Limit theorems for integer partitions and their generalisations

by

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Declaration

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Abstract

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Various properties of integer partitions are studied in this work, in particular the number of summands, the number of ascents and the multiplicities of parts. We work on random partitions, where all partitions from a certain family are equally likely, and determine moments and limiting distributions of the different parameters.

The thesis focuses on three main problems: the first of these problems is concerned with the length of prime partitions (i.e., partitions whose parts are all prime numbers), in particular restricted partitions (i.e., partitions where all parts are distinct). We prove a central limit theorem for this parameter and obtain very precise asymptotic formulas for the mean and variance.

The second main focus is on the distribution of the number of parts of a given multiplicity, where we obtain a very interesting phase transition from a Gaussian distribution to a Poisson distribution and further to a degenerate distribution, not only in the classical case, but in the more general context of λ-partitions: partitions where all the summands have to be elements of a given sequence λ of integers.

Finally, we look into another phase transition from restricted to unrestricted partitions (and from Gaussian to Gumbel-distribution) as we study the number of summands in partitions with bounded multiplicities.

Uittreksel

Limietstellings vir heelgetal-partisies en hulle veralgemenings

("Limit theorems for integer partitions and their generalisations")

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Verskillende eienskappe van heelgetal-partisies word in hierdie tesis bestudeer, in die besonder die aantal terme, die aantal stygings en die veelvoudighede van terme. Ons werk met stogastiese partisies, waar al die partisies in 'n sekere familie ewekansig is, en ons bepaal momente en limietverdelings van die verskillende parameters.

Die teses fokusseer op drie hoofprobleme: die eerste van hierdie probleme gaan oor die lengte van priemgetal-partisies (d.w.s., partisies waar al die terme priemgetalle is), in die besonder beperkte partisies (d.w.s., partisies waar al die terme verskillend is). Ons bewys 'n sentrale limietstelling vir hierdie parameter en verkry baie presiese asimptotiese formules vir die gemiddelde en die variansie.

Die tweede hooffokus is op die verdeling van die aantal terme van 'n gegewe veelvoudigheid, waar ons 'n baie interessante fase-oorgang van 'n normaalverdeling na 'n Poisson-verdeling en verder na 'n ontaarde verdeling verkry, nie net in die klassieke geval nie, maar ook in die meer algemene konteks van sogenaamde λ-partities: partisies waar al die terme elemente van 'n gegewe ry λ van heelgetalle moet wees.

Laastens beskou ons ’n ander fase-oorgang van beperkte na onbeperkte partisies (en van normaalverdeling na Gumbel-verdeling) wanneer ons die aantal terme in partisies met begrensde veelvoudighede bestudeer.

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Dedications

To Sanda.

Contents

Chapter 1

Introduction

1.1 Partitions

1.1.1 Definitions and examples

We are mainly focussing on the study of partitions of a positive integer n into positive integer parts, so let us formally define what a partition is.

Definition 1.1.1. A partition of n is a non-decreasing sequence of positive integers c_1, c_2, $\cdots$, c_t such that

$$n = c_1 + c_2 + \cdots + c_t.$$

We usually denote a partition of n as $(c_1, c_2, \cdots, c_t)$.

Another way to represent a partition of n is to associate a graphical representation called *Ferrers diagram* to it, as illustrated in the example in Figure 1.1. In the example we give the Ferrers diagram associated to the partition $(1, 2, 5, 5, 7)$ of 20. Each part of the partition is represented by a vertical row of dots. If we count the number of dots in the horizontal rows, we obtain $(1, 1, 3, 3, 3, 4, 5)$. This is also a partition of 20, and we say that $(1, 1, 3, 3, 3, 4, 5)$ is the *conjugate* of $(1, 2, 5, 5, 7)$.

1.1.2 Restricted partitions

Various types of partitions that occur in the literature are considered in this thesis, making their study more interesting. For example we can only consider partitions which have no parts of multiplicity more than one, we call these *restricted partitions*.

$$\begin{array}{cccccc}
 & & & & \bullet & 1 \\
 & & & & \bullet & 1 \\
 & & \bullet & \bullet & \bullet & 3 \\
 & & \bullet & \bullet & \bullet & 3 \\
 & & \bullet & \bullet & \bullet & 3 \\
 & \bullet & \bullet & \bullet & \bullet & 4 \\
\bullet & \bullet & \bullet & \bullet & \bullet & 5 \\
1 & 2 & 5 & 5 & 7 &
\end{array}$$

Figure 1.1: Ferrers diagram

We can also consider partitions whose parts are members of a given sequence λ of positive integers. We call these partitions *λ-partitions*. Furthermore, in the case of λ-partitions, we distinguish between *restricted λ-partitions* if parts are not allowed to repeat and *unrestricted λ-partitions* otherwise. We use the term *ordinary partitions* for the case where λ is equal to the set of positive integers $\mathbb{N}$.

In this work λ is always an unbounded non-decreasing sequence of positive integers. It is possible to study partitions were each part is a member of a finite sequence of positive integers but we are not going to consider those.

Example 1. Consider the partitions of $n = 5$:

$$\begin{aligned}
5 &= 1 + 1 + 1 + 1 + 1 \\
&= 1 + 1 + 1 + 2 \\
&= 1 + 1 + 3 \\
&= 1 + 2 + 2 \\
&= 1 + 4 \\
&= 2 + 3 \\
&= 5.
\end{aligned}$$

So there are 7 unrestricted partitions of 5 in total, but only 3 of them are restricted partitions. If we consider only λ-partitions where λ is the set of all odd positive integers then there are only 3 partitions in total and only 1 restricted.

There are many questions that one might want to understand, such as: how many partitions are there in total, or how many partitions are there with a given restriction? It is difficult to get an answer to this kind of questions in general. So various asymptotic results have been given by means of analytic methods to help us understand the structure of partitions.

1.1.3 Generating functions for partitions

One of the most efficient ways to study partitions is the use of so-called *generating functions*. Generating functions are elements of the ring of powers series that encode sequences of numbers. For example consider the generating function for $p(n)$ (the total number of partitions of n). By definition the generating function is

$$F(z) = \sum_{n\geq 1} p(n) z^n. \tag{1.1.1}$$

It is not hard to see that $F(z)$ can be written in the form of an infinite product:

$$F(z) = \prod_{n\geq 1} \frac{1}{1-z^n}. \tag{1.1.2}$$

The infinite product on the right hand side is convergent whenever $|z| < 1$. Therefore, $F(z)$ can be regarded as an analytic function in the unit disc.

Now to extract the coefficient of z^n in $F(z)$ we may use the Cauchy integral formula and obtain

$$p(n) = \frac{1}{2\pi i} \int_\gamma F(z) \frac{dz}{z^{n+1}}, \tag{1.1.3}$$

where γ is any curve around zero oriented in anticlockwise direction that is contained in the interior of the unit disc.

Note that when we consider restricted partitions then the corresponding generating function is

$$F^*(z) = \prod_{n\geq 1} (1+z^n). \tag{1.1.4}$$

Furthermore, if we consider λ-partitions then we only need to take the products over the sequence λ instead of the whole set of positive integers.

1.2 Asymptotic results

Various asymptotic results have been given to understand the structure of partitions of large integers. We shall state some important results.

1.2.1 The Hardy-Ramanujan-Rademacher formula, and the Meinardus scheme

Let us first state the well known result by Hardy and Ramanujan [12] on the number of partitions of n when n is large.

Theorem 1.2.1. *The total number of partitions of n admits the asymptotic formula:*

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \tag{1.2.1}$$

as $n \to \infty$.

Then later Rademacher (see [1], page 69) provided an exact formula for $p(n)$ as we see in the next theorem.

Theorem 1.2.2. *We have, for any n,*

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k\geq 1} A_k(n)\sqrt{k} \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}(x-\frac{1}{24})}\right)}{\sqrt{(x-\frac{1}{24})}} \right]_{x=n} \tag{1.2.2}$$

where

$$A_k(n) = \sum_{\substack{h \bmod k \\ (h,k)=1}} \omega_{h,k} e^{-2\pi i n h/k} \tag{1.2.3}$$

and $\omega_{h,k}$ is a $24k$th root of unity.

There is a powerful generalisation of Theorem 1.2.1 to λ-partitions under certain conditions on the sequence λ known as the Meinardus scheme, see [19]. Let λ be a non-decreasing and unbounded sequence of positive integers, and consider the Dirichlet series:

$$D(s) = \sum_{\lambda} \frac{1}{\lambda^s}.$$

Definition 1.2.3. We say that a sequence λ satisfies the Meinardus scheme if the following three conditions are satisfied:

(M1) The Dirichlet series $D(s)$ converges in the half-plane $\mathrm{Re}(s) > \alpha > 0$, and can be analytically continued into the half-plane $\mathrm{Re}(s) \geq -\alpha_0$ for some $\alpha_0 > 0$. For $\mathrm{Re}(s) \geq -\alpha_0$, $D(s)$ is analytic except for a simple pole at $s = \alpha$ with residue A.

(M2) There is a constant $c > 0$ such that $D(s) \ll |t|^c$ uniformly for $\mathrm{Re}(s) \geq -\alpha_0$ as $|\,\mathrm{Im}(s)| = |t| \to \infty$.

(M3) Let $\chi(\tau) = \sum_\lambda e^{-\lambda\tau}$, where $\tau = r + iy$ with $r > 0$. Then

$$\chi(r) - \mathrm{Re}(\chi(\tau)) \gg \left(\log \frac{1}{r}\right)^2$$

uniformly for $r^{1+\frac{\alpha}{2}} \leq |y| \leq \pi$ as $r \to 0$.

There are many sequences of positive integers satisfying the Meinardus scheme including the sequence $\lambda = \mathbb{Z}^+$, sequence of odd positive integers, squares, and many others.

We can now state the general theorem of Meinardus on the number of λ-partitions of n.

Theorem 1.2.4. *If λ satisfies the Meinardus scheme then*

$$p_\lambda(n) \sim \kappa_1 n^{\kappa_2} \exp\left(\kappa_3 n^{\alpha/(\alpha+1)}\right) \tag{1.2.4}$$

as $n \to \infty$, where the κ_i's are constants that can be determined in terms of α and A.

1.2.2 Limit theorems

The first central limit results in the theory of partitions were given by Erdős and Lehner in their paper in 1941 (see [6]), in which they studied the number of summands in partitions. More precisely, if all partitions of n are equally likely, then they gave a central limit theorem for the number of summands in a random partition when n is large. Their results were later extended and generalised, and the method also adapted to work for other parameters. The following theorem was proved by Erdős and Lehner:

Theorem 1.2.5. *If ϖ_n is the number of summands in a random unrestricted partition (i.e., parts are allowed to repeat) of n then for any real number x*

$$\mathbb{P}\Big(\varpi_n \leq \mu_n + x\sigma_n\Big) \sim e^{-e^{-\left(\gamma + \frac{\pi}{\sqrt{6}}x\right)}} \tag{1.2.5}$$

as $n \to \infty$, where μ_n and σ_n are the mean and the standard deviation of ϖ_n. Furthermore we have the following asymptotic estimates:

$$\mu_n = \frac{\sqrt{6n}}{2\pi}\Big(\log n + 2\gamma - \log(\pi^2/6)\Big) + \mathcal{O}(\log n),$$

and

$$\sigma_n^2 = n + \mathcal{O}(\sqrt{n}\log^2 n).$$

Similarly, if ϖ_n^* is the number of summands in a random restricted partition of n then for any real number x we have

$$\mathbb{P}\Big(\varpi_n^* \leq \mu_n^* + x\sigma_n^*\Big) \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt, \tag{1.2.6}$$

where μ_n^* and σ_n^* are the mean and the standard deviation of ϖ_n^* respectively. Furthermore,

$$\mu_n^* = \frac{2\log 2}{\pi}\sqrt{3n} + \frac{3\log 2}{\pi^2} - \frac{1}{4} + \mathcal{O}(n^{-1/2})$$

and

$$\sigma_n^{*2} = \left(\frac{\sqrt{3}}{\pi} - \frac{12\sqrt{3}\log^2 2}{\pi^3}\right)\sqrt{n} + \mathcal{O}(1)$$

as $n \to \infty$.

Many generalisations of these results were given, let us just mention a few that are directly related to this thesis. Haselgrove and Temperly in [13] gave a limit theorem for the number of summands of unrestricted λ-partitions, under certain technical conditions. Their results were further extended by Richmond [22] and Lee [17]. For the restricted λ-partitions, the paper by Hwang [15] deserved to be mentioned, in which convergence to the Gaussian distribution is proved for λ satisfying the Meinardus scheme. Many of the techniques that are used here are adapted from the methods of Hwang, which also proved to be powerful to study other parameters.

Further examples of central limit theorems in the context of partitions include those by [10] for the number of distinct parts and by [3] for ascents of size d or more (equivalently, parts of multiplicity d or more).

1.3 Tools and techniques

We mainly use two asymptotic techniques, namely the Mellin transform and the saddle point method. In this section, we are going to give a brief overview of these two methods.

1.3.1 Mellin transform

A nice presentation of this technique can be found in [7]. The basic idea of this method is the use of the fact that the asymptotic expansion of a given function $f(x)$ at $x = 0$ or $+\infty$ is related to the singularities of the Mellin transform of f.

Let $f(x)$ be a function defined on the interval $[0, +\infty)$. The Mellin transform of f is defined as

$$\mathcal{M}(f, s) = \int_0^\infty f(x) x^{s-1} dx \tag{1.3.1}$$

if it exists, where s is a complex variable. The range of s where the integral (1.3.1) is convergent has to be a strip and its interior is called the fundamental strip. The Mellin transform has many useful properties, of which we only mention a few that are particularly important to us.

Proposition 1.3.1 (Inversion formula). *We have*

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}(f, s) x^{-s} ds \tag{1.3.2}$$

for any real number c in the fundamental strip if f is continuous at x.

It is not surprising to have such a formula since the Mellin transform is closely related to the Laplace transform. Let us next provide a formula for the Mellin transform of a derivative. We have

$$\mathcal{M}\left(\frac{d}{dx} f(x), s\right) = -(s-1)\mathcal{M}(f, s-1). \tag{1.3.3}$$

Finally if λ is a sequence of positive integers and $D(s)$ the Dirichlet series associated to λ then

$$\mathcal{M}\left(\sum_\lambda f(\lambda x), s\right) = D(s)\mathcal{M}(f, s). \tag{1.3.4}$$

We will use the following theorem very frequently and refer to it as "the Mellin transform method":

Theorem 1.3.2. *Let $\phi(x)$ be a continuous function on $(0, \infty)$ with Mellin transform $\phi^*(s)$ having a non empty fundamental strip $\langle \alpha, \beta \rangle$. Assume that $\phi^*(s)$ admits a meromorphic continuation to the strip $\langle \gamma, \beta \rangle$ for $\gamma < \alpha$ with a finite number of poles there, which is analytic on $\mathrm{Re}(s) = \gamma$. Assume also that there exists a real number $\eta \in (\alpha, \beta)$ such that*

$$\phi^*(s) = O(|s|^{-c}) \tag{1.3.5}$$

with $c > 1$ as $|s| \to \infty$ in the strip $\gamma \leq \mathrm{Re}(s) \leq \eta$. If $\phi^*(s)$ admits the singular expansion for $s \in \langle \gamma, \alpha \rangle$

$$\phi^*(s) \asymp \sum_{(\xi,k)\in A} \frac{d_{\xi,k}}{(s-\xi)^k},$$

then an asymptotic expansion of $\phi(x)$ at 0, $x > 0$, is

$$\phi(x) = \sum_{(\xi,k)\in A} \frac{(-1)^{k-1} d_{\xi,k}}{(k-1)!} x^{-\xi} (\log x)^{k-1} + O(x^{-\gamma}).$$

1.3.2 Saddle-point method

Let $F(z)$ be a generating function that is analytic at the origin. The saddle-point method is a technique to estimate the coefficient $[z^n]F(z)$ for large n. So let us describe the general procedure of the saddle-point method. We have

$$[z^n]F(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} F(z)\frac{dz}{z^{n+1}}, \tag{1.3.6}$$

where $\mathcal{C}$ is a circle around the origin oriented in anticlockwise direction. We make a change of variable $z = e^{-(r+it)}$ where r is positive and t within the interval $[-\pi, \pi)$. Then

$$[z^n]F(z) = \frac{e^{nr}}{2\pi}\int_{-\pi}^{\pi} F(e^{-(r+it)})e^{nit}dt. \tag{1.3.7}$$

Now we cut this integral as follows: the central part, i.e., the integral over $[-t_0, t_0]$ where t_0 is to be chosen later, and the rest which are called the tails. In order to use the saddle-point method it is important that we have a condition on $F(z)$ to guarantee that the tails are small compared to the central part. If we set

$$f(r+it) := \log F(e^{-(r+it)}), \tag{1.3.8}$$

then for $|t| \le t_0$ we have

$$f(r+it) = f(r) + if'(r)t - f''(r)\frac{t^2}{2} + \mathcal{O}(|t_0|^3 \max_{|\eta|\le t_0}|f'''(r+i\eta)|). \tag{1.3.9}$$

We chose r to be the solution of the equation

$$n = -f'(r). \tag{1.3.10}$$

In most of the cases we encounter in this thesis this gives us an r that tends to zero as n goes to infinity. Now t_0 should be chosen in such way that in Equation (1.3.9) the error term tends to zero while the term with t^2 tends to infinity.

If all of these assumptions are satisfied then we deduce that

$$[z^n]F(z) = \frac{e^{nr+f(r)}}{2\pi}\int_{-t_0}^{t_0} e^{-f''(r)t^2/2}dt(1 + \mathcal{O}(|t_0|^3 \max_{|\eta|\le t_0}|f'''(r+i\eta)|)) \tag{1.3.11}$$

and

$$\begin{aligned}\int_{-t_0}^{t_0} e^{-f''(r)t^2/2}dt &= \int_{-\infty}^{+\infty} e^{-f''(r)t^2/2}dt - 2\int_{t_0}^{+\infty} e^{-f''(r)t^2/2}dt \\ &= \frac{1}{\sqrt{2\pi f''(r)}} + \mathcal{O}\left(\frac{e^{-f''(r)t_0^2/2}}{t_0 f''(r)}\right).\end{aligned}$$

To illustrate the two methods we are going to present a proof of Theorem 1.2.4.

1.3.3 Proof of the Meinardus result

Let λ be an unbounded non-decreasing sequence of positive integers satisfying the Meinardus scheme. We have the generating function for the number of λ-partitions

$$F(z) = \prod_{\lambda} \frac{1}{1 - z^{\lambda}}. \tag{1.3.12}$$

We can see that $F(z)$ is analytic in the interior of the unit disc. We use the saddle-point method to estimate the number of λ-partitions $p_{\lambda}(n) = [z^n]F(z)$ for large n.

Proposition 1.3.3. *The number of λ-partitions of n satisfies the asymptotic formula*

$$p_{\lambda}(n) = \frac{e^{nr+f(r)}}{\sqrt{2\pi f''(r)}} \left(1 + \mathcal{O}(n^{-\delta})\right) \tag{1.3.13}$$

for some constant $\delta > 0$, as $n \to +\infty$, where r is the solution of the equation

$$n = \sum_{\lambda} \frac{\lambda}{e^{\lambda r} - 1} \tag{1.3.14}$$

and

$$f(\tau) = \log F(e^{-\tau}). \tag{1.3.15}$$

Proof. Recall the integral for the coefficient

$$[z^n]F(z) = \frac{e^{nr}}{2\pi} \int_{-\pi}^{\pi} \exp(nit + f(r + it))dt, \tag{1.3.16}$$

where r is defined in the statement of the Proposition. Since the series on the right hand side of Equation (1.3.14) is a monotone decreasing function of r, the solution exists and it tends to zero as $n \to \infty$. We choose $t_0 = r^{1+\beta}$ where β is any constant such that $\alpha/3 < \beta < \alpha/2$ and α is defined as in statement (M1) of the Meinardus scheme.

Now suppose that $|t| \leq t_0$, then the function $f(r+it)$ admits the expansion

$$f(r + it) = f(r) + if'(r)t - f''(r)\frac{t^2}{2} + O(|t_0|^3 \sup_{0 \leq \eta \leq t_0} |f'''(r + i\eta)|). \tag{1.3.17}$$

We use Theorem 1.3.2 to find the dependency between n and r as well as estimates for the derivatives of $f(r)$. By the derivative formula we have

$$\mathcal{M}\left(\frac{d^k}{dr^k} f(r), s\right) = (-1)^k \zeta(s - k + 1)\Gamma(s)D(s - k).$$

Before we use Theorem 1.3.2 we have to make sure that all the conditions are satisfied. This is where the condition (M2) of Meinardus scheme comes in: in guarantees that the condition in (1.3.5) is satisfied. Here we used the fact that on a vertical line the gamma function decreases exponentially while the zeta function increases at most polynomially. Therefore Theorem 1.3.2 applies and we get

$$n \sim A\zeta(\alpha+1)\Gamma(\alpha+1)r^{-(\alpha+1)}. \tag{1.3.18}$$

We also have

$$f''(r) \sim A\zeta(\alpha+1)\Gamma(\alpha+2)r^{-(\alpha+2)} \tag{1.3.19}$$

as $r \to 0^+$. To estimate the error term in (1.3.17), note that

$$|f'''(r+i\eta)| \ll \sum_{\lambda} \frac{\lambda^3 e^{-\lambda r}}{|1-e^{-\lambda(r+i\eta)}|^3}.$$

Since

$$|1-e^{-\lambda(r+i\eta)}| \geq 1-e^{-\lambda r},$$

we have

$$|f'''(r+i\eta)| \ll \sum_{\lambda} \frac{\lambda^3 e^{-\lambda r}}{(1-e^{-\lambda r})^3} \ll r^{-\alpha+3}$$

as $r \to 0^+$ uniformly for $|\eta| \leq t_0$. These are enough to show that the contribution from the central integral is

$$\frac{e^{nr+f(r)}}{\sqrt{2\pi f''(r)}}\left(1+\mathcal{O}(r^{3\beta-\alpha})\right). \tag{1.3.20}$$

It remains to estimate the tails: we have

$$\frac{|F(e^{-(r+it)})|}{F(e^{-r})} = \exp\Big(-\sum_{k\geq 1}\frac{1}{k}\sum_{\lambda} e^{-\lambda k r}(1-\cos(\lambda k t))\Big) \tag{1.3.21}$$

$$\leq \exp\Big(-\sum_{\lambda} e^{-\lambda r}(1-\cos(\lambda t))\Big). \tag{1.3.22}$$

By condition (M3) of the Meinardus scheme (1.3.22) is bounded above by $\exp(-K(\log r)^2)$, for some constant $K > 0$, which is smaller that any power of n. This proves that the contribution from the tails is small and completes the proof. □

We now derive Theorem 1.2.4 from Proposition 1.3.3. We only need to expand nr and $f'(r)$ further is such a way that we have a $o(1)$ error term. For that we use Theorem 1.3.2. We have

$$n = A\zeta(\alpha+1)\Gamma(\alpha+1)r^{-(\alpha+1)} + D(0)r^{-1} + \mathcal{O}(r^{-1+\alpha_0})$$

Let $X = r^{-1}$ then we have the equation

$$n = C_1 X^{\alpha+1} + C_2 X + \mathcal{O}(X^{1-\alpha_0}), \tag{1.3.23}$$

where

$$C_1 = A\zeta(\alpha+1)\Gamma(\alpha+1)$$

and

$$C_2 = D(0).$$

We need X in terms of n so assume that

$$X = \left(\frac{n}{C_1}\right)^{\frac{1}{\alpha+1}}(1+Y)$$

and thus

$$\begin{aligned} C_1 X^{\alpha+1} &= n(1+Y)^{\alpha+1} \\ &= n + (\alpha+1)nY + \mathcal{O}(nY^2). \end{aligned}$$

By Equation (1.3.23) we have

$$C_2\left(\frac{n}{C_1}\right)^{\frac{1}{\alpha+1}}(1+Y) + \mathcal{O}(n^{\frac{1-\alpha_0}{1+\alpha}}) = -(\alpha+1)nY + \mathcal{O}(nY^2)$$

which implies

$$Y = -C_2\frac{\left(\frac{n}{C_1}\right)^{\frac{1}{\alpha+1}}}{(\alpha+1)n} + \mathcal{O}(n^{\frac{-2\alpha}{\alpha+1}} + n^{\frac{-(\alpha+\alpha_0)}{\alpha+1}}).$$

Now we have

$$\begin{aligned} \frac{n}{X} &= C_1^{\frac{1}{\alpha+1}} n^{\frac{\alpha}{\alpha+1}}(1 - Y + \mathcal{O}(n^{\frac{-2\alpha}{\alpha+1}} + n^{\frac{-(\alpha+\alpha_0)}{\alpha+1}})) && (1.3.24) \\ &= C_1^{\frac{1}{\alpha+1}} n^{\frac{\alpha}{\alpha+1}} + \frac{C_2}{\alpha+1} + \mathcal{O}(n^{\frac{-\alpha}{\alpha+1}} + n^{\frac{-\alpha_0}{\alpha+1}}). && (1.3.25) \end{aligned}$$

Similarly we have

$$\begin{aligned} f(r) &= A\zeta(\alpha+1)\Gamma(\alpha)r^{-\alpha} + D(0)\log\tfrac{1}{r} + D'(0) + \mathcal{O}(r^{\alpha_0}) \\ &= \frac{C_1}{\alpha}X^{\alpha} + C_2\log X + D'(0) + \mathcal{O}(X^{-\alpha_0}). \end{aligned}$$

We assume that $\alpha_0 < 1$. From Equation (1.3.23) we have

$$\begin{aligned} C_1 X^{\alpha} &= \frac{n}{X} - C_2 + \mathcal{O}(X^{-\alpha_0}) \\ &= C_1^{\frac{1}{\alpha+1}} n^{\frac{\alpha}{\alpha+1}} - \frac{C_2\alpha}{\alpha+1} + \mathcal{O}(n^{\frac{-\alpha}{\alpha+1}} + n^{\frac{-\alpha_0}{\alpha+1}}). \end{aligned}$$

Therefore

$$f(r) = \frac{C_1^{\frac{1}{\alpha+1}}}{\alpha} n^{\frac{\alpha}{\alpha+1}} + \frac{C_2}{1+\alpha} \log n + \frac{C_2(1-\log C_1)}{\alpha+1} + D'(0) + \mathcal{O}(n^{\frac{-\alpha}{\alpha+1}} + n^{\frac{-\alpha_0}{\alpha+1}}).$$

Finally, from (1.3.19) we obtain

$$f''(r) \sim (\alpha+1) C_1^{\frac{-1}{\alpha+1}} n^{\frac{\alpha+2}{\alpha+1}}.$$

Substituting these asymptotic estimates into (1.3.13) we obtain the Meinardus theorem with

$$\kappa_1 = e^{D'(0)} \frac{C_1^{\frac{1-2D(0)}{2(\alpha+1)}}}{\sqrt{2\pi(\alpha+1)}},$$

$$\kappa_2 = \frac{-1}{2} + \frac{2D(0)-1}{2(\alpha+1)}$$

and

$$\kappa_3 = (\alpha^{-1}+1) C_1^{\frac{1}{\alpha+1}}.$$

1.4 Structure of the thesis

The thesis is structured as follows: in Chapter 2 we study the length (number of summands) in partitions of an integer into primes, both in the restricted (unequal summands) and unrestricted case. It is shown how one can obtain asymptotic expansions for the mean and variance (and potentially higher moments) and we also establish the normality of the limiting distribution, a problem left open by Hwang in [15]. In Chapter 3, the number of parts of given multiplicity in λ-partitions is considered. We improve on earlier results by Corteel et al. [4] by proving a central limit theorem for this number, and we observe a phase transition (Gaussian-Poisson-degenerate) as the multiplicity increases. A detailed presentation of the proof is given for the case of ordinary partitions followed by the generalization to λ-partitions, and in the last section of Chapter 3 we give a generalization of the main result obtained by Brennan et al. [3] on ascents in ordinary partitions to λ-partitions. In Chapter 4, we look at the behaviour of the distribution of the number of summands in an ordinary partition between the restricted case and the unrestricted.

Chapter 2

The number of summands in prime partitions

2.1 Introduction

The presentation of this chapter is based on the published version (see [21]) of our results on prime partitions.

Prime partitions are the special case of λ-partitions where λ is the sequence of primes. Prime partitions are harder to count that ordinary partitions, and the analogue of Theorem 1.2.1 for primes is quite complicated as the asymptotic formula cannot be expressed in terms of elementary functions, see [25] for more details. However, in this case Hardy and Ramanujan [11, 12] proved the following asymptotic formula

$$\log pp(n) \sim 2\pi\sqrt{n/(3\log n)},$$

where $pp(n)$ is the number of partitions of n into primes.

Recall from Chapter 1 that the distribution of the length of a random unrestricted partition is asymptotically the double-exponential (Gumbel) distribution, as shown by Erdős and Lehner. The generalisation of this result to λ-partitions by Haselgrove and Temperley (see [13]) applies to the sequence of primes as well, see also Section 2.4.

On the other hand, Hwang's general theorem for restricted λ-partitions, which builds on the Meinardus scheme, is not applicable to prime partitions, as it was also mentioned explicitly in Hwang's paper [15]. Efforts have been made to reduce these conditions, see for instance [18], but yet the sequence of primes fails to satisfy these conditions.

With some modifications of the method used in [15] we will prove the following result on prime partitions in this chapter:

Theorem 2.1.1. *The number of summands in a random restricted partition of n into distinct primes is asymptotically normally distributed with mean and variance satisfying the following asymptotic formulas:*

$$\mu_n = \frac{2\log 2}{\pi}\sqrt{\frac{6n}{\log n}}\Big(1 - \frac{\log\log n}{2\log n} + \mathcal{O}\Big(\frac{1}{\log n}\Big)\Big)$$

and

$$\sigma_n^2 = \frac{\sqrt{6}}{\pi}\Big(1 - \frac{12\log^2 2}{\pi^2}\Big)\sqrt{\frac{n}{\log n}}\Big(1 - \frac{\log\log n}{2\log n} + \mathcal{O}\Big(\frac{1}{\log n}\Big)\Big)$$

respectively as $n \to \infty$.

In principle, the presented method can be used to determine even more terms of an asymptotic expansion. Let us mention that the main asymptotic term of μ_n (and also of mean and variance in the unrestricted case) already occurs in [23], but with the factor $\log n$ missing.

We shall prove this theorem in detail in Section 2.3, and then later on, in Section 2.4, we give an asymptotic formula for mean and variance for the distribution of the number of summands in a random unrestricted prime partition of n. Finally, in Section 2.5, we discuss how our results can be generalized to a wider variety of sequences involving primes.

2.2 Definitions and preliminary results

In this section we gather important information about primes that are fundamental for the proof of our main results. But first, let us agree on notations and conventions that will be used throughout this chapter.

Notation. Assume that all partitions of n into distinct primes are equally likely. Let ϖ_n denote the number of summands in a random partition, the mean and standard deviation are denoted by μ_n and σ_n respectively. The random variable ϖ_n^* , its mean μ_n^* and variance σ_n^* are defined analogously for unrestricted prime partitions. We shall use $\prod_p$ and $\sum_p$ as abbreviations for the product and sum over all primes respectively. The Dirichlet series associated to the sequence of primes is defined by

$$D(s) := \sum_p p^{-s}$$

for complex numbers s with $\mathrm{Re}(s) > 1$.

The first result that we need is the following about the exponential sum

$$g(\tau) := \sum_p e^{-p\tau}.$$

The result states that:

Lemma 2.2.1. *For any constant* $1/3 < c < 1/2$ *there is a constant* $c_1 > 0$ *such that*

$$g(r) - \operatorname{Re} g(\tau) \geq c_1 \log^2 \tfrac{1}{r}$$

for $\tau = r + iy$ *with* $r^{1+c} \leq |y| \leq \pi$ *as* $r \to 0^+$.

A major part of this result has already been proved by Roth and Szekeres in [24, last section]. Since it is a fundamental result and its proof is not too long, we shall give a complete proof here.

Proof. First, let us assume that $\pi r \leq |y| \leq \pi$. We have

$$g(r) - \operatorname{Re} g(\tau) = \sum_p e^{-pr}(1 - \cos py) \tag{2.2.1}$$

$$\geq \sum_{p \leq r^{-1}} e^{-pr}(1 - \cos p|y|) \tag{2.2.2}$$

$$\geq 8e^{-1} \sum_{p \leq r^{-1}} \left\| \frac{p|y|}{2\pi} \right\|^2, \tag{2.2.3}$$

where $\|.\|$ denotes the distance from the nearest integer. To simplify our notation let us define $\alpha = \alpha(y)$ to be $|y|/2\pi$, then we have $r/2 \leq |\alpha| \leq 1/2$. First, if α is rational, say $\alpha = a/q$, where a and q (with $q > 1$) are coprime, then

$$\sum_{p \leq r^{-1}} \left\| \frac{pa}{q} \right\|^2 \gg \sum_{p \leq r^{-1}} \frac{1}{q^2} \gg \frac{1}{q^2 r \log 1/r}$$

for sufficiently small r. This is sufficient for small values of q, say $q \leq r^{-1/3}$. Now suppose that $2r^{-1} \geq q \geq r^{-1/3}$. Then there are $\gg 1/(r \log 1/r)$ elements in $\{pa : p \leq r^{-1}\}$, and each residue class modulo q contains at most $\lceil \frac{1}{qr} \rceil$ elements from this set . It follows that at least $c_2 q/(\log(1/r))$ distinct residue classes occur in $\{pa : p \leq r^{-1}\}$, where c_2 is a positive constant. Therefore, the sum in (2.2.3) is at least

$$\frac{1}{q^2} \sum_{j=1}^{\lfloor c_3 q/\log 1/r) \rfloor} j^2 > c_4 \frac{r^{-1/3}}{\log^3 1/r}$$

for some positive constants c_3 and c_4. That settles the case where α is a rational of the form a/q with $q \leq 2r^{-1}$. Otherwise we approximate α by a rational in the following way: we choose relatively prime integers a and q such that $q \leq 2\lfloor r^{-1} \rfloor$ and

$$\left| \alpha - \frac{a}{q} \right| \leq \frac{1}{2q\lfloor r^{-1} \rfloor}.$$

Then we claim that for any $p \le r^{-1}$

$$\|p\alpha\| \geq \frac{1}{2}\|pa/q\|.$$

The claim follows from the fact that if

$$|x-y| \leq \frac{\|y\|}{2}$$

then $\|x\| \geq \frac{\|y\|}{2}$, using the triangle inequality for $\|.\|$. Now the desired estimate follows from the case of rational α.

For the remaining part, that is for $r^{1+c} \le |y| \le \pi r$

$$g(r) - \operatorname{Re} g(\tau) = \sum_p e^{-pr}(1-\cos py) \geq \sum_{p \le r^{-1}} e^{-pr}(1-\cos p|y|).$$

The latter sum can be estimated as follows: since $p \le r^{-1}$ we have $p|y| \le \pi$, and so

$$1 - \cos p|y| \geq \frac{2}{\pi^2}p^2y^2.$$

Therefore,

$$\sum_{p \le r^{-1}} e^{-pr}(1-\cos p|y|) \geq \sum_{p \le r^{-1}} e^{-1}\frac{2}{\pi^2}p^2y^2 \geq c_1 r^{-1+2c}/\log\tfrac{1}{r},$$

which completes the proof. □

In this chapter, we have to determine asymptotic expansions for a number of harmonic sums over primes, for which the Dirichlet series $D(s)$ plays a fundamental role. First of all, note that the series $D(s)$ is absolutely convergent in the half-plane $\operatorname{Re}(s) > 1$ and therefore analytic in that region. But we can also express $D(s)$ as

$$D(s) = \log\zeta(s) + \sum_p (\log(1-p^{-s}) + p^{-s}) \tag{2.2.4}$$

where $\zeta(s)$ is the Riemann zeta function, so $D(s)$ can be continued analytically to some bigger cut-plane not containing any zeros of $\zeta(s)$. The sum on the right-hand side of (2.2.4) is absolutely convergent for $\operatorname{Re}(s) > 1/2$.

Recall from Proposition 1.3.1 that if $\mathcal{M}(f,s)$ is the Mellin transform of a function $f(x)$ then the Mellin inversion formula says that

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{M}(f,s)x^{-s}ds \tag{2.2.5}$$

for any real c in the fundamental strip of $\mathcal{M}(f,s)$. The next lemma gives us asymptotic formulas for some integrals of type (2.2.5).

Lemma 2.2.2. *Let $F(s)$ be an analytic function in $\operatorname{Re}(s) > 1/2$ admitting the following Taylor expansion around $s = 1$:*

$$F(s) = a_0 + \sum_{k\geq 1}^{N} \frac{a_k}{k!}(s-1)^k + O((s-1)^{N+1}),$$

and assume furthermore that $F(\sigma + it)$ decays exponentially when $|t| \to \infty$, uniformly for $\delta^{-1} \leq \sigma \leq \delta$ for some fixed $\delta > 1$. Then we have

$$\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} F(s)D(s)r^{-s}ds = \sum_{k=0}^{N} \frac{(-1)^k a_k}{r(\log\frac{1}{r})^{k+1}} + \mathcal{O}(\log\log\tfrac{1}{r}/(r(\log\tfrac{1}{r})^{N+2}))$$

for any $c > 1$.
If $F(s)$ is meromorphic admitting only a single pole at $s = 1$ with residue 1 and if the rest of the above conditions are satisfied then we have

$$\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} F(s)D(s)r^{-s}ds = \frac{\log\log\frac{1}{r}}{r} + \frac{B_1}{r} + \mathcal{O}(1/(r\log\tfrac{1}{r}))$$

where B_1 is Mertens's constant, defined as

$$B_1 := \gamma + \sum_p \left(\log(1-1/p) + 1/p\right),$$

where γ is the Euler–Mascheroni constant.

Similar results are quite common in this context, so we do not give a detailed proof here and only provide the main steps.

Proof. The Dirichlet series $D(s)$ admits the following bound

$$|D(\sigma + it)| = \mathcal{O}(\log\log(t))$$

for large t, uniformly for $\sigma \geq 1$. This comes from the fact that $\zeta(\sigma + it)$ is bounded above and below by powers of $\log t$ for $\sigma \geq 1$. Therefore, the function $|F(s)D(s)|$ decays exponentially along a vertical line and uniformly for $\operatorname{Re}(s) \geq 1$, so we may safely shift the path of integration to the left without changing the value of the integral. Hence, for sufficiently small r we may consider the path P defined as the union of the following parts (see Figure 2.1):

- $P_0 = \{s = 1 + \frac{1}{\log\frac{1}{r}}e^{i\theta} : -\pi/2 \leq \theta \leq \pi/2\} \cup \{s = 1 + x \pm i\frac{1}{\log\frac{1}{r}} : \frac{-1}{\sqrt{\log\frac{1}{r}}} \leq x \leq 0\}$,

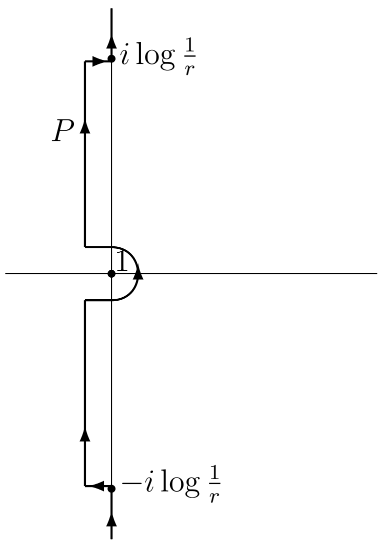

Figure 2.1: Path of integration P.

- $P_1 = \{s = 1 - \frac{1}{\sqrt{\log \frac{1}{r}}} + it : \frac{1}{\log \frac{1}{r}} \leq |t| \leq \log \frac{1}{r}\}$,
- $P_2 = \{s = 1 + it : |t| \geq \log \frac{1}{r}\} \cup \{s = 1 + x \pm i \log \frac{1}{r} : \frac{-1}{\sqrt{\log \frac{1}{r}}} \leq x \leq 0\}$.

For r sufficiently small, this path is included in the interior of the zero-free region of the Riemann zeta function (for more details on the zero-free region of the Riemann zeta function see [9]).

On P_1, one can say that $|D(s)|$ also grows slowly, since the logarithmic derivative $\zeta'(\sigma + it)/\zeta(\sigma + it)$ is bounded above by a power of $\log t$ close to the line $\mathrm{Re}(s) = 1$ (see for instance [2]). Therefore, the contribution from the integrals over P_1 and P_2 are both exponentially small in $\log 1/r$.

On P_0, one uses Taylor expansion for the first case, or Laurent expansion for the second case of the function $F(s)$ around $s = 1$. Thus the integrals that we need to compute are integrals of the form

$$\frac{1}{2\pi i} \int_{P_0} \log(s-1)(s-1)^k r^{-s} ds$$

for $k = -1, 0, 1, 2, 3, \cdots$. Then use the change of variable

$$s = 1 + \frac{z}{\log \frac{1}{r}}.$$

to change the path of integration into the so-called Hankel contour (see for example [8]). The rest of the proof consists of straightforward calculations. $\square$

2.3 Proof of the main theorem

To obtain the central limit theorem, we basically follow the ideas in Hwang's paper [15] but first we would like to compute asymptotic formulas for the mean and variance of the random variable ϖ_n. Recall that restricted partitions are partitions without repetitions, so the following bivariate generating function is the generating function for the number of restricted partitions with given length:

$$Q(u,z) = \prod_p (1 + uz^p),$$

i.e., the coefficient of $u^k z^n$ in $Q(u,z)$ is the number of ways of writing n as a sum of exactly k distinct primes. In other words

$$[z^n]Q(u,z) = pq(n)\mathbb{E}(u^{\varpi_n}),$$

where $pq(n) = [z^n]Q(1,z)$ is the total number of ways of writing n as sum of distinct primes. Note that for any u in a fixed bounded interval containing 1, the infinite product $Q(u,z)$ is convergent for $|z| < 1$ and so it is analytic in the unit disc. Let us then define the following function

$$f(u,\tau) := \log Q(u, e^{-\tau}) = \sum_p \log(1 + ue^{-p\tau}), \tag{2.3.1}$$

where we always use the principal branch of the logarithm function. We write $f_k(r)$ for the kth derivative of $f(u,\tau)$ with respect to τ at $\tau = r$. The next lemma provides an asymptotic formula for $pq(n)$ for large n.

Proposition 2.3.1. *The number of unequal partitions of n into primes has the asymptotic formula*

$$pq(n) = \frac{e^{nr}}{\sqrt{2\pi f_2(r)}} Q(1, e^{-r})(1 + \mathcal{O}(n^{-1/7})) \tag{2.3.2}$$

as $n \to \infty$, where $r > 0$ is the solution of the equation

$$n = \sum_p \frac{pe^{-pr}}{1 + e^{-pr}}. \tag{2.3.3}$$

Proof. We know that $pq(n)$ is the coefficient of z^n in $Q(1,z)$, so

$$pq(n) = \frac{e^{nr}}{2\pi} \int_{-\pi}^{\pi} \exp(nit + f(1, r + it))dt, \tag{2.3.4}$$

for any $r > 0$. Then we choose r as defined in Equation (2.3.3). The series in (2.3.3) is a monotone decreasing function of r, so the solution $r = r(n)$ of

(2.3.3) exists and is unique, and it tends to zero as n tends to infinity. Now, we split the integral in (2.3.4) as follows: first the integral in the center for $|t| \leq r^{1+\beta}$ and then the tails for $r^{1+\beta} < |t| \leq \pi$, where β is any constant such that $1/3 < \beta < 1/2$. For $|t| \leq r^{1+\beta}$ the function $f(1, r+it)$ admits an expansion

$$f(1, r+it) = f(1, r) + i f_1(r) t - f_2(r) \frac{t^2}{2} + \mathcal{O}(t^3 \sup_{0 \leq t_0 \leq t} |f_3(r + it_0)|).$$

We use Lemma 2.2.2 to find an asymptotic formula for $f_k(r)$. The Mellin transform of $f_k(r)$ is given by

$$\mathcal{M}(f_k(r), s) = (-1)^{k+1} \operatorname{Li}_{s-k+1}(-1) \Gamma(s) D(s-k),$$

where the function $\operatorname{Li}_s(-1)$ is the polylogarithm function regarded as a function of s. For $\operatorname{Re}(s) > 0$ one can represent $\operatorname{Li}_s(-1)$ as

$$\operatorname{Li}_s(-1) = \sum_{k \geq 1} \frac{(-1)^k}{k^s}$$

and it can be continued analytically to the whole s-plane as

$$\operatorname{Li}_s(-1) = (2^{1-s} - 1)\zeta(s).$$

Therefore by Lemma 2.2.2 one gets

$$f_k(r) = (-1)^k k! \frac{\pi^2/12}{r^{k+1} \log \frac{1}{r}} \left(1 + \mathcal{O}\left(\frac{1}{\log \frac{1}{r}}\right)\right) \tag{2.3.5}$$

as $r \to 0$. To estimate the third derivative $f_3(r+it)$ we have

$$|f_3(r+it)| \ll \sum_p \frac{p^3 e^{-pr}}{|1 + e^{-p(r+it)}|^3}.$$

For any prime p and $|t| \leq r^{1+\beta}$ we have

$$|1 + e^{-p(r+it)}| \geq 1 - e^{-1}$$

since if $p \geq r^{-1}$, then

$$|1 + e^{-p(r+it)}| \geq 1 - e^{-pr} \geq 1 - e^{-1}$$

and if $p < r^{-1}$ then

$$|1 + e^{-p(r+it)}| \geq 1 + \operatorname{Re}(e^{-p(r+it)}) = 1 + e^{-pr}\cos(pt) > 1.$$

Therefore

$$|f_3(r+it)| \ll \sum_p p^3 e^{-pr} \ll \frac{1}{r^4 \log 1/r}$$

Hence one has

$$nit + f(1, r+it) = f(1,r) - f_2(r)\frac{t^2}{2} + \mathcal{O}(r^{3\beta-1}/\log\tfrac{1}{r}).$$

Thus

$$\int_{-r^{1+\beta}}^{r^{1+\beta}} e^{nit+f(1,r+it)}dt = \int_{-r^{1+\beta}}^{r^{1+\beta}} e^{-f_2(r)t^2/2}dt(1+\mathcal{O}(r^{3\beta-1}/\log\tfrac{1}{r})),$$

and

$$\int_{-r^{1+\beta}}^{r^{1+\beta}} e^{-f_2(r)t^2/2}dt = \int_{-\infty}^{+\infty} e^{-f_2(r)t^2/2}dt + 2\int_{r^{1+\beta}}^{+\infty} e^{-f_2(r)t^2/2}dt.$$

The first integral on the right hand side gives the asymptotic formula, the second integral is smaller than any power of r, as $r \to 0$. It remains to show that the tails are small, and for that we make use of Lemma 2.2.1. In fact,

$$\begin{aligned}\frac{|Q(1, e^{-(r+it)})|^2}{Q(1,e^{-r})^2} &= \prod_p \Big(1 - \frac{2e^{-pr}(1-\cos(pt))}{(1+e^{-pr})^2}\Big)\\ &\le \exp\Big(-\frac{1}{2}\sum_p e^{-pr}(1-\cos(pt))\Big).\end{aligned}$$

Thus, for $r^{1+\beta} < |t| \le \pi$ Lemma 2.2.1 applies and we deduce that the tails of the integral in (2.3.4) are exponentially smaller than the main term. Then the asymptotic formula in Equation (2.3.2) follows by choosing any $\beta > 3/7$. □

Mean and variance

The mean μ_n of the random variable ϖ_n can be obtained as

$$\mu_n = \frac{\partial}{\partial u}\mathbb{E}(u^{\varpi_n})|_{u=1}.$$

So we may express μ_n in terms of $Q(u,z)$ as follows:

$$pq(n)\mu_n = [z^n]Q(1,z)\sum_p \frac{z^p}{1+z^p}. \tag{2.3.6}$$

By the Cauchy formula we have

$$pq(n)\mu_n = \frac{e^{nr}}{2\pi}\int_{-\pi}^{\pi} \exp(nit + f(1, r+it))g(r+it)dt \tag{2.3.7}$$

for any $r > 0$, where

$$g(\tau) := \sum_p \frac{e^{-p\tau}}{1+e^{-p\tau}}.$$

We choose $r = r(n)$ as defined in Proposition 2.3.1. We shall now estimate the following integral rather than working directly on the integral in (2.3.7):

$$pq(n)(\mu_n - g(r)) = \frac{e^{nr}}{2\pi}\int_{-\pi}^{\pi} \exp(nit + f(1, r+it))(g(r+it) - g(r))dt. \quad (2.3.8)$$

In fact, we want to show that the integral in Equation (2.3.8) is of small order compared to the order of $g(r)$. We apply the saddle point technique again to the integral (2.3.8), it is not hard to show that the tails here are also small. So the main term comes from the integral in the center for which $|t| \le r^{1+\beta}$ with the same β defined above. Then $g(r+it)$ admits the following expansion

$$g(r+it) - g(r) = ig_1(r)t - g_2(r)\frac{t^2}{2} + \mathcal{O}(t^3 \sup_{0\le t_1 \le t} |g_3(r+it_1)|). \qquad (2.3.9)$$

By means of the Mellin transform method we can show that the kth derivative $g_k(r)$ of $g(\tau)$ satisfies the following asymptotic formula

$$g_k(r) = (-1)^k k! \frac{\log 2}{r^{k+1}\log\frac{1}{r}}\Big(1 + \mathcal{O}\Big(\frac{1}{\log\frac{1}{r}}\Big)\Big)$$

for small r. One can also prove in a similar way as we did for $f_3(r+it)$, that the error term in Equation (2.3.9) is a $\mathcal{O}(r^{3\beta-1}/\log\frac{1}{r})$. On the other hand we have

$$e^{nit+f(1,r+it)} = e^{-f_2(r)t^2/2}\left(1 - if_3(r)\frac{t^3}{6} + \mathcal{O}(r^{6\beta-2}/\log\tfrac{1}{r}^2)\right).$$

Therefore, as in the proof of Proposition 2.3.1, we can extend the integration to the whole range of real numbers, and we deduce that

$$pq(n)(\mu_n - g(r)) = pq(n)\left(\frac{f_3(r)g_1(r) - f_2(r)g_2(r)}{2f_2(r)^2} + \mathcal{O}(r^{7\beta-3}/\log\tfrac{1}{r}))\right).$$

To make the error term small, we choose $3/7 < \beta < 1/2$, which implies that

$$\mu_n = \sum_p \frac{e^{-pr}}{1+e^{-pr}} + \frac{3\log 2}{\pi^2} + \mathcal{O}\Big(\frac{1}{\log\frac{1}{r}}\Big). \qquad (2.3.10)$$

Note that we have the following estimate for r:

$$r = r(n) = \frac{\pi}{\sqrt{6n\log n}}\Big(1 - \frac{\log\log n}{2\log n} + \mathcal{O}\Big(\frac{1}{\log n}\Big)\Big)$$

as $n \to \infty$. It is possible to expand the formula even further and get more terms in the expansion. Then the estimate of μ_n in (2.3.10) implies the formula in Theorem 2.1.1. We do the same for the variance, which is given by the formula

$$\sigma_n^2 = \frac{\partial^2}{\partial^2 u}\mathbb{E}(u^{\varpi_n})|_{u=1} - \mu_n^2 + \mu_n.$$

The second derivative of the function $Q(u, z)$ with respect to u is

$$\frac{\partial^2}{\partial^2 u} Q(u,z)|_{u=1} = Q(1,z)\left(\sum_p \frac{z^p}{1+z^p}\right)^2 - Q(1,z)\sum_p \frac{z^{2p}}{(1+z^p)^2}.$$

One can derive the following integral, using equation (2.3.6) followed by the Cauchy theorem:

$$pq(n)(\sigma_n^2+\mu_n^2) = \frac{e^{nr}}{2\pi}\int_{-\pi}^{\pi} \exp(int + f(1, r+it))(g(r+it)^2 + h(r+it))dt,$$

where

$$h(\tau) = \sum_p \frac{e^{-p\tau}}{(1+e^{-p\tau})^2}.$$

Then by same method that we used for μ_n, one may show that the variance satisfies the asymptotic formula

$$\sigma_n^2 = h(r) - \frac{g_1(r)^2}{f_2(r)} + \mathcal{O}(r^{3\beta-2}/\log^2 \tfrac{1}{r}),$$

which in turn implies the formula for σ_n^2 in Theorem 2.1.1.

Distribution function

Just like the mean and variance, one can also represent the moment generating function of the normalized random variable $(\varpi_n - \mu_n)/\sigma_n$ in terms of $Q(u, z)$. The moment generating function is by definition

$$M_n(t) = \mathbb{E}(e^{(\varpi_n-\mu_n)t/\sigma_n}) \tag{2.3.11}$$

$$= \exp\left(-\frac{\mu_n t}{\sigma_n}\right)\frac{Q_n(e^{t/\sigma_n})}{Q_n(1)} \tag{2.3.12}$$

where $Q_n(u) = [z^n]Q(u, z)$, and so $Q_n(1) = pq(n)$. We shall study the behavior of $Q_n(u)$ for u in a fixed bounded interval containing 1, say $1-\delta \le u \le 1+\delta$ for a fixed small $\delta > 0$. Throughout this section we will always assume that u is as such, and say that an approximation is uniform in u if it is uniform for u in that interval. We start by an analogue of Proposition 2.3.1.

Proposition 2.3.2. *The following asymptotic formula holds for the coefficient of z^n in $Q(u, z)$:*

$$Q_n(u) = \frac{1}{\sqrt{2\pi f_2(u,r)}} e^{nr+f(u,r)}\left(1+\mathcal{O}\left(n^{-1/7}\right)\right).$$

uniformly in u, as $n \to \infty$. Here, $r = r(u,n)$ is now the unique positive solution of the equation

$$n = \sum_p \frac{pue^{-pr}}{1 + ue^{-pr}}.$$

Before we prove this result, let us first introduce the function $Y(u,s)$ defined to be the Mellin transform of the function $\log(1 + ue^{-x})$. Then the following lemma can be found in [15, Lemma 1]:

Lemma 2.3.3. *For any fixed u lying in the cut-plane $\mathbb{C} \setminus (-\infty, -1]$, the function $Y(u,s)$ can be meromorphically continued to the whole s-plane with simple poles at $s = 0, -1, -2, -3, \cdots$. Moreover, $Y(u,s)$ satisfies the estimate*

$$|Y(u, \sigma + it)| \ll e^{-(\pi/2-\varepsilon)|t|}$$

for any $\varepsilon > 0$ and $|t| \to +\infty$, uniformly as σ and u are restricted to compact sets.

This property follows from the fact that the function $Y(u,s)$ can be written as a product of a polylogarithm and the Gamma function.

Proof. (*of Proposition 2.3.2*) We follow the lines in the proof of Proposition 2.3.1, so by the Cauchy theorem we have

$$Q_n(u) = \frac{e^{nr}}{2\pi} \int_{-\pi}^{\pi} \exp(int + f(u, r + it))dt$$

for any $r > 0$. The saddle-point method suggests to choose r a solution of the equation

$$n = -f_1(u,r) = \sum_p \frac{p}{u^{-1}e^{rp} + 1}. \tag{2.3.13}$$

The sum on the right hand side is a strictly decreasing function of r tending to 0 when $r \to +\infty$ and ∞ when $r \to 0$. Therefore, the solution $r = r(u,n)$ of the equation (2.3.13) exists and it is unique for u and n fixed. Also, $r(u,n)$ tends to 0 uniformly in u as $n \to \infty$. The next step is to split the integral into a central part which is the integral over the interval $[-r^{1+\beta}, r^{1+\beta}]$, where β is a constant such that $1/3 < \beta < 1/2$, and the tails. Let us first evaluate the integral in the center, for which $|t| \leq r^{1+\beta}$ and the function $int + f(u, r + it)$ admits the Taylor expansion

$$int + f(u, r + it) = f(u,r) - f_2(u,r)\frac{t^2}{2} + \mathcal{O}(|t^3| \sup_{0 \leq t_0 \leq t} |f_3(u, r + it_0)|).$$

Note that

$$Y(u,1) = \int_0^{+\infty} \log(1 + ue^{-t})dt$$

is strictly positive for any value of $u > 0$. By the Mellin transform method and Lemma 2.3.3 along with this observation, the function $f_2(u, r)$ is of order $r^{-3}/\log\frac{1}{r}$ and $|f_3(u, r + it)|$ is a $\mathcal{O}(r^{-4}/\log\frac{1}{r})$; these estimates are uniform in u. One can use the same technique as in the proof of Proposition 2.3.1 to justify the bound on $|f_3(u, r+it)|$ provided that u is close enough to 1 (that is to choose a relatively small δ). Therefore, as in Proposition 2.3.1 the integral in the center gives the term we want. The tails are small as a result of the following observation combined with Lemma 2.2.1:

$$\begin{aligned}\frac{|Q(u, e^{-(r+it)})|^2}{Q(u, e^{-r})^2} &= \prod_p \left(1 - \frac{2ue^{-pr}(1-\cos(pt))}{(1+ue^{-pr})^2}\right)\\ &\le \exp\Big(-\frac{2u}{(1+u)^2}\sum_p e^{-pr}(1-\cos(pt)\Big).\end{aligned}$$

Finally, from Equation (2.3.13) and from the Mellin transform method we derive the asymptotic formula for r

$$r = r(u, n) \sim \sqrt{\frac{2Y(u, 1)}{n \log n}}$$

uniformly in u as $n \to \infty$. Thus, the result follows by choosing $\beta > 3/7$. $\square$

Until the end of this section let us use the following abbreviations: $r = r(u, n)$, $r_0 := r(1, n)$, $u = e^{t/\sigma_n}$ and

$$f_{ij}(u, r) = \frac{\partial^i}{\partial^i \tau}\frac{\partial^j}{\partial^j u} f(u, \tau)|_{\tau=r}.$$

Then it follows easily from Proposition 2.3.2 that

$$\frac{Q_n(u)}{Q_n(1)} = \exp\Big(n(r - r_0) + f(u, r) - f(1, r_0)\Big)\Big(1 + \mathcal{O}\Big(\frac{|t|}{\sigma_n} + n^{-1/7}\Big)\Big).$$

By implicit differentiation we have

$$r - r_0 = -\frac{g_1(r_0)}{f_2(1, r_0)}(u-1) + \mathcal{O}(r_0(u-1)^2) = \mathcal{O}(r_0(u-1)).$$

Therefore,

$$f(u, r) - f(u, r_0) = f_1(u, r_0)(r - r_0) + f_2(u, r_0)\frac{(r-r_0)^2}{2} + \mathcal{O}(t^3\sqrt{r_0 \log\tfrac{1}{r_0}}).$$

Also,

$$\begin{aligned}f_1(u, r_0)(r - r_0) &= f_1(1, r_0)(r - r_0) + f_{11}(1, r_0)(u-1)(r - r_0)\\ &\quad + \mathcal{O}(t^3\sqrt{r_0 \log\tfrac{1}{r_0}})\\ &= -n(r - r_0) + g_1(r_0)(u-1)(r - r_0) + \mathcal{O}(t^3\sqrt{r_0\log\tfrac{1}{r_0}}),\end{aligned}$$

and for the second term we have

$$f_2(u, r_0)\frac{(r-r_0)^2}{2} = f_2(1, r_0)\frac{(r-r_0)^2}{2} + \mathcal{O}(t^3\sqrt{r_0 \log \tfrac{1}{r_0}}).$$

Finally,

$$f(u, r_0) - f(1, r_0) = g(r_0)(u-1) + f_{02}(1, r_0)\frac{(u-1)^2}{2} + \mathcal{O}(t^3\sqrt{r_0 \log \tfrac{1}{r_0}}).$$

Thus the function in the exponent can be written as

$$g(r_0)(u-1) + \left(f_{02}(1, r_0) - \frac{g_1(r_0)^2}{f_2(1, r_0)}\right)\frac{(u-1)^2}{2} + \mathcal{O}(t^3\sqrt{r_0 \log \tfrac{1}{r_0}}).$$

On the other hand

$$u - 1 = \frac{t}{\sigma_n} + \frac{t^2}{2\sigma_n^2} + \mathcal{O}(t^3/\sigma_n^3)$$

so the exponent becomes

$$\begin{aligned} &g(r_0)\frac{t}{\sigma_n} + \left(g(r_0) + f_{02}(1, r_0) - \frac{g_1(r_0)^2}{f_2(1, r_0)}\right)\frac{t^2}{2\sigma_n^2} + \mathcal{O}(t^3\sqrt{r_0 \log \tfrac{1}{r_0}}) \\ &= g(r_0)\frac{t}{\sigma_n} + \left(h(r_0) - \frac{g_1(r_0)^2}{f_2(1, r_0)}\right)\frac{t^2}{2\sigma_n^2} + \mathcal{O}(t^3\sqrt{r_0 \log \tfrac{1}{r_0}}). \end{aligned}$$

The error terms in the above expansions can be verified using the same method we used to bound the $|f_3(r+it)|$ in the proof of Proposition 2.3.1. Replacing μ_n and σ_n by their respective values in Equation (2.3.12), we get the asymptotic formula we expected:

$$M_n(t) = e^{t^2/2}(1 + \mathcal{O}((|t| + |t|^3)n^{-1/4+\epsilon} + n^{-1/7})) \tag{2.3.14}$$

as $n \to \infty$, for any $\epsilon > 0$. By Curtiss' theorem [5] the limit distribution is indeed Gaussian.

Remark. The asymptotic formula we get in (2.3.14) is very similar to those we find in [15] or [18] and therefore, the following bounds hold for the tails

$$\mathbb{P}\left(\frac{\varpi_n - \mu_n}{\sigma_n} \geq x\right) \leq \begin{cases} e^{-x^2/2}\left(1 + \mathcal{O}(1/\log^3 n)\right) & \text{if } 0 \leq x \leq n^{1/12-\epsilon}, \\ e^{-n^{1/12-\epsilon}x/2}\left(1 + \mathcal{O}(1/\log^3 n)\right) & \text{if } x \geq n^{1/12-\epsilon}, \end{cases}$$

for any small constant $\epsilon > 0$. Similar bounds hold for

$$\mathbb{P}\left(\frac{\varpi_n - \mu_n}{\sigma_n} \leq -x\right).$$

2.4 Unrestricted partitions

Recall that unrestricted partitions are those whose parts are allowed to repeat. The appropriate bivariate generating function for the unrestricted case is given by

$$Q(u,z) = \prod_p (1 - uz^p)^{-1}.$$

The infinite product converges only if $|uz| < 1$, unlike the restricted case where we had convergence for $|z| < 1$, for any u restricted into a bounded interval containing 1. As before, we consider the logarithm of the above infinite product

$$f(u,\tau) = \log Q(u, e^{-\tau}) = \sum_p \log(1 - ue^{-p\tau}).$$

Mean and Variance

For the mean and variance, we have formulas rather similar to those for the restricted case. The analogue of the integral representation for the mean is

$$pp(n)\mu_n^* = \frac{e^{nr}}{2\pi} \int_{-\pi}^{\pi} \exp(int + f(1, r+it))g(r+it)dt \tag{2.4.1}$$

for any $r > 0$, where $pp(n)$ is the total number of ways of writing n as a sum of primes, and

$$g(\tau) := \sum_p \frac{e^{-p\tau}}{1 - e^{-p\tau}}.$$

For the variance,

$$pp(n)(\sigma_n^{*2} + \mu_n^{*2}) = \frac{e^{nr}}{2\pi} \int_{-\pi}^{\pi} \exp\Big(int + f(1, r+it)\Big)(g(r+it)^2 + h(r+it))dt \tag{2.4.2}$$

where

$$h(\tau) = \sum_p \frac{e^{-p\tau}}{(1 - e^{-p\tau})^2}.$$

The Mellin transform of the kth derivatives of $f(1,\tau)$ and $g(\tau)$ at $\tau = r$ are given by

$$\mathcal{M}(f_k(1,r), s) = (-1)^k \zeta(s-k+1)\Gamma(s)D(s-k)$$

and

$$\mathcal{M}(g_k(r), s) = (-1)^k \zeta(s-k)\Gamma(s)D(s-k).$$

So, by Lemma 2.2.2 the orders of $f_k(1,r)$ and $g_k(r)$ differ from $r^{-(k+1)}$ only by factors of $\log 1/r$ or $\log\log \frac{1}{r}$, where we take $r = r(n) > 0$ to be the solution of the equation

$$n = \sum_p \frac{pe^{-rp}}{1 - e^{-rp}}.$$

Thus, we only have to repeat the procedures in the previous section to compute the mean. For the variance, the Mellin transform of the function $h(r)$ is given by

$$\mathcal{M}(h(r), s) = \zeta(s-1)\Gamma(s)D(s)$$

which has a simple pole at $s = 2$ from the zeta function. Therefore, the order of $h(r)$ is r^{-2} which is greater than the contribution from the $g(\tau)^2$ in the integral for the variance. The asymptotic formulas for the mean and variance follow:

Theorem 2.4.1. *The mean and variance of the distribution of the number of summands in an unrestricted partition of an integer n into primes satisfy the following asymptotic formulas:*

$$\mu_n^* = \sum_p \frac{e^{-rp}}{1 - e^{-rp}} + \mathcal{O}(\log^2 \tfrac{1}{r}).$$

As a function of n,

$$\mu_n^* = \frac{\sqrt{3}}{\pi} (\log\log n + B_1 - \log 2) \sqrt{n \log n} \Big(1 + \frac{\log\log n}{2\log n} + \mathcal{O}\Big(\frac{1}{\log n}\Big)\Big).$$

Likewise,

$$\sigma_n^{*2} = \sum_p \frac{e^{-rp}}{(1 - e^{-rp})^2} + \mathcal{O}\Big(\frac{\log^2 \frac{1}{r}}{r}\Big).$$

As a function of n,

$$\sigma_n^{*2} = \frac{3D(2)n\log n}{\pi^2} \Big(1 + \frac{\log\log n}{\log n} + \mathcal{O}\Big(\frac{1}{\log n}\Big)\Big)$$

as $n \to \infty$.

For comparison, the mean number of summands of a partition into arbitrary parts is

$$\frac{\sqrt{6n}}{2\pi} \Big(\log n + 2\gamma - \log(\pi^2/6)\Big) + \mathcal{O}(\log n),$$

see [14].

Remark. We also have a central limit theorem in the unrestricted case: it is known from [13] that the limit as $n \to \infty$ of the normalized random variable $\frac{\varpi_n^* - \mu_n^*}{\sigma_n^*}$ has the following moment generating function:

$$M(t) = \prod_p \Big(1 - \frac{\tilde{t}}{p}\Big)^{-1} e^{-\frac{\tilde{t}}{p}}$$

where $\tilde{t} = t/\sqrt{D(2)}$. See Appendix A for a further study of this distribution.

2.5 Generalization

A natural question one can ask is whether the result remains true for powers of primes or more generally for polynomials $f(p)$ of primes. But one needs to be careful here since, for example, $p^2 + p$ is always even for any prime p so we need to impose some additional conditions on the polynomial. From the result of [24] and a slight modification of our proof of Lemma 2.2.1 we can get

Lemma 2.5.1. *Let $f(x)$ be a strictly increasing polynomial which takes only integral values for integer x and has the property that for every prime p there is a positive integer x such that $p \nmid xf(x)$. For any constant $1/3 < c < 1/2$ and $r^{1+c} \leq |y| \leq \pi$ we have the inequality*

$$\sum_p e^{-f(p)r}(1 - \cos f(p)y) \geq c' \log^2 \tfrac{1}{r}$$

for an absolute constant $c' > 0$ as $r \to 0^+$.

The associated Dirichlet series is closely related to the Dirichlet series of primes. Suppose that the dominant term in our polynomial $f(x)$ is of the form ax^d where a is a positive integer and d is the degree of $f(x)$. Then

$$D_f(s) - a^{-s}D(ds) = a^{-s} \sum_p \left(\frac{a^s p^{ds} - (f(p))^s}{p^{ds}(f(p))^s} \right).$$

The series on the right hand side is absolutely convergent for $\mathrm{Re}(s) > 1/(2d)$. Therefore our method applies, and the limit distribution of the number of summands in partitions of n into distinct primes is Gaussian with mean and variance

$$\mu_n \sim C_1(a,d) \left(\frac{n}{\log^d n} \right)^{\frac{1}{1+d}} \quad \text{and } \sigma_n^2 \sim C_2(a,d) \left(\frac{n}{\log^d n} \right)^{\frac{1}{1+d}},$$

where $C_i(a,d)$ can be determined explicitly. As for unrestricted partitions, the mean and variance follow the asymptotic formulas:

$$\mu_n^* \sim C_1'(a,d)\,(n \log n)^{\frac{d}{1+d}} \quad \text{and } \sigma_n^{*2} \sim C_2'(a,d)\,(n \log n)^{\frac{2d}{1+d}}$$

when $d \geq 2$, again the $C_i'(a,d)$ can be determined explicitly.

Chapter 3

The number of parts of given multiplicity

3.1 Introduction

Let d be a positive integer. An ascent of size d in a partition $(c_1, c_2, \cdots, c_t)$ of an integer n is a succession of two parts c_i, c_{i+1} such that $c_{i+1} - c_i = d$. If $c_1 = d$ then we assume that the partition has already one ascent of size d. Then the number of ascents of size d in a given partition is exactly the number of parts having multiplicity d in its conjugate partition.

Multiplicities in partitions were studied, amongst others, by Corteel et al [4], who showed that a randomly selected part of a random partition has multiplicity d with probability tending to $\frac{1}{d(d+1)}$. As a main step in their proof, they provide an asymptotic formula for the average number of parts of multiplicity d. A similar result was found by Knopfmacher and Munagi [16] for the number of ascents (successions) of size d. Here we improve on these results by proving a central limit theorem which can be stated as follows:

Theorem 3.1.1. *The limit distribution of the number of parts having multiplicity d (or ascents of size d) in a random partition of n is Gaussian with mean and variance given by the asymptotic formulas:*

$$\mu_n = \frac{\sqrt{6n}}{\pi d(d+1)} + \frac{3}{\pi^2 d(d+1)} + o(1) \tag{3.1.1}$$

and

$$\sigma_n^2 \sim \left(\frac{1}{\pi d(d+1)} - \frac{1}{2\pi d(d+1)(2d+1)} - \frac{3}{\pi^3 d^2 (d+1)^2} \right) \sqrt{6n} \tag{3.1.2}$$

respectively as $n \to \infty$.

A similar limit theorem was shown by Brennan, Knopfmacher and Wagner [3] for ascents of size d or more (equivalently, parts of multiplicity d or more). Later in this paper we give a generalisation of Theorem 3.1.1 and the results in [3] to λ-partitions satisfying the Meinardus scheme as defined in Section 1.2.1.

In the above results, d was considered fixed. But when we let d increase with n and $d \to \infty$ as $n \to \infty$ then the following phase transition can be observed:

Theorem 3.1.2. *The limit distribution of the number of parts of multiplicity d is:*

- *Gaussian with mean and variance asymptotically equal to $\frac{\sqrt{6n}}{\pi d(d+1)}$ for $d = o(n^{1/4})$,*
- *Poisson with parameter $\frac{\sqrt{6}}{\pi\alpha^2}$ for $d \sim \alpha n^{1/4}$,*
- *degenerate at zero for $dn^{-1/4} \to \infty$.*

We present our results in the following way: We shall give a detailed proof of Theorem 3.1.1 in Section 3.2, and in Section 3.3 we discuss how the proof of Theorem 3.1.1 can be adapted to prove Theorem 3.1.2. In these proofs we use methods that can be generalised to the case of λ-partitions. Then Section 3.4 gives a generalisation of Theorem 3.1.2 to the case of λ-partitions and finally in Section 3.5 we discuss the generalisation of the results in [3].

3.2 Proof of Theorem 3.1.1

Throughout this section, d is a fixed positive integer. For a large positive integer n, we assign a uniform probability measure to the set of all partitions of n. Then the random variable ϖ_n is the number of parts of multiplicity d (which is the same as the number of ascents of size d in this case) in a random partition, its mean and standard deviation will be denoted by μ_n and σ_n respectively. We shall use $\prod_\lambda$ and $\sum_\lambda$ as abbreviations for the product and sum over all positive integers respectively. The reader should take note of the change in the other sections as we shall use the same notation but with different meaning.

The following is the generating function for the distribution of the number of parts of multiplicity d:

$$Q(u,z) = \prod_{\lambda} \left(\frac{1}{1-z^{\lambda}} + (u-1)z^{\lambda d} \right), \tag{3.2.1}$$

that is

$$\frac{Q_n(u)}{Q_n(1)} = \mathbb{E}(u^{\varpi_n}), \tag{3.2.2}$$

where $Q_n(u)$ is the coefficient of z^n in $Q(u, z)$. Note that $Q_n(1)$ is the total number of partitions of n. We also introduce the following functions:

$$f(\tau) := -\sum_{\lambda} \log(1 - e^{-\lambda\tau}) \tag{3.2.3}$$

and

$$\phi(v, \tau) := \sum_{\lambda} \log(1 + ve^{-d\lambda\tau}(1 - e^{-\lambda\tau})). \tag{3.2.4}$$

Then we have

$$\log(Q(u, e^{-\tau})) = f(\tau) + \phi(u - 1, \tau). \tag{3.2.5}$$

For simplicity we shall use the following abbreviation: again if $F(\tau)$ is a function of a complex variable τ, and if it is analytic in some domain containing an element τ_0 in its interior, then we write $F_k(\tau_0)$ for

$$\frac{\partial^k}{\partial^k \tau} F(\tau)\Big|_{\tau=\tau_0}.$$

Let us first recall the asymptotic formula for $Q_n(1)$.

Lemma 3.2.1. *The number of partitions of n is given by the following asymptotic formula*

$$Q_n(1) = \frac{e^{nr}}{\sqrt{2\pi f_2(r)}} Q(1, e^{-r})(1 + \mathcal{O}(n^{-1/7})) \tag{3.2.6}$$

as $n \to \infty$, where r is the positive solution of the equation

$$n = \sum_{\lambda} \frac{\lambda}{e^{\lambda r} - 1}. \tag{3.2.7}$$

Note that the above asymptotic formula implies the well known Hardy-Ramanujan formula (Theorem 1.2.1), but one does not require it explicitly to prove our results. The proof of Lemma 3.2.1 is based on the use of the saddle point method as outlined in Section 1.3. We also use a similar approach to obtain the asymptotic formulas for the mean and variance given in Theorem 3.1.1. As mentioned earlier, results on the mean and variance can already be found in the literature (see [4, 16]), but the method we apply here is easier to generalise to λ-partitions.

3.2.1 Mean and Variance

By definition, the mean of the random variable ϖ_n is

$$\begin{aligned}\mu_n &= \frac{\partial}{\partial u}\mathbb{E}(u^{\varpi_n})\Big|_{u=1}\\ &= \frac{1}{Q_n(1)}\frac{\partial}{\partial u}Q_n(u)\Big|_{u=1}.\end{aligned}$$

In order to find an asymptotic formula for μ_n we shall consider the following instead

$$Q_n(1)(\mu_n - g(r)) = \frac{e^{nr}}{2\pi}\int_{-\pi}^{\pi}\exp(int + f(r+it))(g(r+it) - g(r))dt \quad (3.2.8)$$

for any $r > 0$, where

$$g(\tau) = \sum_{\lambda} e^{-d\lambda\tau}(1 - e^{-\lambda\tau}).$$

We approximate this integral by means of the saddle point method, where r is chosen to be the same as defined in Lemma 3.2.1, that is the solution of the equation

$$n = \sum_{\lambda}\frac{\lambda}{e^{\lambda r} - 1}.$$

The series on the right hand side is a monotone decreasing function of r therefore the solution exists and it tends to zero as $n \to \infty$. Now we claim that the integral (3.2.8) can be approximated by

$$\frac{e^{nr}}{2\pi}\int_{-r^{1+\beta}}^{r^{1+\beta}}\exp(int + f(r+it))(g(r+it) - g(r))dt \quad (3.2.9)$$

when $1/3 < \beta < 1/2$. This is not surprising since we know that without the term $g(r+it) - g(r)$ the estimate holds from the fact that for $|t| > r^{1+\beta}$

$$\begin{aligned}\frac{|Q(1, e^{-(r+it)})|}{Q(1, e^{-r})} &= \exp\Big(-\sum_{k\geq 1}\frac{1}{k}\sum_{\lambda} e^{-\lambda k r}(1 - \cos(\lambda k t))\Big)\\ &\leq \exp\Big(-\sum_{\lambda} e^{-\lambda r}(1 - \cos(\lambda t))\Big)\\ &\ll \exp\Big(-c|\log r|^2\Big)\end{aligned}$$

is smaller than any power of r^{-1} as $r \to 0$ (this is in fact the case for any sequence satisfying the condition (M3) in the Meinardus scheme). Note also for any t

$$|g(r+it) - g(r)| \ll \sum_{\lambda} e^{-\lambda r} \ll r^{-1}$$

as $r \to 0$. The claim follows from these two observations. Now for $|t| \le r^{1+\beta}$ we have

$$f(r+it) = f(r) + if_1(r)t - f_2(r)\frac{t^2}{2!} - if_3(r)\frac{t^3}{3!} + \tag{3.2.10}$$

$$f_4(r)\frac{t^4}{4!} + if_5(r)\frac{t^5}{5!} + \mathcal{O}\Big(|t|^6 \sup_{|\eta|\le|t|} |f_6(r+i\eta)|\Big). \tag{3.2.11}$$

In order to obtain asymptotic estimates for $f_k(r)$ (and later also other quantities) we apply the method of Mellin transforms, in particular Theorem 1.3.2. Specifically, the Mellin transform of $f_k(r)$ is

$$(-1)^k \zeta(s-k+1)\Gamma(s)\zeta(s-k).$$

Therefore, Theorem 1.3.2 gives

$$f_k(r) \sim (-1)^k k! \frac{\pi^2}{6r^{k+1}} \tag{3.2.12}$$

as $r \to 0$, in particular $n \sim \frac{\pi^2}{6} r^{-2}$. Furthermore, to estimate the error term in Equation (3.2.10) we have

$$|f_6(r+i\eta)| \ll \sum_\lambda \frac{\lambda^6 e^{-\lambda r}}{|1-e^{-\lambda(r+i\eta)}|^6} \ll \sum_\lambda \frac{\lambda^6 e^{-\lambda r}}{|1-e^{-\lambda r}|^6} \ll r^{-7}.$$

Hence for $|\eta| \le r^{1+\beta}$

$$|f_6(r+i\eta)| \ll r^{-7},$$

and we have

$$e^{nit+f(r+it)} = e^{f_0(r)-f_2(r)t^2/2}\Big(1 - if_3(r)\frac{t^3}{3!} + f_4(r)\frac{t^4}{4!} + if_5(r)\frac{t^5}{5!} + \mathcal{O}(r^{6\beta-2})\Big).$$

Similarly we have

$$g(r+it) - g_0(r) = ig_1(r)t - g_2(r)\frac{t^2}{2} + \mathcal{O}(r^{3\beta-1}), \tag{3.2.13}$$

and we also have the following asymptotic formula:

$$g_k(r) = \frac{(-1)^k k!}{d(d+1)} \frac{1}{r^{k+1}} \Big(1 + \mathcal{O}(r)\Big), \tag{3.2.14}$$

this can also be obtained elementarily since $g(\tau)$ is a difference of geometric series in this case. Therefore, one can approximate $Q_n(1)(\mu_n - g_0(r))$ by

$$\frac{e^{nr+f_0(r)}}{2\pi} \int_{-r^{1+\beta}}^{r^{1+\beta}} e^{-f_2(r)t^2/2} \left(-g_2(r)\frac{t^2}{2} + f_3(r)g_1(r)\frac{t^4}{3!} + \mathcal{O}(r^{7\beta-3})\right) dt$$

with an exponentially small error term, since integrals involving an odd power of t are identically 0. We may now change the range of integration to $(-\infty, +\infty)$ with another exponentially small error term and then apply the formula for the Gaussian integral. Then we get the following expression for the mean in terms of r:

$$\mu_n = g_0(r) - \frac{g_2(r)}{2f_2(r)} + \frac{f_3(r)g_1(r)}{2f_2^2(r)} + \mathcal{O}(r^{7\beta-3}), \tag{3.2.15}$$

which gives

$$\mu_n = \frac{1}{d(d+1)}r^{-1} + \frac{3}{2\pi^2 d(d+1)} + \mathcal{O}(r^{7\beta-3}). \tag{3.2.16}$$

Now for the variance we have

$$\sigma_n^2 = \frac{\partial^2}{\partial^2 u}\mathbb{E}(u^{\varpi_n})|_{u=1} - \mu_n^2 + \mu_n. \tag{3.2.17}$$

So we need to find an approximation of the second factorial moment

$$\frac{\partial^2}{\partial^2 u}\mathbb{E}(u^{\varpi_n})|_{u=1} = \frac{e^{nr}}{2\pi Q_n(1)} \int_{-\pi}^{\pi} \exp(int + f(r+it))\psi(r+it)dt, \tag{3.2.18}$$

where $\psi(\tau) = g^2(\tau) - h(\tau)$ and

$$h(\tau) = \sum_{\lambda} e^{-2d\lambda\tau}(1 - e^{-\lambda\tau})^2.$$

Now we use the same method as for the mean: we obtain

$$g^2(r+it) = g_0^2(r) + 2ig_0(r)g_1(r)t - (g_1^2(r) + g_0(r)g_2(r))t^2 + \mathcal{O}(r^{3\beta-2})$$

and also

$$h(r+it) = h_0(r) + ih_1(r)t - h_2(r)\frac{t^2}{2} + \mathcal{O}(r^{3\beta-1}).$$

Since $h_k(r)$ has the same order as $g_k(r)$, the contribution from $-h(r+it)$ in the integral is $-h_0(r)$ with an error of at most constant order. For $g^2(r+it)$ we proceed as we did for the mean, and the main term of the integral comes from

$$\begin{aligned}&(g^2(r+it) - g_0^2)\left(1 - if_3(r)\frac{t^3}{3!} + f_4(r)\frac{t^4}{4!} + if_5(r)\frac{t^5}{5!} + \mathcal{O}(r^{6\beta-2})\right) = \\ &2ig_0(r)g_1(r)t - (g_1^2(r) + g_0(r)g_2(r))t^2 + 2g_0(r)g_1(r)f_3(r)\frac{t^4}{3!} + \mathcal{O}(r^{7\beta-4}) \\ &\quad + \text{ terms involving odd powers of } t.\end{aligned}$$

When we apply the integral we get

$$\begin{aligned}&\sigma_n^2 + \mu_n^2 - \mu_n - g_0^2(r) + h_0(r) = \\ &\qquad \frac{-g_1^2(r) - g_0(r)g_2(r)}{f_2(r)} + \frac{g_0(r)g_1(r)f_3(r)}{f_2^2(r)} + \mathcal{O}(r^{7\beta-4}),\end{aligned}$$

which implies that

$$\sigma_n^2 = \mu_n - h_0(r) - \frac{g_1^2(r)}{f_2(r)} + \mathcal{O}(r^{7\beta-4}), \tag{3.2.19}$$

which gives

$$\sigma_n^2 = \left(\frac{1}{d(d+1)} - \frac{1}{2d(d+1)(2d+1)} - \frac{3}{\pi^2 d^2(d+1)^2}\right) r^{-1} + \mathcal{O}(r^{7\beta-4}). \tag{3.2.20}$$

Having these formulas for the mean and variance we may use Theorem 1.3.2 to deduce asymptotic formulas as a function of n. First we need an asymptotic formula for $r = r(n)$. We have already mentioned an asymptotic dependence between n and r as a consequence of (3.2.12), and expanding further, using Theorem 1.3.2, we get

$$n = \frac{\pi^2}{6} r^{-2} - \frac{1}{2} r^{-1} + \mathcal{O}(1),$$

which implies that

$$r^{-1} = \frac{\sqrt{6}}{\pi}\sqrt{n} + \frac{3}{2\pi^2} + \mathcal{O}(n^{-1/2}). \tag{3.2.21}$$

So the equations (3.2.16) and (3.2.20) give the asymptotic formulas for the mean and variance

$$\mu_n = \frac{\sqrt{6}}{\pi d(d+1)}\sqrt{n} + \frac{3}{\pi^2 d(d+1)} + \mathcal{O}(n^{-\epsilon}) \tag{3.2.22}$$

and

$$\sigma_n^2 \sim \left(\frac{1}{\pi d(d+1)} - \frac{1}{2\pi d(d+1)(2d+1)} - \frac{3}{\pi^3 d^2(d+1)^2}\right)\sqrt{6n} \tag{3.2.23}$$

as $n \to \infty$, where ϵ is a positive constant. These prove the asymptotic formulas in Theorem 3.1.1.

3.2.2 Moment Generating Function

We saw that the mean and variance are both tending to infinity, so in order to determine the limiting distribution we need to consider the normalised random variable

$$X_n := \frac{\varpi_n - \mu_n}{\sigma_n}. \tag{3.2.24}$$

The moment generating function of X_n is defined as

$$M_n(x) := \mathbb{E}(e^{xX_n}) \tag{3.2.25}$$

for a fixed real number x. To complete the proof of Theorem 3.1.1 we need to show that $M_n(x)$ converges pointwise to $e^{x^2/2}$ within a fixed interval containing 0. Note that $M_n(x)$ can also be written as follows:

$$M_n(x) = e^{-x\mu_n/\sigma_n} \frac{Q_n(e^{x/\sigma_n})}{Q_n(1)}. \tag{3.2.26}$$

Recall the formula for the coefficient

$$Q_n(u) = \frac{e^{nr}}{2\pi} \int_{-\pi}^{\pi} \exp(int + f(r+it) + \phi(u-1, r+it))dt. \tag{3.2.27}$$

We use the saddle point method again to find an asymptotic formula of the latter integral for u suitably close to 1, for now let us just say that $|u-1| \le \delta$ for some fixed small $\delta > 0$. We shall be able to provide an asymptotic formula for $Q_n(u)$ by using a series of lemmas. We begin with the following which allows us to ignore the tails of the integral in (3.2.27).

Lemma 3.2.2. *There is a positive constant c_1, such that if $\pi > |t| > r^{1+c}$ where $1/3 < c < 1/2$ then*

$$\frac{|Q(u, e^{-(r+it)}|}{Q(u, e^{-r})} \ll e^{-c_1|\log r|^2}$$

as $r \to 0^+$.

Proof. In fact this proof works for any sequence of positive integers λ satisfying the condition (M3) of the Meinardus scheme ($\frac{\alpha}{3} < c < \frac{\alpha}{2}$ for the general case), but also for arbitrary d. First we claim that for any complex number z such that $|z| \le 2$ we have

$$\frac{|1+z|}{1+|z|} \le e^{-\frac{1}{9}(|z|-\mathrm{Re}(z))}.$$

Indeed, for $|z| \le 2$ we have

$$\begin{aligned}
\frac{|1+z|^2}{(1+|z|)^2} &= 1 - 2\frac{|z| - \mathrm{Re}(z)}{(1+|z|)^2} \\
&\le 1 - \frac{2}{9}(|z| - \mathrm{Re}(z)) \\
&\le e^{-\frac{2}{9}(|z|-\mathrm{Re}(z))}.
\end{aligned}$$

Now for any l that is a member of the sequence λ, and any z such that $|z| \le 1$

$$|1 + z^l + z^{2l} + \cdots + z^{(d-1)l} + uz^{dl} + z^{(d+1)l} + \cdots| \tag{3.2.28}$$
$$\le |1 + z^l| + |z^{2l} + z^{3l}| + \cdots . \tag{3.2.29}$$

Note that only one of the terms in (3.2.29) involves u, it is either $|uz^{dl} + z^{(d+1)l}|$ or $|z^{(d-1)l} + uz^{dl}|$ depending on the parity of d. We may assume that $1/2 \leq u \leq 2$. Using the inequality above, we find that for all positive real a and b such that $1/2 \leq b/a \leq 2$,

$$|az^{kl} + bz^{(k+1)l}| \leq e^{-\frac{1}{18}(|z|^l - \mathrm{Re}(z^l))}\Big(a|z|^{kl} + b|z|^{(k+1)l}\Big),$$

which implies that (3.2.28) is at most

$$e^{-\frac{1}{18}(|z^l| - \mathrm{Re}(z^l))}(1 + |z|^l + |z|^{2l} + \cdots + |z|^{(d-1)l} + u|z|^{dl} + |z|^{(d+1)l} + \cdots).$$

Hence,

$$\frac{|Q(u, e^{-(r+it)}|}{Q(u, e^{-r})} \leq \exp\Big(-\frac{1}{18}\sum_{\lambda} e^{-\lambda r}(1 - \cos(\lambda r t))\Big),$$

which proves the lemma if the sequence λ satisfies the condition (M3) of the Meinardus scheme. □

Note that the saddle point is chosen to be the solution of the equation

$$n = -f_1(r) - \phi_1(v, r). \tag{3.2.30}$$

So far nothing is known about the solution of Equation (3.2.30), we do not even know if a solution exists. For that we need the following lemma:

Lemma 3.2.3. *For any integer $j \geq 1$,*

$$\phi_j(v, r) \sim (-1)^j j! C(v, d) r^{-(j+1)}, \tag{3.2.31}$$

where

$$C(v, d) = \int_0^1 \frac{\log(1 + vx^d(1 - x))}{x} dx, \tag{3.2.32}$$

these estimates are all uniform for $|v| \leq \delta$.

Proof. We write $\phi(v, \tau)$ as

$$\begin{aligned}\phi(v, \tau) &= \sum_{\lambda}\sum_{k\geq 1}(-1)^{k+1}e^{-kd\lambda\tau}(1 - e^{-\lambda\tau})^k\frac{v^k}{k}\\ &= \sum_{k\geq 1}(-1)^{k+1}\frac{v^k}{k}\sum_{\lambda}e^{-kd\lambda\tau}(1 - e^{-\lambda\tau})^k.\end{aligned}$$

Then

$$\phi_j(v, r) = \sum_{k\geq 1}(-1)^{k+1}\frac{v^k}{k}\frac{\partial^j}{\partial^j\tau}\sum_{\lambda}e^{-kd\lambda\tau}(1 - e^{-\lambda\tau})^k|_{\tau=r}$$

and we have the following Mellin transform:

$$\mathcal{M}\Big(\phi_j(v,r),s\Big) = (-1)^j \alpha(k,d,s-j)\Gamma(s)\zeta(s-j),$$

where

$$\alpha(k,d,s) = \sum_{k\geq 1}(-1)^{k+1}\frac{v^k}{k}\sum_{l=0}^{k}(-1)^l\binom{k}{l}\frac{1}{(kd+l)^s},$$

which is a Dirichlet series uniformly convergent in the right half-plane if $|v| < 1/2$. Applying Theorem 1.3.2 to the function $\phi_j(v,r)$ for fixed v and j, this gives us the asymptotic formula in (3.2.31) with

$$C(v,d) = \alpha(k,d,1) = \int_0^1 \frac{\log(1+vx^d(1-x))}{x}dx.$$

□

Lemma 3.2.3 along with the approximation of $f_k(r)$ in (3.2.12) imply the following:

$$f_k(r) + \phi_k(v,r) \sim (-1)^k k!\Big(\frac{\pi^2}{6} - C(v,d)\Big)r^{-(k+1)} \tag{3.2.33}$$

for any $k \geq 1$. Furthermore, the constant $C(v,d)$ can be made arbitrarily small by making $v = u-1$ small. From these observations, it follows that for fixed small v the function on the right hand side of (3.2.30) is a monotone decreasing function of r for $0 < r < \epsilon$ for some $\epsilon > 0$, and so there is a unique positive $r = r(u,n,d)$ satisfying Equation (3.2.30) provided that n is sufficiently large. One can already deduce an asymptotic relation

$$r^{-1} \sim \sqrt{\frac{6n}{\pi^2 - 6C(v,d)}} \tag{3.2.34}$$

as $n \to \infty$. We are now able to apply the saddle-point method.

Theorem 3.2.4. *The following asymptotic formula holds:*

$$Q_n(v+1) = \frac{1}{\sqrt{2\pi(f_2(r)+\phi_2(v,r))}}e^{nr+f(r)+\phi(v,r)}(1+\mathcal{O}(n^{1/7})) \tag{3.2.35}$$

as $n \to \infty$, uniformly for $|v| \leq \delta$.

Proof. Use Lemma 3.2.2 and Lemma 3.2.3 and apply the saddle point method in the same way as before. □

Now we go back to the formula for the moment generating function given in Equation (3.2.26). We shall adopt some new notations for the remaining part of this section so x will denote a fixed real number, $v = e^{x/\sigma_n} - 1$, $r = r(v)$ and $r_0 = r(0)$. From Theorem 3.2.4 it is not hard to show that

$$\frac{Q_n(v+1)}{Q_n(1)} \sim \exp\Big(nr + f(r) + \phi(v,r) - nr_0 - f(r_0)\Big). \tag{3.2.36}$$

It remains to estimate the exponent of (3.2.36). We recall the relation between n, v and r:

$$n = -f_1(r) - \phi_1(v,r).$$

Then by means of implicit differentiation we get

$$\frac{\partial}{\partial v} r(v)\Big|_{v=\eta} = \frac{\frac{\partial}{\partial v}\phi_1(v, r(\eta))\Big|_{v=\eta}}{f_2(r(\eta)) + \phi_2(\eta, r(\eta))}. \tag{3.2.37}$$

If $|\eta| \le e^{x/\sigma_n} - 1$, then r_0 and $r(\eta)$ are asymptotically equal. Therefore

$$f_2(r(\eta)) + \phi_2(\eta, r(\eta)) \gg r_0^{-3}.$$

Also by a similar technique as in the proof of Lemma 3.2.3 one may get

$$\frac{\partial}{\partial v}\phi_1(v, r(\eta))\Big|_{v=\eta} = \mathcal{O}(r_0^{-2}).$$

Thus the difference $r - r_0$ is a $\mathcal{O}(r_0|v|)$, that is of order $n^{-3/4}$ in terms of n. And so

$$f(r) - f(r_0) = f_1(r_0)(r - r_0) + f_2(r_0)\frac{(r-r_0)^2}{2} + \mathcal{O}(r_0^{1/2}). \tag{3.2.38}$$

For $\phi(v,r)$ we use Taylor expansion with two variables

$$\begin{aligned}\phi(v,r) &= g(r_0)v + \phi_1(0,r_0)(r-r_0)\\ &\quad + h(r_0)\frac{v^2}{2} + g_1(r_0)(r-r_0)v + \phi_2(0,r_0)\frac{(r-r_0)^2}{2} + \mathcal{O}(r_0^{1/2}).\end{aligned}$$

Adding up everything, we remain with

$$\frac{Q_n(v+1)}{Q_n(1)} \sim \exp\Big(g(r_0)v + \Big(-h(r_0) - \frac{g_1^2(r_0)}{f_2(r_0)}\Big)\frac{v^2}{2}\Big). \tag{3.2.39}$$

Moreover,

$$v = \frac{x}{\sigma_n} + \frac{x^2}{2\sigma_n^2} + \mathcal{O}\Big(\frac{x^3}{\sigma_n^3}\Big). \tag{3.2.40}$$

Therefore,

$$M_n(x) \sim \exp\Big((g(r_0) - \mu_n)\frac{x}{\sigma_n} + \frac{1}{2}\Big(g(r_0) - h(r_0) - \frac{g_1^2(r_0)}{f_2}\Big)\frac{x^2}{\sigma_n^2}\Big) \tag{3.2.41}$$

as $n \to \infty$. Then Theorem 3.1.1 follows by using the asymptotic formulas for the mean and variance we proved in the first part of this section, together with Curtiss's Theorem [5].

3.3 Proof of Theorem 3.1.2

Here in this section $d = d(n)$ is an increasing function of n, and we assume that $d(n) \to \infty$ as $n \to \infty$. We shall keep the notations in Section 3.2 and note that the function $\phi(v, \tau)$, and thus also $g(\tau)$ and $h(\tau)$ are now functions of n as d is a function of n.

Let us first assume that $dn^{-1/4} \to \infty$, then it is sufficient to show that that the mean μ_n tends to 0, since we are dealing with a nonnegative random variable ϖ_n, and Markov's inequality will give us the desired result. Indeed, we still have

$$\mu_n \sim \frac{e^{nr}}{2\pi Q_n(1)} \int_{-r^{1+\beta}}^{r^{1+\beta}} \exp\Big(nit + f(r+it)\Big) g(r+it) dt,$$

where r is determined by the equation

$$n = \sum_\lambda \frac{\lambda}{e^{\lambda r} - 1}.$$

So it suffices to show that $g(r+it)$ goes to 0 uniformly in t ($|t| < r^{1+\beta}$). We have

$$|g(r+it)| \leq \sum_\lambda e^{-d\lambda r} |1 - e^{-\lambda(r+it)}|,$$

for $\lambda \geq r^{-1}$ we have

$$\sum_{\lambda \geq r^{-1}} e^{-d\lambda r} |1 - e^{-\lambda(r+it)}| \ll \sum_{\lambda \geq r^{-1}} e^{-d\lambda r} \ll r^{-1} e^{-d},$$

and the latter is smaller than any power of n^{-1}. For $\lambda < r^{-1}$,

$$\sum_{\lambda < r^{-1}} e^{-d\lambda r} |1 - e^{-\lambda(r+it)}| \ll \sum_\lambda e^{-d\lambda r}(1 - e^{-\lambda r}) \ll \frac{1}{d^2 r}.$$

Therefore, $\mu_n \to 0$ as $n \to \infty$, and by Markov's inequality we have

$$\lim_{n\to\infty} \mathbb{P}\Big(\varpi_n \geq \epsilon\Big) = 0 \tag{3.3.1}$$

for any $\epsilon > 0$, which proves the convergence in probability to the degenerate random variable with support at 0.

Now for the remaining case $d = \mathcal{O}(n^{1/4})$, we follow the lines in Section 3.2, but one needs asymptotic estimates for the functions $g_k(r)$ and $h_k(r)$. We cannot directly use Theorem 1.3.2 since d and r are somehow related and this might affect our estimates. But we use the same approach as in the proof of Theorem 1.3.2. We have the Mellin transform

$$\mathcal{M}(g_k(r), s) = (-1)^k (d^{-s+k} - (d+1)^{-s+k}) \Gamma(s) \zeta(s-k).$$

We want to show that the main term in the asymptotic formula of $g_k(r)$ is still

$$\frac{(-1)^k k!}{d(d+1)} r^{-(k+1)}.$$

That is the case if

$$\Big|\int_{c-i\infty}^{c+i\infty}(d^{-s+k}-(d+1)^{-s+k})\Gamma(s)\zeta(s-k)r^{-s}ds\Big| = o\Big(\frac{1}{d^2r^{k+1}}\Big)$$

for $c<1$. Let us just prove this for the case $k=0$, and the other cases are obtained in a similar way. So let $0<c<1$; then we have

$$\begin{aligned}\Big|\int_{c-i\infty}^{c+i\infty}&(d^{-s}-(d+1)^{-s})\Gamma(s)\zeta(s)r^{-s}ds\Big| \\ &\le (dr)^{-c}\int_{c-i\infty}^{c+i\infty}\Big|\Big(1-\frac{d^s}{(d+1)^s}\Big)\Gamma(s)\zeta(s)\Big|ds \\ &\ll \frac{1}{d^{c+1}r^c}.\end{aligned}$$

To check the last line, for every real number t we have

$$|\Gamma(c+it)\zeta(c+it)| \le c_1 e^{-c_2 t}$$

for some positive constants c_1 and c_2, and

$$\Big|1-\frac{d^{c+it}}{(d+1)^{c+it}}\Big| = \Big|1-\Big(1-\frac{1}{d+1}\Big)^c e^{it\log(d/(d+1))}\Big|$$

which is $\mathcal{O}(\max\{t/d, 1/d\})$ if $|t|\le\sqrt{d}$ and $\mathcal{O}(1)$ for $|t|>\sqrt{d}$ as $n\to\infty$ where the implied constants are independent of t. Therefore we have

$$g(r) = \frac{1}{d^2r} + \mathcal{O}\Big(\frac{1}{d^{c+1}r^c}\Big)$$

as $(n,r)\to(\infty,0)$. Similarly for $h(r)$, we have the Mellin transform

$$\mathcal{M}(h_k(r),s) = (-1)^k((2d)^{-s+k}-2(2d+1)^{-s+k}+(2d+2)^{-s+k})\Gamma(s)\zeta(s-k).$$

Again for the case $k=0$

$$\begin{aligned}\Big|\frac{1}{(2d)^{c+it}}&-2\frac{1}{(2d+1)^{c+it}}+\frac{1}{(2d+2)^{c+it}}\Big| \\ &\le \frac{1}{(2d)^c}\Big|1-2\Big(1-\frac{1}{2d+1}\Big)^{c+it}+\Big(1-\frac{1}{d+1}\Big)^{c+it}\Big|,\end{aligned}$$

then for $|t|\le\sqrt{d}$, the latter is a $\mathcal{O}((t^2+1)/d^{c+2})$ and $\mathcal{O}(1/d^c)$ for $|t|>\sqrt{d}$. Therefore

$$h(r) = \Big(\frac{1}{2d}-\frac{2}{2d+1}+\frac{1}{2d+2}\Big)r^{-1}+\mathcal{O}\Big(\frac{1}{d^{c+2}r^c}\Big)$$

as $(n,r) \to (\infty, 0)$.

Therefore the mean and variance are asymptotically equal as n goes to infinity, more precisely:

$$\mu_n \sim \frac{\sqrt{6n}}{\pi d(d+1)} \quad \text{and} \quad \sigma_n^2 \sim \frac{\sqrt{6n}}{\pi d(d+1)}. \tag{3.3.2}$$

We shall now establish an asymptotic formula for $Q_n(u)$. Note first that the statement of Lemma 3.2.2 is valid in this case, and for Lemma 3.2.3 one may easily show (by using the same idea in the proof) that for a fixed positive integer j and sufficiently large n we have

$$\phi_j(u,r) = \mathcal{O}(vr^{-1})$$

as $r \to 0$, and the implied constant is independent of n. Therefore, for some positive constant δ the following asymptotic formula still holds:

$$Q_n(v+1) = \frac{1}{\sqrt{2\pi(f_2(r) + \phi_2(v,r))}} e^{nr+f(r)+\phi(v,r)}(1 + \mathcal{O}(n^{1/7})) \tag{3.3.3}$$

as $n \to \infty$ uniformly for $|v| \le \delta$, where $r = r(v,n)$ is the unique positive solution of the equation

$$n = -f_1(r) - \phi_1(v,r).$$

If $d = o(n^{1/4})$, then both the mean and variance tend to infinity so we shall consider the normalised random variable whose moment generating function can be expressed as

$$M_n(x) = e^{-x\mu_n/\sigma_n} \frac{Q_n(e^{x\varpi_n/\sigma_n})}{Q_n(1)}.$$

By the same arguments that we used to deduce Equation (3.2.39) we obtain the asymptotic formula

$$M_n(x) \sim \exp\Big(- \frac{x\mu_n}{\sigma_n} + g(r)(e^{x/\sigma_n} - 1)\Big) \tag{3.3.4}$$

as $n \to \infty$. Thus Equation (3.3.4) along with the formulas for the mean and variance implies

$$M_n(x) \sim e^{x^2/2}$$

which proves convergence to the normalised Gaussian distribution by using Curtiss's Theorem again.

If $d \sim \alpha n^{1/4}$, then both the mean and variance tend to a constant $\frac{\sqrt{6}}{\pi\alpha^2}$. We want to estimate the probability generating function, so let us fix a sufficiently small $\delta > 0$ and assume that $|u-1| \le \delta$. Then we have

$$\mathbb{E}(u^{\varpi_n}) = \frac{Q_n(u)}{Q_n(1)} \sim \exp\Big(n(r-r_0) + f(r) - f(r_0) + \phi(u-1,r)\Big)$$

as $n \to \infty$, here we use the same notations as in the previous section. First we need to estimate the difference $r - r_0$, so let $|\eta| \leq |u-1| \leq \delta$. Then

$$\begin{aligned}\frac{\partial}{\partial v}\phi_1(v, r(\eta))\Big|_{v=\eta} &= -\sum_{\lambda} \frac{\lambda e^{-d\lambda r(\eta)}\left(d-(d+1)e^{-\lambda r(\eta)}\right)}{(1+\eta e^{-d\lambda r(\eta)} - \eta e^{-(d+1)\lambda r(\eta)})^2} \\ &\ll g_1(r(\eta)) \\ &\ll \frac{1}{r_0}.\end{aligned}$$

Hence $|r - r_0| \ll n^{-1}$. Since $f_1(r_0) = -n$,

$$n(r-r_0) + f(r) - f(r_0) \ll f_2(r_0)n^{-2} \ll n^{-1/2}$$

and

$$\begin{aligned}\phi(u-1, r) &= (u-1)g(r) + \mathcal{O}(n^{-1/4}) \\ &= \frac{\sqrt{6}}{\pi\alpha^2}(u-1) + o(1).\end{aligned}$$

Finally we deduce that

$$\mathbb{E}(u^{\varpi_n}) \sim e^{\frac{\sqrt{6}}{\pi\alpha^2}(u-1)}$$

as $n \to \infty$, which proves the convergence to the Poisson distribution with parameter $\frac{\sqrt{6}}{\pi\alpha^2}$. This completes the proof of Theorem 3.1.2.

3.4 Generalisation

As we mentioned in the introduction we shall see how these results change when we deal with partitions into elements of an arbitrary sequence λ. So from now on λ is a sequence of nondecreasing positive integers $(\lambda_1, \lambda_2, \lambda_3 \cdots)$ such that λ_k tends to infinity when k tends to infinity. The notations $\sum_\lambda$ and $\prod_\lambda$ now stand for the sum and product taken over the sequence λ. Then we have the following theorem:

Theorem 3.4.1. *If the sequence λ satisfies the conditions (M1) to (M3) of the Meinardus scheme then the number of parts of multiplicity d in a random λ-partition of n is asymptotically normally distributed where the mean and variance are given by the asymptotic formulas:*

$$\mu_n \sim \left(\frac{1}{d^\alpha} - \frac{1}{(d+1)^\alpha}\right)\frac{\Gamma(\alpha)A}{(A\zeta(\alpha+1)\Gamma(\alpha+1))^{\alpha/(\alpha+1)}}n^{\alpha/(\alpha+1)}$$

and

$$\sigma_n^2 \sim \Big(\frac{1}{d^\alpha} - \frac{1}{(d+1)^\alpha} - \frac{1}{(2d)^\alpha} + \frac{2}{(2d+1)^\alpha} - \frac{1}{(2d+2)^\alpha} \\ - \Big(\frac{1}{d^\alpha} - \frac{1}{(d+1)^\alpha}\Big)^2 \frac{\alpha}{(\alpha+1)\zeta(\alpha+1)}\Big) \frac{A\Gamma(\alpha) n^{\alpha/(\alpha+1)}}{(A\zeta(\alpha+1)\Gamma(\alpha+1))^{\alpha/(\alpha+1)}}$$

respectively, if $d = o(n^{\alpha/(\alpha+1)^2})$.

If $d \sim a n^{\alpha/(\alpha+1)^2}$, *then the limiting distribution is Poisson with parameter*

$$\frac{A\Gamma(\alpha+1)}{a^{\alpha+1}\Big(A\zeta(\alpha+1)\Gamma(\alpha+1)\Big)^{\alpha/(\alpha+1)}}.$$

And if $dn^{-\alpha/(\alpha+1)^2} \to \infty$, *then the limiting distribution is degenerate at zero.*

We shall not present the proof of this theorem since it is essentially the same as for ordinary partitions, and conditions (M1) to (M3) provide us with the necessary tools we need. More precisely conditions (M1) and (M2) allow us to apply the Mellin transform method to obtain asymptotic estimates for the functions f, ϕ, g, h and their derivatives. The condition (M3) is needed for the tail estimates in the saddle point method (just like in Lemma 3.2.2). See for instance [15] for a similar use of these conditions.

The result in Theorem 3.4.1 works for fairly large varieties of sequences of positive integer. For example all integer valued polynomials, where $\lambda_n = P(n)$ with an additional technical condition: $\gcd(P(n) : n \in \mathbb{Z}) = 1$, satisfies the Meinardus scheme and so Theorem 3.4.1 applies. However, as we saw in the previous chapter, there are some interesting sequences that fail to satisfy the Meinardus conditions such as the sequence of primes. Condition (M3) is satisfied by the sequence of primes, as shown in Lemma 2.2.1. (M1) is clearly not satisfied, but we can apply the techniques of the previous chapter to obtain the required estimates.

Theorem 3.4.2. *The number of parts with multiplicity d in a random prime partition is:*

- *asymptotically normally distributed with mean and variance*

$$\mu_n \sim \frac{1}{\pi d(d+1)}\sqrt{\frac{12n}{\log n}}$$

and

$$\sigma_n^2 \sim \left(\frac{1}{\pi d(d+1)} - \frac{1}{2\pi d(d+1)(2d+1)} - \frac{3}{\pi^3 d^2(d+1)^2}\right)\sqrt{\frac{12n}{\log n}}$$

respectively, if $d = o((n/\log n)^{1/4})$,

- *Poisson with parameter* $\frac{\sqrt{12}}{a^2\pi}$ *if* $d \sim a(n/\log n)^{1/4}$,
- *degenerate at zero for* $d(n/\log n)^{-1/4} \to \infty$.

3.5 Parts with multiplicity d or more in λ-partitions

The number of ascents of size d or more in a random partition of an integer n has already been treated in [3] for fixed d, a result that can be expressed in the language of multiplicities since there is a one-to-one correspondence between partitions having parts of multiplicity d and partitions with ascents of size d. So for completeness we shall give a generalisation of this result for λ-partitions. For this case we have the bivariate generating function

$$Q^*(u,z) = \prod_{\lambda} \left(\tfrac{1+(u-1)z^{\lambda d}}{1-z^{\lambda}}\right), \tag{3.5.1}$$

where the product is taken over the sequence λ. The logarithm

$$\phi^*(v,\tau) = \sum_{\lambda} \log(1 + ve^{-\lambda d\tau}), \tag{3.5.2}$$

is for our purposes actually easier to handle than the function $\phi(v,\tau)$ but the technique remains the same. There is a slight change though: a phase transition occurs when $d \sim an^{1/(1+\alpha)}$, and the limiting distribution in this case is not Poisson. This can be shown by the following simple argument: the probability generating function can be expressed as

$$\frac{Q_n^*(u)}{Q_n^*(1)} \sim \frac{e^{nr}}{2\pi Q_n^*(1)} \int_{-r^{1+\beta}}^{r^{1+\beta}} e^{\phi^*(u-1,r+it)} \exp\Big(nit + f(r+it)\Big)dt \tag{3.5.3}$$

as $n \to \infty$, where r is the unique positive solution of the equation

$$n = \sum_{\lambda} \frac{\lambda}{e^{r\lambda} - 1},$$

and β is an arbitrary constant such that $\frac{\alpha}{3} < \beta < \frac{\alpha}{2}$. It follows that

$$r \sim \Big(A\zeta(\alpha+1)\Gamma(\alpha)\Big)^{1/(\alpha+1)} n^{-1/(\alpha+1)}$$

Moreover, we have

$$\phi^*(u-1, r+it) = \phi^*(u-1,r) + \mathcal{O}(r^{\beta})$$

uniformly for $|t| \leq r^{1+\beta}$. Therefore for a fixed real number u we have

$$\frac{Q_n^*(u)}{Q_n^*(1)} \to \prod_{\lambda}(1+(u-1)e^{-\lambda\kappa})$$

as $n \to \infty$, where

$$\kappa = a(A\zeta(\alpha+1)\Gamma(\alpha+1))^{1/(\alpha+1)}.$$

Hence the final result reads as follows:

Theorem 3.5.1. *If the sequence λ satisfies the conditions (M1) to (M3) of the Meinardus scheme, then the number of parts of multiplicity d or more in a random λ-partition of n is asymptotically normally distributed, where the mean and variance are given by the asymptotic formulas:*

$$\mu_n \sim \frac{1}{d^\alpha}\frac{\Gamma(\alpha)A}{(A\zeta(\alpha+1)\Gamma(\alpha+1))^{\alpha/(\alpha+1)}}n^{\alpha/(\alpha+1)}$$

and

$$\sigma_n^2 \sim \left(\frac{1}{d^\alpha} - \frac{1}{(2d)^\alpha} - \frac{\alpha}{d^{2\alpha}(\alpha+1)\zeta(\alpha+1)}\right)\frac{A\Gamma(\alpha)n^{\alpha/(\alpha+1)}}{(A\zeta(\alpha+1)\Gamma(\alpha+1))^{\alpha/(\alpha+1)}}$$

respectively, if $d = o(n^{1/(\alpha+1)})$.

If $d \sim an^{1/(\alpha+1)}$, then the limiting distribution is a sum of Bernoulli variables

$$\sum_{\lambda} \mathrm{Be}(e^{-\lambda\kappa}),$$

this series converges almost surely.

If $dn^{-1/(\alpha+1)} \to \infty$ then the limiting distribution is degenerate at zero.

For the case of ordinary partitions, $\alpha = 1$ and this gives the result proved in [3, last section]. But also, for $d = 1$, the number of parts having multiplicity d or more is equal to the total number of distinct parts in random λ-partitions, a case that has already been treated in [10] and [15].

Chapter 4

A phase transition from unrestricted to restricted partitions

4.1 Introduction and preliminary results

The number of summands in a random partition of an integer n was first studied by Erdős and Lehner [6] as mentioned in the introduction. Their results in Theorem 1.2.5 were generalised and extended in many directions: for instance, analogous limit theorems were proved for general λ-partitions. See Haselgrove-Temperley [13], Richmond [22], and Lee [17] on unrestricted partitions, Hwang [15] on restricted partitions. We will closely follow the ideas of Hwang who proved that the distribution of the length of a random restricted λ-partition is asymptotically Gaussian. Let us also mention a result by Mutafchiev in [20] which states that if $d \sim \alpha\sqrt{n}$ then among all partitions of n the set of partitions with no parts with multiplicity greater than d has a positive density asymptotically equal to

$$\prod_{\lambda}(1 - e^{-\alpha\lambda})^{-1}. \tag{4.1.1}$$

In this chapter, we also consider those partitions with no parts of multiplicity greater than d, and we show that when d is asymptotically equal to $\sqrt{n}$, then we observe a phase transition in the distribution of the number of summands in such a partition. More precisely we prove the following theorem:

Theorem 4.1.1. *Let $S_{d,n}$ be the set of partitions of an integer n with no parts of multiplicity greater than d (d may be a function of n) and assume that all partitions in $S_{d,n}$ are equally likely. Then we have the following behaviour for the limit distribution of the number of summands in a random partition:*

- *if* $d = o(\sqrt{n})$ *then it is asymptotically Gaussian,*
- *if* $d \gg \sqrt{n}$ *then it is asymptotically Gumbel,*
- *if* $d \sim b\sqrt{n}$ *where* b *is a positive constant, then when normalized, the distribution of the number of summands convergences to a distribution with moment generating function given by*

$$M(x) = \prod_{\lambda} \frac{e^{-\frac{a}{\lambda}}}{1 - \frac{a}{\lambda}} \prod_{\lambda} \Big(\frac{1 - e^{-(\lambda - a)\vartheta}}{1 - e^{-\lambda\vartheta}} \Big) e^{a\vartheta/(e^{\lambda\vartheta}-1)}$$

where the product is taken over the set of positive integers, and

$$a = \frac{x}{\sqrt{\frac{\pi^2}{6} - \kappa}}, \quad \vartheta = \frac{\pi}{\sqrt{6}} b,$$

and

$$\kappa = \sum_{\lambda} \frac{\vartheta^2 e^{-\lambda\vartheta}}{(1 - e^{-\lambda\vartheta})^2}.$$

These results were obtained by analysing the corresponding generating function. We are interested in the number of summands, and so the generating function for our problem is the following: for a positive integer d,

$$Q(d, u, z) = \prod_{\lambda} \sum_{j=0}^{d-1} u^j z^{j\lambda}, \tag{4.1.2}$$

where the product is taken over the set of positive integers, the second variable u counts the number of summands. Let $Q_{d,n}(u)$ be the coefficient of z^n in $Q(d, u, z)$ and let $\varpi_{d,n}$ be the random variable counting the number of summands in a random partition, then the probability distribution of the random variable $\varpi_{d,n}$ can be expressed in terms of $Q_{d,n}(u)$ as follows:

$$\mathbb{E}(u^{\varpi_{d,n}}) = \frac{Q_{d,n}(u)}{Q_{d,n}(1)}. \tag{4.1.3}$$

We shall use this observation to study the limiting distribution of $\varpi_{d,n}$. One can already express the mean and variance in terms of $Q_{d,n}(u)$. Let $\mu_{d,n}$ and $\sigma_{d,n}$ be the mean and the standard deviation of the random variable $\varpi_{d,n}$ respectively. Then we have

$$\mu_{d,n} = \frac{\partial}{\partial u} \frac{Q_{d,n}(u)}{Q_{d,n}(1)} \Big|_{u=1} \tag{4.1.4}$$

and

$$\sigma_{d,n}^2 = \frac{\partial^2}{\partial^2 u} \frac{Q_{d,n}(u)}{Q_{d,n}(1)} \Big|_{u=1} + \mu_{d,n} - \mu_{d,n}^2. \tag{4.1.5}$$

Notation. For what follows we introduce certain notations and abbreviations. The variable τ is a complex variable usually written in the form $r + it$ where r is a positive number. So let

$$F(d, u, \tau) := \log Q(d, u, e^{-\tau}).$$

This function as well as other functions that we will use depend on the parameter d but we will frequently omit the variable d, for example we will write $F(u, \tau)$ for $F(d, u, \tau)$. The following functions will also be used several times:

$$f(u, \tau) := -\sum_{\lambda} \log(1 - ue^{-\lambda\tau}),$$

$$g(\tau) := \frac{\partial}{\partial u} f(u, \tau)\Big|_{u=1} = \sum_{\lambda} \frac{e^{-\lambda\tau}}{1 - e^{-\lambda\tau}},$$

and

$$h(\tau) := \sum_{\lambda} \frac{e^{-\lambda\tau}}{(1 - e^{-\lambda\tau})^2}.$$

The function $F(u, \tau)$ can be written as follows:

$$F(u, \tau) = f(u, \tau) - f(u^d, d\tau),$$

and we also write

$$G(\tau) = g(\tau) - dg(d\tau)$$

and

$$H(\tau) = h(\tau) - d^2 h(d\tau).$$

The hardest part of the proof of our main result in this chapter will be to understand the behaviour of these functions when $\mathrm{Re}(\tau) = r$ is closed to zero.

Finally if $X(u, \tau)$ is an analytic function of τ in a certain domain containing τ_0 then we will write

$$X_\tau(u, \tau_0) = \frac{\partial}{\partial \tau} X(u, \tau)\Big|_{\tau=\tau_0}$$

and $X_{\tau\tau}$, $X_{u\tau\tau}$, $X_{\tau\tau\tau}$, $\cdots$ are defined similarly.

One can express the mean and variance in terms of integrals involving the above functions:

$$\mu_{d,n} = \frac{e^{nr}}{2\pi Q_{d,n}(1)} \int_{-\pi}^{\pi} \exp\left(nit + F(1, r + it)\right) G(r + it) dt \tag{4.1.6}$$

and

$$\sigma^2_{d,n} + \mu^2_{d,n} = \frac{e^{nr}}{2\pi Q_{d,n}(1)} \int_{-\pi}^{\pi} \exp\left(nit + F(1, r + it)\right) \left(G(r + it)^2 + H(r + it)\right) dt \tag{4.1.7}$$

for any $r > 0$.

Most of our functions are expressed in the form of harmonic sums, and we use the Mellin transform method to estimate them. More precisely, we are using Theorem 1.3.2, and we will also allow $\phi(s)$ to be complex-valued where necessary. The advantage that we have is that most of our functions have a nicely behaved Mellin transform, for example:

$$\begin{aligned}\mathcal{M}(f(1,r),s) &= \zeta(s+1)\Gamma(s)\zeta(s),\\ \mathcal{M}(g(r),s) &= \zeta^2(s)\Gamma(s),\\ \mathcal{M}(h(r),s) &= \zeta(s-1)\Gamma(s)\zeta(s).\end{aligned}$$

The above functions are all expressed in terms of the Riemann zeta function $\zeta(s)$ and the gamma function $\Gamma(s)$. We know that $\zeta(s)$ admits a simple pole at $s = 1$ with residue 1 and is analytic everywhere else in the complex plane, also $\Gamma(s)$ is analytic everywhere except for simple poles at $s = 0, -1, -2, \cdots$. Furthermore, all the above Mellin transforms satisfy the hypothesis of Theorem 1.3.2 therefore one has

$$f(1,r) = \frac{\pi^2}{6}r^{-1} - \frac{1}{2}\log\tfrac{1}{r} + \mathcal{O}(1), \tag{4.1.8}$$

$$g(r) = (\log\tfrac{1}{r} + 2\gamma)r^{-1} + \mathcal{O}(1) \tag{4.1.9}$$

$$h(r) = \frac{\pi^2}{6}r^{-2} - \frac{1}{2}r^{-1} + \mathcal{O}(1) \tag{4.1.10}$$

where γ is the Euler-Mascheroni constant. In order to estimate $f(e^{ar}, r)$ for fixed a within the interval $(-1, 1)$, we will also need the Hurwitz zeta function

$$\zeta(s, 1-a) = \sum_{\lambda} \frac{1}{(\lambda - a)^s}.$$

Note that the Mellin transform of the difference $f(e^{ar}, r) - f(1, r)$ is

$$\mathcal{M}(f(e^{ar}, r) - f(1,r), s) = \zeta(s+1)\Gamma(s)(\zeta(s, 1-a) - \zeta(s)).$$

The Hurwitz zeta function admits a simple pole at $s = 1$ with residue 1, therefore the pole $s = 1$ of the Mellin transform cancels out, and the pole at $s = 0$ becomes important. All we need to know for our purposes is that

$$\lim_{s\to 0} \frac{(\zeta(s, 1-a) - \zeta(s) - as\zeta(s+1))}{s} = \sum_{\lambda}\left(-\log\left(1 - \frac{a}{\lambda}\right) - \frac{a}{\lambda}\right),$$

and so we have the equation

$$f(e^{ar}, r) - f(1,r) = a\,r g(r) + \sum_{\lambda}\left(-\log\left(1 - \frac{a}{\lambda}\right) - \frac{a}{\lambda}\right) + o(1) \tag{4.1.11}$$

as $r \to 0^+$, the term $a\,r g(r)$ is the inverse Mellin transform of $a\zeta(s+1)\Gamma(s+1)\zeta(s+1)$.

Our main technique is the use of the saddle point method where we are estimating integrals of the form

$$I(n) = \frac{e^{nr}}{2\pi} \int_{-\pi}^{\pi} \exp\left(nit + \Psi(r+it)\right) \Phi(r+it)dt. \tag{4.1.12}$$

Here $\Psi(\tau)$ and $\Phi(\tau)$ are analytic in the open disk of radius 1. The following lemma plays an important role as we shall see in the next sections.

Lemma 4.1.2. *Let $2 \le d \le n$, and suppose that there are positive constants c_1 and c_2 such that $\frac{c_1}{\sqrt{n}} \le r \le \frac{c_2}{\sqrt{n}}$. If furthermore $\tau = r + iy$ with $\pi \ge |y| \ge r^{1+c}$, where c is any number within $(\frac{1}{3}, \frac{1}{2})$, and $\frac{1}{2} \le u \le 2$, then there are positive constants c_3 and δ depending only on c such that*

$$\frac{|Q(d,u,e^{\tau})|}{Q(d,u,e^{r})} \le e^{-c_3 n^{\delta}}$$

for sufficiently large n.

Proof. First we are going to estimate the quantity

$$\mathrm{Re}\left(\sum_{\lambda}(e^{-\lambda r} - e^{-\lambda\tau})\right),$$

which can be written in the following form:

$$\begin{aligned}\frac{1}{1-e^{-r}} - \mathrm{Re}\left(\frac{1}{1-e^{-\tau}}\right) &= \frac{e^{-r}(1+e^{-r})(1-\cos y)}{(1-e^{-r})(1-2e^{-r}\cos y + e^{-2r})} \\ &\gg \frac{|y|^2}{r(\max\{r,|y|\})^2} \gg r^{2c-1}.\end{aligned}$$

as $r \to 0^+$. Now if $|z| \le 2$ then we claim that there are positive constants c_4 and c_5 such that

$$\frac{|1+z|}{1+|z|} \le e^{-c_4(|z|-\mathrm{Re}(z))}$$

and

$$\frac{|1+z+z^2|}{1+|z|+|z|^2} \le e^{-c_5(|z|-\mathrm{Re}(z))}.$$

Indeed for $|z| \le 2$ we have

$$\begin{aligned}\frac{|1+z|^2}{(1+|z|)^2} &= 1 - 2\frac{|z|-\mathrm{Re}(z)}{(1+|z|)^2} \\ &\le 1 - \frac{2}{9}(|z|-\mathrm{Re}(z)) \\ &\le e^{-\frac{2}{9}(|z|-\mathrm{Re}(z))}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{|1+z+z^2|^2}{(1+|z|+|z|^2)^2} &= 1-2(|z|-\mathrm{Re}(z))\frac{1+|z|^2+(2-\mathrm{Re}(z))(\mathrm{Re}(z)+|z|)}{(1+|z|+|z|^2)^2}\\ &\le 1-\frac{2}{49}(|z|-\mathrm{Re}(z))\\ &\le e^{-\frac{2}{49}(|z|-\mathrm{Re}(z))}.\end{aligned}$$

Hence for any $2 \le d \le n$

$$|1+z+z^2+z^3+\cdots+z^{d-1}| \le |1+z|+|z|^2|1+z|+\cdots,$$

where the last term is either $|z|^{d-2}|1+z|$ or $|z|^{d-3}|1+z+z^2|$ depending on the parity of d. Therefore by the claim we have

$$|1+z+z^2+\cdots+z^{d-1}| \le e^{-c_6(|z|-\mathrm{Re}(z))}(1+|z|+|z|^2+\cdots+|z|^{d-1}),$$

where $c_6 = \min\{c_4, c_5\}$. Now we set $z = ue^{-\lambda\tau}$ and take the product over all $\lambda \ge 1$ to obtain

$$\frac{|Q(u,e^{\tau})|}{Q(u,e^{r})} \le \exp\Big(-c_6\sum_{\lambda}(e^{-\lambda r}-\mathrm{Re}(e^{-\lambda\tau}))\Big).$$

This completes the proof. $\square$

4.2 The Case $d \gg \sqrt{n}$

We split the proof of Theorem 4.1.1 into two parts: throughout this section, we assume that $d \gg \sqrt{n}$.

Mean and variance

We shall estimate the mean and variance expressed in the equations (4.1.6) and (4.1.7) respectively by means of the saddle point technique. We will present this approach in a series of lemmas.

Lemma 4.2.1. *The equation*

$$n = -F_{\tau}(1,r) \tag{4.2.1}$$

admits a unique positive solution $r := r(d,n)$; furthermore, the solution r satisfies the asymptotic expansion

$$r = \frac{\pi}{\sqrt{6n}}(1+\mathcal{O}(n^{-1/2}\log n)) \tag{4.2.2}$$

as $n \to \infty$.

Proof. We know that

$$F_{\tau\tau}(1,r) = \sum_{\lambda} \lambda^2 \frac{\sum_{j=0}^{d-1} j^2 e^{-\lambda jr} \sum_{j=0}^{d-1} e^{-\lambda jr} - \left(\sum_{j=0}^{d-1} j e^{-\lambda jr}\right)^2}{\left(\sum_{j=0}^{d-1} e^{-\lambda jr}\right)^2}$$

which is always positive by the Cauchy-Schwarz inequality. Therefore, the right hand side of Equation (4.2.1) is a monotone decreasing function for $r > 0$. This implies that the solution exists, it is unique and it tends to zero as n tends to infinity. Note that

$$F_\tau(1,r) = f_\tau(1,r) - df_\tau(1,dr).$$

By Theorem 1.3.2 we have

$$f_\tau(1,r) = \frac{\pi^2}{6} r^{-2} + \mathcal{O}(r^{-1} \log \tfrac{1}{r})$$

as $r \to 0^+$. The a priori estimate $r^{-1} = \mathcal{O}(\sqrt{n})$ follows immediately. Note further that

$$df_\tau(1,dr) = \sum_{\lambda} \frac{d\lambda}{e^{\lambda dr} - 1} = \mathcal{O}(r^{-1})$$

as $r \to 0^+$ uniformly for $d \gg \sqrt{n}$. Hence the solution r must be of order $n^{-1/2}$ and therefore by (4.2.1) we have

$$n = \frac{\pi^2}{6} r^{-2} + \mathcal{O}(r^{-1} \log \tfrac{1}{r}),$$

which implies (4.2.2). □

We are going to estimate the mean $\mu_{d,n}$ first by using the integral in (4.1.6). Let r be as defined in Lemma 4.2.1, then we have

$$Q_{d,n}(\mu_{d,n} - G(r)) = \frac{e^{nr}}{2\pi} \int_{-\pi}^{\pi} \exp\Big(nit + F(1, r+it)\Big)(G(r+it) - G(r))dt.$$

We split this integral as follows: we take an arbitrary constant c in $(\frac{1}{3}, \frac{1}{2})$ and we consider the integral over $[-r^{1+c}, r^{1+c}]$, which we call the central integral, and the integral over $[-\pi, -r^{1+c}) \cup (r^{1+c}, \pi]$, which we call the tails.

Lemma 4.2.2 (Central integral estimate)**.** *We have*

$$\int_{-r^{1+c}}^{r^{1+c}} e^{nit+F(1,r+it)}(G(r+it) - G(r))dt = \int_{-r^{1+c}}^{r^{1+c}} e^{F(1,r) - F_{\tau\tau}(1,r)t^2/2} dt \times$$
$$\left(\frac{F_{\tau\tau\tau}(1,r)G_\tau(r) - F_{\tau\tau}(1,r)G_{\tau\tau}(r)}{2F_{\tau\tau}^2(1,r)} + \mathcal{O}\Big(r^{7c-3} \log \frac{1}{r}\Big)\right).$$

Proof. If $|t| \leq r^{1+c}$ then we can Taylor-expand the integrand, so

$$\begin{aligned} nit + F(1, r+it) =& F(1,r) - F_{\tau\tau}(1,r)\frac{t^2}{2} - iF_{\tau\tau\tau}(1,r)\frac{t^3}{6} \\ & \mathcal{O}(r^{4+4c} \max_{|\eta| \leq r^{1+c}} |F_{\tau\tau\tau\tau}(1, r+i\eta)|). \end{aligned}$$

Note that Equation (4.2.1) has been used here. In order to get an estimate for the integral one needs to estimate all of the functions involved in the expansions, so we need to estimate $F(1,r)$ and its derivatives. These functions can be expressed as a difference of two functions as we mentioned in the introduction, and we will see that the contribution from the second term is always small in this case. For instance we have seen that

$$|F_\tau(1,r) - f_\tau(1,r)| \ll r^{-1}.$$

Hence, we also have that $F_{\tau\tau}(1,r)$ and $F_{\tau\tau\tau}(1,r)$ are asymptotically equal to $f_{\tau\tau}(1,r)$ and $f_{\tau\tau}(1,r)$ respectively. Applying Theorem 1.3.2 we have

$$F_{\tau\tau}(1,r) \sim \frac{\pi^2}{6} r^{-3},$$

and $F_{\tau\tau\tau}(1,r)$ is of order r^{-4}. To estimate the error term, we first deal with the contribution from $f_{\tau\tau\tau\tau}(r+it)$, so if $|\eta| \leq r^{1+c}$ then we have

$$|f_{\tau\tau\tau\tau}(r+i\eta)| \ll \sum_{\lambda} \frac{\lambda^4 e^{-\lambda r}}{|1 - e^{-\lambda(r+i\eta)}|^4} \ll \sum_{\lambda} \frac{\lambda^4 e^{-\lambda r}}{(1-e^{-\lambda r})^4} \ll r^{-5},$$

and the remaining term can be estimated as follows:

$$d^4 |f_{\tau\tau\tau\tau}(dr + id\eta)| \ll \frac{1}{r^4}.$$

We finally deduce that

$$nit + F(1, r+it) = F(1,r) - F_{\tau\tau}(1,r)\frac{t^2}{2} - iF_{\tau\tau\tau}(1,r)\frac{t^3}{6} + \mathcal{O}(r^{4c-1}).$$

Therefore,

$$e^{nit + F(1,r+it)} = e^{F(1,r) - F_{\tau\tau}(1,r)\frac{t^2}{2}} (1 - iF_{\tau\tau\tau}(1,r)\frac{t^3}{6} + \mathcal{O}(r^{6c-2})). \tag{4.2.3}$$

One can also expand the function $G(r+it) - G(r)$ and obtains

$$G(r+it) - G(r) = itG_\tau(r) - G_{\tau\tau}(r)\frac{t^2}{2} + \mathcal{O}(r^{3c-1}\log\frac{1}{r}). \tag{4.2.4}$$

Multiplying (4.2.4) and (4.2.3) the integrand can be written as

$$e^{F(1,r)-F_{\tau\tau}(1,r)\frac{t^2}{2}}\Big(itG_\tau(r)-G_{\tau\tau}(r)\frac{t^2}{2} + G_\tau(r)F_{\tau\tau\tau}(1,r)\frac{t^4}{6}+\mathcal{O}(r^{7c-3}\log\tfrac{1}{r})\Big).$$

To make the error term small we need to adjust our arbitrary constant c to be within the interval $(3/7, 1/2)$. Since each term in the above expansion grows at most polynomially in r^{-1} as r goes to 0 uniformly for $|t|\le r^{1+c}$ and we also have $F_{\tau\tau}(1,r)\gg r^{-3}$, we can change the range of integration to $(-\infty,+\infty)$ with an exponentially small error term. To see this, let us for example take

$$\int_{-r^{1+c}}^{r^{1+c}} e^{-F_{\tau\tau}(1,r)\frac{t^2}{2}}G_{\tau\tau}(r)\frac{t^2}{2}dt = \int_{-\infty}^{+\infty} e^{-F_{\tau\tau}(1,r)\frac{t^2}{2}}G_{\tau\tau}(r)\frac{t^2}{2}dt + 2\int_{r^{1+c}}^{+\infty} e^{-F_{\tau\tau}(1,r)\frac{t^2}{2}}G_{\tau\tau}(r)\frac{t^2}{2}dt,$$

and after a change of variable $u = tr^{-1-c}$ the second term on the right can be written in the form

$$B(r)\int_1^{+\infty} e^{-A(r)\frac{u^2}{2}}u^2du \ll \left|B(r)\int_1^{+\infty} e^{-A(r)\frac{u}{2}}u^2du\right|.$$

Here $|B(r)|$ is at most a polynomial in r^{-1}, and we have $A(r)\gg r^{2c-1}$ (the power $2c-1$ is negative by our choice of c), so that the above integral is smaller than any power of r. The rest of the proof is a computation of integrals of the form

$$\int_{-\infty}^{+\infty} t^k e^{-at^2/2}dt = \begin{cases} 0 & \text{if } k \text{ is odd,}\\ \left(\frac{2}{a}\right)^{(k+1)/2}\Gamma\left(\frac{k+1}{2}\right) & \text{if } k \text{ is even,}\end{cases}$$

which we shall not include here. □

The next step is to estimate the tails, which is done in the following lemma:

Lemma 4.2.3 (Tails estimate)**.** *The expression*

$$\frac{1}{Q(1,e^{-r})}\Big|\int_{|t|>r^{1+c}} e^{nit+F(1,r+it)}dt\Big|$$

goes faster to 0 *than any power of* r*.*

Proof. This is a direct corollary of Lemma 4.1.2. Indeed, since $|G(r+it)-G(r)|$ is bounded by some polynomial in r^{-1}, by Lemma 4.1.2 we have

$$\frac{|Q(d,u,e^{r+it})||G(r+it)-G(r)|}{Q(d,u,e^r)} \ll e^{-c_7n^\delta}$$

when $|t| > r^{1+c}$. This implies that

$$\Big|\int_{r^{1+c}}^{\pi} \exp(nit + F(1, r + it))(G(r + it) - G(r))dt\Big| \ll e^{-c_7 n^{\delta}} e^{F(1,r)},$$

where c_7 is a positive constant depending only in c. □

Note that one can also apply the same technique to estimate $Q_{d,n}(1)$ (for this case we replace $G(r + it) - G(r)$ in the above by 1) and we obtain

$$Q_{d,n}(1) = e^{nr+F(1,r)} \int_{-r^{1+c}}^{r^{1+c}} e^{-F_{\tau\tau}(1,r)t^2/2} dt \left(1 + \mathcal{O}(r^{3c-1})\right).$$

We can now write the mean as

$$\mu_{d,n} = G(r) + \frac{F_{\tau\tau\tau}(1, r)G_{\tau}(r) - F_{\tau\tau}(1, r)G_{\tau\tau}(r)}{2F_{\tau\tau}^2(1, r)} + \mathcal{O}\Big(r^{7c-3} \log \tfrac{1}{r}\Big) \qquad (4.2.5)$$

as $n \to \infty$. Thus we have

$$\mu_{d,n} = G(r) + \mathcal{O}(\log \tfrac{1}{r}). \qquad (4.2.6)$$

Similarly for the variance one has to estimate the integral in Equation (4.1.7). We can use the same method we used for the mean $\mu_{d,n}$. We are only interested in the main term of the variance (it is possible to get more terms with a bit more work). Using the same method as before we get the following:

$$\sigma_{d,n}^2 + \mu_{d,n}^2 = \sigma_{d,n}^2 + G^2(r) + \mathcal{O}\left(G(r) \log \tfrac{1}{r}\right) = H(r) + G^2(r) + \mathcal{O}(r^{-1} \log^2 \tfrac{1}{r}).$$

Then we finally find

$$\sigma_{d,n}^2 = H(r) + \mathcal{O}(r^{-1} \log^2 \tfrac{1}{r}). \qquad (4.2.7)$$

If we want to estimate the mean and variance in terms of n then we need to estimate the formula that we found the by Mellin transform method to get asymptotic formulas in terms of r, and then substitute by its expansion in terms of n. The main term in the expansion is

$$\mu_{d,n} = \frac{\sqrt{6n}}{2\pi} \log n + \mathcal{O}(\sqrt{n})$$

and

$$\sigma_{d,n}^2 = \left(\tfrac{\pi^2}{6} - (dr)^2 h(dr)\right) \frac{6n}{\pi^2} + \mathcal{O}(\sqrt{n} \log^2 n)$$

as $n \to \infty$. Note that the term $(dr)^2 h(dr)$ is a $\mathcal{O}(1)$ in this case. Knowing the mean and variance we are now going to find the limiting distribution.

Moment generating function

To get the limit distribution we consider the normalized random variable

$$X_n = \frac{\varpi_{d,n} - \mu_{d,n}}{\sigma_{d,n}},$$

and we want to estimate the moment generating function

$$M_n(x) = \mathbb{E}(e^{xX_n}) = e^{-x\mu_{d,n}/\sigma_{d,n}} \frac{Q_{d,n}(e^{x/\sigma_{d,n}})}{Q_{d,n}(1)}. \tag{4.2.8}$$

It remains to determine an asymptotic formula for the coefficient $Q_{d,n}(u)$ for certain values of u. So we use the following integral representation:

$$Q_{d,n}(u) = \frac{e^{nr}}{2\pi} \int_{-\pi}^{\pi} \exp\Big(nit + F(u, r+it)\Big) dt. \tag{4.2.9}$$

From now on we set $u = e^{ar}$ where a is within some fixed interval around zero; a is always as such until the end of this section. We use the saddle point method again and choose $r = r(a, n)$ as the positive solution of the equation

$$n = -F_\tau(e^{ar}, r). \tag{4.2.10}$$

It is not hard to check that the function on the right hand side is also a monotone decreasing function of r for $r > 0$. So the solution exists and is unique. To obtain the asymptotic behaviour of the solution in terms of n we need the next result.

Lemma 4.2.4. *We have the estimates*

$$F_\tau(e^{ar}, r) = -\frac{\pi^2}{6} r^{-2} + \mathcal{O}(r^{-1} \log \tfrac{1}{r})$$

and

$$F_{\tau\tau}(e^{ar}, r) = \frac{\pi^2}{3} r^{-3} + \mathcal{O}(r^{-2} \log \tfrac{1}{r})$$

as $r \to 0^+$.

Proof. We start with $F_\tau(e^{ar}, r)$, which can be written as a difference of two sums:

$$\sum_\lambda \frac{d\lambda}{e^{(\lambda-a)dr} - 1} - \sum_\lambda \frac{\lambda}{e^{(\lambda-a)r} - 1}.$$

We estimate these sums separately. First we have

$$\sum_\lambda \frac{\lambda}{e^{(\lambda-a)r} - 1} = \sum_\lambda \frac{\lambda - a}{e^{(\lambda-a)r} - 1} + a \sum_\lambda \frac{1}{e^{(\lambda-a)r} - 1},$$

and hence the Mellin transform can be computed as

$$\zeta(s)\Gamma(s)\left(\zeta(s-1,1-a)+a\zeta(s,1-a)\right).$$

The dominant singularity is at $s=2$ which is a simple pole, and the next singularity is at $s=1$ which is a double pole, therefore by Theorem 1.3.2 we have

$$\sum_{\lambda}\frac{\lambda}{e^{(\lambda-a)r}-1}=\frac{\pi^2}{6}r^{-2}+\mathcal{O}(r^{-1}\log\tfrac{1}{r})$$

as $r\to 0^+$. It also follows that

$$\begin{aligned} d\sum_{\lambda}\frac{\lambda}{e^{(\lambda-a)dr}-1}&=\mathcal{O}(d(dr)^{-2})\\ &=\mathcal{O}(d^{-1}r^{-2}). \end{aligned}$$

Now we get from (4.2.10) that r is of order $n^{-1/2}$, and by the assumption that $d\gg\sqrt{n}$, dr is bounded below. Hence

$$d\sum_{\lambda}\frac{\lambda}{e^{(\lambda-a)dr}-1}=\mathcal{O}(r^{-1})$$

and the first part of the lemma follows. The second part is proved analogously. □

As direct corollary of this lemma, we find that the solution r admits the asymptotic expansion

$$r=\frac{\pi}{\sqrt{6n}}(1+\mathcal{O}(n^{-1/2}\log n))$$

as $n\to\infty$, uniformly in a and $d\gg\sqrt{n}$. Then we proceed as before by splitting the integral into three parts. The tails are small by Lemma 4.1.2 just by repeating the argument we had before. So we shall only concentrate on the central integral. For $|t|\le r^{1+c}$, where again c is chosen to be within the range $(1/3,1/2)$, we have

$$\begin{aligned} nit+F(u,r+it)=&F(u,r)-\frac{t^2}{2}F_{\tau\tau}(u,r)+\\ &O\Big(|t|^3\max_{|\eta|\le r^{1+c}}\Big|F_{\tau\tau\tau}(u,r+i\eta)\Big|\Big). \end{aligned}$$

Our arguments in the computation of the mean are still valid since r is of order $n^{-1/2}$, more precisely we still have the following bound on the error term

$$\max_{|\eta|\le r^{1+c}}\Big|F_{\tau\tau\tau}(e^{ar},r+i\eta)\Big|\ll r^{-4}.$$

These observations are all we need to prove the asymptotic formula

$$Q_{d,n}(e^{ar}) = \frac{e^{nr}Q(e^{ar}, e^{-r})}{\sqrt{2\pi F_{\tau\tau}(e^{ar}, r)}}(1 + \mathcal{O}(n^{-\frac{(3c-1)}{2}})) \tag{4.2.11}$$

as $n \to \infty$, uniformly in a.

Now we use the latter asymptotic formula to derive an estimate of the moment generating function $M_n(x)$ for some fixed value of x. So let $u = e^{x/\sigma_{d,n}}$, which can be written in the form e^{ar}, where a is a bounded function of d and n. Before we begin our calculations, we call r_0 the value of r when $a = 0$ ($u = 1$). Then we deduce from (4.2.11) that

$$\frac{Q_{d,n}(e^{ar})}{Q_{d,n}(1)} = \exp(n(r - r_0) + F(e^{ar}, r) - F(1, r_0))(1 + o(1)) \tag{4.2.12}$$

as $n \to \infty$ uniformly in a.

The rest of the section is to estimate the exponent of (4.2.12) and to apply the result to estimate (4.2.8). We first need to estimate the difference $|r - r_0|$.

Lemma 4.2.5. *We have*

$$|r - r_0| \ll \frac{\log n}{n}$$

as $n \to \infty$, uniformly in a.

Proof. Since $r = r(a) := r(a, d, n)$ is uniquely determined by a, d, and n, we can apply implicit differentiation on (4.2.10) and we get

$$\frac{\partial}{\partial a} r(a)\Big|_{a=a_1} = -\frac{\frac{\partial}{\partial a}F_\tau(e^{ar(a_1)}, r(a_1))\Big|_{a=a_1}}{\frac{\partial}{\partial r}F_\tau(e^{a_1 r}, r)\Big|_{r=r(a_1)}}. \tag{4.2.13}$$

We can compute the numerator:

$$\frac{\partial}{\partial a}F_\tau(e^{ar}, r)\Big|_{a=a_1} = r\left(\sum_\lambda \frac{\lambda e^{-(\lambda-a)r}}{(1 - e^{-(\lambda-a)r})^2} - d^2 \sum_\lambda \frac{\lambda e^{-(\lambda-a)dr}}{(1 - e^{-(\lambda-a)dr})^2}\right).$$

By using the Mellin transform we can show that

$$\sum_\lambda \frac{\lambda e^{-(\lambda-a)r}}{(1 - e^{-(\lambda-a)r})^2} \ll r^{-2}\log\frac{1}{r}.$$

For the second term, we know that $dr \gg 1$, therefore we have

$$d^2 \sum_\lambda \frac{\lambda e^{-(\lambda-a)dr}}{(1 - e^{-(\lambda-a)dr})^2} \ll r^{-2}\sum_\lambda (\lambda dr)^2 e^{-\lambda dr} \ll r^{-2}.$$

The denominator can also be estimated in the same way and we have

$$\left|\frac{\partial}{\partial r}F_\tau(e^{a_1 r}, r)\Big|_{r=r(a_1)}\right| \gg r^{-3}.$$

Therefore,

$$|r - r_0| \ll \sup_{a_1} \frac{\partial}{\partial a} r(a)\Big|_{a=a_1} \ll \mathcal{O}(r^2 \log \tfrac{1}{r})$$

which completes the proof. □

We can approximate $F(1, r)$ by means of the Taylor expansion around r_0. From Lemma 4.2.5, we get

$$F(1, r_0) = F(1, r) + n(r - r_0) + \mathcal{O}(n^{-1/2} \log^2 n). \tag{4.2.14}$$

Note here that $F_{tau}(1, r_0) = -n$ by our choice r_0. Hence the exponent of (4.2.12) is reduced to

$$F(e^{ar}, r) - F(1, r) + \mathcal{O}(n^{-1/2} \log^2 n)$$

and this estimate is uniform in a. By the estimate in (4.1.11) we have

$$\begin{aligned} F(e^{ar}, r) - F(1, r) =& a\, rG(r) + \sum_\lambda \left(-\log\left(1 - \frac{a}{\lambda}\right) - \frac{a}{\lambda}\right) \\ &+ \underbrace{f(1, dr) - f(e^{adr}, dr) + adr \cdot g(dr)}_{\text{(at most of constant order)}} + o(1). \end{aligned}$$

Now we are going to use the latter equation to estimate (4.2.8). For a fixed value of x, we define a and r such that r is the solution of

$$n = -F_\tau(e^{ar}, r) \text{ and } ar = \frac{x}{\sigma_{d,n}}.$$

This equation has a solution when x is within some appropriate fixed interval containing zero since $\sigma_{d,n}$ is of order $\sqrt{n}$. From the estimate (4.2.6) we have

$$\begin{aligned} \frac{x\mu_{d,n}}{\sigma_{d,n}} &= a\, rG(r_0) + \mathcal{O}(r \log \tfrac{1}{r}) \\ &= a\, rG(r) + \mathcal{O}(r \log^2 \tfrac{1}{r}), \end{aligned}$$

since,

$$|G(r) - G(r_0)| \ll |G_\tau(r)||r - r_0| \ll \log^2 r.$$

Furthermore if we set $\vartheta := dr$, which is a function of x, d and n, then we finally have

$$M_n(x) \sim \prod_\lambda \frac{e^{-\frac{a}{\lambda}}}{1 - \frac{a}{\lambda}} \cdot \prod_\lambda \left(\frac{1 - e^{-(\lambda - a)\vartheta}}{1 - e^{-\lambda\vartheta}}\right) e^{a\vartheta/(e^{\lambda\vartheta}-1)} \tag{4.2.15}$$

as $n \to \infty$ and $d \gg \sqrt{n}$. Let us remark here that if $dn^{-1/2}$ goes to infinity then ϑ, which is a function of n, also goes to infinity, therefore

$$M_n(x) \to \prod_{\lambda} \frac{e^{-\frac{a}{\lambda}}}{(1-\frac{a}{\lambda})} \quad \text{and } a \sim \frac{\sqrt{6}}{\pi}x$$

as $n \to \infty$, which is the moment generating function of the Gumbel distribution. By Curtiss's theorem [5], the normalised random variable X_n converges in distribution to the Gumbel distribution as $n \to \infty$ just like in the case of unrestricted partitions ($d = n+1$). This is not surprising since almost all partitions are covered in this case.

If now $dn^{-1/2}$ converges to some positive number b, then ϑ is asymptotically constant, more precisely $\vartheta \sim \frac{\pi}{\sqrt{6}}b$. These observations prove the second and the third part of our main theorem.

4.3 The case $d = o(n^{1/2})$

We will follow the lines in the previous section though there are several differences where we have to use other techniques. Again we start by finding the mean and variance.

Mean and variance

The technique used to compute the mean and variance in the previous section can be copied here and so we choose r such that the equation

$$n = -F_\tau(1, r) \tag{4.3.1}$$

is satisfied.

Lemma 4.3.1. *The unique positive solution r of the Equation (4.3.1) satisfies the asymptotic formula*

$$r = \pi\sqrt{\frac{d-1}{6dn}}\,(1 + \mathcal{O}(n^{-1/2}\log n)) \tag{4.3.2}$$

as $n \to \infty$.

Proof. The solution r goes to 0 as n goes to infinity since the function on the right hand side of (4.3.1) is continuous and decreasing on $(0,\infty)$ and it tends to zero as r tends to infinity. So we need to estimate $F_\tau(1, r)$ for r going to zero. We first claim that

$$|df_\tau(1, dy)| \le \frac{1}{dy^2} + \mathcal{O}(y^{-1})$$

as $y \to 0^+$ uniformly in d. Indeed,

$$\begin{aligned}|f_\tau(1, dy)| &= \sum_\lambda \frac{\lambda}{e^{\lambda dy} - 1}\\ &= \sum_{\lambda \le \frac{1}{dy}} \frac{\lambda}{e^{\lambda dy} - 1} + \sum_{\frac{1}{dy} < \lambda \le \frac{2}{dy}} \frac{\lambda}{e^{\lambda dy} - 1} + \sum_{\frac{2}{dy} < \lambda \le \frac{3}{dy}} \frac{\lambda}{e^{\lambda dy} - 1} + \cdots\\ &\le \frac{1}{d^2y^2} + \frac{1}{dy}\sum_{k\ge 1}\frac{k+1}{e^k - 1}.\end{aligned}$$

To get the last line from the second line, we used the inequality $e^{\lambda dy} - 1 \ge \lambda dy$ for any λ. Putting this into (4.3.1) we get

$$n \ll r^{-2}$$

since $f_\tau(1, r)$ is of order r^{-2} uniformly in d, and so dr is a $o(1)$. Now we can use the Mellin transform to estimate the correct magnitude of $f_\tau(1, dr)$ and we get

$$f_\tau(1, dr) = \frac{-\pi^2}{6d^2} r^{-2} + \mathcal{O}(d^{-1}r^{-1}\log \tfrac{1}{dr}),$$

and so

$$n = \frac{\pi^2(d-1)}{6d} r^{-2} + \mathcal{O}(r^{-1}\log\tfrac{1}{r}),$$

which implies (4.3.2). □

The fact that dr goes to zero as n goes to infinity is going to be used several times in everything that follows. We can also use the Mellin transform technique to estimate the other involved functions $G(r)$, $H(r)$ and their derivatives. Let us compute $G(r)$ and $H(r)$ as examples, we have

$$G(r) = g(r) - dg(dr),$$

and

$$g(y) = \frac{\log y^{-1}}{y} + \frac{\gamma}{y} + \frac{1}{4} + \mathcal{O}(y)$$

as $y \to 0^+$, and so

$$G(r) = \frac{\log d}{r} - \frac{d-1}{4} + \mathcal{O}(d^2 r).$$

To estimate $H(r)$, we have

$$H(r) = h(r) - d^2 h(dr)$$

where

$$h(y) = \frac{\pi^2}{6} y^{-2} - \frac{1}{2} y^{-1} + \mathcal{O}(1)$$

as $y \to 0^+$. Therefore,

$$H(r) = \frac{d-1}{2r} + \mathcal{O}(d^2).$$

The derivatives of $G(r)$ and $H(r)$ can also be estimated in this way and we can deduce that

$$\frac{\partial^k}{\partial^k \tau} G(\tau)\Big|_{\tau=r} = \Theta(r^{-(k+1)} \log d)$$

and

$$\frac{\partial^k}{\partial^k \tau} H(\tau)\Big|_{\tau=r} = \Theta(dr^{-(k+1)}).$$

Therefore, in a similar way as we used to prove the estimate (4.2.5), we have the estimate of the mean

$$\mu_{d,n} = G(r) + \frac{F_{\tau\tau\tau}(1,r)G_\tau(r) - F_{\tau\tau}(1,r)G_{\tau\tau}(r)}{2F_{\tau\tau}^2(1,r)} + \mathcal{O}\Big(r^{7c-3} \log \tfrac{1}{r}\Big) \tag{4.3.3}$$

where c is any constant in the range $(\frac{1}{3}, \frac{1}{2})$. For the variance one needs to do a bit more work than in the previous section but the idea is still the same. So we start with Equation (4.1.7), but this time we need more terms in the expansion of $G^2(r+it)$ and $H(r+it)$. For $|t| \le r^{1+c}$ we have

$$G^2(r+it) - G^2(r) = 2iG_\tau(r)G(r)t - (G_{\tau\tau}(r)G(r) + G_\tau^2(r))t^2 + \mathcal{O}(r^{3c-2} \log^2 \tfrac{1}{r}).$$

We also need estimate $H(r+it)$. The Mellin transform of the function $h(y(1+\beta i))$ where y is a real positive variable is

$$\frac{1}{(1+\beta i)^s} \zeta(s-1)\Gamma(s)\zeta(s).$$

Therefore, by applying Theorem 1.3.2 we have

$$h(\tau) = \frac{\pi^2}{6}\tau^{-2} - \frac{1}{2}\tau^{-1} + \mathcal{O}(1)$$

if $\tau = y + y\beta i$ as $y \to 0^+$, uniformly for β in a fixed interval around 0. We can also use a similar calculation to estimate $h(d(r+it))$. Putting all the estimates together we get

$$|H(r+it) - H(r)| = \mathcal{O}(dr^{c-1}).$$

Then we apply the integral to get

$$\begin{aligned}\sigma_{d,n}^2 + \mu_{d,n}^2 =& H(r) + G^2(r) \\ &+ \frac{2F_{\tau\tau\tau}(1,r)(G_\tau(r)G(r)) - 2F_{\tau\tau}(1,r)(G_{\tau\tau}(r)G(r) + G_\tau^2(r))}{2F_{\tau\tau}^2(1,r)} \\ &+ \mathcal{O}(dr^{c-1} + r^{7c-4} \log^2 \tfrac{1}{r})\end{aligned}$$

Therefore, using Equation (4.3.3), we have

$$\sigma_{d,n}^2 = H(r) - \frac{G_\tau^2(r)}{F_{\tau\tau}(1,r)} + \mathcal{O}(dr^{c-1} + r^{7c-4}\log^2 \tfrac{1}{r}). \tag{4.3.4}$$

Thus by choosing c such that $c > 3/7$, we can deduce from Equation (4.3.3) and Equation (4.3.4) that

$$\mu_{d,n} = \frac{\log d}{r} + \mathcal{O}(d) \tag{4.3.5}$$

and

$$\sigma_{d,n}^2 = \left(\frac{d-1}{2} - \frac{3d\log^2 d}{\pi^2(d-1)}\right) r^{-1} + \mathcal{O}\left(\frac{dr^c + r^{7c-3}\log^2 \frac{1}{r}}{r}\right). \tag{4.3.6}$$

as $n \to \infty$.

Moment generating function

Again we take the normalized random variable

$$X_n = \frac{\varpi_{d,n} - \mu_{d,n}}{\sigma_{d,n}},$$

and we need to have an estimate of $Q_{d,n}(u)$ for u within an interval containing 1 to understand the limit behaviour of X_n. Let $r = r(u,d,n)$ be the unique positive solution of the equation

$$n = -F_\tau(u,r). \tag{4.3.7}$$

The right hand side of (4.3.7) is a decreasing function of r if $r > 0$, this follows from the fact that

$$F_{\tau\tau}(u,r) = \sum_\lambda \lambda^2 \frac{\sum_{j=0}^{d-1} j^2 u^j e^{-\lambda jr} \sum_{j=0}^{d-1} u^j e^{-\lambda jr} - \left(\sum_{j=0}^{d-1} j u^j e^{-\lambda jr}\right)^2}{\left(\sum_{j=0}^{d-1} u^j e^{-\lambda jr}\right)^2}$$

is always positive by the Cauchy-Schwarz inequality. Furthermore, the solution r goes to zero as n goes to infinity. We shall now find the asymptotic relation between r and n.

Lemma 4.3.2. *If $u = e^{x/\sigma_{d,n}}$ where x is a fixed real number then*

$$F_\tau(u,y) = F_\tau(1,y)(1 + \mathcal{O}(\sqrt{d}\,n^{-1/4})) \tag{4.3.8}$$

as $n \to \infty$, uniformly for $y > 0$.

Proof. We have

$$F_\tau(u,y) = \sum_\lambda \frac{\partial}{\partial \tau} \log\Big(\sum_{j=0}^{d-1} u^j e^{-\lambda j \tau}\Big)\Big|_{\tau=y}$$
$$= \sum_\lambda \frac{-\sum_{j=0}^{d-1} j\lambda u^j e^{-\lambda j y}}{\sum_{j=0}^{d-1} u^j e^{-\lambda j y}}.$$

Since $\sigma_{d,n}$ is of order $\sqrt{d}n^{1/4}$, we have

$$u^j = 1 + \mathcal{O}(\sqrt{d}n^{-1/4}),$$

uniformly for $0 \le j < d$. Therefore we have

$$F_\tau(u,y) = F_\tau(1,y)(1 + \mathcal{O}(\sqrt{d}\,n^{-1/4})),$$

where the error term is independent of y. □

This lemma implies that the solution of (4.3.7) is also of order $n^{-1/2}$. Therefore, dr is tending to zero as n tends to infinity. Furthermore, we have

$$n = \tfrac{\pi^2(d-1)}{6d} r^{-2} + \mathcal{O}(\sqrt{d}n^{3/4}). \tag{4.3.9}$$

We also need to estimate $F_{\tau\tau}(u,r)$ and $|F_{\tau\tau\tau}(u,r+it)|$ for $|t| \le r^{1+c}$.

Lemma 4.3.3. *If $u = e^{x/\sigma_{d,n}}$ where x is a fixed real number then we have the estimates*

$$F_{\tau\tau}(u,r) \sim \frac{\pi^2(d-1)}{3d} r^{-3} \tag{4.3.10}$$

and

$$|F_{\tau\tau\tau}(u,r+it)| \ll r^{-4} \tag{4.3.11}$$

uniformly for $|t| \le r^{1+c}$.

Proof. Let

$$A := [\tfrac{x}{r\sigma_{d,n}}] \quad \text{and} \quad a := \tfrac{x}{r\sigma_{d,n}} - A,$$

where [.] denotes the nearest integer. For $\lambda \le A$ and for a fixed non-negative integer k, there are positive constants K_1 and K_2 depending only on k such that

$$K_1 d^{k+1} \le \sum_{j=0}^{d-1} j^k u^j e^{-\lambda j r} \le K_2 d^{k+1}, \tag{4.3.12}$$

since $u^j = 1 + \mathcal{O}(\sqrt{d}n^{-1/4})$ as before and $\lambda j r \ll \sqrt{d}n^{-1/4}$ as well. Now we split the series $F_{\tau\tau}(u,r)$ into two parts and we denote by S_1 the sum over $\lambda \le A$

and by S_2 the sum over $\lambda > A$. We are going to estimate them separately: we have

$$S_1 = \sum_{\lambda \le A} \lambda^2 \frac{\sum_{j=0}^{d-1} j^2 u^j e^{-\lambda jr} \sum_{j=0}^{d-1} u^j e^{-\lambda jr} - \left(\sum_{j=0}^{d-1} j u^j e^{-\lambda jr}\right)^2}{\left(\sum_{j=0}^{d-1} u^j e^{-\lambda jr}\right)^2}$$

and so

$$S_1 \le \sum_{\lambda \le A} \lambda^2 \frac{\sum_{j=0}^{d-1} j^2 u^j e^{-\lambda jr}}{\sum_{j=0}^{d-1} u^j e^{-\lambda jr}} \ll A^3 d^2 \ll n$$

by (4.3.12). For S_2 we shift the summation so that we can write the sum as

$$S_2 = \sum_{\lambda \ge 1} (\lambda + A)^2 \left(\frac{e^{-(\lambda-a)r}}{(1-e^{-(\lambda-a)r})^2} - \frac{d^2 e^{-(\lambda-a)dr}}{(1-e^{-(\lambda-a)dr})^2} \right).$$

Now we expand $(\lambda + A)^2$. Then the term with λ^2 is equal to $F_{\tau\tau}(e^{ar}, r)$, and the term with A^2 is almost the same as $H(r)$: the difference is that the sum is taken over a slightly shifted sequence, where the shift a is at most $\frac{1}{2}$ in absolute value. Since the Dirichlet series of the shifted sequence is $\zeta(s, 1-a)$, the term with A^2 contributes only $\mathcal{O}(A^2 H(r))$. The term with $2A\lambda$ can be written as

$$2A \sum_{\lambda} (\lambda - a) \left(\frac{e^{-(\lambda-a)r}}{(1-e^{-(\lambda-a)r})^2} - \frac{d^2 e^{-(\lambda-a)dr}}{(1-e^{-(\lambda-a)dr})^2} \right) + \mathcal{O}(AH(r)).$$

The sum can be estimated by using the Mellin transform method, and we have

$$\sum_{\lambda} (\lambda - a) \frac{e^{-(\lambda-a)r}}{(1-e^{-(\lambda-a)r})^2} = \frac{\log \frac{1}{r}}{r^2} + \mathcal{O}(r^{-2})$$

and

$$d^2 \sum_{\lambda} (\lambda - a) \frac{e^{-(\lambda-a)dr}}{(1-e^{-(\lambda-a)dr})^2} = \frac{\log \frac{1}{dr}}{r^2} + \mathcal{O}(r^{-2}).$$

Putting everything together we get

$$F_{\tau\tau}(u, r) = F_{\tau\tau}(e^{ar}, r) + \mathcal{O}\left(\frac{n^{5/4} \log d}{\sqrt{d}} \right),$$

and we can estimate $F_{\tau\tau}(e^{ar}, r)$ again by Theorem 1.3.2 to get the estimate in (4.3.10).

The estimate in (4.3.11) is done in a similar manner. □

Lemma 4.3.3 and Lemma 4.1.2 allow us to derive the following asymptotic formula by means of the saddle point method: if $u = e^{x/\sigma_{d,n}}$ for a fixed real number x, then

$$Q_{d,n}(u) \sim \frac{1}{\sqrt{2\pi F_{\tau\tau}(u, r)}} \exp\Big(nr + F(u, r)\Big) \tag{4.3.13}$$

as $n \to \infty$. This immediately implies that

$$\frac{Q_{d,n}(u)}{Q_{d,n}(1)} \sim \exp\Big(n(r-r_0) + F(u,r) - F(1,r_0)\Big) \tag{4.3.14}$$

as $n \to \infty$, since $F_{\tau\tau}(u,r)$ and $F_{\tau\tau}(1,r)$ are asymptotically equal uniformly in u. It now remains to estimate the exponent of the right hand side of (4.3.14). As always at this stage we let r_0 be $r(1,d,n)$.

Lemma 4.3.4. *We have*

$$F(u,r) = F(1,r) + \frac{x}{r\sigma_{d,n}}\log d + \frac{(d-1)x^2}{4r\sigma_{d,n}^2} + o(1). \tag{4.3.15}$$

Proof. Let v be $\frac{x}{\sigma_{d,n}}$, so that $u = e^v$. Moreover, we let

$$S_1' = \sum_{\lambda \le A} \log\Big(\sum_{j=0}^{d-1} e^{j(v-\lambda r)}\Big) \quad \text{and} \quad S_2' = \sum_{\lambda > A} \log\Big(\sum_{j=0}^{d-1} e^{j(v-\lambda r)}\Big),$$

so that $F(u,r) = S_1' + S_2'$. We estimate S_1' and S_2' separately. We know that v is of order $d^{-1/2}r^{1/2}$ and that $A = [\frac{v}{r}]$. Hence

$$\begin{aligned}
S_1' &= \sum_{\lambda \le A} \log\Big(d + \frac{d(d-1)}{2}(v-\lambda r) + \mathcal{O}(d^2 r)\Big)\\
&= A\log d + \frac{(d-1)vA}{2} - \frac{r(d-1)A(A+1)}{4} + \mathcal{O}(\sqrt{dr})\\
&= A\log d + \frac{(d-1)x^2}{2r\sigma_{d,n}^2} - \frac{(d-1)x^2}{4r\sigma_{d,n}^2} + \mathcal{O}(\sqrt{dr})\\
&= A\log d + \frac{(d-1)x^2}{4r\sigma_{d,n}^2} + \mathcal{O}(\sqrt{dr}).
\end{aligned}$$

To estimate S_2' we use the same trick as in the proof of Lemma 4.3.3 by shifting the sum and we get

$$\begin{aligned}
S_2' - F(1,r) &= F(e^{ar}, r) - F(1,r)\\
&= f(e^{ar}, r) - f(1,r) - (f(e^{adr}, dr) - f(1,dr))\\
&= a\log d + o(1),
\end{aligned}$$

where $a = \frac{v}{r} - A$. Here we used Equation (4.1.11) to derive the last line from the second line. Combining the two, we get

$$\begin{aligned}
F(u,r) = S_1' + S_2' &= F(1,r) + (A+a)\log d + \frac{(d-1)x^2}{4r\sigma_{d,n}^2} + o(1)\\
&= F(1,r) + \frac{v}{r}\log d + \frac{(d-1)x^2}{4r\sigma_{d,n}^2} + o(1)
\end{aligned}$$

which completes the proof. $\square$

On the other hand, we have

$$n(r-r_0) - F(1,r_0) = -F(1,r) + F_{\tau\tau}(1,r_0)\frac{(r-r_0)^2}{2} + \mathcal{O}(r_0^{-4}|r-r_0|^3)$$

since $F_{tau}(1,r) = -n$ by definition, so we need an estimate of the difference $|r-r_0|$.

Lemma 4.3.5. *We have*

$$r - r_0 \sim (u-1)\frac{3d\log d}{\pi^2(d-1)}r_0 \tag{4.3.16}$$

if d is fixed, and

$$|r-r_0| = \mathcal{O}\Big(\frac{\log d}{\sqrt{d}}n^{-3/4}\Big). \tag{4.3.17}$$

if d goes to infinity with n.

Proof. Let us assume first that d goes to infinity with n. As in the proof of Lemma 4.2.5 we use implicit differentiation and we get

$$\frac{\partial}{\partial u}r = -\frac{\frac{\partial}{\partial u}F_\tau(u,r)}{\frac{\partial}{\partial r}F_\tau(u,r)}. \tag{4.3.18}$$

Then we apply our routine calculation to estimate the numerator and the denominator. For the numerator we split the sum at $A = [\frac{\log u}{r}]$, and the first sum is

$$\sum_{\lambda\le A}\frac{\lambda}{u}\frac{\sum_{j=0}^{d-1}j^2u^je^{-\lambda jr}\sum_{j=0}^{d-1}u^je^{-\lambda jr} - \sum_{j=0}^{d-1}ju^je^{-\lambda jr}\sum_{j=0}^{d-1}ju^je^{-\lambda jr}}{\Big(\sum_{j=0}^{d-1}u^je^{-\lambda jr}\Big)^2}$$

which is of order $\mathcal{O}(A^2d^2)$ by the same argument that we used in Lemma 4.3.3. After shifting the summation, the sum over $\lambda > A$ can be written as

$$\sum_{\lambda}\frac{\lambda+A}{u}\left(\frac{e^{-(\lambda-a)r}}{(1-e^{-(\lambda-a)r})^2} - \frac{d^2e^{-(\lambda-a)dr}}{(1-e^{-(\lambda-a)dr})^2}\right).$$

Here we can see that this sum can be Mellin-transformed, and we can use Theorem 1.3.2 to prove that this sum is a $\mathcal{O}(r^{-2}\log d)$. We have already seen that the denominator admits the asymptotic estimate

$$F_{\tau\tau}(u,r) \gg r^{-3}.$$

These completes the case where d tends to infinity since

$$|r-r_0| \ll |u-1|(r\log d) \ll \frac{r\log d}{\sigma_{d,n}}.$$

If d is fixed then we have from Equation (4.3.7) that

$$-n = F_\tau(u, r) = F_\tau(1, r_0)$$

which implies that

$$F_\tau(u, r) - F_\tau(1, r) = -(F_\tau(1, r) - F_\tau(1, r_0)). \tag{4.3.19}$$

We estimate both sides of Equation (4.3.19). The right hand side is easier, and we get

$$-(F_\tau(1, r) - F_\tau(1, r_0)) = -F_{\tau\tau}(1, r_0)(r - r_0) + \mathcal{O}(r_0^{-4}|r - r_0|^2).$$

To estimate the left hand side, note that for any $0 \le j < d$

$$u^j = 1 + j(u - 1) + \mathcal{O}(dr),$$

and so for any positive integer λ we have

$$\begin{aligned}
&\frac{\sum_{j=0}^{d-1} j u^j e^{-\lambda jr}}{\sum_{j=0}^{d-1} u^j e^{-\lambda jr}} - \frac{\sum_{j=0}^{d-1} j e^{-\lambda jr}}{\sum_{j=0}^{d-1} e^{-\lambda jr}} \\
&= \frac{\sum_{j=0}^{d-1} j u^j e^{-\lambda jr} \sum_{j=0}^{d-1} e^{-\lambda jr} - \sum_{j=0}^{d-1} j e^{-\lambda jr} \sum_{j=0}^{d-1} u^j e^{-\lambda jr}}{\left(\sum_{j=0}^{d-1} e^{-\lambda jr}\right)^2 \left(1 + \mathcal{O}(\sqrt{dr})\right)} \\
&= (u-1) \frac{\sum_{j=0}^{d-1} j^2 e^{-\lambda jr} \sum_{j=0}^{d-1} e^{-\lambda jr} - \left(\sum_{j=0}^{d-1} j e^{-\lambda jr}\right)^2}{\left(\sum_{j=0}^{d-1} e^{-\lambda jr}\right)^2} \left(1 + \mathcal{O}(\sqrt{dr})\right) \\
&+ \mathcal{O}\left(dr \frac{\sum_{j=0}^{d-1} j e^{-\lambda jr}}{\sum_{j=0}^{d-1} e^{-\lambda jr}}\right).
\end{aligned}$$

Summing over all positive integers we have

$$\begin{aligned}
F_\tau(u, r) - F_\tau(1, r) &= (u - 1) F_{u\tau}(1, r) \\
&\quad + \mathcal{O}\left(\sqrt{dr}|u - 1||F_{u\tau}(1, r)| + dr|F_\tau(1, r)|\right).
\end{aligned}$$

Since r and r_0 are asymptotically equal, we have the asymptotic formulas

$$\begin{aligned}
u - 1 &\sim \frac{x}{\sigma_{d,n}}, \\
F_\tau(1, r) &\sim \frac{-\pi^2(d-1)}{6d} r^{-2}, \\
F_{u\tau}(1, r) &= G_\tau(r) \sim -(\log d) r^{-2}, \\
F_{\tau\tau}(1, r_0) &\sim \frac{\pi^2(d-1)}{3d} r^{-3}.
\end{aligned}$$

Finally we obtain

$$r - r_0 = \frac{-(u-1)F_{u\tau}(1,r)}{F_{\tau\tau}(1,r_0)} + \mathcal{O}(r^2 + r^{-1}|r-r_0|^2)$$

This gives the asymptotic formula in the statement of the lemma since r is asymptotically equal to r_0. □

We deduce that

$$\begin{aligned} n(r-r_0) + F(1,r) - F(1,r_0) &= F_{\tau\tau}(1,r_0)\frac{(r-r_0)^2}{2} + \mathcal{O}(d^{-3/2}(\log d)^3\sqrt{r}) \\ &= \frac{3d(\log d)^2}{\pi^2(d-1)} \times \frac{x^2}{2r_0\sigma_{d,n}^2} + o(1) \end{aligned}$$

and so the exponent the right hand side of (4.3.14) can be estimated as follows:

$$\begin{aligned} n(r-r_0) + F(u,r) - F(1,r_0) =& \frac{\log d}{r}\frac{x}{\sigma_{d,n}} \\ &+ \Big(\frac{d-1}{2} + \frac{3d(\log d)^2}{\pi^2(d-1)}\Big)\frac{x^2}{2r_0\sigma_{d,n}^2} + o(1). \end{aligned}$$

Hence, by using the estimates for $\mu_{d,n}$ and $\sigma_{d,n}^2$ in equations (4.3.5) and (4.3.6) respectively (note that the r in the formula is now r_0), we have, for a fixed real number x,

$$\begin{aligned} M_{d,n}(x) &= \mathbb{E}\Big(e^{\frac{x(\varpi_{d,n}-\mu_{d,n})}{\sigma_{d,n}}}\Big) \\ &= e^{-x\mu_{d,n}/\sigma_{d,n}}\frac{Q_{d,n}(e^{x/\sigma_{d,n}})}{Q_{d,n}(1)} \\ &= \exp\Big(\frac{-x}{\sigma_{d,n}}\Big(\frac{1}{r_0} - \frac{1}{r}\Big)\log d \\ &+ \Big(\frac{d-1}{2} + \frac{3d(\log d)^2}{\pi^2(d-1)}\Big)\frac{x^2}{2r_0\sigma_{d,n}^2} + o(1)\Big) \\ &= \exp\Big(\frac{x^2}{2r_0\sigma_{d,n}^2}\Big(\frac{d-1}{2} - \frac{3d(\log d)^2}{\pi^2(d-1)}\Big) + o(1)\Big) \\ &= e^{\frac{x^2}{2}}(1+o(1)) \end{aligned}$$

as $n \to \infty$. This and Curtiss's theorem in [5] prove that if $d = o(n^{1/2})$ then we have convergence in law to the Gaussian distribution. That completes the proof of our main theorem.

Appendices

Appendix A

Asymptotic behaviour of a distribution function in the prime partition problem

In Chapter 2 we mentioned a result by Haselgrove and Temperley [13] that the distribution of the number of summands in a random prime partition converges to some distribution whose moment generating function is closely related to the function:

$$\varphi(s) = \prod_p (1 - s/p)^{-1} e^{-s/p}.$$

In the same paper Haselgrove and Temperley mentioned, without proof, a tail estimate of the limiting distribution. So the purpose the analysis in this appendix is to prove their claim. For that we need to consider the logarithm

$$\Psi(s) = \log \varphi(s) = \sum_p \left(-\log(1 - s/p) - s/p\right),$$

and the inverse Laplace transform

$$f(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \varphi(s) e^{-sx} ds$$

where c is a real number between 0 and 1. Note that the function f is the density function of a random variable whose moment generating function is φ.

The following lemma shows the exponential decay of the function $\varphi(s)$ along a vertical line:

Lemma A.0.6. *For any fixed real number a, there is a constant $c_1 > 0$ such that*

$$|\varphi(a + it)| \ll e^{-c_1 |t|^{1-\epsilon}}$$

as $|t| \to \infty$, for any $\epsilon > 0$. Furthermore, this estimate is uniform if a is contained in a bounded closed interval.

Proof. First, let us consider $a = 0$, then

$$\begin{aligned}\mathrm{Re}(\Psi(it)) &= -\frac{1}{2}\sum_{p}\log(1+t^2/p^2)\\ &\leq -\frac{1}{2}\sum_{p\leq|t|}\log(1+t^2/p^2)\\ &\leq -\frac{C|t|}{\log|t|}\end{aligned}$$

For some constant $C > 0$. Now if $a \neq 0$, then

$$\mathrm{Re}(\Psi(a+it)) = -\frac{1}{2}\sum_{p\neq a}\log(1+t^2/(p-a)^2) + g(t,a) \tag{A.0.1}$$

where

$$g(t,a) = \sum_{p\neq a} -\log|1-a/p| - a/p + (-\log|t/a| - 1)\mu(a),$$

and $\mu(a) = 1$ if a is a prime and zero otherwise. Then we can see that the first term on the right hand side of (A.0.1) is the dominant term and it is of order at least $|t|/(\log|t|)$ as $t \to \infty$. □

The next lemma provides an asymptotic formula for $\Psi(-r)$ as $r \to +\infty$ along the real axis.

Lemma A.0.7. *There are two absolute positive constants c_1 and c_2 such that*

$$c_1\frac{r}{\log r} \leq r\sum_{p}\left(\frac{1}{p} - \frac{1}{p+r}\right) - \Psi(-r) \leq c_2\frac{r}{\log r}$$

for sufficiently large $r > 0$. Furthermore

$$\sum_{p}\left(\frac{1}{p} - \frac{1}{p+r}\right) = \log\log r + B_1 + o(1)$$

as $r \to \infty$, where B_1 is Mertens's constant.

Proof. Let us prove the second estimate first. We have

$$\begin{aligned}\sum_{p}\left(\frac{1}{p} - \frac{1}{p+r}\right) &= \sum_{p\leq r\sqrt{\log r}}\frac{1}{p} - \sum_{p\leq r\sqrt{\log r}}\frac{1}{p+r}\\ &\quad - \sum_{p> r\sqrt{\log r}}\frac{1}{p}\left(\frac{1}{1+r/p} - 1\right).\end{aligned}$$

The second sum on the right hand side is a $o(1)$. For the third sum, the absolute value of the term in brackets is less than r/p so the sum is a $o(1)$. The asymptotic formula for the first sum is a well known sum of the reciprocals of the primes, and proves the second estimate. For the first inequality, let p_n denote the nth prime number and consider the sequence of functions

$$u_n(r) = \log\left(1 + \frac{r}{p_n}\right) - \frac{r}{p_n + r}.$$

Then the function that we want to estimate is

$$\sum_{n\geq 1} u_n(r) = \sum_{n\geq 1} n(u_n(r) - u_{n+1}(r)) \tag{A.0.2}$$

$$= r^2 \sum_{n\geq 1} \int_{p_n}^{p_{n+1}} \frac{n}{t(t+r)^2} dt \tag{A.0.3}$$

$$= r^2 \int_0^\infty \frac{\pi(t)}{t(t+r)^2} dt \tag{A.0.4}$$

where $\pi(t)$ is the prime-counting function. To investigate the last integral, we proceed as follows: by the prime number theorem, for $\epsilon > 0$ there exits some $t_\epsilon > 2$ such that for $t > t_\epsilon$ we have

$$(1-\epsilon)t/\log t \leq \pi(t) \leq (1+\epsilon)t/\log t \tag{A.0.5}$$

Then we split the integral

$$\int_0^\infty \frac{\pi(t)}{t(t+r)^2} dt = \int_0^{t_\epsilon} \frac{\pi(t)}{t(t+r)^2} dt + \int_{t_\epsilon}^\infty \frac{\pi(t)}{t(t+r)^2} dt.$$

The first integral is a $\mathcal{O}(r^{-2})$. For the second integral, we can bound $\pi(t)$ from above and below as is (A.0.5), and make the change of variable $t = ru$. Now we need to estimate the integral

$$\int_{t_\epsilon}^\infty \frac{1}{(t+r)^2 \log t} dt = \frac{1}{r} \int_{\frac{t_\epsilon}{r}}^\infty \frac{1}{(1+u)^2 \log(ru)} du.$$

We can also split the integral on the right hand side:

$$\int_{\frac{t_\epsilon}{r}}^\infty \frac{1}{(1+u)^2 \log(ru)} du = \int_{\frac{t_\epsilon}{r}}^1 \frac{1}{(1+u)^2 \log(ru)} du + \int_1^\infty \frac{1}{(1+u)^2 \log(ru)} du. \tag{A.0.6}$$

Now we claim that for fixed ϵ,

$$\int_{\frac{t_\epsilon}{r}}^1 \frac{1}{(1+u)^2 \log(ru)} du \sim \frac{1}{2\log r} \tag{A.0.7}$$

as $r \to \infty$. Indeed

$$\int_{\frac{t_\epsilon}{r}}^{1} \frac{1}{(1+u)^2 \log(ru)} du \geq \frac{1}{\log r} \int_{\frac{t_\epsilon}{r}}^{1} \frac{1}{(1+u)^2} du = \frac{1}{2\log r} + \mathcal{O}((r\log r)^{-1}).$$

For the lower bound, we take an arbitrary ϵ_1 between 0 and 1 then

$$\int_{\frac{t_\epsilon}{r}}^{1} \frac{1}{(1+u)^2 \log(ru)} du \leq \int_{\frac{t_\epsilon}{r^{\epsilon_1}}}^{1} \frac{1}{(1+u)^2 \log(ru)} du + \int_{\frac{t_\epsilon}{r}}^{\frac{t_\epsilon}{r^{\epsilon_1}}} \frac{1}{(1+u)^2 \log(ru)} du$$
$$\leq \frac{1}{2(1-\epsilon_1)\log r + \log t_\epsilon} + \mathcal{O}(r^{-\epsilon_1})$$

which proves the claim in (A.0.7).

For the second integral on the right hand side of (A.0.6), we take an arbitrary δ between 0 and 1, then we have

$$\int_{1}^{r^\delta} \frac{1}{(1+u)^2 \log(ru)} du \leq \int_{1}^{\infty} \frac{1}{(1+u)^2 \log(ru)} du \leq \frac{1}{\log r} \int_{1}^{\infty} \frac{1}{(1+u)^2} du,$$

the upper bound is asymptotically equal to $(2\log r)^{-1}$, and also we have for the lower bound

$$\int_{1}^{r^\delta} \frac{1}{(1+u)^2 \log(ru)} du \geq \frac{1}{(1+\delta)\log r} \int_{1}^{r^\delta} \frac{1}{(1+u)^2} du,$$

which completes the proof. □

Note that Lemma A.0.7 implies the following asymptotic formula:

$$\Psi(-r) = r\log\log r + B_1 r + o(r) \qquad \text{(A.0.8)}$$

as $r \to +\infty$. This means that $\varphi(-r)$ grows very rapidly in the same way as $r \to +\infty$. We can also estimate the behaviour of $\varphi(r)$ when r is not too close to a prime, that is if there is a constant $a > 0$ such that $|r - p| > a$ for all primes p. Note that

$$\varphi(r)\varphi(-r) = \prod_p (1 - r^2/p^2)^{-1} \qquad \text{(A.0.9)}$$

and that

$$\sum_p \log|1 - r^2/p^2| = \sum_{p \leq 2r} \log|1 - r^2/p^2| + \sum_{p > 2r} \log|1 - r^2/p^2|.$$

The first sum on the right hand side is a $\mathcal{O}(r)$, since the maximum value of the summands corresponds to either p small or p close to r, but with our assumption on r, we know that $|\log|1-r^2/p^2||$ is at most $\mathcal{O}(\log r)$, for all p in that range. For the second sum i.e., $r > 2p$, since the function $|\log(1-t)|$ is a convex function for $0 < t < 1$, and since $r^2/p^2 < 1/4$, we have

$$|\log(1-r^2/p^2)| \ll r^2/p^2$$

uniformly for $p > 2r$. Thus

$$\Big|\sum_{p>2r} \log|1-r^2/p^2|\Big| \ll r^2 \sum_{p>2r} \frac{1}{p^2} = \mathcal{O}(r).$$

Therefore, from equations (A.0.9) and (A.0.8) we have

$$\Psi(r) = -r\log\log r + \mathcal{O}(r)$$

as $r \to +\infty$, which implies that $\varphi(r)$ goes rapidly to 0, asymptotically like $(\log r)^{-r}$.

Now we are going to look at the asymptotic behaviour of $f(x)$ for large x.

Lemma A.0.8. *The following equality is true for any $x > 0$:*

$$f(x) = -\sum_{p} \alpha_p e^{-px} \tag{A.0.10}$$

where α_p is the residue of $\varphi(s)$ at $s = p$.

Proof. We basically shift the path of integration in the definition of f to the right to prove the this result. More precisely, by the Cauchy residue theorem we have

$$f(x) - \frac{1}{2\pi i}\int_{2k-i\infty}^{2k+i\infty} \varphi(s)e^{-sx}ds = -\sum_{p<2k} \alpha_p e^{-px} \tag{A.0.11}$$

for any positive integer $k > 1$. Since the series on the right hand side of (A.0.10) is convergent for any x, we need to prove that the second term on the left hand side of (A.0.11) tends to zero as k tends to infinity. As in the proof of Lemma A.0.6 we can show

$$|\varphi(2k+it)| \le e^{-c(\pi(2k+|t|)-\pi(2k-|t|))}|\varphi(2k)|$$

for some constant $c > 0$. Again by the prime number theorem (to bound $\pi(2k+|t|) - \pi(2k-|t|)$ from below) we obtain

$$\begin{aligned}\Big|\int_{2k-i\infty}^{2k+i\infty} \varphi(s)e^{-sx}ds\Big| &\ll e^{-2kx}|\varphi(2k)| \int_{-\infty}^{\infty} e^{-c(\pi(2k+|t|)-\pi(2k-|t|))}dt \\ &\ll k^M e^{-2kx}|\varphi(2k)|\end{aligned}$$

where M is a constant, since the integral on the right hand side of the first line contributes at most a polynomial in k. Hence, we obtain the required equality by letting $k \to \infty$. $\square$

In particular, $f(x)$ behaves like e^{-2x} as $x \to +\infty$. Now we look at the behaviour of $f(x)$ for negative x, by the change of variable $s = -r(1-it)$ we have

$$f(x) = \frac{r}{2\pi}\int_{-\infty}^{+\infty} e^{\Psi(-r(1-it))+r(1-it)x}dt$$

for any $r > 0$. From this formula we will be able to prove the following:

Proposition A.0.9. *For $x < 0$ we have the asymptotic formula*

$$f(x) = \frac{e^{xr+\Psi(-r)}}{\sqrt{2\pi\Psi''(-r)}}\left(1 + \mathcal{O}(r^{-2/7})\right),$$

where $r = r(x) > 0$ is the unique solution of the equation

$$x = \Psi'(-r) = \sum_p \left(\frac{1}{p+r} - \frac{1}{p}\right).$$

Proof. Let $r > 0$ be as it is defined in the above statement. Note that $r = r(x)$ tends to infinity as $x \to -\infty$. Also let $t_0 = r^{-\beta}$ where $1/3 < \beta < 1/2$. Now we can split the integral: first, let $-t_0 < t < t_0$. In this range we have the expansion

$$\Psi(-r(1-it)) = \Psi(-r) + ri\Psi'(-r)t - r^2\Psi''(-r)\frac{t^2}{2} + \mathcal{O}(r^3|\Psi'''(-r)|t_0^3),$$

so

$$\Psi(-r(1-it)) + r(1-it)x = \Psi(-r) + rx - r^2\Psi''(-r)\frac{t^2}{2} + \mathcal{O}(r^3|\Psi'''(-r)|t_0^3).$$

We now investigate the behaviour of the harmonic sums that occur in the derivatives of Ψ. For the second derivative , we only need a lower bound

$$\begin{aligned}\Psi''(-r) &= \sum_p \frac{1}{(p+r)^2}\\ &= 2\int_0^{\infty}\frac{\pi(t)}{(t+r)^3}dt.\end{aligned}$$

Then, just as we did in proof of Lemma A.0.7 we get the following lower bound

$$\Psi''(-r) \gg \frac{1}{r\log r}.$$

We can do the same thing for the third derivative but this time we need an upper bound. We have

$$\begin{aligned}\Psi'''(-r) &= \sum_p \frac{2}{(p+r)^3}\\ &= 6\int_0^{\infty}\frac{\pi(t)}{(t+r)^4}dt,\end{aligned}$$

and one can show that it is a $\mathcal{O}(1/(r^2 \log r))$. We deduce that

$$\int_{-r^{-\beta}}^{r^{-\beta}} e^{\Psi(-r(1-it))+r(1-it)x} dt = \frac{\sqrt{2\pi} e^{\Psi(-r)+rx}}{\sqrt{r^2 \Psi''(-r)}} \left(1 + \mathcal{O}(r^{1-3\beta})\right).$$

For the tails, note that if $|t| \geq r^{-\beta}$ then

$$\frac{|\varphi(-r(1-it))|}{|\varphi(-r)|} = \exp\left(-\frac{1}{2}\sum_p \log(1 + (rt)^2/(p+r)^2)\right),$$

by the same calculation as in the proof of Lemma A.0.6. Therefore,

$$\frac{|\varphi(-r(1-it))|}{|\varphi(-r)|} \leq \exp\left(-\frac{1}{2}\sum_p \log(1 + r^{2-2\beta}/(p+r)^2)\right) \ll \exp(-c_1 r^{1-2\beta-\epsilon})$$

for some constant $c_1 > 0$ and for any $\epsilon > 0$. This proves that the tails go to zero more quickly than any power of r^{-1}. □

As a corollary we deduce the following estimate for $f(x)$ for negative x.

Corollary A.0.10. *We have*

$$f(x) = e^{-e^{e^{-(x+B_1+o(1))}}}$$

as $x \to -\infty$.

Proof. Let us assume the notation we used in the proof of Proposition A.0.9. So we have

$$\Psi(-r) + xr = \sum_p \left(-\log\left(1 + \frac{r}{p}\right) + \frac{r}{p+r}\right).$$

By the calculation we have already done in the proof of Lemma A.0.7, we get

$$\Psi(-r) + xr = -r^2 \int_0^\infty \frac{\pi(t)}{t(t+r)^2} dt = -\Theta\left(\frac{r}{\log r}\right).$$

Furthermore, by Lemma A.0.7 we have

$$x = -\log\log(r) - B_1 + o(1).$$

Hence,

$$r = e^{e^{-(x+B_1+o(1))}},$$

which completes the proof. □

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