A many-dimensional approach to simulations in modal logic

by

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Thesis presented in partial fulfilment of the requirements for the degree of Master of Science in Mathematics at Stellenbosch University

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January 2012

Declaration

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Date: 13 January 2012

Abstract

Truth preservation is an important topic in model theory. However a brief examination of the models for a logic often show that isomorphism is needlessly restrictive as a truth preserving construction. In the case of modal logics with Kripke semantics the notions of simulation and bisimulation prove far more practical and interesting than isomorphism. We present and study these various notions, followed by a discussion of Shehtman's frame product as semantics for certain many-dimensional modal logics. We show how simulations and bisimulations can be interpreted inside models over frame products. This is followed by a discussion on a category-theoretic setting for frame products, where the arrows may run between frames with different types.

Opsomming

Die behou van waarheid is 'n prominente onderwerp in modelteorie. 'n Vlugtige ondersoek van die modelle vir 'n besondere logika wys egter dat isomorfisme onnodig beperkend as waarheid-behoudende konstruksie is. In die geval van modale logika met Kripke se semantiek is simulasie en bisimulasie heelwat meer prakties en interessant as isomorfisme. Na die bekendstel en studie van hierdie onderskeie begrippe bespreek ons Shehtman se raamproduk as semantiek vir sekere meer-dimensionele modale logikas. Ons wys ons hoe simulasies en bisimulasies binne modelle oor sulke raamprodukte geïnterpreteer kan word. Daarna bespreek ons 'n kategorie-teoretiese konteks vir raamprodukte, waar die pyle tussen rame met verskillende tipes mag loop.

Acknowledgments

I wish to sincerely thank my supervisor Prof. Ingrid Rewitzky for her help in developing me as a mathematician and for assisting me in the development of this work.

Furthermore I would like to thank the other staff members of the Mathematics division for their assistance and inspiration in developing this work and in encouraging my study of mathematics.

I also wish to thank my friends for their advice and encouragement. Without you I would never have finished this.

Laastens wil ek my gesin, en in die besonder my pa, bedank vir hulle bystand en ondersteuning.

Contents

Introduction

Following the development of a semantics for modal logic by Saul Kripke (Kripke (1959)), Krister Segerberg developed structure preserving functions between Kripke's models (Segerberg (1968), Segerberg (1971)). These functions that Goldblatt (1989) renamed *bounded morphisms* led to some of the first preservation results for modal logic. Later work by Johan van Benthem (van Benthem (1976)) generalised these functions to what would later be called *bisimulations*, and led to Van Benthem's famous characterisation theorem, stating that a first order formula in one variable is invariant under bisimulations exactly when it is equivalent to some modal formula. This strengthened bisimulations from a preservation tool to a way to decide which first order formulae belong to the modal fragment of first order logic, further emphasizing the importance of bisimulations.

Around the same time that Krister Segerberg and Johan van Benthem introduced bounded morphisms and bisimulations, independent work in theoretical computer science defined *simulations* (Milner (1971)) and also bisimulations (Park (1981)) between automata. Park (1981) also proved that if two deterministic automata are related by a bisimulation then they accept the same inputs. Although modal logicians and theoretical computer scientists developed their ideas separately it was soon noticed that they coincide. Today bounded morphisms, simulations and bisimulations are standard tools in both modal logic and theoretical computer science.

Our first chapter presents a background on these developments, formalising Kripke semantics and providing the basic tools and motivation for the study of simulations and bisimulations. We examine frame products and product logics as defined by Shehtman (1978) and discuss his axiomatisation of certain product logics. The frame products provide our "many-dimensional" setting, where points are pairs rather than abstract entities and relations are partly determined by this pair structure. Correspondingly we motivate a plane-like intuition for the behaviour of the modalities on product logics as well.

Chapter 2 examines models on frame products where the original frames have the same type. We exhibit a modal formula, involving one propositional variable, that holds in such a model exactly when the set of points of that model where the particular propositional variable is true, is a simulation. We also adapt and prove the result for bisimulations. Then we continue to prove similar results for some approximations to simulations and bisimulations that are traditionally defined via games, instead of via first order properties. The motivating work in this particular chapter is Brink *&* Rewitzky (2004), who exhibited similar characterisations for simulations and bisimulations, but did not work on frame products and defined their modalities differently. To the best of our knowledge we are the first to propose these characterisations for the approximations, but the approximations themselves and their corresponding preservation results are standard.

In Chapter 3 we re-examine frame products, and with a category-theoretic approach in mind we ask the question: *What are products of frames?* Using some results due to Gumm *&* Schröder (2001), we examine category-theoretic products in two well-known categories of frames. After motivating why these categories are not sufficient to examine the frame products of Shehtman (1978), we propose a third category where arrows may run between frames with different types. In keeping with our theme of simulations and bisimulations we prove invariance results for the arrows of this category. We also show that, although some products in this new category do not exist, the frame products of Shehtman (1978) give this category a monoidal structure. To the best of our knowledge we are the first to investigate our proposed category, although it is not an unusual variation on the category of frames and bounded morphisms.

Chapter 1

Many-dimensional modal logic

1.1 Introduction

In this chapter we introduce some basic tools of modal logic namely: modal language; the frames and models where we interpret that language; and the constructions between frames or models that preserve validity of formulae. We also introduce the frame product — a "two-dimensional" construction in the sense that its underlying set consists of pairs rather than abstract points, with the relations partially determined by the pair structure. A product logic is then defined based on frame products. And although frame products are not the only frames where this logic holds, we demonstrate that the behaviour of the modalities on the product logic seem to suggest a strong "two-dimensional" intuition, which we fomalise in an axiomatisation.

1.2 Kripke semantics and truth preservation

We assume a basic familiarity with propositional logic. For the sake of brevity and to keep our notation simple we fix Φ as the countable[1] set of atomic propositional variables. With a *type* we mean a set[2] τ, such that a nonnegative integer $\rho(i) \in \mathbb{N}$ is assigned to each $i \in \tau$, this integer is called the *arity* of i. To simplify our presentation we will often assume that different types are disjoint. This is technically easy to guarantee and we will motivate this assumption later.

[1]The set of propositional variables is usually assumed to be countable to prove completeness results by appealing to Lindenbaum's Lemma (See Blackburn *et al.* (2001)). This assumption is not technically required for our purposes, but it is not very restrictive either, so we merely assume it out of habit.

[2]It is common to assume that types are non-empty, see for example Blackburn *et al.* (2001). This assumption is not generally vital, and for technical reasons that will become apparent in Chapter 3 we will allow empty types.

Definition 1.2.1 (Blackburn *et al.* (2001))**.** Given a type τ, a τ-(*modal*) *formula* is a finite sequence of atoms and connectives $\perp$ (*falsum*); $\rightarrow$ (*implication*); $\Diamond_i$ (*diamond* i), for every $i \in \tau$, of one of the following forms (where $p \in \Phi$; $i \in \tau$; and $\phi, \psi, \phi_1, \phi_2, \ldots, \phi_{\rho(i)}$ are τ-formulae):

- $\perp$
- p
- $\phi \rightarrow \psi$
- $\Diamond_i \phi_1 \, \phi_2 \, \ldots \, \phi_{\rho(i)}$

We use the standard abbreviations denoted by $\top$ (*truth*), $\neg$ (*negation*), $\wedge$ (*conjunction*), $\vee$ (*disjunction*), $\leftrightarrow$ (*bi-implication*) and $\Box_i$ (*box* i):

$$\begin{aligned}
\neg\phi &:= \phi \rightarrow \perp \\
\top &:= \neg\perp \\
\phi \wedge \psi &:= \neg(\phi \rightarrow \neg\psi) \\
\phi \vee \psi &:= \neg\phi \rightarrow \psi \\
\phi \leftrightarrow \psi &:= (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi) \\
\Box_i \phi_1 \, \phi_2 \, \ldots \, \phi_{\rho(i)} &:= \neg\left(\Diamond_i \neg\phi_1 \, \neg\phi_2 \, \ldots \, \neg\phi_{\rho(i)}\right)
\end{aligned}$$

Notation 1.2.2. Our language, the set of all τ-formulae, is denoted by ML_τ (our mnemonic is ***M**odal **L**anguage of type* τ.) Operator binding is treated in the standard way, and if needed we will use round brackets to clarify the order of evaluation. We denote the set of all *positive existential* τ-(*modal*) *formulae*, i.e. those formulae built using only members of Φ, and the connectives $\wedge$, $\vee$ and $\Diamond_i$ for $i \in \tau$, by PE_τ.

Definition 1.2.3 (Blackburn *et al.* (2001))**.** Given a type τ, we call a pair

$$\mathfrak{F} = \left\langle F, (R_i)_{i \in \tau} \right\rangle$$

a (*Kripke*) τ-*frame* if F is a set and $R_i \subseteq F^{\rho(i)+1}$ (for each $i \in \tau$). We call F the *universe* of $\mathfrak{F}$, the elements of F are called the *points* of $\mathfrak{F}$, and every R_i is called an (*accessibility*) *relation* of $\mathfrak{F}$. The *cardinality* of $\mathfrak{F}$ is the cardinality of F. Correspondingly a (*Kripke*) τ-*model* over $\mathfrak{F}$ is a pair $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ where the function $V : \Phi \rightarrow 2^F$ is called a *valuation*. The *universe* of $\mathfrak{M}$ is defined as the *universe* of $\mathfrak{F}$. Similary, for the *points*, (*accessibility*) *relations* and *cardinality* of $\mathfrak{M}$.

Remark 1.2.4. For the sake of brevity we will often refer to a *frame* (resp. *model*) rather than a τ-*frame* (resp. τ-*model*) whenever the type is obvious or unimportant.

Notation 1.2.5. For an n-ary relation R we use the abbreviation $Rv_0v_1 \ldots v_{n-1}$ to mean $\langle v_0, v_1, \ldots, v_{n-1}\rangle \in R$. In the case where R is binary we may write v_0Rv_1 to abbreviate $\langle v_0, v_1\rangle \in R$ instead.

Terminology 1.2.6. For a binary relation $R \subseteq F^2$ and some $v_0, v_1 \in F$ such that v_0Rv_1, we will call v_1 an *R-successor of* v_0.

The following definition formalizes how we interpret the language ML_τ inside τ-frames and τ-models.

Definition 1.2.7 (Blackburn *et al.* (2001))**.** For a τ-frame $\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau}\right\rangle$ and a Kripke τ-model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ with some $v \in F$ we define the notion of a τ-formula being *valid* (denoted by $\Vdash$) inductively as follows (for all $p \in \Phi$, all $\phi, \psi, \phi_1, \phi_2, \ldots, \phi_{\rho(i)} \in \mathsf{ML}_\tau$, and all $i \in \tau$):

- $\mathfrak{M}, v \Vdash \bot$ never holds
- $\mathfrak{M}, v \Vdash p$ if $v \in V(p)$
- $\mathfrak{M}, v \Vdash \phi \to \psi$ if $\mathfrak{M}, v \Vdash \phi$ implies $\mathfrak{M}, v \Vdash \psi$
- $\mathfrak{M}, v \Vdash \Diamond_i\phi_1\phi_2 \ldots \phi_{\rho(i)}$ if there are $v_1, v_2, \ldots, v_{\rho(i)} \in F$ such that

$$R_ivv_1v_2 \ldots v_{\rho(i)} \quad \text{and} \quad \mathfrak{M}, v_1 \Vdash \phi_1 \ ; \ \mathfrak{M}, v_2 \Vdash \phi_2 \ ; \ \ldots \ ; \ \mathfrak{M}, v_{\rho(i)} \Vdash \phi_{\rho(i)}$$

Stronger notions of validity, based on this, are now defined as follows (for all $\phi \in \mathsf{ML}_\tau$):

- $\mathfrak{M} \Vdash \phi$ if for every $w \in F$ we have $\mathfrak{M}, w \Vdash \phi$
- $\mathfrak{F}, v \Vdash \phi$ if for every τ-model $\mathfrak{N}$ over $\mathfrak{F}$ we have $\mathfrak{N}, v \Vdash \phi$
- $\mathfrak{F} \Vdash \phi$ if for every τ-model $\mathfrak{N}$ over $\mathfrak{F}$ we have $\mathfrak{N} \Vdash \phi$

Given a set of τ-formulae $\mathbf{L}$, then $\mathfrak{F} \Vdash \mathbf{L}$ if for all $\varphi \in \mathbf{L}$ we have that $\mathfrak{F} \Vdash \varphi$.

For the sake of thoroughness we also state when a formula of the form $\Box_i\phi_1\phi_2 \ldots \phi_{\rho(i)}$ is valid at a point v in a τ-model $\mathfrak{M}$:

> It holds that $\mathfrak{M}, v \Vdash \Box_i\phi_1\phi_2 \ldots \phi_{\rho(i)}$ iff for all $v_1, v_2, \ldots, v_{\rho(i)} \in F$ we have that $R_ivv_1v_2 \ldots v_{\rho(i)}$ implies that $\mathfrak{M}, v_k \Vdash \phi_k$ for some k.

This condition is easily verified from the definition of the abbreviation $\Box_i$ together with Definition 1.2.7.

We can also reason about formulae without using frames, the sets of formulae that are closed under such "reasoning" are called logics.

Definition 1.2.8 (Blackburn *et al.* (2001))**.** A τ-(*modal*) *logic* is a set of τ-formulae $\mathbf{L}$ that satisfies the following properties

- **L** contains all propositional tautologies.
- If $\phi \in \mathbf{L}$ and $\phi \to \psi \in \mathbf{L}$ then $\psi \in \mathbf{L}$.
- If $\phi \in \mathbf{L}$ and ψ is obtained from ϕ by uniformly replacing propositional variables in ϕ with arbitrary τ-formulae, then $\psi \in \mathbf{L}$.

We say that **L** is *normal* if it also has the following properties

- $$\begin{array}{l} \Diamond_i \perp p_2 \ldots p_{\rho(i)} \leftrightarrow \perp \in \mathbf{L} \\ \Diamond_i\, p_1 \perp \ldots p_{\rho(i)} \leftrightarrow \perp \in \mathbf{L} \\ \vdots \\ \Diamond_i\, p_1\, p_2 \ldots \perp \leftrightarrow \perp \in \mathbf{L} \end{array}$$
- $$\Diamond_i p_1 \ldots p_n \vee q_n \ldots p_{\rho(i)} \leftrightarrow \Diamond_i p_1 \ldots p_n \ldots p_{\rho(i)} \vee \Diamond_i p_1 \ldots q_n \ldots p_{\rho(i)} \in \mathbf{L}$$
- If $p_1 \to q_1,\ p_2 \to q_2,\ \ldots,\ p_{\rho(i)} \to q_{\rho(i)} \in \mathbf{L}$ then $\Diamond_i p_1\, p_2 \ldots p_{\rho(i)} \to \Diamond_i q_1\, q_2 \ldots q_{\rho(i)} \in \mathbf{L}$.

We denote the smallest normal τ-logic by $\mathbf{K}_\tau$.

Remark 1.2.9. Although the notation $\mathbf{K}_\tau$ is fairly standard (see Blackburn *et al.* (2001), Kurucz *et al.* (2003), Goldblatt (2003)) it may seem arbitrary to readers unfamiliar with the field. The letter "K" is chosen in honour of Saul Kripke (Goldblatt (2003)).

Notation 1.2.10. For a type τ and a class F of τ-frames we denote the *logic of* F by

$$\mathbf{Log}\,(\mathsf{F}) := \{\phi \in \mathsf{ML}_\tau |\text{ for every } \mathfrak{F} \in \mathsf{F} \text{ we have that } \mathfrak{F} \Vdash \phi\}$$

It can be verified that $\mathbf{Log}\,(\mathsf{F})$ is indeed a normal τ-logic (see Kurucz *et al.* (2003)).

Definition 1.2.11 (Kurucz *et al.* (2003))**.** For a type τ, we say a normal τ-logic **L** is *Kripke complete* if there is a class of τ-frames F such that $\mathbf{Log}\,(\mathsf{F}) = \mathbf{L}$.

Remark 1.2.12. Some texts use different terminology to capture the notion of Kripke complete and say **L** is both *sound* and *complete* with respect to F (see Blackburn *et al.* (2001)). One important consequence of Kripke completeness is that frames can be used to reason about logics, and vice versa. The logic $\mathbf{K}_\tau$ is an example of a Kripke complete normal τ-logic since it can be shown that if F is the class of all τ-frames then $\mathbf{Log}\,(\mathsf{F}) = \mathbf{K}_\tau$ (see Blackburn *et al.* (2001)).

Given the validity conditions of Definition 1.2.7 we may ask how much of the structure of frames or models need to be preserved to preserve validity of formulae. This leads to the notions of homomorphism, strong homomorphism, isomorphism, simulation, bounded morphism and bisimulation.

Definition 1.2.13 (Blackburn *et al.* (2001))**.** Given τ-frames

$$\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle \quad \text{and} \quad \mathfrak{G} = \left\langle G, (S_i)_{i\in\tau} \right\rangle;$$

two τ-models $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ and $\mathfrak{N} = \langle \mathfrak{G}, U \rangle$; and a function $f : F \to G$.

- We say f is a (τ*-frame*) *homomorphism from* $\mathfrak{F}$ *to* $\mathfrak{G}$ if for every $v_0, v_1, \ldots, v_{\rho(i)} \in F$ we have that

$$R_i v_0 v_1 \ldots v_{\rho(i)} \quad \text{implies} \quad S_i f(v_0) f(v_1) \ldots f\left(v_{\rho(i)}\right).$$

- We say f is a (τ*-model*) *homomorphism from* $\mathfrak{M}$ *to* $\mathfrak{N}$ if f is a homomorphism from $\mathfrak{F}$ to $\mathfrak{G}$ and for all $v \in F$ and $p \in \Phi$ we have

$$\mathfrak{M}, v \Vdash p \quad \text{implies} \quad \mathfrak{N}, f(v) \Vdash p.$$

- We say f is a *strong* (τ*-frame*) *homomorphism from* $\mathfrak{F}$ *to* $\mathfrak{G}$, if for all $v_0, v_1, \ldots, v_{\rho(i)} \in F$

$$R_i v_0 v_1 \ldots v_{\rho(i)} \quad \text{if and only if} \quad S_i f(v_0) f(v_1) \ldots f\left(v_{\rho(i)}\right).$$

- We say f is a *strong* (τ*-model*) *homomorphism from* $\mathfrak{M}$ *to* $\mathfrak{N}$ if f is a strong homomorphism from $\mathfrak{F}$ to $\mathfrak{G}$ and for all $v \in F$ and $p \in \Phi$ we have

$$\mathfrak{M}, v \Vdash p \quad \text{if and only if} \quad \mathfrak{N}, f(v) \Vdash p.$$

- If f is a bijective strong homomorphism from $\mathfrak{F}$ to $\mathfrak{G}$ then we say f is a (τ*-frame*) *isomorphism from* $\mathfrak{F}$ *to* $\mathfrak{G}$.

- If f is a bijective strong homomorphism from $\mathfrak{M}$ to $\mathfrak{N}$ then we say f is a (τ*-model*) *isomorphism from* $\mathfrak{M}$ *to* $\mathfrak{N}$.

The notion of a homomorphism can be further generalized to a structure-preserving relation in the following way.

Definition 1.2.14 (Blackburn *et al.* (2001))**.** Given two τ-frames

$$\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle \quad \text{and} \quad \mathfrak{G} = \left\langle G, (S_i)_{i\in\tau} \right\rangle,$$

a relation $Z \subseteq F \times G$ is called a (τ*-frame*) *simulation from* $\mathfrak{F}$ *to* $\mathfrak{G}$ if $R_i v_0 v_1 \ldots v_{\rho(i)}$ and $v_0 Z w_0$ implies that there are $w_1, w_2, \ldots, w_{\rho(i)} \in G$ such that $S_i w_0 w_1 \ldots w_{\rho(i)}$ and

$$\begin{gathered} v_1 Z w_1 \\ v_2 Z w_2 \\ \vdots \\ v_{\rho(i)} Z w_{\rho(i)}. \end{gathered}$$

We say $\langle \mathfrak{F}, v \rangle$ is *similar* to $\langle \mathfrak{G}, w \rangle$ if v and w are points of $\mathfrak{F}$ and $\mathfrak{G}$ respectively and there is some simulation Y from $\mathfrak{F}$ to $\mathfrak{G}$ such that vYw.

Given two τ-models

$$\mathfrak{M} = \langle \mathfrak{F}, V \rangle \quad \text{and} \quad \mathfrak{N} = \langle \mathfrak{G}, U \rangle$$

then Z is called a *(τ-model) simulation from $\mathfrak{M}$ to $\mathfrak{N}$* if Z is a simulation from $\mathfrak{F}$ to $\mathfrak{G}$ and for all points v and w of $\mathfrak{M}$ and $\mathfrak{N}$ respectively we have that vZw implies that

$$\mathfrak{M}, v \Vdash p \quad \text{implies} \quad \mathfrak{N}, w \Vdash p \quad (\text{for all } p \in \Phi).$$

We say $\langle \mathfrak{M}, v \rangle$ is *similar* to $\langle \mathfrak{N}, w \rangle$ if v and w are points of $\mathfrak{M}$ and $\mathfrak{N}$ respectively and there is some simulation Y from $\mathfrak{M}$ to $\mathfrak{N}$ such that vYw.

Remark 1.2.15. We may informally state the notion that $\langle \mathfrak{F}, v \rangle$ is similar to $\langle \mathfrak{G}, w \rangle$ as the condition that *for every transition in $\mathfrak{F}$ from v there is a corresponding transition in $\mathfrak{G}$ from w.* The transitions "correspond" in the sense that they must be made via relations with the same index, and end at points which are similar once again. The same can be said of similarity of a point in a model to a point in another model.

Note that in general similarity is not a symmetric relation.

The following proposition motivates why simulations are of interest in the study of modal logic.

Proposition 1.2.16 (Blackburn *et al.* (2001))**.** *Given two τ-models*

$$\mathfrak{M} = \langle \mathfrak{F}, V \rangle \quad \textit{and} \quad \mathfrak{N} = \langle \mathfrak{G}, U \rangle$$

and two points v and w of $\mathfrak{M}$ and $\mathfrak{N}$ respectively such that $\langle \mathfrak{M}, v \rangle$ is similar to $\langle \mathfrak{N}, w \rangle$. Then

$$\mathfrak{M}, v \Vdash \phi \quad \textit{implies} \quad \mathfrak{N}, w \Vdash \phi \quad (\textit{for all } \phi \in \mathsf{PE}_\tau).$$

Remark 1.2.17. Proposition 1.2.16 can in fact be strengthened to characterise the modal formulae that are equivalent to positive existential formulae (see Blackburn *et al.* (2001)).

Corollary 1.2.18 (Blackburn *et al.* (2001))**.** *Given two τ-models*

$$\mathfrak{M} = \left\langle F, (R_i)_{i \in \tau}, V \right\rangle, \qquad \mathfrak{N} = \left\langle G, (S_i)_{i \in \tau}, U \right\rangle;$$

and a homomorphism f from $\mathfrak{M}$ to $\mathfrak{N}$. Then for every point v of $\mathfrak{M}$ we have

$$\mathfrak{M}, v \Vdash \phi \quad \textit{implies} \quad \mathfrak{N}, f(v) \Vdash \phi \quad (\textit{for all } \phi \in \mathsf{PE}_\tau)$$

.

We want a notion of morphism between frames or models, that yields preservation and reflection of ML_τ, but homomorphisms do not give this, as seen in the following examples.

Example 1.2.19. Suppose that models $\mathfrak{M} = \langle F, R, V\rangle$ and $\mathfrak{N} = \langle G, S, U\rangle$ are given by

$$\mathfrak{M}: \quad 0 \Vdash p \qquad\qquad \mathfrak{N}: \quad 1 \Vdash p \xrightarrow{S} 2 \Vdash p$$

For simplicity sake we let $\Phi = \{p\}$. Also consider the function f defined as follows.

$$\begin{aligned} f : F &\to G \\ 0 &\mapsto 1 \end{aligned}$$

Now it is easily seen that f is a homomorphism from $\mathfrak{M}$ to $\mathfrak{N}$. We also make three observations regarding Proposition 1.2.16 and Corollary 1.2.18.

(1) $\mathfrak{N}, f(0) \Vdash \Diamond p$ and $\mathfrak{M}, 0 \nVdash \Diamond p$. So model homomorphisms merely preserve positive existential formulae, and do not reflect them. Hence the one-way implication of Corollary 1.2.18 (and consequently also of Proposition 1.2.16) cannot be strengthened to equivalance.

(2) $\mathfrak{M}, 0 \Vdash \neg\Diamond p$ and $\mathfrak{N}, f(0) \nVdash \neg\Diamond p$, also $\neg\Diamond p \in \mathsf{ML}_\tau \setminus \mathsf{PE}_\tau$. So the preservation of formulae in Corollary 1.2.18 (and consequently also in Proposition 1.2.16) cannot be generalised to all formulae in ML_τ.

(3) Note that for $v \in F$ we have

$$\mathfrak{M}, v \Vdash p \text{ iff } \mathfrak{N}, f(v) \Vdash p$$

which is stronger than the requirement on atoms of Definition 1.2.13 for f to be a homomorphism, yet by (2) there is a formula that is not positive existential and that is not preserved by f. So it is insufficient to only strengthen the condition on atoms to obtain preservation and reflection of all formulae in ML_τ.

Example 1.2.20. For strong τ-model homomorphisms we can improve Corollary 1.2.18 to preservation and reflection of the entire ML_τ. Strong homomorphisms are very restrictive however. Note for example the following two models $\mathfrak{M} = \langle F, R, V\rangle$ and $\mathfrak{N} = \langle G, S, U\rangle$ given by

$$\begin{tikzcd} \mathfrak{M}: & 0 \Vdash p \arrow[r, "R"] & 1 \Vdash p \arrow[r, "R"] & 2 \Vdash p \arrow[r, "R"] & \cdots & \mathfrak{N}: & 0 \Vdash p \arrow[loop above, "S"] \end{tikzcd}$$

For simplicity sake we let $\Phi = \{p\}$. Now consider the function f defined as follows.

$$\begin{aligned} f : F &\to G \\ x &\mapsto 0 \quad (\text{for } x \in F) \end{aligned}$$

Although we will prove it later (see Corollary 1.2.29), it seems that f preserves and reflects ML_τ. Note also that f is not a strong homomorphism: it is the case that $f(0) = 0$, $0\ S\ 0$, and $f(2) = 0$ but not that $0\ R\ 2$. The question now becomes whether f belongs to some class of functions that can be easily described using the structure of $\mathfrak{M}$ and $\mathfrak{N}$; and that preserve and reflect ML_τ. We give such a class that f belongs to in Definition 1.2.21.

Note that the formulae examined in observations (1) and (2) from Example 1.2.19 are merely each other's negation. So although these observations point out different shortcomings of Proposition 1.2.16 and Corollary 1.2.18, it seems that they both identify the same issue namely that the point 0 of $\mathfrak{M}$ has no R-successors whereas the point $f(0)$ of $\mathfrak{N}$ has S-successors. But it is not merely good enough to add arbitrary R-successors for 0, we need R-successors for 0 that behave like the S-successors of $f(0)$. More precisely for every S-successors of $f(0)$ there must be an R-successors of 0 mapping to it via f. This is formalised in the following definition.

Definition 1.2.21 (Blackburn *et al.* (2001))**.** Given τ-frames

$$\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle \quad \text{and} \quad \mathfrak{G} = \left\langle G, (S_i)_{i\in\tau} \right\rangle .$$

Then a function $f : F \to G$ is called a *bounded* (τ*-frame*) *morphism from* $\mathfrak{F}$ *to* $\mathfrak{G}$ if the following two conditions hold

- **forward**: $R_i v_0 v_1 \dots v_{\rho(i)}$ implies $S_i f(v_0) f(v_1) \dots f\left(v_{\rho(i)}\right)$.
- **back**: $S_i f(v_0) w_1 w_2 \dots w_{\rho(i)}$ implies that there are $v_1, v_2, \dots, v_{\rho(i)} \in F$ such that $R_i v_0 v_1 \dots v_{\rho(i)}$ and

$$\begin{aligned} f(v_1) &= w_1 \\ f(v_2) &= w_2 \\ &\vdots \\ f\left(v_{\rho(i)}\right) &= w_{\rho(i)}. \end{aligned}$$

Given models two τ-models $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ and $\mathfrak{N} = \langle \mathfrak{G}, U \rangle$. We say f is a *bounded* (τ*-model*) *morphism from* $\mathfrak{M}$ *to* $\mathfrak{N}$ if f is a bounded morphism from $\mathfrak{F}$ to $\mathfrak{G}$ and for all $v \in F$ and $p \in \Phi$ we have

$$\mathfrak{M}, v \Vdash p \quad \text{if and only if} \quad \mathfrak{N}, f(v) \Vdash p$$

.

Remark 1.2.22. Although bounded morphisms were initially referred to as *pseudo epimorphisms* (Segerberg (1968)) and later *p-morphisms* (Segerberg (1971)), we use the name credited to Goldblatt (1989).

Remark 1.2.23. Just as homomorphisms were generalised to simulations, we can generalise bounded morphisms to *bisimulations*. Before we define them however it should be noted that our definition does not completely agree with Blackburn *et al.* (2001), in that we do not require that bisimulations should be non-empty. The reasons for this will become clearer in Chapter 3, particularly in the case where the only bisimulation between two frames is the empty one.

Definition 1.2.24 (Blackburn *et al.* (2001))**.** Given two τ-frames

$$\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle \quad \text{and} \quad \mathfrak{G} = \left\langle G, (S_i)_{i\in\tau} \right\rangle,$$

a relation $Z \subseteq F \times G$ is called a *(τ-frame) bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$* if the following two conditions hold

- **forward**: If $R_i v_0 v_1 \ldots v_{\rho(i)}$ and $v_0 Z w_0$ then we have $w_1, w_2, \ldots, w_{\rho(i)} \in G$ such that $S_i w_0 w_1 \ldots w_{\rho(i)}$ and

$$\begin{gathered} v_1 Z w_1 \\ v_2 Z w_2 \\ \vdots \\ v_{\rho(i)} Z w_{\rho(i)}. \end{gathered}$$

- **back**: If $S_i w_0 w_1 \ldots w_{\rho(i)}$ and $v_0 Z w_0$ then we have $v_1, v_2, \ldots, v_{\rho(i)} \in F$ such that $R_i v_0 v_1 \ldots v_{\rho(i)}$ and

$$\begin{gathered} v_1 Z w_1 \\ v_2 Z w_2 \\ \vdots \\ v_{\rho(i)} Z w_{\rho(i)}. \end{gathered}$$

We say $\langle \mathfrak{F}, v\rangle$ and $\langle \mathfrak{G}, w\rangle$ are *bisimilar* if v and w are points of $\mathfrak{F}$ and $\mathfrak{G}$ respectively and there is some bisimulation Y between $\mathfrak{F}$ and $\mathfrak{G}$ such that vYw, we denote this by $\langle \mathfrak{F}, v\rangle \sim \langle \mathfrak{G}, w\rangle$.

Given two τ-models

$$\mathfrak{M} = \langle \mathfrak{F}, V\rangle \quad \text{and} \quad \mathfrak{N} = \langle \mathfrak{G}, U\rangle$$

then Z is called a *(τ-model) bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$* if Z is a bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$ and Z preserves and reflects truth of atoms, i.e. for all points v and w of $\mathfrak{M}$ and $\mathfrak{N}$ respectively we have that vZw implies that

$$\mathfrak{M}, v \Vdash p \text{ iff } \mathfrak{N}, w \Vdash p \quad (\text{for all } p \in \Phi)$$

.

We say $\langle \mathfrak{M}, v\rangle$ and $\langle \mathfrak{N}, w\rangle$ are *bisimilar* if v and w are points of $\mathfrak{M}$ and $\mathfrak{N}$ respectively and there is some bisimulation Y between $\mathfrak{M}$ and $\mathfrak{N}$ such that vYw, we denote this by $\langle \mathfrak{M}, v\rangle \sim \langle \mathfrak{N}, w\rangle$.

Remark 1.2.25. Observe that both the forward and back conditions of Definition 1.2.24 are reformulations of the condition in Definition 1.2.14. The forward condition of Definition 1.2.24 merely requires that Z be a simulation from $\mathfrak{F}$ to $\mathfrak{G}$, whereas the back condition of Definition 1.2.24 requires that Z^{op} should be a simulation from $\mathfrak{G}$ to $\mathfrak{F}$.

Remark 1.2.26. Observe that bisimulations indeed generalise bounded morphisms:

- For any bounded morphism f from a τ-frame $\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau}\right\rangle$ to a τ-frame $\mathfrak{G}$ the set $\{\langle v, f(v)\rangle | \, v \in F\}$ is a bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$.
- For any bounded morphism f from a τ-model $\mathfrak{M} = \left\langle F, (R_i)_{i\in\tau}, V\right\rangle$ to a τ-model $\mathfrak{N}$ the set $\{\langle v, f(v)\rangle | \, v \in F\}$ is a bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$.

Proposition 1.2.27 (Blackburn *et al.* (2001)). *Given two τ-models*

$$\mathfrak{M} = \left\langle F, (R_i)_{i\in\tau}, V\right\rangle \quad \textit{and} \quad \mathfrak{N} = \left\langle G, (S_i)_{i\in\tau}, U\right\rangle$$

and two points v and w of $\mathfrak{M}$ and $\mathfrak{N}$ respectively such that $\langle\mathfrak{M}, v\rangle \sim \langle\mathfrak{N}, w\rangle$, then

$$\mathfrak{M}, v \Vdash \phi \textit{ iff } \mathfrak{N}, w \Vdash \phi \quad \textit{(for all } \phi \in \mathsf{ML}_\tau\textit{)}.$$

Proof. Suppose that $\langle\mathfrak{M}, v\rangle \sim \langle\mathfrak{N}, w\rangle$ as stated, then there is a bisimulation Z between $\mathfrak{M}$ and $\mathfrak{N}$ such that vZw. Let $\phi \in \mathsf{ML}_\tau$. The result is now proved using structural induction on ϕ.

By Definition 1.2.24 we have that $\langle\mathfrak{M}, v\rangle$ and $\langle\mathfrak{N}, w\rangle$ agree on the truth of atoms. A simple appeal to Definition 1.2.7 and the induction hypothesis shows that the propositional connectives $(\bot, \to)$ are preserved and reflected.

The remaining case is when ϕ is of the form $\Diamond_i\phi_1\phi_2\ldots\phi_{\rho(i)}$ for some $i \in \tau$ and some $\phi_1, \phi_2, \ldots, \phi_{\rho(i)} \in \mathsf{ML}_\tau$, so suppose this is true.

To prove the left to right implication suppose that $\mathfrak{M}, v \Vdash \Diamond_i\phi_1\phi_2\ldots\phi_{\rho(i)}$. By Definition 1.2.7 we have $v_1, v_2, \ldots, v_{\rho(i)} \in F$ such that $R_ivv_1v_2\ldots v_{\rho(i)}$ and

$$\mathfrak{M}, v_1 \Vdash \phi_1 \,;\, \mathfrak{M}, v_2 \Vdash \phi_2 \,;\, \ldots \,;\, \mathfrak{M}, v_{\rho(i)} \Vdash \phi_{\rho(i)}.$$

By the forward condition of Definition 1.2.24 there are $w_1, w_2, \ldots, w_{\rho(i)} \in G$ such that $S_iww_1w_2\ldots w_{\rho(i)}$ and

$$\begin{array}{c} v_1Zw_1 \\ v_2Zw_2 \\ \vdots \\ v_{\rho(i)}Zw_{\rho(i)}. \end{array}$$

Observe that $\phi_1, \phi_2, \ldots, \phi_{\rho(i)}$ have smaller length than ϕ so the induction hypothesis can be applied to $\phi_1, \phi_2, \ldots, \phi_{\rho(i)}$, to give

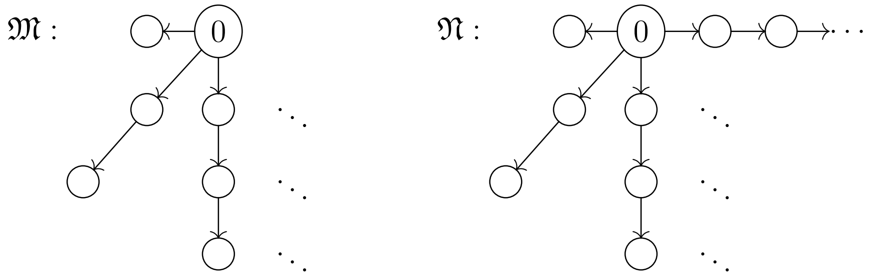
$$\mathfrak{N}, w_1 \Vdash \phi_1 \,;\, \mathfrak{N}, w_2 \Vdash \phi_2 \,;\, \ldots \,;\, \mathfrak{N}, w_{\rho(i)} \Vdash \phi_{\rho(i)}.$$

We already chose $w_1 w_2 \ldots w_{\rho(i)}$ such that $S_i w w_1 w_2 \ldots w_{\rho(i)}$, hence by Definition 1.2.7 we have $\mathfrak{N}, w \Vdash \Diamond_i \phi_1 \phi_2 \ldots \phi_{\rho(i)}$.

The converse can be proven similarly by appealing to the back condition of Definition 1.2.24 instead. This completes the proof. $\square$

Although Proposition 1.2.27 states that bisimilarity of two points in two models implies that those points satisfy the same modal formulae, the converse is not true as we see in the following example from Blackburn *et al.* (2001).

Example 1.2.28. Consider the followong two models, where for the sake of simplicity we assume $\Phi = \{p\}$, and that p is valid at every point of $\mathfrak{M}$ and at every point of $\mathfrak{N}$.



Now point 0 of $\mathfrak{M}$ and point 0 of $\mathfrak{N}$ satisfy the same modal formulae, but they are not bisimilar since 0 in $\mathfrak{M}$ has no infinite chain of successors.

Corollary 1.2.29 (Blackburn *et al.* (2001))**.** *Let two τ-models*

$$\mathfrak{M} = \left\langle F, (R_i)_{i\in\tau}, V \right\rangle \quad \textit{and} \quad \mathfrak{N} = \left\langle G, (S_i)_{i\in\tau}, U \right\rangle$$

be given. Suppose that we also have a bounded morphism f from $\mathfrak{M}$ to $\mathfrak{N}$ and a τ-formula ϕ. Then the following hold.

(1) *For every point v of $\mathfrak{M}$ we have $\mathfrak{M}, v \Vdash \phi$ if and only if $\mathfrak{N}, f(v) \Vdash \phi$.*

(2) *If $\mathfrak{N} \Vdash \phi$ then $\mathfrak{M} \Vdash \phi$.*

(3) *If f is also surjective then $\mathfrak{M} \Vdash \phi$ implies $\mathfrak{N} \Vdash \phi$.*

Proof. (1) This is a direct consequence of Proposition 1.2.27 since τ-model bisimulations generalise bounded τ-model morphisms.

(2) Let v be a point of $\mathfrak{M}$ and suppose that $\mathfrak{N} \Vdash \phi$. Then by Definition 1.2.7 we have $\mathfrak{N}, f(v) \Vdash \phi$, and as a consequence of part (1) also that $\mathfrak{M}, v \Vdash \phi$. But v was an arbitrary point of $\mathfrak{M}$ so by Definition 1.2.7 it follows that $\mathfrak{M} \Vdash \phi$.

(3) Suppose that f is a surjective bounded morphism from $\mathfrak{M}$ to $\mathfrak{N}$ and that $\mathfrak{M} \Vdash \phi$. Now let w be a point of $\mathfrak{N}$. Since f is surjective there is a point v of $\mathfrak{M}$ such that $f(v) = w$. By the assumption that $\mathfrak{M} \Vdash \phi$ together

with Definition 1.2.7 we have that $\mathfrak{M}, v \Vdash \phi$. Then by part (1) we have that $\mathfrak{N}, f(v) \Vdash \phi$, or rather $\mathfrak{N}, w \Vdash \phi$. Since w was an arbitary point of $\mathfrak{N}$, this holds for every point of $\mathfrak{N}$. Now the result follows.

□

Corollary 1.2.30 (Blackburn *et al.* (2001))**.** *Let two τ-frames*

$$\mathfrak{F} = \langle F, (R_i)_{i\in\tau} \rangle \quad \textit{and} \quad \mathfrak{G} = \langle G, (S_i)_{i\in\tau} \rangle$$

be given. Suppose also that we have a bounded morphism f from $\mathfrak{F}$ to $\mathfrak{G}$, and a τ-formula ϕ. Then the following hold.

(1) *For every point v of $\mathfrak{F}$ we have that $\mathfrak{F}, v \Vdash \phi$ implies $\mathfrak{G}, f(v) \Vdash \phi$.*

(2) *If f is surjective then we have that $\mathfrak{F} \Vdash \phi$ implies $\mathfrak{G} \Vdash \phi$.*

(3) *If f is injective then for every point v of $\mathfrak{F}$ we have that $\mathfrak{G}, f(v) \Vdash \phi$ implies $\mathfrak{F}, v \Vdash \phi$. Consequently then we also have that $\mathfrak{G} \Vdash \phi$ implies $\mathfrak{F} \Vdash \phi$.*

Proof. (1) Suppose that f is a bounded morphism from $\mathfrak{F}$ to $\mathfrak{G}$. To prove the contrapositive suppose that v is a point of $\mathfrak{F}$ such that $\mathfrak{G}, f(v) \nVdash \phi$. Now by Definition 1.2.7 there is a valuation $U : \Phi \to 2^G$ such that $\langle \mathfrak{G}, U \rangle, f(v) \nVdash \phi$. Define a valuation V by

$$\begin{aligned} V : \Phi &\to 2^F \\ p &\mapsto \{v_1 \in F | \, f(v_1) \in U(p)\} \end{aligned}$$

Now observe that for every $p \in \Phi$ we have $v \in V(p)$ if and only if $f(v) \in U(p)$, or rather

$$\langle \mathfrak{F}, V \rangle, v \Vdash p \quad \text{if and only if} \quad \langle \mathfrak{G}, U \rangle, f(v) \Vdash p.$$

Hence f is a bounded morphism from $\langle \mathfrak{F}, V \rangle$ to $\langle \mathfrak{G}, U \rangle$. Corollary 1.2.29 together with the choice of v imply that $\langle \mathfrak{F}, V \rangle, v \nVdash \phi$. Now by Definition 1.2.7 we have that $\mathfrak{F}, v \nVdash \phi$. This proves the claim.

(2) Suppose that f is a surjective bounded morphism from $\mathfrak{F}$ to $\mathfrak{G}$. To prove the contrapositive suppose that $\mathfrak{G} \nVdash \phi$. By Definition 1.2.7 there is a point w of $\mathfrak{G}$ such that $\mathfrak{G}, w \nVdash \phi$, and since f is surjective we can find a point v of $\mathfrak{F}$ such that $f(v) = w$. Now by part (1) we have that $\mathfrak{F}, v \nVdash \phi$, and hence by Definition 1.2.7 also that $\mathfrak{F} \nVdash \phi$ as needed.

(3) Suppose that f is an injective bounded morphism from $\mathfrak{F}$ to $\mathfrak{G}$. To prove the contrapositive suppose that v is a point of $\mathfrak{F}$ such that $\mathfrak{F}, v \nVdash \phi$. By

Definition 1.2.7 there is a valuation $V : \Phi \to 2^F$ such that $\langle \mathfrak{F}, V \rangle, v \not\Vdash \phi$. Now define a valuation U as follows.

$$\begin{aligned} U : \Phi &\to 2^G \\ p &\mapsto \{ f(v_1) | \, v_1 \in V(p) \} \end{aligned}$$

Now we claim that U is such that for every $p \in \Phi$ we have

$$\langle \mathfrak{F}, V \rangle, v \Vdash p \quad \text{if and only if} \quad \langle \mathfrak{G}, U \rangle, f(v) \Vdash p.$$

Using the definition of U and Definition 1.2.7 this claim is seen to be equivalent to the claim that

$$v \in V(p) \quad \text{if and only if} \quad f(v) \in \{ f(v_1) | \, v_1 \in V(p) \}.$$

The left to right implication is trivial, and the right to left implication follows from the assumption that f is injective. This shows that f is a bounded morphism from $\langle \mathfrak{F}, V \rangle$ to $\langle \mathfrak{G}, U \rangle$, so by Corollary 1.2.29 and the choice of v we have that $\langle \mathfrak{G}, U \rangle, f(v) \not\Vdash \phi$, and in turn $\mathfrak{G}, f(v) \not\Vdash \phi$ by Definition 1.2.7. This proves the first part of the result. To complete the proof we note that the point v indicated above, will always exist if $\mathfrak{F} \not\Vdash \phi$, and that by the above argument together with Definition 1.2.7, the existence of this point is sufficient to show that $\mathfrak{G} \not\Vdash \phi$.

□

Remark 1.2.31. According to the previous result, bounded τ-frame morphisms that are both surjective and injective have particularly powerful preservation and reflection. The next result shows that this is structural overkill, since such bounded morphisms are in fact isomorphisms.

Lemma 1.2.32 (Blackburn *et al.* (2001))**.**

(1) *Bijective bounded τ-frame morphisms are precisely τ-frame isomorphisms.*

(2) *Bijective bounded τ-model morphisms are precisely τ-model isomorphisms.*

Proof. (1) Given two τ-frames

$$\mathfrak{F} = \left\langle F, (R_i)_{i \in \tau} \right\rangle \quad \text{and} \quad \mathfrak{G} = \left\langle G, (S_i)_{i \in \tau} \right\rangle,$$

suppose f is a bijective bounded morphism from $\mathfrak{F}$ to $\mathfrak{G}$. We need to show that f is a strong τ-frame homomorphism. The forward condition of Definition 1.2.21 already gives that f is a τ-frame homomorphism, so we need to show that

$$S_i f(v_0) f(v_1) \ldots f\left(v_{\rho(i)}\right) \quad \text{implies} \quad R_i v_0 v_1 \ldots v_{\rho(i)}.$$

So suppose that for some $v_0, v_1, \ldots, v_{\rho(i)} \in F$ and $i \in \tau$ we have

$$S_i f(v_0) f(v_1) \ldots f\left(v_{\rho(i)}\right)$$

then the back condition of Definition 1.2.21 gives $w_1, w_2, \ldots, w_{\rho(i)} \in F$ such that $R_i v_0 w_1 w_2 \ldots w_{\rho(i)}$ and

$$\begin{array}{rcl} f(w_1) & = & f(v_1) \\ f(w_2) & = & f(v_2) \\ & \vdots & \\ f\left(w_{\rho(i)}\right) & = & f\left(v_{\rho(i)}\right). \end{array}$$

However the assumed injectivity of f gives that

$$\begin{array}{rcl} w_1 & = & v_1 \\ w_2 & = & v_2 \\ & \vdots & \\ w_{\rho(i)} & = & v_{\rho(i)}, \end{array}$$

so that $R_i v_0 v_1 \ldots v_{\rho(i)}$ as required. Hence f is an τ-frame isomorphism.

For the converse suppose f is an isomorphism from $\mathfrak{F}$ to $\mathfrak{G}$. The forward condition of Definition 1.2.21 is already satisfied, so it suffices to show that f satisfies the back condition of Definition 1.2.21 as well. So suppose that for some $v_0 \in F$, $w_1, w_2, \ldots, w_{\rho(i)} \in G$ and $i \in \tau$ it holds that

$$S_i f(v_0) w_1 w_2 \ldots w_{\rho(i)}$$

Then the assumed surjectivity of f gives some $v_1, v_2, \ldots, v_{\rho(i)} \in F$ such that

$$\begin{array}{rcl} f(v_1) & = & w_1 \\ f(v_2) & = & w_2 \\ & \vdots & \\ f\left(v_{\rho(i)}\right) & = & w_{\rho(i)}. \end{array}$$

Hence, according to the choice of $w_1, w_2, \ldots, w_{\rho(i)}$, we have

$$S_i f(v_0) f(v_1) \ldots f\left(v_{\rho(i)}\right).$$

Now the assumption that f is a strong τ-frame homomorphism implies

$$R_i v_0 v_1 \ldots v_{\rho(i)},$$

so that $v_1, v_2, \ldots, v_{\rho(i)}$ are as required for the back condition of Definition 1.2.21 to hold.

(2) The proof for bounded τ-model morphisms is similar. □

To prove our later results we need some basic properties of bisimulations.

Lemma 1.2.33 (Blackburn *et al.* (2001))**.** *Given three τ-frames*

$$\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle, \qquad \mathfrak{G} = \left\langle G, (S_i)_{i\in\tau} \right\rangle \quad \text{and} \quad \mathfrak{H} = \left\langle H, (T_i)_{i\in\tau} \right\rangle$$

the following hold:

(1) *If Z is a bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$ then its relational opposite,*

$$Z^{op} := \{ \langle w, v \rangle \mid vZw \}$$

is a bisimulation between $\mathfrak{G}$ and $\mathfrak{F}$.

(2) *If Z is a bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$, and Y is a bisimulation between $\mathfrak{G}$ and $\mathfrak{H}$, then their relational composite,*

$$Z \circ Y := \{ \langle v, w \rangle \mid \text{there is } x \in G \text{ such that } vZx \text{ and } xYw \}$$

is a bisimulation between $\mathfrak{F}$ and $\mathfrak{H}$.

(3) *The empty relation is a bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$.*

(4) *If $(Z_j)_{j\in I}$ is a family of bisimulations between $\mathfrak{F}$ and $\mathfrak{G}$ then $\bigcup_{j\in I} Z_j$ is a bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$.*

(5) *There is a maximum bisimulation, with regard to the subset inclusion order, between $\mathfrak{F}$ and $\mathfrak{G}$.*

Proof. (1) Observe that the forward condition of Definition 1.2.24 on Z is exactly the back condition for Z^{op}, and similarly the back condition on Z is exactly the forward condition for Z^{op}.

(2) To show that $Z\circ Y$ satisfies the forward condition of Definition 1.2.24 suppose that $R_i v_0 v_1 \ldots v_{\rho(i)}$ and $v_0 Z\circ Y w_0$. Then by definition of $Z\circ Y$ there is some $x_0 \in G$ such that $v_0 Z x_0$ and $x_0 Y w_0$. Since Z is a τ-frame bisimulation the forward condition of Definition 1.2.24 gives $x_1, x_2, \ldots, x_{\rho(i)} \in G$ such that $S_i x_0 x_1 \ldots x_{\rho(i)}$ and

$$\begin{gathered} v_1 Z x_1 \\ v_2 Z x_2 \\ \vdots \\ v_{\rho(i)} Z x_{\rho(i)}. \end{gathered}$$

Since Y is a τ-frame bisimulation, we appeal to the forward condition of Definition 1.2.24 to obtain $w_1, w_2, \ldots, w_{\rho(i)} \in H$ such that $T_i w_0 w_1 \ldots w_{\rho(i)}$ and

$$\begin{gathered} x_1 Y w_1 \\ x_2 Y w_2 \\ \vdots \\ x_{\rho(i)} Y w_{\rho(i)}. \end{gathered}$$

By definition of $Z \circ Y$ we now have

$$\begin{gathered} v_1 Z \circ Y w_1 \\ v_2 Z \circ Y w_2 \\ \vdots \\ v_{\rho(i)} Z \circ Y w_{\rho(i)}, \end{gathered}$$

so that $Z \circ Y$ satisfies the forward condition of Definition 1.2.24. A similar proof shows that $Z \circ Y$ also satisfies the back condition of Definition 1.2.24, and hence that $Z \circ Y$ is a τ-frame bisimulation as required.

(3) For the empty relation both the forward condition and the back condition of Definition 1.2.24 are vacuous truths.

(4) Suppose that $R_i v_0 v_1 \ldots v_{\rho(i)}$ and $v_0 \left(\bigcup_{j \in I} Z_j\right) w_0$, then there is $j_0 \in I$ such that $v_0 Z_{j_0} w_0$. Then since Z_{j_0} is a τ-frame bisimulation, the forward condition of Definition 1.2.24 gives $w_1, w_2, \ldots, w_{\rho(i)} \in G$ such that $S_i w_0 w_1 \ldots w_{\rho(i)}$ and

$$\begin{gathered} v_1 Z_{j_0} w_1 \\ v_2 Z_{j_0} w_2 \\ \vdots \\ v_{\rho(i)} Z_{j_0} w_{\rho(i)}. \end{gathered}$$

Hence we know that

$$\begin{gathered} v_1 \left(\bigcup_{j \in I} Z_j\right) w_1 \\ v_2 \left(\bigcup_{j \in I} Z_j\right) w_2 \\ \vdots \\ v_{\rho(i)} \left(\bigcup_{j \in I} Z_j\right) w_{\rho(i)}, \end{gathered}$$

which confirms that $\bigcup_{j \in I} Z_j$ satisfies the forward condition of Definition 1.2.24. A similar proof shows that $\bigcup_{j \in I} Z_j$ satisfies the back condition of Definition 1.2.24 so that $\bigcup_{j \in I} Z_j$ is a τ-frame bisimulation as required.

(5) By part (3) we know that there is at least one bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$. According to part (4) we know that the union of all the bisimulations between $\mathfrak{F}$ and $\mathfrak{G}$ is a bisimulation, this is clearly the maximum.

□

Notation 1.2.34. Now that we know that for any two τ-frames $\mathfrak{F}$ and $\mathfrak{G}$ there is a maximum bisimulation between them we denote this maximum bisimulation by $\sim_{\mathfrak{F},\mathfrak{G}}$.

Corollary 1.2.35 (Blackburn *et al.* (2001)). *Bisimilarity of points in frames is an equivalence relation.*

Proof. To verify that bisimilarity is reflexive observe that for any τ-frame the identity function on its universe is a an isomorphism. That bisimilarity is symmetric follows from part (1) of Lemma 1.2.33, and that bisimilarity is transitive follows from part (2) of Lemma 1.2.33. □

Remark 1.2.36. It is worth noting that counterparts to Lemma 1.2.33 and Corollary 1.2.35 for τ-model bisimulations, can also be proved. We omit these since we will be focussing mainly on frames later on, but this means that a lot of our later work can be adapted for models too.

Proposition 1.2.37 (Aczel & Mendler (1989)). *A relation $Z \subseteq F \times G$ is a bisimulation between two τ-frames $\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle$ and $\mathfrak{G} = \left\langle G, (S_i)_{i\in\tau} \right\rangle$, if and only if there is a τ-frame $\mathfrak{Z} := \left\langle Z, (T_i)_{i\in\tau} \right\rangle$ such that the projection functions $p_1 : Z \to F$ and $p_2 : Z \to G$ are bounded morphisms from $\mathfrak{Z}$ to $\mathfrak{F}$ and $\mathfrak{Z}$ to $\mathfrak{G}$ respectively.*

Proof. Suppose that $Z \subseteq F \times G$ is a bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$. Define the τ-frame $\mathfrak{Z} := \left\langle Z, (T_i)_{i\in\tau} \right\rangle$, with T_i (for every $i \in \tau$) being such that for all $\langle v_0, w_0\rangle \langle v_1, w_1\rangle \dots \left\langle v_{\rho(i)}, w_{\rho(i)}\right\rangle \in Z$

$$T_i \langle v_0, w_0\rangle \langle v_1, w_1\rangle \dots \left\langle v_{\rho(i)}, w_{\rho(i)}\right\rangle \text{ iff } R_i v_0 v_1 \dots v_{\rho(i)} \text{ and } S_i w_0 w_1 \dots w_{\rho(i)}.$$

We show that the projection function $p_1 : Z \to F$ is a bounded morphism from $\mathfrak{Z}$ to $\mathfrak{F}$ as required. That p_1 satisfies the forward condition of Definition 1.2.21 is immediate from the definition of T_i. To show that p_1 satisfies the back condition of Definition 1.2.21 suppose that $R_i p_1 (\langle v_0, w_0\rangle) v_1 v_2 \dots v_{\rho(i)}$. Evaluating $p_1 (\langle v_0, w_0\rangle)$ gives $R_i v_0 v_1 \dots v_{\rho(i)}$, and since Z is the domain of p_1 we have $v_0 Z w_0$. Since Z is a bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$ the forward condition of Definition 1.2.24 gives $w_1, w_2, \dots, w_{\rho(i)} \in G$ such that $S_i w_0 w_1 \dots w_{\rho(i)}$ and

$$\begin{gathered} v_1 Z w_1 \\ v_2 Z w_2 \\ \vdots \\ v_{\rho(i)} Z w_{\rho(i)}. \end{gathered}$$

From these latter memberships together with the definition of p_1 we have that

$$\begin{aligned} p_1(\langle v_1, w_1\rangle) &= v_1 \\ p_1(\langle v_2, w_2\rangle) &= v_2 \\ &\vdots \\ p_1\left(\left\langle v_{\rho(i)}, w_{\rho(i)}\right\rangle\right) &= v_{\rho(i)}. \end{aligned}$$

We have also shown that $R_i v_0 v_1 \ldots v_{\rho(i)}$ and $S_i w_0 w_1 \ldots w_{\rho(i)}$, so it follows from the definition of T_i that $T_i \langle v_0, w_0\rangle \langle v_1, w_1\rangle \ldots \left\langle v_{\rho(i)}, w_{\rho(i)}\right\rangle$. We conclude that p_1 satisfies the back condition of Definition 1.2.21, and hence p_1 is shown to be a bounded τ-frame morphism. A similar proof shows that p_2 is a bounded morphism from $\mathfrak{Z}$ to $\mathfrak{G}$.

For the converse suppose that $\mathfrak{Z} := \left\langle Z, (T_i)_{i\in\tau}\right\rangle$ is given and that the set projections are bounded τ-frame morphisms as stated above. Then by Remark 1.2.26 it follows that $\{\langle\langle v,w\rangle, v\rangle | \, vZw\}$ and $\{\langle\langle v,w\rangle, w\rangle | \, vZw\}$ are bisimulations between $\mathfrak{Z}$ and $\mathfrak{F}$, and between $\mathfrak{Z}$ and $\mathfrak{G}$, respectively. It now follows from Lemma 1.2.33 that $\{\langle v, \langle v,w\rangle\rangle | \, vZw\} \circ \{\langle\langle v,w\rangle, w\rangle | \, vZw\}$ is a bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$. Observe that

$$\{\langle v, \langle v,w\rangle\rangle | \, vZw\} \circ \{\langle\langle v,w\rangle, w\rangle | \, vZw\} = \{\langle v,w\rangle | \, vZw\} = Z$$

so that the result follows. □

Given the truth-preserving relations and functions presented so far, we may also ask how we can construct new frames out of old frames without spoiling validity of formulae. One such construction that we will use is the "disjoint union".

Definition 1.2.38 (Kurucz *et al.* (2003))**.** Given a family of frames

$$(\mathfrak{F}_j)_{j\in I}$$

$$\mathfrak{F}_j = \left\langle F_j, (R_{i,j})_{i\in\tau}\right\rangle$$

with pairwise disjoint universes, we define their *disjoint union* as the frame

$$\bigoplus_{j\in I} \mathfrak{F}_j := \left\langle \bigcup_{j\in I} F_j, (R_i)_{i\in\tau} \right\rangle,$$

with $R_i := \bigcup_{j\in I} R_{i,j}$ (for $i \in \tau$).

We can also define the disjoint union of a family of models, but we will not be needing that construction. The assumption that the universes of the frames are pairwise disjoint is only made for simplicity of our presentation. The disjoint union of a family of frames for which this does not hold can be constructed by creating new frames isomorphic to the given frames that do have pairwise disjoint universes. One way to accomplish this is by indexing the elements for each of the frames; the relations can then be redefined according to these disjoint universes to construct the disjoint union as above. The following example demonstrates the idea in this case.

Example 1.2.39. Consider the frames $\mathfrak{F}_1 = \langle F_1, R_{1,1} \rangle$ and $\mathfrak{F}_2 = \langle F_2, R_{1,2} \rangle$, given by

$$\mathfrak{F}_1 : \begin{tikzcd} 0 \arrow[r, "R_{1,1}"] & 1 \arrow[r, "R_{1,1}"] & 2 \end{tikzcd} \qquad \mathfrak{F}_2 : \begin{tikzcd} 0 \arrow[loop above, "R_{1,2}"] \end{tikzcd}$$

The disjoint union $\mathfrak{F}_1 \oplus \mathfrak{F}_2$ is given by

$$\mathfrak{F}_1 \oplus \mathfrak{F}_2 : \begin{tikzcd} \langle 0,1 \rangle \arrow[r, "R_1"] & \langle 1,1 \rangle \arrow[r, "R_1"] & \langle 2,1 \rangle & \langle 0,2 \rangle \arrow[loop above, "R_1"] \end{tikzcd}$$

The following result motivates the use of disjoint unions.

Proposition 1.2.40 (Kurucz *et al.* (2003))**.** *Given a family of frames*

$$(\mathfrak{F}_j)_{j \in I}$$

$$\mathfrak{F}_j = \left\langle F_j, (R_{i,j})_{i \in \tau} \right\rangle$$

and a τ-formula ϕ such that for every $j \in I$ we have $\mathfrak{F}_j \Vdash \phi$, then it follows that $\bigoplus_{j \in I} \mathfrak{F}_j \Vdash \phi$.

1.3 The frame product

In this section we introduce the frame product. Traditionally it has been of interest for the sake of product logics, which we will also define here. However, since our interest is mainly in the structure of frame products and its usefulness for model theory, our treatment of product logics is very sparse and limited to this section. We mention and discuss some basic results on product logics, but for us this merely serves as a motivation to study product logics and in turn frame products.

Definition 1.3.1 (Shehtman (1978))**.** Suppose τ and σ are arbitrary (possibly different) types, and that a τ-frame and a σ-frame are given.

$$\mathfrak{F} = \left\langle F, (R_i)_{i \in \tau} \right\rangle, \qquad \mathfrak{G} = \left\langle G, (S_i)_{i \in \sigma} \right\rangle$$

The *frame product* of $\mathfrak{F}$ and $\mathfrak{G}$ is defined as the $\tau \uplus \sigma$-frame

$$\mathfrak{F} \otimes \mathfrak{G} := \left\langle F \times G, (R_i^{\leftrightarrow})_{i \in \tau}, \left(S_j^{\updownarrow}\right)_{j \in \sigma} \right\rangle,$$

with $F \times G$ denoting the cartesian product, and for every $i \in \tau$

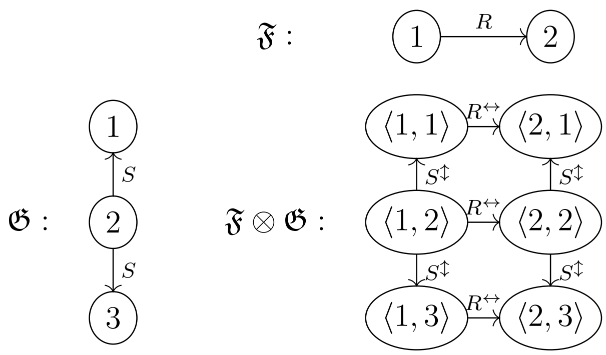
$$R_i^{\leftrightarrow} \langle v_0, w_0 \rangle \langle v_1, w_1 \rangle \ldots \langle v_{\rho(i)}, w_{\rho(i)} \rangle \text{ iff}$$
$$R_i v_0 v_1 \ldots v_{\rho(i)} \text{ and } w_0 = w_1 = \ldots = w_{\rho(i)}$$

and for every $j \in \sigma$

$$S_j^{\updownarrow} \langle v_0, w_0 \rangle \langle v_1, w_1 \rangle \ldots \langle v_{\rho(j)}, w_{\rho(j)} \rangle \text{ iff}$$
$$v_0 = v_1 = \ldots = v_{\rho(j)} \text{ and } S_j w_0 w_1 \ldots w_{\rho(j)}$$

Remark 1.3.2. Definition 1.2.3 states that a frame has a single family of relations, so to adhere to that definition we may combine the families $(R_i^{\leftrightarrow})_{i \in \tau}$ and $\left(S_j^{\updownarrow}\right)_{j \in \sigma}$ into a single family of relations indexed by the disjoint union $\tau \uplus \sigma$, as suggested by the claim that $\mathfrak{F} \otimes \mathfrak{G}$ is a $\tau \uplus \sigma$-frame. A purist may then insist that the relations on a frame product should be indexed with explicit reference to the injection maps into the disjoint union. For the sake of simplicity however, we will often assume that the types τ and σ are disjoint already and hence that the type of the frame product $\mathfrak{F} \otimes \mathfrak{G}$ can be taken to be $\tau \cup \sigma$. This assumption simplifies our presentation without a loss of generality. Also note that the arities of R_i and $R_i^{\leftrightarrow}$ are the same so that $\rho(i) + 1$ unambiguously specifies the arity of both, similarly the arities of S_j and $S_j^{\updownarrow}$ are unambiguously specified by $\rho(j) + 1$.

Remark 1.3.3. The construction of the $R_i^{\leftrightarrow}$ (and similarly the $S_i^{\updownarrow}$) seems very natural, even if only informally: it states that any single transition in the frame product always corresponds to a transition in exactly one of the two original frames, with an unchanged position in the other frame. In terms of expressiveness this also means that any sequence of transitions in the original frames — simultaneous or consecutive — can be expressed in the frame product using appropriate compositions of its relations. The idea that a transition in the frame product corresponds to a transition in one of its factors also motivates the notation used for $R_i^{\leftrightarrow}$ and $S_i^{\updownarrow}$: we think of transitions inside the first factor as "horizontal transitions" in the frame product, and think of transitions inside the second factor as "vertical transitions" in the frame product. We demonstrate this with the following picture.



We can also show that the frame product inherits several properties from the cartesian product, namely commutativity and associativity up to isomorphism, which we will reuse and strengthen in Chapter 3.

Lemma 1.3.4. *For frames*

$$\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle, \quad \mathfrak{G} = \left\langle G, (S_i)_{i\in\sigma} \right\rangle \quad \textit{and} \quad \mathfrak{H} = \left\langle H, (T_i)_{i\in\mu} \right\rangle,$$

we have the following τ-frame isomorphisms

(1) $\mathfrak{F} \otimes \mathfrak{G} \cong \mathfrak{G} \otimes \mathfrak{F}$

(2) $\mathfrak{F} \otimes (\mathfrak{G} \otimes \mathfrak{H}) \cong (\mathfrak{F} \otimes \mathfrak{G}) \otimes \mathfrak{H}$

To prove the associativity in Lemma 1.3.4 we need to take particular care in distinguishing the order in which relations are constructed. For example in the notation used so far, both frame products $\mathfrak{F} \otimes (\mathfrak{G} \otimes \mathfrak{H})$ and $\mathfrak{F} \otimes \mathfrak{G}$ will have relations called $R_i^{\leftrightarrow}$ (for $i \in \tau$). So to prevent confusion we first introduce more expressive (but bulky) notation for the relations on the frame product. After the proof of Lemma 1.3.4 is completed we return to our standard notation.

Notation 1.3.5. For any set X and any integer n let $\Delta_{X,n}$ denote the n-ary diagonal relation on X, i.e.

$$\Delta_{X,n} := \left\{ \langle \overbrace{v, v, \ldots, v}^{n \text{ times}} \rangle \;\middle|\; v \in X \right\}$$

Now for two arbitrary relations R and S of the same arity (say n) let $R * S$ be the relation defined by

$$R * S \langle v_0, w_0\rangle \langle v_1, w_1\rangle \ldots \langle v_n, w_n\rangle \text{ iff } Rv_0v_1\ldots v_n \text{ and } Sw_0w_1\ldots w_n.$$

With this notation, for example, the relation $R_i^{\leftrightarrow}$ on $\mathfrak{F} \otimes \mathfrak{G}$ (for some $i \in \tau$) can be written as $R_i^{\leftrightarrow} = R_i * \Delta_{G,\rho(i)}$, and the relation $R_i^{\leftrightarrow}$ on $\mathfrak{F} \otimes (\mathfrak{G} \otimes \mathfrak{H})$ can be written as $R_i^{\leftrightarrow} = R_i * \Delta_{G\times H,\rho(i)}$.

Now we can prove the result.

Proof of Lemma 1.3.4.

(1) To show that $\mathfrak{F} \otimes \mathfrak{G} \cong \mathfrak{G} \otimes \mathfrak{F}$, define the function

$$\begin{aligned} f : F \times G &\to G \times F \\ \langle v, w\rangle &\mapsto \langle w, v\rangle \end{aligned}$$

It is well-known that f is a bijection, so we only need to show that f is a strong τ-frame homomorphism in the sense of Definition 1.2.13. To do this we need to show that for $i \in \tau$

$$R_i * \Delta_{G,\rho(i)} \langle v_0, w_0 \rangle \langle v_1, w_1 \rangle \ldots \left\langle v_{\rho(i)}, w_{\rho(i)} \right\rangle$$

$$\text{if and only if}$$

$$\Delta_{G,\rho(i)} * R_i f\left(\langle v_0, w_0 \rangle\right) f\left(\langle v_1, w_1 \rangle\right) \ldots f\left(\left\langle v_{\rho(i)}, w_{\rho(i)} \right\rangle\right),$$

and that for $i \in \sigma$

$$\Delta_{F,\rho(i)} * S_i \langle v_0, w_0 \rangle \langle v_1, w_1 \rangle \ldots \left\langle v_{\rho(i)}, w_{\rho(i)} \right\rangle$$

$$\text{if and only if}$$

$$S_i * \Delta_{F,\rho(i)} f\left(\langle v_0, w_0 \rangle\right) f\left(\langle v_1, w_1 \rangle\right) \ldots f\left(\left\langle v_{\rho(i)}, w_{\rho(i)} \right\rangle\right).$$

We only show the case where $i \in \tau$, the other case follows similarly. Note that the condition that $R_i * \Delta_{G,\rho(i)} \langle v_0, w_0 \rangle \langle v_1, w_1 \rangle \ldots \left\langle v_{\rho(i)}, w_{\rho(i)} \right\rangle$, is equivalent to the condition that $R_i v_0 v_1 \ldots v_{\rho(i)}$ and $w_0 = w_1 = \ldots = w_{\rho(i)}$ (by the definition of $*$ and $\Delta_{G,\rho(i)}$). It follows from the definition of $*$ and $\Delta_{G,\rho(i)}$ that the latter condition is equivalent to $\Delta_{G,\rho(i)} * R_i \langle w_0, v_0 \rangle \langle w_1, v_1 \rangle \ldots \left\langle w_{\rho(i)}, v_{\rho(i)} \right\rangle$, which by the definition of f is equivalent to $\Delta_{G,\rho(i)} * R_i f\left(\langle v_0, w_0 \rangle\right) f\left(\langle v_1, w_1 \rangle\right) \ldots f\left(\left\langle v_{\rho(i)}, w_{\rho(i)} \right\rangle\right)$.

(2) To show that $\mathfrak{F} \otimes (\mathfrak{G} \otimes \mathfrak{H}) \cong (\mathfrak{F} \otimes \mathfrak{G}) \otimes \mathfrak{H}$, define the function

$$\begin{aligned} g : F \times (G \times H) &\to (F \times G) \times H \\ \langle v, \langle w, x \rangle \rangle &\mapsto \langle \langle v, w \rangle, x \rangle \end{aligned}$$

It is well-known that g is a bijection, so it suffices to show that g is a strong τ-frame homomorphism in the sense of Definition 1.2.13. Constructing the frame products and using g we see that we are required to verify three conditions (one for each of the original types):

For $i \in \tau$:

$$R_i * \Delta_{G \times H,\rho(i)} \langle v_0, \langle w_0, x_0 \rangle \rangle \langle v_1, \langle w_1, x_1 \rangle \rangle \ldots \left\langle v_{\rho(i)}, \left\langle w_{\rho(i)}, x_{\rho(i)} \right\rangle \right\rangle$$

$$\text{if and only if}$$

$$\left(R_i * \Delta_{G,\rho(i)}\right) * \Delta_{H,\rho(i)} \langle \langle v_0, w_0 \rangle, x_0 \rangle \langle \langle v_1, w_1 \rangle, x_1 \rangle \ldots \left\langle \left\langle v_{\rho(i)}, w_{\rho(i)} \right\rangle, x_{\rho(i)} \right\rangle$$

For $i \in \sigma$:

$$\Delta_{F,\rho(i)} * \left(S_i * \Delta_{H,\rho(i)}\right) \langle v_0, \langle w_0, x_0 \rangle \rangle \langle v_1, \langle w_1, x_1 \rangle \rangle \ldots \left\langle v_{\rho(i)}, \left\langle w_{\rho(i)}, x_{\rho(i)} \right\rangle \right\rangle$$

$$\text{if and only if}$$

$$\left(\Delta_{F,\rho(i)} * S_i\right) * \Delta_{H,\rho(i)} \langle \langle v_0, w_0 \rangle, x_0 \rangle \langle \langle v_1, w_1 \rangle, x_1 \rangle \ldots \left\langle \left\langle v_{\rho(i)}, w_{\rho(i)} \right\rangle, x_{\rho(i)} \right\rangle$$

For $i \in \mu$:

$$\Delta_{F,\rho(i)} * \left(\Delta_{G,\rho(i)} * T_i\right) \langle v_0, \langle w_0, x_0\rangle\rangle \langle v_1, \langle w_1, x_1\rangle\rangle \ldots \left\langle v_{\rho(i)}, \left\langle w_{\rho(i)}, x_{\rho(i)}\right\rangle\right\rangle$$

if and only if

$$\Delta_{F\times G,\rho(i)} * T_i \langle\langle v_0, w_0\rangle, x_0\rangle \langle\langle v_1, w_1\rangle, x_1\rangle \ldots \left\langle\left\langle v_{\rho(i)}, w_{\rho(i)}\right\rangle, x_{\rho(i)}\right\rangle$$

We only demonstrate the case where $i \in \tau$, the other two cases can be done similary. It follows from the definition of $*$ and $\Delta_{G\times H,\rho(i)}$ that $R_i * \Delta_{G\times H,\rho(i)} \langle v_0, \langle w_0, x_0\rangle\rangle \langle v_1, \langle w_1, x_1\rangle\rangle \ldots \left\langle v_{\rho(i)}, \left\langle w_{\rho(i)}, x_{\rho(i)}\right\rangle\right\rangle$ is equivalent to the condition that $R_i v_0 v_1 \ldots v_{\rho(i)}$ and $\langle w_0, x_0\rangle = \langle w_1, x_1\rangle = \ldots = \left\langle w_{\rho(i)}, x_{\rho(i)}\right\rangle$. This is in turn equivalent to $R_i v_0 v_1 \ldots v_{\rho(i)}$, $w_0 = w_1 = \ldots = w_{\rho(i)}$ and $x_0 = x_1 = \ldots = x_{\rho(i)}$, because of a property of ordered pairs. Using the definition of $*$ and $\Delta_{G,\rho(i)}$ we rewite it as $R_i * \Delta_{G,\rho(i)} \langle v_0, w_0\rangle \langle v_1, w_1\rangle \ldots \left\langle v_{\rho(i)}, w_{\rho(i)}\right\rangle$ and $x_0 = x_1 = \ldots = x_{\rho(i)}$. Which by the definition of $*$ and $\Delta_{H,\rho(i)}$ is equivalent to

$$\left(R_i * \Delta_{G,\rho(i)}\right) * \Delta_{H,\rho(i)} \langle\langle v_0, w_0\rangle, x_0\rangle \langle\langle v_1, w_1\rangle, x_1\rangle \ldots \left\langle\left\langle v_{\rho(i)}, w_{\rho(i)}\right\rangle, x_{\rho(i)}\right\rangle.$$

□

In Remark 1.3.2 we discussed our assumption that types are disjoint and made it clear that we will not state the coproduct injections into disjoint unions explicitly. However when we discuss the logics on frame products we will make a very explicit distinction between the modalities from the two factor frames. Similar to the intuition used for the relations on a frame product (Remark 1.3.3), we will use *horizontal* and *vertical* modalities in the logics on frame products. To demonstrate this let two frames

$$\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau}\right\rangle \quad \text{and} \quad \mathfrak{G} = \left\langle G, (S_i)_{i\in\sigma}\right\rangle$$

be given. Consider a $\tau \uplus \sigma$-model $\mathfrak{M} = \langle \mathfrak{F} \otimes \mathfrak{G}, V\rangle$ over the frame product $\mathfrak{F} \otimes \mathfrak{G}$, and recall the validity conditions for the standard modal operators (Definition 1.2.7). We rewrite these validity conditions in terms of the accesibility relations of $\mathfrak{F}$ and $\mathfrak{G}$.

Notation 1.3.6. We obtain a diamond and corresponding box modality for every $R_i^{\leftrightarrow}$, we denote these by $\diamondminus_i$ (*horizontal diamond* i) and $\boxminus_i$ (*horizontal box* i) respectively:
$\mathfrak{M}, \langle v_0, w_0\rangle \Vdash \diamondminus_i \phi_1 \phi_2 \ldots \phi_{\rho(i)}$ if and only if there are

$$\langle v_1, w_1\rangle, \langle v_2, w_2\rangle, \ldots, \left\langle v_{\rho(i)}, w_{\rho(i)}\right\rangle \in F \times G$$

such that

$$R_i^{\leftrightarrow} \langle v_0, w_0\rangle \langle v_1, w_1\rangle \ldots \left\langle v_{\rho(i)} w_{\rho(i)}\right\rangle$$

and

$$\mathfrak{M}, \langle v_1, w_1\rangle \Vdash \phi_1 \, ; \, \mathfrak{M}, \langle v_2, w_2\rangle \Vdash \phi_2 \, ; \, \ldots \, ; \, \mathfrak{M}, \left\langle v_{\rho(i)}, w_{\rho(i)}\right\rangle \Vdash \phi_{\rho(i)}.$$

According to the definition of $R_i^{\leftrightarrow}$ (Definition 1.3.1) this holds exactly when there are $\langle v_1, w_1\rangle, \langle v_2, w_2\rangle, \ldots, \left\langle v_{\rho(i)}, w_{\rho(i)}\right\rangle \in F \times G$ such that $R_i v_0 v_1 \ldots v_{\rho(i)}$ and $w_0 = w_1 = \ldots = w_{\rho(i)}$ and

$$\mathfrak{M}, \langle v_1, w_1\rangle \Vdash \phi_1 \, ; \, \mathfrak{M}, \langle v_2, w_2\rangle \Vdash \phi_2 \, ; \, \ldots \, ; \, \mathfrak{M}, \left\langle v_{\rho(i)}, w_{\rho(i)}\right\rangle \Vdash \phi_{\rho(i)}.$$

Using the equality of the points in $\mathfrak{G}$ we may conclude that:

- $\mathfrak{M}, \langle v_0, w_0\rangle \Vdash \diamondminus_i \phi_1 \phi_2 \ldots \phi_{\rho(i)}$ if and only if there are $v_1, v_2, \ldots, v_{\rho(i)} \in F$ such that $R_i v_0 v_1 \ldots v_{\rho(i)}$ and

$$\mathfrak{M}, \langle v_1, w_0\rangle \Vdash \phi_1 \, ; \, \mathfrak{M}, \langle v_2, w_0\rangle \Vdash \phi_2 \, ; \, \ldots \, ; \, \mathfrak{M}, \left\langle v_{\rho(i)}, w_0\right\rangle \Vdash \phi_{\rho(i)}.$$

A similar argument shows that

- $\mathfrak{M}, \langle v_0, w_0\rangle \Vdash \boxminus_i \phi_1 \phi_2 \ldots \phi_{\rho(i)}$ if and only if for every $v_1, v_2, \ldots, v_{\rho(i)} \in F$ we have that $R_i v_0 v_1 \ldots v_{\rho(i)}$ implies that $\mathfrak{M}, \langle v_k, w_0\rangle \Vdash \phi_k$ for some k.

We can also obtain a diamond and corresponding box modality for every $S_i^{\updownarrow}$, we denote these by $\diamondbar_i$ (*vertical diamond* i) and $\boxbar_i$ (*vertical box* i) respectively. An argument similar to the one given above shows that the semantics for these two modalities are given by:

- $\mathfrak{M}, \langle v_0, w_0\rangle \Vdash \diamondbar_i \phi_1 \phi_2 \ldots \phi_{\rho(i)}$ if and only if there are $w_1, w_2, \ldots, w_{\rho(i)} \in G$ such that $S_i w_0 w_1 \ldots w_{\rho(i)}$ and

$$\mathfrak{M}, \langle v_0, w_1\rangle \Vdash \phi_1 \, ; \, \mathfrak{M}, \langle v_0, w_2\rangle \Vdash \phi_2 \, ; \, \ldots \, ; \, \mathfrak{M}, \left\langle v_0, w_{\rho(i)}\right\rangle \Vdash \phi_{\rho(i)}.$$

- $\mathfrak{M}, \langle v_0, w_0\rangle \Vdash \boxbar_i \phi_1 \phi_2 \ldots \phi_{\rho(i)}$ if and only if for all $w_1, w_2, \ldots, w_{\rho(i)} \in G$ we have that $S_i w_0 w_1 \ldots w_{\rho(i)}$ implies that $\mathfrak{M}, \langle v_0, w_k\rangle \Vdash \phi_k$ for some k.

Terminology 1.3.7. We will call a $\tau \uplus \sigma$-formula that contains no vertical modalities (resp. horizontal modalities) a *horizontal formula* (resp. *vertical formula*).

It is natural to investigate how the logic on a frame product relates to the logics of its factor frames. In this spirit Shehtman (1978) posed the following question.

> For types τ and σ suppose that a class of τ-frames F and a class of σ-frames G are given. Let the axiomatisations of $\mathbf{Log}\,(\mathsf{F})$ and $\mathbf{Log}\,(\mathsf{G})$ be known, now axiomatize
>
> $$\mathbf{Log}\left(\{\mathfrak{F} \otimes \mathfrak{G} | \, \mathfrak{F} \in \mathsf{F}, \mathfrak{G} \in \mathsf{G}\}\right).$$

However it has been remarked by Gabbay *&* Shehtman (1998) that the logic in question is not uniquely determined by $\mathbf{Log}\,(\mathsf{F})$ and $\mathbf{Log}\,(\mathsf{G})$. So a more modern approach is to axiomatize the following logic instead.

Definition 1.3.8 (Gabbay *&* Shehtman (1998))**.** For types τ and σ, suppose that a τ-logic $\mathbf{L}_1$ and a σ-logic $\mathbf{L}_2$ are given. We define the *product logic of* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ as

$$\mathbf{L}_1 \otimes \mathbf{L}_2 := \mathbf{Log}\,(\{\mathfrak{F} \otimes \mathfrak{G} \,|\, \mathfrak{F} \text{ is a } \tau\text{-frame, } \mathfrak{F} \Vdash \mathbf{L}_1,\ \mathfrak{G} \text{ is a } \sigma\text{-frame, } \mathfrak{G} \Vdash \mathbf{L}_2\})\,.$$

Despite the fact that the question posed by Shehtman (1978) has been superceded by the axiomatisation of a product logic, Shehtman (1978) provided axiomatisations for the products of several popular normal modal logics. One logic that was axiomatized by Shehtman (1978) is the logic $\mathbf{K}_\tau \otimes \mathbf{K}_\sigma$ when τ and σ only contain unary members. We briefly discuss this axiomatisation. Most of the following work is due to Shehtman (1978), except where we make explicit reference to another source. As stated before our treatment is very sparse, and serves only as a motivation to study frame products. A more thorough treatment, that takes a more modern approach than Shehtman (1978), can be obtained from Kurucz *et al.* (2003).

As stated, we assume that τ and σ only have unary members. To keep our presentation simple we also assume that τ and σ each have only one member. In this case we may omit the subscripts and simply write $\mathbf{K} \otimes \mathbf{K} = \mathbf{K}_\tau \otimes \mathbf{K}_\sigma$, however to emphasize that the modalities of $\mathbf{K}_\tau$ (resp. $\mathbf{K}_\sigma$) will correspond to horizontal modalities (resp. vertical modalities) in the product logic, we write $\mathbf{K}_\leftrightarrow = \mathbf{K}_\tau$ (resp. $\mathbf{K}_\updownarrow = \mathbf{K}_\sigma$) instead. Hence, we want to axiomatize $\mathbf{K}_\leftrightarrow \otimes \mathbf{K}_\updownarrow$.

Since $\mathbf{K}_\leftrightarrow \otimes \mathbf{K}_\updownarrow$ consists only of formulae that are valid in certain frame products, we consider which formulae hold in all frame products. First of all note the following result.

Lemma 1.3.9 (Shehtman (1978))**.** *Suppose that disjoint types τ and σ are given. Consider a τ-logic $\mathbf{L}_1$ and a σ-logic $\mathbf{L}_2$, together with a τ-frame $\mathfrak{F}$ such that $\mathfrak{F} \Vdash \mathbf{L}_1$ and a σ-frame $\mathfrak{G}$ such that $\mathfrak{G} \Vdash \mathbf{L}_2$. Then $\mathbf{L}_1 \cup \mathbf{L}_2 \subseteq \mathbf{Log}\,(\mathfrak{F} \otimes \mathfrak{G})$.*

This result is easily proved by showing that discarding the relations that are indexed by σ from the frame product $\mathfrak{F} \otimes \mathfrak{G}$, gives a frame that is the disjoint union (Definition 1.2.38) of $|\mathfrak{G}|$-many frames isomorphic to $\mathfrak{F}$. Similary discarding the relations that are indexed by τ from the frame product $\mathfrak{F} \otimes \mathfrak{G}$ gives a frame that is the disjoint union of $|\mathfrak{F}|$-many frames isomorphic to $\mathfrak{G}$. Then by Proposition 1.2.40 we have that $\mathfrak{F} \otimes \mathfrak{G}$ satisfies horizontal formulae corresponding to the logic of $\mathfrak{F}$ and vertical formulae corresponding to the logic of $\mathfrak{G}$.

Since $\mathbf{K}_\leftrightarrow$ and $\mathbf{K}_\updownarrow$ are Kripke complete (see Remark 1.2.12) we can now use Lemma 1.3.9 to conclude that an axiomatisation of $\mathbf{K}_\leftrightarrow \otimes \mathbf{K}_\updownarrow$ should include the axioms of $\mathbf{K}_\leftrightarrow$ and $\mathbf{K}_\updownarrow$. However this lemma does not describe the interaction between the horizontal and vertical modalities. To capture this interaction we re-examine frame products.

Suppose that two frames $\mathfrak{F} = \langle F, R\rangle$ and $\mathfrak{G} = \langle G, S\rangle$ each with a binary relation, and their frame product $\mathfrak{F} \otimes \mathfrak{G} = \left\langle F \times G, R^{\leftrightarrow}, S^{\updownarrow}\right\rangle$ are given. Now observe that for every $\langle v_1, w_1\rangle, \langle v_2, w_2\rangle, \langle v_3, w_3\rangle \in F \times G$ such that $\langle v_1, w_1\rangle R^{\leftrightarrow} \langle v_2, w_2\rangle$ and $\langle v_2, w_2\rangle S^{\updownarrow} \langle v_3, w_3\rangle$ it follows from Definition 1.3.1 that v_1Rv_2, $w_1 = w_2$, w_2Sw_3 and $v_2 = v_3$; which implies that v_1Rv_3 and w_1Sw_3 (again by Definition 1.3.1). Hence $\langle v_1, w_3\rangle \in F \times G$ has the property that $\langle v_1, w_1\rangle S^{\updownarrow} \langle v_1, w_3\rangle$ and $\langle v_1, w_3\rangle R^{\leftrightarrow} \langle v_3, w_3\rangle$. Observe that the converse also holds: given $\langle v_1, w_1\rangle, \langle v_3, w_3\rangle, \langle v_4, w_4\rangle \in F \times G$ such that $\langle v_1, w_1\rangle S^{\updownarrow} \langle v_4, w_4\rangle$ and $\langle v_4, w_4\rangle R^{\leftrightarrow} \langle v_3, w_3\rangle$, then $\langle v_3, w_1\rangle \in F\times G$ is such that $\langle v_1, w_1\rangle R^{\leftrightarrow} \langle v_3, w_1\rangle$ and $\langle v_3, w_1\rangle S^{\updownarrow} \langle v_3, w_3\rangle$. This shows that the frame product $\mathfrak{F} \otimes \mathfrak{G}$ has the following properties.

Definition 1.3.10 (Kurucz *et al.* (2003))**.** Given a frame with two binary relations $\mathfrak{K} = \langle K, Q_1, Q_2\rangle$.

- $\mathfrak{K}$ is *right commutative* if for every $v_1, v_2, v_3 \in K$, such that $v_1Q_1v_2$ and $v_2Q_2v_3$ there is $v_4 \in K$ such that $v_1Q_2v_4$ and $v_4Q_1v_3$.
- $\mathfrak{K}$ is *left commutative* if for every $v_1, v_3, v_4 \in K$, such that $v_1Q_2v_4$ and $v_4Q_1v_3$ there is $v_2 \in K$ such that $v_1Q_1v_2$ and $v_2Q_2v_3$.
- $\mathfrak{K}$ is *commutative* if $\mathfrak{K}$ is left commutative and right commutative.

As shown above, all of the frame products we consider are commutative. We can visualize right commutativity and left commutativity of the frame $\mathfrak{K}$ with the following two diagrams:

$$\begin{tikzcd} v_1 \arrow[r, "Q_1"] \arrow[d, dotted, "Q_2"'] & v_2 \arrow[d, "Q_2"] \\ v_4 \arrow[r, dotted, "Q_1"'] & v_3 \end{tikzcd} \qquad \begin{tikzcd} v_1 \arrow[r, dotted, "Q_1"] \arrow[d, "Q_2"'] & v_2 \arrow[d, dotted, "Q_2"] \\ v_4 \arrow[r, "Q_1"'] & v_3 \end{tikzcd}$$

Returning to the frame product $\mathfrak{F}\otimes\mathfrak{G}$, let $\langle v_1, w_1\rangle, \langle v_2, w_2\rangle, \langle v_3, w_3\rangle \in F\times G$ be such that $\langle v_1, w_1\rangle R^{\leftrightarrow} \langle v_2, w_2\rangle$ and $\langle v_1, w_1\rangle S^{\updownarrow} \langle v_3, w_3\rangle$, it follows from Definition 1.3.1 that v_1Rv_2, $w_1 = w_2$, w_1Sw_3 and $v_1 = v_3$. Hence v_3Rv_2 and w_2Sw_3, which implies that $\langle v_2, w_3\rangle \in F \times G$ has the property that $\langle v_3, w_3\rangle R^{\leftrightarrow} \langle v_2, w_3\rangle$ and $\langle v_2, w_2\rangle S^{\updownarrow} \langle v_2, w_3\rangle$ (by Definition 1.3.1). This shows that the frame product $\mathfrak{F} \otimes \mathfrak{G}$ has the following property as well.

Definition 1.3.11. A frame with two binary relations $\mathfrak{K} = \langle K, Q_1, Q_2\rangle$ is *Church-Rosser* if for every $v_1, v_2, v_3 \in K$ such that $v_1Q_1v_2$ and $v_1Q_2v_3$ there is $v_4 \in K$ such that $v_2Q_2v_4$ and $v_3Q_1v_4$.

As shown above, all of the frame products we consider are Church-Rosser. We can visualize the Church-Rosser property of the frame $\mathfrak{K}$ with the following diagram:

$$\begin{tikzcd}
v_1 \arrow[r, "Q_1"] \arrow[d, "Q_2"'] & v_2 \arrow[d, dotted, "Q_2"] \\
v_3 \arrow[r, dotted, "Q_1"'] & v_4
\end{tikzcd}$$

To axiomatize the product logic $\mathbf{K}_{\leftrightarrow} \otimes \mathbf{K}_{\updownarrow}$ we now translate commutativity and the Church-Rosser property to modal formulae.

Lemma 1.3.12 (Gabbay *&* Shehtman (1998))**.** *A frame with two binary relations $\mathfrak{K} = \langle K, Q_1, Q_2 \rangle$ is commutative if and only if $\mathfrak{K} \Vdash \Diamond_2\Diamond_1 p \leftrightarrow \Diamond_1\Diamond_2 p$.*

Proof. We show that the right commutativity of $\mathfrak{K}$ is equivalent to $\mathfrak{K} \Vdash \Diamond_1\Diamond_2 p \to \Diamond_2\Diamond_1 p$. First suppose that $\mathfrak{K}$ is not right commutative, then there are $v_1, v_2, v_3 \in K$, such that $v_1 Q_1 v_2$ and $v_2 Q_2 v_3$, and for all $v_4 \in K$ it does not hold that both $v_1 Q_2 v_4$ and $v_4 Q_1 v_3$. Now let $V : \Phi \to 2^K$ be a valuation such that $V(p) = \{v_3\}$. Now by the choice of v_1, v_2, v_3 and V it follows that $\langle \mathfrak{K}, V \rangle, v_1 \Vdash \Diamond_1\Diamond_2 p$ and $\langle \mathfrak{K}, V \rangle, v_1 \nVdash \Diamond_2\Diamond_1 p$. Hence $\langle \mathfrak{K}, V \rangle, v_1 \nVdash \Diamond_1\Diamond_2 p \to \Diamond_2\Diamond_1 p$, so that $\mathfrak{K} \nVdash \Diamond_1\Diamond_2 p \to \Diamond_2\Diamond_1 p$ as required.

For the converse suppose that $\mathfrak{K} \nVdash \Diamond_1\Diamond_2 p \to \Diamond_2\Diamond_1 p$, then there is $v_1 \in K$ and a valuation $V : \Phi \to 2^K$ such that $\langle \mathfrak{K}, V \rangle, v_1 \nVdash \Diamond_1\Diamond_2 p \to \Diamond_2\Diamond_1 p$. Hence by Definition 1.2.7 $\langle \mathfrak{K}, V \rangle, v_1 \Vdash \Diamond_1\Diamond_2 p$ and $\langle \mathfrak{K}, V \rangle, v_1 \nVdash \Diamond_2\Diamond_1 p$. So there are $v_2, v_3 \in K$ such that $v_1 Q_1 v_2$, $v_2 Q_2 v_3$ and $\langle \mathfrak{K}, V \rangle, v_3 \Vdash p$, and there is no $v_4 \in K$ such that $v_1 Q_2 v_4$ and $v_4 Q_1 v_3$ for otherwise it would be that $\langle \mathfrak{K}, V \rangle, v_1 \nVdash \Diamond_2\Diamond_1 p$. This shows that $\mathfrak{K}$ is not right commutative.

In a similar fashion it can be shown that left commutativity of $\mathfrak{K}$ is equivalent to $\mathfrak{K} \Vdash \Diamond_2\Diamond_1 p \to \Diamond_1\Diamond_2 p$, this completes the proof. $\Box$

Lemma 1.3.13 (Gabbay *&* Shehtman (1998))**.** *Given a frame with two binary relations $\mathfrak{K} = \langle K, Q_1, Q_2 \rangle$, the following are equivalent.*

(1) *$\mathfrak{K}$ is Church-Rosser.*

(2) $\mathfrak{K} \Vdash \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$

(3) $\mathfrak{K} \Vdash \Diamond_2\Box_1 p \to \Box_1\Diamond_2 p$

Proof. We show that (1) and (2) are equivalent.

Suppose that $\mathfrak{K}$ is not Church-Rosser, then there are $v_1, v_2, v_3 \in K$ such that $v_1 Q_1 v_2$ and $v_1 Q_2 v_3$, and that for every $v_4 \in K$ we have that if $v_3 Q_1 v_4$ then it is not the case that $v_2 Q_2 v_4$. So let $V : \Phi \to 2^K$ be a valuation such that $V(p) = \{v \in K | \, v_2 Q_2 v\}$. We show $\langle \mathfrak{K}, V \rangle, v_1 \nVdash \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$. By the choice of v_1, v_2 and V we have that $\langle \mathfrak{K}, V \rangle, v_1 \Vdash \Diamond_1\Box_2 p$. To verify that $\langle \mathfrak{K}, V \rangle, v_1 \nVdash \Box_2\Diamond_1 p$ recall that $v_1 Q_2 v_3$, and observe that $\langle \mathfrak{K}, V \rangle, v_3 \nVdash \Diamond_1 p$ since by the choice of v_2, v_3 and V for every $v_4 \in K$ we have that if $v_3 Q_1 v_4$ then

$v_4 \notin V(p)$ or rather $\langle \mathfrak{K}, V \rangle, v_4 \nVdash p$. It now follows that $\mathfrak{K} \nVdash \Diamond_1 \Box_2 p \to \Box_2 \Diamond_1 p$ as required.

For the converse suppose that $\mathfrak{K} \nVdash \Diamond_1 \Box_2 p \to \Box_2 \Diamond_1 p$. Now there are $v_1 \in K$ and a valuation $V : \Phi \to 2^K$ such that $\langle \mathfrak{K}, V \rangle, v_1 \nVdash \Diamond_1 \Box_2 p \to \Box_2 \Diamond_1 p$. For the latter to hold it must be the case that

$$\langle \mathfrak{K}, V \rangle, v_1 \Vdash \Diamond_1 \Box_2 p \tag{1.3.14}$$

and

$$\langle \mathfrak{K}, V \rangle, v_1 \nVdash \Box_2 \Diamond_1 p \tag{1.3.15}$$

From (1.3.14) we obtain $v_2 \in K$ such that $v_1 Q_1 v_2$ and $\langle \mathfrak{K}, V \rangle, v_2 \Vdash \Box_2 p$. And from (1.3.15) we obtain $v_3 \in K$ such that $v_1 Q_2 v_3$ and $\langle \mathfrak{K}, V \rangle, v_3 \nVdash \Diamond_1 p$. Since we know that $v_1 Q_1 v_2$ and $v_1 Q_2 v_3$, for $\mathfrak{K}$ to be Church-Rosser there must be $v_4 \in K$ such that $v_2 Q_2 v_4$ and $v_3 Q_1 v_4$. However since $\langle \mathfrak{K}, V \rangle, v_2 \Vdash \Box_2 p$, any $v_4 \in K$ such that $v_2 Q_2 v_4$ must be an element of $V(p)$; and contrary to this since $\langle \mathfrak{K}, V \rangle, v_3 \nVdash \Diamond_1 p$ any $v_4 \in K$ such that $v_3 Q_1 v_4$ cannot be an element of $V(p)$. We conclude that $\mathfrak{K}$ is not Church-Rosser.

A similar argument shows that (1) and (3) are equivalent which completes the proof. $\Box$

It is easily proved that any normal logic contains $\diamondminus \boxbar p \to \boxbar \diamondminus p$ if and only if it contains $\diamondbar \boxminus p \to \boxminus \diamondbar p$, so it suffices to include only $\diamondminus \boxbar p \to \boxbar \diamondminus p$ in our axiomatisation.

We have now shown that all of the frame products that we consider validate $\diamondbar \diamondminus p \leftrightarrow \diamondminus \diamondbar p$ and $\diamondminus \boxbar p \to \boxbar \diamondminus p$. Our proof is by no means the shortest possible proof, but it identifies a hazard: interpreting these formulae as commutativity and the Church-Rosser property strongly suggests a plane-like intuition. And with this intuition in mind one might be tempted to conjecture that commutativity and the Church-Rosser property together characterise frame products. This is not the case, as we will see in a moment, but they are enough to complete the axiomatisation of $\mathbf{K}_{\leftrightarrow} \otimes \mathbf{K}_{\updownarrow}$.

Notation 1.3.16. Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be normal logics, each with a single unary operator. Now let $[\mathbf{L}_1, \mathbf{L}_2]$ denote the normal logic axiomatized by the following

- The axioms of $\mathbf{L}_1$ (with horizontal modalities)
- The axioms of $\mathbf{L}_2$ (with vertical modalities)
- $\diamondbar \diamondminus p \leftrightarrow \diamondminus \diamondbar p$
- $\diamondminus \boxbar p \to \boxbar \diamondminus p$

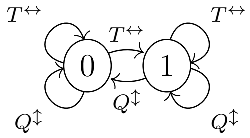
Now we briefly describe how Shehtman (1978) showed that $[\mathbf{K}_{\leftrightarrow}, \mathbf{K}_{\updownarrow}] = \mathbf{K}_{\leftrightarrow} \otimes \mathbf{K}_{\updownarrow}$.

Since we chose axioms satisfied by all the frames in the set

$$\left\{\mathfrak{F}\otimes\mathfrak{G}\,|\,\mathfrak{F}\text{ is a }\tau\text{-frame, }\mathfrak{F}\Vdash\mathbf{K}_{\leftrightarrow},\ \mathfrak{G}\text{ is a }\sigma\text{-frame, }\mathfrak{G}\Vdash\mathbf{K}_{\updownarrow}\right\}$$

it is immediate that $\left[\mathbf{K}_{\leftrightarrow},\mathbf{K}_{\updownarrow}\right]\subseteq\mathbf{K}_{\leftrightarrow}\otimes\mathbf{K}_{\updownarrow}$. However it is not yet clear that the axiomatisation generates the entire logic $\mathbf{K}_{\leftrightarrow}\otimes\mathbf{K}_{\updownarrow}$, in fact Gabbay *&* Shehtman (1998) provide the following frame $\mathfrak{H}=\left\langle H,T^{\leftrightarrow},Q^{\updownarrow}\right\rangle$, which is not isomorphic to a frame product, but where $\left[\mathbf{K}_{\leftrightarrow},\mathbf{K}_{\updownarrow}\right]$ is valid.

$\mathfrak{H}$:



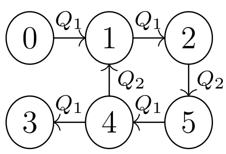
The frame $\mathfrak{H}$ shows that commutativity and the Church-Rosser property do not characterize frame products, however we will see that they do characterize the frames that specify the product logic we are interested in.

Observe that in Definition 1.2.7 the validity of formulae at a specific point in a model or frame is unaffected by the validity of formulae in points that are not related to that point. This suggests that it may often be sufficient to investigate only the frames that are "generated" from some point, as formalised in the following definition.

Definition 1.3.17 (Blackburn *et al.* (2001))**.** Suppose that a frame with two binary relations $\mathfrak{K}=\langle K,Q_1,Q_2\rangle$ is given, together with a point v_0 of $\mathfrak{K}$. Let Q denote the transitive closure of the relation $Q_1\cup Q_2$. We say $\mathfrak{K}$ is *generated by* v_0 if for every $v_1\in K$ we have that v_0Qv_1. We say $\mathfrak{K}$ is *generated* if there is some point that generates it.

Example 1.3.18. The following frame $\mathfrak{K}=\langle K,Q_1,Q_2\rangle$ is generated by 0, but by no other point.

$\mathfrak{K}$:



Definition 1.3.17 is by no means in its most general form, but it is sufficient for our purposes.

To show that $\mathbf{K}_{\leftrightarrow}\otimes\mathbf{K}_{\updownarrow}\subseteq\left[\mathbf{K}_{\leftrightarrow},\mathbf{K}_{\updownarrow}\right]$, we use the following two results, which we state without proof.

Proposition 1.3.19 (Gabbay *&* Shehtman (1998))**.** $\left[\mathbf{K}_{\leftrightarrow},\mathbf{K}_{\updownarrow}\right]$ *is Kripke complete. In fact, the logic of the countable generated commutative Church-Rosser frames with two binary relations is exactly* $\left[\mathbf{K}_{\leftrightarrow},\mathbf{K}_{\updownarrow}\right]$*.*

Proposition 1.3.20 (Shehtman (1978))**.** *For any countable generated commutative Church-Rosser $\tau\uplus\sigma$-frame $\mathfrak{K}$, there is a τ-frame $\mathfrak{F}$ and a σ-frame $\mathfrak{G}$ together with a surjective bounded $\tau\uplus\sigma$-frame morphism from $\mathfrak{F}\otimes\mathfrak{G}$ to $\mathfrak{K}$.*

Proposition 1.3.20 shows that although the frame $\mathfrak{H}$ given by Gabbay *&* Shehtman (1998) is not isomorphic to a frame product, it is the image of a frame product via a surjective bounded $\tau \uplus \sigma$-frame morphism, so according to Corollary 1.2.30 $\mathfrak{H}$ must validate the same formulae as some frame product. Before we use this to prove the final result, it should be noted that the frame product $\mathfrak{F} \otimes \mathfrak{G}$ given by Proposition 1.3.20 can be quite large. In fact although Shehtman (1978), Gabbay *&* Shehtman (1998) and Kurucz *et al.* (2003) all prove Proposition 1.3.20 in different ways, all their proofs use induction arguments to construct the required bounded morphism. With these results however, the axiomatisation is easily shown to work.

Theorem 1.3.21 (Shehtman (1978))**.**

$$\mathbf{K}_{\leftrightarrow} \otimes \mathbf{K}_{\updownarrow} = \left[\mathbf{K}_{\leftrightarrow}, \mathbf{K}_{\updownarrow}\right]$$

Proof. It is already shown that $\left[\mathbf{K}_{\leftrightarrow}, \mathbf{K}_{\updownarrow}\right] \subseteq \mathbf{K}_{\leftrightarrow} \otimes \mathbf{K}_{\updownarrow}$. So to show that $\mathbf{K}_{\leftrightarrow} \otimes \mathbf{K}_{\updownarrow} \subseteq \left[\mathbf{K}_{\leftrightarrow}, \mathbf{K}_{\updownarrow}\right]$, suppose that a $\tau \uplus \sigma$-formula ϕ is given such that $\phi \notin \left[\mathbf{K}_{\leftrightarrow}, \mathbf{K}_{\updownarrow}\right]$. Then by Proposition 1.3.19 there is a countable generated commutative Church-Rosser $\tau \uplus \sigma$-frame $\mathfrak{K}$ such that $\mathfrak{K} \nVdash \phi$. By Proposition 1.3.20, there is a τ-frame $\mathfrak{F}$ and a σ-frame $\mathfrak{G}$ together with a surjective bounded morphism f from $\mathfrak{F} \otimes \mathfrak{G}$ to $\mathfrak{K}$. Since $\mathfrak{K} \nVdash \phi$ it follows from Corollary 1.2.30 that $\mathfrak{F} \otimes \mathfrak{G} \nVdash \phi$. Since $\mathfrak{F} \Vdash \mathbf{K}_{\leftrightarrow}$ and $\mathfrak{G} \Vdash \mathbf{K}_{\updownarrow}$ (by Remark 1.2.12) it follows that $\phi \notin \mathbf{K}_{\leftrightarrow} \otimes \mathbf{K}_{\updownarrow}$. $\square$

As stated before, analogues of Theorem 1.3.21 can also be proved if τ and σ have more members, or if certain axioms are added to any one of the original logics. The following theorem mentions some of the axioms that can be added in this way.

Theorem 1.3.22 (Shehtman (1978))**.** *Let* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *be normal logics, each with a single unary operator, that are axiomatized by any combinations of the following formulae*

(4) $\Diamond\Diamond p \to \Diamond p$

(D) $\Diamond\top$

(B) $p \to \Box\Diamond p$

(T) $p \to \Diamond p$

Then $\mathbf{L}_1 \otimes \mathbf{L}_2 = [\mathbf{L}_1, \mathbf{L}_2]$.

This theorem was further generalized by Gabbay *&* Shehtman (1998), and subsequently even further by Kurucz *et al.* (2003) to describe the family of axioms that can be added without negating the equality $\mathbf{L}_1 \otimes \mathbf{L}_2 = [\mathbf{L}_1, \mathbf{L}_2]$. There are however product logics that can not be axiomatised in this simple way: Gabbay & Shehtman (1998) proves that there are continuum many such pairs of logics.

In practice frame products and product logics can be used to study interactions between modal operators representing time, space, knowledge, actions, etc. by combining frames or logics representing each of these (Kurucz *et al.* (2003)).

We will not explore product logics further and conclude our discussion here. Our further examination of frame products focuses on their structure, not their logics, but their importance for modal logic stays our motivation for their study.

Chapter 2

Model-theoretic characterisations of simulations and bisimulations

2.1 Introduction

In this chapter we examine a question suggested by Brink & Rewitzky (2004) on how frame simulations and bisimulations may be studied inside a frame that has a universe consisting of pairs of points. We study this question in a model over a frame product. We also give two-dimensional criteria of approximations to frame simulations and frame bisimulations after discussing games corresponding to each.

2.2 Simulations and bisimulations

For the remainder of the chapter we fix two τ-frames

$$\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle \quad \text{and} \quad \mathfrak{G} = \left\langle G, (S_i)_{i\in\tau} \right\rangle .$$

For reasons that will become clear later we restrict ourselves to the case where τ only contains unary members, and correspondingly that all the relations on $\mathfrak{F}$ and $\mathfrak{G}$ are binary. We also fix a $\tau \uplus \tau$-model $\mathfrak{M} = \langle \mathfrak{F} \otimes \mathfrak{G}, V \rangle$ over the frame product of $\mathfrak{F}$ and $\mathfrak{G}$.

Considering a propositional variable $p \in \Phi$, observe that $V(p) \subseteq F \times G$. One may enquire when will $V(p)$ be a simulation from $\mathfrak{F}$ to $\mathfrak{G}$. We demonstrate a $\tau \uplus \tau$-formula that is valid in $\mathfrak{M}$ exactly when $V(p)$ is a τ-frame simulation from $\mathfrak{F}$ to $\mathfrak{G}$. The dependence on p is only for the sake of expressing a relation between F and G, but in the case where V is surjective we can in principle strengthen our results to find all simulations from $\mathfrak{F}$ to $\mathfrak{G}$ by examining all the members of Φ in turn. We will merely demonstrate the formula in question.

In Remark 1.3.3 we have stated that the relations and consequently also the modalities on a frame product capture single transitions in the factor frames. So keeping this intuition in mind we may read the formula $\boxminus_i \Diamond\!\!\!|_i \phi$ as *for every* *transition in $\mathfrak{F}$ there is a transition in $\mathfrak{G}$ to a point*[1] *where ϕ is valid.* We have also stated (see Remark 1.2.15) that a simulation from $\mathfrak{F}$ to $\mathfrak{G}$ captures the notion that *for every transition in $\mathfrak{F}$ there is a corresponding transition in $\mathfrak{G}$*. The resemblance between these two observations motivates the following proposition.

Proposition 2.2.1. *Consider a model $\mathfrak{M} = \langle \mathfrak{F} \otimes \mathfrak{G}, V \rangle$ as before. Let a propositional variable $p \in \Phi$ be given. Now the following hold.*

(1) *$V(p)$ is a simulation from $\mathfrak{F}$ to $\mathfrak{G}$ if and only if for every $i \in \tau$ we have that $\mathfrak{M} \Vdash p \to \boxminus_i \diamondbar_i p$.*

(2) *$V(p)$ is a bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$ if and only if for every $i \in \tau$ we have that $\mathfrak{M} \Vdash p \to (\boxminus_i \diamondbar_i p \wedge \boxbar_i \diamondminus_i p)$.*

Proof. (1) For the sake of brevity we prove this only for the case where τ has only one member, i. It is trivial to generalise this. Suppose that $\mathfrak{M} \Vdash p \to \boxminus_i \diamondbar_i p$, this is true exactly when it holds that for every $\langle v, w \rangle \in F \times G$ we have $\mathfrak{M}, \langle v, w \rangle \Vdash p \to \boxminus_i \diamondbar_i p$ (Definition 1.2.7).
By appealing to the semantics of $\to$ (Definition 1.2.7) we see that this is equivalent to the condition that for every $\langle v, w \rangle \in F \times G$ we have that $\mathfrak{M}, \langle v, w \rangle \Vdash p$ implies $\mathfrak{M}, \langle v, w \rangle \Vdash \boxminus_i \diamondbar_i p$.
We use the validity conditions for atoms (Definition 1.2.7) and for $\boxminus_i$ (Notation 1.3.6) to obtain the equivalent statement that for every $\langle v, w \rangle \in F \times G$ we have that $\langle v, w \rangle \in V(p)$ implies that for every $v_1 \in F$ if vR_iv_1 then also $\mathfrak{M}, \langle v_1, w \rangle \Vdash \diamondbar_i p$.
This can be reformulated using the validity condition for $\diamondbar_i$ (Notation 1.3.6) to see that for every $\langle v, w \rangle \in F \times G$ we have that $\langle v, w \rangle \in V(p)$ implies that for every $v_1 \in F$ if vR_iv_1 then there is $w_1 \in G$ such that wS_iw_1 and $\mathfrak{M}, \langle v_1, w_1 \rangle \Vdash p$.
We use the validity condition for atoms (Definition 1.2.7) to reformulate the latter part of the statement to show that for every $\langle v, w \rangle \in F \times G$ we have that $\langle v, w \rangle \in V(p)$ implies that for every $v_1 \in F$ if vR_iv_1 then there is $w_1 \in G$ such that wS_iw_1 and $\langle v_1, w_1 \rangle \in V(p)$.
This is exactly when $V(p)$ is a simulation from $\mathfrak{F}$ to $\mathfrak{G}$ (Definition 1.2.14).

(2) Recall the comparison between simulations and bisimulations offered by Remark 1.2.25. In light of this observation together with part (1) it is sufficient to show that $(V(p))^{\mathrm{op}}$ is a simulation from $\mathfrak{G}$ to $\mathfrak{F}$ if and only if $\mathfrak{M} \Vdash p \to \boxbar_i \diamondminus_i p$. This can be done in a similar fashion to the proof of part 1.

□

Remark 2.2.2. Unfortunately it does not seem possible to generalize Proposition 2.2.1 to arbitrary types. The problem occurs because the semantics of the

[1] Note that this point is not in $\mathfrak{G}$ but rather in $\mathfrak{M}$.

box modalities does not guarantee the existence of points satisfying every one of its arguments, as required for a τ-frame simulation.

2.3 Simulation and bisimulation games

In first order model theory Ehrenfeucht-Fraïssé games yield an equivalence that is weaker than isomorphism, but stronger than elementary equivalence (see Hodges (1993)). In this same way we have shown that bisimilarity is weaker than isomorphism (see Example 1.2.20) and stronger than validating the same formulae (see Corollary 1.2.29 and Example 1.2.28). We now show how similarity and bisimilarity can be interpreted as two-player games of perfect information. Since Proposition 2.2.1 only applies to types that have only unary members, we maintain its assumption that τ only has unary members, and we only consider binary relations.

Recall that we defined similarity and bisimilarity for points of frames and points of models, so we present four kinds of games. The definitions for these games are from Brink & Rewitzky (2004) and Goranko & Otto (2007).

Simulation game between frames

Suppose that two points v_0 and w_0 of $\mathfrak{F}$ and $\mathfrak{G}$ respectively, are given. Now consider two players, *Spoiler* and *Duplicator*, that take turns at selecting more points from the τ-frames $\mathfrak{F}$ and $\mathfrak{G}$ together with members of τ to construct a sequence (called a *play starting at* $\langle v_0, w_0\rangle$)

$$\langle v_0, w_0\rangle, \langle v_1, w_1, i_1\rangle, \langle v_2, w_2, i_2\rangle, \ldots$$

while obeying only the following rules:

(1) In the $(m+1)$-th round Spoiler selects an $i_{m+1} \in \tau$, together with a point v_{m+1} from $\mathfrak{F}$.

(2) In the $(m+1)$-th round Duplicator selects a point w_{m+1} from $\mathfrak{G}$.

(3) At the end of the $(m+1)$-th round it must hold that $v_m R_{i_{m+1}} v_{m+1}$ and $w_m S_{i_{m+1}} w_{m+1}$.

The game continues until one of the players is unable to select a point satisfying the rules, at which point he loses the game. If the game does not end after finitely many rounds then Duplicator wins. If Duplicator lost the game we mark his forfeit with a $-$ in the play.

Simulation game between models

The simulation game can also be played between τ-models, say

$$\mathfrak{N}_1 = \langle \mathfrak{F}, U_1 \rangle \qquad \text{and} \qquad \mathfrak{N}_2 = \langle \mathfrak{G}, U_2 \rangle .$$

Now the simulation game from $\mathfrak{N}_1$ to $\mathfrak{N}_2$ starting at $\langle v_0, w_0 \rangle$ is played like the simulation game from $\mathfrak{F}$ to $\mathfrak{G}$ starting at $\langle v_0, w_0 \rangle$, but with contraint 2 strengthened to

(2*) In the $(m+1)$-th round Duplicator selects a point w_{m+1} from $\mathfrak{G}$ such that for every $p \in \Phi$ we have

$$\mathfrak{N}_1, v_m \Vdash p \quad \text{implies} \quad \mathfrak{N}_2, w_m \Vdash p$$

.

We say Duplicator has a *winning strategy* in the simulation game from $\mathfrak{F}$ to $\mathfrak{G}$ (resp. $\mathfrak{N}_1$ to $\mathfrak{N}_2$) starting at $\langle v_0, w_0 \rangle$ if Duplicator can take his turns in some way that guarantees that he will win a play of the simulation game from $\mathfrak{F}$ to $\mathfrak{G}$ (resp. $\mathfrak{N}_1$ to $\mathfrak{N}_2$) starting at $\langle v_0, w_0 \rangle$ (either because Spoiler ends up unable to move, or because Duplicator can take turns indefinitely).

Bisimulation game between frames

A bisimulation game between τ-frames is played like a simulation game between τ-frames except that Spoiler may select points from any of the two τ-frames and Duplicator selects from the remaining τ-frame. So the contraints for a bisimulation game between $\mathfrak{F}$ and $\mathfrak{G}$ are:

(a) In the $(m+1)$-th round Spoiler selects an $i_{m+1} \in \tau$, and either selects a point v_{m+1} from $\mathfrak{F}$ or a point w_{m+1} from $\mathfrak{G}$.

(b) If Spoiler selected v_{m+1} from $\mathfrak{F}$, then Duplicator selects w_{m+1} from $\mathfrak{G}$, otherwise Duplicator selects v_{m+1} from $\mathfrak{F}$.

(c) At the end of the $(m+1)$-th round it must hold that $v_m R_{i_{m+1}} v_{m+1}$ and $w_m S_{i_{m+1}} w_{m+1}$.

Winning conditions are the same as for the simulation game; the bisimulation game continues until one of the players is unable to select a point satisfying the rules, in which case the current player loses the game. If the game does not end after finitely many rounds then Duplicator wins.

Bisimulation game between models

Like the simulation game the bisimulation game can also be played between τ-models, say

$$\mathfrak{N}_1 = \langle \mathfrak{F}, U_1 \rangle \qquad \text{and} \qquad \mathfrak{N}_2 = \langle \mathfrak{G}, U_2 \rangle\,.$$

Now the bisimulation game between $\mathfrak{N}_1$ and $\mathfrak{N}_2$ starting at $\langle v_0, w_0 \rangle$ is played like the bisimulation game from $\mathfrak{F}$ to $\mathfrak{G}$ starting at $\langle v_0, w_0 \rangle$, but with contraint b strengthened to

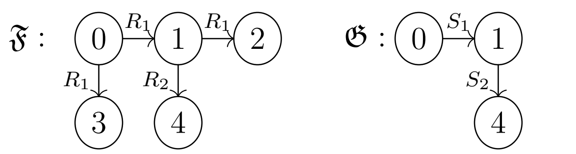
(b*) If Spoiler selected v_{m+1} from $\mathfrak{F}$, then Duplicator selects w_{m+1} from $\mathfrak{G}$, otherwise Duplicator selects v_{m+1} from $\mathfrak{F}$. For every $p \in \Phi$ it must hold that

$$\mathfrak{N}_1, v_m \Vdash p \quad \text{if and only if} \quad \mathfrak{N}_2, w_m \Vdash p.$$

The notion of a winning strategy for Duplicator in the bisimulation game between $\mathfrak{F}$ and $\mathfrak{G}$ (resp. $\mathfrak{N}_1$ and $\mathfrak{N}_2$) starting at $\langle v_0, w_0 \rangle$ is defined as for the simulation game: Duplicator has a *winning strategy* in the bisimulation game between $\mathfrak{F}$ and $\mathfrak{G}$ (resp. $\mathfrak{N}_1$ and $\mathfrak{N}_2$) starting at $\langle v_0, w_0 \rangle$ if Duplicator can take his turns in some way that guarantees that he will win a play of the bisimulation game between $\mathfrak{F}$ and $\mathfrak{G}$ (resp. $\mathfrak{N}_1$ and $\mathfrak{N}_2$) starting at $\langle v_0, w_0 \rangle$.

In a simulation game, from $\mathfrak{F}$ to $\mathfrak{G}$ starting at $\langle v_0, w_0 \rangle$, the intuition behind the players' roles is that Spoiler looks for differences between $\mathfrak{F}$ and $\mathfrak{G}$, whereas Duplicator looks for ways in which the frames resemble each other. More formally Duplicator tries to show that $\langle \mathfrak{F}, v_0 \rangle$ is similar to $\langle \mathfrak{G}, w_0 \rangle$, while Spoiler tries to disprove this claim. The following example illustrates the idea.

Example 2.3.1. Consider the frames $\mathfrak{F} = \langle F, R_1, R_2 \rangle$ and $\mathfrak{G} = \langle G, S_1, S_2 \rangle$ given by the diagrams

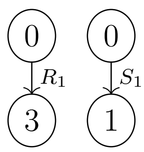


We consider three plays of the simulation game from $\mathfrak{F}$ to $\mathfrak{G}$ starting at $\langle 0, 0 \rangle$.

(1) In the first play that we consider, Spoiler uses his first turn to pick $1 \in \tau$ and the point 3 of $\mathfrak{F}$. Now Duplicator chooses the point 1 of $\mathfrak{G}$. In his next turn Spoiler is unable to move, and hence Duplicator wins. This play is given by:

$$\langle 0, 0 \rangle\,, \langle 3, 1, 1 \rangle$$

Or more graphically given by:



To put Spoiler's loss in the context of similarity, we may observe that $\langle \mathfrak{F}, 0\rangle$ is not similar to $\langle \mathfrak{G}, 0\rangle$, but $\langle \mathfrak{F}, 3\rangle$ is similar to $\langle \mathfrak{G}, 1\rangle$. This suggests that Spoiler's mistake was to move in a way in which Duplicator could respond with a similar point.

(2) In a different play of the game Spoiler picks $1 \in \tau$ together with the point 1 of $\mathfrak{F}$. Now the rules of the game force Duplicator to pick the point 1 of $\mathfrak{G}$. Next Spoiler picks $2 \in \tau$ and the point 4 of $\mathfrak{F}$. Duplicator responds by picking 4 in $\mathfrak{G}$. Now Spoiler is unable to make another move and hence he loses the game. This play is given by

$$\langle 0, 0\rangle, \langle 1, 1, 1\rangle, \langle 4, 4, 2\rangle$$

Or more graphically given by:

$$\begin{tikzcd} 0 \arrow[d, "R_1"] & 0 \arrow[d, "S_1"] \\ 1 \arrow[d, "R_2"] & 1 \arrow[d, "S_2"] \\ 4 & 4 \end{tikzcd}$$

Once again we can examine the game with an appeal to similarity: $\langle \mathfrak{F}, 0\rangle$ is not similar to $\langle \mathfrak{G}, 0\rangle$; $\langle \mathfrak{F}, 1\rangle$ is not similar to $\langle \mathfrak{G}, 1\rangle$; but $\langle \mathfrak{F}, 4\rangle$ is similar to $\langle \mathfrak{G}, 4\rangle$. This reinforces the intuition that Spoiler's mistake was to move in a way in which Duplicator could respond with a similar point.

(3) In the final play that we examine, the first round takes place in the same way as in the previous play, and in his second turn Spoiler picks $1 \in \tau$ together with the point 2 of $\mathfrak{F}$. Duplicator cannot respond to this move and hence he loses the game. This play is given by

$$\langle 0, 0\rangle, \langle 1, 1, 1\rangle, \langle 2, -, 1\rangle$$

Or more graphically given by:

$$\begin{tikzcd} 0 \arrow[d, "R_1"] & 0 \arrow[d, "S_1"] \\ 1 \arrow[d, "R_2"] & 1 \\ 2 & - \end{tikzcd}$$

In the context of similarity we observe that Spoiler was able to prevent Duplicator from choosing similar points to his choices: $\langle \mathfrak{F}, 1\rangle$ is not similar to $\langle \mathfrak{G}, 1\rangle$, and there is no point w of $\mathfrak{G}$ such that $1\ S_1\ w$ and that $\langle \mathfrak{F}, 2\rangle$ is similar to $\langle \mathfrak{G}, w\rangle$.

If the game started with points that are similar then the rules of the game together with Definition 1.2.14 would guarantee that Duplicator can always respond with a point to which any point chosen by Spoiler is similar. Therefore in that case Duplicator can extend the game indefinitely, leading to a win for him. If the game started with dissimilar points then Spoiler can continue to choose dissimilar points until Duplicator is unable to move. We formalise this intuition for all four of the games in the following proposition.

Proposition 2.3.2 (Goranko & Otto (2007)). *Given two points v_0 and w_0 of $\mathfrak{F}$ and $\mathfrak{G}$ respectively, and two τ-models*

$$\mathfrak{N}_1 = \langle \mathfrak{F}, U_1 \rangle \qquad \textit{and} \qquad \mathfrak{N}_2 = \langle \mathfrak{G}, U_2 \rangle$$

the following hold:

(1) $\langle \mathfrak{F}, v_0 \rangle$ *is similar to* $\langle \mathfrak{G}, w_0 \rangle$ *if and only if Duplicator has a winning strategy in the simulation game from* $\mathfrak{F}$ *to* $\mathfrak{G}$ *starting at* $\langle v_0, w_0 \rangle$.

(2) $\langle \mathfrak{N}_1, v_0 \rangle$ *is similar to* $\langle \mathfrak{N}_2, w_0 \rangle$ *if and only if Duplicator has a winning strategy in the simulation game from* $\mathfrak{N}_1$ *to* $\mathfrak{N}_2$ *starting at* $\langle v_0, w_0 \rangle$.

(3) $\langle \mathfrak{F}, v_0 \rangle$ *and* $\langle \mathfrak{G}, w_0 \rangle$ *are bisimilar if and only if Duplicator has a winning strategy in the bisimulation game between* $\mathfrak{F}$ *and* $\mathfrak{G}$ *starting at* $\langle v_0, w_0 \rangle$.

(4) $\langle \mathfrak{N}_1, v_0 \rangle$ *and* $\langle \mathfrak{N}_2, w_0 \rangle$ *are bisimilar if and only if Duplicator has a winning strategy in the bisimulation game between* $\mathfrak{F}$ *and* $\mathfrak{G}$ *starting at* $\langle v_0, w_0 \rangle$.

The various notions of simulation and bisimulation may be generalized, or rather *approximated*, by restricting the length of their games. This can be done by halting an otherwise unfinished game after a predetermined number of rounds, and defaulting the game immediately to a win for Duplicator. We call such a modified simulation game (resp. bisimulation game) an *n-round simulation game* (resp. *n-round bisimulation game*) if its play is limited to n rounds. Winning strategies may then be defined analogously.

Definition 2.3.3 (Goranko & Otto (2007)). Given two points v_0 and w_0 of $\mathfrak{F}$ and $\mathfrak{G}$ respectively, and two τ-models $\mathfrak{N}_1 = \langle \mathfrak{F}, U_1 \rangle$ and $\mathfrak{N}_2 = \langle \mathfrak{G}, U_2 \rangle$.

- $\langle \mathfrak{F}, v_0 \rangle$ is *n-similar* to $\langle \mathfrak{G}, w_0 \rangle$ if Duplicator has a winning strategy for the n-round simulation game from $\mathfrak{F}$ to $\mathfrak{G}$ starting at $\langle v_0, w_0 \rangle$.
- $\langle \mathfrak{N}_1, v_0 \rangle$ is *n-similar* to $\langle \mathfrak{N}_2, w_0 \rangle$ if Duplicator has a winning strategy for the n-round simulation game from $\mathfrak{N}_1$ to $\mathfrak{N}_2$ starting at $\langle v_0, w_0 \rangle$.
- $\langle \mathfrak{F}, v_0 \rangle$ and $\langle \mathfrak{G}, w_0 \rangle$ are *n-bisimilar* if Duplicator has a winning strategy for the n-round bisimulation game between $\mathfrak{F}$ and $\mathfrak{G}$ starting at $\langle v_0, w_0 \rangle$.

- $\langle \mathfrak{N}_1, v_0 \rangle$ and $\langle \mathfrak{N}_2, w_0 \rangle$ are *n-bisimilar* if Duplicator has a winning strategy for the n-round bisimulation game between $\mathfrak{N}_1$ and $\mathfrak{N}_2$ starting at $\langle v_0, w_0 \rangle$.

Remark 2.3.4. Observe that 0-similarity and 0-bisimilarity does not say anything about the relation structure of the τ-frames or τ-models involved: all points in all τ-frames are considered 0-similar and 0-bisimilar; also 0-similarity and 0-bisimilarity for points of models merely restricts the truth values of atoms at those points.

Just as similarity, n-similarity is not a symmetric relation in general.

In light of Proposition 2.3.2 and Definition 2.3.3, it is immediate that n-similarity (resp. n-bisimilarity) is implied by similarity (resp. bisimilarity), for $n \in \mathbb{N}$. And also that n-similarity (resp. n-bisimilarity) implies m-similarity (resp. m-bisimilarity) if $m \leq n$, for $m, n \in \mathbb{N}$. The converse of these are not true, as exhibited by the following example.

Example 2.3.5. Suppose that $\mathfrak{F}$ and $\mathfrak{G}$ are given by

$$\mathfrak{F}: \quad 0 \xrightarrow{R} 1 \xrightarrow{R} 2 \qquad\qquad \mathfrak{G}: \quad \begin{tikzcd} 0 \arrow[loop right, "S"] \end{tikzcd}$$

Now $\langle \mathfrak{F}, 0 \rangle$ and $\langle \mathfrak{G}, 0 \rangle$ are 2-bisimilar but not 3-bisimilar or bisimilar.

Definition 2.3.3 yields counterparts to Proposition 1.2.16 and Proposition 1.2.27, but before we can formulate these we need a notion for the number of consecutive transitions that a formula observes.

Definition 2.3.6 (Goranko *&* Otto (2007))**.** For any type τ, we define the *nesting depth* δ of a τ-formula recursively as follows (where $p \in \Phi$; $i \in \tau$; and $\phi, \psi, \phi_1, \phi_2, \ldots, \phi_{\rho(i)} \in \mathsf{ML}_\tau$)

- $\delta(\bot) = \delta(p) = 0$
- $\delta(\phi \to \psi) = \max\{\delta(\phi), \delta(\psi)\}$
- $\delta\left(\Diamond_i \phi_1\, \phi_2\, \ldots\, \phi_{\rho(i)}\right) = 1 + \max\left\{\delta(\phi_1), \delta(\phi_2), \ldots \delta\left(\phi_{\rho(i)}\right)\right\}$.

Proposition 2.3.7 (Goranko *&* Otto (2007))**.** *Given two points v_0 and w_0 of $\mathfrak{F}$ and $\mathfrak{G}$ respectively, and two τ-models*

$$\mathfrak{N}_1 = \langle \mathfrak{F}, V \rangle \quad \textit{and} \quad \mathfrak{N}_2 = \langle \mathfrak{G}, U \rangle,$$

then the following holds for every $n \in \mathbb{N}$.

(1) *If $\langle \mathfrak{N}_1, v_0 \rangle$ is n-similar to $\langle \mathfrak{N}_2, w_0 \rangle$ then for all $\phi \in \mathsf{PE}_\tau$ with $\delta(\phi) \leq n$ we have*

$$\mathfrak{N}_1, v_0 \Vdash \phi \quad \textit{implies} \quad \mathfrak{N}_2, w_0 \Vdash \phi.$$

(2) *If $\langle \mathfrak{N}_1, v_0\rangle$ and $\langle \mathfrak{N}_2, w_0\rangle$ are n-bisimilar, then for all $\phi \in \mathsf{ML}_\tau$ with $\delta(\phi) \leq n$ we have*

$$\mathfrak{N}_1, v_0 \Vdash \phi \quad \textit{if and only if} \quad \mathfrak{N}_2, w_0 \Vdash \phi.$$

We now return to the model $\mathfrak{M} = \langle \mathfrak{F} \otimes \mathfrak{G}, V\rangle$ and express n-similarity and n-bisimilarity of points in $\mathfrak{F}$ and $\mathfrak{G}$ in terms of formulae satisfied in $\mathfrak{M}$. We have stated before that the formula $\boxminus_i \diamondbar_i \phi$ can be read as *for every transition in $\mathfrak{F}$ there is a transition in $\mathfrak{G}$ to a point where ϕ is valid.* In the simulation game from $\mathfrak{F}$ to $\mathfrak{G}$ we may interpret this as *for every possible selection that Spoiler can make in a certain round, Duplicator can respond with a selection that satisfies the rules of the game.* We now use this interpretation to inductively construct a formula θ_n that we will show characterizes n-similarity.

Notation 2.3.8. Suppose that τ is finite, then for $n \in \mathbb{N}$ define

$$\begin{aligned} \theta_0 &= \top \\ \theta_{n+1} &= \bigwedge_{j\in\tau} \boxminus_j \diamondbar_j \theta_n. \end{aligned}$$

The assumption that τ is finite is necessary since otherwise θ_n (for $n > 0$) will not be a formula in ML_τ. Observe that this complication cannot be readily addressed by replacing θ_n with say $|\tau|$ many formulae. The canonical way to do this would be to move the conjunction over τ from the language into the meta language. This is not always possible as the expression for θ_2 shows:

$$\begin{aligned} \theta_2 &= \bigwedge_{j\in\tau} \boxminus_j \diamondbar_j \theta_1 \quad \text{(by definition of } \theta_2\text{, Notation 2.3.8)} \\ &= \bigwedge_{j\in\tau} \boxminus_j \diamondbar_j \left(\bigwedge_{k\in\tau} \boxminus_k \diamondbar_k \theta_0 \right) \quad \text{(by definition of } \theta_1\text{, Notation 2.3.8)} \\ &= \bigwedge_{j\in\tau} \boxminus_j \diamondbar_j \left(\bigwedge_{k\in\tau} \boxminus_k \diamondbar_k \top \right) \quad \text{(by definition of } \theta_0\text{, Notation 2.3.8)} \end{aligned}$$

Now it is well known that the $\diamondbar_j$ operator cannot be distributed over the conjunction $\bigwedge_{k\in\tau}$; this can be verified by inspecting their semantics (Definition 1.2.7).

We also define a formula γ_n that we will show characterizes n-bisimilarity.

Notation 2.3.9. For $n \in \mathbb{N}$ define

$$\begin{aligned} \gamma_0 &= \top \\ \gamma_{n+1} &= \bigwedge_{j\in\tau} \left(\boxminus_j \diamondbar_j \gamma_n \wedge \boxbar_j \diamondminus_j \gamma_n \right) \end{aligned}$$

Proposition 2.3.10. *Consider the model $\mathfrak{M} = \langle \mathfrak{F} \otimes \mathfrak{G}, V\rangle$, together with a point $\langle v_0, w_0\rangle$ of $\mathfrak{M}$. The following hold for every $n \in \mathbb{N}$:*

(1) $\mathfrak{M}, \langle v_0, w_0 \rangle \Vdash \theta_n$ *if and only if* $\langle \mathfrak{F}, v_0 \rangle$ *is* n*-similar to* $\langle \mathfrak{G}, w_0 \rangle$.

(2) $\mathfrak{M}, \langle v_0, w_0 \rangle \Vdash \gamma_n$ *if and only if* $\langle \mathfrak{F}, v_0 \rangle$ *and* $\langle \mathfrak{G}, w_0 \rangle$ *are* n*-bisimilar.*

Proof. (1) We use induction on n. For $n = 0$ the left-hand side of the claim becomes $\mathfrak{M}, \langle v_0, w_0 \rangle \Vdash \top$, which always holds. The right-hand side of the claim is that $\langle \mathfrak{F}, v_0 \rangle$ is 0-similar to $\langle \mathfrak{G}, w_0 \rangle$, but according to Remark 2.3.4 this is also always true, which proves the case when $n = 0$.

Now suppose that $\mathfrak{M}, \langle v_0, w_0 \rangle \Vdash \theta_{n+1}$. According to the definition of θ_{n+1} (Notation 2.3.8) this is $\mathfrak{M}, \langle v_0, w_0 \rangle \Vdash \bigwedge_{j \in \tau} \boxminus_j \diamondminus_j \theta_n$. According to the semantics of $\bigwedge_{j \in \tau}$ (Definition 1.2.7) we have that for every $j \in \tau$ it holds that $\mathfrak{M}, \langle v_0, w_0 \rangle \Vdash \boxminus_j \diamondminus_j \theta_n$. According to the semantics of $\boxminus_j$ and $\diamondminus_j$ (Notation 1.3.6) we can rewrite the assumption as: for every $j \in \tau$ and for every $v_1 \in F$ such that $v_0 R_j v_1$ there is some $w_1 \in G$ such that $w_0 S_j w_1$ and $\mathfrak{M}, \langle v_1, w_1 \rangle \Vdash \theta_n$. According to the induction hypothesis the latter part of this statement claims that Duplicator has a winning strategy for the n-round simulation game from $\mathfrak{F}$ to $\mathfrak{G}$ starting at $\langle v_1, w_1 \rangle$; while the former part outlines the rules for the first round of the simulation game starting at $\langle v_0, w_0 \rangle$. Together this is equivalent to Duplicator having a winning strategy in the $n+1$-round simulation game from $\mathfrak{F}$ to $\mathfrak{G}$ starting at $\langle v_0, w_0 \rangle$.

(2) This can be proved analogously to part 1.

$\square$

Chapter 3

A category-theoretic view of frame products

3.1 Introduction

Categorical products abound throughout mathematics: simple examples of categorical products include the cartesian product of sets, the direct product of groups and the product of topological spaces. Hence we re-examine the frame product, and coming from a category theoretic viewpoint we ask whether it can be seen as a categorical product. So we aim to answer the following:

(1) Is the frame product a categorical product in some "useful" category?

And if this question proves false we may also ask:

(2) What is the categorical product of two frames?

(3) In which way does the frame product mimic a categorical product?

So far we have avoided one very crucial detail to answering these three questions: what constitutes a "useful" category for studying frames? In the next section we give some background on category theory and show two categories of frames that seem to be useful. After attempting to answer these questions in these two categories we suggest a third category in Section 3.4.

3.2 Categorical background

For this chapter we assume familiarity with the very basic category-theoretic definitions and tools, but we will spend a little time to remind the reader of some standard definitions that a non-category theorist may not be familiar with. We also outline some of our own conventions for our presentation.

To prevent confusion with the homomorphisms of Definition 1.2.13 we will consistently use the term *arrow* to refer to the arrows or morphisms of a category. We will refer to a τ-frame isomorphism in the sense of Definition 1.2.13 as a *frame-isomorphism* and when stating that two frames are *isomorphic* we will mean this according to the category theoretic definition. To emphasise that the latter is relative to a specific category, say $\mathbb{C}$, we will say the frames are $\mathbb{C}$-isomorphic. We will adopt a similar approach to classifying many of the other concepts that a category theorist would define relative to a specific category, so we may also refer to $\mathbb{C}$-products, $\mathbb{C}$-automorphisms etc.

In the spirit of Mac Lane (1997) we denote the *is an object of* relation by "$\in$" and the *is an arrow of* relation by "in".

To address the questions posed at the beginning of this chapter we need to specify the categories that we will consider. To this end we first consider the following lemma.

Lemma 3.2.1 (Blackburn *et al.* (2001))**.**

(1) *The composite of two τ-frame homomorphisms is a τ-frame homomorphism, and this composition is associative.*

(2) *For any τ-frame the identity function on its universe is a τ-frame homomorphism between that τ-frame and itself.*

(3) *The composite of two bounded τ-frame morphisms is a bounded τ-frame morphism, and this composition is associative.*

(4) *For any τ-frame the identity function on its universe is a bounded τ-frame morphism between that τ-frame and itself.*

Notation 3.2.2. Let **Set** denote the category with sets as objects and functions as arrows. In light of Lemma 3.2.1 the following categories are well-defined:

- Let $\mathbf{FH}_\tau$ denote the category with τ-frames as objects and τ-frame homomorphisms as arrows (our mnemonic is ***F**rames with **H**omomorphisms of type τ*).

- Let $\mathbf{FBM}_\tau$ denote the category with τ-frames as objects and bounded τ-frame morphisms as arrows (our mnemonic is ***F**rames with **B**ounded **M**orphisms of type τ*).

Remark 3.2.3. Lemma 1.2.32 can be easily used to show that the notions of frame-isomorphism, $\mathbf{FH}_\tau$-isomorphism and $\mathbf{FBM}_\tau$-isomorphism coincide.

Since we will be working mostly with binary categorical products the following terminology will be useful.

Definition 3.2.4 (Mac Lane (1997))**.** Given a category $\mathbb{C}$, a $\mathbb{C}$-*span* is a pair of arrows in $\mathbb{C}$ with common domain. Since the domain is specified by the arrows we can also denote a $\mathbb{C}$-span as a triple $\langle C, c_1, c_2\rangle$, where C is the domain of the arrows c_1, c_2 in $\mathbb{C}$. This means that for some $A, B \in \mathbb{C}$ the following is a diagram in $\mathbb{C}$

$$A \xleftarrow{c_1} C \xrightarrow{c_2} B$$

We will also refer to $\langle C, c_1, c_2\rangle$ as being a $\mathbb{C}$-*span between* A *and* B, and call C the *domain* of the $\mathbb{C}$-span.

The notion of a span generalizes the notion of a binary relation. In this same spirit one may prove the well known result that a (binary) categorical product is the terminal object in an appropriate category of spans.

In contrast to standard treatments of category theory we will not denote general categorical products using $\times$. Instead we reserve the use of this notation only for the cartesian product of sets, for induced functions between such products of sets, and for products of categories. This is unambiguous. The reason we avoid this notation for categorical products of frames is that we will consider different categories where categorical products of the same pair of frames may differ.

The definition of when a functor is called an embedding is not universally agreed on, we will use the following:

Definition 3.2.5 (Adámek *et al.* (2004))**.** A functor $\mathcal{F} : \mathbb{C} \to \mathbb{D}$ is called an *embedding* if it is injective on arrows.

Note that since an embedding functor is also injective on identity arrows it will be injective on objects as well, in fact embedding functors are exactly those functors which are faithful and injective on objects. Furthermore, given an embedding functor $\mathcal{F} : \mathbb{C} \to \mathbb{D}$, it can be shown that the image of $\mathbb{C}$ under $\mathcal{F}$ is a subcategory of $\mathbb{D}$ that is isomorphic to $\mathbb{C}$ in the category of categories.

Another notion that we will need later is a formalism for "functor in two variables" — this is called a bifunctor.

Definition 3.2.6 (Mac Lane (1997))**.** A *bifunctor* $\mathcal{F} : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$ is a functor from the product of two categories $\mathbb{A} \times \mathbb{B}$ to a category $\mathbb{C}$.

Note that here $\times$ is used to denote the product of two categories in the category of categories. Mac Lane (1997) compares the way that the product of categories is used to define a bifunctor, to the way the product of topological spaces can be used to define a continuous function in two variables. Although we will not need to construct the product of categories, it can be constructed using pairs of objects and pairs of arrows, so using this definition for "functor in two variables" makes sense. For the sake of simplicity, when using bifunctors we will appeal to the following result that states how bifunctors are characterized by their "component-wise" behaviour.

Lemma 3.2.7 (Mac Lane (1997))**.** *Given categories $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$, together with two families of functors*

$$(\mathcal{L}_B : \mathbb{A} \to \mathbb{C})_{B \in \mathbb{B}} \qquad (\mathcal{R}_A : \mathbb{B} \to \mathbb{C})_{A \in \mathbb{A}}$$

such that for all objects $A \in \mathbb{A}$, $B \in \mathbb{B}$ and all arrows $f : A_1 \to A_2$ in $\mathbb{A}$ and $g : B_1 \to B_2$ in $\mathbb{B}$ we have

- $\mathcal{L}_B(A) = \mathcal{R}_A(B)$
- $(\mathcal{L}_{B_2}(f)) \circ (\mathcal{R}_{A_1}(g)) = (\mathcal{R}_{A_2}(g)) \circ (\mathcal{L}_{B_1}(f))$

There is a unique bifunctor $\mathcal{F} : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$ such that $\mathcal{F}(-, B) = \mathcal{L}_B$ and $\mathcal{F}(A, -) = \mathcal{R}_A$ (for all $A \in \mathbb{A}$, $B \in \mathbb{B}$).

For a bifunctor $\mathcal{F} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ (for some category $\mathbb{C}$) one may ask whether its behaviour on $\mathbb{C}$ is similar to the behaviour of the multiplication on a monoid. A classical example of such a bifunctor is the cartesian product of sets in the category of sets (here the singleton set is the "unity element"). However it should be noted that the cartesian product of sets is only associative *up to* **Set**-*isomorphism*. So it is reasonable that the appropriate notion of "a category with multiplication", or rather *monoidal category*, should include the appropriate isomorphisms as well. The following definition captures this idea. Note that here we use the notation 1_A to denote the identity arrow on an object A.

Definition 3.2.8 (Mac Lane (1997))**.** A tuple

$$\langle \mathbb{C}, \boxtimes, E, \alpha, \lambda, \varrho \rangle$$

is called a *monoidal category* if the following hold:

- $\mathbb{C}$ is a category.
- $\boxtimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a bifunctor.
- $E \in \mathbb{C}$,
- α is a natural $\mathbb{C}$-isomorphism:

$$\alpha := (\alpha_{A,B,C} : A \boxtimes (B \boxtimes C) \to (A \boxtimes B) \boxtimes C)_{A,B,C \in \mathbb{C}}.$$

- λ and ϱ are natural $\mathbb{C}$-isomorphisms:

$$\lambda := (\lambda_A : E \boxtimes A \to A)_{A \in \mathbb{C}} \qquad \varrho := (\varrho_A : A \boxtimes E \to A)_{A \in \mathbb{C}}$$

- For every $A, B, C, D \in \mathbb{C}$ the following pentagon commutes:

$$\begin{tikzcd}
& A \boxtimes (B \boxtimes (C \boxtimes D)) \arrow[dl, "1_A \boxtimes \alpha_{B,C,D}"'] \arrow[dr, "\alpha_{A,B,C\boxtimes D}"] & \\
A \boxtimes ((B \boxtimes C) \boxtimes D) \arrow[d, "\alpha_{A,B\boxtimes C,D}"'] & & (A \boxtimes B) \boxtimes (C \boxtimes D) \arrow[d, "\alpha_{A\boxtimes B,C,D}"] \\
(A \boxtimes (B \boxtimes C)) \boxtimes D \arrow[rr, "\alpha_{A,B,C} \boxtimes 1_D"'] & & ((A \boxtimes B) \boxtimes C) \boxtimes D
\end{tikzcd}$$

  i.e. that

$$(\alpha_{A,B,C} \boxtimes 1_D) \circ (\alpha_{A,B\boxtimes C,D}) \circ (1_A \boxtimes \alpha_{B,C,D}) = (\alpha_{A\boxtimes B,C,D}) \circ (\alpha_{A,B,C\boxtimes D})$$

- For every $A, B \in \mathbb{C}$ the following triangle commutes:

$$\begin{tikzcd}
A \boxtimes (E \boxtimes B) \arrow[rr, "\alpha_{A,E,B}"] \arrow[dr, "1_A \boxtimes \lambda_B"'] & & (A \boxtimes E) \boxtimes B \arrow[dl, "\varrho_A \boxtimes 1_B"] \\
& A \boxtimes B &
\end{tikzcd}$$

  i.e. that

$$1_A \boxtimes \lambda_B = (\varrho_A \boxtimes 1_B) \circ \alpha_{A,E,B}$$

We will refer to $\boxtimes$ as the *multiplication* of $\langle \mathbb{C}, \boxtimes, E, \alpha, \lambda, \varrho \rangle$.

Remark 3.2.9. The example of a monoidal category using the cartesian product of sets, as suggested above, is only one example of a much more general result. In our next result, Proposition 3.2.11, we see that any category with finite products (and a terminal object) is monoidal category with multiplication given by fixing a product operation. Our motivation for introducing Proposition 3.2.11 is twofold; firstly, it motivates why monoidal categories are important to us, since it formalizes how the notion of *monoidal category* is a generalisation of *category with finite products*. So if the frame product fails to be a categorical product in all our categories, then exhibiting a monoidal category with the frame product as its multiplication would yield a partial answer to the question posed at the beginning of this chapter.

> (3) In which way does the frame product mimic a categorical product?

Our second reason for introducing Proposition 3.2.11 is that we will need it again in Section 3.5 to prove that the frame product is the multiplication of a particular monoidal category.

Convention 3.2.10. Our next result involves some complicated diagrams with category-theoretic products. To make them less cluttered and to reduce the number of names for arrows we have opted not to label the "obvious" projections, but all other arrows are named.

Proposition 3.2.11 (Mac Lane (1997))**.** *Let $\mathbb{C}$ be a category with finite $\mathbb{C}$-products and $\mathbb{C}$-terminal object E. For any $A, B \in \mathbb{C}$ let $A \boxtimes B$ denote a (fixed) $\mathbb{C}$-product of A and B. For any two arrows $A \xrightarrow{f} B, C \xrightarrow{g} D$ in $\mathbb{C}$ let $f \boxtimes g$ denote the unique arrow such that the following diagram commutes*

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \boxtimes C \arrow[l] \arrow[r] \arrow[d, dotted, "f\boxtimes g"] & C \arrow[d, "g"] \\
B & B \boxtimes D \arrow[l] \arrow[r] & D
\end{tikzcd}$$

For any $A, B, C \in \mathbb{C}$ define $\alpha_{A,B,C} : A \boxtimes (B \boxtimes C) \to (A \boxtimes B) \boxtimes C$ as the unique arrow such that the following diagram commutes

$$\begin{tikzcd}
A \boxtimes (B \boxtimes C) \arrow[r] \arrow[d] \arrow[ddrr, dotted, bend left, "\alpha_{A,B,C}"] & (B \boxtimes C) \arrow[d] \arrow[dr] & \\
A & B & C \\
& (A \boxtimes B) \arrow[ul] \arrow[u] & (A \boxtimes B) \boxtimes C \arrow[l] \arrow[u]
\end{tikzcd}$$

For any $A \in \mathbb{C}$ define $\lambda_A : E \boxtimes A \to A$ as the second projection from the $\mathbb{C}$-product $E \boxtimes A$, similarly define $\varrho_A : A \boxtimes E \to A$ as the first projection from the $\mathbb{C}$-product $A \boxtimes E$. Let $\alpha := (\alpha_{A,B,C})_{A,B,C \in \mathbb{C}}$, $\lambda := (\lambda_A)_{A \in \mathbb{C}}$ and $\varrho := (\varrho_A)_{A \in \mathbb{C}}$. Then α, λ and ϱ are natural $\mathbb{C}$-isomorphisms. Moreover, $\langle \mathbb{C}, \boxtimes, E, \alpha, \lambda, \varrho \rangle$ is a monoidal category.

Remark 3.2.12. Not all monoidal categories are induced by binary products. The most prominent examples of monoidal categories that are not induced by a product are those given by "tensor products": the tensor product in the category of abelian groups and the tensor product in the category of vector spaces are only two examples of this. Mac Lane (1997) discusses more examples in further detail.

3.3 Categorical products using frame homomorphisms and bounded frame morphisms

Corollary 1.2.18 and Corollary 1.2.30 motivate our interest in $\mathbf{FH}_\tau$ and $\mathbf{FBM}_\tau$. Recall the following question posed at the beginning of this chapter.

> (1) Is the frame product a categorical product in some "useful" category?

It follows from Remark 1.3.2 that the type of the frame product almost always differs from the types of the original frames. The exception being when all the types are empty, but $\mathbf{FH}_\varnothing$, $\mathbf{FBM}_\varnothing$ and **Set** are isomorphic categories, so although we can prove that $\mathbf{FH}_\varnothing$-products and $\mathbf{FBM}_\varnothing$-products are frame products it is quite uninteresting. In any other case however any kind of "projection arrows" from a frame product to its factors would not be τ-frame homomorphisms or bounded τ-frame morphisms, so frame products are not $\mathbf{FH}_\tau$-products or $\mathbf{FBM}_\tau$-products.

In Section 3.4 we will consider a way of introducing projections to obtain a category that behaves somewhat like $\mathbf{FBM}_\tau$, however this will still not lead to the frame product being a categorical product. For now we consider the next question.

> (2) What is the categorical product of two frames?

To answer the question in $\mathbf{FH}_\tau$ we need the following construction.

Definition 3.3.1 (Brink *&* Rewitzky (2004))**.** Given two frames

$$\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle \quad \text{and} \quad \mathfrak{G} = \left\langle G, (S_i)_{i\in\tau} \right\rangle,$$

their *BR-product* is defined as the frame

$$\mathfrak{F} \circledast \mathfrak{G} := \left\langle F \times G, (R_i * S_i)_{i\in\tau} \right\rangle$$

with $F \times G$ denoting the cartesian product of the sets F and G, and for every $i \in \tau$ we define $R_i * S_i \subseteq (F \times G)^{\rho(i)+1}$ as follows:

$$R_i * S_i \left\langle v_0, w_0 \right\rangle \left\langle v_1, w_1 \right\rangle \ldots \left\langle v_{\rho(i)}, w_{\rho(i)} \right\rangle \text{ iff } R_i v_0 v_1 \ldots v_{\rho(i)} \text{ and } S_i w_0 w_1 \ldots w_{\rho(i)}$$

Remark 3.3.2. At first the BR-product may seem to generalize the frame product of Definition 1.3.1, by replacing the equality in the construction of, say $R_i^{\leftrightarrow}$, with an arbitrary relation. However, since the BR-product only operates on frames with the same similarity type and constructs a frame having the same similarity type once again, it will rarely happen that the frame product of two frames is also their BR-product.

Proposition 3.3.3. *For two τ-frames*

$$\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle \quad \textit{and} \quad \mathfrak{G} = \left\langle G, (S_i)_{i\in\tau} \right\rangle,$$

their $\mathbf{FH}_\tau$-product is given by $\langle \mathfrak{F} \circledast \mathfrak{G}, \pi_1, \pi_2 \rangle$, where

$$\mathfrak{F} \xleftarrow{\pi_1} \mathfrak{F} \circledast \mathfrak{G} \xrightarrow{\pi_2} \mathfrak{G}$$

is defined by:

$$v \xleftarrow{\pi_1} \langle v, w \rangle \xmapsto{\pi_2} w$$

Proof. First we show that π_1 is a τ-frame homomorphism. Suppose that

$$R_i * S_i \langle v_0, w_0 \rangle \langle v_1, w_1 \rangle \ldots \left\langle v_{\rho(i)}, w_{\rho(i)} \right\rangle,$$

then by definition of $*$ (Definition 3.3.1) we have that $R_i v_0 v_1 \ldots v_{\rho(i)}$, which by definition of π_1 is equivalent to $R_i \pi_1 (\langle v_0, w_0 \rangle) \pi_1 (\langle v_1, w_1 \rangle) \ldots \pi_1 \left(\left\langle v_{\rho(i)}, w_{\rho(i)} \right\rangle \right)$. Similarly, π_2 is a τ-frame homomorphism.

Suppose now that we have another τ-frame $\mathfrak{H} = \left\langle H, (T_i)_{i \in \tau} \right\rangle$ with two τ-frame homomorphisms $\mathfrak{F} \xleftarrow{h_1} \mathfrak{H} \xrightarrow{h_2} \mathfrak{G}$ in $\mathbf{FH}_\tau$. We need a τ-frame homomorphism $\mathfrak{H} \xrightarrow{h} \mathfrak{F} \circledast \mathfrak{G}$ such that the diagram below commutes

$$\begin{tikzcd}
& \mathfrak{H} \arrow[dl, "h_1"'] \arrow[d, dotted, "h"] \arrow[dr, "h_2"] & \\
\mathfrak{F} & \mathfrak{F} \circledast \mathfrak{G} \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \mathfrak{G}
\end{tikzcd}$$

That is, we need to have $h_1 = \pi_1 \circ h$ and $h_2 = \pi_2 \circ h$. The unique function with this property is defined by

$$\begin{aligned} h : H &\to F \times G \\ v &\mapsto \langle h_1(v), h_2(v) \rangle \end{aligned}$$

We now need to show that this definition makes $\mathfrak{H} \xrightarrow{h} \mathfrak{F} \circledast \mathfrak{G}$ a τ-frame homomorphism so that existence is verified as well.

Suppose that $T_i v_0 v_1 \ldots v_{\rho(i)}$, then since both $\mathfrak{H} \xrightarrow{h_1} \mathfrak{F}$ and $\mathfrak{H} \xrightarrow{h_2} \mathfrak{G}$ are τ-frame homomorphisms (Definition 1.2.13), we have $R_i h_1 (v_0) h_1 (v_1) \ldots h_1 \left(v_{\rho(i)} \right)$ and $S_i h_2 (v_0) h_2 (v_1) \ldots h_2 \left(v_{\rho(i)} \right)$. By definition of $*$ (Definition 3.3.1), this is equivalent to $R_i * S_i \langle h_1 (v_0), h_2 (v_0) \rangle \langle h_1 (v_1), h_2 (v_1) \rangle \ldots \left\langle h_1 \left(v_{\rho(i)} \right), h_2 \left(v_{\rho(i)} \right) \right\rangle$.

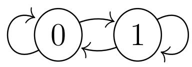
This completes the proof. □

Now that we have a conclusive result for $\mathbf{FH}_\tau$-products, we turn our attention to $\mathbf{FBM}_\tau$. Observe that by Proposition 1.2.37 the $\mathbf{FH}_\tau$-span used in Proposition 3.3.3 will only be a $\mathbf{FBM}_\tau$-span if $F \times G$ is a bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$. This is rarely true, and as we will see in Example 3.3.4, even if it is true the BR-product may still not be a $\mathbf{FBM}_\tau$-product.

In general $\mathbf{FBM}_\tau$-products are significantly more complicated than $\mathbf{FH}_\tau$-products, and we will see that only some exist. We will not attempt to characterize the $\mathbf{FBM}_\tau$-products that do exist and instead provide three examples given by Example 3.3.4, Proposition 3.3.6 and Proposition 3.3.12. The first of these exhibits a case where a binary $\mathbf{FBM}_\tau$-product does not exist, while the two propositions construct particular $\mathbf{FBM}_\tau$-products.

To understand $\mathbf{FBM}_\tau$-products we need a proper understanding of $\mathbf{FBM}_\tau$-spans. Before we defined bounded τ-frame morphisms we stated that they need to both preserve and reflect relation structure. Therefore the projections from a $\mathbf{FBM}_\tau$-product would force the $\mathbf{FBM}_\tau$-product to have structure that is almost like that of its factors. However the universal property that all other $\mathbf{FBM}_\tau$-spans should factor through the $\mathbf{FBM}_\tau$-product means that the $\mathbf{FBM}_\tau$-product should have structure that also resembles that of the domain of every possible $\mathbf{FBM}_\tau$-span between its factors. This latter requirement can be very strict, and the following example due to Gumm & Schröder (2001) demonstrates in a formal way that it is not always possible to account for all $\mathbf{FBM}_\tau$-spans in one τ-frame, consequently not all binary $\mathbf{FBM}_\tau$-products exist.

Example 3.3.4. Let τ be the type specifying a single binary relation. Let $\mathfrak{F} \in \mathbf{FBM}_\tau$ be the complete τ-frame with two points, i.e. let $\mathfrak{F} = \langle F, R \rangle$ with $F = \{0, 1\}$ and $R = \{\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle, \langle 1, 1 \rangle\}$. We can picture this frame as



We will show that the binary $\mathbf{FBM}_\tau$-product of $\mathfrak{F}$ with itself does not exist. We do this in the following four steps:

(1) Construct a τ-frame $\mathfrak{K}$ with arbitrarily large universe.

(2) Show that there is a $\mathbf{FBM}_\tau$-span between $\mathfrak{F}$ and itself that has $\mathfrak{K}$ as domain.

(3) Show that if the $\mathbf{FBM}_\tau$-product in question exists, then there is an injective bounded τ-frame morphism from $\mathfrak{K}$ to said $\mathbf{FBM}_\tau$-product.

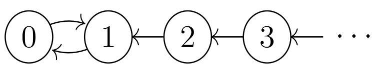
(4) Conclude that the $\mathbf{FBM}_\tau$-product has a cardinality that is greater than any set.

(1) To construct $\mathfrak{K}$, let κ be any infinite ordinal. Now define $\mathfrak{K} = \langle K, Q \rangle \in \mathbf{FBM}_\tau$ with

$$K = \kappa$$

$$vQw \text{ if and only if } \begin{cases} w < v \text{ or} \\ w \leq 1 \end{cases}$$

For example if $\kappa = \omega$ then the τ-frame $\mathfrak{K}$ is the smallest τ-frame with a transitive relation containing



(2) To construct the desired $\mathbf{FBM}_\tau$-span define the functions k_1, k_2 by

$$\begin{array}{rcll} k_1 : K & \to & F & \\ 0 & \mapsto & 1 & \\ v & \mapsto & 0 & \text{for all } v \neq 0 \\ k_2 : K & \to & F & \\ 1 & \mapsto & 1 & \\ v & \mapsto & 0 & \text{for all } v \neq 1 \end{array}$$

We verify that these are bounded τ-frame morphisms, so that we have a $\mathbf{FBM}_\tau$-span $\mathfrak{F} \xleftarrow{k_1} \mathfrak{K} \xrightarrow{k_2} \mathfrak{F}$. This demonstration is not very instructive, but we include it for the sake of completeness.

Since all of the points in $\mathfrak{F}$ are related via R, both k_1 and k_2 immediately satisfy the forward condition of Definition 1.2.21.

To prove that k_1 satisfies the back condition of Definition 1.2.21 suppose that for some $v \in K$ and $x \in F$ we have $k_1(v)\, Rx$. By the definition of R this yields three distinct cases:

- If $k_1(v) = 0$ and $x = 0$ then from the definition of Q it holds that $vQ1$ (regardless of the value of v) and the definition of k_1 gives $k_1(1) = 0 = x$. Hence the back condition holds in this case.
- If $k_1(v) = 0$ and $x = 1$ then from the definition of Q it holds that $vQ0$ (regardless of the value of v) and the definition of k_1 gives $k_1(0) = 1 = x$. Hence the back condition holds in this case.
- If $k_1(v) = 1$ and $x = 0$ then from the definition of k_1 it must hold that $v = 0$. Now observe that $0Q1$ and that $k_1(1) = 0 = x$ so that the back condition holds in this case.
- If $k_1(v) = 1$ and $x = 1$ then from the definition of k_1 it must hold that $v = 0$. Now observe that $0Q0$ and that $k_1(0) = 1 = x$ so that the back condition holds in this case.

We conclude that k_1 is a bounded τ-frame morphism.

To prove that k_2 satisfies the back condition of Definition 1.2.21 suppose that for some $v \in K$ and $x \in F$ we have $k_2(v)\, Rx$. By the definition of R this yields three distinct cases:

- If $k_2(v) = 0$ and $x = 0$ then from the definition of Q it holds that $vQ0$ (regardless of the value of v) and the definition of k_2 gives $k_2(0) = 0 = x$. Hence the back condition holds in this case.
- If $k_2(v) = 0$ and $x = 1$ then from the definition of Q it holds that $vQ1$ (regardless of the value of v) and the definition of k_2 gives $k_2(1) = 1 = x$. Hence the back condition holds in this case.

- If $k_2(v) = 1$ and $x = 0$ then from the definition of k_2 it must hold that $v = 1$. Now observe that $1Q0$ and that $k_2(0) = 0 = x$ so that the back condition holds in this case.
- If $k_2(v) = 1$ and $x = 1$ then from the definition of k_2 it must hold that $v = 1$. Now observe that $1Q1$ and that $k_2(1) = 1 = x$ so that the back condition holds in this case.

We conclude that k_2 is a bounded τ-frame morphism.

Now we have a $\mathbf{FBM}_\tau$-span $\mathfrak{F} \xleftarrow{k_1} \mathfrak{K} \xrightarrow{k_2} \mathfrak{F}$, as needed.

(3) For a contradiction suppose that the $\mathbf{FBM}_\tau$-product of $\mathfrak{F}$ with itself is given by

$$\mathfrak{F} \xleftarrow{g_1} \mathfrak{G} = \langle G, S\rangle \xrightarrow{g_2} \mathfrak{F}$$

Then from the definition of a $\mathbf{FBM}_\tau$-product there must be a bounded τ-frame morphism $\mathfrak{K} \xrightarrow{k} \mathfrak{G}$ such that $k_1 = g_1 \circ k$ and $k_2 = g_2 \circ k$, i.e. the following diagram commutes

$$\begin{tikzcd}
& \mathfrak{K} \arrow[dl, "k_1"'] \arrow[d, dotted, "k"] \arrow[dr, "k_2"] & \\
\mathfrak{F} & \mathfrak{G} \arrow[l, "g_1"] \arrow[r, "g_2"'] & \mathfrak{F}
\end{tikzcd}$$

To show that k is an injective function suppose that it is not, and let $v < \kappa$ be the minimal point in $\mathfrak{K}$ such that there is some $c > 0$ satisfying $k(v) = k(v + c)$. Let $x = k(v)$. A standard result on ordinal arithmetic shows $v < v + c$ (for more details and a proof see Jech (2003)), hence $(v + c)\,Qv$. Using the forward condition on k (Definition 1.2.21) yields $k(v + c)\,Sk(v)$, according to the choice of x this is xSx. We can interpret xSx as $k(v)\,Sx$ instead, and apply the back condition on k (Definition 1.2.21) to obtain a $w < \kappa$ such that vQw and $k(w) = x$. The definition of Q now yields two mutually exclusive cases:

- If $w < v$, then a standard result on ordinal arithmetic states that a $d > 0$ exists such that $w + d = v$ (for more details and a proof see Jech (2003)). Now applying k gives $k(w + d) = k(v) = x$, and w was chosen such that $k(w) = x$, hence we have $k(w + d) = k(w)$. Now the existence of w contradicts the minimality of v.
- If $w \leq 1$ and $v \leq w$ then either $v = 0$ or $v = 1$. We show that both of these cases yield contradictions:
  - Suppose that $v = 0$. We chose v and c such that

$$k(v) = k(v + c)$$

so composition with g_1 yields

$$(g_1 \circ k)(v) = (g_1 \circ k)(v + c)$$

and since k was chosen such that $k_1 = g_1 \circ k$ we have

$$k_1(v) = k_1(v + c)$$

However evaluating this expression with the assumption that $v = 0$ gives a contradiction:

$$1 = k_1(0) = k_1(c) = 0$$

– Suppose that $v = 1$. We chose v and c such that

$$k(v) = k(v + c)$$

so composition with g_2 yields

$$(g_2 \circ k)(v) = (g_2 \circ k)(v + c)$$

and since k was chosen such that $k_2 = g_2 \circ k$ we have

$$k_2(v) = k_2(v + c)$$

However evaluating this expression with the assumption that $v = 1$ gives a contradiction:

$$1 = k_2(1) = k_2(1 + c) = 0$$

Having contradicted the existence of v we conclude that k is an injective function.

(4) Now we have an injection from any cardinal κ into the universe of the $\mathbf{FBM}_\tau$-product. So that for every cardinal κ we have $|\kappa| \leq |G|$. We conclude that the universe of $\mathfrak{G}$ must be a proper class, not a set, and hence that the $\mathbf{FBM}_\tau$-product of $\mathfrak{F}$ with itself does not exist.

Remark 3.3.5. Although Example 3.3.4 shows that some $\mathbf{FBM}_\tau$-products do not exist there are some $\mathbf{FBM}_\tau$-products that do exist. One way to guarantee the existence of a $\mathbf{FBM}_\tau$-product is to choose factors that have few enough $\mathbf{FBM}_\tau$-spans between them. Possibly the most trivial example of this is when there is exactly one $\mathbf{FBM}_\tau$-span between two τ-frames. The $\mathbf{FBM}_\tau$-span that will always exist is the one with empty universe for its domain. This situation is characterized by the following proposition.

Proposition 3.3.6. *For two τ-frames*

$$\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle \quad \textit{and} \quad \mathfrak{G} = \left\langle G, (S_i)_{i\in\tau} \right\rangle,$$

their $\mathbf{FBM}_\tau$-product has empty universe if and only if the maximum bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$ is empty.

Before we can prove Proposition 3.3.6 we need a better understanding of the relationship between τ-frame bisimulations and $\mathbf{FBM}_\tau$-spans. Proposition 1.2.37 already showed that any τ-frame bisimulation gives rise to a $\mathbf{FBM}_\tau$-span. Also since the domain of a $\mathbf{FBM}_\tau$-span obtained in this way is exactly the original τ-frame bisimulation, it follows that the construction demonstrated by Proposition 1.2.37 is injective. This injectivity shows that there are at least as many $\mathbf{FBM}_\tau$-spans between two given τ-frames as there are bisimulations between them. We now show that $\mathbf{FBM}_\tau$-spans also τ-frame specify bisimulations, although this is not in a way that gives a general inverse to the construction of Proposition 1.2.37.

Example 3.3.7. We construct a τ-frame bisimulation from a $\mathbf{FBM}_\tau$-span. To demonstrate this, consider a τ-frame $\mathfrak{H} = \left\langle H, (T_i)_{i\in\tau} \right\rangle$, and a $\mathbf{FBM}_\tau$-span $\mathfrak{F} \xleftarrow{h_1} \mathfrak{H} \xrightarrow{h_2} \mathfrak{G}$. According to Remark 1.2.26 we have that $\{\langle h_1(v), v\rangle | \, v \in H\}$ is a bisimulation between $\mathfrak{F}$ and $\mathfrak{H}$, and also that $\{\langle v, h_2(v)\rangle | \, v \in H\}$ is a bisimulation between $\mathfrak{H}$ and $\mathfrak{G}$. Now it follows from Lemma 1.2.33 that $\{\langle h_1(v), h_2(v)\rangle | \, v \in H\}$ is a bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$.

Now we can prove Proposition 3.3.6.

Proof of Proposition 3.3.6. Recalling Notation 1.2.34 suppose that $\sim_{\mathfrak{F},\mathfrak{G}} = \varnothing$. If there was a $\mathbf{FBM}_\tau$-span between $\mathfrak{F}$ and $\mathfrak{G}$ that had non-empty domain it would contradict the maximality of $\sim_{\mathfrak{F},\mathfrak{G}}$ since the construction of Example 3.3.7 would then yield a non-empty bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$. Now that we know that any $\mathbf{FBM}_\tau$-span between $\mathfrak{F}$ and $\mathfrak{G}$ must have empty domain and since there is exactly one function from the empty set to F (resp. G) there is only one $\mathbf{FBM}_\tau$-span between $\mathfrak{F}$ and $\mathfrak{G}$ (that the empty functions are bounded τ-frame morphisms are vacuously true). Now this "empty $\mathbf{FBM}_\tau$-span" must be the $\mathbf{FBM}_\tau$-product of $\mathfrak{F}$ and $\mathfrak{G}$, since it factors through itself only via the empty function.

For the converse we assume that the $\mathbf{FBM}_\tau$-product of $\mathfrak{F}$ and $\mathfrak{G}$ has empty domain. Proposition 1.2.37 showed that $\sim_{\mathfrak{F},\mathfrak{G}}$ is the domain of a $\mathbf{FBM}_\tau$-span, so the definition of the $\mathbf{FBM}_\tau$-product gives a bounded τ-frame morphism that is a function from $\sim_{\mathfrak{F},\mathfrak{G}}$ to $\varnothing$, which is impossible unless $\sim_{\mathfrak{F},\mathfrak{G}} = \varnothing$. This completes the proof. □

Remark 3.3.8. Proposition 3.3.6 also motivates why we allow empty τ-frame bisimulations in Definition 1.2.24. And although an argument can be made that the statement "$\sim_{\mathfrak{F},\mathfrak{G}} = \varnothing$" may otherwise be rewritten as "there is no bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$", the exclusion of empty τ-frame bisimulations would spoil the relationship between $\mathbf{FBM}_\tau$-spans and τ-frame bisimulations, since we have already allowed empty frames in Definition 1.2.3.

Next we show a class of non-empty $\mathbf{FBM}_\tau$-products.

We have stated in Remark 3.3.5 that we may guarantee the existence of a $\mathbf{FBM}_\tau$-product by choosing factors that have few enough $\mathbf{FBM}_\tau$-spans between them. Considering the relationship identified between τ-frame bisimulations and $\mathbf{FBM}_\tau$-spans we opt to limit the number of bisimilar points between the factors to accomplish this. One way to do this is to take a τ-frame, say $\mathfrak{F}$, and construct the appropriate quotient τ-frame with regard to bisimilarity, say $\mathfrak{F}/\sim$. Now if any of the points in $\mathfrak{F}$ is only bisimilar to its own equivalence class in $\mathfrak{F}/\sim$ then this will allow us to construct the $\mathbf{FBM}_\tau$-product of $\mathfrak{F}$ and $\mathfrak{F}/\sim$. This approach is formalized in the following.

Recall that in Corollary 1.2.35 we have shown that bisimilarity is an equivalence relation, and in Notation 1.2.34 we let $\sim_{\mathfrak{F},\mathfrak{G}}$ denote the maximum bisimulation between any two τ-frames $\mathfrak{F}$ and $\mathfrak{G}$. We will use $[v]_Z$ to denote the equivalence class of v with respect to the equivalence relation Z.

Definition 3.3.9 (Blackburn *&* van Benthem (2007))**.** Given a τ-frame $\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau}\right\rangle$ the *bisimulation quotient of* $\mathfrak{F}$ is defined as the τ-frame

$$\mathfrak{F}/\sim\; = \left\langle F/\sim_{\mathfrak{F},\mathfrak{F}}, (R_i/\sim_{\mathfrak{F},\mathfrak{F}})_{i\in\tau}\right\rangle$$

with

$$R_i/\sim_{\mathfrak{F},\mathfrak{F}} := \left\{ \left\langle [v_0]_{\sim_{\mathfrak{F},\mathfrak{F}}}, [v_1]_{\sim_{\mathfrak{F},\mathfrak{F}}}, \ldots, \left[v_{\rho(i)}\right]_{\sim_{\mathfrak{F},\mathfrak{F}}} \right\rangle \middle|\, R_i v_0 v_1 \ldots v_{\rho(i)} \right\}.$$

Remark 3.3.10. In the previous definition it should be noted that although $\sim_{\mathfrak{F},\mathfrak{F}}$ is not in general a congruence for every R_i, the definition of $R_i/\sim_{\mathfrak{F},\mathfrak{F}}$ is still a sensible one since the forward (or back) condition of Definition 1.2.24 implies that if

$$R_i/\sim_{\mathfrak{F},\mathfrak{F}}\ [v_0]_{\sim_{\mathfrak{F},\mathfrak{F}}} [v_1]_{\sim_{\mathfrak{F},\mathfrak{F}}} \cdots \left[v_{\rho(i)}\right]_{\sim_{\mathfrak{F},\mathfrak{F}}}$$

then for every $w_0 \in [v_0]_{\sim_{\mathfrak{F},\mathfrak{F}}}$ there are

$$w_1 \in [v_1]_{\sim_{\mathfrak{F},\mathfrak{F}}}, w_2 \in [v_2]_{\sim_{\mathfrak{F},\mathfrak{F}}}, \ldots, w_{\rho(i)} \in \left[v_{\rho(i)}\right]_{\sim_{\mathfrak{F},\mathfrak{F}}}$$

such that $R_i w_0 w_1 \ldots w_{\rho(i)}$.

Lemma 3.3.11. *Let* $\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau}\right\rangle$ *be a* τ*-frame.*

(1) *The canonical map for quotient defined by*

$$\begin{array}{rcl} \pi : F & \to & F/\sim_{\mathfrak{F},\mathfrak{F}} \\ v & \mapsto & [v]_{\sim_{\mathfrak{F},\mathfrak{F}}} \end{array}$$

is a bounded morphism from $\mathfrak{F}$ *to* $\mathfrak{F}/\sim$*.*

(2) *Let Y be a bisimulation between $\mathfrak{F}$ and $\mathfrak{F}/\sim$. If $vY[w]_{\sim_{\mathfrak{F},\mathfrak{F}}}$ then*

$$[v]_{\sim_{\mathfrak{F},\mathfrak{F}}} = [w]_{\sim_{\mathfrak{F},\mathfrak{F}}}$$

Proof. (1) That π satisfies the forward condition of Definition 1.2.21 is immediate from the definition of $R_i/\sim_{\mathfrak{F},\mathfrak{F}}$. The observation of Remark 3.3.10 shows that π also satisfies the back condition of Definition 1.2.21.

(2) We first show that the relation $Z \subseteq F \times F$ defined by

$$xZy \text{ iff } xY[y]_{\sim_{\mathfrak{F},\mathfrak{F}}}$$

is a bisimulation between $\mathfrak{F}$ and itself. To show that Z satisfies the forward condition of Definition 1.2.24 suppose that vZw and $R_ivv_1v_2\dots v_{\rho(i)}$. By the definition of Z the former is equivalent to $vY[w]_{\sim_{\mathfrak{F},\mathfrak{F}}}$, now since Y is a τ-frame bisimulation the forward condition of Definition 1.2.24 gives

$$[w_1]_{\sim_{\mathfrak{F},\mathfrak{F}}}, [w_2]_{\sim_{\mathfrak{F},\mathfrak{F}}}, \dots, \left[w_{\rho(i)}\right]_{\sim_{\mathfrak{F},\mathfrak{F}}} \in F/\sim_{\mathfrak{F},\mathfrak{F}}$$

such that

$$R_i/\sim_{\mathfrak{F},\mathfrak{F}} [w]_{\sim_{\mathfrak{F},\mathfrak{F}}} [w_1]_{\sim_{\mathfrak{F},\mathfrak{F}}} [w_2]_{\sim_{\mathfrak{F},\mathfrak{F}}} \dots \left[w_{\rho(i)}\right]_{\sim_{\mathfrak{F},\mathfrak{F}}}$$

and

$$\begin{gathered} v_1Y[w_1]_{\sim_{\mathfrak{F},\mathfrak{F}}} \\ v_2Y[w_2]_{\sim_{\mathfrak{F},\mathfrak{F}}} \\ \vdots \\ v_{\rho(i)}Y\left[w_{\rho(i)}\right]_{\sim_{\mathfrak{F},\mathfrak{F}}}. \end{gathered}$$

According to the observation of Remark 3.3.10 there are

$$w_1' \in [w_1]_{\sim_{\mathfrak{F},\mathfrak{F}}}, w_2' \in [w_2]_{\sim_{\mathfrak{F},\mathfrak{F}}}, \dots, w_{\rho(i)}' \in \left[w_{\rho(i)}\right]_{\sim_{\mathfrak{F},\mathfrak{F}}}$$

such that $R_iww_1'w_2'\dots w_{\rho(i)}'$. The former of these give

$$[w_1']_{\sim_{\mathfrak{F},\mathfrak{F}}} = [w_1]_{\sim_{\mathfrak{F},\mathfrak{F}}}, [w_2']_{\sim_{\mathfrak{F},\mathfrak{F}}} = [w_2]_{\sim_{\mathfrak{F},\mathfrak{F}}}, \dots, \left[w_{\rho(i)}'\right]_{\sim_{\mathfrak{F},\mathfrak{F}}} = \left[w_{\rho(i)}\right]_{\sim_{\mathfrak{F},\mathfrak{F}}},$$

so that

$$\begin{gathered} v_1Y[w_1']_{\sim_{\mathfrak{F},\mathfrak{F}}} \\ v_2Y[w_2']_{\sim_{\mathfrak{F},\mathfrak{F}}} \\ \vdots \\ v_{\rho(i)}Y\left[w_{\rho(i)}'\right]_{\sim_{\mathfrak{F},\mathfrak{F}}} \end{gathered}$$

i.e.

$$\begin{gathered} v_1Zw_1' \\ v_2Zw_2' \\ \vdots \\ v_{\rho(i)}Zw_{\rho(i)}'. \end{gathered}$$

This shows that Z satisfies the forward condition of Definition 1.2.24.

To show that Z satisfies the forward condition of Definition 1.2.24 suppose that vZw and $R_i v v_1 v_2 \ldots v_{\rho(i)}$. By the definition of Z the former is equivalent to $vY[w]_{\sim_{\mathfrak{F},\mathfrak{F}}}$, now since Y is a τ-frame bisimulation the forward condition of

To show that Z satisfies the back condition of Definition 1.2.24 as well, suppose that vZw and $R_i w w_1 w_2 \ldots w_{\rho(i)}$. By the definition of Z the former is equivalent to $vY[w]_{\sim_{\mathfrak{F},\mathfrak{F}}}$, and by definition of $R_i/\sim_{\mathfrak{F},\mathfrak{F}}$ (Definition 3.3.9) the latter implies

$$R_i/\sim_{\mathfrak{F},\mathfrak{F}} [w]_{\sim_{\mathfrak{F},\mathfrak{F}}} [w_1]_{\sim_{\mathfrak{F},\mathfrak{F}}} [w_2]_{\sim_{\mathfrak{F},\mathfrak{F}}} \ldots \left[w_{\rho(i)}\right]_{\sim_{\mathfrak{F},\mathfrak{F}}}.$$

Since Y is a τ-frame bisimulation the back condition of Definition 1.2.24 gives $v_1 v_2 \ldots v_{\rho(i)} \in F$ such that $R_i v v_1 v_2 \ldots v_{\rho(i)}$ and

$$\begin{gathered} v_1 Y [w_1]_{\sim_{\mathfrak{F},\mathfrak{F}}} \\ v_2 Y [w_2]_{\sim_{\mathfrak{F},\mathfrak{F}}} \\ \vdots \\ v_{\rho(i)} Y \left[w_{\rho(i)}\right]_{\sim_{\mathfrak{F},\mathfrak{F}}} \end{gathered}$$

By definition of Z the latter is equivalent to

$$\begin{gathered} v_1 Z w_1 \\ v_2 Z w_2 \\ \vdots \\ v_{\rho(i)} Z w_{\rho(i)}. \end{gathered}$$

This proves that Z satisfies the back condition of Definition 1.2.24. We conclude that Z is a bisimulation between $\mathfrak{F}$ and itself, and hence that $\langle \mathfrak{F}, v\rangle \sim \langle \mathfrak{F}, w\rangle$ so that $[v]_{\sim_{\mathfrak{F},\mathfrak{F}}} = [w]_{\sim_{\mathfrak{F},\mathfrak{F}}}$. This completes the proof. ∎

Given the background on bisimulation quotients we are now able to demonstrate our final example of a $\mathbf{FBM}_\tau$-product.

Proposition 3.3.12. *Let $\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau}\right\rangle$ be a τ-frame. The $\mathbf{FBM}_\tau$-product of $\mathfrak{F}$ and $\mathfrak{F}/\sim$ is given by*

$$\begin{array}{ccccc} \mathfrak{F} & \xleftarrow{1_F} & \mathfrak{F} & \xrightarrow{\pi} & \mathfrak{F}/\sim \\ v & \leftarrow\!\shortmid & v & \mapsto & [v]_{\sim_{\mathfrak{F},\mathfrak{F}}} \end{array}$$

Proof. Lemma 3.2.1 and Lemma 3.3.11 prove that $\mathfrak{F} \xleftarrow{1_F} \mathfrak{F} \xrightarrow{\pi} \mathfrak{F}/\sim$ is a $\mathbf{FBM}_\tau$-span.

Suppose now that $\mathfrak{F} \xleftarrow{h_1} \mathfrak{H} \xrightarrow{h_2} \mathfrak{F}/\sim$ is another $\mathbf{FBM}_\tau$-span. We will show that h_1 is the unique bounded τ-frame morphism making the diagram below commute.

$$\begin{tikzcd} & \mathfrak{H} \arrow[dl, "h_1"'] \arrow[d, dotted, "h_1"] \arrow[dr, "h_2"] & \\ \mathfrak{F} & \mathfrak{F} \arrow[l, "1_F"] \arrow[r, "\pi"'] & \mathfrak{F}/\sim \end{tikzcd}$$

Since 1_F is the identity function it is clear that h_1 is the unique choice for the vertical arrow that makes the left hand side triangle in the above diagram commute. Now we only need to verify that $h_2 = \pi \circ h_1$. Example 3.3.7 showed that for any point v of $\mathfrak{H}$ we have that $\langle \mathfrak{F}, h_1(v) \rangle \sim \langle \mathfrak{F}/\sim, h_2(v) \rangle$. Now by part (2) of Lemma 3.3.11 it follows that $h_2(v) = [h_1(v)]_{\sim_{\mathfrak{F},\mathfrak{F}}}$. Together with the definition of π we now obtain $h_2(v) = \pi(h_1(v))$ from which the result follows. □

The examples given by Example 3.3.4, Proposition 3.3.6 and Proposition 3.3.12 all dismiss the possibility of using a cartesian product as universe for the $\mathbf{FBM}_\tau$-product of two τ-frames. Apart from these examples, that are all adapted from Gumm *&* Schröder (2001), the literature does not seem to offer a clear picture of which $\mathbf{FBM}_\tau$-products exist or what they may look like. However the poor behaviour of $\mathbf{FBM}_\tau$-products as demonstrated in particular by Example 3.3.4 and Proposition 3.3.6 may motivate why $\mathbf{FBM}_\tau$-products are not investigated for semantics of any modal logic.

3.4 Type restriction bounded morphisms

At the beginning of the chapter we have posed the following question:

> (1) Is the frame product a categorical product in some "useful" category?

We immediately made the observation that, for non-empty types, there will not be any projections from the frame product in $\mathbf{FBM}_\tau$. The reason we gave was that the types of a frame product and its factors differ. In this section we propose a generalisation of the notion of bounded morphism to overcome this restriction. Although we will show that with this generalisation frame products still do not arise as categorical products, we lay the foundation for answering the alternative question:

> (3) In which way does the frame product mimic a categorical product?

Definition 3.4.1. Suppose τ and σ are arbitrary (possibly different) types, and that a τ-frame and a σ-frame are given.

$$\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle, \qquad \mathfrak{G} = \left\langle G, (S_j)_{j\in\sigma} \right\rangle$$

A *type restriction bounded morphism* from $\mathfrak{F}$ to $\mathfrak{G}$ is a pair $\langle f, t\rangle$, where t is a function $t : \sigma \to \tau$ that preserves arities (i.e. $\rho(j) = \rho(t(j))$ for all $j \in \sigma$), and f is a bounded σ-frame morphism:

$$\left\langle F, \left(R_{t(j)}\right)_{j\in\sigma} \right\rangle \xrightarrow{f} \left\langle G, (S_j)_{j\in\sigma} \right\rangle$$

i.e. the following conditions hold for every $j \in \sigma$

- **forward**: $R_{t(j)}v_0v_1 \ldots v_{\rho(j)}$ implies $S_j f(v_0) f(v_1) \ldots f\left(v_{\rho(j)}\right)$.
- **back**: $S_j f(v_0) w_1w_2 \ldots w_{\rho(j)}$ implies that there are $v_1, v_2, \ldots, v_{\rho(j)} \in F$ such that $R_{t(j)}v_0v_1 \ldots v_{\rho(j)}$ and

$$\begin{aligned} f(v_1) &= w_1 \\ f(v_2) &= w_2 \\ &\vdots \\ f\left(v_{\rho(j)}\right) &= w_{\rho(j)}. \end{aligned}$$

We call the functions f and t the *point function* and the *type function* respectively.

For arbitrary frames $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ the composite of two type restriction bounded morphisms $\mathfrak{F} \xrightarrow{\langle f,t\rangle} \mathfrak{G} \xrightarrow{\langle g,u\rangle} \mathfrak{H}$ is defined as $\langle g, u\rangle \circ \langle f, t\rangle := \langle g \circ f, t \circ u\rangle$.

It may be curious initially that we have defined a type restriction bounded morphism using functions that go in opposite directions between the structure of two frames, however this is intentional and our motivation for doing this will become clearer in the course of our presentation.

As is usual for category theoretic diagrams we will often denote a type restriction bounded morphism $\langle f, t\rangle$ from $\mathfrak{F}$ to $\mathfrak{G}$ by an arrow

$$\mathfrak{F} \xrightarrow{\langle f,t\rangle} \mathfrak{G}$$

Also note that chasing a diagram of type restriction bounded morphisms will require chasing two diagrams: one for the composites of the point functions, and one for the composites of the type functions. So the above type restriction bounded morphism will involve two functions represented by

$$F \xrightarrow{f} G \qquad\qquad \tau \xleftarrow{t} \sigma$$

Observe that any bounded frame morphism specifies a type restriction bounded morphism with an identity type function, so that type restriction bounded morphisms can be regarded as a generalisation of bounded frame morphisms. We defer the details on this to Proposition 3.4.9.

Part of the reason why homomorphisms and bounded morphisms are important is that they preserve truth of certain formulae, as formalised in Corollary 1.2.18 and Corollary 1.2.29 (and Corollary 1.2.30). So when generalizing to type restriction bounded morphisms we would like a corresponding result on truth preservation. To obtain such a result we need to be able to translate formulae between different modal languages, since type restriction bounded morphisms run between frames of possibly different types.

Notation 3.4.2. So suppose that $t : \sigma \to \tau$ is a type function obtained from some type restriction bounded morphism. Recall Notation 1.2.2. Then for any $\phi \in \mathsf{ML}_\sigma$ we define a formula $\hat{t}(\phi) \in \mathsf{ML}_\tau$ inductively by:

$$\begin{aligned}
\hat{t}(\bot) &= \bot \\
\hat{t}(p) &= p && (\text{for } p \in \Phi) \\
\hat{t}(\neg\phi_1) &= \neg\hat{t}(\phi_1) && (\text{for } \phi_1 \in \mathsf{ML}_\sigma) \\
\hat{t}(\phi_1 \vee \phi_2) &= \hat{t}(\phi_1) \vee \hat{t}(\phi_2) && (\text{for } \phi_1, \phi_2 \in \mathsf{ML}_\sigma) \\
\hat{t}\left(\Diamond_i\left(\phi_1, \phi_2, \ldots, \phi_{\rho(i)}\right)\right) &= \Diamond_{t(i)}\left(\hat{t}(\phi_1), \hat{t}(\phi_2), \ldots, \hat{t}\left(\phi_{\rho(i)}\right)\right) \\
& && \left(\text{for } \phi_1, \phi_2, \ldots, \phi_{\rho(i)} \in \mathsf{ML}_\sigma \text{ and } i \in \sigma\right)
\end{aligned}$$

We see that $\hat{t}(\phi)$ is the formula obtained from ϕ by replacing every $\Diamond_i$ with $\Diamond_{t(i)}$ (for every $i \in \sigma$), and that the nesting depths of formulae are unaffected by $\hat{t}$.

Now we see that type restriction bounded morphisms offer truth preservation results that generalize the results for bounded model morphisms and for bounded frame morphisms (Corollary 1.2.29 and Corollary 1.2.30).

Proposition 3.4.3. *Let two frames*

$$\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle \quad \text{and} \quad \mathfrak{G} = \left\langle G, (S_j)_{j\in\sigma} \right\rangle,$$

be given, together with a type restriction bounded morphism $\langle f, t\rangle$ from $\mathfrak{F}$ to $\mathfrak{G}$. If two models

$$\mathfrak{M} := \langle \mathfrak{F}, V\rangle \quad \text{and} \quad \mathfrak{N} := \langle \mathfrak{G}, U\rangle$$

over $\mathfrak{F}$ and $\mathfrak{G}$ respectively are given, such that f preserves the truth of atoms from $\mathfrak{M}$ to $\mathfrak{N}$ i.e. that

$$\mathfrak{M}, v \Vdash p \text{ iff } \mathfrak{N}, f(v) \Vdash p$$

for all $p \in \Phi$ and $v \in F$, then the following hold for all $\phi \in \mathsf{ML}_\sigma$

(1) *For all $v \in F$ we have $\mathfrak{M}, v \Vdash \hat{t}(\phi)$ if and only if $\mathfrak{N}, f(v) \Vdash \phi$*

(2) *If* $\mathfrak{N} \Vdash \phi$ *then* $\mathfrak{M} \Vdash \hat{t}(\phi)$

(3) *If* f *is surjective, then* $\mathfrak{M} \Vdash \hat{t}(\phi)$ *implies* $\mathfrak{N} \Vdash \phi$

Proof. (1) The essence of the proof is the same as that of Proposition 1.2.27, however for the sake of clarity we go through the details regarding modalities once again, so for the induction on the length of ϕ we only consider the case where ϕ has the form $\Diamond_i\left(\phi_1, \phi_2, \ldots, \phi_{\rho(i)}\right)$ (with $i \in \sigma$).

Suppose that $\mathfrak{M}, v \Vdash \hat{t}\left(\Diamond_i\left(\phi_1, \phi_2, \ldots, \phi_{\rho(i)}\right)\right)$, by definition of $\hat{t}$ (Notation 3.4.2) this is $\mathfrak{M}, v \Vdash \Diamond_{t(i)}\left(\hat{t}(\phi_1), \hat{t}(\phi_2), \ldots, \hat{t}\left(\phi_{\rho(i)}\right)\right)$. Hence, by Definition 1.2.7, there are $v_1, v_2, \ldots, v_{\rho(i)} \in F$ such that
$R_{t(i)} v v_1 v_2 \ldots v_{\rho(i)}$ and

$$\begin{array}{c} \mathfrak{M}, v_1 \Vdash \hat{t}(\phi_1) \\ \mathfrak{M}, v_2 \Vdash \hat{t}(\phi_2) \\ \vdots \\ \mathfrak{M}, v_{\rho(i)} \Vdash \hat{t}\left(\phi_{\rho(i)}\right) \end{array}$$

Now, because $\phi_1, \phi_2, \ldots, \phi_{\rho(i)}$ have smaller length than ϕ the induction hypothesis together with the forward condition of Definition 3.4.1 implies that $S_i f(v) f(v_1) \ldots f\left(v_{\rho(i)}\right)$ and

$$\begin{array}{c} \mathfrak{N}, f(v_1) \Vdash \phi_1 \\ \mathfrak{N}, f(v_2) \Vdash \phi_2 \\ \vdots \\ \mathfrak{N}, f\left(v_{\rho(i)}\right) \Vdash \phi_{\rho(i)} \end{array}$$

So it follows from Definition 1.2.7 that $\mathfrak{N}, f(v) \Vdash \Diamond_i\left(\phi_1, \phi_2, \ldots, \phi_{\rho(i)}\right)$

For the converse suppose that $\mathfrak{N}, f(v) \Vdash \Diamond_i\left(\phi_1, \phi_2, \ldots, \phi_{\rho(i)}\right)$. Then by Definition 1.2.7 there are $w_1, w_2, \ldots, w_{\rho(i)} \in G$ such that $S_i f(v) w_1 w_2 \ldots w_{\rho(i)}$ and

$$\begin{array}{c} \mathfrak{N}, w_1 \Vdash \phi_1 \\ \mathfrak{N}, w_2 \Vdash \phi_2 \\ \vdots \\ \mathfrak{N}, w_{\rho(i)} \Vdash \phi_{\rho(i)} \end{array}$$

The back condition of Definition 3.4.1 now gives $v_1, v_2, \ldots, v_{\rho(i)} \in F$ such that $R_{t(i)} v v_1 v_2 \ldots v_{\rho(i)}$ and

$$\begin{array}{rcl} f(v_1) & = & w_1 \\ f(v_2) & = & w_2 \\ & \vdots & \\ f\left(v_{\rho(i)}\right) & = & w_{\rho(i)} \end{array}$$

Hence it is the case that

$$\begin{array}{c} \mathfrak{N}, f(v_1) \Vdash \phi_1 \\ \mathfrak{N}, f(v_2) \Vdash \phi_2 \\ \vdots \\ \mathfrak{N}, f\left(v_{\rho(i)}\right) \Vdash \phi_{\rho(i)} \end{array}$$

Now, because $\phi_1, \phi_2, \ldots, \phi_{\rho(i)}$ have smaller length than ϕ the induction hypothesis implies that
$\mathfrak{M}, v_1 \Vdash \hat{t}\,(\phi_1)$
$\mathfrak{M}, v_2 \Vdash \hat{t}\,(\phi_2)$
$\vdots$
$\mathfrak{M}, v_{\rho(i)} \Vdash \hat{t}\,(\phi_{\rho(i)})$
Since we have already shown that $R_{t(i)}vv_1v_2\ldots v_{\rho(i)}$ Definition 1.2.7 now implies that $\mathfrak{M}, v \Vdash \Diamond_{t(i)}\left(\hat{t}\,(\phi_1), \hat{t}\,(\phi_2), \ldots, \hat{t}\,(\phi_{\rho(i)})\right)$. The definition of $\hat{t}$ (Notation 3.4.2) finally gives that $\mathfrak{M}, v \Vdash \hat{t}\left(\Diamond_i\left(\phi_1, \phi_2, \ldots, \phi_{\rho(i)}\right)\right)$.

(2) This can be proved similar to Corollary 1.2.29.

(3) This can be proved similar to Corollary 1.2.29. □

Corollary 3.4.4. *Let two frames*

$$\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle \quad \textit{and} \quad \mathfrak{G} = \left\langle G, (S_j)_{j\in\sigma} \right\rangle,$$

be given, together with a type restriction bounded morphism $\langle f, t\rangle$ from $\mathfrak{F}$ to $\mathfrak{G}$. Then the following holds for all $\phi \in \mathsf{ML}_\sigma$

(1) *For every point v of $\mathfrak{F}$ we have that $\mathfrak{F}, v \Vdash \hat{t}\,(\phi)$ implies $\mathfrak{G}, f\,(v) \Vdash \phi$.*

(2) *If f is surjective then we have that $\mathfrak{F} \Vdash \hat{t}\,(\phi)$ implies $\mathfrak{G} \Vdash \phi$.*

(3) *If f is injective then for every point v of $\mathfrak{F}$ we have that $\mathfrak{G}, f\,(v) \Vdash \phi$ implies $\mathfrak{F}, v \Vdash \hat{t}\,(\phi)$. Consequently, when f is injective we have that $\mathfrak{G} \Vdash \phi$ implies $\mathfrak{F} \Vdash \hat{t}\,(\phi)$.*

Proof. This can be proved similarly to Corollary 1.2.30. □

Notation 3.4.5. For any frame $\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle$ we denote the pair $\langle 1_F, 1_\tau\rangle$ by $\mathit{1}_{\mathfrak{F}}$, where 1_F and 1_τ are the identity functions on F and τ respectively. We note that $\mathit{1}_{\mathfrak{F}}$ is a type restriction bounded morphism from $\mathfrak{F}$ to itself: 1_τ as type function associates each relation with itself so arity preservation is immediate and 1_F is the identity bounded morphism on $\mathfrak{F}$ as shown by Lemma 3.2.1.

Since our aim is to construct an alternative category for studying frames we now require the following lemma.

Lemma 3.4.6.

(1) *Composition of type restriction bounded morphisms is well defined.*

(2) *Composition of type restriction bounded morphisms is associative.*

(3) *For any frame $\mathfrak{F}$ its identity arrow is given by $\mathit{1}_{\mathfrak{F}}$.*

Proof.

(1) Suppose that two composable type restriction bounded morphisms are given

$$\mathfrak{F} \xrightarrow{\langle f,t\rangle} \mathfrak{G} \xrightarrow{\langle g,u\rangle} \mathfrak{H}$$

That $t \circ u$ preserves arities follows since both t and u preserve arities (by Definition 3.4.1). That $\langle g \circ f, t \circ u\rangle$ satisfies the required forward (resp. back) condition of Definition 3.4.1 can be verified by applying the forward (resp. back) conditions of $\langle f, t\rangle$ and $\langle g, u\rangle$ in turn.

(2) Note that composition of type restriction bounded morphisms is defined in terms of component-wise composition of functions, which is associative.

(3) Both functions used to define $1_{\mathfrak{F}}$ are identity functions. This together with the component-wise definition of composition of type restriction bounded morphisms gives the desired result. □

Notation 3.4.7. We can therefore define a category with all frames (of arbitrary types) as objects, and type restriction bounded morphisms as arrows. We denote this category by **FGBM** (our mnemonic is ***F****rames with* ***G****eneralized* ***B****ounded* ***M****orphisms*.) We denote the identity functor of **FGBM** by $\mathbf{1}_{\mathbf{FGBM}}$.

As we mentioned after defining type restriction bounded morphisms, any bounded frame morphism can be made into a type restriction bounded morphism by using the identity type function. This enables us to see type restriction bounded morphisms as a kind of generalisation of bounded frame morphisms. Our next result will formalize this generalisation by showing that $\mathbf{FBM}_\tau$ can be thought of as a subcategory of **FGBM**. We will also characterize when **FGBM** is "significantly more general" than $\mathbf{FBM}_\tau$ by characterizing the fullness of this subcategory.

Notation 3.4.8. To study this relationship between $\mathbf{FBM}_\tau$ and **FGBM** let $\mathcal{E}_\tau$ be defined by

$$\begin{array}{rcll} \mathcal{E}_\tau : \mathbf{FBM}_\tau & \to & \mathbf{FGBM} & \\ \mathfrak{F} & \mapsto & \mathfrak{F} & \text{(for } \mathfrak{F} \in \mathbf{FBM}_\tau) \\ (\mathfrak{F} \xrightarrow{f} \mathfrak{G}) & \mapsto & (\mathfrak{F} \xrightarrow{\langle f, 1_\tau\rangle} \mathfrak{G}) & \text{(for } f \text{ in } \mathbf{FBM}_\tau) \end{array}$$

Proposition 3.4.9.

(1) *$\mathcal{E}_\tau$ is an embedding functor.*

(2) *$\mathcal{E}_\tau$ is full if and only if $\rho(i)$ is distinct for every $i \in \tau$.*

Proof.

(1) To show that $\mathcal{E}_\tau$ is a functor, we note that for any τ-frame $\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau}\right\rangle$ we have

$$\mathcal{E}_\tau(1_F) = \langle 1_F, 1_\tau\rangle = 1_{\mathfrak{F}}.$$

Also, for any bounded τ-frame morphisms f and g such that $g \circ f$ exists we have

$$\mathcal{E}_\tau(g\circ f) = \langle g\circ f, 1_\tau\rangle = \langle g\circ f, 1_\tau \circ 1_\tau\rangle = \langle g, 1_\tau\rangle \circ \langle f, 1_\tau\rangle = \mathcal{E}_\tau(g)\circ \mathcal{E}_\tau(f).$$

To show that $\mathcal{E}_\tau$ is an embedding in the sense of Definition 3.2.5 we observe that for two bounded τ-frame morphisms f, g we have:

$$\begin{array}{ll} & \mathcal{E}_\tau\left(\mathfrak{F} \xrightarrow{f} \mathfrak{G}\right) = \mathcal{E}_\tau\left(\mathfrak{H} \xrightarrow{g} \mathfrak{K}\right) \\ \text{iff} & \left(\mathfrak{F} \xrightarrow{\langle f,1_\tau\rangle} \mathfrak{G}\right) = \left(\mathfrak{H} \xrightarrow{\langle g,1_\tau\rangle} \mathfrak{K}\right) \\ \text{iff} & \mathfrak{F} = \mathfrak{H} \text{ and } \mathfrak{G} = \mathfrak{K} \text{ and } \langle f, 1_\tau\rangle = \langle g, 1_\tau\rangle \\ \text{iff} & \mathfrak{F} = \mathfrak{H} \text{ and } \mathfrak{G} = \mathfrak{K} \text{ and } f = g \\ \text{iff} & \left(\mathfrak{F} \xrightarrow{f} \mathfrak{G}\right) = \left(\mathfrak{H} \xrightarrow{g} \mathfrak{K}\right) \end{array}$$

(2) Suppose that $\rho(i)$ is distinct for every $i \in \tau$, this assumption means exactly that ρ restricted to τ is injective. Let $t : \tau \longrightarrow \tau$ be some type function. To show that $\mathcal{E}_\tau$ is full we need to show that $t = 1_\tau$. If $\tau = \varnothing$ then the result follows immediately, so assuming $\tau \neq \varnothing$ let $i \in \tau$. Definition 3.4.1 requires that $\rho(t(i)) = \rho(i)$, and since ρ restricted to τ is assumed to be injective it follows that $t(i) = i$.

For the converse, suppose that there are distinct $j, k \in \tau$ such that $\rho(j) = \rho(k)$. Now let $\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau}\right\rangle$ be some τ-frame. To show that $\mathcal{E}_\tau$ is not full we construct a frame $\mathfrak{G}$ and a type restriction bounded morphism from $\mathfrak{F}$ to $\mathfrak{G}$ that does not have the identity function as type function. Let $\mathfrak{G} = \left\langle F, (S_i)_{i\in\tau}\right\rangle$ with

$$S_i = \begin{cases} R_j & \text{if } i = k \\ R_k & \text{if } i = j \\ R_i & \text{otherwise} \end{cases}$$

And let t be defined by

$$\begin{array}{rcl} t:\tau & \to & \tau \\ j & \mapsto & k \\ k & \mapsto & j \\ i & \mapsto & i \text{ otherwise} \end{array}$$

Now for every $i \in \tau$ we have

$$R_{t(i)}v_0v_1\ldots v_{\rho(i)} \text{ iff } S_iv_0v_1\ldots v_{\rho(i)}.$$

Hence we conclude that $\langle 1_F, t\rangle$ is a type restriction bounded morphism from $\mathfrak{F}$ to $\mathfrak{G}$. □

The result that $\mathcal{E}_\tau$ is not necessarily full leads us to consider the possibility that, between any two fixed τ-frames, there may be "more" **FGBM**-isomorphisms than $\mathbf{FBM}_\tau$-isomorphisms. This leads us to the following characterisation of **FGBM**-isomorphism.

Lemma 3.4.10. *Frames $\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau}\right\rangle$ and $\mathfrak{G} = \left\langle G, (S_i)_{i\in\sigma}\right\rangle$ are* **FGBM**-*isomorphic if and only if there is an arity-preserving bijection between their types, say $t : \sigma \to \tau$, such that*

$$\left\langle F, \left(R_{t(i)}\right)_{i\in\sigma}\right\rangle \quad \text{and} \quad \left\langle G, (S_i)_{i\in\sigma}\right\rangle$$

are frame-isomorphic.

Proof. Suppose $\mathfrak{F}$ and $\mathfrak{G}$ are **FGBM**-isomorphic, so we have a type restriction bounded morphism from $\mathfrak{F}$ to $\mathfrak{G}$, say $\langle f, t\rangle$, that has an inverse say $\langle g, u\rangle$. Then t has an inverse, and hence it is a bijection, we also know t is arity-preserving. So we need to show that f gives the desired frame-isomorphism. Since f has inverse g, it is a bijection. This together with the requirement in Definition 3.4.1 that f should be a bounded σ-frame morphism leads us to conclude from Lemma 1.2.32 that f is a frame-isomorphism.

For the converse suppose there is an arity-preserving bijection between the types of $\mathfrak{F}$ and $\mathfrak{G}$, say $t : \sigma \to \tau$, and a frame-isomorphism

$$\left\langle F, \left(R_{t(i)}\right)_{i\in\sigma}\right\rangle \xrightarrow{f} \left\langle G, (S_i)_{i\in\sigma}\right\rangle.$$

Then $\langle f, t\rangle$ is a type restriction bounded morphism, so we need to construct its inverse. Both components of $\langle f, t\rangle$ are bijections, and hence have inverses say g and u respectively. Now it is sufficient to show that $\langle g, u\rangle$ is a type restriction bounded morphism. Since t preserves arities, so does u. To see that g is in fact a strong τ-frame homomorphism note that for all $i \in \tau$ we have

$$\begin{array}{lll} S_{u(i)} v_0 v_1 \ldots v_{\rho(i)} & \text{iff} & R_{t(u(i))} f^{-1}(v_0)\, f^{-1}(v_1) \ldots f^{-1}\left(v_{\rho(i)}\right) \\ & \text{iff} & R_i g(v_0)\, g(v_1) \ldots g\left(v_{\rho(i)}\right). \end{array}$$

□

The power of this result should not be overestimated. It is possible that there may be many appropriate arity-preserving bijections, and so we may have several non-trivial **FGBM**-automorphisms with the same point function.

Remark 3.4.11. Also note that in practice the weakening of "isomorphism" from frame-isomorphism to **FGBM**-isomorphism should be very carefully considered, as it may either be useful or undesired. For example we may be studying frames where all of the relations describe spatial properties, like *point v is north of point w*, or *point v is east of point w*). Then it may be reasonable to treat such spatial relations as indistinguishable: we may think of a transition *north* in one frame corresponding to a transition *east* in another frame. In this example the frames may seem different, but an **FGBM**-isomorphism between them claims that the spatial structure simply corresponds to a kind of relabeling of "axes" (although the universes may not be relations, so there may not be real axes to speak of). If instead, we are interested in frames with some relations describing spatial properties as before, but together with relations describing temporal properties, like *point v occurs before point w*. Then for practical considerations it may be unreasonable to treat spatial relations as indistinguishable from temporal relations. And **FGBM**-isomorphisms will not make that distinction.

Now that we have established some background on **FGBM** we return to the frame product, and attempt to answer the question:

> (1) Is the frame product a categorical product in some "useful" category?

We observe that we are able to construct type restriction bounded morphisms from a frame product to its factors using the standard set projections in the following way. As usual we consider two frames

$$\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle \quad \text{and} \quad \mathfrak{G} = \left\langle G, (S_i)_{i\in\sigma} \right\rangle.$$

Once again, as motivated in Remark 1.3.2, we assume that the types τ and σ are disjoint, so that the type of $\mathfrak{F} \otimes \mathfrak{G}$ is given by $\tau \cup \sigma$. Now we can show that $\langle p_1, e_1 \rangle$ is a type restriction bounded morphism from $\mathfrak{F} \otimes \mathfrak{G}$ to $\mathfrak{F}$, when p_1 and e_1 are defined as follows:

$$\begin{array}{rcl} p_1 : F \times G & \to & F \\ \langle v, w \rangle & \mapsto & v \end{array} \qquad \text{and} \qquad \begin{array}{rcl} e_1 : \tau & \to & \tau \cup \sigma \\ i & \mapsto & i \end{array} \tag{3.4.12}$$

It is immediate that e_1 preserves arities. To show that $\langle p_1, e_1 \rangle$ satisfies the forward condition of Definition 3.4.1, suppose that

$$R^{\leftrightarrow}_{e_1(i)} \langle v_0, w_0 \rangle \langle v_1, w_1 \rangle \ldots \left\langle v_{\rho(i)}, w_{\rho(i)} \right\rangle.$$

Then the definition of the relations on the frame product (Definition 1.3.1) requires that

$$R_{e_1(i)} v_0 v_1 \ldots v_{\rho(i)} \text{ and } w_0 = w_1 = \ldots = w_{\rho(i)}.$$

Applying the definitions of e_1 and p_1, the former requirement is exactly

$$R_i p_1 \left(\langle v_0, w_0 \rangle\right) p_1 \left(\langle v_1, w_1 \rangle\right) \ldots p_1 \left(\left\langle v_{\rho(i)}, w_{\rho(i)} \right\rangle\right),$$

as required. To show that $\langle p_1, e_1 \rangle$ satisfies the back condition of Definition 3.4.1, suppose that

$$R_i p_1 (\langle v_0, w_0 \rangle) \, v_1 v_2 \ldots v_{\rho(i)}.$$

Applying the definition of p_1 we get

$$R_i v_0 v_1 \ldots v_{\rho(i)}.$$

Now using the definition of the relations on the frame product (Definition 1.3.1) we obtain

$$R_i^{\leftrightarrow} \langle v_0, w_0 \rangle \langle v_1, w_0 \rangle \ldots \left\langle v_{\rho(i)}, w_0 \right\rangle.$$

This is sufficient since $R^{\leftrightarrow}_{e_1(i)} = R^{\leftrightarrow}_i$ and

$$\begin{array}{rcl} p_1 (\langle v_0, w_0 \rangle) & = & v_0 \\ p_1 (\langle v_1, w_0 \rangle) & = & v_1 \\ & \vdots & \\ p_1 \left(\left\langle v_{\rho(i)}, w_0 \right\rangle\right) & = & v_{\rho(i)}. \end{array}$$

A similar argument shows that $\langle p_2, e_2 \rangle$, with

$$\begin{array}{rcl} p_2 : F \times G & \to & G \\ \langle v, w \rangle & \mapsto & w \end{array} \quad \text{and} \quad \begin{array}{rcl} e_2 : \sigma & \to & \tau \cup \sigma \\ i & \mapsto & i, \end{array} \tag{3.4.13}$$

is a type restriction bounded morphism from $\mathfrak{F} \otimes \mathfrak{G}$ to $\mathfrak{G}$.

Now we have constructed a **FGBM**-span

$$\mathfrak{F} \xleftarrow{\langle p_1, e_1 \rangle} \mathfrak{F} \otimes \mathfrak{G} \xrightarrow{\langle p_2, e_2 \rangle} \mathfrak{G} \tag{3.4.14}$$

Observe that the existence of this **FGBM**-span does not violate Proposition 1.2.37, this is because p_1 and p_2 are bounded frame morphisms with regard to different types. The existence of this span is also part of our motivation for the way we have defined type restriction bounded morphisms, in particular it motivates the directions we chose for the point function and type function. For this **FGBM**-span we also have the property that if another **FGBM**-span factors through it, then the factoring is unique, the argument follows. Suppose that another **FGBM**-span is given, say

$$\mathfrak{F} \xleftarrow{\langle h_1, u_1 \rangle} \mathfrak{H} = \left\langle H, (T_i)_{i \in \mu} \right\rangle \xrightarrow{\langle h_2, u_2 \rangle} \mathfrak{G}$$

Then since the cartesian product is a **Set**-product, there is a unique function $h : H \to F \times G$ such that $h_1 = p_1 \circ h$ and $h_2 = p_2 \circ h$ i.e. that the following diagram commutes

$$\begin{tikzcd} & H \arrow[dl, "h_1"'] \arrow[d, dotted, "h"] \arrow[dr, "h_2"] & \\ F & F \times G \arrow[l, "p_1"] \arrow[r, "p_2"'] & G \end{tikzcd}$$

Similarly disjoint union is a **Set**-coproduct so there is a unique function $u : \tau \uplus \sigma \to \mu$ such that $u_1 = u \circ e_1$ and $u_2 = u \circ e_2$ i.e. that the following diagram commutes

$$\begin{tikzcd} & \mu & \\ \tau \arrow[ur, "u_1"] \arrow[r, "e_1"'] & \tau \uplus \sigma \arrow[u, dotted, "u"'] & \sigma \arrow[ul, "u_2"'] \arrow[l, "e_2"] \end{tikzcd}$$

Observe that for every $i \in \tau$ we have

$$\begin{aligned} \rho\left(e_1\left(i\right)\right) &= \rho\left(i\right) \quad \text{(since } e_1 \text{ is arity preserving)} \\ &= \rho\left(u_1\left(i\right)\right) \quad \text{(since } u_1 \text{ is arity preserving)} \\ &= \rho\left(u\left(e_1\left(i\right)\right)\right) \quad \text{(since } u_1 = u \circ e_1\text{)} \end{aligned}$$

Hence u is arity preserving on the image of e_1, similarly it can be shown that u is arity preserving on the image of e_2 so that u is arity preserving. We conclude that if $\langle h, u \rangle$ is a type restriction bounded morphism from $\mathfrak{H}$ to $\mathfrak{F} \otimes \mathfrak{G}$, more specifically if it satisfies the forward and back conditions of Definition 3.4.1, then it is the unique type restriction bounded morphism such that $\langle h_1, u_1 \rangle = \langle p_1, e_1 \rangle \circ \langle h, u \rangle$ and $\langle h_2, u_2 \rangle = \langle p_2, e_2 \rangle \circ \langle h, u \rangle$ i.e. that the following diagram in **FGBM** commutes

$$\begin{tikzcd} & \mathfrak{H} \arrow[dl, "\langle h_1, u_1 \rangle"'] \arrow[d, dotted, "\langle h, u \rangle"] \arrow[dr, "\langle h_2, u_2 \rangle"] & \\ \mathfrak{F} & \mathfrak{F} \otimes \mathfrak{G} \arrow[l, "\langle p_1, e_1 \rangle"] \arrow[r, "\langle p_2, e_2 \rangle"'] & \mathfrak{G} \end{tikzcd}$$

As we will show at the end of this section, the **FGBM**-span (3.4.14) is not a **FGBM**-product in general. The only case that we are aware of when this is a **FGBM**-product is the trivial case where a **FGBM**-product is taken with the **FGBM**-terminal object. To show this we first identify the **FGBM**-terminal object.

Notation 3.4.15. Let $\mathfrak{E}$ be the frame with one point and the empty type, i.e. $\mathfrak{E} := \langle \{0\} \rangle$.

Lemma 3.4.16. *$\mathfrak{E}$ is the* **FGBM**-*terminal object.*

Proof. Observe that the universe of $\mathfrak{E}$, the single point set, is the **Set**-terminal object; and that the type of $\mathfrak{E}$, the empty set, is the **Set**-initial object. Therefore for any frame $\mathfrak{F} = \left\langle F, (R_i)_{i \in \tau} \right\rangle \in$ **FGBM** we have unique functions

$$f : F \to \{0\} \quad \text{and} \quad t : \varnothing \to \tau.$$

The result that $\langle f, t \rangle$ is a type restriction bounded morphism from $\mathfrak{F}$ to $\mathfrak{E}$ is immediate since arity-preservation and the foward and back conditions of Definition 3.4.1 are vacuous truths. $\square$

Lemma 3.4.16 yields a partial motivation why we allowed empty types at the beginning of Chapter 1.

The following result is a special case of a standard category theoretic result (see Adámek *et al.* (2004)) on products with terminal objects.

Corollary 3.4.17. *Given a frame* $\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle$ *its* **FGBM**-*product with* $\mathfrak{E}$ *is given by*

$$\mathfrak{F} \xleftarrow{\langle 1_F, 1_\tau \rangle} \mathfrak{F} \xrightarrow{\langle c, \varnothing \rangle} \mathfrak{E}$$

Here $c : F \to \{0\}$ *denotes the constant function, and* $\varnothing : \varnothing \to \tau$ *denotes the empty function.*

Remark 3.4.18. We may consider this result as a counterpart to Proposition 3.3.12 in **FGBM**. Since there is no $\mathbf{FBM}_\tau$-terminal object we appealed to Lemma 3.3.11 to guarantee the existence and uniqueness of a bounded τ-frame morphism to the relevant bisimulation quotient, in this sense a bisimulation quotient mimics a terminal object for $\mathbf{FBM}_\tau$.

Proposition 3.4.19. *Given a frame* $\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle$. *The* **FGBM**-*span*

$$\mathfrak{F} \xleftarrow{\langle 1_F, 1_\tau \rangle} \mathfrak{F} \xrightarrow{\langle c, \varnothing \rangle} \mathfrak{E}$$

(as defined in Corollary 3.4.17) factors through the **FGBM**-*span*

$$\mathfrak{F} \xleftarrow{\langle p_1, e_1 \rangle} \mathfrak{F} \otimes \mathfrak{E} \xrightarrow{\langle p_2, e_2 \rangle} \mathfrak{E}$$

(as defined for (3.4.14) *by* (3.4.12) *and* (3.4.13)*) via an* **FGBM**-*isomorphism.*

Proof. After we constructed (3.4.14) we characterised when a **FGBM**-span factors via $\mathfrak{F} \overset{\langle p_1,e_1\rangle}{\longleftarrow} \mathfrak{F} \otimes \mathfrak{E} \overset{\langle p_2,e_2\rangle}{\longrightarrow} \mathfrak{E}$. By that argument we only need to show that the pair $\langle h, u \rangle$ defined by

$$\begin{array}{rcl} h : F & \to & F \times \{0\} \\ v & \mapsto & \langle 1_F(v), c(v) \rangle \end{array} \qquad \text{and} \qquad \begin{array}{rcl} u : \tau & \to & \tau \\ i & \mapsto & i \end{array}$$

is an **FGBM**-isomorphism of $\mathfrak{F}$ and $\mathfrak{F} \otimes \mathfrak{E}$. Observe that, by the definitions of 1_F and c, we have for any $v \in F$ that $h(v) = \langle 1_F(v), c(v) \rangle = \langle v, 0 \rangle$. It is now immediate that h and u are bijections; note also that for every $i \in \tau$ we have

$$\begin{array}{rcl} R_i v_0 v_1 \ldots v_{\rho(i)} & \text{iff} & R_{u(i)} v_0 v_1 \ldots v_{\rho(i)} \text{ and } \underbrace{0 = 0 = \ldots = 0}_{\rho(i) \text{ times}} \\ & \text{iff} & R^{\leftrightarrow}_{u(i)} \langle v_0, 0 \rangle \langle v_1, 0 \rangle \ldots \left\langle v_{\rho(i)}, 0 \right\rangle \end{array}$$

We conclude that $\langle h, u \rangle$ is the required **FGBM**-isomorphism. $\square$

The next result follows immediately.

Corollary 3.4.20. *Given a frame* $\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle$ *its* **FGBM**-*product with* $\mathfrak{E}$ *is given by*

$$\mathfrak{F} \xleftarrow{\langle p_1, e_1\rangle} \mathfrak{F} \otimes \mathfrak{E} \xrightarrow{\langle p_2, e_2\rangle} \mathfrak{E}$$

With $\langle p_1, e_1\rangle$ *and* $\langle p_2, e_2\rangle$ *defined by*

$$\begin{array}{rcl} p_1 : F \times \{0\} & \to & F \\ \langle v, 0\rangle & \mapsto & v \end{array} \qquad \textit{and} \qquad \begin{array}{rcl} e_1 : \tau & \to & \tau \\ i & \mapsto & i \end{array}$$

$$\begin{array}{rcl} p_2 : F \times \{0\} & \to & \{0\} \\ \langle v, 0\rangle & \mapsto & 0 \end{array} \qquad \textit{and} \qquad \begin{array}{rcl} e_2 : \varnothing & \to & \tau \end{array}$$

As claimed before we now show that the **FGBM**-span (3.4.14) does not give a **FGBM**-product in general. To show this we demonstrate the stronger result that not all **FGBM**-products exist. This is sufficient since the **FGBM**-span (3.4.14) can always be constructed. To demonstrate a **FGBM**-product that does not exist we modify Example 3.3.4 as in the following example.

Example 3.4.21. Consider the frame $\mathfrak{F}$ as defined in Example 3.3.4. We show that the **FGBM**-product of $\mathfrak{F}$ with itself does not exist. Recall the $\mathbf{FBM}_\tau$-span $\mathfrak{F} \xleftarrow{k_1} \mathfrak{K} \xrightarrow{k_2} \mathfrak{F}$ from Example 3.3.4. Applying $\mathcal{E}_\tau$ (Notation 3.4.8) to this $\mathbf{FBM}_\tau$-span yields the following **FGBM**-span

$$\mathfrak{F} \xleftarrow{\langle k_1, 1_\tau\rangle} \mathfrak{K} \xrightarrow{\langle k_2, 1_\tau\rangle} \mathfrak{F}$$

Suppose for a contradiction that the **FGBM**-product of $\mathfrak{F}$ with itself is given by some **FGBM**-span, say

$$\mathfrak{F} \xleftarrow{\langle g_1, u_1\rangle} \mathfrak{G} = \left\langle G, (S_i)_{i\in\sigma} \right\rangle \xrightarrow{\langle g_2, u_2\rangle} \mathfrak{F}$$

The definition of a **FGBM**-product now yields a type restriction bounded morphism $\mathfrak{K} \xrightarrow{\langle k,s\rangle} \mathfrak{G}$ such that $\langle k_1, 1_\tau\rangle = \langle g_1, u_1\rangle \circ \langle k, s\rangle$ and $\langle k_2, 1_\tau\rangle = \langle g_2, u_2\rangle \circ \langle k, s\rangle$ i.e. the following diagram commutes

$$\begin{tikzcd} & \mathfrak{K} \arrow[dl, "\langle k_1, 1_\tau\rangle"'] \arrow[dr, "\langle k_2, 1_\tau\rangle"] \arrow[d, dotted, "\langle k,s\rangle"] & \\ \mathfrak{F} & \mathfrak{G} \arrow[l, "\langle g_1, u_1\rangle"] \arrow[r, "\langle g_2, u_2\rangle"'] & \mathfrak{F} \end{tikzcd}$$

This only holds if

$$\begin{array}{rcl} k_1 & = & g_1 \circ k \\ k_2 & = & g_2 \circ k \end{array} \quad \text{and} \quad \begin{array}{rcl} 1_\tau & = & s \circ u_1 \\ 1_\tau & = & s \circ u_2 \end{array}$$

i.e. the following two diagrams commute

$$\begin{tikzcd} & K \arrow[dl, "k_1"'] \arrow[dr, "k_2"] \arrow[d, "k"] & \\ F & G \arrow[l, "g_1"] \arrow[r, "g_2"'] & F \end{tikzcd} \qquad \begin{tikzcd} & \tau & \\ \tau \arrow[ur, "1_\tau"] \arrow[r, "u_1"'] & \sigma \arrow[u, "s"] & \tau \arrow[ul, "1_\tau"'] \arrow[l, "u_2"] \end{tikzcd}$$

Keeping in mind that we have no information about σ other than the information given by the type restriction bounded morphisms, we now construct an injective bounded morphism from $\mathfrak{K}$ to a frame with G as universe. To do this let i_0 denote the single element of τ, and let $j_0 = u_1(i_0)$. Then it follows that

$$\begin{aligned} s(j_0) &= (s \circ u_1)(i_0) \quad \text{(by the choice of } j_0\text{)} \\ &= 1_\tau(i_0) \quad \text{(since } s \text{ was chosen such that } 1_\tau = s \circ u_1\text{)} \\ &= i_0. \end{aligned}$$

The fact that $\langle k, s\rangle$ is a type restriction bounded morphism (Definition 3.4.1) now implies that k is a bounded morphism from $\mathfrak{K} = \langle K, Q\rangle$ to $\langle G, S_{j_0}\rangle$. Now we may follow the same argument as in Example 3.3.4 to show that k is an injective function and once again conclude that the universe of $\mathfrak{G}$ must be a proper class, not a set, and hence that the **FGBM**-product of $\mathfrak{F}$ with itself does not exist.

This completes the example.

As in $\mathbf{FH}_\tau$ and $\mathbf{FBM}_\tau$ we conclude that frame products are not **FGBM**-products, which gives partial negative answer to the question:

> (1) Is the frame product a categorical product in some "useful" category?

Unfortunately "useful" is an inherently vague notion, and as such this question may never be convincingly answered. For us the categories $\mathbf{FH}_\tau$, $\mathbf{FBM}_\tau$ and **FGBM** are considered "useful" in part because their arrows preserve truth in some way (Corollary 1.2.18, Corollary 1.2.29 and Proposition 3.4.3).

3.5 Re-examining the frame product using type restriction bounded morphisms

In the previous section we have showed that frame products are not **FGBM**-products, so in this section we consider the alternative question:

> (3) In which way does the frame product mimic a categorical product?

In Section 3.2 it was stated that monoidal categories generalize categories with finite products. So, to answer this question we show that **FGBM** can be made into a monoidal category with the frame product as its multiplication.

At the moment it is technically impossible that the frame product is the multiplication of a monoidal category since it is only defined for pairs of frames, not pairs of arrows, and hence not a bifunctor as required by Definition 3.2.6.

So we need to suitably extend the behaviour of $\otimes$ to type restriction bounded morphisms, and then show that $\otimes$ is a bifunctor.

In Definition 1.3.1 we have defined the frame product of two frames using a **Set**-product for its universe and a **Set**-coproduct for its type, this suggests that we also use **Set**-products and **Set**-coproducts to define the frame product of two type restriction bounded morphisms.

Definition 3.5.1. Suppose we have two type restriction bounded morphisms

$$\begin{array}{ccc} \mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle & & \mathfrak{G} = \left\langle G, (S_i)_{i\in\sigma} \right\rangle \\ \Big\downarrow{\scriptstyle \langle f,t\rangle} & \text{and} & \Big\downarrow{\scriptstyle \langle g,u\rangle} \\ \mathfrak{H} = \left\langle H, (T_i)_{i\in\mu} \right\rangle & & \mathfrak{K} = \left\langle K, (Q_i)_{i\in\nu} \right\rangle \end{array}$$

We define $\langle f,t\rangle \otimes \langle g,u\rangle$ as $\langle f\times g, t\uplus u\rangle$, with $f\times g$ and $t\uplus u$ defined as the unique functions such that the following two diagrams in the category of sets commute

$$\begin{tikzcd} F \arrow[d, "f"'] & F\times G \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dotted, "f\times g"] & G \arrow[d, "g"] \\ H & H\times K \arrow[l, "q_1"] \arrow[r, "q_2"'] & K \end{tikzcd} \qquad \begin{tikzcd} \tau \arrow[r, "i_1"] & \tau\uplus\sigma & \sigma \arrow[l, "i_2"'] \\ \mu \arrow[u, "t"] \arrow[r, "j_1"'] & \mu\uplus\nu \arrow[u, dotted, "t\uplus u"'] & \nu \arrow[l, "j_2"] \arrow[u, "u"'] \end{tikzcd}$$

Here p_1, p_2, q_1 and q_2 are the projection functions from the **Set**-products, while i_1, i_2, j_1 and j_2 are the injections into the disjoint unions (or **Set**-coproducts.)

Remark 3.5.2. Observe that in Definition 3.5.1 the definitions of $f\times g$ and $t\uplus u$ both correspond to the definition of $\boxtimes$ in Proposition 3.2.11: for $f\times g$ the category $\mathbb{C}$ of Proposition 3.2.11 is taken to be **Set**; and for $t\uplus u$ the category $\mathbb{C}$ of Proposition 3.2.11 is $\mathbf{Set}^{\mathrm{op}}$. This immediately necessitates that $\times$ and $\uplus$ are bifunctors and also suggests candidates for the choice of the natural **FGBM**-isomorphisms that we will need to completely specify a monoidal category. However these suggestions essentially come from the category $\mathbf{Set}\times\mathbf{Set}^{\mathrm{op}}$, not **FGBM**, so we will need to verify that all of these suggestions are type restriction bounded morphisms and indeed **FGBM**-isomorphisms.

As stated in Remark 1.3.2 we will assume, without loss of generality, that the factors of a frame product have disjoint types, so that the type of the frame product can be taken to be the union of the types of the factors. This allows us to construct $\langle f\times g, t\uplus u\rangle$ as in Definition 3.5.1 as follows.

$$\begin{array}{rcl} f\times g : F\times G & \to & H\times K \\ \langle v,w\rangle & \mapsto & \langle f(v), g(w)\rangle \end{array} \qquad \begin{array}{rcl} t\uplus u : \mu\cup\nu & \to & \tau\cup\sigma \\ i & \mapsto & \begin{cases} t(i) & \text{if } i\in\mu \\ u(i) & \text{if } i\in\nu \end{cases} \end{array}$$

It is well known and easily verified (see for example Adámek *et al.* (2004)), that $f \times g$ and $t \uplus u$ as constructed here satisfy Definition 3.5.1.

It is not immediate from Definition 3.5.1 that $\langle f, t\rangle \otimes \langle g, u\rangle$ will be a type restriction bounded morphism, so we verify this in our next result.

Lemma 3.5.3. *For two type restriction bounded morphisms*

$$\begin{array}{ccc} \mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle & & \mathfrak{G} = \left\langle G, (S_i)_{i\in\sigma} \right\rangle \\ \Big\downarrow{\scriptstyle\langle f,t\rangle} & \textit{and} & \Big\downarrow{\scriptstyle\langle g,u\rangle} \\ \mathfrak{H} = \left\langle H, (T_i)_{i\in\mu} \right\rangle & & \mathfrak{K} = \left\langle K, (Q_i)_{i\in\nu} \right\rangle \end{array}$$

$\langle f, t\rangle \otimes \langle g, u\rangle$ *is a type restriction bounded morphism from* $\mathfrak{F} \otimes \mathfrak{G}$ *to* $\mathfrak{H} \otimes \mathfrak{K}$

Proof. It is immediate that $t \uplus u$ preserves arities since both t and u do.

To show that $\langle f, t\rangle \otimes \langle g, u\rangle$ satisfies the forward condition of Definition 3.4.1 suppose that we have $i \in \mu$ such that $R^{\leftrightarrow}_{t(i)} \langle v_0, w_0\rangle \langle v_1, w_1\rangle \ldots \left\langle v_{\rho(i)}, w_{\rho(i)}\right\rangle$. By the definition of $R^{\leftrightarrow}_{t(i)}$ (Definition 1.3.1) we have $R_{t(i)} v_0 v_1 \ldots v_{\rho(i)}$ and $w_0 = w_1 = \ldots = w_{\rho(i)}$. Since $\langle f, t\rangle$ satisfies the forward condition of Definition 3.4.1, from $\mathfrak{F}$ to $\mathfrak{H}$, we have $T_i f(v_0) f(v_1) \ldots f\left(v_{\rho(i)}\right)$; and evaluating g at $w_0, w_1, \ldots, w_{\rho(i)}$ yields $g(w_0) = g(w_1) = \ldots = g\left(w_{\rho(i)}\right)$. Now the definition of $T^{\leftrightarrow}_i$ (Definition 1.3.1) gives $T^{\leftrightarrow}_i \langle f(v_0), g(w_0)\rangle \langle f(v_1), g(w_1)\rangle \ldots \left\langle f\left(v_{\rho(i)}\right), g\left(w_{\rho(i)}\right)\right\rangle$, which is exactly $T^{\leftrightarrow}_i (f\times g)(\langle v_0, w_0\rangle)(f\times g)(\langle v_1, w_1\rangle) \ldots (f\times g)\left(\left\langle v_{\rho(i)}, w_{\rho(i)}\right\rangle\right)$, by definition of $f \times g$. We deduce that $\langle f, t\rangle \otimes \langle g, u\rangle$ satisfies the forward condition of Definition 3.4.1 for every $i \in \mu$.

A similar proof shows that $\langle f, t\rangle \otimes \langle g, u\rangle$ also satisfies the forward condition of Definition 3.4.1 for every $i \in \nu$ and consequently for every $i \in \mu \cup \nu$.

To show that $\langle f, t\rangle \otimes \langle g, u\rangle$ satisfies the back condition of Definition 3.4.1 suppose that $i \in \mu$ and that $T^{\leftrightarrow}_i (f\times g)(\langle v_0, w_0\rangle) \langle x_1, y_1\rangle \langle x_2, y_2\rangle \ldots \left\langle x_{\rho(i)}, y_{\rho(i)}\right\rangle$, then the definition of $f \times g$ implies that

$$T^{\leftrightarrow}_i \langle f(v_0), g(w_0)\rangle \langle x_1, y_1\rangle \langle x_2, y_2\rangle \ldots \left\langle x_{\rho(i)}, y_{\rho(i)}\right\rangle.$$

The definition of $T^{\leftrightarrow}_i$ now implies that $T_i f(v_0) x_1 x_2 \ldots x_{\rho(i)}$ and $g(w_0) = y_1 = y_2 = \ldots = y_{\rho(i)}$. Since $\langle f, t\rangle$ satisfies the back condition of Definition 3.4.1 (from $\mathfrak{F}$ to $\mathfrak{H}$) there are $v_1, v_2, \ldots, v_{\rho(i)} \in F$ such that

$$\begin{aligned} f(v_1) &= x_1 \\ f(v_2) &= x_2 \\ &\vdots \\ f\left(v_{\rho(i)}\right) &= x_{\rho(i)} \end{aligned}$$

and $R_{t(i)}v_0v_1\dots v_{\rho(i)}$. Hence $\langle v_1, w_0\rangle, \langle v_2, w_0\rangle, \dots, \langle v_{\rho(i)}, w_0\rangle \in F \times G$ and, by definition of $f \times g$ and $R^{\leftrightarrow}_{t(i)}$, also

$$\begin{array}{rcccl}
(f \times g)(\langle v_1, w_0\rangle) & = & \langle f(v_1), g(w_0)\rangle & = & \langle x_1, y_1\rangle \\
(f \times g)(\langle v_2, w_0\rangle) & = & \langle f(v_2), g(w_0)\rangle & = & \langle x_2, y_2\rangle \\
 & \vdots & & & \\
(f \times g)(\langle v_{\rho(i)}, w_0\rangle) & = & \langle f(v_{\rho(i)}), g(w_0)\rangle & = & \langle x_{\rho(i)}, y_{\rho(i)}\rangle
\end{array}$$

and $R^{\leftrightarrow}_{t(i)}\langle v_0, w_0\rangle \langle v_1, w_0\rangle \dots \langle v_{\rho(i)}, w_0\rangle$. We deduce that $\langle f, t\rangle \otimes \langle g, u\rangle$ satisfies the back condition of Definition 3.4.1 for every $i \in \mu$.

A similar proof shows that $\langle f, t\rangle \otimes \langle g, u\rangle$ also satisfies the back condition of Definition 3.4.1 for every $i \in \nu$ and consequently for every $i \in \mu \cup \nu$. Now the result follows. □

Having established that $\otimes$ maps type restriction bounded morphisms to type restriction bounded morphisms between frame products we can continue to show that $\otimes$ is a bifunctor. The essence of the proof relies on the fact that $\times$ and $\uplus$ are both bifunctors, as motivated in Remark 3.5.2.

Proposition 3.5.4. $\otimes : \mathbf{FGBM} \times \mathbf{FGBM} \to \mathbf{FGBM}$ *is a bifunctor.*

Proof. To use Lemma 3.2.7 consider the families

$$(\mathcal{L}_{\mathfrak{G}})_{\mathfrak{G}\in\mathbf{FGBM}} \qquad\qquad (\mathcal{R}_{\mathfrak{F}})_{\mathfrak{F}\in\mathbf{FGBM}}$$

defined by $\mathcal{L}_{\mathfrak{G}} := - \otimes \mathfrak{G}$ and $\mathcal{R}_{\mathfrak{F}} := \mathfrak{F} \otimes -$, i.e.

$$\begin{array}{rcl}
\mathcal{L}_{\mathfrak{G}} : \mathbf{FGBM} & \to & \mathbf{FGBM} \\
\mathfrak{H} & \mapsto & \mathfrak{H} \otimes \mathfrak{G} \quad (\text{for } \mathfrak{H} \in \mathbf{FGBM}) \\
\langle f, t\rangle & \mapsto & \langle f, t\rangle \otimes \mathit{1}_{\mathfrak{G}} \quad (\text{for } \langle f, t\rangle \text{ in } \mathbf{FGBM}) \\
\mathcal{R}_{\mathfrak{F}} : \mathbf{FGBM} & \to & \mathbf{FGBM} \\
\mathfrak{H} & \mapsto & \mathfrak{F} \otimes \mathfrak{H} \quad (\text{for } \mathfrak{H} \in \mathbf{FGBM}) \\
\langle f, t\rangle & \mapsto & \mathit{1}_{\mathfrak{F}} \otimes \langle f, t\rangle \quad (\text{for } \langle f, t\rangle \text{ in } \mathbf{FGBM})
\end{array}$$

We verify that these are functors. So suppose that an arbitrary frame $\mathfrak{G} = \langle G, (S_i)_{i\in\sigma}\rangle$ is given. Now to show that $\mathcal{L}_{\mathfrak{G}}$ preserves identities consider the image of the identity arrow $\mathit{1}_{\mathfrak{F}}$ of some frame $\mathfrak{F} := \langle F, (R_i)_{i\in\tau}\rangle$:

$$\begin{aligned}
\mathcal{L}_{\mathfrak{G}}(\mathit{1}_{\mathfrak{F}}) &= \mathit{1}_{\mathfrak{F}} \otimes \mathit{1}_{\mathfrak{G}} \quad (\text{by definition of } \mathcal{L}_{\mathfrak{G}}) \\
&= \langle 1_F, 1_\tau\rangle \otimes \langle 1_G, 1_\sigma\rangle \quad (\text{by definition of } \mathit{1}_{\mathfrak{F}} \text{ and } \mathit{1}_{\mathfrak{G}}, \text{ Notation 3.4.5}) \\
&= \langle 1_F \times 1_G, 1_\tau \uplus 1_\sigma\rangle \quad (\text{by definition of } \otimes, \text{ Definition 3.5.1}) \\
&= \langle 1_{F\times G}, 1_{\tau\uplus\sigma}\rangle \quad (\text{since } \times \text{ and } \uplus \text{ are bifunctors, Remark 3.5.2}) \\
&= \mathit{1}_{\mathfrak{F}\otimes\mathfrak{G}} \quad (\text{by definition of } \mathit{1}_{\mathfrak{F}\otimes\mathfrak{G}}, \text{ Notation 3.4.5; and } \otimes, \text{ Definition 3.5.1}) \\
&= \mathit{1}_{\mathcal{L}_{\mathfrak{G}}(\mathfrak{F})} \quad (\text{by definition of } \mathcal{L}_{\mathfrak{G}})
\end{aligned}$$

To prove that $\mathcal{L}_{\mathfrak{G}}$ preserves composition as well, let two composable type restriction bounded morphisms be given:

$$\begin{array}{c} \mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle \\ \downarrow{\scriptstyle \langle f,t\rangle} \\ \mathfrak{H} = \left\langle H, (T_i)_{i\in\mu} \right\rangle \\ \downarrow{\scriptstyle \langle g,u\rangle} \\ \mathfrak{K} = \left\langle K, (Q_i)_{i\in\nu} \right\rangle \end{array}$$

We need to show that

$$\mathcal{L}_{\mathfrak{G}}\left(\langle g, u\rangle \circ \langle f, t\rangle\right) = \mathcal{L}_{\mathfrak{G}}\left(\langle g, u\rangle\right) \circ \mathcal{L}_{\mathfrak{G}}\left(\langle f, t\rangle\right). \tag{3.5.5}$$

The left-hand side of (3.5.5) is given by

$$\begin{aligned} \mathcal{L}_{\mathfrak{G}}\left(\langle g, u\rangle \circ \langle f, t\rangle\right) &= \left(\langle g, u\rangle \circ \langle f, t\rangle\right) \otimes \mathit{1}_{\mathfrak{G}} \\ &\quad \text{(by definition of } \mathcal{L}_{\mathfrak{G}}) \\ &= \langle g \circ f, t \circ u\rangle \otimes \langle 1_G, 1_\sigma\rangle \\ &\quad \text{(by definition of } \circ\text{, Definition 3.4.1;} \\ &\quad\quad \text{and definition of } \mathit{1}_{\mathfrak{G}}\text{, Notation 3.4.5)} \\ &= \langle (g \circ f) \times 1_G, (t \circ u) \uplus 1_\sigma\rangle \\ &\quad \text{(by definition of } \otimes\text{, Definition 3.5.1)} \end{aligned}$$

Now since $\times$ and $\uplus$ are bifunctors (by Remark 3.5.2) it follows that

$$(g \circ f) \times 1_G = (g \times 1_G) \circ (f \times 1_G) \quad \text{and} \quad (t \circ u) \uplus 1_\sigma = (t \uplus 1_\sigma) \circ (u \uplus 1_\sigma)$$

so that

$$\begin{aligned} \mathcal{L}_{\mathfrak{G}}\left(\langle g, u\rangle \circ \langle f, t\rangle\right) &= \langle (g \times 1_G) \circ (f \times 1_G), (t \uplus 1_\sigma) \circ (u \uplus 1_\sigma)\rangle \\ &= \langle g \times 1_G, u \uplus 1_\sigma\rangle \circ \langle f \times 1_G, t \uplus 1_\sigma\rangle \\ &\quad \text{(by definition of } \circ\text{, Definition 3.4.1)} \\ &= \left(\langle g, u\rangle \otimes \langle 1_G, 1_\sigma\rangle\right) \circ \left(\langle f, t\rangle \otimes \langle 1_G, 1_\sigma\rangle\right) \\ &\quad \text{(by definition of } \otimes\text{, Definition 3.5.1)} \\ &= \left(\langle g, u\rangle \otimes \mathit{1}_{\mathfrak{G}}\right) \circ \left(\langle f, t\rangle \otimes \mathit{1}_{\mathfrak{G}}\right) \\ &\quad \text{(by definition of } \mathit{1}_{\mathfrak{G}}\text{, Notation 3.4.5)} \\ &= \mathcal{L}_{\mathfrak{G}}\left(\langle g, u\rangle\right) \circ \mathcal{L}_{\mathfrak{G}}\left(\langle f, t\rangle\right) \\ &\quad \text{(by definition of } \mathcal{L}_{\mathfrak{G}}) \end{aligned}$$

This shows that $\mathcal{L}_{\mathfrak{G}}$ preserves composition. We deduce that $(\mathcal{L}_{\mathfrak{G}})_{\mathfrak{G}\in\mathbf{FGBM}}$ is a family of functors. A similar proof shows that $(\mathcal{R}_{\mathfrak{F}})_{\mathfrak{F}\in\mathbf{FGBM}}$ is a family of functors as well.

From the definition of these functors it is immediate that

$$\mathcal{L}_{\mathfrak{G}}(\mathfrak{F}) = \mathfrak{F}\otimes\mathfrak{G} = \mathcal{R}_{\mathfrak{F}}(\mathfrak{G}) \quad \text{(for any frames } \mathfrak{F}, \mathfrak{G})$$

Now we need to show that these families of functors are commutative in the sense of Lemma 3.2.7. So suppose that we have two arbitrary type restriction bounded morphisms

$$\begin{array}{c} \mathfrak{F} = \left\langle F, (R_i)_{i\in\tau}\right\rangle \\ \Big\downarrow{\scriptstyle\langle f,t\rangle} \\ \mathfrak{H} = \left\langle H, (T_i)_{i\in\mu}\right\rangle \end{array} \qquad \text{and} \qquad \mathfrak{G} = \left\langle G, (S_i)_{i\in\sigma}\right\rangle \xrightarrow{\langle g,u\rangle} \mathfrak{K} = \left\langle K, (Q_i)_{i\in\nu}\right\rangle$$

We need to show that

$$(\mathcal{L}_{\mathfrak{K}}(\langle f,t\rangle)) \circ (\mathcal{R}_{\mathfrak{F}}(\langle g,u\rangle)) = (\mathcal{R}_{\mathfrak{H}}(\langle g,u\rangle)) \circ (\mathcal{L}_{\mathfrak{G}}(\langle f,t\rangle)). \tag{3.5.6}$$

This is equivalent to showing that the following diagram (in **FGBM**) commutes

$$\begin{tikzcd} \mathcal{L}_{\mathfrak{G}}(\mathfrak{F}) = \mathcal{R}_{\mathfrak{F}}(\mathfrak{G}) \arrow[r, "\mathcal{R}_{\mathfrak{F}}(\langle g,u\rangle)"] \arrow[d, "\mathcal{L}_{\mathfrak{G}}(\langle f,t\rangle)"'] & \mathcal{R}_{\mathfrak{F}}(\mathfrak{K}) = \mathcal{L}_{\mathfrak{K}}(\mathfrak{F}) \arrow[d, "\mathcal{L}_{\mathfrak{K}}(\langle f,t\rangle)"] \\ \mathcal{L}_{\mathfrak{G}}(\mathfrak{H}) = \mathcal{R}_{\mathfrak{H}}(\mathfrak{G}) \arrow[r, "\mathcal{R}_{\mathfrak{H}}(\langle g,u\rangle)"'] & \mathcal{R}_{\mathfrak{H}}(\mathfrak{K}) = \mathcal{L}_{\mathfrak{K}}(\mathfrak{H}) \end{tikzcd}$$

By evaluating the functors we see that this diagram is equivalent to

$$\begin{tikzcd} \mathfrak{F}\otimes\mathfrak{G} \arrow[r, "1_{\mathfrak{F}}\otimes\langle g,u\rangle"] \arrow[d, "\langle f,t\rangle\otimes 1_{\mathfrak{G}}"'] & \mathfrak{F}\otimes\mathfrak{K} \arrow[d, "\langle f,t\rangle\otimes 1_{\mathfrak{K}}"] \\ \mathfrak{H}\otimes\mathfrak{G} \arrow[r, "1_{\mathfrak{H}}\otimes\langle g,u\rangle"'] & \mathfrak{H}\otimes\mathfrak{K} \end{tikzcd}$$

For this diagram to commute it is necessary that the following diagrams, for the point function and type function respectively, commute.

$$\begin{tikzcd} F\times G \arrow[r, "1_F\times g"] \arrow[d, "f\times 1_G"'] & F\times K \arrow[d, "f\times 1_K"] \\ H\times G \arrow[r, "1_H\times g"'] & H\times K \end{tikzcd} \qquad \begin{tikzcd} \tau\uplus\sigma & \tau\uplus\nu \arrow[l, "1_\tau\uplus u"'] \\ \mu\uplus\sigma \arrow[u, "t\uplus 1_\sigma"] & \mu\uplus\nu \arrow[l, "1_\mu\uplus u"] \arrow[u, "t\uplus 1_\nu"'] \end{tikzcd}$$

These two diagrams are equivalent to the equations

$$(1_H\times g)\circ(f\times 1_G) = (f\times 1_K)\circ(1_F\times g)$$

and

$$(t \uplus 1_\sigma) \circ (1_\mu \uplus u) = (1_\tau \uplus u) \circ (t \uplus 1_\nu),$$

which hold since $\times$ and $\uplus$ are bifunctors on the category of sets (by Remark 3.5.2).

Now the families of functors, $(\mathcal{L}_{\mathfrak{G}})_{\mathfrak{G}\in\mathbf{FGBM}}$ and $(\mathcal{R}_{\mathfrak{F}})_{\mathfrak{F}\in\mathbf{FGBM}}$, satisfy the conditions of Lemma 3.2.7 so that $\otimes : \mathbf{FGBM} \times \mathbf{FGBM} \to \mathbf{FGBM}$ is a bifunctor as required. □

In Remark 3.5.2 we stated that the use of **Set**-products and **Set**-coproducts to define the bifunctor $\otimes$ suggests possible natural **FGBM**-isomorphisms to complete the specification of our monoidal category. We now exhibit these suggestions and show that they are in fact type restriction bounded morphisms and **FGBM**-isomorphisms as needed, to do it we will use Convention 3.2.10 once again.

Notation 3.5.7. Given sets F, G, H define the function

$$ap_{F,G,H} : F \times (G \times H) \to (F \times G) \times H$$

as the unique function such that the following diagram commutes

$$\begin{tikzcd}
F \times (G \times H) \arrow[r] \arrow[d] \arrow[ddrr, dotted, "ap_{F,G,H}"] & (G \times H) \arrow[d] \arrow[dr] & \\
F & G & H \\
& (F \times G) \arrow[ul] \arrow[u] & (F \times G) \times H \arrow[l] \arrow[u]
\end{tikzcd}$$

It is well known that since we have taken $\times$ to be the cartesian product we can construct $ap_{F,G,H}$ as follows

$$\begin{aligned} ap_{F,G,H} : F \times (G \times H) &\to (F \times G) \times H \\ \langle v, \langle w, x \rangle \rangle &\mapsto \langle \langle v, w \rangle, x \rangle \end{aligned}$$

Similarly, given types τ, σ, μ we define the function

$$at_{\tau,\sigma,\mu} : (\tau \uplus \sigma) \uplus \mu \to \tau \uplus (\sigma \uplus \mu)$$

as the unique function such that the following diagram commutes

$$\begin{tikzcd}
\tau \uplus (\sigma \uplus \mu) & (\sigma \uplus \mu) \arrow[l] & \\
\tau \arrow[u] \arrow[dr] & \sigma \arrow[u] \arrow[d] & \mu \arrow[ul] \arrow[d] \\
& (\tau \uplus \sigma) \arrow[r] & (\tau \uplus \sigma) \uplus \mu \arrow[uull, dotted, "at_{\tau,\sigma,\mu}"']
\end{tikzcd}$$

The construction of $at_{\tau,\sigma,\mu}$ is slightly more technical than that of $ap_{F,G,H}$. If we assume, as motivated in Remark 1.3.2, that τ, σ and μ are disjoint sets and let $\uplus$ simply give their union then $\tau \uplus (\sigma \uplus \mu) = (\tau \uplus \sigma) \uplus \mu$ and $at_{\tau,\sigma,\mu}$ will be the identity function. We will assume this is true, although this assumption causes no loss in generality. Our mnemonics for the names *ap* and *at* are ***a**ssociativity isomorphism **p**oint function* and ***a**ssociativity isomorphism **t**ype function* respectively. For any frames

$$\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle, \mathfrak{G} = \left\langle G, (S_i)_{i\in\sigma} \right\rangle, \mathfrak{H} = \left\langle H, (T_i)_{i\in\mu} \right\rangle$$

we now define $\alpha_{\mathfrak{F},\mathfrak{G},\mathfrak{H}} = \left\langle ap_{F,G,H}, at_{\tau,\sigma,\mu} \right\rangle$, and let $\alpha = (\alpha_{\mathfrak{F},\mathfrak{G},\mathfrak{H}})_{\mathfrak{F},\mathfrak{G},\mathfrak{H}\in\mathbf{FGBM}}$.

Proposition 3.5.8. *α is a natural* **FGBM***-isomorphism from $- \otimes (- \otimes -)$ to $(- \otimes -) \otimes -$.*

That every $\alpha_{\mathfrak{F},\mathfrak{G},\mathfrak{H}}$ is an **FGBM**-isomorphism from $\mathfrak{F}\times(\mathfrak{G}\times\mathfrak{H})$ to $(\mathfrak{F}\times\mathfrak{G})\times\mathfrak{H}$ was already proved in Lemma 1.3.4. However there we did not distinguish between the types of $\mathfrak{F}\times(\mathfrak{G}\times\mathfrak{H})$ and $(\mathfrak{F}\times\mathfrak{G})\times\mathfrak{H}$, this is very much a standard approach. In fact Kurucz *et al.* (2003) states Lemma 1.3.4 without mentioning the types at all. Type restriction bounded morphisms allow us to make this distinction between slightly different types, but also give us a formal motivation in the form of Lemma 3.4.10 to neglect the distinction under suitable conditions. The components of the members of α were chosen according to the conditions laid out in Proposition 3.2.11, which necessitates their naturality. This implies that α is natural as well.

Notation 3.5.9. Given a set F define the functions

$$lp_F : \{0\} \times F \to F \qquad \text{and} \qquad rp_F : F \times \{0\} \to F$$

as the second and first **Set**-product projections respectively. Just as in the case of $ap_{F,G,H}$ we can easily construct lp_F and rp_F, we do it as follows.

$$\begin{array}{rcl} lp_F : \{0\} \times F & \to & F \\ \langle 0, v\rangle & \mapsto & v \end{array} \qquad \begin{array}{rcl} rp_F : F \times \{0\} & \to & F \\ \langle v, 0\rangle & \mapsto & v \end{array}$$

Similarly, given a type τ we define the functions

$$lt_\tau : \tau \to \varnothing \uplus \tau \qquad \text{and} \qquad rt_\tau : \tau \to \tau \uplus \varnothing$$

as the second and first **Set**-coproduct injections respectively. Once again it is harder to construct lt_τ and rt_τ, but as stated repeatedly we may assume here that $\varnothing \uplus \tau = \varnothing \cup \tau = \tau$, and let $lt_\tau = 1_\tau$. A similar assumption also allows us to construct rt_τ as 1_τ. Our mnemonics for lp and lt are ***l**eft unit isomorphism **p**oint function* and ***l**eft unit isomorphism **t**ype function* respectively. Similarly our mnemonics for rp and rt are ***r**ight unit isomorphism **p**oint function* and ***r**ight unit isomorphism **t**ype function* respectively.

For any frame $\mathfrak{F} = \left\langle F, (R_i)_{i\in\tau} \right\rangle$ we now define $\lambda_{\mathfrak{F}} = \left\langle lp_F, lt_\tau \right\rangle$ and $\varrho_{\mathfrak{F}} = \left\langle rp_F, rt_\tau \right\rangle$ and we let $\lambda = (\lambda_{\mathfrak{F}})_{\mathfrak{F}\in\mathbf{FGBM}}$ and $\varrho = (\varrho_{\mathfrak{F}})_{\mathfrak{F}\in\mathbf{FGBM}}$.

Recall the frame $\mathfrak{E}$ from Notation 3.4.15. It is easily seen that $\lambda_{\mathfrak{F}}$ is an **FGBM**-isomorphism from $\mathfrak{E} \otimes \mathfrak{F}$ to $\mathfrak{F}$, and that $\varrho_{\mathfrak{F}}$ is an **FGBM**-isomorphism from $\mathfrak{F} \otimes \mathfrak{E}$ to $\mathfrak{F}$. In fact we have already proved a similar result in Proposition 3.4.19. To verify that λ and ϱ are both natural we use the same argument as for the naturality of α: note that the components of the members of both families were chosen according to the conditions laid out in Proposition 3.2.11, which necessitates their naturality. We conlude that λ and ϱ are natural as well. We have now motivated have the following result.

Proposition 3.5.10. *The frame $\mathfrak{E}$ is a left unit for $\otimes$ up to natural* **FGBM**-*isomorphism λ, and a right unit for $\otimes$ up to natural* **FGBM**-*isomorphism ϱ.*

We now state our final result which shows that $\otimes$ is the multiplication of a monoidal category.

Theorem 3.5.11. *The tuple $\langle \mathbf{FGBM}, \otimes, \mathfrak{E}, \alpha, \lambda, \varrho \rangle$ is a monoidal category.*

Proof. Recalling Proposition 3.5.4, Proposition 3.5.8 and Proposition 3.5.10 we note that we only need to verify that the two diagrams in Definition 3.2.8 both commute. In Notation 3.5.7 and Notation 3.5.9 we defined point functions and type functions of α, λ and ϱ in terms of the isomorphisms in Proposition 3.2.11. Hence the point functions and type functions has been chosen such that the two diagrams in Definition 3.2.8 commute. $\square$

Theorem 3.5.11 gives part of our motivation for the way we have defined type restriction bounded morphisms, in particular it motivates the directions we chose for the point function and type function. Observe that if the two functions had the same direction and the type of the frame product was left as it is then it would void Proposition 3.5.8 and consequently also Theorem 3.5.11. Theorem 3.5.11 also motivates why we allow empty types since the beginning of Chapter 1.

Having shown that $\otimes$ is the multiplication of a monoidal category we have gleaned some understanding of how $\otimes$ mimics a categorical product. This result should not be surprising in light of how much of the construction was obtained from the monoidal structure induced by products on the category $\mathbf{Set} \times \mathbf{Set}^{\mathrm{op}}$. Apart from **FGBM** using frames for objects rather than pairs of sets as in $\mathbf{Set} \times \mathbf{Set}^{\mathrm{op}}$, we have also required that arrows in **FGBM** satisfy the forward and back conditions of Definition 3.4.1. Our contribution therefore, was to show that adding these conditions does not spoil the monoidal category structure obtained from the $\mathbf{Set} \times \mathbf{Set}^{\mathrm{op}}$-products.

Concluding remarks and suggested further work

The study of **FGBM** still poses several questions.

- In the course of our investigation we have shown that some **FGBM**-products do not exist (Example 3.4.21) and we have exhibited a **FGBM**-terminal object (Lemma 3.4.16), but it remains to be seen whether any non-trivial **FGBM**-products exist. More generally we may want to characterise the **FGBM**-limits that exist. The existence of **FGBM**-colimits is also something that we have not discussed, although **FGBM**-coproducts can be constructed with rather little effort.

- Our motivation in chapter 3 was to provide a category where the frame product can be studied, and although we have shown that the frame product gives a multiplication on **FGBM** (Theorem 3.5.11), and that this monoidal category is symmetric (Lemma 1.3.4) it is not yet clear whether this monoidal category is closed, or whether the monoids and comonoids in the the category are of any interest (see Mac Lane (1997) for details on *closed categories* and *monoids*).

- Other simpler category-theoretic questions about **FGBM** also remain unanswered, for example what are epis, monos, and the various other "special" arrows.

- We may define a "forgetful" functor, $\mathcal{U} : \mathbf{FGBM} \to \mathbf{Set} \times \mathbf{Set}^{\mathrm{op}}$ by mapping every frame to the pair of sets denoting its universe and type; and mapping every type restriction bounded morphism to the pair of functions denoting its point function and type function. We did not investigate the properties of $\mathcal{U}$. For example it is not clear to us whether it has a left adjoint, and our poor understanding of **FGBM**-products dismisses the chance of quickly providing a **FGBM**-limit that is not preserved by $\mathcal{U}$.

- We have only investigated type restriction bounded morphisms between frames, but a variation for models is obvious. Furthermore, a generalisation to *type restriction simulations* by replacing point functions with the appropriate relations, seems like a topic for further work. It may be insightful to examine which properties of simulations are maintained by this generalisation. A similar study can be done for *type restriction bisimulations*.

- The popular category theoretic setting to study modal logic is via coalgebras (as done in Venema (2007)), but for brevity of our exposition we preferred not to take that route. Since the functor that determines a category of coalgebras also fixes their type it is immediately hard to use coalgebras to study frame products as we did, but it may still be interesting to find an equivalence or isomorphism between **FGBM** and some category of coalgebras. Although we received the work of Sano (2011) quite late during our own research, and are therefore not very familiar with it, they still seem to rely on different categories for each type.

List of References

Aczel, P. & Mendler, N.P. 1989. A final coalgebra theorem. In *Category Theory and Computer Science*, pages 357–365.

Adámek, J., Herrlich, H. & Strecker, G.E. 2004. *Abstract and Concrete Categories.* Copyright held by the authors, Published under GNU Free Documentation License. `http://katmat.math.uni-bremen.de/acc`.

Blackburn, P., de Rijke, M. & Venema, Y. 2001. *Modal Logic*, volume 53 of *Cambridge tracts in theoretical computer science.* Cambridge University Press.

Blackburn, P. & van Benthem, J. 2007. *Handbook of Modal Logic*, chapter 1 Modal Logic: A Semantic Perspective. Elsevier.

Brink, C. & Rewitzky, I. 2004. Simulations as modalities. Unpublished.

Gabbay, D. & Shehtman, V. 1998. Products of modal logics, part 1. *Logic Journal of IGPL*, 6(1):73–146.

Goldblatt, R. 1989. Varieties of Complex Algebras. *Annals of Pure and Applied Logic*, 44:173–242.

Goldblatt, R. 2003. Mathematical modal logic: A view of its evolution. *Journal of Applied Logic*, 1(5-6):309–392.

Goranko, V. & Otto, M. 2007. *Handbook of Modal Logic*, chapter 5 Model theory of modal logic. Elsevier.

Gumm, H.P. & Schröder, T. 2001. Products of coalgebras. *Algebra Universalis*, 46:163–185.

Hodges, W. 1993. *Model Theory.* Cambridge University Press.

Jech, T. 2003. *Set Theory.* The third millennium edition. Springer-Verlag.

Kripke, S.A. 1959. A completeness theorem in modal logic. *The Journal of Symbolic Logic*, 24:1–14.

Kurucz, A., Wolter, F., Zakharyaschev, M. & Gabbay, D. 2003. *Many Dimensional Modal Logics: Theory and Applications.* Studies in Logic and the Foundations of Mathematics. Elsevier.

Mac Lane, S. 1997. *Categories for the Working Mathematician.* Number 5 in Graduate Texts in Mathematics, 2nd edition. Springer-Verlag.

Milner, R. 1971. An algebraic definition of simulation between programs. In *Proceedings of the 2nd international joint conference on Artificial intelligence*, pages 481–496.

Park, D. 1981. Concurrency and automata on infinite sequences. In *Proceedings of the 5th GI-Conference on Theoretical Computer Science*, pages 167–183.

Sano, K. 2011. Generalized product of coalgebraic hybrid logics. In *Proceeedings of Algebra and Coalgebra in Computer Science - 4th International Conference, CALCO 2011*, pages 329–343.

Segerberg, K. 1968. Decidability of S4.1. *Theoria*, 34(1):7–20.

Segerberg, K. 1971. An essay in classical modal logic. *Filosofiska Studier*, 13.

Shehtman, V. 1978. Two-dimensional modal logic. *Mathematical notes of the Academy of Sciences of the USSR.*

van Benthem, J.F.A.K. 1976. Modal correspondence theory. Ph.D. thesis, University of Amsterdam.

Venema, Y. 2007. *Handbook of Modal Logic*, chapter 6 Algebras and Coalgebras. Elsevier.