned}$$

5.2 Fourier Pseudospectral Discretisation

In this section, we apply the Fourier pseudospectral method to the problem (5.1)-(5.3). We approximate the exact solution $u(x,t)$ by

$$u_N(x,t) = \sum_{j=1}^{N} u_j(t)\phi_j(x), \quad j = 1, \dots N. \tag{5.11}$$

The substitution of (5.11) into (5.1) yields at the collocation points:

$$\frac{du_k(t)}{dt} = -\gamma u_k(t) \sum_{j=1}^{N} u_j(t) d_{kj}^{(1)} - \kappa \sum_{j=1}^{N} u_j(t) d_{kj}^{(3)}, \tag{5.12}$$

where $u_k(t) = u(x_k, t)$, $d_{kj}^{(1)}$ and $d_{kj}^{(3)}$ are defined by (4.27). By considering all interior points, a system of ODEs is obtained

$$\frac{d\mathbf{u}}{dt} = G(\mathbf{u}, t), \tag{5.13}$$

where $G(\mathbf{u}, t) = -\gamma \mathbf{u} * D\mathbf{u} - \kappa D^3 \mathbf{u}$, D and D^3 are matrices of entries $d_{kj}^{(1)}$ and $d_{kj}^{(3)}$ defined by (4.27). Applying the fourth order Runge-Kutta method on (5.38) we obtain the following scheme

$$\begin{aligned}
\Delta_1 &= G\left(t_n, \mathbf{u}^n\right), && (5.14)\\
\Delta_2 &= G\left(t_n + k/2, \mathbf{u}^n + k\Delta_1/2\right), && (5.15)\\
\Delta_3 &= G\left(t_n + \Delta t/2, \mathbf{u}^n + k\Delta_2/2\right), && (5.16)\\
\Delta_4 &= G\left(t_n + \Delta t, \mathbf{u}^n + k\Delta_3\right), && (5.17)\\
\mathbf{u}^{n+1} &= \mathbf{u}^n + \frac{k}{6}\left(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4\right). && (5.18)
\end{aligned}$$

5.3 Numerical Results of the third order KdV equation

In order to test the accuracy of our methods, we first define the maximum error and the discrete L^2-norm error respectively as follows

$$\|E(t)\|_\infty = \max_{0 \le j \le N-1} |u_j(t) - u_j^{num}(t)| \tag{5.19}$$

and

$$\|E(t)\|_2 = \frac{1}{N}\left(\sum_{j=0}^{N-1}\left|u_j(t) - u_j^{num}(t)\right|^2\right)^{1/2}, \tag{5.20}$$

where $u_j(t)$ is the exact solution of (5.1) and $u_j^{num}(t)$ is the numerical solution of the finite difference or the Fourier pseudospectral scheme at the point x_j.

5.3.1 Single soliton solutions

We consider the KdV equation (5.1) with $\gamma = 6$ and $\kappa = 1$. We use periodic boundary conditions and the initial condition is

$$u(x,0) = \frac{k_1^2}{2}\mathrm{sech}^2\left[\frac{1}{2}(k_1 x + x_0)\right], \quad -L \le x \le L. \tag{5.21}$$

From Chapter 3, equation (5.1) has an exact solution of the form

$$u(x,t) = \frac{k_1^2}{2}\mathrm{sech}^2\left[\frac{1}{2}\left(k_1 x - k_1^3 t + x_0\right)\right], \; -L \le x \le L. \tag{5.22}$$

Equation (5.22) represents a single soliton with amplitude $a = \dfrac{k_1^2}{2}$ moving to the right with a constant speed $v = k_1^2$. The computation is performed with $k_1 = 0.5$, $L = 100$, $x_0 = 0$, $\Delta t = 0.01$ and $N = 250$.

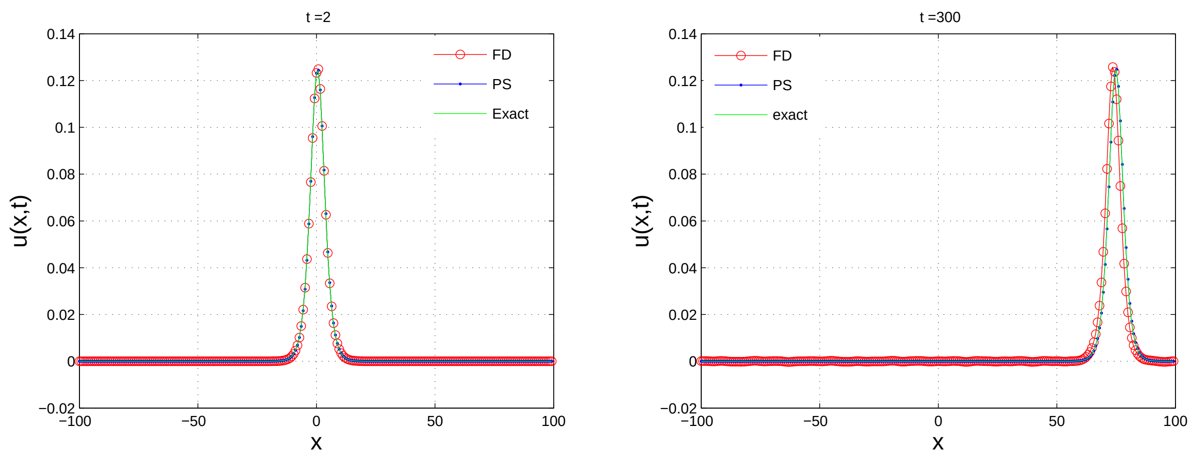
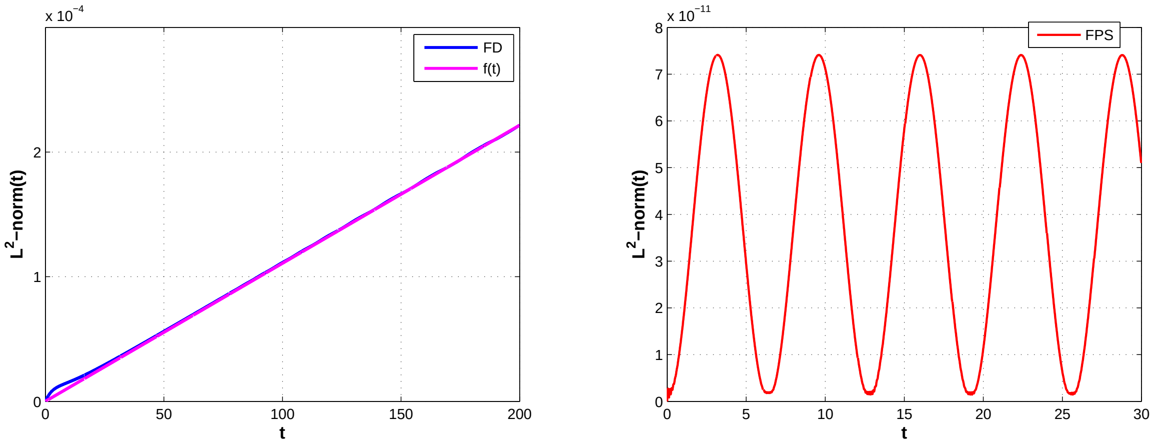


Figure 5.1: Numerical and Analytical single soliton solutions at $t = 0$ and $t = 300$

The numerical solutions obtained by the finite difference and the Fourier pseudospectral method are compared to the analytical solution. As seen in Figure 5.1, the graph of the analytical and the numerical solutions of a single soliton for a small time t are in good agreement with each other, although when t is large enough the finite difference solution seems to travel slower than the exact and the Fourier pseudospectral solutions.

Error as a function of t

Figure 5.2 shows that as time goes by, the L^2-norm error of the finite difference method follows a linear growth $f(t) \approx 1.12 \times 10^{-6}t$, whereas the L^2-norm error of the Fourier pseudospectral oscillates approximatively from 1.52×10^{-12} to 7.38×10^{-11} with a period $T = \frac{h}{v} = 6.4$ as long as the analytical solution is inside the grid. Therefore the L^2-norm error of the Fourier pseudospectral is bounded as long as the soliton inside the grid. Note that $k \ll h$. This means that we almost solve a semi discrete problem. The error in time is then negligible. As we increase the number of grid points, we observe that oscillations become smaller and the L^2-norm error increases as a function of time t. In this case, the semi discrete problem then becomes a fully discrete problem. This is illustrated in Figure 5.3



(a) Finite difference method

(b) Fourier pseudospectral method

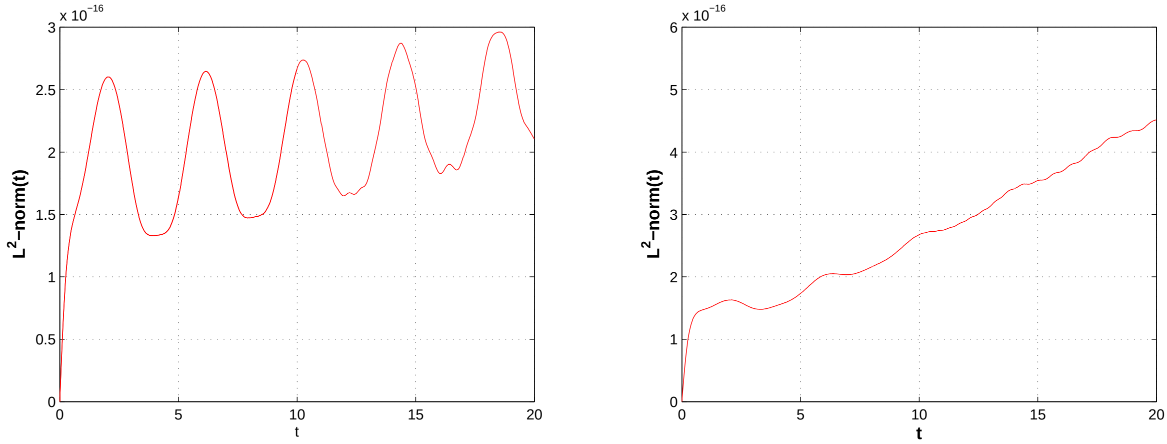
Figure 5.2: L^2-norm error as a function of time of the KdV3 with a single soliton solution as initial condition.

Absolute error

The absolute value of the difference between the analytical and the numerical solutions referred to as the error is represented in Figure 5.4. We observe that the Fourier pseudospectral method is better than the finite difference method in terms of maximum error, although the finite difference method seems to be better than Fourier pseudospectral method around the boundary of the soliton where the error using the finite difference method is zero as long as the soliton remains around the center of the grid.

Error as a function of N

We investigate the accuracy of the finite difference against the accuracy of Fourier pseudospectral method. We observe from Figure 5.5 that the Fourier pseudospectral method



$N = 300$ $N = 400$

Figure 5.3: The PS-L^2 norm error of a single soliton solution

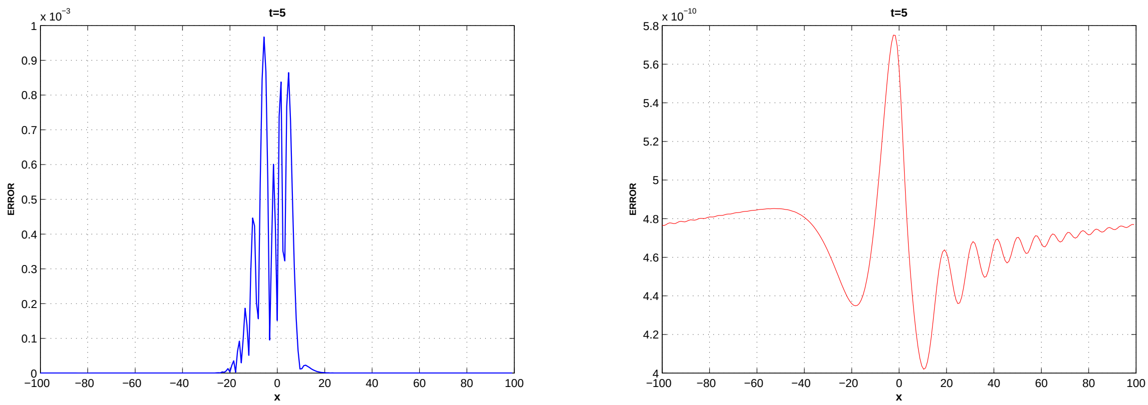


Figure 5.4: Errors of the FD(blue) and the PS(red) at $t = 5$

outperforms the finite difference method in terms of accuracy. For instance, in order to achieve $\log_{10}(\|E\|_2) = -5.3$, it is required to $N = 220$ for the FD against only $N = 101$ for the PS approximation. It is also observed that the L^2-norm error of the PS is seen to converge much faster than the L^2-norm error of the FD. Figure 5.6 shows that for sufficiently large values of N (100 to 300) the PS L^2-norm error in logarithmic base is bounded by the function $f_1(N) \approx a_1 N + b_1$ where $a_1 \approx -0.0433$ and $b_2 \approx 0.8673$. This means that the PS L^2-norm converges exponentially to zero in agreement with [48]. The inverse of the FD L^2-norm error converges quadratically to zero and fits the function $f_2(N) \approx a_2 + b_2 N + c_2 N^2$ where $a_2 \approx 5.4813$, $b_2 \approx -7304$ and $c_2 \approx 6$. This means, as N increases, the PS method converges much faster than the FD method. Therefore, if high accuracies (from 10^{-8} to 10^{-16}) are required for the third order KdV equation, the FD method requires an extremely large number of grid points.

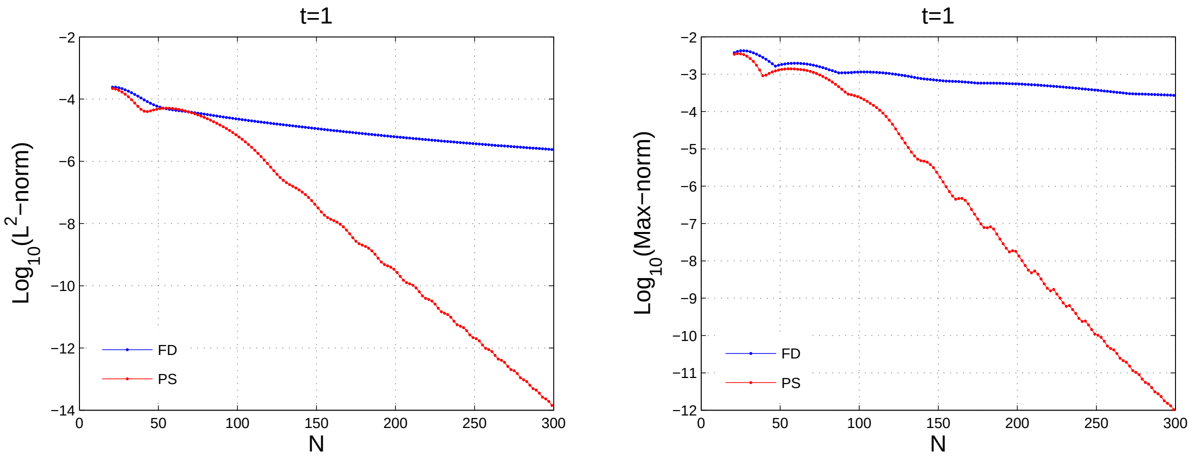


Figure 5.5: L^2-norm errors of the FD(blue) and the PS(red) at $t = 1$

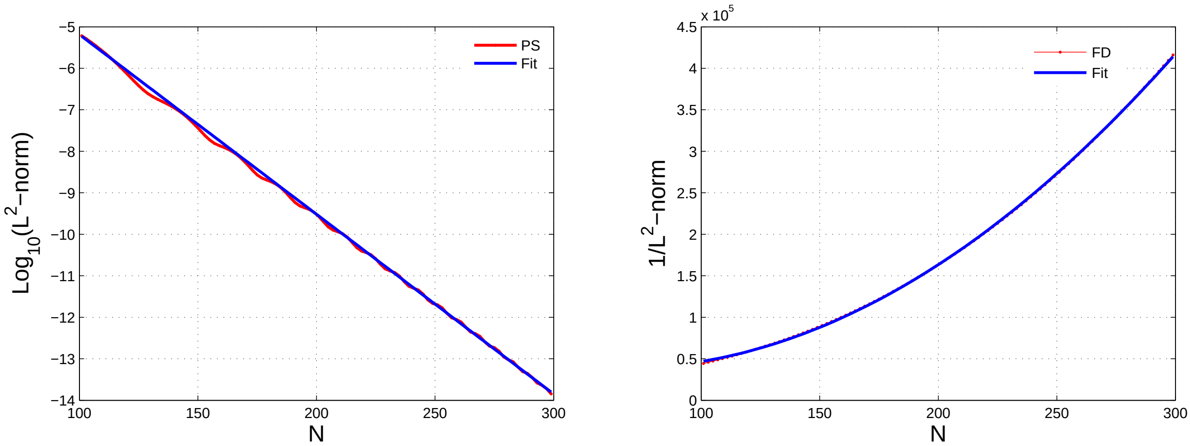


Figure 5.6: Curve fitting of the L^2-norm errors of the FD(blue) and the PS(red) at $t = 1$

.

5.3.2 Two soliton solution

For this problem, we consider the KdV equation (5.1) with $\gamma = 6$ and $\kappa = 1$ with periodic boundary conditions and the initial condition is obtained from the exact solutions given in Chapter 3, namely

$$u(x,t) = 2\left(\log f\right)_{2x} \tag{5.23}$$

where

$$f = 1 + \mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2} + \left(\frac{\xi_1 - \xi_2}{\xi_1 + \xi_2}\right)^2 \mathrm{e}^{\xi_1+\xi_2} \tag{5.24}$$

$$\xi_i = k_i x - k_i^3 t + x_i, \quad i = 1, 2. \tag{5.25}$$

Numerical solutions

We study here the interaction of two soliton solutions to the third order KdV equation. The numerical solutions of the third order KdV equation for this problem are obtained by the FD and the PS method, and are compared with the analytical solution in Figure 5.7 for $k_1 = 0.7$, $k_2 = 0.5$, $d_1 = 20$, $d_2 = 0$, $N = 250$ and $L = 200$. Numerical methods generate solutions that

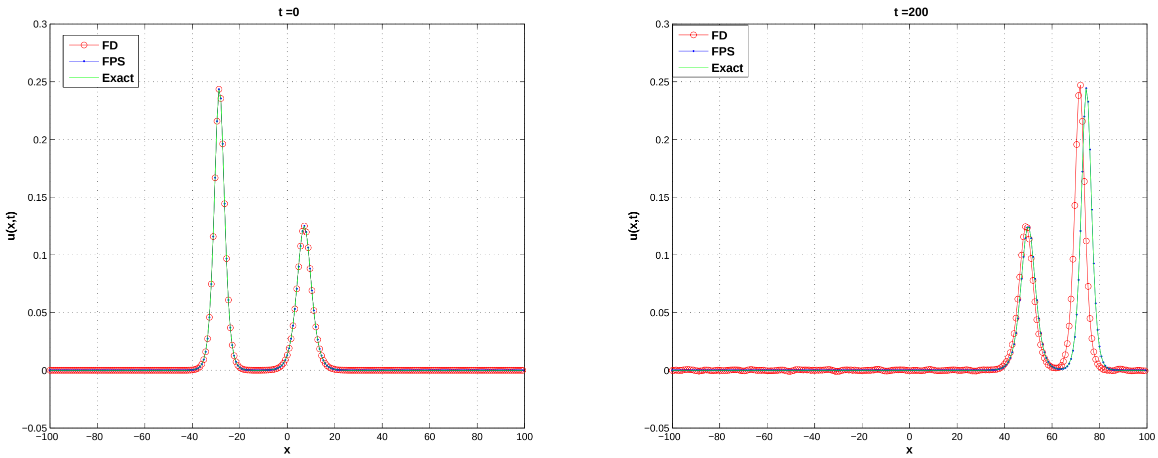


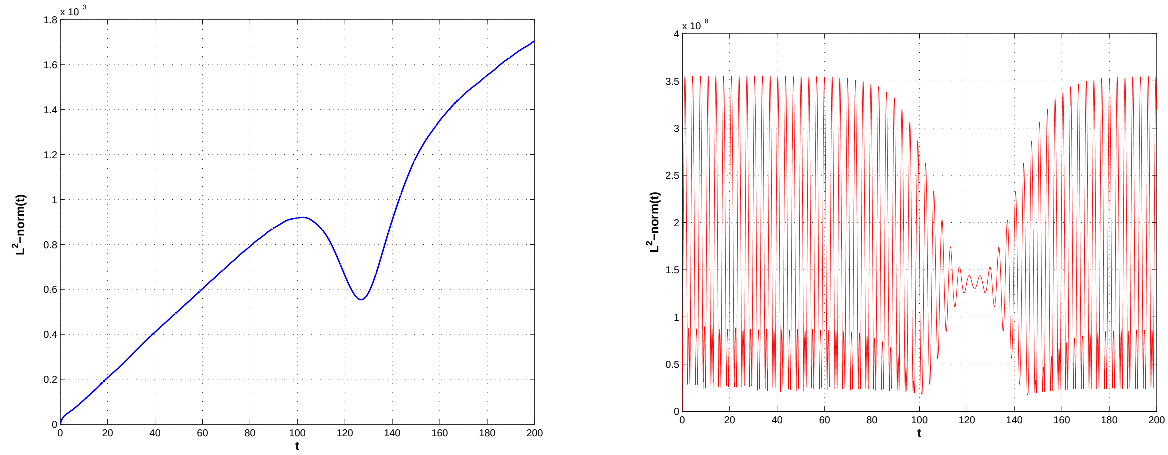
Figure 5.7: Numerical and analytical solutions at $t = 0$ and $t = 300$

have the same behaviour as soliton solutions, i.e., these solutions remain unchanged before and after their interaction. We call them numerical solitons. Nevertheless, the numerical solitons differ from analytical solitons (obtained by the simplified Hirota method) in terms of their speed. In fact, numerical solitons obtained by the FD method travel at a lower speed than those obtained by the PS method and analytical solitons. Numerical solitons obtained by the PS method move almost at the same speed than the analytical solitons.

Error as a function of t

Figure 5.8 displays the L^2-norm error of the FD and the Fourier-PS method. One can see that the L^2-norm (of magnitude 10^{-8}) the PS is very small compared to the L^2-norm (of magnitude 10^{-3}) of the FD; this means PS method is more accurate than the FD method. Unlike the case of the one soliton solution where the L^2-norm error of the FD grows linearly and the L^2-norm error evolves periodically, we observe here two phases in the evolution of the L^2-norm error in the case of the two soliton solution. These two phases represent the L^2-norm error after and before the interaction. The L^2-norm error of the FD increases linearly from $t = 0$ to $t \approx 100$ and then decreases from time $t \approx 100$ to $t \approx 127$ and finally decreases from $t \approx 127$ to $t \approx 200$ and achieves the same linear growth around $t \approx 190$. The time $t \approx 127$ corresponds to the time when the two solitons have nearly the same height.

The L^2-norm error of the PS in Figure 5.8, oscillates periodically from $t = 0$ to $t \approx 80$ with the same amplitude. From $t \approx 80$ to $t \approx 125$ the amplitude of oscillations decrease to



FD PS

Figure 5.8: L^2-norm error of a two soliton solution as a function of t.

reach their minimum at $t \approx 152$ and then increases in likewise oscillatory fashion until it reaches the same type of periodicity. Recall that in the case where there is no interaction between the two soliton solutions, the behaviour of the L^2-norm error is the same as in the case of one soliton solution.

Error as a function of N

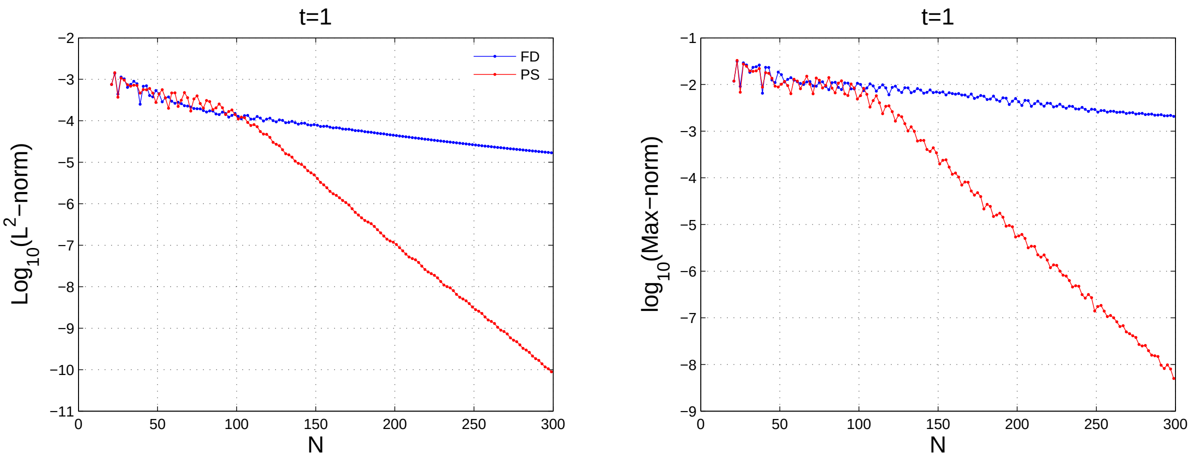


Figure 5.9: L^2-norm of the FD(blue) and the PS(red) at $t = 1$.

We investigate the accuracy of the FD and the PS method as illustrated by Figure 5.9. It is observed that for N large enough, the PS with faster convergence outperforms the FD method; this is in good agreement with Chapter 4. In fact, we have seen that the PS converges exponentially while the FD has a quadratic convergence. Figure 5.10 shows the convergence of L^2-norms of the FD and the PS methods.

For N large enough, the PS L^2-norm error in the logarithmic base is bounded by the

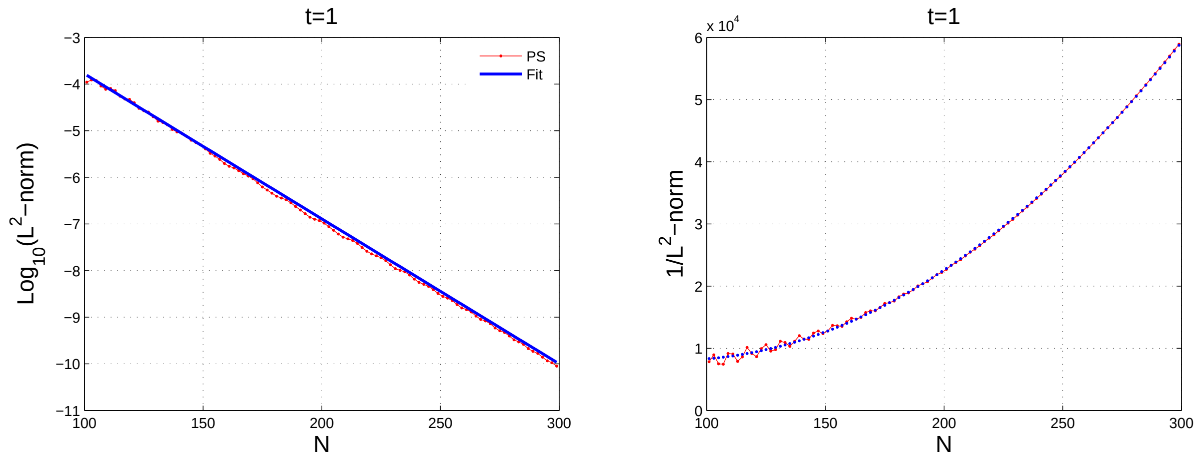


Figure 5.10: Errors of the FD(blue) and the PS(red) at $t = 1$

function $g_1(N) = a_1N + b_1$ where $a_1 = -0.0311$ and $b_1 = -0.67$. This means that the PS L^2-norm error has a exponential convergence. The FD L^2-norm error in the inverse base converges quadratically and fits the curve of the function $g_2(N) = (a_2 + b_2N + c_2N^2)^{-1}$ where $a_2 = 51843.8$, $b_2 = -567.063$ and $c_2 = 3.183$.

5.4 Numerical Results of the fifth order KdV equation

The well-known fifth order KdV equation,

$$u_t + \alpha u^2 u_x + \beta u u_{3x} + \gamma u_x u_{2x} + \kappa u_{5x} = 0, \tag{5.26}$$

where α, β, γ and κ are real constants, admits N-soliton solutions depending on suitable values of the real constants as studied in Chapter 3. As the constants α, β, γ and κ take different values, the properties of (5.26) change radically. We are only interested in the cases for which (5.26) admits soliton solutions, in particular for $\alpha = 20$, $\beta = 10$, $\gamma = 25$ and $\kappa = 1$ the fifth order KdV equation is known as Kaup-Kupershmidt (KK) equation and admits N-soliton solutions obtained in Chapter 3.

In recent years, using a simplified version of the Hirota method, Hereman and Nuseir [16, 25] explicitly constructed multi soliton solutions of the KK equation for which soliton solutions were not previously known. Unfortunately, their method provides no clues to formulating the generic form of the N-soliton solutions of the KK equation. More recently, using the relationship between the bilinear form of the Sawada-Kotera and KK equations, Parker [49, 50] determined explicitly the general form of the N-soliton solution of the KK equations, those he described as 'anomalous' soliton solutions. Most recently, Inc [51] used the Adomian decomposition method (ADM) and Saucez et al. [52] developed a method of lines solution strategy, using an adaptive mesh refinement algorithm based on the equidistribution principle and spatial regularization techniques for obtaining soliton solutions to the Kaup-Kupershmidt (KK) equation.

In this section, numerical single and double soliton solutions of the KK equation are obtained using the the finite difference or the PS methods in space and the fourth order Runge Kutta method for time integration and these solutions are compared to the analytical solutions.

5.4.1 Finite Difference Discretisation

Consider the KK equation with periodic boundary conditions

$$\begin{aligned} u_t + 20u^2 u_x + 10uu_{3x} + 25u_x u_{2x} + u_{5x} &= 0, \quad x \in [-L \quad L]\,,\ t \in [-0 \quad T] \qquad (5.27)\\ u(x+2L, t) &= u(x), \qquad (5.28)\\ u(x, 0) &= f(x). \qquad (5.29) \end{aligned}$$

In order to solve this problem numerically, we define the spatial and the time grid by $x_j = j\Delta x$ with $\Delta x = 2L/N$ and $t_n = n\Delta t$ respectively. The numerical approximation of $u(x,t)$ at grid point $x = x_j$ is denoted by $v_j(t)$. Consequently the semi-discretisation by the central difference scheme leads to

$$\begin{aligned} \frac{dv_j(t)}{dt} &= -\frac{10}{h}\left(v_{j+1}(t) - v_{j-1}(t)\right)\left(v_j(t)\right)^2 - \frac{5}{h^3}\left(v_{j+2}(t) - 2v_{j+1}(t) + 2v_{j-1}(t) - v_{j-2}(t)\right)v_j(t)\\ &\quad -\frac{25}{2h^3}\left(v_{j+1}(t) - v_{j-1}(t)\right)\left(v_{j+1}(t) - 2v_j(t) + v_{j-1}(t)\right)\\ &\quad -\frac{\kappa}{2h^5}\left(v_{j+3}(t) - 4v_{j+2}(t) + 5v_{j+1}(t) - 5v_{j-1}(t) + 4v_{j-2}(t) + v_{j-3}(t)\right). \qquad (5.30) \end{aligned}$$

Taking all interior points into consideration, a system of ODEs is obtained

$$\frac{d\mathbf{v}}{dt} = F(\mathbf{v}, t), \qquad (5.31)$$

where $F(\mathbf{v}, t) = -20\mathbf{v} * \mathbf{v} * D_1\mathbf{v} - 10\mathbf{v} * D_3\mathbf{v} - 25D_1\mathbf{v} * D_2\mathbf{v} - D_5\mathbf{v}$, D_1, D_2, D_3 and D_5 are defined in Chapter 4. The fully discrete form is obtained by using the fourth order Runge-Kutta method

$$\begin{aligned} \Delta_1 &= F\left(t_n, \mathbf{v}^n\right), \qquad (5.32)\\ \Delta_2 &= F\left(t_n + k/2, \mathbf{v}^n + k\Delta_1/2\right), \qquad (5.33)\\ \Delta_3 &= F\left(t_n + k/2, \mathbf{v}^n + k\Delta_2/2\right), \qquad (5.34)\\ \Delta_4 &= F\left(t_n + k, \mathbf{v}^n + k\Delta_3\right), \qquad (5.35)\\ \mathbf{v}^{n+1} &= \mathbf{v}^n + \frac{k}{6}\left(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4\right). \qquad (5.36) \end{aligned}$$

5.4.2 Fourier Pseudospectral Discretisation

The computational domain of (5.27) is discretised according to the Fourier pseudospectral scheme. At collocation points, we obtain

$$\begin{aligned}\frac{dv_k(t)}{dt} &= -20\,(u_k(t))^2\sum_{j=1}^{N}u_j(t)d_{kj}^{(1)} - 10u_k(t)\sum_{j=1}^{N}d_{kj}^{(3)}u_j(t)\\ &\quad -25\sum_{j=1}^{N}d_{kj}^{(1)}u_j(t)\sum_{j=1}^{N}d_{kj}^{(2)}u_j(t) - \sum_{j=1}^{N}d_{kj}^{(5)}u_j(t) \qquad (5.37)\end{aligned}$$

here $u_k(t) = u(x_k, t)$, $d_{kj}^{(1)}$, $d_{kj}^{(2)}$, $d_{kj}^{(3)}$ and $d_{kj}^{(5)}$ are defined by (4.27). By considering all interior points, a system of ODEs is obtained

$$\frac{d\mathbf{u}}{dt} = G(\mathbf{u}, t), \qquad (5.38)$$

where $G(\mathbf{u}, t) = -20\mathbf{u} * \mathbf{u} * D\mathbf{u} - 10\mathbf{u} * D^3\mathbf{u} - 25D\mathbf{u} * D^2\mathbf{v} - D^5\mathbf{v}$, D,D^2,D^3 and D^5 are matrices of respective entries $d_{kj}^{(1)}$, $d_{kj}^{(2)}$, $d_{kj}^{(3)}$ and $d_{kj}^{(5)}$ defined by (4.27). Applying the fourth order Runge-Kutta method on (5.38) we obtain the following scheme

$$\begin{aligned}\Delta_1 &= G\left(t_n, \mathbf{u}^n\right), && (5.39)\\ \Delta_2 &= G\left(t_n + k/2, \mathbf{u}^n + k\Delta_1/2\right), && (5.40)\\ \Delta_3 &= G\left(t_n + k/2, \mathbf{u}^n + k\Delta_2/2\right), && (5.41)\\ \Delta_4 &= G\left(t_n + k, \mathbf{u}^n + k\Delta_3\right), && (5.42)\\ \mathbf{u}^{n+1} &= \mathbf{u}^n + \frac{k}{6}\left(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4\right). && (5.43)\end{aligned}$$

We use the fully numerical schemes of subsections 5.4.1 and 5.4.2 in order to solve the problem (5.27) where the initial condition is chosen to be a single soliton solution and a double soliton solution. Numerical solutions are compared with analytical solutions.

5.4.3 Single soliton solution

In this section, we investigate numerically the behaviour of the soliton solutions of the problem (5.27) with periodic boundary conditions. The initial conditions are deduced from the exact solutions

$$u(x, t) = \frac{3}{2}\left(\log f\right)_{2x} \qquad (5.44)$$

where

$$f = 1 + \mathrm{e}^{\xi_1} + \frac{1}{16}\mathrm{e}^{2\xi_1} \quad \text{and} \quad \xi_1 = k_1 x - k_1^5 t + x_1. \qquad (5.45)$$

We begin our investigation of the KK equation taking as initial condition the soliton (5.44) at $t = 0$ with $k_1 = 0.5$, $L = 100$, $x_1 = 0$, $\Delta t = 0.002$ and $N = 250$. We observe from Figure 5.11 that our soliton moves along the spatial direction and seems to retain its initial profile for

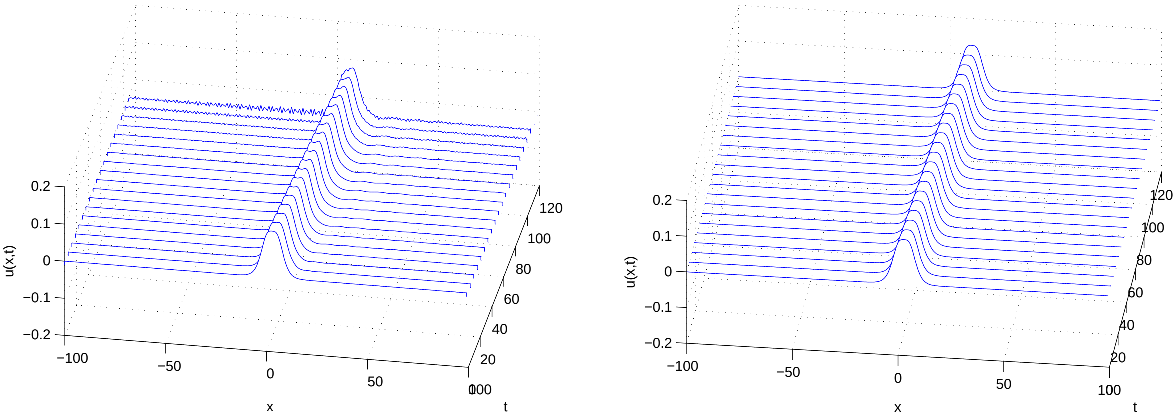


Figure 5.11: Numerical integration of the KK equation: left FD and right PS

a relatively long period of time $t = 100$. In order to investigate the agreement between the numerical and the analytical solution of the KK equation, we capture the propagation of the soliton at two different times $t = 0.1$ and $t = 100$. It can be seen from Figure 5.12 that for a small period of time $t = 0.1$, numerical solutions are in good agreement with analytical ones. For a long period of time $t = 100$, the shape of the soliton solutions obtained with the PS remains virtually unchanged, while oscillations appear at the base of the soliton obtained with the FD. These oscillations grow as time increases, affecting the shape of the FD soliton solutions so that it becomes unbounded around the time $t = 105$, not only because of the FD scheme, but also because of the nonlinear nature of the KK equation.

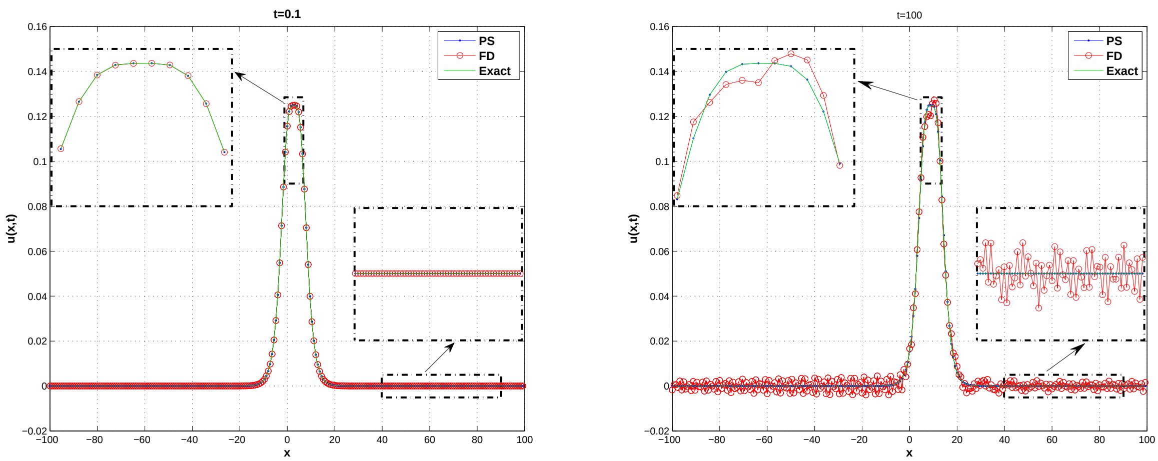


Figure 5.12: Numerical integration of the KK equation: left $t = 0.1$ and right $t = 100$

Error as a function of time t

Figure 5.13 shows the evolution of the L^2-norm as time increases. It illustrates why the FD soliton reaches nonlinear instability while the PS soliton retains its initial profile. In fact, as time goes on, the FD L^2-norm grows rapidly until $t \approx 5$ and then acquires a linear growth for a reasonable period of time. After a certain period of time $t \approx 40$, this growth loses its linearity and the FD L^2-norm grows abnormally until a blow-up occurs. By contrast, the PS

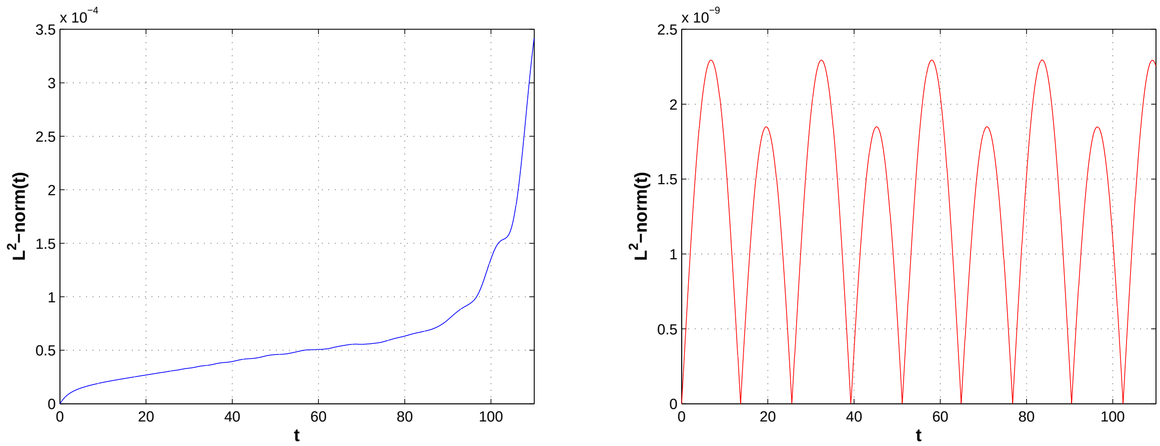


Figure 5.13: The L^2-norm of the KK equation as a function of t: left FD and right PS

L^2-norm error remains bounded as time increases; this explains why the PS soliton solutions remain stable. We also notice that as the speed k_1 of the soliton increases, the instability of the soliton solutions becomes more severe. A good study on the stability of soliton solutions of the KdV is given by Tzirtzilakis et. al in [45, 46, 47].

Error as a function of N

In order to verify whether the PS method is more accurate than the FD method, we can evaluate L^2-norms and Max-norms in the logarithm base as functions of grid points N. From Figure 5.14 it is observed that for N large enough, the PS method is more accurate than the FD method. The PS method converges much faster than FD method, and we can observe that in order to achieve the same accuracy (0.5×10^{-5}) as the PS method for 110 points, the FD needs about 250 points.

The FD method converges quadratically to zero with fitting curve $f_2(N) = a_2 + b_2N + c_2N^2$ where $a_2 \approx 92001.411$, $b_2 \approx -923.651$ and $c_2 \approx 5.313$ whereas PS method converges exponentially to zero since its L^2-norm in the logarithmic base is bounded by the function $f_1(N) = a_1N + b_1$ where $a_1 \approx -0.0331$ and $b_1 \approx -0.8763$.

We shall not show the results of the two soliton solution here, since their are qualitatively the same as found those for the one soliton solutions. The interaction of the two soliton solution of the KK equation can be interpreted as the same as the interaction of the two

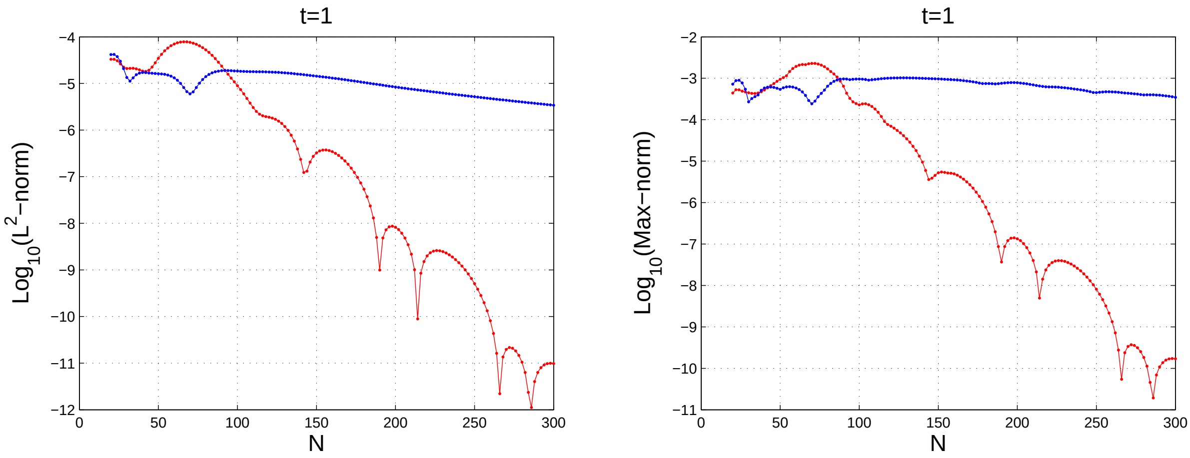
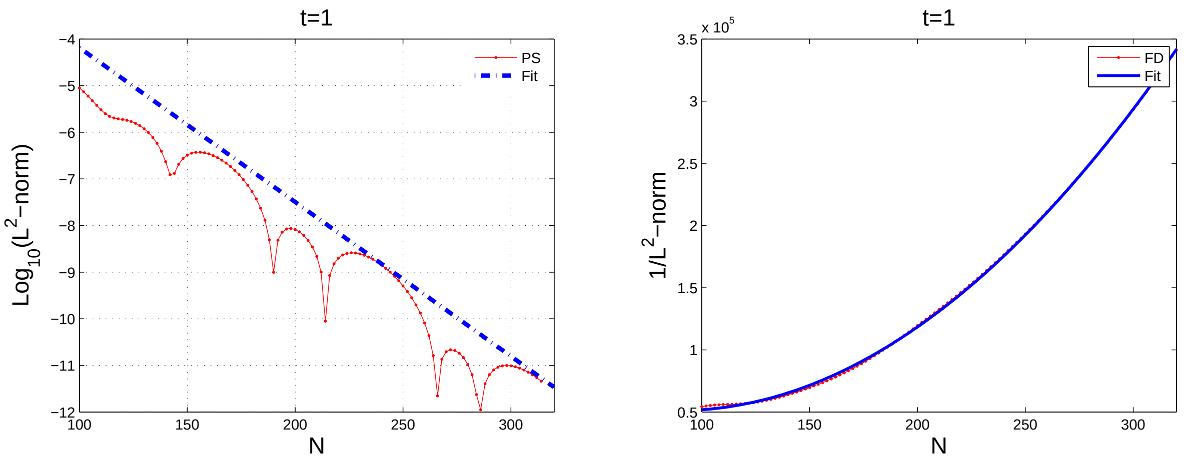


Figure 5.14: Left L^2-norm and right Max-norm



PS FD

Figure 5.15: L^2-norm error fitting curves

soliton solution of the third order KdV equation.

5.5 Chapter Summary

In this chapter, we investigate numerical solutions of the the third and the fifth order KdV equation. As a result, the numerical soliton solutions are in good agreement with the analytical soliton solutions. Numerical soliton solutions have also the elastic properties of analytical soliton solutions. However, numerical soliton solutions obtained by the FD method move slower than those obtained by the PS method and the analytical solutions. The PS method is more accurate than the FD and enjoys exponential accuracy, while the FD has quadratic accuracy.

Chapter 6

Conclusion

6.1 Thesis Summary

Solitons have been important phenomena in physics since the pioneering observations of John Scott Russell [54] in 1934. However, the issue of solitons was solved only in 1895 when Diederik Korteweg and Gustav de Vries proposed a model of shallow water that exhibits soliton solutions. Nevertheless, scientists did not pay attention to the model (KdV equation). It was the work of Norman Zabusky and Martin Kruskal that provided the framework for competitive studies of the KdV equation.

A hierarchy of KdV equations that can admit soliton solutions is described by Newell [59]. The KdV hierarchy have been an important class of NLPDEs with various applications in the physical science and engineering field. For example, in plasma physics, these equations give rise to ion acoustic solitons [60]. In geophysical fluid dynamics, they describe a long wave in shallow seas and deep oceans [61, 62]. They are present in cluster physics, super deformed nuclei, fission, thin films, radar and rheology [63, 64].

In order to understand and describe these phenomena better, a number of sophisticated methods have been developed for constructing analytical solutions of the KdV hierarchy. Among them we can cite, the sinh-cosh-tanh method [15], the Jacobi elliptic functions method [14], the Fan-sub-equation method [2, 3, 4], Painlevé analysis [16], the inverse scattering transform (IST) [9, 12], the Hirota bilinear method [13, 17] and the simplified Hirota method [25]. However, these methods do not often provide exact solutions to PDEs. Therefore the use of numerical methods becomes relevant. Finite element methods [65], finite difference methods [66] and spectral methods [47, 58] have been successful methods for solving the KdV hierarchy.

The purpose of this thesis was to study and implement the PS method to solve the initial value problem of the KdV hierarchy of PDEs. However, solutions obtained by the PS spectral method are not always a faithful representation of the exact solutions. In order to fulfil this main objective, we construct travelling waves and soliton solutions of these equations to test the performance of the PS method. We also did a comparative study the PS and the FD. The issues of stability, consistency and convergence of the PS and FD method were addressed.

Using the Fan method in Chapter 2, we successfully recovered travelling wave solutions of the third and the fifth order KdV equations. The solutions of the third order KdV have been previously found in [1]. We recall that the Fan method is only used to find travelling wave solution of a PDE but it can be also used to test whether a given PDE admits travelling wave solutions as illustrated in §2.2.2. It was shown in §2.1.4 that travelling wave solutions of the KdV hierarchy are only possible if there is a perfect balance between dispersion and non-linearity.

A PDE is expected to be integrable if it passes the Painlevé test. In Chapter 3 we successfully verified that the third order KdV equation unconditionally pass the Painlevé test §3.1.2, while the fifth order KdV equation requires suitable choice of its parameters §3.1.3. Particular attention was paid to the KK, Lax and SK equations. In fact, these equations admit two sets of resonances. However, only one set of the two sets of resonances allow these three equations to pass the Painlevé test. This is a sufficient condition to pass the test. We also found the Bäcklund transformation used in determining soliton solution of the KdV hierarchy. For the third order KdV, there exists only one Bäcklund transformation, whereas the fifth order KdV admits two Bäcklund transformations. With the aid of MATHEMATICA, multi-soliton solutions of the third KdV equation have been found. Similar results have been found in [25] with the help of symbolic computation software. In the case of the one soliton solution, similar results have been found in §2.2.3 using the Fan method. This proves the validity of the two methods. Only one of the two Bäcklund transformations helps to find soliton solutions. The one and the two soliton solutions of the KK equation have determined with the help of MATHEMATICA.

Chapter 4 reviewed and motivated the use of numerical schemes employed to approximate solutions of the initial values problem studied in this thesis. The first numerical method used was the finite difference method §4.1, approximates the exact solution with local polynomials of low order. The second numerical method used was the Fourier pseudospectral method §4.2, which uses global representations by Fourier series. In the time t, the Runge-Kutta method was employed (§4.3). We noticed that the second order Runge Kutta method is an unstable method for solving the KdV hierarchy. We successfully located the regions of linear stability of the KdV hierarchy using these methods. Computational tests show the validity of these regions. While investigating the accuracy and convergence, we established that the PS converges exponentially and is more accurate than the FD which has a quadratic convergence.

Chapter 5 presented numerical results of these method for the KdV hierarchy. We first investigated the third order KdV with periodic boundary conditions. The initial conditions used were the one and the two soliton solutions. We also investigated the KK equation with initial and boundary conditions. The maximum and the L^2 norm errors were used to test the performance of the numerical methods. We graphically found that the PS methods enjoy exponential accuracy, while the FD methods have quadratic accuracy. The FD method is less accurate than the PS method as seen in Chapter 4. Numerical soliton solutions are in good agreement with analytical soliton solutions for a small period of time. For a long period of time, the FD numerical soliton solutions propagated slower than the PS and the exact soliton solutions.

6.2 Future Directions

Symbolic Software have been developed for the Painlevé test [67], for the sinh-cosh-tanh [68]. In the literature, no one designed symbolic softwares to solve NLPDEs with the Fan method. This is a direction for future work to solve NLPDEs faster using the Fan method.

Bibliography

[1] PL Bhatnagar, *Nonlinear waves in one-dimensional dispersive systems*, Oxford mathematical monographs, 31-37, (1979)

[2] R Sabry, MA Zahran, E Fan *A new generalized expansion method and its application in finding explicit exact solutions for a generalized variable coefficients KdV equation*, Physics Letters A, 236, 93-101, (2004).

[3] E Yomba, *The modified extended Fan sub-equation method and its application to the (2+1)-dimensional Broer-Kaup-Kupershmidt equation*, Chaos, Solitons and Fractals 27, 187-196, (2006)

[4] AA Soliman, MA Abdou *Exact travelling wave solutions of nonlinear partial differential equations*, Chaos, Solitons and Fractals, **32**, 808-815, (2007).

[5] JE Pearson, *Complex patterns in a simple system*, Science, **261**, 189, (1993).

[6] JL Synge, BA Griffith, *Principles of Mechanics*, 3, (1981).

[7] WN Reynolds, JE Pearson and S Ponce-Dawson, *Dynamics of self-replicating patterns in reaction diffusion systems,* Phys. Rev. Let. 72, (17), 2797-2800, (1994).

[8] Y Shigihara,Wave dispersion study in the Indian Ocean Tsunami of December 26, 2004, Science of Tsunami Hazards, Vol. 25, No. 1, page 42 (2006).

[9] G Teschl, *Inverse Scattering Transform for the Toda Hierarchy*, Math. Nach., 202, 1-8, (1999).

[10] S Karigiannis *Inverse Scattering Transform and Integrability of the nonlinear evolution equations*, Minor Thesis, 1-31, (1995)

[11] T Aktosun, *Inverse Scattering Transform, KdV, and Solitons*, Mathematics Subject Classification, (2000)

[12] AR Osborne, *Inverse Scattering Transform: Tools for the Nonlinear Fourier Analysis and Filtering of Ocean Surface Waves*, Chaos, Solitons and Fractals, 5, 12, 2623-2637, (1995)

[13] F Druitt, *Hirota's Direct Method and Sato's Formalism in Soliton Theory*, Honours Thesis, 2005.

[14] AH Khater, MM Hassan, RS Temsah, *Cnoidal wave solutions for a class of fifth-order KdV equations*, Mathematics and Computers in Simulation, 70, 221-226, (2005).

[15] A-M Wazwaz, *Analytic study on the generalized fifth-order KdV equation: New solitons and periodic solutions*, Communications in Nonlinear Science and numerical Simulation, 12, 1172-1180, (2007).

[16] A Nuseir, *Symbolic Computation of Exact Solutions of Nonlinear Partial Differential Equations Using Directs Methods.*

[17] R Hirota. *The Direct Method in Soliton Theory.* Cambridge University Press, 2004.

[18] G-q Xu and Z-b Li,*PDEPtest: a package for the Painlevé test of nonlinear partial differential equations*, Applied Mathematics and Computation 169 (2005) 13641379

[19] A Bekir, Painlevé test for some (2+1)-dimensional nonlineaar equations, Chaos, Solitons and Fractals 32 (2007) 449-455.

[20] J Weiss, M Tabor, G Carnevale, *The Painlevé property of partial differential equations*, Journal of Mathematical Physics. 24 (1983) 522526.

[21] M Jimbo, M D Kruskal, T Miwa, The Painlevé test for the Self-dual YangMills equations,Phys. Lett. A 92 (1982) 59..

[22] S Y Lou, C L Chen, X Y Tang, *(2 + 1)-Dimensional (M + N)-Component AKNS system:Painlevé integrability, infinitely many symmetries and similarity reductions*, Journal of Mathematical Physics 43 (2002) 40784109.

[23] A Roy Chowdhury, The Painlevé Analysis and its Applications, Chapman and Hall/CRC, Baton Rouge, FL, 2000.

[24] T Alagesan, Y Chung, K Nakkeeran. Painlevé test for the certain (2+1)-dimensional nonlear evolution equations. Chaos, Solitons and Fractals 26 (2005) 1203-9

[25] W Hereman, A Nuseir, *Symbolic methods to construct exact solutions of nonlinear partial differential equations*, Mathematics and Computaters in Simulation 43 (1997) 13-27

[26] P Lax, *Integrals of nonlinear equations of evolution and solitary waves, Comm. Pure Appl. Math. 21 (1968)467-490.*

[27] K sawada and T Kotera, *A method of finding N-soliton solutions of the KdV and the KdV-like equation, Progr. Theoret. Phys. 51 (1974) 1355-1367*

[28] A R Mitchell, D F Griffiths, *The Finite Difference Method in Partial Differential Equations*, John Willey and Sons Ltd, (1980), 9-10

[29] T Sogabe, *A fast numerical algorithm for the determinant of a pentadiagonal matrix,* Applied Mathematics and Computation, (2007), 1-7

[30] E Kilic, M El-Milkkawy, *A computational algorithm for special nth-order pentadiagonal Toeplitz determinants*, Applied Mathematics and Computation, (2007), 1-3

[31] L N Trefethen, *Finite difference and spectral methods for ordinary and partial differential equations*,(1996),

[32] J C Butcher, *The Numerical Analysis of Ordinary Differential Equations: Runge-Kutta and General Linear Methods* (Wiley). (1987)

[33] E Hairer, S P Norsett, G Wanner, Solving ordinary differential equations

[34] M D Kruskal, N Joshi and R Halburd , *Analytic and Asymptotic Methods for Nonlinear Singularity Analysis: A Review and Extensions of Tests for the Painlevé Property*, in Proceedings of CIMPA Summer School on Nonlinear Systems, Editors: Grammaticos B and Tamizhmani K, Lecture Notes in Physics 495, Springer Verlag, Heidelberg, (1997), 171205.

[35] R Conte, A P Fordy and A Pickering , *A Perturbative Painlevé Approach to Nonlinear Differential Equations*, Physica 69D(1-2) (1993), 3358

[36] B Fornberg, *A practical guide to pseudospectral methods*, Cambridge University Press, 1996

[37] E Tadmor, The exponential accuracy of Fourier and Chebyshev differencing methods. SIAM Journal of Numerical Analysis,23,1, (1986), 1-10

[38] M Azaiez, S Boudoux-Dagonat, M A Fikri and G Labrosse, *The computational fluid dynamics by spectral methods : some illustrative results*, Computers and Chemical Engineering, 17,1, (1993),523-529

[39] C Talbot and A Crampton, *Pseudo Spectral Methods Applied to Problems in Elasticity*,Journal of Scientific Computing 27 , 1-3 (2006),443 - 454

[40] M T Darvishi, F Khani and S Kheybari, *A numerical solution of the KdV-Burgers' equation by spectral collocation method and Darvishi's Preconditionings*, Int. J. Comtep. Math. Sciences, 2, 22, (2007), 1085-1095

[41] M T Darvishi, F Khani and S Kheybari, *A Numerical Solution of the Laxs 7th-order KdV Equation by Pseudospectral Method and Darvishis Preconditioning*, Int. J. Comtep. Math. Sciences, 2, 22, (2007), 1097 - 1106

[42] J A C Weideman and S C Reddy, *A MATLAB differention matrix suite*, ACM Transactions on Mathematical Software, 26, 4, (2000), 465-519.

[43] D Gottlieb and S A Orszag, *Numerical analysis of spectral methods*, Society for Industrial and Applied Mathematics, Philadelphia (1977)

[44] D. Gottlieb, M.Y. Hussaini, S.A. Orszag, "Theory and application of spectral methods" R.G. Voigt (ed.) D. Gottlieb (ed.) M.Y. Hussaini (ed.), Spectral Methods for Partial Differential Equations , SIAM (1984)

[45] E Tzirtzilakis, V Marinakis, C Apokis and T C Bountis, *Soliton-like solutions of the higher order wave equations of the Korteweg-de Vies type*, Journal of Mathematical Physics 43, 12, (2002), 6151-6165.

[46] E Tzirtzilakis, M Xenos, V Marinakis and T C Bountis, *Interaction and stability of solitary waves in shallow water*, Chaos, Solitons and Fractals, 14, (2002), 87-95

[47] E Tzirtzilakis, C D Skokos and TC bountis, *Numerical solution of the Boussinesq equation using spectral methods and stability of solitary wave propagation*,Scientific computing to computational engineering, (2004), 928-933

[48] E Tadmor, *The exponential accuracy of Fourier and Chebyshev Differencing Methods*, Journal on Numerical analysis, 23, 1, (1986), 1-10

[49] A Parker, *On soliton solutions of the Kaup-Kupershmidt equation.I. Direct bilinearisation and solitary wave*, Physica D, 137,(2000), 25-33

[50] A Parker, *On soliton solutions of the Kaup-Kupershmidt equation.II. 'Anomalous' N-soliton solutions*, Physica D, 137,(2000), 34-48

[51] M Inc, *On numerical soliton solution of the KaupKupershmidt equation and convergence analysis of the decomposition method*, Applied Mathematics and Computation 172 (2006), 7285

[52] P Saucez, A V Wouwer, W E Schiesserc, P Zegeling, *Method of lines study of nonlinear dispersive waves*, Journal of Computational and Applied Mathematics, 168, (2004), 413423

[53] A Constantin, V S Gerdjikov, R I Ivanov, *Inverse Scattering Transform for the Camassa-Holm equation* ,

[54] Chris Eilbeck, 'Jonh Scott Russell and solitary waves, $http : //www.ma.hw.ac.uk/\ chris/scott_russell.html$

[55] C S Gardner , J M Greene, M D Kruskal and R M Miura, *Method for Solving the Korteweg-deVries Equation*, Phys. Rev. Lett., 19, (1967),1095-97.

[56] R Hirota, *Exact Solution of the Korteweg De Vries Equation for Multiple Collisions of Solitons*, Phys. Rev. Lett., 27, (1971), 1192-1194.

[57] R Courant, K Friedrichs and H Lewy, *On partial difference equations of mathematical physics, IBM journal of Research and Development*, 11, (1967), 215-234

[58] E Rachid, *Numerical solution of the Korteweg-de Vries equation by the Fourier pseudospectral method*, Bulletin of Belgium Society Simon Stevin 14(2007), 709-721.

[59] A C Newell, *Solitons in mathematics a physics*, society for industrial and applied mathematics, (1985), 1-244

[60] G Das and J Sarma, *A new mathematical approach for finding the solitary waves in dusty plasma*, Physics Plasmas, 6, (1999) 4394-4397

[61] A Osborne, *The inverse scattering transform: Tools for finding the nonlinear Fourier analysis and filtering of ocean surface wave*, Choas Solitons and Fractals, 5, (1985), 422-443.

[62] L Ostrousky and Y A Stepanyants, *Do internal solitons exist in the ocean?*, Review of Geophysics,43, 3, (1998), 293-310.

[63] L Reatto and D Galli, *What is ROTON?*, International Journal of Physics, B13, (1999), 607-616.

[64] A Ludu and J P Draayer, *Nonlinear modes of liquid drops as solitary waves*, Physical review letters, 80, (1998), 2125-2128.

[65] E N Aksan and A Özdes, Numerical solution of Korteweg-de Vries equation by Garlekin B-spline finite element method, Applied mathematics and computation, 175, (2006), 1256-1265.

[66] A R Bahadir, Exponencial finite element method applied to Korteweg-de Vries equation for small times, Applied Mathematics and Computation, 160, (2005), 675-682.

[67] D Baldwin and W Hereman, Symbolic Sofware for the Painlevé Test of Nonlinear Ordinary and Partial Differential Equations, 1, 13, (2006),1-21.

[68] D Baldwin, Ü Göktas, W Hereman, L Hong, R S Martino and J C Miller, Symbolic computation of the exact solutions expressible in hyperbolic and elliptic functions for nonlinear PDEs, Journal of Symbolic Computation 37, (2004) 669-705.