Polynomial containment in refinement spaces and wavelets based on local projection operators.

Désirée V. Moubandjo

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Promoter: Prof. J. M. de Villiers
Department of Mathematical Sciences
Stellenbosch University

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Declaration

I, the undersigned, hereby declare that this dissertation contains no material which has been accepted for a degree or diploma by the University of Stellenbosch or any other institution, except by way of background information and duly acknowledged in the dissertation, and that, to the best of my knowledge and belief, this dissertation contains no material previously published or written by another person, except where due acknowledgment is made in the text of the dissertation.

Moubandjo

Désirée V. Moubandjo

7 March 2007

Date

Summary

The study of refinement pairs $(a\,,\phi)$, with a denoting a refinement mask, and where ϕ is the corresponding refinable function, is of fundamental importance in both subdivision and wavelet analysis, as used extensively in the application areas of, respectively, geometric modelling and signal analysis.

In this thesis we start with a given refinement pair $(a\,,\phi)$, and build a theory based entirely on time-domain methods, to eventually yield a wavelet decomposition technique with finite algorithms, and with the ability to efficiently detect local irregularities (or non-smoothness) in a given signal.

After presenting, in Chapter 1, results from the literature on refinement pairs, we proceed in Chapter 2 to prove by means of entirely time-domain methods that, for every resolution level $r \in \mathbb{Z}$, the presence of zero of order N at -1 in the corresponding refinement mask symbol A guarantees that all polynomials of degree $\leq N-1$ are contained in the refinement space $V^{(r)}$ spanned by the integer shifts of $\phi(2^r\cdot)$. In the process, a fundamental identity for the ϕ-commutator operator, as well as a generalised Marsden identity, are obtained. Our results hold for a larger class of refinement pairs than was established before by means of Fourier transform methods where ϕ was assumed to also satisfy properties like integer shift independence, Riesz-stability and/or bounded variation.

A quasi-interpolation operator $\mathcal{Q}_r$ mapping , for every $r \in \mathbb{Z}$, real-valued functions on $\mathbb{R}$ into $V^{(r)}$, such that polynomials in Π_{N-1} are reproduced , is then explicitly constructed in Chapter 3, by using the results of Chapter 2.

Next, in Chapter 4, we characterise a local linear projection operator sequence $\{\mathcal{P}_r : r \in \mathbb{Z}\}$, where $\mathcal{P}_r : V^{(r+1)} \to V^{(r)}$, $r \in \mathbb{Z}$, by means of the Laurent polynomial solution Λ of a certain Bezout identity based on the refinement mask symbol A. In particular, it is shown that, provided A possesses no symmetric zeros, and $A(0) \neq 0$, there does indeed always exist such a Laurent polynomial Λ of minimal length.

Next, in Chapter 5, we define the error space sequence $\{W^{(r)} : r \in \mathbb{Z}\}$ by $W^{(r)} = \{f - \mathcal{P}_r f : f \in V^{(r+1)}\}$, $r \in \mathbb{Z}$, and constructively, by means of solving more Bezout identities, show the existence of a finitely supported function $\psi \in V^{(1)}$ such that, for every $r \in \mathbb{Z}$, $W^{(r)}$ is spanned by the integer shifts of $\psi(2^r \cdot)$. According to our definition, we then call ψ a wavelet. The wavelet decomposition algorithm based on the quasi-interpolation operator $\mathcal{Q}_r$, the projection operator $\mathcal{P}_r$, and the wavelet ψ, is then based on finite sequences, and is shown to possess, for a given signal f, the essential property of yielding relatively small wavelet coefficients in regions where the support interval of $\psi(2^r \cdot -j)$ overlaps with a C^N-smooth region of f.

Finally, in Chapter 7, we show how our theory can be used to deduce the known results on the orthonormal Daubechies wavelets.

The specific example of cardinal B-splines is given prominence throughout. Also, the issue of preservation of symmetry is analysed rigorously. Graphical examples are provided to illustrate the theoretical results.

Opsomming

Die studie van verfyningspare $(a\,,\phi)$, waar a 'n verfyningsmasker aandui, is van fundamentele belang in beide subdivisie en golfie analise, soos wyd gebruik word in die toepassingsgebiede van, onderskeidelik, geometriese modellering en seinanalise.

In hierdie tesis begin ons met 'n gegewe verfyningspaar $(a\,,\phi)$, en bou ons dan 'n teorie wat geheel en al gebaseer is op tydgebied metodes, om uiteindelik 'n golfie dekomposisie tegniek met eindige algoritmes te verkry, en met die vermoë om doeltreffend lokale onreëlmatighede (of nie-gladheid) in 'n gegewe sein op te spoor en uit te wys.

Nadat ons in Hoofstuk 1 resultate uit die literatuur oor verfynbare funksies gee, gaan ons in Hoofstuk 2 voort om, alleenlik met behulp van tydgebied metodes, te bewys dat, vir elke resolusievlak $r \in \mathbb{Z}$, die teenwoordigheid van 'n nulpunt van orde N by -1 in die ooreenkomstige verfyningsmaskersimbool A 'n waarborg is dat alle polinome van graad $\leq N-1$ bevat word in die verfyningsruimte $V^{(r)}$ wat onderspan word deur die heelgetal skuiwe van $\phi(2^r\cdot)$. In die proses word verkry 'n fundamentele identiteit vir die ϕ-kommutator operator, sowel as 'n veralgemeende Marsden identiteit. Ons resultate geld vir 'n groter klas verfynbare pare as wat vantevore daargestel is met behulp van Fourier transform metodes waar ϕ veronderstel was om ook eienskappe soos heelgetal skuif onafhanklikheid, Riesz-stabiliteit en/of begrensde variasie te bevredig.

'n Kwasi-interpolant operator Q_r wat, vir elke $r \in \mathbb{Z}$, reëlwaardige funksies op $\mathbb{R}$ afbeeld in $V^{(r)}$, sodat polinome in Π_{N-1} gereproduseer word, word dan eksplisiet gekonstrueer in Hoofstuk 3, met behulp van Hoofstuk 2 se resultate.

Vervolgens, in Hoofstuk 4, karakteriseer ons 'n lokale lineêre projeksie operator ry $\{\mathcal{P}_r : r \in \mathbb{Z}\}$, waar $\mathcal{P}_r : V^{(r+1)} \to V^{(r)}$, $r \in \mathbb{Z}$, met behulp van 'n Laurent polinoom oplossing Λ van 'n sekere Bezout identiteit gebaseer op die verfyningsmaskersimbool A. In die besonder word getoon dat, indien A geen simmetriese nulpunte besit nie, en $A(0) \neq 0$, dan bestaan daar inderdaad altyd so 'n Laurent polinoom Λ van minimale lengte.

Volgende, in Hoofstuk 5, definieer ons die foutruimte-ry $\{W^{(r)} : r \in \mathbb{Z}\}$ deur $W^{(r)} = \{f - \mathcal{P}_r f \ : \ f \in V^{(r+1)}\}$, $r \in \mathbb{Z}$, waarna ons konstruktief, deur nog Bezout identiteite op te los, die bestaan aantoon van 'n eindig-ondersteunde funksie $\psi \in V^{(1)}$ sodanig dat, vir elke $r \in \mathbb{Z}, W^{(r)}$ onderspan word deur die heelgetal skuiwe van $\psi(2^r\cdot)$. Volgens ons definisie noem ons ψ dan 'n golfie. Die golfie dekomposisie algoritme gebaseer op die kwasi-interpolasie operator $\mathcal{Q}_r$, die projeksie operator $\mathcal{P}_r$, en die golfie ψ, is dan gebaseer op eindige algoritmes, en word dan aangetoon om, vir 'n gegewe sein f, die essensiële eienskap te besit om relatief klein golfie koëffisiënte op te lewer in gebiede waar die steuninterval van $\psi(2^r \cdot -j)$ oorvleuel met 'n C^N-gladde gebied van f.

Ten slotte, in Hoofstuk 7, toon ons aan hoedat ons teorie gebruik kan word om die bekende resulte oor die ortonormale Daubechies golfies af te lei.

Die spesifieke voorbeeld van kardinale B-latfunksies geniet deurentyd prominensie. Verder word die kwessie van die behoud van simmetrie deeglik ondersoek. Grafiese voorbeelde word verskaf om die teoretiese resultate te illustreer.

A MA FILLE ALEXANDRA,

POUR TES ANNEES INVESTIES A CET OUVRAGE ...

Contents

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Preface

In this thesis, our main building block is a so-called refinement pair $(a\,,\phi)$, with $a = \{a_j : j \in \mathbb{Z}\}$ denoting a finitely-supported bi-infinite real-valued sequence, called the refinement mask, and where ϕ is a finitely-supported continuous real-valued function in $\mathbb{R}$, called the refinable function, such that the two-scale functional equation

$$\phi = \sum_{j=-\infty}^{\infty} a_j \phi(2 \cdot -j), \tag{0.1}$$

called the refinement equation, is satisfied.

In the mathematical analysis of both subdivision and wavelets, with their many and varied applications in, respectively, geometric modelling and signal analysis, the existence and construction of refinement pairs play a fundamental role. In particular, the basic questions of, given a finitely supported sequence $a = \{a_j : j \in \mathbb{Z}\}$, whether there exists a corresponding finitely-supported continuous solution ϕ of the refinement equation (0.1), and if so, to determine the degree of smoothness, and other properties, of ϕ, as well as an efficient computational method for ϕ, have produced a vast literature over the last two decades, with in particular noteworthy pioneering work by Ingrid Daubechies and various collaborators (see e.g. **[21]**, **[20]**, **[22]**, **[23]**).

In Chapter 1, we present an introduction to refinement pairs, in particular focussing on those results on which we shall rely in the rest of the thesis. The examples we mention, and continue to use in the rest of the thesis, are the cardinal B-splines, the Daubechies orthonormal refinement pairs, as well as a certain two-parameter family of refinement pairs.

For a given refinement pair (a, ϕ), it is important in wavelet decomposition applications to obtain the largest possible value of the positive integer N such that the polynomial containment result

$$\Pi_{N-1} \subset V^{(r)}, \quad r \in \mathbb{Z}, \tag{0.2}$$

holds, with Π_{N-1} denoting the set of polynomials of degree $\leq N-1$, and where, for every $r \in \mathbb{Z}$, $V^{(r)}$ is the linear span of the integer shift sequence $\{\phi(2^r \cdot -j) : j \in \mathbb{Z}\}$, so that also $\{V^{(r)} r \in \mathbb{Z}\}$ is a nested sequence of spaces, by virtue of the fact that ϕ is a refinable function.

Previous work (see e.g. **[6]**, **[3]**, **[4]**, **[19]**, **[30]**, **[55]**) based on Fourier transform techniques has shown that, provided ϕ possesses linearly independent shifts, or satisfies a Riesz-stability condition, the condition (0.2) holds if and only if the Laurent polynomial A defined by $A(z) = \sum_j a_j z^j$, $z \in \mathbb{C} \setminus \{0\}$, known as the corresponding refinement mask symbol, has a zero of order N at -1, or, equivalently, if ϕ satisfies the Strang-Fix condition of order N.

In our Chapter 2, we show that, by working entirely in the time domain, and specifically not using Fourier transform methods, the same "zero of order N at -1" condition on A is sufficient for the polynomial containment result (0.2) to hold, even if ϕ satisfies no integer shift linear independence, or Riesz-stability property. Hence our results enlarge the class of refinement pairs (a, ϕ) for which it is known that the above mentioned zero property of A guarantees (0.2). In the process, we also show, in Section 2.4, that the fundamental identity

$$\sum_j p(j)\phi(\cdot - j) = \sum_j \phi(j)p(\cdot - j), \quad p \in \Pi_{N-1}, \tag{0.3}$$

holds for a larger class of refinable functions than was obtained in previous work (see e.g. **[14]**, **[6]**), where the Poisson summation formula was applied to the function ϕ, the validity of which depends on ϕ to also satisfy, e.g., the property of bounded variation (see e.g. **[7]**), which is not required by our approach. Also, we do not require the subdivision scheme corresponding to the refinement mask to be convergent.

Moreover, we use (0.3) to establish, in Section 2.5, a generalised Marsden identity, which now also holds for a larger class of refinement pairs (a, ϕ) than can be deduced from the similar result in [3], where ϕ was required to possess linearly independent integer shifts. The resulting expansions for the monomials in Π_{N-1}, as given in our Corollary 2.6, then immediately yields the polynomial containment result (0.2).

Using the results of Chapter 2, we show in Chapter 3, for every $r \in \mathbb{Z}$, the explicit construction of a quasi-interpolation operator $\mathcal{Q}_r$ mapping the real-valued functions on $\mathbb{R}$ into $V^{(r)}$ in such a way that polynomials in Π_{N-1} are reproduced, as is useful in a variety of approximation contexts. The fundamental property of exactness in reproducing polynomials of quasi-interpolants allows a large range of constructions in a space containing polynomials as seen e.g. in **[56]**, **[41]**, **[43]**, **[53]**, **[17]**, as well as **[2]**. Our quasi-interpolation operator $\mathcal{Q}_r$ can be used as the first step in the wavelet decomposition algorithm of a given signal, as described in Section 6.1. Although we do not subject $\mathcal{Q}_r$ to a rigorous error analysis in this thesis, our numerical results for a specific example, as illustrated graphically in Section 3.2, supports the fact that $\mathcal{Q}_r$ has optimal approximation order for $r \to \infty$, as could be shown, e.g., by means of a Peano kernel type argument.

In Chapters 4 to 6, we introduce a general multi-resolutional-like framework, based on local projection operators, for the construction of a wavelet decomposition technique with the basic feature detection property of being able to locally detect irregular (or non-smooth, or non-polynomial-like) behaviour in a given signal f.

To this end, we first characterise, in Theorem 4.1, a local projection operator sequence $\{\mathcal{P}_r : r \in \mathbb{Z}\}$, with $\mathcal{P}_r : V^{(r+1)} \to V^{(r)}$, $r \in \mathbb{Z}$, in terms of a Laurent polynomial Λ satisfying the Bezout identity

$$A(z)\Lambda(z) + A(-z)\Lambda(-z) = 2, \quad z \in \mathbb{C} \setminus \{0\}. \tag{0.4}$$

Using standard results from polynomial algebra, we proceed to prove, in Theorem 4.3 and

Corollary 4.4, the existence of a Laurent polynomial Λ of minimal length satisfying (0.4), provided only that A possesses no symmetric zeros, and $A(0) \neq 0$. Our method of proof in fact provides an explicit construction method, as given by Algorithm 4.1, based on the Euclidean algorithm, for Λ.

Devoting special attention to the case of the cardinal B-spline refinement pair of order m, we show, in Section 4.3, that the associated Laurent polynomial $\Lambda = \Lambda_m$ can for this case be efficiently computed recursively with respect to the spline order m.

For a refinement pair $(a\,,\phi)$, and a local projection operator sequence $\{\mathcal{P}_r : r \in \mathbb{Z}\}$, with $\mathcal{P}_r : V^{(r+1)} \rightarrow V^{(r)}$, $r \in \mathbb{Z}$, as in Theorem 4.1, we proceed to define, in Section 5.1, the linear space sequence $\{W^{(r)} : r \in \mathbb{Z}\}$ by $W^{(r)} = \{f - \mathcal{P}_r f : f \in V^{(r+1)}\}$, $r \in \mathbb{Z}$, i.e. $W^{(r)}$ can be interpreted as carrying the error with respect to the projection operator $\mathcal{P}_r$.
We then define a wavelet ψ as a finitely supported continuous function in $V^{(1)}$ such that, for every $r \in \mathbb{Z}$, the integer shift sequence $\{\psi(2^r \cdot -j)\}$ spans the linear space $W^{(r)}$.

After showing that, for every $r \in \mathbb{Z}$, $f \in W^{(r)}$ if and only if $\mathcal{P}_r f = 0$, we characterise, in Section 5.2, the function ψ of least possible support in $V^{(1)}$ such that $\mathcal{P}_0\psi = 0$, in terms of a Bezout identity, which we then proceed to solve explicitly. The function ψ thus obtained, as described in Theorem 5.4, is then proved, by means of the basic decomposition result of Theorem 5.6, to be indeed a wavelet, as defined above.

Next, in Chapter 6, we formulate our wavelet decomposition algorithm based on a given refinement pair $(a\,,\phi)$, the quasi-interpolation operator $\mathcal{Q}_r$, the local linear projection operator sequence $\mathcal{P}_r$, and the resulting wavelet ψ. Of particular importance is Theorem 6.1, from which we can deduce that, for every $r\,,\, j \in \mathbb{Z}$, the coefficient of the wavelet $\psi(2^r \cdot -j)$ in the wavelet expansion of a signal f can be expected to be relatively small if the support interval of $\psi(2^r \cdot -j)$ overlaps with a region where f exhibits C^N-smooth (or Π_{N-1} polynomial-like) behaviour, and thereby confirming the above mentioned local feature detection property of our wavelet ψ.

In the literature on wavelet construction methods based on multi-resolutional analyses (see e.g. **[51]**, **[24]**, **[38]**, **[57]**, **[60]**, **[36]**, **[4]**, **[5]**, **[11]**, **[10]**, **[12]**, **[13]**, **[14]**, **[16]**), and in which Fourier transform methods are mostly used, related conditions like integer shift linear independence, Riesz-stability, and/or orthogonality are often imposed on either the refinement pair $(a\,,\phi)$ or the generating linear spaces. The purely time-domain approach employed in this thesis shows that a finite wavelet decomposition algorithm with the essential property of localised feature detection can in fact be obtained without demanding any type of stability for our generating refinement pair $(a\,,\phi)$.

It should be emphasized that our wavelet decomposition algorithm uses only the *finite* sequences obtained from A and Λ. As a special case, we therefore also obtain cardinal spline wavelet decomposition algorithms that are finite, as opposed to those obtained from the bi-orthogonal cardinal spline wavelets of Chui and Wang (see **[16]**, **[9]**, **[10]**), where, due to the fact that the spline wavelet was chosen there to be orthogonal to the cardinal B-spline itself, an infinite decomposition algorithm was obtained, with the subsequent complication in practical applications of having to resort to truncation of sequences.

Special attention is devoted in Chapters 4 to 6 to the issue of symmetry, in the sense that it is consistently investigated whether, if the refinement mask symbol A is a symmetric polynomial, some form of symmetry is preserved by the Laurent polynomial Λ of minimal length, and the resulting wavelet ψ of least possible support.

Throughout Chapters 3 to 6, numerical examples, including the cardinal B-spline cases of orders 2 to 5, are used to graphically illustrate the theoretical results.

Finally, in Chapter 7, we show how our results of Chapters 4 to 6 can be used to deduce the well-known results on Daubechies wavelet decomposition.

Regarding our notation: the symbols we use are defined either in the "List of Symbols" section, or in the main text of the thesis.

List of symbols

Symbol	Meaning
$\mathbb{N}$	the set of natural numbers.
$\mathbb{Z}$	the set of integers.
$\mathbb{Z}_+$	the set of non-negative integers.
$\mathbb{R}$	the set of real numbers.
$\mathbb{R}_+$	the set of non-negative real numbers.
$\mathbb{R}_-$	the set of negative real numbers.
$\mathbb{C}$	the set of complex numbers.
Π_k	the linear space of polynomials of degree $\leq k, \quad k \in \mathbb{Z}_+$.
$\sum\limits_j$	$= \sum\limits_{j\in\mathbb{Z}}$
$\sup\limits_x$	$= \sup\limits_{x\in\mathbb{R}}$
$\mathcal{M}(\mathbb{Z})$	the space of bi-infinite real-valued sequences.
$\mathcal{M}(\mathbb{R})$	the space of real-valued functions on $\mathbb{R}$.
$\mathcal{M}_0(\mathbb{Z})$	the space of sequences $c \in \mathcal{M}(\mathbb{Z})$ such that c is finitely supported, i.e. if $c = \{c_j : j \in \mathbb{Z}\}$, then there exist integers J and K, with $J \leq K$, such that $c_j = 0\,,\ j \notin \{J, \ldots, K\}$.
$\mathcal{C}(\mathbb{R})$	$\{f \in \mathcal{M}(\mathbb{R}) : f \text{ is continuous on } \mathbb{R}\}$.
$\mathcal{C}_{\mathrm{u}}(\mathbb{R})$	$\{f \in \mathcal{C}(\mathbb{R}) : f \text{ is uniformly bounded on } \mathbb{R}\}$.
$\mathcal{C}_0(\mathbb{R})$	the space of functions $f \in \mathcal{C}(\mathbb{R})$ such that f is finitely supported, i.e. there exists a bounded interval $[\alpha, \beta] \subset \mathbb{R}$, such that $f(x) = 0\,,\ x \notin [\alpha, \beta]$.
$\mathcal{C}^k(\mathbb{R})$	$\{f \in \mathcal{M}(\mathbb{R}) : f^{(k)} \in \mathcal{C}(\mathbb{R})\}\,, \quad k \in \mathbb{N}$
$\mathcal{C}_0^k(\mathbb{R})$	$\mathcal{C}^k(\mathbb{R}) \bigcap \mathcal{C}_0(\mathbb{R})$.
$\\| \ \\|_\infty$	the sup norm $\\|f\\|_\infty = \sup\limits_x \|f(x)\|\,,\ f \in \mathcal{C}_{\mathrm{u}}(\mathbb{R})$.

deg	the degree of a polynomial.
$\delta_{k,j}$	the Kronecker symbol, $\delta_{k,j} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \quad j, k \in \mathbb{Z}$
δ_j	$= \delta_{j,0}, \quad j \in \mathbb{Z}$
$\binom{n}{j}$	the binomial coefficient, with the convention that, for $n \in \mathbb{Z}_+$, $j \in \mathbb{Z}$, we have $\binom{n}{j} = 0, \quad$ if $j > n$ or $j < 0$.
$\lceil x \rceil$	the smallest integer larger than x
$\lfloor x \rfloor$	the largest integer smaller than x
(a, ϕ)	a refinement pair
a	the refinement mask sequence in (a, ϕ)
ϕ	the refinable function in (a, ϕ)
A	the refinement mask symbol polynomial $A(z) = \sum_j a_j z^j = \sum_{j=0}^{n} a_j z^j$, $z \in \mathbb{C}$
n	the degree of A
N	the order of the zero at -1 of A
$V^{(r)}$	the linear space $V^{(r)} = \left\{ \sum_j c_j \phi(2^r \cdot -j) : c \in \mathcal{M}(\mathbb{R}) \right\}$
$\mathcal{Q}_r$	the quasi-interpolation operator mapping $\mathcal{M}(\mathbb{R})$ into $V^{(r)}$
$\mathcal{P}_r$	the local projection operator mapping $V^{(r+1)}$ into $V^{(r)}$
$W^{(r)}$	$\{f - \mathcal{P}_r f \, : \, f \in V^{(r+1)}\}$
ψ	a wavelet

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Chapter 1

Refinement pairs

1.1 Preliminaries

If a sequence $a \in \mathcal{M}_0(\mathbb{Z})$ and a function $\phi \in C_0(\mathbb{R})$, with $\phi \neq 0$, satisfy the equation

$$\phi = \sum_j a_j \phi(2 \cdot -j)\,, \tag{1.1}$$

we shall say that $(a\,, \phi)$ is a *refinement pair*; the sequence a is the *refinement mask* , the function ϕ is the *refinable function* and the equation (1.1) is called the *refinement equation.*

For a refinement pair $(a\,, \phi)$, since $a \in \mathcal{M}_0(\mathbb{Z})$, there exist integers n_1 and n_2, with $n_2 > n_1$, such that $a_j = 0,\ j \notin \{n_1, \ldots, n_2\}$, with $a_{n_1} \neq 0\,,\ a_{n_2} \neq 0$. Now define the sequence $\tilde{a} \in \mathcal{M}_0(\mathbb{Z})$ by $\tilde{a}_j = a_{j+n_1}\,,\ j \in \mathbb{Z}$, so that $\tilde{a}_j = 0,\ j \notin \{0, 1, \ldots, n_2 - n_1\}$, with $\tilde{a}_0 \neq 0\,,\ \tilde{a}_{n_2-n_1} \neq 0\,,$ and let the function $\tilde{\phi} \in C_0(\mathbb{R})$ be defined by $\tilde{\phi} = \phi(\cdot + k)$, where $k \in \mathbb{R}$, so that also $\phi = \tilde{\phi}(\cdot - k)$. Then (1.1) can be used to obtain

$$\begin{aligned}
\sum_j \tilde{a}_j \tilde{\phi}(2 \cdot -j) &= \sum_j a_{j+n_1} \phi(2 \cdot -j + k) \\
&= \sum_j a_j \phi(2 \cdot +n_1 + k - j) \\
&= \phi\left(\cdot + \frac{n_1 + k}{2}\right) = \tilde{\phi}\left(\cdot + \frac{n_1 - k}{2}\right).
\end{aligned}$$

It follows that if we choose $k = n_1$, so that $\tilde{\phi} = \phi(\cdot + n_1)$, then

$$\tilde{\phi} = \sum_j \tilde{a}_j \tilde{\phi}(2 \cdot -j) = \sum_{j=0}^{n_2-n_1} \tilde{a}_j \tilde{\phi}(2 \cdot -j)\,,$$

i.e. $(\tilde{a}, \tilde{\phi})$ is a refinement pair.

Throughout this thesis we shall therefore assume, without loss of generality, that if (a, ϕ) *is a refinement pair, then, for an integer* $n \in \mathbb{N}$, *we have*

$$a_j = 0\,, \quad j \notin \{0, 1, \ldots, n\}\,, \tag{1.2}$$

with also

$$a_0 \neq 0 \quad , \quad a_n \neq 0\,. \tag{1.3}$$

For a given refinement pair $(a\,,\phi)$, we define the *refinement mask symbol* A by

$$A(z) = \sum_j a_j z^j = \sum_{j=0}^{n} a_j z^j\,, \quad z \in \mathbb{C}\,. \tag{1.4}$$

Note from (1.2), (1.3) and (1.4) that A is a polynomial of degree n, with

$$A(0) \neq 0\,. \tag{1.5}$$

The following necessary conditions for the existence of a refinement pair are immediate consequences of results in [**22**].

Theorem 1.1. *For* $n \in \mathbb{N}$, *suppose that* $(a\,,\phi)$ *is a refinement pair. Then the following hold:*

(a) $n \geq 2$;

(b) ϕ *is unique up to a multiplicative constant;*

(c) there exists an integer $k \in \mathbb{N}$ *such that*

$$A(1) = 2^k\,; \tag{1.6}$$

(d) if $k \geq 2$ *in (c), then there exists a function* $\tilde{\phi} \in C_0^{k-1}(\mathbb{R})$ *such that* $(2^{1-k}a\,,\tilde{\phi})$ *is a refinement pair, with* $\tilde{\phi}^{(k-1)} = \phi$;

(e)

$$\phi(x) = 0\,, \quad x \notin (0, n)\,; \tag{1.7}$$

(f)

$$\int_{-\infty}^{\infty} \phi(x)dx \neq 0$$

if and only if $k = 1$ *in* (c).

In view of Theorem 1.1 (a), (c) and (d), *we shall furthermore assume throughout this thesis that, if* $(a\,,\phi)$ *is a given refinement pair, then* $n \geq 2$, *and*

$$A(1) = \sum_j a_j = 2\,. \tag{1.8}$$

1.2 Refinement spaces

For a given refinement pair $(a\,,\phi)$, the *refinement space sequence* $\{V^{(r)} = V_\phi^{(r)} : r \in \mathbb{Z}\}$ defined by

$$V^{(r)} = \left\{ \sum_j c_j \phi(2^r \cdot -j) : c \in \mathcal{M}(\mathbb{R}) \right\}, \quad r \in \mathbb{Z}\,, \tag{1.9}$$

is a nested sequence of spaces, as proven below.

Theorem 1.2. *Let* $(a\,,\phi)$ *denote a refinement pair. Then*

$$V^{(r)} \subset V^{(r+1)}, \quad r \in \mathbb{Z}\,, \tag{1.10}$$

where $\{V^{(r)} : r \in \mathbb{Z}\}$ *is the refinement space sequence in (1.9).*

Proof. For $r \in \mathbb{Z}$, suppose $f \in V^{(r)}$. Then there exists a sequence $c \in \mathcal{M}(\mathbb{Z})$ such that $f = \sum\limits_j c_j \phi(2^r \cdot -j)$. Using (1.1), we obtain

$$\begin{aligned} f &= \sum_j c_j \sum_k a_k \phi(2^{r+1} \cdot -2j - k) \\ &= \sum_j c_j \sum_k a_{k-2j} \phi(2^{r+1} \cdot -k) \\ &= \sum_k \left[\sum_j a_{k-2j} c_j \right] \phi(2^{r+1} \cdot -k)\,, \end{aligned}$$

and thus $f = \sum\limits_k d_k \phi(2^{r+1} \cdot -k)$, with the sequence $d \in \mathcal{M}(\mathbb{Z})$ defined by $d_k = \sum\limits_j a_{k-2j} c_j$, $k \in \mathbb{Z}$. Hence $f \in V^{(r+1)}$, and it follows that (1.10) holds. ∎

1.3 A denseness result

In practical wavelet decomposition applications, it is important that, if $f \in \mathcal{C}_0(\mathbb{R})$, then f can be approximated arbitrarily well in the $||\cdot||_\infty$ norm by a function in $V^{(r)}$, with r sufficiently large. Our following result gives a sufficient condition on the refinable function ϕ in the refinement pair $(a\,,\phi)$ for this denseness result to hold.

Theorem 1.3. *Suppose $(a\,,\phi)$ is a refinement pair such that the partition of unity condition*

$$\sum_j \phi(x-j) = 1\,, \quad x \in \mathbb{R}\,, \tag{1.11}$$

is satisfied. Then, if $f \in \mathcal{C}_0(\mathbb{R})$, there exists a sequence $\{f_r : r \in \mathbb{Z}_+\}$, with $f_r \in V^{(r)}$, $r \in \mathbb{Z}_+$, such that

$$||f - f_r||_\infty \to 0\,, \quad r \to \infty\,. \tag{1.12}$$

Proof. We shall rely on the Schoenberg-type operator sequence $\{\mathcal{V}_r : \mathcal{M}(\mathbb{R}) \to V^{(r)}\,,\ r \in \mathbb{Z}\}$ (see e.g. **[53]**, **[46]**, p. 64 and **[29]**, Corollary 5.9, for the case of cardinal splines), as defined by

$$\mathcal{V}_r f = \sum_j f\left(\frac{j+n/2}{2^r}\right)\phi(2^r\cdot -j)\,, \quad r \in \mathbb{Z}\,, \quad f \in \mathcal{M}(\mathbb{R})\,. \tag{1.13}$$

Let $f \in \mathcal{C}_0(\mathbb{R})$, and choose $\varepsilon > 0$. Our proof will be complete if we can show that there exists an integer $R(\varepsilon) \in \mathbb{N}$ such that

$$||f - \mathcal{V}_r f||_\infty \leq \varepsilon\,, \quad r \geq R(\varepsilon)\,, \tag{1.14}$$

for then $||f - \mathcal{V}_r f||_\infty \to 0\,,\ r \to \infty$, and the desired result (1.12) follows by choosing $f_r = \mathcal{V}_r f\,,\ r \in \mathbb{Z}_+$.

To this end, we first note that, since $f \in \mathcal{C}_0(\mathbb{R})$, we also know that f is uniformly continuous on $\mathbb{R}$, i.e. there exists a positive number $\delta(\varepsilon)$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{n||\phi||_\infty} \text{ for all } x, y \in \mathbb{R} \text{ such that } |x-y| < \delta(\varepsilon)\,, \tag{1.15}$$

and where δ is independent of x and y.

Let $x \in \mathbb{R}$ and $r \in \mathbb{Z}_+$ be fixed, and denote by k the (unique) integer such that $x \in$

$\left[\frac{k}{2^r}, \frac{k+1}{2^r}\right)$. Using (1.13) and (1.11), as well as (1.7), we obtain

$$\begin{aligned}
|f(x) - (\mathcal{V}_r f)(x)| &= \left| \sum_j \left[f(x) - f\left(\frac{j+\frac{n}{2}}{2^r}\right)\right] \phi(2^r x - j) \right| \\
&= \left| \sum_{j=k-n+1}^{k} \left[f(x) - f\left(\frac{j+\frac{n}{2}}{2^r}\right)\right] \phi(2^r x - j) \right| \\
&\leq \|\phi\|_\infty \sum_{j=k-n+1}^{k} \left| f(x) - f\left(\frac{j+\frac{n}{2}}{2^r}\right)\right| . \qquad (1.16)
\end{aligned}$$

Now observe that, since $x \in \left[\frac{k}{2^r}, \frac{k+1}{2^r}\right)$, we have, for $j \in \{k-n+1, \ldots, k\}$,

$$-\frac{n}{2^{r+1}} = \frac{k}{2^r} - \frac{k+\frac{n}{2}}{2^r} \leq x - \frac{j+\frac{n}{2}}{2^r} \leq \frac{k+1}{2^r} - \frac{k-n+1+\frac{n}{2}}{2^r} = \frac{n}{2^{r+1}},$$

and thus

$$\left| x - \frac{j+\frac{n}{2}}{2^r} \right| \leq \frac{n}{2^{r+1}}, \quad j = k-n+1, \ldots, k. \qquad (1.17)$$

Hence, if we choose the integer $R(\varepsilon)$ sufficiently large such that

$$R(\varepsilon) > \log_2 \left(\frac{n}{2\delta(\varepsilon)} \right),$$

it follows from (1.15), (1.16) and (1.17) that, for $r \geq R(\varepsilon)$, we have

$$|f(x) - (\mathcal{V}_r f)(x)| < \frac{\varepsilon}{n\|\phi\|_\infty} \|\phi\|_\infty (k - (k-n+1) + 1) = \varepsilon .$$

Since the positive number ε is independent of k, it follows that

$$\sup_x |f(x) - (\mathcal{V}_r f)(x)| \leq \varepsilon, \quad r \geq R(\varepsilon),$$

thereby yielding the desired result (1.14). ■

1.4 Sufficient mask conditions for denseness

We proceed to establish a sufficient condition on a given refinement pair (a, ϕ) for the partition of unity result (1.11) to hold. We shall need the following preliminary results.

Proposition 1.4. *Let (a, ϕ) be a refinement pair. Then*

$$\sum_j \phi\left(\frac{j}{2^r}\right) = 2^r \sum_j \phi(j), \quad r \in \mathbb{Z}_+ . \qquad (1.18)$$

Proof. We prove (1.18) by induction. After noting that (1.18) trivially holds for $r = 0$, suppose next that (1.18) holds for a fixed integer $r \in \mathbb{Z}_+$. Then (1.1) gives

$$\begin{aligned}
\sum_j \phi\left(\frac{j}{2^{r+1}}\right) &= \sum_j \sum_k a_k\, \phi\left(\frac{j}{2^r} - k\right) \\
&= \sum_j \sum_{l=0}^{2^r-1} \sum_k a_k\, \phi\left(\frac{2^r j + l}{2^r} - k\right) \\
&= \sum_{l=0}^{2^r-1} \sum_k a_k \sum_j \phi\left(\frac{l}{2^r} + j - k\right) \\
&= \sum_{l=0}^{2^r-1} \sum_k a_k \sum_j \phi\left(\frac{l}{2^r} + j\right) \\
&= \left[\sum_k a_k\right]\left[\sum_j \sum_{l=0}^{2^r-1} \phi\left(\frac{l}{2^r} + j\right)\right] \\
&= 2\sum_j \phi\left(\frac{j}{2^r}\right) = 2^{r+1}\sum_j \phi(j)\,,
\end{aligned}$$

thereby concluding our inductive proof. ■

The following relationship can now be proven.

Theorem 1.5. *If $(a\,, \phi)$ is a refinement pair, then*

$$\int_{-\infty}^{\infty} \phi(x)dx = \sum_j \phi(j)\,. \tag{1.19}$$

Proof. Using Proposition 1.4, and Theorem 1.1(e), we deduce that

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(x)dx = \int_0^n \phi(x)dx &= \lim_{r\longrightarrow\infty} \frac{1}{2^r} \sum_{j=0}^{n2^r-1} \phi\left(\frac{j}{2^r}\right) \\
&= \lim_{r\longrightarrow\infty} \frac{1}{2^r} \sum_j \phi\left(\frac{j}{2^r}\right) \\
&= \lim_{r\longrightarrow\infty} \left[\sum_j \phi(j)\right] = \sum_j \phi(j)\,.
\end{aligned}$$

■

Combining Theorem 1.5 and Theorem 1.1(f), we immediately obtain the following result.

Corollary 1.6. *If $(a\,,\phi)$ is a refinement pair, then*

$$\sum_j \phi(j) \neq 0\,.$$

In view of Theorem 1.1(b) and Corollary 1.6, *we henceforth assume in this thesis that the normalising condition*

$$\sum_j \phi(j) = 1 \tag{1.20}$$

holds.

We shall also rely on the following relationship between a sequence $a \in \mathcal{M}_0(\mathbb{Z})$ and its corresponding Laurent polynomial A.

Proposition 1.7. *Suppose $a \in \mathcal{M}_0(\mathbb{Z})$, and let the Laurent polynomial A be defined by $A(z) = \sum_j a_j z^j\,,\ z \in \mathbb{C} \setminus \{0\}$. Then the following statements are equivalent:*

(a) the sum rules

$$\sum_j a_{2j} = \sum_j a_{2j+1} = 1, \tag{1.21}$$

hold;

(b) the condition (1.8) and

$$A(-1) = 0 \tag{1.22}$$

are satisfied;

(c)

$$\sum_j a_{k-2j} = 1\,, \quad k \in \mathbb{Z}\,. \tag{1.23}$$

Proof. The equivalence of (a) and (b) follows immediately from the equations

$$\left.\begin{aligned} A(1) &= \sum_j a_{2j} + \sum_j a_{2j+1}\,, \\ A(-1) &= \sum_j a_{2j} - \sum_j a_{2j+1}\,. \end{aligned}\right\}$$

Next, suppose that (c) holds. The sum rules (1.21) are then obtained by setting $k = 0$ and $k = 1$ in (1.23), i.e. (c) implies (a).

It remains to show that (a) implies (c). To this end, suppose first $k = 2l$, with $l \in \mathbb{Z}$. Then

$$\sum_j a_{k-2j} = \sum_j a_{2l-2j} = \sum_j a_{2j} = 1\,,$$

from (1.21). Similarly, if $k = 2l + 1$, with $l \in \mathbb{Z}$, then

$$\sum_j a_{k-2j} = \sum_j a_{2l+1-2j} = \sum_j a_{2j+1} = 1\,,$$

again from (1.21), and thereby completing the proof of (1.23). ■

Next, we prove an important implication for a refinement pair satisfying the sum rules (1.21).

Theorem 1.8. *Let $(a\,,\phi)$ denote a refinement pair such that the sum rules (1.21) hold. Then*

$$\sum_j \phi(x-j) = \sum_j \phi(j)\,, \quad x \in \mathbb{R}\,. \tag{1.24}$$

Proof. Since the dyadic set $\{k/2^r : k \in \mathbb{Z}\,,\ r \in \mathbb{Z}_+\}$ is dense in $\mathbb{R}$, and $\phi \in C_0(\mathbb{R})$, so that $\sum_j \phi(\cdot - j) \in C(\mathbb{R})$, it will suffice to prove that

$$\sum_j \phi\left(\frac{k}{2^r} - j\right) = \sum_j \phi(j)\,, \quad k \in \mathbb{Z}\,,\ r \in \mathbb{Z}_+\,. \tag{1.25}$$

To this end, we let $k \in \mathbb{Z}\,,\ r \in \mathbb{Z}_+$ and repeatedly use (1.1), together with the equivalence of (a) and (c) in Proposition 1.7, to obtain

$$\begin{aligned}\sum_j \phi\left(\frac{k}{2^r}-j\right) &= \sum_j\sum_l a_l\phi\left(\frac{k}{2^{r-1}}-2j-l\right)\\ &= \sum_j\sum_l a_{l-2j}\phi\left(\frac{k}{2^{r-1}}-l\right)\\ &= \sum_l\left[\sum_j a_{l-2j}\right]\phi\left(\frac{k}{2^{r-1}}-l\right)\\ &= \sum_l \phi\left(\frac{k}{2^{r-1}}-l\right)\\ &= \ldots = \sum_l \phi(k-l) = \sum_j \phi(j),\end{aligned}$$

thereby proving the desired result (1.25). ∎

We henceforth assume in this thesis that, if $(a\,,\phi)$ is a refinement pair, then the sum rules (1.21) are satisfied.

The following result then follows immediately from Theorem 1.8 and the assumption (1.20).

Theorem 1.9. *If $(a\,,\phi)$ is a refinement pair, then the partition of unity property (1.11) of Theorem 1.3 holds.*

In summary, *we shall assume in this thesis that if $(a\,,\phi)$ is a refinement pair, then the refinement mask a satisfies (1.2) and (1.3) for an integer $n \geq 2$, as well as the sum rules (1.21), whereas, the corresponding refinable function ϕ satisfies (1.7), (1.11), (1.19) and (1.24).*

1.5 An existence result

An example of an existence result for a refinement pair $(a\,,\phi)$ is given by the following theorem on positive masks, for the proof of which we refer to **[48]** and **[49]**, (see also **[6]** for an extension to the multivariate case).

Theorem 1.10. *Suppose $a \in \mathcal{M}_0(\mathbb{Z})$ is a sequence satisfying, for an integer $n \geq 2$, the properties (1.2), (1.3) and (1.21), as well as the positivity condition*

$$a_j > 0\,, \quad j = 0, 1, \ldots, n\,. \tag{1.26}$$

Then there exists a function $\phi \in C_0(\mathbb{R})$ such that $(a\,,\phi)$ is a refinement pair. Moreover, ϕ satisfies the positivity property

$$\phi(x) > 0\,, \quad x \in (0\,,n)\,. \tag{1.27}$$

1.6 Symmetric refinement pairs

Our next result shows that, for a refinement pair $(a\,,\phi)$, the property of symmetry in the refinement mask a is preserved by the corresponding refinable function ϕ.

Theorem 1.11. *For a refinement pair $(a\,,\phi)$, suppose the refinement mask a is symmetric in the sense that*

$$a_{n-j} = a_j\,, \quad j \in \mathbb{Z}\,. \tag{1.28}$$

Then ϕ satisfies the symmetry condition

$$\phi(n - \cdot) = \phi\,. \tag{1.29}$$

Proof. Let $\tilde{\phi} = \phi(n-\cdot)$. We then have, from (1.28), together with the refinement equation (1.1),

$$\begin{aligned}
\sum_j a_j \tilde{\phi}(2\cdot -j) &= \sum_j a_{n-j}\tilde{\phi}(2\cdot -j) \\
&= \sum_j a_j \tilde{\phi}(2\cdot + j - n) \\
&= \sum_j a_j \phi(n - (2\cdot + j - n)) \\
&= \sum_j a_j \phi(2(n - \cdot) - j) \\
&= \phi(n - \cdot) = \tilde{\phi}\,,
\end{aligned}$$

whereas, since also Theorem 1.9 shows that (1.11) holds,

$$\sum_j \tilde{\phi}(j) = \sum_j \phi(n-j) = \sum_j \phi(j) = 1\,.$$

From the uniqueness result of Theorem 1.1(*b*), there exists a non-zero real number c such that $\tilde{\phi} = c\phi$, and thus

$$1 = \sum_j \tilde{\phi}(j) = c \sum_j \phi(j) = c\,,$$

so that $c = 1$ and therefore $\phi(n - \cdot) = \tilde{\phi} = \phi$. ■

1.7 The cardinal B-splines

An example of a refinement pair as in Theorem 1.10 is provided by the cardinal B-splines $\{N_m : m \in \mathbb{N}\}$, as defined recursively by

$$\left.\begin{array}{l} N_1(x) = \begin{cases} 1, & x \in [0\,,1)\,, \\ 0, & x \in \mathbb{R} \setminus [0\,,1)\,, \end{cases} \\ \\ N_{m+1} = \displaystyle\int_0^1 N_m(\cdot - t)dt\,, \quad m \in \mathbb{N}\,. \end{array}\right\} \tag{1.30}$$

As shown in [9], Chapter 4, the sequence $\{N_m : m \in \mathbb{N}\}$ satisfies the recursive formula

$$N_{m+1} = \frac{\cdot}{m} N_m + \frac{m+1-\cdot}{m} N_m(\cdot - 1)\,, \quad m \in \mathbb{N}\,. \tag{1.31}$$

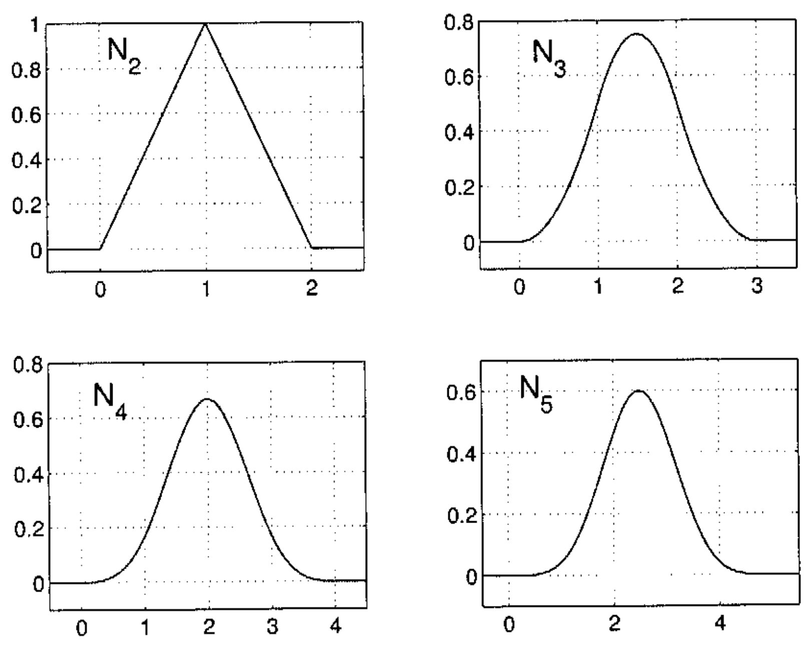


Figure 1.1: The cardinal B-splines $\{N_m : m = 2, \ldots, 5\}$

In Figure 1.1, we have drawn the cardinal B-splines N_m, $m = 2,3,4,5$, by means of (1.31), together with the explicit formula

$$N_2(x) = \begin{cases} x\,, & 0 \le x \le 1\,, \\ 2-x\,, & 1 \le x \le 2\,, \\ 0\,, & x \in \mathbb{R} \setminus [0\,,2)\,. \end{cases} \tag{1.32}$$

It is also shown in [9], Chapter 4, that, for each integer $m \ge 1$, the function N_m is a finitely-supported piecewise polynomial of degree at most $(m-1)$, with breakpoints at $\{0, 1, \ldots, m\}$, and with $N_m \in \mathcal{C}^{m-2}(\mathbb{R})$ (if $m \ge 2$); also

$$N_m = \sum_j \frac{1}{2^{m-1}} \binom{m}{j} N_m(2 \cdot -j)\,, \tag{1.33}$$

which can be seen to be consistent with Theorem 1.10 by setting $n = m$ in that result and choosing

$$a_j = a_j^{(m)} = \frac{1}{2^{m-1}} \binom{m}{j}, \quad j \in \mathbb{Z}\,, \tag{1.34}$$

so that also then $\phi = N_m$.

The cardinal B-splines are the only known examples of refinable functions with closed formulas in terms of elementary functions.

1.8 The cascade algorithm

An efficient computational algorithm for the refinable function $\phi \in \mathcal{C}_0(\mathbb{R})$ in a refinement pair $(a\,,\phi)$ is provided by the following.

For $a \in \mathcal{M}_0(\mathbb{Z})$, we define the *cascade operator* $\mathcal{T}_a : \mathcal{M}(\mathbb{R}) \longrightarrow \mathcal{M}(\mathbb{R})$ by

$$\mathcal{T}_a f = \sum_j a_j f(2 \cdot -j)\,, \quad f \in \mathcal{M}(\mathbb{R})\,.$$

For a given initial function $g \in \mathcal{M}(\mathbb{R})$, the *cascade algorithm* generates the sequence $\{f_r : r \in \mathbb{Z}_+\} \subset \mathcal{M}(\mathbb{R})$ recursively by means of

$$f_0 = g\,, \quad f_{r+1} = \mathcal{T}_a f_r\,, \quad r \in \mathbb{Z}_+\,.$$

This iterative scheme is one of the time-domain methods presented in [22] for the study of solutions of the refinement equation (1.1) for a given sequence $a \in \mathcal{M}_0(\mathbb{Z})$. Moreover, as shown in [22], p 1402, the cascade algorithm does not always converge to a C_0-refinable function, even when it exists. The paper [21] provides some sets of conditions for the convergence of the cascade algorithm. In the subsequent substantial amount of literature on refinable function existence (see e.g. [18], [6], [50], [58], [61], [27], [44], [37], [8], [28]), methods based on Fourier transforms were shown to be particularly convenient in especially the multi-dimensional case, often leading to joint spectral radius type conditions, with resulting prohibitively expensive computation, on the refinement mask. In our univariate setting, we have available direct and easily checkable conditions on the refinement mask for cascade algorithm convergence and refinable function existence, as given , for example, by the following result that was proved in [25], Theorem 2.2 .

Theorem 1.12. *Let $(a\,,\phi)$ denote the refinement pair of Theorem 1.10. Then the sequence $\{\phi_r : r \in \mathbb{Z}_+\}$ generated by the cascade algorithm*

$$\phi_0 = N_2\,, \quad \phi_{r+1} = \mathcal{T}_a\phi_r\,, \quad r \in \mathbb{Z}_+\,,$$

with N_2 given by (1.32), converges uniformly, and at a geometric rate, to ϕ, in the sense that

$$\|\phi - \phi_r\|_\infty \leq \frac{\rho^r}{1-\rho} \to 0\,, \quad r \to \infty\,,$$

where

$$0 < \rho \leq 1 - \min\{a_0, \ldots, a_n\} < 1\,.$$

1.9 The Daubechies refinement pairs

A well-studied family of refinement pairs whose mask sequences have some negative coefficients is given, for $N \geq 2$, by the *Daubechies refinement pairs* $(a^{D,N}\,,\,\phi_N^D)$ of order N, where, in this case, we have $n = 2N - 1$. The existence and properties of this family are extensively analysed in [21] and [20].

For $N = 2, 3, 4$, the refinement masks $a^{D,N}$ are given by

$$\left.\begin{array}{l}
\begin{cases} a_0^{D,2} = \frac{1+\sqrt{3}}{4} \;\; ; \;\; a_1^{D,2} = \frac{3+\sqrt{3}}{4} \;\; ; \;\; a_2^{D,2} = \frac{3-\sqrt{3}}{4} \\ a_3^{D,2} = \frac{1-\sqrt{3}}{4} \;\; ; \;\; a_j^{D,2} = 0 \;\; , \;\; j \notin \{0, \dots, 3\} \, , \end{cases} \\
\begin{cases} a_0^{D,3} = \frac{1+\sqrt{10}+\sqrt{5+2\sqrt{10}}}{16} \;\; ; \;\; a_1^{D,3} = \frac{5+\sqrt{10}+3\sqrt{5+2\sqrt{10}}}{16} \\ a_2^{D,3} = \frac{5-\sqrt{10}+\sqrt{5+2\sqrt{10}}}{8} \;\; ; \;\; a_3^{D,3} = \frac{5-\sqrt{10}-\sqrt{5+2\sqrt{10}}}{8} \\ a_4^{D,3} = \frac{5+\sqrt{10}-3\sqrt{5+2\sqrt{10}}}{16} \;\; ; \;\; a_5^{D,3} = \frac{1+\sqrt{10}-\sqrt{5+2\sqrt{10}}}{16} \\ a_j^{D,3} = 0 \, , \quad j \notin \{0, \dots, 5\} \, , \;\; . \end{cases} \\
\begin{cases} a_0^{D,4} = .325803428051 \;\; ; \;\; a_1^{D,4} = 1.01094571509 \;\; ; \;\; a_2^{D,4} = .892200138247 \\ a_3^{D,4} = -.039575026235 \;\; ; \;\; a_4^{D,4} = -.264507167369 \;\; ; \;\; a_5^{D,4} = .043616300474 \\ a_6^{D,4} = .046503601071 \;\; ; \;\; a_7^{D,4} = -.014986989330 \;\; ; \;\; a_j^{D,4} = 0 \, , \; j \notin \{0, \dots, 7\} \end{cases}
\end{array}\right\} \tag{1.35}$$

The refinable function ϕ_N^D can also be calculated by the cascade algorithm: the sequence $\{\phi_r : r \in \mathbb{Z}_+\}$ defined by

$$\phi_0 = N_2 \, , \quad \phi_{r+1} = \mathcal{T}_a \phi_r \, , \quad r \in \mathbb{Z}_+ \, ,$$

converges uniformly to ϕ_N^D, i.e.

$$\left\| \phi_N^D - \phi_r \right\|_\infty \to 0 \, , \quad r \to \infty \, .$$

The Daubechies refinable functions ϕ_N^D , $N = 2, 3, 4$, are plotted in Figure 1.2.

The refinable function ϕ_N^D has orthonormal integer shifts in the sense that

$$\int_{-\infty}^{\infty} \phi_N^D(x-j) \phi_N^D(x-k) dx = \delta_{j,k} \, , \quad j \, , k \, \in \mathbb{Z} \, .$$

The Daubechies refinement pair $(a^{D,N} \, , \, \phi_N^D)$ will be further discussed in Chapter 7.

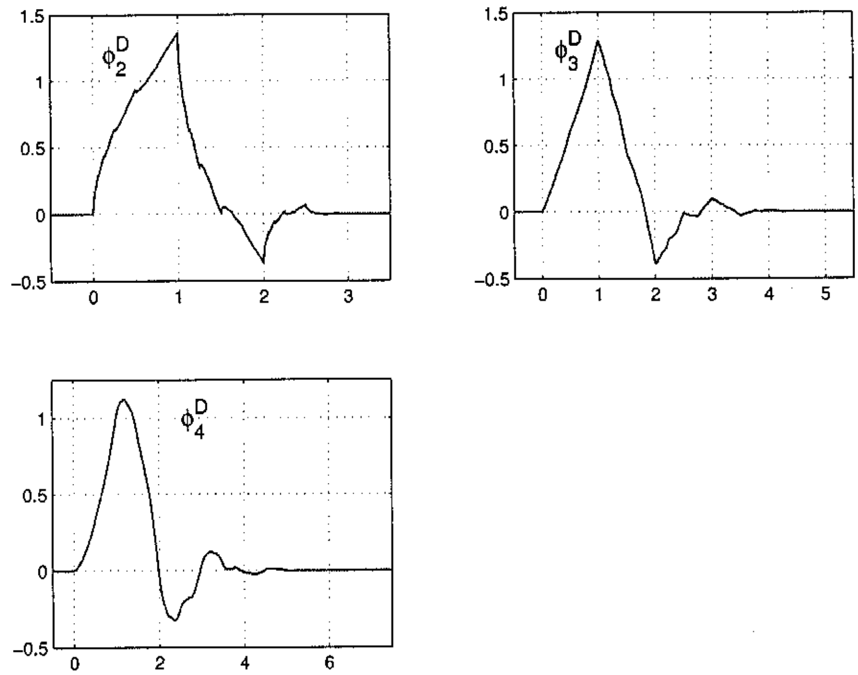


Figure 1.2: The Daubechies scaling functions $\{\phi_N^D : N = 2, 3, 4\}$

1.10 A two-parameter family of refinement pairs

Consider the two-parameter family $\{A_N(\cdot|t) : N \in \mathbb{N}\,,\, t \in \mathbb{R}\}$ of refinement mask symbols as defined by

$$A_N(z|t) = \frac{1}{2^{N-1}(1+t)}(1+z)^N(t+z)\,, \quad t \in \mathbb{R} \setminus \{0,1\}\,, \quad N \in \mathbb{N}\,,\ z \in \mathbb{C}\,. \qquad (1.36)$$

Note that, in this case, we have $n = N + 1$.

For $t \in \mathbb{R}_+ \setminus \{0,1\}$, in which case the corresponding refinement mask satisfies the positivity condition (1.26), Theorem 1.10 shows that, for every integer $N \in \mathbb{N}$, there exists a function $\phi_N(\cdot|t) \in C_0(\mathbb{R})$ such that $(a^N(t)\,, \phi_N(\cdot|t))$ is a refinement pair, where $A_N(z|t) = \sum_j a_j^N(t)z^j\,,\ z \in \mathbb{C}$.

In [**26**], it has been shown that, for every integer $N \in \mathbb{N}$, if $t \in \mathbb{R}_- \setminus [-3, -\frac{1}{3}]$, in which case one or more negative refinement mask coefficients are also allowed, then there also exists a function $\phi_N(\cdot|t) \in C_0(\mathbb{R})$ such that $(a^N(t)\,, \phi_N(\cdot|t))$ is a refinement pair.

An example is drawn by means of the cascade algorithm in Figure 1.3.

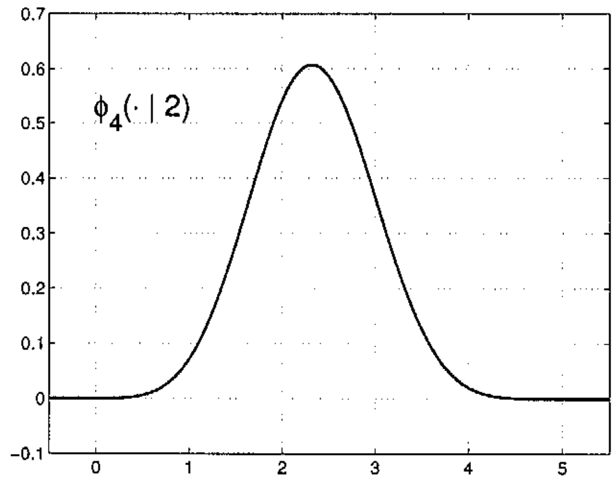


Figure 1.3: The refinable function $\phi_4(\cdot|2)$

Chapter 2

Polynomial containment in refinement spaces

For a given refinement pair (a, ϕ), we now proceed to investigate some structural properties of the nested refinement space sequence $\{V^{(r)} : r \in \mathbb{Z}\}$, as defined by (1.9). Recall from Chapter 1 that we assume in this thesis that, if (a, ϕ) is a given refinement pair, then the following hold:

(a) there is an integer $n \geq 2$ such that

$$a_j = 0\,, \quad j \notin \{0, 1, \ldots, n\}\,, \quad \text{with } a_0 \neq 0\,,\ a_n \neq 0\,; \tag{2.1}$$

(b) the sum rules

$$\sum_j a_{2j} = \sum_j a_{2j+1} = 1\,, \tag{2.2}$$

or, equivalently, the refinement mask symbol conditions

$$A(1) = 2\,, \quad A(-1) = 0\,, \tag{2.3}$$

are satisfied, and we denote by N the positive integer, with $N \leq n$, which is such that

$$\left.\begin{aligned} &A(z) = \frac{1}{2^{N-1}}(1+z)^N B(z)\,, \quad z \in \mathbb{C}\,,\\ &\text{where} \quad B(-1) \neq 0\,, \quad B(1) = 1\,; \end{aligned}\right\} \tag{2.4}$$

(c)

$$\phi(x) = 0\,, \quad x \notin (0, n)\,; \tag{2.5}$$

(d)

$$\int_{-\infty}^{\infty} \phi(t)dt = \sum_j \phi(j) = \sum_j \phi(x-j) = 1\,, \quad x \in \mathbb{R}\,. \tag{2.6}$$

As will prove to be important in our eventual wavelet decomposition technique, we focus in this chapter on finding the largest possible value of $k \in \mathbb{Z}_+$ so that the polynomial containment result

$$\Pi_k \subset V^{(r)}\,, \quad r \in \mathbb{Z}\,, \tag{2.7}$$

is satisfied. Observe from (2.6) that (2.7) does indeed hold for $k = 0$. We proceed to study in Section 2.1 below the implication of the assumption (2.4) on the mask a.

2.1 The higher order sum rules

As a first step towards finding the largest integer k for which (2.7) holds, we prove the following equivalence.

Proposition 2.1. *For a refinement pair $(a\,, \phi)$, the following two statements are equivalent:*

(i) the condition (2.4) holds for a positive integer N ;

(ii) the higher order sum rules

$$\sum_j (2j)^k a_{2j} = \sum_j (2j+1)^k a_{2j+1}\,, \quad k = 1, \ldots, N-1\,, \text{ if N} \geq 2\,, \tag{2.8}$$

hold; and

$$\sum_j (2j)^N a_{2j} \neq \sum_j (2j+1)^N a_{2j+1}\,. \tag{2.9}$$

Proof. First observe from (1.4) that, for $k = 1, 2, \ldots, N$,

$$A^{(k)}(-1) = (-1)^k \left[\sum_j Q_k(2j) a_{2j} - \sum_j Q_k(2j+1) a_{2j+1} \right]\,, \tag{2.10}$$

where Q_k is the polynomial of degree k defined by

$$Q_k(x) = x(x-1)\cdots(x-k+1)\,, \quad x \in \mathbb{R}\,, \quad k = 1, 2, \ldots, N\,. \tag{2.11}$$

We see from (2.11) that, for $k \in \{1, 2, \ldots, N\}$, there exist coefficients $\{c_{k,l} : l = 0, 1, \ldots, k-1\}$ such that

$$Q_k(x) = x^k + \sum_{l=0}^{k-1} c_{k,l} x^l, \quad x \in \mathbb{R}, \quad k = 1, \ldots, N. \tag{2.12}$$

Using (2.12) and (2.10), and recalling also the equivalence of (2.2) and (2.3), we deduce that the statement (i) is equivalent to the following statement:

(iii)

$$\sum_j (2j)^k a_{2j} - \sum_j (2j+1)^k a_{2j+1} = \sum_{l=0}^{k-1} c_{k,l} \left[\sum_j (2j+1)^l a_{2j+1} - \sum_j (2j)^l a_{2j} \right]$$
$$k = 1, \ldots, N-1, \text{ if } N \geq 2; \tag{2.13}$$

$$\sum_j (2j)^N a_{2j} - \sum_j (2j+1)^N a_{2j+1} \neq \sum_{l=0}^{N-1} c_{k,l} \left[\sum_j (2j+1)^l a_{2j+1} - \sum_j (2j)^l a_{2j} \right]. \tag{2.14}$$

Hence, if we can show that (iii) holds if and only if (ii) holds , our proof of the equivalence of (i) and (ii) would be complete.

Suppose therefore that (ii) holds. Then, if $N = 1$, (2.9), together with (2.2), show that (2.14) holds, and it follows that (iii) holds. If $N \geq 2$, it follows from (2.2) and (2.8) that (2.13) holds, with both sides equal to zero. Moreover, (2.8) and (2.9) imply that the right-hand-side of (2.14) equals 0, whereas the left-hand-side $\neq 0$, i.e. (2.14) also holds. Thus (ii) implies (iii).

To prove the converse, i.e. that (iii) implies (ii), we first note that, if $N = 1$, then (2.2), together with (2.14), yield (2.9), so that (iii) implies (ii).

Suppose next that $N \geq 2$. We first prove inductively that (2.13) implies (2.8), as is immediately clear for $N = 2$. Suppose therefore, for a fixed integer $N \geq 2$, that (2.13) implies (2.8). Now suppose that (2.13) be satisfied with N replaced by $N + 1$. But then (2.2) and (2.13) hold, and it follows from the inductive hypothesis that (2.8) hold. Now set $k = N$ in (2.13), and use (2.8) for $k = 1, \ldots, N-1$, to deduce that (2.8) also holds for $k = N$, so that (2.8) holds with N replaced by $N + 1$, and thereby concluding our inductive proof. The inequality (2.9) with N replaced by $N + 1$, is now an immediate consequence of (2.8) and (2.14), both with N replaced by $N + 1$. We have completed our proof of the fact that (iii) implies (ii). ■

2.2 The discrete moments

For a given refinement pair $(a\,,\phi)$, we define the *discrete moments* $\{\mu_k : k \in \mathbb{Z}_+\}$ by

$$\mu_k = \sum_j j^k \phi(j)\,, \quad k \in \mathbb{Z}_+\,. \tag{2.15}$$

Observe in particular, from (2.6) and (2.15), that

$$\mu_0 = 1\,. \tag{2.16}$$

From the refinement equation (1.1), we have

$$\phi(k) = \sum_j a_j \phi(2k-j) = \sum_j a_{2k-j}\phi(j)\,, \quad k \in \mathbb{Z}\,,$$

and thus

$$\sum_{j=1}^{n-1} a_{2k-j}\phi(j) = \phi(k)\,, \quad k = 1, \dots, n-1\,. \tag{2.17}$$

Hence the discrete moments $\{\mu_k : k \in \mathbb{Z}_+\}$ can be computed by first computing the eigenvector $\{\phi(j) : j = 1, \dots, n-1\}$ corresponding to the eigenvalue 1 of the coefficient matrix of the linear system (2.17), before using (2.15).

We proceed to show that, alternatively, the discrete moments $\{\mu_k : k = 0, 1, \dots, N-1\}$ can be computed more efficiently from a recursive formula.

We use the notation

$$\alpha_k = \sum_j (2j)^k a_{2j}\,, \quad \beta_k = \sum_j (2j+1)^k a_{2j+1}\,, \quad k \in \mathbb{Z}_+\,. \tag{2.18}$$

Our result is as follows:

Proposition 2.2. *Suppose $(a\,,\phi)$ is a refinement pair. Then the discrete moments sequence $\{\mu_k : k = 0, 1, \dots, N-1\}$ has the recursive formulation consisting of* (2.16), *together with the formula*

$$\mu_k = \frac{1}{2^k - 1} \sum_{j=0}^{k-1} \binom{k}{j} \alpha_{k-j}\mu_j\,, \quad k = 1, \dots, N-1\,, \tag{2.19}$$

where the sequence $\{\alpha_j\}$ is given as in (2.18) .

Remark. Since the statements (i) and (ii) in Proposition 2.1 are equivalent, we observe that (2.19) can be rewritten as

$$\mu_k = \frac{1}{2^k - 1}\sum_{j=0}^{k-1}\binom{k}{j}\beta_{k-j}\mu_j\,, \quad k = 1,\ldots,N-1\,,$$

with the sequence $\{\beta_j\}$ given as in (2.18).

Proof of Proposition 2.2. For $k \in \{1,\ldots,N-1\}$, we use, consecutively, (2.15), (1.1), Proposition 2.1 , (2.8), and (2.18), to deduce that

$$\begin{aligned}
\mu_k &= \sum_j j^k\phi(j) \\
&= \sum_j j^k \sum_l a_l\phi(2j-l) \\
&= \sum_j j^k \sum_l a_{2j-l}\phi(l) \\
&= \sum_l \left[\sum_j j^k a_{2j-l}\right]\phi(l) \\
&= \sum_l \left[\sum_j j^k a_{2j-2l}\right]\phi(2l) + \sum_l \left[\sum_j j^k a_{2j-2l-1}\right]\phi(2l+1) \\
&= \sum_l \left[\sum_j (j+l)^k a_{2j}\right]\phi(2l) + \sum_l \left[\sum_j (j+l)^k a_{2j-1}\right]\phi(2l+1) \\
&= \frac{1}{2^k}\left\{\sum_l \left[\sum_j (2j+2l)^k a_{2j}\right]\phi(2l) + \sum_l \left[\sum_j [(2j-1)+(2l+1)]^k a_{2j-1}\right]\phi(2l+1)\right\} \\
&= \frac{1}{2^k}\left\{\sum_l \left[\sum_j \sum_{r=0}^{k}\binom{k}{r}(2j)^{k-r}(2l)^r a_{2j}\right]\phi(2l)\right. \\
&\qquad \left. + \sum_l \left[\sum_j \sum_{r=0}^{k}\binom{k}{r}(2j-1)^{k-r}(2l+1)^r a_{2j-1}\right]\phi(2l+1)\right\} \\
&= \frac{1}{2^k}\sum_{r=0}^{k}\binom{k}{r}\left[\sum_l (2l)^r\phi(2l)\sum_j a_{2j}(2j)^{k-r}\right. \\
&\qquad \left. + \sum_l (2l+1)^r\phi(2l+1)\sum_j a_{2j+1}(2j+1)^{k-r}\right] \\
&= \frac{1}{2^k}\sum_{r=0}^{k}\binom{k}{r}\left[\sum_l (2l)^r\phi(2l)\alpha_{k-r} + \sum_l (2l+1)^r\phi(2l+1)\beta_{k-r}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^k}\sum_{r=0}^{k}\binom{k}{r}\alpha_{k-r}\sum_{l} l^r\phi(l) \\
&= \frac{1}{2^k}\sum_{r=0}^{k}\binom{k}{r}\alpha_{k-r}\mu_r \\
&= \frac{1}{2^k}\mu_k + \frac{1}{2^k}\sum_{r=0}^{k-1}\binom{k}{r}\alpha_{k-r}\mu_r\,, \qquad (2.20)
\end{aligned}
$$

since (2.18) and (2.2) give $\alpha_0 = 1$, from which the desired result (2.19) then immediately follows. ■

2.3 The subdivision operator

For a given sequence $a \in \mathcal{M}_0(\mathbb{Z})$, we define the *subdivision operator* $S_a : \mathcal{M}(\mathbb{Z}) \to \mathcal{M}(\mathbb{Z})$ by

$$
(S_a\lambda)_j = \sum_k a_{j-2k}\lambda_k\,, \quad j \in \mathbb{Z}, \quad \lambda \in \mathcal{M}(\mathbb{Z})\,. \qquad (2.21)
$$

We proceed to give a sufficient condition on a sequence $a \in \mathcal{M}_0(\mathbb{Z})$ for which the operator S_a maps polynomial sequences in Π_{N-1} into explicitly formulated polynomial sequences in Π_{N-1}.

Proposition 2.3. *Suppose $a \in \mathcal{M}_0(\mathbb{Z})$ is a sequence such that the condition (2.4) holds for a positive integer N, and, for $l \in \{0, \ldots, N-1\}$, let λ_l denote the monomial sequence in $\mathcal{M}(\mathbb{Z})$ defined by*

$$
\lambda_{l,j} = j^l\,, \quad j \in \mathbb{Z}\,. \qquad (2.22)
$$

Then , with S_a defined by (2.21), we have

$$
(S_a\lambda_l)_j = P_l(j)\,, \quad j \in \mathbb{Z}\,, \qquad (2.23)
$$

where P_l is the polynomial in Π_l given by

$$
P_l(x) = \sum_{k=0}^{l} p_{l,k}x^k\,, \quad x \in \mathbb{R}\,, \qquad (2.24)
$$

with

$$
p_{l,k} = \frac{(-1)^{l-k}}{2^l}\binom{l}{k}\alpha_{l-k}\,, \quad k = 0, 1, \ldots, l\,. \qquad (2.25)
$$

Moreover, P_l is the unique polynomial in Π_l such that (2.23) holds.

Proof. Let $l \in \{0, 1, \dots, N-1\}$, and choose $j \in \mathbb{Z}$. Then, from (2.21) , (2.22) and (2.18), we get

$$
\begin{aligned}
(S_a\lambda_l)_{2j} &= \sum_k a_{2j-2k}k^l \\
&= \sum_k a_{2k}(j-k)^l \\
&= \frac{1}{2^l}\sum_k a_{2k}(2j-2k)^l \\
&= \frac{1}{2^l}\sum_k a_{2k}\sum_{r=0}^{l}\binom{l}{r}(2j)^r(-1)^{l-r}(2k)^{l-r} \\
&= \frac{1}{2^l}\sum_{r=0}^{l}(-1)^{l-r}\binom{l}{r}\left[\sum_k a_{2k}(2k)^{l-r}\right](2j)^r \\
&= \frac{1}{2^l}\sum_{r=0}^{l}(-1)^{l-r}\binom{l}{r}\alpha_{l-r}(2j)^r, \qquad (2.26)
\end{aligned}
$$

whereas, from the equivalence of (i) and (ii) in Proposition 2.1, and using also (2.8), we obtain

$$
\begin{aligned}
(S_a\lambda_l)_{2j+1} &= \sum_k a_{2j+1-2k}k^l \\
&= \frac{1}{2^l}\sum_k a_{2k+1}\left[(2j+1)-(2k+1)\right]^l \\
&= \frac{1}{2^l}\sum_k a_{2k+1}\sum_{r=0}^{l}\binom{l}{r}(2j+1)^r(-1)^{l-r}(2k+1)^{l-r} \\
&= \frac{1}{2^l}\sum_{r=0}^{l}(-1)^{l-r}\binom{l}{r}\left[\sum_k a_{2k+1}(2k+1)^{l-r}\right](2j+1)^r \\
&= \frac{1}{2^l}\sum_{r=0}^{l}(-1)^{l-r}\binom{l}{r}\alpha_{l-r}(2j+1)^r. \qquad (2.27)
\end{aligned}
$$

The results (2.23) , (2.24) and (2.25) then follow immediately from (2.26) and (2.27). The uniqueness statement of the proposition follows from the fact that integer evaluations on $\mathbb{Z}$ determines a polynomial uniquely. ■

We proceed to prove a fundamental identity for refinable functions.

2.4 The ϕ-commutator operator

For a given refinement pair (a, ϕ), we define the *ϕ-commutator operator* $\Gamma_\phi : \mathcal{M}(\mathbb{R}) \longrightarrow \mathcal{M}(\mathbb{R})$, as first introduced in the cardinal B-spline setting in **[14]**, by

$$\Gamma_\phi f = \sum_j f(j)\phi(\cdot - j) - \sum_j \phi(j)f(\cdot - j), \quad f \in \mathcal{M}(\mathbb{R}). \tag{2.28}$$

According to our next result, the ϕ-commutator vanishes on the polynomials space Π_{N-1}. This result is also proven in **[14]**, **[6]**, **[19]** by means of Fourier transform methods, and where in particular the use of the Poisson summation formula requires the refinable function to satisfy a sufficient condition like bounded variation.

Theorem 2.4. *Suppose (a, ϕ) is a refinement pair. Then*

$$\Gamma_\phi f = 0, \quad f \in \Pi_{N-1}. \tag{2.29}$$

Proof. Since ϕ is continuous on $\mathbb{R}$, and since the dyadic set $\left\{\frac{k}{2^r} : k \in \mathbb{Z}, r \in \mathbb{Z}_+\right\}$ is dense in $\mathbb{R}$, it will suffice to prove that

$$\sum_j f(j)\phi\left(\frac{k}{2^r} - j\right) = \sum_j \phi(j) f\left(\frac{k}{2^r} - j\right), \quad k \in \mathbb{Z}, r \in \mathbb{Z}_+, \quad f \in \Pi_{N-1}, \tag{2.30}$$

or, equivalently,

$$\sum_j j^l \phi\left(\frac{k}{2^r} - j\right) = \sum_j \phi(j)\left(\frac{k}{2^r} - j\right)^l, \quad k \in \mathbb{Z}, r \in \mathbb{Z}_+, \ l = 0, 1, \ldots, N-1, \tag{2.31}$$

which we proceed to prove by induction on the integer r.

Since (2.31) trivially holds for $r = 0$, it remains to prove that, if $r \in \mathbb{Z}_+$ is such that (2.31) holds, then (2.31) also holds with r replaced by $r+1$, i.e.

$$\sum_j j^l \phi\left(\frac{k}{2^{r+1}} - j\right) = \sum_j \phi(j)\left(\frac{k}{2^{r+1}} - j\right)^l, \quad k \in \mathbb{Z}, \quad l = 0, 1, \ldots, N-1. \tag{2.32}$$

First, using the equivalence of (2.30) and (2.31), we observe that the inductive hypothesis (2.31) implies that

$$\sum_j P_l(j)\phi\left(\frac{k}{2^r} - j\right) = \sum_j \phi(j)P_l\left(\frac{k}{2^r} - j\right), \quad k \in \mathbb{Z}, \quad l = 0, 1, \ldots, N-1, \tag{2.33}$$

with P_l denoting the polynomial in Π_l defined in Proposition 2.3. Next, we use, consecutively, (1.1), (2.21), (2.23), (2.33), (2.24), (2.15) and (2.25), to obtain, for $k \in \mathbb{Z}$ and $l \in \{0, 1, \ldots, N-1\}$, and with the definition $p_{l,j} = 0,\ j \notin \{0, 1, \ldots, l\}$,

$$\begin{aligned}
\sum_j j^l \phi\left(\frac{k}{2^{r+1}} - j\right) &= \sum_j j^l \sum_i a_i \phi\left(\frac{k}{2^r} - 2j - i\right) \\
&= \sum_j j^l \sum_i a_{i-2j} \phi\left(\frac{k}{2^r} - i\right) \\
&= \sum_i \sum_j a_{i-2j} j^l \phi\left(\frac{k}{2^r} - i\right) \\
&= \sum_i P_l(i) \phi\left(\frac{k}{2^r} - i\right) \\
&= \sum_i \phi(i) P_l\left(\frac{k}{2^r} - i\right) \\
&= \sum_i \phi(i) \sum_j p_{l,j} \left(\frac{k}{2^r} - i\right)^j \\
&= \sum_i \phi(i) \sum_j p_{l,j} \sum_n \binom{j}{n} \left(\frac{k}{2^r}\right)^n (-1)^{j-n} i^{j-n} \\
&= \sum_n (-1)^n \left(\frac{k}{2^r}\right)^n \sum_j (-1)^j \binom{j}{n} p_{l,j} \sum_i i^{j-n} \phi(i) \\
&= \sum_n (-1)^n \left(\frac{k}{2^r}\right)^n \sum_j (-1)^j \binom{j}{n} p_{l,j} \mu_{j-n} \\
&= \sum_n (-1)^n \left(\frac{k}{2^r}\right)^n \sum_j (-1)^j \binom{j}{n} \frac{(-1)^{l-j}}{2^l} \binom{l}{j} \alpha_{l-j} \mu_{j-n} \\
&= \frac{(-1)^l}{2^l} \sum_n (-1)^n \left(\frac{k}{2^r}\right)^n \sum_j \binom{j}{n} \binom{l}{j} \alpha_{l-j} \mu_{j-n} \,.
\end{aligned} \tag{2.34}$$

For the right-hand-side of (2.32) we have, using also (2.20) ,

$$\begin{aligned}
\sum_j \phi(j) \left(\frac{k}{2^{r+1}} - j\right)^l &= \sum_j \phi(j) \sum_n \binom{l}{n} \left(\frac{k}{2^{r+1}}\right)^n (-1)^{l-n} j^{l-n} \\
&= (-1)^l \sum_n \frac{(-1)^n}{2^n} \left(\frac{k}{2^r}\right)^n \binom{l}{n} \sum_j j^{l-n} \phi(j) \\
&= (-1)^l \sum_n \frac{(-1)^n}{2^n} \left(\frac{k}{2^r}\right)^n \binom{l}{n} \mu_{l-n} \\
&= (-1)^l \sum_n \frac{(-1)^n}{2^n} \left(\frac{k}{2^r}\right)^n \binom{l}{n} \frac{1}{2^{l-n}} \sum_j \binom{l-n}{j} \alpha_{l-n-j} \mu_j
\end{aligned}$$

$$= \frac{(-1)^l}{2^l}\sum_n (-1)^n \left(\frac{k}{2^r}\right)^n \binom{l}{n} \sum_j \binom{l-n}{j} \alpha_{l-n-j}\mu_j$$
$$= \frac{(-1)^l}{2^l}\sum_n (-1)^n \left(\frac{k}{2^r}\right)^n \binom{l}{n} \sum_j \binom{l-n}{j-n} \alpha_{l-j}\mu_{j-n}$$
$$= \frac{(-1)^l}{2^l}\sum_n (-1)^n \left(\frac{k}{2^r}\right)^n \sum_j \binom{l}{n} \binom{l-n}{j-n} \alpha_{l-j}\mu_{j-n} \,. \tag{2.35}$$

But, for $l \geq n$, $j \geq n$, and $l \geq j$, it holds that

$$\binom{l}{n}\binom{l-n}{j-n} = \frac{l!}{n!(l-n)!}\frac{(l-n)!}{(j-n)!(l-j)!} = \frac{l!}{n!(j-n)!(l-j)!}$$

and

$$\binom{j}{n}\binom{l}{j} = \frac{j!}{n!(j-n)!}\frac{l!}{j!(l-j)!} = \frac{l!}{n!(j-n)!(l-j)!}\,,$$

and thus

$$\binom{l}{n}\binom{l-n}{j-n} = \binom{j}{n}\binom{l}{j}\,. \tag{2.36}$$

It then follows from (2.34), (2.35), and (2.36), that (2.32) holds, thereby concluding our inductive proof of (2.31) . ■

We proceed in the next section to show that, if $f \in \Pi_{N-1}$, then there exists a sequence $c \in \mathcal{M}(\mathbb{Z})$ such that

$$f = \sum_j c_j \phi(\cdot - j)\,, \tag{2.37}$$

and thereby showing that (2.7) holds with $k = N-1$.

2.5 A generalised Marsden identity

According to **[46]**, p 65, the Marsden identity

$$(\cdot + t)^{m-1} = \sum_j Q_m(j+t)N_m(\cdot - j)\,, \quad t \in \mathbb{R}\,, \tag{2.38}$$

where Q_m is the polynomial of degree $m-1$ defined by

$$Q_m = \prod_{k=1}^{m-1}(\cdot + k)\,,$$

holds, with N_m denoting the cardinal B-spline of order m, as defined by (1.30). Recall also from (1.33) that $(a^{(m)}, N_m)$ is a refinement pair, with $a^{(m)} \in \mathcal{M}_0(\mathbb{Z})$ given by (1.34).

Also, in [29], Corollary 5.8, this identity is generalised to splines with arbitrary knots.

Our next result generalizes (2.38) to the setting of any refinement pair $(a\,,\phi)$ of the type considered in this thesis.

Theorem 2.5. *Let $(a\,,\phi)$ denote a refinement pair. Then the identity*

$$(\cdot+t)^{N-1}=\sum_j Q(j+t)\phi(\cdot-j)\,,\quad t\in\mathbb{R}\,, \tag{2.39}$$

holds, where Q is the polynomial of degree $N-1$ as given by

$$Q(z)=\sum_{j=0}^{N-1} q_j z^j\,,\quad z\in\mathbb{C}\,, \tag{2.40}$$

with the coefficient sequence $\{q_j : j=0,1,\ldots,N-1\}$ satisfying the (backwards) recursive formulation

$$\left.\begin{aligned} q_{N-1}&=1\,,\\ q_j&=(-1)^{j+1}\sum_{k=j+1}^{N-1}(-1)^k\binom{k}{j}\mu_{k-j}\,q_k\,,\quad j=N-2,N-3,\ldots,0\,,\quad \text{if } N\geq 2\,,\end{aligned}\right\} \tag{2.41}$$

and where the discrete moments $\{\mu_k : k=0,1,\ldots,N-1\}$ are as in Proposition 2.2. Moreover, Q is the unique polynomial in Π_{N-1} for which the identity (2.39) holds.

Proof. Let $t\in\mathbb{R}$ be fixed and let Q be a polynomial in Π_{N-1} of the form (2.40). Then the function $P\in\mathcal{M}(\mathbb{R})$ defined by $P=Q(\cdot+t)$ is also a polynomial in Π_{N-1}, and it follows from Theorem 2.4, (2.40), (2.19) and (2.16) that

$$\begin{aligned}\sum_j Q(j+t)\phi(\cdot-j) &= \sum_j P(j)\phi(\cdot-j)\\ &= \sum_j \phi(j)P(\cdot-j)\\ &= \sum_j \phi(j)Q(\cdot+t-j)\\ &= \sum_j \phi(j)\sum_{k=0}^{N-1} q_k\sum_{i=0}^{k}\binom{k}{i}(\cdot+t)^i(-1)^{k-i}j^{k-i}\\ &= \sum_{k=0}^{N-1} q_k\sum_{i=0}^{k}\binom{k}{i}(\cdot+t)^i(-1)^{k-i}\sum_j j^{k-i}\phi(j)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{N-1} q_k \sum_{i=0}^{k} \binom{k}{i} (\cdot + t)^i (-1)^{k-i} \mu_{k-i} \\
&= \sum_{i=0}^{N-1} (-1)^i \left[\sum_{k=i}^{N-1} (-1)^k \binom{k}{i} \mu_{k-i}\, q_k \right] (\cdot + t)^i \\
&= q_{N-1}(\cdot + t)^{N-1} \\
&\quad + \sum_{i=0}^{N-1} (-1)^i \left[\sum_{k=i}^{N-2} (-1)^k \binom{k}{i} \mu_{k-i}\, q_k \right] (\cdot + t)^i \\
&= q_{N-1}(\cdot + t)^{N-1} \\
&\quad + \sum_{j=0}^{N-2} \left[(-1)^j q_j + \sum_{k=j+1}^{N-2} (-1)^k \binom{k}{j} \mu_{k-j}\, q_k \right] (\cdot + t)^j ,
\end{aligned} \tag{2.42}
$$

with the convention that $\sum_{j=J}^{K} c_j = 0$ if $K < J$.

It follows from (2.42) that the identity (2.39) is satisfied if and only if the sequence $\{q_j : j = 0, 1, \ldots, N-1\}$ has the (backwards) recursive formulation (2.41), and thereby showing also the stated uniqueness of the polynomial Q. ■

An equation similar to (2.41) can be found in **[3]**, Theorem 3(d), where it is linked to the accuracy of order $N-1$ of the refinable function, but in which the class of admissible refinement pairs is smaller; e.g. it is required there that, in addition to our assumed conditions on a refinement pair $(a\,,\phi)$, the refinable function ϕ also possesses linearly independent integer shifts.

Differentiating the equation (2.39) with respect to t yields the formula

$$
\frac{(N-1)!}{(N-1-k)!}(\cdot + t)^{N-1-k} = \sum_j Q^{(k)}(j+t)\phi(\cdot - j), \quad t \in \mathbb{R}, \quad k = 0, 1, \ldots, N-1,
$$

in which we now set t=0 to obtain the following consequence of Theorem 2.5.

Corollary 2.6. *Let $(a\,,\phi)$ denote a refinement pair. Then*

$$
x^k = \frac{k!}{(N-1)!} \sum_j Q^{(N-1-k)}(j)\phi(x-j), \quad x \in \mathbb{R}, \quad k = 0, 1, \ldots, N-1, \tag{2.43}
$$

where the polynomial Q is defined by (2.40) and (2.41).

It immediately follows from Corollary 2.6 that the desired polynomial inclusion result (2.7) does indeed hold with $k = N - 1$, as follows:

Corollary 2.7. *Let* $(a\,,\phi)$ *denote a refinement pair. Then*

$$\Pi_{N-1} \subset V^{(r)}, \quad r \in \mathbb{Z}.$$

Chapter 3

The quasi-interpolation operator

In our eventual wavelet decomposition technique, we shall require, for $r \in \mathbb{Z}$, a local approximation operator $\mathcal{Q}_r : \mathcal{M}(\mathbb{R}) \to V^{(r)}$ with the polynomial reproduction property

$$\mathcal{Q}_r f = f\,, \quad f \in \Pi_{N-1}\,. \tag{3.1}$$

Such an operator $\mathcal{Q}_r$ is sometimes referred to in the literature as a quasi-interpolation operator.

To this end, we shall first show that there exists a function $u \in V^{(0)} \cap \mathcal{C}_0(\mathbb{R})$ such that

$$\sum_j f(j+\tau)\, u(\cdot - j) = f\,, \quad f \in \Pi_{N-1}\,, \quad \tau \in \mathbb{R}\,. \tag{3.2}$$

Since $u \in V^{(0)} \cap \mathcal{C}_0(\mathbb{R})$, it is sufficient that we seek a sequence $\{u_j : j \in \mathbb{Z}\} \in \mathcal{M}_0(\mathbb{Z})$ such that the function u defined by

$$u = \sum_j u_j\, \phi(\cdot - j)\,, \tag{3.3}$$

satisfies the condition (3.2).

Regarding alternative construction methods in the literature: in **[40]**, quasi-interpolants are built from convergent subdivision schemes, whereas **[15]** provides an example of construction by means of knots insertion for splines with arbitrary knots. In **[9]**, Chapter 4, a method for quasi-interpolant construction in the cardinal spline setting is introduced.

3.1 The explicit construction

Let $\tau \in \mathbb{R}$ be fixed. Now observe that (3.2) is equivalent to the condition

$$x^k = \sum_j (j+\tau)^k\, u(x-j)\,, \quad x \in \mathbb{R}\,, \quad k = 0,1,\ldots,N-1\,. \tag{3.4}$$

Using (3.3), we find that , for $x \in \mathbb{R}$, $k \in \{0,1,\ldots,N-1\}$, we have

$$\begin{aligned}\sum_j (j+\tau)^k\, u(x-j) &= \sum_j (j+\tau)^k \sum_l u_l\, \phi(x-j-l) \\ &= \sum_j (j+\tau)^k \sum_l u_{l-j}\, \phi(x-l) \\ &= \sum_l \left(\sum_j (j+\tau)^k\, u_{l-j}\right) \phi(x-l)\,. \end{aligned} \tag{3.5}$$

It follows from (3.5), together with (2.43) in Corollary 2.6, that the condition (3.4) holds if and only if

$$\sum_l \left[\sum_j (j+\tau)^k\, u_{l-j} - \frac{k!}{(N-1)!} Q^{(N-1-k)}(l)\right] \phi(\cdot - l) = 0\,, \quad k = 0,1,\ldots,N-1\,. \tag{3.6}$$

Thus, if we can find a sequence $\{u_j : j \in \mathbb{Z}\} \in \mathcal{M}_0(\mathbb{Z})$ satisfying the condition

$$\sum_j (j+\tau)^k\, u_{l-j} = \frac{k!}{(N-1)!} Q^{(N-1-k)}(l)\,, \quad l \in \mathbb{Z}\,, \quad k = 0,1,\ldots,N-1\,, \tag{3.7}$$

then (3.6), and therefore also (3.4), hold.

A necessary condition for (3.7) to hold is obtained by setting $l = 0$ in (3.7), thereby yielding

$$\sum_j (j-\tau)^k\, u_j = \frac{(-1)^k k!}{(N-1)!} Q^{(N-1-k)}(0)\,, \quad k = 0,1,\ldots,N-1\,. \tag{3.8}$$

To obtain a minimally supported sequence $\{u_j : j \in \mathbb{Z}\} \in \mathcal{M}_0(\mathbb{Z})$ satisfying (3.8) , we set

$$u_j = 0\,, \quad j \notin \{0,1,\ldots,N-1\}\,, \tag{3.9}$$

so that (3.8) becomes the $N \times N$ linear system

$$\sum_{j=0}^{N-1} (j-\tau)^k\, u_j = \frac{(-1)^k k!}{(N-1)!} Q^{(N-1-k)}(0)\,, \quad k = 0,1,\ldots,N-1\,, \tag{3.10}$$

or, equivalently, in matrix-vector notation,

$$W\boldsymbol{u} = \boldsymbol{f}, \tag{3.11}$$

where

$$W = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ x_0 & x_1 & x_2 & \dots & x_{N-1} \\ x_0^2 & x_1^2 & x_2^2 & \dots & x_{N-1}^2 \\ \vdots & & & & \vdots \\ x_0^{N-1} & x_1^{N-1} & x_2^{N-1} & \dots & x_{N-1}^{N-1} \end{bmatrix}, \tag{3.12}$$

where the point sequence $\{x_j : j \in \{0, 1, \dots, N-1\}\}$ is given by

$$x_j = j - \tau, \quad j \in \{0, 1, \dots, N-1\}, \tag{3.13}$$

where the vector $\boldsymbol{u} \in \mathbb{R}^N$ is given by

$$\boldsymbol{u} = [u_0, u_1, \dots, u_{N-2}, u_{N-1}]^T, \tag{3.14}$$

and where $\boldsymbol{f} \in \mathbb{R}^N$ is the vector

$$\begin{aligned} &\boldsymbol{f} = [f_0, f_1, \dots, f_{N-2}, f_{N-1}]^T, \\ &\text{with} \quad f_k = \frac{(-1)^k\, k!}{(N-1)!} Q^{(N-1-k)}(0), \quad k = 0, 1, \dots, N-1, \end{aligned} \tag{3.15}$$

Now observe that W^T is a Vandermonde matrix with, since $x_0, \dots, x_{N-1}$ are distinct points, explicit inverse given by

$$\left(W^T\right)^{-1} = \begin{bmatrix} L_0(0) & L_1(0) & \dots & \dots & L_{N-1}(0) \\ L_0'(0) & L_1'(0) & \dots & \dots & L_{N-1}'(0) \\ \dfrac{L_0''(0)}{2!} & \dfrac{L_1''(0)}{2!} & \dots & \dots & \dfrac{L_{N-1}''(0)}{2!} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dfrac{L_0^{(N-1)}(0)}{(N-1)!} & \dfrac{L_1^{(N-1)}(0)}{(N-1)!} & \dots & \dots & \dfrac{L_{N-1}^{(N-1)}(0)}{(N-1)!} \end{bmatrix}, \tag{3.16}$$

so that W is also an invertible matrix with, from (3.16),

$$W^{-1} = \begin{bmatrix} L_0(0) & L_0'(0) & \dfrac{L_0''(0)}{2!} & \cdots & \dfrac{L_0^{(N-1)}(0)}{(N-1)!} \\ L_1(0) & L_1'(0) & \dfrac{L_1''(0)}{2!} & \cdots & \dfrac{L_1^{(N-1)}(0)}{(N-1)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{N-1}(0) & L_{N-1}'(0) & \dfrac{L_{N-1}''(0)}{2!} & \cdots & \dfrac{L_{N-1}^{(N-1)}(0)}{(N-1)!} \end{bmatrix}, \tag{3.17}$$

where $\{L_j : j = 0, 1, \ldots, N-1\}$ are the Lagrange fundamental polynomials of degree $N-1$ defined by

$$L_j = \prod_{j \neq k=0}^{N-1} \frac{\cdot - x_k}{x_j - x_k}, \quad j = 0, 1, \ldots, N-1, \tag{3.18}$$

with $\{x_j : j \in \{0, 1, \ldots, N-1\}\}$ as in (3.13).

Combining (3.10)-(3.17), we deduce that the unique solution of (3.10) is given by the formula

$$u_j = \frac{1}{(N-1)!} \sum_{k=0}^{N-1} (-1)^k L_j^{(k)}(0)\, Q^{(N-1-k)}(0), \quad j = 0, 1, \ldots, N-1. \tag{3.19}$$

We proceed to show that, if the sequence $\{u_j : j \in \mathbb{Z}\} \in \mathcal{M}_0(\mathbb{Z})$ is defined by (3.19) and (3.9), then the condition (3.7) is also satisfied.

To this end, we use (3.8) to obtain, for $k \in \{0, 1, \ldots, N-1\}$ and $l \in \mathbb{Z}$,

$$\begin{aligned} \sum_j (j+\tau)^k u_{l-j} &= \sum_j [l - (j-\tau)]^k u_j \\ &= \sum_j \sum_{n=0}^{k} \binom{k}{n} l^n (-1)^{k-n} (j-\tau)^{k-n} u_j \\ &= \sum_{n=0}^{k} \binom{k}{n} l^n (-1)^{k-n} \sum_j (j-\tau)^{k-n} u_j \\ &= \sum_{n=0}^{k} \binom{k}{n} l^n (-1)^{k-n} \frac{(-1)^{k-n}(k-n)!}{(N-1)!} Q^{(N-1-k+n)}(0) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{k} \frac{k!\, l^n}{n!(k-n)!} \frac{(k-n)!}{(N-1)!} Q^{(N-1-k+n)}(0) \\
&= \frac{k!}{(N-1)!} \sum_{n=0}^{k} \frac{\left(Q^{(N-1-k)}\right)^{(n)}(0)}{n!} l^n \\
&= \frac{k!}{(N-1)!} Q^{(N-1-k)}(l)\,,
\end{aligned}
$$

since $Q^{(N-1-k)}$ is a polynomial of degree k, and thereby showing that (3.7) is indeed satisfied.

Suppose now that the Lagrange fundamental polynomials $\{L_j : j = 0, 1, \ldots, N-1\}$ in (3.18) are given by

$$
L_j(x) = \sum_{k=0}^{N-1} l_{j,k} x^k\,, \quad x \in \mathbb{R}\,, \quad j = 0, 1, \ldots, N-1\,, \tag{3.20}
$$

so that

$$
l_{j,k} = \frac{1}{k!} L_j^{(k)}(0)\,, \quad k\,, j = 0, 1, \ldots, N-1\,. \tag{3.21}
$$

Also, (2.40) gives

$$
Q^{(N-1-k)}(0) = (N-1-k)!\, q_{N-1-k}\,, \quad k = 0, 1, \ldots, N-1\,. \tag{3.22}
$$

Substituting (3.21) and (3.22) into (3.19) then yields the formula

$$
u_j = \sum_{k=0}^{N-1} \frac{(-1)^k}{\binom{N-1}{k}} l_{j,k} q_{N-1-k}\,, \quad j = 0, 1, \ldots, N-1\,. \tag{3.23}
$$

We have therefore proved the first part of the following result.

Theorem 3.1. *Let $(a\,, \phi)$ denote a refinement pair and suppose $\tau \in \mathbb{R}$. Then the function $u \in V^{(0)} \cap C_0(\mathbb{R})$ defined by (3.3), where the coefficient sequence $\{u_j : j \in \mathbb{Z}\} \in \mathcal{M}_0(\mathbb{Z})$ is defined by (3.23) and (3.9), satisfies the polynomial reproduction identity (3.2). Moreover*

,

$$
u(x) = 0\,, \quad x \notin (0\,, N+n-1)\,. \tag{3.24}
$$

Proof. It remains to prove the finite support property (3.24). Indeed, combining (3.3), (3.9) and (2.5) , we conclude that (3.24) does indeed hold. ∎

We proceed to show how the function $u \in V^{(0)} \cap C_0(\mathbb{R})$ of Theorem 3.1 can be used to construct a quasi-interpolation operator $\mathcal{Q}_r : \mathcal{M}(\mathbb{R}) \to V^{(r)}$ for which (3.1) holds. To this end, we let $f \in \mathcal{M}(\mathbb{R})$, and $\tau \in \mathbb{R}$, and observe from (3.3) that, for $r \in \mathbb{Z}$, we have

$$\begin{aligned}\sum_j f\left(\frac{j+\tau}{2^r}\right) u(2^r \cdot -j) &= \sum_j f\left(\frac{j+\tau}{2^r}\right) \sum_k u_k \phi(2^r \cdot -j-k)\\ &= \sum_j f\left(\frac{j+\tau}{2^r}\right) \sum_k u_{k-j} \phi(2^r \cdot -k)\\ &= \sum_k \left[\sum_j u_{k-j} f\left(\frac{j+\tau}{2^r}\right)\right] \phi(2^r \cdot -k)\,. \qquad (3.25)\end{aligned}$$

Hence, if we choose $f \in \Pi_{N-1}$, so that, if we define $g = f\left(\frac{\cdot}{2^r}\right)$, i.e. $f = g(2^r \cdot)$, then also $g \in \Pi_{N-1}$, it follows from (3.25) and Theorem 3.1 that

$$\begin{aligned}\sum_k \left[\sum_j u_{k-j} f\left(\frac{j+\tau}{2^r}\right)\right] \phi(2^r \cdot -k) &= \sum_j g(j+\tau) u(2^r \cdot -j)\\ &= g(2^r \cdot) = f\,. \qquad (3.26)\end{aligned}$$

The following result is then a consequence of (3.26) and Theorem 3.1.

Theorem 3.2. *Suppose $(a\,,\phi)$ is a refinement pair and let $\tau \in \mathbb{R}$. Then, if we define the linear functional sequence $\{\Upsilon_{r,k} : \mathcal{M}(\mathbb{R}) \to \mathbb{R}\,; r\,,k \in \mathbb{Z}\}$ by*

$$\Upsilon_{r,k} f = \sum_j u_{k-j} f\left(\frac{j+\tau}{2^r}\right), \quad k\,,r \in \mathbb{Z}\,, \quad f \in \mathcal{M}(\mathbb{R})\,, \qquad (3.27)$$

with the sequence $\{u_j : j \in \mathbb{Z}\} \in \mathcal{M}_0(\mathbb{Z})$ as in Theorem 3.1 , then the approximation operator sequence $\{\mathcal{Q}_r : \mathcal{M}(\mathbb{R}) \to V^{(r)}\,,\, r \in \mathbb{Z}\}$ defined by

$$\mathcal{Q}_r f = \sum_k (\Upsilon_{r,k} f)\, \phi(2^r \cdot -k)\,, \quad r \in \mathbb{Z}\,, \qquad (3.28)$$

is a quasi-interpolation operator sequence in the sense of (3.1).

Remarks. (*a*) Observe from (3.25) that the operator $\mathcal{Q}_r$ has the alternative formulation

$$\mathcal{Q}_r f = \sum_j f\left(\frac{j+\tau}{2^r}\right) u(2^r \cdot -j)\,, \quad r \in \mathbb{Z}\,, \quad f \in \mathcal{M}(\mathbb{R}). \qquad (3.29)$$

(*b*) A rigorous Peano-kernel type error analysis for the error $\|f - Q_r f\|_\infty$, with $f \in C_0^N(\mathbb{R})$ is not included in this thesis, but can be used to prove that Q_r has optimal approximation order for $r \to \infty$. Here we merely rely on the reproduction property (3.1) of Q_r, and we illustrate graphically the approximation power of the quasi-interpolation operator Q_r.

Using (3.29) and (3.24), and the polynomial reproduction property (3.1) of the quasi-interpolation operator Q_r, we next prove the following result which shows that, for a given function $f \in \mathcal{M}(\mathbb{R})$, the operator Q_r also preserves, for sufficiently large r, local Π_{N-1} polynomial behaviour of f, in a sense made precise in Theorem 3.3 below. This local Π_{N-1} polynomial reproduction property of Q_r will prove useful in our eventual wavelet decomposition technique.

Motivated by the definition (1.13) of the operator $\mathcal{V}_r$, in conjunction with (1.7), we now use (3.24) to choose the real number τ in the operator definition (3.27), (3.28) as

$$\tau = \tau_0 = \frac{N+n-1}{2}. \tag{3.30}$$

Observe from (3.30) and (3.24) that, if $\tau = \tau_0$ as in (3.30), then

$$u(x) = 0\,, \quad x \notin (0\,, 2\tau_0)\,. \tag{3.31}$$

Our result is as follows.

Theorem 3.3. *For a refinement pair $(a\,,\phi)$, let $\{Q_r : \mathcal{M}(\mathbb{R}) \to V^{(r)}\,,\ r \in \mathbb{Z}\}$ denote the quasi-interpolation operator sequence of Theorem 3.2, with $\tau = \tau_0$, as defined by (3.30). Suppose moreover, that the function $f \in \mathcal{M}(\mathbb{R})$ satisfies, for a bounded interval $[\alpha\,,\beta]$, the property*

$$(f-p)|_{[\alpha\,,\beta]} = 0\,, \tag{3.32}$$

with p denoting a polynomial in Π_{N-1}. Then

$$(f - Q_r f)|_{[\alpha+\frac{\tau_0}{2^r}\,,\beta-\frac{\tau_0}{2^r}]} = 0\,, \tag{3.33}$$

for all integers r such that

$$r > 1 + \log_2 \frac{\tau_0}{\beta - \alpha} . \tag{3.34}$$

Remark. Observe that the inequality (3.34) ensures that $\beta - \frac{\tau_0}{2^r} > \alpha + \frac{\tau_0}{2^r}$ in (3.33).

Proof of Theorem 3.3. Let r denote an integer satisfying the inequality (3.34). It follows from (3.29) and (3.31) that

$$(Q_r f)(x) = \sum_{j=\lceil 2^r x - 2\tau_0 \rceil}^{\lfloor 2^r x \rfloor} f\left(\frac{j+\tau}{2^r}\right) u(2^r x - j) , \quad x \in \mathbb{R} ,$$

and thus

$$(Q_r f)|_{[\alpha + \frac{\tau_0}{2^r}, \beta - \frac{\tau_0}{2^r}]} = \sum_{j=\lceil 2^r \alpha - \tau_0 \rceil}^{\lfloor 2^r \beta - \tau_0 \rfloor} f\left(\frac{j+\tau}{2^r}\right) u(2^r \cdot - j) . \tag{3.35}$$

Now observe that

$$\alpha \leq \frac{j + \tau_0}{2^r} \leq \beta \quad \text{for} \quad j = \lceil 2^r \alpha - \tau_0 \rceil , \ldots , \lfloor 2^r \beta - \tau_0 \rfloor . \tag{3.36}$$

The desired result (3.33) is then a consequence of (3.35), (3.36), (3.32) and (3.1). ■

3.2 Example A: the cardinal B-spline case

We proceed to consider the case where the refinement pair (a , ϕ) is chosen, for $m \geq 2$, as the cardinal B-spline refinement pair $(a^{(m)} , N_m)$ of Section 1.7, in which case, according to (1.4) and (1.34), the corresponding refinement mask symbol $A = A_m$ is given by

$$A_m(z) = \frac{1}{2^{m-1}}(1+z)^m , \quad z \in \mathbb{C} . \tag{3.37}$$

Hence, in the notation (2.1) and (2.4), we have here $n = N = m$.

We shall explicitly calculate, for $m = 2, 3, 4, 5$, the sequence $\{u_j = u_{m,j} : j = 0, \ldots , m-1\}$ and the function $u = u_m$ of Theorem 3.1, thereby obtaining also the quasi-interpolation operator sequence $\{Q_r = Q_{m,r} : r \in \mathbb{Z}\}$ of Theorem 3.2. We choose here, as in Theorem 3.3, the real number $\tau = \tau_0$ as given by (3.30), so that here

$$\tau = \tau_0 = m - \frac{1}{2} . \tag{3.38}$$

First, consecutively using (1.34), (2.18), (2.16), (2.19), (2.41), (3.38), (3.13), (3.20) and (3.23), we calculate the following Table 3.1 and Table 3.2. The intermediate calculations are displayed in Appendix A (Tables A.1 to A.4).

Table 3.1: The mask sequence $\{a_j^{(m)} : j = 0, \ldots, m\}$

$a_j^{(m)}$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$m=2$	$\frac{1}{2}$	1	$\frac{1}{2}$	X	X	X
$m=3$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	X	X
$m=4$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	X
$m=5$	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{5}{16}$	$\frac{1}{16}$

Table 3.2: The sequence $\{u_j = u_{m,j} : j = 0, 1, \ldots, m-1\}$

$u_{m,j}$	$m=2$	$m=3$	$m=4$	$m=5$
$j=0$	$\frac{1}{2}$	$-\frac{1}{8}$	$-\frac{7}{48}$	$\frac{47}{1152}$
$j=1$	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{31}{48}$	$-\frac{107}{288}$
$j=2$	X	$-\frac{1}{8}$	$\frac{31}{48}$	$\frac{319}{192}$
$j=3$	X	X	$-\frac{7}{48}$	$-\frac{107}{288}$
$j=4$	X	X	X	$\frac{47}{1152}$

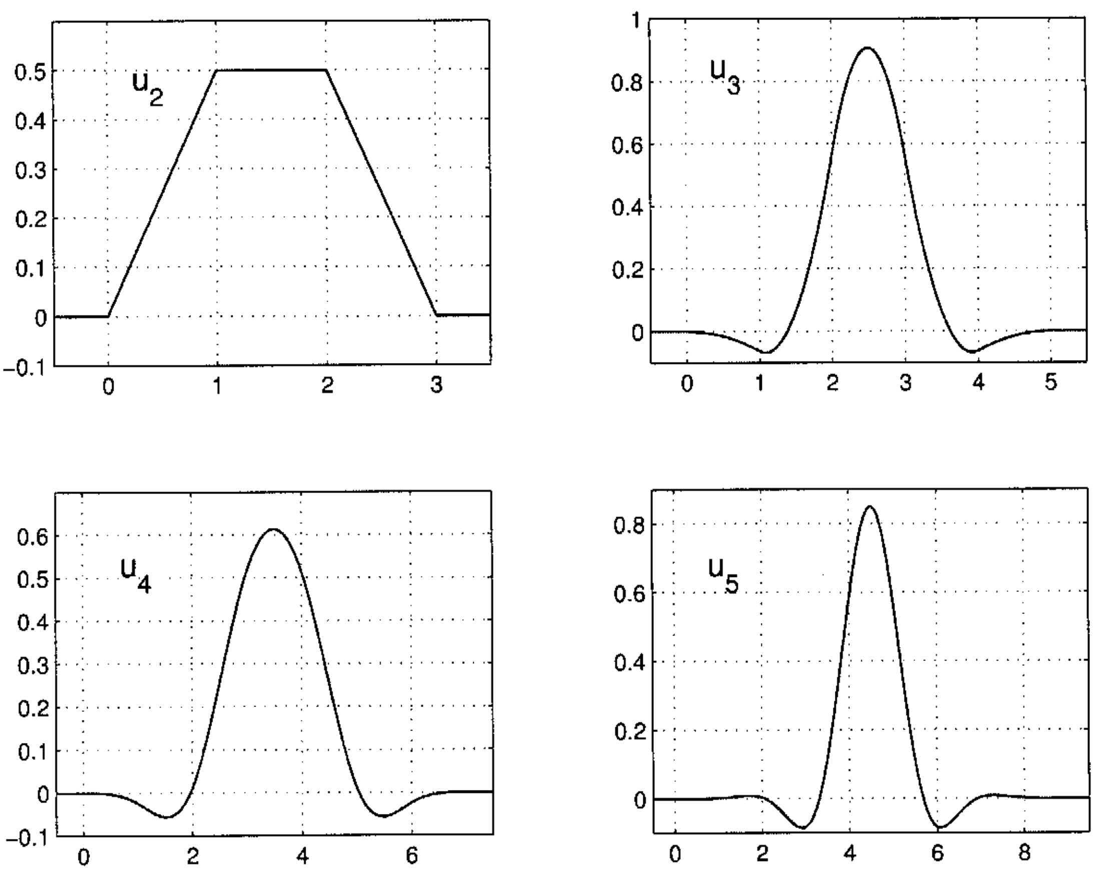


Figure 3.1: The functions $\{u_m : m = 2, \ldots, 5\}$

Using the values of Table 3.2 in the definition (3.3), we obtain the graphs of the function u_m, $m = 2, 3, 4, 5$, as shown in Figure 3.1.

We proceed to illustrate Theorem 3.3 by choosing, for $m \in \{2, 3, 4, 5\}$, the function $f_m \in \mathcal{M}(\mathbb{R})$ as

$$f_m(x) = \begin{cases} x^{m-1}, & x \in [0\,,\,1]\,, \\ \sin\left(\frac{\pi}{2}x\right)\,, & x \in \mathbb{R} \setminus [0\,,\,1]\,, \end{cases} \tag{3.39}$$

as plotted in Figure 3.2.

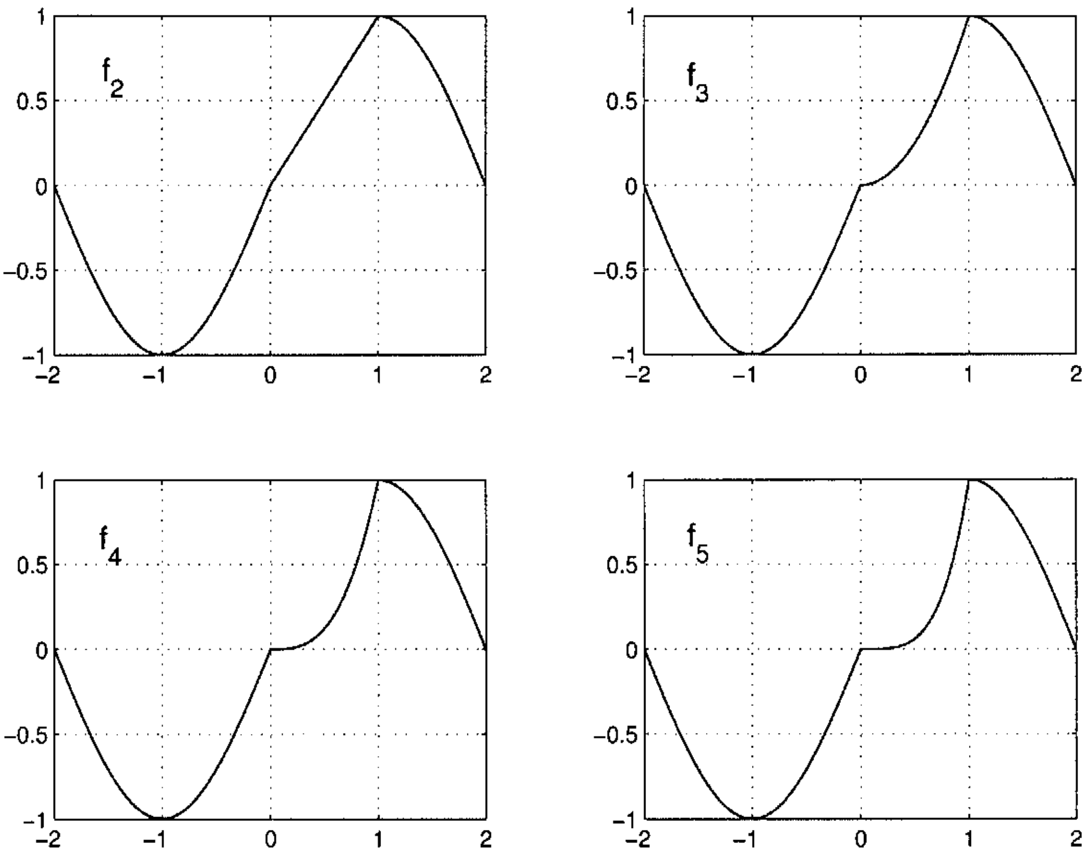


Figure 3.2: The functions $\{f_m : m = 2, \ldots, 5\}$

Observe from (3.38) that the inequality (3.34) is here given by

$$r > \log_2(2m-1)\,; \tag{3.40}$$

having noted also from (3.32) and the top line of (3.39) that , in the notation of Theorem 3.3, we have here $\alpha = 0$ and $\beta = 1$. Also, note that the result (3.33) for the function $f \in \mathcal{M}(\mathbb{R})$ defined by (3.39) is given , in the notation $\mathcal{Q}_r = \mathcal{Q}_{m,r}$, $r \in \mathbb{Z}$, by

$$(\mathcal{Q}_{m,r} f_m)(x) = x^{m-1}, \quad \frac{2m-1}{2^{r+1}} \leq x \leq 1 - \frac{2m-1}{2^{r+1}}. \tag{3.41}$$

Hence if, according to (3.40), we choose

$$r = \begin{cases} 3, & m = 2, \\ 4, & m = 3,4, \\ 6, & m = 5, \end{cases}$$

then (3.41) gives

$$\left.\begin{aligned} (\mathcal{Q}_{2,3} f_2)(x) &= x, & \frac{3}{16} \leq x \leq \frac{13}{16}, \\ (\mathcal{Q}_{3,4} f_3)(x) &= x^2, & \frac{5}{32} \leq x \leq \frac{27}{32}, \\ (\mathcal{Q}_{4,4} f_4)(x) &= x^3, & \frac{7}{32} \leq x \leq \frac{25}{32}, \\ (\mathcal{Q}_{5,6} f_5)(x) &= x^4, & \frac{9}{128} \leq x \leq \frac{119}{128}. \end{aligned}\right\} \tag{3.42}$$

The results (3.42) are illustrated in Figures 3.3 to 3.6, where we have used (3.21), (3.20), (1.31), (3.38), together with (3.39) and Table 3.2, to compute the approximating function $\mathcal{Q}_{m,r} f_m$ for the shown values of m and r. The error functions defined by $E_{m,r} = \mathcal{Q}_{m,r} f_m - f_m$ are also given graphically.

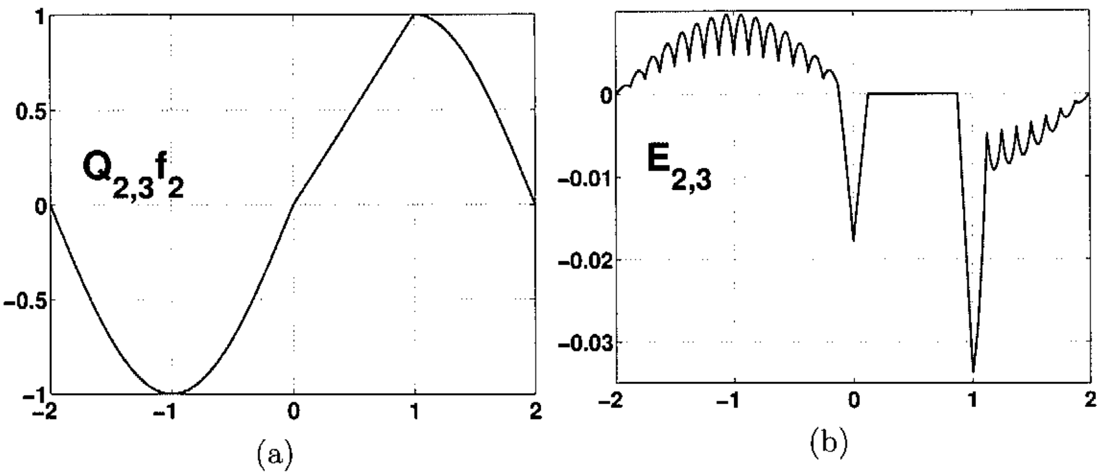


Figure 3.3: The functions $\mathcal{Q}_{2,3}f_2$ and the error $E_{2,3}$

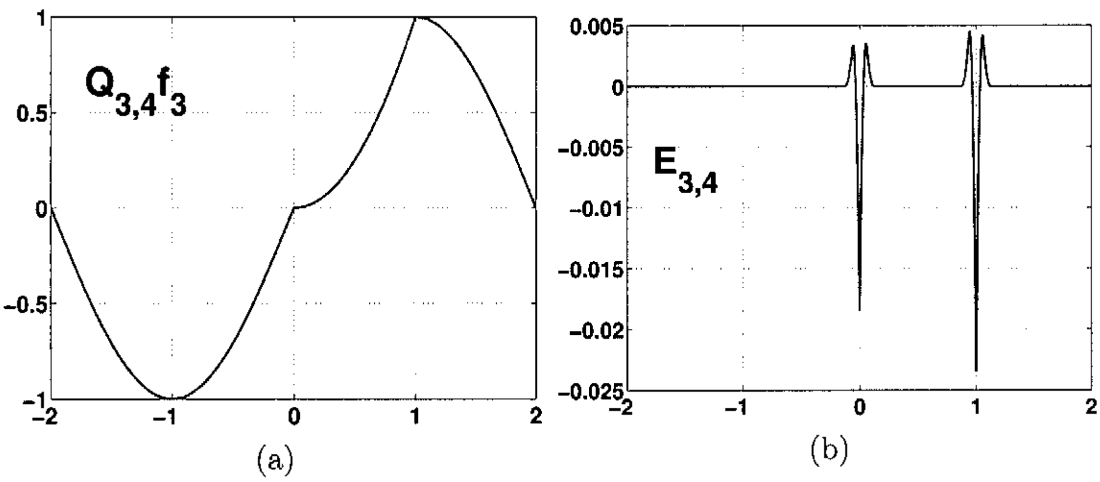
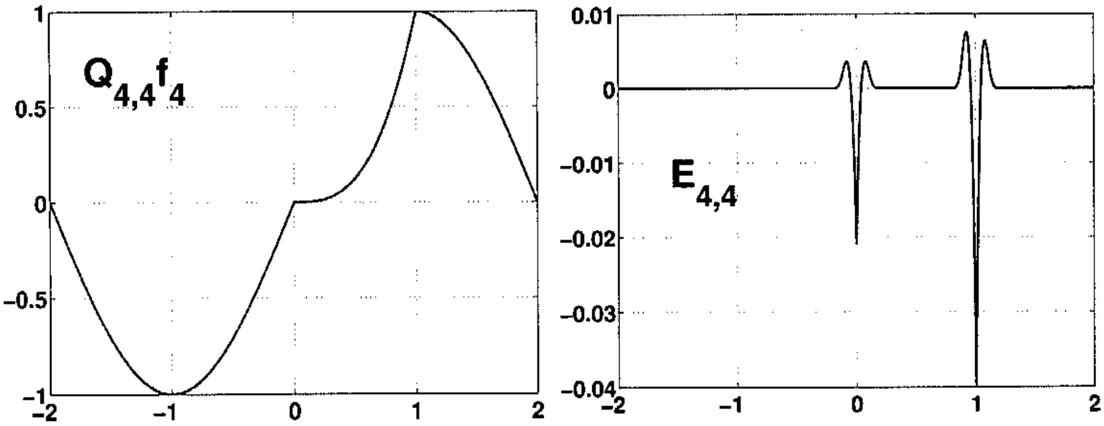
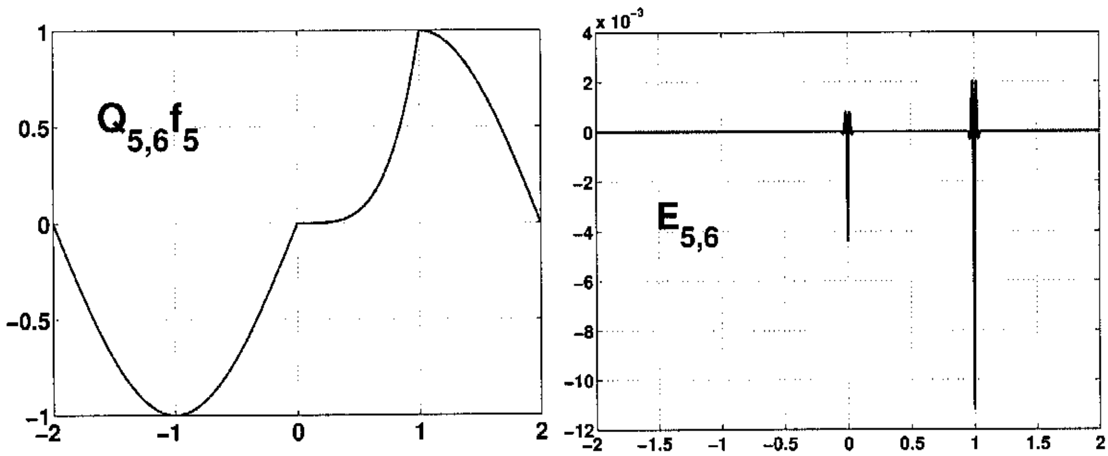


Figure 3.4: The functions $\mathcal{Q}_{3,4}f_3$ and the error $E_{3,4}$



(a) (b)

Figure 3.5: The functions $Q_{4,4}f_4$ and the error $E_{4,4}$



(a) (b)

Figure 3.6: The functions $Q_{5,6}f_5$ and the error $E_{5,6}$

3.3 Example B: the case $N = 4$, $t = 2$ of Section 1.10

For $n = 5$, consider the mask sequence $a \in \mathcal{M}_0(\mathbb{Z})$ obtained by setting $N = 4$ and $t = 2$ in (1.36), yielding, for $A = A_4(\cdot|2)$, the formula

$$A(z) = \frac{1}{24}(1+z)^4(2+z), \quad z \in \mathbb{C},$$

and thus, using (1.4), we find that the corresponding refinement sequence $a = a^4(2) \in \mathcal{M}_0(\mathbb{Z})$ is given by

$$a_0 = \frac{1}{12};\ a_1 = \frac{3}{8};\ a_2 = \frac{2}{3};\ a_3 = \frac{7}{12};\ a_4 = \frac{1}{4};\ a_5 = \frac{1}{24}; \quad a_j = 0,\ j \notin \{0, \ldots, 5\}. \tag{3.43}$$

Since the mask a above satisfies the conditions of Theorem 1.10, it follows that there exists a refinable function $\phi = \phi_4(\cdot|2) \in C_0(\mathbb{R})$ such that $(a\,,\,\phi)$ is a refinement pair.

In Table 3.3 below, we show the corresponding sequence $\{u_j : j \in \mathbb{Z}\} \in \mathcal{M}_0(\mathbb{Z})$ of Theorem 3.1, and where the intermediate calculations are given in Tables A.5 and A.6 in Appendix A.

Note that, since here $n = 5$, it follows from (3.30) that $\tau_0 = 4$ for this case. Also, in Figure 3.7, we show the graph of the function u.

Table 3.3: The sequence $\{u_j : j = 0, \ldots, 3\}$

	$j=0$	$j=1$	$j=2$	$j=3$
u_j	$-\frac{131}{1134}$	$\frac{23}{63}$	$\frac{359}{378}$	$-\frac{113}{567}$

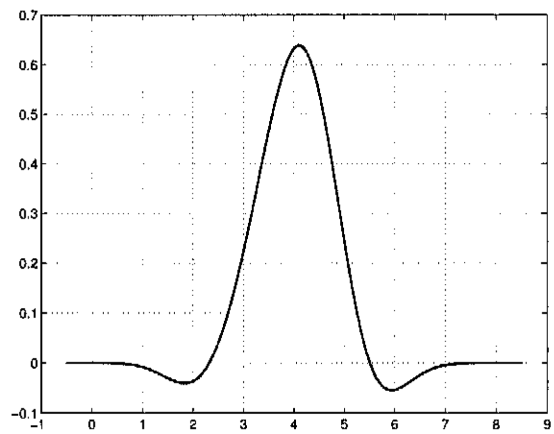


Figure 3.7: The function u for the mask (3.43).

In this case, the inequality (3.34) is given by $r > 3$. Hence, if we define the function $f_4 \in \mathcal{M}(\mathbb{R})$ as given by (3.39), then (3.33) gives

$$(\mathcal{Q}_4 f_4)(x) = x^3\,, \quad \frac{1}{4} \leq x \leq \frac{3}{4}\,,$$

which is verified by Figure 3.8.

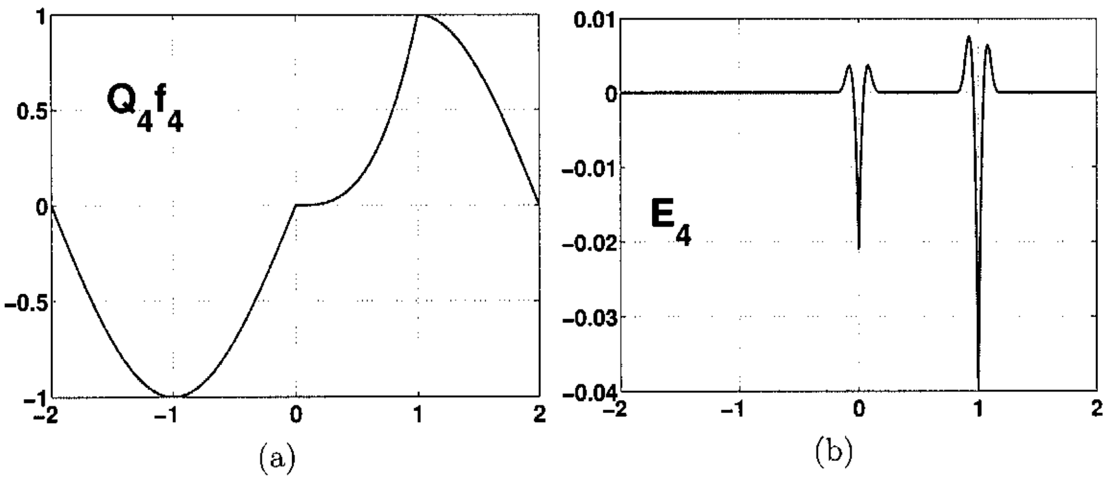


Figure 3.8: The functions $Q_4 f_4$ and the error $E_4 = f_4 - Q_4 f_4$

Observe also from Figures 3.4, 3.5, 3.6 and 3.8 that, for $x \notin [0, 1]$, where (3.39) gives $f_m(x) = \sin(\frac{\pi}{2}x)$, the quasi-interpolant approximates f with excellent accuracy if $x \leq -\varepsilon$ and $x \geq 1 + \varepsilon$ for some $\varepsilon > 0$. The peaks close to $x = 0$ and $x = 1$ are a result of the discontinuities at these points in the derivative $f_m^{(m-1)}$. Our numerical evidence therefore suggests that a C^∞-smooth function like sin is approximated extremely well by our quasi-interpolant for sufficiently large r.

3.4 Example C: the Daubechies case

Next, we consider the Daubechies refinement pair $(a^{D,N}, \phi_N^D)$ for $N = 2, 3, 4$, of Section 1.9, where the mask is given by for $N = 2, 3, 4$, where the mask is given by (1.35). Our results are shown in Table 3.4 and Figure 3.9, and the intermediate tables are given in Appendix A, Tables A.8 to A.10.

Note from (3.30) that, since $n = 2N - 1$, we have here $\tau = \tau_0 = \frac{3N}{2} - 1$.

Table 3.4: The sequence $\{u_j = u_j^{D,N} : j = 0, \ldots, 2N-1\}$

$u_j^{(D,N)}$	$j=0$	$j=1$	$j=2$	$j=3$
$N=2$	$\frac{1-\sqrt{3}}{2}$	$\frac{1+\sqrt{3}}{2}$	X	X
$N=3$	.5742699999	-1.831138832	2.256868832	X
$N=4$	-1.00829255	4.0168024	-6.0033339	3.99482406

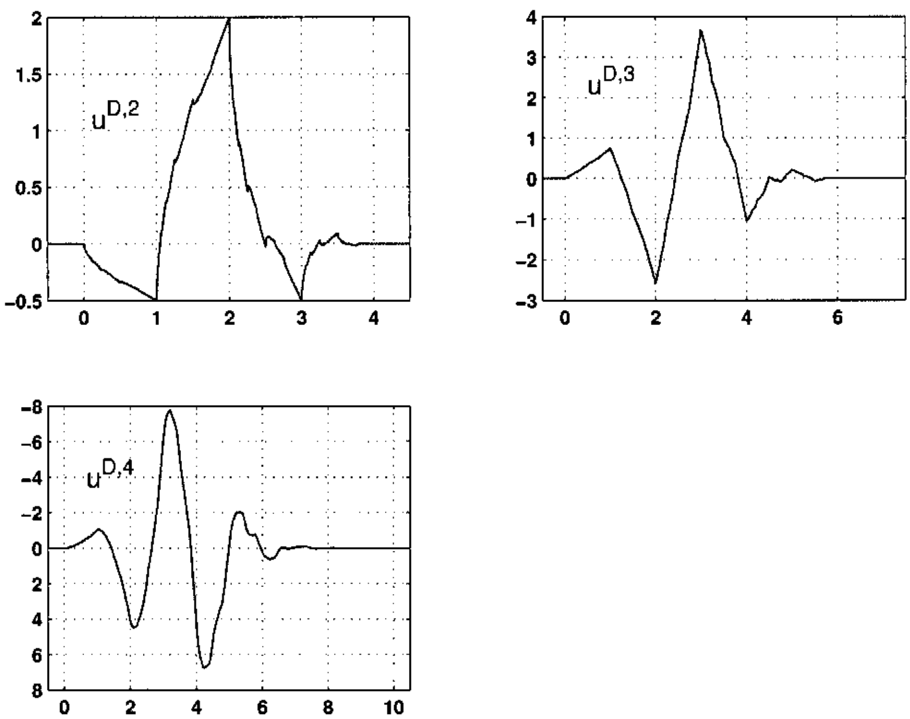


Figure 3.9: The functions $\{u = u^{D,N} : N = 2, 3, 4\}$

Chapter 4

The projection operator

For a given refinement pair (a, ϕ), and corresponding refinement spaces $\{V^{(r)} = V_{\phi}^{(r)} : r \in \mathbb{Z}\}$, as defined by (1.9), our wavelet construction method of the forthcoming Chapter 5 will rely on our ability to construct a sequence $\{\mathcal{P}_r : r \in \mathbb{Z}\}$, with $\mathcal{P}_r : V^{(r+1)} \longrightarrow V^{(r)}$, $r \in \mathbb{Z}$, of local linear projection operators.

4.1 The fundamental Bezout identity

Our first result, in Theorem 4.1 below, provides a characterisation of such a local linear projection operator sequence in terms of the solution of a Bezout identity.

For a Laurent polynomial P defined by $P(z) = \sum_j p_j z^j$, $z \in \mathbb{C} \setminus \{0\}$, we define the *even part* $P^{(e)}$ and the *odd part* $P^{(o)}$ respectively by

$$P^{(e)}(z) = \sum_j p_{2j} z^{2j} \quad \text{and} \quad P^{(o)}(z) = \sum_j p_{2j+1} z^{2j+1}, \quad z \in \mathbb{C} \setminus \{0\}, \tag{4.1}$$

so that

$$\left.\begin{aligned} P(z) &= P^{(e)}(z) + P^{(o)}(z) \\ P(-z) &= P^{(e)}(z) - P^{(o)}(z) \end{aligned}\right\}, \quad z \in \mathbb{C} \setminus \{0\}.$$

Thus,

$$\left.\begin{aligned} P^{(e)}(z) &= \frac{P(z) + P(-z)}{2} \\ P^{(o)}(z) &= \frac{P(z) - P(-z)}{2} \end{aligned}\right\}, \quad z \in \mathbb{C} \setminus \{0\}. \tag{4.2}$$

Theorem 4.1. *For a given refinement pair (a, ϕ), let A denote the corresponding refinement mask symbol polynomial of degree n , as given by (1.4). Then, if Λ is a Laurent polynomial satisfying the Bezout identity*

$$A(z)\Lambda(z) + A(-z)\Lambda(-z) = 2\,, \quad z \in \mathbb{C} \setminus \{0\}\,, \tag{4.3}$$

the local linear operator sequence $\{\mathcal{P}_r : r \in \mathbb{Z}\}$ where $\mathcal{P}_r : V^{(r+1)} \to V^{(r)}$, $r \in \mathbb{Z}$, as defined by

$$\mathcal{P}_r f = \sum_j \left[\sum_k \lambda_{2j-k} c_k\right] \phi(2^r \cdot -j) \ \textit{for}\ f = \sum_j c_j \phi(2^{r+1} \cdot -j)\,, \quad r \in \mathbb{Z}\,, \tag{4.4}$$

and with the sequence $\{\lambda_j : j \in \mathbb{Z}\} \in \mathcal{M}_0(\mathbb{Z})$ given by

$$\Lambda(z) = \sum_j \lambda_j z^j\,, \quad z \in \mathbb{C} \setminus \{0\}\,, \tag{4.5}$$

satisfies the reproduction property

$$\mathcal{P}_r f = f\,, \quad f \in V^{(r)}\,, \quad r \in \mathbb{Z}\,, \tag{4.6}$$

so that $\mathcal{P}_r$ is a projection on $V^{(r)}$ for every $r \in \mathbb{Z}$.

Proof. Let $r \in \mathbb{Z}$ be fixed, and suppose $f \in V^{(r)}$, i.e. there exists a sequence $\tilde{c} \in \mathcal{M}(\mathbb{Z})$ such that $f = \sum_j \tilde{c}_j \phi(2^r \cdot -j)$. Using also the refinement equation (1.1), we deduce that

$$\begin{aligned} f &= \sum_j \tilde{c}_j \sum_k a_k \phi(2^{r+1} \cdot -2j-k) \\ &= \sum_j \tilde{c}_j \sum_k a_{k-2j} \phi(2^{r+1} \cdot -k) \\ &= \sum_k \left[\sum_j a_{k-2j} \tilde{c}_j\right] \phi(2^{r+1} \cdot -k)\,. \end{aligned} \tag{4.7}$$

It follows from (4.4) and (4.7) that if, for a sequence $\{\lambda_j : j \in \mathbb{Z}\} \in \mathcal{M}_0(\mathbb{Z})$, the operator sequence $\{\mathcal{P}_r : r \in \mathbb{Z}\}$ is defined by (4.4), we have, for $r \in \mathbb{Z}$,

$$\begin{aligned} \mathcal{P}_r f &= \sum_j \left[\sum_k \lambda_{2j-k} \sum_l a_{k-2l} \tilde{c}_l\right] \phi(2^r \cdot -j) \\ &= \sum_j \left[\sum_l \left\{\sum_k a_{k-2l} \lambda_{2j-k}\right\} \tilde{c}_l\right] \phi(2^r \cdot -j)\,. \end{aligned} \tag{4.8}$$

Hence $\mathcal{P}_r f = f$ if and only if

$$\sum_j \left[\tilde{c}_j - \sum_l \left\{\sum_k a_{k-2l}\lambda_{2j-k}\right\}\tilde{c}_l\right]\phi(2^r\cdot-j) = 0\,,$$

or, equivalently,

$$\sum_j \left\{\sum_l \left[\delta_{j,l} - \sum_k a_{k-2l}\lambda_{2j-k}\right]\tilde{c}_l\right\}\phi(2^r\cdot-j) = 0\,.$$

It follows that, if the sequence $\{\lambda_j : j\in\mathbb{Z}\}$ in the definition (4.4) is chosen to satisfy the condition

$$\sum_k a_{k-2l}\lambda_{2j-k} = \delta_{j,l}, \quad j\,,\, l\in\mathbb{Z}\,, \tag{4.9}$$

then the reproduction property (4.6) is satisfied. Our proof will therefore be complete if we can prove that $\{\lambda_j : j\in\mathbb{Z}\}$ is a sequence in $\mathcal{M}_0(\mathbb{Z})$ satisfying (4.9) if and only if the Laurent polynomial Λ satisfies the Bezout identity (4.3), with $\{\lambda_j : j\in\mathbb{Z}\}$ and Λ related by (4.5).

To this end, we first note from (1.4) and (4.5) that, for a sequence $\{\lambda_j : j\in\mathbb{Z}\}\in\mathcal{M}_0(\mathbb{Z})$, and for $j\in\mathbb{Z}\,,\ z\in\mathbb{C}\setminus\{0\}$, we have

$$\begin{aligned}
\sum_l\sum_k a_{k-2l}\lambda_{2j-k}z^{2l} &= \sum_l\sum_k a_{2k-2l}\lambda_{2j-2k}z^{2l-2k}z^{2k}\\
&\quad+\sum_l\sum_k a_{2k+1-2l}\lambda_{2j-2k-1}z^{2l-2k-1}z^{2k+1}\\
&= \sum_k \lambda_{2j-2k}\left[\sum_l a_{2k-2l}z^{-(2k-2l)}\right]z^{2k}\\
&\quad+\sum_k \lambda_{2j-2k-1}\left[\sum_l a_{2k+1-2l}z^{-(2k+1-2l)}\right]z^{2k+1}\\
&= A^{(e)}(z^{-1})\left[\sum_k \lambda_{2j-2k}z^{-(2j-2k)}\right]z^{2j}\\
&\quad+A^{(o)}(z^{-1})\left[\sum_k \lambda_{2j-2k-1}z^{-(2j-2k-1)}\right]z^{2j}\\
&= z^{2j}\left[A^{(e)}(z^{-1})\Lambda^{(e)}(z^{-1}) + A^{(o)}(z^{-1})\Lambda^{(o)}(z^{-1})\right]\\
&= z^{2j}\left[\frac{A(z^{-1})+A(-z^{-1})}{2}\,\frac{\Lambda(z^{-1})+\Lambda(-z^{-1})}{2}\right.\\
&\qquad\left.+\frac{A(z^{-1})-A(-z^{-1})}{2}\,\frac{\Lambda(z^{-1})-\Lambda(-z^{-1})}{2}\right]\\
&= \frac{1}{2}z^{2j}\left[A(z^{-1})\Lambda(z^{-1}) + A(-z^{-1})\Lambda(-z^{-1})\right]\,,
\end{aligned} \tag{4.10}$$

whereas

$$\sum_l \delta_{j,l} z^{2l} = z^{2j}\,, \quad z \in \mathbb{C}\,. \tag{4.11}$$

It follows from (4.10) and (4.11) that, for $z \in \mathbb{C} \setminus \{0\}$ and $j \in \mathbb{Z}$,

$$\sum_l \left[\sum_k a_{k-2l}\lambda_{2j-k} - \delta_{j,l}\right] z^{2l} = z^{2j}\left[\frac{A(z^{-1})\Lambda(z^{-1}) + A(-z^{-1})\Lambda(-z^{-1})}{2} - 1\right]. \tag{4.12}$$

According to (4.12), $\{\lambda_j : j \in \mathbb{Z}\}$ is a sequence in $\mathcal{M}_0(\mathbb{Z})$ satisfying the linear system (4.9) if and only if Λ is a Laurent polynomial satisfying the Bezout identity

$$A(z^{-1})\Lambda(z^{-1}) + A(-z^{-1})\Lambda(-z^{-1}) = 2\,, \quad z \in \mathbb{C} \setminus \{0\}\,,$$

which is equivalent to the Bezout identity (4.3). ■

Next, we investigate the solvability of the Bezout identity (4.3).

We shall say that a polynomial p has a *symmetric zero* at $z_0 \in \mathbb{C} \setminus \{0\}$ if and only if $p(z_0) = p(-z_0) = 0$.

First, we derive the following necessary condition.

Proposition 4.2. *For a polynomial A of degree $n \in \mathbb{N}$, suppose there exists a Laurent polynomial Λ satisfying the Bezout identity (4.3). Then A has no symmetric zeros, and $A(0) \neq 0$.*

Proof. Suppose A has a symmetric zero $z_0 \in \mathbb{C} \setminus \{0\}$, i.e. $A(z_0) = A(-z_0) = 0$. But then, if we set $z = z_0$ in (4.3), a contradiction is obtained. Similarly, the assumption $A(0) = 0$ leads to a contradiction. ■

We see from Proposition 4.2 that, for a given A, the Bezout identity (4.3) possesses no Laurent polynomial solution Λ if A has a symmetric zero, or if $A(0) = 0$.

Our following polynomial algebra result is of fundamental importance in the solving of the Bezout identity (4.3).

Theorem 4.3. *Suppose A is a polynomial of degree $n \in \mathbb{N}$, with $n \geq 2$, and such that A possesses no symmetric zeros, and $A(0) \neq 0$. Then there exists a unique polynomial $S \in \Pi_{n-2}$ satisfying the Bezout identity*

$$A(z)S(z) - A(-z)S(-z) = z^{\mu}, \quad z \in \mathbb{C}, \tag{4.13}$$

where μ is the odd integer given by

$$\mu = \begin{cases} n-1 & , \text{ if } n \text{ is even}, \\ n-2 & , \text{ if } n \text{ is odd}. \end{cases} \tag{4.14}$$

Proof. For a polynomial p, we define the polynomial p_- by $p_-(z) = p(-z)$, $z \in \mathbb{C}$. Since the polynomial A has no symmetric zeros, and $A(0) \neq 0$, it follows that the two polynomials A and A_- have no common factors, i.e. $\gcd\{A, A_-\} = 1$. According to Bezout's theorem (see **[20]**, p.169 for the proof), there therefore exist polynomials U and V such that

$$A(z)U(z) + A(-z)V(z) = 1, \quad z \in \mathbb{C}. \tag{4.15}$$

Now use the polynomial division theorem to deduce, since also $\deg(A) = n$, that there exist polynomials Q and R, with $R \in \Pi_{n-1}$, and with Q and R uniquely determined by V and A, such that

$$z^{\mu}V(z) = Q(z)A(z) + R(z), \quad z \in \mathbb{C}. \tag{4.16}$$

Multiplying the equation (4.15) by z^{μ}, and using (4.16), we find that

$$A(z)S(z) + A(-z)R(z) = z^{\mu}, \quad z \in \mathbb{C}, \tag{4.17}$$

with S defined by

$$S(z) = z^{\mu}U(z) + Q(z)A(-z), \; z \in \mathbb{C}. \tag{4.18}$$

Observe that anyone of the two assumptions $S = 0$ or $R = 0$, together with the definition of μ given in (4.14), contradicts (4.17). Hence $S \neq 0$ and $R \neq 0$.
Now rewrite (4.17) in the form

$$A(z)S(z) = z^{\mu} - A(-z)R(z), \quad z \in \mathbb{C},$$

from which, together with the fact that (4.14) gives $\mu \leq n-1$, we see that

$$n + \deg(S) = \deg(AS) = \deg(z^{\mu} - A_-R) = \deg(AR) \leq 2n - 1,$$

and thus $\deg(S) \leq n-1$, i.e. $S \in \Pi_{n-1}$.

We claim that $\{S, R\}$ is the unique solution pair in Π_{n-1} of the Bezout identity (4.17). To prove this claim, suppose $\tilde{S}$, $\tilde{R} \in \Pi_{n-1}$ are such that

$$A(z)\tilde{S}(z) + A(-z)\tilde{R}(z) = z^{\mu}, \quad z \in \mathbb{C}. \tag{4.19}$$

Substracting (4.19) from (4.17) then yields

$$A\left(S - \tilde{S}\right) = A_{-}\left(\tilde{R} - R\right). \tag{4.20}$$

Suppose $S \neq \tilde{S}$, so that also, from (4.20), $R \neq \tilde{R}$. Since the polynomials A and A_{-} have no common factors, it follows from (4.20) that there exists a non-zero polynomial W such that

$$S - \tilde{S} = WA_{-},$$

and thus, since also S , $\tilde{S} \in \Pi_{n-1}$,

$$n - 1 \geq \deg(S - \tilde{S}) = \deg(W) + \deg(A) \geq \deg(A) = n,$$

a contradiction. It follows that $S = \tilde{S}$ and thus also, from (4.20), we have $R = \tilde{R}$, thereby proving our uniqueness claim.

Since (4.17) holds for all $z \in \mathbb{C}$, we may replace z by $-z$ in (4.17) to obtain

$$A(z)[-R(-z)] + A(-z)[-S(-z)] = z^{\mu}, \quad z \in \mathbb{C}, \tag{4.21}$$

after having recalled also that μ is an odd integer. Noting furthermore that the polynomials $-R_{-}$ and $-S_{-}$ both belong to Π_{n-1}, it follows from (4.17) and (4.21), together with the uniqueness result above, that $S = -R_{-}$, $z \in \mathbb{C}$, which , when substituted into (4.17), yields the desired Bezout identity (4.13).

Hence, we have shown that there exists a unique polynomial $S \in \Pi_{n-1}$ satisfying (4.13). It therefore remains to prove that $S \in \Pi_{n-2}$.

To this end, we write $A(z) = \sum_{j=0}^{n} a_j z^j$, $z \in \mathbb{C}$, and $S(z) = \sum_{j=0}^{n-1} s_j z^j$, $z \in \mathbb{C}$, according to which

$$A(z)S(z) - A(-z)S(z) = 2a_n s_{n-1} z^{2n-1} + T(z), \quad z \in \mathbb{C}, \tag{4.22}$$

where $T \in \Pi_{2n-2}$. Since, moreover, (4.14) and the fact that $n \geq 2$, give

$$\mu \leq n-1 < 2n-1\,,$$

it follows from (4.22) and (4.13) that $a_n s_{n-1} = 0$. But $\deg(A) = n$ implies $a_n \neq 0$, and thus $s_{n-1} = 0$, so that $S \in \Pi_{n-2}$. ■

Remark. The uniqueness statement in Theorem 4.3 implies that S is also the polynomial of minimal degree satisfying the Bezout identity (4.13).

The following result on the solvability of the Bezout identity (4.3) is now an immediate consequence of Theorem 4.3.

Corollary 4.4. *For $n \in \mathbb{N}$, with $n \geq 2$, let the polynomials A and S, and the odd integer μ, be as in Theorem 4.3. Then the Laurent polynomial Λ defined by*

$$\Lambda(z) = \frac{2S(z)}{z^{\mu}}\,, \quad z \in \mathbb{C} \setminus \{0\}\,, \tag{4.23}$$

is the unique Laurent polynomial of the form (4.5), with

$$\lambda_j = 0\,, \quad j \notin \begin{cases} \{-n+1,\ldots,-1\}\,, & \text{if } n \text{ is even},\\ \{-n+2,\ldots,0\}\,, & \text{if } n \text{ is odd}, \end{cases} \tag{4.24}$$

and such that the Bezout identity (4.3) is satisfied.

The proof of Theorem 4.3, together with (4.23), yield the following algorithm for the construction of the Laurent polynomial Λ of Corollary 4.4.

Algorithm 4.1.

1. *Use the Euclidean algorithm to find polynomials U and V such that (4.15) holds.*
2. *Define the integer μ by (4.14).*
3. *Use polynomial division to find the polynomials Q and R, with $R \in \Pi_{n-1}$, such that (4.16) is satisfied.*
4. *Define*

$$\Lambda(z) = -\frac{2}{z^{\mu}}R(-z)\,, \quad z \in \mathbb{C} \setminus \{0\}\,. \tag{4.25}$$

4.2 The symmetric case

Let p be a polynomial of degree k. Then there exists an integer $l \in \mathbb{Z}_+$, with $l \leq k$, and a polynomial q of degree $k-l$ with $q(0) \neq 0$, such that $p(z) = z^l q(z)$, $z \in \mathbb{C}$, so that $q = p$ if and only if $l = 0$, and where l and q are uniquely determined by p. We shall say that p is a *symmetric polynomial* if and only if, in the notation $q(z) = \sum_{j=0}^{k-l} q_j z^j$, $z \in \mathbb{C}$, we have $q_{k-l-j} = q_j$, $j = 0, 1, \ldots, k-l$, or equivalently, $z^{k-l} q(z^{-1}) = q(z)$, $z \in \mathbb{C}$.

Our following result shows the sense in which polynomial symmetry is preserved in the context of Theorem 4.3.

Theorem 4.5. *For $n \in \mathbb{N}$, let the polynomial A be as in Theorem 4.3. Suppose furthermore that A is a symmetric polynomial.*

(a) If n is even, the polynomial S of Theorem 4.3 is a symmetric polynomial;

(b) if n is odd, the polynomial $\tilde{S} \in \Pi_n$ defined by

$$\tilde{S} = \frac{1}{2}(1+z)S(z), \quad z \in \mathbb{C}, \tag{4.26}$$

where $S \in \Pi_{n-1}$ is the polynomial obtained as in Theorem 4.3, but with n replaced by the even integer $n+1$, and with A replaced by the polynomial $\tilde{A}$ of degree $n+1$ defined by

$$\tilde{A} = \frac{1}{2}(1+z)A(z), \quad z \in \mathbb{C}, \tag{4.27}$$

is a symmetric polynomial satisfying the Bezout identity

$$A(z)\tilde{S}(z) - A(-z)\tilde{S}(-z) = z^n, \quad z \in \mathbb{C}. \tag{4.28}$$

Proof. (a) If n is even, it follows from (4.14), and (4.13) with z replace by z^{-1}, that

$$A(z^{-1})S(z^{-1}) - A(-z^{-1})S(-z^{-1}) = z^{-n+1}, \quad z \in \mathbb{C} \setminus \{0\},$$

and thus

$$\left[z^n A(z^{-1})\right]\left[z^{n-2}S(z^{-1})\right] - \left[(-z)^n A(-z^{-1})\right]\left[(-z)^{n-2}S(-z^{-1})\right] = z^{n-1}, \quad z \in \mathbb{C} \setminus \{0\}. \tag{4.29}$$

Since A is a symmetric polynomial of degree n, with $A(0) \neq 0$, it follows that $z^n A(z^{-1}) = A(z)$, $z \in \mathbb{C}$. Thus, if we define the polynomial $\hat{S}$ by $\hat{S} = z^{n-2}S(z^{-1})$, $z \in \mathbb{C} \setminus \{0\}$, from which $S \in \Pi_{n-2}$ implies $\tilde{S} \in \Pi_{n-2}$, we find that the Bezout identity

$$A(z)\hat{S}(z) - A(-z)\hat{S}(-z) = z^{n-1}, \quad z \in \mathbb{C}, \tag{4.30}$$

is satisfied. By comparing (4.13) and (4.30), and recalling also the uniqueness statement in Theorem 4.3, we deduce that $S = \hat{S}$, i.e.

$$z^{n-2}S(z^{-1}) = S(z), \quad z \in \mathbb{C} \setminus \{0\}, \tag{4.31}$$

which, in the notation $S(z) = \sum_{j=0}^{n-2} s_j z^j$, $z \in \mathbb{C}$, is equivalent to

$$s_{n-2-j} = s_j \quad j = 0, 1, \ldots, n-2. \tag{4.32}$$

Suppose first that $S(0) \neq 0$. Since $S(0) = s_0$, it follows that $s_0 \neq 0$. But (4.32) gives $s_0 = s_{n-2}$ and thus $s_{n-2} \neq 0$, so that $\deg(S) = n-2$. It then follows from (4.32) and (4.31) that S is symmetric.

Next, suppose $S(0) = 0$, and denote by σ the largest non-negative integer such that

$$s_0 = \ldots = s_\sigma = 0. \tag{4.33}$$

It follows from (4.32) and (4.31) that then also

$$s_{n-2-\sigma} = \ldots = s_{n-2} = 0. \tag{4.34}$$

Moreover, since $S \neq 0$ from (4.13), we deduce from (4.32) and (4.34) that $n-3-\sigma \geq \sigma$, i.e. $\sigma \leq \frac{1}{2}(n-3)$, and thus, since also n is even, so that $n \geq 4$, we have the inequality $\sigma \leq \frac{1}{2}(n-4)$. Also, note from (4.34) that $\deg(S) = n-3-\sigma$.

Denote by T the polynomial with, according to (4.33), the properties $\deg(T) = n-4-\sigma$, $T(0) \neq 0$, and $S(z) = z^{\sigma+1}T(z)$, $z \in \mathbb{C}$. Now use (4.31) to obtain, for $z \in \mathbb{C} \setminus \{0\}$,

$$z^{\sigma+1}T(z) = z^{n-2}S(z^{-1}) = z^{n-3-\sigma}T(z^{-1}),$$

and thus

$$z^{n-4-2\sigma}T(z^{-1}) = T(z), \quad z \in \mathbb{C} \setminus \{0\},$$

showing that S is also symmetric if $S(0) = 0$, and thereby completing our proof of (a).

(b) Suppose next that n is odd. We see from the definition (4.27) that $\tilde{A}$ is a polynomial of even degree $n+1$, with no symmetric zeros; also $\tilde{A}(0) = \frac{1}{2}A(0) \neq 0$, and $\tilde{A}$ satisfies, for $z \in \mathbb{C}$,

$$\begin{aligned} z^{n+1}\tilde{A}(z^{-1}) &= \frac{1}{2}z^{n+1}(1+z^{-1})A(z^{-1}) \\ &= \frac{1}{2}(1+z)\left[z^n A(z^{-1})\right] \\ &= \frac{1}{2}(1+z)A(z) = \tilde{A}(z), \end{aligned}$$

since A is symmetric, and it follows that $\tilde{A}$ is also a symmetric polynomial. We may therefore appeal to Theorem 4.3 to deduce that there does indeed exist a (unique) polynomial $S \in \Pi_{n-1}$ satisfying the Bezout identity

$$\tilde{A}(z)S(z) - \tilde{A}(-z)S(-z) = z^n, \quad z \in \mathbb{C}. \tag{4.35}$$

Also, according to (a) above, S is a symmetric polynomial, with, from (4.31),

$$z^{n-1}S\left(\frac{1}{z}\right) = S(z), \quad z \in \mathbb{C} \setminus \{0\}. \tag{4.36}$$

Next we substitute (4.27) into (4.35) to obtain

$$A(z)\left[\frac{1}{2}(1+z)S(z)\right] - A(-z)\left[\frac{1}{2}(1-z)S(-z)\right] = z^n, \quad z \in \mathbb{C},$$

from which it follows that the polynomial $\tilde{S} \in \Pi_n$, as defined by (4.26), does indeed satisfy the Bezout identity (4.28). Moreover, using (4.26) and (4.36), we obtain, for $z \in \mathbb{C}$,

$$\begin{aligned} z^n\tilde{S}(z^{-1}) &= \frac{1}{2}z^n(1+z^{-1})S(z^{-1}) \\ &= \frac{1}{2}(1+z)\left[z^{n-1}S(z^{-1})\right] \\ &= \frac{1}{2}(1+z)S(z) = \tilde{S}(z), \end{aligned} \tag{4.37}$$

from which the symmetry of the polynomial $\tilde{S}$ follows as in the proof of the symmetry of the polynomial S in (a) above. ■

If n is odd, it follows from (4.28) in Theorem 4.5 (b) that the following extension of Corollary 4.4 holds.

Corollary 4.6. *Suppose $n \in \mathbb{N}$, with n an odd integer, and let the symmetric polynomials A and $\tilde{S}$ be as in Theorem 4.5. Then the Laurent polynomial $\Lambda = \tilde{\Lambda}$ defined by*

$$\tilde{\Lambda}(z) = \frac{2\tilde{S}(z)}{z^n}, \quad z \in \mathbb{C} \setminus \{0\}, \tag{4.38}$$

satisfies the Bezout identity (4.3), with, in the notation $\tilde{\Lambda}(z) = \sum_j \tilde{\lambda}_j z^j$, $z \in \mathbb{C} \setminus \{0\}$,

$$\tilde{\lambda}_j = 0 \; j \notin \{-n, \ldots, 0\}. \tag{4.39}$$

4.3 Example A: the cardinal B-spline case

The polynomial solution S of the Bezout identity (4.13) can always be obtained by means of Algorithm 4.1, the first step of which involves an application of the Euclidean algorithm.

We proceed to show that, for the cardinal B-spline case $(a, \phi) = (a^{(m)}, N_m)$, $m \geq 2$, for which $A = A_m$, as given by (3.37), the polynomials $S = S_m$ of Theorem 4.3 can be computed recursively with respect to the order m.

First, we see from Theorem 4.3 and (3.37) that, for $m \geq 2$, S_m is the unique polynomial in Π_{m-2} satisfying the Bezout identity

$$(1+z)^m S_m(z) - (1-z)^m S_m(-z) = 2^{m-1} z^{\mu}, \quad z \in \mathbb{C}, \tag{4.40}$$

where

$$\mu = \begin{cases} m-2 & \text{, if } m \text{ is odd}, \\ m-1 & \text{, if } m \text{ is even}. \end{cases} \tag{4.41}$$

Setting $m = 2$ in (4.40) and (4.41), we get

$$S_2(z) = \frac{1}{2}, \quad z \in \mathbb{C}. \tag{4.42}$$

For a fixed integer $k \in \mathbb{N}$, we next successively set $m = 2k$, $m = 2k+1$ and $m = 2k+2$ in (4.40), to obtain, after using also (4.41),

$$(1+z)^{2k} S_{2k}(z) - (1-z)^{2k} S_{2k}(-z) = 2^{2k-1} z^{2k-1}, \quad z \in \mathbb{C}, \tag{4.43}$$

$$(1+z)^{2k+1} S_{2k+1}(z) - (1-z)^{2k+1} S_{2k+1}(-z) = 2^{2k} z^{2k-1}, \quad z \in \mathbb{C}, \tag{4.44}$$

$$(1+z)^{2k+2}S_{2k+2}(z)-(1-z)^{2k+2}S_{2k+2}(-z)=2^{2k+1}z^{2k+1}, \quad z\in\mathbb{C}. \tag{4.45}$$

It follows from (4.43) and (4.44) that

$$(1+z)^{2k}\left[(1+z)S_{2k+1}(z)-2S_{2k}(z)\right] = (1-z)^{2k}\left[(1-z)S_{2k+1}(-z)-2S_{2k}(-z)\right], \quad z\in\mathbb{C}. \tag{4.46}$$

Since the two polynomials $(1+z)^{2k}$ and $(1-z)^{2k}$ have no common factors, and since $S_m\in\Pi_{m-2}$, it follows from (4.46) that we must have

$$(1+z)S_{2k+1}(z)-2S_{2k}(z)=c(1-z)^{2k}, \quad z\in\mathbb{C}, \tag{4.47}$$

for some constant $c\in\mathbb{R}$. Setting $z=-1$ in (4.47) then yields $c=-2^{1-2k}S_{2k}(-1)$, which, inserted into (4.47), gives

$$S_{2k+1}(z)=\frac{2S_{2k}(z)-2^{1-2k}S_{2k}(-1)(1-z)^{2k}}{1+z}, \quad z\in\mathbb{C}. \tag{4.48}$$

Next, we use (4.44) and (4.45) to deduce that

$$\begin{aligned}(1+z)^{2k+1}&\left[(1+z)S_{2k+2}(z)-2z^2S_{2k+1}(z)\right]\\ &=(1-z)^{2k+1}\left[(1-z)S_{2k+2}(-z)-2z^2S_{2k+1}(-z)\right], \quad z\in\mathbb{C}.\end{aligned} \tag{4.49}$$

As above, we see that we must have

$$(1+z)S_{2k+2}(z)-2z^2S_{2k+1}(z)=\tilde{c}(1-z)^{2k+1}, \quad z\in\mathbb{C}, \tag{4.50}$$

for some constant $\tilde{c}\in\mathbb{R}$. Setting $z=-1$ in (4.50) then gives $\tilde{c}=-2^{-2k}S_{2k+1}(-1)$, which, inserted into (4.50), yields

$$S_{2k+2}(z)=\frac{2z^2S_{2k+1}(z)-2^{-2k}S_{2k+1}(-1)(1-z)^{2k+1}}{1+z}, \quad z\in\mathbb{C}. \tag{4.51}$$

Hence we have proved the following result.

Theorem 4.7. *If, in Theorem 4.3 , we have, for an integer $m\geq 2$, that $A=A_m$, as given by (3.37), then the (unique) polynomial $S=S_m$ in Π_{m-2} solving the Bezout identity*

(4.13) can be computed by means of the recursive formulation

$$\left.\begin{aligned} S_2(z) &= \frac{1}{2}, \\ S_{2k+1}(z) &= \frac{2S_{2k}(z) - 2^{1-2k}S_{2k}(-1)(1-z)^{2k}}{1+z}, \\ S_{2k+2}(z) &= \frac{2z^2 S_{2k+1}(z) - 2^{-2k}S_{2k+1}(-1)(1-z)^{2k+1}}{1+z}, \end{aligned}\right\}, \quad z \in \mathbb{C}, \quad k \in \mathbb{N}. \tag{4.52}$$

Observe in particular in Table 4.1 that the polynomials $\{S_m : m = 2, 4\}$ are symmetric, which, since A_m is a symmetric polynomial , as can be verified from (3.37), is in accordance with Theorem 4.5(a).

Table 4.1: The polynomials $\{S_m : m = 2, \ldots, 5\}$ and $\{\tilde{S}_m : m = 3, 5\}$

	$S_m(z)$	$\tilde{S_m}(z)$
$m = 2$	$\frac{1}{2}$	X
$m = 3$	$-\frac{1}{4}z + \frac{3}{4}$	$-\frac{1}{8}z^3 + \frac{3}{8}z^2 + \frac{3}{8}z - \frac{1}{8}$
$m = 4$	$-\frac{1}{4}z^2 + z - \frac{1}{4}$	X
$m = 5$	$\frac{3}{32}z^3 - \frac{15}{32}z^2 + \frac{25}{32}z - \frac{5}{32}$	$\frac{3}{32}z^5 - \frac{15}{32}z^4 + \frac{5}{8}z^3 + \frac{5}{8}z^2 - \frac{15}{32}z + \frac{3}{32}$

Using Theorem 4.7, Corollary 4.4, and (4.5), we now explicitly calculate, for $m = 2, \ldots, 5$, the sequence $\lambda = \lambda_m \in \mathcal{M}_0(\mathbb{Z})$ to be given by Table 4.2.

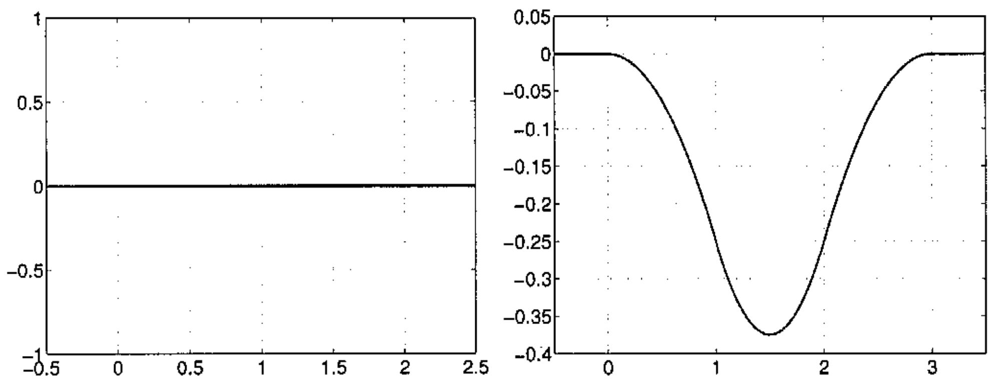
Table 4.2: The sequence $\{\lambda_m : m = 2, \ldots, 5\}$

$\lambda_{m,j}$	$m=2$	$m=3$	$m=4$	$m=5$
$j=0$	X	$-\frac{1}{2}$	X	$\frac{3}{8}$
$j=-1$	1	$\frac{3}{2}$	$-\frac{1}{2}$	$-\frac{15}{8}$
$j=-2$	X	X	2	$\frac{25}{8}$
$j=-3$	X	X	$-\frac{1}{2}$	$-\frac{5}{8}$

With the notation $\Lambda_m(z) = \sum_j \lambda_{m,j} z^j$, $z \in \mathbb{C}\setminus\{0\}$, and $\tilde{\Lambda}_m(z) = \sum_j \tilde{\lambda}_{m,j} z^j$, $z \in \mathbb{C}\setminus\{0\}$, we see that the operator sequence $\{\mathcal{P}_r = \mathcal{P}_{m,r} : r \in \mathbb{Z}\}$ in (4.4) can be expressed in this special case by

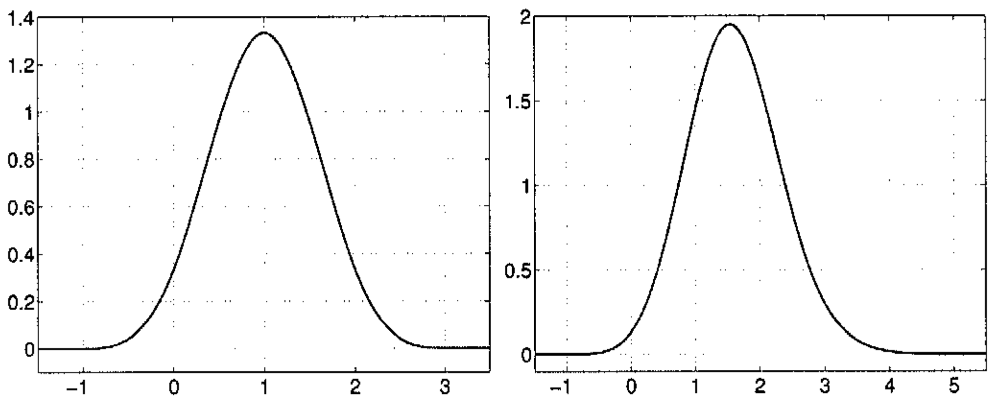
$$\mathcal{P}_{m,r} f = \sum_j \left[\sum_k \lambda_{m,\,2j-k} c_k \right] N_m(2^r \cdot -j)\,,\ r \in \mathbb{Z}\,,\ \text{for } f = \sum_j c_j N_m(2^{(r+1)} \cdot -j)\,. \quad (4.53)$$

By choosing, respectively, $c_j = \delta_j$, $j \in \mathbb{Z}$, and $c_j = \delta_{j-1}$, $j \in \mathbb{Z}$, in (4.53), we can now use (4.53), together with Table 4.2, to compute the functions $\mathcal{P}_{m,0}N_m(2\cdot)$ and $\mathcal{P}_{m,0}\left(N_m(2 \cdot -1)\right)$ for $m = 2, \ldots, 5$. The resulting graphs are shown in Figures 4.1 and 4.2, and 4.3 and 4.4, respectively.



(a) $\mathcal{P}_{2,0}\,(N_2(2\cdot))$ (b) $\mathcal{P}_{3,0}\,(N_3(2\cdot))$

Figure 4.1: $\mathcal{P}_{m,0}(N_m(2\cdot))\,,\quad m=2,3$



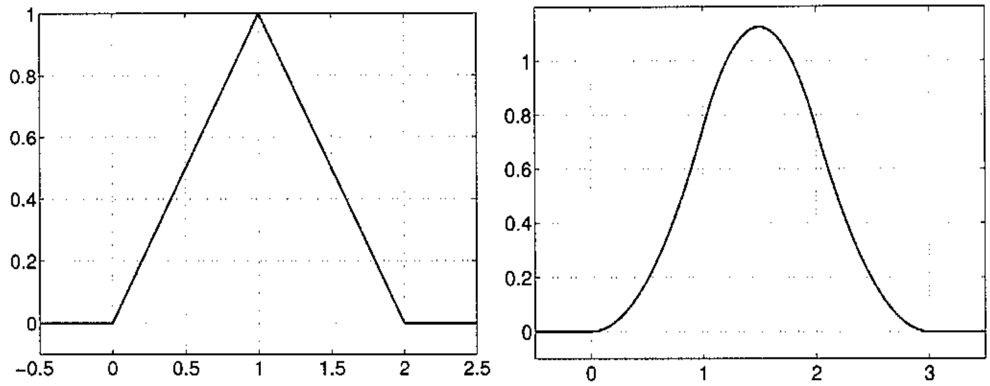
(a) $\mathcal{P}_{4,0}\,(N_4(2\cdot))$ (b) $\mathcal{P}_{5,0}\,(N_5(2\cdot))$

Figure 4.2: $\mathcal{P}_{m,0}(N_m(2\cdot))\,,\quad m=4,5$

For odd order m, we appeal to the symmetrizing Corollary 4.6 to rewrite (4.53) in the form

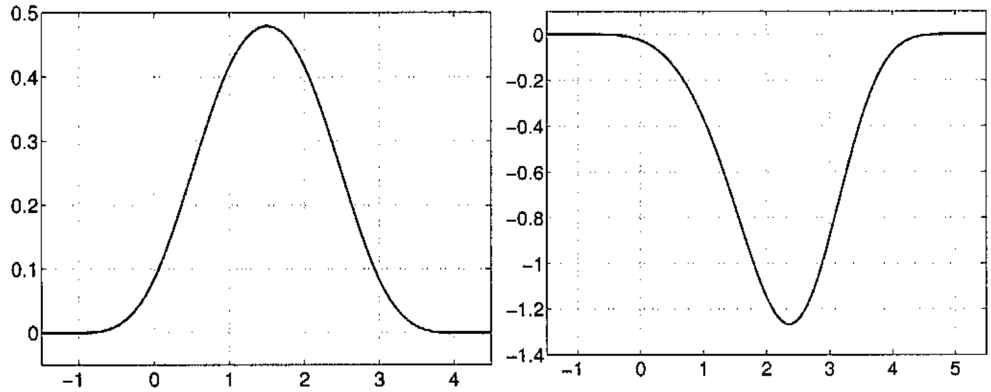
$$\tilde{\mathcal{P}}_{m,r}f=\sum_j\left[\sum_k\tilde{\lambda}_{m,\,2j-k}c_k\right]N_m(2^r\cdot-j)\,,\ r\in\mathbb{Z}\,,\ \text{for } f=\sum_j c_jN_m(2^{(r+1)}\cdot-j)\,. \quad (4.54)$$

Using (4.54), together with Table 4.3, we show in Figures 4.5 and 4.6 the graphs of the functions $\tilde{\mathcal{P}}_{m,0}N_m$ and $\tilde{\mathcal{P}}_{m,0}\,(N_m(\cdot-1))$ for $m=3,5$.



(a) $\mathcal{P}_{2,0}\left(N_2(2\cdot -1)\right)$ (b) $\mathcal{P}_{3,0}\left(N_3(2\cdot -1)\right)$

Figure 4.3: $\mathcal{P}_{m,0}\left(N_m(2\cdot -1)\right)\ , \quad m=2,3$

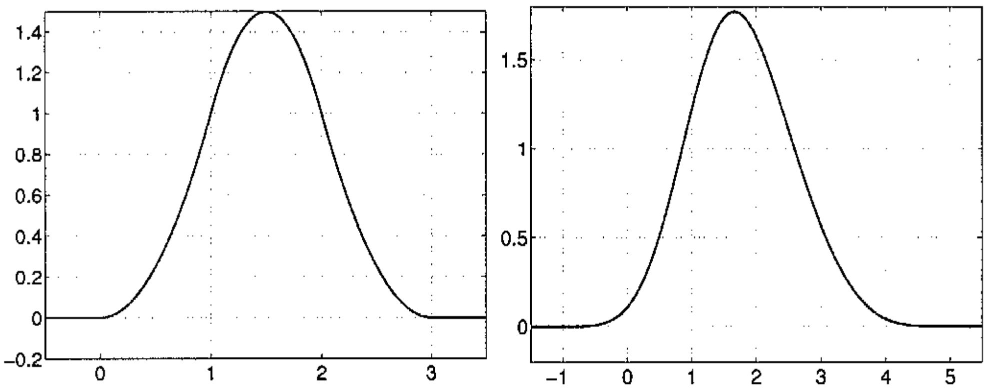


(a) $\mathcal{P}_{4,0}\left(N_4(2\cdot -1)\right)$ (b) $\mathcal{P}_{5,0}\left(N_5(2\cdot -1)\right)$

Figure 4.4: $\mathcal{P}_{m,0}\left(N_m(2\cdot -1)\right)\ , \quad m=4,5$

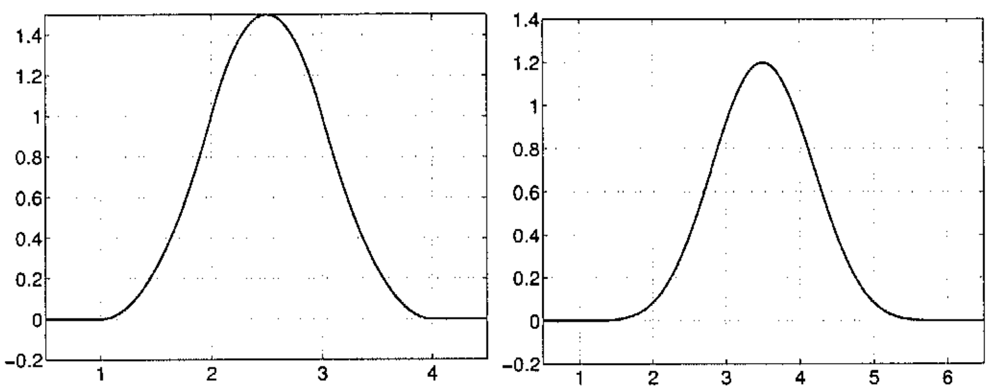
Table 4.3: The sequence $\{\tilde{\lambda}_m : m=3,5\}$

$\tilde{\lambda}_{m,j}$	$j=-5$	$j=-4$	$j=-3$	$j=-2$	$j=-1$	$j=0$
$m=3$	X	X	$-\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$
$m=5$	$\frac{3}{16}$	$-\frac{15}{16}$	$\frac{5}{4}$	$\frac{5}{4}$	$-\frac{15}{16}$	$\frac{3}{16}$



(a) $\tilde{\mathcal{P}}_{3,0}N_3$ (b) $\tilde{\mathcal{P}}_{5,0}N_5$

Figure 4.5: $\tilde{\mathcal{P}}_{m,0}N_m$, $m = 3, 5$



(a) $\tilde{\mathcal{P}}_{3,0}\left(N_3(\cdot - 1)\right)$ (b) $\tilde{\mathcal{P}}_{5,0}\left(N_5(\cdot - 1)\right)$

Figure 4.6: $\tilde{\mathcal{P}}_{m,0}\left(N_m(\cdot - 1)\right)$, $m = 3, 5$

4.4 Example B

For Example B of Sections 3.3 and 4.4, we obtain the following table.

Table 4.4: The polynomials for Example B in Sections 3.3

$U(z)$	$\frac{97}{108}z^4 - \frac{97}{18}z^3 + \frac{149}{12}z^2 - \frac{725}{54}z + 6$
$V(z)$	$\frac{97}{108}z^4 + \frac{97}{18}z^3 + \frac{149}{12}z^2 + \frac{725}{54}z + 6$
$S(z)$	$\frac{13}{108}z^3 - \frac{13}{18}z^2 - \frac{17}{12}z - \frac{17}{54}$
$\Lambda(z)$	$\frac{13}{54} - \frac{13}{9}z^{-1} + \frac{17}{6}z^{-2} - \frac{17}{27}z^{-3}$

4.5 Example C

For the Daubechies refinement pair we compute the polynomial S whose coefficients are given in Table 4.5.

Table 4.5: The coefficients $\{s_j\,,\ j=0,\ldots,2N-3\}$ in the polynomial S associated with $(a^{D,N}\,,\phi_N^D)$

s_j	$N=2$	$N=3$	$N=4$
$j=0$	.3169873002	−.02591462488	.02116436174
$j=1$	.1830127022	.06285585123	−.06567156440
$j=2$	X	.2031176194	−.01293570229
$j=3$	X	.1859042617	.2225484619
$j=4$	X	X	.2532712325
$j=5$	X	X	.08162321377

Chapter 5

Wavelet construction

The local linear projection sequence $\{\mathcal{P}_r : r \in \mathbb{Z}\}$ of Chapter 4 can now be used as the basis for a general multi-resolutional-like framework for wavelet construction.

5.1 Preliminaries

For a given refinement pair $(a\,,\phi)$, let the nested sequence $\{V^{(r)} = V_\phi^{(r)} : r \in \mathbb{Z}\}$ be as defined by (1.9), and suppose that $\{\mathcal{P}_r : r \in \mathbb{Z}\}$ is the local linear projection sequence, with $\mathcal{P}_r : V^{(r+1)} \to V^{(r)}\,,\ r \in \mathbb{Z}$, as in Theorem 4.1. Now introduce the linear space sequences $\{W^{(r)} : r \in \mathbb{Z}\}$ and $\{X^{(r)} : r \in \mathbb{Z}\}$ by the definitions

$$W^{(r)} = \{\, f - \mathcal{P}_r f \,:\, f \in V^{(r+1)} \,\}\,, \quad r \in \mathbb{Z}\,, \tag{5.1}$$

and

$$X^{(r)} = \{\, f \in V^{(r+1)} \,:\, \mathcal{P}_r f = 0 \,\}\,, \quad r \in \mathbb{Z}\,. \tag{5.2}$$

The following result then holds.

Proposition 5.1. *The linear space sequences $\{W^{(r)} : r \in \mathbb{Z}\}$ and $\{X^{(r)} : r \in \mathbb{Z}\}$ defined by (5.1), (5.2) satisfy the properties*

$$W^{(r)} \subset V^{(r+1)}\,, \quad r \in \mathbb{Z}\,, \tag{5.3}$$

and

$$W^{(r)} = X^{(r)}\,, \quad r \in \mathbb{Z}\,. \tag{5.4}$$

Proof. The inclusion (5.3) is immediately clear from (5.1), together with the fact that $\mathcal{P}_r : V^{(r+1)} \to V^{(r)} \subset V^{(r+1)}$, $r \in \mathbb{Z}$, as proved in Theorem 1.2.

To prove (5.4), we fix $r \in \mathbb{Z}$ and supose first that $f \in W^{(r)}$. Note from (5.3) that then $f \in V^{(r+1)}$. Also, according to the definition (5.1),there exists a function $g \in V^{(r+1)}$ such that $f = g - \mathcal{P}_r g$. Since $\mathcal{P}_r : V^{(r+1)} \to V^{(r)}$, and since the reproduction property (4.6) holds, we obtain

$$\mathcal{P}_r f = \mathcal{P}_r(g - \mathcal{P}_r g) = \mathcal{P}_r g - \mathcal{P}_r^2 g = \mathcal{P}_r g - \mathcal{P}_r g = 0\,,$$

so that, according to the definition (5.2), we have $f \in X^{(r)}$. Hence $W^{(r)} \subset X^{(r)}$.

Next, suppose $f \in X^{(r)}$, so that, from definition (5.2), we have that $f \in V^{(r+1)}$ with $f - \mathcal{P}_r f = f$ and thus $f \in W^{(r)}$, so that $X^{(r)} \subset W^{(r)}$, and thereby completing our proof of (5.4). ■

If for a given refinement pair $(a\,, \phi)$, with corresponding local linear projection operator sequence $\{\mathcal{P}_r : r \in \mathbb{Z}\}$ as in Theorem 4.1, a function $\psi \in C_0(\mathbb{R})$ is such that the condition

$$W^{(r)} = \left\{ \sum_j c_j \psi(2^r \cdot -j) : c \in \mathcal{M}(\mathbb{R}) \right\}, \quad r \in \mathbb{Z}\,, \tag{5.5}$$

holds, where the linear space sequence $\{W^{(r)} : r \in \mathbb{Z}\}$ is defined by (5.1), then we shall call ψ the *wavelet generated by the refinement pair* $(a\,, \phi)$ *and its corresponding projection operator sequence* $\{\mathcal{P}_r : r \in \mathbb{Z}\}$.

Hence we proceed, in Section 5.2 below, to seek a function $\psi \in C_0(\mathbb{R})$ satisfying (5.5).

5.2 The Bezout identity for wavelets

Motivated by the result (5.4) of Proposition 5.1, we first seek to obtain a function $\psi \in V^{(1)}$ of least possible support such that

$$\mathcal{P}_0 \psi = 0\,; \tag{5.6}$$

before showing that (5.5) then does indeed hold.

To this end, we first note that ψ is a finitely-supported function in $V^{(1)}$ if there exists a sequence $\gamma \in \mathcal{M}_0(\mathbb{Z})$ such that

$$\psi = \sum_j \gamma_j \phi(2 \cdot -j) . \tag{5.7}$$

We then have the following sufficient condition for (5.6) to hold.

Proposition 5.2. *For a refinement pair (a, ϕ), let Λ denote a Laurent polynomial as in Theorem 4.1. Then, if Γ is a Laurent polynomial satisfying the Bezout identity*

$$\Lambda(z)\Gamma(z) + \Lambda(-z)\Gamma(-z) = 0, \quad z \in \mathbb{C} \setminus \{0\} , \tag{5.8}$$

and the sequence $\gamma \in \mathcal{M}_0(\mathbb{Z})$ is defined by

$$\Gamma(z) = \sum_j \gamma_j z^j , \quad z \in \mathbb{C} \setminus \{0\} , \tag{5.9}$$

the finitely-supported function $\psi \in V^{(1)}$, as given by (5.7), satisfies the condition (5.6).

Proof. Using (4.4) and (5.7), we obtain

$$\mathcal{P}_0 \psi = \sum_j \left[\sum_k \lambda_{2j-k} \gamma_k \right] \phi(\cdot - j) ,$$

from which we see that if the sequence $\gamma \in \mathcal{M}_0(\mathbb{Z})$ is chosen such that

$$\sum_k \lambda_{2j-k} \gamma_k = 0, \quad j \in \mathbb{Z} , \tag{5.10}$$

then the condition (5.6) holds. Our proof will therefore be complete if we can show that (5.10) is equivalent to the Bezout identity (5.8).

To this end, we use (4.5) and (5.9) to deduce that (5.10) holds if and only if we have, for

$z \in \mathbb{C} \setminus \{0\}$, that

$$
\begin{aligned}
0 &= \sum_j \left[\sum_k \lambda_{2j-k}\gamma_k \right] z^{2j} \\
&= \sum_j \left[\sum_k \lambda_{2j-2k}\gamma_{2k} \right] z^{2j} + \sum_j \left[\sum_k \lambda_{2j-2k-1}\gamma_{2k+1} \right] z^{2j} \\
&= \sum_k \left[\sum_j \lambda_{2j-2k} z^{2j-2k} \right] \gamma_{2k} z^{2k} + \sum_k \left[\sum_j \lambda_{2j-2k-1} z^{2j-2k-1} \right] \gamma_{2k+1} z^{2k+1} \\
&= \left[\sum_k \gamma_{2k} z^{2k} \right] \left[\sum_j \lambda_{2j-2k} z^{2j-2k} \right] + \left[\sum_k \gamma_{2k+1} z^{2k+1} \right] \left[\sum_j \lambda_{2j-2k-1} z^{2j-2k-1} \right] \\
&= \Lambda^{(e)}(z)\Gamma^{(e)}(z) + \Lambda^{(o)}(z)\Gamma^{(o)}(z) \\
&= \frac{\Lambda(z) + \Lambda(-z)}{2} \frac{\Gamma(z) + \Gamma(-z)}{2} + \frac{\Lambda(z) - \Lambda(-z)}{2} \frac{\Gamma(z) - \Gamma(-z)}{2} \\
&= \frac{1}{2} \left[\Lambda(z)\Gamma(z) + \Lambda(-z)\Gamma(-z) \right] ,
\end{aligned}
$$

i.e the Bezout identity (5.8) is satisfied. ∎

We proceed to solve the Bezout identity (5.8) as follows.

Proposition 5.3. *In Proposition 5.2, a Laurent polynomial of minimal length satisfying the Bezout identity (5.8) is given by*

$$\Gamma(z) = K z^{2n_0+1} \Lambda(-z) , \quad z \in \mathbb{C} \setminus \{0\} , \tag{5.11}$$

where $K \in \mathbb{R} \setminus \{0\}$ and $n_0 \in \mathbb{Z}$ can be chosen arbitrarily.

Proof. First, rewrite (5.8) in the form

$$\Lambda(z)\Gamma(z) = -\Lambda(-z)\Gamma(-z) , \quad z \in \mathbb{C} \setminus \{0\} . \tag{5.12}$$

Next, we note that the Laurent polynomial Λ possesses no symmetric zeros in $\mathbb{C} \setminus \{0\}$, for if we assume that $z_0 \in \mathbb{C} \setminus \{0\}$ is such that $\Lambda(z_0) = \Lambda(-z_0) = 0$, we obtain a contradiction by setting $z = z_0$ in the Bezout identity (4.3). It follows from (5.12) that a Laurent polynomial Γ of shortest possible length satisfying (5.12) necessarily has the form (5.11).

∎

Combining the results of Propositions 5.2 and 5.3, we deduce that the following result holds.

Theorem 5.4. *Let (a, ϕ) denote a refinement pair, and suppose Λ is a Laurent polynomial as in Theorem 4.1. If, for $K \in \mathbb{R} \setminus \{0\}$ and $n_0 \in \mathbb{Z}$, we define the finitely supported function ψ by means of (5.7), (5.9), (5.11) and (4.5), then ψ is a minimally supported function in $V^{(1)}$ such that the condition (5.6) is satisfied.*

A specific finitely supported $\psi \in V^{(1)}$ satisfying (5.6) can now be formulated as follows.

Theorem 5.5. *Let (a, ϕ) denote a refinement pair, with corresponding mask symbol A as in Theorem 4.3, let the polynomial S of Theorem 4.3 be given by*

$$S(z) = \sum_{j=0}^{n-2} s_j z^j, \quad z \in \mathbb{C} \setminus \{0\}, \tag{5.13}$$

and denote by μ the odd integer defined by (4.14). Then the function ψ given by

$$\psi = 2\sum_{j=0}^{n-2}(-1)^j s_j \phi(2\cdot -j) \tag{5.14}$$

corresponds to the choices $K = -1$ and $n_0 = \frac{\mu-1}{2}$ in (5.11) of Theorem 5.4. Moreover,

$$\psi(x) = 0, \quad x \notin (0, n-1). \tag{5.15}$$

Proof. Choosing the Laurent polynomial Λ of Theorem 4.1 as in Corollary 4.4, and letting $K = -1$ and $n_0 = \frac{\mu-1}{2}$ in (5.11), we see from (5.11), (5.9), (4.23), (4.14) and (5.13) that

$$\sum_j \gamma_j z^j = 2\sum_{j=0}^{n-2}(-1)^j s_j z^j, \quad z \in \mathbb{C},$$

from which it follows that

$$\gamma_j = 0, \quad j \notin \{0, 1, \ldots, n-2\}, \tag{5.16}$$

and

$$\gamma_j = 2(-1)^j s_j, \quad j = 0, 1, \ldots, n-2. \tag{5.17}$$

Inserting (5.16) and (5.17) into (5.7) then yields (5.14).

The finite support property (5.15) is a direct consequence of (5.14), together with condition (2.5). ■

To prove that the function ψ of Theorem 5.4 is a wavelet generated by (a, ϕ), it therefore remains to show that (5.5) holds.

5.3 The basic decomposition result

Our next result shows that the function ψ of Theorem 5.7 satisfies (5.5), i.e. ψ is indeed a wavelet.

Theorem 5.6. *For a refinement pair (a, ϕ), and a projection sequence $\{\mathcal{P}_r : r \in \mathbb{Z}\}$ as in Theorem 4.1, let the finitely supported function $\psi \in V^{(1)}$ be as in Theorem 5.4. Then*

$$f - \mathcal{P}_r f = \sum_j \left[\sum_k \omega_{2j-k} c_k \right] \psi(2^r \cdot -j) \quad \textit{for} \quad f = \sum_j c_j \phi(2^{r+1} \cdot -j)\,, \quad r \in \mathbb{Z}\,, \tag{5.18}$$

where the sequence $\omega \in \mathcal{M}_0(\mathbb{Z})$ is defined by

$$\omega_j = \frac{1}{K}(-1)^{j+1} a_{j+2n_0+1}\,, \quad j \in \mathbb{Z}\,. \tag{5.19}$$

Proof. Let $r \in \mathbb{Z}$ be fixed, and suppose $f = \sum_j c_j \phi(2^{r+1} \cdot -j)$, where $c \in \mathcal{M}(\mathbb{Z})$. We proceed to investigate the existence of a sequence $\omega \in \mathcal{M}_0(\mathbb{Z})$ such that (5.18) holds. Using (4.4) and the refinement equation (1.1) we obtain

$$\begin{aligned}
f - \mathcal{P}_r f &= \sum_j c_j \phi(2^{r+1} \cdot -j) - \sum_j \left[\sum_k \lambda_{2j-k} c_k \right] \sum_l a_l \phi(2^{r+1} \cdot -2j - l) \\
&= \sum_j c_j \phi(2^{r+1} \cdot -j) - \sum_j \left[\sum_k \lambda_{2j-k} c_k \right] \sum_l a_{l-2j} \phi(2^{r+1} \cdot -l) \\
&= \sum_j c_j \phi(2^{r+1} \cdot -j) - \sum_l \left[\sum_k \lambda_{2l-k} c_k \right] \sum_j a_{j-2l} \phi(2^{r+1} \cdot -j) \\
&= \sum_j c_j \phi(2^{r+1} \cdot -j) - \sum_j \sum_l \left[\sum_k \lambda_{2l-k} c_k \right] a_{j-2l} \phi(2^{r+1} \cdot -j) \\
&= \sum_j \left\{ \sum_k \left[\delta_{j,k} - \sum_l \lambda_{2l-k} a_{j-2l} \right] c_k \right\} \phi(2^{r+1} \cdot -j)\,.
\end{aligned} \tag{5.20}$$

Also, (5.7) and the refinement equation (1.1) yields, for $\omega \in \mathcal{M}_0(\mathbb{Z})$,

$$\begin{aligned}
\sum_j \left[\sum_k \omega_{2j-k}c_k\right] \psi(2^r \cdot -j) &= \sum_j \sum_k \omega_{2j-k}c_k \sum_l \gamma_l \phi(2^{r+1} \cdot -2j-l) \\
&= \sum_j \sum_k \omega_{2j-k}c_k \sum_l \gamma_{l-2j} \phi(2^{r+1} \cdot -l) \\
&= \sum_l \sum_k \omega_{2l-k}c_k \sum_j \gamma_{j-2l} \phi(2^{r+1} \cdot -j) \\
&= \sum_j \left\{\sum_k \left[\sum_l \gamma_{j-2l}\omega_{2l-k}\right] c_k\right\} \phi(2^{r+1} \cdot -j) \,. \quad (5.21)
\end{aligned}$$

It follows from (5.20) and (5.21) that a sequence $\omega \in \mathcal{M}_0(\mathbb{Z})$ satisfies (5.18) if and only if

$$\sum_j \left\{\sum_k \left[\delta_{j,k} - \sum_l \lambda_{2l-k}a_{j-2l} - \sum_l \gamma_{j-2l}\omega_{2l-k}\right] c_k\right\} \phi(2^{r+1} \cdot -j) = 0 \,. \tag{5.22}$$

Hence, if we can find a sequence $\omega \in \mathcal{M}_0(\mathbb{Z})$ such that

$$\sum_l \lambda_{2l-k}a_{j-2l} + \sum_l \gamma_{j-2l}\omega_{2l-k} = \delta_{j,k}\,, \quad j\,,\, k \in \mathbb{Z}\,, \tag{5.23}$$

the equation (5.22), and therefore also the condition (5.18), will be satisfied.

We proceed to show that (5.23) can be reformulated as a pair of Bezout identities. To this end, we define the Laurent polynomial Ω by

$$\Omega(z) = \sum_j \omega_j z^j\,, \quad z \in \mathbb{C} \setminus \{0\}\,. \tag{5.24}$$

Using (1.4), (4.5), (5.9) and (5.24), we see that (5.23) holds if and only if, for $j \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \{0\}$, we have

$$\begin{aligned}
z^j &= \sum_k \delta_{j,k} z^k = \sum_k \left[\sum_l \lambda_{2l-k}a_{j-2l}\right] z^k + \sum_k \left[\sum_l \gamma_{j-2l}\omega_{2l-k}\right] z^k \\
&= \sum_l \left[\sum_k \lambda_{2l-k} z^{k-2l}\right] a_{j-2l} z^{2l} + \sum_l \left[\sum_k \omega_{2l-k} z^{k-2l}\right] \gamma_{j-2l} z^{2l} \\
&= \sum_l \left[\sum_k \lambda_k (z^{-1})^k\right] a_{j-2l} z^{2l-j} z^j + \sum_l \left[\sum_k \omega_k (z^{-1})^k\right] \gamma_{j-2l} z^{2l-j} z^j \\
&= z^j \left\{\left[\sum_l a_{j-2l} z^{2l-j}\right] \Lambda(z^{-1}) + \left[\sum_l \gamma_{j-2l} z^{2l-j}\right] \Omega(z^{-1})\right\}\,,
\end{aligned}$$

and thus (5.23) holds if and only if, for $j \in \mathbb{Z}$, the Bezout identity

$$\left[\sum_l a_{j-2l} z^{2l-j}\right] \Lambda(z^{-1}) + \left[\sum_l \gamma_{j-2l} z^{2l-j}\right] \Omega(z^{-1}) = 1\,, \quad z \in \mathbb{C} \setminus \{0\}\,,$$

or, equivalently, the Bezout identity

$$\left[\sum_l a_{j-2l} z^{j-2l}\right] \Lambda(z) + \left[\sum_l \gamma_{j-2l} z^{j-2l}\right] \Omega(z) = 1\,, \quad z \in \mathbb{C} \setminus \{0\}\,, \tag{5.25}$$

holds. But (5.25) holds for $j \in \mathbb{Z}$ if and only if it holds for even integers j and for odd integers j. Hence (5.25) is equivalent to the pair of Bezout identites

$$\left.\begin{aligned} A^{(e)}(z)\Lambda(z) + \Gamma^{(e)}(z)\Omega(z) &= 1\,, \\ A^{(o)}(z)\Lambda(z) + \Gamma^{(o)}(z)\Omega(z) &= 1\,, \end{aligned}\right\} z \in \mathbb{C} \setminus \{0\}\,,$$

or, equivalently,

$$\left.\begin{aligned} \left[A(z) + A(-z)\right]\Lambda(z) + \left[\Gamma(z) + \Gamma(-z)\right]\Omega(z) &= 2\,, \\ \left[A(z) - A(-z)\right]\Lambda(z) + \left[\Gamma(z) - \Gamma(-z)\right]\Omega(z) &= 2\,, \end{aligned}\right\} z \in \mathbb{C} \setminus \{0\}\,. \tag{5.26}$$

Now observe that (5.26), and therefore also the condition (5.23), is satisfied if and only if the pair of Bezout identities

$$\left.\begin{aligned} A(z)\Lambda(z) + \Gamma(z)\Omega(z) &= 2\,, \\ A(-z)\Lambda(z) + \Gamma(-z)\Omega(z) &= 0\,, \end{aligned}\right\} z \in \mathbb{C} \setminus \{0\}\,, \tag{5.27}$$

hold.

Observe next from (5.11) that

$$\Gamma(-z) = -Kz^{2n_0+1}\Lambda(z)\,, \quad z \in \mathbb{C} \setminus \{0\}\,. \tag{5.28}$$

It now follows from (5.28) that (5.27), and therefore also the condition (5.23), holds if and only if the pair of Bezout identities

$$\left.\begin{aligned} A(z)\Lambda(z) + Kz^{2n_0+1}\Lambda(-z)\Omega(z) &= 2\,, \\ \Lambda(z)\left[A(-z) - Kz^{2n_0+1}\Omega(z)\right] &= 0\,, \end{aligned}\right\} z \in \mathbb{C} \setminus \{0\}\,, \tag{5.29}$$

are satisfied.

The second line of (5.29) is satisfied by a Laurent polynomial Ω if and only if

$$\Omega(z) = \frac{1}{K} z^{-2n_0-1} A(-z)\,, \quad z \in \mathbb{C} \setminus \{0\}\,, \tag{5.30}$$

after having recalled also from Proposition 5.3 that $K \neq 0$.

Substituting (5.30) into the left-hand-side of the first line of (5.29), we obtain

$$A(z)\Lambda(z) + Kz^{2n_0+1}\Lambda(-z)\Omega(z) = A(z)\Lambda(z) + A(-z)\Lambda(-z), \quad z \in \mathbb{C} \setminus \{0\},$$

which, together with the Bezout identity (4.3) satisfied by A and Λ, shows that the choice (5.30) of Ω also satisfies the first line of (5.29). We have therefore shown that the condition (5.23) is satisfied by a sequence $\omega \in \mathcal{M}_0(\mathbb{Z})$ if and only if the corresponding Laurent polynomial Ω is given by the formula (5.30); which can, according to (5.24) and (1.4), be rewritten as

$$\sum_j \omega_j z^j = \frac{1}{K}\sum_j (-1)^j a_j z^{j-2n_0-1} = \frac{1}{K}\sum_j (-1)^{j+1} a_{j+2n_0+1} z^j, \quad z \in \mathbb{Z} \setminus \{0\},$$

which, in turn, holds if and only if the sequence $\omega \in \mathcal{M}_0(\mathbb{Z})$ is given by (5.19).

It follows that, if we choose the sequence $\omega \in \mathcal{M}_0(\mathbb{Z})$ as in (5.19), then (5.22), and therefore also the desired result (5.18), hold. ■

We see from Theorem 5.6, together with the definition (5.1), that the condition (5.5) is indeed satisfied by the finitely supported function $\psi \in V^{(1)}$ as given in Theorem 5.4. Hence, we can now state the following.

Corollary 5.7. *The finitely supported function $\psi \in V^{(1)}$ given in Theorem 5.4 is a wavelet generated by the refinement pair (a, ϕ) and corresponding local linear projection operator sequence $\{\mathcal{P}_r : r \in \mathbb{Z}\}$ of Theorem 4.1.*

The results of Theorem 5.6 and Corollary 5.7 can now be specialised to the specific context of Theorem 5.5, as follows.

Corollary 5.8. *The finitely supported function $\psi \in V^{(1)}$ of Theorem 5.5 is a wavelet generated by the refinement pair (a, ϕ) and the projection operator sequence $\{\mathcal{P}_r : r \in \mathbb{Z}\}$ based on Theorem 4.1 and Corollary 4.4. Moreover, the decomposition result (5.19) of Theorem 5.6 holds in this case with*

$$\omega_j = (-1)^j a_{j+\mu}, \quad j \in \mathbb{Z}. \tag{5.31}$$

5.4 Symmetry properties in wavelets

We proceed to investigate whether, if the refinement mask symbol A corresponding to a refinement pair $(a\,,\phi)$ is a symmetric polynomial, the corresponding wavelet ψ of Theorem 5.6 also satisfies a symmetry property.

Theorem 5.9. *For a symmetric refinement pair $(a\,,\phi)$ as in Theorem 1.11, with n an even integer, the wavelet ψ of Theorem 5.5 is a symmetric function in the sense that*

$$\psi(n-1-\cdot) = \psi\,. \tag{5.32}$$

Proof. Since n is an even integer, it follows from (4.32) in the proof of Theorem 4.5(a), (1.29) in Theorem 1.11, together with (5.14), and with the definition $s_j = 0,\ j \notin \{0,\ldots,n-2\}$, that

$$\begin{aligned}
\psi(n-1-\cdot) &= 2\sum_j (-1)^j s_{n-2-j}\phi(2n-2-2\cdot-j) \\
&= 2\sum_j (-1)^j s_j \phi(n-2\cdot+j) \\
&= 2\sum_j (-1)^j s_j \phi(n-(n-2\cdot+j)) \\
&= 2\sum_j (-1)^j s_j \phi(2\cdot-j) = \psi\,,
\end{aligned}$$

thereby yielding (5.32). ■

If n is an odd integer, a wavelet possessing a symmetry property , but with a larger support interval than (5.15), is obtained as follows.

Theorem 5.10. *Suppose, in Theorem 5.4, with n denoting an odd integer, the refinement pair $(a\,,\phi)$ is symmetric in the sense of Theorem 1.11, and the Laurent polynomial $\Lambda = \tilde{\Lambda}$, as in Corollary 4.6. Then the choices $K = -1$ and $n_0 = \frac{n-1}{2}$ produces the wavelet formulation*

$$\psi = \tilde{\psi} = 2\sum_{j=0}^{n}(-1)^j \tilde{s}_j \phi(2\cdot-j)\,, \tag{5.33}$$

where the sequence $\{\tilde{s}_j : j = 0, 1, \ldots, n\}$ is given by

$$\tilde{S}(z) = \sum_{j=0}^{n} \tilde{s}_j z^j\,, \quad z \in \mathbb{C}\,, \tag{5.34}$$

with $\tilde{S} \in \Pi_n$ as in Theorem 4.5 (b). Moreover,

$$\tilde{\psi}(x) = 0\,, \quad x \notin (0,\, n)\,, \tag{5.35}$$

and $\tilde{\psi}$ satisfies the symmetry condition

$$\tilde{\psi}(n - \cdot) = -\tilde{\psi}\,. \tag{5.36}$$

Proof. Using (5.7), (5.11), (5.9), (4.38) and (5.34), we deduce that (5.33) holds. The finite support property (5.35) of $\tilde{\psi}$ is then an immediate consequence of (5.33) and (2.5). Next, we use (5.33) and (1.29), together with the fact that (4.37) in the proof of Theorem 4.5(b) yields $\tilde{s}_{n-j} = \tilde{s}_j$, $j = 0, 1, \ldots, n$, to obtain

$$\begin{aligned}
\tilde{\psi}(n - \cdot) &= 2\sum_j (-1)^j \tilde{s}_{n-j}\phi(2n - 2 \cdot - j) \\
&= 2\sum_j (-1)^{1+j} \tilde{s}_j \phi(n - 2 \cdot + j) \\
&= -2\sum_j (-1)^j \tilde{s}_j \phi(n - (n - 2 \cdot + j)) \\
&= -2\sum_j (-1)^j \tilde{s}_j \phi(2 \cdot - j) = -\tilde{\psi}\,,
\end{aligned}$$

and thereby proving (5.36). ■

5.5 Examples

For the cardinal B-spline refinement pair $(a^{(m)}\,, N_m)$, where $m \geq 2$, we define the *cardinal spline wavelet* ψ_m of order m, according to (5.14) in Theorem 5.5, by

$$\psi_m = 2\sum_{j=0}^{m-2} (-1)^j s_{m,j} N_m(2 \cdot - j)\,, \tag{5.37}$$

where the sequence $\{s_{m,j} : j = 0, 1, \ldots, m-2\} \subset \mathbb{R}$ is defined by

$$S_m(z) = \sum_{j=0}^{m-2} s_{m,j} z^j\,, \quad z \in \mathbb{C}\,, \tag{5.38}$$

with the polynomial S_m as in Theorem 4.7.

For an odd integer m, the wavelet $\tilde{\psi}_m$ of Theorem 5.10 is given by

$$\tilde{\psi}_m = 2\sum_{j=0}^{m} (-1)^j \tilde{s}_{m,j} N_m(2 \cdot - j)\,, \tag{5.39}$$

where the sequence $\{\tilde{s}_{m,j} : j = 0, 1, \ldots, n\}$ is defined by

$$\tilde{S}_m(z) = \sum_{j=0}^{n} \tilde{s}_{m,j} z^j , \quad z \in \mathbb{C} ,$$

with the symmetric polynomial $\tilde{S} = \tilde{S}_m$ as in Theorem 4.5(*b*).

As graphically illustrated in Figures 5.1 and 5.2 for $m = 2, \ldots, 5$, the wavelets ψ_2 , ψ_4 , $\tilde{\psi}_3$ and $\tilde{\psi}_5$ satisfy the symmetry properties (5.32) and (5.36), with $n = m$. Observe in particular from Figure 5.1 that the wavelets ψ_m , $m = 2, \ldots, 5$, have no sign changes.

The corresponding polynomials $\Gamma = \Gamma_m$, $m = 2, \ldots, 5$, and $\tilde{\Gamma} = \tilde{\Gamma}_m$, $m = 3, 5$, are given in Tables 5.1. Also, in Table 5.2, we give the Laurent polynomials $\Omega = \Omega_m$, $m = 2, \ldots, 5$.

Table 5.1: The polynomials $\{\Gamma_m : m = 2, \ldots, 5\}$ and $\{\tilde{\Gamma}_m : m = 3, 5\}$

	Γ_m	$\tilde{\Gamma}_m$
$m = 2$	1	X
$m = 3$	$\frac{1}{2}z + \frac{3}{2}$	$\frac{1}{4}z^3 + \frac{3}{4}z^2 - \frac{3}{4}z - \frac{1}{4}$
$m = 4$	$-\frac{1}{2}z^2 - 2z - \frac{1}{2}$	X
$m = 5$	$-\frac{3}{8}z^3 - \frac{15}{8}z^2 - \frac{25}{8}z - \frac{5}{8}$	$-\frac{3}{16}z^5 - \frac{15}{16}z^4 - \frac{5}{4}z^3 + \frac{5}{4}z^2 + \frac{15}{16}z + \frac{3}{16}$

Table 5.2: The polynomials $\{\Omega_m : m = 2, \ldots, 5\}$

$m = 2$	$-\frac{1}{2}z + 1 - \frac{1}{2}z^{-1}$
$m = 3$	$\frac{1}{4}z - \frac{3}{4} + \frac{3}{4}z^{-1} - \frac{1}{4}z^{-2}$
$m = 4$	$-\frac{1}{8}z^2 + \frac{1}{2}z - \frac{3}{4} + \frac{1}{2}z^{-1} - \frac{1}{8}z^{-2}$
$m = 5$	$\frac{1}{16}z^2 - \frac{5}{16}z + \frac{5}{8} - \frac{5}{8}z^{-1} + \frac{5}{16}z^{-2} - \frac{1}{16}z^{-3}$

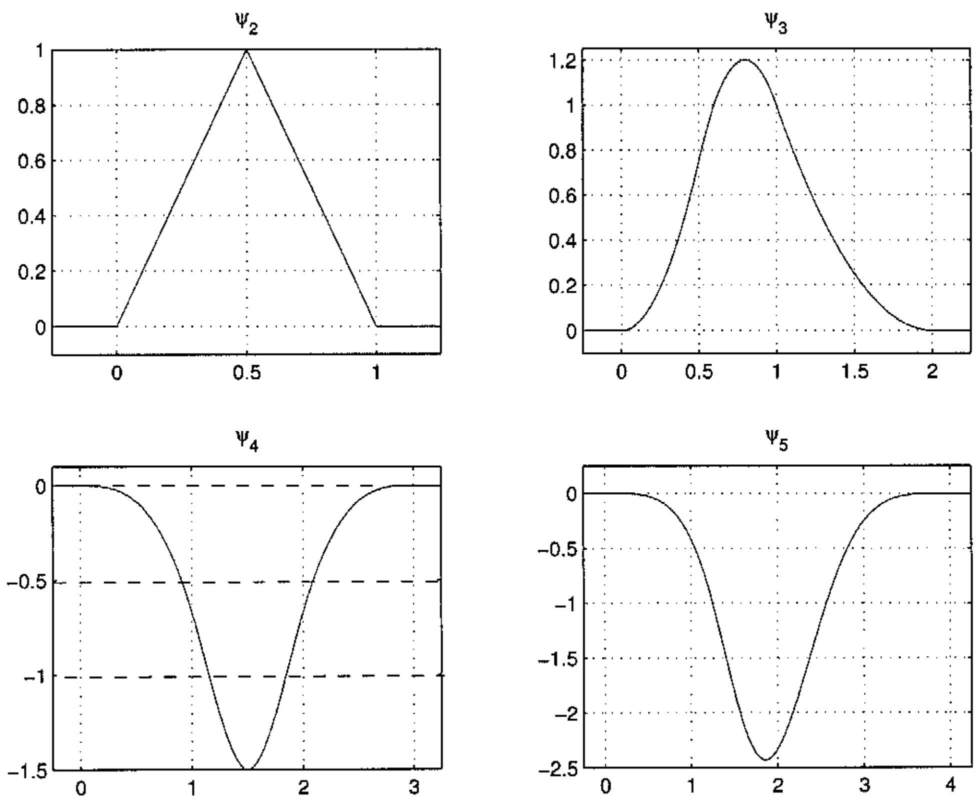


Figure 5.1: The cardinal spline wavelets $\{\psi_m : m = 2, \ldots, 5\}$

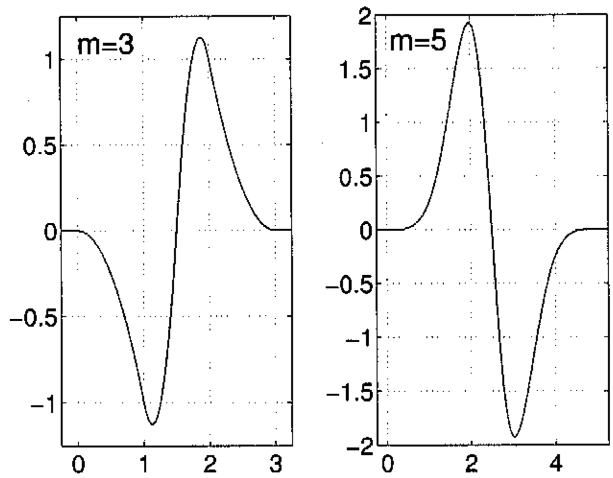


Figure 5.2: The wavelet $\tilde{\psi}_m$ for $m = 3, 5$

Next, for Example B, as in Sections 3.3 and 4.4 , our calculations yield the results of Table 5.3 .

Table 5.3: The polynomials associated with Example B

$\Gamma(z)$	$-\frac{13}{54}z^3 - \frac{13}{9}z^2 + \frac{17}{6}z - \frac{17}{27}$
$\Omega(z)$	$\frac{1}{24}z^2 - \frac{1}{4}z + \frac{7}{12} - \frac{2}{3}z^{-1} + \frac{3}{8}z^{-2} - \frac{1}{12}z^{-3}$

The corresponding wavelet $\psi_4(\cdot|2)$ is shown in Figure 5.3.

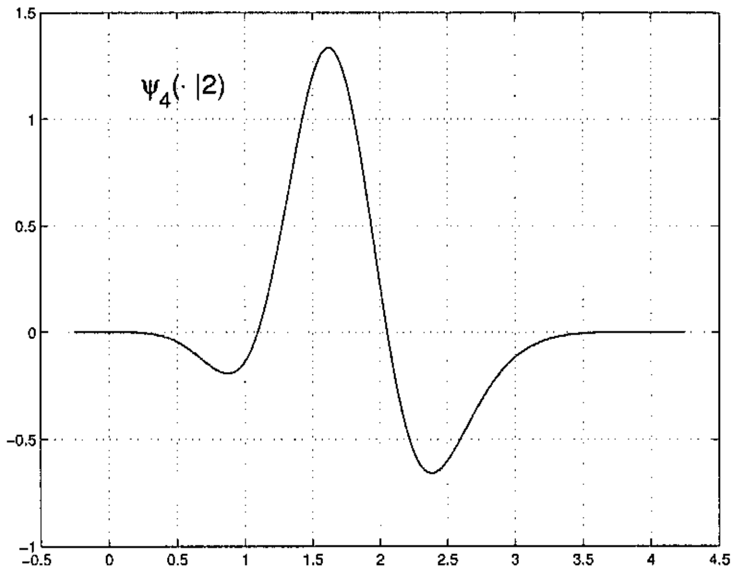


Figure 5.3: The wavelet $\psi_4(\cdot|2)$

Chapter 6

Decomposition and reconstruction algorithms

6.1 Decomposition

Given a signal $f \in \mathcal{M}(\mathbb{R})$, and a refinement pair $(a\,, \phi)$, we proceed to use Theorems 3.2, 4.1 and 5.6 to decompose f into detail components at increasingly coarse resolution levels by using the wavelet ψ of Theorem 5.4.

For appropriately chosen values of $R_0\,,\, R_1 \in \mathbb{Z}$, with $R_0 < R_1$, and $K \in \mathbb{R} \setminus \{0\}$, $n_0 \in \mathbb{Z}$, and $\tau \in \mathbb{R}$, and for a given signal $f \in \mathcal{M}(\mathbb{R})$ for which the data set $\left\{ f\left(\frac{j+\tau}{2^{R_1}}\right) : j \in \mathbb{Z} \right\}$ is known, we define the sequences $\{c^{(r)} : r = R_1\,,\, R_1 - 1, \ldots, R_0\} \subset \mathcal{M}(\mathbb{Z})$ and $\{d^{(r)} : r = R_1, R_1 - 1, \ldots, R_0\} \subset \mathcal{M}(\mathbb{Z})$ by

$$c_j^{(R_1)} = \sum_k u_{j-k} f\left(\frac{k+\tau}{2^{R_1}}\right), \quad j \in \mathbb{Z}\,, \tag{6.1}$$

$$c_j^{(r)} = \sum_k \lambda_{2j-k} c_k^{(r+1)}, \quad j \in \mathbb{Z}\,, \quad r = R_1 - 1, \ldots, R_0\,, \tag{6.2}$$

$$d_j^{(r)} = \frac{1}{K} \sum_k (-1)^{k+1} a_{2j+2n_0+1-k} c_k^{(r+1)}, \quad j \in \mathbb{Z}\,, \quad r = R_1 - 1, \ldots, R_0\,, \tag{6.3}$$

with the sequence $\{u_j : j \in \mathbb{Z}\} \in \mathcal{M}_0(\mathbb{Z})$ defined as in Theorem 3.1, and the function sequences $\{f_r : r = R_1, R_1 - 1, \ldots, R_0\} \subset \mathcal{M}(\mathbb{R})$ and $\{g_r : r = R_1 - 1, \ldots, R_0\} \subset \mathcal{M}(\mathbb{R})$ by

$$f_r = \sum_j c_j^{(r)} \phi(2^r \cdot - j)\,, \quad r = R_1, \ldots, R_0\,, \tag{6.4}$$

$$g_r = \sum_j d_j^{(r)} \psi(2^r \cdot - j)\,, \quad r = R_1 - 1, \ldots, R_0\,, \tag{6.5}$$

so that $f_r \in V^{(r)}$, $r = R_1, \ldots, R_0$, and $g_r \in W^{(r)}$, $r = R_1 - 1, \ldots, R_0$. Then, from Theorems 4.1 and 5.6, the decomposition result

$$f_{r+1} = f_r + g_r\,, \quad r = R_1 - 1, \ldots, R_0\,, \tag{6.6}$$

holds.

It follows from (6.6) that

$$f_{R_1} = f_{R_0} + \sum_{r=R_0}^{R_1-1} g_r\,. \tag{6.7}$$

For $r \in \mathbb{Z}$, we say that $\{d_j^{(r)} : j \in \mathbb{Z}\}$ are the *wavelet coefficients of the signal* f *at resolution level* r, whereas the function g_r is called the *wavelet component of the signal* f. We shall say that (6.1), (6.2), (6.3) is the *wavelet decomposition algorithm* based on the refinement pair $(a\,, \phi)$ and the local projection operator sequence $\{\mathcal{P}_r : r \in \mathbb{Z}\}$.

6.2 The singularity detection property

We proceed to demonstrate the efficiency in feature detection of our decomposition algorithm.

Theorem 6.1. *If, in (6.1), we have* $f \in \Pi_{N-1}$, *then the wavelet coeficients* $\{d_j^{(r)} : j \in \mathbb{Z}\}$, $r = R_1 - 1, \ldots, R_0$ *as obtained from (6.1), (6.2), (6.3), satisfy*

$$d_j^{(r)} = 0\,, \quad j \in \mathbb{Z}\,, \quad r = R_1 - 1, \ldots, R_0\,. \tag{6.8}$$

Proof. Let $\sigma \in \{0, 1, \ldots, N-1\}$. It will suffice to prove our theorem for the choice

$$f(x) = x^\sigma\,, \quad x \in \mathbb{R}\,. \tag{6.9}$$

First, for $r = R_1 - 1$, we see from (6.3), (6.1) and (6.9) that

$$\begin{aligned} d_j^{(R_1-1)} &= \frac{1}{K}\sum_k (-1)^k a_{2j+2n_0+1-k} \sum_l u_{k-l}\left(\frac{l+\tau}{2^{R_1}}\right)^\sigma \\ &= \frac{\sigma!}{2^{\sigma R_1}K(N-1)!}\sum_k (-1)^k a_{2j+2n_0+1-k} Q^{(N-1-\sigma)}(k)\,, \end{aligned} \tag{6.10}$$

from (3.7), where Q is the polynomial of degree $N-1$ as in Theorem 2.5.

We claim that

$$\sum_k (-1)^k a_{2j+2n_0+1-k} p(k) = 0\,, \quad j \in \mathbb{Z}\,, \quad p \in \Pi_{N-1}\,. \tag{6.11}$$

To prove (6.11), we fix $l \in \{0, 1, \ldots, N-1\}$, to deduce, for $j \in \mathbb{Z}$, that

$$\begin{aligned}
\sum_k (-1)^k a_{2j+2n_0+1-k} k^l &= \sum_k a_{2j+2n_0+1-2k} (2k)^l - \sum_k a_{2j+2n_0-2k} (2k+1)^l \\
&= \sum_k a_{2k+1} (2j + 2n_0 - 2k)^l - \sum_k a_{2k} (2j + 2n_0 + 1 - 2k)^l \\
&= \sum_k a_{2k+1} \left[(2j + n_0 + 1) - (2k+1)\right]^l - \sum_k a_{2k} \left[(2j + 2n_0 + 1) - 2k\right]^l \\
&= \sum_k a_{2k+1} \sum_{q=0}^{l} \binom{l}{q} (2j + 2n_0 + 1)^{l-q} (-1)^q (2k+1)^q \\
&\qquad - \sum_k a_{2k} \sum_{q=0}^{l} \binom{l}{q} (2j + 2n_0 + 1)^{l-q} (-1)^q (2k)^q \\
&= \sum_{q=0}^{l} (-1)^q \binom{l}{q} (2j + 2n_0 + 1)^{l-q} \left[\sum_k a_{2k+1} (2k+1)^q - \sum_k a_{2k} (2k)^q\right] \\
&= 0\,,
\end{aligned}$$

by virtue of the equivalence of the statements (i) and (ii) of Proposition 2.1, which is equivalent to the desired result (6.11) .

Since $Q^{(N-1-\sigma)} \in \Pi_{N-1}$, $\sigma = 0, 1, \ldots, N-1$, it follows from (6.10) and (6.11) that

$$d_j^{(R_1-1)} = 0\,, \quad j \in \mathbb{Z}\,. \tag{6.12}$$

Next, for $r = R_1 - 2$, we deduce from (6.3), (6.2), (6.1), (6.9) and (3.7) that, for $j \in \mathbb{Z}$, we have

$$\begin{aligned}
d_j^{(R_1-2)} &= \frac{\sigma!}{2^{\sigma R_1} K (N-1)!} \sum_k (-1)^k a_{2j+2n_0+1-k} \sum_l \lambda_{2k-l} c_l^{(R_1)} \\
&= \frac{\sigma!}{2^{\sigma R_1} K (N-1)!} \sum_k (-1)^k a_{2j+2n_0+1-k} \sum_l \lambda_{2k-l} Q^{(N-1-\sigma)}(l) \\
&= \frac{\sigma!}{2^{\sigma R_1} K (N-1)!} \left[\sum_k (-1)^k a_{2j+2n_0+1-k} \sum_l \lambda_{2k-2l} Q^{(N-1-\sigma)}(2l)\right. \\
&\qquad \left. + \sum_k (-1)^k a_{2j+2n_0+1-k} \sum_l \lambda_{2k-2l-1} Q^{(N-1-\sigma)}(2l+1)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sigma!}{2^{\sigma R_1}K(N-1)!}\left[\sum_k(-1)^k a_{2j+2n_0+1-k}\sum_l \lambda_{2l}Q^{(N-1-\sigma)}(2k-2l)\right.\\
&\qquad\left.+\sum_k(-1)^k a_{2j+2n_0+1-k}\sum_l \lambda_{2l-1}Q^{(N-1-\sigma)}(2k-2l+1)\right]\\
&= \frac{\sigma!}{2^{\sigma R_1}K(N-1)!}\left[\sum_l \lambda_{2l}\sum_k(-1)^k a_{2j+2n_0+1-k}Q^{(N-1-\sigma)}(2k-2l)\right.\\
&\qquad\left.+\sum_l \lambda_{2l-1}\sum_k(-1)^k a_{2j+2n_0+1-k}Q^{(N-1-\sigma)}(2k-2l+1)\right]\\
&= 0, \qquad\qquad (6.13)
\end{aligned}
$$

as obtained by choosing, for a fixed $l \in \mathbb{Z}$, the polynomial $p \in \Pi_{N-1}$ in (6.11) as, respectively,

$$p = Q^{(N-1-\sigma)}(2\cdot -2l) \quad\text{and}\quad p = Q^{(N-1-\sigma)}(2\cdot -2l+1)\,.$$

Repeated application of the above method yields

$$d_j^{(r)} = 0\,,\quad j \in \mathbb{Z}\,,\quad r = R_1-3,\ldots,R_0\,. \qquad (6.14)$$

Together, (6.12), (6.13) and (6.14) then imply (6.8) . ■

The result of Theorem 6.1 has the following important implication with respect to the wavelet decomposition algorithm (6.1), (6.2), (6.3). If the signal f is C^N-smooth in a certain region, so that, according to Taylor's theorem, f is locally well approximated by a polynomial in Π_{N-1} in that region, it follows from Theorem 6.1 that the wavelet coefficients $d_j^{(r)}$ can be expected to be relatively small if the support interval of the corresponding wavelet $\psi(2^r\cdot -j)$, e.g. the interval $\left[\frac{j}{2^r}, \frac{j+n-1}{2^r}\right]$ for the wavelet of Theorem 5.5, overlaps with this C^N-smooth region of f, thereby providing localised information, at each resolution level r, on the smoothness (or the lack thereof) of f.

6.3 Reconstruction

In practical application, after the decomposition phase has been completed, and after the wavelet coefficients have been processed according to the specific needs of the application, there is a need to reconstruct the function f_{R_1} from the processed coefficients. For this purpose, we deduce the following reconstruction algorithm.

For $r \in \{R_0, R_0+1, \ldots, R_1\}$, we see from (6.4), and the refinement equation (1.1), that

$$\begin{aligned} f_r &= \sum_k c_k^{(r)} \sum_j a_j \phi(2^{r+1} \cdot -2k - j) \\ &= \sum_k c_k^{(r)} \sum_j a_{j-2k} \phi(2^{r+1} \cdot -j) \\ &= \sum_j \left[\sum_k a_{j-2k} c_k^{(r)} \right] \phi(2^{r+1} \cdot -j), \end{aligned} \tag{6.15}$$

whereas (6.5) and (5.7) give, for $r \in \{R_0, R_0+1, \ldots, R_1 - 1\}$, and with the sequence $\{\gamma_j : j \in \mathbb{Z}\} \in \mathcal{M}_0(\mathbb{Z})$ given , from (5.11), (5.9) and (4.5), by

$$\gamma_j = K(-1)^{j+1} \lambda_{j-2n_0-1}, \quad j \in \mathbb{Z},$$

the result

$$\begin{aligned} g_r &= \sum_k d_k^{(r)} \sum_j \gamma_j \phi(2^{r+1} \cdot -2k - j) \\ &= \sum_k d_k^{(r)} \sum_j \gamma_{j-2k} \phi(2^{r+1} \cdot -j) \\ &= \sum_j \left[\sum_k \gamma_{j-2k} d_k^{(r)} \right] \phi(2^{r+1} \cdot -j). \end{aligned} \tag{6.16}$$

Combining (6.15), (6.16) and (6.4), then show that (6.6) holds if and only if

$$\sum_j \left[c_j^{(r+1)} - \left\{ \sum_k a_{j-2k} c_k^{(r)} + \sum_k \gamma_{j-2k} d_k^{(r)} \right\} \right] \phi(2^{r+1} \cdot -j) = 0. \tag{6.17}$$

Hence, if, for $r \in \{R_0, R_0+1, \ldots, R_1 - 1\}$, we let

$$c_j^{(r+1)} = \sum_k a_{j-2k} c_k^{(r)} + (-1)^{j+1} K \sum_k \lambda_{j-2n_0-1-2k} d_k^{(r)}, \quad j \in \mathbb{Z}, \tag{6.18}$$

then (6.17), and therefore also (6.6), are satisfied. Formula (6.18) is called the *wavelet reconstruction algorithm* based on the refinement pair (a, ϕ) and the local projection operator sequence $\{\mathcal{P}_r : r \in \mathbb{Z}\}$.

6.4 Example

Let us consider the signal $f \in \mathcal{C}_0(\mathbb{R})$ defined by

$$f(x) = \begin{cases} \dfrac{1}{1+x^2} & , \quad 0 \leq x \leq 1 , \\ \dfrac{1}{2}x(x-2)^2 & , \quad 1 \leq x \leq 2 , \\ 0 & , \quad x \geq 2 , \end{cases} \tag{6.19}$$

and

$$f(x) = f(-x) , \quad x \in \mathbb{R} . \tag{6.20}$$

Then $f \in \mathcal{C}_0^1(\mathbb{R}) \setminus \mathcal{C}_0^2(\mathbb{R})$, with discontinuities in the second derivative f'' at $x \in \{-2, -1, 1, 2\}$.

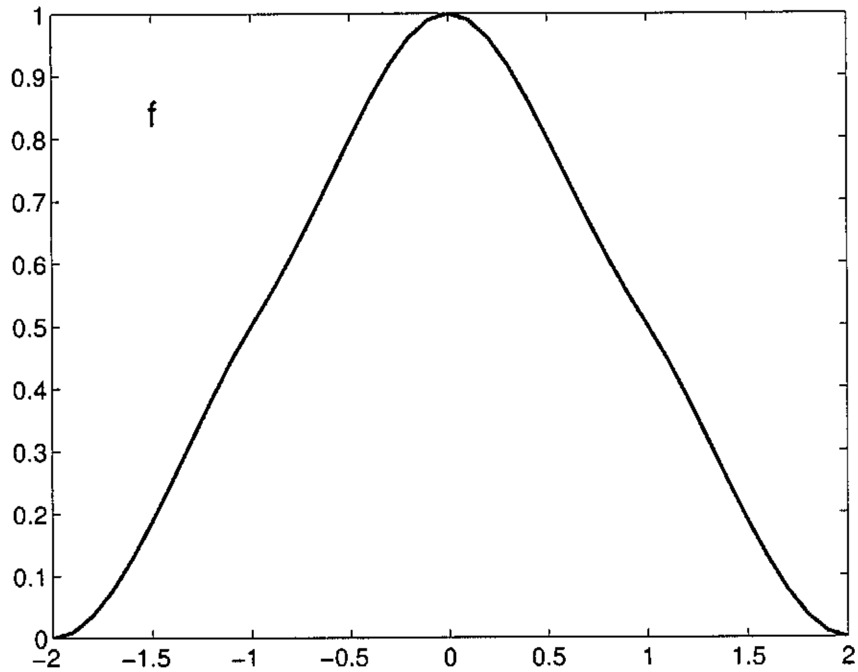


Figure 6.1: The signal f

We now use the wavelet decomposition algorithm (6.1), (6.2), (6.3) based on the cardinal B-spline refinement pair $(a^{(4)}, N_4)$, the local projection operator sequence $\{\mathcal{P}_{4,r} : r \in \mathbb{Z}\}$, as given by (4.53), with, from Table 4.2,

$$\lambda_{4,-3} = \lambda_{4,-1} = -\frac{1}{2} , \quad \lambda_{4,-2} = 2 , \quad \lambda_{4,j} = 0 , \; j \notin \{-3, -2, -1\} ,$$

and where the corresponding wavelet ψ_4 is given , according to (5.37) and Table 4.1, by

$$\psi_4 = -\frac{1}{2}N_4(2\cdot) + 2N_4(2\cdot -1) - \frac{1}{2}N_4(2\cdot -2),$$

as drawn in Figure 5.1. Also, we choose $R_1 = 10$ and $R_0 = 6$. The quasi-interpolant approximation $f_{10} = Q_{4,10}f$ is shown in Figure 6.2.

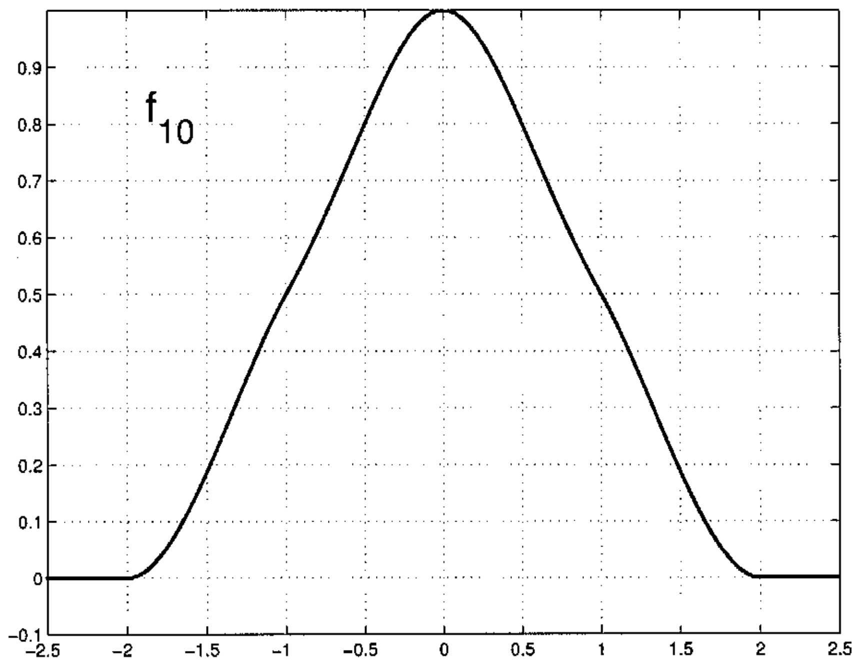


Figure 6.2: The approximation $f_{10} = Q_{4,10}f$

In our graphs in Figures 6.3 and 6.4, we plot, for $r = R_1, \ldots, R_0$, the coefficients sequences $\{c_j^{(r)} : j \in \mathbb{Z}\}$ and $\{d_j^{(r)} : j \in \mathbb{Z}\}$ against the indexes j.

We see in Figures 6.3 and 6.4 that the singularities in the second derivatives f'' at $x \in \{-2, -1, 1, 2\}$ are efficiently detected by our decomposition algorithm, with sharply defined localisation.

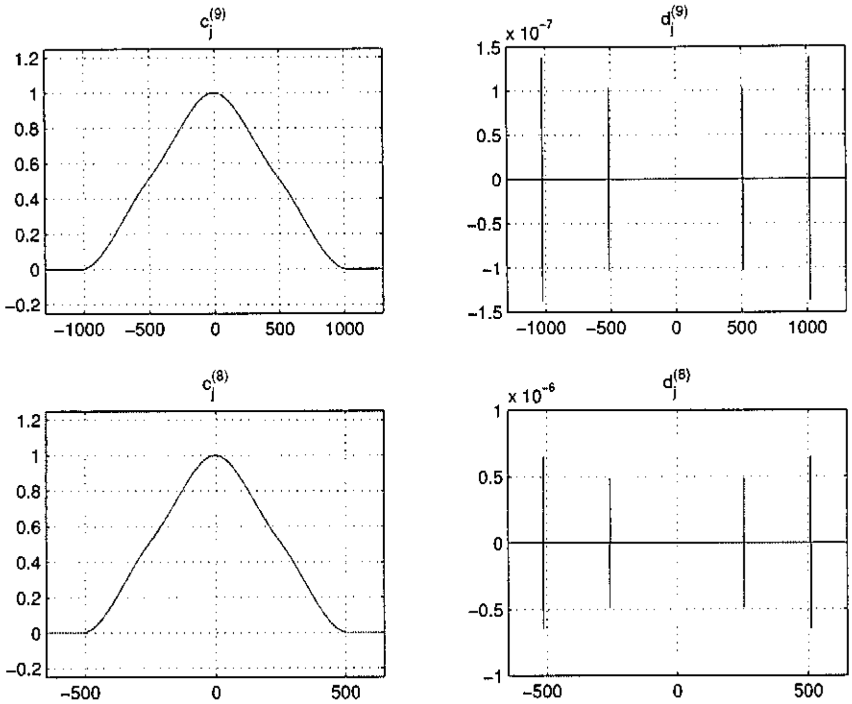


Figure 6.3: The coefficients $\{c_j^{(r)} : j \in \mathbb{Z}\}$ and $\{d_j^{(r)} : j \in \mathbb{Z}\}$ at level $r = 9, 8$

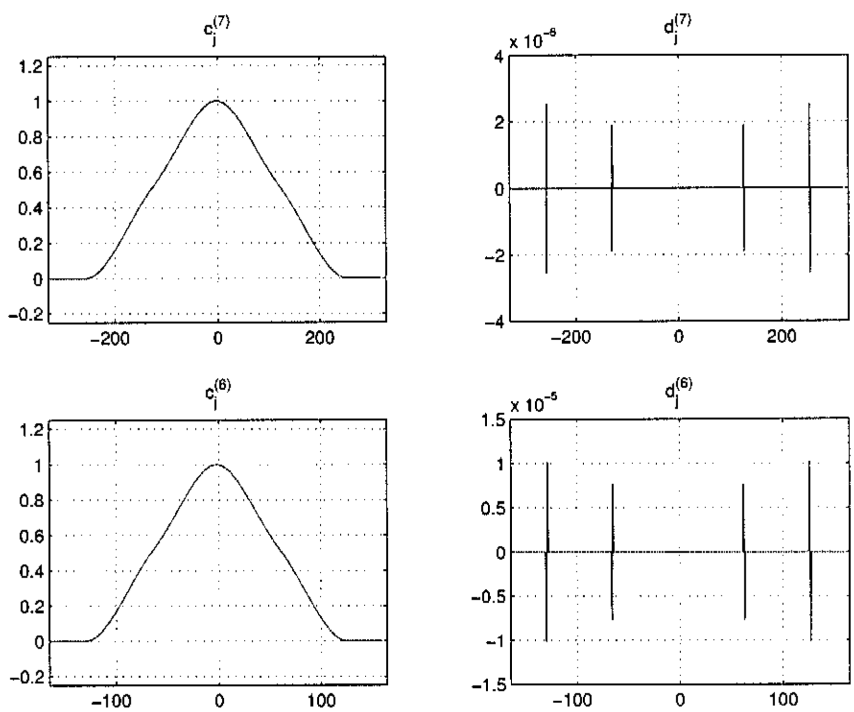


Figure 6.4: The coefficients $\{c_j^{(r)} : j \in \mathbb{Z}\}$ and $\{d_j^{(r)} : j \in \mathbb{Z}\}$ at level $r = 7, 6$

Chapter 7

Orthogonal refinement pairs and wavelets

In this chapter, we show how our wavelet decomposition technique, as developed in Chapters 4, 5 and 6, can be specialised to yield the well-known results (see **[20]**, **[21]**) on Daubechies wavelet decomposition.

7.1 A necessary condition for orthonormality

We consider here orthogonality with respect to the inner product space consisting of the linear space $\mathcal{C}_0(\mathbb{R})$, with inner product $\langle\cdot,\cdot\rangle : \mathcal{C}_0(\mathbb{R}) \to \mathbb{R}$ defined by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)dx\,, \quad f, g \in \mathcal{C}_0(\mathbb{R})\,. \tag{7.1}$$

If a refinement pair $(a\,,\phi)$ is such that

$$\langle \phi(\cdot - j), \phi(\cdot - k)\rangle = \delta_{j,k}\,, \quad j, k \in \mathbb{Z}\,, \tag{7.2}$$

i.e. ϕ has orthonormal integer shifts, we shall say that $(a\,,\phi)$ is an *orthonormal refinement pair.* Observe from (7.1) that the orthonormality condition (7.2) has the equivalent formulation

$$\langle \phi, \phi(\cdot - j)\rangle = \delta_j\,, \quad j \in \mathbb{Z}\,. \tag{7.3}$$

For an orthonormal refinement pair $(a\,,\phi)$, it holds that the refinable function ϕ has linearly independent integer shifts, as is evident from the following result.

Proposition 7.1. *Suppose $\phi \in C_0(\mathbb{R})$ has orthonormal integer shifts in the sense of (7.2). Then, if $c \in \mathcal{M}(\mathbb{Z})$ is a sequence such that*

$$\sum_j c_j \phi(\cdot - j) = 0\,, \tag{7.4}$$

we have

$$c_j = 0\,, \quad j \in \mathbb{Z}\,. \tag{7.5}$$

Proof. For $k \in \mathbb{Z}$, it follows from (7.4) and (7.1)that

$$\begin{aligned} 0 &= \left\langle \sum_j c_j \phi(\cdot - j), \phi(\cdot - k) \right\rangle \\ &= \sum_j c_j \langle \phi(\cdot - j), \phi(\cdot - k) \rangle \\ &= \sum_j c_j \delta_{j,k} = c_k\,, \end{aligned}$$

and thus the desired result (7.5) holds. ∎

Next we derive the following necessary condition for a refinement pair to be orthonormal.

Theorem 7.2. *If a refinement pair $(a\,,\phi)$, with refinement mask symbol A given by the polynomial of degree n as defined by (1.4), is orthonormal in the sense of (7.2), then the polynomial A satisfies the Bezout identity*

$$A(z)A(z^{-1}) + A(-z)A(-z^{-1}) = 4\,, \quad z \in \mathbb{C} \setminus \{0\}\,. \tag{7.6}$$

Proof. For an orthonormal refinement pair $(a\,,\phi)$ as described in the theorem , we can use (7.3), the refinement equation (1.1), the definition (7.1), as well as (7.2), to deduce that, for $j \in \mathbb{Z}$, we have

$$\begin{aligned} \delta_j = \langle \phi, \phi(\cdot - j) \rangle &= \left\langle \sum_k a_k \phi(2\cdot -k), \sum_l a_l \phi(2\cdot -2j - l) \right\rangle \\ &= \left\langle \sum_k a_k \phi(2\cdot -k), \sum_l a_{l-2j} \phi(2\cdot -l) \right\rangle \\ &= \sum_k a_k \sum_l a_{l-2j} \langle \phi(2\cdot -k), \phi(2\cdot -l) \rangle \\ &= \frac{1}{2} \sum_k a_k \sum_l a_{l-2j} \langle \phi(\cdot - k), \phi(\cdot - l) \rangle \\ &= \frac{1}{2} \sum_k a_k \sum_l a_{l-2j} \delta_{k,l} = \frac{1}{2} \sum_k a_k a_{k-2j}\,, \end{aligned}$$

and thus

$$\sum_k a_k a_{k-2j} = 2\delta_j\,, \quad j \in \mathbb{Z}\,. \tag{7.7}$$

But, using (1.4), we have, for $z \in \mathbb{C} \setminus \{0\}$, that

$$\begin{aligned}
\sum_j \sum_k a_k a_{k-2j} z^{2j} &= \sum_j \sum_k a_{2k} a_{2k-2j} z^{2j} + \sum_j \sum_k a_{2k+1} a_{2k+1-2j} z^{2j} \\
&= \sum_k a_{2k} \left[\sum_j a_{2k-2j} (z^{-1})^{2k-2j} \right] z^{2k} \\
&\quad + \sum_k a_{2k+1} \left[\sum_j a_{2k+1-2j} (z^{-1})^{2k+1-2j} \right] z^{2k+1} \\
&= \sum_k a_{2k} \left[\sum_j a_{2j} (z^{-1})^{2j} \right] z^{2k} \\
&\quad + \sum_k a_{2k+1} \left[\sum_j a_{2j+1} (z^{-1})^{2j+1} \right] z^{2k+1} \\
&= A^{(e)}(z^{-1}) A^{(e)}(z^{-1}) + A^{(o)}(z^{-1}) A^{(o)}(z^{-1}) \\
&= \frac{A(z) + A(-z)}{2} \frac{A(z^{-1}) + A(-z^{-1})}{2} \\
&\quad + \frac{A(z) - A(-z)}{2} \frac{A(z^{-1}) - A(-z^{-1})}{2} \\
&= \frac{1}{2} \left[A(z) A(z^{-1}) + A(-z) A(-z^{-1}) \right] .
\end{aligned} \tag{7.8}$$

Since also

$$\sum_j 2\delta_j z^{2j} = 2\,, \quad z \in \mathbb{C}\,, \tag{7.9}$$

the desired result (7.6) is an immediate consequence of (7.7), (7.8) and (7.9). ■

The following result is then an immediate consequence of Theorem 7.2.

Corollary 7.3. *If, in Theorem 7.2, the polynomial A also satisfies the condition (2.4) for an integer $N \leq n$, with B a polynomial of degree $n - N$, then the Bezout identity (7.6) has the equivalent formulation*

$$\left(\frac{1+z}{2}\right)^{2N} C(z) - \left(\frac{1-z}{2}\right)^{2N} C(-z) = z^{2N-1}\,, \quad z \in \mathbb{C} \setminus \{0\}\,, \tag{7.10}$$

where

$$C(z) = z^{N-1} B(z) B(z^{-1})\,, \quad z \in \mathbb{C} \setminus \{0\}\,. \tag{7.11}$$

We now proceed, in Section 7.2 below, to find, for a given integer $N \geq 2$, the minimum value of n, where $n \geq N$, and for which there exists a polynomial B of degree $n - N$ satisfying (7.10), (7.11), as well as the condition (2.4). In the process, we shall also derive an explicit formulation for B. (see also **[20]**, **[21]**, **[9]**)

7.2 The Daubechies refinement pair

Let the integer $N \geq 2$ be given. By comparing the Bezout identities (4.40) and (7.10), we deduce from Theorems 4.3 and 4.7, and Corollary 7.3, that, if there exists a polynomial $B \in \Pi_{N-1}$ such that

$$z^{N-1}B(z)B(z^{-1}) = 2S_{2N}(z)\,, \quad z \in \mathbb{C} \setminus \{0\}\,, \tag{7.12}$$

with S_{2N} denoting the polynomial in Π_{2N-2} as described in Theorem 4.7, then the polynomial B satisfies (7.10), (7.11).

Since (7.12) holds if and only if

$$z^{N-1}B(z)B(z^{-1}) = 2S_{2N}(z)\,, \quad |z| = 1\,, \tag{7.13}$$

and since

$$B(z)B(z^{-1}) = B(z)B(\overline{z}) = B(z)\overline{B(z)} = |B(z)|^2\,, \quad |z| = 1\,,$$

we deduce that a polynomial B in Π_{N-1} satisfies (7.12) if and only if B satisfies the condition

$$|B(z)|^2 = 2z^{1-N}S_{2N}(z)\,, \quad |z| = 1\,. \tag{7.14}$$

Introducing the notation

$$S_{2N}(z) = \sum_{j=0}^{2N-2} s_{N,j}z^j\,, \quad z \in \mathbb{C}\,, \tag{7.15}$$

where the coefficients of the polynomial S_{2N} now have a different first subscript to what was used in (5.38), and noting from Theorem 4.5(a) that S_{2N} is a symmetric polynomial, with, from (4.32),

$$s_{N,2N-2-j} = s_{N,j}\,, \quad j = 0, 1, \ldots, 2N-2\,, \tag{7.16}$$

we deduce that, for $z \in \mathbb{C} \setminus \{0\}$, we have

$$\begin{aligned} z^{1-N} S_{2N}(z) &= \sum_{j=0}^{2N-2} s_{N,j} z^{j+1-N} \\ &= \sum_{j=-N+1}^{N-1} s_{N,j+N-1} z^j \\ &= \sum_{j=-N+1}^{-1} s_{N,j+N-1} z^j + s_{N,N-1} + \sum_{j=1}^{N-1} s_{N,j+N-1} z^j \\ &= \sum_{j=1}^{N-1} s_{N,-j+N-1} z^{-j} + s_{N,N-1} + \sum_{j=1}^{N-1} s_{N,j+N-1} z^j \\ &= s_{N,N-1} + \sum_{j=1}^{N-1} s_{N,j+N-1} [z^j + z^{-j}]\,. \end{aligned} \tag{7.17}$$

It follows from (7.17) that, on the unit circle $z = e^{ix}$, $x \in \mathbb{R}$, we have

$$(e^{ix})^{1-N} S_{2N}(e^{ix}) = s_{N,N-1} + 2 \sum_{j=1}^{N-1} s_{N,j+N-1} \cos(jx)\,, \quad x \in \mathbb{R}\,. \tag{7.18}$$

Using de Moivre's Theorem, it can be shown that there exists a sequence $\{\beta_{j,k} : k = 0, 1, \ldots, j, \quad j \in \mathbb{N}\} \subset \mathbb{R}$ such that

$$\cos(jx) = \sum_{k=0}^{j} \beta_{j,k} (\cos x)^k\,, \quad x \in \mathbb{R}\,, \quad j \in \mathbb{N}\,. \tag{7.19}$$

Subtituting (7.19) into (7.18) then yields

$$(e^{ix})^{1-N} S_{2N}(e^{ix}) = s_{N,N-1} + 2 \sum_{j=1}^{N-1} s_{N,j+N-1} \sum_{k=0}^{j} \beta_{j,k} \left(1 - 2\sin^2 \frac{x}{2}\right)^k\,, \quad x \in \mathbb{R}\,. \tag{7.20}$$

Now define the polynomial $p_N \in \Pi_{N-1}$ by

$$p_N(z) = 2 \left[s_{N,N-1} + 2 \sum_{j=1}^{N-1} s_{N,j+N-1} \sum_{k=0}^{j} \beta_{j,k} (1-2z)^k \right]\,, \quad z \in \mathbb{C}\,. \tag{7.21}$$

It then follows from (7.20) and (7.21) that

$$2(e^{ix})^{1-N} S_{2N}(e^{ix}) = p_N \left(\sin^2 \frac{x}{2}\right)\,, \quad x \in \mathbb{R}\,, \tag{7.22}$$

and thus also

$$2z^{1-N}S_{2N}(z) = p_N\left(\frac{1}{2}\left[1 - \frac{z+z^{-1}}{2}\right]\right), \quad z = e^{ix}, \quad x \in \mathbb{R},$$

which holds if and only if

$$2z^{1-N}S_{2N}(z) = p_N\left(\frac{1}{2}\left[1 - \frac{z+z^{-1}}{2}\right]\right), \quad z \in \mathbb{C} \setminus \{0\}. \tag{7.23}$$

Also, note from (7.22) that

$$2(-e^{ix})^{1-N}S_{2N}(-e^{ix}) = (e^{i(x+\pi)})^{1-N}S_{2N}(e^{i(x+\pi)}) = p_N\left(\sin^2\left(\frac{x+\pi}{2}\right)\right) = p_N\left(1-\sin^2\frac{x}{2}\right), \quad z = e^{ix}, \quad x \in \mathbb{R}. \tag{7.24}$$

According to (4.40), the polynomial S_{2N} solves the Bezout identity

$$\left(1+\frac{z+z^{-1}}{2}\right)^N [2z^{1-N}S_{2N}(z)] + \left(1-\frac{z+z^{-1}}{2}\right)^N [2(-z)^{1-N}S_{2N}(-z)] = 2^N, \quad z \in \mathbb{C} \setminus \{0\}. \tag{7.25}$$

Hence , from (7.22), (7.24) and (7.25),the polynomial p_N defined by (7.21) solves the Bezout identity

$$(1+\cos x)^N p_N\left(\sin^2\frac{x}{2}\right) + (1-\cos x)^N p_N\left(1-\sin^2\frac{x}{2}\right) = 2^N, \quad x \in \mathbb{R},$$

which is equivalent to

$$\left(1-\sin^2\frac{x}{2}\right)^N p_N\left(\sin^2\frac{x}{2}\right) + \left(\sin^2\frac{x}{2}\right)^N p_N\left(1-\sin^2\frac{x}{2}\right) = 1, \quad x \in \mathbb{R},$$

i.e.

$$(1-z)^N p_N(z) + z^N p_N(1-z) = 1, \quad z \in [0,1],$$

which in turn , holds if and only if

$$(1-z)^N p_N(z) + z^N p_N(1-z) = 1, \quad z \in \mathbb{C}. \tag{7.26}$$

The Bezout identity (7.26) is a particular case of the one studied in [**42**]. Also, in the latter a compilation of Bezout identities encountered in the study of scaling functions and wavelets is presented with some examples.

We proceed to solve explicitly for the polynomial $p_N \in \Pi_{N-1}$ from (7.26). To this end, we first note from (7.26) that

$$p_N(z) = \frac{1}{(1-z)^N}[1 - z^N p_N(1-z)], \quad z \in \mathbb{C} \setminus \{1\}. \tag{7.27}$$

By repeated differentiation of the convergent geometric power series

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j, \quad |z| < 1,$$

we obtain the formula

$$\frac{1}{(1-z)^N} = \sum_{j=0}^{\infty} \binom{N+j-1}{j} z^j, \quad |z| < 1, \tag{7.28}$$

which we now substitute into (7.27) to deduce that there exists a sequence $\{\alpha_j : j \in \mathbb{Z}_+\} \subset \mathbb{R}$ such that

$$p_N(z) = \sum_{j=0}^{N-1} \binom{N+j-1}{j} z^j + z^N \sum_{j=0}^{\infty} \alpha_j z^j, \quad |z| < 1. \tag{7.29}$$

But $p_N \in \Pi_{N-1}$,which implies, together with (7.29), that $\alpha_j = 0$, $j \in \mathbb{Z}_+$, and it follows from (7.29) that the polynomial p_N has explicit formulation

$$p_N(z) = \sum_{j=0}^{N-1} \binom{N+j-1}{j} z^j, \quad z \in \mathbb{C}. \tag{7.30}$$

Observe in particular from (7.30) that

$$p_N\left(\sin^2 \frac{x}{2}\right) = \sum_{j=0}^{N-1} \binom{N+j-1}{j} \left(\sin^2 \frac{x}{2}\right)^j \geq 1 > 0, \quad x \in \mathbb{R}. \tag{7.31}$$

Combining (7.23), (7.30), (7.22) and (7.31), we have therefore now established, except for (7.34) below, the following result.

Proposition 7.4. *For $N \geq 2$, the polynomial S_{2N} of Theorem 4.7 is a polynomial of degree $2N-2$ with the explicit formulation*

$$S_{2N}(z) = \frac{1}{2} z^{N-1} \sum_{j=0}^{N-1} \binom{N+j-1}{j} \left(\frac{1}{2}\left[1 - \frac{z+z^{-1}}{2}\right]\right)^j, \quad z \in \mathbb{C}, \tag{7.32}$$

and satisfying the properties

$$S_{2N}(z) \neq 0\,, \quad |z| = 1\,, \tag{7.33}$$

and

$$S_{2N}(0) \neq 0\,. \tag{7.34}$$

Proof. It remains to prove (7.34). But the fact that $\deg(S_{2N}) = 2N - 2$ implies that $s_{N,2N-2} \neq 0$ in the representation (7.15). It then follows from (7.16) that also $s_{N,0} \neq 0$, thereby yielding the desired result (7.34). ■

The explicit formula (7.32) provides an alternative to the recursion formula (4.52) for the computation of the polynomial S_m of Theorem 4.7 if m is an even integer.

We proceed to show how the property (7.33) of S_{2N} allows us to find an explicit computational method for the construction of a polynomial B of degree $N-1$ satisfying the condition (7.12), as well as the second line of (2.4).

Suppose $z_0 \in \mathbb{C}$ is such that $S_{2N}(z_0) = 0$. Then (7.33) and (7.34) show that $|z_0| \neq 1$ and $z_0 \neq 0$. According to Theorem 4.5(a), we know that S_{2N} is a symmetric polynomial, with, from (4.31),

$$z^{2N-2} S_{2N}(z^{-1}) = S_{2N}(z)\,, \quad z \in \mathbb{C} \setminus \{0\}\,,$$

from which it follows that $S_{2N}(z_0^{-1}) = 0$, i.e. the zeros of S_{2N} occur in symmetric pairs. Note in particular also that, since $|z_0| \neq 1$, we have that $z_0^{-1} \neq \overline{z_0}$.

We deduce that there exist integers $K, L \in \mathbb{Z}_+$, with

$$K + 2L = N - 1\,, \tag{7.35}$$

and sequences $\{r_1, \ldots, r_K\} \subset \mathbb{R}$ and $\{z_1, \ldots, z_L\} \subset \mathbb{C} \setminus \mathbb{R}$, such that

$$S_{2N}(z) = c \left[\prod_{j=1}^{K} (z - r_j)\left(z - \frac{1}{r_j}\right)\right] \left[\prod_{k=1}^{L} (z - z_k)(z - \overline{z}_k)\left(z - \frac{1}{z_k}\right)\left(z - \frac{1}{\overline{z}_k}\right)\right], \quad z \in \mathbb{C}\,, \tag{7.36}$$

with c denoting a real non-zero constant, where

$$\left.\begin{aligned} &|r_j| > 1\,, && j = 1, \ldots, K, \\ &|z_j| > 1\,, \ \operatorname{Im}(z_j) > 0\,, && j = 1, \ldots, L, \end{aligned}\right\} \tag{7.37}$$

and with the convention that, if either $K = 0$, or $L = 0$, then respectively, $\{r_1, \ldots, r_K\} = \emptyset$, or $\{z_1, \ldots, z_L\} = \emptyset$. It follows from (7.36) that the equation (7.14) can be written as

$$|B(z)|^2 = 2|c| \left[\prod_{j=1}^{K} |z - r_j| \left|z - \frac{1}{r_j}\right|\right] \left[\prod_{k=1}^{L} |z - z_k||z - \overline{z}_k| \left|z - \frac{1}{z_k}\right| \left|z - \frac{1}{\overline{z}_k}\right|\right], \quad |z| = 1. \tag{7.38}$$

Now observe that, if $|z| = 1$ and $\alpha \in \mathbb{C} \setminus \{0\}$, then

$$\left|z - \frac{1}{\overline{\alpha}}\right| = \left|\overline{z - \frac{1}{\overline{\alpha}}}\right| = \left|\overline{z} - \frac{1}{\alpha}\right| = \left|\frac{1}{z} - \frac{1}{\alpha}\right| = \left|\frac{1}{z} - \frac{1}{\alpha}\right||z| = \left|1 - \frac{z}{\alpha}\right| = \left|\frac{z}{\alpha} - 1\right| = \frac{1}{|\alpha|}|z - \alpha|,$$

and thus

$$|z - \alpha| \left|z - \frac{1}{\overline{\alpha}}\right| = \frac{1}{|\alpha|}|z - \alpha|^2, \quad |z| = 1, \quad \alpha \in \mathbb{C} \setminus \{0\}. \tag{7.39}$$

Using (7.39) in (7.38), we obtain

$$|B(z)|^2 = \tilde{c} \left|\left[\prod_{j=1}^{K}(z - r_j)\right] \left[\prod_{k=1}^{L}(z - z_k)(z - \overline{z}_k)\right]\right|^2, \quad |z| = 1, \tag{7.40}$$

with

$$\tilde{c} = 2|c| \left[\prod_{j=1}^{K} \frac{1}{|r_j|}\right]^{-1} \left[\prod_{k=1}^{L} \frac{1}{|z_k|^2}\right]^{-1}. \tag{7.41}$$

Now choose the constant c in (7.41) such that (7.40), (7.41) yield the condition $B(1) = 1$, thereby giving, from (7.40), the formula

$$|B(z)|^2 = \left|\left[\prod_{j=1}^{K} \frac{z - r_j}{1 - r_j}\right] \left[\prod_{k=1}^{L} \frac{z^2 - \{2\mathrm{Re}(z_k)\}z + |z_k|^2}{1 - 2\mathrm{Re}(z_k) + |z_k|^2}\right]\right|^2, \quad |z| = 1. \tag{7.42}$$

It follows from (7.42) that the polynomial $B = B_N^D$ of degree $N - 1$ defined by

$$B(z) = B_N^D(z) = \left[\prod_{j=1}^{K} \frac{z - r_j}{1 - r_j}\right] \left[\prod_{k=1}^{L} \frac{z^2 - \{2\mathrm{Re}(z_k)\}z + |z_k|^2}{1 - 2\mathrm{Re}(z_k) + |z_k|^2}\right], \quad z \in \mathbb{C}, \tag{7.43}$$

satisfies the condition (7.42), and therefore also the condition (7.13). Moreover, $B_N^D(1) = 1$, and, from (7.14), $|B(-1)|^2 = 2(-1)^{1-N}S_{2N}(-1) \neq 0$, by virtue of (7.14) and (7.33), so that the polynomial $B = B_N^D$ also satisfies the conditions in the second line of (2.4).

Appealing also to Corollary 7.3 , we have therefore proved the following result.

Theorem 7.5. *For an integer $N \geq 2$, suppose $A = A_N^D$ is the polynomial of degree $2N-1$ as defined by (2.4), with the polynomial $B = B_N^D$ of degree $N-1$ given by (7.43), where the sequences $\{r_1, \ldots, r_K\} \subset \mathbb{R}$ and $\{z_1, \ldots, z_L\} \subset \mathbb{C} \setminus \mathbb{R}$, with K , $L \in \mathbb{Z}_+$ satisfying (7.35), denote the zeros (7.37) of the symmetric polynomial S_{2N} of degree $2N-2$, as obtained, either recursively from (4.52), or by means of the explicit formulation (7.32). Then the conditions in the second line of (2.4) are satisfied by $B = B_N^D$, and the polynomial $A = A_N^D$ defined by the first line of (2.4) is a polynomial of minimal degree satisfying the Bezout identity (7.6), and such that the conditions (2.4) hold.*

Now define the sequence $\{a_{N,j}^D : j \in \mathbb{Z}\} \in \mathcal{M}_0(\mathbb{Z})$ by

$$A_N^D(z) = \sum_j a_{N,j}^D z^j = \sum_{j=0}^{2N-1} a_{N,j}^D z^j\,, \quad z \in \mathbb{C}\,, \tag{7.44}$$

with the polynomial A_N^D as in Theorem 7.5, so that the condition (2.1) is satisfied by the sequence $a = a_N^D \in \mathcal{M}_0(\mathbb{Z})$, with $n = 2N-1$, i.e.

$$a_{N,j}^D = 0\,, \quad j \notin \{0, 1, \ldots, 2N-1\}\,, \quad \text{with } a_{N,0}^D \neq 0\,,\ a_{N,2N-1}^D \neq 0\,. \tag{7.45}$$

The existence of a function $\phi_N^D \in C_0(\mathbb{R})$ such that $(a_N^D\,, \phi_N^D)$ is an orthonormal refinement pair was first shown by Ingrid Daubechies in **[21]**. Hence $(a_N^D\,, \phi_N^D)$ is indeed the Daubechies refinement pair introduced in Section 1.9.

7.3 The Daubechies wavelet

For an integer $N \geq 2$, let $(a_N^D\,, \phi_N^D)$ denote the Daubechies refinement pair of order N. Note in particular that the corresponding refinement mask symbol $A = A_N^D$, as given by (7.43), possesses no symmetric zeros in $\mathbb{C} \setminus \{0\}$, since, if we assume that $z_0 \in \mathbb{C} \setminus \{0\}$ is such that $A_N^D(z_0) = A_N^D(-z_0) = 0$, a contradiction is obtained by setting $z = z_0$ in the Bezout identity (7.6).

By comparing the two Bezout identities (7.6) and (4.3), it follows from Theorem 7.5 that the Laurent polynomial $\Lambda = \Lambda_N^D$ defined by

$$\Lambda(z) = \Lambda_N^D(z) = \frac{1}{2}A(z^{-1}), \quad z \in \mathbb{C} \setminus \{0\}, \tag{7.46}$$

satisfies the Bezout identity (4.3). Hence, if we define the Daubechies refinement space sequence of order N by

$$V^{(r)} = V_{D,N}^{(r)} = \left\{ \sum_j c_j \phi_N^D(2^r \cdot -j) : c \in \mathcal{M}(\mathbb{Z}) \right\}, \quad r \in \mathbb{Z}, \tag{7.47}$$

it follows from Theorem 4.1 that, if we define the sequence $\{\lambda_{N,j}^D : j \in \mathbb{Z}\} \in \mathcal{M}_0(\mathbb{Z})$ by

$$\Lambda_N^D(z) = \sum_j \lambda_{N,j}^D z^j, \quad z \in \mathbb{C} \setminus \{0\}, \tag{7.48}$$

so that (7.46), (7.44) and (7.48) give

$$\lambda_{N,j}^D = \frac{1}{2} a_{N,-j}^D, \quad j \in \mathbb{Z}, \tag{7.49}$$

the linear operator sequence $\{\mathcal{P}_{N,r}^D : r \in \mathbb{Z}\}$ where $\mathcal{P}_{N,r}^D : V_{D,N}^{(r+1)} \to V_{D,N}^{(r)}$, $r \in \mathbb{Z}$, as defined by

$$\mathcal{P}_{N,r}^D f = \sum_j \left[\sum_k \frac{1}{2} a_{-2j+k} c_k \right] \phi_N^D(2^r \cdot -j) \ \text{ for } \ f = \sum_j c_j \phi_N^D(2^{r+1} \cdot -j), \quad r \in \mathbb{Z}, \tag{7.50}$$

has the reproduction property

$$\mathcal{P}_{N,r}^D f = f, \quad f \in V_{D,N}^{(r)}, \quad r \in \mathbb{Z},$$

i.e. $\mathcal{P}_{N,r}^D$ is a local linear projection operator on $V_{D,N}^{(r)}$ for every $r \in \mathbb{Z}$.

To obtain a wavelet $\psi = \psi_N^D$ based on the Daubechies refinement pair (a_N^D, ϕ_N^D) and the local linear projection operator sequence $\{\mathcal{P}_{N,r}^D : r \in \mathbb{Z}\}$, as given in (7.50), we appeal to Theorem 5.4, with the choices $K = 2$ and $n_0 = N - 1$, so that

$$\Gamma(z) = \Gamma_N^D(z) = \sum_j \gamma_{N,j}^D z^j, \quad z \in \mathbb{C}, \tag{7.51}$$

from which, together with (7.46), we get

$$\gamma_{N,j}^D = (-1)^{j+1} a_{N,2N-1-j}^D, \quad j \in \mathbb{Z}, \tag{7.52}$$

to deduce that the corresponding wavelet is given, according to (5.7), by the formula

$$\psi = \psi_N^D = \sum_{j=0}^{2N-1} (-1)^{j+1} a_{N,2N-1-j}^D \phi_N^D(2\cdot -j)\,. \tag{7.53}$$

The wavelet $\psi_N^D \in V_{D,N}^{(1)}$ as defined by (7.53), is called the *Daubechies wavelet of order N* in the literature.

It can be shown (see **[21]**, **[20]**) that ψ_N^D preserves the orthonormal integer shifts property of the Daubechies refinable function ϕ_N^D. Also, the Daubechies decomposition algorithm of order N is an orthogonal decomposition , in the sense that

$$V_{D,N}^{(r)} \perp W_{D,N}^{(r)}\,, \quad r \in \mathbb{Z}\,, \tag{7.54}$$

where $\{W_{D,N}^{(r)} : r \in \mathbb{Z}\}$ is the *Daubechies wavelet space sequence of order N* defined by

$$W_{D,N}^{(r)} = \left\{ \sum_j c_j \psi_N^D(2^r \cdot -j) : c \in \mathcal{M}(\mathbb{Z}) \right\}, \quad r \in \mathbb{Z}\,. \tag{7.55}$$

Moreover, since here $n = 2N-1$, we have from (2.5), together with (7.53), that

$$\psi_N^D(x) = 0\,, \quad x \notin (0, 2N-1)\,. \tag{7.56}$$

The following result shows that our wavelet construction method of Theorem 5.5 yields a wavelet with smaller support than the Daubechies wavelet ψ_N^D.

Theorem 7.6. *In the setting of Theorem 5.5, suppose $(a\,,\phi)$ is the Daubechies refinement pair $(a_N^D\,,\phi_N^D)$. Then the wavelet $\psi = \tilde{\psi}_N^D$ given by (5.14), i.e.*

$$\tilde{\psi}_N^D = 2\sum_{j=0}^{2N-3} (-1)^j s_j \phi_N^D(2\cdot -j)\,, \tag{7.57}$$

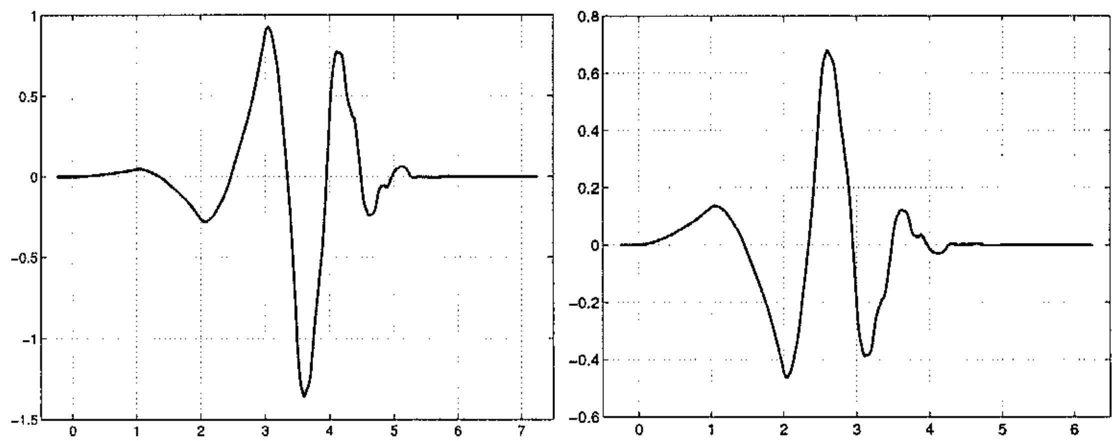
satisfies, according to (5.15), the finite support property

$$\tilde{\psi}_N^D(x) = 0\,, \quad x \notin (0,\, 2N-2)\,. \tag{7.58}$$

Note however that the wavelet $\tilde{\psi}_N^D$ does not possess orthonormal integer shifts as does the Daubechies wavelet ψ_N^D. Also, the wavelet $\tilde{\psi}_N^D$ is not orthogonal to the refinable function ϕ_N^D.

Finally, it should be pointed out that, since the polynomial A_N^D is not a symmetric polynomial, the wavelets ψ_N^D and $\tilde{\psi}_N^D$ do not possess any properties of symmetry as do the cardinal spline wavelets for ψ_m for m even, and $\tilde{\psi}_m$ for odd m.

Figure 7.1 illustrates the two types of Daubechies wavelet for $N = 4$.



(a) ψ_4^D (b) $\tilde{\psi}_4^D$

Figure 7.1: The wavelets ψ_4^D and $\tilde{\psi}_4^D$

7.4 Example

Finally, we compare the performance of the two wavelets ψ_4^D and $\tilde{\psi}_4^D$ with respect to the wavelet decomposition of the signal f given by (6.19). Using the wavelet decomposition algorithm (6.1), (6.2), (6.3), our graphical results are shown in Figures 7.2 to 7.5.
From Figures 7.3 to 7.5, we observe that both the wavelets ψ_4^D and $\tilde{\psi}_4^D$ sharply detects the discontinuities in the second derivative f'', thereby further demonstrating, for $\tilde{\psi}_4^D$, the efficiency of our wavelets decomposition algorithm (6.1), (6.2) and (6.3) as based on Theorem 5.5.

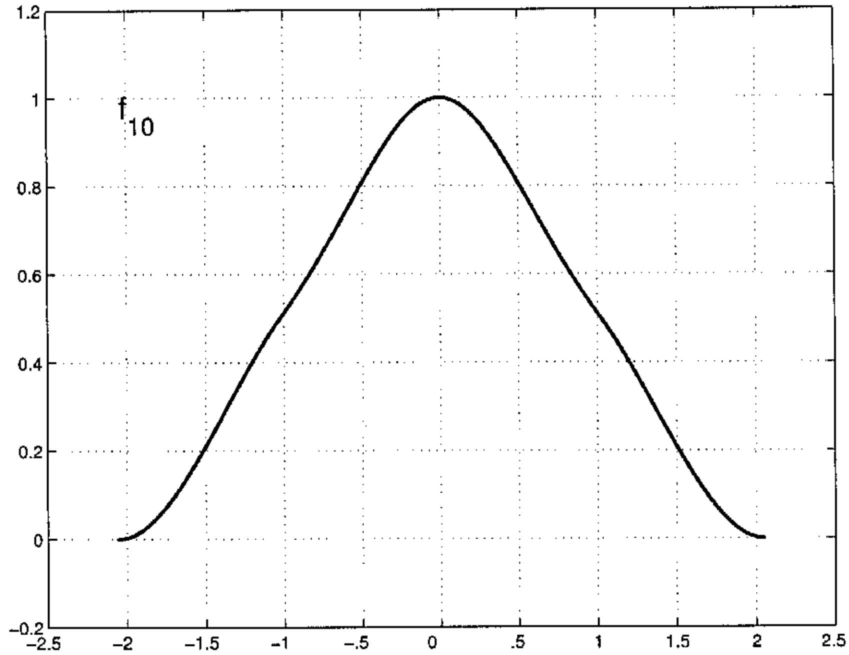


Figure 7.2: The approximation f_{10}

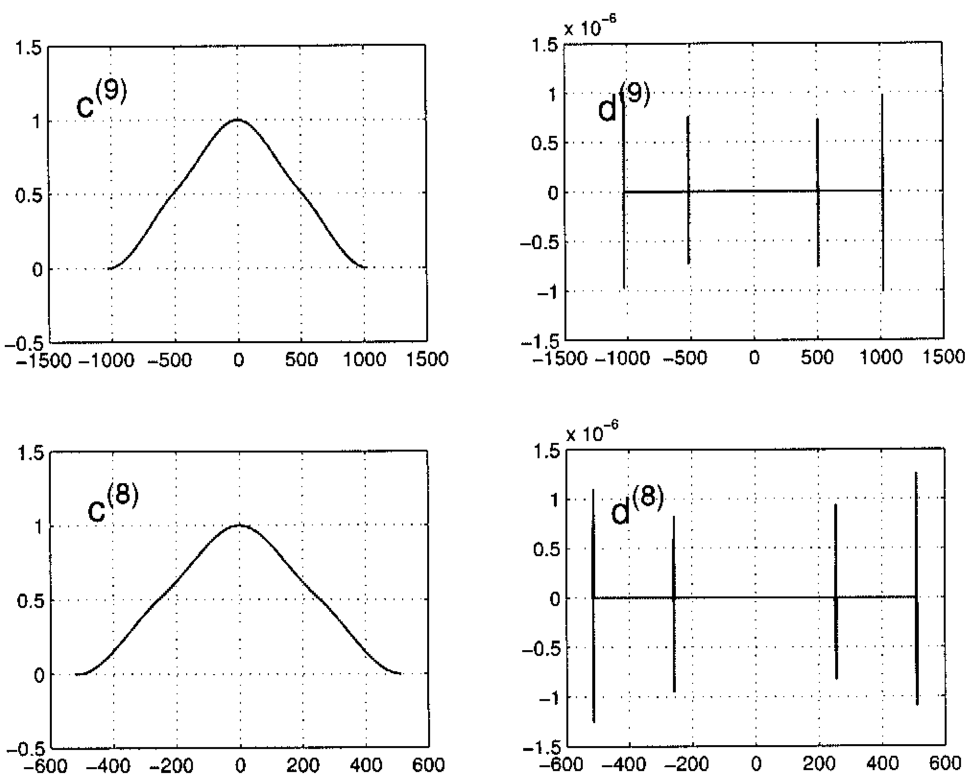


Figure 7.3: The coefficients at level $r = 9, 8$ using $\tilde{\psi}_4^D$

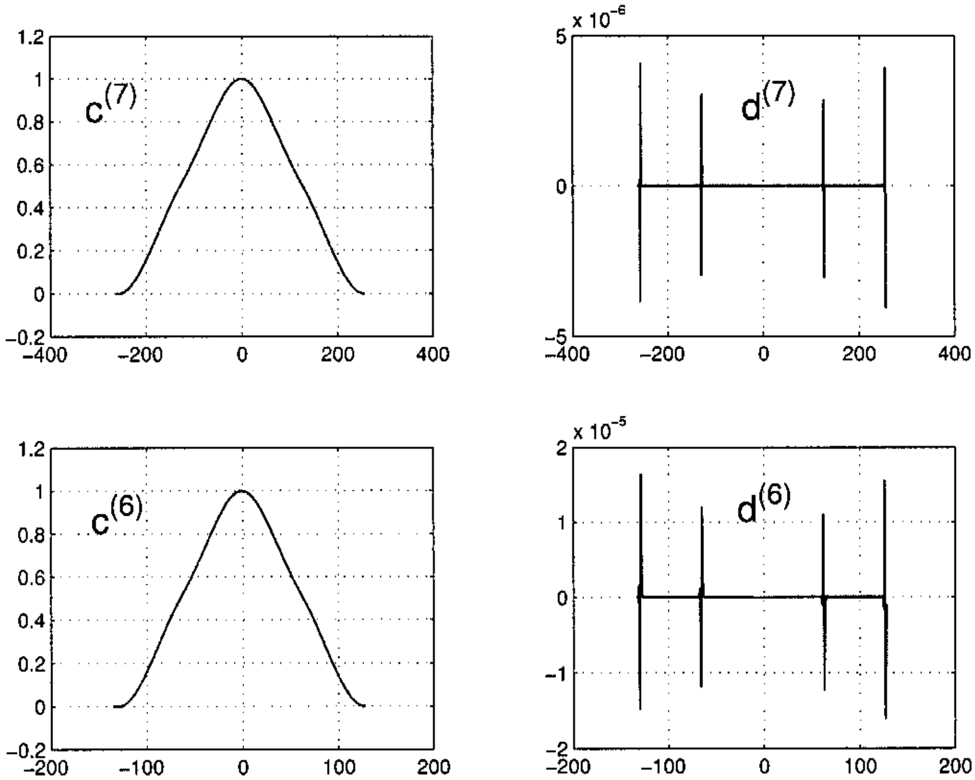


Figure 7.4: The coefficients at level $r = 7, 6$ using $\tilde{\psi}_4^D$

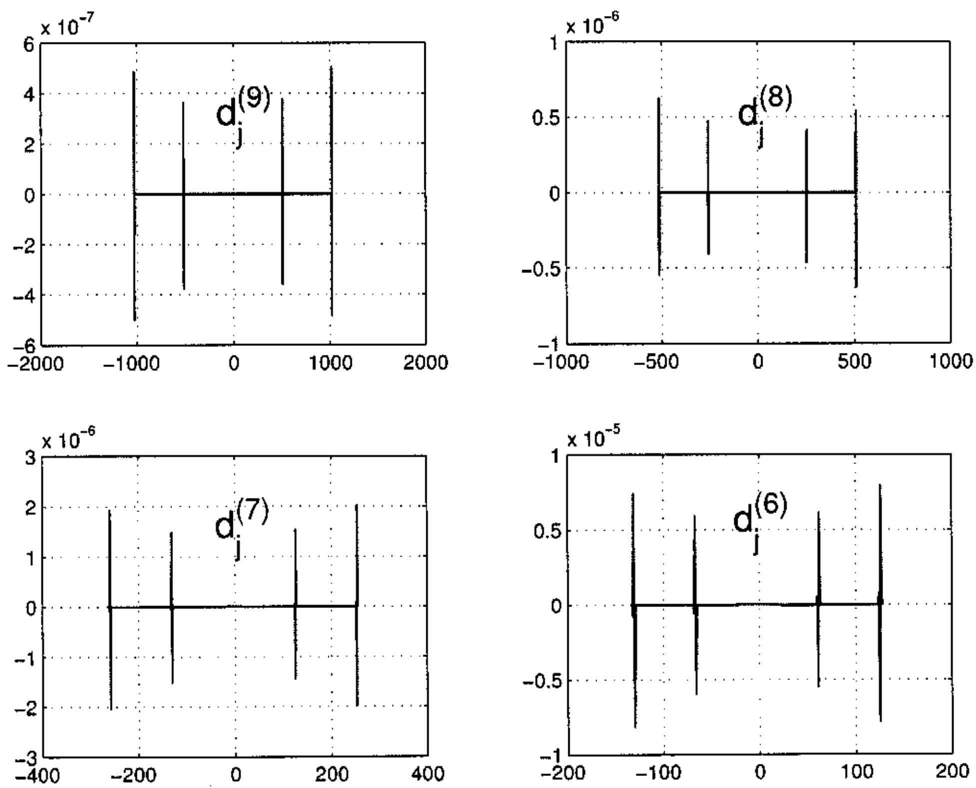


Figure 7.5: The coefficients at level $r = 9, 8, 7, 6$ using ψ_4^D

Appendix A

Intermediate tables

Table A.1: The sequence $\{\alpha_j = \alpha_{m,j} : j = 1, \ldots, m-1\}$

$\alpha_{m,j}$	$m=2$	$m=3$	$m=4$	$m=5$
$j=1$	1	$\frac{3}{2}$	2	$\frac{5}{2}$
$j=2$	X	3	5	$\frac{15}{2}$
$j=3$	X	X	14	25
$j=4$	X	X	X	90

Table A.2: The discrete moments $\{\mu_k = \mu_{m,k} : k = 1, \ldots, m-1\}$

$\mu_{m,k}$	$m=2$	$m=3$	$m=4$	$m=5$
$k=1$	1	$\frac{3}{2}$	2	$\frac{5}{2}$
$k=2$	X	$\frac{5}{2}$	$\frac{13}{3}$	$\frac{20}{3}$
$k=3$	X	X	10	$\frac{75}{4}$
$k=4$	X	X	X	$\frac{331}{6}$

Table A.3: The sequence $\{q_j = q_{m,j} : j = 0, 1, \ldots, m-1\}$

$q_{m,j}$	$m=2$	$m=3$	$m=4$	$m=5$
$j=4$	X	X	X	1
$j=3$	X	X	1	10
$j=2$	X	1	6	35
$j=1$	1	3	11	50
$j=0$	1	2	6	24

Table A.4: The sequences $\{l_{j,k} : j, k = 0, 1, \ldots, m-1\}$ with τ as in (3.38).

(a) $m = 2$

$l_{j,k}$	$k=0$	$k=1$
$j=0$	$-\frac{1}{2}$	-1
$j=1$	$\frac{3}{2}$	1

(b) $m = 3$

$l_{j,k}$	$k=0$	$k=1$	$k=2$
$j=0$	$\frac{3}{8}$	1	$\frac{1}{2}$
$j=1$	$-\frac{5}{4}$	-3	-1
$j=2$	$\frac{15}{8}$	2	$\frac{1}{2}$

(c) $m = 4$

$l_{j,k}$	$k=0$	$k=1$	$k=2$	$k=3$
$j=0$	$-\frac{5}{16}$	$-\frac{23}{24}$	$-\frac{3}{4}$	$-\frac{1}{6}$
$j=1$	$\frac{21}{16}$	$\frac{31}{8}$	$\frac{11}{4}$	$\frac{1}{2}$
$j=2$	$-\frac{35}{16}$	$-\frac{47}{8}$	$-\frac{13}{4}$	$-\frac{1}{2}$
$j=3$	$\frac{35}{16}$	$\frac{71}{24}$	$\frac{5}{4}$	$\frac{1}{6}$

(d) $m = 5$

$l_{j,k}$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$j=0$	$\frac{35}{128}$	$\frac{11}{12}$	$\frac{43}{48}$	$\frac{1}{3}$	$\frac{1}{24}$
$j=1$	$-\frac{45}{32}$	$-\frac{37}{8}$	$-\frac{13}{3}$	$-\frac{3}{2}$	$-\frac{1}{6}$
$j=2$	$\frac{189}{64}$	$\frac{75}{8}$	$\frac{65}{8}$	$\frac{5}{2}$	$\frac{1}{4}$
$j=3$	$-\frac{105}{32}$	$-\frac{229}{24}$	$-\frac{41}{6}$	$-\frac{11}{6}$	$-\frac{1}{6}$
$j=4$	$\frac{315}{128}$	$\frac{31}{8}$	$\frac{103}{48}$	$\frac{1}{2}$	$\frac{1}{24}$

Table A.5: the sequences for Example B

j	0	1	2	3	4	5
a_j	$\frac{1}{12}$	$\frac{3}{8}$	$\frac{2}{3}$	$\frac{7}{12}$	$\frac{1}{4}$	$\frac{1}{24}$
α_j	1	$\frac{7}{3}$	$\frac{20}{3}$	$\frac{64}{3}$	X	X
μ_j	1	$\frac{7}{3}$	$\frac{158}{27}$	$\frac{2942}{189}$	X	X
q_j	$\frac{1864}{189}$	$\frac{136}{9}$	7	1	X	X

Table A.6: The sequence $\{l_{j,k} : j, k = 0, 1, \ldots, 3\}$ for Example B, with $\tau = 4$

$l_{j,k}$	$k=0$	$k=1$	$k=2$	$k=3$
$j=0$	-1	$-\frac{11}{6}$	-1	$-\frac{1}{6}$
$j=1$	4	7	$\frac{7}{2}$	$\frac{1}{2}$
$j=2$	-6	$-\frac{19}{2}$	-4	$-\frac{1}{2}$
$j=3$	4	$\frac{13}{3}$	$\frac{3}{2}$	$\frac{1}{6}$

Table A.7: The sequence $\{\alpha_j^{D,N} : N = 2, 3, 4\}$

$\alpha_j^{D,N}$	$j=1$	$j=2$	$j=3$
$N=2$	$\frac{3-\sqrt{3}}{2}$	X	X
$N=3$	.817401168	.668144670	X
$N=4$	1.005393212	1.010815514	.253920226

Table A.8: The sequence $\{\mu_j^{D,N} : N = 2, 3, 4\}$

$\mu_j^{D,N}$	$j = 1$	$j = 2$	$j = 3$
$N = 2$	$\frac{3-\sqrt{3}}{2}$	X	X
$N = 3$	.817401168	.6681446696	X
$N = 4$	1.005393212	1.010815512	.9073603654

Table A.9: The sequence $\{q_j^{D,N} : N = 2, 3, 4\}$

$q_j^{D,N}$	$j = 0$	$j = 1$	$j = 2$	$j = 3$
$N = 2$	$\frac{3-\sqrt{3}}{2}$	1	X	X
$N = 3$	.6681446694	1.634802336	1	X
$N = 4$	.9073603574	3.032446528	3.016179636	1

Table A.10: The sequences $\{l_{j,k} : j, k = 0, 1, \ldots, N-1\}$ for the Daubechies case with $\tau = \frac{3N}{2} - 1$.

(a) $N = 2$

$l_{j,k}$	$k=0$	$k=1$
$j=0$	-1	-1
$j=1$	2	1

(b) $N = 3$

$l_{j,k}$	$k=0$	$k=1$	$k=2$
$j=0$	$\frac{15}{8}$	2	$\frac{1}{2}$
$j=1$	$-\frac{21}{4}$	-5	-1
$j=2$	$\frac{35}{8}$	3	$\frac{1}{2}$

(c) $N = 4$

$l_{j,k}$	$k=0$	$k=1$	$k=2$	$k=3$
$j=0$	-4	$-\frac{13}{3}$	$-\frac{3}{2}$	$-\frac{1}{6}$
$j=1$	15	$\frac{31}{2}$	5	$\frac{1}{2}$
$j=2$	-20	-19	$-\frac{11}{2}$	$-\frac{1}{2}$
$j=3$	10	$\frac{47}{6}$	2	$\frac{1}{6}$

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