A Survey of Computational Methods for Pricing Asian Options

by

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Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

Signature: .......................  
F.D. Nieuwveldt.

Date: ............................
Abstract

In this thesis, we investigate two numerical methods to price financial options. We look at two types of options, namely European options and Asian options. The numerical methods we use are the finite difference method and numerical inversion of the Laplace transform. We apply finite difference methods to partial differential equations with both uniform and non-uniform spatial grids. The Laplace inversion method we use is due to Talbot. It is based on the midpoint-type approximation of the Bromwich integral on a deformed contour. When applied to Asian options, we have the problem of computing the hypergeometric function of the first kind. We propose a new method for numerically calculating the hypergeometric function. This method too is based on using Talbot contours. Throughout the thesis, we use the Black-Scholes equation as our benchmark problem.
Opsomming

In hierdie tesis ondersoek ons twee numeriese metodes vir die pryssvasstelling van finansiële opsies. Ons sal kyk na twee tipes opsies, naamlik Europese opsies en Asiatishe opsies. Die numeriese metodes wat ons gaan gebruik is eindige verskille en die numeriese omkering van die Laplace transform. Eindige verskille word toegepas op parsiale differensiaalvergelykings met beide uniforme en nie-uniforme ruimtelike roosters. Die Laplace inverse metode wat ons gebruik is te danke aan Talbot. Dit is gebaseer op die middelpunt-tipe benadering van die Bromwich integraal, maar op 'n vernormde kontoer. Wanneer Talbot se metode toegepas word op Asiatishe opsies het ons die probleem van die berekening van die hypergeometriese funksie van die eerste soort. Ons stel 'n nuwe metode voor vir die numeriese benadering van die hypergeometriese funksie wat gebaseer is op Talbot se kontoere. Ons sal deurgaans die Black-Scholes vergelyking gebruik as toets-probleem.
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Dedicated to my Father, Jimmy Nieuwveldt
(1951-2008)
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Chapter 1

Introduction

This chapter is a discussion of the financial concepts we shall be concerned with in this thesis. Before we proceed to computational finance we need to define what we mean by pricing options.

1.1 Forward contracts and options.

A forward contract is an agreement to buy or sell one unit of an underlying asset at a specified future time, known as the delivery time, for a price specified in advance, called the forward price. A trader who agrees to buy the asset is said to be taking a long forward position and the other party has a short forward position.

An option is a security giving the right, without any obligation, to buy or sell an asset within a prescribed time period. It is only the option holder who has the choice of exercising this right; if he or she decides to exercise the option, then the seller of the option is obliged to trade. There are different kinds of options. An American option is one that can be exercised at any time up until the the option expires. A European option can only be exercise at a specified exercise date. The price to be paid for the asset when the option is exercised is called the exercise price or the strike price. The day which the option is exercised is called the expiration date or maturity date.

There are different kinds of options, but we shall only consider call and put options. A call option gives its holder the right, without any obligation, to buy one unit of the underlying asset at a predetermined strike price. The holder of a put option has the right to sell one unit of the underlying asset at the strike price without being obliged to do so.
Chapter 1. Introduction

At first glance a call option might resemble a long forward position. Both cases involve buying an asset at a future date for a price fixed in advance. The difference between the two is that the holder of a long forward contract is committed to buying the asset for the fixed price, whereas the owner of a call option is not obliged to do so.

The following convenient terminology is often used. We say that at time $t$ a call option with strike price $K$ is
- in-the-money if $S(t) > K$,
- at-the-money if $S(t) = K$,
- out-of-the-money if $S(t) < K$.

Similarly for a put option we say that it is
- in-the-money if $S(t) < K$,
- at-the-money if $S(t) = K$,
- out-of-the-money if $S(t) > K$.

1.1.1 Definition of Arbitrage

One of the fundamental concepts for the pricing of financial derivatives is that of arbitrage. Arbitrage is defined as the opportunity to make money without the risk of loss. We have two scenarios: making an immediate profit with no risk of future loss, and no immediate cost of future loss but the possibility of future gain. Such opportunities cannot exist for significant length of time before prices move to eliminate them. If there exist two securities, both with the same payoff, then the securities must have the same price, which is known as the law of one price [31, p. 34]. If this is not the case, then an investor could buy the cheaper and sell the more expensive one, thus making an immediate profit with no future cost.

1.2 Asian options

Asian options are securities with payoffs that depend on the average value of an underlying stock price over its lifespan. It is a contract given the holder the right to buy an asset for its average price over some prescribed period. The average may be taken geometrically or arithmetically, which
may be measured either continuously or discretely. There are several reasons why Asian options became popular in the marketplace. For example, Asian options are less sensitive to market fluctuations near the expiry date. However, these options have proved to be much more difficult to price than other options. No general analytical solution for the price of the Asian option is known. Therefore we need accurate numerical methods for pricing these options. We list a few approaches:

- Numerical solution of a two-dimensional partial differential equation (PDE), as in [12].
- A rescaled one-dimensional PDE as in [20, 28] and more recently [7].
- Monte Carlo simulations with variance reduction techniques, see for example [14].
- Rogers and Shi derived lower and upper bounds for the price [20] and more recently Thompson improved on these bounds [26].
- Linetsky used spectral methods [17].
- Geman and Yor used Bessel processes and the Laplace transform to get an expression where one needs to do Laplace inversion, see [10]. For a survey of techniques for the inversion see [5]. Other transform approaches include [2, 16, 24].

The methods we shall concentrate on in this thesis are the Laplace transform approach and the PDE approach. We shall derive a two-dimensional PDE for pricing arithmetic Asian options in Chapter 2. We shall see that this PDE will become very important in the chapters that follow where we will derive all our main methods. Both Vecer’s PDE and Geman and Yor’s Laplace transform approach can be derived from this PDE without using any previous knowledge of Bessel processes.

1.3 Outline of Thesis

The famous Black-Scholes model is a convenient way to calculate the value of an option. In this thesis an accurate numerical method to solve this equation will be presented. In spite of the fact that an analytic solution exists, this helps to develop a general numerical scheme to price different types of options. In particular, Asian options are not solvable in an analytic sense. The numerical methods we shall use are the finite difference method with
uniform and non-uniform grids and the numerical inversion of the Laplace transform. We shall now describe the contents of this thesis.

Chapter 2 discusses the derivation of the mathematics of options and their prices. Section 2.1 describes the stochastic model for asset prices. The derivation of the Black-Scholes equation is based on this model and is given in Section 2.2. Thereafter, in Section 2.3, we shall derive a two space dimensional PDE for Asian options in the Black-Scholes framework. In Section 2.4 we derive well known one dimensional PDEs in the literature for Asian options. Section 2.5 deals with application of the Laplace transform in option pricing. We conclude the chapter with a formula for the European option and the Asian option price as a Laplace transform.

Chapter 3 consists of our numerical setup for some of the equations derived in Chapter 2. Section 3.1 deals with time and spatial discretization for a finite difference scheme for a general parabolic equation. In Section 3.2 we give a von Neumann stability analysis for the finite difference scheme. We apply the finite difference method with uniform spacing on the Black-Scholes PDE in Section 3.3. In Section 3.4, spatial transformations will be discussed to generate non-uniform grids. In conclusion we give numerical results in Section 3.5 for the Black-Scholes equation and for Asian options in Section 3.6.

Chapter 4 deals with numerical inversion of the Laplace transform. In particular, we shall use Talbot’s method as our inversion algorithm. In Section 4.1 we give an overview of the numerical inversion of the Laplace transform and the Talbot method. In Section 4.2, we apply our numerical inversion algorithm to price European options. The chapter will conclude with the numerical inversion method for Asian options. It consists of an alternative method for numerically computing the hypergeometric function of the first kind. Two contour representations of the hypergeometric function will be proposed: one based on the Barnes-Mellin transform and the other based on a Laplace transform. We shall apply Talbot’s method in the latter case and compare it with existing software packages. Numerical results and timings will be given. Thereafter, we apply it to price Asian options numerically. It consist of two inversion problems: one for high volatility and the other for low volatility of which both are double inversion problems.

The thesis ends with conclusions and recommendations for further study in Chapter 5. In the Appendix we give the theory of mathematical finance that we use throughout the thesis. Appendix A consists of a background
in probability and martingale theory with a subsection for Ito’s Lemma. In Appendix B we solve the Black-Scholes equation analytically. Finally, in Appendix C we give the theory and properties of the Laplace transform.
Chapter 2

Black-Scholes analysis

In this chapter the Black-Scholes equation for option pricing will be derived. We shall consider two different options namely call and put options and give the analytical solutions of European options. We also derive a two-dimensional PDE for pricing arithmetic average Asian options from which we shall derive all our main methods. From this two-dimensional PDE we shall derive one-dimensional PDEs as in [20, 28, 29]. Lastly we derive a semi-analytical formula for pricing Asian options, see [10]. The method we shall use differ from that in [10], who used the Laplace transform and knowledge of Bessel processes for their derivation. We shall use the method in [6] which is a PDE based method using the Laplace transform. No knowledge of Bessel processes is needed.

2.1 A model for asset prices.

Let \( S \) denote an asset, say for example a stock. The return on an asset price is defined by

\[
\frac{dS_t}{S_t},
\]

that is, the return is the change of the asset’s price divided by the original price. This poses the question as to how we should model it. We can divide the contributions into two parts. The one is a predictable, deterministic return. It gives a contribution

\[
\mu dt,
\]

where \( \mu \) is a measure of the average rate of growth of the asset price, known as the drift. In simple models \( \mu \) is taken to be constant. The second contribution to \( \frac{dS_t}{S_t} \) is a random change due to uncertainties in the asset
price. The non-deterministic contribution to \( \frac{dS_t}{S_t} \) may be modeled as

\[
\sigma dB_t,
\]

where \( \sigma \) is called the volatility of the stock and \( B_t \) is a Brownian motion, see (2.1.4). The volatility expresses how much the price fluctuates. We now have a model for the return of an asset price namely

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dB_t,
\]

or

\[
dS_t = \mu S_t dt + \sigma S_t dB_t. \tag{2.1.4}
\]

Note that if \( \sigma = 0 \), it leaves us with a normal deterministic differential equation. In essence, we have a definite asset price, because there is no fluctuation. The model is called geometric Brownian motion, see (2.1.6). By Itô’s lemma (2.1.5) we have

\[
d(ln S_t) = \left( \mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dB_t. \tag{2.1.5}
\]

We conclude that \( ln S_t \) is a Brownian motion with drift. The solution to this stochastic differential equation is given by [11, p. 56, (6.3)]

\[
S_t = S_0 e^{(\mu - \frac{1}{2} \sigma^2) t + \sigma B_t}. \tag{2.1.6}
\]

This is one of few stochastic differential equations that have an explicit solution. In Figure 2.1 we see a graph of a stock price following geometric Brownian motion. It is always positive since \( S_0 > 0 \).
Figure 2.1: Simulation of stock prices following geometric Brownian motion.

2.2 Black Scholes PDE

In this section we derive the well-known PDE of Black and Scholes for option pricing. In order to explain the Black-Scholes analysis, we shall make the following assumptions

- The asset price follows the geometric Brownian motion.
- The risk-free rate $r$ and the asset volatility $\sigma$ are known and constant over the life of the option.
- There are no transaction costs associated with hedging a portfolio.
- The underlying asset pays no dividends during the life of an option.
- There are also no arbitrage opportunities.
- Trading of the underlying asset can take place continuously.
- Short selling is permitted and the assets are divisible. This means we can buy and sell any fractional amount of the underlying asset, and we may sell assets that we do not own.
If the option depends on two variables, $S$ and $t$, the value of the option will be denoted by $V(S, t)$. Recall that Ito’s lemma is given by

$$df(S, t) = \left( \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S \frac{\partial f}{\partial S} dB,$$

see (A.0.5). Applying Ito’s lemma on $V(S, t)$, we have

$$dV = \sigma S \frac{\partial V}{\partial S} dB + \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt. \quad (2.2.1)$$

This is the random walk process of $V$. The steps thus far were only mathematical. To complete our model we need some financial arguments. We now construct a portfolio consisting of one option with value $V$ and a number $-\Delta$ of the underlying asset. At this point $\Delta$ is unspecified. The value of this portfolio will be

$$\Pi = V - \Delta S, \quad (2.2.2)$$

and the change of the portfolio is

$$d\Pi = dV - \Delta dS. \quad (2.2.3)$$

Combining (2.1.4), (2.2.1), (2.2.2) and (2.2.3) we have

$$d\Pi = \sigma S \left( \frac{\partial V}{\partial S} - \Delta \right) dB + \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right) dt. \quad (2.2.4)$$

Next, we can eliminate the random component by choosing

$$\Delta = \frac{\partial V}{\partial S}. \quad (2.2.5)$$

With this choice of $\Delta$ the change of the portfolio is deterministic with

$$d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \quad (2.2.6)$$

Had we invested an amount $\Pi$ in riskless assets our growth should equal $r\Pi dt$. Thus

$$r\Pi dt = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \quad (2.2.7)$$

Substituting (2.2.2) and (2.2.5) in (2.2.7) and dividing by $dt$ we get

$$\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV + \frac{\partial V}{\partial t} = 0. \quad (2.2.8)$$
This is the Black-Scholes PDE, see for example [31, p. 43]. It is a parabolic PDE with non-constant coefficients. Let us denote the value of the European call option on the share $S$ at time $t$ by $C(S,t)$ and the European put option by $P(S,t)$. To obtain an option price we need to combine equation (2.2.8) with boundary conditions. This will be given in the next section.

### 2.2.1 European call option.

In the European situation for a call option we know that at expiry time, $t = T$, the payoff is $\max(S(T) - K, 0)$. If $S(T) < K$, then the option is worthless as it is cheaper to buy the share on the open market. If $S(T) \geq K$, then one could buy the option for $K$ and sell it immediately for $S$ and make a profit of $S - K$, otherwise the option value is zero. Therefore the payoff is

$$C(S,T) = \max(S(T) - K, 0).$$

We also have at $S = 0$ that

$$C(0,t) = 0, \quad \text{for} \quad 0 \leq t \leq T. \quad (2.2.10)$$

If the holder had instead invested his/her money in the bank at the risk-free interest rate $r$, the boundary condition at infinity would have been

$$C(S,t) \rightarrow S - Ke^{-r(T-t)}, \quad \text{as} \quad S \rightarrow \infty. \quad (2.2.11)$$

The complete system is thus

$$\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} - r C + \frac{\partial C}{\partial t} = 0$$

$$C(0,t) = 0$$

$$\lim_{S \to \infty} C(S,t) = S - Ke^{-r(T-t)}$$

$$C(S,T) = \max (S - K, 0). \quad (2.2.12)$$

The analytic formula for the value of a European call option is the solution to (2.2.12), given by

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \quad (2.2.13)$$
where $d_1$, $d_2$ and $N(d)$ are given by

$$
d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}},$$

$$
d_2 = \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}},$$

$$
N(d) = \int_{-\infty}^{d} \exp(-\frac{1}{2}x^2)dx. \tag{2.2.14}
$$

See Appendix B for a derivation of the exact solution. Figure 2.3 shows a graph of the final condition for a European call option.

### 2.2.2 European put option.

A European put option gives the holder the right to sell the underlying asset for the strike price $K$ at maturity time $T$, with payoff $\max(K - S(T), 0)$. If $S(T) > K$, then the option is worthless as it is better to sell the share on the open market. If $S(T) \leq K$, then one could sell the share for $K$
Figure 2.3: Final value and solution of a European call option. The parameters are $K = 10$; $\sigma = 0.3$; $r = 0.03$ and $T = 1$.

and make a profit of $K - S(T)$. This condition should hold, otherwise the option value is zero. The complete system is

$$\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + r S \frac{\partial P}{\partial S} - r P + \frac{\partial P}{\partial t} = 0$$

\[ P(0, t) = Ke^{-r(T-t)} \]
\[ P(S, t) = 0 \quad \text{as} \quad S \to \infty, \]
\[ P(S, T) = \max (K - S, 0). \quad (2.2.15) \]

The solution of the European put option is then given by

$$P(S, t) = Ke^{-r(T-t)} N(d_2) - SN(-d_1), \quad (2.2.16)$$

see Appendix (B) with $d_1$, $d_2$ and $N(d)$ defined as above in (2.2.14). Figure 2.5 shows a graph of the final condition for European put option.
Figure 2.4: Payoff diagram for a European put option.

2.2.3 Put-Call parity.

Here we shall show that call and put options are perfectly correlated. We demonstrate this with the following argument. Suppose we buy one unit of the asset and put option and sell one call option. The call and put both have the same exercise price, $K$, and expiry date, $T$. We have

$$\Pi = S + P - C,$$

where $\Pi$ denote the value of the portfolio and $P$ and $C$ denote the values of the put and call respectively. The payoff for this portfolio at time $T$ is

$$S + \max(K - S, 0) - \max(S - K, 0).$$

Whether $S$ is greater or less than $K$ at time $T$, the payoff for both cases is the same, i.e., $K$. By discounting the final value of the portfolio we are sure to receive a guaranteed amount $K$ at time $t = T$. At time $t$ the portfolio is worth $Ke^{-r(T-t)}$. We have

$$S + P - C = Ke^{-r(T-t)}. \tag{2.2.17}$$

This relationship is known as the put-call parity.
2.3 Arithmetic average Asian options

In this section we derive a PDE for Asian options. We follow the same strategy as in the case of European options. In this case our final condition depends on the average over the lifetime of the asset. The final condition for an arithmetic average fix strike Asian option is

$$\max \left( \frac{1}{T} \int_0^T S(\tau) d\tau - K, 0 \right),$$

where the integral divided by $T$ represents the average of the option. By contrast, a floating strike Asian option has a payoff based on the difference between the underlying asset at the expiration $T$ and the average, say $I$, of the underlying prior to the expiration. We have the final condition for a floating strike Asian option as

$$\max \left( \frac{1}{T} \int_0^T S(\tau) d\tau - S_T, 0 \right).$$
In this thesis we shall consider the fixed strike Asian option. Let us define a new variable

\[ I(t) = \int_0^t S(\tau) \, d\tau, \quad (2.3.3) \]

which is the average of the history of the asset price, where we know that this is independent of the current price. We can therefore treat \( I, S \) and \( t \) as independent variables. The value at time \( t < T \) of the option depends on \( I, S \) and \( t \), which we write as \( V(S, I, t) \).

In order to apply Ito’s lemma to \( V \), the stochastic differential equation for \( I \) must be determined. We have

\[ dI = S(t) \, dt. \]

When applying Ito’s lemma to \( V(S, I, t) \) we get

\[ dV = \sigma S \frac{\partial V}{\partial S} dB + \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial I} + S \frac{\partial V}{\partial I} \right) dt, \]

where \( dB \) is a standard Brownian motion. By the Black-Scholes assumption we have

\[ dS = rS \, dt + \sigma S \, dB_t. \]

We can now write

\[ dV = dS \frac{\partial V}{\partial S} + \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial I} \right) dt \]

or

\[ dV - dS \frac{\partial V}{\partial S} = \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial I} + S \frac{\partial V}{\partial I} \right) dt. \]

Since the option is European, we can set up the usual risk-free portfolio as in the Black-Scholes case. Let us long one Asian option and short an amount \( \frac{\partial V}{\partial S} \) of the underlying asset. The value of this portfolio is

\[ \Pi = V + S \frac{\partial V}{\partial S}, \]

or

\[ d\Pi = dV + dS \frac{\partial V}{\partial S} = \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial I} + S \frac{\partial V}{\partial I} \right) dt, \]

which is riskless. Since we invested an amount \( \Pi \) in riskless assets our growth should equal \( r\Pi dt \), we then have

\[ r\Pi dt = \left( rV - rS \frac{\partial V}{\partial S} \right) dt = \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial I} + S \frac{\partial V}{\partial I} \right) dt. \]
This gives the two-dimensional PDE for arithmetic Asian option
\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + S \frac{\partial V}{\partial I} - rV = 0. \tag{2.3.4}
\]
This equation is similar to the Black-Scholes equation, except for the new term \( S \frac{\partial V}{\partial I} \), see [31, p. 215]. The parameters \( r \) and \( \sigma \) are the interest rate and volatility as before. The PDE (2.3.4) will be the cornerstone for most of the methods and derivations concerning Asian options that we shall study in the remainder of this thesis.

2.4 One-Dimensional PDEs for Asian options

2.4.1 Pricing arithmetic average Asian options with the PDE approach

In this section we shall derive one-dimensional PDEs for pricing Asian options as in [20, 23, 29]. These equations are well-known PDEs in the literature. Our method of derivation differs from Vecer’s approach who used options on a traded account. We shall use a change of variables approach on (2.3.4) as in [7].

2.4.2 Rogers and Shi’s PDE for pricing Asian options

Recall that the PDE for an Asian call option with value \( C(S, I, T) \) is
\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + S \frac{\partial C}{\partial I} - rC = 0, \tag{2.4.1}
\]
with final condition
\[
C(S, I, T) = \max \left( \frac{I}{T} - K, 0 \right). \tag{2.4.2}
\]

We use the change of variables
\[
x = \frac{K - \frac{I}{T}}{S}, \quad C(S, I, t) = Sf(t, x), \tag{2.4.3}
\]
by which Rogers and Shi [20] have reduced the PDE from two variables to one. The new one space dimensional PDE is
\[
\frac{\partial f}{\partial t} + \sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} - \left( \frac{1}{T} + rx \right) \frac{\partial f}{\partial x} = 0, \tag{2.4.4}
\]
with final condition
\[ f(T, x) = \max(-x, 0). \] (2.4.5)

The PDE is defined on the whole real axis. Note that in Geman and Yor’s paper [10], a formula was obtained for the case \( x \leq 0 \) (equivalently \( I \geq KT \)) as
\[ C(S, I, t) = S \left( \frac{1 - e^{-r(T-t)}}{rT} \right) + e^{-r(T-t)} (I/T - K). \] (2.4.6)

By making the change of variables as in (2.4.3), we get
\[ f(x, t) = \frac{1}{rT} (1 - e^{-r(T-t)}) - xe^{-r(T-t)}. \] (2.4.7)

We consider the solution of the PDE only for \( x \geq 0 \) using (2.4.7) for the boundary condition at \( x = 0 \). The complete system of Rogers and Shi’s PDE is therefore
\[ \frac{\partial f}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 f}{\partial x^2} - \left( \frac{1}{T} + rx \right) \frac{\partial f}{\partial x} = 0, \]
\[ f(x, T) = 0, \]
\[ f(0, t) = \frac{1}{rT} (1 - e^{-r(T-t)}), \]
\[ f(L, t) = 0. \] (2.4.8)

The numerical solution of the PDE in (2.4.8) is problematic for the cases of low volatility. For example, Dubois and Lelièvre note that the reason for the poor results is due to the fact that when \( x \) is close to zero, the advection term is larger than the diffusion term [7]. In the same manner one can derive the PDEs by Vecer [28, 29] as shown in the next section.

### 2.4.3 Vecer’s PDEs for pricing Asian options.

One can derive Vecer’s PDE, see [28] [29], without using any concepts of options on a traded account. Let us make the following change of variables on (2.4.4):
\[ x = y - \frac{t}{T}, \quad q(t, y) = f(t, x). \] (2.4.9)

This transforms the PDE to
\[ \frac{\partial q}{\partial t} + \frac{\sigma^2(y - t/T)^2}{2} \frac{\partial^2 q}{\partial y^2} - r(y - t/T) \frac{\partial q}{\partial y} = 0, \]
\[ q(T, y) = \max(1 - y, 0). \] (2.4.10)
This equation has been obtained in Vecer [28]. We now have an equation with coefficients depending on time and space. We can get rid of the advective term by considering the change of variables
\[ g(t, y) = f(t, \frac{1}{rT}(ye^{r(T-t)} - 1)). \]  
(2.4.11)

One can show that \( g \) is a solution of
\[ \frac{\partial g}{\partial t} + \frac{\sigma^2}{2} \left( y - e^{-r(T-t)} \right)^2 \frac{\partial^2 g}{\partial y^2} = 0. \]  
(2.4.12)

This PDE was derived in [29]. Let us make a final transformation
\[ y = 1 - rT\zeta, \quad \tau = T - t. \]

The final form of our PDE we would like to solve is
\[ \frac{\partial g}{\partial \tau} = \frac{\sigma^2}{2} \left( \zeta - \frac{1 - e^{-\tau \sigma}}{rT} \right)^2 \frac{\partial^2 g}{\partial \zeta^2}, \]
\[ g(\zeta, 0) = \max(\zeta, 0). \]  
(2.4.13)

In the next chapter we solve these PDEs numerically.

## 2.5 Option pricing in the Laplace domain

In this section we apply the Laplace transform in option pricing. We start by taking the Laplace transform of the Black-Scholes equation in time, see [23], and then move on to Asian options. The Geman and Yor formula will also be derived by taking a different route based on the Laplace transform of a PDE [6].

### 2.5.1 Laplace transform and the Black-Scholes PDE

To apply the Laplace transform we first make the change of variables
\[ \tau = T - t \quad \text{and} \quad x = \log(S/K), \]
on (2.2.8). This transformation gives
\[ \frac{\partial C}{\partial \tau} = \frac{1}{2} \sigma^2 \frac{\partial^2 C}{\partial x^2} + \mu \frac{\partial C}{\partial x} - rC, \]
with \( \mu = r - \frac{1}{2} \sigma^2 \). Applying the Laplace transform, see (C.0.1), to (2.5.1) gives
\[
\frac{1}{2} \sigma^2 \partial^2 \hat{C} \frac{\partial \hat{C}}{\partial x^2} + \mu \frac{\partial \hat{C}}{\partial x} - (r - p) \hat{C} = -C(x, 0). \tag{2.5.1}
\]
For \( x \leq 0 \) and \( x \geq 0 \) we have
\[
C(x, 0) = 0 \quad \text{and} \quad C(x, 0) = -\psi(e^x - 1),
\]
respectively. The variable \( \psi = 1 \) and \( \psi = -1 \) denote the call and put options respectively. Let us first solve the homogeneous ODE in (2.5.1). We assume that the solution has the form \( \hat{C}(x, p) = e^{\xi x} \). Plugging it into (2.5.1), we get the characteristic equation
\[
\frac{1}{2} \sigma^2 \xi^2 + \mu \xi - (r + p) = 0,
\]
which has two roots
\[
\xi_1 = \frac{\mu + \sqrt{\mu^2 + 2 \sigma^2 p}}{\sigma^2}, \quad \xi_2 = \frac{\mu - \sqrt{\mu^2 + 2 \sigma^2 p}}{\sigma^2}.
\]
If \( p > 0 \), we have \( \xi_2 < 0 < \xi_1 \), since \( \sqrt{\mu^2 + 2 \sigma^2 p} > \mu \). Next we find the particular solution for the non-homogeneous ODE, which hold if
\[
\psi(e^x - 1) > 0.
\]
We assume that the solution has the form \( \hat{C}(x, p) = ae^{\xi x} + b \). Plugging in into (2.5.1) and matching the left- and right-hand sides, we obtain
\[
a = \frac{\psi}{p}, \quad \text{and} \quad b = -\frac{\psi}{p + r}.
\]
Our general solution is
\[
\hat{C}(x, p) = \begin{cases} 
B_1 e^{\xi_1 x} + B_2 e^{\xi_2 x} + \frac{\psi - 1}{2} \left[ \frac{e^x}{p} - \frac{1}{p + r} \right], & x \leq 0 \\
B_2 e^{\xi_1 x} + B_4 e^{\xi_2 x} + \frac{\psi + 1}{2} \left[ \frac{e^x}{p} - \frac{1}{p + r} \right], & x > 0.
\end{cases}
\]
We need the option value to go to zero as \( x \) becomes large and negative, and the value should approach the current value of the non-zero payoff as \( x \) becomes large and positive. This lead to \( B_3 = B_4 = 0 \). To determine \( B_1 \)
and $B_2$, we require that the option and its derivative with respect to $x$ are continuous at $x = 0$. We then have that the solutions to $B_1$ and $B_2$ are

\[
B_1 = \frac{1}{\xi_1 - \xi_2} \left[ \frac{\xi_2}{p + r} + 1 \right] = 0,
\]

\[
B_2 = \frac{1}{\xi_1 - \xi_2} \left[ \frac{\xi_1}{p + r} + 1 \right].
\]

We now have, in the Laplace domain, the value of an European option as

\[
\hat{C}(x, p) = \begin{cases} 
B_1 e^{x_1 x} + \frac{\psi - 1}{2} \left[ \frac{e^x}{p} - 1 \right], & x \leq 0 \\
B_2 e^{x_1 x} + \frac{\psi + 1}{2} \left[ \frac{e^x}{p} - 1 \right], & x > 0.
\end{cases}
\]

The option value is given by the inverse Laplace transform:

\[
V(t, S) = K \mathcal{L}^{-1}(\hat{C}(x, p)).
\]

In Chapter 4, the formula (2.5.3) will play the role as our test problem for testing the quality of our numerical inversion methods.

## 2.6 Asian options as a Laplace transform

In this section we derive the Laplace transform expression for Asian options [10]. We shall consider a different approach based on a PDE whereas Geman and Yor used knowledge of Bessel processes. Consider again the PDE for Asian options (2.3.4)

\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial t} + \frac{\partial C}{\partial I} + \frac{S}{S} \frac{\partial C}{\partial I} - rC = 0,
\]

with

\[
C(S, I, T) = \max \left( \frac{I}{T} - K, 0 \right).
\]

Let us make the following transformations

\[
\phi = \frac{C}{S}, \quad \eta = \frac{I - KT}{TS}.
\]

We can show that $\phi$ satisfies the following PDE

\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} \sigma^2 \eta^2 \frac{\partial^2 \phi}{\partial \eta^2} + \left( \frac{1}{T} - r \eta \right) \frac{\partial \phi}{\partial \eta} = 0.
\]
Let us set
\[ \tau = T - t, \quad \psi = e^{-\tau r} \phi, \quad (2.6.5) \]
to obtain the PDE
\[ \frac{\partial \psi}{\partial \tau} = \frac{1}{2} \sigma^2 \eta^2 \frac{\partial^2 \psi}{\partial \eta^2} + \left( \frac{1}{T} - r \eta \right) \frac{\partial \psi}{\partial \eta}, \quad (2.6.6) \]
with initial condition
\[ \psi(\eta, 0) = \max(\eta, 0). \]
The PDE in (2.6.6) is posed on \( \tau > 0 \) and \( -\infty < \eta < \infty \). We shall now give an explanation of reducing the domain, see [16]. With a plain vanilla European-style option, we are never certain that we shall end up in-the-money until the expiration date. Asian options differ. With an Asian option, up until time \( t \), say, if \( I_t/t \) exceeds the strike, then it will certainly exceed the strike at expiration since it is a non-decreasing function of time. But \( I/T > K \), yields \( \eta > 0 \), we are thus in-the-money. Say for instance that \( \eta \) starts at a negative value and it has a positive probability of reaching at-the-money, i.e. \( \eta = 0 \). If it does, then it leaves the \( \eta < 0 \) region and stays positive until expiration. We can then reduce our domain to \( \eta < 0 \) and taking the boundary condition at \( \eta = 0 \). With this reduced domain we have an initial condition of zero. This is perfect for applying a Laplace transform.

### 2.6.1 Derivation of the Geman-Yor formula for Asian options.

In this section we shall derive the formula of Geman and Yor for an Asian option, see [10]. Let us make the following transformations on the PDE in (2.6.4)
\[ \nu = \frac{2r}{\sigma^2} - 1, \quad \tau = \frac{1}{4} \sigma^2 \tau_1, \quad \alpha = \frac{-1}{4} \sigma^2 T \eta, \quad C = \frac{1}{4} \sigma^2 e^{r \tau_1} T \psi. \]
The PDE becomes on the region \( \alpha > 0, \tau > 0 \)
\[ \frac{\partial C}{\partial \tau} = 2 \alpha^2 \frac{\partial^2 C}{\partial \alpha^2} - (1 + 2(\nu + 1)\alpha) \frac{\partial C}{\partial \alpha} + 2(1 + \nu)C, \quad (2.6.7) \]
with initial condition zero and boundary condition
\[ C(0, \tau) = \frac{e^{2(1+\nu)\tau} - 1}{2(1 + \nu)}. \]
Now let us use the Laplace transform as in (C.0.1), i.e.,
\[ \hat{C}(\alpha, p) = \int_0^\infty e^{-p\tau} C(\alpha, \tau) d\tau. \]

Applying it on the PDE yields the transformed equation
\[ 2\alpha^2 \frac{\partial^2 \hat{C}}{\partial \alpha^2} - (1 + 2(\nu + 1)\alpha) \frac{\partial \hat{C}}{\partial \alpha} + (2(\nu + 1) - p)\hat{C} = 0, \]

with the boundary condition transformed to
\[ \hat{C}(0, p) = \frac{1}{p(p - 2(1 + \nu))} \quad \text{for} \quad \Re(p) > \max(2(1 + \nu), 0). \quad (2.6.8) \]

The solution can be expressed in terms of the hypergeometric function, \(_1 F_1\),
\[ \hat{C}(\alpha, p) = C_1(p)A_1(\alpha, p) + C_2(p)A_2(\alpha, p), \]

where \( \mu = \sqrt{2p + \nu^2} \) and then
\[ A_1(\alpha, p) = (2\alpha)^{(2+\nu+\mu)}/2\Gamma(1 + \mu) \frac{\Gamma(b)}{\Gamma(2 + \frac{1}{2}(\mu + \nu))} (-z)^{-a}, \quad \text{and} \quad \gamma(b) \Gamma(b - a) (\gamma(b) \Gamma(b - a))^{-1}, \quad \text{see [1] (13.2)]. } \]

We need to choose a solution that is analytic in the right half-plane for a valid Laplace transform. Due to the series expansion of the hypergeometric function this excludes the function \( A_1 \). We now have the solution as
\[ \hat{C}(\alpha, p) = C_2(p)A_2(\alpha, p). \]

To compute the form of \( C_2(p) \) we have to impose the boundary condition (2.6.8). We have the following identity, that if \( \Re(z) < 0 \) and as \( |z| \to \infty \), then
\[ \frac{1}{p(p - 2(1 + \nu))} = C_2(p) \frac{\Gamma(1 + \mu)}{\Gamma(2 + \frac{1}{2}(\mu + \nu))}. \]

Solving for \( C_2(p) \), we have
\[ \hat{C}(\alpha, p) = \frac{(2\alpha)^{(2+\nu+\mu)}/2\Gamma(2 + \frac{1}{2}(\mu + \nu))}{p(p - 2(1 + \nu))\Gamma(1 + \mu)} \Gamma(1 + \mu) \frac{\Gamma(b)}{\Gamma(2 + \frac{1}{2}(\mu + \nu))} (-z)^{-a}. \quad (2.6.10) \]
Applying the inverse Laplace transform and undoing the change of variables, we get the formula

\[
C(S,I,t) = \frac{4S}{\sigma^2 T} e^{-\tau(T-t)} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\tau\gamma} \hat{C}(\alpha,p) dp,
\]

(2.6.11)

where

\[
\tau = \frac{\sigma^2}{4} (T-t), \quad \alpha = \frac{\sigma^2}{4} \frac{KT}{S}.
\]

There is very little chance that this can be evaluated analytically. In Chapter 4 we shall apply numerical inversion methods to this problem.
Chapter 3

Finite difference methods and Option pricing

PDEs form the basis of many mathematical models in physics, biology and more recently in financial markets. We first solve the Black-Scholes equation numerically. In spite of the fact that an analytic solution exists, this helps to develop a general numerical scheme to price Asian options. The numerical method we shall use is based on the finite difference method. Both uniform and non-uniform grids will be considered. We know that for arithmetic average Asian options a closed form solution does not exist. We therefore have to price it numerically. This chapter also deals with the PDE method for pricing Asian options.

3.1 Finite difference schemes

The material in this section is taken from [1]. The idea behind finite difference methods is to replace the partial derivatives by approximations based on Taylor expansions. The Taylor expansions of \( u(x + h, t) \) and \( u(x - h, t) \) are

\[
    u(x + h, t) = u(x, t) + \frac{\partial u}{\partial x} h + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} h^2 + O(h^3),
\]

(3.1.1)

and

\[
    u(x - h, t) = u(x, t) - \frac{\partial u}{\partial x} h + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} h^2 + O(h^3).
\]

(3.1.2)

The forward difference approximation is given by

\[
    \frac{\partial u}{\partial x}(x, t) = \frac{u(x + h, t) - u(x, t)}{h} + O(h)
\]

(3.1.3)
and the backward difference by
\[
\frac{\partial u}{\partial x}(x, t) = \frac{u(x, t) - u(x - h, t)}{h} + O(h).
\] (3.1.4)

Subtracting (3.1.1) and (3.1.2) yields
\[
\frac{\partial u}{\partial x}(x, t) = \frac{u(x + h, t) - u(x - h, t)}{2h} + O(h^2),
\] (3.1.5)

which is the central difference formula. When we add (3.1.1) and (3.1.2) we obtain the approximation to the second partial derivative, namely
\[
\frac{\partial^2 u}{\partial x^2}(x, t) = \frac{u(x + h, t) - 2u(x, t) + u(x - h, t)}{h^2} + O(h^2).\] (3.1.6)

In a similar way one can get the partial derivative with respect to time as
\[
\frac{\partial u}{\partial t}(x, t) = \frac{u(x, t + k) - u(x, t)}{k} + O(k),\] (3.1.7)

which is a forward difference approximation.

### 3.1.1 Discretization

We will start with a simple PDE, the heat equation, defined by
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \text{for} \quad 0 \leq x \leq L \quad \text{and} \quad 0 \leq t \leq T,
\] (3.1.8)

subject to an initial condition
\[
u(x, 0) = g(x)\] (3.1.9)

and boundary conditions given by
\[
u(0, T) = a(t) \quad \text{and} \quad \nu(L, t) = b(t).
\] (3.1.10)

To approximate the solution of the PDE, we first have to discretise our domain. The space axis will be divided in \(N + 1\) uniformly spaced points \(\{jh\}_{j=0}^{N}\), where \(h = L/N\) and \(L\) is the truncation parameter if our problem is defined on the whole real axis. The time axis will be divided also in \(M + 1\) uniformly spaced points \(\{ik\}_{i=0}^{M}\) with \(k = T/M\). The points \((jh, ik)\) is called the mesh. Let \(u_j^i\) denote the approximate solution on the grid, i.e.,
\[
u_j^i \approx u(jh, ik).
\] (3.1.11)
Chapter 3. Finite difference methods and Option pricing

It follows from (3.5), (3.6) and (3.7) that
\[
\frac{\partial u}{\partial t}(x_j, t_i) \approx \frac{u_{j}^{i+1} - u_{j}^{i}}{k},
\]
\[
\frac{\partial u}{\partial x}(x_j, t_i) \approx \frac{u_{j+1}^{i} - u_{j-1}^{i}}{2h},
\]
\[
\frac{\partial^2 u}{\partial x^2}(x_j, t_i) \approx \frac{u_{j+1}^{i} - 2u_{j}^{i} + u_{j-1}^{i}}{h^2}. \tag{3.1.12}
\]

Applying a forward difference scheme to the time variable in (3.1.8) gives
\[
\frac{u_{j}^{i+1} - u_{j}^{i}}{k} = \frac{u_{j+1}^{i} - 2u_{j}^{i} + u_{j-1}^{i}}{h^2}, \tag{3.1.13}
\]
which we can write as
\[
u u_{j+1}^{i} = \nu u_{j}^{i} + (1 - 2\nu) u_{j}^{i} + \nu u_{j-1}^{i}, \tag{3.1.14}
\]
where \(\nu = k/h^2\). Now suppose that all the approximate values at time \(i\), i.e., \(\{u_{j}^{i}\}_{j=0}^{N}\) are known. Note that on the next time level \(u_{0}^{i+1} = a((i+1)k)\) and \(u_{N}^{i+1} = b((i+1)k)\) are given by the boundary conditions. A formula for computing all the other approximate values at time \(i+1\) is given by equation (3.1.14), i.e., \(\{u_{j+1}^{i+1}\}_{j=1}^{N-1}\). Since we have initial condition \(u_{0}^{0} = g(jh)\), it means that the complete set of approximations \(\{u_{j}^{i}\}_{j=0,i=0}^{N,M}\) can be computed by stepping forward in time. As a vector our solution set looks like
\[
\mathbf{u}^{i} = \begin{bmatrix}
    u_{1}^{i} \\
    u_{2}^{i} \\
    \vdots \\
    u_{N-1}^{i}
\end{bmatrix}. \tag{3.1.15}
\]

In matrix vector form one can write the forward difference in time and central difference in space as
\[
\mathbf{u}^{i+1} = F\mathbf{u}^{i} + \mathbf{p}^{i}, \quad \text{for} \quad 0 \leq i \leq N - 1, \tag{3.1.16}
\]
with
\[
\mathbf{u}^{0} = \begin{bmatrix}
    g(h) \\
    g(2h) \\
    \vdots \\
    g((N-1)h)
\end{bmatrix}. \tag{3.1.17}
\]
Here the matrix $F$ takes the form
\[
F = \begin{bmatrix}
1 - 2\nu & \nu & 0 & \cdots & \cdots & 0 \\
\nu & 1 - 2\nu & \nu & 0 & \cdots & \\
0 & \cdots & \cdots & \cdots & \cdots & \\
\vdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \\
0 & \cdots & \cdots & 1 - 2\nu & \nu & \\
\end{bmatrix}, \quad (3.1.18)
\]
and the vector $p^i$ contains the boundary conditions in its first and last entries
\[
p^i = \begin{bmatrix}
\nu a(ik) \\
0 \\
\vdots \\
\nu b(ik)
\end{bmatrix}. \quad (3.1.19)
\]
The forward difference in time can be replaced with a backward difference scheme. The approximation to the heat equation then becomes
\[
u_{j+1}^i = \nu_j^i + \nu \left( u_{j+1}^{i+1} - 2u_j^{i+1} + u_{j-1}^{i+1} \right). \quad (3.1.20)
\]
This may be written as
\[
Bu_{i+1}^i = u_i^i + q_i^i, \quad \text{for} \quad 0 \leq i \leq N - 1, \quad (3.1.21)
\]
where the matrix $B$ is given by
\[
B = \begin{bmatrix}
1 + 2\nu & -\nu & 0 & \cdots & \cdots & 0 \\
-\nu & 1 + 2\nu & -\nu & 0 & \cdots & \\
0 & \cdots & \cdots & \cdots & \cdots & \\
\vdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \cdots & \cdots & -\nu & 1 + 2\nu & -\nu \\
0 & \cdots & \cdots & 0 & -\nu & 1 + 2\nu
\end{bmatrix}, \quad (3.1.22)
\]
along with the vector
\[
q^i = \begin{bmatrix}
\nu a((i + 1)k) \\
0 \\
\vdots \\
0 \\
\nu b((i + 1)k)
\end{bmatrix}. \quad (3.1.23)
\]
Note that the scheme requires the solution of a tri-diagonal system at each time level. It is an implicit scheme.
3.1.2 Crank-Nicolson.

The Crank-Nicolson method is just the average of the forward- and backward difference schemes, and the formula is given by

\[
\frac{1}{2} (I + B) u^{i+1} = \frac{1}{2} (I + F) u^i + \frac{1}{2} (p^i + q^i).
\]  

(3.1.24)

The Crank-Nicolson method is an implicit scheme, and also requires the solution of a tri-diagonal system at each time level.

3.2 Stability.

The stability analysis we shall use is the von Neumann stability which is based on Fourier methods. This means we are restricted to linear and constant coefficient PDEs. In this subsection \(i\) denote the imaginary unit.

To illustrate the method, we substitute \(u^n_j = \alpha^n e^{ij\beta h}\) in (3.1.14), this gives

\[
\alpha^{n+1} e^{ij\beta h} = \nu \alpha^n e^{ij\beta h} + (1 - 2\nu) \alpha^n e^{ij\beta h} + \nu \alpha^n e^{ij\beta h} e^{-i\beta h}.
\]

Cancellation of \(\alpha^n e^{ij\beta h}\) leads to

\[
\alpha = \begin{cases} 
(1 - 2\nu) + \nu (e^{i\beta h} + e^{-i\beta h}), \\
1 - 2\nu (1 - \cos(\beta h)), \\
1 - 4\nu \sin^2(\frac{\beta h}{2}).
\end{cases}
\]

The parameter \(\alpha\) is called the amplification factor. For stability we need \(|\alpha| \leq 1\), see [18], which leads to

\[-1 \leq 1 - 4\nu \sin^2\left(\frac{\beta h}{2}\right) \leq 1,\]

and which simplifies to

\[0 \leq \nu \sin^2\left(\frac{\beta h}{2}\right) \leq \frac{1}{2}.
\]

We therefore have the stability condition

\[
\nu \leq \frac{1}{2}, \quad (3.2.1)
\]

or

\[
k \leq \frac{1}{2} h^2 \quad (3.2.2)
\]
which can restrict the time step \( k \) significantly for small values of \( h \). We will now show that the Crank-Nicolson scheme is unconditionally stable. Let us take the average of equations (3.1.14) and (3.1.20), which gives

\[
2(1 + \nu)u_j^{n+1} = \nu u_j^{n+1} + \nu u_{j-1}^{n+1} + \nu u_j^n + 2(1 - \nu)u_j^n + \nu u_{j-1}^n. \tag{3.2.3}
\]

After making the substitution \( u_j^n = \alpha^n e^{i\beta j} \) and re-organizing we get the amplification factor

\[
|\alpha| \leq 1 \iff -1 < \frac{1 - 2\nu \sin^2(\frac{\beta h}{2})}{1 + 2\nu \sin^2(\frac{\beta h}{2})} < 1, \tag{3.2.4}
\]

which lead to \( 1 + 2\nu \sin^2(\frac{\beta h}{2}) > 1 \). Note that \( \sin^2(\frac{\beta h}{2}) \) takes on values between 0 and 1. The Crank-Nicolson scheme is therefore stable for all \( \nu > 0 \). Because the Black-Scholes equation does not have constant coefficients this stability analysis is not directly applicable to it. The general result, namely that implicit methods are more stable than explicit methods, can be shown to be true for Black-Scholes equation as well.

### 3.3 Finite difference methods for the Black-Scholes PDE on a uniform grid.

In Chapter 2, we discussed the Black-Scholes PDE for option pricing. We now discuss a finite difference method for solving this equation. First, we will make a substitution \( \tau = T - t \) so that \( \tau \) runs from \( T \) to 0. The Black-Scholes PDE becomes

\[
\frac{\partial V}{\partial \tau} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + rV = 0. \tag{3.3.1}
\]

The condition at \( t = T \) now becomes the condition at \( \tau = 0 \). For the European call option the initial condition is

\[
C(S, 0) = \max(S(0) - K, 0), \tag{3.3.2}
\]

and for the European put it is

\[
P(S, 0) = \max(K - S(0), 0). \tag{3.3.3}
\]
Chapter 3. Finite difference methods and Option pricing

The Black-Scholes PDE is defined on the infinite domain $S \in [0, \infty)$. We need a finite domain, so we will truncate the domain to $S \in [0, L]$. The choice of the parameter $L$ will be discussed later, in subsection 3.5.1. The boundary conditions for the call option is

$$C(0, \tau) = 0 \quad \text{and} \quad C(L, \tau) = L - Ke^{-r\tau}. \quad (3.3.4)$$

For the European put option we have

$$P(0, \tau) = Ke^{-r\tau} \quad \text{and} \quad P(L, \tau) = 0. \quad (3.3.5)$$

We denote the numerical solution at time level $i$ by

$$V^i = \begin{bmatrix} V_1^i \\ V_2^i \\ \vdots \\ V_{N-1}^i \end{bmatrix}, \quad (3.3.6)$$

where $V^0$ is specified by the initial conditions (3.3.2) or (3.3.3). The boundary values, $V_0^i$ and $V_{N}^i$ are given by (3.3.4) and (3.3.5). We can write (3.3) in discretization form using the forward difference scheme as

$$\frac{V_{j+1}^i - V_j^i}{k} = - \frac{1}{2} \sigma^2 (jh)^2 \frac{(V_{j+1}^i - 2V_j^i + V_{j-1}^i)}{h^2} - rjh \frac{V_{j+1}^i - V_{j-1}^i}{2h} + rV_j^i = 0. \quad (3.3.7)$$

Using a backward difference scheme gives

$$\frac{V_{j+1}^i - V_j^i}{k} = - \frac{1}{2} \sigma^2 (jh)^2 \frac{(V_{j+1}^i + 2V_{j+1}^i + V_{j-1}^i)}{h^2} - rjh \frac{V_{j+1}^i - V_{j-1}^i}{2h} + rV_j^i = 0. \quad (3.3.8)$$

We can now redefine $F$ in (3.1.18) as follows

$$F = (1 - rk)I + \frac{1}{2} k\sigma^2 D_2 T_2 + \frac{1}{2} kr D_1 T_1, \quad (3.3.9)$$

and

$$p^i = \begin{bmatrix} \frac{1}{2} k(\sigma^2 - r)V_0^i \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (3.3.10)$$
$D_1$ and $D_2$ are diagonal matrices

$$D_1 = \begin{bmatrix} 1 & 0 & \ldots & \ldots & 0 \\ 0 & 2 & 0 & \ddots & \vdots \\ \vdots & 0 & 3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & N-1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1^2 & 0 & \ldots & \ldots & 0 \\ 0 & 2^2 & 0 & \ddots & \vdots \\ \vdots & 0 & 3^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & (N-1)^2 \end{bmatrix},$$

and $T_1$ and $T_2$ are tri-diagonal matrices

$$T_1 = \begin{bmatrix} 0 & 1 & 0 & \ldots & \ldots & 0 \\ -1 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & -1 & 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & -1 & 0 & 1 \\ 0 & \ldots & \ldots & 0 & -1 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -2 & 1 & 0 & \ldots & \ldots & 0 \\ 1 & -2 & 1 & \ddots & \vdots & \vdots \\ \vdots & 1 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \ddots \vdots \\ 0 & \ldots & \ldots & 0 & 1 & -2 \end{bmatrix}.$$  

Similarly, for the backward difference approximation the matrix $B$ of (3.1.22) is modified to

$$B = (1 + rk)I - \frac{1}{2} k \sigma^2 D_2 T_2 - \frac{1}{2} k r D_1 T_1,$$

and

$$q^i = \begin{bmatrix} \frac{1}{2} k (\sigma^2 - r) V_0^{i+1} \\ \vdots \\ 0 \\ \frac{1}{2} k (N - 1) (\sigma^2 (N - 1) + r) V_N^{i+1} \end{bmatrix}. \quad (3.3.11)$$

By taking the average of the forward and backward difference formulas we obtain the Crank-Nicolson scheme:

$$\frac{1}{2} (I + B) V^{i+1} = \frac{1}{2} (I + F) V^i + \frac{1}{2} (p^i + q^i), \quad (3.3.12)$$

see [11]. See Figure 3.1 for a three dimensional graph for the numerical solution corresponding to a European call option.
Figure 3.1: The plot shows the solution of the European call option in three dimensions using a Crank-Nicolson scheme. The parameters are $K = 10, \sigma = 0.3$, $r = 0.03$ and time to expiration is $T = 1$.

3.4 Solving PDEs with finite difference methods and non-uniform grids.

The final condition in option pricing is not differentiable, and so we will make grid refinements to attain better accuracy. In the case of option pricing this region is near $S = K$. We have another reason for grid refinement in the neighbourhood of the exercise price. This is the fact that option prices of practical interest are situated near the exercise price.

3.4.1 Non-uniform grids: Direct method.

In this section we introduce a non-uniform finite-difference mesh for the spatial domain and keep the timestep uniform, see for instance [2]. The reason for using non-uniform meshes is due to the non-smooth final condition in option pricing; see Figure 2.2 for example. The idea behind non-uniform meshes is to distribute more grid points at singularities or sharp edges, which ought to improve the accuracy.
The space step of the scheme will become \( h_j = x_j - x_{j-1} \). We will have to derive new difference approximations for non-uniform grids. For example consider the Taylor expansion

\[
\begin{align*}
    u(x_{j+1}, t) &= u(x_j, t) + h_{j+1} \frac{\partial u}{\partial x} + \frac{1}{2} h_{j+1}^2 \frac{\partial^2 u}{\partial x^2} + O(h_{j+1}^3), \\
    u(x_{j-1}, t) &= u(x_j, t) + h_j \frac{\partial u}{\partial x} + \frac{1}{2} h_j^2 \frac{\partial^2 u}{\partial x^2} + O(h_j^3).
\end{align*}
\]

By adding the two expressions above, we have

\[
\frac{\partial u}{\partial x}(x_j, t) \approx \frac{u(x_{j+1}, t) - u(x_j, t)}{h_{j+1} + h_j}. \tag{3.4.2}
\]

Multiplying and adding we have for the second derivative

\[
\frac{\partial^2 u}{\partial x^2}(x_j, t) \approx 2 \frac{h_j (u_{j+1} - u_j) - h_{j+1} (u_j - u_{j-1})}{h_j h_{j+1} (h_{j+1} + h_j)}. \tag{3.4.3}
\]

The Black-Scholes equation becomes

\[
\frac{u_{j+1}^i - u_j^i}{k} = \frac{1}{2} \sigma^2 S_j^2 (h_j^+ (u_{j+1}^i - u_j^i) - h_j^- (u_j^i - u_{j-1}^i)) + r S_j \frac{u_{j+1}^i - u_j^i}{h_j + h_{j+1}} - r u_j^i, \tag{3.4.4}
\]

where

\[
h_j^+ = \frac{2}{h_{j+1} + h_j}, \quad h_j^- = \frac{2}{h_j + h_{j+1}}.
\]

Let equidistant points be given by

\[
x_j = \sinh^{-1}(-K/c) + j \Delta x, \tag{3.4.5}
\]

where

\[
\Delta x = \frac{1}{N} \sinh^{-1}((L - K)/c) - \sinh^{-1}(-K/c). \tag{3.4.6}
\]

Then a non-uniform mesh, \( 0 = S_0 < S_1 < \cdots < S_N = L \), is defined by

\[
S_j = K + c \sinh(x_j), \quad 0 \leq j \leq N. \tag{3.4.7}
\]

The transformation above was used in \[4\]. The parameter \( c \) controls the fraction of mesh points \( S_j \) that lies in the neighbourhood of the strike \( K \). In the next section we shall look at an alternative grid refinement strategy, namely coordinate transformation.
3.4.2 Non-Uniform grids: Coordinate transformation.

In this section we adopt the method of coordinate transformation as in [19]. We make an analytic transformation and apply an equidistant grid discretization on the transformed equation. The method will be explained for a general parabolic PDE with non-constant coefficients:

\[
\frac{\partial u}{\partial t} = \alpha(s) \frac{\partial^2 u}{\partial s^2} + \beta(s) \frac{\partial u}{\partial s} + \gamma(s)u(s, t) \tag{3.4.8}
\]

\[
u(a, t) = L(t), \quad u(b, t) = R(t), \quad u(s, 0) = \varphi(s). \tag{3.4.9}
\]

Now consider a coordinate transformation \( y = \psi(s) \), which must be one-to-one, with an inverse \( s = \phi(y) = \psi^{-1}(y) \). Also let \( \hat{u}(y, t) := u(s, t) \), so that we can distinguish the original and transformed equation; the hat will define the solution on the transformed grid. If we apply the chain rule, the first derivative becomes

\[
\frac{\partial u}{\partial s} = \frac{\partial \hat{u}}{\partial y} \frac{\partial y}{\partial s} = \frac{1}{\phi'(y)} \frac{\partial \hat{u}}{\partial y}, \tag{3.4.10}
\]

and for the second derivative we have

\[
\frac{\partial^2 u}{\partial s^2} = \frac{1}{(\phi'(y))^2} \frac{\partial^2 \hat{u}}{\partial y^2} - \frac{\phi''(y)}{(\phi'(y))^3} \frac{\partial \hat{u}}{\partial y}. \tag{3.4.11}
\]

The coefficients of the PDE become

\[
\hat{\alpha}(y) = \frac{\alpha(\phi(y))}{(\phi'(y))^2},
\hat{\beta}(y) = \frac{\beta(\phi(y))}{\phi'(y)} - \alpha(\phi(y)) \frac{\phi''(y)}{(\phi'(y))^3},
\hat{\gamma}(y) = \gamma(\phi(y)). \tag{3.4.12}
\]

The boundaries \( s = a \) and \( s = b \) are also transformed into \( y = \psi(a) \) and \( y = \psi(b) \), respectively. We need the function \( \psi \) to be monotonically increasing such that the new transformed equation has equidistant grid size, i.e., \( h = (\psi(b) - \psi(a))/N \).

The spatial transformation we are going to use is a normalized version of (3.4.5)

\[
y = \psi(s) = \frac{\sinh^{-1}((s - \kappa)/c) - c_{1}}{c_{2} - c_{1}}, \tag{3.4.13}
\]

where

\[
c_1 = \sinh^{-1}((a - \kappa)/c), \quad c_2 = \sinh^{-1}((b - \kappa)/c).
\]
Figure 3.2: Plot of the transformation function and the distribution of points as in equation (3.4.13) with parameters $K = 10$, $r = 0.03$, $\sigma = 0.3$, $T = 1$ and $c = K/50$. 
These constants normalize the function such that \( y \) lies in \([0,1]\). The grid is then refined around \( s = \kappa \), which we shall set equal to \( K \). The parameter \( c \) determines the rate of stretching. We now have the inverse and the derivatives of the transformation

\[
\begin{align*}
\phi(y) &= c \sinh(c_2 y + c_1 (1 - y)) + \kappa, \\
\phi'(y) &= c (c_2 - c_1) \cosh(c_2 y + c_1 (1 - y)), \\
\phi''(y) &= c (c_2 - c_1)^2 \sinh(c_2 y + c_1 (1 - y)).
\end{align*}
\]  

Since we transformed the original Black-Scholes equation to an initial value problem, we now have the transformation of the initial condition for a European call option as

\[
\hat{u}(y,0) = \max(c \sinh(c_2 y + c_1 (1 - y)) + \kappa - K, 0).
\]  

With this transformation we have a grid refinement at \( s = K \); see Figure 3.2 where \( K = 10 \). Note that the sharp edge in the initial condition does not disappear. After applying the transformation the Black-Scholes equation for an European option becomes

\[
\frac{\partial C}{\partial \tau} = \frac{1}{2} \sigma^2 \left( \frac{\phi'(y)^2}{\phi(y)} \right) \frac{\partial^2 C}{\partial y^2} + \left( \frac{r \phi(y)}{\phi'(y)} - \frac{1}{2} \sigma^2 \phi'(y)^2 \frac{\phi''(y)}{\phi(y)^3} \right) \frac{\partial C}{\partial y} - r C. 
\]  

We apply a uniform grid to discretize the variable \( y \) in the above PDE. In doing this we get a non-uniform grid on \( S \), see Figure 3.2.
3.5 Numerical results for European options

The numerical experiments performed in this section are based on the Black-Scholes PDE. We used the parameters

- \( K = 10 \)
- \( \sigma = 30\% \)
- \( r = 3\% \)
- \( T = 1 \).

We use the maximum absolute error at time \( \tau = T \), i.e.,

\[
\text{error} := \max_{1 \leq j \leq N} |C^M_j - C(jh, \tau = T)|.
\] (3.5.1)

3.5.1 European call option.

The European call option will be used as our test problem since a closed form solution exists. We shall compare our numerical values with the exact solution as in equation (2.2.13). We applied a second order finite difference scheme (Crank-Nicolson) on both uniform and non-uniform grids. The outer boundary has been placed at three times the strike price, i.e., \( S = 3K \); see [13].

In Figure 3.3, we plot the analytical solution against the numerical solution with non-uniform spacing with different values of \( c \). We can see that more points are allocated at the strike price. The smaller the value \( c \), more points are distributed at the strike and less if \( c \) gets larger. Unfortunately, we have not found mathematical results in the literature for an optimal value of \( c \).

Figure 3.4 compares the two different non-uniform grid methods as in Section 3.4.1 and 3.4.2. We see that the coordinate transformation outperforms the direct method. Although we have a grid refinement at the strike, the maximum error still occurs at \( S = K \). The errors are only plotted on the interval \([K/2, 3K/2]\) since the most important option prices lie near the strike price. Figure 3.5 shows the absolute errors for both uniform and non-uniform grids. We can see that the maximum error in both methods occurs at the strike. With the grid refinement at the strike our error is decreased at \( S = K \).
3.5.2 Error analysis

In this section we give numerical evidence for the effectiveness of non-uniform grids. Tables 3.1 and 3.2 list the errors when using the Crank-Nicolson method on uniform and non-uniform grids. We used at most a 50 × 50 grid since we want to compute the solution to good accuracy using as few points as possible. In Table 3.1 we used $c = K/5$ and in Table 3.2
Figure 3.4: Comparison of the two different non-uniform grid methods. The circles represent the coordinate transformation method and the triangles the direct method.

we increased it to \( c = 2K/5 \). With a \( 10 \times 10 \) grid we obtained three-digit accuracy with \( c = 2K/5 \). For both values of \( c \) and our different grid sizes (10, 20, 40, 50), the non-uniform grid outperforms the equidistant grid. Note that if we increase the number of grid points the errors between equidistant and non-uniform grids will become equal. This is to be expected since the numerical solution will converge if the number of grid points increases.
Figure 3.5: Figure shows the absolute error over the stock price. Note that we only plot the errors in the region of interest.

<table>
<thead>
<tr>
<th>Grid</th>
<th>Error(uniform grid)</th>
<th>Error(non-uniform(1))</th>
<th>Error(non-uniform(2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 × 10</td>
<td>3.8 × 10^{-2}</td>
<td>2.5 × 10^{-2}</td>
<td>4.1 × 10^{-2}</td>
</tr>
<tr>
<td>20 × 20</td>
<td>7.7 × 10^{-3}</td>
<td>5.8 × 10^{-3}</td>
<td>6.9 × 10^{-3}</td>
</tr>
<tr>
<td>40 × 40</td>
<td>2.1 × 10^{-3}</td>
<td>1.6 × 10^{-3}</td>
<td>2.0 × 10^{-3}</td>
</tr>
<tr>
<td>50 × 50</td>
<td>1.2 × 10^{-3}</td>
<td>8.5 × 10^{-4}</td>
<td>9.8 × 10^{-4}</td>
</tr>
</tbody>
</table>

Table 3.1: The errors for equidistant vs. non-uniform grids where \( c = K/5 \). Non-uniform (1) and (2) corresponds to the coordinate transformation and direct method respectively.

The results above was done for a European call option. We expect the results for the put option to be of similar accuracy.
Table 3.2: The errors for equidistant vs. non-uniform grids where $c = 2K/5$. Non-uniform (1) and (2) corresponds to the coordinate transformation and direct method respectively.

<table>
<thead>
<tr>
<th>Grid</th>
<th>Error(Uniform Grid)</th>
<th>Error(non-uniform(1))</th>
<th>Error(non-uniform(2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 × 10</td>
<td>3.8 × 10^{-2}</td>
<td>6.9 × 10^{-3}</td>
<td>2.3 × 10^{-2}</td>
</tr>
<tr>
<td>20 × 20</td>
<td>7.7 × 10^{-3}</td>
<td>4.0 × 10^{-3}</td>
<td>1.0 × 10^{-2}</td>
</tr>
<tr>
<td>40 × 40</td>
<td>2.1 × 10^{-3}</td>
<td>1.1 × 10^{-3}</td>
<td>1.5 × 10^{-3}</td>
</tr>
<tr>
<td>50 × 50</td>
<td>1.2 × 10^{-3}</td>
<td>2.9 × 10^{-4}</td>
<td>5.5 × 10^{-4}</td>
</tr>
</tbody>
</table>

3.6 Numerical results for Asian Options

3.6.1 Asian options on uniform grid.

In this section we investigate the numerical implementation of Asian options. In Chapter 2 we derived the two PDEs of Veer [28, 29], namely (2.4.10) and (2.4.13). Here we shall solve them numerically through finite differences using Crank-Nicolson time integration. We shall first solve the PDE defined on $z \in [-1, 1]$

$$
\begin{align*}
\frac{\partial u}{\partial \tau} &= rT(\tau - z) \frac{\partial u}{\partial z} + \frac{T}{2} (\tau - z)^2 \sigma^2 \frac{\partial^2 u}{\partial z^2}, \\
u(z, 0) &= \max(z, 0), \\
u(-1, \tau) &= 0, \\
\frac{\partial^2 u}{\partial z^2} &= 0,
\end{align*}

(3.6.1)

which is derived from the PDE (2.4.10) using the following change of variables: $z = 1 - y$ and $\tau = 1 - t/T$. We discretize our domain with uniform spacing by

$$z_j = -1 + j\Delta z, \quad \tau_i = i\Delta t,$$

for $0 \leq j \leq N$, $0 \leq i \leq M$ and $\tau_M = T$. The approximate solution will be denoted by $u^i_j = u(z_j, \tau_i)$. The approximation to the PDE is

$$
\frac{u^i_{j+1} - u^i_j}{k} = rT(\tau_i - z_j) \frac{u^i_{j+1} - u^i_{j-1}}{2h} + \frac{T}{2} \sigma^2 (\tau_i - z_j)^2 \frac{u^i_{j+1} - 2u^i_j + u^i_{j-1}}{h^2},
$$

where $k = T/M$ and $h = 2/N$. The initial condition is

$$u^0_j = \max(z_j, 0),$$
and the boundary conditions are

\[ u_0^i = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial z^2} = 0 \rightarrow u_N^i = 2u_{N-1}^i - u_{N-2}^i. \]

Let us discuss how to treat the right hand boundary condition given by linear interpolation. We have at the boundary \( z = 1 \)

\[ \frac{\partial^2 u_N^i}{\partial z^2} \approx \frac{u_N^i - 2u_{N-1}^i + u_{N-2}^i}{h^2}. \]

Substituting \( u_N^i = 2u_{N-1}^i - u_{N-2}^i \) in (3.6.1) we have

\[ \frac{\partial^2 u_N^i}{\partial z^2} = 0, \]

which represents a free or natural boundary condition at the right endpoint. The matrix \( T_2 \) for the second derivative reduces to

\[
T_2 = \begin{bmatrix}
-2 & 1 & 0 & \ldots & \ldots & 0 \\
1 & -2 & 1 & \ddots & \ddots & \vdots \\
0 & 1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 1 & -2 & 1 \\
0 & \ldots & \ldots & 0 & 0 & 0
\end{bmatrix},
\]

i.e., the last two entries in the last row becomes zero in the matrix. Now for the first derivative we have

\[ \frac{\partial u_N^i}{\partial z} \approx \frac{2u_{N-1}^i - 2u_{N-2}^i}{2h}, \]

so that the matrix \( T_1 \) becomes

\[
T_1 = \begin{bmatrix}
0 & 1 & 0 & \ldots & \ldots & 0 \\
-1 & 0 & 1 & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & -1 & 0 & 1 \\
0 & \ldots & \ldots & 0 & -2 & 2
\end{bmatrix}.
\]

The changes in the matrices are indicated by the bold digits in the last row. This means that we have incorporated the boundary condition in the
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differentiation matrices.

Recall the PDE in (2.4.13):

\[
\frac{\partial g}{\partial \tau} = \sigma^2 \left( \zeta - \frac{1 - e^{-rt}}{rT}\right)^2 \frac{\partial^2 g}{\partial \zeta^2},
\]

(3.6.2)

which has the same initial and boundary conditions as in (3.6.1). In discretization form the PDE is

\[
\frac{g_{j}^{i+1} - g_{j}^{i}}{k} = \frac{\sigma^2}{2} (\zeta_j - \alpha(\tau_i))^2 g_{j+1}^{i} - 2g_{j}^{i} + g_{j-1}^{i},
\]

where

\[
\alpha(\tau) = \frac{1 - e^{-rt}}{rT}.
\]

We have discretized the PDEs using uniform grids. The results of the two PDEs above will be compared in the next section. In the next section we shall do the same for non-uniform grids. We shall concentrate on the PDE in (3.6.2) for numerical purposes on the non-uniform grid.

### 3.6.2 Asian options on non-uniform grids.

In the Black-Scholes PDE we had a singularity at the exercise price. The PDE we are dealing with in this section has a singularity at \( \zeta = 0 \), see Figure 3.6. We now need a grid refinement at zero and not at the exercise price. We applied the direct method as in section (3.4.1). The complete system for the PDE defined on \( \zeta \in [-1, 1] \) is

\[
\frac{\partial g}{\partial \tau} - \frac{\sigma^2}{2} \left( \zeta - \frac{1 - e^{-rt}}{rT}\right)^2 \frac{\partial^2 g}{\partial \zeta^2} = 0,
\]

\[
g(\zeta, 0) = \max(\zeta, 0),
\]

\[
g(-1, \tau) = 0,
\]

\[
\frac{\partial^2 g}{\partial \zeta^2} = 0.
\]

(3.6.3)

Since we have a singularity at zero, we will need a new function to get a non-uniform scheme. Let equidistant points be given by

\[
\xi_j = j\Delta \xi + \sinh^{-1}(-1/c),
\]

(3.6.4)

where

\[
\Delta \xi = \frac{1}{N} \left( \sinh^{-1}(1/c) - \sinh^{-1}(-1/c) \right)
\]

(3.6.5)
A non-uniform mesh is defined through the transformation

$$\zeta_j = c \sinh(\xi_j).$$  \hspace{2cm} (3.6.6)

See Figure 3.7 for a plot of the transformation function. Our PDE in discretized form is

$$g_j^{i+1} - g_j^i = \frac{\sigma^2}{2} \left( \zeta_j - \frac{1 - e^{-r\tau_i}}{rT} \right)^2 \left( h_j^{+}(g_{j+1}^i - g_j^i) - h_j^{-}(g_j^i - g_{j-1}^i) \right),$$  \hspace{2cm} (3.6.7)

where

$$h_j^{+} = \frac{2}{h_{j+1} + h_j}, \quad h_j^{-} = \frac{2}{h_j(h_{j+1} + h_j)}. \hspace{2cm} (3.6.8)$$

The initial condition is now

$$g(\zeta, 0) = \max(c \sinh(\xi_j), 0).$$

Now we have all the ingredients for numerical implementation. See Figure 3.8 for a solution on a non-uniform grid.

Table 3.3 compares the results between uniform and non-uniform grids. The parameters we use for the comparison are when $r = 0.15$, $S_0 = 100$ and $T = 1$ with low and high volatilities. To be consistent with the papers [28, 29] we use 200 space points and 400 time points on the above methods.
Figure 3.7: Plot of the transformation function (3.6.6).

Figure 3.8: A plot of the solution of a Asian call option on a non-uniform grid.
Chapter 3. Finite difference methods and Option pricing

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$K$</th>
<th>Non-uniform grid</th>
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<th>PDE2</th>
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<th>upper</th>
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</thead>
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<td>6.810</td>
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<td>2.750</td>
<td>2.7484</td>
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<td></td>
</tr>
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<td>16.533</td>
<td>16.5137</td>
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<td>5.748</td>
<td>5.7288</td>
<td>5.728161</td>
<td>5.735488</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.3: Computing Asian options on a non-uniform grid with $c = 1/100$ with 200 spatial grid points and 400 time grid points. The parameter values are $r = 0.15$, $S_0 = 100$ and $T = 1$. PDE1 and PDE2 corresponds to the PDEs in (3.6.1) and (3.6.2) respectively. The non-uniform grid column results are those of the PDE in (3.6.2) along with a non-uniform spatial grid. The lower and upper bounds are taken from [26].

The column PDE1 corresponds to the PDE in (3.6.1) with a uniform grid.
The column PDE2 is the one in equation (3.6.2) also on a uniform grid. We use the non-uniform grid on the PDE in (3.6.2) for our numerical results.
We took the stretching parameter as $c = 1/100$.

The values for the lower and upper bounds were taken from the paper [7] who used the bounds of Thompson [26]. We see that with the uniform grid some values may lie outside the lower and upper bounds, particularly for small volatility. With the non-uniform grid, however, all values are within the bounds.

3.7 Conclusion.

Option pricing can be valued by solving PDEs. We used European style options as our test model since the exact solution is known. Not all PDEs have an exact solution, however, for example in the case of arithmetic average Asian options. In this chapter we constructed a method based on non-uniform grids to solve these equations with less points and obtained better results. We have proposed two methods, the direct method and analytical coordinate transformation for constructing grid stretching due to the final conditions. We saw that with a $10 \times 10$ grid, we obtained three digits accuracy with $c = 2K/5$ for European style options. The parameter $c$ is very important for accuracy. In all finite difference methods we used a Crank-Nicolson scheme for time integration. For Asian options we used the direct method for the simplicity of the right-hand boundary condition.
In the case of Asian options, an exact solution does not exist. We used the lower and upper bounds of Thompson [20] since it is an improvement on the bounds by Rogers and Shi [20]. The transformation function had to be modified for Asian options since we are working on an interval $[-1, 1]$. The initial condition thus have a singularity at zero. In all cases of the Asian options (low and high volatility), using a $200 \times 400$ grid with $c = 1/100$, all values lie in the desired bounds when using a non-uniform grid.
Chapter 4

The inversion of the Laplace transform in option valuation

This chapter deals with the numerical inversion of the Laplace transform for computing option prices. The inversion algorithm is based on the method of Talbot [25]. The method is based on choosing a suitable contour to replace the usual Bromwich contour. The Black-Scholes model will serve as our test problem for testing the quality of the numerical inversion of the Laplace transform as an exact solution is known. The main problem we address here is inverting the Laplace transform formula for pricing Asian options as an exact solution does not exist. The Laplace transform approach for Asian options is problematic for the low volatility cases, which is due to the numerical computation for large negative arguments of the hypergeometric function. We shall develop new methods to compute numerically the hypergeometric function, using Barnes contour integral representations. The method will be tested and compared to existing software packages. These methods will be applied to price Asian options. The methods will be tested on a wide range of problem sets.

4.1 Numerical inversion of the Laplace transform

The methods for inverting the Laplace transform that we shall consider are all based on numerical integration of the Bromwich contour integral

\[ f(t) = \frac{1}{2\pi i} \int_{\gamma - it\infty}^{\gamma + it\infty} e^{zt} F(z) dz, \quad \gamma > \gamma_0. \]  

(4.1.1)
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Here $F(z)$ is the function that needs to be inverted and $\gamma_0$ is the convergence absicissa with $\gamma > \gamma_0$. The singularities of $F(z)$ lie in the open half-plane $\Re z < \gamma$. The integral (4.1.1) is not well suited for numerical integration. It can be seen from the Bromwich line, $z = \gamma + iy$, $-\infty < y < \infty$, that the exponential factor $e^{zt}$ is highly oscillatory. The other reason is that the transform $F(z)$ decays slowly as $|y| \to \infty$. Talbot [25] suggested that the Bromwich line be deformed into a contour that begins and ends in the left half plane, i.e., $\Re z \to -\infty$ at both ends, see Figure [4.1] for a general Talbot contour. Due to the exponential factor the integrand decays rapidly on such a contour. In such situations the trapezoidal or midpoint rules converge extraordinarily rapidly, see [30]. Such a deformation is permissible if the contour encloses all singularities of $F(z)$ and providing that $|F(z)| \to 0$ uniformly in $\Re z < \gamma$ as $|z| \to \infty$.

Talbot’s contour is given by

$$s(z) = \frac{z}{1 - e^{-z}} + \left(\frac{\nu - 1}{2}\right) z, \quad z \in (-2\pi i, 2\pi i),$$

with $\nu$ constant. If we parameterize this contour with $z = 2i\theta$, we obtain

$$z(\theta) = \theta \cot(\theta) + iv\theta, \quad -\pi \leq \theta \leq \pi. \quad (4.1.2)$$

One can verify that the contour $z(\theta)$ approaches $-\infty$ as $\theta$ tends to $\pm\pi$. A more general contour suggested by Talbot is

$$z(\theta) = \gamma + \mu(\theta \cot(\theta) + iv\theta), \quad -\pi \leq \theta \leq \pi. \quad (4.1.3)$$

Here $\mu$ controls the width of the contour and the constant $\gamma$ translates the contour to the left or right. We shall also need the derivative

$$z'(\theta) = \mu(\cot(\theta) - \csc^2(\theta) + iv).$$

We can now write (4.1.1) as

$$f(t) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} e^{z(\theta)t} F(z(\theta)) z'(\theta) d\theta. \quad (4.1.4)$$

In the interval $[-\pi, \pi]$ we take $N$ points $\theta_k$ spaced uniformly at a distance $h = 2\pi/N$,

$$\theta_k = -\pi + \left(k - \frac{1}{2}\right) h, \quad 1 \leq k \leq N.$$

Our midpoint approximation to (4.1.4) becomes

$$f_N(t) = \frac{h}{2\pi i} \sum_{k=1}^{N} e^{z_k t} F(z_k) w_k, \quad (4.1.5)$$
with \( z_k = z(\theta_k), \ w_k = z'(\theta_k) \). The parameters proposed in [27] for the Talbot contour amount to

\[
z(\theta) = \frac{N}{t} (0.5017\theta \cot(0.6407\theta) - 0.6122 + 0.2645i\theta), \tag{4.1.6}
\]

with convergence rate approximately \( O(3.89^{-N}) \). In the next section we shall apply this method to option evaluation.

### 4.2 Numerical results for European options

In this section we shall give numerical results for Talbot’s numerical inversion method for pricing European options. Again, as in Chapter 3, we shall use the Black-Scholes formula as our benchmark model. Recall from Section 2.5.1 that we derived a formula for European options in the Laplace domain as

\[
\hat{C}(x, p) = \begin{cases} 
B_1 e^{\xi_1 x}, & x \leq 0 \\
B_2 e^{\xi_1 x} + \frac{e^x}{p} - \frac{1}{p + r}, & x > 0,
\end{cases}
\]
where \( p \) denotes the Laplace variable, with

\[
B_1 = \frac{1}{\xi_1 - \xi_2} \left[ \frac{\xi_2}{p + r} + \frac{1 - \xi_2}{p} \right], \quad (4.2.1)
\]

\[
B_2 = \frac{1}{\xi_1 - \xi_2} \left[ \frac{\xi_1}{p + r} + \frac{1 - \xi_1}{p} \right],
\]

and

\[
\xi_1 = \frac{\mu + \sqrt{\mu^2 + 2\sigma^2p}}{\sigma^2}, \quad \xi_2 = \frac{\mu - \sqrt{\mu^2 + 2\sigma^2p}}{\sigma^2}.
\]

Note, we made the transformation \( x = \log(S/K) \), \( \tau = T - t \) on the original Black-Scholes PDE, see Section (2.5.1). We took the Laplace transform with respect to \( \tau \). To compute the value of the option we need to approximate

\[
C(S, T) = K \frac{1}{2\pi i} \int_{\Gamma} e^{pt} \hat{C}(x, p) \, dp. \quad (4.2.2)
\]

We shall apply the contour \((4.1.6)\), i.e.,

\[
\Gamma: \quad p(\theta) = N/\tau (0.5017\theta \cot(0.6407\theta) - 0.6122 + 0.2645i\theta), \quad -\pi \leq \theta \leq \pi.
\]

(4.2.3)

We need to be certain that our contour encloses all singularities. In the case of a European call option we have two poles at \( p = 0 \) and \( p = -r \) and a branch-point at \( p = -2\mu^2/\sigma^2 \), all on \( \mathbb{R}^+ \); see Figure 4.2 for a typical plot.

We have to do numerical inversion for three cases, namely, \( x < 0 \), \( x = 0 \) and \( x > 0 \) since \( x = \log(S/K) \) and the fact that \( S > K \), \( S = K \) or \( S < K \).

For the comparison between the numerical and theoretical errors we shall use the exact solution of a call option given by

\[
C(S, t) = SN(d_1) - Ke^{-(T-t)}N(d_2), \quad (4.2.4)
\]

where \( d_1, d_2 \) and \( N(d) \) are given in (2.2.14).

### 4.2.1 Option is out-of-the-money: \( x < 0 \)

For numerical purposes we shall only consider the call option. The problem we shall attack first is when the option is to be out-of-the-money, i.e., when the asset price \( S \) is less then the strike \( K \). This means we have \( x = \log(S/K) < 0 \). Numerical experiments reported in this section are based on the parameters
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\[ p = -2 \mu^2 / \sigma^2 \]
\[ p = -r \]
\[ p = 0 \]

\[ \text{Figure 4.2: Plot of a typical Talbot contour.} \]

- \( S = 8 \)
- \( K = 10 \)
- \( \sigma = 30\% \)
- \( r = 3\% \)
- \( T = 1 \)

We have the inversion formula as

\[ C(S, t) = K \frac{1}{2\pi i} \int_{-\pi}^{\pi} e^{p(\theta)t} \left( B_1(p(\theta)) e^{\xi_1(p(\theta)x)} \right) p'(\theta) d\theta, \quad (4.2.5) \]

where \( p(\theta) \) is given as in \((4.2.3)\). The approximation of the call option value is given by

\[ C(S, t) \approx K \frac{h}{2\pi i} \sum_{k=1}^{N} e^{p(\theta_k)t} \left( B_1(p(\theta_k)) e^{\xi_1(p(\theta_k)x)} \right) p'(\theta_k). \quad (4.2.6) \]

Figure 4.3 shows the numerical error and theoretical errors for different
values of \( N \). We expect a theoretical convergence rate of \( O(3.89^{-N}) \) as in [27], which is the dashed line in Figure 4.3. We find our best approximation at \( N = 28 \), with an error \( 2.69 \times 10^{-16} \). For larger \( N \) we see rounding errors coming into effect, which is associated with the fact that the inversion of the Laplace transform is an ill-conditioned problem.

4.2.2 Option is in-the-money: \( x > 0 \)

We shall calculate the value of the option when it is in-the-money, i.e., \( x > 0 \) or \( S > K \). We use the parameters

- \( S = 11 \)
- \( K = 10 \)
- \( \sigma = 30\% \)
- \( r = 3\% \)
- \( T = 1 \)
We have the inversion formula as
\[
C(S, t) = K \frac{1}{2\pi i} \int_{-\pi}^{\pi} e^{p(\theta)\tau} \left( B_2(p(\theta)) e^{\xi_1(p(\theta))x} + \frac{e^x}{p(\theta)} - \frac{1}{p(\theta) + r} \right) p'(\theta) d\theta,
\]
which we approximate by the midpoint rule as
\[
C(S, t) \approx K \frac{h}{2\pi i} \sum_{k=1}^{N} e^{p(\theta_k)\tau} \left( B_2(p(\theta_k)) e^{\xi_2(p(\theta_k))x} + \frac{e^x}{p(\theta_k)} - \frac{1}{p(\theta_k) + r} \right) p'(\theta_k).
\]

In Figure 4.4 we have plotted the numerical error vs. theoretical error over \( N \), the number of function evaluations. Again, we see that the numerical errors agree with the theoretical errors. The best approximation is at \( N = 31 \) with an error approximately \( 1.9 \times 10^{-15} \). So far we used \( T = 1 \). We can vary the time, say in an interval \( 1 \leq \tau \leq 100 \). In Figure 4.5 we plotted the errors over the time interval \( 1 \leq \tau \leq 100 \). We have used \( N = 20 \), yielding errors approximately \( 10^{-11} \) over the whole time interval.
Figure 4.5: Numerical errors for in the interval $1 \leq \tau \leq 100$.

4.3 Numerical results for Asian options

In Chapter 2 we derived a Laplace transform formula to price Asian options. In this section we shall solve this problem numerically since an exact solution does not exist. Geman and Yor [10] gave their Laplace transform in terms of an integral, which involves the integral representation of the hypergeometric function. In the literature Shaw [24] achieved accurate results for this inversion problem by keeping the contour along the Bromwich line and using the software package Mathematica for the numerical integration. Numerically this is not the best method to use as the length of the truncated Bromwich line increases for low volatility.

Here we shall compute the inversion formula through a contour deformation as in the previous section. Recall from Section 2.6.1 that the Laplace transform of the option price is given by

$$
\hat{C}(\alpha, p) = \frac{(2\alpha)^{2+\nu-\mu}/\Gamma(2 + \frac{1}{2}(\mu + \nu))}{p(p - 2(1 + \nu))\Gamma(1 + \mu)} {}_1F_1\left(\frac{1}{2}(\mu - \nu - 2), 1 + \mu, -\frac{1}{2\alpha} \right).
$$

(4.3.1)
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To compute the value of the option, we need to take the inverse Laplace transform

\[ C(S, I, t) = \frac{4S}{\sigma^2 T} e^{-r(T-t)} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\tau p} \hat{C}(\alpha, p) dp, \quad (4.3.2) \]

where

\[ \tau = \frac{\sigma^2}{4} (T-t), \quad \alpha = \frac{\sigma^2 TK}{4S}, \quad \mu = \sqrt{\nu^2 + 2p}. \]

The contour integral in (4.3.2) has two poles, one at the origin \( p = 0 \) and the other at \( p = 2\nu + 2 \) with a branch-point at \( p = -\frac{\nu^2}{2} \), see Figure 4.6[24]. We stated that the computational inversion for low volatility cases is slow to compute. This is due to the computation of the hypergeometric function for large negative arguments. In the next section we shall give two contour integral representations of the hypergeometric function that we shall use for computing it more efficiently.
4.3.1 Computation of the confluent hypergeometric function using Talbot contours

The numerical values of the confluent hypergeometric function \( {}_1F_1(a, b, x) \), is accurate or fast computable for certain choices of parameters \( a, b \) and \( x \), in software packages such as Mathematica or Maple. Numerical problems start when \( a \) is large, the algorithms typically loses accuracy and the computational time increases. When the variable \( x \) is such that \(|ax| < 1\), the numerical computation is fast and accurate. In this section we propose a different and new method for the numerical computation of \( {}_1F_1(a, b, x) \) for large values of \( a \) and large negative values of \( x \). The method is based on Mellin-Barnes type contour representation of the hypergeometric functions, which one can approximate with Talbot’s method. We shall compare our results with the built-in function of the software package Mathematica.

4.3.2 Barnes-type contour integrals for the confluent hypergeometric function

The hypergeometric function \( {}_1F_1(a, b, x) \) has a series expansion given by

\[
{}_1F_1(a, b, x) = 1 + \frac{ax}{b1!} + \frac{a(a + 1)x^2}{b(b + 1)2!} + \frac{a(a + 1)(a + 2)x^3}{b(b + 1)(b + 2)3!} + \ldots . \tag{4.3.3}
\]

It is normally written as

\[
{}_1F_1(a, b, x) = \sum_{k=0}^{\infty} \frac{(a)_k x^k}{(b)_k k!}, \tag{4.3.4}
\]

where

\[
(a)_k = \lambda(\lambda + 1)(\lambda + 2)\ldots(\lambda + k - 1) = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)}, \tag{4.3.5}
\]

where \((a)_k\) is the Pochhammer symbol. When \( a \) or \( x \) is large, we need to take more and more terms in the series to obtain the desired accuracy. This results in an increase in computational time and we have the effect of accumulation of rounding errors. To avoid this we propose two contour integrals. First, we have

\[
{}_1F_1(a, b, -x) = \frac{1}{2\pi i} \frac{\Gamma(b)}{\Gamma(a)} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(-s)\Gamma(a + s)}{\Gamma(b + s)} x^s ds, \tag{4.3.6}
\]
for $\left|\text{arg}(-x)\right| < \frac{\pi}{2}$, $a, b \neq 0, -1, -2, \ldots$, see [1, p. 506,(13.2.9)] for a restriction on $\gamma$. An alternative formula given in [21] is

$$1F_1(a, b, -x) = \frac{\Gamma(b)}{\Gamma(a)} \frac{x^{-a}}{2\pi i} \int_{-\infty}^{i\infty} \frac{\Gamma(z)\Gamma(a-z)}{\Gamma(b+z)} x^{-z} dz. \quad (4.3.7)$$

We also have a Laplace inversion type contour representation given by

$$1F_1(a, b, x) = \frac{\Gamma(b)}{2\pi i} x^{1-b} \int_{\psi-i\infty}^{\psi+i\infty} e^{zs} s^{-b}(1-s^{-1})^{-a} ds, \quad (4.3.8)$$

with $\text{Re}$(\psi) > 1, see [8] p. 273]. Let us make the change of variables $s = z/x$, to obtain

$$1F_1(a, b, x) = \frac{\Gamma(b)}{2\pi i} \int_{\psi-i\infty}^{\psi+i\infty} e^{z} z^{-b} \left(1 - \frac{x}{z}\right)^{-a} dz. \quad (4.3.9)$$

This result is similar to the Hankel integral for the complex gamma function $\Gamma(c)$ given in [22]

$$\frac{1}{\Gamma(c)} = \frac{1}{2\pi i} \int_B z^{-c} e^z dz. \quad (4.3.10)$$

When we apply Talbot’s method to (4.3.7), we get

$$1F_1(a, b, -x) \approx \frac{\hbar}{\Gamma(a)} x^{-a} \sum_{k=1}^{N} \frac{\Gamma(z(\theta_k))\Gamma(a-z(\theta_k))}{\Gamma(b+z(\theta_k))} x^{-z(\theta_k)} z'(\theta_k), \quad (4.3.11)$$

where $z(\theta)$ is defined by a contour similar to (4.1.6). If we apply Talbot’s method to (4.3.9), we get

$$1F_1(a, b, x) = \frac{\hbar}{2\pi i} \sum_{k=1}^{N} e^{z(\theta_k)} z'(\theta_k)^{-b} \left(1 - \frac{x}{z(\theta_k)}\right)^{-a} \quad (4.3.12)$$

In the next subsection we shall give numerical results using the method (4.3.12) and compare it with existing software packages. In Figure 4.7 we have plotted the hypergeometric function on the interval $t \in [-2, 0]$ with $a = 1000$ and $b = 1$.

### 4.3.3 Numerical implementation of the hypergeometric function

In this subsection we give numerical results for the method described in the previous subsection. We have some closed form expressions for the
hypergeometric function, i.e., when $a = b$ and positive. We have for the simple case when $a = b = 1$,

$$\,_{1}F_{1}(1, 1, x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x. \quad (4.3.13)$$

It is valid for all real or complex values of $x$. In Figure 4.8 we have plotted the theoretical error against the numerical error. We observe geometric convergence rates of $O(3.89^{-N})$ which agrees with the analysis of Trefethen et al [27]. We have another closed form expression; this is when $a = 1$ and $b = 3/2$, in which case the hypergeometric function can be written in terms of the complementary error function as

$$\,_{1}F_{1}(1, 3/2, x) = -\frac{1}{2} \frac{\sqrt{\pi} \text{erf}(\sqrt{-x}) - 1}{e^{x} \sqrt{-x}}. \quad (4.3.14)$$

The complementary error function is given by

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-x^2} \, dx, \quad (4.3.15)$$

see for example [1]. In Figure 4.9 we have plotted the error for $a = 1, b = 3/2$ and $x = 1$, and we again have geometric convergence rates. In both cases we get 15 digits of accuracy. When we choose the parameter $x = 6$, Talbot's
method appear to suffer more severely from roundoff errors, see Figure 4.10.

We shall now compare the timing and accuracy obtained between the Talbot method and the Mathematica built-in function, Hypergeometric1F1[a, b, x]. In Table 4.1 we give a few test problems. A comparison of the timing obtained with the Mathematica built-in function and Talbot’s method can be seen in Table 4.2. The timings were made in Mathematica version 5.0 on a Pentium 4 1.5 Ghz processor equipped with 256MB RAM. In case 1 we see that the timing is fast in both cases. The Talbot method becomes more advantageous when the absolute value of |ax| increases. In case 3, the timing was 796.4 seconds to compute with the Mathematica method. We see that the Talbot method takes 5.1 seconds for the computation of case 3. When we increase the real part of a we see in case 4 that the Mathematica method takes almost 4 hours to compute whereas the Talbot method computes it in less than 20 seconds. In all the five cases we have 16 significant digits accuracy.
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Figure 4.9: Same as Figure 4.8 but the function is \( \text{}_1 F_1(1, 3/2, 1) \).

In this section we gave a new method for the computation of the hypergeometric function using Talbot’s method. In the next section we shall apply these ideas to price Asian options.

4.3.4 High volatility analysis of Asian options

In the previous section we proposed two contour representations for the confluent hypergeometric function, namely (4.3.7) and (4.3.9). We shall use it here for numerical purposes for pricing Asian options for high volatility cases. Recalling that the Laplace transform of the option price is

\[
\hat{C}(\alpha, p) = \frac{(2\alpha)^{(2+\nu-\mu)/2}\Gamma(2 + \frac{1}{2}(\mu + \nu))}{p(p - 2(1 + \nu))\Gamma(1 + \mu)} \text{}_1 F_1 \left( \frac{1}{2}(\mu - \nu - 2), 1 + \mu, -\frac{1}{2\alpha} \right),
\]

From (4.3.9), we have

\[
\text{}_1 F_1(a, b, x) = \frac{\Gamma(b)}{2\pi i} \int_{\psi-i\infty}^{\psi+i\infty} e^{zs} s^{-b} \left( 1 - \frac{x}{z} \right)^{-a} ds,
\]
and plugging it in (4.3.16), we have
\[
\hat{C}(\alpha, p) = \frac{\Gamma(2 + \frac{1}{2}(\mu + \nu)) (2\alpha)^{-a}}{p(p - 2(1 + \nu))} \int_{\psi=-i\infty}^{\psi=i\infty} e^{z} z^{-b} (1 - x/z)^{-a} dz, \tag{4.3.17}
\]
where
\[a = (\mu - \nu - 2)/2, \quad b = \mu + 1 \quad \text{and} \quad x = -\frac{1}{2\alpha}.\]

Taking the inverse Laplace transform we have
\[
C(S, I, T) = \frac{1}{2\pi i} \int_{\Gamma_1} e^{\pi r} F(p) \left( \frac{1}{2\pi i} \int_{\Gamma_2} e^{z} z^{-b} G(p, z) dz \right) dp, \tag{4.3.18}
\]
Table 4.2: Timing comparison (in sec.) for evaluating cases 1 → 5 in Table 4.1 where \( N \) denotes the number of function evaluations.

<table>
<thead>
<tr>
<th>Case</th>
<th>Mathematica method</th>
<th>Talbot ((N))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.08 (100)</td>
</tr>
<tr>
<td>2</td>
<td>81.5</td>
<td>1.6 (3000)</td>
</tr>
<tr>
<td>3</td>
<td>796.4</td>
<td>5.1 (10000)</td>
</tr>
<tr>
<td>4</td>
<td>&gt;14000</td>
<td>18.2 (30000)</td>
</tr>
<tr>
<td>5</td>
<td>517.2</td>
<td>0.75 (1500)</td>
</tr>
</tbody>
</table>

with
\[
p(\theta) = N/\tau \left(0.5017\theta \cot(0.6407\theta) - 0.6122 + 0.2645i\theta\right), \quad -\pi \leq \theta \leq \pi,
\]

with \( z(\phi) \) as in (4.16) and
\[
F(p(\theta)) = \frac{(2\alpha)^{-a(p(\theta))}\Gamma(2 + \frac{1}{2}(\mu + \nu))}{p(\theta)(p(\theta) - 2(\nu + 1))},
\]

and
\[
G(p(\theta), z(\phi)) = z(\phi)^{-b(p(\theta))}(1 - x/z(\phi))^{-a(p(\theta))}.
\]

For the remaining of this chapter we shall denote \( C = C(S, I, T) \). Applying the midpoint rule to both integrals in (4.3.18) we get
\[
C = \frac{h_1 h_2}{(2\pi i)^2} \sum_{k=1}^{N} e^{\rho(\theta_k)} F(p(\theta_k)) p'(\theta_k) \left( \sum_{j=1}^{M} e^{z(\phi_j)} G(p(\theta_k), z(\phi_j)) z'(\phi_j) \right),
\]

with \( h_1 = 2\pi/N \) and \( h_2 = 2\pi/M \). For the high volatility cases we shall use the above described method. In the next section we shall give a Mellin transform approach for the low volatility case.

### 4.3.5 Low volatility analysis

Here we consider the cases of low volatility. Recall from (4.3.7), the Barnes-Mellin contour integral
\[
\begin{align*}
_1F_1(a, b, -x) &= \frac{\Gamma(b)}{\Gamma(a)} \frac{x^{-a}}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(z) \Gamma(a - z)}{\Gamma(b + z)} x^{-z} dz.
\end{align*}
\]

We can now write (4.3.16) as
\[
\hat{C}(\alpha, p) = \frac{1}{p(p - 2\nu - 2)} \frac{\Gamma(c)}{2\pi i} \frac{1}{\Gamma(a)} \int_{-i\infty}^{i\infty} x^{-z} \frac{\Gamma(z) \Gamma(a - z)}{\Gamma(c + z)} dz,
\]

\[\tag{4.3.22}\]

\[\tag{4.3.21}\]
where
\[ a = \frac{\mu - \nu}{2} - 1, \quad c = \frac{\mu + \nu}{2} + 2 \quad \text{and} \quad x = 2\alpha. \]

By taking the Laplace inversion, we get the double transform as
\[
C = \frac{1}{(2\pi i)^2} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\rho \tau} \frac{1}{p(p-2\nu-2)} \left( \frac{\Gamma(c)}{\Gamma(a)} \int_{-\infty}^{i\infty} x^{-z} \frac{\Gamma(a-z) \Gamma(a+\nu+\rho)}{\Gamma(c+z)} \, dz \right) \, dp.
\]
(4.3.23)

We have a double transform depending on \( p \) and \( z \). The Mellin transform also depends on \( p \) since \( a \) and \( c \) depends on \( p \). Again we shall use the contour in (4.3.19) for \( p(\theta) \) and for \( z(\phi) \) we use (4.1.6). We can now write the option value as
\[
C \approx \frac{h_1 h_2}{(2\pi i)^2} \sum_{k=1}^{N} e^{p(\theta_k) \tau} F(p(\theta_k)) p'(\theta_k) \left( \sum_{j=1}^{M} x^{-z(\phi_j)} G(p(\theta_k), z(\phi_j)) z'(\phi_j) \right),
\]
where
\[
F(p(\theta)) = \frac{1}{p(\theta)(p(\theta) - 2\nu - 2)} \frac{\Gamma(c(p(\theta)))}{\Gamma(a(p(\theta)))},
\]
and
\[
G(p(\theta), z(\phi)) = \frac{\Gamma(z(\phi)) \Gamma(a(p(\theta) - z(\phi)))}{\Gamma(c(p(\theta)) + z(\phi))}.
\]

For the cases where \( \sigma \to 0 \), we shall use the above presentation for our numerical computations. In the next section we shall give numerical results for the methods described here and compare it with the existing methods in the literature.

### 4.3.6 Numerical results

In this section we shall compare the results when we use Talbot’s method and the results obtained in [24] and [6]. Shaw computed the numerical inversion problem along a truncated Bromwich contour. The test problems we shall consider are the same as in Shaw. The GYShaw results in Table 4.4 are based on the Mathematica approach given in [24] with a larger truncation parameter. The GY-Talbot results were computed using (4.3.20). In all seven cases the results of our method agrees with those of Shaw. When solving the inversion problem on a Bromwich contour, Shaw needed to truncate the contour from 1000 to 1500. For the difficult case 4, the truncation parameter needs to be taken larger at 40000 to obtain 6 digit accuracy.
Table 4.3: The test cases considered here, with \( q = 0 \).

\[
\begin{array}{cccccc}
\text{Cases} & S & K & r & \sigma & T \\
1 & 1.9 & 2 & 0.05 & 0.5 & 1 \\
2 & 2 & 2 & 0.05 & 0.5 & 1 \\
3 & 2.1 & 2 & 0.05 & 0.5 & 1 \\
4 & 2 & 2 & 0.02 & 0.1 & 1 \\
5 & 2 & 2 & 0.18 & 0.3 & 1 \\
6 & 2 & 2 & 0.0125 & 0.25 & 2 \\
7 & 2 & 2 & 0.05 & 0.5 & 2
\end{array}
\]

Table 4.4: The results GYSWShaw comes from the paper [6] where the inversion was done along a truncated Bromwich contour. The GYTalbot column is based on the numerical inversion on a Talbot contour as in (4.3.20), where \((M, N)\) are the number of function evaluations.

### 4.3.7 Low volatility problems

Here we shall concentrate on the difficult case for smaller volatility. We took case 4 as in Table 4.3 with varying volatility, see Table 4.3. For this section we are going to use the double inversion presentation based on the Mellin transform as in (4.3.23)

\[
C = \frac{1}{(2\pi i)^2} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\rho \pi t} \frac{1}{p(p - 2\nu - 2)} \left( \frac{\Gamma(c)}{\Gamma(a)} \int_{-\infty}^{i\infty} \frac{\Gamma(z)\Gamma(a - z)}{\Gamma(c + z)} x^{-z} dz \right) dp,
\]

where

\[
a = \frac{\mu - \nu}{2} - 1, \quad \text{and} \quad c = \frac{\mu + \nu}{2} + 2.
\]

In this case we found that the Talbot contour in (4.1.6) is not optimal for the Mellin-Barnes contour integral. The contour was originally developed for integrals with decaying factor \( e^x \). We find that as the volatility decreases, we needed to widen the contour with a factor say \( \rho \). By taking \( \rho \) to be
Table 4.5: The same parameters as in Table 4.3, but with smaller volatilities.

<table>
<thead>
<tr>
<th>Cases</th>
<th>$S$</th>
<th>$K$</th>
<th>$r$</th>
<th>$\sigma$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2</td>
<td>2</td>
<td>0.02</td>
<td>0.1</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>2</td>
<td>0.02</td>
<td>0.05</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>2</td>
<td>0.02</td>
<td>0.01</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
<td>2</td>
<td>0.02</td>
<td>0.005</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
<td>2</td>
<td>0.02</td>
<td>0.001</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 4.6: The results GYS-Mellin comes from the paper Shaw [24] using the Mellin transform and Gamma quotients. The GY Talbot-Mellin column is the method as described in the previous section, where $N$ and $M$ are the number of function evaluations. In the case 11 GYS-Mellin yield unstable results whereas GY Talbot-Mellin are correct. Actually, in all cases our values are correct to six figures, when compared to the results in [6].

Inversely proportional to the volatility $\sigma$, a sufficient value obtained by experimenting is $\rho = 1/(5\sigma)$. We also truncated the interval to $[-\pi/2, \pi/2]$ with $M$ points $\phi_k$ spaced at a distance $h = \pi/M$,

$$\phi_k = -\pi/2 + \left(k - \frac{1}{2}\right)h, \quad 1 \leq k \leq M.$$ 

The contours we shall use here are thus

$$p(\theta) = \rho \frac{N}{\tau} \left(0.5017\theta \cot(0.6407\theta) - 0.6122 + 0.2645i\theta\right), \quad (4.3.25)$$

$$z(\phi) = \rho M \left(0.5017\phi \cot(0.6407\phi) - 0.6122 + 0.2645i\phi\right).$$

For numerical purposes we took $M = N$ in all the cases. In Table 4.6 the GYS-Mellin column denotes the use of asymptotics of Gamma quotients as in [24] and GY Talbot-Mellin column are based on the method in (4.3.24). In the case with $\sigma = 0.005$, Shaw’s method is unstable, whereas the GY Talbot-Mellin method is stable and correct to 6 significant figures.
4.3.8 Problems for $q > r$

Here we took the parameters as in Table 4.1, but with $q > r$ with $q = 2r$, see Table 4.7. Recall that we have singularities at $p = 0$ and $p = 2\nu + 2$, where $\nu = \frac{2(r-q)}{\sigma^2} - 1$. We have that $\nu < -1$, which means that the singularity at $p = 2\nu + 2$ moves from the positive real axis to the negative real axis. The results in Table 4.8 obtained with Talbot’s contour agree with Shaw’s.

<table>
<thead>
<tr>
<th>Cases</th>
<th>$S$</th>
<th>$K$</th>
<th>$r$</th>
<th>$\sigma$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>1.9</td>
<td>2</td>
<td>0.05</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>14</td>
<td>2</td>
<td>2</td>
<td>0.05</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>15</td>
<td>2.1</td>
<td>2</td>
<td>0.05</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>16</td>
<td>2</td>
<td>2</td>
<td>0.02</td>
<td>0.1</td>
<td>1</td>
</tr>
<tr>
<td>17</td>
<td>2</td>
<td>2</td>
<td>0.18</td>
<td>0.3</td>
<td>1</td>
</tr>
<tr>
<td>18</td>
<td>2</td>
<td>2</td>
<td>0.0125</td>
<td>0.25</td>
<td>2</td>
</tr>
<tr>
<td>19</td>
<td>2</td>
<td>2</td>
<td>0.05</td>
<td>0.5</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 4.7: The same as in Table 4.3, but with $q = 2r$.

<table>
<thead>
<tr>
<th>Cases</th>
<th>GYShaw</th>
<th>GYTalbot</th>
<th>$(M, N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>0.147562</td>
<td>0.147562</td>
<td>(32, 32)</td>
</tr>
<tr>
<td>14</td>
<td>0.191747</td>
<td>0.191747</td>
<td>(32, 32)</td>
</tr>
<tr>
<td>15</td>
<td>0.242316</td>
<td>0.242316</td>
<td>(32, 32)</td>
</tr>
<tr>
<td>16</td>
<td>0.0357853</td>
<td>0.0357853</td>
<td>(100, 100)</td>
</tr>
<tr>
<td>17</td>
<td>0.0522607</td>
<td>0.0522607</td>
<td>(80, 80)</td>
</tr>
<tr>
<td>18</td>
<td>0.145308</td>
<td>0.145308</td>
<td>(50, 50)</td>
</tr>
<tr>
<td>19</td>
<td>0.240495</td>
<td>0.240495</td>
<td>(32, 32)</td>
</tr>
</tbody>
</table>

Table 4.8: Table shows the results for the case when $q > r$. The Talbot method agrees with the results of Shaw in all cases.
4.3.9 Problems for $q = r$

In this section we shall use the data set in Table 4.3, but with $q = r$, see Table 4.9. Again the contour changes since when $r = q$ gives $\nu = -1$ such that we have a double singularity at $p = 0$. In Table 4.9 The results

<table>
<thead>
<tr>
<th>Cases</th>
<th>$S$</th>
<th>$K$</th>
<th>$r$</th>
<th>$q$</th>
<th>$\sigma$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1.9</td>
<td>2</td>
<td>0.05</td>
<td>0.05</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>21</td>
<td>2</td>
<td>2</td>
<td>0.05</td>
<td>0.05</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>22</td>
<td>2.1</td>
<td>2</td>
<td>0.05</td>
<td>0.05</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>23</td>
<td>2</td>
<td>2</td>
<td>0.02</td>
<td>0.02</td>
<td>0.1</td>
<td>1</td>
</tr>
<tr>
<td>24</td>
<td>2</td>
<td>2</td>
<td>0.18</td>
<td>0.18</td>
<td>0.3</td>
<td>1</td>
</tr>
<tr>
<td>25</td>
<td>2</td>
<td>2</td>
<td>0.0125</td>
<td>0.0125</td>
<td>0.25</td>
<td>2</td>
</tr>
<tr>
<td>26</td>
<td>2</td>
<td>2</td>
<td>0.05</td>
<td>0.05</td>
<td>0.5</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 4.9: Same parameters as in Table 4.3, but with $q = r$.

<table>
<thead>
<tr>
<th>Cases</th>
<th>CI Bess</th>
<th>GY Talbot</th>
<th>$(M, N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.169202</td>
<td>0.169202</td>
<td>(32, 32)</td>
</tr>
<tr>
<td>21</td>
<td>0.217815</td>
<td>0.217815</td>
<td>(32, 32)</td>
</tr>
<tr>
<td>22</td>
<td>0.272924</td>
<td>0.272924</td>
<td>(32, 32)</td>
</tr>
<tr>
<td>23</td>
<td>0.0451431</td>
<td>0.0451431</td>
<td>(100, 100)</td>
</tr>
<tr>
<td>24</td>
<td>0.115188</td>
<td>0.115188</td>
<td>(80, 80)</td>
</tr>
<tr>
<td>25</td>
<td>0.158380</td>
<td>0.158380</td>
<td>(50, 50)</td>
</tr>
<tr>
<td>26</td>
<td>0.291315</td>
<td>0.291315</td>
<td>(32, 32)</td>
</tr>
</tbody>
</table>

Table 4.10: The results labeled as CI Bess comes from [6] with the use of the contour integral of the Bessel function form of a hypergeometric function.

labeled as CI Bess use the contour integral of the Bessel function form of the hypergeometric function, see [6]. Again our method is accurate to 6 decimals.

4.3.10 Problems for $q = r$ for low volatility cases

Here we use our double Laplace Mellin transform for numerical purposes. Since we are dealing with small volatility we used the contour in (4.3.24). In Table 4.12 we can see that the Bessel function form does not give any results where the double transform approach is stable for all the cases.
### Table 4.11: Table shows the parameters when $q = r$ for low volatility cases.

<table>
<thead>
<tr>
<th>Cases</th>
<th>$S$</th>
<th>$K$</th>
<th>$r$</th>
<th>$q$</th>
<th>$\sigma$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>27</td>
<td>1</td>
<td>2</td>
<td>0.02</td>
<td>0.02</td>
<td>0.1</td>
<td>1</td>
</tr>
<tr>
<td>28</td>
<td>2</td>
<td>2</td>
<td>0.02</td>
<td>0.02</td>
<td>0.05</td>
<td>1</td>
</tr>
<tr>
<td>29</td>
<td>2</td>
<td>2</td>
<td>0.02</td>
<td>0.02</td>
<td>0.01</td>
<td>1</td>
</tr>
<tr>
<td>30</td>
<td>2</td>
<td>2</td>
<td>0.02</td>
<td>0.02</td>
<td>0.005</td>
<td>1</td>
</tr>
<tr>
<td>31</td>
<td>2</td>
<td>2</td>
<td>0.02</td>
<td>0.02</td>
<td>0.001</td>
<td>1</td>
</tr>
</tbody>
</table>

### Table 4.12: We used the double transform method for numerical purposes.

<table>
<thead>
<tr>
<th>Cases</th>
<th>CLBess</th>
<th>GYTalbot-Mellin</th>
<th>$(M, N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>27</td>
<td>0.0451431</td>
<td>0.0451431</td>
<td>(40, 40)</td>
</tr>
<tr>
<td>28</td>
<td>0.0225755</td>
<td>0.0225755</td>
<td>(40, 40)</td>
</tr>
<tr>
<td>29</td>
<td>NA</td>
<td>0.00451536</td>
<td>(80, 80)</td>
</tr>
<tr>
<td>30</td>
<td>NA</td>
<td>0.00225768</td>
<td>(120, 120)</td>
</tr>
<tr>
<td>31</td>
<td>NA</td>
<td>0.000451537</td>
<td>(250, 250)</td>
</tr>
</tbody>
</table>

### 4.4 Conclusion

In this section we valued options with the Laplace transform. We used the Talbot method for numerically inverting the Laplace transform. The application of Talbot’s method for option pricing is an untreated problem in the literature. As our test problem we used the Black-Scholes equation as our benchmark model. Using Talbot’s contour and the optimal parameters proposed in Trefethen et al. [27] we showed that the convergence rate corresponds to the theory with a geometric convergence rate $O(3.89^{-N})$. In the case of Asian options we have no analytical solution to our disposal. We have the problem of computing the hypergeometric function of the first kind. Two contour representations for the hypergeometric function was proposed which can be approximated using Talbot’s method. The method was compared with Mathematica’s built-in function Hypergeometric1F1$[a, b, t]$. We showed that our method is much faster then the existing Mathematica built-in function and our method yields the same accuracy. One can expand Talbot’s method to the hypergeometric function of the second kind since contour representation similar to that of $1F_1(a, b, t)$, exists. Using the contour representations of $1F_1(a, b, t)$ and Talbot’s method we priced Asian options effectively. For the low volatility case we gave a double transform formula which result in a double inversion problem. In some cases [24] gave unstable results for low volatility, whereas the double transform inversion gave stable and accurate results in all the test problems.
Chapter 5

Conclusion and Further Study

Options can be valued by using partial differential equations. An exact solution for European style options is known. Numerical experiments are necessary to determine the value of Asian options as an exact solution does not exist. The computational time must be minimized but still yields a small error. In this thesis we used finite difference methods and numerical inversion of the Laplace transform.

In Chapter 3 we used finite difference methods for pricing options. The final condition for option pricing is not differentiable. We used a spatial grid refinement near the area where the kink occurs. In the Black-Scholes equation this area is at $S = K$ when $t = T$. We made grid stretching by the means of a coordinate transformation and by a direct method. The fraction of grid points distributed near $S = K$ depends on $c$, the stretching parameter. We found that the value $c = 2K/5$ is satisfactory for the European case. In the case of Asian options, the kink in the final condition does not occur at the exercise price, but rather at zero. In this case we needed a new generating function for our non-uniform grid. Since the Asian option has no closed form solution, we only had good lower and upper bounds to our disposal.

Let us summarize the results using finite difference methods:

- For European options, only $10 \times 10$ space and time-steps are needed to obtain 3 digits of accuracy on a non-uniform grid with $c = 2K/5$. As $K$ increases, the domain on which the PDE is to be solved also increases.

- For the Asian option case we needed a new grid refinement function to generate a non-uniform grid. We used $200 \times 400$ space and time
steps using uniform and non-uniform grids with \( c = 1/100 \). Using a uniform grid not all the prices lie in the desired bounds, whereas with our non-uniform grid we have satisfactory results that lie in the bounds.

In chapter 4 we used the numerical inversion of the Laplace transform to compute option prices. In particular we used Talbot’s method which is based on a deformation of the Bromwich contour. Again the Black-Scholes model served as our benchmark model. Asian options can also be valued using Laplace inversion. The Laplace problem we solved was that obtained by Geman and Yor \cite{10}. Let us summarize the results using the inversion of the Laplace transform:

- For European options we obtained 15 digits accuracy with only 28 function evaluations. These results were obtained for in-the money and out-of-the money cases. When we fixed the number of function evaluations at 20 and varying the time interval between \( 1 \leq t \leq 100 \), we obtained 12 digits accuracy over the whole time interval.

- For the Asian option case we had the problem of computing the hypergeometric function of the first kind. Here we proposed alternative ways for numerically computing the hypergeometric function by means of a Laplace inversion method and one based on Barnes-Mellin transforms. The Laplace inversion method was inverted using Talbot’s method and compared to the existing Mathematica built-in function. We showed that our method is much faster than the existing methods and our results agrees with Mathematica to 16 significant figures.

- Using the alternative methods for computing the hypergeometric function we efficiently priced Asian options. For high volatility cases we used the Laplace inversion method and for low volatility we used the Barnes-Mellin transform. In both cases we had double inversion problems. For the low volatility cases we needed to modify the contour for better accuracy. The approach we used are all new methods not studied in the literature and our results are the most accurate in solving the Geman and Yor \cite{10} Laplace inversion problem.

Next we shall give some further studies one can pursue.

5.1 Further study

This section is devoted to some extended ideas one can pursue that were not handled in this thesis. We list a few:
• Geman and Yor’s [10] Laplace transform for Asian options was derived from \((2.3.4)\) after making a suitable change of variables. One can start by taking the Laplace transform to time and the Mellin transform to one of the spatial variables on \((2.3.4)\). This will result in a double inversion transform.

• Another problem not studied in this thesis are the hedging parameters. A comparison can be made for instance between the finite difference method and the Laplace inversion method. The Black-Scholes model can again be taken as benchmark. As we have the Laplace transform of the Black-Scholes PDE, we only need to take the respective partial derivatives and do numerical inversion.

• We developed a new method for computing \(_1F_1(a, b, z)\) for certain values of the parameters. The Talbot method can be extended to compute the hypergeometric function of the second kind which can be written in terms of \(_1F_1(a, b, z)\). As we have developed a new method for computing \(_1F_1(a, b, z)\) function we now have a new method for the computation of the hypergeometric function of the second kind. The Hermite function [15] is another special function that can be written in terms of hypergeometric functions.

• Bessel functions can also be written in terms of hypergeometric functions. This gives rise to an alternative method for computing Bessel functions [15].

• Talbot’s method for the hypergeometric function proposed in this thesis can be extended to problems in other fields which also have slow computation times.

We shall now give mathematical formulas for some of the problems in the above list. As stated above, the hypergeometric function of the second kind is given by

\[
U(a, b, z) = \frac{\Gamma(1 - b)}{\Gamma(1 + a - b)} F_1(a, b, z) + \frac{\Gamma(b - 1)}{\Gamma(a)} z^{(1-b)} F_1(1 + a - b, 2 - b, z),
\]

which we can solve by solving the hypergeometric function of the first kind by our Laplace inversion method. The function \(U(a, b, z)\) will thus suffer the same computational time issue with Mathematica. Another hypergeometric function one can also look at is the \(2F_1(a, b, c, z)\) function which can be written as a Mellin-Barnes integral as [1]

\[
2F_1(a, b, c, -x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{\Gamma(z)\Gamma(a - z)\Gamma(b - z)}{\Gamma(c - z)} x^{-z} dz,
\]
The modified Bessel function of the first kind can be written as

\[ I_p(x) = \frac{(x/2)^p}{\Gamma(p + 1)} e^{-x} \Gamma(p + 1/2; 2p + 1; 2x), \]

see [13] for formulas for the other Bessel function families in terms of the \( _1F_1(a; b; z) \) function. This does not guarantee that it is a better method, but gives an alternative way for computing Bessel functions.
Appendix A

Background Theory

We will look at the background results of probability theory, martingale theory and stochastic differential equations, which will play an important role in the chapters that follow. Most of the results in this appendix comes from [3]. First, we will look at the problem from probability theory.

A.0.1 Probability theory

Let $\Omega$ be a non-empty set. A $\sigma$-algebra $\mathcal{F}$ on $\Omega$ is a family of subsets of $\Omega$ such that

1. the empty set $\emptyset$ belongs to $\mathcal{F}$;
2. if $A$ belongs to $\mathcal{F}$, then so does the complement $\Omega \setminus A$;
3. if $A_1, A_2, \ldots$ is a sequence of sets in $\mathcal{F}$, then their union $A_1 \cup A_2 \ldots$ also belongs to $\mathcal{F}$.

A probability measure $\mathbb{P}$ is a function mapping $\mathcal{F}$ into $[0, 1]$ with the following properties:

1. $\mathbb{P}(\Omega) = 1$,
2. if $A_1, A_2, \ldots$ are pairwise disjoint sets belonging to $\mathcal{F}$, then

$$\mathbb{P}(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mathbb{P}(A_k).$$

We shall interpret probability measures as follows. If $\Omega$ is the set of all possible outcomes of an experiment, then $\mathcal{F}$ represents all events $A \subseteq \Omega$ of interest. For each such $A \in \mathcal{F}$, the probability $\mathbb{P}(A)$ is a number between 0 and 1 (inclusive) denoting the probability that the event $A$ takes place.
A.0.2 Random variables

In order to define a random variable, we need to define what a Borel set is. The Borel $\sigma$-algebra on $\mathbb{R}$, denoted by $\mathcal{B}(\mathbb{R})$ is the smallest $\sigma$-algebra containing all the open intervals in $\mathbb{R}$. These sets are called Borel sets.

Since $\mathcal{B}(\mathbb{R})$ is a $\sigma$-algebra, it follows that any union of open intervals is also contained in $\mathcal{B}(\mathbb{R})$. We can now give the definition of a random variable as follows.

A function $\xi$ which take values in $\mathbb{R}$ is $\mathcal{F}$-measurable if the pre-image of every $B \in \mathcal{B}(\mathbb{R})$ is an $\mathcal{F}$-measurable set, i.e. $\{\omega \in \Omega : \xi(\omega) \in B\} \in \mathcal{F}$ for every $B \in \mathcal{B}(\mathbb{R})$. If $(\Omega, \mathcal{F}, P)$ is a probability space, then such a function is called a random variable. A random variable $\xi$ is said to be integrable if

$$\int_{\Omega} |\xi|dP < \infty. \quad (A.0.2)$$

If this is the case

$$E(\xi) = \int_{\Omega} \xi dP \quad (A.0.3)$$

exists and is called the expectation of $\xi$.

A.0.3 Martingales

We shall assume that time is continuous and we deal with continuous time processes. Let the continuous process be denoted by $S_t, t \in [0, \infty)$. Let $\mathcal{F}_t, t \in [0, \infty)$, represent a family of information sets that becomes continuously available to the decision maker as time passes. This family of information sets will satisfy

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \ldots, \quad (A.0.4)$$

for $s < t < T$. The set $\mathcal{F}_t, t \in [0, \infty)$ is called a filtration. Consider the random price process $S_t$ during a finite interval $[0, T]$. At a point $t = \tau$, the value of the price process will be $S_\tau$. If the value of $S_t$ is included in the set $\mathcal{F}_\tau$, then it is said that $S_t, t \in [0, \infty)$ is adapted to $\mathcal{F}_\tau, t \in [0, \infty)$. This means the value of $S_t$ is known, given the information set $\mathcal{F}_\tau$, which can be regarded as the stock price record.

Using different information sets one can generate different values of $S_t$. This is expressed using conditional expectations

$$E_t[S_T] = E[S_T|\mathcal{F}_t], \quad t < T. \quad (A.0.5)$$
This denotes the forecast of a future price, $S_T$ of $S_t$, using the information up to time $t$. Now let us define the conditions of a martingale.

**Definition:** A process $S_t, t \in [0, \infty)$ is a **martingale** with respect to the family of information sets $\mathcal{F}_t$ and with respect to the probability $\mathbb{P}$, if, for all $t > 0$,

1. $S_t$ is known, given $\mathcal{F}_t$, that is, $S_t$ is $\mathcal{F}_t$ adapted, and
2. unconditional forecasts are finite
   
   $$\mathbb{E}|S_t| < \infty,$$
3. and if
   
   $$\mathbb{E}_t[S_T] = S_t, \text{ for all } t < T,$$

   with probability 1. We took all expectations with respect to the probability $\mathbb{P}$.

### A.0.4 Properties of Brownian motion

The concept of Brownian motion will now be defined, as it is the core concept of the Black-Scholes and many other financial models. We will use Brownian motion to model the uncertainties, which is used to express the randomness of the stochastic process.

A standard Brownian motion is a real valued, continuous stochastic process $(B_t)_{t \geq 0}$, with independent and stationary increments defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We formally defined it as follows:

**Definition:** A random process $B_t, t \in [0, T]$ is a standard Brownian motion if:

1. The process begins at zero, $B_0 = 0$.
2. $B_t$ has stationary, independent increments.
3. $B_t$ is continuous in $t$.
4. The increments $B_t = B_s$, have a normal distribution with mean zero and variance $|t - s|:
   
   $$(B_t - B_s) \sim N(0, |t - s|).$$

Appendix A. Background Theory

In Figure A.1 we show a simple path of a Brownian motion. Although $B_t$ is continuous everywhere, it is not differentiable anywhere with probability one. A stochastic process $S_t$ is said to follow a geometric Brownian motion if it satisfies the following stochastic differential equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t.$$  \hfill (A.0.6)

For an initial value $S_0$ the equation has the analytic solution

$$S_t = S_0 e^{(\mu - \frac{1}{2} \sigma^2)t + \sigma B_t}.$$  \hfill (A.0.7)

With the Brownian motion now defined, we shall focus on the Ito calculus.

**A.0.5 Ito formula**

The following is an intuitive proof of the Ito formula. If a stock price $S$ follows the equation by Taylor expansion, then the following will occur:

$$df(S,t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS)^2 + \ldots$$
But we have
\[ dS/S = \mu dt + \sigma dB, \]
see (2.1.4). It follows that
\[
(dS)^2 = (S\mu dt + S\sigma dB)^2 \\
= \mu^2 S(dt)^2 + 2\mu\sigma SdBdt + \sigma^2 S(dB)^2.
\]
Using the rules that \( dt^2 = 0 \) and \( dt dB = 0 \), we have
\[ dS^2 = \sigma^2 S^2(dB)^2 = \sigma^2 S^2 dt. \]
We can now write the Ito formula as
\[
df(S, t) = \left( \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S \frac{\partial f}{\partial S} dB.
\]
Appendix B

Derivation of the exact solution of European options

We shall solve the Black-Scholes equation for a call option and use the put-call parity relation to compute the exact solution of the put option, see [31]. The Black-Scholes equation for a European call with value $C(S, t)$ is

$$
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} - r C + \frac{\partial C}{\partial t} = 0.
$$

We can transform the above PDE to the heat equation through the following change of variables

$$
x = \log(S/K), \quad \tau = \frac{1}{2} \sigma^2 (T - t) \quad \text{and} \quad C = Ke^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2 \tau} U(x, \tau),
$$

where $k = \frac{2r}{\sigma^2}$. We now have

$$
\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2} \quad \text{for} \quad -\infty < x < \infty, \quad \tau > 0,
$$

with initial condition

$$
U(x, 0) = \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0) = U_0(x).
$$

The PDE is defined on the whole real axis. For the solution we only need the initial condition. The solution to the heat equation is

$$
U(x, \tau) = \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{\infty} U_0(s) e^{-(x-s)^2/4\tau} ds.
$$

Let us make the change of variables $x' = (s - x)/\sqrt{2\tau}$, such that

$$
U(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2}(k+1)(x+x'/\sqrt{2\tau})} e^{-x'^2/2} dx' - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2}(k-1)(x+x'/\sqrt{2\tau})} e^{-x'^2/2} dx'.
$$
The value of the integrals on the right-hand side is
\[ I_1 = e^{\frac{1}{2}(k+1)x + \frac{2}{2}(k+1)^2 \tau} N(d_1), \]
\[ I_2 = e^{\frac{1}{2}(k-1)x + \frac{2}{2}(k-1)^2 \tau} N(d_2), \]
where
\[ d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k + 1)\sqrt{2\tau}, \]
\[ d_2 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k - 1)\sqrt{2\tau}, \]
and
\[ N(d) = \frac{1}{2\pi} \int_{-\infty}^{d} \exp(-\frac{1}{2}s^2)ds \]
is the cumulative distribution function for the normal distribution. Undoing the change of variables we recover our original function
\[ C(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \]
where
\[ d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T-t}}, \]
\[ d_2 = \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T-t}}, \]
(B.0.1)
We can use the put-call parity relation (2.2.17) to calculate the put option
\[ P(S, t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1). \]
Appendix C

Laplace transform

The Laplace transform plays an important role in mathematical applications in physics and engineering. It is a powerful method for solving differential and integral equations. In this appendix we shall give the basic theory of the Laplace transform. For more results see [21]. Suppose that $f$ is a real- or complex valued function of the variable $t > 0$ and $s$ is a real or complex parameter. We define the Laplace transform of $f$ as

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt,$$  \hspace{1cm} (C.0.1)

where $\mathcal{L}(.)$ is the Laplace transform operator. The parameter $s$ belongs to a domain on the real line or in the complex plane. We always use the notation $s = x + iy$ when $s$ is complex.

Although a function $f(t)$ may be defined for all values of $t$, its Laplace transform is not influenced by values of $f(t)$, when $t < 0$. The Laplace transform of $f(t)$ is actually defined for the function $P(t)$ given by

$$P(t) = \begin{cases} 
    f(t), & t \geq 0 \\
    0, & t < 0.
\end{cases} \hspace{1cm} (C.0.2)$$

A sufficient condition for the existence of the Laplace transform is that $|f(t)|$ does not grow too rapidly as $t \to \infty$. We say that the function $f(t)$ is of exponential order $\alpha$ if there exist constants $M > 0$ and $\alpha$ such that for some $t_0 \geq 0$,

$$|f(t)| \leq Me^{\alpha t}, \quad t \geq t_0.$$  

The Laplace transform $F(s) = F(x + iy)$ exists for values of $s = x + iy$ in a domain that includes the right half-plane $\Re(s) > \alpha$.  

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Appendix C. Laplace transform

We can also apply the Laplace transform to differential equations. This leads to more restrictions for the Laplace transform. Let us define a function consisting of a spatial and time parameter by $u = u(x, t)$, with the time variable $t > 0$. Denote by $U(x, s)$ the Laplace transform of $u$ with respect to $t$. We have

$$U(x, s) = \mathcal{L}(u(x, t)) = \int_0^\infty e^{-st}u(x, t)dt, \quad (C.0.3)$$

where $x$ is the untransformed variable. Let us consider the different partial derivatives and their Laplace transforms. For the first derivative with respect to space we have

$$\mathcal{L} \left( \frac{\partial u}{\partial x} \right) = \int_0^\infty e^{-st} \frac{\partial}{\partial x} u(x, t)dt$$

$$= \frac{\partial}{\partial x} \left( \int_0^\infty e^{-st}u(x, t)dt \right) = \frac{\partial U}{\partial x}. \quad (C.0.4)$$

We have that the transform of the derivative is the derivative of the transform. Similarly we have

$$\mathcal{L} \left( \frac{\partial^2 u}{\partial x^2} \right) = \frac{\partial^2 U}{\partial x^2}.$$  

The derivative with respect to time is different. We have

$$\mathcal{L} \left( \frac{\partial u}{\partial t} \right) = s\mathcal{L}(u(x, t)) - u(x, 0). \quad (C.0.5)$$

Let us consider the heat equation given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$  

Applying the Laplace transform to the PDE we get

$$\frac{\partial^2 U}{\partial x^2} = sU(x, s) - u(x, 0).$$

By applying the Laplace transform we reduced the PDE to an ODE (ordinary differential equation).

In order to apply the Laplace transform, it is necessary to invoke the inverse transform. If $\mathcal{L}(f(t)) = F(s)$, then the inverse Laplace transform is given by

$$\mathcal{L}^{-1}(F(s)) = f(t), \quad t \geq 0, \quad (C.0.6)$$
which maps the Laplace transform of a function back to the original function. The complex inversion formula for computing the inverse Laplace transform is given by

\[ f(t) = \mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{ts} F(s) ds, \quad (C.0.7) \]

and the vertical line at \( \Re(s) = \gamma \) is known as the Bromwich line. We have that \( \Re(s) = \gamma > \gamma_0 \), i.e., that the Bromwich line should lie to the right of all singularities of the function \( F(s) \). The function \( F(s) \) is then analytic in the half-plane \( \Re s > \gamma_0 \).
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