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Analysis of the Effects of Growth-Fragmentation-Coagulation in Phytoplankton Dynamics

by

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Abstract

An integro-differential equation describing the dynamical behaviour of phytoplankton cells is considered in which the effects of cell division and aggregation are incorporated by coupling the coagulation-fragmentation equation with growth, and the McKendrick-von Foerster renewal model of an age-structured population. Under appropriate conditions on the model parameters, the associated initial-boundary value problem is shown to be well posed in a physically relevant Banach space using the theory of strongly continuous semigroups of operators, the theory of perturbation of positive semigroups and the semilinear abstract Cauchy problems theory. In particular, we provide sufficient conditions for honesty of the model. Finally, the results on the effects of the growth-fragmentation-coagulation on the overall evolution of the phytoplankton population are summarised.

Uittreksel

'n Integro-differensiaalvergelyking wat die dinamiese ontwikkeling van fitoplanktonselle beskryf, word beskou. Die uitwerking van seldeling en -aggregasie is geïnkorporeer deur die vergelyking van koagulasie en fragmentasie met groei-aan die McKendrick-von Foerster hernuwingsmodel van 'n ouderdomsgestruktureerde populasie te koppel. Die teorie van sterk kontinue semigroepe van operatore, steuringsteorie van positiewe semigroepe en die teorie van semilineêre abstrakte Cauchy probleme word aangewend om, onder gepaste voorwaardes met betrekking tot die model se parameters, te bewys dat die geassosieerde beginwaarde-probleem met randvoorwaardes 'goed gestel' is in 'n fisies relevante Banach-ruimte. In die besonder word voldoende voorwaardes vir eerlikheid van die model verskaf. Ten slotte word 'n opsomming van die resultate met betrekking tot die gekombineerde uitwerking van groei-fragmentasie- koagulasie op die gesamentlike ontwikkeling van die fitoplanktonpopulasie verskaf.

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Dedications

I dedicate this thesis to my beloved Parents, Sister and Brother.

Contents

Chapter 1

Introduction

In many processes of the natural sciences including Physics, Chemistry, Biology and so on, the description of the evolution of a system is usually made by considering a state function $(t, \xi) \to u(t, \xi)$, where t is the time and ξ is an element of some state space Ω, which can be the mass, length, or the number of basic 'building bricks', uniquely identifying the state of an individual. Depending on a specific area of science, the function u is often called density, concentration or distribution, and describes the state of complex systems at time t. Since the state of a system changes as time evolves, then the variable describing time plays a special role.

Mathematically, the state u is given by a relation that can be formulated as a difference equation (when time is regarded as a discrete variable) or differential equation (when time is regarded as a continuous variable). Such equations are called evolution equations. These equations are built by balancing the change of the system in time against its 'spatial' behaviour. The function u has an important property, namely, all the elements of the state space Ω must be accounted for, in other words,

$$\int_{\Omega} u(t, \xi) d\mu_{\xi} = \int_{\Omega} u(0, \xi) d\mu_{\xi}, \tag{1.0.1}$$

for any time t, where $d\mu_{\xi}$ is an appropriate measure in the state space. Therefore, from a physical point of view, the natural spaces for studying such problems are L_1 spaces.

1.1 Fragmentation and Coagulation

Fragmentation and coagulation processes are two natural phenomena that occur in many fields of applied sciences and engineering. Recently, the theory of these processes has been developed by various techniques. The main are by probabilistic (Markov processes) and the functional analytic technique. The aim of these techniques is to describe the dynamical behaviour of physical systems that undergo changes due to fragmentation and coagulation.

1.2 Pure Fragmentation

The process of fragmentation occurs in a large variety of situations, for instance, rock fracture, droplet break-up, degradation of large polymer chains, DNA fragmentation, splitting of phytoplankton aggregates. The fragmentation process results in change of size distribution of particles in time. In real life, the fragmentation process is commonly accompanied by other mechanisms such as growth, decay or coalescence. In absence of any of these mechanisms, the fragmentation process is called 'pure fragmentation'.

The fragmentation model was first introduced by Blatz and Tobolsky [15] to model depolymerisation. Melzak [23], introduced the fragmentation equation that had the form:

$$\frac{\partial u(t,x)}{\partial t} = (\mathcal{F}u)(t,x),$$

for almost all (a.a.) $x > 0$ and $t > 0$, where the fragmentation operator $\mathcal{F}$ is given by

$$(\mathcal{F}u)(t,x) = \int_x^\infty \gamma(y,x)u(t,y)dy - u(t,x)\int_0^x \frac{y}{x}\gamma(x,y)dy. \tag{1.2.1}$$

An underlying assumption of models described by (1.2.1) is that the number of particles in the system is large enough to enable us to interpret $u(t,x)$ as a density function. Consequently, $u(t,x)dx$ is the average number of particles of mass x in the interval $(x, x+dx)$ at time t. The multiple fragmentation kernel $\gamma(x,y)$ describes the formation rate of particles of mass y due to the fragmentation of a particle of mass x.

Subsequently, Ziff and McGrady's [28, 29] formulation of fragmentation was based on a different form of the fragmentation kernel γ. If we consider γ as the product of two separate functions,

$$a(x) := \int_0^x \frac{y}{x}\gamma(x,y)dy, \qquad x > 0, \tag{1.2.2}$$

and

$$b(x|y) = \frac{\gamma(y,x)}{a(x)}, \tag{1.2.3}$$

then the function a describes the overall break-up rate of an x-particle and the function b describes the distribution of particles of size x formed during the break-up of particles of size y. If we substitute (1.2.2) and (1.2.3) into (1.2.1), then for a.a. $x > 0$ and $t \geq 0$, the operator $\mathcal{F}$ is given by

$$(\mathcal{F}u)(t,x) = -a(x)u(t,x) + \int_x^\infty a(y)b(x|y)u(t,y)dy. \tag{1.2.4}$$

As early as 1980, Ziff and his students, [29], adopted and provided explicit solutions to a large class of continuous models of fragmentations of the form

(1.2.4). Also, they used the mono-disperse initial condition $u(x,0) = \delta(x-l)$ for $l > 0$, where δ denotes the Dirac delta and the power law fragmentation rates given by $a(x) = x^{\alpha}$, $\alpha \in \mathbb{R}$. $b(x|y)$ was also given by a power law:

$$b(x|y) = (\nu + 2)\frac{x^{\nu}}{y^{\nu+1}},$$

with $\nu \in (-2, 0]$, see also [22] for a more detailed discussion of this case.

1.2.1 Pure Coagulation

The most important process corresponding to the evolution of disperse systems, which is known as the study of solid or liquid particles suspended in a medium, usually in a gas, is the process of the coagulation of particles. This process occurs in various situations, for example, volcanic dust, condensation of water vapour in the atmosphere, meteoritic dust, spores and seeds from plants, and coalescence of phytoplankton aggregates.

As in the pure fragmentation, the coagulation process is called 'pure coagulation' when the evolution of a system is due only to the mechanism of coalescence. Pure coagulation was firstly studied by Smoluchowski [27] who derived the following infinite set of nonlinear differential equations:

$$\frac{\partial}{\partial t}u_i(t) = \frac{1}{2}\sum_{j=1}^{i-1} k_{i-j,j}u_{i-j}(t)u_j(t) - u_i(t)\sum_{j=1}^{\infty} k_{i,j}u_j(t),$$

The function $k_{i,j}$ is called a coagulation kernel which describes the intensity of interaction between particles of mass i and j and is supposed to be known. The unknown non-negative function $u_i(t)$ is the concentration of particles with mass i, $i \geq 1$, [27].

Afterwards, the results of Smoluchowski were extended by Müller [17] to a continuous equation where he adopted a continuous mass density function. This was the first example in which the pure coagulation was considered as a continuous problem and modelled as an integro-differential equation

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) = &\frac{1}{2}\int_0^x k(x-y,y)u(t,x-y)u(t,y)dy \\ &- u(t,x)\int_0^{\infty} k(x,y)u(t,y)dy. \end{aligned} \tag{1.2.5}$$

This equation describes the evolution of the particle mass density function $u(t,x)$ in time t. The amount $u(t,x)dx$ is the average number of particles at time t whose masses lie between x and $x+dx$. The function $k(x,y)$ (coagulation kernel) is introduced by assuming that the average number of coalescences between particles of mass x to $x+dx$ and those of mass y to $y+dy$, is $u(t,x)u(t,y)k(x,y)dxdydt$ during the time interval $(t, t+dt)$.

1.3 Phytoplankton Aggregates

1.3.1 Motivation

Phytoplankton are microscopic plants and organisms living in oceans, lakes and ponds around the world. They are important because the food chain begins with them and all life on earth is basically dependent upon their existence. Also, in the process of photosynthesis, phytoplankton produce half of the world's oxygen. Moreover, they are the only food available for many species of fish in their larval stage. Since larvae do not move on their own, they survive only if they are in the vicinity of the aggregates; that is, groups of phytoplankton cells living together. Then, the best situation is when the larva is near a phytoplankton aggregate. For these reasons, it turns out that the description of the density and distribution of aggregates is important in connection with the study of fish recruitment [1].

1.3.2 Modelling Approach

Modelling the distribution of aggregates can be made by different methods. One approach is called individual-based models based on individual behaviour of cells. It can be thought of as providing 'microscopic' properties, track the random motion and division of individual cells [26]. Another approach, based on a 'macroscopic' description known to ecologists as advection-diffusion-reaction equations, which describe the spatial densities of cells concentrations [21] and is heavily used in simulations [5]. On the other hand, it has been observed that modelling cell division within aggregates is rather difficult. In this direction, a few modelling efforts have focused on special cases of cell division: It has been assumed that either the cells in the aggregate are dead and thus do not divide, or all daughter-cells remain in the aggregate of the mother cell, or all daughter cells break off an aggregate and join the single cell population. For example, Ackleh et al., [4], used an individual-based model to describe cell division in aggregates and assumed that all daughter-cells fall off the aggregates and join the single cell population, hence leaving the aggregates size unchanged despite the cell division. The resulting problem, analyzed in papers [3, 4], can be regarded as a combination of the classical coagulation equation of the form (1.2.5) with the McKendrick-von Foerster renewal model of an age-structured population. In [6, 26], the authors mentioned that due to external forces such as currents, the aggregates may also undergo splitting into two, or more, aggregates of arbitrary size. Thus, it seems natural to take into account in the model the multiple-fragmentation equation (1.2.4). As a result, if we denote by x_0 the smallest size of an aggregate, the combination of the terms of the growth, death, fragmentation and coagulation leads to the full fragmentation-coagulation equation coupled with the McKendrick-von

Foerster renewal model

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) =& -\frac{\partial}{\partial x}[r(x)u(t,x)] - d(x)u(t,x) - a(x)u(t,x)\\ &+ \int_{x+x_0}^{x_1} a(y)b(x|y)u(t,y)\,dy - u(t,x)\int_{x_0}^{x_1} k(x,y)u(t,y)dy \\ &+ \frac{1}{2}\int_{x_0}^{x-x_0} k(x-y,y)u(t,x-y)u(t,y)dy,\end{aligned} \tag{1.3.1}$$

with initial and boundary conditions

$$u(0,x) = u_0(x), \quad \lim_{x\to x_0^+} r(x)u(t,x) = \int_{x_0}^{x_1} \beta(y)u(t,y)dy,$$

where d and r denote the death and the growth rate for an aggregate, respectively, and β represents the rate at which daughter cells enter the single cell population.

1.3.3 A Mathematical Survey

In [4] the mathematical analysis of (1.3.1) without fragmentation and death terms is carried out using semigroups of linear operators in $L_2([x_0,x_1])$, which has no direct biological interpretation in this context. Also, the proof of non-negativity of the solution, and thus of global-in-time existence, is unclear. The results of [4] were extended in [3] to cover time-dependent coefficients r and d with $x_0 = 0$ and $x_1 = \infty$. However, the global-in-time existence results there depend upon the existence of appropriate upper and lower solutions. Arino and R. Rudnicki [6] have been concerned with a problem of the form (1.3.1) in the space

$$X_1 = L_1([x_0,x_1),xdx) = \left\{u;\ \int_{x_0}^{x_1} |u(x)|xdx < +\infty\right\}, \tag{1.3.2}$$

with $x_0 = 0$ and $x_1 = \infty$, where its norm gives the total mass of aggregates in the system. They considered only a binary bounded fragmentation and introduced a coagulation term different from (1.2.5). The authors used a growth coefficient r, proportional to x, and therefore no boundary conditions at $x = x_0 = 0$ were required. They proved that if the fragmentation rate depends on the size x, for example, proportional to x, and that it is an increasing function, then the distribution of the size of aggregates converges to a stationary distribution as time goes to infinity. On the other hand, the average size of aggregates tends to zero if the fragmentation rate is larger than the growth and coagulation rates and to infinity otherwise. In [10], the author disregarded the coagulation term as their aim was to analyze the inter-relation between the growth and fragmentation of aggregates in X_1. The mathematical analysis of (1.3.1) in [12] is performed via the semigroup theory approach.

Under the assumption that the fragmentation rate was linearly bounded and the number of daughter particles is bounded, the associated initial boundary value problem was shown to be well-posed in the Banach space

$$X_{0,1} = L_1([x_0, x_1), (1+x)dx) = \left\{ u;\ \int_{x_0}^{x_1} |u(x)|(1+x)dx < +\infty \right\},$$

with $x_0 \geq 0$ and $x_1 \leq \infty$, which keeps track of both the number of aggregates and of the total number of cells in the ensemble. The choice of this space is dictated by the fact that the fragmentation operator is known to behave well in the space $L_1([x_0, x_1), xdx)$, whereas the transport and coagulation terms have nice properties in $L_1([x_0, x_1), dx)$. In [13], the authors aimed to extend the earlier results of [12] to more general fragmentation and growth operators.

1.4 Outline of Thesis

In this thesis, our main concern is combining the results of [10, 12, 13] to present a unified mathematical analysis of phytoplankton dynamics. In this direction, we aim to examine basic questions of well-posedness of (1.3.1) in the space X_1 with $x_0 > 0$ and $x_1 = \infty$. For this, we assume that a is an arbitrary fragmentation rate. Our results generalize earlier works on fragmentation-coagulation models with linearly bounded fragmentation in $X_{0,1}$. The strategy we adopt is based on the theory of semigroup of linear operators [16]. In particular, we use the approach developed by Voigt [18] to analyse the linear part of (1.3.1). This approach is crucial for the investigation of the existence of solution as it establishes that, under appropriate assumptions, a perturbation $T + B$ of a generator T of a substochastic semigroup by a positive (unbounded) operator B has an extension G that also generates a substochastic semigroup. In addition, the non-linear part, that is, the coagulation operator requires to satisfy certain Lipschitz and Fréchet differentiability conditions to establish the local existence and uniqueness of a strongly differentiable solution to (1.3.1); see [14] for details. In this regard, an overview of the theory of semigroup of linear operators as well as of the semilinear abstract Cauchy problems is given in Chapter 2.

It is worthwhile to mention that during the fragmentation process the total mass of aggregates in the system should be conserved throughout the evolution, that is, the total mass of the described quantity contained in all the aggregates before and after a fragmentation event should be the same. Thus, if the fragmentation (splitting) of aggregates occurs alongside another process of growth determined by the conservation law, then the evolution of the total mass should follow this law due to the conservativity of the fragmentation process. If this is the case, then such a process is said to be 'honest'. In this context, the mass in the system is expected to evolve according to the following equation

$$\frac{\partial}{\partial t}\int_{x_0}^{\infty} u(t,x)x\,dx = -\int_{x_0}^{\infty}\frac{\partial}{\partial x}[r(x)u(t,x)]x\,dx - \int_{x_0}^{\infty} d(x)u(t,x)x\,dx.$$

However, this equation is not always valid as it depends on some proprieties of the parameters of the model. In this regards, Arlotti and Banasiak [8] have obtained a number of results which can be used to determine whether the semigroup associated with the fragmentation process is honest or dishonest. An account of this approach is presented in Chapter 2. Making use of these results we show, under fairly mild conditions on the parameters, of the model that the related semigroup is honest. More precisely, we give conditions which guarantee that G is the closure of $T + B$. In Chapter 3, we provide a brief outline of the analysis of pure fragmentation equation so as to give the reader a preliminary insight of the mathematical analysis that we use. Next, in Chapter 4 we present the mathematical analysis of phytoplankton dynamics that is based on the approach mentioned above. Finally, the results are summarised in the last chapter.

Chapter 2

Analytical Background

This chapter is a brief introduction to some functional analysis concepts and presents a short summary of the vast literature that has been produced on the theory of semigroups of linear, and semi-linear operators and on the perturbation theory in Banach spaces. The reader can find further results in the classic texts for semigroups [14, 16, 25].

2.1 Operators

Definition 2.1.1. *Let X, Y be real or complex Banach spaces. An operator from X to Y is a linear rule $A : D(A) \longrightarrow Y$, where $D(A)$ is a linear subspace of X, called the domain of A.*

Definition 2.1.2. *A linear operator A from X into Y is said to be bounded on $D(A)$ if there exists a positive number M such that for all $\psi \in D(A)$*

$$\|A\psi\|_Y \leq M\|\psi\|_X. \tag{2.1.1}$$

The smallest possible M such that (2.1.1) holds is denoted by $\|A\|$ and is called the operator norm of A. An equivalent definition of $\|A\|$ is

$$\|A\| = \sup_{\|\psi\|\leq 1} \|A\psi\| = \sup_{\|\psi\|=1} \|A\psi\|.$$

Definition 2.1.3. *Let X and Y be a Banach spaces and $A : D(A) \longrightarrow Y$ a linear operator with domain $D(A) \subset X$. Then*

- *A is densely defined if $\overline{D(A)} = X$.*
- *A is called a closed linear operator if its graph*

$$G(A) = \{(\psi, \phi) \in X \times Y; \psi \in D(A), A\psi = \phi\}$$

is closed in the normed space $X \times Y$.

Theorem 2.1.4. *Suppose that A is a linear operator from X into Y. Then A is bounded on $D(A)$ if and only if it is continuous.*

Proof. [19, Theorem 2.5] □

Theorem 2.1.5. *Let $A : X \longrightarrow X$ be a compact linear operator on a normed space X. Then $I - A$ is injective if and only if it is surjective. If $I - A$ is injective (and therefore also bijective), then the inverse operator $(I - A)^{-1} : X \longrightarrow X$ is bounded.*

Proof. [19, Theorem 3.4] □

We introduce the notion of a positive cone and positive operators.

Definition 2.1.6. *Let $X = L_1(\mathbb{R}_+, d\mu)$.*

i) The positive cone of X, denoted by X_+, is the set of functions in X that are positive a.e.

ii) Let $A : X \supseteq D(A) \longrightarrow X$. If for all $\psi \in D(A) \cap X_+ = D(A)_+$ we have that $A\psi \in X_+$, then the operator A is called positive (with respect to X_+).

Definition 2.1.7. *A Banach space X is of type L if it consists of equivalence classes of numerically-valued functions defined on a set Ω and if it has the two following properties*

(1) *If ψ is a continuous X-valued function defined on $I = [\alpha, \beta]$, then there exists a function ϕ measurable on the product $I \times \Omega$ such that $\psi(t) = \phi(t, \cdot)$ (equality in X) for each $t \in [\alpha, \beta]$.*

(2) *If ψ is continuous on $I = [\alpha, \beta]$ and ϕ is any function that is measurable on $I \times \Omega$ and satisfies $\psi(t) = \phi(t, \cdot)$ for each $t \in [\alpha, \beta]$, then*

$$\left[\int_\alpha^\beta \psi(t) dt(\cdot) \right] \approx \int_\alpha^\beta \phi(t, \cdot) dt.$$

Lemma 2.1.8. (Gronwall's Inequality - differential form)
Let $I = [t_0, t_1]$. Suppose $\psi : I \longrightarrow \mathbb{R}$ is in $C^1(I)$ and $a : I \longrightarrow \mathbb{R}$ is continuous. If

$$\psi'(t) \leq a(t)\psi(t)$$

for $t \in I$, and $\psi(t_0) = \psi_0$. Then

$$\psi(t) \leq \psi_0 \exp\left(\int_{t_0}^t a(s) ds \right).$$

Proof. [2, Internet sources] □

Theorem 2.1.9. (Gronwall's Inequality - integral form)
Let ψ, a be two positive continuous functions on $I = [t_0, t_1]$ and let $C \geq 0$. If

$$\psi(t) \leq C + \int_{t_0}^{t} a(s)\psi(s)ds$$

for $t \in I$, then

$$\psi(t) \leq C \exp\left(\int_{t_0}^{t} a(s)ds\right).$$

Proof. [2, Internet sources] □

2.2 Linear Semigroup Theory

In this section, we deal with methods of the semigroup theory to find solutions of a Cauchy problem.

Definition 2.2.1. *Let X be a Banach space and A a linear operator with domain $D(A)$ and range ImA contained in X such that*

$$\begin{aligned} \frac{du}{dt}(t) &= Au(t), \quad t > 0, \\ u(0) &= u_0, \end{aligned} \tag{2.2.1}$$

where $u_0 \in X$. A function is called the classical (or strict) solution of (2.2.1) if it satisfies the following conditions

- *$u(t)$ is continuous on $[0, \infty)$ and continuously differentiable on $(0, \infty)$,*
- *for each $t > 0$, $u(t) \in D(A)$ and $u(t)$ satisfies (2.2.1).*
- $$\lim_{t \to 0} u(t) = u_0 \text{ in the norm of } X.$$

Definition 2.2.2. *A family $(S(t))_{t \geq 0}$ of bounded linear operators on X is called a C_0-semigroup, or a strongly continuous semigroup, if*

- *$S(0) = I$;*
- *$S(t + s) = S(t)S(s)$ for all $t, s \geq 0$;*
- *$\lim_{t \to 0^+} S(t)\psi = \psi$ for any $\psi \in X$.*

Theorem 2.2.3. *Assume that the family $(S(t))_{t \geq 0}$ forms a C_0-semigroup on X, then there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|S(t)\| \leq Me^{\omega t}, \quad t \geq 0. \tag{2.2.2}$$

Proof. [25, Theorem 2.2] □

Definition 2.2.4. *If $M = 1$ and $\omega = 0$, then $(S(t))_{t\geq 0}$ is called a contraction semigroup on X.*

With each C_0-semigroup $(S(t))_{t\geq 0}$ we can associate an operator A which is called the generator of this semigroup.

Definition 2.2.5. *The operator $A : X \supseteq D(A) \longrightarrow X$ defined by*

$$D(A) := \left\{ \psi \in X \ / \ A\psi := \lim_{h\to 0^+} \frac{S(h)\psi - \psi}{h} \text{ exists in } X \right\}, \tag{2.2.3}$$

is called the infinitesimal generator of $(S(t))_{t\geq 0}$. Typically the semigroup generated by A is denoted by $(S_A(t))_{t\geq 0}$.

Definition 2.2.6. *If the family $(S(t))_{t\geq 0}$ generated by A satisfies (2.2.2) for given M and ω, then we write*

$$A \in \mathcal{G}(M, \omega).$$

Lemma 2.2.7. *Let A be the infinitesimal generator of $(S(t))_{t\geq 0}$.*

(i) The operator $A : X \supseteq D(A) \longrightarrow X$ is a closed and densely defined linear operator that determines the semigroup uniquely.

(ii) If $\psi \in D(A)$, then for all $t > 0$, $S(t)\psi \in D(A)$ and

$$\frac{d}{dt} S(t)\psi = S(t)A\psi = AS(t)\psi.$$

Proof. [16, Lemma 1.3, Theorem 1.4]. □

For $\psi \in X \setminus D(A)$, the function $u(t) = S(t)\psi$ is continuous but, in general, neither differentiable, nor $D(A)$-valued, and therefore it is not a classical solution to (2.2.1). Nevertheless, the integral $v(t) = \int_0^t u(s)ds \in D(A)$ and it is a strict solution of the integrated version of (2.2.1),

$$\begin{aligned} \frac{dv}{dt}(t) &= A(v(t)) + \psi \\ \lim_{t\to 0^+} v(t) &= 0, \end{aligned} \qquad t \geq 0 \tag{2.2.4}$$

or equivalently,

$$u(t) = A \int_0^t u(s)ds + \psi. \tag{2.2.5}$$

We say that a function u satisfying (2.2.4) (or, equivalently, (2.2.5)) is a mild solution or integral solution of (2.2.1).

Proposition 2.2.8. *Let $(S(t))_{t\geq 0}$ be the semigroup generated by $(A, D(A))$. Then $t \to S(t)\psi$, $\psi \in D(A)$, is the only solution of (2.2.1) taking values in $D(A)$. Similarly, for $\psi \in X$, the function $t \to S(t)\psi$ is the only mild solution to (2.2.1).*

Proof. [11, Proposition 3.4] □

In light of Proposition 2.2.8, it is natural to ask which operators A generate C_0-Semigroups. In the case when the operator A is bounded, it is the infinitesimal generator of $(S(t))_{t\geq 0}$ given by

$$S(t) = e^{At} := \sum_{k=0}^{\infty} \frac{(At)^k}{k!}, \quad t \geq 0.$$

However, convergence of this series is not likely when A is unbounded. But, in light of viewing the exponential formula as

$$e^{tA} = \lim_{n\to\infty} \left(1 - \frac{tA}{n}\right)^{-n} = \lim_{n\to\infty} \left(\frac{n}{t}\left(\frac{n}{t}I - A\right)^{-1}\right)^{n}, \tag{2.2.6}$$

we can ensure that under some assumptions on the operator A, one can prove that the above limit exists and A is the infinitesimal generator of a C_0-semigroup. We begin with a necessary definition.

Definition 2.2.9. *Let $A : X \supseteq D(A) \longrightarrow X$ be a linear operator.*

- *The resolvent set, denoted by $\rho(A)$, of A is the set of complex numbers defined by*

$$\rho(A) = \{\lambda \in \mathbb{C}; \quad \lambda I - A : D(A) \to X \text{ is invertible}\}.$$

- *The spectrum of A, denoted by $\sigma(A)$ is the complement in $\mathbb{C}$ of $\rho(A)$.*

- *For $\lambda \in \rho(A)$, the operator $R(\lambda, A)$ defined by*

$$R(\lambda, A) := (\lambda I - A)^{-1},$$

is called the resolvent operator.

Lemma 2.2.10. *Let $(A, D(A))$ be a closed, densely defined operator. Suppose there exist $\omega \in \mathbb{R}$ and $M > 0$ such that $[\omega, \infty) \subset \rho(A)$ and $\|\lambda R(\lambda, A)\| \leq M$ for all $\lambda \geq \omega$. Then the following convergence statements hold for $\lambda \to \infty$:*

$$\lambda R(\lambda, A)\psi \to \psi, \quad \text{for all } \psi \in X.$$

Proof. [16, Lemma 3.4]. □

The next theorem states necessary and sufficient conditions that characterise a linear operator A to be the infinitesimal generator of a C_0-semigroup in a Banach space setting.

Theorem 2.2.11. (Hille-Yosida Theorem, 1948)

$A \in \mathcal{G}(M, \omega)$ if and only if

- *A is closed and densely defined,*
- *there exist $M > 0, \omega \in \mathbb{R}$ such that $(\omega, \infty) \in \rho(A)$ and for all $n \geq 1$, $\lambda > \omega$,*

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}. \tag{2.2.7}$$

In the contraction case, (2.2.7) is equivalent to

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda},$$

for $\lambda > 0$.

Proof. [11, Theorem 3.5]. □

2.3 Perturbations of Semigroups

Let $(A, D(A))$ be a generator of a C_0-semigroup on a Banach space X and $(B, D(B))$ be another operator in X. The purpose of the perturbation theory is to find conditions which ensure that there is an extension G of $A + B$ that generates a C_0-semigroup on X and to characterize this extension.

2.3.1 Bounded Perturbation Theorem

The simplest and possibly the most often used perturbation result can be obtained if the operator B is bounded. The following theorem holds:

Theorem 2.3.1. (Bounded perturbation)
Let $(A, D(A)) \in \mathcal{G}(M, \omega)$; that is, it generates a C_0-semigroup $(S_A(t))_{t\geq 0}$ satisfying $\|S_A(t)\| \leq Me^{\omega t}$ for some $\omega \in \mathbb{R}$ and $M \geq 1$. If $B \in \mathcal{L}(X)$, then

$$(A + B, D(A)) \in \mathcal{G}(M, \omega + M\|B\|).$$

Proof. [11, Theorem 4.9]. □

2.3.2 Kato's Perturbation Theorem

The Kato's Perturbation Theorem is useful in the sense that it allows us to establish the existence of a *smallest* substochastic semigroup (i.e. the smallest extension that generates a substochastic semigroup) associated with a specific Cauchy problem. We begin with the definition of the terms *stochastic* and *substochastic* semigroups.

Definition 2.3.2. *The strongly continuous semigroup of operators $(S(t))_{t\geq 0}$ on the Banach space $X = L_1(\Omega, d\mu)$ is said to be*

- *substochastic if $S(t) \geq 0$ and $\|S(t)\| \leq 1$ for all $t \geq 0$,*

- *stochastic if in addition, it satisfies* $\|S(t)\psi\| = \|\psi\|$ *for all non-negative* $\psi \in X$.

Proof. [11, Theorem 5.13]. □

Corollary 2.3.3. *Let* $(S(t))_{t\geq 0}$ *be the semigroup generated by* $(A+B, D(A))$. *Then* $(S(t))_{t\geq 0}$ *satisfies the Duhamel equation*

$$S(t)\psi = S_A(t)\psi + \int_0^t S(t-s)BS_A(s)\psi ds, \quad \psi \in D(A). \tag{2.3.1}$$

Proof. [11, Corollary 5.15]. □

Theorem 2.3.4. (Kato's Perturbation Theorem in L_1 Setting)
Let $X = L_1(\Omega, d\mu)$ *and suppose that the operators* A *and* B *satisfy:*

- $(A, D(A))$ *generates a substochastic semigroup* $(S_A(t))_{t\geq 0}$;
- $D(B) \supset D(A)$ *and* $Bu \geq 0$ *for* $u \in D(B)_+$;
- *For all* $u \in D(A)_+$,

$$\int_\Omega (Au + Bu) d\mu \leq 0. \tag{2.3.2}$$

Then, there exists a smallest substochastic semigroup, $(S_G(t))_{t\geq 0}$, *generated by an extension,* G, *of* $A + B$. *Moreover,* G *is characterized by*

$$(I - G)^{-1}\psi = \sum_{n=0}^{\infty} (I - A)^{-1}[B(I - A)^{-1}]^n \psi, \quad \forall \psi \in X. \tag{2.3.3}$$

Proof. [11, Corollary 5.17]. □

Theorem 2.3.4 gives the existence of a smallest substochastic semigroup $(S_G(t))_{t\geq 0}$ generated by an extension G of the operator $A+B$. This semigroup, for arbitrary $\psi \in D(G)$ and $t > 0$, satisfies

$$\frac{d}{dt} S_G(t)\psi = GS_G(t)\psi. \tag{2.3.4}$$

In what follows, we provide an interesting theory pertaining to honesty and dishonesty of substochastic semigroups. Before proceeding, we note that there is another theory treating this issue which uses tools of spectral theory [11, Theorem 4.3].

2.4 Generator Characterization of Substochastic Semigroups

In this section, we present a brief summary of techniques that allow us to characterise the generator of substochastic semigroups. Further details on this theory can be found in [11].

Let X denote the Banach space $L_1(\Omega, d\mu)$ endowed with the standard norm $\|\cdot\|$. Let A be the generator of a substochastic semigroup on $L_1(\Omega, d\mu)$ and let $B : D(A) \to L_1(\Omega, d\mu)$ be a positive linear operator such that

$$\int_\Omega (A+B)u d\mu = -c(u), \quad u \in D(A)_+, \tag{2.4.1}$$

where c is a nonnegative (possibly zero) functional defined on $D(A)$, which can be written as an integral functional; that is,

$$c(u) = \int_\omega \varsigma(x)u(x)d\mu_x^{'},$$

for some positive measurable function ς and positive measure $\mu^{'}$. We do not assume that c is bounded or closed.

Definition 2.4.1. *A positive semigroup $(S_G(t))_{t\geq 0}$ generated by an extension G of the operator $A+B$ is said to be strictly substochastic if (2.4.1) holds with $c \neq 0$.*

Definition 2.4.2. *We say that a positive semigroup $(S_G(t))_{t\geq 0}$ generated by an extension G of the operator $A+B$ is honest if c extends to $D(G)$ and for any $0 \leq \psi \in D(G)$ the solution $u(t) = S_G(t)\psi$ of (2.3.4) satisfies*

$$\frac{d}{dt}\int_\Omega u(t)d\mu = \frac{d}{dt}\|u(t)\| = -c(u(t)).$$

Hence, if $c = 0$, then honest semigroups are the same as stochastic semigroups.

Theorem 2.4.3. *The semigroup $(S_G(t))_{t\geq 0}$ is honest if and only if $G = \overline{A+B}$.*

Proof. [11, Theorem 6.13]. □

Corollary 2.4.4. *The semigroup $(S_G(t))_{t\geq 0}$ is honest if and only if for any $u \in D(G)_+$ we have*

$$\int_\Omega Gud\mu \geq -c(u).$$

The statement also holds true if we replace $D(G)_+$ by $R(\lambda, G)X_+$ for some/any $\lambda > 0$.

Proof. [11, Corollary 6.14]. □

The problem is that in most cases we do not have any direct characterisation of G. Thus the previous theory has a limited practical value. In what follows, we introduce a technique that relies on the extension of operators.

2.4.1 Extension Techniques for Honesty

Let $\mathsf{E} := L_0(\Omega, d\mu)$ denote the set of μ-measurable functions that are defined on Ω and take values in the extended set of real numbers, and by $\mathsf{E_f}$ the subspace of E consisting of functions that are finite almost everywhere.

Let $\mathsf{F} \subset \mathsf{E}$ be defined by the condition: $\psi \in \mathsf{F}$ if and only if for any non-negative and nondecreasing sequence $(\psi_n)_{n\in\mathbb{N}}$ satisfying $\sup_{n\in\mathbb{N}} \psi_n = |\psi|$, we have $\sup_{n\in\mathbb{N}}(I - A)^{-1}\psi_n \in X$.

Under some natural assumptions on B (that are satisfied if, e.g., B is an integral operator with non-negative kernel), [7], we construct another subset of E, say G, defined as the set of all functions $\psi \in X$ such that for any nonnegative, nondecreasing sequence $(\psi_n)_{n\in\mathbb{N}}$ of elements of $D(B)$ such that $\sup_n \psi_n = |\psi|$, we have $\sup_n B\psi_n < \infty$ almost everywhere. We can then define mappings $\mathsf{L} : \mathsf{F}_+ \to X_+$ and $\mathsf{B} : \mathsf{G}_+ \to \mathsf{E}_+$ by

$$\mathsf{L}f := \sup_{n\in\mathbb{N}} R(1, A)\psi_n, \qquad \psi \in \mathsf{F}_+,$$

$$\mathsf{B}\psi := \sup_{n\in\mathbb{N}} B\psi_n, \qquad \psi \in \mathsf{G}_+,$$

where $0 \le \psi_n \le \psi_{n+1}$ for any $n \in \mathbb{N}$, and $\sup_{n\in\mathbb{N}} \psi_n = \psi$. We extend the mappings L and B onto F and G, respectively, by linearity [11, Theorem 2.64]. By [11, Lemma 6.18] L is one-to-one therefore, we can define the operator A with $D(\mathsf{A}) = \mathsf{LF} \subset X$ by

$$\mathsf{A}u = u - \mathsf{L}^{-1}u, \tag{2.4.2}$$

so that A is an extension of A . The central theorem of this paragraph reads:

Theorem 2.4.5. *Let $X = L_1(\Omega, d\mu)$ and suppose that the operators A and B satisfy*

- *$(A, D(A))$ generates a substochastic semigroup $(S_A(t))_{t\ge0}$;*
- *$D(B) \supset D(A)$ and $Bu \ge 0$ for $u \in D(B)_+$;*
- *For all $u \in D(A)_+$,*

$$\int_\Omega (Au + Bu)d\mu \le 0, \tag{2.4.3}$$

for all $u \in D(A)_+$.

Then the extension G of $A + B$ that generates the smallest substochastic semigroup on X described by Theorem 2.3.4, is given by

$$Gu = \mathsf{A}u + \mathsf{B}u, \tag{2.4.4}$$

$$D(G) = \left\{u \in D(\mathsf{A}) \cap D(\mathsf{B}) : \mathsf{A}u + \mathsf{B}u \in X, \ \textit{and} \lim_{n\to\infty} ||(\mathsf{LB})^n u|| = 0\right\}.$$

Proof. [11, Theorem 6.20]. □

If we now consider $u \in D(G)$ then, by (2.4.4) and the definition of $D(\mathsf{A})$, we see that $u \in D(\mathsf{A}) = \mathsf{LF}$ and therefore there exists a unique $\psi \in \mathsf{F}$ satisfying $u = \mathsf{L}\psi$. For such u, we can write $Gu = \mathsf{AL}\psi + \mathsf{BL}\psi$ and, using (2.4.2), we obtain a representation theorem for Gu,

$$Gu = \mathsf{L}\psi - \psi + \mathsf{BL}\psi. \tag{2.4.5}$$

In particular, in this case $\mathsf{L}\psi = u \in D(G)$ is integrable and thus $-\psi + \mathsf{BL}\psi$ is also integrable. Moreover, if $D(G)_+ \ni u = R(1, G)g$, with $g \in X_+$, then from (2.4.5) we get $\psi = \mathsf{L}\psi - Gu + \mathsf{BL}\psi = (I - G)u + \mathsf{B}u \geq 0$, that is, $\psi \in \mathsf{F}_+$. Finally, for any $u \in D(G)$, we can find elements $\bar{u}_\pm \in D(G)_+$ such that $u = \bar{u}_+ - \bar{u}_-$ and $G\bar{u}_\pm = \mathsf{L}\bar{\psi}_\pm - \bar{\psi}_\pm + \mathsf{BL}\bar{\psi}_\pm$.

Theorem 2.4.6. *If for any $\psi \in \mathsf{F}_+$ such that $-\psi + \mathsf{BL}\psi \in X$ and $c(\mathsf{L}\psi)$ exists,*

$$\int_\Omega (\mathsf{L}\psi - \psi + \mathsf{BL}\psi)\, d\mu \geq -c(\mathsf{L}\psi),$$

then $G = \overline{A + B}$.

Proof. [11, Theorem 6.22]. □

However, though we do not know G and the particular extension G that we introduced above can be difficult to construct, by using a general extension of G, sufficient criteria can be obtained for honesty and dishonesty.

Theorem 2.4.7. *Let $\mathcal{G}$ be any extension of G. Then*

- (a) *If $\displaystyle\int_\Omega \mathcal{G}u\, d\mu \geq -c(u)$ for all $u \in D(\mathcal{G})_+$, then the semigroup is honest.*
- (b) *If there exists $u \in D(\mathcal{G})_+ \cap X$ such that for some $\lambda > 0$,*
  - *(i) $\lambda u(x) - [\mathcal{G}u](x) = \psi(x) \geq 0$, a.e.,*
  - *(ii) $c(u)$ is finite and $\displaystyle\int_\Omega \mathcal{G}u\, d\mu < -c(u)$, then the semigroup $(S_G(t))t \geq 0$ is not honest.*

Proof. [11, Theorem 6.23]. □

2.5 Semilinear Semigroups

The success of linear semigroup theory in solving linear evolution equations has stimulated extensions of the linear ideas to examine semilinear problems. Unlike the linear case, semilinear semigroup theory is not complete, yet it remains a useful and powerful method of analyzing more difficult evolution equations.

Definition 2.5.1. (**Semilinear Abstract Cauchy Problem**)
Let X be a Banach space and let $(G, D(G))$ be an operator in X with associated semigroup $(S_G(t))_{t\geq 0}$. Furthermore, let N be a nonlinear operator which maps a subset D of X into X where $D(G) \cap D$ is not empty. Then the abstract problem,

$$\frac{du}{dt}(t) = Gu(t) + Nu(t), \quad (t \geq 0); \quad u(0) = u_0 \in D(G) \cap D, \tag{2.5.1}$$

is called a semilinear *abstract Cauchy problem (ACP).*

Definition 2.5.2. *A function u is said to be a strong solution to the semilinear ACP (2.5.1) on $[0, t_0)$ if u is continuous on $[0, t_0)$, differentiable on $(0, t_0)$, $u(t) \in D(G) \cap D$ for all $t \in [0, t_0)$ and satisfies (2.5.1).*

Proposition 2.5.3. *Let u be a strong solution on $[0, t_0)$ of the semilinear ACP (2.5.1). Then u satisfies the integral equation*

$$u(t) = S_G(t)u_0 + \int_0^t S_G(t-s)N(u(s))ds, \quad 0 \leq t < t_0, \tag{2.5.2}$$

where $(S_G(t))_{t\geq 0}$ is the semigroup associated with the linear operator G.

Proof. [14, p. 108]. □

Definition 2.5.4. *$u : [0, t_0) \to X$ is said to be a mild solution to the semilinear ACP (2.5.1) if*

1. *u is continuous on $[0, t_0)$,*
2. *$u(t) \in D$ for all $t \in [0, t_0)$,*
3. *u satisfies (2.5.2).*

We now introduce some definitions which are required in the theorems that follow.

Definition 2.5.5. (**Local Lipschitz Condition**)
An operator N on a Banach space X is said to satisfy a local Lipschitz condition if, for any given $u_0 \in X$, there exists a closed ball

$$\overline{B}(u_0, r) = \{\psi \in X : \quad \|\psi - u_0\| \leq r\},$$

and a constant C such that $\|N\psi - N\phi\| \leq C\|\psi - \phi\|$ for all $\psi, \phi \in \overline{B}(u_0, r)$, where C may depend on u_0 and r.

Theorem 2.5.6. *Let $(G, D(G))$ be the generator of the strongly continuous semigroup $(S_G(t))_{t\geq 0}$ on X, let N be a nonlinear operator, and let X be a Banach space. If N satisfies a local Lipschitz condition on X, then the semilinear ACP has a unique, local in time, mild solution.*

Proof. [14, Theorem 3.20, p. 119]. □

Definition 2.5.7. (**Fréchet Derivative**)
If a linear operator $N_\psi \in \mathcal{L}(X)$ exists such that $N(\psi+\delta) = N\psi+N_\psi\delta+\mathcal{H}(\psi,\delta)$ where $\mathcal{H}$ satisfies

$$\lim_{\delta\to 0}\left(\frac{\|\mathcal{H}(\psi,\delta)\|}{\|\delta\|}\right) = 0,$$

then we say that N is Fréchet differentiable at f, and N_ψ is the Fréchet derivative.

Theorem 2.5.8. *Let $(G, D(G))$ generate the strongly continuous semigroup $(S_G(t))_{t\geq 0}$ on X and let N satisfy the local Lipschitz condition*

$$\|N(\psi) - N(\phi)\| \leq \kappa\|\psi - \phi\|$$

for all ψ, ϕ in the closed ball $\overline{B}(u_0, r) \subseteq D = D(N)$. If

1. *N is Fréchet differentiable at any $\psi \in B(u_0, r)$ and the Fréchet derivative N_ψ is such that $\|N_\psi\phi\| \leq \kappa_1\|\phi\|$ for all $\psi \in B(u_0, r)$, $\phi \in X$ where κ_1 is a positive constant independent of ψ and ϕ,*

2. *the Fréchet derivative is continuous with respect to $\psi \in B(u_0, r)$ such that*

$$\|N_{\psi_1}\phi - N_{\psi_2}\phi\| \to 0 \quad \textit{as} \quad \|\psi_1 - \psi_2\| \to 0 \quad \textit{where} \quad \psi_1, \psi_2 \in B(u_0, r),$$

   for any given $\phi \in X$,

3. *$u_0 \in D(G)$,*

then there exists $t_1 > 0$ such that the continuous solution on $[0, t_1)$ of (2.5.2) is strongly differentiable on $[0, t_1)$ and satisfies the equation (2.5.1).

Proof. [14, Theorems 3.30 and 3.32]. □

Chapter 3

Pure Fragmentation Equation

3.1 Description of the Model and Assumptions

The classical fragmentation equation describing the evolution of the particle-mass distribution function for a continuous system undergoing only fragmentation can be derived by balancing the loss and gain of particles of mass x over a short period of time. The initial value problem for this kinetic type rate equation is

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= -a(x)u(t,x) + \int_x^\infty a(y)b(x|y)u(t,y)dy, \quad x,t \geq 0 \\ u(0,x) &= u_0(x), \end{aligned} \tag{3.1.1}$$

which describes the evolution of the density u of particles having mass x at time t. The coefficient a describes the rate of fragmentation; that is, the number of splitting events per unit time. We assume that a is a positive and continuous function on $(0,\infty)$ and throughout this chapter we consider that a is (essentially) bounded on compact subsets of $(0,\infty)$; that is,

$$a \in L_{\infty,\,loc}\left((0,\infty)\right). \tag{3.1.2}$$

The function b describes the distribution of particle masses x, also called daughter particles, spawned by the fragmentation of a parent particle of mass $y > x$. Thus the first term on the right-hand side, called the loss term, gives the rate at which mass x particles vanish by fragmenting to particles of a smaller mass. The second term on the right-hand side, called the gain term, gives the rate at which the class of mass x particles gains new particles of mass x by fragmentation of particles of mass $y > x$.

For the total mass in the system to remain constant during fragmentation in absence of any other mechanism, i.e, for the mass of all daughter particles to be equal to the mass of the parent, b must satisfy the condition of the conservation mass principle which is mathematically expressed by

$$\int_0^y xb(x|y)dx = y. \tag{3.1.3}$$

The expected number of daughter particles produced by fragmentation of a mass y particle is, by definition, given by

$$n(y) = \int_0^y b(x|y)dx.$$

Here, we mention that $n(y)$ may be infinite. If u is a solution to (3.1.1), the total mass of the ensemble at a time t is given by the first moment of u; that is, $M(t) = \displaystyle\int_0^\infty xu(t,x)dx$. From the physical point of view the total mass of fragmenting particles cannot increase, thus fragmentation equations are usually investigated in the space X_1, see (1.3.2).

It is also worthwhile to note that the function b plays an important role in the analysis of the model, and various forms of this function have been applied by some authors. Very often, the form of power law, that is,

$$b(x|y) = (\nu + 2)\frac{x^\nu}{y^{\nu+1}}, \tag{3.1.4}$$

with $\nu > -2$, has been utilised. Another common form, called homogeneous fragmentation, is given by

$$b(x|y) = \frac{1}{y}h\left(\frac{x}{y}\right).$$

Here, we observe that in the case when $h(r) = (\nu + 2)r^\nu$, the coefficient b is nothing but the power law mentioned above.

Another form, which is a generalisation of (3.1.4), is given by

$$b(x|y) = \beta(x)\gamma(y), \tag{3.1.5}$$

where, to satisfy the local principle of mass conservation,

$$\gamma(y) = \frac{y}{\int_0^y s\beta(s)ds}.$$

Here, we assume that β is a non-negative continuous function on $(0, \infty)$. Equation (3.1.5) is a natural generalization of the power law b described in (3.1.4) and has the advantage of allowing the number of daughter particles,

$$n(y) = \frac{y\int_0^y \beta(s)ds}{\int_0^y s\beta(s)ds},$$

to vary with the parent size y, [9]. An important role in the analysis is played by the function

$$b(x|x) = \beta(x)\gamma(x) = \frac{x\beta(x)}{\int_0^x s\beta(s)ds} = \frac{d}{dx}\ln\int_0^x s\beta(s)ds.$$

see [11, Theorem 8.13, Theorem 8.18].

3.2 Dishonesty in the Pure Fragmentation Model

The process of pure fragmentation should simply rearrange the distribution of masses of the particles without altering the total mass of the system. Thus the total mass should be conserved and should be accounted for. In other words, the density u of particles should satisfy the conservation equation

$$\frac{d}{dt}M(t) = \int_0^\infty \frac{\partial}{\partial t}u(t,x)x dx = 0, \tag{3.2.1}$$

which follows formally from (3.1.1) and (3.1.3), as the expected mass rate equation can be found by multiplying (3.1.1) by x and integrating over $[0,\infty)$. If the equation (3.2.1) is satisfied by all nonnegative solutions of (3.1.1), then the semigroup describing the evolution is conservative for positive initial data and is a stochastic semigroup. In other words, the process is *honest* in the space X_1. However, the semigroup may turn out not to be conservative even though the model is formally conservative. In fact, by analysing equation (3.1.1) with specific coefficients, it has been observed that, if the fragmentation rate is unbounded as $x \to 0$, then (3.2.1) is not valid. This indicates that the described quantity, 'mass', leaks out from the system. This phenomenon was termed 'shattering fragmentation' and was attributed to the phase transition in which a 'dust' of particles with zero size and non-zero mass is formed. In such a case the global conservation principles are not always satisfied, and the process is called dishonest. In the next section we present sufficient conditions for the fragmentation semigroup to be honest.

3.3 Analysis of the Model

In this section we give a summary of results that have been presented in [11]. First, we start with well-posedness of the pure fragmentation equation.

By $\mathcal{A}$ and $\mathcal{B}$ we denote the expressions appearing on the right-hand side of the equation (3.1.1) as

$$[\mathcal{A}u](x) = -a(x)u(x), \qquad [\mathcal{B}u](x) = \int_x^\infty a(y)b(x|y)u(y)dy,$$

defined on all measurable and finite almost everywhere functions u for which they make pointwise (almost everywhere) sense. The formal expressions $\mathcal{A}$ and $\mathcal{B}$ may define various operators. With these expressions, we associate operators A and B in X_1 defined by $Au = \mathcal{A}u$, $Bu = \mathcal{B}u$ and set

$$D(A) = \{u \in X_1;\ au \in X_1\}.$$

Lemma 3.3.1. *$(A+B, D(A))$ is a well-defined operator.*

Proof. The result follows directly from the fact that $\|Bu\|_{X_1} \leq \|Au\|_{X_1}$. □

With this lemma, the fragmentation equation can be expressed as an abstract Cauchy problem (ACP),

$$\begin{aligned} \frac{du}{dt} &= Au + Bu, \quad t > 0, \\ u(0) &= u_0. \end{aligned} \tag{3.3.1}$$

Clearly, if $a \in L_\infty(\mathbb{R}_+)$, then both A and B are bounded on X_1 and it follows immediately that, for each $u_0 \in X_1$, (3.3.1) has a unique strong solution $u : \mathbb{R}_+ \to X_1$ given by $u(t) = e^{(A+B)t}u_0$. Generally, when a is unbounded, $(A + B, D(A))$ does not necessarily generate a semigroup on X_1. However, by assuming that a verifies (3.1.2), it is still possible to construct a solution to (ACP) by adopting an approach based on a perturbation result due to Kato.

Theorem 3.3.2. *Under the assumptions of this section, there exists a smallest substochastic semigroup $(S_G(t))_{t\geq 0}$ generated by an extension G of $A + B$.*

Proof. [9, Theorem 8.3]. □

The semigroup $(S_G(t))_{t\geq 0}$ can be obtained as the strong limit in X_1 of semigroups $(S_{G_r}(t))_{t\geq 0}$ generated by $(A + rB, D(A))$ as $r \nearrow 1^-$; the limit is monotonic on non-negative data. The fact that, in general, G is a proper extension of $\overline{A + B}$ has far reaching consequences which we explain below.

It should be noted that the Theorem 3.3.2 can be used to deduce the existence and uniqueness of a solution to the ACP (3.3.1) associated with the extended operator G. Unfortunately, the mass conservation (honesty in X_1) cannot be deduced from the Kato's Perturbation Theorem alone, unless by imposing some additional constraints. Here, we state the following theorem that addresses this question.

Theorem 3.3.3. *If*

$$\lim_{x\to 0^+} \sup a(x) < +\infty,$$

then $G = \overline{A + B}$ and so, $(S_G(t))_{t\geq 0}$ is honest.

Proof. [9, Theorem 8.5]. □

Chapter 4

Mathematical Analysis of Phytoplankton Dynamics

The mathematical model we introduce in this chapter describes the dynamical behaviour of phytoplankton. Phytoplankton consists of aggregates of all possible sizes/masses and the aggregates are structured by size. Moreover, the aggregate size can change due to splitting, death, growth or combining of aggregates into bigger ones. To include the effects of cell division, the McKendrick-von Foerster renewal condition is incorporated.

4.1 Description of the Model and Assumptions

The processes of growth in the evolution of aggregates of phytoplankton populations occur due to division of cells forming the aggregate. Thus, if we assume that the minimum mass of an aggregate, corresponding to the mass of a single cell, is $x_0 > 0$, then the mass of an aggregate should be an integer multiple of x_0. But, due to the relative mass of a typical aggregate against the mass of a single cell, the mass of an aggregate can be in principle any real number $x \geq x_0 > 0$. With this, we introduce the density function $u(t, x)$ which gives the number density of aggregates of mass x at time t. Thus,

$$\int_{x_0}^{\infty} u(t, x)dx$$

is the number of aggregates having a mass in the range $[x_0, \infty)$, whereas

$$\int_{x_0}^{\infty} u(t, x)\, xdx$$

is the mass contained in the aggregates having a mass within this range.

Since we are interested in the evolution of the mass of aggregates in the system, the natural space seems to be

$$X_1 = L_1([x_0, \infty), xdx) = \left\{ u : \quad \|u\|_1 := \int_{x_0}^{\infty} |u(x)|\, xdx < \infty \right\}. \tag{4.1.1}$$

In this case, the norm of a non-negative u, that is, the integral over $[x_0, \infty)$ according to xdx, gives the total mass of aggregates in the system.

Before introducing the model of phytoplankton dynamics, let us begin with a description of all the mechanisms that are involved in the processes of evolution of phytoplankton dynamics, namely, the growth, the mortality, the fragmentation and the coagulation process.

4.1.1 Growth and Mortality

Phytoplankton cells may die, for example, by sinking to the seabed, or from other causes. We denote by d the death rate. Generally, we assume that it is a non-negative function and

$$d \in L_\infty((x_0, \infty)). \tag{4.1.2}$$

Aggregates grow as a result of division of phytoplankton cells or by aggregations. We define $r(x(t)) = dx/dt$ as the growth rate for an aggregate of time-dependent mass $x(t)$. We note that the function r can take various forms, in biological applications, typically we have $r(x) \sim x$ as the growth rate is proportional to the number of cells in the aggregate. Thus, we assume that r is a non-negative function, differentiable at x_0 and

$$r \in AC((x_0, \infty)), \tag{4.1.3}$$

where $r \in AC((x_0, \infty))$ means that r is absolutely continuous in the standard sense on each compact subinterval of (x_0, ∞). If growth and mortality were the only processes taking place, the equation for the dynamics would read

$$\frac{\partial}{\partial t}u(t,x) = -\partial_x[r(x)u(t,x)] - d(x)u(t,x).$$

The streaming term $-\partial_x[r(x)u(t,x)]$, where $r \geq 0$, describes processes where aggregates gain mass due to division of cells but which nevertheless can undergo fragmentation caused by an external agent.

4.1.2 Fragmentation

During a small time interval Δt, a fraction $a(x)\Delta t$ of the aggregates of mass x undergo breakup, where a is the fragmentation rate. We consider the multiple fragmentation process in which an aggregate may split into more than two pieces, and we assume that a is a non-negative function, that is,

$$a \in L_{\infty,loc}((x_0, \infty)). \tag{4.1.4}$$

An aggregate of mass less than x_0 does not fragment since the minimum mass of an aggregate is x_0. Therefore, we assume that

$$a(x) = 0 \quad \text{for} \quad x < 2x_0. \tag{4.1.5}$$

The mass distribution of daughter particles after fragmentation is denoted by b. We assume that

$$b(x|y) = 0 \quad \text{for} \quad y < x + x_0,$$

and

$$\int_{x_0}^{y-x_0} xb(x|y)dx = y, \quad y > 2x_0, \tag{4.1.6}$$

which accounts for mass conservation after any fragmentation event. If the dynamic was just the result of fragmentation, the equation would read

$$\frac{\partial}{\partial t}u(t,x) = -a(x)u(t,x) + \int_{x+x_0}^{\infty} a(y)b(x|y)u(t,y)\,dy.$$

4.1.3 Coagulation

The coagulation process is the processes in which two distinct aggregates join together to form a single one. We introduce the "stickiness function", namely, the coagulation kernel $k(x,y)$ which describes the rate at which an aggregate of mass x sticks to an aggregate of mass y. The dynamical behaviour of phytoplankton undergoing only coagulation can be obtained by balancing loss and gain of aggregates of mass x over a short period of time. The coagulation process is given by the integro-differential equation

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) = \frac{\chi_U(x)}{2} &\int_{x_0}^{x-x_0} k(x-y,y)u(t,x-y)u(t,y)dy \\ - u(t,x) &\int_{x_0}^{\infty} k(x,y)u(t,y)dy, \end{aligned} \tag{4.1.7}$$

where χ_U is the characteristic function of the interval $U = [2x_0, \infty)$ which ensures that no aggregate of mass $x < 2x_0$ can emerge as a result of coagulation. We assume as well that the coagulation kernel k is a non-negative function in $L_\infty([x_0,\infty) \times [x_0,\infty))$ with

$$k_0 := \operatorname{ess\,sup}\{k(x,y); \quad (x,y) \in [x_0,\infty) \times [x_0,\infty)\}.$$

The first integral on the right side of (4.1.7) expresses the fact that an aggregate of mass x can only come into existence if two aggregates with masses $x - y$ and y coalesce. The second term accounts for the loss of aggregates of mass x because they have coalesced with aggregates of mass y, $y \geq x_0$. Note that the factor $1/2$ takes into account that either an aggregate of mass $x - y$ coalesces with one of mass y or vice versa.

4.1.4 Full Model

Taking into account all mechanisms described above, the full equation supplemented with the initial condition that describes the evolution of phytoplankton dynamics has the following form:

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) &= -\frac{\partial}{\partial x}[r(x)u(t,x)] - d(x)u(t,x) - a(x)u(t,x) \\ &\quad + \int_{x+x_0}^{\infty} a(y)b(x|y)u(t,y)\,dy - u(t,x)\int_{x_0}^{\infty} k(x,y)u(t,y)dy \\ &\quad + \frac{\chi_U(x)}{2}\int_{x_0}^{x-x_0} k(x-y,y)u(t,x-y)u(t,y)dy,\end{aligned} \tag{4.1.8}$$

$$u(0,x) = u_0(x) \in X_1.$$

4.1.5 Boundary Conditions

In many cases single cells, which are the product of the division inside the aggregate, leave it to form new aggregates. In this subsection, we model this process by the McKendrick-von Foerster renewal boundary condition, and discuss its assumptions and meaning. In particular, we show that imposing such a boundary condition is related to the integrability of $1/r(x)$ at x_0. Firstly, we define the growth operator

$$[\mathcal{T}u](x) = -[r(x)\,u(x)]_x\,. \tag{4.1.9}$$

By using the method of characteristics, we find

$$\frac{dx}{dt} = r(x),$$

which implies

$$\int_{x_0+\epsilon}^{x} \frac{ds}{r(s)} = t + C, \tag{4.1.10}$$

where $\epsilon > 0$ is a given positive number and C is an arbitrary constant. We denote by $R(x)$ the fixed antiderivative of $1/r(x)$, say

$$\int_{x_0+\epsilon}^{x} \frac{ds}{r(s)}. \tag{4.1.11}$$

To ensure global existence of characteristics we need to impose more assumptions on the growth rate r. For this purpose, let us first introduce the dual X_∞ to X_1, that is,

$$X_\infty = \left\{\psi, \psi \in \mathcal{E} \text{ for which } \|\psi\|_\infty = \text{ess} \sup_{x_0<x<\infty} \frac{|\psi(x)|}{x} < \infty\right\},$$

where $\mathcal{E}$ is the set of measurable functions. With this identification, the duality pairing is the integral

$$< f, \psi >= \int_{x_0}^{\infty} f(x)\psi(x)\,dx. \tag{4.1.12}$$

Hence, it is clear that if $f \in X_\infty$, then it is bounded (a.e.) by an affine function.

Lemma 4.1.1. *If $r \in X_\infty$ and $1/r(x)$ is non-integrable at x_0, then*

$$\lim_{x\to\infty} R(x) = \infty \ \textit{and} \ \lim_{x\to x_0} R(x) = -\infty.$$

Proof. As $r \in X_\infty$, then $r(x) \leq \|r\|_\infty x$. Hence,

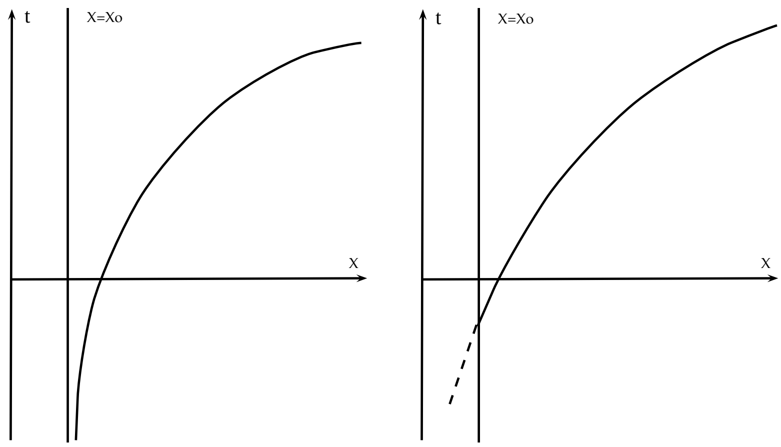
$$\lim_{x\to\infty} R(x) \geq \lim_{x\to\infty} \int_{x_0+\epsilon}^{x} \frac{ds}{\|r\|_\infty s} = +\infty.$$

Furthermore,

$$\lim_{x\to x_0} R(x) = \lim_{x\to x_0} -\int_{x}^{x_0+\epsilon} \frac{ds}{r(s)} = -\infty,$$

where we made use of the non-integrability of $1/r(x)$ at x_0. □

Depending on integrability of $1/r$ at $x = x_0$, two different cases arise at the boundary $x = x_0$.



(a) Characteristic functions does not reach the line $x = x_0$

(b) Characteristic functions does reach the line $x = x_0$

- If $1/r(x)$ is not integrable at x_0 then, by considering (4.1.3) and Lemma 4.1.1, the characteristics (4.1.10) have a vertical asymptote at the line $x = x_0$, which means that the characteristics of $\mathcal{T}$ do not reach the line $x = x_0$ (see figure 4.1a). Here, the transport equation (4.1.9) does not require a boundary condition.

- If $1/r(x)$ is integrable at x_0 then, the characteristics (4.1.10) are defined for all x in $[x_0, \infty)$, which means that the characteristics of $\mathcal{T}$ do reach the line $x = x_0$ (see figure 4.1b). Physically, it means that mass enters the system through the boundary $x = x_0$. Thus, to allow single cells to enter to the system as new aggregates and start to grow, the transport equation (4.1.9) requires a boundary condition at $x = x_0$.

We now define the boundary condition at $x = x_0$. Firstly, we observe that due to the form of the growth term, which may have zero limit at $x = x_0$, the simplest homogeneous condition should be written as

$$\lim_{x \to x_0^+} r(x)u(t,x) = 0.$$

If r is continuous at x_0 with non-zero limit, this is the same as saying that $u(t, x_0) = 0$, then we have the standard no-influx condition. However, we have to take into consideration that in fragmentation events large aggregates split creating an array of smaller aggregates [12]. These smaller aggregates may have mass x_0 and they are created at the rate $\int_{x_0}^{\infty} a(y)b(x_0|y)u(t,y)dy$. To let these aggregates enter into population, we should have

$$\lim_{x \to x_0^+} r(x)u(t,x) = \int_{x_0}^{\infty} a(y)b(x_0|y)u(t,y)dy,$$

which means that the smallest aggregates having mass x_0 enter the population at the rate $r(x)u(t,x)|_{x=x_0}$. Also, we have to take into consideration the creation of daughter-cells having mass x_0 that fall off the aggregate joining the single cell population. In general, we consider the following boundary condition which covers all these cases

$$\lim_{x \to x_0^+} r(x)u(t,x) = \int_{x_0}^{\infty} \beta(y)u(t,y)dy,$$

where β represents the rate at which single cells enter the single cell population as new aggregates and start to grow. We assume that β is a positive measurable function satisfying $\beta \in X_\infty$.

4.1.6 Abstract Reformulation

The remaining part of the thesis is devoted to the analysis of the full non-linear problem of phytoplankton. Generally, the idea and the techniques that we will use for the analysis is to convert the integro-differential equation (4.1.8) to an abstract Cauchy problem so to employ the theory introduced in Chapter 2 in the framework of the space X_1. To proceed, our starting point is to identify the right-hand side of the equations (4.1.8). For this, we denote by $\mathcal{A}$, $\mathcal{B}$ and N, the expressions appearing on the right-hand side of the equations (4.1.8). Thus

$$[\mathcal{A}u](x) = -\frac{d}{dx}[r(x)u(x)] - q(x)u(x), \tag{4.1.13}$$

where $q = a + d$,

$$[\mathcal{B}u](x) = \int_{x+x_0}^{\infty} a(y)b(x|y)u(y)dy. \tag{4.1.14}$$

Furthermore, we define the coagulation operator N on X_1 by,

$$\begin{aligned}
[Nu](x) &:= \frac{\chi_U(x)}{2}\int_{x_0}^{x-x_0} k(x-y,y)u(x-y)u(y)dy - u(x)\int_{x_0}^{\infty} k(x,y)u(y)dy, \\
&= \mathcal{N}_1[u,u](x) - \mathcal{N}_2[u,u](x) \qquad (4.1.15)\\
&= \mathcal{N}[u,u](x),
\end{aligned}$$

where for $\psi, \phi \in X_1$,

$$\begin{aligned}
\mathcal{N}_1[\psi,\phi](x) &= \frac{\chi_U(x)}{2}\int_{x_0}^{x-x_0} k(x-y,y)\psi(x-y)\phi(y)dy \\
\mathcal{N}_2[\psi,\phi](x) &= \psi(x)\int_{x_0}^{\infty} k(x,y)\phi(y)dy.
\end{aligned}$$

For each fixed $t \geq 0$, we define a function $u(t) : (x_0, \infty) \to \mathbb{R}$ of the "mass" variable x by,

$$u(t)(x) = u(t,x), \quad \text{for a.e. } x > x_0,\ t \geq 0. \qquad (4.1.16)$$

Hence, u is the function from $[0,\infty)$ into the space X_1. Since X_1 is a Banach space of type L, see Definition (2.1.7), $\dfrac{\partial u}{\partial t}$ can be thought of as the derivative with respect to t of the function $u : [0,\infty) \to X_1$ defined by (4.1.16). Therefore, for fixed $t \geq 0$, we can rewrite equation (4.1.8) defined on its maximal domain as

$$\begin{aligned}
\frac{d}{dt}u(t) &= [\mathcal{A} + \mathcal{B} + N]\, u(t), \\
u(0) &= u_0.
\end{aligned}$$

4.2 Analysis in Case of r^{-1} Non-Integrable at x_0

4.2.1 Streaming Semigroup

With respect to the above, the transport problem reads

$$\begin{aligned}
\frac{du}{dt}(t) &= Au(t) \\
u(0) &= u_0,
\end{aligned}$$

where A is the realization of $\mathcal{A}$ defined via (4.1.13) on

$$D(A) = \{u \in X_1;\ qu \in X_1,\ \ ru \in AC((x_0,\infty)) \text{ and } (ru)_x \in X_1\}.$$

It turns out that direct estimates of the resolvent of A are not easy. For this reason, we start dealing with the following equation given by

$$\frac{du}{dt}(t) = Tu(t) \qquad (4.2.1)$$

where $[Tu](x) := -\frac{d}{dx}[r(x)u(x)], \;\; x \in (x_0, \infty)$, defined on the domain

$$D(T) = \{u \in X_1; \;\; ru \in AC((x_0, \infty)) \text{ and } (ru)_x \in X_1\}.$$

Generally, the idea is to find a solution to (4.2.1) which will be used to prove the existence of the resolvent of A. The first step is to solve this equation by applying the method of characteristics. The goal of this method, when applied to (4.2.1), is to find curves in the (x, t) plane where the PDE becomes an ODE. Such curves, along which the solution of the PDE reduces to an ODE, are called the characteristic curves. According to (4.2.1), if s is the auxiliary variable characterizing these curves such that $x(s)$, $t(s)$ and $u(x(s), t(s))$, it follows that the characteristic curve that goes through the point $(x(0), t(0)) = (\xi, 0)$ is the graph of the function $x(s)$ that satisfies the ODEs

$$\begin{cases} \dfrac{dx(s)}{ds} = r(x(s)), \\ \\ x(0) = \xi, \quad \xi > x_0, \end{cases} \qquad \begin{cases} \dfrac{dt(s)}{ds} = 1, \\ \\ t(0) = 0, \end{cases} \tag{4.2.2}$$

and

$$\frac{du(t(s), x(s))}{ds} = -\frac{dr(x(s))}{dx(s)} u(x(s), t(s)), \tag{4.2.3}$$

describes the value of $u(t(s), x(s))$ along a characteristic curve. Direct integration of equation (4.2.2) gives

$$\int_{\xi}^{x(t)} \frac{dz}{r(z)} = t. \tag{4.2.4}$$

Hence, (4.2.4) becomes

$$R(\xi) = R(x(t)) - t,$$

where R was defined by (4.1.11). Since $1/r(x)$ is not integrable at x_0, by Lemma 4.1.1 and the monotonicity of R (increasing function), we deduce that R is globally invertible on $\mathbb{R}$. Hence, we define

$$\xi := R^{-1}(R(x) - t), \quad x > x_0, \; t \geq 0.$$

We note that the characteristic curve depends essentially on the initial condition ξ as we change (t, x). Hence, it is worthwhile to set $\xi = Y(t, x)$ and write

$$Y(t, x) := R^{-1}(R(x) - t), \quad x > x_0, \; t \geq 0.$$

Now, direct integration of (4.2.3) leads to the solution

$$u(t, x) = u(0, x(0)) \exp\left(-\int_0^t r'(Y(-s, \xi))ds\right), \tag{4.2.5}$$

where $x(s) = Y(-s, \xi)$. Furthermore, we have

$$\frac{dY(-s, \xi)}{ds} = r(Y(-s, \xi)) \tag{4.2.6}$$

Using the previous equation, we find

$$\begin{aligned}\frac{d}{ds}\ln r(Y(-s,\xi)) &= \frac{d\ln r(Y(-s,\xi))}{dY}\frac{dY(-s,\xi)}{ds}\\ &= \frac{r'(Y(-s,\xi))}{r(Y(-s,\xi))}\frac{dY(-s,\xi)}{ds}\\ &= r'(Y(-s,\xi)).\end{aligned}$$

Therefore, (4.2.5) becomes

$$u(t,x) = \frac{r(Y(t,x))u_0(Y(t,x))}{r(x)}, \quad t\geq 0, x>x_0.$$

Generally, we can prove, as in [11, Theorem 9.4], that $(T,\, D(T))$ generates a C_0-semigroup $(S_T(t))_{t\geq 0}$ expressed by

$$[S_T(t)u_0](x) = \frac{r(Y(t,x))u_0(Y(t,x))}{r(x)}, \quad t\geq 0, x>x_0,$$

where u_0 is any fixed element of $D(T)$. In particular, we have

$$\|S_T(t)u_0\|_1 \leq \int_{x_0}^{\infty}\frac{r(Y(t,x))u_0(Y(t,x))}{r(x)}xdx. \tag{4.2.7}$$

Next, we have

$$\frac{d}{dx}R(Y(t,x)) = \frac{d}{dx}(R(x)-t) = \frac{1}{r(x)}$$

also

$$\begin{aligned}\frac{d}{dx}R(Y(t,x)) &= \frac{dR(Y(t,x))}{dY}\frac{dY(t,x)}{dx}\\ &= \frac{1}{r(Y(t,x))}\frac{dY(t,x)}{dx}.\end{aligned}$$

Thus, $\dfrac{dY(t,x)}{dx} = \dfrac{r(Y(t,x))}{r(x)}$. If we set $\xi = Y(t,x)$, then it is easy to see that

$$\frac{d\xi}{r(\xi)} = \frac{dx}{r(x)} \quad \text{and} \quad Y(t,x_0)=x_0,\ Y(t,\infty)=\infty \quad \text{by Lemma 4.1.1. Thus,}$$

$$\|S_F(t)u_0\|_1 \leq \int_{x_0}^{\infty} u_0(\xi)Y(-t,\xi)d\xi,$$

where $x(t) = Y(-t,\xi)$. Since $x(t) = Y(-t,\xi)$ is the solution to the Cauchy problem

$$\frac{dx}{dt} = r(x), \qquad x(0)=\xi,$$

so that

$$x(t) = \xi + \int_0^t r(x(s))ds \leq \xi + \int_0^t \|r\|_\infty x(s)ds$$

then, by Gronwall's inequality, see Lemma 2.1.9, we obtain

$$Y(-t, \xi) \leq \xi e^{\|r\|_\infty t}.$$

Therefore,

$$\|S_T(t)u_0\|_1 \leq e^{t\|r\|_\infty}\|u_0\|_1.$$

In particular, by the Hille-Yosida Theorem, we obtain for $\psi \in X_1$ and $\lambda > \|r\|_\infty$,

$$\|R(\lambda, T)\psi\|_1 \leq \frac{1}{\lambda - \|r\|_\infty}\|\psi\|_1. \tag{4.2.8}$$

Let us now revert to the operator A defined on

$$D(A) = \{u \in X_1;\ qu \in X_1,\ \ ru \in AC((x_0, \infty)) \text{ and } (ru)_x \in X_1\}.$$

Theorem 4.2.1. *Let $\lambda > \|r\|_\infty$. The resolvent $R(\lambda, A)$ of the operator A is expressed as follows:*

$$[R(\lambda, A)\psi](x) = \frac{e^{-\lambda R(x) - Q(x)}}{r(x)} \int_{x_0}^x e^{\lambda R(y) + Q(y)}\psi(y)dy, \tag{4.2.9}$$

where $\lambda R + Q$ is a fixed anti-derivative of $(\lambda + q(s))/r(s)$, say

$$R(x) = \int_{x_0+\varepsilon}^x \frac{ds}{r(s)} \quad \text{and} \quad Q(x) = \int_{x_0+\varepsilon}^x \frac{q(s)}{r(s)}ds, \quad \varepsilon > 0.$$

Furthermore, the operator A generates a positive semigroup, say, $(S_A(t))_{t\geq 0}$, satisfying, for any $\psi \in X_1$,

$$\|S_A(t)\psi\|_1 \leq e^{\|r\|_\infty t}\|\psi\|_1. \tag{4.2.10}$$

Proof. The first step in this direction is to find the resolvent of A, which is formally given by the solution of the equation

$$\lambda u(x) + \frac{d}{dx}[r(x)u(x)] + q(x)u(x) = \psi(x), \quad \lambda > 0,\ \psi \in X_1. \tag{4.2.11}$$

Let us start with possible eigenfunctions of (4.2.11). By direct integration we find that the general solution to the differential equation

$$\lambda u(x) + \frac{d}{dx}[r(x)u(x)] + q(x)u(x) = 0, \quad \lambda > 0,$$

is given by

$$u(x) = Cv_\lambda(x) = C\frac{e^{-\lambda R(x) - Q(x)}}{r(x)}.$$

Making use of the method of variation of constants, the general formal solution of the resolvent equation (4.2.11) is given by

$$u(x) = Cv_\lambda(x) + [R_\lambda\psi](x),$$

where

$$[R_\lambda\psi](x) = \frac{e^{-\lambda R(x)-Q(x)}}{r(x)} \int_{x_0}^{x} e^{\lambda R(y)+Q(y)}\psi(y)dy.$$

Since

$$\begin{aligned} \|v_\lambda\|_1 &= \int_{x_0}^{\infty} \frac{e^{-\lambda R(x)-Q(x)}}{r(x)} x\, dx \\ &\geq \int_{x_0}^{x_0+\varepsilon} \frac{e^{-\lambda R(x)-Q(x)}}{r(x)} x\, dx \\ &\geq x_0 \int_{x_0}^{x_0+\varepsilon} \frac{dx}{r(x)} = \infty, \end{aligned} \tag{4.2.12}$$

where we used the monotonicity, non-negativity of $e^{-\lambda R(x)-Q(x)}$ in the interval $(x_0, x_0+\varepsilon)$ and non-integrability of r^{-1} at x_0. Thus, no eigenfunction of A corresponding to an eigenvalue $\lambda > 0$ belongs to X_1. This suggests that a good candidate for the resolvent of A is given by

$$[R_\lambda\psi](x) = \frac{e^{-\lambda R(x)-Q(x)}}{r(x)} \int_{x_0}^{x} e^{\lambda R(y)+Q(y)}\psi(y)dy. \tag{4.2.13}$$

Next, we need to check that this solution $R_\lambda\psi$ fulfils all conditions of $D(A)$. By the Fubini Theorem

$$\begin{aligned} \|R_\lambda\psi\|_1 &= \int_{x_0}^{\infty} |[R_\lambda\psi](x)| x\, dx \\ &\leq \int_{x_0}^{\infty} \frac{e^{-\lambda R(x)-Q(x)}}{r(x)} \left(\int_{x_0}^{x} e^{\lambda R(y)+Q(y)}|\psi(y)|\, dy\right) x\, dx \\ &\leq \frac{1}{(\lambda - \|r\|_\infty)} \int_{x_0}^{\infty} |\psi(y)| y\, dy, \end{aligned}$$

where we made use of (4.2.8) and the monotonicity of $e^{Q(x)}$. Hence, R_λ is a bounded operator on X_1 with $\|R_\lambda\psi\|_1 \leq \dfrac{1}{(\lambda - \|r\|_\infty)}\|\psi\|_1$.

Furthermore, we have

$$\|qR_\lambda\psi\|_1 \leq \int_{x_0}^{\infty} \left(\frac{e^{\lambda R(y)+Q(y)}}{y} \int_{y}^{\infty} \frac{xq(x)e^{-\lambda R(x)-Q(x)}}{r(x)} dx\right) |\psi(y)| y\, dy.$$

Since

$$\begin{aligned} \frac{xq(x)}{r(x)} e^{-\lambda R(x)-Q(x)} &\leq \frac{x(\lambda + q(x))}{r(x)} e^{-\lambda R(x)-Q(x)} \\ &= e^{-\lambda R(x)-Q(x)} - \frac{d}{dx}\left(xe^{-\lambda R(x)-Q(x)}\right), \end{aligned} \tag{4.2.14}$$

we deduce that

$$\begin{aligned}
\|qR_\lambda\psi\|_1 &\leq \int_{x_0}^{\infty}\left(1+\frac{e^{\lambda R(y)+Q(y)}}{y}\int_y^{\infty} e^{-\lambda R(x)-Q(x)}\,dx\right)|\psi(y)|y\,dy \\
&\leq \int_{x_0}^{\infty}\left(1+\|r\|_\infty\frac{e^{\lambda R(y)+Q(y)}}{y}\int_y^{\infty}\frac{xe^{-\lambda R(x)-Q(x)}}{r(x)}\,dx\right)|\psi(y)|y\,dy \\
&\leq (1+\|r\|_\infty(\lambda-\|r\|_\infty)^{-1})\|\psi\|_1,
\end{aligned}$$

where again we used (4.2.8) the monotonicity of $e^{-Q(x)}$ and $1/\|r\|_\infty \leq x/r(x)$.

In addition, we notice that for $\psi \in X_1$

$$r(x)[R_\lambda\psi](x) = e^{-\lambda R(x)-Q(x)}\int_{x_0}^{x} e^{\lambda R(y)+Q(y)}\psi(y)\,dy, \tag{4.2.15}$$

and both $e^{-\lambda R(x)-Q(x)}$ and the integral (as a function of its upper limit) are absolutely continuous and bounded over any fixed interval $[\alpha,\beta] \subset (x_0,\infty)$. Hence, it follows that the product is absolutely continuous on $[\alpha,\beta]$ and therefore, $r[R_\lambda\psi]$ is absolutely continuous there. Thus, it can be differentiated at any $x \in (x_0,\infty)$ and

$$\begin{aligned}
(r(x)[R_\lambda\psi](x))_x &= -\frac{\lambda+q(x)}{r(x)}e^{-\lambda R(x)-Q(x)}\int_{x_0}^{x} e^{\lambda R(y)+Q(y)}\psi(y)\,dy+\psi(x). \\
&= -(\lambda+q(x))[R_\lambda\psi](x)+\psi(x) \in X_1.
\end{aligned}\tag{4.2.16}$$

Combining all these properties proved above, we infer that $R_\lambda(X_1) \subset D(A)$.

In addition, according to (4.2.16) we have,

$$(r(x)[R_\lambda\psi](x))_x + (\lambda+q(x))[R_\lambda\psi](x) = \psi(x) \in X_1,$$

which implies that $[(\lambda I - A)R_\lambda]\psi = \psi$.

In order to show that R_λ is the resolvent of A, it remains to be shown that $\lambda I - A$ is injective on $D(A)$. As before, the only solution of $\lambda u(x)+q(x)u(x)+(r(x)u(x))_x = 0$ is

$$u(x) = C\,v_\lambda = C\,\frac{e^{-\lambda R(x)-Q(x)}}{r(x)}.$$

Adopting the same argument used in (4.2.12), it follows that $u \notin X_1$ and therefore, $\lambda I - A$ is injective. Hence, the resolvent $R(\lambda, A)$ of the operator A is equal to R_λ which is given by (4.2.13).

Since the resolvent is a positive operator for $\psi \in X_{1+}$, then by the Hille-Yosida Theorem, $(A, D(A))$ generates a positive semigroup satisfying (4.2.10). □

4.2.2 Growth-Fragmentation Equation

We have shown that A generates a C_0-semigroup. We now intend to prove the existence of a solution of the growth-fragmentation equation, namely, $A+B$ in X_1. For this, we make use of Kato's Perturbation Theorem. Let us define B as the realization of $\mathcal{B}$, (see (4.1.14)) on the domain

$$D(B) = D(A) = \{u \in X_1;\ qu \in X_1,\ \ ru \in AC((x_0,\infty)) \text{ and } (ru)_x \in X_1\}.$$

The corresponding Cauchy problem reads,

$$\begin{aligned} \frac{du}{dt}(t) &= [A+B]u(t), \quad t \geq 0, \\ u(0) &= u_0, \end{aligned} \tag{4.2.17}$$

where

$$[(A+B)u](x) = -\frac{d}{dx}[r(x)u(x)] - q(x)u(x) + \int_{x+x_0}^{\infty} a(y)b(x|y)u(y)\,dy.$$

Lemma 4.2.2. *For any $u \in D(A)_+$, we have*

$$\int_{x_0}^{\infty} [Au+Bu](x)\,x\,dx = \int_{x_0}^{\infty} r(x)u(x)\,dx - \int_{x_0}^{\infty} d(x)u(x)x\,dx. \tag{4.2.18}$$

Proof. By the Fubini Theorem,

$$\begin{aligned} \int_{x_0}^{\infty}[Bu](x)xdx &= \int_{x_0}^{\infty}\left(\int_{x+x_0}^{\infty} a(y)b(x|y)u(y)\,dy\right)xdx \\ &= \int_{2x_0}^{\infty} a(y)u(y)\left(\int_{x_0}^{y-x_0} xb(x|y)\,dx\right)dy \\ &= \int_{2x_0}^{\infty} a(y)u(y)ydy, \end{aligned}$$

where we made use of (4.1.6). Furthermore, we have

$$\int_{x_0}^{\infty}[qu](x)xdx = \int_{x_0}^{\infty}[a(x)+d(x)]u(x)xdx.$$

Hence, by (4.1.5) and combining the last two terms, we get the following:

$$\int_{x_0}^{\infty}[-qu+Bu](x)xdx = -\int_{x_0}^{\infty} d(x)u(x)\,xdx, \tag{4.2.19}$$

To complete the proof it suffices to show that

$$\int_{x_0}^{\infty}[Tu](x)\,x\,dx = \int_{x_0}^{\infty} r(x)u(x)\,dx,$$

where T is the operator described by (4.2.1). The approach we consider is similar to the analysis performed in the proof of [11, Lemma 9.7] for the model of fragmentation with decay.

Let $\lambda > \|r\|_\infty$ and $u \in D(A)_+$. Then $u = R(\lambda, A)\psi$ for some $\psi \in X_1$. Since the resolvent $R(\lambda, A)\psi$ satisfies the equation

$$(\lambda I - A)R(\lambda, A)\psi = (\lambda I - T + q)R(\lambda, A)\psi = \psi,$$

then it follows directly that

$$[TR(\lambda, A)\psi](x) = -\psi(x) + (\lambda + q(x))[R(\lambda, A)\psi](x).$$

Now,

$$\begin{aligned}\int_{x_0}^{\infty} &((\lambda + q(x))[R(\lambda, A)\psi](x))\, x dx \\ &= \int_{x_0}^{\infty} e^{\lambda R(y)+Q(y)}\psi(y) \left(\int_y^{\infty} \frac{x(\lambda + q(x))}{r(x)} e^{-\lambda R(x)-Q(x)} dx \right) dy.\end{aligned}$$

Also, for any $y > x_0$ we have

$$\begin{aligned}\int_y^{\infty} \frac{x(\lambda + q(x))}{r(x)} e^{-\lambda R(x)-Q(x)} dx &= -\int_y^{\infty} x \left(\frac{d}{dx} e^{-\lambda R(x)-Q(x)} \right) dx \\ &= \int_y^{\infty} e^{-\lambda R(x)-Q(x)} dx + y e^{-\lambda R(y)-Q(y)} - \lim_{x\to\infty} x e^{-\lambda R(x)-Q(x)},\end{aligned}$$

where we used integration by parts. Note that $\lim_{x\to\infty} x e^{-\lambda R(x)-Q(x)} = 0$. In fact,

$$x e^{-\lambda R(x)-Q(x)} \le x e^{-\lambda R(x)} = x \exp\left(-\lambda \int_{x_0+\varepsilon}^{x} \frac{ds}{r(s)} \right) \le x \left| \frac{x_0 + \varepsilon}{x} \right|^{\frac{\lambda}{\|r\|_\infty}} = \frac{C}{x^{\frac{\lambda - \|r\|_\infty}{\|r\|_\infty}}} \to 0,$$

as $x \to \infty$ where $C = (x_0 + \varepsilon)^{\frac{\lambda}{\|r\|_\infty}}$, and we made use of $r(x) \le \|r\|_\infty x$ and $\lambda > \|r\|_\infty$, respectively. Hence

$$\begin{aligned}\int_{x_0}^{\infty} [Tu](x)\, x\, dx &= \int_{x_0}^{\infty} [TR(\lambda, A)\psi](x)\, x\, dx \\ &= \int_{x_0}^{\infty} e^{\lambda R(y)+Q(y)}\psi(y) \left(\int_y^{\infty} e^{-\lambda R(x)-Q(x)} dx \right) dy \\ &= \int_{x_0}^{\infty} e^{-\lambda R(x)-Q(x)} \left(\int_{x_0}^{x} e^{\lambda R(y)+Q(y)}\psi(y) dy \right) dx \\ &= \int_{x_0}^{\infty} r(x)u(x) dx,\end{aligned}$$

where the last equality comes from (4.2.15). $\square$

Next, we notice that we need to rescale the semigroup $(S_A(t))_{t\geq 0}$ to a substochastic semigroup so that Kato's Perturbation Theorem becomes applicable.

Lemma 4.2.3. *The operator*

$$(\widetilde{A}, D(A) = (A - \|r\|_\infty I, D(A))$$

generates a positive semigroup of contractions given by

$$S_{\widetilde{A}}(t)\psi = e^{-\|r\|_\infty t} S_A(t)\psi, \quad \psi \in X_1.$$

Proof. Consider the operator $\widetilde{A} = A - \|r\|_\infty I$. Then

$$R(\lambda, \widetilde{A}) = R\left(\lambda + \|r\|_\infty, A\right).$$

Let $\lambda = \dfrac{n}{t}$. Then the terms of the sequence in (2.2.6) for the operator $\widetilde{A}$ can be written as

$$\left(\frac{n}{t} R\left(\frac{n}{t}, \widetilde{A}\right)\right)^n \psi = \left(\left(1 - \frac{\|r\|_\infty t}{k}\right) \frac{k}{t} R\left(\frac{k}{t}, A\right)\right)^k \left(\frac{k - \|r\|_\infty t}{t} R\left(\frac{k}{t}, A\right)\right)^{\|r\|_\infty t} \psi,$$

where $k = n + \|r\|_\infty t$, $t > 0$ fixed. Since $\frac{k - \|r\|_\infty t}{t} \sim \frac{k}{t}$ as $k \longrightarrow \infty$, then for $\|r\|_\infty t$ fixed, Lemma 2.2.10 results in the following

$$\left(\frac{k - \|r\|_\infty t}{t} R\left(\frac{k}{t}, A\right)\right)^{\|r\|_\infty t} \psi \longrightarrow \psi \text{ as } k \longrightarrow \infty.$$

Furthermore, we have

$$\lim_{k\to\infty} \left(1 - \frac{\|r\|_\infty t}{k}\right)^k = e^{-\|r\|_\infty t}.$$

Therefore,

$$S_{\widetilde{A}}(t)\psi = \lim_{n\to\infty} \left(\frac{n}{t} R\left(\frac{n}{t}, \widetilde{A}\right)\right)^n \psi = e^{-\|r\|_\infty t} S_A(t)\psi,$$

with

$$\|S_{\widetilde{A}}(t)\psi\|_1 \leq \|\psi\|_1.$$

□

Now, we are ready to state the following result:

Proposition 4.2.4. *There is an extension $\widetilde{G}$ of the operator $\widetilde{A} + B$ that generates a substochastic semigroup $(S_{\widetilde{G}}(t))_{t\geq 0}$ of bounded linear operators on X_1. This semigroup, for arbitrary $\psi \in D(\widetilde{G})$ and $t \geq 0$, satisfies,*

$$\frac{d}{dt} S_{\widetilde{G}}(t)\psi = \widetilde{G} S_{\widetilde{G}}(t)\psi.$$

$(S_{\widetilde{G}}(t))_{t\geq 0}$ can be obtained as a strong limit in X_1 of semigroups $(S_r(t))_{t\geq 0}$ generated by $(\widetilde{A}+rB, D(\widetilde{A}))$ as $r\to 1^-$; if $\psi\in X_{1+}$, then the limit is monotonic. The generator $\widetilde{G}$ of $(S_{\widetilde{G}}(t))_{t\geq 0}$ is characterized by:

$$(\lambda I-\widetilde{G})^{-1}\psi=\sum_{n=0}^{\infty}(\lambda I-\widetilde{A})^{-1}[B(\lambda I-\widetilde{A})^{-1}]^n\psi \tag{4.2.20}$$

for $\psi\in X_1$.

Proof. By Lemma 4.2.3, the operator $(\widetilde{A}, D(\widetilde{A}))$ generates a substochastic semigroup $(S_{\widetilde{A}}(t))_{t\geq 0}$. We obviously have that $Bu\geq 0$ for any $u\in D(\widetilde{A})_+$. Also, by (4.2.18),

$$\int_{x_0}^{\infty}[Au+Bu](x)\,x\,dx=\int_{x_0}^{\infty}r(x)u(x)\,dx-\int_{x_0}^{\infty}d(x)u(x)x\,dx. \tag{4.2.21}$$

If we add $-\displaystyle\int_{x_0}^{\infty}\|r\|_\infty xu(x)\,dx$ to both sides of (4.2.21), then

$$\int_{x_0}^{\infty}(\widetilde{A}u+Bu)x\,dx=-\int_{x_0}^{\infty}(\|r\|_\infty x-r(x))u(x)\,dx-\int_{x_0}^{\infty}d(x)u(x)x\,dx.$$

Using the fact that $r(x)\leq\|r\|_\infty x$ for every $x\in(x_0,+\infty)$, it is clear that

$$\int_{x_0}^{\infty}(\widetilde{A}u+Bu)x\,dx\leq 0.$$

Consequently, the assumptions of Kato's Perturbation Theorem 2.3.4 are fulfilled. □

Theorem 4.2.5. *There is an extension G of the operator $A+B$ given by*

$$(G,D(G))=(\widetilde{G}+\|r\|_\infty I, D(\widetilde{G})),$$

which generates a positive semigroup $(S_G(t))_{t\geq 0}=(e^{\|r\|_\infty t}S_{\widetilde{G}}(t))_{t\geq 0}$ in X_1. Moreover, the generator G is characterized by:

$$(\lambda I-G)^{-1}\psi=\sum_{n=0}^{\infty}(\lambda I-A)^{-1}[B(\lambda I-A)^{-1}]^n\psi, \tag{4.2.22}$$

for $\psi\in X_1$ and $\lambda>\|r\|_\infty$.

Proof. The argument used follows similar lines to those used in [11, Proposition 9.29]. Formula (4.2.22) is obtained directly from (4.2.20). In fact, since $\lambda I-G=(\lambda-\|r\|_\infty)I-\widetilde{G}$, it is clear that $R(\lambda,G)=R(\lambda',\widetilde{G})$ for $\lambda>\|r\|_\infty$, where $\lambda'=\lambda-\|r\|_\infty$. To prove the first part of the theorem, we note that the operator $\widetilde{A}$ was constructed from A by the substraction of the bounded operator $\|r\|_\infty I$. Also the approximating semigroups $(S_r(t))_{t\geq 0}$ mentioned in the previous proposition are generated by $(A - \|r\|_\infty I + rB, D(A))$, $0 < r < 1$. In addition,

$$\lim_{r\to 1^-} S_r(t)\psi = S_{\widetilde{G}}(t)\psi \tag{4.2.23}$$

in X_1, uniformly in t on bounded intervals. We introduce the semigroups $(S'_r(t))_{t\geq 0} := (e^{\|r\|_\infty t}S_r(t))_{t\geq 0}$ generated by $A + rB$. Since multiplication by $e^{\|r\|_\infty t}$ does not affect convergence, (4.2.23) implies that $(S'_r(t))_{t\geq 0}$ converges strongly to the semigroup $(S_G(t))_{t\geq 0} = (e^{\|r\|_\infty t}S_{\widetilde{G}}(t))_{t\geq 0}$ generated by $G = \widetilde{G} + \|r\|_\infty I$. □

4.3 Analysis in Case of r^{-1} Integrable at x_0

In this section the approach that we will use is analogous to that of [12] and [24]. The first used the abstract space $X_{0,1}$ and assumed linear boundedness of the fragmentation rate. The second adopted a similar approach and extended the work in the bigger space X_1 to general fragmentation rate kernels.

4.3.1 Streaming Semigroup

The transport problem reads

$$\begin{aligned} \frac{du}{dt}(t) &= Au(t) \\ \lim_{x\to x_0^+} r(x)[u(t)(x)] &= \int_{x_0}^{\infty} \beta(y)[u(t)(y)]\,dy, \\ u(0) &= u_0. \end{aligned} \tag{4.3.1}$$

The first step is to restrict the operator A to a domain in which the boundary condition is satisfied. In this direction we introduce A_β as A restricted to

$$D(A_\beta) = \left\{ u \in D(A) : \quad \lim_{x\to x_0^+} r(x)u(x) = \int_{x_0}^{\infty} \beta(y)u(y)\,dy \right\}. \tag{4.3.2}$$

Our starting point towards finding a solution to (4.3.1) is to solve the resolvent equation

$$\lambda u(x) + \frac{d}{dx}[r(x)u(x)] + q(x)u(x) = \psi(x), \quad \lambda > 0,\ \psi \in X_1. \tag{4.3.3}$$

Hence, we first start with possible eigenfunctions of (4.3.3). By direct integration we find that the general solution to the differential equation

$$\lambda u(x) + \frac{d}{dx}[r(x)u(x)] + q(x)u(x) = 0, \quad \lambda > 0, \tag{4.3.4}$$

is given by

$$u(x) = c\,\frac{e^{-\lambda\widetilde{R}(x)-\widetilde{Q}(x)}}{r(x)}$$

where c is an arbitrary scalar. Making use of the method of variation of constants, the general formal solution of the resolvent equation is in the form

$$u_\beta(x) = [\widetilde{R}_\lambda\psi](x) + c\frac{e^{-\lambda\widetilde{R}(x)-\widetilde{Q}(x)}}{r(x)},$$

where

$$[\widetilde{R}_\lambda\psi](x) = \frac{e^{-\lambda\widetilde{R}(x)-\widetilde{Q}(x)}}{r(x)}\int_{x_0}^{x} e^{\lambda\widetilde{R}(y)+\widetilde{Q}(y)}\psi(y)\,dy,$$

and

$$\widetilde{R}(x) = \int_{x_0}^{x}\frac{ds}{r(s)}, \quad \text{and} \quad \widetilde{Q}(x) = \int_{x_0}^{x}\frac{q(s)}{r(s)}\,ds.$$

We note that A_0 will denote the realization of A_β with zero-boundary condition (standard no-influx condition) at $x = x_0$, that is, $\lim_{x\to x_0} r(x)u(x) = 0$. Thus, using this limit, the natural choice for the solution to the problem $\lambda u - A_0 u = \psi$ is

$$u_0(x) = \frac{e^{-\lambda\widetilde{R}(x)-\widetilde{Q}(x)}}{r(x)}\int_{x_0}^{x} e^{\lambda\widetilde{R}(y)+\widetilde{Q}(y)}\psi(y)\,dy. \tag{4.3.5}$$

The following lemma is an auxiliary result in proving other results.

Lemma 4.3.1. *Let $\lambda > \|r\|_\infty$. Then, for any $x_0 \le x < \infty$,*

$$I(x,\infty) := \int_{x}^{\infty}\frac{e^{-\lambda\widetilde{R}(s)}}{r(s)}s\,ds \le \frac{1}{\lambda - \|r\|_\infty}xe^{-\lambda\widetilde{R}(x)}. \tag{4.3.6}$$

Proof. [12, Lemma 2.1]. □

Lemma 4.3.2. *Under the adopted assumptions, if $\lambda > \|r\|_\infty$, then $R(\lambda, A_0) = \widetilde{R}_\lambda$ defines the resolvent of $(A_0, D(A_0))$ and satisfies the estimate,*

$$\|R(\lambda, A_0)\|_1 \le \frac{1}{\lambda - \|r\|_\infty}. \tag{4.3.7}$$

Proof. We first mention that formula (4.3.7) can be proved as in the proof performed in the Theorem 4.2.1 by using estimate (4.3.6).

We note that since r^{-1} is integrable at x_0, then $\lim_{x\to x_0}\widetilde{R}(x) = 0$. Hence, we easily get from (4.3.5) that

$$\lim_{x\to x_0} r(x)u_0(x) = 0.$$

Since $e^{-\lambda\widetilde{R}-\widetilde{Q}}$ is a bounded function that is differentiable on (x_0,∞), and $\displaystyle\int_{x_0}^{x} e^{\lambda\widetilde{R}(y)+\widetilde{Q}(y)}\psi(y)\,dy$ is absolutely continuous on (x_0,∞), then $ru_0 \in AC((x_0,\infty))$. Furthermore, we have

$$\begin{aligned}
\|q\widetilde{R}_\lambda\psi\|_1 &\le \int_{x_0}^{\infty}\left(\frac{e^{\lambda\widetilde{R}(y)+\widetilde{Q}(y)}}{y}\int_y^{\infty}\frac{xq(x)e^{-\lambda\widetilde{R}(x)-\widetilde{Q}(x)}}{r(x)}\,dx\right)|\psi(y)|y\,dy,\\
&\le \int_{x_0}^{\infty}\left(1+\frac{e^{\lambda\widetilde{R}(y)+\widetilde{Q}(y)}}{y}\int_y^{\infty}e^{-\lambda\widetilde{R}(x)-\widetilde{Q}(x)}\,dx\right)|\psi(y)|y\,dy,
\end{aligned}$$

where we made use of the formula (4.2.14) and $\lim_{x\to\infty} e^{-\lambda\widetilde{R}(x)-\widetilde{Q}(x)} = 0$. Using the fact that $r(x) \le x\|r\|_\infty$, we obtain

$$\begin{aligned}
\|q\widetilde{R}_\lambda\psi\|_1 &\le \int_{x_0}^{\infty}\left(1+\|r\|_\infty\frac{e^{\lambda\widetilde{R}(y)+\widetilde{Q}(y)}}{y}\int_y^{\infty}\frac{xe^{-\lambda\widetilde{R}(x)-\widetilde{Q}(x)}}{r(x)}\,dx\right)|\psi(y)|y\,dy\\
&\le (1+\|r\|_\infty(\lambda-\|r\|_\infty)^{-1})\|\psi(y)\|_1,
\end{aligned}$$

where we used (4.3.7) in the last inequality. Hence, $u_0 \in D(A_0)$. Furthermore, direct substitution shows that $(\lambda I - A_0)u_0 = \psi$.

Now, it remains to show that $(\lambda I - A_0)$ is injective on $D(A_0)$ so $R(\lambda, A_0)$ is the resolvent of A_0. The solution of the homogeneous equation (4.3.4) is

$$u(x) = c\,\frac{e^{-\lambda\widetilde{R}(x)-\widetilde{Q}(x)}}{r(x)},$$

where c is an arbitrary constant. Then $u \in D(A_0)$ means that $\lim_{x\to x_0} r(x)u(x) = 0$. Hence, $\lim_{x\to x_0} c\,e^{-\lambda\widetilde{R}(x)-\widetilde{Q}(x)} = 0$. Since $\lim_{x\to x_0}\widetilde{R}(x) = \lim_{x\to x_0}\widetilde{Q}(x) = 0$ then $c = 0$. Therefore, for every $u \in D(A_0)$, $(\lambda I - A_0)$ is injective. As a conclusion,

$$R(\lambda, A_0) = u_0(x) = \frac{e^{-\lambda\widetilde{R}(x)-\widetilde{Q}(x)}}{r(x)}\int_{x_0}^{x} e^{\lambda\widetilde{R}(y)+\widetilde{Q}(y)}\psi(y)\,dy$$

is the resolvent of A_0. □

Next, we turn our attention to the problem with $\beta \ne 0$ and we set

$$\kappa := x_0\|\beta\|_\infty + \|r\|_\infty.$$

Our aim is to characterise the resolvent of the operator $(A_\beta, D(A_\beta))$. This will enable us to prove the generation result which will be used in the the next subsection to show the existence of solutions to the growth-fragmentation equation by means of Kato's Perturbation Theorem.

The solution u_β of the resolvent equation $\lambda u - A_\beta u = \psi$ is given by

$$u_\beta(x) = [\widetilde{R}_\lambda \psi](x) + c\frac{e_\lambda(x)}{r(x)},$$

where $e_\lambda(x) = e^{-\lambda\widetilde{R}(x)-\widetilde{Q}(x)}$. We observe that since

$$\lim_{x\to x_0} r(x)u_\beta(x) \;=\; \lim_{x\to x_0}\left[r(x)[\widetilde{R}_\lambda\psi](x) + c\, e_\lambda(x)\right],$$

we obtain

$$\int_{x_0}^{\infty} \beta(x)u_\beta(x)\,dx \;=\; c$$

or, in short,

$$< \beta, u_\beta > \;=\; c,$$

where we used the fact that $u_\beta \in D(A_\beta)$, see (4.3.2), and the duality pairing property (4.1.12). Hence, multiplying by β and integrating both sides of

$$u_\beta(x) = [\widetilde{R}_\lambda\psi](x) + < \beta, u_\beta > \frac{e_\lambda(x)}{r(x)}, \tag{4.3.8}$$

we easily get

$$< \beta, u_\beta > = \frac{< \beta, \widetilde{R}_\lambda\psi >}{1- < \beta, r^{-1}e_\lambda >},$$

provided $< \beta, r^{-1}e_\lambda > \neq 1$. In such a case, (4.3.8) becomes

$$u_\beta(x) = [\widetilde{R}_\lambda\psi](x) + \frac{e_\lambda(x)}{r(x)}\frac{< \beta, \widetilde{R}_\lambda\psi >}{1- < \beta, r^{-1}e_\lambda >}. \tag{4.3.9}$$

Lemma 4.3.3. *Let $\lambda > \kappa$, then*

$$< \beta, r^{-1}e_\lambda > < 1 \qquad \textit{and} \qquad \Psi_{\lambda,\beta}\psi = \frac{e_\lambda}{r}\frac{< \beta, \widetilde{R}_\lambda\psi >}{(1- < \beta, r^{-1}e_\lambda >)}$$

is a positive, bounded and compact linear operator on X_1.

Proof. We have

$$\begin{aligned} < \beta, r^{-1}e_\lambda > &\le \|\beta\|_\infty \int_{x_0}^{\infty} \frac{e^{-\lambda\widetilde{R}(x)-\widetilde{Q}(x)}}{r(x)}\, x dx \\ &\le \|\beta\|_\infty \int_{x_0}^{\infty} r^{-1}e^{-\lambda\widetilde{R}(x)}\, x dx \\ &\le \|\beta\|_\infty I(x_0,\infty) \\ &\le \frac{\|\beta\|_\infty x_0}{\lambda - \|r\|_\infty} < 1, \end{aligned}$$

where we made use of the monotonicity of $e^{-\widetilde{Q}(x)}$, and (4.3.6) with $\widetilde{R}(x_0) = 0$, respectively.

It is clear that $\Psi_{\lambda,\beta}$ is linear. Furthermore, for $\psi \in X_1$, $< \beta, \widetilde{R}_\lambda \psi >$ is finite, thus $\Psi_{\lambda,\beta}$ is continuous and so is bounded. Finally, since this operator is of finite rank (rank one), therefore, $\Psi_{\lambda,\beta}$ is compact. □

Lemma 4.3.4. *For any $\lambda > \kappa$, the resolvent $R(\lambda, A_\beta)$ of the operator $(A_\beta, D(A_\beta))$ is given by*

$$R(\lambda, A_\beta) = R(\lambda, A_0) + \Psi_{\lambda,\beta} R(\lambda, A_0). \tag{4.3.10}$$

Furthermore, the resolvent $R(\lambda, A_\beta)$ satisfies the estimate

$$\|R(\lambda, A_\beta)\|_1 \leq \frac{1}{\lambda - \kappa}. \tag{4.3.11}$$

Proof. We know from the previous lemma that $\Psi_{\lambda,\beta}$ is bounded and compact. Also, since $(I + \Psi_{\lambda,\beta})f = 0$ admits uniquely the trivial solution $f = 0$, then by Theorem 2.1.5 we conclude that for $\lambda > \kappa$, $I + \Psi_{\lambda,\beta}$ is invertible. A straightforward calculation shows that the inverse is

$$(I + \Psi_{\lambda,\beta})^{-1} g = g - \frac{e_\lambda}{r} < \beta, g >, \quad g \in X_1.$$

Furthermore, let $g \in D(A_0)$. Since $\lim_{x \to x_0} r(x) g(x) = 0$, we have

$$\begin{aligned} \lim_{x \to x_0} r(x)(I + \Psi_{\lambda,\beta}) g(x) &= \lim_{x \to x_0} r(x) g(x) + \lim_{x \to x_0} e_\lambda(x) \frac{< \beta, g >}{(1 - < \beta, r^{-1} e_\lambda >)} \\ &= < \beta, u_\beta > . \end{aligned}$$

Thus, we have $(I + \Psi_{\lambda,\beta}) D(A_0) \subseteq D(A_\beta)$.

Now, let $g \in D(A_\beta)$. Then

$$\begin{aligned} \lim_{x \to x_0} r(x)(I + \Psi_{\lambda,\beta})^{-1} g(x) &= \lim_{x \to x_0} r(x) g(x) - \lim_{x \to x_0} e_\lambda(x) < \beta, g > \\ &= 0. \end{aligned}$$

Hence, $(I + \Psi_{\lambda,\beta})^{-1} D(A_\beta) \subseteq D(A_0)$ and therefore, $D(A_\beta) = (I + \Psi_{\lambda,\beta}) D(A_0)$.

Now, for $g \in D(A_0)$ there exists $f \in D(A_\beta)$ and $\psi \in X_1$ such that $R(\lambda, A_0)\psi = g$ and $R(\lambda, A_\beta)\psi = f$ for which

$$R(\lambda, A_\beta)\psi = (I + \Psi_{\lambda,\beta}) R(\lambda, A_0)\psi,$$

which proves the equality (4.3.10).

According to (4.3.9) and using (4.3.6) we have

$$\begin{aligned}
\|u_\beta\|_1 &\leq \|[\widetilde{R}_\lambda\psi]\|_1 + I(x_0,\infty)\left|\frac{<\beta,\widetilde{R}_\lambda\psi>}{1-<\beta,r^{-1}e_\lambda>}\right| \\
&\leq \|[\widetilde{R}_\lambda\psi]\|_1\left[1+\frac{x_0\|\beta\|_\infty}{(\lambda-\|r\|_\infty)(1-\frac{x_0\|\beta\|_\infty}{\lambda-\|r\|_\infty})}\right] \\
&\leq \|[\widetilde{R}_\lambda\psi]\|_1\left[\frac{\lambda-\|r\|_\infty}{\lambda-\|r\|_\infty-x_0\|\beta\|_\infty}\right] \\
&\leq \|\psi\|_1\left[\frac{1}{\lambda-\|r\|_\infty-x_0\|\beta\|_\infty}\right]
\end{aligned}$$

which ends the proof. □

4.3.2 Growth-Fragmentation Equation

The growth-fragmentation equation reads:

$$\begin{aligned}
\frac{du}{dt}(t) &= [A_\beta+B]u(t), \quad t\geq 0. \qquad (4.3.12)\\
\lim_{x\to x_0^+} r(x)[u(t)(x)] &= \int_{x_0}^{\infty}\beta(y)[u(t)(y)]\,dy,\\
u(0) &= u_0.
\end{aligned}$$

Before proceeding with the analysis, let us make a general survey related to the operator B and the existence of a solution to (4.3.12).

Lemma 4.3.5.

$$D(A_\beta)\subseteq D(B). \qquad (4.3.13)$$

Proof. Indeed, let $u\in D(A_\beta)$ then, by the definition of $D(A_\beta)$, it is clear that $au\in X_1$. Then it follows that

$$\begin{aligned}
\|Bu\|_1 &= \int_{x_0}^{\infty}\left|\int_{x+x_0}^{\infty}a(y)b(x|y)u(y)\,dy\right|x\,dx\\
&\leq \int_{2x_0}^{\infty}a(y)|u(y)|\left(\int_{x_0}^{y-x_0}b(x|y)x\,dx\right)dy\\
&= \int_{x_0}^{\infty}a(y)|u(y)|y\,dy = \|au\|_1<\infty,
\end{aligned}$$

where we used (4.1.5) and (4.1.6). Therefore, $u\in D(B)$. □

The previous lemma shows in particular that $\|Bu\|_1\leq\|au\|_1$. Hence, if we assume that $a\in L_\infty((x_0,\infty))$, then the operator B is bounded and the operator A_β is perturbed by a bounded operator, which means that the sum of these operators always generates a semigroup $(S_{A_\beta+B}(t))_{t\geq 0}$ on X_1 associated with (4.3.12).

In the general case, if we consider $a \in L_{\infty,\mathrm{loc}}((x_0, \infty))$, then the operator B might be unbounded. Then the existence of a solution to (4.3.12) requires Kato's Perturbation Theorem.

Let us define

$$(\widetilde{A}_\beta, D(A_\beta)) = (A_\beta - \kappa I, D(A_\beta)). \tag{4.3.14}$$

Lemma 4.3.6. *For any $u \in D(A_\beta)_+$, the operator $\widetilde{A}_\beta + B := A_\beta - \kappa I + B$ satisfies*

$$\int_{x_0}^{\infty} [(\widetilde{A}_\beta + B)(u)](x) x \, dx \le 0.$$

Proof. Let $u \in D(A_\beta)_+$. We have

$$\begin{aligned}\int_{x_0}^{\infty} (A_\beta u + Bu) x \, dx &= -\int_{x_0}^{\infty} \frac{d}{dx}[r(x)u(x)] x \, dx - \int_{x_0}^{\infty} q(x)u(x) x \, dx \\ &\quad + \int_{x_0}^{\infty} Bu(x) x \, dx.\end{aligned}$$

By (4.2.19), we get

$$\int_{x_0}^{\infty} (A_\beta u + Bu) x \, dx = -\int_{x_0}^{\infty} \frac{d}{dx}[r(x)u(x)] x \, dx - \int_{x_0}^{\infty} d(x)u(x) x \, dx.$$

Now, integrating by parts, we have

$$\int_{s}^{\eta} \frac{d}{dx}[r(x)u(x)] \, x \, dx = \eta r(\eta)u(\eta) - s r(s)u(s) - \int_{s}^{\eta} r(x)u(x) \, dx,$$

for any $x_0 < s < \eta < \infty$.
Because $(ru)_x \in X_1$, the left-hand side converges to $\int_{x_0}^{\infty} \partial_x[r(x)u(x)] x \, dx$ and $sr(s)u(s) \to x_0 \int_{x_0}^{\infty} \beta(x)u(x) dx$ as $s \to x_0$ and $\eta \to \infty$.
Since $r \in X_\infty$ and $u \in X_1$, ru is integrable on (x_0, ∞) and so the last integral on the right-and side converges to $\int_{x_0}^{\infty} r(x)u(x) dx$. Then it follows that $\eta r(\eta)u(\eta)$ converges to a limit l as $\eta \to \infty$.

Suppose that $l \neq 0$. Then $r(x)u(x) \ge \sigma x^{-1}$ for some $\sigma > 0$ and for large enough x, which contradicts the integrability of ru. Thus

$$\lim_{\eta \to \infty} \eta r(\eta)u(\eta) = 0. \tag{4.3.15}$$

Therefore,

$$\begin{aligned}\int_{x_0}^{\infty}(A_\beta u + Bu)x\,dx &= x_0\int_{x_0}^{\infty}\beta(x)u(x)\,dx + \int_{x_0}^{\infty} r(x)u(x)\,dx \\ &\quad - \int_{x_0}^{\infty} d(x)u(x)x\,dx. \end{aligned} \tag{4.3.16}$$

Now, we have

$$x_0\int_{x_0}^{\infty}\beta(x)u(x)\,dx + \int_{x_0}^{\infty} r(x)u(x)\,dx \le (x_0\|\beta\|_\infty + \|r\|_\infty)\int_{x_0}^{\infty} u(x)x\,dx = \kappa\|u\|_1.$$

Hence

$$\begin{aligned}\int_{x_0}^{\infty}(\widetilde{A}_\beta u + Bu)x\,dx &\le -\int_{x_0}^{\infty}(\kappa x - x_0\beta(x) - r(x))u(x)\,dx - \int_{x_0}^{\infty} d(x)u(x)x\,dx \\ &\le 0.\end{aligned}$$

□

As was proved earlier, $\widetilde{A}_\beta$ generates a substochastic semigroup. In addition, according to Lemma 4.3.5 and 4.3.6, Kato's Perturbation Theorem ascertains that there is an extension $\widetilde{G}_\beta$ of the operator $\widetilde{A}_\beta + B$ that generates a substochastic semigroup $(S_{\widetilde{G}_\beta}(t))_{t\ge 0}$ of bounded linear operators on X_1.

Theorem 4.3.7. *There is an extension G_β of $A_\beta + B$ given by*

$$(G_\beta, D(G_\beta)) = (\widetilde{G}_\beta + \kappa I, D(\widetilde{G}_\beta))$$

that generates a positive semigroup $(S_{G_\beta}(t))_{t\ge 0} = (e^{\kappa t}S_{\widetilde{G}_\beta}(t))_{t\ge 0}$. Moreover, the generator G_β is characterized by:

$$(\lambda I - G_\beta)^{-1}\psi = \sum_{n=0}^{\infty}(\lambda I - A_\beta)^{-1}[B(\lambda I - A_\beta)^{-1}]^n\psi$$

for $\psi \in X_1$ and $\lambda > \kappa$.

Proof. Similar to the proof of Theorem 4.2.5. □

4.4 Honesty of the Semigroups

In the sequel, we combine some earlier results to provide a characterization of honest substochastic semigroups to our model in X_1 space. The theory introduced in Chapter 2 requires us to know the generator. Unfortunately, Theorems 4.2.5 and 4.3.7 do not provide any characterisation of the domain of the generators G and G_β. However, there is an interesting technique providing conditions for honesty in terms of known operators. This theory has mostly been used in a series of Banasiak papers by using extensions of operators [10, 11, 12]. This method is based on the fact that we know at least one extension of the generators G and G_β, namely G_{max} and $(G_\beta)_{max}$, respectively. Using the results obtained in this theory we conclude our work in this section by giving sufficient conditions for honesty of the semigroups $(S_G(t))_{t\geq 0}$ and $(S_{G_\beta}(t))_{t\geq 0}$ defined via Theorem 4.2.5 and Theorem 4.3.7, respectively. We note that the ideas and calculations for honesty of $(S_G(t))_{t\geq 0}$ and $(S_{G_\beta}(t))_{t\geq 0}$ are analogous to the analysis in [10] and [12] and [24]. To achieve this goal we begin by specifying the extensions of the operators which we will be working with.

4.4.1 Operator Extensions

Define by $\mathcal{E}$ the set of measurable functions that are defined on (x_0, ∞) and take values in $\mathbb{R} \cup \{-\infty, \infty\}$. By E_f we define the subspace of $\mathcal{E}$ consisting of functions that are finite almost everywhere where $X_1 \subset E_f \subset \mathcal{E}$.

In the case when $1/r$ is non-integrable at x_0, we consider the operator $\mathcal{A}$ given by (4.1.13) on the domain

$$D(\mathcal{A}) = \{u \in X_1;\ \ ru \in \mathrm{AC}((x_0,\infty)) \text{ and } \partial_x[ru] + qu \in E_f\}.$$

In the case when $1/r$ is integrable at x_0, we define $\mathcal{A}_\beta$ as $\mathcal{A}$ restricted to

$$D(\mathcal{A}_\beta) = \{u \in D(\mathcal{A});\ \ \lim_{x\to x_0^+} r(x)u(x) = \int_{x_0}^{\infty} \beta(y)u(y)\,dy\}.$$

Similarly, by $\mathcal{B}$ we denote the operator defined by the expression (4.1.14) on $D(\mathcal{B}) = \{u \in X_1;\ \ \mathcal{B}u \in E_f\}$. Using these concepts, we can define operators that can be thought of as the maximal extension of $A + B$ and $A_\beta + B$ in X_1 as follows:

$$[\mathcal{G}u](x) := [\mathcal{A}u](x) + [\mathcal{B}u](x), \quad [\mathcal{G}_\beta u](x) := [\mathcal{A}_\beta u](x) + [\mathcal{B}u](x),$$

defined on the domain $D(\mathcal{G}) = \{u \in D(\mathcal{A}) \cap D(\mathcal{B});\ \ x \to [\mathcal{G}u](x)\ \in X_1\}$ and $D(\mathcal{G}_\beta) = \{u \in D(\mathcal{A}_\beta) \cap D(\mathcal{B});\ \ x \to [\mathcal{G}_\beta u](x)\ \in X_1\}$, respectively.

Accordingly, for $\lambda > \|r\|_\infty$, we consider the operator $\mathcal{R}(\lambda)$ extending $R(\lambda, A)$ defined by the following expression:

$$[\mathcal{R}(\lambda)\psi](x) = \frac{e^{-\lambda R(x) - Q(x)}}{r(x)} \int_{x_0}^{x} e^{\lambda R(y) + Q(y)}\psi(y)\,dy, \tag{4.4.1}$$

on the domain $D(\mathcal{R}(\lambda)) = \{\psi \in \mathcal{E};\ \ x \to [\mathcal{R}(\lambda)\psi](x)\ \in E_f\}$.

Also, for $\lambda > \kappa$, $\mathcal{R}_\beta(\lambda)$ is the operator extension of $R(\lambda, A_\beta)$ defined by

$$\mathcal{R}_\beta(\lambda)\psi = \mathcal{R}_0(\lambda)\psi + \frac{e_\lambda}{r}\frac{<\beta, \mathcal{R}_0(\lambda)\psi>}{1- <\beta, r^{-1}e_\lambda>} \tag{4.4.2}$$

on $D(\mathcal{R}_\beta(\lambda)) = \{\psi \in \mathcal{E};\quad x \to [\mathcal{R}_\beta(\lambda)\psi](x)\ \in E_f\}$, where

$$[\mathcal{R}_0(\lambda)\psi](x) = \frac{e^{-\lambda\widetilde{R}(x)-\widetilde{Q}(x)}}{r(x)}\int_{x_0}^{x} e^{\lambda\widetilde{R}(y)+\widetilde{Q}(y)}\psi(y)\,dy.$$

Note that since the kernels of $\mathcal{B}$, $\mathcal{R}(\lambda)$ and $\mathcal{R}_\beta(\lambda)$ are non-negative, the existence of the respective integrals is equivalent to the existence of the positive and negative parts of the integrands. It can be shown as in [11, Section 9.3] that $G \subset \mathcal{G}$ and $G_\beta \subset \mathcal{G}_\beta$, so that the extensions are defined correctly.

4.4.2 Honesty of the Semigroup $(S_G(t))_{t\geq 0}$

We begin this subsection by stating an auxiliary result that allows us to prove the honesty of the semigroup $(S_G(t))_{t\geq 0}$.

Lemma 4.4.1. *Let $0 \leq \psi \in \mathcal{E}$ and $\lambda \geq \|r\|_\infty$ such that $\mathcal{R}(\lambda)\psi \in X_1$. If d is essentially bounded close to x_0, then $\psi \in L_1([x_0, \eta], xdx)$ for any $\eta < \infty$.*

Proof. For every $f \in X_{1+}$ and $\lambda > \|r\|_\infty$ such that $R(\lambda, G)f = u \in D(G)_+ \subset X_1$, there is $\psi \in \mathcal{E}_+$ such that $u = \mathcal{R}(\lambda)\psi$. By (4.4.1) and the Tonelli Theorem, we obtain:

$$\begin{aligned}\int_{x_0}^{\infty}(\mathcal{R}(\lambda)\psi)(x)\,xdx &= \int_{x_0}^{\infty} y\psi(y)\left(\frac{e^{\lambda R(y)+Q(y)}}{y}\int_{y}^{\infty}\frac{xe^{-\lambda R(x)-Q(x)}}{r(x)}\,dx\right)dy \\ &= \int_{x_0}^{\infty} y\psi(y)\varphi(y)dy.\end{aligned}$$

The function φ is continuous and non-negative, and the only points where it may be zero are at $y = x_0$ or as $y \to \infty$. As $y \to x_0$, the integral term of φ tends to infinity, see (4.2.12), and the other term tends to 0.

Now, let us take a continuous function $d_c \geq d$ a.e on $[x_0, \eta]$ for some $\eta > x_0$

(e.g. the essential bound of d_c there). Then

$$
\begin{aligned}
\varphi(y) &= \frac{e^{\lambda R(y)+Q(y)}}{y}\int_y^\infty \frac{xe^{-\lambda R(x)-Q(x)}}{r(x)}\,dx,\\
&= \frac{e^{\lambda R(y)+\int_{x_0+\epsilon}^{y}\frac{a(s)+d_c(s)}{r(s)}ds}\;e^{\int_{x_0+\epsilon}^{y}\frac{d(s)-d_c(s)}{r(s)}ds}}{y}\\
&\quad\times\int_y^\infty \frac{xe^{-\lambda R(x)-\int_{x_0+\epsilon}^{x}\frac{a(s)+d_c(s)}{r(s)}ds}e^{\int_{x_0+\epsilon}^{x}\frac{d_c(s)-d(s)}{r(s)}ds}}{r(x)}\,dx,\\
&\geq \frac{e^{\lambda R(y)+\int_{x_0+\epsilon}^{y}\frac{a(s)+d_c(s)}{r(s)}ds}\;e^{\int_{x_0+\epsilon}^{y}\frac{d(s)-d_c(s)}{r(s)}ds}}{y}\\
&\quad\times e^{\int_{x_0+\epsilon}^{y}\frac{d_c(s)-d(s)}{r(s)}ds}\int_y^\infty \frac{xe^{-\lambda R(x)-\int_{x_0+\epsilon}^{x}\frac{a(s)+q_c(s)}{r(s)}ds}}{r(x)}\,dx,\\
&= \frac{e^{\lambda R(y)+\int_{x_0+\epsilon}^{y}\frac{a(s)+d_c(s)}{r(s)}ds}}{y}\int_y^\infty \frac{xe^{-\lambda R(x)-\int_{x_0+\epsilon}^{x}\frac{a(s)+d_c(s)}{r(s)}ds}}{r(x)}\,dx,\\
&= \Theta(y),
\end{aligned}
$$

where we used the fact that, $d_c(s) - d(s)$ is positive. Since d_c is continuous in some neighbourhood of x_0, the l'Hospital rule gives,

$$
\lim_{y\to x_0}\Theta(y) = \lim_{y\to x_0}\frac{1}{\frac{-r(y)}{y}+\lambda+a(y)+d_c(y)} > 0,
$$

where we made use of $\dfrac{r(y)}{y} \leq \|r\|_\infty < \lambda$ and the assumption (4.1.5). Thus, we see that φ is bounded away from zero close to $x = x_0$ and therefore, $\psi \in L_1([x_0,\eta], xdx)$ for any $\eta < +\infty$. $\square$

Theorem 4.4.2. *If d is essentially bounded close to x_0, then $G = \overline{A+B}$ and thus the semigroup $(S_G(t))_{t\geq 0}$ is honest.*

Proof. For every $f \in X_{1+}$ and $\lambda > \|r\|_\infty$ such that $R(\lambda, G)f = u \in D(G)_+ \subset X_1$, there is $\psi \in \mathcal{E}_+$ such that $u = \mathcal{R}(\lambda)\psi$. Thus, according to (2.4.5), we obtain

$$
Gu = \lambda\mathcal{R}(\lambda)\psi - \psi + \mathcal{B}\mathcal{R}(\lambda)\psi. \tag{4.4.3}
$$

If $u \in D(G) \subset X_1$ then, by Lemma 4.4.1, we have $\psi \in L_1([x_0,\eta], xdx)$ and therefore also $\mathcal{B}\mathcal{R}(\lambda)\psi \in L_1([x_0,\eta], xdx)$. Hence, we can integrate (4.4.3) as follows

$$
\begin{aligned}
\int_{x_0}^\infty [Gu](x)x\,dx &= \lim_{\eta\to\infty}\int_{x_0}^\eta [Gu](x)x\,dx\\
&= \lim_{\eta\to\infty}\int_{x_0}^\eta \left(\lambda\mathcal{R}(\lambda)\psi - \psi + \mathcal{B}\mathcal{R}(\lambda)\psi\right)x\,dx,
\end{aligned}
$$

where the limit on the right-hand side exists. Using the fact that the integral over $[x_0, \eta]$ of each term within this limit exists, changing the order of integration by the Fubini Theorem, and the change of the variable of integration, we obtain

$$\begin{aligned}\int_{x_0}^{\eta}[\mathcal{BR}(\lambda)\psi](x)x dx &= \int_{2x_0}^{\eta} a(y)[\mathcal{R}(\lambda)\psi](y)\left(\int_{x_0}^{y-x_0} xb(x|y)dx\right)dy \\ &\quad+\int_{\eta}^{\infty} a(y)[\mathcal{R}(\lambda)\psi](y)\left(\int_{x_0}^{\eta-x_0} xb(x|y)dx\right)dy, \\ &= \int_{2x_0}^{\eta} a(y)[\mathcal{R}(\lambda)\psi](y))ydy \\ &\quad+\int_{\eta}^{\infty} a(y)[\mathcal{R}(\lambda)\psi](y)\left(\int_{x_0}^{\eta-x_0} xb(x|y)dx\right)dy.\end{aligned}$$

Thus, we can write

$$\begin{aligned}&\int_{x_0}^{\eta}[\mathcal{BR}(\lambda)\psi](x)x dx \\ &= \int_{x_0}^{\eta}(\lambda+q(y))[\mathcal{R}(\lambda)\psi](y)ydy - \int_{x_0}^{\eta} d(y)[\mathcal{R}(\lambda)\psi](y)ydy \\ &\quad -\lambda\int_{x_0}^{\eta}[\mathcal{R}(\lambda)\psi](y)ydy + \int_{\eta}^{\infty} a(y)[\mathcal{R}(\lambda)\psi](y)\left(\int_{x_0}^{\eta-x_0} xb(x|y)dx\right)dy.\end{aligned} \tag{4.4.4}$$

Next,

$$\begin{aligned}I &= \int_{x_0}^{\eta}\left(\frac{(\lambda+q(y))e^{-\lambda R(y)-Q(y)}}{r(y)}\int_{x_0}^{y} e^{\lambda R(s)+Q(s)}\psi(s)ds\right)y\,dy, \\ &= \int_{x_0}^{\eta} e^{\lambda R(s)+Q(s)}\psi(s)\left(\int_{\eta}^{s} y\frac{d}{dy}e^{-\lambda R(y)-Q(y)}dy\right)ds, \\ &= \int_{x_0}^{\eta} e^{\lambda R(s)+Q(s)}\psi(s)\left(se^{-\lambda R(s)-Q(s)} - \eta e^{-\lambda R(\eta)-Q(\eta)} + \int_{s}^{\eta} e^{-\lambda R(y)-Q(y)}dy\right)ds.\end{aligned}$$

It follows that

$$\begin{aligned}I &= \int_{x_0}^{\eta}\psi(s)sds - \eta e^{-\lambda R(\eta)-Q(\eta)}\int_{x_0}^{\eta} e^{\lambda R(s)+Q(s)}\psi(s)ds, \\ &\quad+\int_{x_0}^{\eta} e^{\lambda R(s)+Q(s)}\psi(s)\left(\int_{s}^{\eta} e^{-\lambda R(y)-Q(y)}dy\right)ds \qquad (4.4.5) \\ &= \int_{x_0}^{\eta}\psi(s)sds - \eta r(\eta)[\mathcal{R}(\lambda)\psi](\eta) + \int_{x_0}^{\eta} r(s)[\mathcal{R}(\lambda)\psi](s)ds,\end{aligned}$$

since

$$\begin{aligned}\int_{x_0}^{\eta} r(y)[\mathcal{R}(\lambda)\psi](y)dy &= \int_{x_0}^{\eta}\left(e^{-\lambda R(y)-Q(y)}\int_{x_0}^{y} e^{\lambda R(s)+Q(s)}\psi(s)ds\right)dy, \\ &= \int_{x_0}^{\eta} e^{\lambda R(s)+Q(s)}\psi(s)\left(\int_{s}^{\eta} e^{-\lambda R(y)-Q(y)}dy\right)ds.\end{aligned}$$

Combining the result of (4.4.4) and (4.4.5), we obtain

$$\begin{aligned}
&\int_{x_0}^{\eta}(-\psi(x)+[\mathcal{B}\mathcal{R}(\lambda)\psi](x)+\lambda[\mathcal{R}(\lambda)\psi](x))xdx \\
= \quad & -\eta r(\eta)[\mathcal{R}(\lambda)\psi](\eta)+\int_{x_0}^{\eta} r(x)[\mathcal{R}(\lambda)\psi](x)dx-\int_{x_0}^{\eta} d(x)[\mathcal{R}(\lambda)\psi](x)xdx \\
&+\int_{\eta}^{\infty} a(y)[\mathcal{R}(\lambda)\psi](y)\left(\int_{x_0}^{\eta-x_0} xb(x|y)dx\right)dy.
\end{aligned}$$

Since $u=\mathcal{R}(\lambda)\psi\in X_1$, then using (4.1.2) and $r\in X_\infty$, respectively, it follows that $d(x)u(x)\in L_1((x_0,\infty),xdx)$ and $r(x)u(x)\in L_1((x_0,\infty),dx)$. Moreover, there exists a sequence $\eta_k\to\infty$ as $k\to\infty$ for which $\eta_k r(\eta_k)u(\eta_k)\to 0$. Indeed, otherwise $xr(x)u(x)\geq\varepsilon>0$ for some ε and all sufficient large x. But then $r(x)u(x)\geq\varepsilon x^{-1}$ which would contradict the integrability of ru. Then, using the fact the limit exists, we can write

$$\begin{aligned}
\int_{x_0}^{\infty}[Gu](x)x\,dx &= \int_{x_0}^{\infty} r(x)u(x)\,dx-\int_{x_0}^{\infty} d(x)u(x)\,xdx \\
&\quad+\lim_{k\to\infty}\left[\int_{\eta_k}^{\infty} a(y)u(y)\left(\int_{x_0}^{\eta_k-x_0} b(x|y)x\,dx\right)dy\right] \\
&\geq \int_{x_0}^{\infty} r(x)u(x)\,dx-\int_{x_0}^{\infty} d(x)u(x)\,xdx.
\end{aligned}$$

□

4.4.3 Honesty of the Semigroup $(S_{G_\beta}(t))_{t\geq 0}$

The following Lemma [12, Lemma 2.6] plays an important role in the proof of the next theorem.

Lemma 4.4.3.

(*a*) *If* $\psi\in D(\mathcal{R}_\beta(\lambda))$, *then*

- (*i*) $\psi\in L_1((x_0,\eta),xdx)$ *for any* $\eta<\infty$;
- (*ii*) $\mathcal{R}_\beta(\lambda)\psi$ *is continuous on* (x_0,∞);
- (*iii*)
$$\lim_{x\to x_0^+} r(x)[\mathcal{R}_\beta(\lambda)\psi](x)=\int_{x_0}^{\infty}\beta(x)[\mathcal{R}_\beta(\lambda)\psi](x)\,dx.$$

(*b*) $r^{-1}e_\lambda\in D(A):=\{\psi\in X_1;\ \ a\psi\in X_1\}$.

Theorem 4.4.4.
$$G_\beta=\overline{A_\beta+B}.$$

Proof. The analysis we intend to use is similar to that in the previous subsection. In other words, we make use of (4.3.16) and [11, Theorem 6.13] to check that

$$\begin{aligned}\int_{x_0}^{\infty}[G_\beta u](x)x\,dx &\geq \int_{x_0}^{\infty} r(x)u(x)\,dx - \int_{x_0}^{\infty} d(x)u(x)x\,dx\\ &+x_0\int_{x_0}^{\infty}\beta(x)u(x)\,dx,\end{aligned}$$

on elements of the form $u = R(\lambda, G_\beta)f,\ f \in X_{1+},\ \lambda > \|r\|_\infty$.

We recall that if $u = R(\lambda, G_\beta)f,\ f \in X_{1+}$, then there exists $\psi \in \mathcal{E}_+$ such that $u = \mathcal{R}_\beta(\lambda)\psi$ and

$$G_\beta u = \lambda\mathcal{R}_\beta(\lambda)\psi - \psi + \mathcal{B}\mathcal{R}_\beta(\lambda)\psi.$$

If $u = \mathcal{R}_\beta(\lambda)\psi \in X_1$, then $\psi \in D(\mathcal{R}_\beta(\lambda))$ and, by Lemma 4.4.3 (a)(i), $\psi \in L_1((x_0,\eta), xdx)$ and therefore, $\mathcal{B}\mathcal{R}_\beta(\lambda)\psi \in L_1((x_0,\eta), xdx)$ as all other terms of the equality above are integrable.

Now, according to Lemma 4.3.4, we consider the decomposition

$$\mathcal{R}_\beta(\lambda)\psi = \mathcal{R}_0(\lambda)\psi + \mathcal{R}_{0,\beta}\psi = \mathcal{R}_0(\lambda)\psi + \frac{e_\lambda}{r}\frac{<\beta, \mathcal{R}_0(\lambda)\psi>}{1-<\beta, r^{-1}e_\lambda>}.$$

By Lemma 4.4.3 (b), $\mathcal{B}\mathcal{R}_{0,\beta}\psi = B\mathcal{R}_{0,\beta}\psi \in X_1$ and therefore, $\mathcal{R}_0(\lambda)\psi \in L_1((x_0,\eta), xdx)$. Hence, we can write

$$G_\beta u = \lambda\mathcal{R}_0(\lambda)\psi + \lambda\mathcal{R}_{0,\beta}\psi - \psi + \mathcal{B}\mathcal{R}_0(\lambda)\psi + \mathcal{B}\mathcal{R}_{0,\beta}\psi,$$

where each of the first three terms on the right-hand side is in $L_1((x_0,\eta), xdx)$ and the last two are both in X_1. Since $G_\beta u \in X_1$, we can write

$$\begin{aligned}\int_{x_0}^{\infty}[G_\beta u](x)x\,dx &= \lim_{\eta\to\infty}\int_{x_0}^{\eta}[G_\beta u](x)x\,dx\\ &= \lim_{\eta\to\infty}\left[\int_{x_0}^{\eta}(\lambda[\mathcal{R}_0(\lambda)\psi](x) - \psi(x) + [\mathcal{B}\mathcal{R}_0(\lambda)\psi](x))x\,dx\right]\\ &+\lim_{\eta\to\infty}\left[\int_{x_0}^{\eta}(\lambda[\mathcal{R}_{0,\beta}\psi](x) + [\mathcal{B}\mathcal{R}_{0,\beta}\psi](x))x\,dx\right], \quad (4.4.6)\end{aligned}$$

where the limit on the right-hand side exists. Since the integral over (x_0,η) of each term within this limit exists, we can evaluate

$$\begin{aligned}&\int_{x_0}^{\eta}[\mathcal{B}\mathcal{R}_0(\lambda)\psi](x)x\,dx = \int_{x_0}^{\eta}\left(\int_{x+x_0}^{\infty}a(y)b(x|y)[\mathcal{R}_0(\lambda)\psi](y)\,dy\right)x\,dx\\ =\ &\int_{x_0}^{\eta}(\lambda+q(y))[\mathcal{R}_0(\lambda)\psi](y)y\,dy - \lambda\int_{x_0}^{\eta}[\mathcal{R}_0(\lambda)\psi](y)y\,dy\\ &-\int_{x_0}^{\eta}d(y)[\mathcal{R}_0(\lambda)\psi](y)y\,dy + \int_{\eta}^{\infty}a(y)[\mathcal{R}_0(\lambda)\psi](y)\left(\int_{x_0}^{\eta-x_0}xb(x|y)\,dx\right)dy.\end{aligned}$$

As before, we write again

$$\begin{aligned} I_1 &= \int_{x_0}^{\eta}\left(\frac{(\lambda+q(y))e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}}{r(y)}\int_{x_0}^{y}e^{\lambda\widetilde{R}(s)+\widetilde{Q}(s)}\psi(s)\,ds\right)y\,dy \\ &= \int_{x_0}^{\eta}e^{\lambda\widetilde{R}(s)+\widetilde{Q}(s)}\psi(s)\left(\int_{\eta}^{s}y\frac{d}{dy}e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}dy\right)ds \\ &= \int_{x_0}^{\eta}e^{\lambda\widetilde{R}(s)+\widetilde{Q}(s)}\psi(s)\left(se^{-\lambda\widetilde{R}(s)-\widetilde{Q}(s)}-\eta e^{-\lambda\widetilde{R}(\eta)-\widetilde{Q}(\eta)}+\int_{s}^{\eta}e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}\,dy\right)ds \\ &= \int_{x_0}^{\eta}\psi(s)s\,ds-\eta r(\eta)[\mathcal{R}_0(\lambda)\psi](\eta)+\int_{x_0}^{\eta}r(y)[\mathcal{R}_0(\lambda)\psi](y)\,dy. \end{aligned}$$

It follows that

$$\begin{aligned} &\int_{x_0}^{\eta}(-\psi(x)+[\mathcal{B}\mathcal{R}_0(\lambda)\psi](x)+\lambda[\mathcal{R}_0(\lambda)\psi](x))x\,dx \\ = \;& -\eta r(\eta)[\mathcal{R}_0(\lambda)\psi](\eta)-\int_{x_0}^{\eta}d(y)[\mathcal{R}_0(\lambda)\psi](y)y\,dy+\int_{x_0}^{\eta}r(y)[\mathcal{R}_0(\lambda)\psi](y)\,dy \\ &+\int_{\eta}^{\infty}a(y)[\mathcal{R}_0(\lambda)\psi](y)\left(\int_{x_0}^{\eta-x_0}xb(x|y)\,dx\right)dy. \end{aligned}$$

Let $u_0 := [\mathcal{R}_0(\lambda)\psi]$. Since $u_0 = u - \mathrm{C}.r^{-1}e_\lambda$, where $C =<\beta, \mathcal{R}_\beta(\lambda)\psi>$, we see that $u_0 \in X_1$. Thus $ru_0 \in L_1((x_0,\infty),dx)$ and $du_0 \in L_1((x_0,\infty),xdx)$. Furthermore, as before, there exists a sequence $\eta_k \to \infty$ as $k \to \infty$ for which $\eta_k r(\eta_k)u_0(\eta_k) \to 0$. Hence

$$\begin{aligned} &\lim_{k\to\infty}\int_{x_0}^{\eta_k}(-\psi(x)+[\mathcal{B}u_0](x)+\lambda u_0(x))x\,dx \\ = \;& -\int_{x_0}^{\infty}d(y)u_0(y)y\,dy+\int_{x_0}^{\infty}r(y)u_0(y)\,dy \\ &+\lim_{k\to\infty}\int_{\eta_k}^{\infty}a(y)u_0(y)\left(\int_{x_0}^{\eta_k-x_0}xb(x|y)\,dx\right)dy. \end{aligned} \tag{4.4.7}$$

Now, for the last two terms in (4.4.6), we first set

$$\mathcal{R}_{0,\beta}\psi = \frac{e_\lambda}{r}\frac{<\beta,\mathcal{R}_0(\lambda)\psi>}{1-<\beta,r^{-1}e_\lambda>} = \frac{e_\lambda}{r}<\beta,\mathcal{R}_\beta(\lambda)\psi>.$$

It turns out that ψ enters the expression through a constant scalar multiplier.

Hence, we first evaluate

$$\begin{aligned}
&\int_{x_0}^{\infty}\left(\int_{x+x_0}^{\infty} a(y)b(x|y)\frac{e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}}{r(y)}dy\right)x\,dx \,+\, \lambda\int_{x_0}^{\infty}\frac{e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}}{r(y)}y\,dy\\
=\;&\int_{x_0}^{\infty} a(y)\frac{e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}}{r(y)}y\,dy \,+\, \lambda\int_{x_0}^{\infty}\frac{e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}}{r(y)}y\,dy\\
=\;&\int_{x_0}^{\infty}(\lambda+q(y))\frac{e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}}{r(y)}y\,dy - \int_{x_0}^{\infty} d(y)\frac{e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}}{r(y)}y\,dy\\
=\;& x_0 + \int_{x_0}^{\infty} e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}\,dy - \int_{x_0}^{\infty} d(y)\frac{e^{-\lambda\widetilde{R}(y)-\widetilde{Q}(y)}}{r(y)}y\,dy,
\end{aligned}$$

where we made use of integration by parts, (4.3.15) and the fact that $\widetilde{R}(x_0) = \widetilde{Q}(x_0) = 0$. Hence, using the linearity of integral, we obtain

$$\begin{aligned}
&\int_{x_0}^{\infty}(\lambda[\mathcal{R}_{0,\beta}\psi](x) + [\mathcal{B}\mathcal{R}_{0,\beta}\psi](x))x\,dx \qquad (4.4.8)\\
=\;& x_0 < \beta, \mathcal{R}_{\beta}(\lambda)\psi > + \int_{x_0}^{\infty} r(y)[\mathcal{R}_{0,\beta}\psi](y)\,dy - \int_{x_0}^{\infty} d(y)[\mathcal{R}_{0,\beta}\psi](y)y\,dy.
\end{aligned}$$

Combining (4.4.7) and (4.4.8), we obtain the final result

$$\begin{aligned}
&\int_{x_0}^{\infty}[G_{\beta}u](x)x\,dx = \lim_{k\to\infty}\int_{x_0}^{\eta_k}[G_{\beta}u](x)x\,dx\\
=\;& x_0\int_{x_0}^{\infty}\beta(x)u(x)\,dx + \int_{x_0}^{\infty} r(x)u(x)\,dx - \int_{x_0}^{\infty} d(x)u(x)x\,dx\\
&+\lim_{k\to\infty}\int_{\eta_k}^{\infty} a(y)u_0(y)\left(\int_{x_0}^{\eta_k} xb(x|y)\,dx\right)dy\\
\geq\;& x_0\int_{x_0}^{\infty}\beta(x)u(x)\,dx + \int_{x_0}^{\infty} r(x)u(x)\,dx - \int_{x_0}^{\infty} d(x)u(x)x\,dx
\end{aligned}$$

which proves the thesis. □

4.5 Local and Global Solution for the Full Non-Linear Equation

This section will make use of the semi-linear abstract Cauchy problem method to prove the local and global existence of solutions in the Banach space X_1 to the full non-linear problem for combined mortality, growth, coagulation and fragmentation. We note that the techniques used throughout this section is similar to that performed in [24].

The full non-linear problem reads,

$$\begin{aligned}
\frac{du(t)}{dt} &= [G_{0,\beta} + N]u(t),\\
u(0) &= u_0,
\end{aligned}$$

where $G_{0,\beta}$ represents G or G_β that generate $S_G(t)$ or $S_{G_\beta}(t)$, respectively. Before proceeding to analyse the full problem, our starting point is to explore relevant properties of the coagulation operator N.

Note that it is easy to prove by a simple calculation that the operator N defined by (4.1.15) is bilinear and bounded.

Proposition 4.5.1. *$N(X_1) \subset X_1$ with $\|N\psi\|_1 \leq 2\frac{k_0}{x_0}\|\psi\|_1^2$ for all $\psi \in X_1$.*

Proof. Let $\psi, \phi \in X_1$, we have

$$\begin{aligned}
&\int_{2x_0}^{\infty}\int_{x_0}^{x-x_0} x|\psi(x-y)\phi(y)|\,dy\,dx \\
&\quad = \int_{x_0}^{\infty}\int_{y+x_0}^{\infty} x|\psi(x-y)\phi(y)|\,dx\,dy \\
&\quad = \int_{x_0}^{\infty}\int_{x_0}^{\infty} (z+y)|\psi(z)\phi(y)|\,dz\,dy. \qquad (4.5.1)
\end{aligned}$$

Since $x_0 \leq y$ and $x_0 \leq z$, it follows that $\dfrac{y+z}{2} \leq \dfrac{yz}{x_0}$. Using this argument with (4.5.1) we have the following

$$\begin{aligned}
\|\mathcal{N}_1[\psi,\phi]\|_1 &\leq \int_{x_0}^{\infty} x\frac{\chi_U(x)}{2}\int_{x_0}^{x-x_0} k(x-y,y)|\psi(x-y)\phi(y)|\,dy\,dx \\
&\leq \frac{k_0}{2}\int_{2x_0}^{\infty}\int_{x_0}^{x-x_0} x|\psi(x-y)\phi(y)|\,dy\,dx \\
&\leq \frac{k_0}{x_0}\left(\int_{x_0}^{\infty}\int_{x_0}^{\infty} zy|\psi(z)\phi(y)|\,dz\,dy\right) \\
&= \frac{k_0}{x_0}\|\psi\|_1\|\phi\|_1,
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|\mathcal{N}_2[\psi,\phi]\|_1 &\leq \int_{x_0}^{\infty}\int_{x_0}^{\infty} xk(x,y)|\psi(x)\phi(y)|dy\,dx \\
&\leq k_0\int_{x_0}^{\infty}\int_{x_0}^{\infty} x|\psi(x)\phi(y)|dy\,dx \\
&\leq \frac{k_0}{x_0}\|\psi\|_1\|\phi\|_1.
\end{aligned}$$

Therefore,

$$\|N\psi\|_1 \leq \|\mathcal{N}_1[\psi,\psi]\|_1 + \|\mathcal{N}_2[\psi,\psi]\|_1 \leq 2\frac{k_0}{x_0}\|\psi\|_1^2.$$

The result follows. $\square$

Proposition 4.5.2. *N is locally Lipschitz on X_1.*

Proof. Let $u_0 \in X_1$ and $\psi, \phi \in \overline{B}(u_0, r) := \{\psi \in X_1 : \|\psi - u_0\|_1 \leq r\}$, then

$$\begin{aligned} \|N\psi - N\phi\|_1 &= \|\mathcal{N}[\psi,\psi] - \mathcal{N}[\phi,\phi]\|_1 \\ &= \|\mathcal{N}[\psi-\phi,\psi] + \mathcal{N}[\phi,\psi-\phi]\|_1 \\ &\leq 2\frac{k_0}{x_0}(\|\psi-\phi\|_1\|\psi\|_1 + \|\phi\|_1\|\psi-\phi\|_1) \\ &= 2\frac{k_0}{x_0}\|\psi-\phi\|_1(\|\psi\|_1 + \|\phi\|_1) \\ &\leq \varrho_{r,u_0}\|\psi-\phi\|_1, \end{aligned}$$

where

$$\varrho_{r,u_0} = 4\frac{k_0}{x_0}(r + \|u_0\|_1), \tag{4.5.2}$$

and we make use of the bi-linearity of $\mathcal{N}$. Thus, N is locally Lipschitz on X_1. □

Proposition 4.5.3. *N is Fréchet differentiable on X_1 and for any $\psi \in X_1$, the Fréchet derivative N_ψ is given by*

$$N_\psi \phi := \mathcal{N}[\psi,\phi] + \mathcal{N}[\phi,\psi], \quad \forall\, \phi \in X_1.$$

Moreover the Fréchet derivative is continuous with respect to ψ.

Proof. Let $\psi, \delta \in X_1$. The bilinearity of $\mathcal{N}$ leads to

$$\begin{aligned} N(\psi+\phi) &= \mathcal{N}[\psi+\phi,\psi+\phi] \\ &= \mathcal{N}[\psi,\psi] + \mathcal{N}[\psi,\phi] + \mathcal{N}[\phi,\psi] + \mathcal{N}[\phi,\phi]. \end{aligned}$$

For fixed ψ, $\mathcal{N}[\psi,\cdot] + \mathcal{N}[\cdot,\psi]$ is a bounded operator on X_1 with

$$||\mathcal{N}[\psi,\delta] + \mathcal{N}[\delta,\psi]\,||_1 \leq 4\frac{k_0}{x_0}\,||\psi||_1\,||\delta||_1 \quad \forall \delta \in X_1\,.$$

Also,

$$\frac{||N\delta||_1}{||\delta||_1} \leq 2\frac{k_0}{x_0}\,||\delta||_1 \;\to 0 \;\text{ as } ||\delta||_1 \to 0\,.$$

Hence, N is Fréchet differentiable at each $\psi \in X_1$ and the Fréchet derivative N_ψ at ψ is given by

$$N_\psi \phi := \mathcal{N}[\psi,\phi] + \mathcal{N}[\phi,\psi] \quad \forall \phi \in X_1\,.$$

Consequently,

$$\|N_\psi \phi\|_1 \leq \varrho_{r,u_0}\|\phi\|_1, \quad \forall\, \phi \in X_1,\; \psi \in \overline{B}(u_0,r).$$

Also, for $\psi_1, \psi_2, \phi \in X_1$,

$$\begin{aligned} \|N_{\psi_1}\phi - N_{\psi_2}\phi\|_1 &= \|\mathcal{N}[\psi_1, \phi] + \mathcal{N}[\phi, \psi_1] - \mathcal{N}[\psi_2, \phi] - \mathcal{N}[\phi, \psi_2]\|_1 \\ &= \|\mathcal{N}[\psi_1 - \psi_2, \phi] + \mathcal{N}[\phi, \psi_1 - \psi_2]\|_1 \\ &\leq 4\frac{k_0}{x_0}\|\phi\|_1\|\psi_1 - \psi_2\|_1 \to 0 \quad \text{as} \quad \|\psi_1 - \psi_2\|_1 \to 0. \end{aligned}$$

Hence, the Fréchet derivative is continuous with respect to ψ. □

4.5.1 Local Existence

Theorem 4.5.4. Local existence of a solution
There exist positive constants r_0, t_0 and a strongly differentiable function

$$u : [0, t_0) \to B(u_0, r_0) := \{\psi \in X_1 : \ \|\psi - u_0\|_1 < r_0\}$$

such that

$$\frac{du}{dt}(t) = G_{0,\beta}[u(t)] + N[u(t)], \ \ 0 < t < t_0;\ u(0) = u_0 \in D(G_{0,\beta}) \cap X_{1+}, \tag{4.5.3}$$

Proof. $G_{0,\beta}$ is the generator of a strongly continuous semigroup. Regarding the properties of the nonlinear operator N, the theorem follows from standard results on semilinear ACPs; see Theorem 2.5.8. □

4.5.2 A Non-Negative Solution

To show that the local (in time) solution is in X_{1+} for all $t \in [0, t_0)$, we adopt the argument used in [12]. First we note that the solution u of (4.5.3) is also the unique strongly differentiable solution of

$$\frac{du}{dt}(t) = (G_{0,\beta}[u(t)] - \alpha u(t)) + (\alpha u(t) + N[u(t)]) \tag{4.5.4}$$

for any $\alpha \in \mathbb{R}$. Hence, u is the unique solution of the integral equation

$$u(t) = e^{-\alpha t} S_{G_{0,\beta}}(t) u_0 + \int_0^t e^{-\alpha(t-s)} S_{G_{0,\beta}}(t-s) N_\alpha[u(s)] ds,\ 0 \leq t < t_0, \tag{4.5.5}$$

where $N_\alpha := N + \alpha I$.

Lemma 4.5.5. *Let $\alpha \geq \frac{k_0}{x_0}(\|u_0\|_1 + r_0)$. Then $N_\alpha \psi \in X_{1+}$ for all $\psi \in B(u_0, r_0) \cap X_{1+}$.*

Proof. By definition, we have

$$N_\alpha \psi = \alpha\psi + \mathcal{N}_1[\psi, \psi] - \mathcal{N}_2[\psi, \psi].$$

Clearly, $\mathcal{N}_1[\psi, \psi] \in X_{1+}$ for all $\psi \in X_{1+}$. Also, for $\psi \in B(u_0, r_0) \cap X_{1+}$,

$$\begin{aligned} \psi(x) \int_{x_0}^{\infty} k(x,y)\psi(y)dy &\leq \frac{k_0}{x_0}\psi(x)\|\psi\|_1 \\ &\leq \frac{k_0}{x_0}(\|u_0\|_1 + r_0)\psi(x). \end{aligned}$$

Hence

$$\begin{aligned} \alpha\psi(x) - \mathcal{N}_2[\psi, \psi](x) &\geq \alpha\psi(x) - \frac{k_0}{x_0}(\|u_0\|_1 + r_0)\psi(x) \\ &\geq 0 \text{ provided that } \alpha \geq \frac{k_0}{x_0}(\|u_0\|_1 + r_0). \end{aligned}$$

□

Theorem 4.5.6. *Let $u_0 \in D(G_{0,\beta}) \cap X_{1+}$ and let $u : [0, t_0) \to B(u_0, r_0)$ be the unique strict solution of (4.5.3). Then there exists $t_1 \in (0, t_0]$ such that $u(t) \in X_{1+}$ for all $t \in [0, t_1)$.*

Proof. The technique we adopt in the sequel is based on the Banach Fixed Point Theorem. To proceed, let $Y := C([0,\tau], X)$ with norm $\|\psi\|_Y := \max\{\|\psi(t)\|_1 : 0 \leq t \leq \tau\}$. Moreover, let

$$\Delta := \{\psi \in Y : \psi(t) \in \overline{B}(u_0, r_1) \cap X_{1+} \ \forall t \in [0, \tau]\}$$

where $0 < r_1 < r_0$, and define

$$\begin{aligned} (\mathcal{Q}\psi)(t) &:= e^{-\alpha t} S_{G_{0,\beta}}(t)u_0 + \int_0^t e^{-\alpha(t-s)} S_{G_{0,\beta}}(t-s) N_\alpha[\psi(s)]ds, \ 0 \leq t \leq \tau, \\ D(\mathcal{Q}) &:= \Delta, \end{aligned}$$

with $\alpha \geq \frac{k_0}{x_0}(\|u_0\|_1 + r_0)$. By continuity of $e^{-\alpha t}$ and in view of the Lemma 4.5.5, we have $\mathcal{Q}(\Delta) \subset Y$ and $(\mathcal{Q}\psi)(t) \in X_{1+}$ for all $t \in [0, \tau]$.

Further, using the honesty of $(S_{G_{0,\beta}}(t))_{t\geq 0}$, it follows that the semigroup satisfies (4.2.21) and (4.3.16) on $D(G_{0,\beta})$. Hence, $(S_{G_{0,\beta}}(t))_{t\geq 0}$ is bounded as follows:

$$\|S_{G_{0,\beta}}(t)\| \leq \begin{cases} e^{(\|r\|_\infty - m)t}, & r^{-1} \text{ is not integrable at } x_0, \\ \\ e^{(\kappa - m)t}, & r^{-1} \text{ is integrable at } x_0, \end{cases} \tag{4.5.6}$$

where $m = \inf_{x \in (x_0, \infty)} d(x)$. We define

$$\theta = \begin{cases} \|r\|_\infty, & r^{-1} \text{ is not integrable at } x_0, \\ \\ \kappa, & r^{-1} \text{ is integrable at } x_0. \end{cases} \tag{4.5.7}$$

Next, for all $\psi, \phi \in \Delta$,

$$\begin{aligned}
\|(\mathcal{Q}\psi)(t) - (\mathcal{Q}\phi)(t)\|_1 &\leq \int_0^t e^{-\alpha(t-s)} \|S_{G_{0,\beta}}(t-s)\|_{\mathcal{L}(X_1)} \|N_\alpha[\psi(s)] - N_\alpha[\phi(s)]\|_1 \, ds \\
&\leq \int_0^t e^{(\theta-\alpha)(t-s)} \|N_\alpha[\psi(s)] - N_\alpha[\phi(s)]\|_1 \, ds \\
&\leq (\varrho_{r_0,u_0} + \alpha) \int_0^t e^{(\theta-\alpha)(t-s)} \|\psi(s) - \phi(s)\|_1 \, ds,
\end{aligned}$$

where ϱ_{r_0,u_0} is defined via (4.5.2), and $\mathcal{L}(X_1)$ is the set of bounded linear operators on X_1. Hence

$$\|\mathcal{Q}\psi - \mathcal{Q}\phi\|_Y \leq (\varrho_{r_0,u_0} + \alpha) e^{\theta\tau} \tau \|\psi - \phi\|_Y.$$

Similarly,

$$\begin{aligned}
\|(\mathcal{Q}\psi)(t) - u_0\|_1 &\leq \|e^{-\alpha t} S_{G_{0,\beta}}(t) u_0 - u_0\|_1 + \int_0^t e^{-\alpha(t-s)} \|S_{G_\beta}(t-s) N_\alpha[\psi(s)]\|_1 \, ds \\
&\leq \|e^{-\alpha t} S_{G_{0,\beta}}(t) u_0 - u_0\|_1 + \int_0^t e^{(\theta-\alpha)(t-s)} \|N_\alpha[\psi(s)]\|_1 \, ds. \quad (4.5.8)
\end{aligned}$$

Now

$$\begin{aligned}
\|N_\alpha[\psi(s)]\|_1 &= \|N_\alpha[\psi(s)] - N_\alpha u_0 + N_\alpha u_0\|_1 \\
&\leq \|N_\alpha[\psi(s)] - N_\alpha u_0\|_1 + \|N_\alpha u_0\|_1 \\
&\leq (\varrho_{r_0,u_0} + \alpha)\|\psi(s) - u_0\|_1 + \|N u_0\|_1 + \alpha\|u_0\|_1 \\
&\leq (\varrho_{r_0,u_0} + \alpha) r_1 + \|N u_0\|_1 + \alpha\|u_0\|_1.
\end{aligned}$$

Hence, the expression in (4.5.8) is bounded above by

$$\|e^{-\alpha t} S_{G_{0,\beta}}(t) u_0 - u_0\|_1 + ((\varrho_{r_0,u_0} + \alpha) r_1 + \|N u_0\|_1 + \alpha\|u_0\|_1) e^{\theta\tau} \tau.$$

If we now define

$$\zeta(\tau) := \frac{1}{r_1} \max_{0 \leq t \leq \tau} \{\|e^{-\alpha t} S_{G_{0,\beta}}(t) u_0 - u_0\|_1\} + \frac{1}{r_1} ((\varrho_{r_0,u_0} + \alpha) r_1 + \|N u_0\|_1 + \alpha\|u_0\|_1) e^{\theta\tau} \tau,$$

then it follows that

$$\begin{aligned}
&\|(\mathcal{Q}\psi)(t) - u_0\|_1 \leq r_1 \zeta(\tau), \quad \forall \in [0, \tau], \qquad \text{and} \\
&\|\mathcal{Q}\psi - \mathcal{Q}\phi\|_Y \leq \zeta(\tau) \|\psi - \phi\|_Y, \quad \forall \psi, \phi \in \Delta.
\end{aligned}$$

Since $\zeta(\tau) \to 0^+$ as $\tau \to 0^+$, we can choose t_1 so that $0 < \zeta(t_1) < 1$ and then $\mathcal{Q}$ becomes a contractive mapping satisfying $\mathcal{Q}(\Delta) \subset \Delta$. Hence, by the Banach Fixed Point Theorem, there exists a unique solution $u \in \Delta$ of $u = \mathcal{Q}u$ and so the integral equation (4.5.5) has a unique solution $u \in C([0, t_1], X_{1+})$. $\square$

Corollary 4.5.7. *Let the maximal interval of existence of the strict solution u of (4.5.3) be $[0, T_{max})$. Then $u(t) \in X_{1+}$ for all $t \in [0, T_{max})$ whenever $u_0 \in D(G_{0,\beta}) \cap X_{1+}$.*

Proof. To proceed, let

$$\tau_{\max} := \sup\{0 < \tau < T_{max} : \ u(t) \in X_{1+} \text{ for all } t \in [0, \tau]\}.$$

Suppose that $\tau_{\max} < T_{max}$ and consider the semi-linear problem,

$$\frac{dv}{dt}(t) = G_{0,\beta}[v(t)] + N[v(t)], \ t > 0; \ v(0) = u(\tau_{\max}). \tag{4.5.9}$$

The solution of (4.5.9) on $[0, T_{max} - \tau_{\max}]$ is $v(t) = u(t + \tau_{\max})$. Since X_{1+} is closed, $u(\tau_{\max}) \in X_{1+}$ and the previous analysis shows that $u(t + \tau_{\max}) \in X_{1+}$ for sufficiently small t. This contradicts the definition of $\tau_{\max}$ and therefore, $u(t) \in X_{1+}$ for all $t \in [0, T_{max})$. □

4.5.3 Global Existence

To prove the global existence of a strict non-negative solutions to (4.5.3) we shall establish that the local solution cannot blow up in finite time [20].

Lemma 4.5.8. *If $u \in D(G_{0,\beta}) \cap X_{1+}$, then*

$$\int_{x_0}^{\infty} x(G_{0,\beta}u)(x)dx \leq \begin{cases} \|r\|_\infty \|u\|_1, & r^{-1} \textit{is not integrable at } x_0, \\ \\ (x_0\|\beta\|_\infty + \|r\|_\infty)\|u\|_1, & r^{-1} \textit{is integrable at } x_0, \end{cases}$$

Proof. The result follows directly from the honesty of the semigroup $(S_{G_{0,\beta}}(t))_{t\geq 0}$ generated by the operator $G_{0,\beta}$, see Theorems 4.4.2 and 4.4.4. □

Lemma 4.5.9. *If $u \in X_{1+}$, then*

$$\int_{x_0}^{\infty} x(Nu)(x)dx = 0.$$

Proof. Let $u \in X_{1+}$. We have

$$\begin{aligned}
&\int_{x_0}^{\infty}\int_{x_0}^{x-x_0} \chi_U(x) x k(x-y,y)u(x-y)u(y)dydx \\
= &\int_{x_0}^{\infty}\int_{y+x_0}^{\infty} xk(x-y,y)u(x-y)u(y)dxdy \\
= &\int_{x_0}^{\infty}\int_{x_0}^{\infty} (z+y)k(z,y)u(z)u(y)dzdy \\
= &\ 2\int_{x_0}^{\infty}\int_{x_0}^{\infty} xk(x,y)u(x)u(y)dydx,
\end{aligned}$$

where we used the fact that $k(x,y) = k(y,x)$. It follows that

$$\begin{aligned}\int_{x_0}^{\infty} x(Nu)(x)dx =& \frac{1}{2}\int_{x_0}^{\infty} \chi_U(x)\left(\int_{x_0}^{x-x_0} xk(x-y,y)u(x-y)u(y)dy\right)dx \\ &- \int_{x_0}^{\infty}\int_{x_0}^{\infty} xk(x,y)u(x)u(y)dydx \\ =& 0.\end{aligned}$$

□

Theorem 4.5.10. *The abstract Cauchy problem (4.5.3) has a unique, global, non-negative solution u for each $u_0 \in D(G_{0,\beta}) \cap X_{1+}$.*

Proof. Because the local solution u is a non-negative solution of (4.5.3), it follows from the previous two lemmas that

$$\begin{aligned}\frac{d}{dt}\|u(t)\|_1 &= \int_{x_0}^{\infty} x(G_{0,\beta}[u(t)])(x)dx + \int_{x_0}^{\infty} x(N[u(t)])(x)dx \\ &\le \begin{cases} \|r\|_\infty \|u\|_1, & r^{-1}\text{is not integrable at } x_0, \\ (x_0\|\beta\|_\infty + \|r\|_\infty)\|u\|_1, & r^{-1}\text{is integrable at } x_0,\end{cases}\end{aligned}$$

for $0 \le t < T_{max}$. Therefore, by Gronwall's inequality, see Lemma 2.1.8, we get

$$\|u(t)\|_1 \le \begin{cases} \|u_0\|_1\, e^{(\|r\|_\infty t)}, & r^{-1}\text{is not integrable at } x_0, \\ \|u_0\|_1\, e^{(x_0\|\beta\|_\infty + \|r\|_\infty)t}, & r^{-1}\text{is integrable at } x_0,\end{cases}$$

for all $t \in [0, T_{max})$. Consequently, u does not blow up in finite time, which ends the proof. □

Chapter 5

Conclusion

In this thesis, our concern has been to establish the existence of non-negative solution to the mathematical model of phytoplankton as well as to investigate the effects of growth-fragmentation-coagulation on the overall evolution of the phytoplankton population. While analysing the model, we realized that the behaviour of r at $x = x_0$ played a vital role in determining whether the transport equation requires a boundary condition at the left end of the size domain. We have shown in the case of $1/r(x)$ is not integrable at x_0 that the Gronwall's inequality and the assumption on the growth rate, that is, $r \in X_\infty$, are crucial for the generation of the strongly continuous semigroup for the linear part of the problem in X_1. On the other hand, if $1/r(x)$ is integrable at x_0, the boundary condition becomes crucial for the investigation of existence of the solution. Furthermore, thanks to the non-zero size ($x_0 > 0$), the coagulation operator behaves well in X_1, which yields the existence of non-negative solution of the full nonlinear problem.

Depending on the model parameters, the main result we deduce from this work is the honesty of the model in X_1. We have shown that if $1/r$ is not integrable, then any anti-derivative satisfies $\lim_{x \to x_0} R(x) = -\infty$ and then the behaviour of the solution at $x = x_0$ must be controlled by imposing more regularity on the coefficients so that the honesty can be verified, see Theorem 4.4.2. This result showed that if the death rate is essentially bounded close to x_0 then the distribution of the aggregates does not influence the evolution of the total mass of the system. This extends the analysis of honesty in [12, 13, 24]. On the other hand, if $1/r$ is integrable, the boundary condition allows the smallest size aggregates to get again into the system and start growing which yields the honesty of the model, see Theorem 4.4.4. As a result, the overall evolution of the total mass of a phytoplankton community depends only on the growth and death rates of the aggregates. Therefore, the fragmentation and coagulation processes do not influence the evolution of the total mass of the phytoplankton population.

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