Applications of change of numéraire for option pricing

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Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and has not previously in its entirety or in part been submitted at any university for a degree.

Signature: .... Date: ....

Abstract

The word **numéraire** refers to the unit of measurement used to value a portfolio of assets. The **change of numéraire technique** involves converting from one measurement to another. The foreign exchange markets are natural settings for interpreting this technique (but are by no means the only examples).

This dissertation includes elementary facts about the change of numeraire technique. It also discusses the mathematical soundness of the technique in the abstract setting of Delbaen and Schachermayer's Mathematics of Arbitrage. The technique is then applied to financial pricing problems. The right choice of numéraire could be an elegant approach to solving a pricing problem or could simplify computation and modelling.

Opsomming

Die woord **numéraire** verwys na die eenheidwaarde waarin finansiële portefeuljes gemeet word. Die **wysiging van numéraire metode** verander die eenheidwaarde na 'n gepaste eenheid vir 'n bepaalde probleem. 'n Natuurlike interpretasie vir die metode is op die buitelandse valuta markte (dit is egter geensins die enigste interpretasie nie).

Hierdie tesis bevat elementêre feite oor die wysiging van numéraire metode. Daar is ook 'n bespreking oor die wiskundige standvastigheid in die abstrakte milieu van Delbaen en Schachermayer se Arbitrage Wiskunde. Die metode word dan toegepas om die prys van sekere finansiële instrumente te bepaal. Die regte keuse van numéraire kan 'n elegante benadering vir probleemoplossing wees, of dit kan berekeninge en modelopstelling vereenvoudig.

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Contents

Notation

Financial instruments and dynamics

t	-	time
T	-	expiry date
$\mathbb{T}$	-	trading dates
$\Upsilon(t, H)$	-	time t price of claim H
$S(t)$	-	spot price of one unit of the underlying stock at time t
$\overline{S}(t)$	-	discounted value of the underlying stock at time t
$V(t)$	-	value of a portfolio at time t of a number of underlying stocks
$\overline{V}(t)$	-	discounted value with respect to a certain numéraire of a portfolio at time t
$E(t)$	-	spot domestic currency price of a unit of foreign exchange at time t
K	-	strike price
K^*	-	foreign currency strike price of an option on domestic currency
$B(t, T), B_T(t)$	-	(domestic currency) price of a pure discount bond at time t which pays one unit (of domestic currency) at time T
$B^*(t, T), B_T^*(t)$	-	foreign currency price of a pure discount bond at time t which pays one unit of foreign exchange at time T
$F_S(t, T)$	-	forward price at time t, matures at time T, for the underlying stock S
$F_{EB^*}(t, T)$	-	forward domestic currency price of a unit of foreign exchange at time t which matures at time T
$G_S(t, T)$	-	futures price at time t, matures at time T, for the underlying stock S
C_t	-	price of an European Call at time t
C_t^*	-	foreign currency price at time t of an European Call option written on one unit of domestic currency
c_t	-	price of an American Call at time t
c_t^*	-	foreign currency price at time t of an American Call option written on one unit of domestic currency
P_t	-	price of an European Put at time t
P_t^*	-	foreign currency price at time t of an European Put option written on one unit of domestic currency
p_t	-	price of an American Put at time t
p_t^*	-	foreign currency price at time t of an American Put option written on one unit of domestic currency
r	-	domestic riskless interest rate
r^*	-	foreign riskless interest rate
δ	-	dividend rate
μ	-	drift
σ	-	volatility

Probability and Stochastic processes

$\equiv$	-	Defined as
$\sim$	-	Equivalent to
a.s.	-	Almost surely
$(\Omega, \mathcal{F}, \mathbb{P})$	-	Complete Probability Space
$\mathbb{P}$	-	Probability measure (usually the market measure)
$\mathbb{Q}$	-	Probability measure (usually the equivalent martingale measure)
$\sigma(\mathcal{A})$	-	Smallest σ-algebra containing $\mathcal{A}$
$\mathbb{T}$	-	Index time set, usually representing intervals $[0,T]$, $\mathbb{R}_+$ or $\overline{\mathbb{R}}_+$
$\mathbb{R}_+ \equiv [0,\infty)$	-	Set of all non-negative real numbers
$\overline{\mathbb{R}}_+ \equiv [0,\infty]$	-	Set of all non-negative real numbers with infinity
$\mathcal{B}(I)$	-	Borel σ-algebra of I
$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$	-	Filtered Complete Probability Space
$\mathbb{F} \equiv (\mathcal{F}_t)_{0\leq t\leq\infty}$	-	Filtration
M_t, N_t	-	Martingale processes
$L^p(\Omega, \mathcal{F}, \mathbb{Q})$	-	L^p space of $(\Omega, \mathcal{F})$ with respect to measure $\mathbb{Q}$
C_t	-	Poisson process
$Po(\lambda)$	-	Poisson distribution with mean λ
B_t, W_t	-	Brownian motion or Wiener process
$\mathcal{N}(\mu, \sigma^2)$	-	Normal distribution with mean μ and variance σ^2
$M^\tau \equiv \{M_{\tau\wedge t}\}$	-	Stopped process at stopping time τ
$m\mathcal{F}$	-	Space of $\mathcal{F}$-measurable random variables
1_A	-	Indicator function on the set A
Leb	-	Lebesgue measure on $\mathbb{R}$
$\langle M, N\rangle_t$	-	Predictable mutual variation process
$\langle M\rangle_t$	-	Predictable quadratic variation process
$[M]_t$	-	Optional quadratic variation process
$[M,N]_t$	-	Optional mutual variation process
$\mathcal{C}^n$	-	Space of functions with continuous n^{th} derivative
$\mathcal{C}^{n,m}$	-	Space of functions in two variables with continuous n^{th} partial derivative in the first and continuous m^{th} partial derivative in the second
$\mathcal{K}_0[S]$ $(\mathcal{K}[S])$	-	All (bounded) claims that can be hedged by admissible strategies with initial value zero in the market S
$\mathcal{C}_0[S]$ $(\mathcal{C}[S])$	-	All (bounded) claims that can be superhedged by admissible strategies with initial value zero in the market S
$(f)_- \equiv \max(0, -f)$	-	Negative part of a function
$\mathcal{M}^e(S)$	-	Set of all equivalent (local) martingale measures for the market S
$\mathbf{e}$	-	The $\mathbb{R}^n$-valued vector of the form $[1,0,...,0]'$
$\Phi(\cdot)$	-	Cumulative distribution of a $\mathcal{N}(0,1)$-distributed random variable

Chapter 0

Background

0.1 Financial background

The genesis of financial market instruments are shares kept in a company. One invests in a company by means of some contribution and receives say in the company's future. Shareholders may also receive dividends as a reflection of their 'share' in the company's profits. Shares (also known as stocks) are *tradeable*. Financial markets like the stock exchange trade stocks and prices are influenced by supply and demand. Stocks are high risk investments. Investors don't have the *certainty* of a definite growth. A poor investment could lead to just that - great losses.

Why invest in stocks?

The higher the risk, the more handsome the *probable* reward. The growth rates of low risk securities like bonds or merely leaving money in the bank are outdone by the growth of stocks. Another strong motive for making an investment in stocks is to find some sort of growth that can beat the 'norm'. The 'norm' definition might mean to beat inflation or have a greater growth than the average growth of shares in the market. This all depends on the investor's intentions.

Derivatives

Another type of security, which one can describe as being on a different 'level', is a derivative. A derivative is a contract that promises some payment or delivery in the future and its value is determined by some underlying variables. These variables may be very risky like stocks, or physical items or assets like machinery, natural resources or harvests. All we require is that the underlying are *tradeable* and there is a way of attaching a market price to the underlying.

There are also different 'levels' of derivatives. Derivatives with underlying like in the above could be called *level 1* derivatives. *Level n* derivatives depend on underlying (*level 0*) and/or any of the 'lower level' derivatives and a *level n-1* derivative. Most of the cases we will consider here will be *level 1* derivatives.

Types of derivatives

There are plenty of derivatives. Some have distinct characteristics. Derivatives are called *European* if the derivative can only be exercised at the end of expiry (also called maturity) and *American* if the derivative can be exercised at any time up to expiry. As a mixture of the two, *Bermudan* derivatives are derivatives that may be exercised at certain specific dates prior to expiration. The origins for the names are unknown, except that Bermuda lies between America and Europe. The *terminal value* is the value of the derivative at the time exercised or expiration. A derivative is *path dependent* if its terminal value depends upon the value of the underlying, not only at that time, but also at prior points in time. In practice the simple and more common type of derivative is classified as *vanilla* and the complicated and more specialized derivative as *exotic*.

This classification is not precise and vary in practice and in literature.
Why use derivatives?
This type of financial instrument gives us some *assurance* about a future event. Thus in a world of highly volatile securities we could lower our risk using derivatives. The fundamental question of mathematics of financial markets now arises : what is a fair price for this *assurance*? It is through probability that we try to answer this question. In the sequel we discuss some basic derivatives:

Forward Contracts - An agreement to buy (or sell) an asset on a specified date, T, for a specified price, K. The buyer is said to hold the long position and the seller the short position.

Futures Contracts - An agreement to trade an asset at a future time, T, at a certain price, but trading takes place on an exchange and is subject to regulation.

Swap Contract - An agreement between two counter-parties to exchange two assets, liabilities or streams of cash flows at a specific date or over a certain time period. Here are some examples:

- **Interest Rate Swap** - Exchanging of two streams of cash flows, both with the same currency and both with payment obligations (interest rates). A *fixed-for-floating* interest rate swap exchange a cash flow with a fixed interest rate for that of a cash flow with a floating interest rate. *Fixed-for-fixed* and *floating-for-floating* are defined similarly.
- **Currency Swap** - The same as an interest rate swap except that the two cash flows are of different currency.

Option Contract - The contract gives the owner the right, but not the obligation, to trade a given number of securities for a fixed price at a future date (having expiry date T). Here are some examples:

- **European Call Option** - The contract gives the owner the right, but not the obligation, to buy an asset at a specified time, T, for a specified price, K.
- **European Put Option** - The contract gives the owner the right, but not the obligation, to sell an asset at a specified time, T, for a specified price, K.
- **American Call Option** - The contract gives the owner the right, but not the obligation, to buy an asset at any time up to the expiry date, T, for a specified price, K.
- **American Put Option** - The contract gives the owner the right, but not the obligation, to sell an asset at any time up to the expiry date, T, for a specified price, K.
- **Binary Option/Digital Option** - There are two types: asset-or-nothing and cash-or-nothing. With an asset-or-nothing binary option the owner receives, at the time he/she exercise the option, the asset if its value is more than that of the strike price and nothing otherwise. For a cash-or-nothing binary option the owner receives a cash payout for the asset if its value is below that of the strike price.
- **Barrier Option** - This is a path dependent option and two features exist. The *knockout* feature causes the option to terminate or be exercised when the underlying reaches a certain barrier level. The *knock-in* feature causes the option to become effective only if the underlying reaches a certain barrier level.
- **Option on an Option** - This is an option, like the above, but the underlying asset is also an option. This is an example of a *level 2* type derivative.
- **Swaption** - This is an option with the underlying asset a swap. This is also an example of a *level 2* type derivative.

0.2 Mathematical background

In this section we describe some of the tools needed in the mathematics of finance. $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space. This serves as the "factory" in which we produce our mathematical "tools".

0.2.1 Stochastic processes

Let $\sigma(\mathcal{C})$, where $\mathcal{C}$ is a subset of the power set of Ω, denote the smallest σ-algebra that contains $\mathcal{C}$. We let $\mathbb{T}$ be some interval in $\mathbb{R}_+$. $\mathcal{B}(\mathbb{T})$ denotes the σ-algebra of Borel subsets of $\mathbb{T}$.

Definition. A *filtration* $\mathbb{F}$ is a family of sub-σ-algebras $(\mathcal{F}_t)_{0\leq t\leq\infty}$ of $\mathcal{F}$ that is increasing, i.e. $\mathcal{F}_s \subset \mathcal{F}_t$ if $s < t$.

Definition. A *stochastic process* X is a function $X : \mathbb{T}\times\Omega \to \mathbb{R}$ such that $X(t,\cdot)$ is $\mathcal{F}$-measurable for all $t \in \mathbb{T}$. A stochastic process is called *measurable* if X is $(\mathcal{B}(\mathbb{T})\times\mathcal{F})$-measurable. A stochastic process is said to be *adapted to a filtration* $\mathbb{F}$ if $X_t \in \mathcal{F}_t$ (that is, X_t is $\mathcal{F}_t$ measurable) for each t.

A stochastic process may also be described as $X : \Omega \to \mathbb{R}^{\mathbb{T}}$. Thus every sample point of X results in a function on $\mathbb{T}$. This function is one of many possible *paths* (one for each $\omega \in \Omega$) for the process. A process X is said to be continuous if it has continuous paths, i.e. $t \mapsto X(t,\omega)$ is a continuous mapping on $\mathbb{T}$ for each $\omega \in \Omega$. In the same way we have right and left continuous processes. Adopted from the French we have that a process is *càdlàg* if it is right continuous with left limits and *càglàd* if it is left continuous with right limits. The left continuous process Y_{t-} is defined pathwise: $Y_{t-}(\omega) = \lim_{s\to t^-} Y_s(\omega)$, and likewise $Y_{t+}(\omega) = \lim_{s\to t^+} Y_s(\omega)$ for the right continuous process Y_{t+}.

Definition. Two processes X and Y are *indistinguishable* if $\mathbb{P}(X_t = Y_t, \forall t) = 1$. A process indistinguishable from the zero process $(X_t(\omega) = 0, \forall\omega, t)$ is called *evanescent*. A process Z is said to be a *version* or *modification* of W if $\mathbb{P}(Z_t = W_t) = 1,\ \forall t$.

If two stochastic processes, X and Y, are modifications of each other and both have right continuous paths a.s., then X and Y are indistinguishable. From this we have that two càdlàg processes which are versions of each other are in fact indistinguishable from each other.

Throughout we assume $\mathbb{F}$ satisfies the 'usual conditions' :
(a) *completeness*: Every $\mathbb{P}$-null set in $\mathcal{F}$ belongs to $\mathcal{F}_0$ and thus to each $\mathcal{F}_t$.
(b) *right-continuity*: $\mathcal{F}_t = \mathcal{F}_{t^+} \equiv \bigcap_{s>t}\mathcal{F}_s\ \forall t$.

Likewise $\mathcal{F}_{t^-} \equiv \sigma(\bigcup_{s<t}\mathcal{F}_s)$.
A filtration which satisfies the 'usual conditions' is called a standard filtration.
The natural filtration of a process X, say $\mathbb{F}$, is constructed in the following manner:
(a) for all t we define $\mathcal{H}_t = \sigma(X_u; u \leq t)$, where $\{X_u; u \leq t\}$ denote the union of the inverse mappings with respect to X_u for all u up to t. Specifically $\mathcal{H}_\infty = \sigma(\bigcup_{0\leq t}\mathcal{H}_t) = \sigma(X_u; 0 \leq u)$.
(b) $\mathcal{G}_t = \bigcap_{s>t}\mathcal{H}_s$ and $\mathcal{G}_\infty = \mathcal{H}_\infty$.
(c) $\mathcal{Z} = \{A \mid \exists B \in \mathcal{G}_\infty, A \subset B \text{ and } \mathbb{P}(B) = 0\}$.
(d) $\mathcal{F}_t = \sigma(\mathcal{G}_t \cup \mathcal{Z})\ \forall t$.
Condition (a) provides that X is adapted to $\mathbb{F}$. Conditions (b) and (c) are there to ensure that the natural filtration is right-continuous and complete.

Definition. A process $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is *progressively measurable* if for each t the restriction of X to $[0,t] \times \Omega$ is $(\mathcal{B}([0,t]) \times \mathcal{F}_t)$-measurable.

If a process is progressively measurable it implies that it is measurable and adapted.

Definition. For $p \geq 1$, an adapted process X is called a [closed] *L^p-martingale* (resp. *supermartingale*, *submartingale*) with respect to the filtration $\mathbb{F}$ if, for all $t \in \mathbb{R}_+$ $[t \in \overline{\mathbb{R}}_+]$,
(i) $X_t \in L^p(\mathbb{P})$; that is, $\mathbb{E}[|X_t|^p] < \infty$
(ii) if $s \leq t$, then $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$, a.s. (resp. $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$ and $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$).

If

$$\sup_{t\in\mathbb{R}_+} \mathbb{E}\left(|X_t|^p\right) < \infty, \tag{1}$$

then X is moreover called *L^p-bounded.* An L^1-martingale is also referred to as a *martingale.* Closed martingales include a final value. On a finite time interval $[0,T]$ all martingales are closed and we say martingale X is closed by the value X_T (or X_∞ in the infinite case).
Note that a process is dependent on the filtration (conditioning) as well as the probability measure (expectation) for it to be a martingale. We can denote this by saying X is a $(\mathbb{F},\mathbb{P})$-martingale.
The right-continuity assumption for a standard filtration ensures càdlàg versions for martingales and certain super-(sub-)martingales. The càdlàg versions are always assumed to be the martingales themselves and is like this throughout this text.

Definition. A stochastic process C is called a Poisson process with parameter λ if
(i) $C_0 = 0$
(ii) for $0 \leq s \leq t$, $C_t - C_s$ is $Po(\lambda(t-s))$-distributed
(iii) for $0 = t_0 < t_1 < ... < t_m$, we have that $\left\{C_{t_k} - C_{t_{k-1}} : k = 1, ..., m\right\}$ is a set of independent random variables.

Definition. A stochastic process B such that
(i) $B_0 = 0$ a.s.
(ii) almost all paths $t \to B_t(\omega)$ are continuous
(iii) for $0 \leq s \leq t$, $B_t - B_s$ is $\mathcal{N}(0, t-s)$-distributed
(iv) for $0 = t_0 < t_1 < ... < t_m$, we have that $\left\{B_{t_k} - B_{t_{k-1}} : k = 1, ..., m\right\}$ is a set of independent random variables
is called a (standard) *Brownian motion (BM).*

0.2.2 Stochastic calculus

Definition. A random variable $\tau : \Omega \to \overline{\mathbb{R}}_+$ is a *stopping time / optional time* with respect to a filtration $\mathbb{F}$ if $\{\tau \leq t\} \in \mathcal{F}_t$ for each $t \in \mathbb{R}_+$.

The associated σ-algebra for a stopping time τ is $\mathcal{F}_\tau$. It can be described as

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \quad \forall t \in \mathbb{T}\}. \tag{2}$$

From this we see that $\mathcal{F}_\tau$ is a σ-algebra and that $\tau \in \mathcal{F}_\tau$.

A sequence of stopping times $\{\tau_k : k \in \mathbb{N}\}$ is called a *localizing sequence* if $\tau_k \uparrow \infty$ a.s.. A process Y is said to retain a certain property *locally* if and only if there exist a localizing sequence such that for each k that property is satisfied for the stopped process $Y^{\tau_k} \equiv \{Y_{\tau_k \wedge t}\}$.

For instance the process $(Z_t)_{0\leq t\leq\infty}$ would be a local martingale if there exist a localizing sequence $(\tau_k)_{k\in\mathbb{N}}$ such that for each k the process $X_t = Z_{\tau_k\wedge t}$ is a martingale.

The integral $\int_0^t K_s ds$ produces a random variable Y on Ω with each sample point having value

$$Y(\omega) = \int_0^t K_s(\omega)ds \tag{3}$$

which is calculated using the normal Lebesgue integral.

When integrating with respect to a stochastic process the normal (Riemann and Lesbesgue) rules are not applicable anymore. This becomes evident when considering a Brownian motion. The total variation,

$$V(B(\cdot,\omega)) = \lim_{\delta(\pi_k)\to 0} \sum_{i=1}^{k} \left| B_{t_i^k}(\omega) - B_{t_{i-1}^k}(\omega) \right| \tag{4}$$

where $\pi_k = \left\{ t_i^k : i = 0, 1, ...k \right\}$ are partitions on $[0, T]$ and $\delta(\pi)$ is the mesh of the partition π, is a.s. infinite. So the paths of B are not of bounded variation. The integral can't be defined pathwise, but the *quadratic variation*

$$\lim_{\delta(\pi_k)\to 0} \sum_{i=1}^{k} \left| B_{t_i^k}(\omega) - B_{t_{i-1}^k}(\omega) \right|^2 \tag{5}$$

does exist if the limit is taken in the L^2-norm. This quantity equals T. This gives a certain hope to define the integral in the L^2-sense. But first some important definitions and constructions for the general stochastic integral.

Note: Throughout the rest of this section we assume that all processes are adapted to a *specific* filtration.

Definition. The *predictable σ-algebra* $\mathcal{P}$ is the σ-algebra on $(\mathbb{R}_+ \times \Omega)$ generated by all left continuous processes. Sets in $\mathcal{P}$ are called *predictable sets.* A process X is a *predictable process* if $X \in m\mathcal{P}$.

First a measure is constructed on $\mathcal{P}$. With $F \in \mathcal{F}_s$ and $s < t$ we have a (left continuous) predictable process $1_{(s,t]\times F}$ and thus a predictable set $((s, t] \times F)$. For M a right continuous L^2-martingale we define a set function for these sets as

$$\nu_M((s, t] \times F) = \mathbb{E}[1_F(M_t - M_s)^2] \quad \text{for} \quad F \in \mathcal{F}_s \quad \text{and} \quad s < t, \tag{6}$$

$$\nu_M(\{0\} \times F_0) = 0 \quad \text{for} \quad F_0 \in \mathcal{F}_0. \tag{7}$$

ν_M can be uniquely extended to a σ-finite measure on $\mathcal{P}$. This measure ν_M has been called the Doléans measure.

Definition. A process X is a *simple process* if it can be expressed as a finite linear combination of indicator functions of the above predictable sets.

A simple process thus has the form

$$X = c_0 1_{\{0\}\times F_0} + \sum_{j=1}^{n} c_j 1_{(s_j,t_j]\times F_j} \tag{8}$$

where $c_j \in \mathbb{R}$, $F_j \in \mathcal{F}_{s_j}$, $s_j < t_j$ for $0 \leq j \leq n$ and $F_0 \in \mathcal{F}_0$.

Secondly the stochastic integral is defined for simple processes. Let $\mathcal{E}$ denote the class of simple processes, then for X in $\mathcal{E}$ we define

$$\int XdM \equiv \sum_{j=1}^{n} c_j 1_{F_j}(M_{t_j} - M_{s_j}). \tag{9}$$

Thirdly the integrands are extended to the set of $L^2(\mathbb{R}_+ \times \Omega, \mathcal{P}, \nu_M)$. This is an easy limit extension when one knows that $\mathcal{E}$ is a dense set of $L^2(\mathbb{R}_+ \times \Omega, \mathcal{P}, \nu_M)$. If we let $\mathcal{I}_M$ denote the stochastic integral with respect to a right continuous L^2-martingale M, then it produces the following mapping

$$\mathcal{I}_M : L^2(\mathbb{R}_+ \times \Omega, \mathcal{P}, \nu_M) \to L^2(\Omega, \mathcal{F}, \mathbb{P}) : X \mapsto \int XdM.$$

Between these two spaces we have the isometry :

$$\mathbb{E}\left[\left(\int XdM\right)^2\right] = \int_{\mathbb{R}_+ \times \Omega} X^2 d\nu_M. \tag{10}$$

Remark: The class of integrands X can be stretched further under certain conditions for M, the integrator. If the Doléans measure is absolutely continuous with respect to $(Leb \times \mathbb{P})$ on $\mathcal{P}$, the space of integrands expands to $L^2(\mathbb{R}_+ \times \Omega, \mathcal{V}, \tilde{\nu}_M)$, where $\mathcal{V}$ is the σ-algebra generated by all adapted and measurable processes and $\tilde{\nu}_M$ is the measure extended to $\mathcal{V}$. If M is a continuous L^2-martingale we can have $L^2(\mathbb{R}_+ \times \Omega, \mathcal{M}, \overline{\nu}_M)$ as our space of integrands, where $\mathcal{M}$ is the σ-algebra generated by all progressively measurable processes and $\overline{\nu}_M$ is the measure extension to $\mathcal{M}$.
We can also include an integrator which is locally a right continuous L^2-martingale. The integral process $\int_{[0,t]} XdM$ inherits properties from the integrator M. If M is a right continuous local L^2-martingale then $\int_{[0,t]} XdM$ will also be a right continuous local L^2-martingale. The class of integrators expands to local martingales and finally to the most general class of integrators, *semimartingales*.

Definition. A *semimartingale* is a process S of the form $S_t = M_t + A_t$ where M is a local martingale and A is of local bounded variation. The stochastic integral with respect to a semimartingale is defined as

$$\int_0^t H_s dS_s = \int_0^t H_s dM_s + \int_0^t H_s dA_s. \tag{11}$$

This decomposition is not unique. (Uniqueness is taken here and in what follows in the sense of uniqueness up to evanescent sets.) The decomposition for the smaller class of *special semimartingales* is unique in this sense.

Definition. A *special semimartingale* is a process X of the form $X_t = M_t + A_t$ where M is a local martingale and A is a predictable process locally of bounded variation that starts at zero.

The time is ripe to look at specific types of bounded variation processes, known as the 'variation processes'. The first definition relates to the predictable processes which could be defined for the class of processes that are locally L^2-bounded martingales, but the class of continuous local martingales is sufficient for the purpose of this discussion.

Definition. The *predictable mutual variation process* $\langle M, N \rangle$ is defined for continuous local martingales M and N as a predictable process, locally of bounded variation and starting at zero, such that $MN - \langle M, N \rangle$ is a local martingale. $\langle M, M \rangle \equiv \langle M \rangle$ is defined as the *predictable quadratic variation process* of M.

Introducing special semimartingales gives a hint to the mutual variation process being unique. Trivially the quadratic variation process is also unique. Unfortunately the above definition is not adequate for the jump processes of general local martingales. To find a fitting 'quadratic variation process' it is important to first look at the decomposition of local martingales and their *orthogonality.*

Definition. Two local martingales M and N are *orthogonal* if their product MN is a local martingale that starts at zero.

It can be shown that for any local martingale M there exist a localizing sequence $\{\tau_k : k \in \mathbb{N}\}$ such that for each k,

$$M^{\tau_k} = M_0 + M^c + M^d + V \tag{12}$$

where M^c and M^d are two bounded orthogonal martingales with M^c continuous and V a process of bounded variation all starting at zero. M^d is a *purely discontinuous martingale* or called the *purely discontinuous part* of M. Likewise M^c is called the *continuous part* of M.

Definition. The *optional quadratic variation process* is defined for local martingales, M, as the following

$$[M, M] \equiv [M]_t = \langle M^c \rangle_t + \sum_{s \leq t} (\Delta M_s)^2 \tag{13}$$

where $\Delta M_s = M_s - M_{s-}$ is the jump at time s and define $\Delta M_0 = M_0$.

If M is continuous and starts at zero, then the predictable quadratic variation and the optional quadratic variation are the same, i.e. $\langle M \rangle = [M]$. $[M]$ is a unique increasing process of bounded variation such that $M^2 - [M]$ is a local martingale and $\Delta[M]_s = (\Delta M_s)^2$ for all $s \in \mathbb{R}_+$. The *optional mutual variation process* is defined via polarization and gives the pleasing result of

$$[M, N]_t = \langle M^c, N^c \rangle_t + \sum_{s \leq t} (\Delta M_s)(\Delta N_s). \tag{14}$$

The predictable and optional variation processes can be extended to also include processes of bounded variation in their definition. In general the optional quadratic variation can be defined for semimartingales.

Definition. The *optional quadratic variation* of a semimartingale S is

$$[S, S] \equiv [S]_t = \langle S^c \rangle_t + \sum_{u \leq t} (\Delta S_u)^2 \tag{15}$$

where S^c is the *continuous martingale part* of S.

Even though the decomposition of semimartingales is not unique, the quadratic variation of the continuous martingale part is independent of the choice of decomposition. From this it can be seen that continuous processes of bounded variation have constant quadratic variation. It can be shown that the quadratic sum for the semimartingale X

$$S_T^n(\omega) = \sum_{i=1}^{n} (X_{t_i^n}(\omega) - X_{t_{i-1}^n}(\omega))^2 \tag{16}$$

where $\pi_k = \left\{ t_i^k : i = 0, 1, ...k \right\}$ are partitions on $[0, T]$, converges in probability to $[X]_T$ as the mesh tends to zero.

Example. As an example we look at the Brownian motion B. The Brownian process is a continuous martingale and thus a local L^2-martingale. The integral $\int XdB$ with respect to Brownian motion is called the Itô integral. The Doléans measure gives us $\nu_B = Leb \times \mathbb{P}$. Applying this to the isometry and using Fubini's theorem we have the famous Itô isometry:

$$\mathbb{E}\left[\left(\int XdB\right)^2\right] = \mathbb{E}\left(\int_{\mathbb{R}_+} X_s^2 ds\right). \tag{17}$$

As we have said before $\langle B \rangle_t = t$. The next lemma provides a sufficient condition for integrability with respect to a Brownian motion.

Lemma. If Z a measurable adapted process such that for all t :

$$\int_0^t (Z_s)^2 ds < \infty \quad a.s. \tag{18}$$

then Z can be integrated with respect to Brownian motion.
An example of a special semimartingale involving Brownian motion is given by:

Definition. An *Itô process* is a stochastic process (X_t) of the form

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dB_s \tag{19}$$

where X_0 is $\mathcal{F}_0$ measurable, K and H are $\mathbb{F}$-adapted and measurable processes with $\int_0^t |K_s|\, ds$ and $\int_0^t H_s^2 ds$ finite a.s. for all t. The integrals $\int_0^t K_s ds$ and $\int_0^t H_s dB_s$ denote the Lebesgue integral with respect to s and Itô integral with respect to the Brownian motion process B respectively.

Remark: The processes K and H are unique a.s. $(Leb \times \mathbb{P})$. To see this, consider two possible decompositions

$$\begin{aligned} X_t &= X_0 + \int_0^t K_s ds + \int_0^t H_s dB_s & (20)\\ &= X_0' + \int_0^t K_s' ds + \int_0^t H_s' dB_s. & (21) \end{aligned}$$

X_0 and X_0' are starting values and must equal each other almost surely. Rearranging,

$$\int_0^t [K_s - K_s'] ds = \int_0^t [H_s' - H_s] dB_s \tag{22}$$

lets a continuous bounded variation process on the left equal a continuous martingale on the right. A continuous martingale of bounded variation is indistinguishable from a constant process and the integral starting from zero implies that it is evanescent. Thus $K = K'$ and $H = H'$ a.s. $(Leb \times \mathbb{P})$.

The next result is fundamental in stochastic calculus. It confirms that a wide range of functions of Itô processes are again Itô processes.

Theorem.(Itô's Formula) Let M be a continuous (local) martingale and V be a continuous process which is locally of bounded variation. Let $f \in \mathcal{C}^{2,1}$. Then a.s., we have for each t

$$f(M_t, V_t) - f(M_0, V_0) = \int_0^t f_x(M_s, V_s) dM_s + \int_0^t f_y(M_s, V_s) dV_s + \frac{1}{2}\int_0^t f_{xx}(M_s, V_s) d\langle M \rangle_s. \tag{23}$$

Corollary 1.(Itô's Formula) Let S be a continuous semimartingale and $g \in \mathcal{C}^2$. Then $g(S)$ is again a semimartingale and the following holds:

$$g(S_t) - g(S_0) = \int_0^t g'(S_u) dS_u + \frac{1}{2}\int_0^t g''(S_u) d\langle S \rangle_u . \tag{24}$$

Corollary 2.(Itô's Formula) Let W denote a Brownian motion and let $h \in \mathcal{C}^{2,1}$. Then a.s., we have for each t

$$h(W_t, t) - h(W_0, 0) = \int_0^t h_x(W_s, s) dW_s + \int_0^t h_y(W_s, s) ds + \frac{1}{2}\int_0^t h_{xx}(W_s, s) ds. \tag{25}$$

Theorem.(Multi-Dimensional Itô's Formula) With $m, n \in \mathbb{N}$, let $M_i(t)$ be a continuous local martingale for $1 \leq i \leq m$ and $V_k(t)$ be a continuous process locally of bounded variation for $1 \leq k \leq n$. Suppose that D is a domain in $\mathbb{R}^{m+n}$ such that a.s. $Z(t) = (M_1(t), ..., M_m(t), V_1(t), ..., V_n(t))$ takes values in D for all t. Let $f(\mathbf{x}, \mathbf{y})$ be a continuous real-valued function of $(\mathbf{x}, \mathbf{y}) \in D$ such that all first and second partial derivatives in $\mathbf{x}$ and all the first partial derivatives in $\mathbf{y}$ exist and are continuous in D. Then a.s. we have for all t:

$$\begin{aligned} f(Z(t)) - f(Z(0)) &= \sum_{i=1}^m \int_0^t \frac{\partial f}{\partial x_i}(Z(s)) dM_i(s) + \sum_{j=1}^n \int_0^t \frac{\partial f}{\partial y_j}(Z(s)) dV_j(s) \\ &+ \frac{1}{2}\sum_{k=1}^m \sum_{l=1}^m \int_0^t \frac{\partial^2 f}{\partial x_k \partial x_l}(Z(s)) d\langle M_k, M_l \rangle (s). \end{aligned} \tag{26}$$

Corollary.(Integration by parts) If we have $Z_t = (M_t, N_t)$ with M and N two continuous local martingales and let $f((x_1, x_2)) = x_1 x_2$ then we have with Itô's formula:

$$M_t N_t - M_0 N_0 = \int_0^t N_s dM_s + \int_0^t M_s dN_s + \int_0^t d\langle M, N \rangle_s \tag{27}$$

$$d(MN)_t = N_t dM_t + M_t dN_t + d\langle M, N \rangle_t . \tag{28}$$

The above Itô's Formula is only for 'neat' continuous processes. The next two theorems are given to allow for jump processes.

Theorem.(Itô's Formula) Let X be a semimartingale and let $f \in \mathcal{C}^2$. Then $f(X)$ is again a semimartingale and the following holds:

$$\begin{aligned} f(X_t) - f(X_0) &= \int_0^t f'(X_{s-}) dX_s + \frac{1}{2}\int_0^t f''(X_{s-}) d[X]_s \\ &+ \sum_{s \leq t} \left[f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s - \frac{1}{2} f''(X_{s-})(\Delta X_s)^2 \right]. \end{aligned} \tag{29}$$

Theorem.(Integration by parts) Let X and Y be two semimartingales. Then XY is a semimartingale and

$$X_t Y_t = \int_0^t Y_{s-} dX_s + \int_0^t X_{s-} dY_s + [X, Y]_t \tag{30}$$

$$d(XY)_t = Y_{t-} dX_t + X_{t-} dY_t + d[X, Y]_t. \tag{31}$$

For further and more in-depth discussion into stochastic calculus, consult [20],[13],[18] and [19]. The discrete parameter martingale world of D. Williams [23] is a good starting point for any novice. A possible sequel is the neat and clean continuous setting of Chung and Williams [4]. Note that the above references are a small part of a large literature.

0.3 Market assumptions and Model dynamics

What does our set-up look like? What are the motivations behind the assumptions? The set-up or financial market model $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{T}, S)$ is concerned with the following concepts:
Trading Dates $\mathbb{T}$
There are four cases to consider,

- $\mathbb{T}_1 = \{0, 1, ..., T\}$ finite discrete time
- $\mathbb{T}_2 = \mathbb{N} = \{0, 1, ...\}$ infinite discrete time
- $\mathbb{T}_3 = [0, T]$ finite horizon continuous time
- $\mathbb{T}_4 = \mathbb{R}_+$ infinite horizon continuous time

Uncertainty
The uncertainty is modeled via a filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ which is our universe of uncertainty. The filtration represents the evolution of available information over time. $\mathbb{P}$ is called the market measure.

Assets to be traded
In the markets there can only be finitely many assets, say $S = (S_0, S_1, ..., S_n)$. An asset traded by its price is a price process and is modeled as a measurable stochastic process. The asset vector $(S_0, S_1, ..., S_n)$ is adapted to the information based filtration, i.e. each price process is adapted to the filtration. Note that the filtration is not necessarily generated by the process S. This means that other sources (e.g. political policy; natural or social climate; steadfast law system) influence price movements. Asset number 0 describes 'cash'. The prices of all the other assets are in terms of this currency. To normalise we divide through by S_0. Hence we assume this 'cash account' or 'money account' is strictly positive almost surely over the trading time $\mathbb{T}$.

Trade procedures
Traders/agents all have access to the same filtration, i.e. information. No agent can research more information than any other or gain access to more information, be it in a legal or illegal way. Insider trading as an example is not possible. Out of a mathematical and economical perspective this is a good assumption, even though in practice this is something not all traders are committed to.
Agents may buy and sell assets and short selling is allowed. Assets may also be traded in fractions, i.e. sold or bought in unnatural quantities. There are no transaction costs. Even though these assumptions seem far removed from what happens in reality, we need them here so that our theory and model may be applied without 'friction'.

A portfolio V is expressed in terms of the underlying assets $S = (S_0, S_1, ..., S_n)$ and their quantities $\theta = (\theta_0, \theta_1, ..., \theta_n)$:

$$V(t) = (\theta(t) \cdot S(t)) \equiv \sum_{k=0}^{n} \theta_k(t) S_k(t) \quad \forall t \in \mathbb{T}, \tag{32}$$

where $(\cdot)$ denotes the Euclidean inner product. The quantities are predictable processes and the vector $\theta = (\theta_0, \theta_1, ..., \theta_n)$ is called the portfolio strategy or trading strategy.

Let us now first consider the discrete time setting, i.e. $\mathbb{T} = \mathbb{T}_1$. In this setting we use our economic intuition to strengthen our ideas and give clarity of what should be done, before moving on to the more difficult and abstract continuous time setting.

Definition. A trading strategy θ is *self-financing* if for $t = 1, ..., T-1$,

$$(\theta(t) \cdot S(t)) \equiv \sum_{k=0}^{n} \theta_k(t) S_k(t) = \sum_{k=0}^{n} \theta_k(t+1) S_k(t) \equiv (\theta(t+1) \cdot S(t)) \tag{33}$$

with initial investment $V(0) = (\theta(1) \cdot S(0))$.

A self-financing strategy is true to its name, there is no input or outflow of money when moving between the time steps. Wealth is only distributed between the assets. Note that for our discrete time setting the trading strategy is predictable, i.e. $\theta(t) \in m\mathcal{F}_{t-1}\ \forall t \in \mathbb{T}$. The reason for this is clear. Traders must decide on their move beforehand without knowledge of what the market will do in the future. Let us now investigate discounted prices. $\overline{S} = (1, S_1/S_0, ..., S_n/S_0) = \left(1, \overline{S}_1, ..., \overline{S}_n\right)$ will denote our discounted asset prices and the portfolio would be

$$\overline{V}(t) = \frac{V}{S_0}(t) = \theta_0(t) + \sum_{k=1}^{n} \theta_k(t) \overline{S}_k(t) = \theta_0(t) + (\theta(t) \cdot \overline{S}(t)). \tag{34}$$

Working in discounted terms the 0'th coordinate becomes superfluous. For discounted assets this coordinate remains constant. Given any $\mathbb{R}^n$ predictable process vector $(\theta_1, ..., \theta_n)$, then there exists exactly one self-financing strategy $(\theta_0, \theta_1, ..., \theta_n)$ such that $\theta_0(1) = 0$. How do we interpret this economically? A trader starts with a portfolio of which the money account has zero quantity. At the next trading date the money account absorbs the gains or losses occurring from the rest of the portfolio. So for the rest of this section we write $\overline{S} = \left(\overline{S}_1, ..., \overline{S}_n\right)$ and $\theta = (\theta_1, ..., \theta_n) \in \Theta$. Here Θ is the set that represents all (self-financing) strategies, i.e. all $\mathbb{R}^n$-valued predictable processes. Another observation,

$$\begin{aligned}
\text{let } \Delta\overline{V}(t+1) &= \overline{V}(t+1) - \overline{V}(t) && (35)\\
&= \left(\theta_0(t+1) + (\theta(t+1) \cdot \overline{S}(t+1))\right) - \left(\theta_0(t) + (\theta(t) \cdot \overline{S}_k(t))\right) && (36)\\
&= \theta_0(t+1) + (\theta(t+1) \cdot \overline{S}(t+1)) - \theta_0(t+1) - (\theta(t+1) \cdot \overline{S}(t)), && (37)\\
&= (\theta(t+1) \cdot \Delta\overline{S}(t+1)). && (38)
\end{aligned}$$

In particular the final value of the portfolio becomes in discounted units

$$\overline{V}_T = \overline{V}_0 + \sum_{t=1}^{T} (\theta(t) \cdot \Delta\overline{S}(t)) = V_0 + (\theta * \overline{S})_T. \tag{39}$$

Here $\overline{V}_0$ is the initial investment of the portfolio and the martingale transform ($*$) is the discrete 'stochastic integral'. This gives a foretaste of similar concepts needed for the continuous-time case. Using a planned strategy the trader can *replicate* a certain claim, i.e. produce the same outcome. These are called *replicating strategies* and claims for which a replicating strategy exists are called *attainable*.

Definition. The subspace $\mathcal{K}$ of $L^0(\Omega, \mathcal{F}, \mathbb{P})$ defined by

$$\mathcal{K} = \left\{(\theta * \overline{S})_T | \theta \in \Theta\right\} \tag{40}$$

is the set of claims attainable at price 0.

The affine space $a + \mathcal{K}$ is the set of claims attainable at price $a \in \mathbb{R}$. Economically this is a claim with initial value a which a trader can replicate with some replicating strategy θ. As described here these strategies replicate the claims exactly. Extending the concept we may search for a self-financing strategy such that the corresponding portfolio does not replicate, but only dominate the claim, i.e. $V^{\theta}(t) \geq f_S(t)$, where f is some claim depending on the underlying asset vector S. V^{θ} is said to *super-replicate* the claim. The price of the claim is the smallest initial investment required to super-replicate the claim.

Definition. The convex cone $\mathcal{C}$ in $L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$ defined by

$$\mathcal{C} = \{g \in L^{\infty}(\Omega, \mathcal{F}, \mathbb{P}) | \exists f \in \mathcal{K} \text{ with } f \geq g\} \tag{41}$$

The stage is now set for the essence of mathematical finance, the notion of *arbitrage*. Arbitrage is the possibility to make a profit without risk and without net investment of capital.

Definition. A financial market $\overline{S}$ satisfies the *no-arbitrage (NA)* condition if

$$\mathcal{K} \cap L^0_+(\Omega, \mathcal{F}, \mathbb{P}) = \{0\} \tag{42}$$

or, equivalently,

$$\mathcal{C} \cap L^0_+(\Omega, \mathcal{F}, \mathbb{P}) = \{0\}\,. \tag{43}$$

An arbitrage opportunity occurs mathematically when there is a trading strategy θ, with initial investment zero, such that the resulting claim $(\theta * \overline{S})_T$ is nonnegative and not identically zero. It is a crucial assumption to make in the market model. An economist declares the NA condition as an absolute necessity for a fair market. NA also has an impact on the deep mathematical tools that need to be used. Our interest is the close relation between the NA condition and the main assumption to be used in this text - the existence of an equivalent martingale measure.

Theorem.(First Fundamental Theorem of Asset Pricing) Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{T}_1}, \mathbb{P})$ be a filtered complete probability space and $\mathbb{T}_1 = \{0, 1, ..., T\}$ for some natural number T. Suppose the $\mathbb{R}^{n+1}$-valued process $S = (S_0, S_1, ..., S_n)$ is adapted to $\mathbb{F}$, with $S_0(t) > 0$ a.s.$(\mathbb{P})$ for each $t \in \mathbb{T}_1$. Then the following are equivalent:
(i) The market model $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{T}_1, S)$ satisfies the no-arbitrage condition.
(ii) There is a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that the discounted price process $\overline{S} = S/S_0$ is a $(\mathbb{F}, \mathbb{Q})$-martingale. This measure $\mathbb{Q}$ is either called the *equivalent martingale measure (EMM)* or the *risk-neutral measure*.

The original proof of this theorem is due to Dalang, Morton and Willinger [5].

What a wonderful connection between the economically meaningful NA and the mathematical martingale theory behind the EMM. Unfortunately in the continuous time setting (like $\mathbb{T} = \mathbb{T}_4$) the NA condition is not enough to ensure an EMM. There needs to be a stronger condition that will give an EMM (or something close enough for mathematically sound results) and still not allow a trader to 'get something from nothing'.

First we need some preparation in the general setting before announcing a sufficient condition. How does one describe mathematically what happens in practice? A buy-and-hold strategy of one asset, $h = f1_{(\tau_1, \tau_2]}$ with $\tau_1, \tau_2 \in \mathbb{T}_3$ and $f \in m\mathcal{F}_{\tau_1}$, has the interpretation of buying f units of asset s at stopping time τ_1 and selling at stopping time τ_2. In practice strategies would consist of a sum of these buy-and-hold strategies of a finite combination of assets. Such strategies we shall call simple strategies. The movement of the portfolio or 'stochastic integral' is defined by $(h * s)_t = (s_{t \wedge \tau_2} - s_{t \wedge \tau_1})f$. The class of simple strategies is not enough to give a desired result and the general class of predictable processes is unavoidable. This requires us to make two constraints, one constraining the character of assets and the other on what we perceive as to be acceptable trading strategies. These constraints need to be justified economically.

Constraint 1
The stochastic integral $(\theta * S)_t$ has to exist. As mentioned before the most general class of integrators is the class of semimartingales. Thus asset processes or price processes are assumed to be semimartingales and as usual with their càdlàg versions. The essence of this constraint is mathematical, though it can be justified economically, which we shall mention later. Self-financing strategies can now be defined with stochastic integration theory and a hint from discrete time in equation (38).

Definition. Let $V(t) = \sum_{k=0}^{n} \theta_k(t)S_k(t)$. A portfolio strategy is called *self-financing* if the stochastic integrals $\int_0^T \theta_k(t)dS_k(t)$ exist for each k and

$$dV(t) = \sum_{k=0}^{n} \theta_k(t)dS_k(t). \tag{44}$$

Our self-financing portfolio's portfolio strategy is called the self-financing strategy for the asset price vector $(S_0, S_1, ..., S_n)$.

As in the discrete time setting the money account S_0 becomes superfluous when working with discounted terms. This can be seen from the above definition and using the integration by parts formula

$$\begin{aligned} d\overline{V} = d\frac{V}{S_0} &= Vd(\frac{1}{S_0}) + \frac{1}{S_0}dV + d\left[V, \frac{1}{S_0}\right] && (45)\\ &= \sum_{k=0}^{n} \theta_k S_k d(\frac{1}{S_0}) + \sum_{k=0}^{n} \frac{\theta_k}{S_0} dS_k + \sum_{k=0}^{n} \theta_k d\left[S_k, \frac{1}{S_0}\right] && (46)\\ &= \sum_{k=0}^{n} \theta_k \left(S_k d(\frac{1}{S_0}) + \frac{1}{S_0}dS_k + d\left[S_k, \frac{1}{S_0}\right]\right) && (47)\\ &= \sum_{k=0}^{n} \theta_k d\frac{S_k}{S_0} = 0 + \sum_{k=1}^{n} \theta_k d\overline{S_k}. && (48) \end{aligned}$$

Constraint 2
With infinite trading dates ($\mathbb{T} \neq \mathbb{T}_1$) the problem of doubling strategies ("les martingales" in French) occur. A lower bound on the losses needs to be introduced. We can justify this economically with an example of a doubling strategy game. When a coin is tossed and it comes up heads, the player is paid twice the amount of his bet. If the coin comes up tails, the player loses everything. The doubling strategy is the following - the player doubles his bet until the first time he wins. If he starts with one Rand and loses, then the next bet is two Rand. If he loses again the following bet is four Rand. The probability that heads will eventually show up is one, even with a biased coin. The player will make a profit of one Rand almost surely. It is a winning strategy if unbounded accumulated losses are allowed. In the practice no institution (Risk managers, Investment company, Casino) has the funds to sustain that.

Definition. If $\theta = (\theta_1, ..., \theta_n)$ is a $\mathbb{R}^n$-valued predictable process such that
- $\theta_0 = 0$,
- θ is $\overline{S}$-integrable
- there exist a positive constant a so that $(\theta * S)_t \geq -a$ for all $t \geq 0$;
then θ is called a-admissible. We denote this class of strategies by Θ_a. The class of admissible

strategies consists of all the a-admissible strategies for which such a exist and is denoted by Θ. Thus if $\theta \in \Theta$, then $\theta \in \Theta_a$ for some positive a.

Note that the formulation of the definition for the 'acceptable' class of trading strategies Θ differs between the continuous time setting and the *finite* discrete time setting.

Let

$$\mathcal{K}_a = \left\{(\theta * \overline{S})_\infty | \ \theta \in \Theta_a \quad \text{and} \quad (\theta * \overline{S})_\infty = \lim_{t\to\infty}(\theta * \overline{S})_t \quad \text{exists a.s.}\right\}, \tag{49}$$

$$\mathcal{K}_0 = \left\{(\theta * \overline{S})_\infty | \ \theta \in \Theta \quad \text{and} \quad (\theta * \overline{S})_\infty = \lim_{t\to\infty}(\theta * \overline{S})_t \quad \text{exists a.s.}\right\}, \tag{50}$$

$$\mathcal{C}_0 = \mathcal{K}_0 - L^0_+, \tag{51}$$

$$\mathcal{K} = \mathcal{K}_0 \cap L^\infty, \tag{52}$$

$$\mathcal{C} = \mathcal{C}_0 \cap L^\infty. \tag{53}$$

The set $\mathcal{K}_a$ consists of all claims that can be replicated by a-admissible strategies. The sets $\mathcal{K}_0$ ($\mathcal{K}$) and $\mathcal{C}_0$ ($\mathcal{C}$) consist of all (bounded) admissible claims that can be respectively replicated and super-replicated by admissible strategies. The stage is now set to introduce a new concept.

Definition. The semi-martingale price process S satisfies the condition
- *no-arbitrage (NA)* if $\mathcal{C} \cap L^\infty_+ = \{0\}$,
- *no free lunch with vanishing risk (NFLVR)* if $\overline{\mathcal{C}} \cap L^\infty_+ = \{0\}$.
Here $\overline{\mathcal{C}}$ denotes the closure of $\mathcal{C}$ with respect to the norm topology of L^∞.

No Free Lunch (NFL)
The notion of NFL was to create a stronger condition than NA and admit a EMM. In general we can describe the family of No Free Lunches as

$$\tilde{\mathcal{C}} \cap L^\infty_+ = \{0\} \tag{54}$$

where $\tilde{\mathcal{C}}$ is the closure of $\mathcal{C}$ with respect to a forthcoming topology. Traditionally NFL denotes $\tilde{\mathcal{C}}$ with respect to the weak*-topology. If $\tilde{\mathcal{C}}$ is formed as the limits of the weak* converging sequences of elements of $\mathcal{C}$, then the above condition is known as *no free lunch with bounded risk (NFLBR).* The concept that is of interest to us is NFLVR.

No Free Lunch with Vanishing Risk (NFLVR)
How does one interpret NFLVR? If S allows a free lunch with vanishing risk then there is a $f \in L^\infty_+ - \{0\}$ and sequences $\{f_m\}_{m=0}^\infty = \left\{(\theta^m * \overline{S})_\infty\right\}_{m=0}^\infty \in \mathcal{K}$, where $\{\theta^m\}_{m=0}^\infty$ is a sequence of admissible strategies and $\{g_m\}_{m=0}^\infty$ satisfying $g_m \leq f_m$, such that

$$\lim_{m\to\infty} \|f - g_m\|_\infty = 0. \tag{55}$$

The *vanishing risk* part refers to the negative parts $\{(f_m)_-\}_{m=0}^\infty$ and $\{(g_m)_-\}_{m=0}^\infty$ that tend to zero uniformly. Economically this means that without an initial investment there is a system of trading strategies in the market that can get 'close enough' to a possible profit and make the possibility of a loss vanish the 'closer' one gets. Reading the previous description again, it feels like there is an underlying sense of risk involved. That is the truth, because the notion of Free Lunches does not lead to clear-cut profits as does the strong notion of arbitrage. Our notion is that a Free Lunch with Vanishing Risk opens up only the slightest margins of risk. In a (strict) fair market even this is disallowed, which makes NFLVR an acceptable assumption to make for a practitioner. Though, it is in supporting the underlying mathematical tools that the NFLVR concept comes into its own.

Theorem.(General Version of the Fundamental Theorem of Asset Pricing) The following are equivalent for an $\mathbb{R}^n$-valued (locally) bounded semimartingale $\overline{S}$ as the asset process in a financial market model $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{T}, S)$:
(i) NFLVR : $\overline{S}$ satisfies the condition of No Free Lunch with Vanishing Risk.
(ii) EMM : there is a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $\overline{S}$ is a (local) martingale under $\mathbb{Q}$.

For a proof of the theorem see Delbaen and Schachermayer [6] or [10]. It is through this theorem that NFLVR becomes an important axiom for the mathematics of finance.

Referring back to simple trading strategies, the stochastic integral $(h * s)_t = (s_{t\wedge\tau_2} - s_{t\wedge\tau_1})f$ is defined without assuming that s is a semimartingale. When assuming that s is a locally bounded càdlàg process satisfying the economic acceptable NFLVR for simple trading strategies, then it forces s to be a semimartingale to begin with. This is the economic justification for the semimartingale assumption of constraint 1. The technical assumption of $(\theta * \overline{S})_\infty = \lim_{t\to\infty}(\theta * \overline{S})_t$ exists a.s. is also implied for admissible trading strategies when assuming NFLVR. These are good examples of how well NFLVR is suited for the theory.

What about unbounded processes?
If one goes beyond the scope of continuous processes - which are all locally bounded - the jumps in the process could be unbounded. Extreme events, like natural disasters, are an interpretation of sudden market shifts for which the magnitude of the jump is unknown or can not be contained within prescribed boundaries.

Definition. An $\mathbb{R}^n$-valued semimartingale X is called a sigma-martingale if there is a predictable process ϕ, taking values in $(0, \infty)$, such that the $\mathbb{R}^n$-valued stochastic integral $\phi * X$ is a martingale.

Now the most general version of the Fundamental Theorem of Asset Pricing can be stated, and again refer to the book by Delbaen and Schachermayer [10] or the original paper [9] for the proof.

Theorem.(The Fundamental Theorem of Asset Pricing for Unbounded Processes) The following are equivalent for an $\mathbb{R}^n$-valued semimartingale $\overline{S}$ as the asset process in a financial market:
(i) NFLVR : $\overline{S}$ satisfies the condition of No Free Lunch with Vanishing Risk.
(ii) ESMM : there is a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $\overline{S}$ is a sigma-martingale under $\mathbb{Q}$.

Most of the topics discussed in this section can be found in the 'Mathematics of Arbitrage' [10]. This book by Delbaen and Schachermayer gives an insightful theoretical and historical account of the development of the Mathematics of Finance. The book also contains the authors' original papers on the fundamentals of the subject, which is the theme and purpose of the book.

0.4 Pricing

This section will provide pricing formulae for certain vanilla derivatives. Pricing is calculated using standard methods and not the Change of Numéraire technique. For a thorough treatment of pricing and replicating derivatives consult the books by Elliott and Kopp [14] and Etheridge [15].

Let us start this section with two examples of asset price dynamics. Prices follow a specific stochastic process, usually described by stochastic differential equations (satisfying all integrability conditions).

(General) Black-Scholes dynamics. The market consists of a riskless cash account, S^a, and a single risky asset, S^b, with dynamics

$$dS_t^a = r_t S_t^a dt, \quad S_0^a = 1, \tag{56}$$
$$dS_t^b = \mu_t S_t^b dt + \sigma_t S_t^b dW_t, \quad S_0^b = x, \tag{57}$$

where W is a $\mathbb{P}$-Brownian motion and $x \in \mathbb{R}$.

Constant elasticity of variance (CEV). The stock price is assumed to be governed by the diffusion process

$$\frac{dS_t}{S_t} = \mu_t dt + \nu_t S_t^{\xi} dW_t, \quad \xi \in [-1, 1], \tag{58}$$

where W is a $\mathbb{P}$-Brownian motion and ν assumed deterministic.

In the rest of this section we assume the Black-Scholes dynamics and that the European claim at time T, C_T, satisfies the technical condition $\mathbb{E}_{\mathbb{Q}}[C_T^2] < \infty$. These dynamics are governed by the market probability $\mathbb{P}$ and the natural filtration generated by the Brownian motion W.
The procedure for pricing is the following:

(1) Find an EMM $\mathbb{Q}$. This means that $\overline{S}_t = S_t^b / S_t^a$ is an $(\mathbb{F}, \mathbb{Q})$-martingale.

With Itô's formula the solutions for the Black-Scholes dynamics are

$$S_t^a = \exp[\int_0^t r_s ds] \quad \text{(thus of bounded variation) and} \tag{59}$$
$$S_t^b = \exp[\int_0^t (\mu_s - \frac{1}{2}\sigma_s^2) ds + \int_0^t \sigma_s dW_s] \tag{60}$$

Considering discounted terms the following expression is obtained,

$$d\overline{S}_t = S_t^b d\frac{1}{S_t^a} + \frac{1}{S_t^a} dS_t^b \tag{61}$$
$$= \frac{S_t^b}{S_t^a}(-r_t)dt + \frac{1}{S_t^a}(\mu_t S_t^b dt + \sigma_t S_t^b dW_t) \tag{62}$$
$$= -r_t \overline{S}_t dt + \overline{S}_t(\mu_t dt + \sigma_t dW_t) \tag{63}$$
$$\frac{d\overline{S}_t}{\overline{S}_t} = (\mu_t - r_t)dt + \sigma_t dW_t \tag{64}$$

To obtain a martingale process we need to eliminate the drift. The next theorem provides a measure $\mathbb{Q}$ which makes the above process driftless.

Theorem.(Girsanov) Let φ_t be a measurable adapted process such that $\int_0^T \varphi_s^2 ds < \infty$ *a.s.* and

$$\Lambda_t = \exp[-\int_0^t \varphi_s dW_t - \frac{1}{2}\int_0^t \varphi_s^2 ds] \tag{65}$$

is an $(\mathbb{F}, \mathbb{P})$-martingale with W as an $\mathbb{P}$-Brownian motion. Define a new measure $\mathbb{Q}$ on $\mathcal{F}_T$ as

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \Lambda_T. \tag{66}$$

Then the process

$$W_t' = W_t + \int_0^t \varphi_s ds \tag{67}$$

is a standard $\mathbb{Q}$-Brownian motion.

A sufficient condition on φ for Λ to be a martingale is Novikov's condition:

$$\mathbb{E}_{\mathbb{P}}\Big[\exp\left(\frac{1}{2}\int_0^T \varphi_s^2 ds\right)\Big] < \infty. \tag{68}$$

Set $\varphi = \sigma^{-1}(\mu - r)$, which obviously satisfy Novikov's condition when μ, r and $\sigma > 0$ are constants. For general processes under Novikov's condition this gives us the desired result of

$$\frac{d\overline{S}_t}{\overline{S}_t} = \sigma_t dW_t'. \tag{69}$$

(2) Form the process $\overline{V}_t = \mathbb{E}_{\mathbb{Q}}[\frac{C_T}{S_T^a}|\mathcal{F}_t]$ and find a predictable process θ_t such that $d\overline{V} = \theta_t d\overline{S}$, i.e. replicate the claim.

The hypothesis is that $C_T \in L^2(\Omega, \mathcal{F}_T)$ and thus the square integrable martingale process

$$\overline{V}_t = \mathbb{E}_{\mathbb{Q}}\Big[\frac{C_T}{S_T^a}|\mathcal{F}_t\Big]. \tag{70}$$

To proceed we need another main result in continuous pricing.

Theorem.(Martingale Representation) From the settings above let N be a square integrable $(\mathbb{F}, \mathbb{Q})$-martingale. Then there is a unique predictable process γ_t such that

$$dN_t = \gamma_t dW_t'. \tag{71}$$

Set $\theta_t = \frac{\phi_t}{\sigma_t \overline{S}_t}$, where ϕ_t is a γ-type process gained from the Martingale Representation Theorem for the process $\overline{V}$ in the following:

$$d\overline{V}_t = \phi_t dW_t' = \theta_t \sigma_t \overline{S}_t dW_t' = \theta_t d\overline{S}_t. \tag{72}$$

Note: To get a predictable θ, our σ must be predictable too. Both $\overline{S}$ and ϕ are predictable, because $\overline{S}$ is a continuous process and ϕ is constructed that way in the Martingale Representation Theorem.

Letting $\eta_t = \overline{V}_t - \theta_t \overline{S}_t$ it becomes clear that the claim is being replicated precisely,

$$C_T = \eta_T S_T^a + \theta_T S_T^b = V_T. \tag{73}$$

(3) The price of the claim is $V_0 = \mathbb{E}_{\mathbb{Q}}[\frac{C_T}{S_T^a}]$.

This can be seen by taking the expectation of

$$\overline{V}_T = V_0 + \int_0^T \theta_s d\overline{S}_s, \tag{74}$$

where the integral is a martingale with mean zero and remembering that $S_0^a = 1$. The price at time t would be (using equation 70)

$$V_t = S_t^a \mathbb{E}_{\mathbb{Q}}\Big[\frac{C_T}{S_T^a}|\mathcal{F}_t\Big]. \tag{75}$$

Let us now consider some well-known claims.

0.4.1 Bond

Bonds are promissory notes sold by governments, states, corporations and other financial institutions. It is a promise to pay a fixed or floating interest, known as the coupon, on regular intervals over the period of time up to maturity when the debt is repaid in full. Bonds are long-term and usually issued to raise capital. Some bonds are muturity-free and interest is paid indefinitely, but one is unlikely to see such bonds in modern times. The two most important properties of bonds are the low default risk and that the instrument itself is tradeable. Governments issue bonds and are unlikely to default on the contract. Bonds are traded on stock exchanges and are low risk investments. It is for this reason that we can relate the bond to the 'riskless asset'. A zero coupon bond is a bond sold at discount with no coupons and redeemed for face value at maturity.

Definition. A *zero coupon bond*, $B(t,T)$ (or $B_T(t)$) maturing at time T is a claim that pays 1 at time T ($B(T,T) = 1$).

Taking the results above, the pricing formula for this claim is

$$B(t,T) = S_t^a \mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{S_T^a}|\mathcal{F}_t\Big] = \mathbb{E}_{\mathbb{Q}}\Big[\exp[-\int_t^T r_s ds]\Big|\mathcal{F}_t\Big]. \tag{76}$$

0.4.2 Forward

A forward contract does not require an initial payment or price. The contingent claim h is traded for the pre-determined price of $F(h,t,T)$, an $\mathcal{F}_t$-measurable random variable, decided at time t and executed at time T. The claim or payoff at time T is $h - F(h,t,T)$. Therefore the initial price of the claim is

$$0 = \mathbb{E}_{\mathbb{Q}}\Big(\frac{h - F(h,t,T)}{S_T^a}|\mathcal{F}_t\Big) = \mathbb{E}_{\mathbb{Q}}\Big(\frac{h}{S_T^a}|\mathcal{F}_t\Big) - F(h,t,T)\mathbb{E}_{\mathbb{Q}}\Big(\frac{1}{S_T^a}|\mathcal{F}_t\Big). \tag{77}$$

Rearranging, we get the T-*forward price* for h as

$$F(h,t,T) = \frac{S_t^a \mathbb{E}_{\mathbb{Q}}[(S_T^a)^{-1} h|\mathcal{F}_t]}{B(t,T)}. \tag{78}$$

Specifically for the risky asset S^b

$$F(S^b,t,T) = \frac{S_t^b}{B(t,T)}. \tag{79}$$

0.4.3 Futures

A futures contract is traded on an exchange. Parties involved need not know each other, so the exchange needs to bear any default risk. Hence the contract requires standardised features, such as daily settlement arrangements known as *marking to market.* The investor is required to pay an initial margin which is adjusted daily to reflect gains and losses since the future price is determined on the exchange by demand and supply. The price is thus paid over the life of the contract in a series of instalments that enable the exchange to balance long and short positions and minimise its exposure to default risk. Consider a finite number of trading times $0 = t_0, t_1, ..., t_n = T$ at which these 'balancing' instalments occur. The price agreed at time t_i, to be paid at time T, for a claim with price h at time T is written as $G(h, t_i, T)$. Trivially $G(h, T, T) = h$. The difference in price at consecutive balancing times is $G(h, t_i, T) - G(h, t_{i-1}, T)$ and indicates the amounts to be paid into the *margin account* which the buyer either receives (if positive) or contributes (if negative). From the pricing fromula above under the risk-neutral measure the best estimate for the margin amount is zero,

$$0 = \mathbb{E}_{\mathbb{Q}}\Big(\frac{G(h, t_i, T) - G(h, t_{i-1}, T)}{S^a_{t_i}/S^a_{t_{i-1}}}|\mathcal{F}_{t_{i-1}}\Big) \tag{80}$$

$$= S^a_{t_{i-1}}\mathbb{E}_{\mathbb{Q}}\Big(\frac{G(h, t_i, T) - G(h, t_{i-1}, T)}{S^a_{t_i}}|\mathcal{F}_{t_{i-1}}\Big) \tag{81}$$

considering only the time period (t_{i-1}, t_i). Hence discounting only for that period.
Heuristically, adding the above in a continuous sense gives

$$0 = S^a_t\mathbb{E}_{\mathbb{Q}}\Big(\int_t^T (S^a_u)^{-1} dG(h, u, T)|\mathcal{F}_t\Big) \quad \forall t \in [0, T] \tag{82}$$

and thus the integral

$$M_t = \int_0^t (S^a_u)^{-1} dG(h, u, T) \tag{83}$$

is a $(\mathbb{F}, \mathbb{Q})$-martingale with mean zero. Hence $G(h, t, T)$ is also a $(\mathbb{F}, \mathbb{Q})$-martingale,

$$G(h, t, T) - G(h, 0, T) = \int_0^t (S^a_u) dM_u \tag{84}$$

with a mean of $G(h, 0, T)$ - the original price agreement. With $G(h, T, T) = h$ we have $G(h, 0, T) = \mathbb{E}(h)$. This all motivates the definition for futures pricing.

Definition. The *futures price* G at time t for a $\mathcal{F}_T$-measurable claim h is

$$G(h, t, T) = \mathbb{E}_{\mathbb{Q}}(h|\mathcal{F}_t). \tag{85}$$

The total of accumulated balancing margins that the buyer received is

$$\int_0^T dG(h, u, T) = G(h, T, T) - G(h, 0, T) = h - G(h, 0, T) \tag{86}$$

and therefore by time T has paid in total $-(h - G(h, 0, T)) + h = G(h, 0, T)$. This is the price originally agreed upon to be paid for the claim h at time 0.

The moral of the definition and these arguments is that the futures price for a claim is the best estimate from our given information under the risk-neutral measure.

Chapter 1

Introduction to the Numéraire

This chapter will define the numéraire and discuss some basic results needed for its application. It relies heavily on the paper by El Karoui, Geman and Rochet [12].
As discussed in the background section *Market assumptions and Model dynamics* there exists a money account which receives the gains from the portfolio and covers the losses. In the study of the Black-Scholes model this price process is called a riskless account - take r to be deterministic and we have a bank account with *known* floating interest r_s. For our calculations in discounted terms this price process becomes superfluous. In that sense it passes almost undetected. Inherently portfolios are valued in terms of a specified benchmark. Even considering the 0-coordinate asset as part of the portfolio, there is still a phantom benchmark that values the portfolio. It may be a riskless bank account or a market index or with respect to a interest rate (inflation / LIBOR / REPO). *We* decide on, what we shall now call by its real name, the *numéraire* to use as benchmark. A *numéraire* is the price process that measures the value of our portfolio. For instance if inflation acted as the numéraire, discounting would reflect the time value of wealth. This shows that an investment has real growth and is not just keeping up with social trends. The character of the numéraire should be represented by a non vanishing asset of non-existing or low default. Thus the following definition:

Definition. *Numéraire* - A price process $X(t)$ that is almost surely strictly positive for each $t \in \mathbb{T}$, in other words $X(t) > 0$ a.s., for all $t \in \mathbb{T}$.

This includes the numéraire being a risky asset. An example of exchanging foreign investments illustrates this. Consider your numéraire as Rands with a foreign investment subject to the exchange rate. At first glance the numéraire doesn't seem to inherit any reasonable risk. This unfortunately is only a smoke screen, like the Indian flute player that amuses a snake. Changing the numéraire from Rands to Sterling one could figuratively describe as 'swapping the flute for the snake'. Now certainly the new numéraire contains all the volatility and risk that goes with foreign exchange. Is this a new problem or a new look at something old? What really happens to risk when we *change the numéraire*? Is it even *safe* to change the numéraire? Do we gain from doing such a thing? This is the main focus of this text. In chapter two we delve more in-depth into the discussion of safety. For now let us just set our minds at ease when working with portfolios.

Proposition 1.1. Self-financing portfolios remain self-financing after a numéraire change.

Proof: Discrete time case

A portfolio is self-financing if $(\theta(t) \cdot S(t)) = (\theta(t+1) \cdot S(t))$ for $t = 1, ..., T-1$ or equivalently $((\Delta\theta(t+1)) \cdot S(t)) = 0 \quad \forall t \geq 0$. Let $X(t)$ be a new numéraire. Dividing through and into the vector we have

$$\Big([\Delta\theta(t+1)] \cdot [\frac{S(t)}{X(t)}]\Big) = 0 \quad \forall t \geq 0. \tag{1.1}$$

Thus the portfolio expressed in the new numéraire remains self-financing.

Proof: Continuous time case
We have portfolio $V(t) = \sum \theta_k(t) S_k(t)$ with the self-financing condition $dV(t) = \sum \theta_k(t) dS_k(t)$. Let $X(t)$ be a new numéraire. Then from Itô's lemma we have

$$d\Big(\frac{S_k(t)}{X(t)}\Big) = S_k(t-) d\Big(\frac{1}{X(t)}\Big) + \frac{1}{X(t-)} dS_k(t) + d\left[S_k, \frac{1}{X}\right]_t \tag{1.2}$$

as well as

$$d\Big(\frac{V(t)}{X(t)}\Big) = V(t-) d\Big(\frac{1}{X(t)}\Big) + \frac{1}{X(t-)} dV(t) + d\left[V, \frac{1}{X}\right]_t. \tag{1.3}$$

Thus with θ predictable and the above equations all imply that

$$\begin{aligned} d\left(\frac{V(t)}{X(t)}\right) &= \sum_{k=0}^{n} \theta_k(t) \left(S_k(t-) d\left(\frac{1}{X(t)}\right) + \frac{1}{X(t-)} dS_k(t) + d\left[S_k, \frac{1}{X}\right]_t\right) && (1.4) \\ &= \sum_{k=0}^{n} \theta_k(t) d\left(\frac{S_k(t)}{X(t)}\right). && (1.5) \end{aligned}$$

Thus the portfolio expressed in the new numéraire remains self-financing.$\diamond$

The rest of this chapter concentrates on developing an intuitive idea of the change of numéraire technique. To simplify, assume all price processes are *continuous*. Therefore an asset price process is a locally bounded semimartingale. The next assumption applies to all price processes and is equivalent to the market satisfying the NFLVR condition.

Assumption 1.1. For some non-dividend paying numéraire $N(t)$ (with $N(0) = 1$) there exists a probability measure π equivalent to $\mathbb{P}$ such that for any S_k without dividends, $S_k(t)/N(t)$ is a local martingale with respect to π.

From our assumption and the proposition we have that the discounted portfolio value

$$\overline{V}(t) = \frac{V(t)}{N(t)} = V(0) + \sum_{k=0}^{n} \int_0^t \theta_k d\overline{S}_k \tag{1.6}$$

is a π-local martingale.
Consider the case of a nonnegative discounted portfolio value ($\overline{V}(t) \geq 0$). Let $\{T_m\}_{m \in \mathbb{N}}$ be a localizing sequence for $\overline{V}$, then for $s < t$

$$\mathbb{E}_\pi(\overline{V}(T_m \wedge t) | \mathcal{F}_s) = \overline{V}(T_m \wedge s), \tag{1.7}$$

because $\overline{V}$ is a π-local martingale and $V(0)$ is non-random. Taking the limits over m and applying Fatou's lemma we have

$$\mathbb{E}_\pi(\overline{V}(t) | \mathcal{F}_s) \leq \overline{V}(s) \tag{1.8}$$

and $\overline{V}$ is a supermartingale. It is easy to see that this is true for all admissible strategies.
If the terminal value is square integrable, i.e. $\mathbb{E}_\pi[\overline{V}_T^2] < \infty$, then the discounted portfolio value is a π-martingale:

$$\mathbb{E}_\pi(\overline{V}(t) | \mathcal{F}_s) = \overline{V}(s). \tag{1.9}$$

To see this, note that the stochastic integral $\sum_{k=0}^{n}\int_0^T \theta_k(u)d\overline{S}_k(u) = V(T) - V(0) \in L^2(\pi)$. Thus

$$\begin{aligned}
\mathbb{E}_\pi\left[\left(\sum_{k=0}^{n}\int_0^T \theta_k d\overline{S}_k\right)^2\right] &= \sum_{k=0}^{n}\mathbb{E}_\pi\left[\left(\int_0^T \theta_k d\overline{S}_k\right)^2\right] + \sum_{i\neq j}\mathbb{E}_\pi\left[\left(\int_0^T \theta_i d\overline{S}_i\right)\left(\int_0^T \theta_j d\overline{S}_j\right)\right] \\
&= \sum_{k=0}^{n}\mathbb{E}_\pi\left(\int_0^T \theta_k^2 d\left\langle \overline{S}_k\right\rangle\right) + \sum_{i\neq j}\mathbb{E}_\pi\left(\int_0^T \theta_i\theta_j d\left\langle \overline{S}_i, \overline{S}_j\right\rangle\right), \\
&\quad \text{because of the Itô Isometry,} \\
&= \mathbb{E}_\pi\left[\sum_{i=0}^{n}\sum_{j=0}^{n}\left(\int_0^T \theta_i\theta_j d\left\langle \overline{S}_i, \overline{S}_j\right\rangle\right)\right] \\
&= \mathbb{E}_\pi\left[\langle V\rangle_T\right].
\end{aligned}$$

So $\mathbb{E}_\pi[\langle V\rangle_T] < \infty$ which implies that $V(t) - V(0)$ is an L^2-bounded martingale.
With the square integrability condition the value process of the replicating portfolio is a martingale under discounting. This portfolio corresponds with the martingale $\mathbb{E}_\pi[H(T)/N(T)|\mathcal{F}_t]$ (H is an attainable claim) and we call it a *hedging portfolio.* The claim H is said to be *hedged* by the hedging portfolio. Let V^Θ and V^Φ be two hedging portfolios for (H, N, π), then for all t

$$\overline{V}_t^\Theta = \mathbb{E}_\pi[\overline{V}^\Theta(T)|\mathcal{F}_t] = \mathbb{E}_\pi\left[\frac{H(T)}{N(T)}|\mathcal{F}_t\right] = \mathbb{E}_\pi[\overline{V}^\Phi(T)|\mathcal{F}_t] = \overline{V}_t^\Phi. \tag{1.10}$$

Taking the expectation results in a fair price for the claim H and it is the same for any hedging portfolio. Even if there exists another equivalent martingale measure π' for the same hedging portfolio, we have

$$\mathbb{E}_{\pi'}\left[\frac{H(T)}{N(T)}|\mathcal{F}_t\right] = \overline{V}_t = \mathbb{E}_\pi\left[\frac{H(T)}{N(T)}|\mathcal{F}_t\right] \tag{1.11}$$

and thus the fair price is independent of the chosen equivalent martingale measure.
The question we pose ourselves now is: which properties remain after numéraire change?
To answer this question, we need the following important assumption.

Assumption 1.2. For any non-dividend paying numéraire, $X(t)$, the risk-neutral measure admits a martingale process with respect to the original numéraire, i.e. $X(t)/N(t)$ is a π-martingale.

This assumption ensures that the corresponding measure for the new numéraire, defined via its Radon-Nikodym derivative with respect to π, is indeed a probability measure. Thus to answer the question stated above, we have the following proposition:

Proposition 1.2. Let $X(t)$ be non-dividend paying numéraire with $X(t)/N(t)$ a π-martingale. Then there exists a probability measure Q_X defined via its Radon-Nikodym derivative relative to π as

$$\left.\frac{dQ_X}{d\pi}\right|_{\mathcal{F}_T} = \frac{X(T)}{X(0)N(T)} \tag{1.12}$$

such that
(i) the discounted securities are Q_X-local martingales.
(ii) if a contingent claim H has a fair price under (N, π), then it has a fair price under (X, Q_X) and the hedging portfolio is the same.

Note: In N. El Karoui, H. Geman and J.C. Rochet [12] it is incorrectly stated that $X(t)$ is a π-martingale and not $X(t)/N(t)$ as it is stated here.

Proof
(i) Let $\overline{S}(t) = S(t)/N(t)$ ($S^*(t) = S(t)/X(t)$) be the relative price of an asset S with respect to the old (new) numéraire N (X). We only consider the case where $\overline{S}(t)$ is a π-martingale, but the localization argument follows the same route. Then

$$\frac{dQ_X}{d\pi}\Big|_{\mathcal{F}_T} = \frac{\overline{X}_T}{X(0)} \quad \text{satisfies} \quad \mathbb{E}_\pi\left(\frac{dQ_X}{d\pi}\Big|\mathcal{F}_t\right) = \frac{\overline{X}(t)}{X(0)}, \tag{1.13}$$

since, by hypothesis, $\overline{X}$ is a π-martingale and X_0 is $\mathcal{F}_t$-measurable for all $t \geq 0$.
Bayes' formula (for a proof refer to Appendix A) gives us

$$\begin{aligned}
\mathbb{E}_{Q_X}\left[S^*(T)|\mathcal{F}_t\right]\frac{\overline{X}(t)}{X(0)} = \mathbb{E}_{Q_X}\left[S^*(T)|\mathcal{F}_t\right]\mathbb{E}_\pi\left[\frac{dQ_X}{d\pi}\Big|\mathcal{F}_t\right] &= \mathbb{E}_\pi\left[\frac{dQ_X}{d\pi}S^*(T)\Big|\mathcal{F}_t\right] \\
&= \mathbb{E}_\pi\left[\left(\frac{X(T)}{N(T)X(0)}\right)\left(\frac{S(T)}{X(T)}\right)\Big|\mathcal{F}_t\right] \\
&= \frac{1}{X_0}\mathbb{E}_\pi\left[\overline{S}(T)|\mathcal{F}_t\right] \\
&= \frac{1}{X_0}\overline{S}_t
\end{aligned}$$

again because of the π-martingale property and X_0 being $\mathcal{F}_t$-measurable for all $t \geq 0$. In the end we have

$$\mathbb{E}_{Q_X}\left[S^*(T)|\mathcal{F}_t\right] = \frac{S(t)}{X(t)} = S^*(t) \tag{1.14}$$

and S^* is a Q_X-(local)martingale.

(ii) If H has a fair price under (N, π), then $\overline{V}_t = \mathbb{E}_\pi[H(T)/N(T)|\mathcal{F}_t]$ is the value process of a self-financing portfolio generating H. With

$$\mathbb{E}_\pi\left[\frac{H(T)}{N(T)}|\mathcal{F}_t\right] = \mathbb{E}_{Q_X}\left[\frac{H(T)}{X(T)}|\mathcal{F}_t\right]\mathbb{E}_\pi\left[\frac{X(T)}{N(T)}|\mathcal{F}_t\right] \tag{1.15}$$

$$= \mathbb{E}_{Q_X}\left[\frac{H(T)}{X(T)}|\mathcal{F}_t\right]\left(\frac{X(t)}{N(t)}\right) \tag{1.16}$$

$$N(t)\mathbb{E}_\pi\left[\frac{H(T)}{N(T)}|\mathcal{F}_t\right] = V(t) = X(t)\mathbb{E}_{Q_X}\left[\frac{H(T)}{X(T)}|\mathcal{F}_t\right] \tag{1.17}$$

we have a fair price and with Proposition 1.1 we have that $\mathbb{E}_{Q_X}[H(T)/X(T)|\mathcal{F}_t]$ is also self-financing and the hedging portfolio is the same.$\diamond$

Corollary. If X and Y are two numéraires, the general numéraire change can be written at any time $t < T$ as

$$X(t)\mathbb{E}_{Q_X}[Y(T)\Xi|\mathcal{F}_t] = Y(t)\mathbb{E}_{Q_Y}[X(T)\Xi|\mathcal{F}_t] \tag{1.18}$$

with Ξ any random $\mathcal{F}_T$-measurable cash flow and

$$\frac{dQ_X}{dQ_Y} = \frac{X(T)}{X(0)}\Big/\frac{Y(T)}{Y(0)}. \tag{1.19}$$

The two measures are equivalent (thus also equivalent to the market measure $\mathbb{P}$), because the numéraires are strictly positive almost surely. The idea is simply to multiply by the old and divide by the new.

Remark: The above proposition holds in general when no constraints are put on the price processes. The continuity assumption ensures that every attainable claim satisfying the square integrability condition can be hedged perfectly. In chapter three we drop this assumption.

The Bond

Now we turn our attention to a well known numéraire: the bond.

Let $n(t)$ be a reinvested short-rate process with instantaneous interest rate process $r(t)$, i.e.

$$n(t) = \exp\left(\int_0^t r(s)ds\right), \tag{1.20}$$

and π be the EMM admitted by this process. Then a bond can be priced as follows

$$\begin{aligned} B(t,T) &= n(t)\mathbb{E}_\pi\left(\frac{1}{n(T)}|\mathcal{F}_t\right) && (1.21)\\ &= \mathbb{E}_\pi\left[\exp\left(-\int_t^T r(u)du\right)\Big|\mathcal{F}_t\right]. && (1.22)\end{aligned}$$

We can now use our bond as a numéraire and price a forward on a non-dividend share S.

$$F_S(t) = \frac{S(t)}{B_T(t)} = \mathbb{E}_{Q_T}\left[\frac{S(T)}{B_T(T)}|\mathcal{F}_t\right] \tag{1.23}$$

where Q_T is the forward measure given by

$$\begin{aligned} \frac{dQ_T}{d\pi} &= \frac{B_T(T)}{B_T(0)}\Big/\frac{n(T)}{n(0)} && (1.24)\\ &= \frac{1}{B_T(0)n(T)}. && (1.25)\end{aligned}$$

The equation (1.23) is true from Proposition 1.2. It is because we assume that $S(T) = \frac{S(T)}{B_T(T)} \in L^2(\Omega, \mathcal{F}_T)$ (square integrability condition) such that the conditional expectation under Q_T is a fair price for the forward contract on S. We have from this principle the forward measure to price a forward $F(h,t,T)$ for any claim h satisfying $h \in L^2(\Omega, \mathcal{F}_T)$:

$$F(h,t,T) = \mathbb{E}_{Q_T}\left[\frac{h}{B_T(T)}|\mathcal{F}_t\right]. \tag{1.26}$$

As an example h could be a bond with maturity date $T^* > T$. Our forward price at time t would be

$$F(B_{T^*}(T),t,T) = \mathbb{E}_{Q_T}\left[\frac{B_{T^*}(T)}{B_T(T)}|\mathcal{F}_t\right] = \frac{B_{T^*}(t)}{B_T(t)}. \tag{1.27}$$

A general pricing formula for the call option with (B_T, Q_T) as numéraire is given by the following

$$\begin{aligned} \frac{C(0)}{B_T(0)} &= \mathbb{E}_{Q_T}\left[\left(\frac{S(T)}{B_T(T)} - K\right)^+\right] && (1.28)\\ &= \mathbb{E}_{Q_T}\left[\frac{S(T)}{B_T(T)}1_A\right] - KQ_T(A) && (1.29)\end{aligned}$$

where $A = \{\omega | S(T)(\omega) > KB_T(T)(\omega)\}$. From the corollary with numéraire changed to S and measure Q_S we have that

$$\begin{aligned} \frac{C(0)}{B_T(0)} &= \frac{S(0)}{B_T(0)}\mathbb{E}_{Q_S}(1_A) - KQ_T(A) && (1.30)\\ C(0) &= S(0)Q_S(A) - KB_T(0)Q_T(A). && (1.31)\end{aligned}$$

This result can be expanded more generally to

$$C(0) = \mathbb{E}_{X_i}\left[\left(\sum_{k=1}^{n}\lambda_k X_k(T)\right)^+\right] \quad (1.32)$$

$$= \sum_{k=1}^{n}\lambda_k X_k(0) Q_{X_k}(A) \quad (1.33)$$

where $A = \{\omega | \sum_{k=1}^{n}\lambda_k X_k(T,\omega) > 0\}$.

1.1 Foreign Exchange

Foreign exchange trades involve assets/securities or currency whose worth or wealth is weighted in a foreign currency. This is a good example of change of numéraire. Options can be priced using domestic currency as numéraire. The flip side to that is changing numéraire to the foreign currency. Using this idea we can find relationships between domestic and foreign derivatives.

Let E be our exchange rate process of domestic currency for one unit of foreign currency. B and B^* are bonds for respectively the domestic and foreign markets. Now the next theorem will price a forward contract on one unit of foreign exchange in terms of our domestic currency.

Theorem 1.1.(Interest Parity Theorem) With E,B and B^* as above the forward of one unit of foreign exchange maturing at time T is

$$F_{EB^*}(t,T) = E(t)\frac{B^*(t,T)}{B(t,T)}. \quad (1.34)$$

Proof
The asset E is not tradeable. However, the asset EB^* is tradeable. So from the formula of (1.23) with $B(t,T)$ as our numéraire and Q as the risk-neutral measure

$$F_{EB^*}(t,T) = \mathbb{E}_Q\left[\frac{E(T)B^*(T,T)}{B(T,T)}\Big|\mathcal{F}_t\right] \quad (1.35)$$

$$= \frac{E(t)B^*(t,T)}{B(t,T)}.\diamond \quad (1.36)$$

European puts and calls on foreign exchange are denoted by $P(E,K,T,r^*,r)$ and $C(E,K,T,r^*,r)$, where K is the strike price on one unit of foreign currency, T the time to maturity, r^* the foreign riskless rate of interest and r the domestic riskless rate of interest. The put-call parity in this context is written as follows

$$E(t)B^*(t,T) - KB(t,T) = C(E,K,T-t,r^*,r) - P(E,K,T-t,r^*,r). \quad (1.37)$$

Applying the Interest Parity Theorem to the above equation

$$F_{EB^*}(t,T)B(t,T) - KB(t,T) = C(E,K,T-t,r^*,r) - P(E,K,T-t,r^*,r) \quad (1.38)$$

$$(F_{EB^*}(t,T) - K)B(t,T) = C(E,K,T-t,r^*,r) - P(E,K,T-t,r^*,r). \quad (1.39)$$

This makes sense. If the call where to be exercised, the put would be declared worthless and the price of the call determined by discounting the payoff with respect to the riskless rate of return.

The payoff for the call is described here as the forward value of one unit of foreign exchange minus its strike price. Similarly if the put where to be exercised, we would have a worthless call and the price of the put determined by discounting the payoff for the put (the strike minus the forward).

In a similar way we can define ordinary American options $c(S,K,T,\delta,r)$ and $p(S,K,T,\delta,r)$, where S is the underlying asset and δ denotes the dividend rate. The next result is due to Carr and Chesney [3] and we shall give a new proof of this result in chapter four:

Theorem 1.2.(American Put-Call Symmetry) Given an American call $c(S_c,K_c,T,\delta,r)$ and an American put $p(S_p,K_p,T,r,\delta)$ with the same time to maturity and different stocks and different strike prices. If the stocks have the same volatility, i.e. $\sigma(S_c)=\sigma(S_p)$, and the call and the put have the same 'moneyness', i.e.

$$\frac{S_p}{K_p}=\frac{K_c}{S_c} \tag{1.40}$$

then the American Put Call Symmetry relates the values of the two options as

$$\frac{c(S_c,K_c,T,\delta,r)}{\sqrt{S_cK_c}}=\frac{p(S_p,K_p,T,r,\delta)}{\sqrt{S_pK_p}}. \tag{1.41}$$

Note: The dividend rate and riskless rate interchange roles for the different options.

Remark:(i) The above symmetry hold for stocks that are driven by a Brownian motion.
(ii) The first criterion mentioned in theorem 1.2 is that the stocks should have the same volatility. The stocks thus show movement to similar shocks in the market. The second criterion is the 'moneyness' condition. Recall that an option is 'at the money' on time t if $K=S_t$. Under this condition the stocks are inversely proportional to each other. They fluctuate around the 'at the money value' of K_cK_p.

Consider these two options:
(i) $c(E,K,T,r^*,r)$ - An American Call on one unit of foreign exchange with exchange rate E and strike K.
(ii) $p^*(\frac{1}{E},\frac{1}{K},T,r,r^*)$ - An American Put (in foreign currency) on one unit of domestic exchange with exchange rate $1/E$ and strike $1/K$.
Here r and r^* denote the domestic riskless rate and foreign riskless rate respectively.

The exchange rate's volatility is the same, whether taking numéraire in domestic or foreign currency. The above also satisfy the 'moneyness' condition

$$\frac{E}{K}=\frac{1/K}{1/E}. \tag{1.42}$$

Thus from the American Put Call Symmetry we have

$$c(E,K,T,r^*,r)=EKp^*(\frac{1}{E},\frac{1}{K},T,r,r^*). \tag{1.43}$$

What this means is that a call option on one unit of foreign exchange with strike K, is the same (taking in consideration the exchange rate) as K foreign priced put options with each strike price 1/K. Thus on the right-hand side we sell one unit of foreign exchange and on the left-hand side we buy one unit.

Chapter 2

No-Arbitrage and the Numéraire

The content of this chapter is taken from the book of Delbaen and Schachermayer [10] and their paper on the subject [7].
Previously the question was posed if the change of numéraire technique is a 'safe' action. To reach a verdict on whether such a technique is legal, one needs to describe the law that governs pricing in financial mathematics. No-Arbitrage (and conditions like NFLVR) enables the financial mathematician to achieve a fair price for a contingent claim. In chapter one it was shown that self-financing portfolios remain intact after numéraire change. Can the same be said for No-Arbitrage? The question is daunting, especially if the NA condition does not remain! It is thus important to delve into these technical difficulties. Intuitively the notion of arbitrage should not depend on whether we do book-keeping in one or another numéraire. Unfortunately intuition is not the best guide in the infinite time setting. In the sequel we consider the two cases (discrete and continuous) separately, but first let us study our premisses. Denote the market with respect to the old numéraire as an $\mathbb{R}^d$-valued process

$$S = (S^1, S^2, ..., S^d). \tag{2.1}$$

What do we allow to be a new numéraire? It should at least be a tradeable asset. The assets in the market S are all possibilities. More generally, we can take a portfolio to act as numéraire. Denote the ratio between the new and the old numéraire by Π. Thus the new numéraire with respect to the old numéraire is a value process $\Pi_t = a + (\Psi * S)_t$, $a \in \mathbb{R}$ the initial value. Let us now establish three characteristics we would like to see in Π, namely
(1) $\Pi_t > 0$ for all $t \in \mathbb{T}$,
(2) $\Pi_0 = 1$,
(3) $\lim_{t\to\infty} \Pi_t = \Pi_\infty$ exists a.s. and is a.s. strictly positive.
Note that $1/\Pi$ also has these characteristics. The first characteristic is inherited from the definition of a numéraire. The second is purely for normalisation. The third is useful for an infinite time horizon and the continuous trading setting, as we shall see. Let us return to the notation used and defined in the section *Market assumptions and Model dynamics* (see the last definition on page 17 and equations (49) - (53)). Thus we have

$$\Pi_t = 1 + (\Psi * S)_t > 0 \;\Rightarrow\; (\Psi * S)_t > -1, \tag{2.2}$$

which allows $\Psi \in \Theta_1 \subset \Theta$ and with the limit condition of the third characteristic lets $(\Psi * S)_t \in \mathcal{K}_0$. Adding the asset Π to the market would not generate any new admissible portfolios. Therefore extending the market from S to the $\mathbb{R}^{d+2}$-valued $X = (S, 1, \Pi)$, would retain the NA condition. Here the constant 1 stands for the old numéraire, for its value remains unchanged with respect to itself. By the same reasoning, the NFLVR condition is also satisfied in the extended market model. Define

$$X = (S, 1, \Pi) \Leftrightarrow Z = (\frac{S}{\Pi}, \frac{1}{\Pi}, 1) \tag{2.3}$$

as two markets that relate through a change in numéraire.

2.1 Discrete time setting

When comparing the two extended markets the first step would be to find connections between the attainable claims in the separate markets. Attainable claims are generated from self-financing portfolios with predictable trading strategies. Recall that Proposition 1.1 states that the self-financing property is invariant under a change in numéraire. Hence we can drop the dummy entries of 1 (see also why the 0-coordinate is superfluous in *Market assumptions and Model dynamics*). The focus is now shifted to claims attained via discounted (with respect to the numéraire) portfolios with $\mathbb{R}^{d+1}$-valued predictable trading strategies. Intuitively a connection between attainable claims in the different markets should be some factor, like the ratio of new and old numéraires. Before we make this precise, we need the following definition and lemma.

Definition. Denote $\mathcal{K}[X]$ as the set of all bounded claims that can be replicated by admissible strategies in the extended market X with initial value zero.

Lemma 2.1. Fix $0 \leq t \leq T$ and let $f \in \mathcal{K}[X]$ be $\mathcal{F}_t$ measurable. Then the random variable $\frac{f}{\Pi_t}$ is of the form $\frac{f'}{\Pi_T}$, where $f' \in \mathcal{K}[X]$.

Proof
By telescoping it is clear that

$$\frac{f}{\Pi_t} - \frac{f}{\Pi_T} = \frac{1}{\Pi_T}\left(\frac{f}{\Pi_t}(\Pi_T - \Pi_t)\right) = \frac{1}{\Pi_T}\sum_{s=t+1}^{T}\frac{f}{\Pi_t}(\Pi_s - \Pi_{s-1}). \tag{2.4}$$

Let $f'' = \sum_{s=t+1}^{T} \frac{f}{\Pi_t}(\Pi_s - \Pi_{s-1})$. Then $f'' \in \mathcal{K}[X]$ with replicating trading strategy in the following way: (i) zero on the trading dates $(1, 2, ..., t)$ and (ii) the $\mathcal{F}_t$-measurable $\frac{f}{\Pi_t}$ on the trading dates $(t+1, t+2, ..., T)$. Hence $f' = f'' + f \in \mathcal{K}[X]$ gives the above result.$\diamond$

Definition. Let $\mathcal{M}^e(S)$ denote the set of all equivalent martingale measures for the market S.

In the main theorem of this section we shall also investigate the corresponding change in risk-neutral measure for a change in numéraire. The first proposition will now show that there exists a one-to-one correspondence between the attainable claims in market X with the attainable claims in market Z.

Proposition 2.1. Let X,Z and Π be defined as above. Then

$$\mathcal{K}[Z] = \left\{\frac{f}{\Pi_T}\,\middle|\, f \in \mathcal{K}[X]\right\} = \frac{1}{\Pi_T}\mathcal{K}[X]. \tag{2.5}$$

Proof

Let $g \in \mathcal{K}[Z]$. Then there exists a $(d+1)$-dimensional predictable process θ, such that

$$g = \sum_{t=1}^{T} (\theta(t) \cdot \Delta Z(t)) = \sum_{t=1}^{T} \sum_{i=1}^{d+1} \theta^i(t) \Delta Z^i(t). \tag{2.6}$$

For convenience denote $S_t^{d+1} = 1 \quad \forall t \in \mathbb{T}$, then for $i = 1, ..., d+1$ and $t = 1, ..., T$,

$$\begin{aligned} \Delta Z_t^i &= \frac{S_t^i}{\Pi_t} - \frac{S_{t-1}^i}{\Pi_{t-1}} && (2.7) \\ &= \frac{\Delta S_t^i}{\Pi_t} + S_{t-1}^i \left(\frac{1}{\Pi_t} - \frac{1}{\Pi_{t-1}} \right) && (2.8) \\ &= \frac{\Delta S_t^i}{\Pi_t} - \frac{S_{t-1}^i}{\Pi_{t-1}} \frac{\Delta \Pi_t}{\Pi_t} && (2.9) \\ &= \frac{1}{\Pi_t} \left(\Delta S_t^i - Z_{t-1}^i \Delta \Pi_t \right). && (2.10) \end{aligned}$$

Summing we get

$$\begin{aligned} (\theta_t \cdot \Delta Z_t) &= \sum_{i=1}^{d+1} \theta_t^i \Delta Z_t^i && (2.11) \\ &= \frac{1}{\Pi_t} \sum_{i=1}^{d+1} \left(\theta_t^i \Delta S_t^i - \left(\theta_t^i Z_{t-1}^i \right) \Delta \Pi_t \right) && (2.12) \\ &= \frac{1}{\Pi_t} \left(\sum_{i=1}^{d} \theta_t^i \Delta S_t^i - \left(\sum_{i=1}^{d+1} \theta_t^i Z_{t-1}^i \right) \Delta \Pi_t \right) && (2.13) \\ &= \frac{1}{\Pi_t} (\phi \cdot \Delta X_t), && (2.14) \end{aligned}$$

where ϕ is a $\mathbb{R}^{d+1}$-valued random variable with each component $\mathcal{F}_{t-1}$ measurable. Hence $(\phi \cdot \Delta X(t)) \in \mathcal{K}[X]$ with the trading strategy zero except for ϕ at time t and therefore $\mathcal{F}_t$ measurable. Applying Lemma 2.1 for each of the trading dates results in

$$\begin{aligned} g = \sum_{t=1}^{T} (\theta(t) \cdot \Delta Z(t)) &= \sum_{t=1}^{T} \frac{1}{\Pi_t} (\phi \cdot \Delta X(t)) && (2.15) \\ &= \sum_{t=1}^{T} \frac{f_t}{\Pi_T} \in \frac{1}{\Pi_T} \mathcal{K}[X], && (2.16) \end{aligned}$$

because *the set of trading dates is finite* and the set of attainable claims is closed under summation. Thus $\mathcal{K}[Z] \subset \frac{1}{\Pi_T} \mathcal{K}[X]$. Through symmetry, changing the numéraire back from the new to the old then yields

$$\mathcal{K}[X] \subset \frac{1}{1/\Pi_T} \mathcal{K}[Z] = \Pi_T \mathcal{K}[Z], \tag{2.17}$$

such that equality $\mathcal{K}[X] = \Pi_T \mathcal{K}[Z]$ is obtained.$\diamond$

The symmetry argument in the previous proof can be interpreted in two ways. The first is applying the ratio of new to old numéraire when moving from the old market to the new market. Moving back from the new market to the old one, the ratio just gets inverted, e.g. $1/\Pi$. Another philosophy is that any price process can only be measured with respect to a chosen numéraire. In this sense Π *is* the new numéraire with the mysterious (old) numéraire hidden within it. The numéraire itself remains a constant 1 throughout and 'mysteriously' disappears as we did above with the

extended markets Z and X. Changing to the new numéraire Π introduces the risky asset $1/\Pi$ to the market, which is the old numéraire in terms of the new one. When moving back to the old market, we would have $1/\Pi$ acting as numéraire.
The next theorem is the main result for this section.

Theorem 2.1.(No-Arbitrage under a Change of Numéraire: Discrete time) Let X,Z and Π be defined as above, with X satisfying the no-arbitrage condition. Then Z also satisfies the no-arbitrage condition. Moreover $\mathbb{Q}$ belongs to $\mathcal{M}^e(X)$ if and only if the measure $\mathbb{Q}'$ defined by $d\mathbb{Q}' = \Pi_T d\mathbb{Q}$ belongs to $\mathcal{M}^e(Z)$.

Proof
The market X satisfies the no-arbitrage condition,

$$\begin{aligned} \{0\} &= \mathcal{K}[X] \cap L^0_+ && (2.18)\\ \{0\} &= \frac{1}{\Pi_T}\Big(\mathcal{K}[X] \cap L^0_+\Big) && (2.19)\\ &= \Big(\frac{1}{\Pi_T}\mathcal{K}[X]\Big) \cap \Big(\frac{1}{\Pi_T}L^0_+\Big). && (2.20) \end{aligned}$$

With proposition 2.1 and the random variable Π_T that is strictly positive almost surely, the market Z satisfies the no-arbitrage condition too:

$$\{0\} = \mathcal{K}[Z] \cap L^0_+. \tag{2.21}$$

An equivalent probability measure $\mathbb{Q}$ is in the set $\mathcal{M}^e(X)$ if and only if $\mathbb{E}_{\mathbb{Q}}[f] = 0,\ \forall f \in \mathcal{K}[X]$ (for a proof refer to Appendix A). This is the same as

$$\mathbb{E}_{\mathbb{Q}}\Big[\Pi_T \frac{f}{\Pi_T}\Big] = 0, \quad \forall f \in \mathcal{K}[X], \tag{2.22}$$

which is equivalent to

$$\mathbb{E}_{\mathbb{Q}'}[g] = 0, \quad \forall g \in \mathcal{K}[Z]. \tag{2.23}$$

The measure $\mathbb{Q}'$ is a probability measure, because of the normalisation characteristic of a numéraire and Π a $\mathbb{Q}$-martingale in the market X, $\mathbb{E}_{\mathbb{Q}}(\Pi_T) = \Pi_0 = 1$. By the definition of $\mathbb{Q}'$ it is easy to see that $\mathbb{Q}'$,$\mathbb{Q}$ and $\mathbb{P}$ are all equivalent. Hence the statement of (2.22) is true if and only if $\mathbb{Q}' \in \mathcal{M}^e(Z)$ (the same as the previous result with the proof in Appendix A) and the theorem is proved.$\diamond$

2.2 Continuous time setting

As was seen in the section *Market assumptions and Model dynamics* some intuitive ideas are contradicted when continuous trading is allowed. This forces economically motivated restrictions to be incorporated, e.g. lower bounds on losses. Does the change of numéraire technique contain its own 'les martingales' (doubling strategy) example?

Example 2.1. Let a market consist of two assets. One is the current numéraire and the other a risky asset satisfying all the characteristics of a numéraire. Denote the market as $(1, E)$. To make this more precise, E could represent the foreign exchange rate of the \$ (dollar) with respect to the £ (sterling). Changing the numéraire the market can be described as $(1/E, 1)$ with $1/E$ representing the £ in terms of the \$. Assume the discrete and infinite time setting $\mathbb{T} = \mathbb{T}_2$ and fix $0 < \alpha < 1$. Let the process E_n be driven by a sequence of independent identically distributed Bernoulli variables $\{\epsilon_n\}_{n\geq 1}$ such that $\mathbb{P}[\epsilon_n = 1] = \mathbb{P}[\epsilon_n = -1] = \frac{1}{2}$ in the following way:

$$E_0 = 1, \tag{2.24}$$

$$E_n = \begin{cases} 2E_{n-1} - \alpha & \text{, if } \epsilon_n = 1, \\ \alpha & \text{, if } \epsilon_n = -1, \end{cases} . \tag{2.25}$$

The three characteristics of a numéraire are all satisfied for the above process. In particular $E_\infty = \alpha > 0$ a.s., because α is an absorbing state. Notice as well that E is an $\mathbb{P}$-martingale. One can see this by taking the natural filtration of E and

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[E_n|\mathcal{F}_{n-1}] &= \mathbb{E}_{\mathbb{P}}[1_{\{\epsilon_n=1\}}(2E_{n-1}-\alpha)|\mathcal{F}_{n-1}] + \mathbb{E}_{\mathbb{P}}[1_{\{\epsilon_n=-1\}}\alpha|\mathcal{F}_{n-1}] & (2.26)\\ &= (2E_{n-1}-\alpha)\mathbb{E}_{\mathbb{P}}[1_{\{\epsilon_n=1\}}] + \alpha\mathbb{E}_{\mathbb{P}}[1_{\{\epsilon_n=-1\}}],\ \epsilon_n \text{ independent of } \mathcal{F}_{n-1}, & (2.27)\\ &= (2E_{n-1}-\alpha)\mathbb{P}[\epsilon_n = 1] + \alpha\mathbb{P}[\epsilon_n = -1] & (2.28)\\ &= E_{n-1} & (2.29) \end{aligned}$$

Our intuition indicates thus that E is a fair game. In the long run though $E_\infty = \alpha < 1$ produces a loss. The British trader could short-sell E and make a profit whenever the process hits the absorbing state. Unfortunately this strategy is not admissible, because the expression $2E_{n-1}-\alpha$ is unbounded outside the absorbing state. From the American trader's perspective, i.e. changing the numéraire, investing in the sterling $1/E$ would produce a profit in the long run, $1/E_\infty = \frac{1}{\alpha} > 1$. Holding stock in $1/E$ is also admissible (from the definition of a numéraire!). The American trader therefore has an arbitrage opportunity. This is profound and of concern for any model with infinitely many trading dates, and thus for continuous trading. Assume from now on that trading is continuous over $\mathbb{T}_3 = [0,T]$ or $\mathbb{T}_4 = [0,\infty)$. How should numéraires be restricted such that the above situation can be avoided?

Definition. An element $f \in \mathcal{K}_a$ ($\mathcal{K}$) is maximal in $\mathcal{K}_a$ ($\mathcal{K}$) if for any $g \in \mathcal{K}_a$ ($\mathcal{K}$) such that $g \geq f$ a.s. implies $g = f$ a.s.

It is easy to see that if f is maximal in $\mathcal{K}$ then it is also maximal in the smaller set $\mathcal{K}_b$ for any $b \in \big[\, \|(f)_-\|_\infty, \infty\big)$. Since f is an admissible claim, $\|(f)_-\|_\infty$ would be finite. Conversely, if f is maximal $\mathcal{K}_a$ then f is also maximal in the bigger set $\mathcal{K}$. To see this suppose that there exists $g \in \mathcal{K}$ such that $g \geq f$. This implies that g is a-admissible and thus $g = f$. The same goes for an element that is maximal in $\mathcal{K}$ is also maximal in $\mathcal{C}$ and vice versa. The NA condition ensures the existence of maximal elements in the sense that the 0 random variable is maximal in $\mathcal{K}$. Any random variable that dominates 0 is excluded from the set by the NA condition. Conversely if 0 is maximal it excludes all dominating elements from the set $\mathcal{K}$, which is the condition of NA.

Theorem 2.2.(No-Arbitrage under a Change of Numéraire: Continuous time) Let S be an $\mathbb{R}^d$-valued semimartingale. Let Π be a semimartingale process with all the characteristics of a numéraire. The two markets under consideration, defined as $X = (S, 1, \Pi)$ and $Z = (\frac{S}{\Pi}, \frac{1}{\Pi}, 1)$, are thus $\mathbb{R}^{d+2}$-valued semimartingale asset processes. The market Z satisfies the NA condition if and only if $\Pi_\infty - 1$ is maximal in $\mathcal{K}_1[X]$. Similarly the market X satisfies the NA condition if and only if $\frac{1}{\Pi_\infty} - 1$ is maximal in $\mathcal{K}_1[Z]$.

Proof
Between the markets we have the relation $X = \Pi Z$. With integration by parts from vector stochastic calculus, we deduce that

$$dX_t = d\Pi_t Z_{t-} + \Pi_{t-} dZ_t + d[\Pi, Z]_t, \tag{2.30}$$

where multiplication describes a scalar times a vector. Let $Y = \Pi(1 + K \cdot Z)$, where K is a 1-admissible predictable strategy for the market Z and therefore making Y non-negative. Note that Y is a portfolio with initial value 1 in the market Z that has been converted through the change of numéraire Π into values that fit in the market X. Hence by integration by parts

$$dY_t = d\Pi_t[1 + (K \cdot Z)_{t-}] + \Pi_{t-}K_t \cdot dZ_t + K_t \cdot d[\Pi, Z]_t. \tag{2.31}$$

Use the expression for dX to get

$$\begin{aligned} dY_t &= d\Pi_t[1 + (K \cdot Z)_{t-}] + K_t \cdot (dX_t - d\Pi_t Z_{t-}) & (2.32)\\ &= d\Pi_t[1 + (K \cdot Z)_{t-} - K_t \cdot Z_{t-}] + K_t \cdot dX_t & (2.33) \end{aligned}$$

which is of the form

$$dY_t = L_t \cdot dX_t. \tag{2.34}$$

Notice that L is the same as K except for an adjustment in the numéraire coordinate. Since K is predictable, we see that so is the adjusted value and therefore L itself is also predictable. L is also 1-admissible for market X, because Y is never negative and has initial value 1. Let us now assume that Z allows an arbitrage opportunity in the form of an admissible strategy K_a. Thus there exists a positive real number a such that $(K_a \cdot Z)_t \geq -a$. Then $K = \frac{K_a}{a}$ is a 1-admissible strategy allowing an arbitrage opportunity. Hence, with strict inequality on a non-negligible set, we have

$$\begin{aligned} (K \cdot Z)_\infty &\geq 0 & (2.35)\\ 1 + (K \cdot Z)_\infty &\geq 1 & (2.36)\\ Y_\infty - 1 &\geq \Pi_\infty - 1 & (2.37) \end{aligned}$$

so that Π is not maximal in $\mathcal{K}_1[X]$. Therefore if $\Pi_\infty - 1$ is maximal, then the market Z satisfies the NA condition. Converting in terms of the numéraires, we can similarly say that maximality of $\frac{1}{\Pi_\infty} - 1$ implies X satisfying the NA condition. Assume $\frac{1}{\Pi_\infty} - 1$ is not maximal in $\mathcal{K}_1[Z]$. Thus there exists a 1-admissible strategy K such that

$$\begin{aligned} (K \cdot Z)_\infty &\geq \frac{1}{\Pi_\infty} - 1 & (2.38)\\ Y_\infty - 1 &\geq 0, & (2.39) \end{aligned}$$

with strict inequality on a non-negligible set. So Y is a portfolio with a 1-admissible strategy that generates an arbitrage in the market X. Therefore if X satisfies the NA condition, then $\frac{1}{\Pi_\infty} - 1$ is maximal. Converting again concludes the theorem with the result that Z satisfying the NA condition implies the maximality of $\Pi_\infty - 1$.$\diamond$

Corollary. X satisfies the NA condition and $\Pi_\infty - 1$ is maximal in $\mathcal{K}[X]$ if and only if Z satisfies the NA condition and $\frac{1}{\Pi_\infty} - 1$ is maximal in $\mathcal{K}[Z]$.

Proof
One applies the above theorem directly. The only difference is the bigger $\mathcal{K}$ sets that are under consideration. From the characteristics of a numéraire the replicating portfolio would be 1-admissible. Maximality is retain in both sets and consequently the corollary is proved.$\diamond$

In the example the market satisfied the NA condition. Thus the 0 random variable is maximal and it also dominates the outcome $E_\infty - 1 = \alpha - 1$. The above theorem thus clarifies the concern that was created in the example. The next theorem (stated without proof) relates the above results to NFLVR. Here we also redefine $\mathcal{M}^e(S)$.

Definition. Let $\mathcal{M}^e(S)$ denote the set of all equivalent *local* martingale measures for the market S.

Theorem 2.3.(No Free Lunch with Vanishing Risk under a Change of Numéraire) Let X and Z be defined as above and suppose that S is locally bounded and satisfies the condition of NFLVR. Write $\Pi_t = 1 + (\Psi \cdot S)_t$ as numéraire with all its characteristics. Then the following are equivalent:
(1) $(\Psi \cdot S)_\infty$ is maximal in $\mathcal{K}$.
(2) There is an element $\mathbb{Q} \in \mathcal{M}^e(S)$ for which the process Π is a $\mathbb{Q}$-uniformly integrable martingale.
(3) There is an element $\mathbb{Q} \in \mathcal{M}^e(S)$ for which $\mathbb{E}_\mathbb{Q}[(\Psi \cdot S)_\infty] = 0$.
(4) There is an element $\mathbb{Q} \in \mathcal{M}^e(S)$ such that $\mathbb{E}_\mathbb{Q}[\Pi_\infty] = \sup \{\mathbb{E}_\mathbb{R}[\Pi_\infty] : \mathbb{R} \in \mathcal{M}^e(S)\}$
(5) Z satisfies the NA condition.
(6) $\mathcal{M}^e(Z) \neq \emptyset$.
(7) Z satisfies the condition of NFLVR.

A proof for the above theorem and a more in-depth look at the matter of NFLVR consult two papers by Delbaen and Schachermayer in [7] and [8].

Chapter 3

General Theory of the Numéraire

The paper by Schroder [21] provided all the results stated in this chapter, as well as the main result that forms the core of this text. The main assumption in his paper (stated as 3.1 below) can be extended to a more general setting as described in the Remark following it.
The setting created in this chapter is more general and extends the ideas created in chapter one. In this chapter the assumption that price processes are continuous will be removed. With no continuity assumption Proposition 1.2(ii) is only valid for claims that can be hedged. All the results in this chapter allow for incomplete markets. Fix the time interval $[0, T]$. Shares are considered to pay dividends. Dividends are reinvested in the share similar to the way interest is reinvested in a short-rate process or riskless bank account. We denote the proportional dividend payout rate by δ, a continuous adapted process and almost surely positive on $[0, T]$. In the case of discrete dividend payouts, say D_i at stopping times $\tau_i, i = 1, 2, ...$, we just replace $\int_0^t \delta_s ds$ with $\sum_{\tau_i \in [0,t]} log(1 + D_i/S(\tau_i))$. Our definition of a numéraire as stated in chapter one remains intact under the general conditions. In the sequel we combine Assumption 1.1 and Assumption 1.2.

Assumption 3.1. There exists a risk-neutral measure, $\mathbb{Q}$, such that every reinvested price process relative to the reinvested short-rate process as numéraire is a $\mathbb{Q}$-martingale.

Remark: This is the assumption used by Schroder [21]. It is a strong assumption in the sense that it demands *every* price process to be a martingale relative to the short-rate process. We know from Delbaen and Schachermayer [10] that for bounded markets this is equivalent to the NFLVR condition. One would like to extend this to markets that are locally bounded - which includes the neat setting of continuous processes modelling markets. For locally bounded markets NFLVR is equivalent to a non-empty set of Equivalent Local Martingale Measures (ELMMs) for the market S, i.e. $\mathcal{M}^e(S) \neq \emptyset$. Unfortunately an ELMM is not enough to ensure that the corresponding measure change in the *change of numéraire technique* would be a probability measure. Though let us restrict ourselves to numéraires that are driven by a maximal process. Theorem 2.3 now provides us with an ELMM $\mathbb{Q}$ such that for a given numéraire, it would be a $\mathbb{Q}$-uniformly integrable martingale with respect to the short-rate process. The maximality restriction also guarantees that the market under the new numéraire remains true to the NFLVR condition. Thus to sum up - for locally bounded markets under the NFLVR condition, we study numéraires that satisfy the maximality restriction. The maximality restriction in practical terms means that the new numéraire can be hedged in terms of the old numéraire. The results that follow are sourced from Schroder [21], but still remain intact under the general setting.

Note: Only the setting created above and the main assumption mentioned are enforced on the study. No dynamics for the market process S is mentioned. In the sequel the discussion will focus on the effect a change of numéraire has on (1) the change in measure and (2) the interchanging of roles between the dividend rate and the riskless rate (compare with the American Put Call Symmetry in chapter one).

Let the reinvested price process, S, be a new numéraire candidate. With R as the reinvested short-rate process, where $R_t = e^{\int_0^t r_s ds}$ with r a continuous adapted process and almost surely positive on $[0,T]$, let Z be the ratio between the new and old numéraires:

$$Z_t \equiv \frac{S_t e^{\int_0^t \delta_s ds}}{S_0} / \frac{R_t}{1} = S_t e^{\int_0^t (\delta_s - r_s) ds} / S_0, \quad t \in [0,T]. \tag{3.1}$$

From Assumption 3.1 Z is a $\mathbb{Q}$-martingale.
Associated with the new numéraire is the probability measure $\mathbb{Q}'$ defined via the Radon-Nikodym derivative:

$$\frac{d\mathbb{Q}'}{d\mathbb{Q}} = Z_T. \tag{3.2}$$

The following theorem forms the core result for this chapter.

Theorem 3.1. Define $S'_t = KS_0/S_t$ and $\mathbb{Q}'$ as above, where K is a constant positive value. Then the time-zero price of an asset with $\mathcal{F}_\tau$-measurable payoff P_τ at the stopping time $\tau \in [0,T]$ is

$$V_0 = \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_0^\tau r_s ds} P_\tau\right) = \mathbb{E}_{\mathbb{Q}'}\left(e^{-\int_0^\tau \delta_s ds} P_\tau S'_\tau / K\right). \tag{3.3}$$

In general the time-ς price of an asset at stopping time $\varsigma \leq \tau$ with $\mathcal{F}_\tau$-measurable payoff P_τ at the stopping time $\tau \in [0,T]$ is

$$V_\varsigma = \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_\varsigma^\tau r_s ds} P_\tau \middle| \mathcal{F}_\varsigma\right) = \mathbb{E}_{\mathbb{Q}'}\left(e^{-\int_\varsigma^\tau \delta_s ds} P_\tau S^*_\tau / K \middle| \mathcal{F}_\varsigma\right), \tag{3.4}$$

where $S^*_t = KS_\varsigma/S_t$ for all stopping times $t \in [\varsigma, \tau]$. The dynamics of S and S' are described by

$$dS_t = S_{t-}(r_t - \delta_t)dt + dM_t \tag{3.5}$$
$$dS'_t = S'_{t-}(\delta_t - r_t)dt + dM'_t, \quad S'_0 = K, \tag{3.6}$$

where M and M' are local martingales under $\mathbb{Q}$ and $\mathbb{Q}'$ respectively. If the difference between the riskless rate r and the dividend rate δ is a bounded process then M and M' are martingales. The quadratic variations of M and M' satisfy

$$\frac{d[M]_t}{(S_t)^2} = \frac{d[M']_t}{(S'_t)^2} \tag{3.7}$$

between jumps.

Proof
Via Assumption 3.1 we have the martingale property with the short-rate as numéraire. We have the time-zero price as

$$\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_0^\tau r_s ds} P_\tau\right) = \mathbb{E}_{\mathbb{Q}}\left(Z_\tau \frac{S_0}{S_\tau} e^{-\int_0^\tau \delta_s ds} P_\tau\right) \tag{3.8}$$
$$= \mathbb{E}_{\mathbb{Q}}\left(Z_\tau e^{-\int_0^\tau \delta_s ds} P_\tau S'_\tau / K\right). \tag{3.9}$$

Again using the assumption, Z is a $\mathbb{Q}$-martingale and noticing that the rest of the terms are all $\mathcal{F}_\tau$-measurable we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_0^\tau r_s ds} P_\tau\right) &= \mathbb{E}_{\mathbb{Q}}\left(Z_\tau e^{-\int_0^\tau \delta_s ds} P_\tau S'_\tau / K\right) && (3.10)\\
&= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(Z_T|\mathcal{F}_\tau) e^{-\int_0^\tau \delta_s ds} P_\tau S'_\tau / K\right) && (3.11)\\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left(Z_T e^{-\int_0^\tau \delta_s ds} P_\tau S'_\tau / K|\mathcal{F}_\tau\right)\right] && (3.12)\\
&= \mathbb{E}_{\mathbb{Q}}\left[Z_T e^{-\int_0^\tau \delta_s ds} P_\tau S'_\tau / K\right] && (3.13)\\
&= \mathbb{E}_{\mathbb{Q}'}\left[e^{-\int_0^\tau \delta_s ds} P_\tau S'_\tau / K\right]. && (3.14)
\end{aligned}$$

The last two steps come from a property of conditional expectation and the change in measure. For the price of the claim at time ς we have

$$\begin{aligned}
e^{\int_0^\varsigma r_s ds}\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_0^\tau r_s ds} P_\tau|\mathcal{F}_\varsigma\right) &= \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_\varsigma^\tau r_s ds} P_\tau|\mathcal{F}_\varsigma\right) && (3.15)\\
&= \frac{\mathbb{E}_{\mathbb{Q}'}\left(\frac{1}{Z_T} e^{-\int_\varsigma^\tau r_s ds} P_\tau|\mathcal{F}_\varsigma\right)}{\mathbb{E}_{\mathbb{Q}'}\left(\frac{1}{Z_T}|\mathcal{F}_\varsigma\right)} && (3.16)\\
&= Z_\varsigma \mathbb{E}_{\mathbb{Q}'}\left(\left(\frac{1}{Z_T} e^{-\int_\varsigma^\tau r_s ds} P_\tau|\mathcal{F}_\tau\right)|\mathcal{F}_\varsigma\right). && (3.17)
\end{aligned}$$

The last two steps come from Bayes' formula and the tower property for conditional expectations. Again, $1/Z$ is a $\mathbb{Q}'$-martingale and the rest of the terms are $\mathcal{F}_\tau$-measurable such that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_\varsigma^\tau r_s ds} P_\tau|\mathcal{F}_\varsigma\right) &= Z_\varsigma \mathbb{E}_{\mathbb{Q}'}\left(\left(\frac{1}{Z_T}|\mathcal{F}_\tau\right) e^{-\int_\varsigma^\tau r_s ds} P_\tau|\mathcal{F}_\varsigma\right) && (3.18)\\
&= \mathbb{E}_{\mathbb{Q}'}\left(\frac{Z_\varsigma}{Z_\tau} e^{-\int_\varsigma^\tau r_s ds} P_\tau|\mathcal{F}_\varsigma\right) && (3.19)\\
&= \mathbb{E}_{\mathbb{Q}'}\left(e^{-\int_\varsigma^\tau \delta_s ds} P_\tau S^*_\tau / K|\mathcal{F}_\varsigma\right). && (3.20)
\end{aligned}$$

From Assumption 3.1 $Z_t \equiv S_t e^{\int_0^t (\delta_s - r_s) ds}/S_0 = S_t X_t$, where $X_t = \frac{1}{S_0}\exp[\int_0^t (\delta_s - r_s) ds]$, is a $\mathbb{Q}$-martingale. Applying Itô's formula and noticing that X_t is a continuous bounded variation process,

$$\begin{aligned}
dZ_t &= S_{t-} dX_t + X_t dS_t && (3.21)\\
&= S_{t-}(\delta_t - r_t) X_t dt + X_t dS_t. && (3.22)
\end{aligned}$$

By rearranging the terms we get

$$\begin{aligned}
dS_t &= S_{t-}(r_t - \delta_t) dt + \frac{1}{X_t} dZ_t && (3.23)\\
&= S_{t-}(r_t - \delta_t) dt + dM_t. && (3.24)
\end{aligned}$$

Similarly with $1/Z_t = S'_t X'_t$, where $X'_t = \frac{1}{K}\exp[\int_0^t (r_s - \delta_s) ds]$, being a $\mathbb{Q}'$-martingale the second result is obtained as follows

$$\begin{aligned}
d(Z_t^{-1}) &= S'_{t-} dX'_t + X'_t dS'_t && (3.25)\\
&= S'_{t-}(r_t - \delta_t) X'_t dt + X'_t dS'_t && (3.26)\\
dS'_t &= S'_{t-}(\delta_t - r_t) dt + \frac{1}{X'_t} d(Z_t^{-1}) && (3.27)\\
&= S'_{t-}(\delta_t - r_t) dt + dM'_t. && (3.28)
\end{aligned}$$

From the above

$$\begin{aligned} dM_t &= \frac{1}{X_t} dZ_t \quad \text{and} && (3.29)\\ dM'_t &= \frac{1}{X'_t} d(Z_t^{-1}). && (3.30)\end{aligned}$$

so that M and M' will be respectively $\mathbb{Q}$- and $\mathbb{Q}'$-martingales if $1/X_t = S_0 \exp[\int_0^t (r_s - \delta_s)ds]$ and $1/X'_t = K \exp[\int_0^t (\delta_s - r_s)ds]$ are bounded. Both of these are satisfied when $(r - \delta)$ is a bounded process. Otherwise M and M' are local martingales.
The quadratic variations of M and M' therefore satisfy

$$\begin{aligned} \frac{d[M]}{(S_t)^2} &= \frac{(1/X_t)^2 d[Z]_t}{(S_t)^2} = \frac{d[Z]_t}{Z_t^2} && (3.31)\\ \frac{d[M']}{(S'_t)^2} &= \frac{(1/X'_t)^2 d\,[1/Z]_t}{(S'_t)^2} = \frac{d[1/Z]_t}{(1/Z_t)^2}. && (3.32)\end{aligned}$$

Between the jumps of S both Z and $1/Z$ are continuous (remember that Z is positive almost surely). Applying Itô's lemma to $1/Z$:

$$\begin{aligned} d\frac{1}{Z_t} &= -\frac{1}{Z_t^2} dZ_t + \frac{1}{Z_t^3} d[Z]_t && (3.33)\\ \frac{d[Z]_t}{Z_t^2} &= \frac{d(1/Z_t)}{1/Z_t} + \frac{dZ_t}{Z_t} = \frac{d[1/Z]_t}{(1/Z_t)^2}, && (3.34)\end{aligned}$$

the last result coming from the symmetry in the equations. Thus

$$\frac{d[M]_t}{(S_t)^2} = \frac{d[M']_t}{(S'_t)^2}.\diamond \qquad (3.35)$$

The first example looks at a numéraire change with the asset S following specified dynamics.

Example 3.1. Let $\mathbf{W} \equiv [W^1, W^2, ..., W^d]$ denote a vector of d independent standard Brownian motions under the risk-neutral measure Q. The asset price, S, and the state variables, $\mathbf{X} \equiv [X^1, ..., X^m]$, are assumed to satisfy the following dynamics

$$\begin{aligned} \frac{dS_t}{S_t} &= (r(t, \mathbf{X}_t) - \delta(t, \mathbf{X}_t))dt + \sigma(S_t, \mathbf{X}_t)dW_t^1 && (3.36)\\ d\mathbf{X}_t &= \mu(\mathbf{X}_t)dt + \phi(\mathbf{X}_t)d\mathbf{W}_t && (3.37)\end{aligned}$$

where μ is $m \times 1$, ϕ is $m \times d$ and r,δ and σ are all real-valued. As above $\widehat{Q}$ is defined via the Radon-Nikodym derivative $d\hat{Q}/dQ = Z_T$. With S having continuous paths the dynamics of Z

$$\begin{aligned} dZ_t &= \frac{S_t}{S_0} d(e^{\int_0^t (\delta(s,\mathbf{X}_s) - r(s,\mathbf{X}_s))ds}) + e^{\int_0^t (\delta(s,\mathbf{X}_s) - r(s,\mathbf{X}_s))ds} d(\frac{S_t}{S_0}) && (3.38)\\ &= Z_t(\delta(t, \mathbf{X}_t) - r(t, \mathbf{X}_t))dt + Z_t(r(t, \mathbf{X}_t) - \delta(t, \mathbf{X}_t))dt + Z_t\sigma(S_t, \mathbf{X}_t)dW_t^1 && (3.39)\\ &= Z_t\sigma(S_t, \mathbf{X}_t)dW_t^1 && (3.40)\end{aligned}$$

has the explicit solution

$$Z_t = exp\big[-\frac{1}{2}\int_0^t \sigma(S_t, \mathbf{X}_t)^2 ds + \int_0^t \sigma(S_t, \mathbf{X}_t)dW_s^1\big]. \qquad (3.41)$$

By Girsanov's theorem

$$\widehat{\mathbf{W}}_t = \mathbf{e}\int_0^t \sigma(S_t, \mathbf{X}_t)ds - \mathbf{W}_t \qquad (3.42)$$

is a d-dimensional standard Brownian motion under $\widehat{Q}$, where $\mathbf{e} = [1, 0, ..., 0]'$. Define $\widehat{S} = KS_0/S_t$, then from Itô's formula and Girsanov's theorem

$$\begin{aligned} d(\frac{1}{S_t}) &= -\frac{1}{S_t^2} dS_t + \frac{1}{S_t^3} d\langle S \rangle_t && (3.43) \\ \frac{d\widehat{S}_t}{\widehat{S}_t} &= -\frac{dS_t}{S_t} + \frac{d\langle S \rangle_t}{S_t^2} && (3.44) \\ &= (\delta(t, \mathbf{X}_t) - r(t, \mathbf{X}_t))dt - \sigma(S_t, \mathbf{X}_t)dW_t^1 + \sigma(S_t, \mathbf{X}_t)^2 dt && (3.45) \\ &= (\delta(t, \mathbf{X}_t) - r(t, \mathbf{X}_t))dt + \sigma(KS_0/\widehat{S}_t, \mathbf{X}_t)d\widehat{W}_t^1 && (3.46) \end{aligned}$$

and

$$d\mathbf{X}_t = [\mu(\mathbf{X}_t) + \phi(\mathbf{X}_t)\sigma(KS_0/\widehat{S}_t, \mathbf{X}_t)\mathbf{e}]dt - \phi(\mathbf{X}_t)d\widehat{\mathbf{W}}_t. \quad (3.47)$$

Note that the modification to the drift term is the instantaneous covariance between asset returns and the increments in the vector of state variables, i.e. $d\langle \mathbf{S}, \mathbf{X} \rangle_t = \phi(\mathbf{X}_t)\sigma(S_t, \mathbf{X}_t)$, where $\mathbf{S} = [S, 0, ..., 0]'$ is a $m \times 1$ vector.

The latter describe the dynamics of $\widehat{S}$, after a numéraire change. To check the instantaneous volatilities of returns:

$$\begin{aligned} \frac{d[M]_t}{S_t^2} &= \frac{S_t^2 \sigma(S_t, \mathbf{X}_t)^2 d[W^1]_t}{S_t^2} = \sigma(S_t, \mathbf{X}_t)^2 dt && (3.48) \\ \frac{d[\widehat{M}]_t}{\widehat{S}_t^2} &= \frac{\widehat{S}_t^2 \sigma(KS_0/\widehat{S}_t, \mathbf{X}_t)^2 d[\widehat{W}^1]_t}{\widehat{S}_t^2} = \sigma(KS_0/\widehat{S}_t, \mathbf{X}_t)^2 dt && (3.49) \end{aligned}$$

and thus

$$\frac{d[M]_t}{(S_t)^2} = \sigma^2 dt = \frac{d[\widehat{M}]_t}{(\widehat{S}_t)^2}. \diamond \quad (3.50)$$

The above setup is used frequently in examples in chapter three and four. In the next example we shall see how a different numéraire changes both the intensity and distribution of jumps in a jump-diffusion model, illustrating that the square returns will generally be different at jumps.

Example 3.2. Assume the riskless rate and dividend yield are both zero, and the asset price S follows a Poisson jump process $Po(\lambda)$ under the risk-neutral probability measure Q. At jumps (τ_i,$i = 1, 2, ...$) we have a Bernoulli distribution

$$S(\tau_i) = \begin{cases} uS(\tau_i-) \text{ , with } & Q\text{-probability} \quad p, \\ dS(\tau_i-) \text{ , with } & Q\text{-probability} \quad 1-p, \end{cases} \quad (3.51)$$

and between jumps,

$$\frac{dS_t}{S_t} = (1-\mu)\lambda dt, \quad \tau_i < t < \tau_{i+1}, \quad i = 0, 1, ... \quad (3.52)$$

where $\mu = pu + (1-p)d$ is the expected price ratio at jumps and $\tau_0 = 0$. Let the underlying asset price S be our new numéraire with new measure Q_S where $\frac{dQ_S}{dQ} = \frac{S_T}{S_0}$. The following identity (see Appendix A for proof) gives us

$$Q_S(S(\tau_i) = uS(\tau_i-)) = pu\mathbb{E}_Q(S(\tau_i-)/S(0)) \quad (3.53)$$

and because the Poisson process is continuous in probability (and thus in law) we have

$$\begin{aligned} Q_S(S(\tau_i) = uS(\tau_i-)) &= pu\mathbb{E}_Q(e^{(1-\mu)\lambda(\tau_i - \tau_{i-1})}) && (3.54) \\ &= pu\left(\frac{\lambda}{\lambda - (1-\mu)\lambda}\right) && (3.55) \\ &= pu\mu^{-1}. && (3.56) \end{aligned}$$

Similar to the above we have the complete result

$$S(\tau_i)^{-1} = \begin{cases} u^{-1}S(\tau_i-)^{-1} \text{ , with } & Q_S\text{-probability } \quad pu\mu^{-1}, \\ d^{-1}S(\tau_i-)^{-1} \text{ , with } & Q_S\text{-probability } \quad (1-p)d\mu^{-1}, \end{cases} \tag{3.57}$$

and between jumps

$$\frac{d(S_t^{-1})}{S_t^{-1}} = (\mu - 1)\lambda dt, \quad \tau_i < t < \tau_{i+1}, \quad i = 0, 1, ... \tag{3.58}$$

The intensity of the jump process under Q_S is $\mu\lambda$, because S is a martingale under Q and

$$Q_S(\tau_1 > t) = \mathbb{E}_Q[1_{(\tau_1>t)}S_T/S_0] = \mathbb{E}_Q[1_{(\tau_1>t)}S_t/S_0] = e^{(1-\mu)\lambda t}Q(\tau_1 > t) \tag{3.59}$$

and the memoryless property of the jump times. Returns of S under Q and S^{-1} under Q_S are identical *only* in the special case when $u = d^{-1}$ and $\mu = 1$.$\diamond$

The theorem as well as its corollaries do not assume any specific dynamics for S, as in the examples. In the sequel our focus will move to pricing futures, forwards and options.

Corollary 1. The value of an European call option on S with strike price K is the same, after numéraire change, as the value of an European put option on S' with strike price S_0, both maturing at time τ,:

$$\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_0^\tau r_s ds}(S_\tau - K)^+\right) = \mathbb{E}_{\mathbb{Q}'}\left(e^{-\int_0^\tau \delta_s ds}(S_0 - S'_\tau)^+\right) \tag{3.60}$$

for any stopping time $\tau \leq T$.

Proof
Set the payoff to be $P_\tau = (S_\tau - K)^+$.$\diamond$

Notice how the riskless rate and the dividend rate interchange roles. Corollary 1 can be extended to American options if the next assumption holds:

Assumption 3.2. The fair price of a American option allowing the holder to exercise and receive, at any stopping time $\tau \in [0, T]$, the payoff P_τ, P an adapted process, is given as

$$\sup_{\tau\in[0,T]} \mathbb{E}_{\mathbb{Q}}(e^{-\int_0^\tau r_s ds}P_\tau). \tag{3.61}$$

Corollary 2. The value of an asset-or-nothing binary option on S with strike price K is the same, after numéraire change, as the value of a cash-or-nothing binary option on S' with strike price S_0, both maturing at time τ,:

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^\tau r_s ds}S_\tau 1_{\{S_\tau \geq K\}}\right] = S_0\mathbb{E}_{\mathbb{Q}'}\left[e^{-\int_0^\tau \delta_s ds}1_{\{S'_\tau \leq S_0\}}\right] \tag{3.62}$$

for any stopping time $\tau \leq T$.

Proof
Take $P_\tau = S_\tau 1_{\{S_\tau \geq K\}}$ and note that $\{S_\tau \geq K\} = \{S'_\tau \leq S_0\}$.$\diamond$

Consider the stopping time $\tau = \min[T, \inf\{t : S_t \geq K\}]$ and assume that S is continuous. The continuity assumption ensures that the asset hits the barrier level and does not 'jump' over it. Then the value of a barrier option on S which pays K when the asset rises to K and nothing

otherwise is the same as a barrier option on S' which pays S_0 when the asset falls to S_0 and nothing otherwise.
With $\delta = 0$, r deterministic and $B_T(t) = exp(-\int_t^T r_s ds)$ the previous results of the formula (1.28) to (1.31) for an European call maturing at time T can be obtained:

$$\begin{aligned} \frac{C(0)}{B_T(0)} &= \mathbb{E}_{Q_T}\left[\left(\frac{S(T)}{B_T(T)} - K\right)^+\right] &(3.63)\\ C(0) &= \mathbb{E}_{Q_T}[B_T(0)(S(T)-K)^+] &(3.64)\\ C(0) &= \mathbb{E}_{Q_T}[B_T(0)S(T)1_A] - KB_T(0)Q_T(A) &(3.65)\end{aligned}$$

with $A = \{\omega | S(T,\omega) \geq K\} \equiv \{\omega | S(0,\omega) \geq S'(T,\omega)\}$. Applying corollary 2 with change in measure from Q_T to Q_S:

$$\begin{aligned} C(0) &= S(0)\mathbb{E}_{Q_S}[1_A] - KB_T(0)Q_T(A) &(3.66)\\ &= S(0)Q_S(A) - KB_T(0)Q_T(A). &(3.67)\end{aligned}$$

This result is a slight improvement on the previous, because here there is no assumption of continuity for S.
The general pricing formula for an European call option on S with strike price S'_0 maturing at the stopping time τ is

$$\begin{aligned} C_0 &= \mathbb{E}_Q\left(e^{-\int_0^\tau r_s ds}(S_\tau - S'_0)^+\right) &(3.68)\\ &= \mathbb{E}_Q\left(e^{-\int_0^\tau r_s ds}S_\tau 1_A\right) - \mathbb{E}_Q\left(e^{-\int_0^\tau r_s ds}S'_0 1_A\right) &(3.69)\\ &= S_0\mathbb{E}_{Q'}\left(e^{-\int_0^\tau \delta_s ds}1_A\right) - S'_0\mathbb{E}_Q\left(e^{-\int_0^\tau r_s ds}1_A\right) &(3.70)\\ &= S_0 e^{-\int_0^t \delta_s ds}Q'(S_t \geq S'_0) - S'_0 e^{-\int_0^t r_s ds}Q(S_t \geq S'_0), &(3.71)\end{aligned}$$

the last equality considering r and δ deterministic and $\tau = t$ constant. Verbally the expression states that the price of a European call is the difference between the current stock price (depreciated by possible future dividends) and the strike price (discounted by the short-rate) all under their separate probabilities of the option being exercised. It is important to note that the probability measures Q and Q' are used and thus that this is a numéraire technique. The same can be done with a European put option on S' with strike price S_0 maturing at stopping time τ:

$$\begin{aligned} P_0 &= \mathbb{E}_Q\left(e^{-\int_0^\tau r_s ds}(S_0 - S'_\tau)^+\right) &(3.72)\\ &= \mathbb{E}_{Q'}\left(e^{-\int_0^\tau \delta_s ds}(S_\tau - S'_0)^+\right) &(3.73)\\ &= S_0\mathbb{E}_Q\left(e^{-\int_0^\tau r_s ds}1_A\right) - S'_0\mathbb{E}_{Q'}\left(e^{-\int_0^\tau \delta_s ds}1_A\right) &(3.74)\\ &= S_0 e^{-\int_0^t r_s ds}Q(S'_t \leq S_0) - S'_0 e^{-\int_0^t \delta_s ds}Q'(S'_t \leq S_0), &(3.75)\end{aligned}$$

the last equality considering r and δ deterministic and $\tau = t$ constant. In the scenario for pricing a European put is the difference between the strike price (discounted by the short-rate) and the current stock price (depreciated by possible future dividends) under their respective probabilities of the option being exercised.

Note: (i) A pricing example of the Black-Scholes-Merton formula can be found in Appendix B.
(ii) The above pricing under Assumption 3.2 can be extended to American put and call options.
(iii) An application to foreign exchange will be discussed in the foreign exchange section.

Corollary 3. The futures price on S maturing at time T, after numéraire change, is given by

$$G_S(t,T) = S_t\mathbb{E}_{\mathbb{Q}'}\left(e^{\int_t^T (r_s - \delta_s)ds}|\mathcal{F}_t\right). \qquad (3.76)$$

Proof
Starting with the price of a future and using Bayes' formula (see Appendix A),

$$G_S(t,T) = \mathbb{E}_{\mathbb{Q}}\left[S_T|\mathcal{F}_t\right] \tag{3.77}$$

$$= \frac{\mathbb{E}_{\mathbb{Q}'}\left[S_T\frac{1}{Z_T}|\mathcal{F}_t\right]}{\mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{Z_T}|\mathcal{F}_t\right]} \tag{3.78}$$

$$= \frac{\mathbb{E}_{\mathbb{Q}'}\left[S_T\frac{S_0}{S_T}e^{\int_0^T(r_s-\delta_s)ds}|\mathcal{F}_t\right]}{\frac{S_0}{S_t}e^{\int_0^t(r_s-\delta_s)ds}} \tag{3.79}$$

$$= S_t\mathbb{E}_{\mathbb{Q}'}\left[e^{\int_t^T(r_s-\delta_s)ds}|\mathcal{F}_t\right].\diamond \tag{3.80}$$

Under the numéraire change the futures price depends on the spot price and the expected increase in interest and loss of dividends over this period of time. The difference between the riskless rate and the dividend rate is known as the cost of carry.

As previously described, bonds are claims that pay out 1 at the end of maturity. Let $B(t,T)$ represent the time t value of a bond that pays out one unit of the short-rate (money account) at time T:

$$B(t,T) = \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}\Big|\mathcal{F}_t\right). \tag{3.81}$$

In a similar way, let $B'(t,T)$ represent the time t value of a bond, measured in units of the asset, that pays out one unit of the asset at time T:

$$B'(t,T) = S_t^{-1}\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}S_T\Big|\mathcal{F}_t\right). \tag{3.82}$$

After changing the numéraire from the short-rate to the asset

$$B'(t,T) = S_t^{-1}\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}S_T\Big|\mathcal{F}_t\right) = \mathbb{E}_{\mathbb{Q}'}\left(e^{-\int_t^T \delta_s ds}\Big|\mathcal{F}_t\right). \tag{3.83}$$

To see this, take the expectation of the bond with any $F \in \mathcal{F}_t$,

$$\mathbb{E}_{\mathbb{Q}}\Big[1_F B'(t,T)\Big] = \mathbb{E}_{\mathbb{Q}}\Big[1_F S_t^{-1}\mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T r_s ds}S_T\Big|\mathcal{F}_t\Big)\Big] \tag{3.84}$$

$$= \mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\Big(1_F S_t^{-1}e^{-\int_t^T r_s ds}S_T\Big|\mathcal{F}_t\Big)\Big] \tag{3.85}$$

$$= \mathbb{E}_{\mathbb{Q}}\Big(1_F S_t^{-1}e^{-\int_t^T r_s ds}S_T\Big). \tag{3.86}$$

Now apply Theorem 3.1 with payoff $P_T = 1_F\frac{S_T}{S_t}$ and note the spot price value of the asset is S_t,

$$\mathbb{E}_{\mathbb{Q}}\Big[1_F B'(t,T)\Big] = \mathbb{E}_{\mathbb{Q}}\Big(1_F S_t^{-1}e^{-\int_t^T r_s ds}S_T\Big) \tag{3.87}$$

$$= \mathbb{E}_{\mathbb{Q}'}\Big(e^{-\int_t^T \delta_s ds}1_F\frac{S_T}{S_t}\frac{S_t}{S_T}\Big) \tag{3.88}$$

$$= \mathbb{E}_{\mathbb{Q}'}\Big(e^{-\int_t^T \delta_s ds}1_F\Big) \tag{3.89}$$

$$= \mathbb{E}_{\mathbb{Q}'}\Big[1_F\mathbb{E}_{\mathbb{Q}'}\Big(e^{-\int_t^T \delta_s ds}\Big|\mathcal{F}_t\Big)\Big]. \tag{3.90}$$

Since both sides are $\mathcal{F}_t$-measurable, (3.83) follows. The forward price on the asset S is given as

$$F_S(t,T) = \mathbb{E}_{\mathbb{Q}}(e^{-\int_t^T r_s ds}S_T|\mathcal{F}_t)/B(t,T) \tag{3.91}$$

and the following result is obtained:

Corollary 4. The forward price on the asset S is given by

$$F_S(t,T) = S_t \frac{B'(t,T)}{B(t,T)}. \tag{3.92}$$

If the short-rate process is deterministic, then the time zero price of a forward and a future are equal and gives the result in corollary 3. When $\delta = r$ then G_S and S have the same movement. In this case the forward contract on the futures contract is equivalent to a forward contract on the asset underlying the futures contract. Here a simple ratio between forwards and futures on the same underlying asset is obtained:

$$F_S(t,T) = G_S(t,T)\frac{B'(t,T)}{B(t,T)} \tag{3.93}$$

$$\frac{F_S(t,T)}{G_S(t,T)} = \frac{B'(t,T)}{B(t,T)} \tag{3.94}$$

with B and B' considering the short rate process with the respected numéraires.

Corollary 5. Let S^a and S^b denote two assets with respective dividend rates δ^a and δ^b. Under Assumption 3.1 and from Theorem 3.1 the assets have the following dynamics

$$dS_t^i = S_{t-}^i(r_t - \delta_t^i)dt + dM_t^i, \quad i \in \{a,b\}, \tag{3.95}$$

where M^i a $\mathbb{Q}$-(local) martingale. Let $A_t = S_t^b S_0^a / S_t^a$ and $\mathbb{Q}^a$ be the corresponding measure change for the change in numéraire from the short-rate process to the asset S^a. Then the value of an option to receive one unit of asset S^b in exchange for one unit of asset S^a is the same, after a change of numéraire, as the value of a call option on A with strike price S_0^a:

$$\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_0^\tau r_s ds}(S_\tau^b - S_\tau^a)^+\right) = \mathbb{E}_{\mathbb{Q}^a}\left(e^{-\int_0^\tau \delta_s^a ds}(A_\tau - S_0^a)^+\right) \tag{3.96}$$

for any stopping time $\tau \leq T$. Furthermore

$$dA_t = A_{t-}(\delta_t^a - \delta_t^b)dt + dN_t, \quad A_0 = S_0^b, \tag{3.97}$$

where N is a local martingale under $\mathbb{Q}^a$.
Note: If $(\delta^a - \delta^b)$ is a bounded process, then N is a martingale.

Remark: When Assumption 3.1 is relaxed to admit a local martingale measure, it does not change the dynamics of the assets mentioned above. Proposition 1.2 states that S^b will remain a local martingale under the new numéraire S^a and Theorem 2.3 provides the corresponding measure $\mathbb{Q}^a$. Excluding the *note*, Corollary 5 remains valid in these general conditions.

Proof of Corollary 5
Apply Theorem 3.1 with $P_\tau = (S_\tau^b - S_\tau^a)^+$, then

$$\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_0^\tau r_s ds}(S_\tau^b - S_\tau^a)^+\right) = \mathbb{E}_{\mathbb{Q}^a}\left(e^{-\int_0^\tau \delta_s^a ds}(S_\tau^b - S_\tau^a)^+(S_0^a/S_\tau^a)\right) \tag{3.98}$$

$$= \mathbb{E}_{\mathbb{Q}^a}\left(e^{-\int_0^\tau \delta_s^a ds}(A_\tau - S_0^a)^+\right). \tag{3.99}$$

Under Assumption 3.1 let us denote the $\mathbb{Q}$-martingale discounted processes by

$$Z_t^i = S_t^i e^{\int_0^t (\delta_s^i - r_s)ds} / S_0^i, \quad i \in \{a,b\}. \tag{3.100}$$

Through Bayes' formula the ratio of Z^b and Z^a is a $\mathbb{Q}^a$-martingale,

$$\begin{aligned} Z_s^b = \mathbb{E}_{\mathbb{Q}}\left[Z_t^b|\mathcal{F}_s\right] &= \mathbb{E}_{\mathbb{Q}}\left[\frac{Z_t^b}{Z_t^a}Z_t^a|\mathcal{F}_s\right] && (3.101)\\ &= \mathbb{E}_{\mathbb{Q}^a}\left[\frac{Z_t^b}{Z_t^a}|\mathcal{F}_s\right]\mathbb{E}_{\mathbb{Q}}\left[Z_t^a|\mathcal{F}_s\right] && (3.102)\\ &= Z_s^a\mathbb{E}_{\mathbb{Q}^a}\left[\frac{Z_t^b}{Z_t^a}|\mathcal{F}_s\right]. && (3.103)\end{aligned}$$

With $\Delta_t = \exp(\int_0^t \delta_s^b - \delta_s^a ds)$, a continuous process of bounded variation, apply integration by parts to get

$$\begin{aligned} d\Big(\frac{Z_t^b}{Z_t^a}\Big) &= d\Big(\Delta_t\frac{S_t^b}{S_0^b}\frac{S_0^a}{S_t^a}\Big) && (3.104)\\ S_0^b d\Big(\frac{Z_t^b}{Z_t^a}\Big) &= A_{t-}d\Delta_t + \Delta_t dA_t && (3.105)\\ &= A_{t-}(\delta_t^b - \delta_t^a)\Delta_t dt + \Delta_t dA_t && (3.106)\\ dA_t &= A_{t-}(\delta_t^a - \delta_t^b)dt + S_0^b\Delta_t^{-1}d\Big(\frac{Z_t^b}{Z_t^a}\Big). && (3.107)\end{aligned}$$

An integrand consisting of a continuous process Δ^{-1} and constant S_0^b with respect to a (local) martingale integrator makes the process defined as

$$dN_t = S_0^b\Delta_t^{-1}d\Big(\frac{Z_t^b}{Z_t^a}\Big) \qquad (3.108)$$

a $\mathbb{Q}^a$-local martingale.$\diamond$

The next example is an application of Corollary 5 and used frequently for specific instruments and for pricing problems, as we shall see in chapter four.

Example 3.3. Let $\mathbf{W} \equiv [W^1, W^2, ..., W^d]$ denote a vector of d independent standard Brownian motions under the risk-neutral measure Q. The risk-neutral price processes of assets a and b satisfy

$$\begin{aligned} \frac{dS_t^a}{S_t^a} &= (r(t,\mathbf{X}_t) - \delta^a(t,\mathbf{X}_t))dt + \sigma^a dW_t^1 && (3.109)\\ \frac{dS_t^b}{S_t^b} &= (r(t,\mathbf{X}_t) - \delta^b(t,\mathbf{X}_t))dt + \sigma^b[\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2], && (3.110)\end{aligned}$$

where the volatility coefficients, σ^a and σ^b, and the instantaneous correlation betweeen asset returns, $\rho \in [-1,1]$, are constants. The riskless rate, r, and the dividend rates, δ^a and δ^b, are functions of the m-dimensional state variable vector, $\mathbf{X}$, which satisfies

$$d\mathbf{X}_t = \mu(\mathbf{X}_t)dt + \phi(\mathbf{X}_t)d\mathbf{W}_t \qquad (3.111)$$

with appropriate dimensions as stated in example 3.1. Let $\widehat{Q}$ denote the change in measure for S^a acting as the new numéraire. By Girsanov's theorem

$$\widehat{\mathbf{W}}_t = t\sigma^a\mathbf{e} - \mathbf{W}_t, \qquad (3.112)$$

is a d-dimensional standard Brownian motion under $\widehat{Q}$, where $\mathbf{e} = [1, 0, ..., 0]'$. Define $\widehat{S}_t^a = S_0^a/S_t^a$ so that we have the following dynamics

$$\frac{d\widehat{S}_t^a}{\widehat{S}_t^a} = (\delta^a(t,\mathbf{X}_t) - r(t,\mathbf{X}_t))dt + \sigma^a d\widehat{W}_t^1. \qquad (3.113)$$

Apply Itô's formula to $A_t = S_t^b \widehat{S}_t^a$ and obtain

$$
\begin{aligned}
dA_t &= S_t^b d\widehat{S}_t^a + \widehat{S}_t^a dS_t^b + d[S^b, \widehat{S}^a]_t && (3.114)\\
&= A_t\big((\delta_t^a - r_t)dt + \sigma^a d\widehat{W}_t^1\big) + A_t\big((r_t - \delta_t^b)dt + \sigma^b(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)\big) && (3.115)\\
&\quad + \sigma^b \rho S_t^b \sigma^a \widehat{S}_t^a d[W^1, \widehat{W}^1]_t + \sigma^b \sqrt{1-\rho^2} S_t^b \sigma^a \widehat{S}_t^a d[W^2, \widehat{W}^1]_t. && (3.116)
\end{aligned}
$$

Substituting equation (3.112) and remembering that W^1 and W^2 are independent Q-Brownian motions as well as noticing that $\sigma^a t$ is of bounded variation, we simplify the dynamics of A to the form in Corollary 5:

$$
\begin{aligned}
\frac{dA_t}{A_t} &= (\delta^a - \delta^b)dt + \sigma^a d\widehat{W}_t^1 + \sigma^b \rho dW_t^1 + \sigma^b \sqrt{1-\rho^2} dW_t^2 && (3.117)\\
&\quad - \sigma^b \sigma^a \rho d[W^1]_t - \sigma^b \sigma^a \sqrt{1-\rho^2} d[W^2, W^1]_t && (3.118)\\
&= (\delta^a - \delta^b)dt + \sigma^a d\widehat{W}_t^1 - (-\sigma^b \sqrt{1-\rho^2} dW_t^2) + \sigma^b \rho (dW_t^1 - \sigma^a dt) && (3.119)\\
&= (\delta^a - \delta^b)dt + (\sigma^a - \sigma^b \rho) d\widehat{W}_t^1 - (\sigma^b \sqrt{1-\rho^2} d\widehat{W}_t^2). && (3.120)
\end{aligned}
$$

Thus the valuation of an exchange option between assets S^a and S^b is reduced to that of a call option on A with riskless rate δ^a and dividend rate δ^b. Note that in this example our stocks are driven by Brownian motions and the respected volatilities are constant.

3.1 Foreign Exchange

An important practical description of the general theory lies in the foreign exchange market. Let E be a replacement for S as the exchange rate between the domestic and foreign currency. The domestic riskless rate r remains and instead of δ we have r^* representing the riskless rate in a foreign country.

From corollary 1 and corollary 2 we are able to express a general formula and relation for foreign exchange Put and Call options maturing at time τ and strike price K:

$$
\begin{aligned}
C(E, K, \tau, r^*, r) &= E_0 \mathbb{E}_{Q'}\left(e^{-\int_0^\tau r_s^* ds} 1_A\right) - K \mathbb{E}_Q\left(e^{-\int_0^\tau r_s ds} 1_A\right) && (3.121)\\
&= E_0 K \left(\frac{1}{K}\mathbb{E}_{Q'}\left(e^{-\int_0^\tau r_s^* ds} 1_A\right) - \frac{1}{E_0}\mathbb{E}_Q\left(e^{-\int_0^\tau r_s ds} 1_A\right)\right) && (3.122)\\
&= E_0 K P^*\left(\frac{1}{E}, \frac{1}{K}, \tau, r, r^*\right), && (3.123)
\end{aligned}
$$

where $A = \{\omega | E(T, \omega) \geq K\}$. The last equality comes from the general formula for the Put option, which shares a relation with its foreign Call as well:

$$
P(E, K, \tau, r^*, r) = K \mathbb{E}_Q\left(e^{-\int_0^\tau r_s ds} 1_A\right) - E_0 \mathbb{E}_{Q'}\left(e^{-\int_0^\tau r_s^* ds} 1_A\right) = E_0 K C^*\left(\frac{1}{E}, \frac{1}{K}, \tau, r, r^*\right). \quad (3.124)
$$

Under Assumption 3.2 the relation remains intact for American options:

$$
c(E, K, \tau, r^*, r) = E_0 K p^*\left(\frac{1}{E}, \frac{1}{K}, \tau, r, r^*\right). \quad (3.125)
$$

The above relation is the same as the one derived from the American Put-Call symmetry at the end of chapter one. Though in this chapter we assume no specific dynamics for the exchange rate

E, as was the case for the American Put-Call symmetry, where the dynamics are driven by a Brownian motion.

The applications of corollary 3 and corollary 4 are straight forward. Corollary 4 provides a general Interest Parity Theorem (no continuity assumption) to the result obtained in chapter one:
(Interest Parity Theorem) The forward of one unit of foreign exchange maturing at time T is

$$F_{EB^*}(t,T) = E(t)\frac{B^*(t,T)}{B(t,T)}. \tag{3.126}$$

Do consult Grabbe [16] for pricing foreign exchange derivatives. In his paper he assumes the Interest Parity Theorem and applies dynamics driven by Brownian motion to the exchange rates and bond structures. The following section expands the concept to any comparative article in the different countries.

Gold as numéraire

In the book by Grant [17], he refers to gold as the oldest measurement of wealth. From Biblical times kings traded in gold. During the rule of the Roman Empire gold and silver coins where used as currency to trade with. The Romans also gave birth to inflation. Traders would scrape thin layers of the edge of the coins and melt the shredded pieces together to form more coins. This created more currency that was originally intended for the market which in turned produced inflation. As inflation was present, larger quantities of gold were needed to trade with. For more convenient business trading, policy introduced promissory notes representing a certain amount of ounces of gold in the bank. These promissory notes are known to us as money. Then came the change - money became the trading currency and gold a commodity. Banks would lend money to citizens and businesses against a borrowing rate. Banks would even short-sell money, i.e. lend money they did not even own. In modern times a corporate bank would borrow its deficit from the reserve bank. Hence hard currency was born and gold became a commodity.

Different currencies are in use in different countries. Their relation to each other we call foreign exchange. To model foreign exchange rates, one could choose a currency of preference (say the rand) and study the change of other currencies (dollar, euro, sterling) with respect to this currency. Considering another preferred currency would only be an application of the change of numéraire technique. A concern in this approach is that one only models the relationship between currencies and not the currencies themselves. Grant [17] suggests using the commodity gold as a benchmark. His argument is not a mathematical approach, but one of economical policy. Though it does beg the question: Is a given currency worth its 'weight in gold'?

A mathematical approach: Applying corollary 5
Let r denote the riskless rate in the United States market. The non-dividend paying commodity G (gold) has dynamics

$$dG_t = r_t G_{t-} dt + dM_t^G, \tag{3.127}$$

where M^G is an Q-martingale. The probability measure Q is a risk-neutral measure associated with the extended foreign exchange market $(G, E^1, E^2, ..., E^k)$, E^i representing the exchange rate of a specified currency with the dollar currency acting as numéraire. Again from Assumption 3.1 and Theorem 3.1 the exchange rates follow the dynamics

$$dE_t^i = E_{t-}^i(r_t - r_t^i)dt + dM_t^i, \quad i \in \{1, 2, ..., k\} \tag{3.128}$$

where r^i is the riskless rate in the country with currency E^i and M^i is an Q-martingale for all $i \in \{1, 2, ..., k\}$. Changing the numéraire to gold, our risk-neutral measure changes to Q_g. The dynamics for the dollar $D_t = G_0/G_t$ are given by

$$dD_t = -r_t D_{t-} dt + dM_t^D \tag{3.129}$$

and dynamics for the other exchange rates $\widehat{E}_t^i = E_t^i G_0/G_t$ are

$$d\widehat{E}_t^i = r_t^i \widehat{E}_{t-}^i dt + d\widehat{M}_t^i, \quad i \in \{1, 2, ..., k\} \tag{3.130}$$

where $\widehat{M}^i$ is an Q_g-martingale for all $i \in \{D, 1, 2, ..., k\}$ and the currencies measured in ounces of gold. Note that the riskless rate of each currency is still present in the above numéraire change. There are two reasons for this. One, without their domestic rates the currencies are not tradeable. Two, movements on foreign exchange markets are not just influenced by change in the market, but also by their domestic economic policy. Inflation, tax and general productivity are all factors that contribute to riskless drift term. In the context discussed above, one could replace gold by any other commodity or by a (weighted) portfolio of commodities. This then relates and expands the mathematical setting described by Grabbe [16] and the economical setting described by Grant [17].

Chapter 4

Applications

This chapter provides a number of examples and applications of the change of numéraire technique. To keep consistency, let Q ($\widehat{Q}$) denote the risk-neutral measure with respect to the old (new) numéraire. Similarly let W denote a Q-Brownian motion and $\widehat{W}$ a $\widehat{Q}$-Brownian motion. Theorem 3.1 and example 3.1 are applied throughout this chapter and on occasions without specific reference.

4.1 Change in dynamics

The source of this section is the paper by Schroder [21]. It focuses on how the change of numéraire technique influences the dynamics of the market.

4.1.1 Constant elasticity of variance (CEV)

Let our risk-neutral asset price process follow constant elasticity of variance dynamics:

$$\frac{dS_t}{S_t} = (r_t - \delta_t)dt + \nu_t S_t^{\xi} dW_t, \quad \xi \in [-1, 1]. \tag{4.1}$$

This model, introduced by Cox in 1975, generates a fluctuating volatility, and the elasticity of changes in the variance of the logarithmic-returns can be shown to be constant. However, Delbaen and Shirakawa [11] showed for this model that the asset price hits zero with positive probability, which is a significant defect in this model.

In the genre of example 3.1 we have $\sigma(S_t) = \nu_t S_t^{\xi}$ and the dynamics for a change in numéraire given by:

$$\frac{d\widehat{S}_t}{\widehat{S}_t} = (\delta_t - r_t)dt + \sigma(KS_0/\widehat{S}_t)d\widehat{W}_t, \tag{4.2}$$

where $\widehat{W}$ is the $\widehat{Q}$-Brownian motion obtained via Girsanov's theorem. In this case $\sigma(KS_0/\widehat{S}_t) =$

$\nu_t(KS_0)^\xi \widehat{S}_t^{-\xi}$ which simplifies to

$$\frac{d\widehat{S}_t}{\widehat{S}_t} = (\delta_t - r_t)dt + \widehat{\nu}_t \widehat{S}_t^{\widehat{\xi}} d\widehat{W}_t, \tag{4.3}$$

where $\widehat{\nu}_t = \nu_t(KS_0)^\xi$ and $\widehat{\xi} = -\xi$. Thus the 'type' of dynamics remains intact after a change in numéraire. What is meant by 'type' is the type of behavior of the parameters, especially ν in the above scenario. ν is deterministic and it remains deterministic after numéraire change. Taking all the parameters constant and considering the special case of $\xi = 0$, the above shows that Black-Scholes dynamics also remains intact after numéraire change.

4.1.2 American Put Call Symmetry

Using the notation from Theorem 1.2, let S_c be driven by the dynamics:

$$\frac{dS_c(t)}{S_c(t)} = (r(t) - \delta(t))dt + \sigma(S_c(t))dW_t, \tag{4.4}$$

where $\sigma(x) = f(|\log x/\sqrt{K_c K_p}|)$ for some bounded function f. The boundedness of f ensures that the stochastic integral part is a martingale. Set

$$S_p(t) = \frac{K_c K_p}{S_c(t)}, \tag{4.5}$$

such that the 'moneyness' condition holds. Given the form of σ we have

$$\begin{aligned} \sigma(S_c) &= f\left(\left|\log \frac{S_c}{\sqrt{K_c K_p}}\right|\right) && (4.6)\\ &= f\left(\left|\log \frac{\sqrt{K_c K_p}}{S_c}\right|\right) && (4.7)\\ &= f\left(\left|\log \frac{S_p}{\sqrt{K_c K_p}}\right|\right) && (4.8)\\ &= \sigma(S_p). && (4.9)\end{aligned}$$

So the two stocks have the same volatility. Applying a change of numéraire (as was done in Theorem 3.1)

$$\frac{dS_p}{S_p} = (\delta(t) - r(t))dt + \sigma(S_p(t))d\widehat{W}_t, \tag{4.10}$$

describes the dynamics for S_p. Girsanov applies here, so the EMM Q is unique, and any claim can be hedged. The fair prices of European put and call options maturing at time τ satisfy the relation

$$\begin{aligned} \mathbb{E}_Q\left(e^{-\int_0^\tau r_s ds}(S_c(\tau) - K_c)^+\right) &= \mathbb{E}_{\widehat{Q}}\left(e^{-\int_0^\tau \delta_s ds}(S_c(\tau) - K_c)^+ \frac{S_c(0)}{S_c(\tau)}\right) && (4.11)\\ &= \mathbb{E}_{\widehat{Q}}\left(e^{-\int_0^\tau \delta_s ds}(K_p - S_p(\tau))^+ \frac{S_c(0)}{K_p}\right) && (4.12)\end{aligned}$$

between the underlying stocks S_c and S_p. With the moneyness condition manipulating the term

$$\begin{aligned} \frac{S_c(0)}{K_p} &= \frac{\sqrt{S_c(0)}\sqrt{S_c(0)}}{K_p} && (4.13)\\ &= \sqrt{\frac{S_c(0)K_c}{S_p(0)K_p}} && (4.14)\end{aligned}$$

we get

$$\frac{\mathbb{E}_Q\left(e^{-\int_0^\tau r_s ds}(S_c(\tau)-K_c)^+\right)}{\sqrt{S_c(0)K_c}} = \frac{\mathbb{E}_{\widehat{Q}}\left(e^{-\int_0^\tau \delta_s ds}(K_p-S_p(\tau))^+\right)}{\sqrt{S_p(0)K_p}}. \tag{4.15}$$

Under Assumption 3.2 taking the supremum over all $\tau \in [0,T]$ the American Put-Call Symmetry is obtained,

$$\frac{c(S_c,K_c,\delta,r)}{\sqrt{S_c(0)K_c}} = \frac{p(S_p,K_p,r,\delta)}{\sqrt{S_p(0)K_p}}. \tag{4.16}$$

Remark: The above proof of the American Put Call Symmetry is much shorter and more direct than the proof used in Carr and Chesney [3]. Their proof relies on a stochastic differential equation interpretation. This proof also allows for stochastic riskless and dividend rates. The same can be said for volatilities that are functions of the underlying assets. As seen above the volatility function contains a logarithmic component. This is in connection with the technical discussion by Carr and Chesney [3] to ensure that the volatility function is chosen in such a way to satisfy a logarithmic symmetry condition.
The American Put Call Symmetry result is used without rigorous proof in Schroder [21] as an example of applications of Theorem 3.1.
Assumption 3.2 is superfluous in the Black-Scholes setting. Hence here the American Put Call Symmetry applies to a broader spectrum of pricing models.
Under the assumption that the Put and Call claims can be hedged and Assumption 3.2 the 'moneyness' condition is enough to imply the American Put Call Symmetry. In a non-Brownian setting, instantaneous volatilities can be described as instantaneous quadratic returns, i.e. $d[S]_t/S_t^2$. From Theorem 3.1 and because the assets are related via the 'moneyness' condition these are the same for the two assets between jumps. The motivation for this can be found in the Brownian setting of Example 3.1, where the instantaneous quadratic returns are derived to be the volatility squared multiplied with a dt increment [equation (3.50)].

4.1.3 Stochastic volatility

Let the risk-neutral asset price process be driven by

$$\frac{dS_t}{S_t} = (r_t - \delta_t)dt + \sqrt{\nu_t}dW_t^1, \tag{4.17}$$

where r and δ are deterministic and the volatility is driven by two independent Q-Brownian motions W^1 and W^2

$$d\nu_t = (\mu - \kappa\nu_t)dt + \psi\sqrt{\nu_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2), \tag{4.18}$$

with μ, κ, ψ and $\rho \in [-1,1]$ constants. Recall that $\rho W_t^1 + \sqrt{1-\rho^2}W_t^2$ is also a Q-Brownian motion and its instantaneous correlation with W^1 is ρ. The parameter ρ thus represents the instantaneous correlation between the asset and its volatility. Under a numéraire change the dynamics is the following

$$\frac{d\widehat{S}_t}{\widehat{S}_t} = (\delta_t - r_t)dt + \sqrt{\nu_t}d\widehat{W}_t^1. \tag{4.19}$$

For the state variable ν the drift term increases by

$$\phi(\nu_t)\sigma(\nu_t)\mathbf{e} = \left[\rho\psi\sqrt{\nu_t}, \sqrt{1-\rho^2}\psi\sqrt{\nu_t}\right]\sqrt{\nu_t}\begin{bmatrix}1\\0\end{bmatrix}, \tag{4.20}$$

so that

$$d\nu_t = (\mu - (\kappa - \rho\psi)\nu_t)dt - \psi\sqrt{\nu_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2). \tag{4.21}$$

The numéraire change affected the drift term of the volatility as well as the instantaneous correlation, which has changed sign.

4.2 Pricing Claims

The source of this section is the paper by Benninga, Björk and Wiener [2]. It applies the change of numéraire technique to a number of pricing problems found in practice.
The right choice of numéraire could have several advantages. A claim could, with respect to the right numéraire, reduce to a well known pricing problem. Choosing a risky asset as numéraire would reduce the amount of underlying assets to model by one (given that the original numéraire - usually the short-rate process - is not an underlying asset itself). As was seen in chapter two the set of underlying shifts from

$$X = (X_0, X_1, ..., X_n) \quad \text{to} \quad Z = (1, Z_1, ..., Z_n). \tag{4.22}$$

In this set 1 denotes the new numéraire process. Therefore the riskless process is zero in Z.
Note: In the pricing problems below, unless otherwise specified, all parameters are assumed constant. All asset processes are driven by Brownian motions and are therefore continuous processes.

4.2.1 Quanto product

This is an example of a foreign exchange pricing problem. A financial instrument is a *quanto product* if it is denominated in a currency other that in which it is traded. The pricing problem under consideration in this section, is an European option on a British asset denominated in U.K. pounds with a U.S. dollar strike price. Our first step in pricing this claim should be to decide whether to price the option in terms of the U.S. dollar currency or in terms of the U.K. pounds currency. The crux of this section lies in this decision. In fact the numéraire technique is not even mentioned or used in the subseqent computations. It is purely an instinctive call that could either lead to a long or short computation. In this example though, with a little thought, the U.S. dollar would seem the best choice. The reason is simple - under this currency our strike price is constant.

Let E denote the pound/dollar exchange rate. E forms a risk-neutral process with respect to the British short-rate under the measure Q. Its dynamics is given by

$$\frac{dE_t}{E_t} = (r^{£} - r^{\$})dt + \sigma^E dW_t^1, \tag{4.23}$$

where $r^{£}$ is the British riskless rate and $r^{\$}$ is the American riskless rate.

Let S denote the stock. S is a British stock. Dividends are reinvested into the stock. Its dynamics is given by

$$\frac{dS_t}{S_t} = (r^{£} - \delta)dt + \sigma^S[\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2], \tag{4.24}$$

where δ denotes the dividend rate and ρ the instantaneous correlation between S and E.

Let K denote the strike price in terms of the U.S. dollar. To write the payoff function for the claim in terms of the U.S. dollar, we first need to convert the British stock into U.S. dollars. The payoff or time T value of the option is given as

$$C^{\$}(T) = \left(\frac{S(T)}{E(T)} - K\right)^{+}. \tag{4.25}$$

We therefore need to determine the dynamics of S/E. Borrowing from example 3.3, the dynamics of $A = S/E$ can be expressed with respect to the American short-rate under the measure Q' as

$$\frac{dA_t}{A_t} = (r^{\$} - \delta)dt + (\sigma^E - \sigma^S\rho)d\widehat{W}_t^1 - (\sigma^S\sqrt{1-\rho^2}d\widehat{W}_t^2).$$

The total volatility of the above dynamics we shall write as

$$\begin{aligned} \sigma &= \sqrt{(\sigma^E - \sigma^S\rho)^2 + (\sigma^S\sqrt{1-\rho^2})^2} && (4.26) \\ &= \sqrt{(\sigma^E)^2 + (\sigma^S)^2 - 2\sigma^E\sigma^S\rho}. && (4.27) \end{aligned}$$

Using the Black-Scholes-Merton formula (Appendix B) the price of the option is

$$\begin{aligned} C^{\$}(t) &= \frac{S(t)}{E(t)}e^{-\delta(T-t)}\Phi(d_1(t)) - Ke^{-r^{\$}(T-t)}\Phi(d_2(t)), \quad \text{where} && (4.28) \\ d_1(t) &= \frac{\log(\frac{S(t)}{E(t)K}) + (r^{\$} - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, && (4.29) \\ d_2(t) &= d_1(t) - \sigma\sqrt{T-t}. && (4.30) \end{aligned}$$

The price of the British claim is obtained by just converting via the exchange rate:

$$\begin{aligned} C^{£}(t) &= E(t)C^{\$}(t) && (4.31) \\ &= S(t)e^{-\delta(T-t)}\Phi(d_1(t)) - KE(t)e^{-r^{\$}(T-t)}\Phi(d_2(t)). && (4.32) \end{aligned}$$

Remark: Benninga, Björk and Wiener [2] did not incorporate dividend payments in the pricing formula. The author also adapted the dynamics of the exchange rate and asset in their paper to suit the setting created in chapter three. The author feels this approach creates a clearer interpretation, both mathematically and economically.

4.2.2 Saving plans with choice of indexing

This is another example of a foreign exchange pricing problem. An investor has the choice at maturity between an indexed linked domestic interest rate growth or a foreign interest rate growth of investment. For example, the investor holds an option between the growth in the rand currency linked to South African inflation or growth in the euro currency. In other words the investor holds partially a swaption between two interest rates of different currencies.

The payoff of this pricing problem for one unit of domestic currency is

$$\Upsilon(T, dom) = \max\left[e^{r_d T}I(T), e^{r_f T}\frac{E(T)}{E(0)}\right],$$

where r_d and r_f denotes respectively the domestic and foreign riskless rates, I the domestic inflation process and E the exchange rate of the domestic currency per foreign currency. In terms of the foreign currency, the payoff is

$$\Upsilon(T, for) = \max\left[e^{r_d T}\frac{I(T)}{E(T)}, \frac{e^{r_f T}}{E(0)}\right] = \left(e^{r_d T}\frac{I(T)}{E(T)} - \frac{e^{r_f T}}{E(0)}\right)^+ + \frac{e^{r_f T}}{E(0)}.$$

Under the risk-neutral measure Q_f for the foreign market numéraire (the short-rate process in the foreign market) the time t price of the claim is

$$\Upsilon(t, for) = e^{r_f t}\mathbb{E}_{Q_f}\left[e^{-r_f T}\left(\left(e^{r_d T}\frac{I(T)}{E(T)} - \frac{e^{r_f T}}{E(0)}\right)^+ + \frac{e^{r_f T}}{E(0)}\right)\Big|\mathcal{F}_t\right] \tag{4.33}$$

$$= e^{r_f t}\mathbb{E}_{Q_f}\left[\left(e^{(r_d - r_f)T}\frac{I(T)}{E(T)} - \frac{1}{E(0)}\right)^+\Big|\mathcal{F}_t\right] + \frac{1}{E(0)}e^{r_f t}. \tag{4.34}$$

This resembles a European call option with strike $E^{-1}(0)$. To price the claim, we need to assign dynamics:

$$\frac{dE_t}{E_t} = (r_d - r_f)dt + \sigma^E dW_t^1, \tag{4.35}$$

$$\frac{dI_t}{I_t} = \alpha_I dt + \sigma^I[\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2], \tag{4.36}$$

where σ^E and σ^I denote the respective volatilities and ρ the instantaneous correlation between E and I. The Brownian motions are with respect to Q_d, the risk-neutral measure for the domestic market numéraire. Using the idea developed in example 3.3, we have for $A = I/E$

$$\frac{dA_t}{A_t} = (\alpha_I - r_d + r_f)dt + (\sigma^E - \sigma^I\rho)d\widehat{W}_t^1 - (\sigma^I\sqrt{1-\rho^2}d\widehat{W}_t^2),$$

under the measure Q_f. It is trivial to see the dynamics for $S_t = \exp[(r_d - r_f)t]A_t$ are

$$\frac{dS_t}{S_t} = \alpha_I dt + \sigma dW_t,$$

where σ is the total volatility

$$\sigma = \sqrt{(\sigma^E)^2 + (\sigma^I)^2 - 2\sigma^E\sigma^I\rho}.$$

Our pricing problem reduces to

$$\Upsilon(t, for) = e^{r_f t}\mathbb{E}_{Q_f}\left[\left(S(T) - E^{-1}(0)\right)^+\Big|\mathcal{F}_t\right] + E^{-1}(0)e^{r_f t},$$

which involves a call option on S - which has dividend rate $-\alpha$ - and strike price $E^{-1}(0)$. The foreign short-rate is already incorporated in the formula. From the Black-Scholes-Merton formula, we can write the price of the claim as

$$\Upsilon(t, for) = e^{r_f t}(S_t e^{\alpha_I(T-t)}\Phi(d_1(t)) - E^{-1}(0)\Phi(d_2(t))) + E^{-1}(0)e^{r_f t} \tag{4.37}$$

$$= e^{(r_d - \alpha_I)t}\frac{I(t)}{E(t)}e^{\alpha_I T}\Phi(d_1(t)) - e^{r_f t}E^{-1}(0)\Phi(d_2(t)) + E^{-1}(0)e^{r_f t}, \tag{4.38}$$

$$d_1(t) = \frac{\log(\frac{I(t)E(0)}{E(t)}) + (r_d - r_f)t + (\alpha_I + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \tag{4.39}$$

$$d_2(t) = d_1(t) - \sigma\sqrt{T-t}. \tag{4.40}$$

Converting the price of the claim in domestic currency:

$$\begin{aligned}\Upsilon(t,dom) &= E(t)\Upsilon(t,for) && (4.41)\\ &= e^{(r_d-\alpha_I)t}I(t)e^{\alpha_I T}\Phi(d_1(t)) - e^{r_f t}\frac{E(t)}{E(0)}\Phi(d_2(t)) + \frac{E(t)}{E(0)}e^{r_f t}. && (4.42)\end{aligned}$$

The time zero price reduces neatly to

$$\begin{aligned}\Upsilon(0,dom) &= I(0)e^{\alpha_I T}\Phi(d_1(0)) - \Phi(d_2(0)) + 1, \quad \text{where} && (4.43)\\ d_1(0) &= \frac{\log(I(0)) + (\alpha_I + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, && (4.44)\\ d_2(0) &= d_1(0) - \sigma\sqrt{T}. && (4.45)\end{aligned}$$

Remark: Benninga, Björk and Wiener [2] only gave the time-zero price of the claim. They also described the dynamics of the domestic inflation rate I in terms of the foreign measure. It was noted that the drift term of I in this setting does not have a known economical interpretation. In the above, the dynamics of I are written with respect to the domestic measure. Here the drift α_I has a clear economical interpretation on domestic shores.

4.2.3 Employee stock ownership plan

One type of employee stock ownership plan gives the holder the right to buy a stock at the minimum value of its price in six months or in one year. The option matures in one year and might include a rebate. To generalise the above, let T_0 and $T_1 > T_0$ denote the times at which the stock prices are considered in this pricing problem. The payoff for the holder is therefore the difference between what he/she would have paid for the stock S at maturity and the discounted price available to the holder:

$$\Upsilon(T_1, \text{ESOP}) = S(T_1) - \beta \min[S(T_1), S(T_0)].$$

Here $(1-\beta) \in [0,1]$ denotes the rebate.
With $\Upsilon(t, \min)$ denoting the value of the payoff $\min[S(T_1), S(T_0)]$, let us express the price at time t as

$$\Upsilon(t, \text{ESOP}) = S(t) - \beta\Upsilon(t, \min),$$

and the dynamics of the underlying asset under the short-rate measure Q as

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sigma dW_t.$$

The difficulty of the problem is to price terminal value $\min[S(T_1), S(T_0)]$, because $S(T_0)$ does not have a natural interpretation as the spot price of a traded asset. Let us introduce a new asset of the form

$$S^0(t) = \begin{cases} S(t) & \text{if} \quad 0 \le t \le T_0, \\ S(T_0)e^{(r-\delta)(t-T_0)} & \text{if} \quad T_0 \le t \le T_1, \end{cases} \tag{4.46}$$

such that the payoff can now be expressed as

$$\Upsilon(T, \min) = \min[S(T_1), K_{T_0}S^0(T_1)],$$

where $K_t = e^{-(r-\delta)(T_1-t)}$. The Q dynamics for S^0 is written as

$$\frac{dS_t^0}{S_t^0} = (r-\delta)dt + \sigma^0(t)dW_t,$$

with

$$\sigma^0(t) = \begin{cases} \sigma \text{ if } & 0 \le t \le T_0, \\ 0 \text{ if } & T_0 \le t \le T_1, \end{cases} \tag{4.47}$$

and this asset S^0 will act as our new numéraire. Let $Y = S/S^0$ denote the asset S in terms of this numéraire with Q^0 as the corresponding measure change. As a special case of example 3.3, it is easy to write the dynamics for Y:

$$\frac{dY_t}{Y_t} = [\sigma - \sigma^0(t)]dW_t^0.$$

The time t price of the claim in terms of this numéraire is

$$\begin{aligned}
\Upsilon(t, \min) &= S^0(t)\mathbb{E}_{Q^0}\left(\min[Y(T_1), K_{T_0}]|\mathcal{F}_t\right) &(4.48)\\
&= S^0(t)\left[\mathbb{E}_{Q^0}\left(Y(T_1)|\mathcal{F}_t\right) - \mathbb{E}_{Q^0}\left((Y(T_1) - K_{T_0})^+|\mathcal{F}_t\right)\right] &(4.49)\\
&= S^0(t)Y(t) - S^0(t)\Upsilon(t, \text{Opt}) &(4.50)\\
&= S(t) - S^0(t)\Upsilon(t, \text{Opt}). &(4.51)
\end{aligned}$$

Here $\Upsilon(t, \text{Opt})$ denotes the time t price of a call option on Y with strike $K_{\widehat{t}}$ ($\widehat{t} = \max[T_0, t]$) and zero riskless rate. Note that the strike varies over the time interval $[T_0, T_1]$. The reason for this will be clear in the pricing formula itself, but one can also interpret the problem as reducing to the usual call pricing problem when time reaches T_0. We therefore have on the interval $[T_0, T_1]$ a call option on asset S maturing at time T_1 with strike $S(T_0)$ (which is known over this time interval). The fluctuating strike price K incorporates the riskless and dividend rates involved with the asset S. Since the volatility varies over time as well, we need the explicit solution of Y:

$$\begin{aligned}
Y_t &= \exp\left[-\frac{1}{2}\int_0^t (\sigma - \sigma_s^0)^2 ds + \int_0^t (\sigma - \sigma_s^0) dW_s^0\right] &(4.52)\\
&= \exp\left[-\frac{1}{2}\sigma^2(t - T_0)^+ + \sigma 1_{\{t \ge T_0\}}(W_t^0 - W_{T_0}^0)\right]. &(4.53)
\end{aligned}$$

Hence

$$\begin{aligned}
\Upsilon(t, \text{Opt}) &= Y_t\Phi(d_1) - K_{\widehat{t}}\Phi(d_2), \quad \text{where} &(4.54)\\
d_1(t) &= \frac{\log(\frac{Y_t}{K_{\widehat{t}}}) + \frac{1}{2}\sigma^2(T - \widehat{t})}{\sigma\sqrt{T - \widehat{t}}} &(4.55)\\
&= \frac{\log(\frac{S_t}{S_t^0}) + (r - \delta + \frac{1}{2}\sigma^2)(T - \widehat{t})}{\sigma\sqrt{T - \widehat{t}}} &(4.56)\\
&= \frac{1_{\{t > T_0\}}\log(\frac{S_t}{S_{T_0}}) + (r - \delta + \frac{1}{2}\sigma^2)(T - \widehat{t})}{\sigma\sqrt{T - \widehat{t}}}, &(4.57)\\
d_2(t) &= d_1(t) - \sigma\sqrt{T - \widehat{t}} \quad \text{and} &(4.58)\\
\widehat{t} &= \max[T_0, t]. &(4.59)
\end{aligned}$$

Therefore the price of the claim results in

$$\begin{aligned}
\Upsilon(t, \text{ESOP}) &= (1 - \beta)S(t) + \beta S(t)\Phi(d_1) - \beta S^0(t)K_{\widehat{t}}\Phi(d_2), \text{ where} &(4.60)\\
d_1(t) &= \frac{1_{\{t > T_0\}}\log(\frac{S_t}{S_{T_0}}) + (r - \delta + \frac{1}{2}\sigma^2)(T - \widehat{t})}{\sigma\sqrt{T - \widehat{t}}}, &(4.61)\\
d_2(t) &= d_1(t) - \sigma\sqrt{T - \widehat{t}} \quad \text{and} &(4.62)\\
\widehat{t} &= \max[T_0, t]. &(4.63)
\end{aligned}$$

Remark: The above pricing includes dividends and the time-t value of the claim for all $t \in [0, T_1]$. In Benninga, Björk and Wiener [2] the claim is only priced for $t \in [0, T_0]$.

Chapter 5

Conclusion

The change of numéraire technique can be used efficiently in both the general theory of financial mathematics and in computations. This dissertation showed at an introductory level how conveniently this technique can be applied. Chapter one is a short chapter that describes pricing models in a continuous setting. Notes and books for beginner students usually contain only the binomial and geometric Brownian motions settings. With the help of the change of numéraire technique chapter one provides an easy translation to the general setting. In chapter three this setting expands to locally bounded markets. Dynamics of the market and pricing formulas for certain derivatives are also studied in this chapter.
Chapter two motivated why the technique is valid in a mathematical context. Further study in workable or unbounded spaces is possible, but lies outside the scope of this dissertation.
Chapter four provides unique pricing problems where the change of numéraire technique can be applied to simplify computation. There are many more such problems where this technique could be applied to quite successfully.

Appendix A

Miscellaneous proofs

Bayes' formula. Given equivalent probability measures $\mathbb{P}$ and $\mathbb{Q}$ on the measurable space $(\Omega, \mathcal{F})$ and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. If $Y \geq 0$ is in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and in $L^1(\Omega, \mathcal{F}, \mathbb{Q})$, then we have the identity

$$\mathbb{E}_{\mathbb{P}}\left[Y \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{G}\right] = \mathbb{E}_{\mathbb{Q}}\left[Y|\mathcal{G}\right] \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{G}\right] \quad a.s.(\mathbb{Q}). \tag{A.1}$$

Proof
To ensure that $X = Z$ a.s. for $X, Z \in m\mathcal{G}$, it is enough to show $\mathbb{E}(1_G X) = \mathbb{E}(1_G Z)$ for all $G \in \mathcal{G}$. Noticing $1_G, \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{G}\right) \in m\mathcal{G}$ and conditioning on $\mathcal{G}$ with respect to $\mathbb{Q}$ we have that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[1_G \mathbb{E}_{\mathbb{Q}}\left(Y|\mathcal{G}\right) \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{G}\right)\right] &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left(1_G Y \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{G}\right)\Big|\mathcal{G}\right)\right] && \text{(A.2)}\\
&= \mathbb{E}_{\mathbb{Q}}\left[1_G Y \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{G}\right)\right]. && \text{(A.3)}
\end{aligned}$$

Using the Radon-Nikodym derivative to change the measure and conditioning on $\mathcal{G}$ with respect to $\mathbb{P}$ we evolve with

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[1_G \mathbb{E}_{\mathbb{Q}}\left(Y|\mathcal{G}\right) \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{G}\right)\right] &= \mathbb{E}_{\mathbb{Q}}\left[1_G Y \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{G}\right)\right] && \text{(A.4)}\\
&= \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} 1_G Y \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{G}\right)\right] && \text{(A.5)}\\
&= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} 1_G Y \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{G}\right)\Big|\mathcal{G}\right)\right]. && \text{(A.6)}
\end{aligned}$$

With $1_G, \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{G}\right), \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}Y|\mathcal{G}\right) \in m\mathcal{G}$ and manipulation

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[1_G \mathbb{E}_{\mathbb{Q}}\left(Y|\mathcal{G}\right) \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{G}\right)\right] &= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} 1_G Y \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{G}\right)\Big|\mathcal{G}\right)\right] && \text{(A.7)}\\
&= \mathbb{E}_{\mathbb{P}}\left[1_G \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{G}\right) \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} Y\Big|\mathcal{G}\right)\right] && \text{(A.8)}\\
&= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left(1_G \frac{d\mathbb{Q}}{d\mathbb{P}} \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} Y\Big|\mathcal{G}\right)\Big|\mathcal{G}\right)\right]. && \text{(A.9)}
\end{aligned}$$

Conditioning on $\mathcal{G}$ with respect to $\mathbb{P}$ in the new form and changing the measure back to $\mathbb{Q}$ we get the desired result:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[1_G \mathbb{E}_{\mathbb{Q}}\left(Y|\mathcal{G}\right)\mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{G}\right)\right] &= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left(1_G\frac{d\mathbb{Q}}{d\mathbb{P}}\mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}Y\Big|\mathcal{G}\right)\Big|\mathcal{G}\right)\right] && \text{(A.10)}\\
&= \mathbb{E}_{\mathbb{P}}\left[1_G\frac{d\mathbb{Q}}{d\mathbb{P}}\mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}Y\Big|\mathcal{G}\right)\right] && \text{(A.11)}\\
&= \mathbb{E}_{\mathbb{Q}}\left[1_G\mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}Y\Big|\mathcal{G}\right)\right].\diamond && \text{(A.12)}
\end{aligned}$$

Result for Chapter 2 Theorem 1 For probability measure $\mathbb{Q} \sim \mathbb{P}$ on $(\Omega, \mathcal{F})$ in the $\mathbb{R}^d$-valued market S the following are equivalent:
(i) $\mathbb{Q} \in \mathcal{M}^e(S)$,
(ii) $\mathbb{E}_{\mathbb{Q}}[f] = 0, \quad \forall f \in \mathcal{K}[S]$.

Proof (ii)⇒(i) Say $\mathbb{E}_{\mathbb{Q}}[f] = 0$, $\forall f \in \mathcal{K}[S]$. Specifically it would hold for trading strategies that are zero everywhere except for the $\mathbb{R}^d$-valued $1_A\mathbf{c}$ at time t. A is any $\mathcal{F}_{t-1}$-measurable set, so the trading strategy is predictable. $\mathbf{c}$ is any constant vector in $\mathbb{R}^d$, but let us only consider the unit vectors $\mathbf{e}_j$ (all coordinates zero except the jth coordinate which is one). Hence for a chosen j

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[(1_A\mathbf{e}_j \cdot \Delta S(t))] &= 0 && \text{(A.13)}\\
\mathbb{E}_{\mathbb{Q}}[(1_A\Delta S^j(t))] &= 0 \quad \forall A \in \mathcal{F}_{t-1} && \text{(A.14)}\\
\mathbb{E}_{\mathbb{Q}}(\Delta S^j(t)|\mathcal{F}_{t-1}) &= 0 && \text{(A.15)}\\
\mathbb{E}_{\mathbb{Q}}(S^j(t)|\mathcal{F}_{t-1}) &= S^j(t-1). && \text{(A.16)}
\end{aligned}$$

This was done for an arbitrary t and j and therefore $\mathbb{Q} \in \mathcal{M}^e(S)$.
(i)⇒(ii)Let $\mathbb{Q} \in \mathcal{M}^e(S)$, then for any $f \in \mathcal{K}[S]$

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[f] &= \mathbb{E}_{\mathbb{Q}}\Big(\sum_{t=1}^{T}(\theta(t)\cdot\Delta S(t))\Big) && \text{(A.17)}\\
&= \mathbb{E}_{\mathbb{Q}}\Big(\sum_{t=1}^{T}\sum_{i=1}^{d}\theta^i(t)\Delta S^i(t)\Big) && \text{(A.18)}\\
&= \sum_{t=1}^{T}\sum_{i=1}^{d}\mathbb{E}_{\mathbb{Q}}[\theta^i(t)\mathbb{E}_{\mathbb{Q}}(\Delta S^i(t)|\mathcal{F}_{t-1})] = 0.\diamond && \text{(A.19)}
\end{aligned}$$

Example 3.2 Identity. As defined in the example we have the identity

$$Q_S(S(\tau_i) = uS(\tau_i-)) = pu\mathbb{E}_Q(S(\tau_i-)/S(0)). \quad \text{(A.20)}$$

Proof

$$\begin{aligned}
Q_S(S(\tau_i) = uS(\tau_i-)) &= \mathbb{E}_Q\left[1_{\{S(\tau_i)=uS(\tau_i-)\}}\frac{S_T}{S_0}\right], \text{ then conditioning on } \mathcal{F}_{\tau_i}, && \text{(A.21)}\\
&= \mathbb{E}_Q\left[1_{\{S(\tau_i)=uS(\tau_i-)\}}\mathbb{E}_Q\left(\frac{S_T}{S_0}|\mathcal{F}_{\tau_i}\right)\right], \text{and } S \text{ a martingale,} && \text{(A.22)}\\
&= \mathbb{E}_Q\left[1_{\{S(\tau_i)=uS(\tau_i-)\}}\frac{S_{\tau_i}}{S_0}\right] && \text{(A.23)}\\
&= Q(\{S(\tau_i) = uS(\tau_i-)\})\mathbb{E}_Q\left(\frac{S_{\tau_i}}{S_0}\,|\,\{S(\tau_i) = uS(\tau_i-)\}\right) && \text{(A.24)}\\
&= p\mathbb{E}_Q(uS(\tau_i-)/S(0)).\diamond && \text{(A.25)}
\end{aligned}$$

Appendix B

Black-Scholes-Merton formula

In this section we shall derive the Black-Scholes-Merton formula for an European call option. The aim is to supply a concrete example of the numéraire technique with which the known standard methods concur. This section also acts as reference for many of the claims priced in *Applications: Pricing Claims*.

Let S be the underlying asset with dynamics driven by a Q-Brownian motion W, so that

$$\frac{dS_t}{S_t} = (r_t - \delta_t)dt + \sigma dW_t, \tag{B.1}$$

where r and δ are assumed deterministic and the volatility σ, assumed constant. From (3.72) the time zero price of an European call with strike price K and maturing at time T is

$$C_0 = S_0 e^{-\int_0^T \delta_s ds} Q'(S_T \geq K) - K e^{-\int_0^T r_s ds} Q(S_T \geq K). \tag{B.2}$$

Applying Theorem 3.1 the time-t price of the claim is

$$\begin{aligned} C_t &= \mathbb{E}_Q \left(e^{-\int_t^T r_s ds} (S_T - K)^+ | \mathcal{F}_t \right) && \text{(B.3)} \\ &= \mathbb{E}_Q \left(e^{-\int_t^T r_s ds} S_T 1_A | \mathcal{F}_t \right) - \mathbb{E}_Q \left(e^{-\int_t^T r_s ds} K 1_A | \mathcal{F}_t \right) && \text{(B.4)} \\ &= S_t e^{-\int_t^T \delta_s ds} \mathbb{E}_{Q'} \left(1_A | \mathcal{F}_t \right) - K e^{-\int_t^T r_s ds} \mathbb{E}_Q \left(1_A | \mathcal{F}_t \right). && \text{(B.5)} \end{aligned}$$

where $A = \{\omega | S(T, \omega) \geq K\}$.

Manipulating the inequality

$$\begin{aligned} S_T &\geq K && \text{(B.6)} \\ \frac{Z_T}{Z_t} &\geq \frac{K}{S_t} \exp\left[\int_t^T (\delta_s - r_s) ds \right] && \text{(B.7)} \end{aligned}$$

introduces the martingale process Z with dynamics

$$dZ_t = Z_t \sigma dW_t. \tag{B.8}$$

The explicit solution for Z is given by

$$Z_t = \exp[-\frac{1}{2}\sigma^2 t + \sigma W_t]. \tag{B.9}$$

Taking the logarithm on both sides of the inequality gives

$$-\frac{1}{2}\sigma^2(T-t)+\sigma(W_T-W_t)\geq \log\left(\frac{K}{S_t}\right)+\int_t^T(\delta_s-r_s)ds. \tag{B.10}$$

Further we have an Q-Brownian motion and an Q'-Brownian motion in the form of the processes W and $\widehat{W}_t=\sigma t-W_t$ respectively, which will permit

$$\log\left(\frac{S_t}{K}\right)+\int_t^T(r_s-\delta_s)ds-\frac{1}{2}\sigma^2(T-t) \;\geq\; -\sigma(W_T-W_t) \quad \text{and} \tag{B.11}$$

$$\log\left(\frac{S_t}{K}\right)+\int_t^T(r_s-\delta_s)ds+\frac{1}{2}\sigma^2(T-t) \;\geq\; \sigma(\widehat{W}_T-\widehat{W}_t) \tag{B.12}$$

to be measured in terms of the cumulative (standard) normal distribution function. Notice that both $-(W_T-W_t)$ and $(\widehat{W}_T-\widehat{W}_t)$ are $\mathcal{N}(0,T-t)$-distributed under Q and Q' respectively, because these random variables are conditioned to $\mathcal{F}_t$. We can now conclude with the price of the option at time t

$$C_t = S_te^{-\int_t^T\delta_sds}\Phi(d_1)-Ke^{-\int_t^T r_sds}\Phi(d_2), \quad \text{where} \tag{B.13}$$

$$d_1(t) = \frac{\log(\frac{S_t}{K})+\int_t^T(r_s-\delta_s)ds+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \tag{B.14}$$

$$d_2(t) = d_1(t)-\sigma\sqrt{T-t}. \tag{B.15}$$

and Φ denotes the cumulative $\mathcal{N}(0,1)$-distribution function.

Taking r and δ as constants the formula becomes the familiar

$$C_t = S_te^{-\delta(T-t)}\Phi(d_1)-Ke^{-r(T-t)}\Phi(d_2), \quad \text{where} \tag{B.16}$$

$$d_1(t) = \frac{\log(\frac{S_t}{K})+(r-\delta+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{B.17}$$

$$d_2(t) = d_1(t)-\sigma\sqrt{T-t}. \tag{B.18}$$

For computational reasons applied in chapter four, we state the formula for the option when there are no short-rate and dividend rate present, i.e. $\delta=r=0$:

$$C_t = S_t\Phi(d_1)-K\Phi(d_2), \quad \text{where} \tag{B.19}$$

$$d_1(t) = \frac{\log(\frac{S_t}{K})+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \tag{B.20}$$

$$d_2(t) = d_1(t)-\sigma\sqrt{T-t}. \tag{B.21}$$

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