Developing Algebraic Reasoning in the Intermediate Phase to Encourage Critical Thinking: A Case Study of Teachers

by

Uvile Oluwadarasimi Asekun

Thesis presented in fulfilment of the requirements for the degree of Masters in Education in the Department of Curriculum Studies in the Faculty of Education at Stellenbosch University

Supervisor: Prof MF Gierdien

March 2023

Declaration

I declare that this thesis's contents are my original work, and I am the sole author except where stated otherwise by reference or acknowledgement. This work has not been submitted for any other degree or processional qualification except as specified. The production or publication in part or whole by Stellenbosch University will not infringe on any third-party rights.

March 2023

Uvile Oluwadarasimi Asekun

ABSTRACT

Mathematics is esteemed in curricula and society as a subject that embodies the highest standard of knowledge. Mathematics is a form of language that can represent a numerical idea using numbers, letters, and symbols in algebra to encourage logical and critical thinking in learners. In addition, algebra is a cognitive process that can be used as a channel to review learners' algebraic reasoning abilities because it is viewed as a cognitive process. As a result, algebraic reasoning requires teachers' attention to assist learners in developing critical thinking.

This study explored how teachers in the intermediate phase use critical thinking (CT) to encourage the development of algebraic reasoning (AR). In addition, this study explores how teachers use pattern tasks to engage and encourage learners to think critically to develop algebraic reasoning when solving problems. This study focused on the Intermediate Phase, which consists of Grade 4, Grade 5 and Grade 6 learners. Only Grades 5 and 6 were used as a sample for the focus group interview. The researcher presumed that Grade 4 learners could be overwhelmed by the concepts of this study, and due to time constraints, they could not be included.

A mixed-method approach of quantitative and qualitative methods was adopted to accomplish the research objective. The qualitative methods included a literature review, lesson observation and interviews with the participating teachers, focusing on evaluation methods provided by CAPS. The quantitative methods include focus groups and post-reflective questionnaires, which helped to understand learners' responses to CT questions in Grades 5 and 6 for AR development and teachers' perception of CT.

The results at the end of the research showed that teachers' perceptions had been stimulated, and they had gained more understanding of what CT is and how it can be implemented in their math lessons. The focus group interview and lesson observations also showed learners' reasoning for AR development when they engaged in the pattern task. Finally, the results showed that both Grade 5 and 6 learners need more practice with their generalisation reasoning.

Consequently, it is recommended that CT questions should be part of every mathematics lesson to develop learners' skills in analysing and justifying its generalisation for the development of algebraic reasoning.

UMXHOLO

Izibalo zixatyiswe kakhulu kwiikharityhulam nakuluntu njengesifundo esibonisa owona mgangatho uphakamileyo wolwazi. Izibalo luhlobo lolwimi olunokumela uluvo lwamanani kusetyenziswa amanani, oonobumba, neesimboli kwialjibra ukukhuthaza ukuba abafundi bacinge ngokunzulu kwaye bazikise ukucinga. Ukongeza, i-aljibra yinkqubo yokuqonda enokusetyenziswa njengejelo lokuphonononga ulwazi lwabafundi malunga nokuqiqa ngealjebra kuba ijongwa njengenkqubo yokuqonda. Ngenxa yoko, ukuqiqa ngealjibra kufuna ukuba utitshala athathele ingqalelo ukunceda abafundi ekuphuhliseni ukucinga nzulu.

Olu phononongo luphononongе indlela ootitshala besigaba esiphakathi abasebenzisa ngayo ukucinga okunzulu (CT) ukukhuthaza uphuhliso lokuqiqa nge-algebraic (AR). Ukongeza, olu phononongo luphononongа indlela ootitshala abasebenzisa ngayo imisebenzi yeepateni ukuzibandakanya nokukhuthaza abafundi ukuba bacinge nzulu ukuze baphuhlise ukuqiqa kwealjibra xa besombulula iingxaki. Olu phononongo lujolise kwiSigaba esiPhakathi, esinabafundi beBanga lesi-4, iBanga lesi-5 neBanga lesi-6. Kuphela ngamaBakala 5 no-6 asetyenziswa njengesampulu kudliwano-ndlebe lweqela ekugxilwe kulo. Umphandi ucingele ukuba abafundi beBanga lesi-4 banokonganyelwa ziikhonsepthi zolu phando, kwaye ngenxa yokunqongophala kwexesha, abanakufakwa.

Indlela exubeneyo yeendlela zobuninzi kunye nekhwalithi yamkelwa ukufezekisa injongo yophando. Iindlela ezisemgangathweni zibandakanya uphononongo loncwadi, ukuqwalaselwa kwezifundo nodliwano-ndlebe nootitshala abathatha inxaxheba, kugxininiswe kwiindlela zovavanyo ezibonelelwa yiCAPS. Iindlela zokubala zibandakanya amaqela ekugxilwe kuwo kunye neekhweshine zasemva kokucamngca, eziye zanceda ekuqondeni iimpendulo zabafundi kwimibuzo ye-CT kumaBakala 5 no-6 kuphuhliso lwe-AR kunye nembono yootitshala nge-CT.

Iziphumo ekupheleni kophando zibonise ukuba iimbono zootitshala ziye zavuselelwa, kwaye baye baqonda ngakumbi ukuba yintoni i-CT kunye nokuba inokuphunyezwa njani kwizifundo zabo zezibalo. Udliwano-ndlebe lweqela ekugxilwe kulo kunye nokuqwalaselwa kwezifundo kwakhona kubonise ukuqiqa kwabafundi kuphuhliso lwe-AR xa besenza umsebenzi wepatheni. Okokugqibela, iziphumo zabonisa ukuba abafundi beBanga lesi-5 nelesi-6 bafuna uqheliselo oluthe kratya ngokuqiqa kwabo ngokubanzi.

Ngako oko, kucetyiswa ukuba imibuzo yeCT ifanele ukuba yinxalenye yesifundo ngasinye semathematika ukuphuhlisa izakhono zabafundi ekuhlalutyeni nasekuthetheleleni ukudityaniswa kwayo ngokubanzi kuphuhliso lokuqiqa ngealjibra.

OPSOMMING

Wiskunde word in kurrikulums en die samelewing geag as 'n vak wat die hoogste standaard van kennis vergestalt. Wiskunde is 'n vorm van taal wat 'n numeriese idee kan verteenwoordig deur syfers, letters en simbole in algebra te gebruik om logiese en kritiese denke by leerders aan te moedig. Omdat dit as 'n kognitiewe proses beskou word, kan algebra as 'n kanaal gebruik word om leerders se algebraïese redenasievermoëns te hersien. Gevolglik vereis algebraïese redenering onderwysers se aandag om leerders te help om kritiese denke te ontwikkel.

Hierdie studie het ondersoek hoe onderwysers in die intermediêre fase kritiese denke (KD (CT in Engels)) gebruik om die ontwikkeling van algebraïese redenasie (AR) aan te moedig. Daarbenewens ondersoek hierdie studie hoe onderwysers patroontake gebruik om leerders te betrek en aan te moedig om krities te dink om algebraïese redenasie te ontwikkel wanneer probleme opgelos word. Hierdie studie het gefokus op die Intermediêre Fase, wat uit graad 4-, graad 5- en graad 6-leerders bestaan. Slegs graad 5 en 6 is as steekproef vir die fokusgroeponderhoud gebruik. Die navorser het aangeneem dat graad 4-leerders oorweldig kon word deur die konsepte van hierdie studie, en weens tydsbeperkings kon hulle nie ingesluit word nie.

'n Gemengde-metode-benadering van kwantitatiewe en kwalitatiewe metodes is gebruik om die navorsingsdoelwit te bereik. Die kwalitatiewe metodes het 'n literatuuroorsig, leswaarneming en onderhoude met die deelnemende onderwysers ingesluit, met die fokus op evalueringsmetodes wat deur die KABV (CAPS in Engels) verskaf is. Die kwantitatiewe metodes sluit fokusgroepe en post-reflektiewe vraelyste in, wat gehelp het om leerders se antwoorde op RT-vrae in graad 5 en 6 vir AR-ontwikkeling en onderwysers se persepsie van RT te verstaan.

Die resultate aan die einde van die navorsing het getoon dat onderwysers se persepsies gestimuleer is, en hulle het meer begrip gekry van wat KD is en hoe dit in hul wiskundelesse geïmplementeer kan word. Die fokusgroeponderhoud en leswaarnemings het ook leerders se redenasie vir AR-ontwikkeling getoon wanneer hulle by die patroontaak betrokke was. Laastens het die resultate getoon dat beide graad 5- en 6-leerders meer oefening nodig het met hul veralgemeningsredenering.

Gevolglik word dit aanbeveel dat KD-vrae deel van elke wiskundeles moet wees om leerders se vaardighede in analisering en die veralgemening daarvan vir die ontwikkeling van algebraïese redenasie te ontwikkel.

ACKNOWLEDGEMENT

I want to express my gratitude to the following people who supported me in making this thesis a success. My appreciation goes to:

- To the Almighty God for giving me strength and for His Grace throughout this research.
- To my loving and supportive husband, Olabanji Asekun. Thank you for your patience, academic guidance, and encouragement throughout the journey.
- To my daughter for her support and patience through this long journey. Thank you for your understanding and for supporting my academic journey.
- To my supervisor Prof MF Gierdien, for your patience and support. Thank you for supporting me throughout this challenging journey and encouraging me.
- To my mother-in-law, Prof Olayinka Asekun, thank you for the constant calls to share nuggets of wisdom and guidance, your prayers and support throughout my academic journey.
- To the rest of my family, thank you for your prayers and support.
- Thanks to the teachers who willingly participated in this research study and gave their valuable time.
- To the members of The Redeemed Christian Church of God, Desire of Nations, Stellenbosch and ministers, for providing spiritual support during this study.

"Thy word is lamp unto my feet, and a light unto my path."

(Psalms 119:105)

TABLE OF CONRTENTS

LIST OF FIGURES

LIST OF TABLE

GLOSSARY

AR: Algebraic reasoning

CAPS: Curriculum Assessment Policy Standards

CT: Critical thinking

DBE: Department of Basic Education

FET: Further Educational Training

HOT: Higher Order Thinking

IP: Intermediate Phase

MKO: More Knowledgeable Other

NCTM: National Council of Teachers of Mathematics

ZPD: Zone of Proximal Development

Chapter 1: INTRODUCTION

1.1 Background

Mathematics holds an esteemed position in curricula and society as a subject that embodies the highest standard of knowledge (Resnick et al., 1982; Roohi, 2015). Mitchener (2016) supports the view that mathematics is beyond merely arithmetic and geometry. Mathematics is around us and applied in different contexts and presents itself in different ways. Moreover, we consciously and unconsciously use mathematics in our daily lives. For example, we apply mathematics unconsciously when we tell the time, count money, or use fractions to share a slice of pizza amongst friends, and consciously when we learn maths formally at school.

Mathematics is a study that deals with arithmetic and symbols (Mitchener, 2016). Other researchers define mathematics as a study of patterns and their relationships, which gives an opportunity to observe, hypothesise, experiment, discover, and create (Devlin, 1994; Warren, 2005). The Curriculum and Assessment Policy Statement (CAPS) document for Grades 4–6 (Intermediate Phase) defines mathematics as:

…a language that makes use of symbols and patterns to formulate an understanding of numerical, geometric and graphical relationships…., development of mental processing that enhances logical and critical thinking, accuracy and problem solving which yields to decision-making (CAPS, 2010, p.8).

From this definition of mathematics in the CAPS document, it can be deduced that mathematics is a form of language that can be used to represent a numerical idea using numbers, letters, and symbols to encourage logical and critical thinking in learners. The use of letters and symbols to represent a number in mathematics is known as algebra. In mathematics, numbers are used differently in arithmetic and algebra. Numbers represent a linear equation, while algebra uses numbers and symbols to represent unknowns. The term ‘unknowns’ is also referred to as a variable; it represents a specific number or letter and is not static (Radford, 1996). These symbols (variables/unknowns) form part of mathematical language.

Algebraic Reasoning (AR) is defined as a generalisation that helps to identify, express and justify consistency, structure, properties and relationships using the symbolic language of algebra that helps in AR development (Kaput, 2008; Kriegler, 2008; Radford, 2007; Lins & Kaput, 2004). Generalisation occurs when one identifies a consistent pattern in several instances and can support the relationship between quantities (Fong, 2004, p.43). Some researchers view AR as algebraic thinking (AT), defined as the capacity to explore patterns and functions (Van de Walle, Karp, & Bay-Williams, 2011, p.262; Driscoll, 1999, p.2). Therefore, pattern activities in the Intermediate Phase (IP) can be used as support for number concept development to develop AR (CAPS, 2010; Twohill, 2019).

International case studies have shown the possibility of developing AR in the Intermediate Phase (IP) (Ng, 2004; Mulligan et al., 2012; Blanton et al., 2015; Barlow et al., 2017). Promoting AR in the classroom involves incorporating conjecture, argumentation and generalisation in purposeful ways so that learners consider arguments as ways to build reliable knowledge (Blanton & Kaput, 2003, p.74). The knowledge that learners build on will have to be questioned, clarified, analysed, and understood, using inferences to make a sound judgement, which is the critical thinking (CT) process. Thus, the key to AR development is not merely about letters and unknowns but “a way of thinking” (Kieran, 2011; Lew, 2004 & Septiani et al., 2018).

However, research has shown that teachers need to articulate algebraic concepts transparently to develop meaningful AR and to encourage the use of different rules when generalising patterns. Teachers are responsible for integrating AR into classroom practice (Bastable & Schifter, 2008; Blanton & Kaput, 2005). Drawing on the research of Lew (2004) and Kieran (2004) who viewed algebra as a way of thinking, this research seeks to understand how teachers stimulate IP learners’ critical thinking to develop AR in their teaching.

This research was conducted with learners in the intermediate phase in a South African context and compared with how other curricula introduce early algebra. Secondly, it was used to study how patterns may be used effectively to encourage CT in the process of developing AR in the IP. Finally, it seeks to contribute to the curriculum of the intermediate phase by presenting guidelines for developing lessons that use CT to develop AR. It also helped teachers develop strategies and processes to ensure that their planned lessons can develop learners’ algebraic proficiency.

1.2 Purpose of the study

This study was motivated by my observations as a Mathematics teacher in both primary (Grades 5–7) and high school (Grades 9–10). I observed that learners in high school struggled to solve algebraic problems, which seemed to be influenced by algebraic reasoning (AR) challenges in the lower grades. In CAPS, the ‘Patterns, functions and algebra’ content area for the primary level focuses on investigating numeric and geometric patterns using function machines and input and output tables (CAPS, 2014, p.10). At the secondary level, algebra falls under the ‘Algebra’ content that focuses on manipulating algebraic expressions (CAPS, 2011, p.11). For example, a problem that has a variable e.g. $x + 3 = 8$ is considered an algebraic problem but if composed only of numbers e.g. $5 + 3 = 8$, it is arithmetical. Usiskin (1999, p.7) notes that the concept of letters/variables is not static but changes over time. From this observation, the question arises as to whether a teacher’s development of AR in the intermediate phase helps to develop learners’ understanding of symbol manipulation and to apply algebra as a strategy for solving a problem critically to deepen learners’ understanding.

Therefore, teachers need to facilitate critical thinking to help learners make a sound judgement of the question or solutions provided (Septiani et al., 2018, p.673; Zielgler & Kapur, 2018, p.2; ACARA, 2013). Thus, the synthesis between AR and CT is that they are both reasoning strategies that help learners as they generalise patterns when they engage in AR and which in the final stage are necessary for critical and sound judgements about their reasoning.

1.3 Definition of concepts

1.3.1 Algebraic Reasoning (AR)

Algebra is used to cultivate algebraic reasoning in learners. Algebra is a language of symbols, functions, and generalising of the relation between quantities, which involves the use of unknowns (CAPS, 2010, p.8). An unknown is a specific number; variables are not fixed but vary from letters to other forms of representation, such as pictures (Radford, 1996). Therefore, generalising patterns using unknowns stimulates the development of algebraic reasoning.

Algebraic reasoning (AR) is a process of generalising problems by exploring concepts of patterns and functions of patterns. Generalisation is when one identifies a consistent pattern for several instances and can support the idea. Kaput, Blanton, and Moreno agree that generalisation and using symbols to express generalisation to give a justifiable reason are critical skills for the development of AR (2008, p.21). Likewise, Sfard (1991) agrees that for learners to understand algebra and algebraic expressions, they first need to understand the use of operations and algebra as thinking that is expressed by algebraic expressions (which comprise letters and numbers) to develop their mathematical understanding (1991, p.3).

1.3.2 Critical thinking (CT)

Critical thinking is a process regarded as a high-order reasoning process, which aims to clarify, analyse, and understand, making inferences depending on the information (ACARA, 2012, 2013; Kong, 2015; Cahyono et al., 2019). Other researchers define CT as a disciplined and active process of skilfully analysing the generalised information that is influenced by observation and reflection to justify and create a logical conclusion (Scriven &Paul, 1996; Ennis, 1989; ACARA, 2012, 2013; Kong, 2015; Tunca, 2015; Pithers & Soden, 2000, p.239). Moreover, Bloom’s Taxonomy (1956) classifies the six critical thinking keys: remembering, analysing, understanding, evaluating, applying and creating.

1.3.3 Intermediate Phase

The South African Curriculum currently used in schools is called the ‘Curriculum and Assessment Policy Statement’ (CAPS)(DBE, 2011). The school system is divided up according to phases: (1) Foundation Phase (Grades 1–3); (2) Intermediate Phase (Grades 4–6); (3) Senior Phase (Grades 7–9); and (4) Further Education Training (Grades 10–12). The CAPS document for mathematics consists of five strands or content areas, each with its objectives. The five strands covered in the Intermediate Phase are (1) Numbers, Operations and Relationships; (2) Patterns, Functions, and Algebra; (3) Space and Shape (Geometry); (4) Measurement; and (5) Data Handling. This study focused on the second strand on how teachers help learners to think critically in the process of algebraic reasoning development in the intermediate phase.

1.4 Problem statement and Research Question

Kieran (2004) and Lew (2004) both view algebra as a cognitive process in mathematics. Cognitive has to do with intellectual activities like perceiving, thinking, problem-solving and remembering (Donald, Lazarus, & Lolwana, 2010, p.363). Rosita (2018) agrees that algebra is a cognitive process that can be used as a channel to review learners' algebraic reasoning abilities because it is viewed as a cognitive process. As a result, AR requires teachers' attention to assist learners in developing CT. Therefore, this study aims to explore how teachers in the intermediate phase use critical thinking to encourage the development of algebraic reasoning. The purpose of this study is to explore how teachers use pattern tasks to engage and encourage learners to think critically to develop algebraic reasoning when solving problems. Therefore, a mixed-method approach was adopted to accomplish the study's purpose. The qualitative part was obtained through in-depth interviews of the participating teachers, focusing on evaluation methods provided by CAPS. Therefore, from learners' responses to teachers' teaching, the findings were give a broader insight into learners' responses to CT in the process of AR development. In addition, the quantitative findings helped to understand learners' responses to CT questions in Grades 5 and 6 for AR development.

Only Western Cape Education Department (WCED) schools was considered for data acquisition in this research project. Due to logistic constraints, data acquisition for this research was be limited to three WCED schools in the Cape Winelands and Metro South District. Teachers were interviewed on how they use critical thinking to cultivate the development of algebraic reasoning when learners engage in pattern tasks.

Moreover, since the study applies to the Intermediate Phase, which consists of Grade 4, Grade 5 and Grade 6, 15 learners in Grade 5 and another 15 in Grade 6 from the three selected schools participated in the focus group interview, making a total of 90 learners that were interviewed. The researcher presumed that Grade 4 learners could be overwhelmed with the concepts of this study, and due to time constraints, they could not be included. A total of 11 teachers participated in the IP's pre- and post- interviews. This contributed to the teachers' perception of CT and helped the researcher understand how teachers prepare lessons that stimulate CT to develop learners' AR skills and how the learners respond to

CT questions. Therefore, considering the explanation above, this research primarily aims to answer:

How do teachers use critical thinking to develop algebraic reasoning in the intermediate phase when they teach?

Secondary questions: The study also addresses the following secondary questions:

a. What do we know about practising IP teachers' understanding of AR?
b. What are the issues these teachers face?
c. What types of tasks are suitable to generate data on AR and CT in the IP?
d. What ways to design AR tasks that can potentially foster CT and AR?
e. How do learners respond to CT questions for AR development in different grades?

This study attempts to find answers to these questions.

1.5 Research Aims and Objectives

This study aims to investigate how teachers use CT to develop AR in the intermediate phase when solving problems.

The specific objectives of this research include the following:

a. To analyse Patterns and Algebraic content (as specified in the CAPS document 2010 date) in the curriculum that encourages the development of algebraic reasoning for learners to engage their critical thinking skills when solving problems.
b. To analyse teachers' lesson plans to see whether and how they engage learners towards the development of AR to encourage CT.

1.6 Theoretical framework

Grant & Osanloo (2014) define the theoretical framework as the manual for the research study. It is a guide that provides support and structures for how to conduct the study; in this case, it answers the main research question. The theory that structures the study is the cognitive constructivism theory developed by Piaget (1936) and Vygotsky (1978). It also draws on Bruner's (1996) 'theory of instruction'. Bruner (1996) describes the theory of instruction as the theory that discourages monologue teaching, whereas more active learning helps learners think critically and construct new ideas.

Furthermore, a learner's analysis and justification of their findings depend on schema structure. The structure of the schema provides meaning and organisation to the information, giving an experience that enables the individual to search for additional information. The cognitive structure is in line with the cognitive constructivism teaching methods that aim to assist in the process of assimilating new information into the exist ing knowledge and helping learners to think critically to fit the new knowledge into their existing knowledge (Piaget, 1936; Vygotsky, 1978; Bruner, 1996). This served as a foundation for the importance of how teachers use pattern tasks to encourage learners to think critically in the process of developing algebraic reasoning.

1.7 Research methodology

The research methodology explores procedures, and the way knowledge is acquired. This research explores how teachers use CT to develop AR in the IP in their teaching. It means that the methodology is about answering the question 'how is research done' to get a response to the research question. The research adopted a mixed-method approach and used a case study because the study is more qualitative than quantitative. Mixed-methods research is a methodology for conducting research that integrates quantitative and qualitative research methods (Creswell, 2003).

The qualitative methods of this research include an extensive study of literature on algebraic reasoning and a desktop analysis of policies or curricula implemented in the intermediate phase, interviews and lesson observations. The quantitative data was collected and analysed through individual and focus-group interviews and a teachers' post-reflective questionnaire. The results obtained from Grades 5 and 6 were compared, revealing learners' errors when generalising patterns and their responses to CT questions. Finally, the qualitative and quantitative data were triangulated for reliability and validity.

1.8 Data analysis and interpretation

Responses to the main research question drive data analysis. The data collected through interviews and observations were triangulated with the focus group interviews and a reflective questionnaire. Triangulation is used when the question uses multiple perspectives to respond to the main question, giving a better meaning to the data. Triangulation is a strategic method used by researchers to analyse and present data for comprehension (Denzin, 1989). Greene, Caracelli and Graham (1989) agree that triangulation helps to find convergence, corroboration and correspondence of results gained from multiple methods for validation.

This qualitatively driven study aims to study how teachers use critical thinking (CT) skills for algebraic reasoning (AR) development in their teaching. Hence, the teachers were the primary unit of analysis. In addition, learners' responses to CT questions for AR development were used as supporting data. Additionally, the study used different methods to collect data; hence triangulation is best suited for this multiple-case study design (Denzin, 2009). First, the teachers in the intermediate phase triangulate data through interviews about the understanding and challenges of fostering AR in learners during their teaching. Secondly, the researcher's and the teachers' (Grades 5 and 6) data were triangulated through observation and task evaluation for their understanding of CT and AR. Lastly, data was triangulated after teachers had presented the tasks and were observed by the researcher to see how they stimulated CT for AR development. This method helped to analyse teachers' perspectives in the intermediate phase when teaching critical thinking skills for algebraic reasoning development.

1.9 Ethical consideration

This research requested ethical clearance for this research, considering that the information gathered is about processes and human subjects. Additionally, relevant permission is required since the study works with teachers and learners. Therefore, permission was obtained from the Western Cape Education Department, Principals, and the Stellenbosch University Research Ethics committee so that the teachers were entirely aware of the study's intentions and were required to sign a consent document. In addition, learners approached their parents, who would fill in consent forms. Finally, all the names of schools, teachers and learners remained anonymous.

1.10 Delimitations of the study

The study draws data from 11 intermediate phase teachers (IP) teachers. There were only three schools in the Cape Winelands District. Although the 11 teachers were from three different schools, there was no control over each teacher's particular grade. The teachers could either teach Grade 4 and 5, Grade 5 and 6 or one of the grades in the IP. Furthermore, the focus group interviews were limited to 15 Grades 5 and 6 learners per school, totalling 90 participants. Additionally, the research was completed in two phases. The first phase in term 1 included the presentation and interviews; the second encompassed the lesson observation, focus group interviews and post-reflective questionnaire. The study was done in phases because the second term's CAPS curriculum tackles numeric and geometric patterns.

1.11 Thesis Outline

This thesis chapters are organised as follows:

- ❖ Chapter 1 introduces the research and outlines the objectives and method of the research that would be carried out. Moreover, it focuses on algebra and the development of AR in the IP on how it plays a role in CT development.
- ❖ Chapter 2 presents a literature review introduced in Chapter 1: On the background of the study. In this chapter, Algebra was discussed and looked at in the context of the South African curriculum compared to other school curricula in the IP to evaluate topics used for AR development.
- ❖ Chapter 3 presents an additional literature review on thinking and CT to the thinking process. The chapter's purpose is to understand the meaning of CT and its connection with AR.
- ❖ Chapter 4 discusses constructivism, the theoretical framework underpinning this study. It includes looking at the role of the teacher as a facilitator of CT and learners engaging in a constructive classroom that nurtures CT.
- ❖ Chapter 5 overviews the research methodology and data gathering method and procedures. The study is a mixed-method study and uses qualitative and quantitative approaches for data collection.
- ❖ Chapter 6 analyses the results of the data collected through the methods discussed in Chapter 5.

- Chapter 7 concludes the research and presents recommendations for potential future research.

In keeping with the outlined layout, the next chapter presents a comprehensive review of the literature on Algebra.

Chapter 2: ALGEBRA

The chapter reviews the literature on algebra in Section 2.1, a general definition of algebra in the school curriculum and the South African context in Section 2.2. Furthermore, an overview of South African curriculum content is compared to other curricula. The curricular content helps to understand AR in Section 2.3 and the meaning of AR in arithmetic in Section 2.4 towards assisting with the transition to algebra to understand variables and how they give meaning to pattern generalisation when developing AR. Lastly, in Section 2.5, a summary of concepts of Algebra and AR are studied.

2.1 Algebra

There are diverse definitions of algebraic terminology. CAPS defines algebra as a mathematical language used for investigation and communication in mathematics, which encompasses a study of symbolising, functions, and generalising relationships in the Seventh Grade (2010, p.8). However, in most cases, people understand algebra as calculations that involve letters representing the 'letters as variables or unknowns. Different strategies must be applied to solve the specific number represented by the letter. An unknown is a particular number; variables are not fixed but vary from letters to other forms of representation, such as pictures (Radford, 1996). The expression of generalisations about numbers and their relationships using functions to justify them is algebra (Watson, 2007, p.3). This means that algebra helps to comprehend quantity representations and use the function to understand the relationship of quantities. However, understanding algebra requires one to understand the meaning of variables.

2.1.1 Variable

Epp (2011) defines variables as placeholders of the unknown. The term 'unknowns' is also referred to as a variable; it represents a specific number or letter and is not static (Radford, 1996). These symbols (variables/unknowns) form part of algebra.

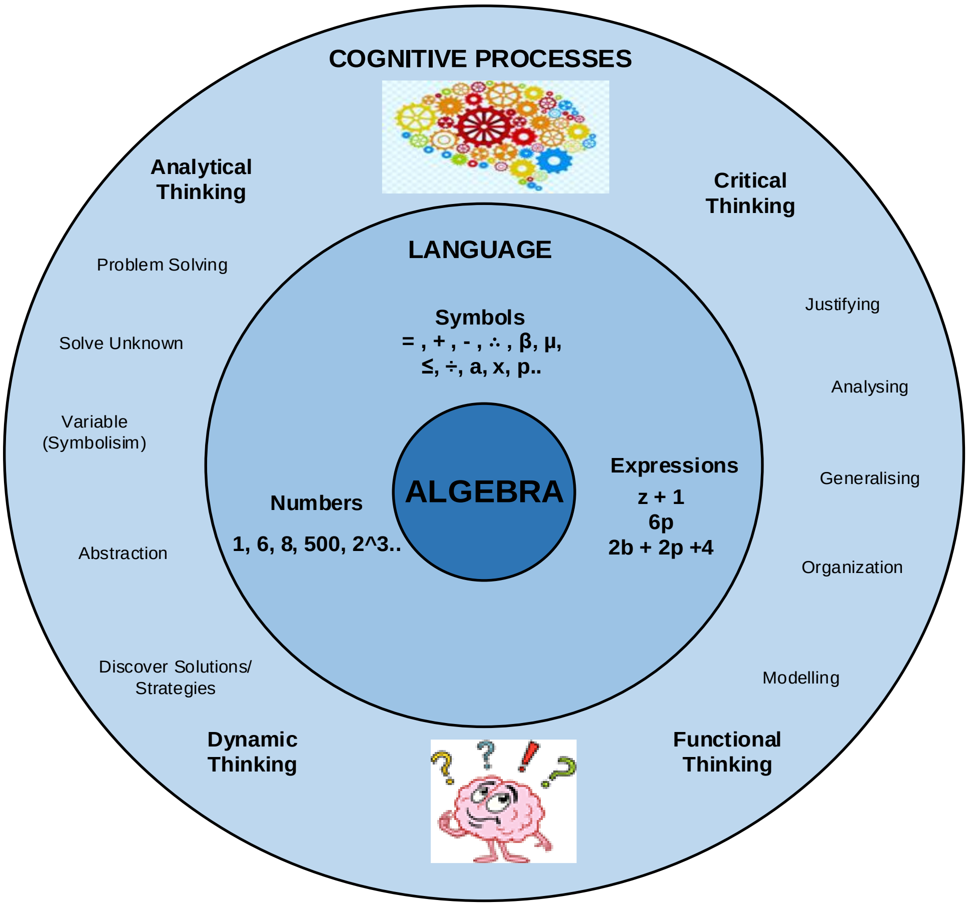


Figure 1: The Structure of Algebra (Adapted from Lew 2004, p.93)

Furthermore, Kieran (2011, p.582) explains algebra as a cognitive process representing the unknown more than understanding letters. Lew (2004) further agrees that algebra is a cognitive process that looks at facts and procedures instead of thinking (cognitive processing) stimulated by six mathematical thinking abilities. They require the skill to Generalise, Abstract, Analyse, think Dynamically, and Model and Organise (Lew 2004, p.93). The researcher used these abilities to analyse the activities toward AR development using the thinking mentioned earlier by Lew (2004). The study helps to understand the thinking process involved in stimulating AR development, as shown in Figure 1.

Moreover, Figure 1 above shows how algebra visualises the thinking processes in the view of the research (Kieran, 2004b & Lew, 2004), as they both view algebra as a cognitive process. The thinking ideologies that are encouraged by engaging in solving problems are: (a) Functional thinking, (b) Dynamic thinking, (c) Analytical thinking, and (d) Critical thinking. The thinking process is all enveloped by critical thinking as it helps to assess, verify, interpret and formulate the results to reach a justifiable conclusion.

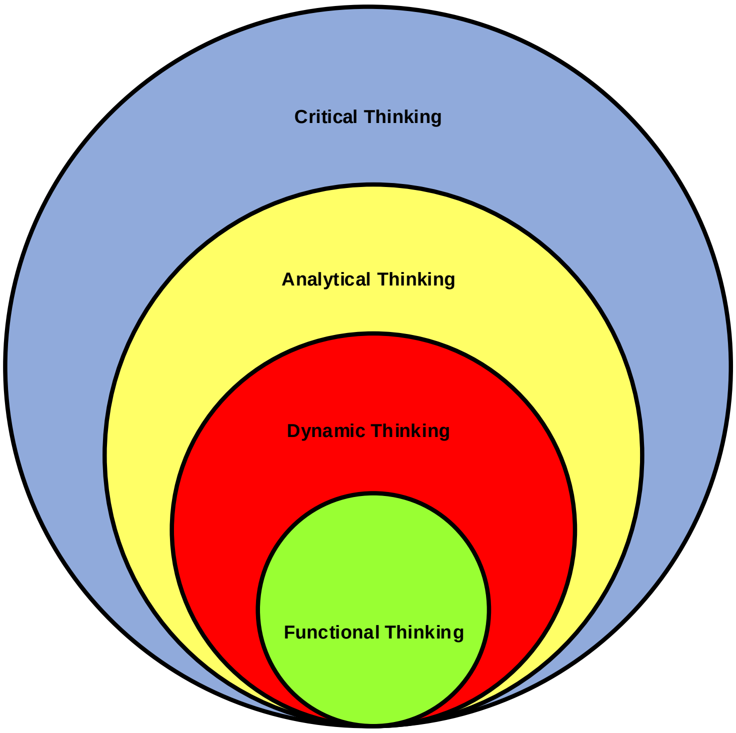


Figure 2: Relationship between the different cognitive processes

The thinking ideologies for problem-solving toward AR development in Figure 2 are defined as follows:

Functional Thinking (FT) is an act of thinking about functions. A function aims to determine a relationship between two sets of quantities [Ministry of Education (Ontario), 2014, p.8]. FT's concept function also aims to engage an individual with an activity to identify the relation between two quantities, which helps build, describe, and reason with and about functions (Smith, 2008 & Blanton, 2008). Smith further notes that FT is representational thinking that aims to distinguish the relation of quantities. More significantly, in thinking about the quantity relation, to generalise quantities in different scenarios (2008, p.143). Driscoll agrees that AR's primary key is to recognise patterns, organise data, and use function rules that clearly show the input and output (1999).

According to Beatty & Bruce 2012, FT helps to observe or identify change when analysing the relationship between the two sets of quantities. FT is a generalising process that allows learners to think about quantities in an arithmetic situation and go beyond that using other forms of representing the information. FT helps to identify and express patterns as a general relation.

According to the Ministry of Education (Ontario), 2014, learners can develop FT when they generalise patterns and use inverse operations.

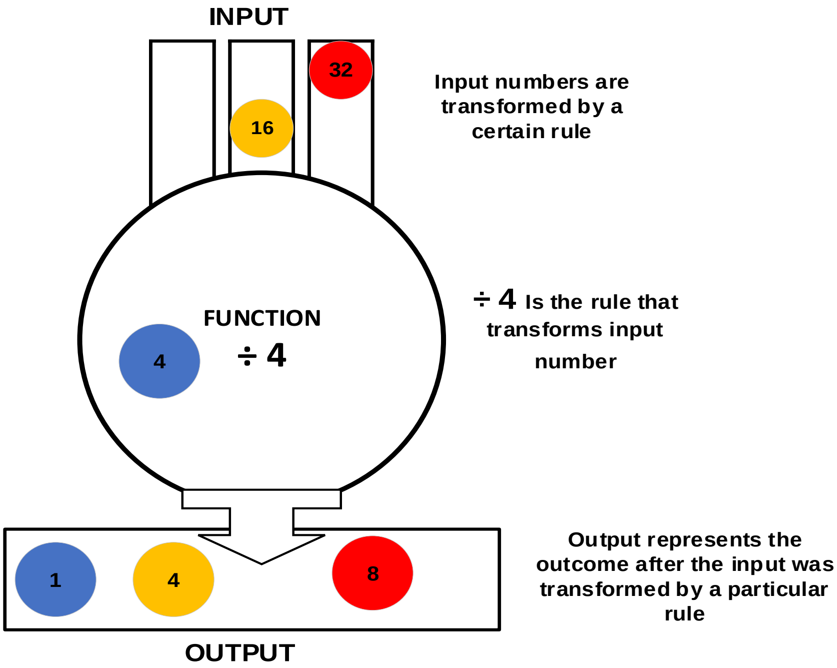


Figure 3: Function Machine

For example, learners can use the function machines presented in Figure 3 or visual patterns to support the symbolic representation. As FT constructs, describes, and reasons in terms of functions, it helps to comprehend variables. Therefore, in generalising patterns, FT helps to model and organise information from patterns to achieve generalisation. However, to develop productive conceptual algebraic reasoning, learners must understand mathematical ideas through functional thinking and develop critical thinking.

Based on Figure 4, the process of generalising patterns and developing reasoning to understand functional thinking, the creative thinking process kicks in to generate ideas and finally judge the ideas through critical thinking. Ulger (2016, p. 696) notes that critical thinking helps solve problems by cultivating logical ideas, views, and perspectives. Algebra is a mathematical strategy applied in the classroom which connects to situations in daily life to develop learners' critical thinking processes.

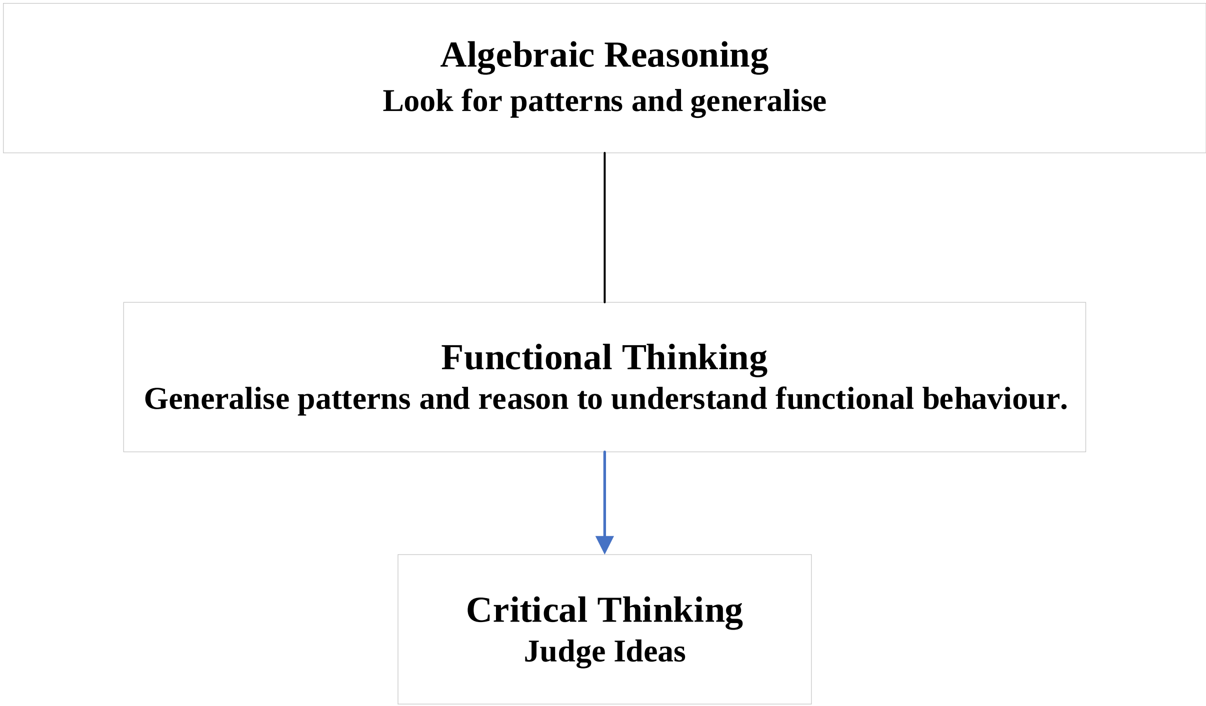


Figure 4: Relationship between AR and CT in connection with FT

Dynamic Thinking (DT): It studies the interconnectedness of relationships on an event and its impact. In algebra, the variable helps to understand a function (Lew 2004, p.93). Additionally, Lannin (2009) notes that patterning is essential for introducing algebra, as patterns represent a variable's dynamic representation. Usiskin further notes the relationship between algebraic conception and the use of variables as a generalisation of patterns that incorporates working with variables to understand the relation between two quantities and understand the structure of giving a justifiable conclusion (1999, p.12).

Therefore, it is critical to comprehend the application of algebra as it creates an understanding of the mathematical role in real-life situations and the learner's personal development to help toward AR development (CAPS 2011, p.9). Modelling variables for better understanding stimulates DT and may contribute to AR development. Lew defines modelling as a strategy of presenting complex problems using mathematical ideas to identify the characteristics of the issues and using a model to reach a justifiable conclusion (2004, p.95). Identifying the traits of a problem and breaking the information down is defined as analytical thinking.

Analytical Thinking (AT)) helps to break down complex information into a simpler format. Lew uses equations to describe AT by decoding the equation to find the unknown and the value (2004, p.93). For example, solving Equation 1 for the unknown (x):

Equation 1

$$6^x = 36$$
$$6^x = 6^2$$
$$x = 2$$

The process of substituting the value to examine if the answer is correct is the process of analytical thinking. But, first, the equation was divided into manageable sizes to understand and draw a proper conclusion about its value.

Critical Thinking (CT) is defined as a cognitive process of carefully evaluating information and determining how to make it meaningful to give sound judgement (Scriven & Paul, 1996; Ennis, 1989; ACARA, 2012, 2013b; Kong, 2015; Tunca, 2015 & Pithers & Soden, 2000, p.239). For instance, a task to draw a pattern or use other methods to represent the number of people (10 people) versus their eyes. The learners may display their pattern using symbols that will drive towards generalisation; as explained earlier, generalisation is a process of discovering a consistent pattern. Critical thinking towards AR development helps a learner make meaningful relations among the independent variables (people versus the number of eyes). The number of eyes (dependent variable) in this example depends on the number of people (independent variable).

Furthermore, CT helps learners organise their thinking. Algebra is a cognitive process that allows learners to learn to organise challenging problems using different organising skills such as tables, flow diagrams, or sentences. Such organisation is essential for many problem-solving activities as it promotes organisational thinking of identifying the relation between quantities (Lew 2004, p.95). In addition, sorting and organising tables is essential for problem comprehension and the connection between independent and dependent variables.

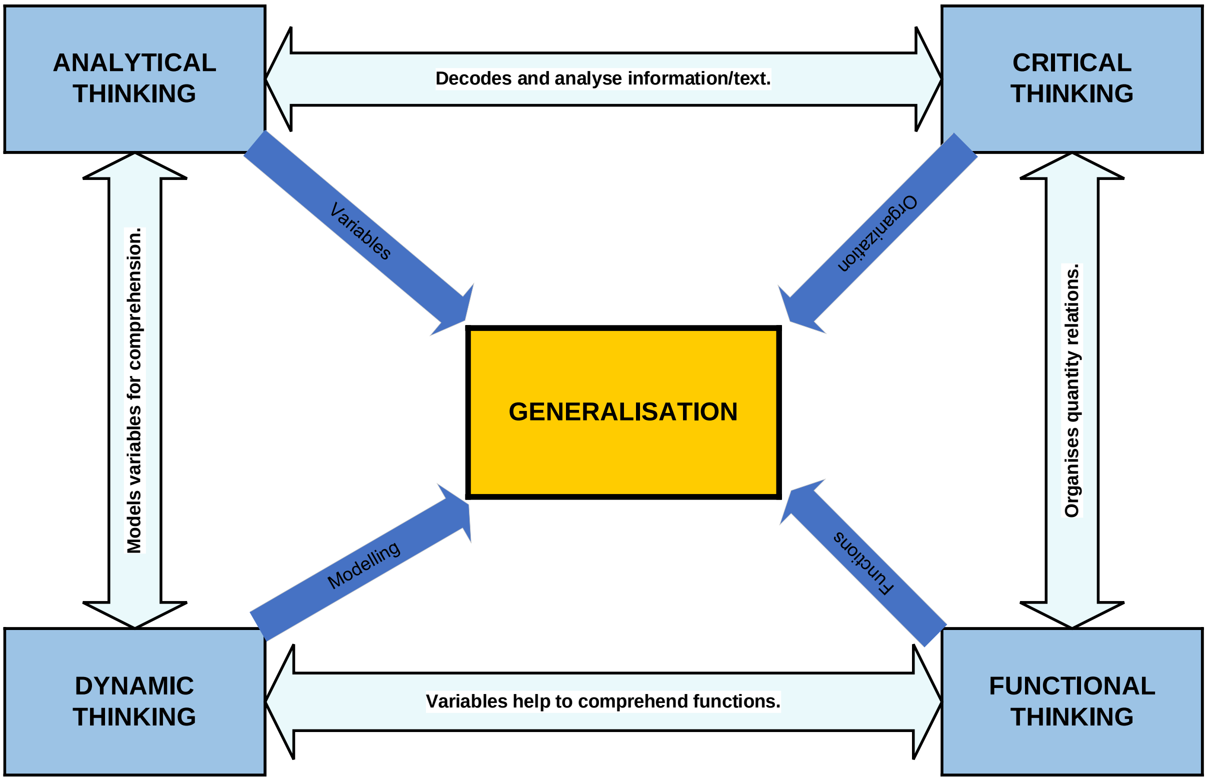


Figure 5: Representation of AT, CT, DT, and FT towards generalisation

In summary, AT, CT, DT, and FT are all thinking processes aiming to understand generalisation. However, the research seeks to understand how teachers use CT to develop AR in their lessons. Therefore, observations helped to see how teachers use CT to create learners' AR and how learners' CT is encouraged through pattern activities. Lannin (2009) notes that generalising through pattern activities helps to transit their arithmetic knowledge towards understanding symbolic representations. Kaput (2008) supports generalisation through patterns because it is fundamental to learners' thinking development in mathematics. Twohill (2016) notes that in generalising patterns for a rule, recursive thinking helps compare quantities to discover the relationship between the quantities, and functional thinking helps to understand the relationship between the two quantities.

Furthermore, Blanton and Kaput (2004) examined how teachers in the Intermediate Phase use FT to build algebraic reasoning in their teaching instruction for AR development. Additionally, the researchers used word problems to develop FT using a number of dogs corresponding to their eyes and tails. FT forms part of the thinking processes shown in Figure 5 towards generalisation. These thinking processes help learners understand, model, and organise problems to reach a justifiable conclusion. Achieving a logical conclusion requires CT, where learners have analysed, generalised and justified their answers.

2.2 Algebra in School Curriculum, South African Context

Algebra in school is a transformation and manipulation of symbols, pattern generalisation, a study of variables, functions and relationships of numbers, and mathematical modelling in our daily lives (Watson 2007, p.20). However, Kieran 2004a & Mason 2011 note that algebra in current schools focuses on rules and procedures instead of encouraging learners to see algebra as a tool that helps comprehend the application in daily-life situations so that it will not lose its meaning. Hodgen, Küchemann & Brown (2010, p. 2) agree that algebra is the central topic within the school mathematics curriculum due to its power within mathematics and beyond.

The school algebra in the Intermediate Phase (IP) defines the relationship between the unknown and known data in the problem covered under the CAPS topic 'Patterns, Functions and Algebra' (Roberts 2012, p.17). Defining algebra in the school curriculum is fundamental. Therefore, this would help to understand algebra in the South African curriculum in IP towards AR development.

The South African Curriculum is called the 'Curriculum and Assessment Policy Statement' (CAPS) and is currently used in schools (DBE, 2011). The school curriculum is divided according to phases: (1) Foundation Phase (Grades 1–3); (2) Intermediate Phase (Grades 4–6); (3) Senior Phase (Grades 7–9); and (4) Further Education Training (Grades 10–12). The CAPS document for mathematics consists of five strands or content areas, each with its own objectives. For example, the five strands covered in the Intermediate Phase are (1) Numbers, Operations and Relationships; (2) Patterns, Functions, and Algebra; (3) Space and Shape (Geometry); (4) Measurement; and (5) Data Handling.

Learners begin formal algebra in Grade 7. According to Roberts (2012, p.17), in the IP, learners are introduced to:

1. Generalising arithmetic
2. Generalising towards the idea of a function; and
3. Modelling as a language of mathematics

In CAPS (2011), the 'Patterns, functions and algebra' is the second aspect of early algebra that investigates patterns (numeric and geometric) using the function machines and inputs and output tables. It also aims to analyse how different presentations describe the problem or relationship of variables.

Another aspect of AR 'Number, operations and relationships content, which is more arithmetic (CAPS 2011), focuses on operations, number properties, inverse relationships, and equivalence. Lastly, the third aspect of early algebra (modelling as a language of mathematics) is only evident through opportunities for problem-solving and encouraging learners to explain their reasoning. Therefore, learners' AR must be developed in the IP to build their understanding of algebra in mathematics and daily-life situations. This means that AR development requires arithmetic and algebra to encourage learners' potential to think critically about mathematics (Carpenter & Levi 2000, p.1).

However, the content for AR development in the IP curriculum aims to (1) describe patterns and relationships using symbolic expressions, graphs, and tables; (2) to find and analyse similarities and changes in patterns and relationships to predict and solve problems (CAPS, 2011 p.10), which is a manipulative skill for algebra. Therefore, understanding school algebra from other curricular perspectives in South Africa helps understand how and when algebraic reasoning is developed in the IP in other curriculums.

2.3 Algebra in the intermediate phase in Different Curricular perspectives

Analysing other curricula gave more perspectives on how the South African curriculum encouraged AR in the primary grades. Researchers Cai et al. (2005) conducted a comparative study to analyse how algebraic concepts are introduced and studied to develop AR. Their research compared five curriculums from China, Russia, Singapore, South Korea, and the United States. The comparison tool for these curriculums is the algebraic goal and standard of the National Council of Teachers of Mathematics (NCTM) Principles and Standards for School Mathematics with the following thinking aims:

1. **Functional Thinking**: rules are alphanumeric and aim to understand patterns, their relations, and functions.

2. **Analytical Thinking**: analyses and represent mathematical situations using algebraic symbols to structure the situation.
3. **Dynamic Thinking**: uses mathematical models to represent and understand quantitative relationships.
4. **Critical Thinking**: analyses and justifies the change in various contexts.

(NCTM 2000)

The IP curriculum was analysed and compared with these curriculums and summarised in a table to see if the various countries share the same views of AR development in the early grades and the roles of cognitive processes (FT, DT, AT and CT) to stimulate generalising patterns.

2.3.1 China

The Chinese curriculum aims to develop three thinking skills in learners in the early grades:

(1) Analyse quantities in different ways; (2) Use inverse operations to solve problems; (3) Generalise from different or specific examples. The focus on quantitative relationships stimulates learners to present findings on relationships numerically and symbolically. Furthermore, the aim is for learners to compare arithmetic to algebraic ways when representing quantities' relationships.

The thinking skills are gradually introduced based on grade levels. Initially, solving equations between Grades 1–4, working with variables and functions sense, is introduced. Variable understanding is characterised as follows: First, in Grades 1–3, variables are used as a placeholder for unknowns in the form of a picture, box, word or brackets when dealing with equations; (b) Secondly, the variables are generalised as representative of different values when solving patterns; and lastly (c) to present relationships as inverse or direct proportions.

Then in Grade 5, equations and the solving of equations are formally introduced. Finally, in Grade 6, the functional relationship between quantities is modelled using different models such as diagrams, tables, graphs, pictures, or equations.

2.3.2 Singapore

Singapore's curriculum has a strong emphasis on problem-solving. According to Chen (2008) and Fong (2004), problem-solving is a crucial component of learning mathematics because it requires learners to apply the knowledge they have learned to solve a problem.

Singapore's curriculum is similar to China's curriculum as they use thinking skills to facilitate AR development. In focusing on AR, the curriculum uses number pattern activities to encourage learners to generalise. In addition, the curriculum uses the "Three Algebraic Thinking Process" to develop algebraic reasoning (AR): (1) Analyse parts and whole; (2) Generalising and specialising; and (3) Doing and Undoing (Driscoll 1999; Cai et al. 2005 & Cai et al. 2011).

Algebraic reasoning processes help learners develop thinking and problem-solving skills. For example, the model method introduced in Grade 2 helps learners to organise their thinking when solving a problem (Fong 2004, p.43). Furthermore, in Grade 6, the formal introduction of algebraic concepts is initiated. For example, variables and creation, solving and assessing expressions are introduced in Grade 6. The functional approach helps to develop an understanding of letters as variables. It further allows learners to understand the operations and apply their knowledge of number bonds using thinking skills to identify patterns and their relationships.

For example, in Figure 6, AR is developed through the model presentation to avoid using abstract letters to represent the unknowns.

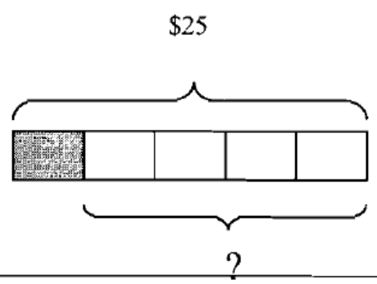
Meili had \$25. She spent $\frac{1}{5}$ of it and saved the rest. How much did she save?	

Figure 6: **Table presentation of the problem (Adopted from Fong 2004, p.44)**

The model method helps learners with no knowledge of formal algebra to construct pictorial equations like that presented in Figure 6 to solve challenging word problems with the part-whole relationship, which requires proportional reasoning (Fong 2004, p44).

2.3.3 South Korea

The South Korean curriculum uses thinking skills for AR development to help with the relation or transition of arithmetic to algebra. The thinking skills are as follows: generalisation, abstraction, analytical thinking, dynamic thinking, modelling, and organisation (Lew 2004). The curriculum aims to provide activities that develop learners' foundation for learning formal algebra.

The foundations in grade levels to develop formal algebra exist in the first grade. For example, in Grade 1, symbols such as "represent unknowns and identify math structures (e.g., commutative law of addition). Also, to work backwards and form tables to present quantity relationships. Secondly, in Grades 3–4, they engage in activities that require the discovery of digits presented by " " or by a vertical alphabetical symbol in multiplication (see example below).

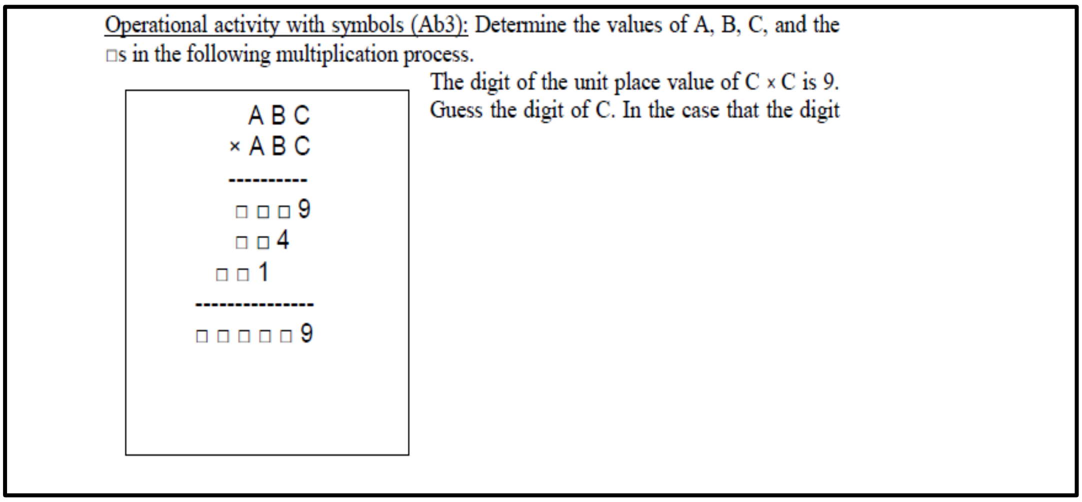


Figure 7: Example of alphabetic manipulation in multiplication (Adapted from Lew 2004, p.98)

Furthermore, learners are encouraged to apply trial and error strategies, simplify inverse operations, direct proportionality, and models when solving problems. Lastly, in Grade 6, learners are introduced to functional thinking. However, the FT relation between variables presentation is symbols like " " or ( ) or diagrams, pictures, graphs, tables, and equations.

Finally, learners are encouraged to use functional thinking to understand the relationship between two quantities and present them symbolically using " " or ( ).

2.3.4 Russia

The Russian curriculum is reviewed based on the work of Davydov, Gorbov, Mikulina, & Saveleva (1999). It focuses on encouraging the use of concrete work with quantities to develop algebraic understanding. The curriculum draws on learners' skills to solve equations and multi-step problems using models to analyse and express quantities. It also encourages manipulating relationships symbolically, which is essential for AR development. Learners also use symbolic representation to transform inequalities into qualities and find missing wholes and parts using operations. Modelling is encouraged as it allows for the diverse use of algebraic representation and symbols.

In Grades 2–3, learners solve equations and two-step word problems (with four operations) and write symbolically the expression represented by word problems. Moreover, proportional reasoning is developed by building on the concepts of quantity, the relation between quantity measuring units, and the concept of the number.

2.3.5 United States

According to Cai, Lew, Anne Morris, John Moyer, Fong & Jean Schmittau (2005), the analysis of the curriculum is based on the investigation done by NTCM, which aims to encourage the implementation of the recommendation of the curriculum and Evaluation Standards for School Mathematics developed by NTCM (1989). The investigations aim to focus on the mathematical change and establish informal algebra-related opportunities, avoiding or postponing formal algebraic representations and procedures. Algebraic ideas such as patterns and pattern relation, representation and modelling are the goals for this change (Cai, Lew, Anne Morris, John Moyer, Fong & Jean Schmittau 2005). Furthermore, the researchers have discovered delays in mathematics content until the learner's understanding of arithmetic is enriched and refined through informal learning (Cai, Lew, Anne Morris, John Moyer, Fong & Jean Schmittau 2005).

Mathematics content from the Common Core State Standards for Mathematics Curriculum in primary grades aims to develop AR as follows:

Grades 1-3: In Grade 1, the learner's number sense and identification of combinations are developed through linear equations (e.g. 3 + 1 = 4). Moreover, both Grade 1 and 2 learners engage in solving linear problems involving either addition or subtraction. Learners are encouraged to understand and apply the relation between these two operations. Then in Grade 2, they begin to work on groups of objects to lay a foundation for multiplication. Lastly, in Grade 3, learners engage in only two operations of mathematics, multiplication and division. They study the relationship between these operations and solve the problems that involve the two operations. Additionally, they solve problems involving the four operations and analyse and justify patterns in the arithmetical method.

Grades 4–6: Grade 4 learners engage in problems using four mathematical operations, identify factors and multiples and generate and analyse their patterns. Then later in the fifth grade, learners begin to generalise patterns and express their rules using variables. In Grade 5, learners begin to analyse patterns and relationships and are encouraged to write and interpret numerical expressions. Finally, algebra is introduced formally in Grade 6. Learners engage in activities involving expressions and equations that encourage applying and extending previous understandings of arithmetic to algebraic expressions and reason, solving singular variable equations and inequalities, and finally, representing and analysing quantity relations to identify dependent and independent variables.

2.3.6 South Africa

Patterns, Functions, and Algebra contents' main progression occurs when learners can complete and extend patterns, represent patterns in different forms, and identify and describe patterns. Thus, it helps learners become familiar with algebraic work, which comprises formulating equations, solving equations, and constructing algebraic patterns utilising variables and expressions, which prepares them for the senior phase (CAPS 2011, p.18).

Furthermore, the content aims to encourage AR development by cultivating an understanding of operations properties (e.g. inverse operations) with whole numbers, functional thinking and writing number sentences (CAPS 2011, p.18). For example, commutative, distributive, and inverse functions help the learner to comprehend operations and whole numbers' properties.

The content objectives of the intermediate phase included under the ‘Patterns, Functions, and Algebra’ strand of the Intermediate Phase CAPS curriculum are outlined in Table 1 below:

TABLE 1: INTERMEDIATE PHASE CAPS CURRICULUM CONTENT OBJECTIVES UNDER THE 'PATTERNS, FUNCTIONS, AND ALGEBRA' STRAND. (CAPS 2011, P.18).

	Grade 4	**Grade 5**	**Grade 6**
Patterns	Geometric **(G)** and Numeric **(N)**	Geometric **(G)** and Numeric **(N)**	Geometric **(G)** and Numeric **(N)**
Pattern Analyses	• Identify relationship and rules from patterns **(G/N) to**: - find a constant difference or ratio **(N)**, not limited to **(G)** patterns. - investigation of learner's creation - presentation in physical or diagram form **(G)** • Learners describe their observation or rules in their own words.	• Identify relationship and rules from patterns **(G/N) to**: - find a constant difference or ratio **(N)**, but not limited to **(G)** patterns. - investigation of learner's creation - presentation in physical or diagram form **(G)** • Learners describe their observation or rules in their own words.	• Identify relationship and rules from patterns **(G/N) to**: -find a constant difference or ratio **(N),** but not limited to **(G)** patterns. - investigation of learner's creation - presentation in physical or diagram form **(G)** - represented in tables **(G&N)** • Learners describe their observation or rules in their own words.
Model for input & output	Determines relationships using flow diagrams and tables **(N)**.	Determines relationships using flow diagrams and tables **(N)**.	Determines relationships using flow diagrams and table **(G&N)**
Equivalence presentation	Presentation of relation or rules either: - verbally, - function machines/flow diagram, - table format **(N)** and - by number sentence	Presentation of relation or rules either: - verbally, - function machines/flow diagram, - table format **(N)** and - by number sentence	Presentation of relation or rules either: - verbally, - function machines/ flow diagram, - table format **(G&N)** and - by number sentence
Number sentences: Algebraic Expressions	Write number sentences to describe the problem **(G&N):** Then inspect by trial and error strategy and finally substitute for verifying the solution.	Write number sentences to describe the problem **(G&N):** Then inspect by trial and error strategy and finally substitute for verifying the solution.	Write number sentences to describe the problem **(G&N):** Then inspect by trial and error strategy and finally substitute for verifying the solution.

TABLE 2: SOUTH AFRICAN EARLY ALGEBRA CURRICULUM IN COMPARISON TO OTHER CURRICULA BASED ON THE NTCM

	Functional Thinking: Alphanumeric presentations	Dynamic Thinking: Modelling	Analytical Thinking: Symbolic structures	Critical Thinking: Analyse/apply in new contexts.
China	•	•	•	•
Russia	•	•		•
Singapore	•	•	•	•
S. Korea	•	•	•	
The U.S.	•	•	•	•
S. A	•	•	•	

The focus on AR before and during the Intermediate Phase ensures early algebraic development, mainly symbolic representation. It is evident from Table 2 that the curriculums encourage symbolic understanding of AR development.

2.4 Algebraic reasoning

Algebraic reasoning (AR) is a generalising problem by exploring concepts of patterns and function patterns. Generalisation is when one identifies a consistent pattern for several instances and can support the idea. Kaput, Blanton, and Moreno agree that generalisation and using symbols to justify generalisation are critical for AR development (2008, p.21). Likewise, Sfard (1991) agrees that, for learners to understand algebra and algebraic expressions, they first need to understand operations and algebra as thinking expressed by algebraic expression (which comprises letters and numbers) to develop their understanding (1991, p.3). Therefore, AR's development requires understanding algebraic concepts to develop learners' problem-solving skills in algebra. Additionally, it helps develop AR by building on an arithmetic foundation using patterns to understand variables representing the unknowns.

Van de Walle, Karp & Bay-Williams also note that AR focuses on mathematics and cuts across the mathematics content (2011, p. 262). However, Van Ameron argues that arithmetic does not give learners opportunities to generalise as it only focuses on solving linear equations with identified unknowns (2003, p.64). Mason (1996, p.23) & Kieran (2004b) agree that arithmetic is procedural learning that uses familiar techniques to solve the unknown, unlike algebra, which begins indirectly with the unknown and proceeds using familiar techniques to solve the problem.

Algebra and arithmetic connections lead to algebraic reasoning, which helps understand patterns and functions and how generalisation is justified. The distinction in Table 3 contrasts the characteristics of arithmetic, algebraic, and algebraic reasoning characteristics. Pre-algebra is for a transition between arithmetic, algebra, and algebraic reasoning in Table 3, which shows generalisation, understanding of variables, meaning or expressions and reasoning with unknowns toward AR development. The Pre-algebra characteristics on the tables help to understand:

a) **Generalisation** is when a learner identifies unique characteristics and common factors to understand the relationship between the two quantities and a consistent pattern to support the idea.
b) **Understanding of variables:** In algebra, variables carry diverse meanings and functions. They are representative of the unknowns. Variable definitions differ because, in arithmetic, they are units or abbreviations; in algebra, the letters substitute a variable or unknown number (Van Ameron 2013, p.30). Usiskin explains that the concept of letters/variables is not static but changes over time (1999, p.7). For example, Usiskin draws attention to variable application in equations such as a formula () or solving an x variable equation ( ). Usiskin emphasises that the variable concept cannot fit into a single conception because this oversimplifies the idea and, as a result, affects the purpose of algebra (1999, p.9). According to researchers, the ideas of variable, expression, and equation are also crucial but have a different meanings in algebra and arithmetic when they are generalized in the process of FT. (Kieran, Pang, Schifter, & Ng, 2016). Moreover, operations are perceived as a command to perform an action or find a numerical outcome. They also help select/brainstorm strategies to avoid assumptions and provide necessary proof (critical thinking).

c) **Meaning of expressions:** An expression is an alphanumeric representation () that comprises operations. The operations are a command to perform an action or find a numerical outcome. Watson also agrees that algebraic symbolism needs comprehension of operations and fluency of alphanumeric rules (2007, p.3).

d) **Reasoning with knowns and unknowns:** Arithmetic and algebra have common characteristics as they either deal with a known or an unknown (Radford 2012, p.676). Kieran (2004), Van Ameron (2003) & Mason 1996 agree that arithmetic is procedural and only focuses on finding the unknown using known numbers, which differs from algebra, which uses the unknowns to discover the known. Twohill asserts that the growth of AR occurs when students reason rationally about the known and unknowns to comprehend their relationship (2013, p.56).

All the above shows the fusion between arithmetic and algebra, influencing the thinking process towards algebraic reasoning. Thus, it pivots around understanding variables and interpreting the pattern they represent. Variables in the case of AR solely help learners to be able to generate accurate solutions. Driscoll notes that in other instances, some focus not exclusively on the variables but on: (1) the vital role functions contribute to algebra, which may characterise AR as the capacity to display relation of quantities so that relations among variables can be straightforward; or (2) how the solver models the problem (1999, p.1). It helps to stimulate the learner to think more critically of the method they apply and, in the end, justify their approach.

Moonpo, Inprasitha & Changsri agree that mathematics aims to develop learners' reasoning and their ability to think (2018). Furthermore, Driscoll (1999) notes that to reason algebraically to solve the problem involves looking at the function and impact the system's structure has on calculations, which is motivated by the habits of mind, namely:

(1) **Doing–Undoing**: The reversibility capacity to work and understand the process of working backwards. For example, the reversibility will be to use the inverse of the operation.

(2) **Building rules to Represent Functions:** Understanding the relationship between the input and the output through the organisation helps build the functional rule. Understanding inverse operations helps verify and deepen understanding of how the functional rule works.

(3) **Abstracting from computation:** Abstract reasoning is essential for observing patterns and relationships of quantities and solving complex problems (DBE, 2014; p.126 & Kieran, 2004, p.149). Abstract reasoning is synonymous with algebraic reasoning as they analyse patterns and relationships. AR is the opposite of concrete reasoning and hands-on learning (Blake & Pope 2008, p. 61). For example, a fraction is an abstract concept, while half an apple is a concrete idea. Therefore, reasoning algebraically is thinking about the operations and variables and their relation to arithmetic, which is regarded as thinking about computations independently from numbers.

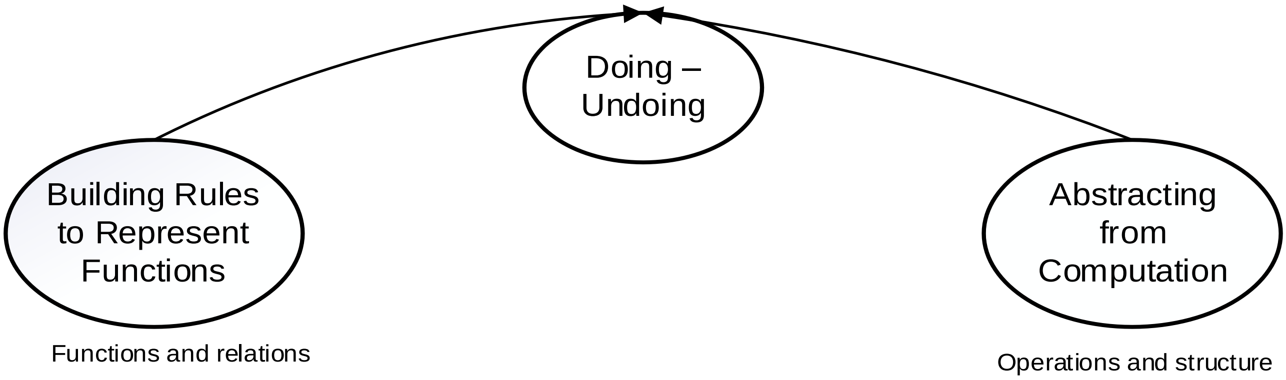


Figure 8: Three Algebraic Thinking Habits of Mind (Adopted from Driscoll 1999, p.2)

Driscoll's habits of the mind in Figure 8, which fosters algebraic reasoning, can critically help learners think of the content. Critical means to analyse and look deep into the material. Hence, learning with understanding is more powerful than learning mechanically (Kilpatrick, Swafford, & Findell, 2001, p.118; Lunenburg, 2012, p. 2). Teaching learners to understand the content transforms their thinking as they begin to internalise, analyse, and be more insightful about the content, leading to critical thinking.

Therefore, in school mathematics, learners must be skilled in applying symbols and rules to generate a correct solution. In addition, the mathematical procedure must connect to in-depth conceptual knowledge to foster firm understanding (Hiebert & Carpenter, 1992).

TABLE 3: ALGEBRAIC REASONING DEVELOPMENT IN CONTRAST TO ARITHMETIC AND ALGEBRA RELATION (ADAPTED FROM VAN AMEROM, 2002, P. 20)

Pre-algebra	Arithmetic	Algebra	Algebraic reasoning
Generalisation	• **Objective:** to find a numerical solution • **Generalising a** specific number of situations • **Table** as a calculational tool	• **Objective:** to generalise and symbolise methods of problem-solving • **The generalisation** of relations between numbers, reduction to uniformity • **Table** as a problem-solving tool	• **Objective:** exploring properties and relationships • **A generalisation** of exploring equality as a relationship between quantities • **Table** as a functional thinking tool
Understanding of variables	• **Manipulation** of fixed numbers • **Letters** are measurement labels or abbreviations of an object	• **Manipulation** of variables • **Letters** are variables or unknowns	• **Manipulation** of alphanumeric expressions • **Letters** are variables that represent the property of a number
Symbolic expressions (SE)	• **SE:** Represent processes • **Operations** refer to actions • **An equal sign** is used to give a result or balance sides.	• **SE:** are outcomes and processes • **Operations** are autonomic objects • **An equal sign** represents equivalence	• **SE**: used to reason with generalisation • **Operations** help to select and use the appropriate strategy • **The equal sign** represents the relationship among alphanumeric
Solving and Reasoning with (u)knowns	• **Reasoning** with known numbers • **Unknown**s as endpoint • **Linear problems** in one unknown	• **Reason** using unknowns • **Unknowns** as a starting point • **Problems** with multiple unknowns: systems of equations	• **Reasoning** using the relation of numbers, variables and operations • **Unknowns** as representative of the posed situation • **Simplify problem**s/expressions using alphanumeric as functions/rules to find the next term

Therefore, teaching and learning should encourage active learning to develop AR. Kaput notes that algebraic teaching and learning that builds on learners' prior knowledge to encourage mathematical thinking should be integrated with other subject matters, providing opportunities to reflect on what they know for sound judgement (2000, p.3). Therefore, AR in the IP can encourage and stimulate learners to think critically by encouraging them to develop vibrant and sound ideas of representing and communicating their ideas (Carpenter & Levi 2000, p.2).

2.5 Summary

This chapter, general research on algebra and different curricula highlights algebra's characteristics towards AR in the IP. The theory of algebra as a 'way of thinking' helps to understand AR development's thinking processes. Furthermore, the thinking process emphasises the need for IP learners to think about ways to improve their AR. Finally, this way of thinking is under discussion in the next chapter.

Chapter 3: CRITICAL THINKING

This chapter presents an understanding of thinking. Section 3.1; in general, to help with an understanding of Section 3.2, where critical thinking defined is reviewed. Then the literature overview of Chapter 4: how teachers can facilitate CT in the classroom and how Bloom's Taxonomy helps develop tasks that encourage CT. Furthermore, the literature reviews CT stimulation in Section 4.3 and how the PAH Framework of learning theories supports CT in learners. Moreover, AR is discussed in Section 2.4 and CT in Section 3.2. Analysis of their traits helps to understand the connection and how CT is encouraged in AR development. Lastly, the chapter concludes with a summary in Section 3.4 of Algebra and CT's AR development use in the IP.

3.1 Thinking

Thinking is a cognitive process. The term 'cognitive' has to do with intellectual activities like perceiving, thinking, problem-solving and remembering (Donald, Lazarus, & Lolwana 2010, p.363). De Bono describes thinking as a purposeful exploration of one's experience, which helps one understand, plan, solve problems and make thoughtful decisions and reason (1976, p.33). The thinking process can be developed by experience or through learning.

Researchers agree that thinking skills can be developed separately or by fusion in school subjects (Swartz, 2001; McGuinness et al., 2003; Rajendran, 2010 & Lipman, 1985). The infusion method integrates other teaching and instils CT skills (Rajendran, 2010 & Swartz, 1992). Swartz (2001) & Butera et al. (2014) support subjects' integration as it helps develop CT in learners. Moreover, the method allows learners to improve their thinking skills by implementing concepts in different contexts. For example, learning three-dimensional shapes in a scientific topic, 'Density', would enable learners to infuse or inquire about the similarities of the contents from different subjects to make sound reasoning.

Researchers view mathematics as one of the subjects to encourage CT skills because thinking is linked to the knowledge, reasoning, and evidence of problem-solving in mathematics (Rajendran, 2010; Aizikovitsh & Amit, 2010). Therefore, it is paramount for teachers to create an environment that fosters developing and applying critical thinking skills.

3.2 Critical thinking

Doğanay and Figan (2006) note that the term CT originates from the word "reasoning", whose root is "reason", which originates from the Latin meaning of ratio, which means "balance" Fisher (, 1990).

Critical thinking is a reasoning process defined as a high-order reasoning process which aims to clarify, analyse, and understand, using inferences depending on the information. (ACARA, 2012, 2013b; Kong, 2015 & Cahyono et al., 2019). Other researchers define CT as a disciplined and active process of skilfully analysing the generalised information that is influenced by observation and reflection to justify and create a logical conclusion (Scriven, 1996; Ennis, 1989; ACARA, 2012, 2013b; Kong, 2015; Tunca, 2015 & Pithers & Soden, 2000, p.239). Moreover, Bloom's Taxonomy notes the six keys of critical thinking: remembering, analysing, understanding, evaluating, applying, and creating.

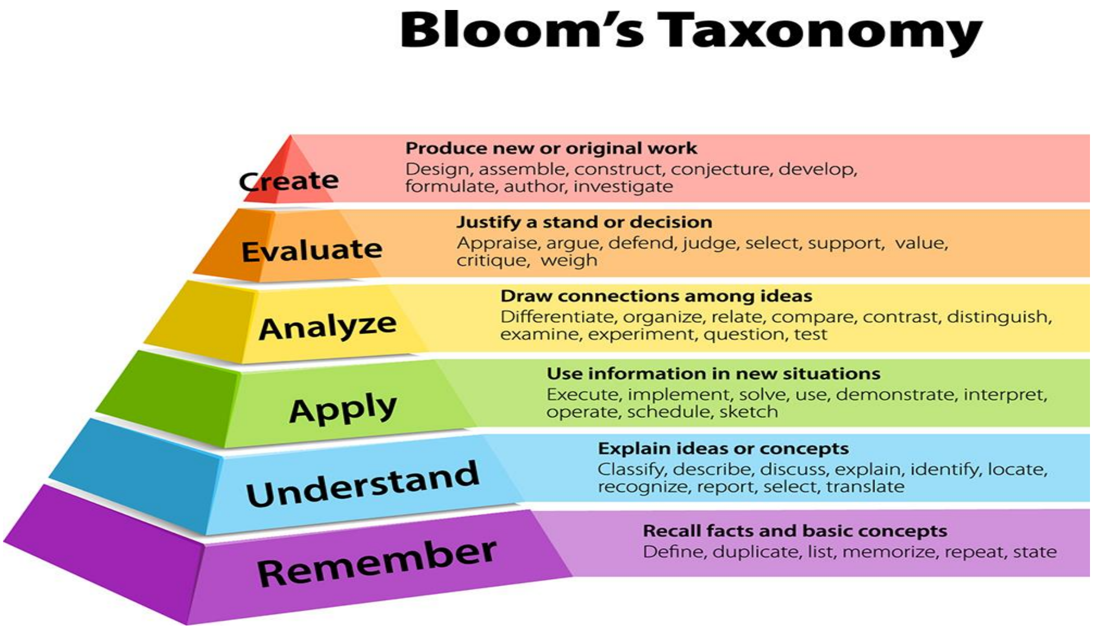


Figure 9: Bloom's Taxonomy (Adapted from http://jan.ucc.nau.edu/d-elc/tutorials/best_practices/best_practices.html)

The Blooms Taxonomy (1956) is commonly used in mathematics education and heavily influences teaching and assessment. The Blooms Taxonomy (BT) categorises questions in activities or tests in different cognitive levels, such as low, medium, and higher thinking levels. Researchers characterise higher thinking level (HTL) as non-sequential thinking, which cannot

be predicted but can be justified and explained if there is more than one solution (Resnick 1987, Stein & Lane 1996, and Senk et al. 1997). According to Su, Ricci, & Mnatsakanian, CT is a HTL because it enhances creative problem-solving skills, encouraging learners to use different methods when solving mathematical problems (2016, p. 190). Problem-solving is when individuals analyse, verify, and justify their answers (Krulik & Rudnick 1995).

Furthermore, Carson (2007) and Ellis (2005) agree that problem-solving is part of the thinking skills teachers encourage learners to apply because it involves critical thinking skills, decision-making, conceptualising, and information processing when teaching the thinking process. CT is an essential general capability to build a learner's confidence (ACARA 2013b). Moreover, Paul & Elder note that CT is a form of thinking that is self-directed, self-disciplined, self-monitored and self-corrective (2000, p. 15). Therefore, to encourage CT during AR development, the learning environment should be more positive and learner oriented.

Therefore, to enhance CT abilities in mathematics classrooms, teachers must establish a supportive learning environment that builds on constructivism's guiding principles (Kong, 2015; Kwan & Wong, 2014; Sun & van Es, 2015; Tunca, 2015; Widyatiningtyas et al., 2015; Yuliani & Saragih, 2015). Constructivist theories of education note the importance of fostering CT in the curricula to motivate and equip learners with reasoning and analytical skills (Fosnot, 1996; Piaget, 1977 & Kelly, 1991). Furthermore, constructivism highlights the idea of knowledge as not being passively received but preferably actively constructed (Donald et al. 2010, p.80). Therefore, this study's constructive approach is essential to highlight transmission and transactional instruction toward CT in the learning environment. In a traditional classroom setting, a teacher is a source and transmitter of knowledge to learners who passively listen and acquire facts. However, transactional instruction allows learners to participate in their learning to actively gain new insight.

3.3 Connection of AR with CT

Thinking and reasoning are cognitive skills that help learners turn their experiences into learning (Donald, Lazarus, & Lolwana 2010, p.363). When learners begin to think outside the box and use their critical thinking to justify their thinking, it deepens understanding rather than following memorised steps. Moeller, Cutler, Fiedler & Weier view the CT process as analysing clues and exploring possibilities without opposing the main ideas as they consider alternatives (2013, p. 58). The learner holds the fact but stretches their thinking to find alternative ways of solving the problem.

CT is an essential skill for problem-solving (Wechsler et al., 2018, p.119). According to Butler et al. 2012; Phan, 2010 & Wechsler et al., 2018, CT skills positively impact learners' personal, social, and academic performance. Moreover, CT stimulation helps cultivate learners to be innovative in their future careers (UNESCO, 2016). CT not only helps in problem-solving but also in AR development.

Kaput (2008) and Glassmeyer & Edwards (2016) argue that it is critical to develop AR in primary grades and to improve mathematics instruction. Kieran (2004b) agrees that developing AR in primary grades stimulates thinking that encourages learners to analyse, generalise, and justify problem-solving. Additionally, in the AR development process, teachers need to encourage CT to further Driscoll's (1999) thinking habits and use Bloom's taxonomy to guide the questions they ask about providing CT opportunities. Finally, for effective teaching instruction, AR modelling should be encouraged using timed pointers that will shift or expand CT. Finally, it is a habit to ask different questions to help learners organise their thinking to develop AR (Driscoll 1999, p.3). Thus, it shows the AR habits to be developed, and consistent CT modelling in the lessons is essential.

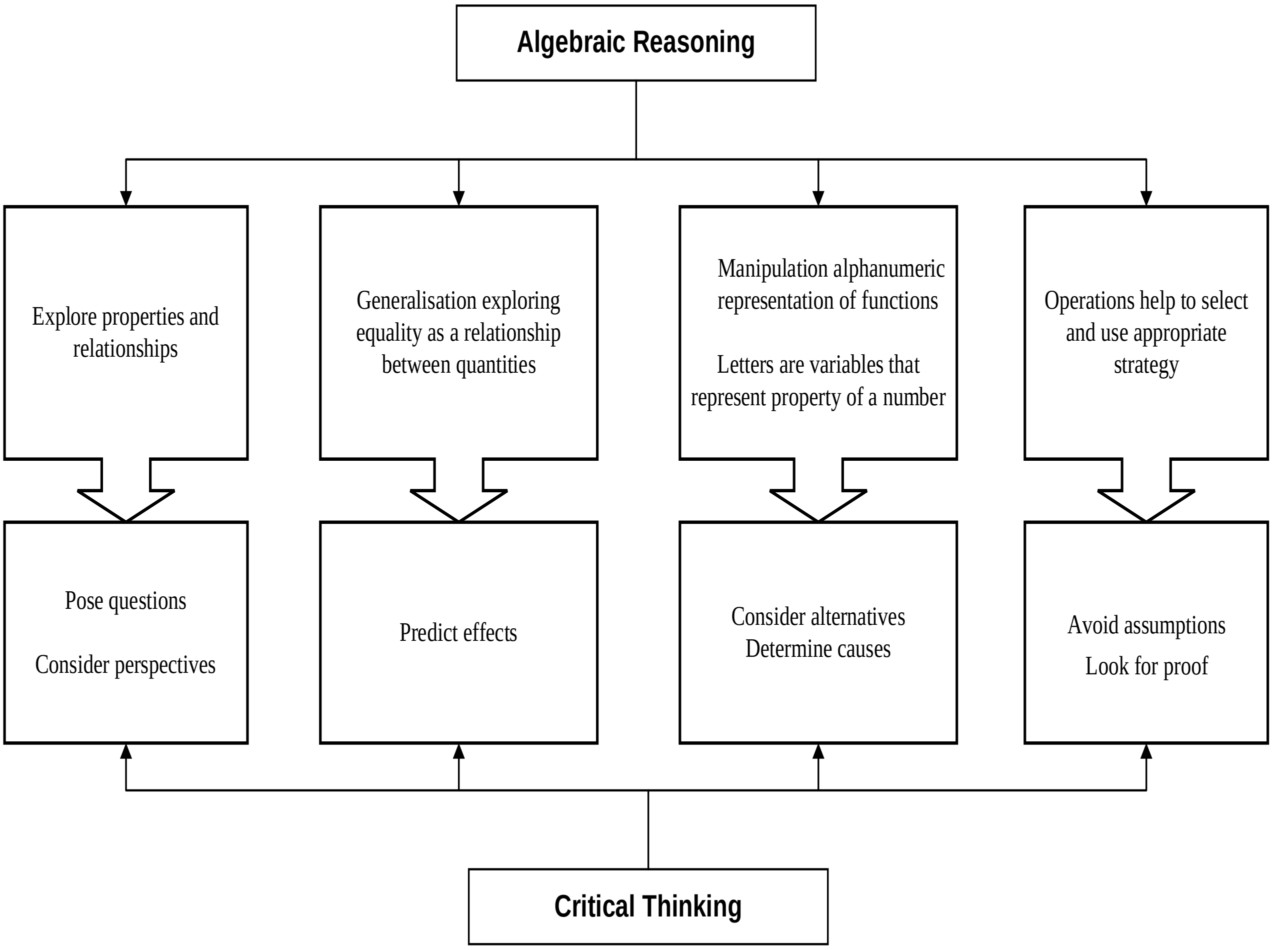


Figure 10: Relationship between AR and CT

The assimilation of AR and CT characteristics represented in Figure 10 goes as follows: (a) Generalisation encourages the identification of common factors; the relationship between two quantities to improvise; predicts results as they explore equality and relationship between quantities; (b) Generalisation encourages the identification of common factors; the relationship between two quantities to improvise; predict effects as they explore equality and relationship between quantities; (c) Generalisation encourages the identification of common factors; the relationship between two quantities to improvise; (b) Alphanumeric representation: each letter represents a different definition.

Usiskin explains that the concept of letters/variables is not static but changes over time (1999, p.7). Figure 10 above aims to express the synthesis between AR and CT. According to Lockhart (2002), mathematical thinking is "an art of explanation". AR is essential in mathematics instruction as it involves mathematical thinking and arithmetic, which allows the exploration of mathematical structure. It aims to develop a more profound knowledge of mathematical concepts, practices, and problem-solving.

The framework expressed by Mason (2008) and Hewitt (2009) is that very young children possess skills that comprise a possible developmental pathway for AR. Their research displayed that, learners can observe and apply some AR skills in mathematics contexts even though they cannot express them as rules or apply them to all situations (Schifter, Bastable, Russell, Seyferth and Riddle, 2008). However, the research has also highlighted that intervention for AR development to develop thinking skills is still required. Hence CT's ability in mathematics will help learners to explore, develop AR and understand mathematics. According to ACARA (2013b), learners reason mathematically when they justify their thinking and adapt the known to the unknown to transfer knowledge into a new context and further analyse and generalise to give justifiable responses. Therefore, AR and CT are reasoning skills that aim to analyse, generalise and justify (Figure 11).

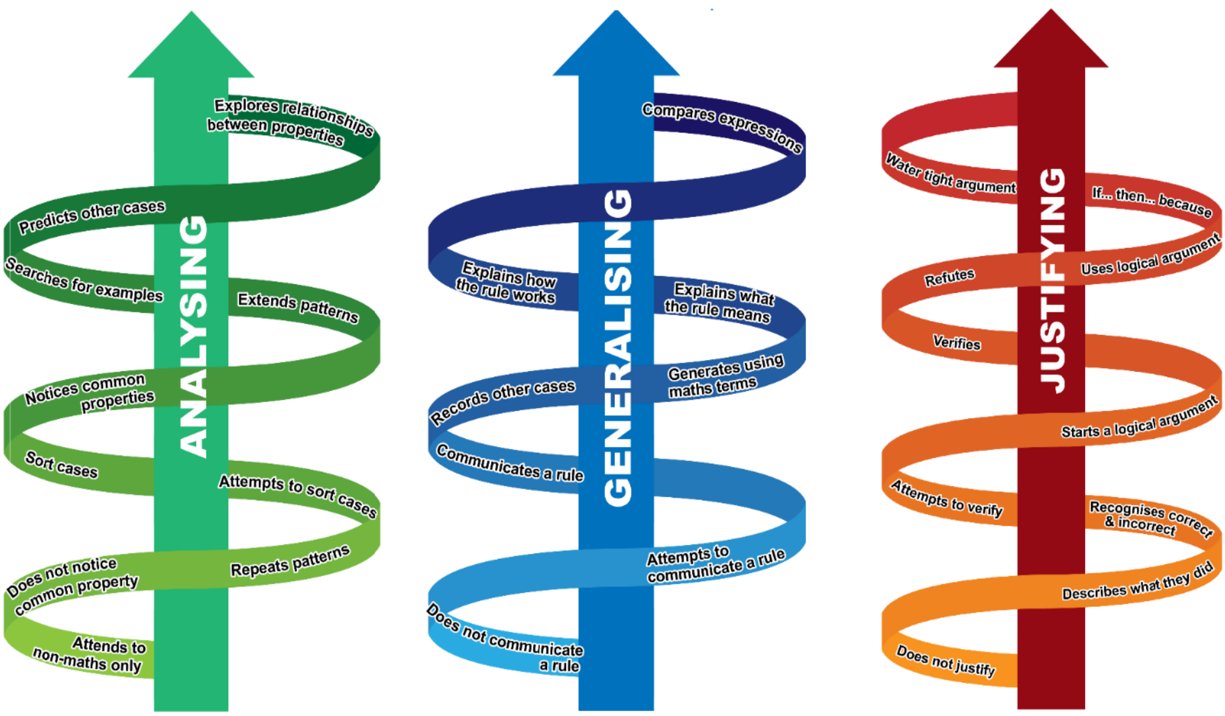


Figure 11: Reasoning Processing Tools (Adopted from reSolve, 2018)

The interpretations of mathematical reasoning differ as they are either associated with logic or summing with valid conclusions (Sternberg, 1999 & Artzt & Yaloz-Femia, 1999; Steen, 1999). This process is supported by Driscoll (1999) in “Thinking Habits to help learners think about their thinking as they apply the doing and undoing process". In the process of algebraic reasoning, a learner will need to be able to (a) analyse, (b) generalise, and (c) justify the patterns to reach a valid conclusion. The reasoning trajectories in the process of AR development are:

a) **Analysing:** The analysis process occurs when one compares a case to discover differences and then sort and classify it. For example, when working with patterns, the learner needs to observe how each pattern differs and build up to form the next pattern. Twohill (2016) explains the process of finding the next pattern as recursive thinking. Teachers need to help stimulate CT to guide learners' thinking when identifying the relationship of quantities in patterns.

b) **A generalisation** is a shift in thinking. In the case of AR, it involves discovering common properties of patterns and finding a rule that describes the pattern (which can be written symbolically). Bastable & Schifter (2008) note that language, diagrams, and story contexts convey generalisation. Moreover, the alphanumeric rules help explain the application of the rule for the pattern.

c) According to Siegler and Lin, **justifying** is to 'self-explain with inferences concerning 'how' and 'why' events happen (2010, p.85). It requires one to convince others of the symbolism used to describe the pattern and support how they help one understand it. Finally, it is defined as the CT process of carefully evaluating and determining how to make the information meaningful to give sound judgment.

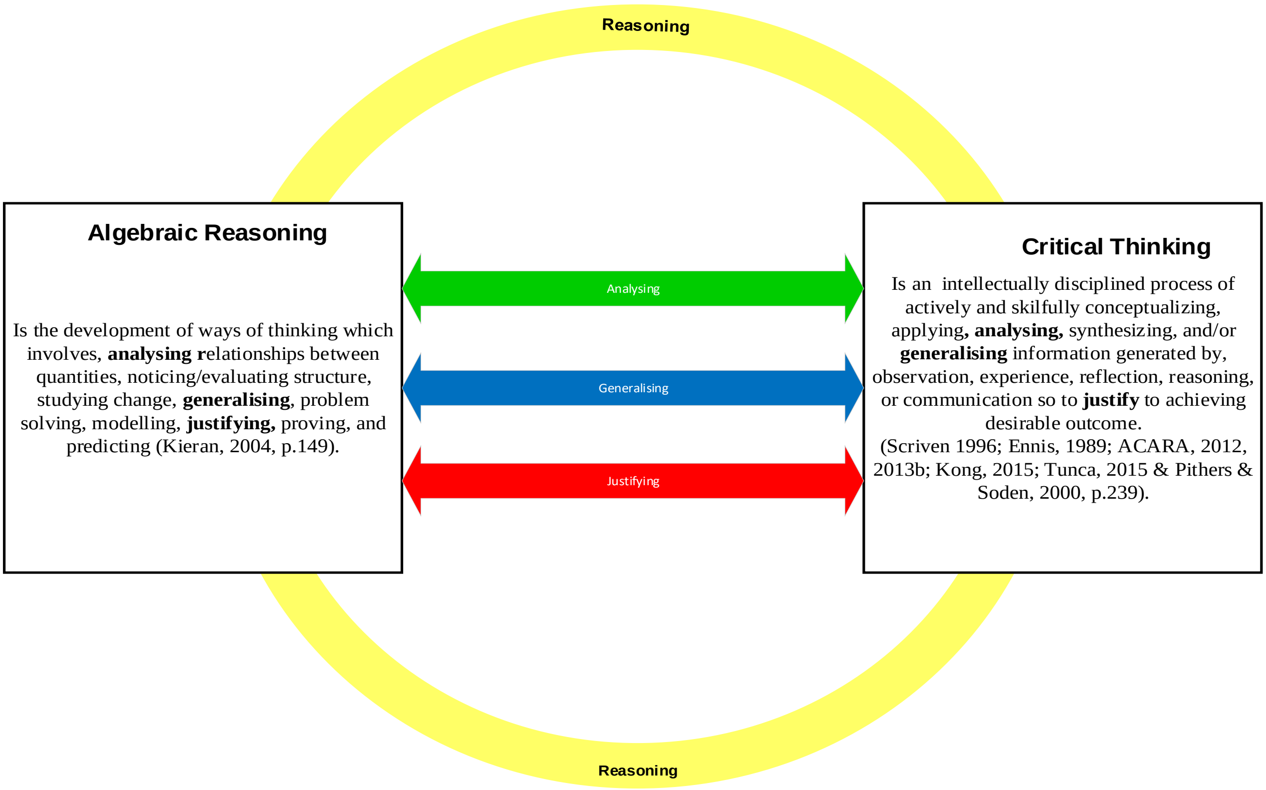


Figure 12: AR and CT as reasoning processes (Adopted from ACARA, 2013)

Henceforth, drawing from the definitions of AR and CT in Figure 12, one can understand that they are reasoning processes that aim to analyse, generalise, and justify. This is based on Kaput's (2008) formulation of Early Algebra and Blanton et al. (2015), who developed four critical mathematical practices that characterise early algebraic thinking. These are important as they outline the development of algebraic reasoning and learner response analysis of their generalisation, modelling, justification, pattern structure and logical reasoning.

3.4 Summary

The research on thinking to understand CT gave a clear view of what thinking is and what distinguishes CT from thinking. Then the CT theory is viewed from the teacher's perspective as facilitators and learners as participants in nurturing CT. This way of thinking helps to understand CT in connection to AR. Their characteristics are explained in Section 1.1. Further, this will help form the methodology and research designs that underpin the research to inform data analysis.

Chapter 4: THEORETICAL FRAMEWORK

"The principal goal of education in the school should be creating individuals capable of doing new things, not simply repeating what other generations have done." Jean Piaget

This chapter reviews the theoretical framework as a guide that supports the research study and provides a lens to view the study (Grant & Osanloo, 2014). Next, the theory of cognitive constructivism theory developed by Piaget (1936) and Vygotsky (1978) and Bruner's (1996) 'theory of instruction' is discussed in Section 4.1. This is followed by a discussion of teachers as facilitators of CT to discourage monologue teaching in Section 4.2, where learner engagement in a constructive classroom that nurtures CT is discussed. Lastly, the chapter concludes with a summary in Section 4.3 of a guide that helps to achieve the study's objective: "To analyse teachers' lesson plans to see whether and how they engage learners towards the development of AR to encourage CT."

4.1 Constructivism

Fosnot notes that constructivist learning theory is when a learner interprets the new knowledge, building upon their prior knowledge by actively collecting, reinventing and assimilating the knowledge as their own, which may change or add to their knowledge as they interact physically and socially (1996, p.30).

Constructivism concerns how individuals construct their knowledge and understanding of the world through experience and reflection. When knowledge is only gained and not comprehended, it does not help with the process of interpretation (Rodgers & Dunn, 1997,p. 16). Hanekom (2019) notes that what is more critical about constructivist teaching is to direct learners to restructure and apply new concepts to their current disjointed knowledge (2019, p. 49). This means that a teacher has to create an active learning experience where learners can deepen their knowledge, providing an opportunity of reflecting and explain how this adds to their existing knowledge.

Therefore, the development of AR to encourage CT reflects and relies on teacher practice and is relatively aligned with the theory that drives the teaching practice, in this case, constructivism. Several theorists discuss the constructivism theory. However, for this research, the theory of cognitive constructivism theory developed by Piaget (1936) and Vygotsky (1978) and Bruner's (1966) 'theory of instruction' is discussed.

4.1.1 Piaget

Piaget held the view that intelligence is stimulated by action, whereby children learn by interacting with their surroundings. Piaget (1953) notes that interaction is responsible for learning. The theorist believes that information given to the individual will not lead to immediate comprehension and application; instead, they will need to construct their knowledge as they interact with their surroundings (Piaget, 1953). He states that information will go through assimilation and accommodation through four stages of development (Wadsworth, 2004). Piaget (1953) defines assimilation as a process whereby learners bring new knowledge to their schemas and accommodation. Then, learners change their schemas to accommodate the latest information or knowledge. However, four stages of development are controlled for a learner to assimilate and accommodate knowledge or information.

According to Piaget (1953), the four stages of development are (1) the Sensorimotor stage (0–2 years); (2) the Preoperational stage (2–7 years old), (3) the Concrete operational stage (7–11 years old), and the (4) Formal operational stage (11 years–adulthood). In the first stage, the child becomes aware of their environment through their senses and physical engagement and language as they develop within the stage. Secondly, they develop their language skills and use symbols or pictures to identify different objects. They also become curious about their environment in the third stage, and they manipulate symbols and concrete objects. That is because, in the first two stages, the child has developed their physical, social, and logical-mathematical knowledge; in the third stage, the child manipulates symbols and concrete objects. Lastly, the child uses symbols and abstract concepts.

In summary, Piaget's theory is about a child's development in different stages. His theory is about how the child assimilates new knowledge to accommodate it into their existing knowledge to search for equilibration (Wadsworth, 2004). According to Piaget (1953), equilibration balances new knowledge with existing knowledge. Therefore, teachers need to understand these stages and teach within the learner's ability to comprehend the concepts. Vygotsky disagrees with Piaget that effective learning occurs in stages but holds that it happens by stimulation of the learner's culture and environment.

4.1.2 Vygotsky

The theorist notes that development occurs from the social environment to the individual and not from the individual to the social environment. He further emphasises that the inclusion of society and culture plays a significant role in cognitive growth (Vygotsky, 1978). Inclusion is about acknowledging and accepting differences between all learners and building on similarities (White Paper 6, 2001). Therefore, learning is a collaborative or inclusive process whereby an individual's knowledge is developed by interacting with their culture and society. Cole and Wertsch further note that the development of the mind is interlinked with biological development, which in most cases is directly influenced by the culture or history, which helps coordinate people with their current physical environment (1996, p. 2). According to the theorist, learning takes place before development, and teaching and learning are more social by nature and facilitated by language.

Therefore, as much cognitive development occurs through social interaction, language plays a mediating role in cognitive development. Green agrees that language facilitates social interaction with adults and peers. Through these interactions, Vygotsky argues that the intellectual attitudes and tools of culture are mediated and internalised by children (2001, p. 83). Vygotsky views language as the most valuable instrument, a way of communication with the outside world. He claims that language plays an essential role in cognitive development as the only means of communicating knowledge to children through adults. Secondly, language is an instrument of intellectual adaptation (Vygotsky 1978). Therefore, Vygotsky recognises the importance of language and social interaction in cognitive development through supervised learning as children and the co-construction knowledge of their peers or older children who are more knowledgeable within the zone of proximal development.

The two fundamental principles underpinning the Vygotsky theory of cognitive development are (1) the More Knowledgeable Other (MKO) and (2) the Zone of Proximal Development (ZPD) (Vygotsky, 1978). The MKO refers to someone with more understanding or a higher level of skill than the learner regarding a specific task, method, or concept. As the learner engages in the task with the help of the MKO, their abilities grow in the range of their learning abilities called ZPD (Vygotsky, 1978). Therefore, this implies that teaching should occur through the scaffolding phase of ZPD learners. Scaffolding is more driven or encouraged by teachers and other MKOs (e.g., peers), providing support structures to achieve the stage. Hence his theory adopts more constructivism-based teaching styles, which marks a deliberate attempt to shift from ‘traditional, objectivist models of didactic, memory-oriented transmission models’ (Cannella & Reiff, 1994) to a more student-centred approach.

4.1.3 Bruner

The scaffolding theory of Bruner originated as part of the social constructivist approach around 1976. It was primarily influenced by the work of Russian psychologist Lev Vygotsky, who argued that we learn best in a social environment where we build meaning through interaction with others. However, Bruner was more concerned about how the scaffolded knowledge is stored or encoded in the learner’s memory.

Bruner (1966) notes that cognitive development occurs in three modes of representation: Enactive, Iconic, and Symbolic. The enactive presentation is the first memory associated with Piaget's sensorimotor stage. Thinking is demonstrated by physical interaction, for example, a symbolic tool to demonstrate knowledge. The iconic representation uses hierarchical structures, spatial signifiers, or photographs to reflect past interactions. For example, it is possible to relate images or concrete representations such as maps or graphs to those that use this form of representation. The last mode is the symbolic representation mode, where information is kept as a code or symbol, such as language. According to Bruner (1966), language is a guiding aid in symbolising the world, and this symbolic representation is crucial to cognitive development. He further notes that learners construct knowledge through organising and categorising information using symbolic representation. Therefore the teacher should adopt a constructive approach to facilitate learning.

4.1.4 A comparison between Piaget, Vygotsky and Bruner’s views on cognitive development.

Bruner (1966) and Vygotsky (1978) both emphasise the environment of a child, especially the social setting, more than Piaget did (Table 4). Both accept that adults should play an active role in helping to educate the child. Furthermore, like Vygotsky, Bruner stressed the social aspect of learning, citing that other individuals can help a child develop skills through scaffolding. However, Bruner’s (1966) constructivist theory integrates an idea of Piaget’s theory. First, he includes the Piagetian idea that cognitive growth occurs in progressive stages and that each stage is integrated and built upon by successive stages. Secondly, Bruner argues with Piaget that categorisation and representation are essential to a person’s cognitive development.

Therefore, fundamental differences between Bruner's (1966) theory and Piaget's (1953) theories are as follows. First, stage theories maintain that cognitive readiness is key to learning and development. According to these, age or biological state dictates what can be learnt and how learning can occur. Second, the constructivist theory says that the translation of the information dictates what type of information can be processed and how learning can occur. Piaget would say that an individual cannot process certain types of information at certain ages or stages. However, Bruner disagrees, stating that certain aspects of any content or principle can be taught to any child.

Therefore, despite these constructivists' similarities and differences, they all agree that learning is an active process. Thus, cognitive development should facilitate the learning process instead of being informed.

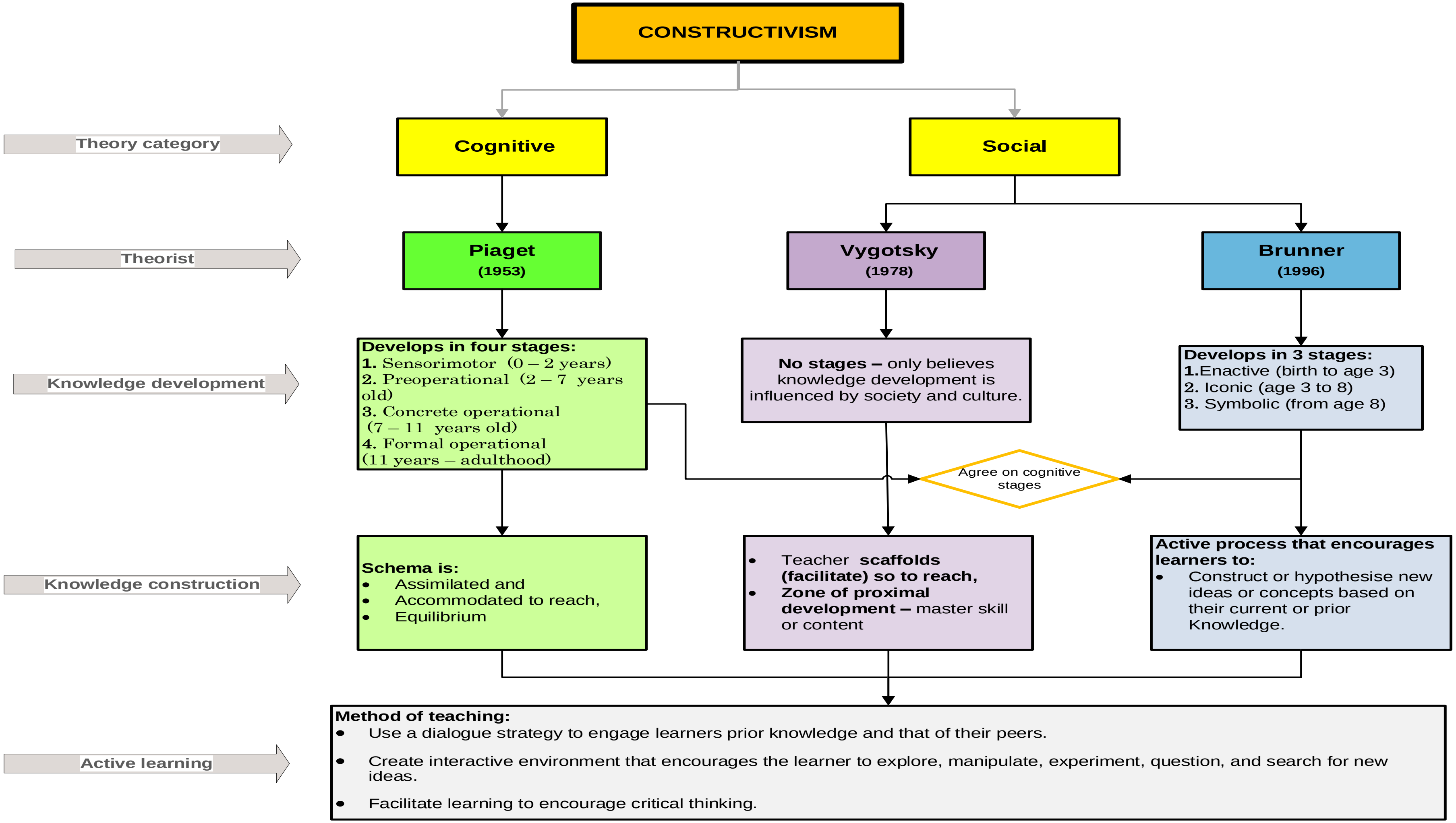


Figure 13: A comparison between the views of Piaget (1936), Vygotsky (1978), and Bruner (1966) on cognitive development and its value on instructional purpose

4.2 Teachers as facilitators of CT

Teachers should create an environment that accommodates different learning styles and encourages thinking. Steward and Felicetti (1992) describe learning styles as a method or condition in which a learner understands. Therefore, teachers must create an environment that encourages learners to view themselves as sources of knowledge during the teaching and learning process and not rely upon them. Researchers describe learning as making meaning or interpreting new experiences based on the learner's previous knowledge (Merizow, 2001; Jarvis, 1995; Merriam & Caffarella, 1991). Learners' understanding develops holistically, meaning that learners do not begin learning in the classroom but from their homes and surroundings.

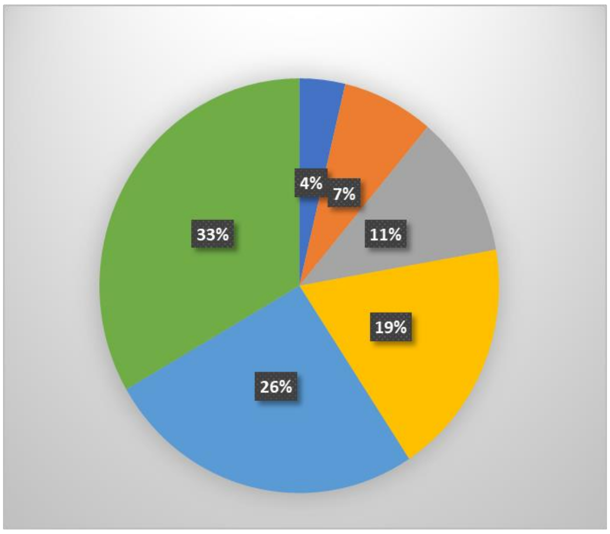
Learning comes in three different forms, namely: (1) Formal learning, which refers to a structured education system (Jarvis 1990, p.165); (2) Non-formal learning is also structured and carried out sometimes within a school system and ends specific certificate (Tudor 2013, p. 822), and (3) Informal learning which is an unorganised and unsystematic process from daily learning experiences. Jarvis describes this as a process in which a learner gains knowledge, skills, and attitudes, which also takes place outside the existing education system (1990, p.165). So, for example, first, learning occurs in homes, media, or situations one finds themselves in, meaning that learning does not begin in a structured system or environment but can happen in an organised educational activity or through experiences.

Henceforth, learning should be more learner-centred than teacher-centred. A teacher-centred classroom, in other terms, is a traditional classroom. A teacher-centred learning environment only acknowledges the teacher as a reliable source of knowledge. Learners depend on the source of knowledge and only memorise the knowledge imparted to them (Gaskaree, Mashhady, & Dousti, 2010). Teacher-centred is a monologue education that separates teacher and learner, thus resulting in mere meaningful dialogue.

The monologue learning environment limits an opportunity to create a meaningful dialogue where learners ask questions and think about their learned contents (Freire 1972). The monologue learning environment encourages more memorisation and hinders any thinking, exploration, and self-actualising opportunities which can further limit teachers to encourage learners to think critically.

Freire (1972) views education as the freedom in which learners can have a dialogue about the content of their learning which can translate to self-discovery and attain something that will be humanitarian and used to transform the world. Hence, teachers must be facilitators and encourage self-corrective learning by continuously asking learners questions about their thinking. That will help them view themselves as sources of knowledge in the classroom. According to Driscoll (1999, p.7), guiding questions help reinforce and extend learners' thinking. Blaschke & Hase (2016) & Uday (2019) further emphasise the importance of the teacher's instructional guide that draws learners holistically to reflect, explore, hypothesise, generate, and apply the concepts when engaging with their learning. Uday 2019 & Donald et al. (2010) describe holistic thinking as more than just about knowledge and thinking; that encompasses all learning traits, such as learner experience and learning by thinking or reasoning critically. Therefore, when teachers encourage CT in the classroom, it helps learners to be creators of knowledge by organising their thinking.

Therefore, the role of the teacher is to co-create knowledge with their learners and not appear as the primary source of knowledge. Giving learners opportunities for hands-on experience to stimulate experiential learning will encourage CT, meaning that CT is an active process in which learners engage, eliminating them from passively accepting knowledge given by the teacher. Thus, it provides learners with the opportunity to engage in thinking about concepts presented to them actively.



Passive	Reading	4%
	Hearing	7%
	Observe	11%
	Observe and Hear	19%
Active	Speaking	26%
	Speaking and Doing	33%

Figure 14: Passive and Active learning Diagram (Adapted from Edgar Dale's cone of experience http://teachernoella.weebly.com/dales-cone-of-experience.html)

Encouraging CT in a mathematics classroom would actively involve learners in discovering possible answers. Researchers agree that learners' thinking skills develop when teachers create an environment that supports the thinking activities (Swartz & Parks, 1994; Rajendran, 2010 & Mason et al., 2010). Therefore, teachers should dominate and control at a minimum level to encourage learners to take an active role, as presented in Figure 14 (Freire, 1972; Henningsen & Stein, 1997). Additionally, Blanton & Kaput note that enriching the development of AR in the classroom requires daily activities that allow learners to analyse, generalise, and justify meaningfully (2003, p.74).

Therefore, the development of algebraic reasoning uses critical thinking to guide learners to build sound knowledge and reasoning. Thus, it shows that an environment that encourages learners to be active helps encourage critical thinking skills.

4.3 Learner engagement in a constructive classroom that nurtures CT

A constructive classroom welcomes a monologue nature. A constructive environment allows learners to be active with their thinking as they gain new insight. Moreover, constructivism rejects the idea that learners are blank slates upon which teachers write new knowledge (Shan 2019, p.4). Sheridan (1993) highlights the apparent paradox of constrained liberty, noting the use of structure in a constructivist classroom:

Freedom is structurally constrained. The objectives of the lesson facilitate learners' thinking. Therefore, learners thinking is not free but structurally supported even if they can use the language of their choice. (p.116).

The researcher notes that learners are under the control barriers, guiding them to accomplish specific lesson goals because they do not know precisely what they are supposed to do—giving learners more liberty to be in control benefits the teacher and the learner. This form of environment encourages heutagogy.

Heutagogy is known as self-determined learning, which is learner-centred and encourages learners' autonomy, capacity, and capability (Hase & Kenyon 2007; Blaschke, 2012). To understand pedagogy, one must comprehend the traits of pedagogy and andragogy. Pedagogy is the art and science of teaching learners (Smith 2012 & Knowles 1973). Andragogy, on the other hand, is the art and science of helping adults learn (Knowles 1984).

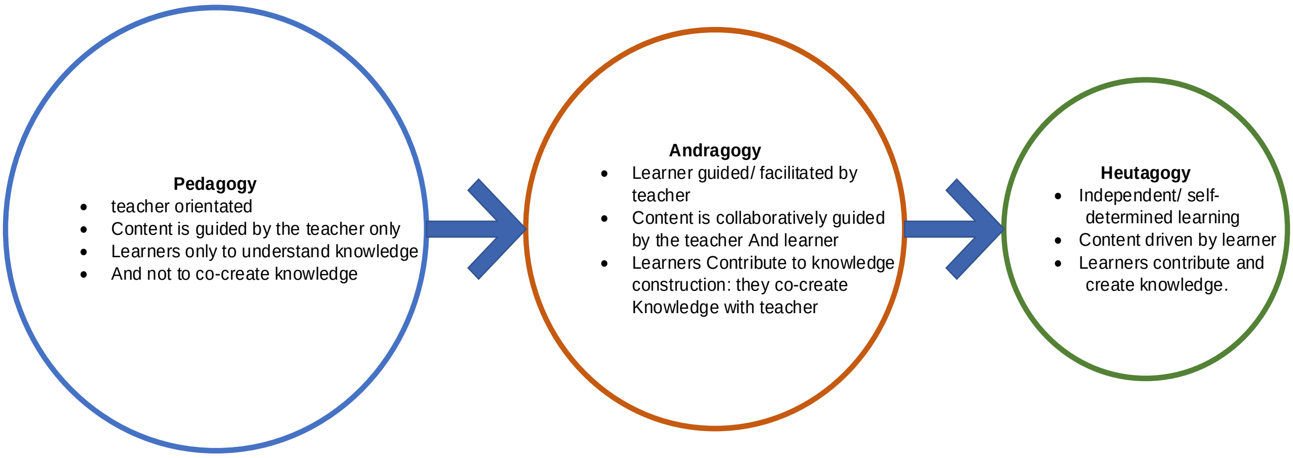


Figure 15: PAH Framework showing the learning design of CT stimulation for learners (Adopted from Hegarty & Thompson, 2019)

The pedagogy and andragogy instructional methods are considered either a limitation or non-sufficient in preparing learners for the workforce (Peters 2001 & 2004; Kamenetz 2010). Figure 15 shows how a more self-directed and self-determined approach helps learners reflect on what and how their learning is called heutagogy. Moreover, researchers agree on the importance of heutagogy as it:

- Encourages CT and reflective thinking.
- Is learner-centred on encouraging learner engagement?
- Stimulates learners' CT skills to investigate and question ideas.
- Encourages the learners to apply knowledge in diverse contexts practically.
- Encourages growth and personal development to promote democratic learning and adaptation to new contexts.

(Canning, 2013; Canning & Callan, 2010; Ashton & Elliott, 2007; Ashton & Newman, 2006; Dick, 2013; Kerry, 2013)

Hence teachers are encouraged to differentiate learning, encouraging learners to learn using heutagogy. In addition, engaging learners as active partners in their learning can positively affect learning (Burke, 2011). Finally, according to Marri (2005), teachers who still practise teacher-centred instruction but use technology, which comprises andragogy and heutagogy traits that encourage CT in learning, have higher achievement or success in their teaching.

4.4 Summary

In this chapter, research on the theoretical framework has helped to highlight the constructivism theory, which is the anchor of the study. The constructivism theory looked at how it can be implemented in the classroom, considering the teacher and learner roles contributing to a constructive learning environment. Furthermore, the theory of algebra as a 'way of thinking' helps to understand AR development's thinking processes. Furthermore, that emphasises the need for IP learners to think about improving their AR. Finally, this way of thinking is under discussion in the next chapter. Further, this will help form the methodology and research designs that underpins the research to inform data analysis.

Chapter 5: RESEARCH METHODOLOGY

The chapter discusses the research methodology approaches adopted in this study. The sections in the chapter are arranged in order and detail the procedure for collecting data for the study. The first step towards selecting the appropriate research methodology in Section 5.1 is the literature study of developing algebraic reasoning in the intermediate phase to encourage critical thinking. Next, Section 5.2 is compared to other curricula content. Next, the research paradigm in Section 5.3 guided this study in the interpretive approach. Then, as the study has adopted a mixed-method approach discussed in Section 5.4, the quantitative and qualitative methods are discussed on how they are used to answer the primary and sub-questions in this study. Next, Section 5.5 discusses the data collection instruments in detail. Then, Sections 5.6 and 5.7 used a triangulation method and Section 5.8 to test the study's reliability and validity. Lastly, the ethical considerations of both the SU ethics and WCED are in Section 5.8, and a summary is provided in Section 5.9.

5.1 Research Methodology

The research methodology is about exploring procedures and the way knowledge is acquired. This research explores how teachers use Critical thinking (CT) to develop algebraic reasoning (AR) in the intermediate phase (IP) of their teaching. It means that the methodology is about answering the question ‘of how research is done to get a response to the research question. This research adopts a mixed-method approach and uses a case study because the study is more qualitative than quantitative. Mixed-methods research is a methodology for conducting research that integrates quantitative and qualitative research methods (Creswell, 2003). The qualitative methods of this research include an extensive study of literature on algebraic reasoning and a desktop analysis of policies or curricula implemented in the intermediate phase. The quantitative aspect would include collecting and analysing data through individual and focus-group interviews, lesson observations and a teachers’ post-reflective questionnaire. The results obtained from different IP classes were compared, revealing learners’ errors when generalising patterns and their responses to CT questions.

5.2 Literature study

The primary research question and the sub-questions drive the literature review of the study of developing algebraic reasoning in the intermediate phase to encourage critical thinking. The literature is analysed to understand algebraic reasoning and the developmental process of using critical thinking to encourage algebra as a way of thinking. In addition, it would contribute to the application of understanding early algebra using CT to develop AR in the Intermediate Phase.

Furthermore, a close study for this project is presented and carried out on the policy document (CAPS, DBE, 2011) and international policies or curricula that have implemented algebraic reasoning in the intermediate phase. The scrutiny of the policies will help to fulfil objective (b). In addition, these analyses will help to understand the advantages, procedures, and requirements for algebraic reasoning in a South African context.

5.3 Interpretivist paradigm

A research paradigm is a conceptual lens through which the researcher examines the methodological aspects of their research project to determine the research methods used and how the data is analysed (Kivunja & Kuyini 2017, p.33). Furthermore, it serves as a guideline for interpreting the outlines of conducting a research study (Bertram and Christiansen, 2014). Additionally, the research paradigm is influenced by the study of epistemology and ontology, which is possible within the research paradigm (Denzin & Lincoln, 2005; Guba & Lincoln, 1994). According to Poni (2014), research paradigms also have a philosophical underpinning that orientates the researchers' point of view. There are currently three major research paradigms in education, social and behavioural sciences, i.e. positivist, interpretive and pragmatic.

The researcher used the interpretive approach as a research paradigm for this study. Willis (2007) and Neuman (2011) note that an interpretive paradigm is socially constructed because a person understands the context or research from experiences and skills gained over time. The theory framework supports this under social constructivist theory, which supports and encourages learners to construct their meaning and justify their findings using their experiences and social interaction opportunities. The focus is mainly on understanding an individual and their interpretation of the world surrounding them (Kivunja & Kuyini 2017, p.33).

The approach helped the researcher understand how mathematics teachers use CT to develop AR in the IP. It also helps in responding to the research question regarding how teachers write, pose, and set up questions and teach in a way that CT develops AR in the Intermediate Phase. That is why the researcher opted to use mixed-method research for the study.

5.4 Mixed-method research

Mixed-method research is integrated research using qualitative and quantitative approaches that aim to generate a more accurate and adequate understanding of social phenomena than is possible using only one of these approaches (Biesta, 2017, p. 160). Leech & Onwuegbuzie (2008) agree that it represents research that involves collecting, analysing, and interpreting quantitative and qualitative data in single or multiple studies investigating a common phenomenon. Mixed-methods research requires collecting and analysing two types of data; this may be time-consuming, but it will give a researcher more evidence to collaborate on the findings. Additionally, the mixed qualitative and quantitative data helps a researcher to use the strength of one data type to mitigate the weakness of the other.

The chapter aims to analyse quantitative and qualitative data to answer the research questions. Table 4 shows the research questions and the qualitative and quantitative data collection methods.

Qualitative and quantitative data collection methods are presented in the quest to answer the questions above. The qualitative method includes an extensive literature study on algebraic reasoning, a desktop analysis of policies and curricula implemented in the intermediate phase, and lesson observations. In addition, the quantitative method includes collecting and analysing data through individual and focus-group interviews and a post-reflective questionnaire for teachers.

The quantitative and qualitative data are triangulated in each section to answer the research questions. Triangulation uses multiple methods, data collection strategies, and data sources to obtain a complete picture of what is being studied and to cross-check information (Gay, Mills & Airasian, 2012). For example, the focus group interviews with learners were used to contribute to understanding the solution strategies observed in the learners' written work when generalising patterns and their responses to CT questions.

TABLE 4: QUANTITATIVE AND QUALITATIVE METHODS APPLIED TO RESEARCH QUESTIONS

Research questions	The primary method for data collection:	Qualitative/Quantitative
a. What do we know about practising IP teachers' understanding of AR?	• Interviews • Observations • Post-reflective questionnaire	• Both qualitative and quantitative.
b. What are the issues these teachers face?	• Interviews • Observations	• Only qualitative
c. What tasks are suitable for generating data on AR and CT in IP?	• Literature	• Only qualitative
d. What ways to design AR tasks that can potentially foster CT and AR?	• Literature	• Only qualitative
e. How do learners respond to CT questions for AR development in different grades?	• Observations • Focus group interviews - Tasks	• Both qualitative and quantitative.

5.4.1 Qualitative research

Qualitative research utilises non-numerical methods such as observations, in-depth interviews, or focus groups. It can also involve analysing content such as documents or records. Qualitative analysis deals with words, meaning, and interpretation (Green & Thorogood 2018, p.7). A qualitative research method draws meaning from learners' experiences and opinions of

participants to gain perspectives on issues by investigating them in their specific context (Denzin and Lincoln, 2005; Creswell, 2014). In this study, the qualitative method helped explore and gain perspectives on how teachers prepare lessons and tasks that engage learners to think critically and respond to critical questions towards AR development.

Additionally, teachers need to employ a didactical role during the lesson. A didactical role is a practical task whereby the mathematics teacher transforms mathematics by providing learners with opportunities to notice, appreciate and develop mathematical concepts as they engage with the tasks (Jaworski & Haung, 2014, p.174). According to Treffers (1987, p. 58), this means encouraging active learning for learners during the lesson and motivating them to solve the problem, showing them different methods as they engage in vertically planned lessons. A vertically planned lesson fits into mathematics instruction or has the primary objective of helping learners to engage critical thinking skills to solve more significant or more complicated problems (Treffers 1987, p. 58). Using pattern tasks helped to understand how teachers transform lessons to engage and facilitate learners to think critically to develop AR.

Working with the qualitative method gives the researcher access to extensive material such as the curriculum documents, relevant tasks for AR development, interview feedback and observation reports. In addition, it grants insight into the teachers and the context and pedagogical practices involved. There is a wide range of qualitative methods for data collection. This study included observation, structured interviews, and post-reflective questionnaires. The objective of each data collection method is explained in more detail under the heading data collection in the next chapter. In addition, these methods further helped to gather quantitative data.

5.4.2 Quantitative research

Quantitative research examines relationships between variables for data containing numbers to be analysed (Creswell, 2014). The data drew on post-task assessment of learners to find how many can analyse, generalise, and justify pattern tasks, which served as evidence of both AR and CT. The results from Grades 5 and 6 were compared to reveal learners' errors when generalising patterns and their responses to CT questions. It was not important to know how many rights or wrongs of the tasks were important for this study, but rather how learners presented their tasks in grades 5 & 6.

Moreover, the fundamental idea of this mixed-method research is to use the same or parallel variables, constructs, or concepts (Creswell, 2014). For example, the teachers and learners can be viewed as parallel variables. The concepts of CT and AR data were acquired through interviews, observations and reflections. The analysis, generalisation, and justification of pattern tasks given to learners by the teacher during the data collection process were measured quantitatively. Therefore, a parallel mixed-method approach was chosen with a case study design to collect data to maximise the benefits of both qualitative and quantitative methods and enable triangulation (Creswell, 2014).

5.5 Multiple-case study design

This research adopts a mixed-method approach and uses a case study because the study is more qualitative than quantitative. A case study can answer 'why' and 'how' research questions rather than 'what' but aims to explain and evaluate why a particular programme did or did not work well (Yin, 2014). However, this study takes an approach of a multiple case study design which may either yield two similar findings or different findings for specific reasons (Yin, 2014). The development of the 4Cs in schools is (1) Critical thinking, (2) Communication, Creativity, and (4) Collaboration (Walser, 2008; Mathis, 2013). However, this study focuses on the critical thinking integrated into lessons by teachers for algebraic reasoning development.

Secondly, a multiple case study design can be stronger than a single case design using various methods to gather data to allow a more comprehensive exploration of research questions and theoretical evolution, even though they require more time and resources (Eisenhardt & Graebner, 2007; Yin, 2014). The researcher could not have access to a group of teachers to interview in the phase as the selected schools differ in subject allocation per teacher. The schools had a minimum of ±4 teachers teaching mathematics in the intermediate phase. The researcher interviewed 11 teachers from three schools selected in the Cape Winelands and a minimum of 15 learners in Grade 5 and Grade 6, totalling 30 learners per school. A total of 90 learners were observed in the three case study schools.

Thirdly, the research question and secondary questions guide the researcher into the type of case study. According to Yin (2014), a case study has a particular ability to answer questions in an:

- Exploratory (‘how’) – seeks to explore a phenomenon and figure out what is happening through new insights
- descriptive ('who', 'where') – to give detailed explanation
- explanatory (‘why’) – explain a problem giving details of what is happening

The main research question for the study is a ‘how’ question which is exploratory, and secondary questions with types of ‘what’ is exploratory as the study is built on a mixed-method design where the quantitative data services support the qualitative data with the case study framework (Creswell & Plano Clark, 2007). According to Denzin and Lincoln (2000), qualitative research includes interpretive and naturalistic approaches. Therefore, the researcher viewed this study as more exploratory. It seeks to understand how teachers help learners think critically, make sense of or interpret patterns to make meaning, and develop algebraic reasoning skills.

Yin (2003) also stated that, given a choice, a multiple-case study is better than a one-case study, as the analytic benefits are much more significant. If conclusions are similar, generalisability is significantly expanded. Multiple case studies that use various methods to gather data allow a more comprehensive exploration of research questions and theoretical evolution (Eisenhardt & Graebner, 2007). Using teachers from different grades helped to analyse how teachers prepare lessons for algebraic reasoning development using critical thinking. Furthermore, the learners’ task response data were collected as evidence to analyse learners’ responses to CT questions.

Lastly, the researcher chose a multiple case study design because data is collected from multiple sources. For example, Yin (2014) identifies two case study forms: holistic, where a case is studied as a whole or only focused on a school, and embedded, where multiple units are studied within the case, such as school, teachers and learners. This study is an embedded case study because it studies the teachers' understanding of critical thinking towards algebraic reasoning development and aims to understand how learners respond to critical questions through solving pattern task problems. Moreover, the study used theocratical frameworks to support the encouragement of critical thinking toward algebraic reasoning development.

5.6 Selection of participants and sampling procedures

Only WCED schools were considered for data acquisition in this research project. Due to logistic constraints, data acquisition for this research was limited to only three WCED schools in the municipality where the researcher resides. Furthermore, the number of participants for the interview was limited to at least one teacher from each grade in the intermediate phase in each school. A total of 11 teachers participated in the study in the intermediate phase from the three selected schools.

This study was applied to the Intermediate Phase, which consists of Grades 4, 5, and 6. However, only Grades 5 and 6 were used for a sample and the focus group interview in this study as the researcher presumes that Grade 4 learners could be overwhelmed by the concepts involved. Therefore, a sample of 15 learners in Grade 5 and 15 in Grade 6 was used from the three selected schools, totalling 90 learners that participated in the focus group interviews.

Furthermore, teachers may not be aware of CT and AR; as used in this study, these terms were well-defined to the participants. It was not clear how long the restrictions because of the coronavirus pandemic would last. Still, the researcher used all available platforms to communicate the definitions to the participants. Additionally, this study did not aim to analyse teachers' planning but rather how they encouraged learners to engage critically in their lessons when solving tasks. Due to COVID-19, the interviews and presentations were done electronically at this stage. Therefore, emails were sent and followed up with a call to the schools requesting permission to conduct the study. In addition, communication between the researcher and the teachers was carried out through WhatsApp and Google forms for interviews.

5.7 Data gathering method and procedure.

The research was focused on teachers' role in the intermediate phase to encourage CT skills. Therefore, the teachers volunteered for selection, and their participation was based on their interests. The researcher did a short presentation on CT and AR so the volunteers would have a basic understanding of the aims and focus of the research. Furthermore, the permission of the Stellenbosch University Ethics Committee, the Western Cape Education Department, and the school principal was requested to gain permission to work with the learners and teachers. Data collection instruments included interviews, AR lessons, observation, and teachers' reflections.

5.7.1 Teacher presentation

Teachers do not use the term CT in their daily teaching practice. Therefore, the researcher first presented the study to the teachers to explain CT, known as Higher Order Thinking (HOT) (see APPENDIX A**: PRESENTATION**). Researchers characterise HOT as non-sequential thinking which cannot be predicted because it has not been taught but can be justified and explained if there is more than one solution (Resnick, 1987; Stein & Lane, 1996; Senk et al., 1997). This study includes cycles of teacher-researcher activity whereby "invention and revision" (Bakker 2004, p. 38) form part of the process. Thus, the researcher intervened by providing a presentation and example of AR tasks and observing the teachers. However, the teachers had a pre-interview before the presentation and a post-interview to capture their exposure and later insights about critical thinking (see APPENDIX B**: INTERVIEW QUESTIONS**). According to CAPS, the questions relating to AR tasks should cater for the following cognitive levels: (1) Knowledge 25%; (2) Routine procedures 45%; (3) Complex procedures 20%; and (4) Problem solving 10% (CAPS, 2011, p.296). Teachers looked at the pattern task questions provided by the researcher using cognitive levels specified in CAPS to assess the higher-order questions.

Furthermore, Su, Ricci, & Mnatsakanian note that CT is HOT because it enhances creative problem-solving skills, encouraging learners to use different methods when solving mathematical problems (2016, p.190). Bloom's Taxonomy (1956) is commonly used in mathematics education and heavily influences teaching and assessment. Bloom's Taxonomy (BT) categorises questions in activities or tests into different cognitive levels, such as low, medium, or higher thinking levels. The low, middle and higher thinking levels in CAPS are tested by the abilities that learners should exhibit for their capacity for mathematical reasoning.

The teachers presented their planned lessons using their tasks for lesson introductions and later incorporated tasks provided by the researcher. The researcher observed how the teacher facilitates learners' thinking to be more critical as they engage in tasks to stimulate their algebraic reasoning. After the lesson presentation, the teachers met with the researcher to discuss their observations of learners' responses to critical thinking questions. This allowed teachers to ask questions and be more active participants in the research.

5.7.2 Lessons

A series of lessons that require critical thinking to facilitate algebraic reasoning were carried out selectively in Grades 5 and 6 in the three selected primary schools. These lessons were carried out by the teachers who were part of the study. The lessons comprised tasks that focused on arithmetic, word problems, and modelling of numbers from arithmetic to algebra and open-ended questions to allow the exploration of mathematics. In these algebraic reasoning tasks (see APPENDIX C**: TASK**)), learners engaged and focused on how learners generalise, analyse, and justify their reasoning when engaged in the following tasks:

Task 1: The task helped learners view different array representations that result in a common value. They are encouraged to do-undo the problem using inverses. The main purpose is to develop learners' capacity to analyse situations, find reasons and develop logical arguments for how the inverse operation works (Hassan, Skelton, & Smit 2002).

Task 2: The task enables learners to re-enforce the inverse skills in a different form of representation (using tables and function machines). This helps learners bring balance to the use of representation as they explore diverse inverse representations to deepen their understanding of the table and function machines' application in patterns (Hassan, Skelton, & Smit 2002).

Task 3: Learners explore the number sentences, encouraging the habit of doing and undoing the sentences to deepen algebraic reasoning. This strengthens basic facts and strategies for computation (Driscoll 1999).

Task 4: The learners are given matchsticks to model the shapes and answer the following questions. The task encouraged visual generalisation towards numerical generalisation.

Learners built rules for the represented functions. Learners learnt to work systematically and keep a record of results to assist them in developing and testing conjectures. As the learners describe and explain patterns, they move from additive to multiplicative reasoning (Driscoll 1999).

Learner task-response was assessed using a rubric (See APPENDIX D**: RUBRIC FOR TASK ASSESSMENT**) to analyse their response for evidence of analysing, generalisation and justification; to show the evidence of AR and CT. Finally, the conscious role of the researcher was applied to prevent bias.

5.7.3 Interview

Interviews are essential for providing insight into participants' thinking and experiences. According to Holstein & Gubrium (2011, p. 152), qualitative research can be a site of interpretive practice whereby social interaction and knowledge co-construction occur between the interviewer and interview. Yeo (2017) agrees that interviews hold rich value in hermeneutic understanding as they allow the interviewee to share their situated experience and the interviewer to ask guiding and probing questions so that they may interpret and make meaning of the experience.

Therefore the researcher conducted interviews before the presentation and after the lesson to gather information about the teachers' views of CT questions based on the activities provided by the researcher. CT is referred to as higher-order thinking because it benefits the enhancement of creative problem-solving options by encouraging learners to seek new strategies when solving mathematical problems (Su et al., 2016, p.190). The teachers used cognitive levels specified in CAPS to assess whether the questions on activities encouraged learners critically. Teachers from Grades 5 and 6 were interviewed to interpret and observe how they plan lessons to encourage critical thinking to develop algebraic reasoning in the intermediate phase.

Learners were indirectly part of the study. Therefore, learners were selected to form part of the focus group task-based interview. The interview helped to observe and analyse how learners respond to critical thinking questions when they solve pattern tasks to develop their algebraic reasoning. They also worked on tasks that indicate CT toward AR development. Teachers who participated in this research facilitated the tasks. Moreover, in this case, teachers were observed how they use CT questions to probe learners' understanding of patterns and how they solve the problems. Learner engagement with tasks was recorded to gain insight into their response to CT questions.

5.7.4 Observations

Observation contributes to the quality of research by giving the researcher more insight into the processes in action in the learning environment (Katz-Buonincontrom & Anderson (2018). Classroom observations were carried out in Grades 5 and 6 to assess the nature of teaching and learning for this study. These observations allowed the researcher to record behaviours in action, interactions, and discussions between teacher and learner (Merriam and Tisdell, 2016).

Through the series of observations, the researcher observed how teachers interact with learners to stimulate their CT and see learners' reasoning behind their thinking as they engage in a lesson. The observation tool was utilised to gather more information as the learners interacted and tried out ideas (see APPENDIX E**: LESSON OBSERVATION**). The focus was mainly on teachers and how they use CT to develop learners' AR.

5.7.5 Reflections

Reflection helps to verify new concepts in context. Robins et al. (2003) describe the reflective practice as a tool that allows teachers to understand themselves, their philosophies and classroom dynamics more deeply. The researcher noted teachers' reflections on the lessons and observations on the learners' engagement throughout the lessons. Additionally, detailed notes and reflections were recorded to prevent errors, bias, and missed opportunities (Merriam, 1998). Finally, teachers were given a post-reflective questionnaire (see APPENDIX F**: POST-REFLECTIVE QUESTIONNAIRE**) to reflect on the series of lessons on using AR to encourage CT.

5.8 Reliability and Validity of instruments

Trustworthy research comprises reliability and validity of data. Merriman (2009, p. 234) also emphasises the importance of reliability and validity in data collection, analysis and interpretation of the findings. Reliability aims to emit consistency in results, while validity aims to ensure the findings are more credible and the tools used to measure what it intends to measure (Tavakol & Dennick, 2011). To ensure the trustworthiness of this study, the researcher chose to triangulate data and ensure triangulation in collecting data for validity and cross-checking methods for reliability. Cresswell (2012) & Mathison (1988) define triangulation as a process of corroborating evidence from different individuals and data collection methods to improve research validity or evaluate findings, descriptions and themes in qualitative research. As mentioned in Section 5.4, this research study uses a mixed-method approach to answer the research questions. Hence, the researcher uses triangulation methods to enhance the validity of the research findings.

This research used qualitative and quantitative data and validation of findings through data triangulation. The individuals involved, teachers and learners, contributed to the qualitative data collected through interviews, observations and focus group interviews and quantitative through task response and reflective questionnaires. Therefore, the study's interdependency between the qualitative and quantitative results enhances the study's interpretation. Furthermore, Bazeley (2016) notes that quantitative and qualitative sources contribute to contextualising any form of data being analysed and presented and consequently to the capacity for analytic generalisation from the results obtained.

The researcher, therefore, utilised the different triangulation categories to analyse data. According to Denzin (1978), there are four categories of triangulation: (1) Data triangulation (use different data sources); (2) Investigator triangulation(evaluator/investigators); (3) Theoretical triangulation (use multiple perspectives to interpret a single set of data) and (4) Methodological triangulation (use of multiple qualitative and quantitative methods to study the programme). First, data triangulation was used to collect data through various instruments, such as task responses, observations, interviews and questionnaires. In addition, in-depth and focus group interviews were used to triangulate findings. Secondly, different theoretical frameworks were used for developing algebraic reasoning to encourage critical thinking and evidence from different curriculums. Therefore, theoretical triangulation was used to interpret data, especially in the observations of lessons and teachers' feedback from questionnaires and interviews. Finally, methodological or method triangulation was selected because a mixed-method approach was used to collect data for this research. Hence, the researcher used the triangulation method to validate the findings.

5.9 Ethical considerations

The researcher requested ethical clearance for this research, considering that the information gathered is about processes and human subjects. Additionally, relevant permission is required since the study works with teachers and learners. Therefore, permission was obtained from the Western Cape Education Department, the Stellenbosch University Research Ethics committee (see APPENDIX G**: ETHICS APPROVAL SU** and APPENDIX H**: RESEARCH APPROVAL LETTER_WCED**) and Principals (see APPENDIX I**: PERMISSION LETTER FOR SCHOOL PRINCIPAL**). Moreover, teachers were entirely aware of the study's intentions and signed a consent document (see APPENDIX J**: ASSENT FORM**).

The research involves interviews and observations of teachers and learners. Focus group interviews were also part of the data collection, used for the sole purpose of this research consent. In addition, parents, legal guardians, and teachers signed a consent form to confirm participation (see APPENDIX K**: PARENT -LEGAL GUARDIAN CONSENT FORM** and APPENDIX L**: CONSENT FORM**). Finally, all the names of schools, teachers and learners were kept anonymous. Instead, the focus was on their work, as presented in the data analysis.

5.10 Summary

This chapter discussed the research design and methodology used for the study. A mixed method was chosen for the study to answer the research questions. The chapter gave a detailed overview of the instruments for data collection, the validity and reliability, and how they each answer the research question. Additionally, the issue of the participants was discussed in consideration of ethics. Further, this helped to form the methodology and research designs that underpin the research to inform data analysis.

Chapter 6: DATA ANALYSIS AND INTERPRETATION

The chapter discusses the role of data analysis in answering the research questions. As this is a mixed-method study, the qualitative data was analysed first, after which the quantitative data was analysed. For the qualitative data, the first analysis is on the pre– and post-interviews in Section 6.1 towards teachers' understanding of critical thinking (CT) towards algebraic reasoning (AR) development. Then followed by the lesson observations in Section 6.2 to see learners' responses to tasks encouraging CT. Next, the quantitative data analysis is presented in Section 6.2 on learners' responses to pattern tasks that were evaluated for analysing, justifying and generalising, which will serve as evidence of AR and CT development. Then, a discussion on the post-reflective interviews discussed in Section 6.4 evaluates the teacher's perception and triangulation of data for reliability and validity of data.

6.1 Qualitative analysis

The quantitative data for this study was collected through a literature study of other curriculums. Then a presentation was done to explain the terminology to the participants; before that, they had to complete a pre-interview and a post-interview question form on Google Forms since teachers were not available for calls. Lastly, the lesson observation was used to see how learners responded to CT questions towards AR development.

6.1.1 Teacher pre-interview questions response presentation

The researcher's selected teachers approached three different schools to get a non-biased view of CT in classroom response. First, before the presentation on CT to the teachers, all participants were required to sign a consent form and answer some questions before the presentation. Next, the interview was conducted to evaluate the number of participants trained on or taught about critical thinking. A total of 11 teachers were interviewed (four in Grade 4, four in Grade 5 and 3 in Grade 6); 64% of them stated that they had been trained and 36% had never been trained nor taught about critical thinking. Additionally, the same percentage noted that they use cognitive levels and Bloom's taxonomy to guide questioning when planning their tasks. Finally, these interviews were carried out to see the input on teacher's pre-knowledge about critical thinking; some teachers understood critical thinking as:

- *Thinking wider*
- *The process where a person can think outside the box, using thinking skills to analyse different things, problems, or data.*
- *Critical thinking involves thought-provoking questions. Using analytical or evaluation while gathering information.*
- *Critical thinking refers to thinking deeper about problems and ways to solve them.*
- *An advanced thinking level of understanding a concept. It analyses information in various ways.*

From the responses above, it is evident that some teachers understand that CT is not only thinking about the solution but involves thinking in-depth and coming up with other ways to clarify and solve a problem. The researcher then explained what critical thinking is, the researcher's purpose, and what the researcher could expect to see occurring between the teachers and learners during the lesson observation.

6.1.2 Post-interview on teacher's understanding of CT after the presentation

The presentation gave teachers a broader understanding of CT, as all the interviewed teachers agreed on the importance of CT in AR development. In addition, the presentation was to prepare the teachers participating in the observation to clearly understand what the researcher would focus on during the lesson. Teachers have indicated that the presentation has improved their understanding of critical thinking in the following way:

- *"Yes, the presentation elaborated more on critical Thinking, how we as teachers should use critical thinking and different strategies to help learners to develop and improve their thinking skills or ways. It's clearer now because the presentation focused on teachers rather than learners. This will make it much easier for us teachers to be more comfortable implementing critical thinking activities."*
- *"The concepts and role of critical theory in the 4th industrial revolution were outlined. However, every problem is an opportunity to allow learners to explore other answers."*
- *"Yes, it has. Critical thinking is a major part of solving problems in mathematics. Cultivating learners' critical thinking skills will help improve our learners' performances; if we as teachers know how to improve and stimulate their critical thinking skills, the healthier the mathematical environment."*

- *"In understanding how important it is to encourage all learners to exercise critical thinking skills through questions that build their maximum participation and understanding more of the concepts. The presentation was helpful indeed to make mathematics educators get more information and in-depth knowledge of the research topic."*
- *"Yes, using complex activities and games that stimulate and enforce learners."*

The response above shows that the presentation has broadened teachers' understanding of CT as they realised after the presentation that CT can solve complex problems and justify the reasoning behind their answers. Although a consensus among the teachers was that CT needs to be included in every lesson or task, the teachers agreed that CT would help learners develop their explorative skills and help them to be strategic when solving mathematical problems. This means the learners do have the ability to think critically, but only if it is encouraged. Therefore, the teachers shared that they would encourage CT in their classrooms through debates, research, peer teaching, and higher-order questions.

Moreover, one of the teachers highlighted that giving learners opportunities to develop their CT skills helps to create a positive environment where they can explore, explain their chosen methods and converse freely with their teacher. However, even though CT is important to teachers, they have also recognised the following that may be a barrier to CT development:

- *"Time management"*
- *"Not committing themselves to whatever he/she must do to me; that's a barrier."*
- *"Language"*
- *"Language of teaching and learning can be a barrier. Mathematics Language can be a barrier. Time can be a barrier. Curriculum requirements. Assessment tasks. Learners with learning difficulties. Overcrowded classroom. Absenteeism. Poor planning and lack of support."*
- *"Lack of knowledge on basic mathematical operations. Having no experience in problem-solving. Route memorisation in learning mathematics."*
- *"I think the lack of effort from learners may make it hard to incorporate critical thinking skills."*
- *"Inefficient time and lack of participation in the classroom."*
- *"Trying to finish the curriculum."*

Despite the challenges above, the teachers have shared that it is one of the skills learners need. In a scenario where the language of teaching-learning is a barrier, code-switching can be used as an educational resource to help learners comprehend and be competent in their second language (Maluleke, 2019). Additionally, "*An environment that is not a traditional classroom. Where learners are allowed help facilitate their learning.*" Engaging learners as active partners in their learning can positively affect learning (Burke, 2011).

6.1.3 Lesson observations

The Grades that were observed in the study are Grades 5 and 6. The lesson observations were done in the three selected schools. However, due to time constraints, some teachers were unable to teach the topic of "patterns on time" and opted to support the learners to answer the pattern task using their previous grade knowledge so that they could complete the allocated concepts for the term. Therefore, the researcher was able to observe these schools: A (Grades 5 & 6), B (only Grade 6), and C (only Grade 6). A total of four lessons were observed per grade. The allocated time for geometric and numeric patterns is 12 hours, including an hour per grade for lesson introduction, making it 15 hours. I have noticed that teachers are treating the topics as separate and not understanding that they can help to build learners' addition, subtraction, multiplication, and division skills and also be able to apply the inverses they learnt under whole numbers. The outcomes of the observations are discussed in the following sections. To protect the school, the researcher has masked the name of the school by giving each school a unique name for easy identification and analysis.

6.1.3.1 Kyanite school: Grade 5 observation

The teacher first gave an example for learners to revise patterns. Then asked learners before the lesson to look for patterns and come back with their examples of patterns as homework the next day. The 3,6, and 9 pattern in Figure 16 below is the example the teacher gave the learners to introduce the geometric pattern terminology.

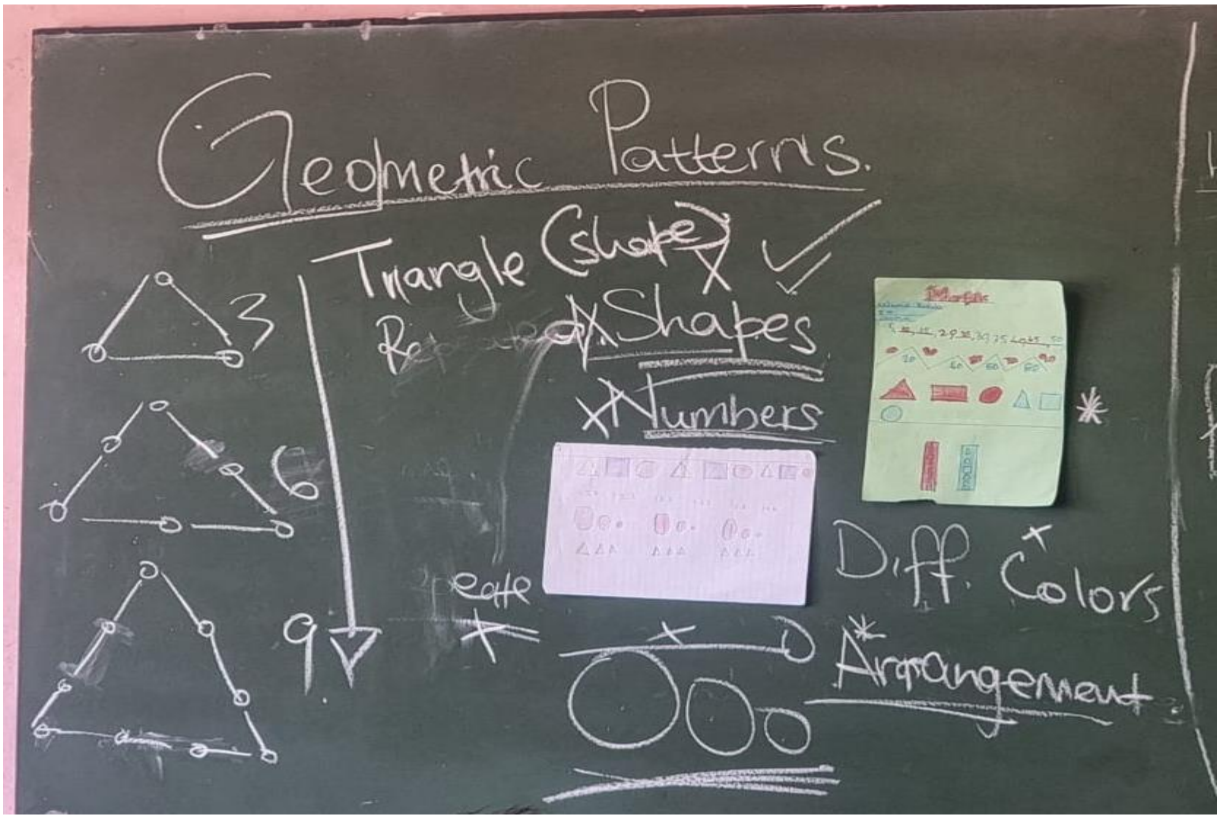
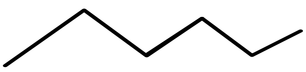


Figure 16: The picture above shows the definition of a geometric pattern

This is how the teacher introduced learners to geometric patterns:

Teacher:  *"Is this a shape?"*

Learner: *"No, teacher, it is a line that is not straight."*

Teacher: *"Look at the patterns on the board. What do you notice about these patterns?"*

Learner: *"We notice shapes, colour, number, structure and arrangement of shapes"*.

Teacher: *"Based on the observation and the keywords written on the board. Define what you think are geometric patterns.*

Learner: *"Geometric patterns involve shapes and numbers that are repeated and arranged in order."*

Teacher: *"Let us look at the tiling of our classroom floor. What is our repeated pattern?"*

Learner: *"It is black and red."*

Teacher: *"Why is it a pattern?"*

Learner: *The tiles are arranged and repeated in colour.*

This class discussion aimed to develop learners' conceptual understanding of how real-life patterns can be translated into mathematical problems. The teacher further describes geometric patterns by breaking down the term to the learners "Geo refers to the soil reminds learners that to find a distance between Cape Town to Johannesburg, we use a scale drawing". Finally, the teacher emphasises that mathematics is not only always numbers but can also be translated into numbers. After the explanation, the teacher guided learners through analysing the board's examples about patterns and asked the learners if that was a pattern. Learners confidently ruled out the examples as incorrect because there was no repetition or arrangement. Moreover, the teacher went ahead to challenge the learners to find the rule of this pattern shown in Figure 17:

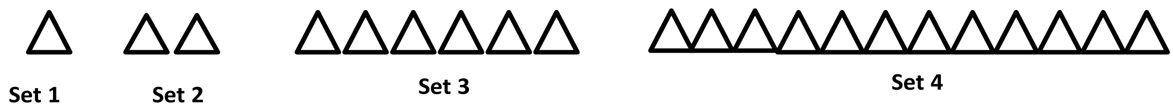


Figure 17: Pattern recognition exercise

Teacher: *"What do you notice about this pattern?"*

Learner1: *"responds that she notice that on set 2 is +1 triangle, set 3 is +5 triangles, then on set four +11 triangles".*

Teacher: *"What is our pattern then?"*

Learner 2: *"Our pattern is: 1, 2, 6, 12."*

Teacher: *"To get the next set, what will be the rule?"*

Learner 3: *"In our group, we notice that to get to the next set, you take the set number and multiply it by the previous set to get the next set. For example, to get the number of triangles in set 3, we multiplied three by the number of triangle triangles equal to 6 in set 3."*

The teacher understood the purpose of stimulating learners' CT by giving them opportunities to look for a relationship in patterns and generalise to find the rules that help grow the pattern. Then, by laying this foundation for the learners, the teacher could move ahead in guiding the learners to participate in the tasks.

6.1.3.2 Kyanite school: Grade 6 observation

In the Grade 6 class, the teacher started with an analysis of the sequence of multiples, for example, the 100th multiple of 5 in 5, 10, 15, 20, 25, 30,….? Learners could easily pick up that to get the 100th multiple, and they will have to multiply 100 by 5 = 500. That is because learners could identify that they were calculating in fives. The teacher then challenged the learners to identify the sequence not multiple in Table 5, and the following conversation between the teacher and learners took place.

Teacher: *"To get the next number, what will be the rule?"*

Learner: *"The rule will be to add 5 (+5) to the previous term to get the next number."*

Teacher: "In that case, what number will be in the 100th position."

Learner: "The number will be 105."

The learners seemed to get more challenged when they were asked to give the 100th term and realised that they could not add up to 100. Therefore the teacher guides learners first to organise the information in a table format, then move on to a flow diagram to test the rule.

TABLE 5: SEQUENCE TABLE

Position no.	**1**	**2**	**3**	**4**	**5**	**6**	**100**
Sequence	6	11	16	21	26	31	**?**

The teacher further explains to the learners how to transition the above information from the table to the function machine, where the input is the position number, and the output is the sequence number.

The rule +5 will only function for the first number but not for the 100th term. Learners could recognise that to get the following sequence, and the rule will be ×5+1 to get the following number. Therefore the 100th number will be 501 and not 105. The transition from numbers to table format and the function machines help the learners find the rule and play around with numbers. Afterwards, the teacher gave learners an activity to complete in their respective groups.

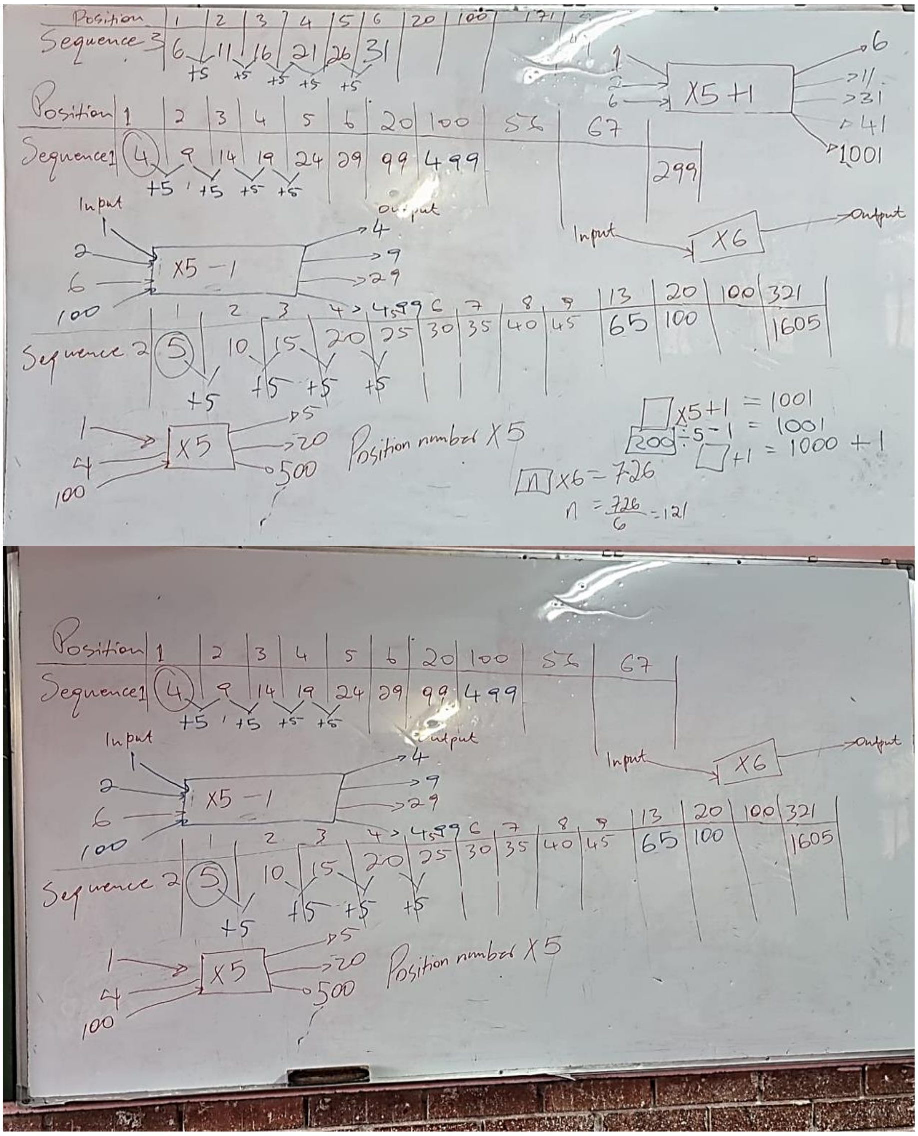


Figure 18: Function machine for learners' group work to find the rule to get the 20th and 100th numbers in the sequence

Figure 18 shows learners' responses as they share their answers in their respective groups. First, the teacher challenged learners to figure out the input if the output is 1001 in sequence 3. Next, the teacher asks learners to devise a calculation plan to find the input. For example, learners could recognise that if they go forward, they will multiply and find the input will divide. Additionally, they shared in their calculation plan that they first divided 1000 by 5 to get 200 and added 1. The teacher was able to prompt learners with questions to support their reasoning and encouraged them to use mathematical concepts that were the keywords for the lesson, such as input, output, rule and inverse operation. Finally, the teacher asked learners to share how they reached the answer, thus reinforcing inverse skills.

6.1.3.3 Serpentine school: Grade 6 observation

The teacher drew two patterns on the board and asked the learners to complete them. Then, the teacher guided the learner's thinking by asking them to share their observations about the patterns and what they needed to get the next figure. For example, in Figure 19 below, learners noticed that for pattern A, they added 2 matchsticks to get the next pattern, and for pattern B, they had to add 3 matchsticks to get the next figure.

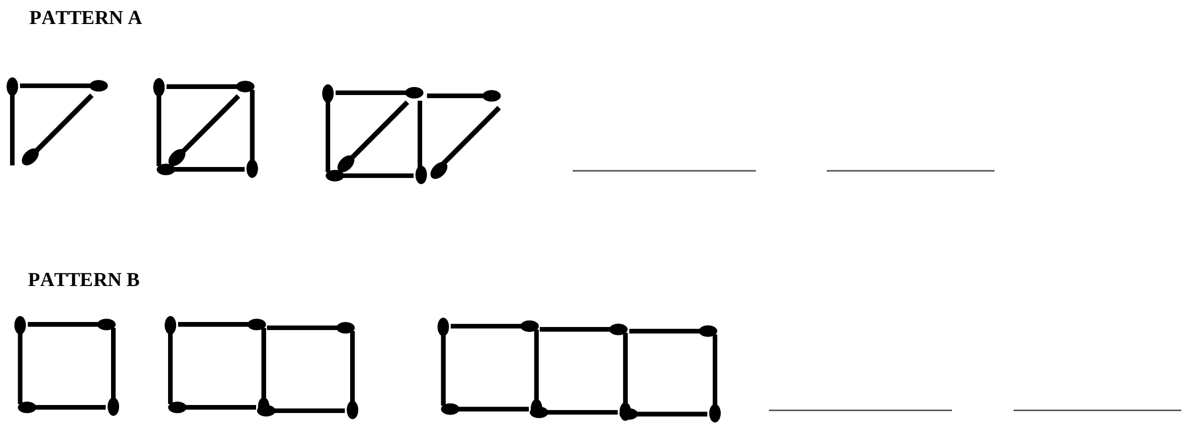
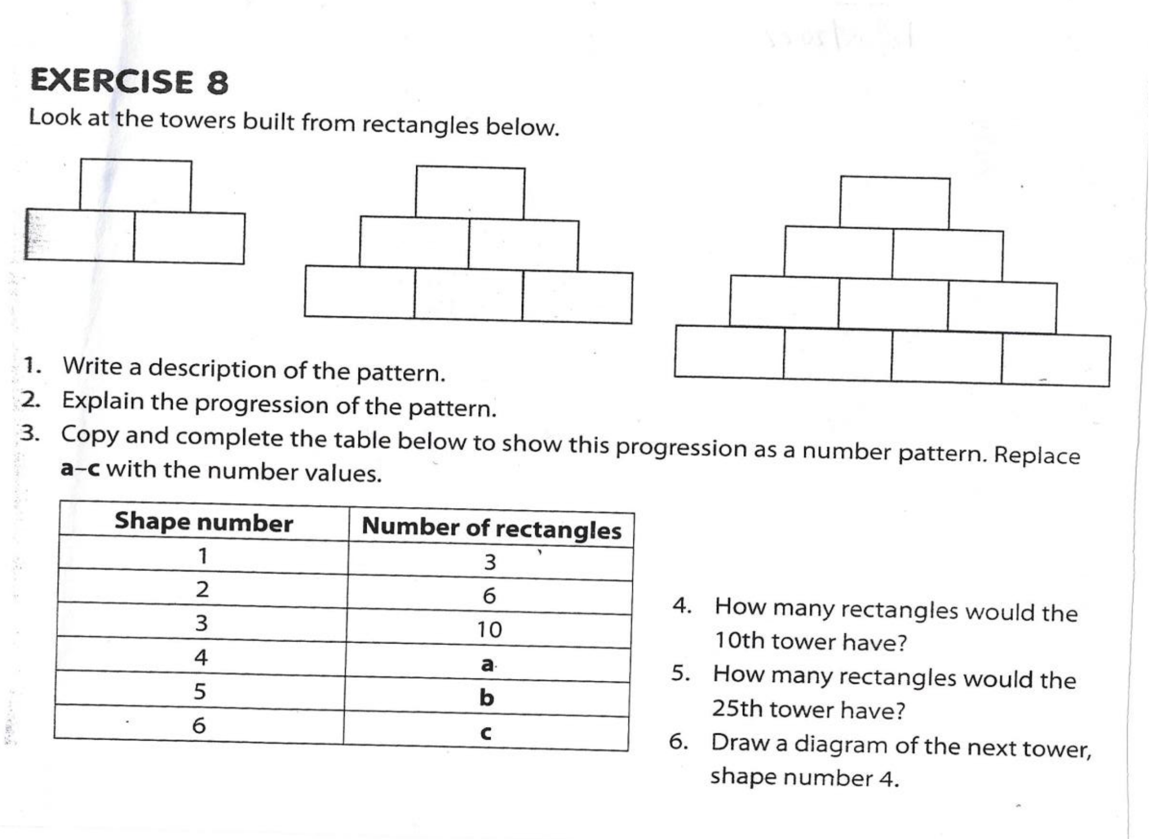


Figure 19: The representation of the pattern used for the introduction

After the learners expanded the pattern, the teacher questioned them regarding the rule they would use to determine the number of matchsticks needed to get to the 20th figure. Since they added matchsticks to Patterns A and B, the teacher reverted the learners to the inverse processes. Teacher further explains that consequently, for us to discover a suitable rule, what would be the inverse of +2 and +3? The rule will be input 2+1 = output for Pattern A and input 3 + 1 = output for Pattern B because learners indicated that they would utilise multiplication as the opposite of division. They also understood that they needed to utilise multiplication on the function machine to ensure their rule was accurate when they translated their observations of adding a +2 and a +3 on the patterns in Figure 19 depicted above. The use of inverse encourages doing and undoing, which is a sign that AR has developed (Driscoll 1999, p.2)

In the other lesson, the teacher challenged learners to solve the problem shown in Figure 20 where a pattern consists of the first and second differences. For example, learners noticed that all the towers have the first top block in common and different levels.



EXERCISE 8

Look at the towers built from rectangles below.

1. Write a description of the pattern.
2. Explain the progression of the pattern.
3. Copy and complete the table below to show this progression as a number pattern. Replace **a–c** with the number values.

Shape number	Number of rectangles
1	3
2	6
3	10
4	**a**
5	**b**
6	**c**

4. How many rectangles would the 10th tower have?
5. How many rectangles would the 25th tower have?
6. Draw a diagram of the next tower, shape number 4.

Figure 20: First and second difference pattern problem

Figure 20 shows the learners' task as a class activity from their textbooks. First, the teacher guided the learners to look at their patterns; the levels will also increase as the towers increase. Finally, the teacher had a class discussion to guide the learners' thinking in finding a solution:

Teacher: "*So here is our pattern in a sequence 3, 6, 10… So what will be next shape look like?*

Learner: "*disagrees that Shape number 3 will have 10 blocks because the increase is by +3 to get the next shape; therefore, the pattern is supposed to be 3, 6, 9.*"

Teacher: "*No, that cannot be because, based on our information from our textbook, shape 3 has 10 blocks and not 9.*

Let us go back to our observation about each tower since we already noticed that each tower has a first block at the top for all the towers. So, therefore, let's first look at the number of levels that each tower has."

Learner: "*For each level, +1 Block/level will be common for all; therefore, to get the next level, I used a rule (shape number +1 = number of level)."*

Teacher: "*You are almost there with your rule. Learners take note once you get a pattern where you have a first and second difference (Figure 21), meaning two numbers in a bracket and multiply with one, something like this n(n+1) where n= shape number."*

3, 6, 10

3 4 1st difference

1 2nd difference

Figure 21: Pattern first and second difference

Learners completed the task through teacher assistance and using the above information, and they could get the number of rectangles for the 25th tower and draw the shape number 4. However, it is evident that learners who find rules still need guidance from their teacher to provide reasonable pattern rules.

6.1.3.4 Tanzanite school: Grade 6 observation

The teacher revised the numeric patterns on the DBE workbooks. Then learners were taken to the computer lab to complete the tasks on a math program used by the school. The learners did most of their work in a computer lab via a math learning program. The teacher walked around to assist learners in their work. When learners got stuck on a particular question, the teacher revisited the question through a class discussion. Therefore, not much interaction occurred between a teacher and a learner

6.1.4 Overall class observation

The class observation was to understand how learners respond to CT questions encouraging AR development. I observed by identifying if the learners were making connections, generalising and being able to justify their reasoning. The teachers encouraged the learners to:

- Observe the picture/ pattern given.
- Write out the sequence.
- Observe whether the pattern is increasing or decreasing by figuring out the constant difference.
- Organise the information in a table format, then,
- Use the function machine to test if the rule is functional.

However, it was only in two schools that the most interaction between the teacher and learner was observed. The third school used a learning program where learners watched videos and answered questions that tested their understanding. Learners did not have sufficient practical engagement with CT because of their minimal conversation in class. It was more evidence that the interaction between the teacher and learner helps with mathematical reasoning and the learners' math vocabulary development. That is because learners are allowed to learn through social interaction to construct meaning and understanding (Krahenbuhl, 2016). In some cases, the teacher tries to help learners move toward the answer by encouraging them to use the guiding steps to find a correct rule for the sequence.

6.2 Quantitative analysis

The quantitative data for this study were collected through focus group interviews and post-reflective questionnaires. Therefore, the quantitative analysis was not given much consideration or focus because the study focused on teachers' rather than learners' responses to AR tasks.

6.2.1 Analysis of focus group interviews

A group of 15 learners was observed in the selected grades (5&6) in three different schools. The purpose of these focus group interviews is to analyse learners' written responses and their level of algebraic reasoning in a grade in comparison with the other. The four-task worksheets were analysed using a rubric. The questions' scoring framework considered the learner's ability to analyse, justify and generalise (see APPENDIX E**: LESSON OBSERVATION**). For example, Task 1 focused on analysing the different array presentations that result in common value; Task 2 helps learners reinforce inverse skills using tables and function machines; Task 3 looks at number sentence transitioned pattern to encourage habits of mind of doing and undoing to deepen AR; and lastly, Task 4 looks at different matchstick pattern presentation to encourage generalisation. The results below are analysed per school, after which the overall performance of Grade 5 and 6 learners' responses to algebraic reasoning tasks are analysed. Finally, the school interviews contributed to identifying different factors that may be helpful for learners or disadvantage them when engaging in AR tasks.

6.2.2 Analysis of AR focus group interviews at Kyanite School

In the Kyanite school, The learners in Grade 5 and Grade 6 were given four tasks each. A total of 40 minutes was allocated to each task. The average result for each grade is shown in Table 6.

TABLE 6: AVERAGE REASONING PERCENTAGE FOR KYANITE SCHOOL.

REASONING:	GRADE 5 AVERAGE	GRADE 6 AVERAGE
ANALYSING	60%	72%
JUSTIFYING	64%	71%
GENERALISE	42%	52%

As shown in Table 6, the average percentage achieved in Grade 5 was 60% for analysing reasoning and 72% for Grade 6. Then in justifying reasoning, Grade 5 had an average of 64% and Grade 6 71%. Finally, Grade 5 was 42% for generalising reasoning, and Grade 6 was 52%. These are the three strands used to show AR development and CT. Based on the average percentages, it is evident that Grade 6 is better at CT than Grade 5. The result is expected considering that Grade 6 learners are higher, have been exposed to solving patterns, and are more familiar with symbol manipulation to derive a suitable rule.

Moreover, both grades showed their capability of justifying reasoning for task 3, which can be seen in Table 7. Learners understood the context first and saw the transition towards moving to a pattern and finding the applicable rule. In Task 1, the learners could comprehend and had knowledge of inverse operations, as shown in Figure 23. The percentage difference between the two grades in analysing, justifying and generalising is 12%, 7% and 10%, respectively. Learners need more support in developing those two skills.

TABLE 7: KYANITE SCHOOL ANALYSIS OF TASK PERFORMANCE.

GRADE 5			
	ANALYSING	JUSTIFYING	GENERALISE
TASK 1 (%)	68%	0%	0%
TASK 2 (%)	65%	42%	53%
TASK 3 (%)	53%	87%	53%
TASK 4 (%)	52%	0%	20%
AVERAGE %	60%	64%	42%

GRADE 6			
	ANALYSING	JUSTIFYING	GENERALISE
TASK 1 (%)	88%	0%	0%
TASK 2 (%)	72%	49%	87%
TASK 3 (%)	56%	93%	33%
TASK 4 (%)	73%	0%	37%
AVERAGE %	72%	71%	52%

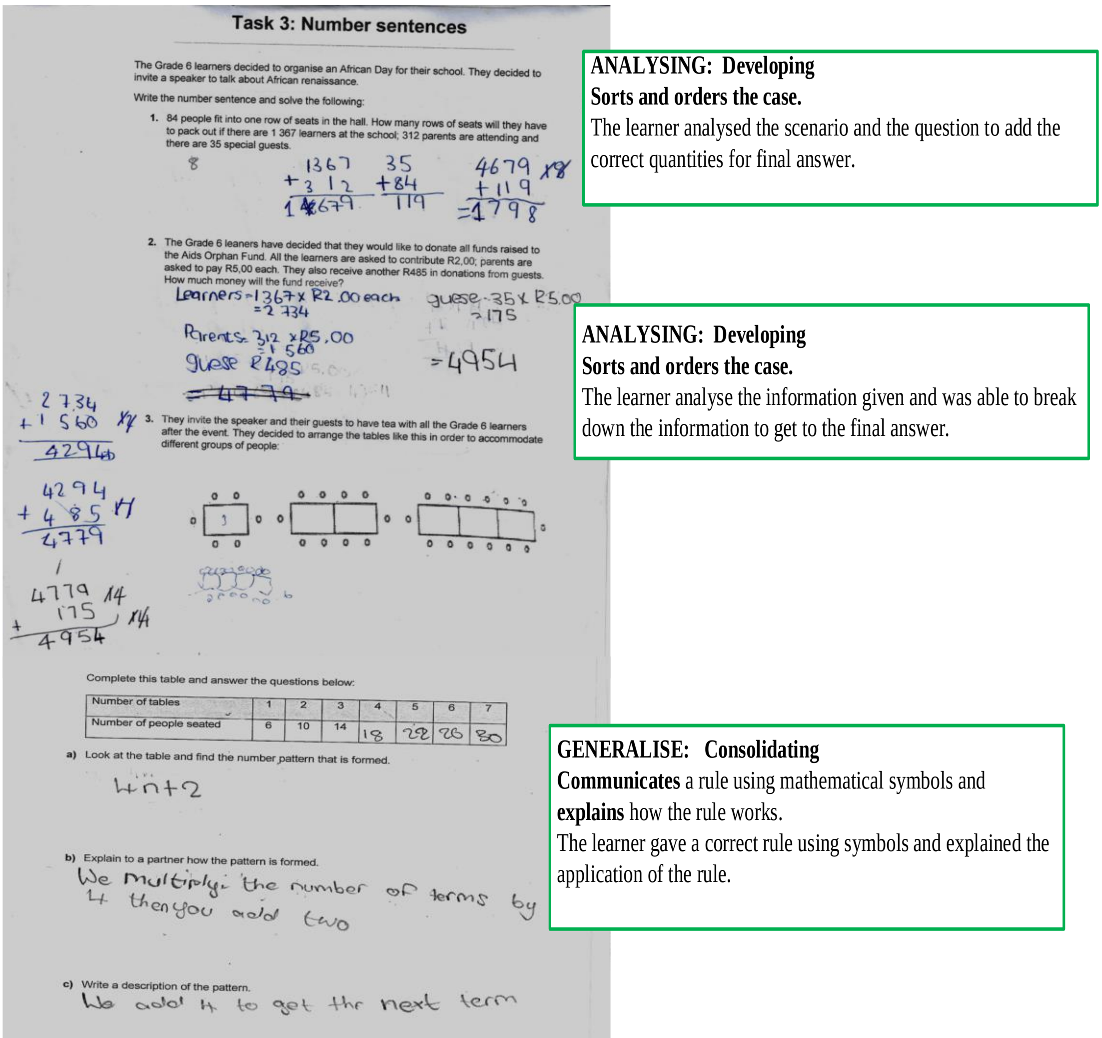


Figure 22: Learner work sample analysing number sentence

6.2.3 Analysis of AR focus group interviews at Serpentine School

The learners in grade 5 and grade 6 were also each given four tasks. A total of 40 minutes was allocated to each task. The average result for each grade is shown in Table 8.

TABLE 8: AVERAGE REASONING PERCENTAGE FOR SERPENTINE SCHOOL.

REASONING:	GRADE 5 AVERAGE	GRADE 6 AVERAGE
ANALYSING	54%	71%
JUSTIFYING	72%	79%
GENERALISE	41%	61%

As shown in Table 8, the average percentage achieved in Grade 5 was 54% for analysing reasoning and 71% for Grade 6. Then in justifying reasoning, Grade 5 had an average of 72% and Grade 6 79%. Finally, Grade 5 was 41% for generalising reasoning, and Grade 6 was 61%. These are the three strands used to show AR development and CT. Based on the average percentages of the Serpentine school, and similar to the previous school, it is evident that Grade 6 is better at CT than Grade 5. The result is to be expected considering that Grade 6 learners are in the concluding grade of IP and have had more exposure to pattern tasks in this grade from the previous two grade levels.

The case of Kyanite and Serpentine schools is similar: the average percentage of grades in the analysis is higher than in Grade 5. Table 8 shows evidence that the average of Grade 5 learners was due to Task 3 and Task 4. As these tasks look to challenge learners' thinking, in tasks three, medium and 4, higher-level learners had generalising problems. Learners need support to improve their skills in generalising. Figure 23 shows an example of a learner's response in Grade 6 on task 2, as their AR development is better than the Grade 5 learners. The percentage differences between the two grades in analysing, justifying and generalising is 17%, 7% and 20%, respectively.

TABLE 9: SERPENTINE SCHOOL ANALYSIS OF TASK PERFORMANCE.

GRADE 5				GRADE 6			
	ANALYSING	JUSTIFYING	GENERALISE		ANALYSING	JUSTIFYING	GENERALISE
TASK 1 (%)	68%	0%	0%	TASK 1 (%)	75%	0%	0%
TASK 2 (%)	60%	58%	53%	TASK 2 (%)	90%	84%	94%
TASK 3 (%)	38%	87%	53%	TASK 3 (%)	47%	73%	53%
TASK 4 (%)	49%	0%	17%	TASK 4 (%)	72%	0%	37%
AVERAGE %	54%	72%	41%	AVERAGE %	71%	79%	61%

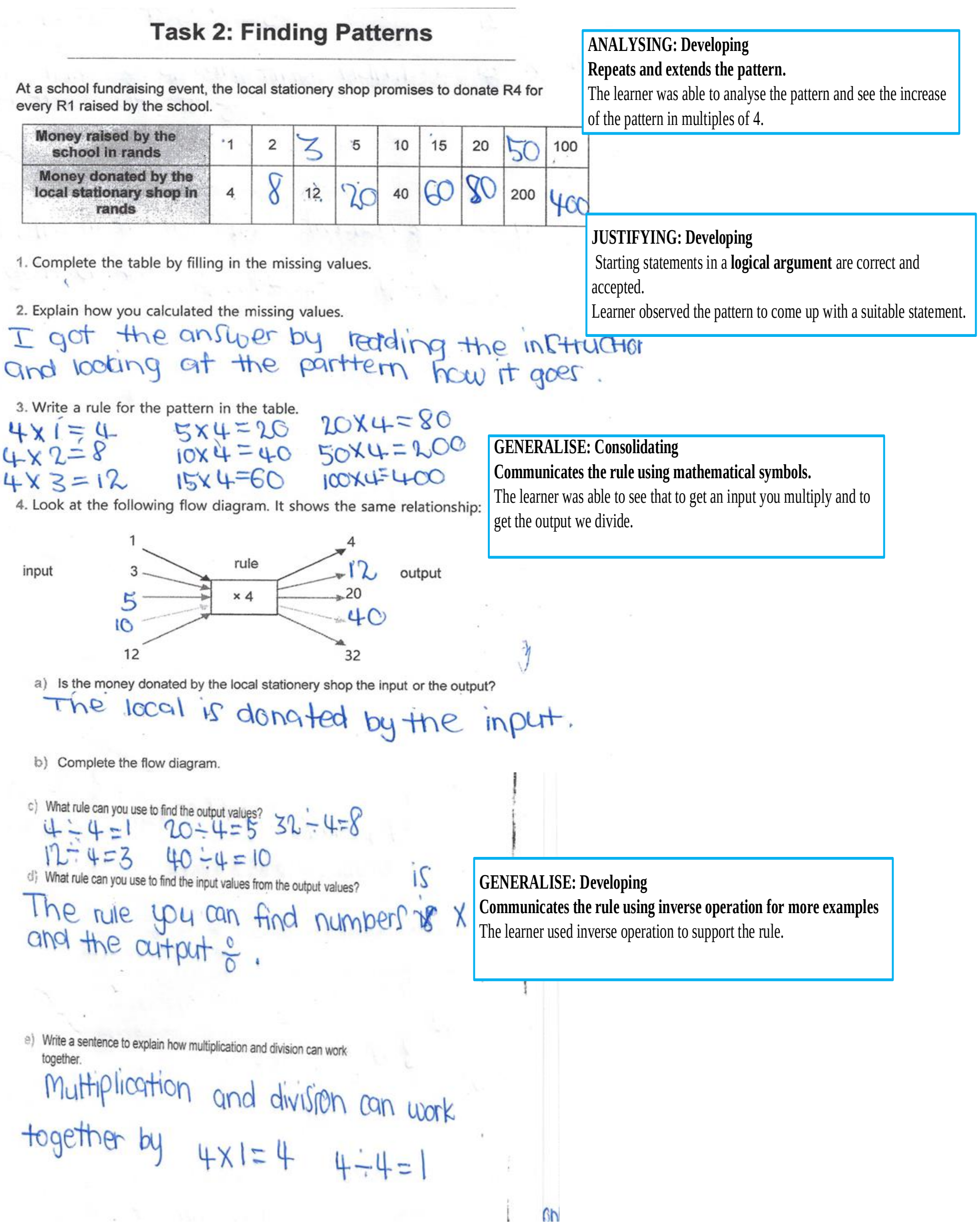


Figure 23: Learner responds by observing and generating a rule for the pattern

6.2.4 Analysis of AR focus group interviews at Tanzanite School

In Tanzanite school, the average percentage achieved in Grade 5 was 63% in analysing and 77% in Grade 6. In reasoning justifying, Grade 5 had an average of 72%, Grade 6 had an average of 87%, and in generalising, Grade 5 had an average of 39%, and Grade 6 had an average of 54% (See Table 10). These are the three strands used to show AR development and CT.

TABLE 10: AVERAGE REASONING PERCENTAGE FOR TANZANITE SCHOOLS

REASONING:	GRADE 5 AVERAGE	GRADE 6 AVERAGE
ANALYSING	63%	77%
JUSTIFYING	72%	87%
GENERALISE	39%	54%

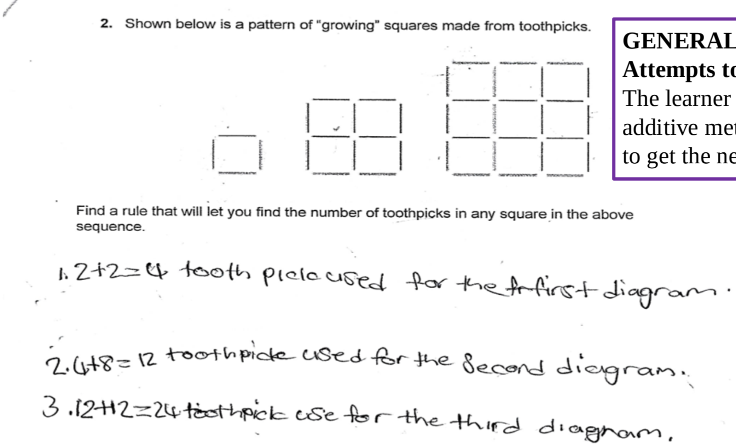
Based on the average percentages, it is again evident that Grade 6 is better in analysis skills than Grade 5. Table 11 shows evidence that learners in Grade 6 and 5 need to be trained in generalising. The percentage difference between the two grades in analysing is 14%, justifying is 15% and generalising is 15%. The learners could recognise the pattern on question 3 in Task 4 and develop different reasoning for their generalising in Figure 24. The three examples below show the learner's response:

TABLE 11:TANZANITE SCHOOL ANALYSIS OF TASK PERFORMANCE

GRADE 5			
	ANALYSING	JUSTIFYING	GENERALISE
TASK 1 (%)	85%	0%	0%
TASK 2 (%)	60%	58%	53%
TASK 3 (%)	56%	87%	53%
TASK 4 (%)	52%	0%	10%
AVERAGE %	63%	72%	39%

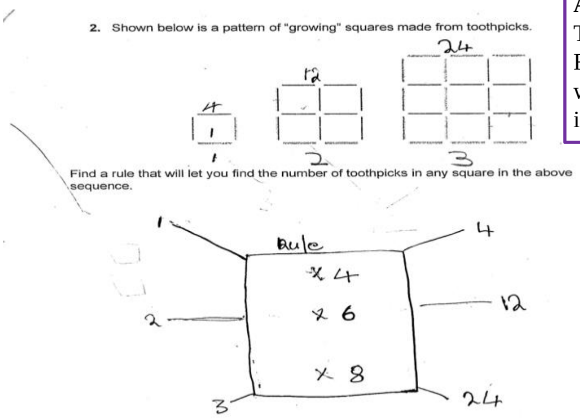
GRADE 6			
	ANALYSING	JUSTIFYING	GENERALISE
TASK 1 (%)	80%	0%	0%
TASK 2 (%)	90%	73%	93%
TASK 3 (%)	69%	100%	40%
TASK 4 (%)	69%	0%	30%
AVERAGE %	77%	87%	54%

LEARNER A



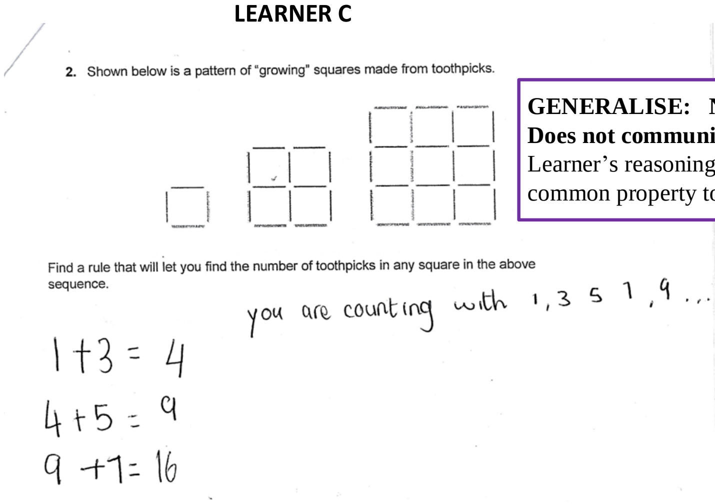
GENERALISE: Beginning
Attempts to communicate a rule for the pattern.
The learner has not given the correct rule but was using an additive method to find the number matchsticks were added to get the next term.

LEARNER B



GENERALISE: Beginning
Attempts to communicate a rule for the pattern.
The learner has not given the correct rule but was using an Function machines to verify the correct number to multiply with. to get the number matchsticks. The method is incorrect because there should only be one rule

LEARNER C



GENERALISE: Not evident
Does not communicate a common property or rule.
Learner's reasoning is unclear and there is no identification of common property towards find the rule. .

Figure 24: Learner's response pattern generalisation

6.2.5 The overall analysis of the three schools

This section aims to analyse all Grades 5 and 6 to analyse the learners' reasoning performance. A comparison of all three schools clearly shows learners' ability in the three categories of CT they have been tested in. Figure 25 shows that the three schools are scattered across the Cape Winelands district, and the results are similar. Grade 5 learners struggle to generalise because they tend to think patterns have only one line of difference.

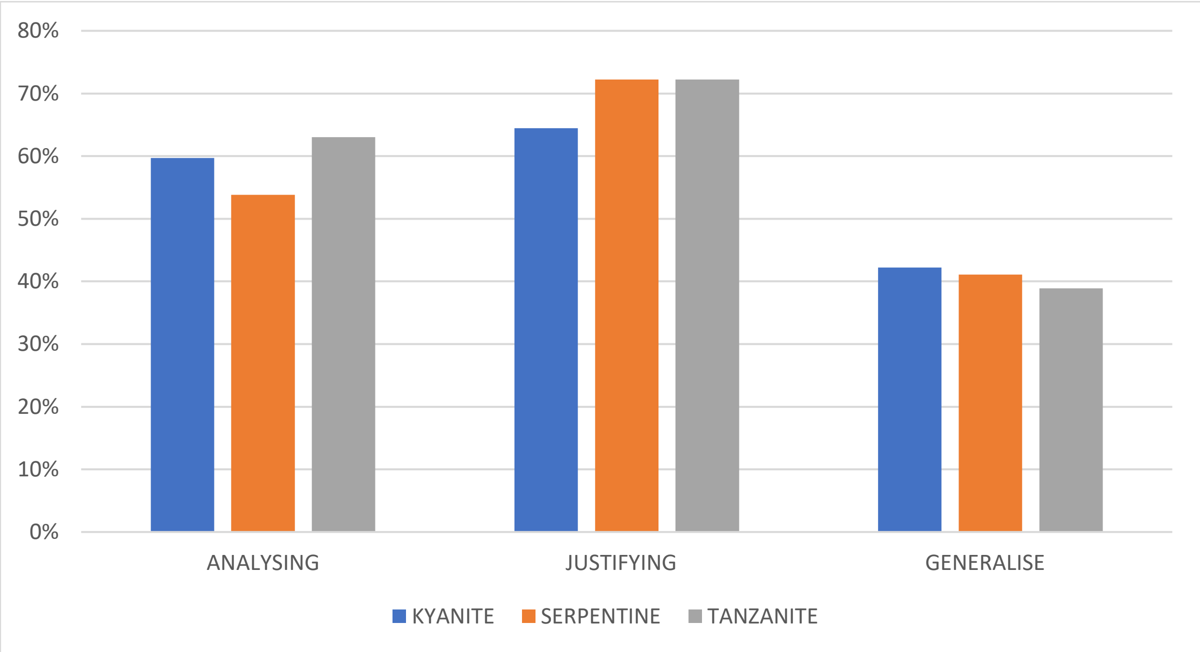


Figure 25: Reasoning average for all schools Grade 5

The same thing can be said about the Grade 6 classes (See Figure 26). Though the performance of Grade 6 learners is better than that of the Grade 5 learners in all schools, there is still evidence that generalising is a skill that needs improvement to meet the other CT skills.

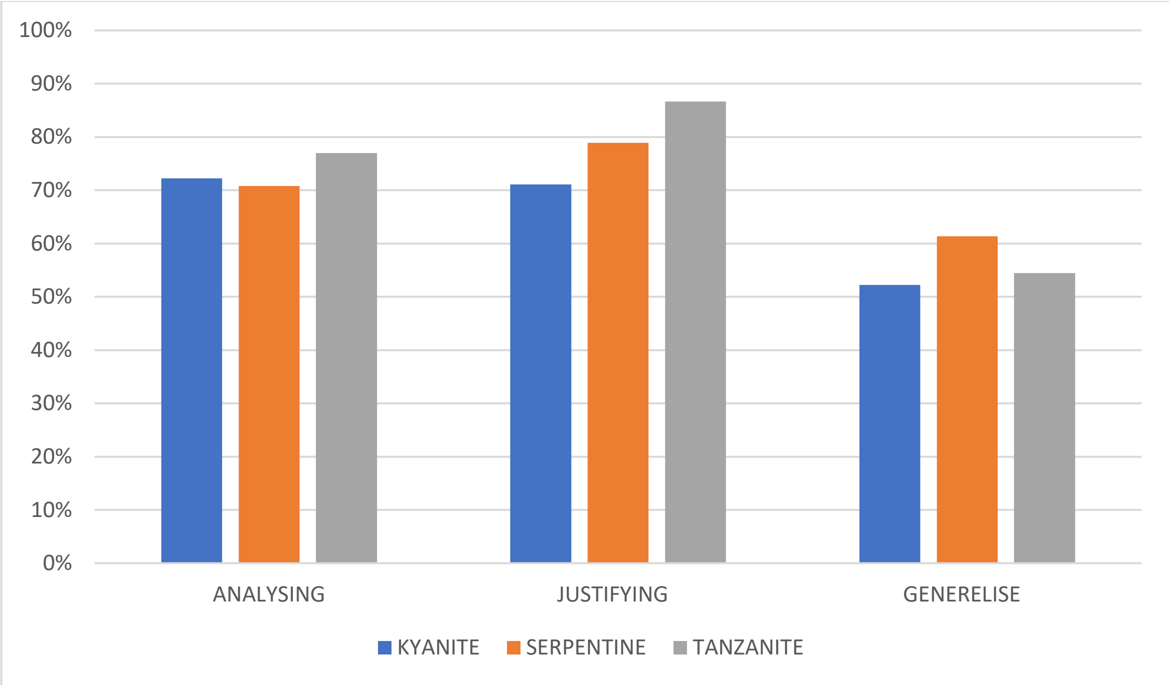


Figure 26: Reasoning average for all schools Grade 6

The graphs below show the Tasks results between Grades 5 and 6 and the relevant AR skills that serve as AR and CT development in learners. The tabs represent the performance percentage and the average per grade on the three skills, namely analysis, generalisation and justification of the learner's response per task. For example, based on the figure below, the average performance in the two grades is as follows (Table 12):

TABLE 12:THE AVERAGE PERCENTAGE OF THE TWO SKILLS ON REASONING SKILLS.

	GRADE 5	GRADE 6
REASONING:	AVERAGE %	AVERAGE %
ANALYSING	58	73
JUSTIFYING	20	39
GENERALISE	19	42

The overview of the reasoning analysis in select grades allows the researcher to see the performance between the two grade levels, and reasoning that is high in this case is justifying and then generalising and lastly to be able to justify. For this study, the focus on the development of AR in IP looks at how learners' CT is encouraged when they engage in pattern tasks between the two selected grades. For example, Grade 6 shows their reasoning skills to be higher in analysing, justifying and generalising than in Grade 5 (see **Figure 30**).

This could be because most of Grade 5 struggled to generalise and justify the patterns with first and second differences. One remarkable observation on the four tasks was that the learners responded to Tasks 1, 2, and 3, and in Task 4, learners attempted only Questions 2 and 4, and most did not answer them. In addition, both grades comprehended the matchsticks problem, encouraging them to think before moving matchsticks around in Figure 27 and Figure 28.

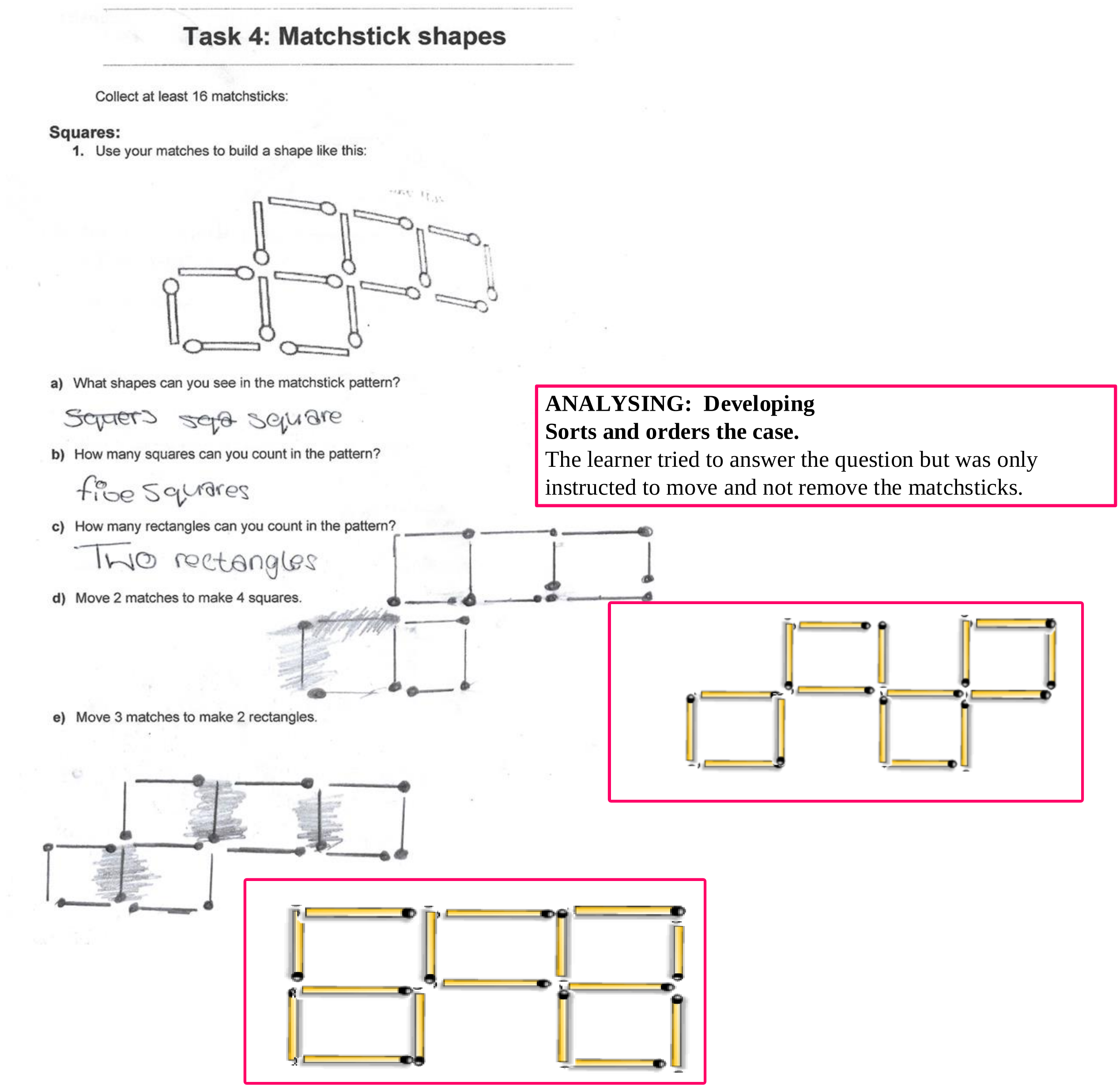
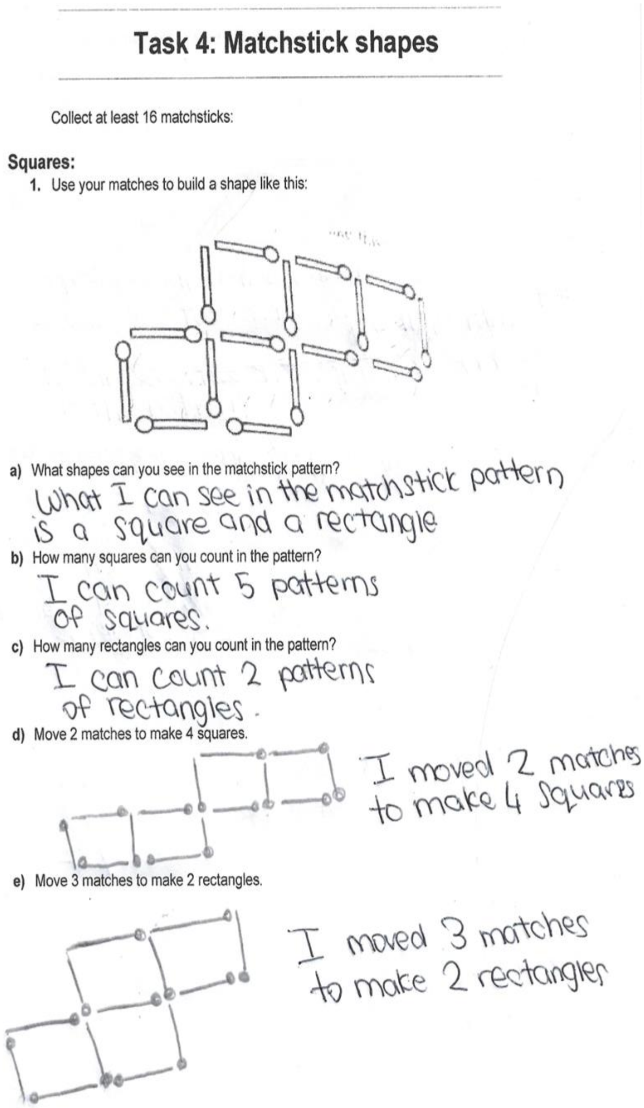


Figure 27: Learner's response to pattern generalisation

The figure above shows the work of the learner. The learner removed the matchsticks instead of moving them around. The above shows the learner's incorrect thinking and not understanding of the question.



ANALYSING: Extending
Notices and explores the relationship between properties. The learner was able to move around matchsticks without removing any matchsticks.

Figure 28: Example of learners working on moving around matchsticks to answer the question

This task was challenging for most learners because some persisted that they had to remove the matchsticks instead of moving them around. Finally, they worked in groups and discussed how to solve numbers d and e. The teacher walked around to observe what the learners were doing and guided them by probing questions as they interacted in their groups. When learners are given opportunities to apply CT skills, their thinking capacity developed, and they are able to find ways of solving real-life problems (Noddings & Brooks, 2016). However, one of the challenges observed is the learner's mathematical vocabulary; hence, their justifying skills are shallow. Learners need more training and are encouraged to use mathematical terms to justify their reasoning.

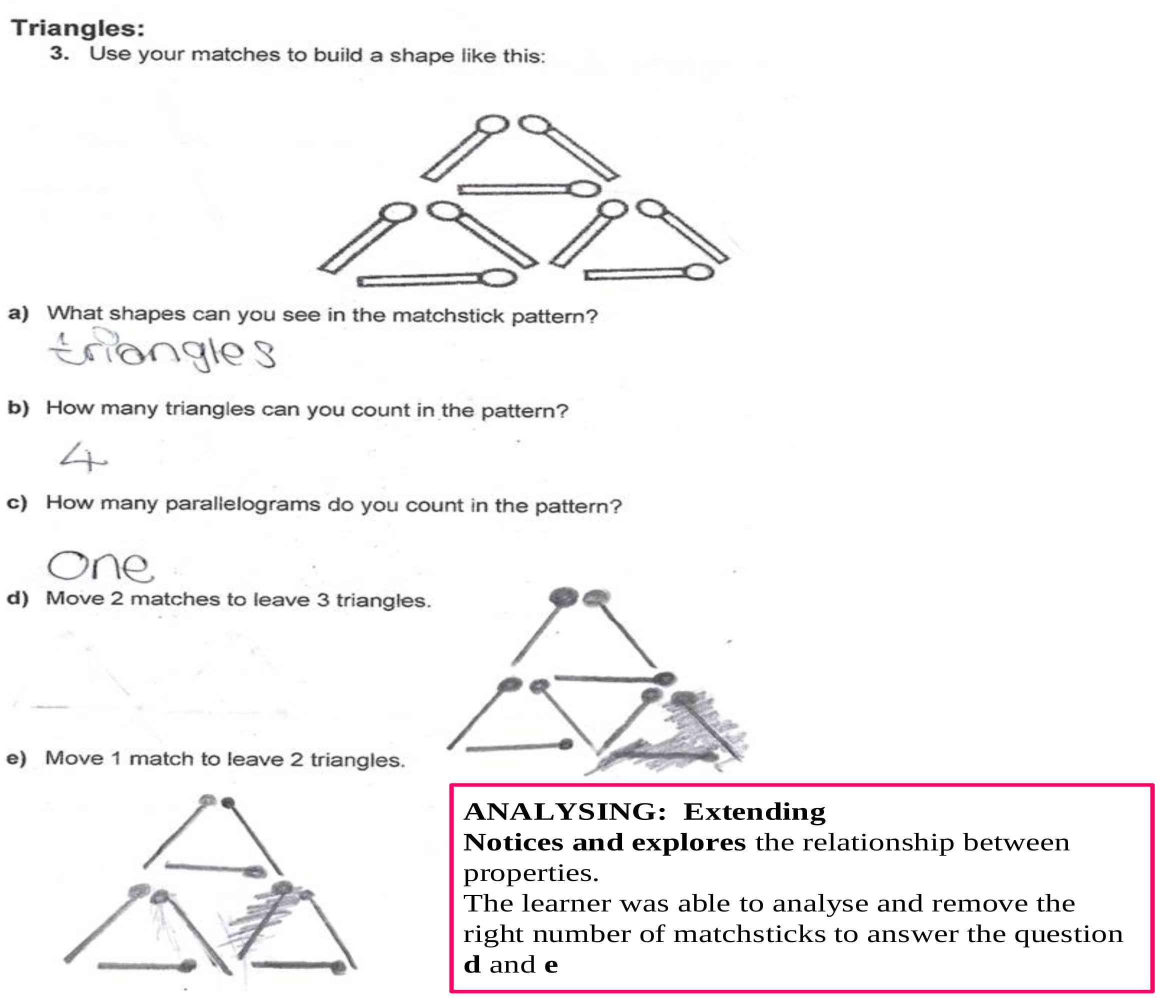


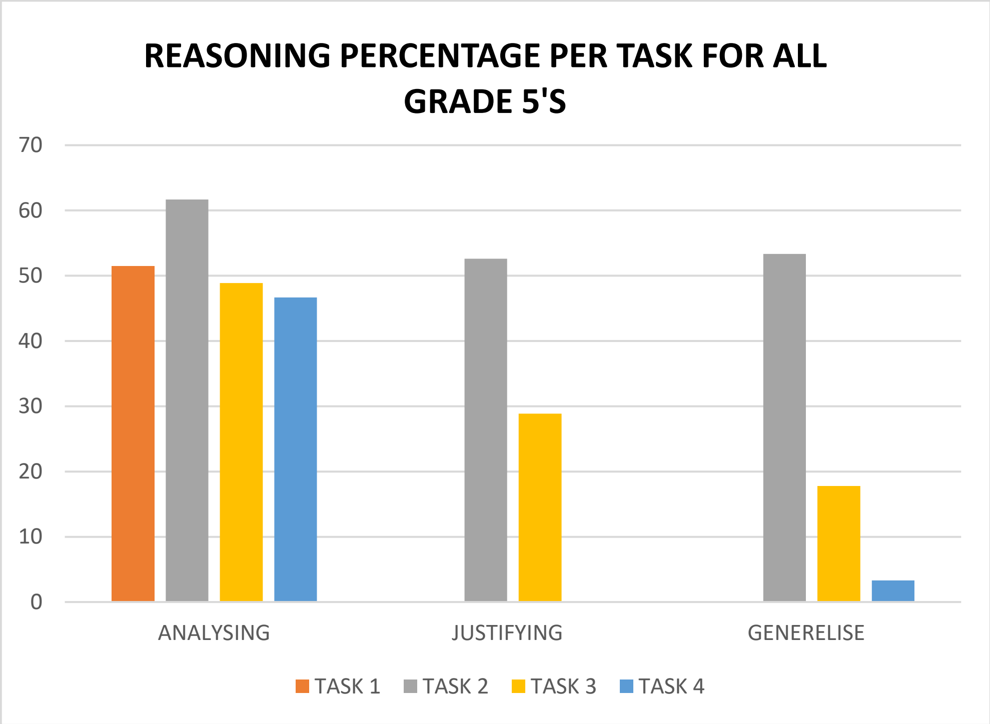
Figure 29: Learner's response pattern generalisation

The task shown in Figure 29 was answered correctly by the two grades. The learners could remove the matchsticks as instructed to get the correct answer. However, the Tasks did not require learners to justify their procedures or steps. This resulted in low levels of justification in the reported data.

Therefore, teachers need to be encouraged to understand that topics on whole numbers do not work in isolation but can also be developed through pattern tasks to improve their skills and reasoning using mathematical concepts.

REASONING PERCENTAGE PER TASK FOR ALL GRADE 5's					
REASONING:	**TASK 1**	**TASK 2**	**TASK 3**	**TASK 4**	**AVERAGE %**
ANALYSING	74	62	49	47	58
JUSTIFYING	0	53	29	0	20
GENERALISE	0	53	18	3	19

REASONING PERCENTAGE PER TASK FOR ALL GRADE 6's					
REASONING:	**TASK 1**	**TASK 2**	**TASK 3**	**TASK 4**	**AVERAGE %**
ANALYSING	81	84	57	71	73
JUSTIFYING	0	69	89	0	39
GENERALISE	0	91	42	34	42



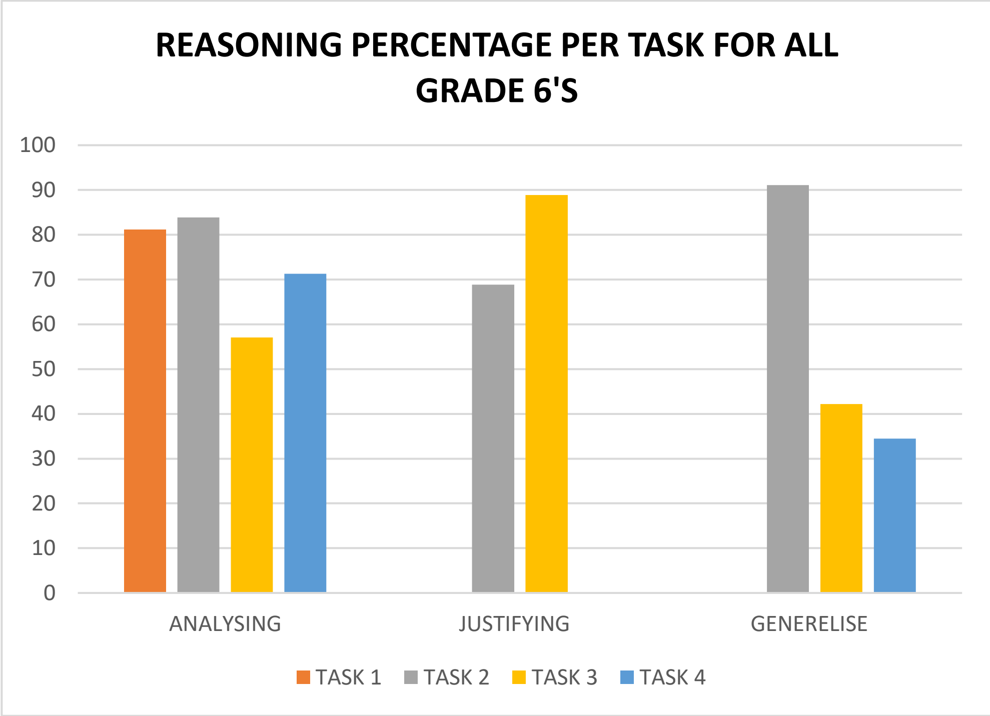


Figure 30: Kyanite school, B and C analysis of task performance

6.3 Post-reflective questionnaires on CT

This post-reflective data was collected from the reflective questionnaires given to teachers after the focused group sessions. The purpose of the questionnaire was to get feedback and reactions from the teachers after a focused group session. It was also to ascertain if CT is a tool they can incorporate daily in their lessons. The questionnaire had eight questions (See APPENDIX F: **POST-REFLECTIVE QUESTIONNAIRE**). 11 teachers completed the questionnaire. Figure 31 shows the teacher's perceptions about the importance of the application of CT in their teaching and learning practice.

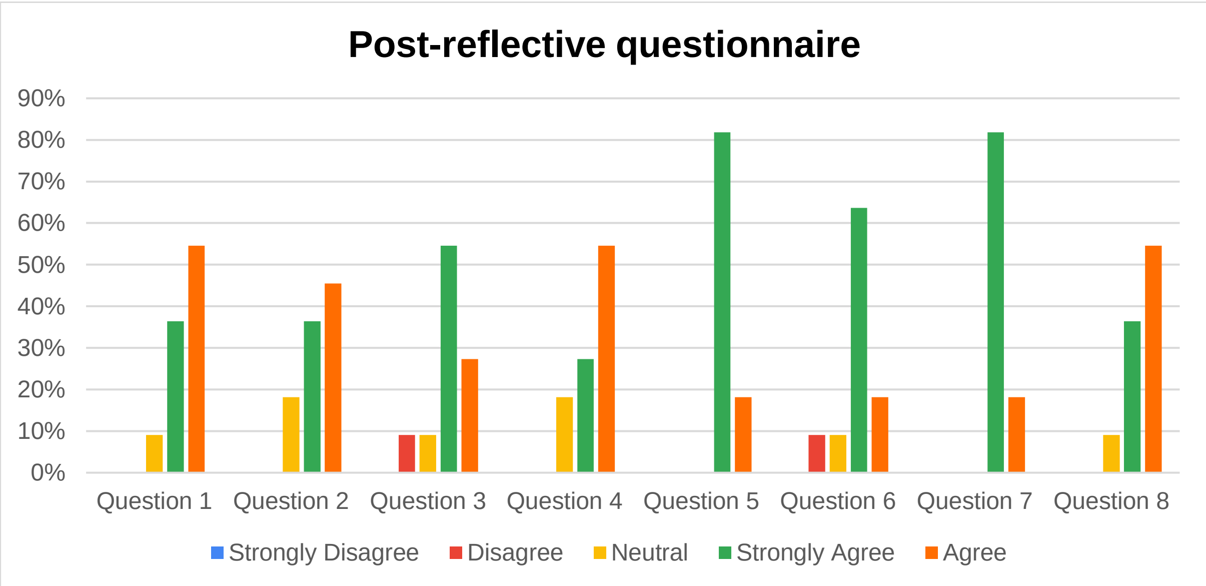


Figure 31: Teacher's perception of CT in their teaching and learning practices

Figure 31 shows that 10 (91) % of teachers agree that they understand the meaning of CT (Question 1), while 9% feel neutral about CT. This could be because the teacher needs more training on how CT can apply to teaching and learning. However, four strongly agree (45%), Five agree(36%), and the rest is neutral (18%) if they are confident in formulating and identifying CT questions. However, even though CT is a thinking process that needs to be encouraged to promote active learning in the classroom, six (55%) teachers strongly agree, three (27%) agree, one (9%) is neutral, and one (9%) disagree. As some teachers have highlighted in the interview, they do not dispute the importance of CT, but it may be time-consuming, affecting syllabus completion. Figure 32 represents teachers' willingness to adapt CT in their lesson plans. As seen in the figure, most teachers are concerned about the time it takes to incorporate CT into lesson tasks and is worried it may affect the completion of the syllabus.

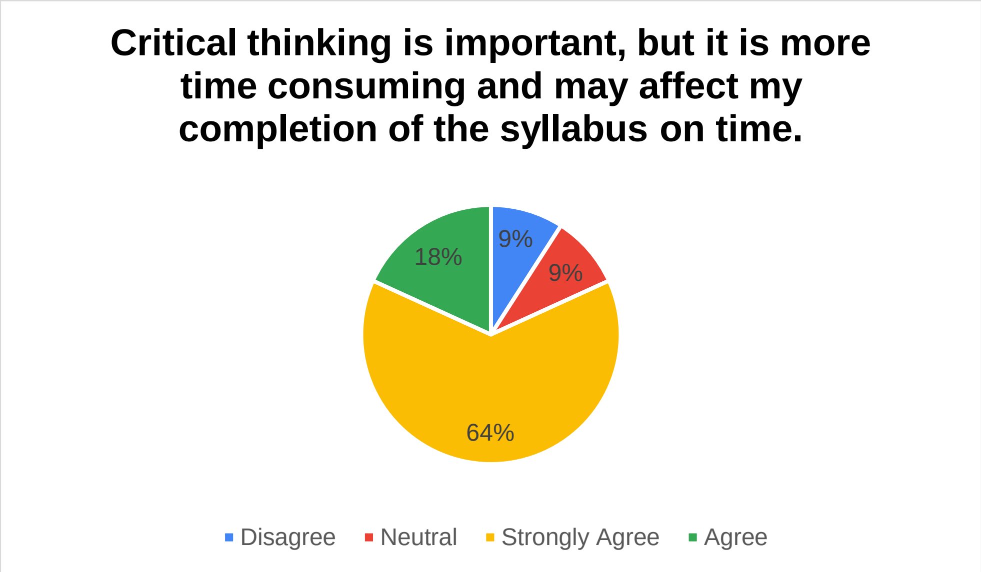


Figure 32: Teacher's perception of the importance of CT

Moreover, two (18%) of teachers are neutral about creating tasks or extending textbook questions to encourage CT in most lessons, while 82% are more confident and sure of their capability. However, all teachers (100%) agree that CT is more helpful for learners to understand CT than merely memorising steps. Furthermore, 10 (91%) teachers agree that their CT understanding has contributed to their teaching practices. A way to address this is for curriculum advisers to incorporate CT in the syllabus so teachers do not see it as an additional task but as part of their lessons. Despite the concerns, 89% of teachers understand and support CT development in the classroom.

The quantitative data in this section are triangulated with the qualitative data in sections 6.1.1 and 6.2.2. The data triangulation exemplifies how the teachers perceive CT towards AR development in their teaching and learning practices. Additionally, all the qualitative data from the focus group interviews showed how learners respond to CT questions and tasks. The qualitative and analysis triangulation has improved the teacher's understanding of CT. The data was triangulated to strengthen the quality of the data analysis.

6.4 Summary

This chapter aimed to provide data analysis of the qualitative and quantitative data to answer the research questions. The data was triangulated to strengthen the quality of the data analysis. Qualitative data, such as pre-and post-interviews and lesson observations and quantitative data, such as focus group interviews and post-reflective questionnaires, were triangulated to answer the research questions. In conclusion, in this chapter, all the findings show that teachers understand the importance of CT for AR development. The next chapter presents a summary, main findings and conclusions, limitations and pedagogical implications, and study suggestions.

Chapter 7: CONCLUSION AND RECOMMENDATION

This chapter provides a summary of the chapter in Section 7.1, followed by the summary of findings, which includes conclusions of the research study and answers to the research questions in Section 7.2. Finally, the study's limitations are highlighted in Section 7.3 and recommendations for further research are presented in Section 7.4.

7.1 Purpose and Summary of Chapters of the Study

This study aimed to investigate how teachers use CT to develop AR in the intermediate phase when solving problems. An extensive literature review was conducted on critical thinking, algebra and algebraic reasoning to answer the primary research question, "How do teachers use critical thinking to develop algebraic reasoning in the intermediate phase when they teach?"

Chapter 1: presented the background and motivation for the study, the problem statement with the research question and sub-research questions, the basic concepts for the study, and a brief overview of the research methodology adopted in the research.

Chapter 2: discusses algebra as a mathematical language and different curricula internationally and locally; the chapter highlights algebra's characteristics towards AR in the IP. Moreover, the theory of algebra as a 'way of thinking helps to understand AR development's thinking processes. Furthermore, literature that emphasises the need for IP learners to think about improving their AR is also discussed.

Chapter 3: presented research on thinking to understand CT. The chapter gave a clear view of thinking and distinguished CT from thinking. The chapter also discussed the CT theory from the teacher's perspectives as facilitators and learners as participants in nurturing CT. This way of thinking helps to understand CT in connection to AR, explained by their characteristics. Further, this will help form the methodology and research designs that underpin the research to inform data analysis.

In Chapter 4:, research on the theoretical framework has helped to highlight the constructivism theory, which is the anchor of the study. The constructivism theory looked at how it can be implemented in the classroom, considering the teacher and learner roles contributing to a constructive learning environment. Furthermore, the theory of algebra as a 'way of thinking helps to understand AR development's thinking processes. Furthermore, that emphasises the need for IP learners to think about improving their AR. Finally, this way of thinking is discussed in the next chapter. Further, this will help form the methodology and research designs that underpins the research to inform data analysis.

Chapter 5: discussed the research design and methodology used for the study. A mixed-method approach was chosen for the study to answer the research questions. The chapter gave a detailed overview of the instruments for data collection, the validity and reliability, and how they will each answer the research question. Additionally, the issue of the participants was discussed in consideration of ethics. Further, this will help form the methodology and research designs that underpins the research to inform data analysis.

Chapter 6: aimed to provide data analysis of the qualitative and quantitative data to answer the research questions. The data was triangulated to strengthen the quality of the data analysis. Qualitative data (pre- and post-interviews, lesson observations), and quantitative data (focus group interviews and post-reflective questionnaires), were triangulated to answer the research questions. In conclusion, in this chapter, all the findings show that teachers understand the importance of CT for AR development. The next section presents a summary, main findings and conclusions, limitations and pedagogical implications, and suggestions for further study.

7.2 Summary of findings

The study aimed to answer this main research question:

How do teachers use critical thinking to develop algebraic reasoning in the intermediate phase when they teach?

Secondary questions: The study also addresses the following secondary questions:

a. What do we know about practising IP teachers' understanding of AR?
b. What are the issues these teachers face?
c. What types of tasks are suitable to generate data on AR and CT in IP?
d. What ways to design AR tasks that can potentially foster CT and AR?
e. How do learners respond to CT questions for AR development in different grades?

The summary of findings is presented in five sections in the same way as the qualitative and quantitative data presentation and analyses were arranged.

7.2.1 What do we know about practising IP teachers' understanding of AR?

The IP teachers are teachers who teach grades between Grade 4 and 6. IP teachers' understanding of AR was understood through qualitative and quantitative data. The pre-and-post interviews helped me understand more about how teachers view AR. In the beginning, teachers thought CT was to think out of the box without considering the process involved in AR's development. The presentation helped to shape teachers' perceptions of AR through the presentation. That AR development gives learners a problem-solving task where they look for patterns and generalise and justify their work. The lesson observation has contributed to the study of how teachers now guide learners in AR tasks to the development of AR. Therefore, the teachers created an instructional guide that draws learners holistically to reflect, explore, hypothesise, generate, and apply the concepts when engaging with their learning (Blaschke & Hase 2019 & Uday 2019). Other researchers agree that an effective mathematics classroom is where both teacher and learner engage in math activities that involve solving problems through logical reasoning, justifying procedures and solutions, employing multiple representations of concepts, and making connections between math and everyday life (Cohen and Ball 2001; Donovan and Bransford 2005; Hiebert 2005; Schielack et al. 2006; Stigler and Hiebert 1999).

7.2.2 The issues that teachers in the IP phase encounter

Time constraints because they cannot dwell much on the Pattern task as it may be time-consuming. The other issue is that teachers tend to think that if their learners struggle with topics before pattern topics, such as addition, subtraction, multiplication and division, they will hinder the introduction of numeric and geometric patterns. So the issue is that teachers treat mathematical topics as disconnected, not considering that pattern tasks can be used to sharpen learners' skills in addition, subtraction, multiplication and division. Moreover, some teachers brush through this topic, especially regarding pattern tasks, because learners struggle to interpret what they are analysing and justify their thinking. Language is one of the issues the IP phase teachers face, even though it is considered one of the resources that creates meaning when learning mathematics (O'Halloran, 2015). Therefore, the teachers need to continuously motivate and support learners to use the correct mathematical terminology.

7.2.3 The tasks suitable for generating data on AR and CT in IP

The literature study was the first drive to make the task more suitable for generating data on AR and CT. It is a task that allows learners to analyse information and then present their work through formalising a pattern to generalise and justify findings. Van de Walle, Karp & Bay-Williams agreed that AR's primary focus is more on patterns and functions and being able to analyse situations using manipulating symbols to support and communicate thinking (2011, p.262). Drawing on this explanation, the researcher chose to select the task in light of the CAPS curriculum on the content of Pattern, Functions and Algebra. Therefore, the task needs to provide learners with an opportunity to analyse, generalise and justify AR and CT development.

7.2.4 A design of AR tasks to potentially foster CT and AR

A task designed to foster CT and AR potentially has to incorporate the three reasoning trajectories: analysing, generalising and justifying (Driscoll, 1999). Romberg and Kaput (1999) suggest that when designing a task that will foster CT and AR, one considers the five questions:

- Does the task lead anywhere?
- Does the task lead to a model building?
- Does the task lead to inquiry and justification?
- Does the task involve the flexible use of technologies?
- Is the task relevant to students?

These questions can serve as an instrumental tool for ensuring that the task design supports analysing, generalising and justifying ensuring that they foster CT and AR.

7.2.5 Response of learners to CT questions for AR development in different grades

The learner response to focus group interviews in Section 6.2 revealed some of the learner's poor reasoning skills towards AR development, where they had to analyse, justify and generalise patterns. The main challenge for most learners was generalising, and communicating the pattern's rule was a problem for some learners. According to the findings of Akkan and Cakıroğlu (2012), learners are more successful in solving number sequence patterns than visual patterns, so this finding corresponds with this study. Hence, the performance level in generalisation for Tasks 3 and 4 was not impressive. Analysis of learners' responses in Section

6.2.5 shows they can analyse and justify but need to be given more opportunities to develop their generalising reasoning skills.

7.3 Limitations to the study

As mentioned in Section 1.10, the study's major limitation is the sample size of Grade 5 and 6 learners and the number of schools. First, the limit was set to 15 learners per grade to attain learner responses to CT questions, as the study focused more on teachers than learners. In the IP, there were only 11 teachers that participated in the study. The second limitation was that the study was limited to one district in the Western Cape and to only three schools. Lastly, the study focused on one of the mathematics strands in the Patterns, Functions and Algebra content area. Lastly, time is one of the most significant research limitations, and the lesson observation was limited to one week per school, one day for the introduction and the other four days for the pattern tasks.

7.4 Recommendations

This study was only limited to the development of AR in the IP and cannot claim implementation across all phases. Additionally, the study was limited to the IP phase teachers to evaluate their understanding of CT and AR development.

Future studies could include:

- An empirical study on teachers' understanding of critical thinking (CT) and implementation in teaching and learning mathematics.
- An analysis of Systemic Test results to evaluate learners' response to pattern questions between the Grade 6 learners and compare it with Grade 9 learners. In this case, compare two concluding grades of the Senior phase (Grades 7–9) and IP. Or compare Grade 3 in the Foundation Phase and Grade 6 to see how learners have progressed in their AR development.

REFERENCES

Aizikovitsh, E., and Amit, M. (2010). Evaluating an infusion approach to the teaching of critical thinking skills through mathematics. Procedia-Social and Behavioral Sciences, 2 (1), 3818–3822.

Akkan Y. and Cakıroğlu U. (2012). Doğrusal ve Đkinci Dereceden Oruntuleri Genellestirme Stratejileri: 6-8. Sınıf Oğrencilerinin Karsılastırılması. Eğitim ve Bilim, 37 (165), 184-194.

Artzt, A. F., & Yaloz-Femia, S. (1999). Mathematical Reasoning during Small-Group Problem Solving. In L. V. Stiff, & F. R. Curcio (Eds.), Developing Mathematical Reasoning K-12 Yearbook (pp. 115-126). Reston, VA: National Council of Teachers of Mathematics.

Ashton, J., & Elliott, R. (2007). Juggling the balls – study, work, family, and play: Student perspectives on flexible and blended heutagogy. European Early Childhood Education Research Journal, 15(2), 167-181.

Ashton, J., & Newman, L. (2006). An unfinished symphony: 21st-century teacher education using knowledge-creating pedagogies. British Journal of Educational Technology, 37(6), 825-840.

Australian Academy of Science and Australian Association of Mathematics Teachers [AAMT]. (2017). reSolve: Mathematics by inquiry. Assessing Reasoning Special Topic 5. http://www.resolve.edu.au/ Canberra: Australian Government Department of Education and Training.

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2012). The Shape of the Australian Curriculum. URL: http://www.acara.edu.au/verve/_resources/the_shape_of_the_australian_curriculum_v4.pdf (accessed 18 February 2016).

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2013b). Critical and Creative Thinking Learning Continuum. URL: http://www.acara.edu.au/verve/_resources/General_capabilities_-CCT__learning_continuum.pdf (accessed 18 February 2016).

Bakker, A. 2004. Design Research in Statistics Education: On Symbolizing and Computer Tools. Dissertation Utrecht University. Utrecht: CD-β Press, Center for Science and Mathematics Education.

Barlow A.T., Prince K.M., Lischka A.E., Duncan M.D. (2017) Developing Algebraic Reasoning through Variation in the US In: Huang R., Li Y. (eds) Teaching and Learning Mathematics through Variation. Mathematics Teaching and Learning. Sense Publishers, Rotterdam

Bastable, V., & Schifter, D. (2008). Classroom stories: Examples of elementary students engaged in early algebra. In J. Kaput, D. Carraher, & M. Blanton (Eds.), Algebra in the early grades (pp. 165 - 184). Mahwah, NJ: Lawrence Erlbaum.

Bazeley, P. (2016). Mixed or merged? Integration as the real challenge for mixed methods. Qualitative Research in Organizations and Management: An International Journal.

Beatty, R. & Bruce, C. (2012) Linear Relationships: From Patterns to Algebra. Toronto: Nelson Publications, Canada. ISBN-13: 9780176519698 (1 Research Text and 1 Practice Text) ISBN-13: 9780176519674 (DVD)

Bertram, C., & Christiansen, I. (2014). Understanding research. An introduction to reading research. Pretoria: Van Schaik Publishers.

Biesta, G. (2017). Mixing methods in educational research. Research methods and methodologies in education, 159-165.

Blanton, M. L. (2008). Algebra and the elementary classroom. Transforming thinking, transforming practice. Portsmouth, NH: Heinemann.

Blanton, M., & Kaput, J. (2003). Developing elementary teachers' algebra eyes and ears. Teaching Children Mathematics, 10(2), 70-77.

Blanton, M., & Kaput, J. (2005). Characterising a classroom practice that promotes algebraic Reasoning. Journal for Research in Mathematics Education, 36 (5), 412-446.

Blanton, M., Stephens, A., Knuth, E., Isler, I., Kim, J. (2015). The development of Children's Algebraic Thinking: The Impact of a Comprehensive Early Algebra Intervention in Third Grade. Journal for Research in Mathematics Education, 46, 39-87.

Blanton, M.L. & Kaput, J.J. (2004). Functional thinking as a route into algebra in the elementary grades. In Cai, J. & Knuth, E. (eds). *Early Algebraization. Advances in Mathematics Education*. Berlin: Springer: 5-23.

Blaschke, L. M. (2012). Heutagogy and lifelong learning: A review of heutagogical practice and self-determined learning. The International Review of Research in Open and Distributed Learning, 13(1), 56-71.

Blaschke, L. M., & Hase, S. (2019). Heutagogy and digital media networks. Pacific Journal of Technology Enhanced Learning, 1(1), 1-14.

Blaschke, L.M. (2012). Heutagogy and Lifelong Learning: A Review of Heutagogical Practice and Self-Determined Learning. European Journal of Open, Distance, and ELearning (EURODL)

Blaschke, L.M., & Hase, S. (2016). Heutagogy: A Holistic Framework for Creating Twenty-First-Century Self-determined Learners.

Bloom, B. (Ed.) (1956). Taxonomy of educational objectives: Book I, cognitive domain. New York: Longman Green.

Bruner, J. S. (1966). Toward a theory of instruction, Cambridge, Mass.: Belkapp Press.

Burke, A. (2011). Group work: How to use groups effectively. Journal of Effective Teaching, 11(2), 87-95.

Butera, G., Amber Friesen, Palmer, S.B., Lieber, J., Horn, E.M., Hanson, M.J, and Czaja, C. (2014). Integrating Mathematics Problem Solving and Critical Thinking Into the Curriculum. Young Children (March), 70-77

Butler, H. A., Dwyer, C. P., Hogan, M. J., Franco, A., Rivas, S. F., Saiz, C., & Almeida, L. S. (2012). The Halpern Critical Thinking Assessment and real-world outcomes: Cross-national applications. Thinking Skills and Creativity, 7(2), 112-121.

Cahyono,B., Kartono, Waluyo, B., Mulyono. (2019). Analysis critical thinking skills in solving problems algebra in terms of cognitive style and gender. Journal of Physics: Conference Series, Volume 1321(2), 1-7.

Cai, J., Fong Ng, S., & Moyer, J. C. (2011). Algebraic thinking in earlier grades: lessons from China and Singapore. In J. Cai, & E. Knuth (Eds.), Early algebraisation: A global dialogue from multiple perspectives. (pp. 25-42). Heidelberg: Springer.

Cai, Jinfa, Hee Chan Lew, Anne Morris, John C. Moyer, Swee Fong Ng, and Jean Schmittau. (2005) "The Development of Students' Algebraic Thinking in Earlier Grades: A Cross-Cultural Comparative Perspective". Zentralblatt fuer Didaktik der 37 pp. 5-15.

Cannella, G. S., & Reiff, J. C. (1994). Individual constructivist teacher education: Teachers as empowered learners. Teacher education quarterly, 27-38.

Canning, N. & Callan, S. (2010). Heutagogy: Spirals of reflection to empower learners in higher education. Reflective Practice, 11(1), 71-82.

Canning, N. (2013). Practitioner development in early years education. In S. Hase & C. Kenyon, Self-determined learning: Heutagogy in action. Sydney, Australia: Bloomsbury Academic.

CAPS: Department of Education. (2010). Curriculum and Assessment Policy for Mathematics Intermediate Phase. Pretoria: Department of Education.

CAPS: Department of Education. (2011). Curriculum and Assessment Policy for Mathematics Further Education Training Phase. Pretoria: Department of Education.

CAPS: Department of Education. (2014). Curriculum and Assessment Policy for Mathematics Further Education Training Phase. Pretoria: Department of Education.

Carpenter, T.P. & Levi, L. (2000). Developing Conceptions of Algebraic Reasoning in the Primary Grades. NCISLA, Wisconsin Center for Education Research, University of Wisconsin, School of Education. Research Report.

Carson, J. (2007). A Problem With Problem Solving: Teaching Thinking Without Teaching Knowledge. The Mathematics Educator, (17), pp.7-14.

Chen, S. 2008. Primary Singapore mathematics, a revolutionary program designed to improve student mathematics achievement: what is Singapore maths?

Cohen, D.K. and Ball, DL (2001) “Making Change: Instruction and Its Improvement.” Phi Delta Kappan 83, pp. 73–77.

Cole, M., & Wertsch, J. V. (1996). Beyond the individual-social antinomy in discussions of Piaget and Vygotsky. Human Development, 39(5), 250–256. https://doi.org/10.1159/000278475

Creswell, J. W. (2003). Research design: qualitative, quantitative, and mixed method approaches. Thousand Oaks, CA: Sage Publications

Creswell, J. W. (2014). Research Design: Qualitative, Quantitative and Mixed Methods Approaches (4th ed.). Thousand Oaks, CA: Sage.

Creswell, J. W., & Plano-Clark, V. L. (2007). Designing and conducting mixed methods research. Thousand Oaks, Calif: SAGE Publications.

Davydov, V. V., Gorbov, S. F., Mikulina, G. G., Saveleva, O. V. (1999). Mathematics Class 1. J. Schmittau (Ed.). Binghamton, NY: State University of New York.

Davydov, V. V.; Gorbov, S. F.; Mikulina, G. G.; Saveleva, O. V. (1999): Mathematics: class 1. J. Schmittau (Ed.) - Binghamton, NY: State University of New York

De Bono, E. (1976). Teaching thinking. Middlesex, England: Penguin Books Ltd.

Denzin, N. K. (1978). The research act: A theoretical introduction to sociological methods. New York: McGraw-Hill.

Denzin, N. K. (2009). The research act: A theoretical introduction to sociological methods (3rd ed.). Englewood Cliffs, NJ: Prentice Hall.

Denzin, N., & Lincoln, Y. S. (Eds.). (2005). Introduction: The Discipline and Practice of Qualitative Research. In The Sage Handbook of Qualitative Research (3rd ed., pp. 1-32). Thousand Oaks, CA: Sage Publications.

Denzin, N.K (1989) The Research Act. Third Edition. New York, McGraw-Hill.

Denzin, N.K. and Lincoln, Y.S. (2000) The Discipline and Practice of Qualitative Research: Handbook of Qualitative Research, Thousand Oaks.

Department of Basic Education(DBE). (2011). Curriculum and Assessment Policy Statement Grades 4-6. Mathematics. Pretoria: Department of Basic Education.

Devlin, K. (1994). Mathematics: The Science of Patterns. New York, NY: Scientific American Library, 1994

Dick, B. (2013). Crafting learner-centred processes using action research and action learning. In S. Hase & C. Kenyon (Eds.), Self-determined learning: Heutagogy in action. Bloomsbury Academic: London.

Doğanay, A., Ü. Figen. (2006). Eleştirel Düşünmenin Öğretimi. Ali, Şimşek (Editor) Teaching Based on the Type of Content. Ankara. Nobel Yayın-Dağıtım.

Donald, D., Lazarus, S., & Lolwana, P. (2010). Educational Psychology in Social Context: Ecosystemic application in Southern Africa. Fourth Edition. Oxford University Press Southern Africa (Pty) Ltd.

Donovan, M.S and Bransford J.D. (2005)How Students Learn. Mathematics in the Classroom. Washington, DC: The National Academies Press.

Driscoll, M.J. (1999). Fostering algebraic thinking: A guide for teachers, grades 6-10. Portsmouth, NH: Heinemann. [p.2]

Edgar Dale's cone of experience http://teachernoella.weebly.com/dales-cone-of-experience.html Available at (07.09.2017)

Eisenhardt, K. M., & Graebner, M. E. (2007). Theory building from cases: Opportunities and challenges. The Academy of Management Journal, 50(1), 25-32.

Ellis, A. K. (2005). Research on educational innovations (4th ed.). Larchmont, NY: Eye on Education.

Ennis, R.H. (1996). Critical Thinking. Upper Saddle River, New Jersey: Prentice Hall, Inc

Epp, S. S. (2011). Variables in mathematics education. In Springer Publishing, 6, pp. 54–61.

Fisher, R. (1990). Teaching Children To Think. London: Stanley Thornes Publishers Ltd.

Fong, N. S. (2004). Developing algebraic thinking in early grades: A case study of the Singapore primary mathematics curriculum. The Mathematics Educator, 8(1), 39–59.

Fosnot, C. (1996). Constructivism: A psychological theory of learning. In C. T. Fosnot (Ed.), Constructivism:Theory, perspectives, and practice. New York: Teachers College Press.

Freire, P. (1972). Pedagogy of the oppressed. Harmondsworth: Penguin.

Gaskaree, B. L., Mashhady, H., & Dousti, M. (2010). Using critical thinking activities as tools to

Gay, L., Mills, G., & Airasian, P. (2012). Overview of qualitative research. Educational research: Competencies for analysis and applications.

Glassmeyer, D. & Edwards, B. (2016). How Middle-Grade Teachers Think about Algebraic Reasoning. Mathematics Teacher Education and Development (Vol. 18), pp.92 – 106.

Grant, C., & Osanloo, A. (2014). Understanding, selecting, and integrating a theoretical framework in dissertation research: Creating the blueprint for your “house.” Administrative Issues Journal: Connecting Education, Practice, and Research, 4(2), 12–26.

Green, J., & Thorogood, N. (2018). Qualitative methods for health research. sage.

Green, L. 2001. Promoting development in the middle childhood. In Engelbrecht, P. & Green, L (Eds.). Promoting learner development: Preventing and working with barriers to learning. Pretoria: Van Schaik

Greene, J. C., Caracelli, V., & Graham, W. F. (1989). Toward a conceptual framework for mixed methods evaluation designs. Educational Evaluation and Policy Analysis, 11, 255–274.

Guba, E. G., & Lincoln, Y. S. (1994). Competing paradigms in qualitative research. Handbook of qualitative research, 2(163-194), 105.

Hanekom, P. (2019). Designing a whatsapp VCOP mpdel to support the effectiveness if blended-learning teacher professional learning sciences short courses, 49.

Hase, S., & Kenyon, C. (2007). Heutagogy: A Child of Complexity Theory. Complicity: An International Journal of Complexity and Education, 4, 111-118.

Hassan, K., Skelton, C., & Smit, S. (2002) Solutions for All. Mathematics Grade 6. Learners Book. Macmillan. (pg.5-6)

Hegarty, B., & Thompson, M. (2019). A teacher's influence on student engagement: using smartphones for creating vocational assessment portfolios. *Journal of Information Technology Education: Research*, *18*, 113–139.

Henningsen, M., and Stein, M.K. (1997). Mathematical tasks and student cognition: classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. Journal for Research in Mathematics Education, 25(5), 524-549.

Hewitt, D. (2009). From before birth to beginning school. In J. Houssart, & J. Mason (Eds.), Listening counts: Listening to young learners of mathematics. (pp. 1-16). Staffordshire, UK: Trentham Books Limited.

Hiebert, E. H. (2005). The effects of text difficulty on second graders' fluency development. Reading Psychology, 26(2), 183-209.

Hiebert, J., & Carpenter, T.P. (1992). Learning and teaching with under- standing. In: D. A. Grouns (Ed.), Handbook of research on mathematics teaching and learning (pp. 65-92). New York: Macmillan.

Hodgen, J., Küchemann, D., & Brown, M. (2010). Textbooks for the teaching of algebra in lower secondary school: are they informed by research? Pedagogies, 5(3), 187-201.http://www.ccds.org/customized/uploads/files/sing_math_ccds.pdf [9 May 2015]

Holstein, J. A., & Gubrium, J. F. (2011). Animating interview narratives. In D. Silverman (Ed.), Qualitative research: Issues of theory, method and practice (3rd ed., pp. 149–167). SAGE Publications Ltd.

Jarvis, P. (1990). International Dictionary of Adult and Continuing Education. Routledge: London.

Jarvis, P. (1995). Adult and continuing education: theory and practice. Second Edition. London: Routledge.

Jaworski, B., & Huang, R. (2014). Teachers and didacticians: Key stakeholders in the processes of developing mathematics teaching. ZDM, 46(2), 173-188.

Kamenetz, A. (2010). Edupunks, edupreneurs, and the coming transformation of higher education. Canada: Chelsea Green Publishing Company.

Kaput, J. J. (2008). What is algebra? What is algebraic Reasoning? In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), Algebra in the early grades (pp. 235–272). New York: Lawrence Erlbaum Associates

Kaput, J. J., Blanton, M. L., & Moreno, L. (2008). Algebra from a symbolisation point of view. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), Algebra in the early grades. (pp. 19-56). New York.: Lawrence Erlbaum Ass.

Kaput, J.J (2000). Teaching and Learning a New Algebra with Understanding. University of Massachusetts Dartmouth.

Katz-Buonincontro J, Anderson RC. How Do We Get From Good to Great? The Need for Better Observation Studies of Creativity in Education. Front Psychol. 2018 Nov 27;9:2342. doi: 10.3389/fpsyg.2018.02342. PMID: 30538657; PMCID: PMC6277490.

Kelly, G.A., (1991). The psychology of personal constructs: Volume one - A theory of personality. London: Routledge.

Kerry, T. (2013). Applying the principles of heutagogy to a postgraduate distance-learning program. In S. Hase & C. Kenyon (Eds.), Self-determined learning: Heutagogy in action. London: Bloomsbury

Kieran C. (2004a) The Core of Algebra: Reflections on its Main Activities. Kluwer Academic Publishers. The Future of the Teaching and Learning of Algebra The 12thICMI Study. New ICMI Study Series, vol 8 (p.21 – 40).

Kieran, C. (2004b). Algebraic thinking in the early grades: What is it? The Mathematics Educator 8, No. 1, 139-151.

Kieran, C. (2011). Overall commentary on early algebraisation: Perspectives for research and teaching. In. J. Cai & E. Knuth (Eds.) Early algebraisation. A global dialogue from multiple perspectives (pp. 579–593). London: Routledge.

Kieran, C., Pang, J., Schifter, D., & Ng, S. F. (2016). Early algebra: Research into its nature, its learning, its teaching. In G. Kaiser (Ed.), ICME-13 Topical surveys. Springer Open. https://doi.org/10.1007/978-3-319-32258-2

Kivunja, C., & Kuyini, A. B. (2017). Understanding and applying research paradigms in educational contexts. International Journal of Higher Education, 6(5), 26. https://doi.org/10.5430/ijhe.v6n5p26

Knowles, M.S. (1973): The adult learner: A neglected species. Houston: Gulf Publishing Company.

Knowles, M.S. (1984). The adult learner: a neglected species. Houston Gulf.

Kong, SC (2015). An experience of a three-year study on the development of critical thinking skills in flipped secondary classrooms with pedagogical and technological support. Computers and Education, 89 (1), 16–31.

Krahenbuhl, K. S. (2016). Student-centred education and constructivism: Challenges, concerns, and clarity for teachers. The Clearing House: A Journal of Educational Strategies, Issues and Ideas, 89(3), 97-105

Kriegler, S. (2008). Just what is algebraic thinking. UCLA: Department of Mathematics. [Online]. Available: http//www.math.ucla.edu/kriegler/ pub/ algebra

Krulik, S., and Rudnick, J.A. (1995). The New Sourcebook for Teaching Reasoning and Problem-Solving in Elementary School. Needham Heights: Allyn dan Bacon.

Kwan, Y. W., & Wong, A. F. (2014). The constructivist classroom learning environment and its associations with critical thinking ability of secondary school students in Liberal Studies. Learning Environments Research, 17(2), 191-207.

Lannin, J. K. (2009). Generalisation and justification: The challenge of introducing algebraic reasoning through patterning activities. Mathematical thinking and learning, 7(3), 231-258.

Leech, N. L., & Onwuegbuzie, A. J. (2008). A typology of mixed methods research designs.

Lew, H.-C. (2004). Developing Algebraic Thinking in Early Grades: Case Study of Korean Elementary School Mathematics 1. In *The Mathematics Educator* 8(1),88-89.

Lins, R., & Kaput, J. (2004). The early development of algebraic reasoning: The current state of the field. In H. Chick & K. Stacy (Eds.), The Future of the Teaching and Learning of Algebra: The 12th ICMI Study. London: Kluwer.

Lipman, M. (1985). Philosophy and the Cultivation of Reasoning. _Thinking: The Journal of Philosophy for Children_ 5 (4):33-41.

Lockhart, P. (2002). A mathematician's lament. Retrievedfromwww.maa.org/devlin/ LockhartsL

Maluleke, M. J. 2019. Using Code-Switching as an Empowerment Strategy in Teaching Mathematics to Students with Limited Proficiency in English in South African Schools. South African Journal of Education, 39(3). Available at: http://search.ebscohost.com.ezproxy.iielearn.ac.za/login.aspx?direct=true&db=eric&AN=EJ1228974&site=ehost-live.

Marri, A.R. (2005). Building a Framework for Classroom-Based Multicultural Democratic Education: Learning From Three Skilled Teachers. Teachers College Record, 107, 1036-1059.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee. (Eds.), Approaches to algebra: Perspectives for learning and teaching (pp. 65-86). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Mason, J. (2008). Making use of children's powers to produce algebraic thinking. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), Algebra in the early grades. (pp. 57-94). New York: Lawrence Erlbaum Ass.

Mason, J. (2011). Commentary on part III. In J. Cai & E. Knuth (Eds.) Early algebraisation. A global dialogue from multiple perspectives (pp. 557–577). Berlin: Springer.

Mason, J., Burton, L., and Stacey, K. (2010). Thinking Mathematically, 2nd edition. London: Pearson Education Limited.

Mathis, W. (2013). Twenty-first century skills and implications for education. Research based options for education policymaking. http://nepc.colorado.edu.

Mathison, S. (1988). Why triangulate? Educational Researcher, 17(2), 13-17.

McGuinnes, C., Sheey, N., Curry, C., and Eakin, A. (2003). ACTs II Sustainable Thinking in Classrooms: A Methodology for Enhancing Thinking Across the Curriculum. Materials available from Professor C. McGuinness, School of Psychology, Queen's University, Belfast, Northern Ireland

Merizow, J. (2001). Learning as transformations: Critical perspectives on a theory in progress. Francisco: Jossey-Bass.

Merriam S. B., Tisdell E. (2016). Qualitative Research: A Guide to Design and Implementation, 4th Edn. San Francisco, CA: Jossey-Bass.

Merriam, S. (1998). Qualitative research and case study application in education. San Francisco: Jossey-Bass.

Merriam, S.B. & Caffarella, R.S. (1991). Learning in adulthood. San Francisco: Jossey-Bass.

Mitchener, W.G. (2016). Nature of Mathematics. A Report to the Nation on the Future of Mathematics Education. ©1989 by the National Academy of Sciences. Courtesy of the National Academy Press, Washington, DC.

Moeller, M., Cutler, K., Fiedler, D. & Weier, L (2013). Visual thinking strategies = creative and critical thinking. Phi Delta Kappan ,95 (3), 56-60.

Moonpo, P., Inprasitha, M., & Changsri, N. (2018). Algebraic Reasoning in Early Grade: Promoting through Lesson Study and Open Approach. Psychology, 09(06), 1558–1569.

Mulligan, J. Cavanagh, M. & Keanan-Brown, D. (2012). The Role of Algebra and Early Algebraic Reasoning in the Australian Curriculum: Mathematics. Engaging the Australian national curriculum: Mathematics -- perspectives from the field (pp. 47-70).

National Council of Teachers of Mathematics (NCTM) (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). 2000. Principles and standards for school mathematics. Reston, Va.: NCTM.

Neuman, W. (2011). Social Research Methods: Qualitative and Quantitative Approaches. Boston: Pearson Education.

Noddings, N., & Brooks, L. (2016). Teaching controversial issues: The case for critical thinking and moral commitment in the classroom. Teachers College Press.

O'Halloran, K. L. (2015). The language of learning mathematics: A multimodal perspective. The Journal of Mathematical Behavior, 40, 63-74.

Ontario Ministry of Education. (2014). K-12 Paying attention to Fractions Support Document for Paying Attention to Mathematics Education. Retrieved from http://www.edu.gov.on.ca/eng/literacynumeracy/PayingAttentiontoAlgebra.pdf

Paul, R., and Elder, L. (2002). Critical thinking: tools for taking charge of your learning and life. Upper Saddle River, N.J.: Prentice-Hall.

Peters, O. (2001). Learning and teaching in distance education: Analyses and interpretations from an international perspective (2nd ed.). London: Kogan Page

Peters, O. (2004). Distance education in transition - New trends and challenges (4th ed., Volume 5). Oldenburg, Germany: Bibliotheks- und Informationssystem der Universität Oldenburg.

Phan, H. P. (2010). Critical thinking as a self-regulatory process component in teaching and learning. Psicothema, 22(2), 284–292.

Piaget, J. (1936). Origins of intelligence in the child. London: Routledge & Kegan Paul.

Piaget, J. (1953). The origins of intelligence in children. New York, NY: Basic Books.

Piaget, J. (1977). The development of thought: Equilibration of cognitive structures. (A. Rosin, Trans). New York: The Viking Press.

Pithers, R.T. & Soden, R. (2000) Critical thinking in education: a review. University of Strathclyde, Glasgow, UK. Educational Research, 42:3, 237-249, DOI: 10.1080/001318800440579

Poni, M. (2014). Research paradigms in education. Journal of Educational and Social Research, 4(1), 407.

Radford, L. (1996). The roles of geometry and arithmetic. In N. Bednarz, C. Kieran & L. Lee (Eds.) Approaches to algebra (pp. 39–53). Dordrecht: Kluwer Academic Publishers.

Radford, L. (2007). Iconicity and Contraction: a semiotic investigation of form of algebraic generalisations of patterns in different contexts. ZDM Mathematics Education. [Online]. Available: DOI 10.1007/s11858-007-0061-0

Rajendran, N.S. (2010). Teaching and Acquiring Higher Order Thinking Skills: Theory and Practice. Tanjong Malim, Perak: Penerbit Universiti Pendidikan Sultan Idris

Resnick, D. P., & Resnick, L. B. (1982). Standards, Curriculum, and Performance: A Historical and Comparative Perspective A report to the National Commission on Excellence in Education.

Resnick, L. (1987). Education and learning to think. Washington, DC: National Academy Press.

Robert K. Yin. (2014). Case Study Research Design and Methods (5th ed.) . Thousand Oaks, CA: Sage. 282 pages. (ISBN 978-1-4522-4256-9) .

Roberts, N. (2012) Patterns, functions, and algebra in the South African curriculum: Towards more detail for South African teachers. In S. Nieuwoudt, D. Laubscher, H. Dreyer Proceedings of the Eighteenth National Congress of the Association for Mathematics Education of South Africa, 24-28 June 2012, North West University, Potchefstroom. pp. 302 – 319

Robins, A., Ashbaker, B., Enriquez, J. and Morgan, J. (2003) Learning to reflect: professional practice for professionals and paraprofessionals. International Journal of Learning, 10: 2555–65.

Rodgers, D., & Dunn, M. (1997). And never the twain shall meet: One student’s practical theory encounters constructivist teacher ed practices. Journal of Early Childhood Teacher Education, 18(3), 10-25.

Romberg, T. A., & Kaput, J. J. (1999). Mathematics worth teaching, mathematics worth understanding. In Mathematics classrooms that promote understanding (pp. 15-30). Routledge.

Roohi, F. (2015). Role of Mathematics in the Development of Society, [Online], 2015, Retrieved from http://www.ncert.nic.in/pdf_files/Final-Article-Role%20of%20 Mathematics%20 in% 20the%20Development%20ofSocietyNCER-.pdf

Rosita, N. T. (2018, March). Analysis of algebraic reasoning ability of cognitive style perspectives on field dependent field independent and gender. In Journal of Physics: Conference Series (Vol. 983, No. 1, p. 012153). IOP Publishing.

Schielack, J.F. et al. (2006) Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence. Reston, VA: The National Council of Teachers of Mathematics.

Schifter, D., Bastable, V., Russell, S. J., Seyferth, L., & Riddle, M. (2008). Algebra in the grades K-5 classroom: Learning opportunities for students and teachers. In C. E. Greenes, & R. Rubenstein (Eds.), Algebra and algebraic thinking in school mathematics. (pp. 263-278). Reston, VA: NCTM.

Scriven, M. & Paul, R. (1996). Defining critical thinking: A draft statement for the National Council for Excellence in Critical Thinking. [On-line]. Available HTTP: http://www.criticalthinking.org/University/univlibrary/library.nclk

Senk, S. L., Beckmann, C. E., & Thompson, D. R. (1997). Assessment and grading in high school mathematics classrooms. Journal for Research in Mathematics Education, 28(2), 187-215

Septiani Y. Maudy, Didi, S., & Endang, M. (2018). Student Algebraic Thinking Level. Internal Journal of Information and Education Technology, 8(9), pp. 672 – 676.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. Educational Studies in Mathematics, 22(1), pp.1-36.

Shah, R. K. (2019). Effective constructivist teaching learning in the classroom. Shah, RK (2019). Effective Constructivist Teaching Learning in the Classroom. Shanlax International Journal of Education, 7(4), 1-13.

Sheridan, D. (1993). Teaching secondary English: Readings and applications. New York: Longman.

Siegler, R. S., & Lin, X. (2010). Self-explanations promote children's learning. In H. S. Waters & W. Schneider (Eds.), Metacognition, strategy use, and instruction (pp. 85–112). Guilford Press.

Smith, E. (2008). Representational thinking as a framework for introducing functions in the elementary curriculum. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), Algebra in the early grades (p. 143). Mahwah, NJ: Lawrence Erlbaum Associates/Taylor &Francis Group

Steen, L. (1999). Twenty questions about mathematical reasoning. In L. V. Stiff (Ed.), Developing mathematical reasoning in grades K-12: 1999 Yearbook (pp. 270-285). Reston, VA: National Council of Teachers of Mathematics.

Stein M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. Educational Research and Evaluation, 2(1), 50-80.

Sternberg, R. (1999). The nature of mathematical reasoning. In L. Stiff (Ed.), Developing mathematical reasoning in grades K-12: 1999 Yearbook, (pp. 37-44). Reston, VA: National Council of Teachers of Mathematics.

Stewart, K.L., Felicetti, L.A. (1992). Learning styles of marketing majors. *Educational Research Quarterly*, 15(2), 15-23.

Stigler, J.W. and Hiebert, J.(1999)The Teaching Gap: Best Ideas from the World's Teachers for Improving Education in the Classroom. New York, NY: The Free Press.

Su, H.F., Ricci, F.A., & Mnatsakanian, M. (2016). Mathematical teaching strategies: Pathways to critical thinking and metacognition. Journal of Research in Education and Science (IJRES), 2 (1), 190-200

Sun, J., & Van Es, E. A. (2015). An exploratory study of the influence that analyzing teaching has on preservice teachers' classroom practice. Journal of teacher education, 66(3), 201-214.

Swartz, R. (1992). Critical thinking, the curriculum, and the problem of transfer. In D. Perkins, J. Bishop, & J. Lochhead (Eds.), Thinking: The Second International Conference (pp. 261-284). Hillsdale, NJ: Erlbaum.

Swartz, R. (2001). Infusing Critical and Creative Thinking into Content Instruction, In A.L. Costa (Ed.). Developing Minds: A Resource Book for Teaching Thinking, 3rd edition. Alexandria, VA: Association for Supervision and Curriculum Development.

Swartz, R., and Parks, S. (1994). Infusing the Teaching of Critical and Creative Thinking into Content Instruction: A Lesson Design Handbook for the elementary grades. New York: Critical Thinking Press and Software.

Tavakol, M., & Dennick, R. (2011). Making sense of Cronbach's alpha. International Journal of Medical Education, 2. DOI: 10.5116/ijme.4dfb.8dfd, 53-55.

Treffers, A. 1987. Three Dimensions: A Model of Goal and Theory Description in Mathematics Instruction – The Wiskobas Project. Dordrecht: Reidel.

Tudor, S.L. (2013). Formal - Non-formal – Informal In Education. 5th International Conference EDU-WORLD 2012 - Education Facing Contemporary World Issues. Social and Behavioral Sciences 76, 821 – 826.

Tunca, N. (2015). The regression level of constructivist learning environment characteristics on classroom environment characteristics supporting critical thinking. Eurasian Journal of Educational Research, 60, 181- 200 Doi: 10.14689/ejer.2015.60.11

Twohill, A. (2016). The approaches to solution of linear figural patterns adopted by children attending Irish primary schools. Presented at the *13thInternational Congress on Mathematical Education (ICME-13).* Hamburg, Germany.

Twohill, A. (2013) Algebraic reasoning in primary school: Developing a framework of growth points. Proceedings of the British Society for Research into Learning Mathematics 33(2): 55-60.

Twohill, A. (2019). Comparison and Critique of Early Algebra in the Curricula of South Africa, Germany and Ireland. 1 - 1, 2019. In NNN (Eds.). Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education, Vol. 1, pp. XX-YY. Pretoria, South Africa: PME.

Uday, Y. (2019). Pedagogy, Andragogy, and Heutagogy – Continuum and Comparison. Research Scholar, Department of Education, Regional Institute of Education (RIE) Mysore, Karnataka, India. 7(8), pp. 1229-1234.

Ulger, K. (2016). The Relationship between Creative Thinking and Critical Thinking Skills of Students. 31(4), pp. 695-710

United Nations Educational, Scientific, and Cultural Organization-UNESCO (2016). Assessment of Transversal competencies in education: Policy and practice in the Asian-Pacific Education

Research Institutes Network (ERI-NET). Paris and Bangkok, UNESCO. http://unesdoc.unesco.org/ images/0024/002440/244022E.pdf (Accessed 12 October 2016.)

Usiskin, Z. (1999). Conceptions of School Algebra and Uses of Variables. In Algebraic Thinking, Grades K–12: Readings from NCTM's School-Based Journals and Other Publications, edited by Barbara Moses, pp. 7–13. Reston, Va.: National Council of Teachers of Mathematics.

Van Amerom, B. A. (2002). Reinvention of early algebra: Developmental research on the transition from arithmetic to algebra. Utrecht: CD-ß Press, Center for Science and Mathematics Education.

Van de Walle, J. A., Karp, K., & Bay-Williams, J. (2011). Elementary and middle school mathematics: Teaching developmentally. Boston, MA: Allyn & Bacon.

Vygotsky, L. S. (1978). Mind in society: The development of higher psychological processes. Cambridge, MA: Harvard University Press.

Wadsworth, B.J. (2004). Piaget's theory of cognition and affective development. Boston, MA: Allyn & Bacon.

Walser, N. (2008). Teaching 21st century skills: what does it look like in practice? Harvard Education Letter. https://www.siprep.org/uploaded/ProfessionalDevelopment/Readings/21stCenturySkills.pdf

Warren, E. (2005). Young children's ability to generalise the pattern rule for growing patterns. In H. Chick & J. Vincent (Eds.), Proceedings of the 29th conference of the International Group for the Psychology of Mathematics Education (Vol. 4, pp. 305- 312). Melbourne: Program Committee.

Watson, A. (2007). Key understandings in mathematics learning. Paper 6: Algebraic reasoning. University of Oxford.

Watson, A. (2009). Mathematics learning A review commissioned by the Nuffield Foundation 6 Paper 6: Algebraic reasoning. In T. Nunes, P. Bryant and A. Watson, University of Oxford: 1-42. Nuffield Foundation: London. Retrieved from www.nuffieldfoundation.org

Wechsler, S.M, Saiz, C., Rivas S.F, Vendramini, C.M.M, Almeida, L.S, Maria Celia Mundima, M.C & Franco, A. (2018). Creative and critical thinking: Independent or overlapping components? 27(1),114 – 122.

Widyatiningtyas, R., Kusumah, Y. S., Sumarmo, U., & Sabandar, J. (2015). The Impact of Problem-Based Learning Approach to Senior High School Students' Mathematics Critical Thinking Ability. Indonesian Mathematical Society Journal on Mathematics Education, 6(2), 30-38.

Willis, J. W. (2007). Foundations of qualitative research: interpretive and critical approaches. London: Sage

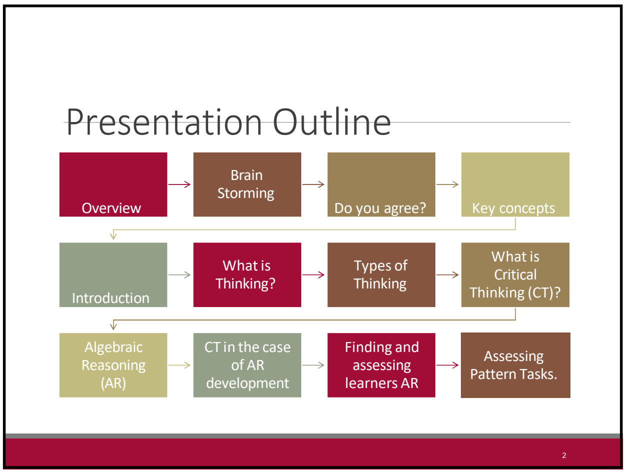
Yin, R. K. (2003). Case study research: Design and methods (3rd ed.). Thousand Oaks, CA: Sage.

Yuliani, K., & Saragih, S. (2015). The Development of Learning Devices Based Guided Discovery Model to Improve Understanding Concept and Critical Thinking Mathematically Ability of Students at Islamic Junior High School of Medan. Journal of education and practice, 6(24), 116-128.

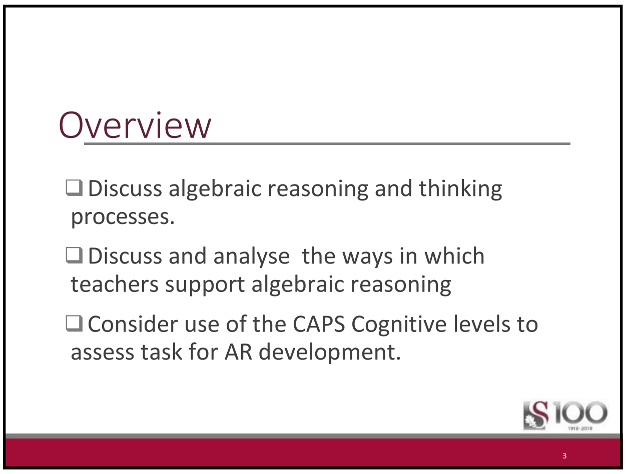
Ziegler, E. & Kapur, M. (2018). The interplay of creativity, failure and learning in generating algebra problems. Institute for Learning Sciences and Higher Education, ETH Zurich, Switzerland, pp. 1 – 31.

APPENDIX A: PRESENTATION

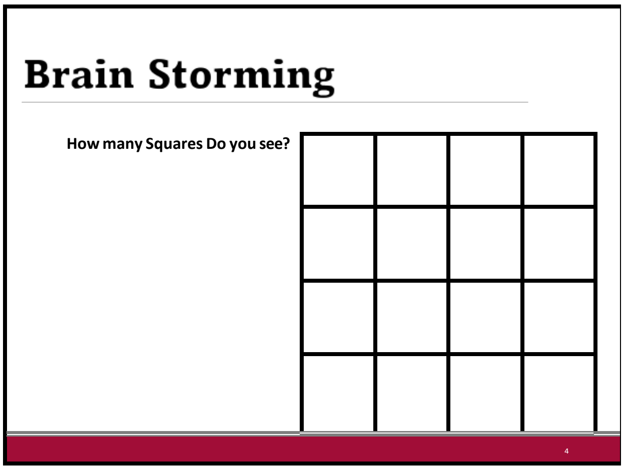
1



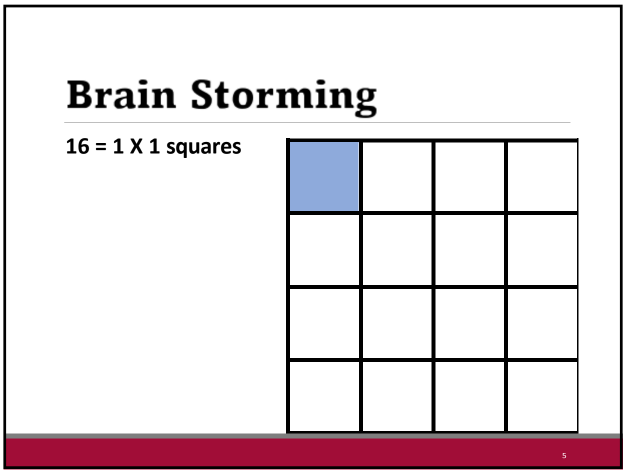
2



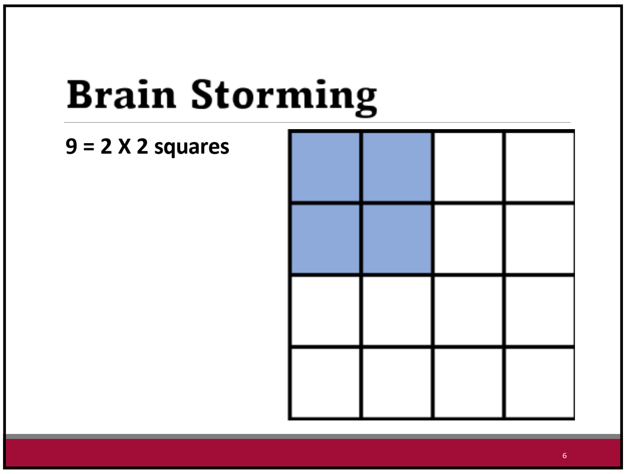
3



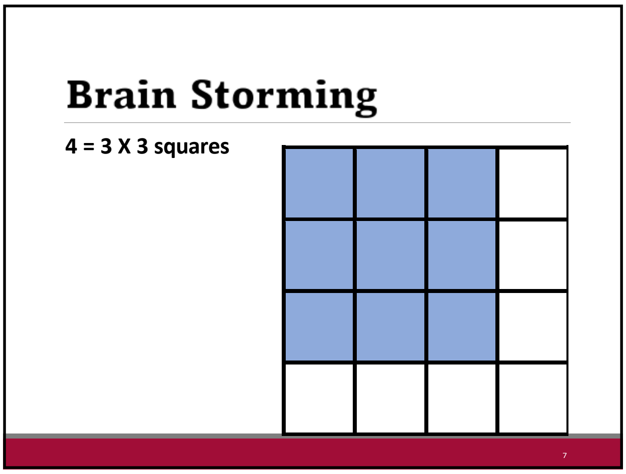
4



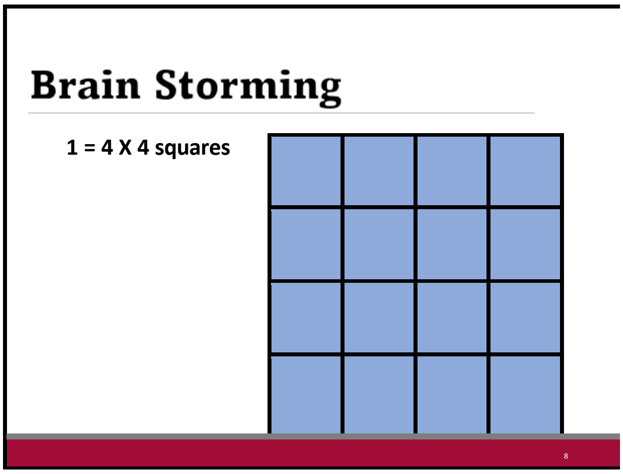
5



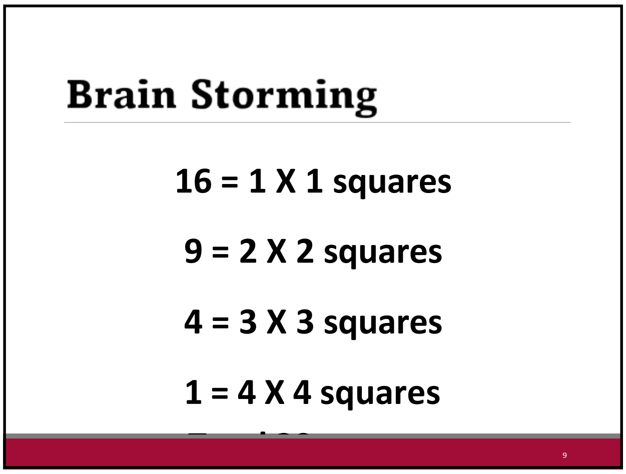
6


Brain Storming
4 = 3 X 3 squares
7

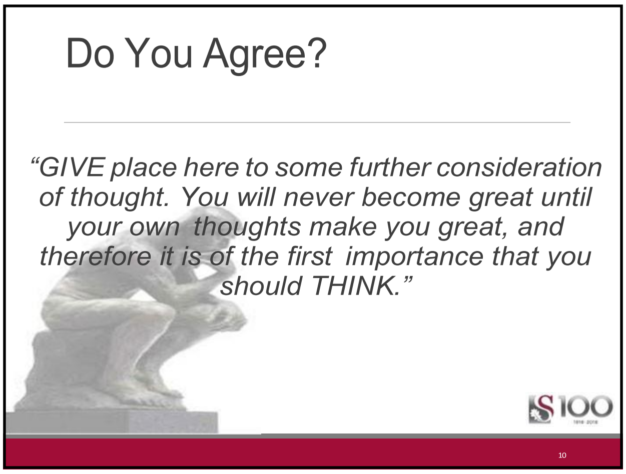
7


Brain Storming
1 = 4 X 4 squares
8

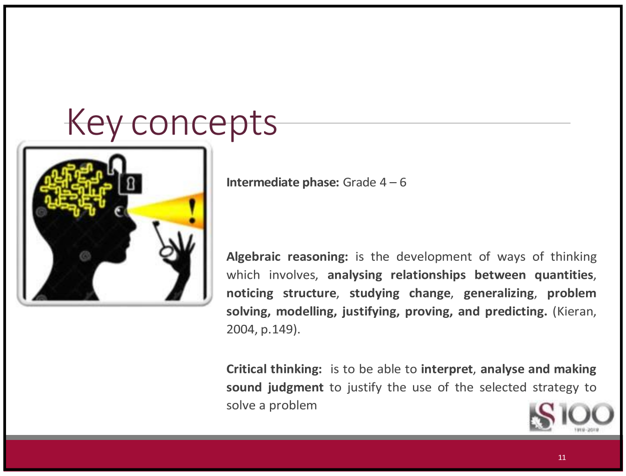
8


Brain Storming
16 = 1 X 1 squares
9 = 2 X 2 squares
4 = 3 X 3 squares
1 = 4 X 4 squares
9

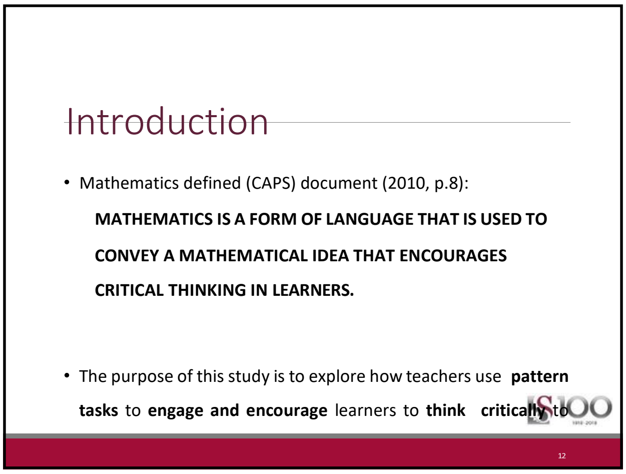
9


Do You Agree?
"GIVE place here to some further consideration of thought. You will never become great until your own thoughts make you great, and therefore it is of the first importance that you should THINK."
S 100
10

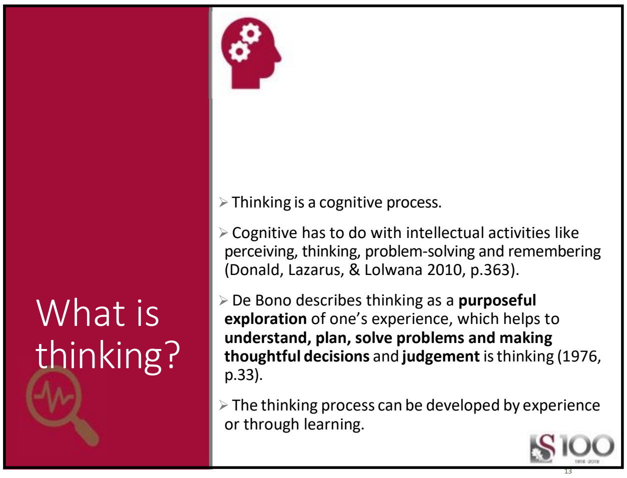
10


Key concepts
Intermediate phase: Grade 4 – 6
Algebraic reasoning: is the development of ways of thinking which involves, analysing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modelling, justifying, proving, and predicting. (Kieran, 2004, p.149).
Critical thinking: is to be able to interpret, analyse and making sound judgment to justify the use of the selected strategy to solve a problem
S 100
11

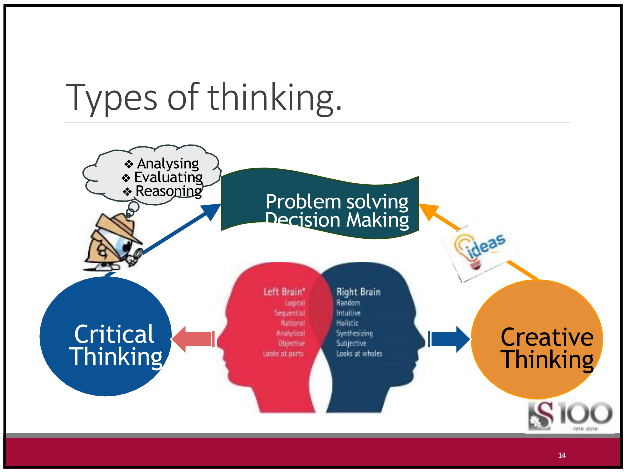
11


Introduction
• Mathematics defined (CAPS) document (2010, p.8):
MATHEMATICS IS A FORM OF LANGUAGE THAT IS USED TO
CONVEY A MATHEMATICAL IDEA THAT ENCOURAGES
CRITICAL THINKING IN LEARNERS.
• The purpose of this study is to explore how teachers use pattern
tasks to engage and encourage learners to think critically to
12

12

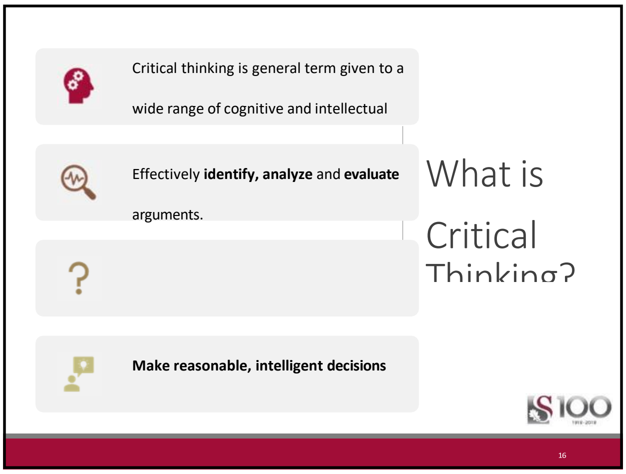


13

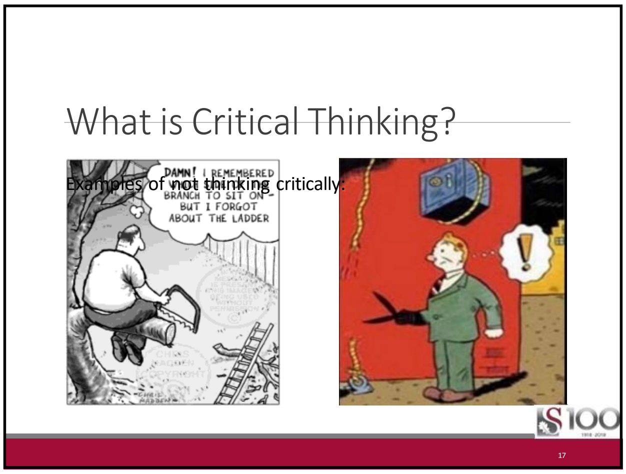


14

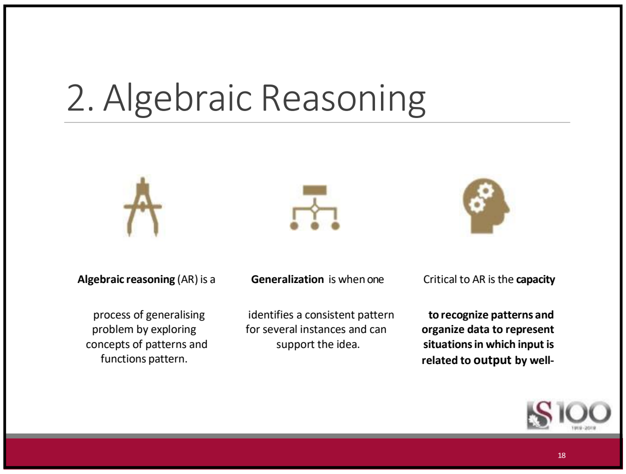
15



16



17



18

Pre - algebra	Arithmetic	Algebra	Algebraic reasoning
Generalisation	• **Objective:** to find a numerical solution • **Generalising a** specific number of situations • **Table** as a calculational tool	• **Objective:** to generalise and symbolise methods of problem-solving • **The generalisation** of relations between numbers, reduction to uniformity • **Table** as a problem-solving tool	• **Objective:** exploring properties and relationships • **A generalisation** of exploring equality as a relationship between quantities • **Table** as a functional thinking tool
Understanding of variables	• **Manipulation** of fixed numbers • **Letters** are measurement labels or abbreviations of an object	• **Manipulation** of variables • **Letters** are variables or unknowns	• **Manipulation** of alphanumeric expressions • **Letters** are variables that represent the property of a number
Symbolic expressions (SE)	• **SE:** Represent processes • **Operations** refer to actions • **Equal sign** used to give a result or balance sides.	• **SE:** are outcomes and processes • **Operations** are autonomic objects • **An equal sign** represents equivalence	• **SE**: used to reason with generalisation • **Operations** help to select and use the appropriate strategy • **The equal sign** represents the relationship among alphanumeric
Solving and Reasoning with (u)knowns	• **Reasoning** with known numbers • **Unknown**s as endpoint • **Linear problems** in one unknown	• **Reason** using unknowns • **Unknowns** as a starting point • **Problems** with multiple unknowns: systems of equations	• **Reasoning** using the relation of numbers, variable and operations • **Unknowns** as representative of the posed situation • **Simplify problem**s/expression using alphanumeric as functions/rule to find next term

19

19

CT in the case of AR development

Critical thinking (CT) is a reasoning process that yields to

high-order reasoning process, which aims to clarify, analyse, and understand making use of inferences

Leaners reason mathematically when they

justify their thinking and **adapt the known to the unknown** in the process of transferring knowledge in a **new context** and further

20

20

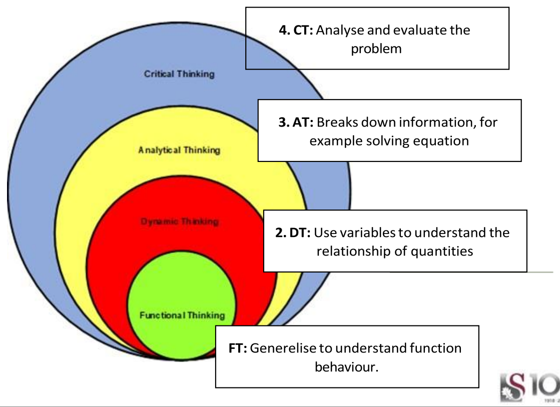
Continue….

In the case of AR development the following thinking processes are involved:

1. Analytical thinking
2. Dynamic thinking
3. Functional thinking

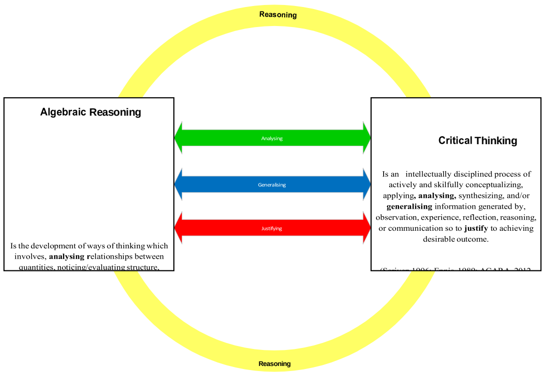
21

21



22

22



23

23

Finding and supporting learners AR.

Teachers must focus on student thinking in order to develop their AR by asking simple questions such as the following:

- Tell me what you were thinking.
- Did you solve this a different way?

Blanton, M.L. & Kaput, J.J. (2003). Developing elementary teachers'

24

24

Assessing Pattern Tasks

According to CAPS, the questions of tasks should cater the following cognitive levels:

- Knowledge 25%;
- Routine procedures 45%;
- Complex procedures 20% and
- Problem solving 10%

(CAPS 2011, p.296)

25

25

Pattern Tasks

For learners to engage in algebraic reasoning, you need:

1. A cognitively stimulating task that provides learners with opportunities to:
   - Look for patterns
   - Make generalizations
   - Use their critical thinking skills

2. The task is differentiated so to challenge all learners in their perspective level.

26

26

Q&A

Thank you for your attention

27

27

APPENDIX B: INTERVIEW QUESTIONS

SEMI – STRUCTURED INTERVIEW QUESTIONS			
PARTICIPANT:		**DATE:**	

Prior presentation:

1. What do you understand about the term critical thinking?
2. Have you been trained or taught about critical thinking skill in any teaching program?
3. When planning for a lesson or task, do you use a blooms taxonomy or CAPS cognitive levels as a guide for questions.

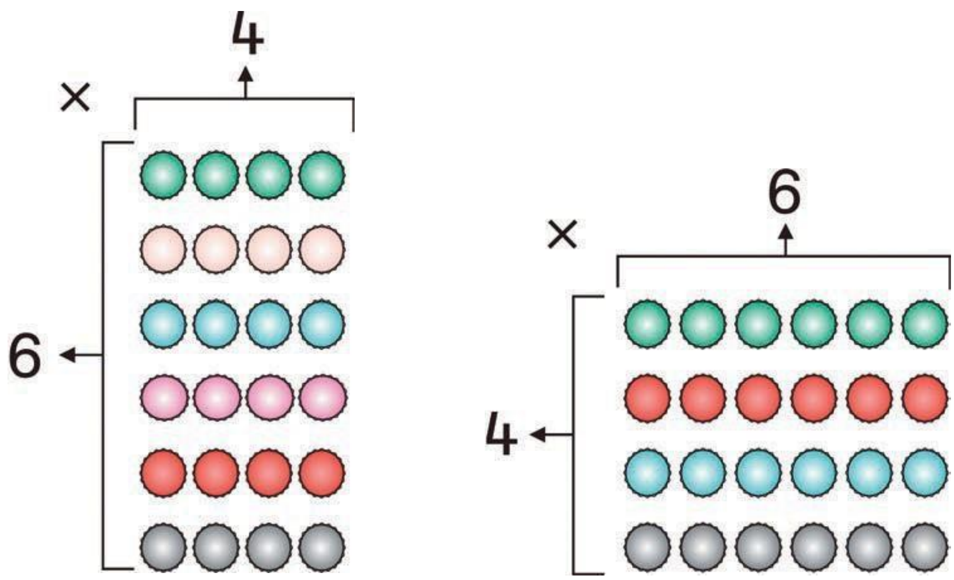
After the presentation:

4. Has the presentation contributed to improving your understanding of how to cultivate learners critical thinking? If yes, how?
5. Do you think critical thinking is very crucial for algebraic reasoning development?
6. Is it possible to incorporate CT in every lesson or tasks? Why?
7. Do you think learners can think critically? If so, then what should a teacher do to cultivate critical thinking in learners?
8. Would you encourage critical thinking in your classroom?
    - If so, how do you encourage it?
    - If not, why you do not encourage it?
9. What are the barriers you think you can encounter in the process of teaching critical thinking in a math classroom?
10. In what way would you monitor and encourage yourself in your practice to ensure that you incorporate critical thinking?
11. How would you define an environment that encourages critical thinking for algebraic reasoning development?
12. Do you think that critical thinking is a skill that can be encouraged across other learning areas?

APPENDIX C: TASK

Task 1: Operating with numbers

Look at the two groups of counters.



1. What do you notice about the two groups of counters?

2. Write a multiplication number sentence for each group of counters.

3. Write a division number sentence for each group of counters.

4. We can write four number sentences for the numbers 3, 5 and 15.
   $3 \times 5 = 15$ $5 \times 3 = 15$ $15 \div 3 = 5$ $15 \div 5 = 3$

Write four numbers sentences using

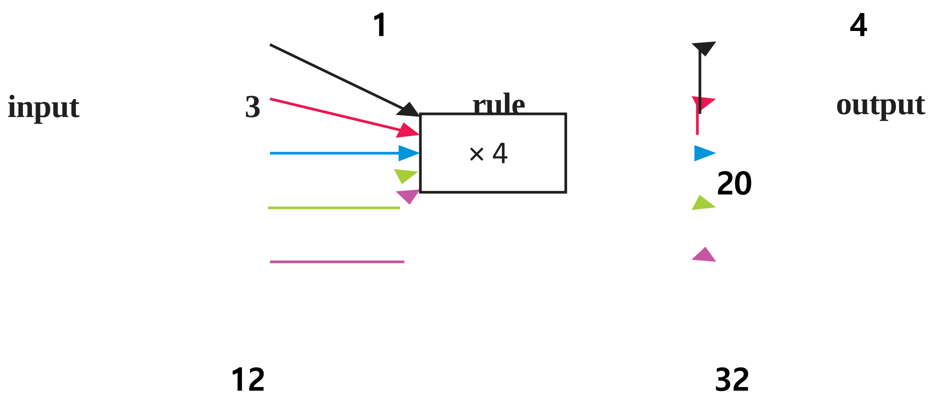
a) 7, 8 and 56 **b) 9, 3 and 27.**

Task 2: Finding Patterns

At a school fundraising event, the local stationery shop promises to donate R4 for every R1 raised by the school.

Money raised by the school in rands	1	2		5	10	15	20		100
Money donated by the local stationary shop in rands	4		12		40			200	

1. Complete the table by filling in the missing values.
2. Explain how you calculated the missing values.
3. Write a rule for the pattern in the table.
4. Look at the following flow diagram. It shows the same relationship:



a) Is the money donated by the local stationery shop the input or the output?

b) Complete the flow diagram.

c) What rule can you use to find the output values?

d) What rule can you use to find the input values from the output values?

e) Write a sentence to explain how multiplication and division can work together.

Task 3: Number sentences

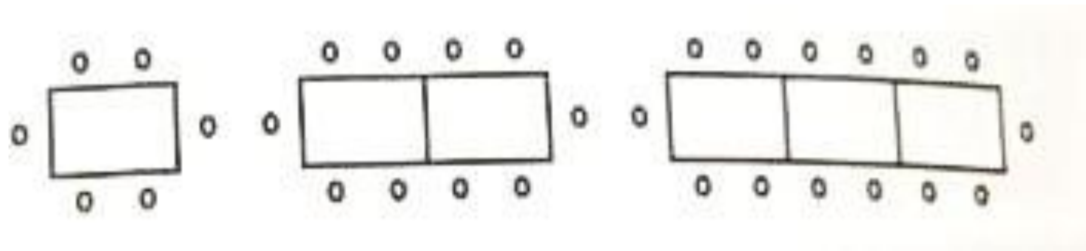
The Grade 6 learners decided to organise an African Day for their school. They decided to invite a speaker to talk about African renaissance.

Write the number sentence and solve the following:

1. 84 people fit into one row of seats in the hall. How many rows of seats will they have to pack out if there are 1 367 learners at the school; 312 parents are attending and there are 35 special guests.

2. The Grade 6 leaners have decided that they would like to donate all funds raised to the Aids Orphan Fund. All the learners are asked to contribute R2,00; parents are asked to pay R5,00 each. They also receive another R485 in donations from guests. How much money will the fund receive?

3. They invite the speaker and their guests to have tea with all the Grade 6 learners after the event. They decided to arrange the tables like this in order to accommodate different groups of people:



Complete this table and answer the questions below:

Number of tables	1	2	3	4	5	6	7
Number of people seated	6	10	14				

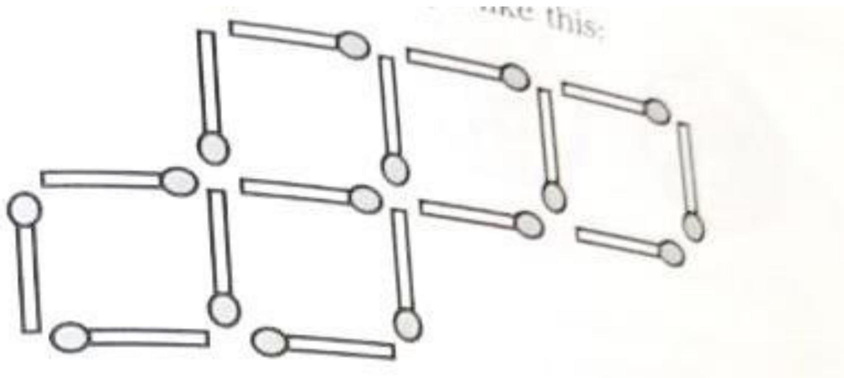
a) Look at the table and find the number pattern that is formed.
b) Explain to a partner how the pattern is formed.
c) Write a description of the pattern.

Task 4: Matchstick shapes

Collect at least 16 matchsticks:

Squares:

1. Use your matches to build a shape like this:



a) What shapes can you see in the matchstick pattern?

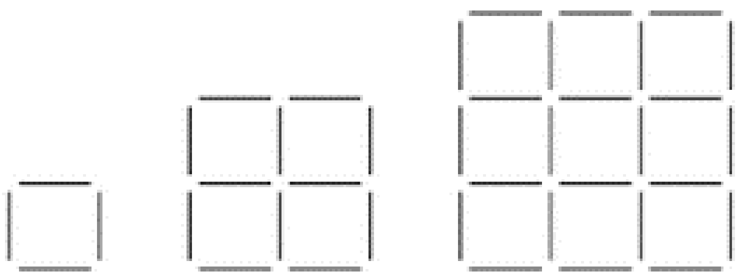
b) How many squares can you count in the pattern?

c) How many rectangles can you count in the pattern?

d) Move 2 matches to make 4 squares.

e) Move 3 matches to make 2 rectangles.

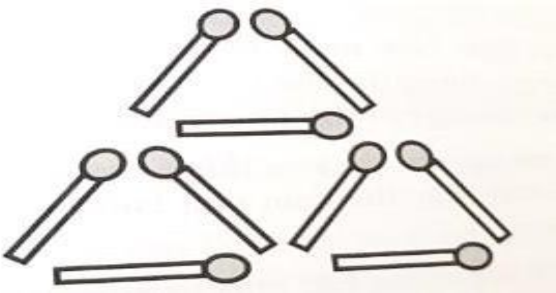
2. Shown below is a pattern of "growing" squares made from toothpicks.



Find a rule that will let you find the number of toothpicks in any square in the above sequence.

Triangles:

3. Use your matches to build a shape like this:



a) What shapes can you see in the matchstick pattern?

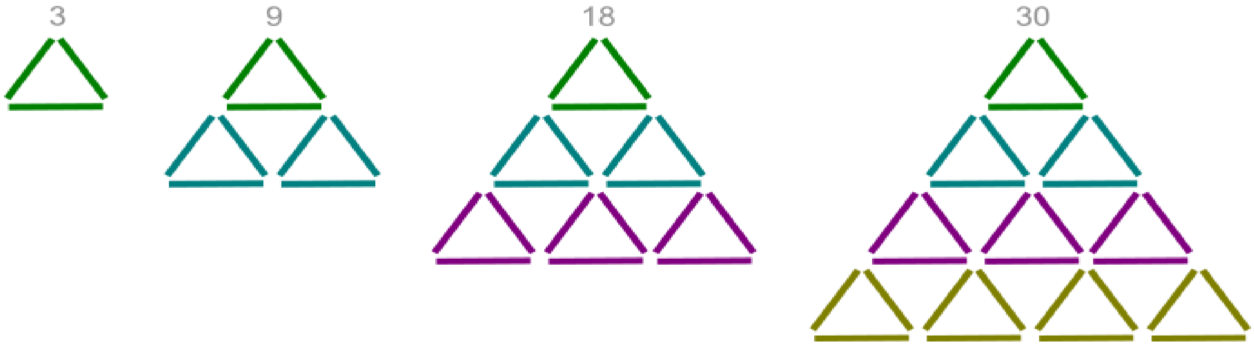
b) How many triangles can you count in the pattern?

c) How many parallelograms do you count in the pattern?

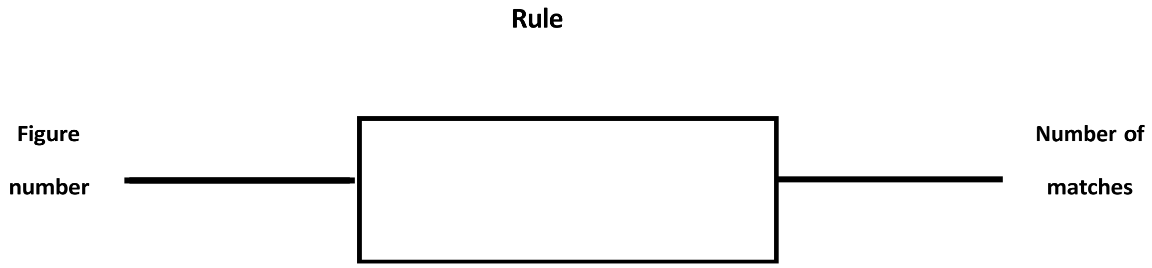
d) Move 2 matches to leave 3 triangles.

e) Move 1 match to leave 2 triangles.

4. Consider the sequence below:



a) Find the rule that will be required to find the number of matchsticks for the tenth diagram in the sequence above:



b) Explain how you would find the number of matches that would be needed for the 100th figure in this sequence.

APPENDIX D: RUBRIC FOR TASK ASSESSMENT

Student Name: **Reasoning Task:** **Date:**

Observation of student's reasoning:

	ANALYSING	GENERALISING	JUSTIFYING
NOT EVIDENT	• **Does not notice** common property or pattern.	• **Does not communicate** a common property or rule (conjecture) for a pattern.	• **Does not justify**.
BEGINNING	• **Recalls** random known facts or **attempts to sort** examples or **repeats** patterns.	• **Attempts to communicate** a common property or rule (conjecture) for a pattern.	• **Describes** what they did and **recognises** what is correct or incorrect. • **Argument** is not coherent or does not include all steps.
DEVELOPING	• **Notices** a common property, or **sorts and orders** cases, or **repeats and extends** patterns. • **Describes** the property or pattern.	• **Generalises: communicates** a rule (conjecture) using mathematical terms, and records other cases or examples.	• **Attempts to verify** by testing cases and **detects and corrects** errors or inconsistencies. • Starting statements in a **logical argument** are correct.
CONSOLIDATING	• **Systematically searches** for examples, extends patterns, or analyses structures, to form a conjecture. • Makes **predictions** about other cases.	• **Generalises: communicates** a rule (conjecture) using mathematical symbols and **explains what** the rule means or **explains how** the rule works using examples.	• **Verifies** truth of statements by confirming all cases or **refutes** a claim by using a counter example. • Uses a correct **logical argument.**
EXTENDING	• **Notices and explores** relationships between properties.	• **Generalises** cases, patterns or properties using mathematical symbols and **applies** the rule. • **Compares** different expressions for the same pattern or property to show	• Uses a **watertight logical argument**. • **Verifies** that the generalisation holds for *all* cases using logical argument.

Comments (feedback, reasoning prompts for further development):

Mathematics by Inquiry is an initiative of, and funded by, the Australian Government Department of Education and Training

Australian Academy of Science

aamt

APPENDIX E: LESSON OBSERVATION

LESSON OBSERVATION			
LEARNING AREA:	Mathematics	**GRADE:**	5 / 6
TOPIC:	Patterns (Task _)	**DATE:**	
TIME:			

Teacher's Questions to Elicit and Challenge Reasoning[1]

	Observed	Comments
A. **Managing:** supporting learners to organising their work. e.g. who is a writer?	□ Yes □ No	
B. **Clarifying:** engage learners' prior knowledge to prompt relational thinking. [illegible] inverse of division.	□ Yes □ No	
C. **Orienting:** Facilitate/motivate learners towards the correct/away from the incorrect answer. e.g. How can you check for the answer?	□ Yes □ No	
D. **Prompting Mathematical Reflection:** ask learners to reflect and explain their thinking so to understand others mathematical ways of thinking so to extend their thinking about the problem. e.g. How do you explain that? /Does anyone have a different way?	□ Yes □ No	
E. **Eliciting Algebraic Reasoning:** Ask learners to: • To undo to build rules to describe functional thinking. • Abstract from computation they made so to give more meaning to their work. • Ask them about what statements are always true about nth terms • Look for relationships in patterns – using inverses (work forward and backward) • Justify generalization	□ Yes □ No	

[1] Driscoll, M.J. (1999). Fostering algebraic thinking: A guide for teachers, grades 6-10. Portsmouth, NH: Heinemann. [p.6]

APPENDIX F: POST-REFLECTIVE QUESTIONNAIRE

ASEKUN U.O

REFLECTIVE QUESTIONNAIRE			
PARTICIPANT :		**DATE:**	
GRADE:	4 / 5 / 6		

	Strongly disagree	Disagree	Fairly Agree	Agree	Strongly agree
1. I understand what critical thinking is or means.					
2. I can formulate and identify critical thinking questions.					
3. Critical thinking questions support my teaching and help learners to be more active in the lesson.					
4. I can create tasks or extend questions from textbooks to encourage critical thinking in most of my lessons.					
5. Encouraging critical thinking helps learners understand mathematics better than to merely memorize steps.					
6. Critical thinking is important, but it is more time consuming and may affect my completion of the syllabus on time.					
7. Critical thinking encourages active learning.					
8. My understanding of CT has helped me to understand and enabled me to teach my learners.					

APPENDIX G: ETHICS APPROVAL SU

UNIVERSITEIT
STELLENBOSCH
UNIVERSITY

CONDITIONAL APPROVAL GRANTED

REC: Social, Behavioural and Education Research (SBER) - Initial Application Form

3 September 2021

Project number: REC-2021-19017

Project title: Developing algebraic reasoning in the intermediate phase to encourage critical thinking: a case study of teachers Dear Ms

UO Asekun

Your REC: Social, Behavioural and Education Research (SBER) - Initial Application Form submitted on 16/08/2021 16:13 was reviewed by the REC: Social, Behavioural and Education Research (REC: SBE) and approved with certain conditions.

This conditional approval means that the researcher may proceed with the envisaged research provided that they respond or adhere to the stipulations/conditions.

Ethics approval period:

Protocol approval date (Humanities)	Protocol expiration date (Humanities)
3 September 2021	2 September 2022

REC STIPULATIONS/CONDITIONS:

1) The main concern is clarity whether learners are going to participate in a focus group session, of which I do not think so. The REC gathers that observations will be conducted when learners do the four planned tasks. At the same time, it seems like learners will participate in group discussion, because the PI mentions that "learners will be selected to form part of the focus group task-based interview". The PI also mentions in the response letter that the "focus group interviews will be done virtually in school during the math period since learners alternate days in other schools" (1.1). It is not clear how many learners will be invited and what they will do during the focus group session. The PI should check whether she indeed wants learners to participate in the focus group session. [RESPONSE REQUIRED]

2) Could the PI also please clarify in a clearer manner and in ONE short paragraph, the following: HOW many minors of potential participants will be identified and contacted? The PI mentioned MS Teams/ Whatsapp/Zoom, will the Grade 5 and 6 also using MS Teams for their interview (what does the PI mean by "virtually communication" in your updated proposal, does that mean is there a facility in the classroom where the PI can virtually talk to these minors)? and the most important, WHAT does the PI want these minors to do? [RESPONSE REQUIRED

HOW TO RESPOND:

Some of these stipulations/conditions may require your response. Where a response is required, you must respond to the REC within

three (3) months of the date of this letter.

Your conditional approval will lapse automatically should your response not be received by the REC within 3 months of the date of this letter.

For instructions on how to respond to these stipulations, please download the FAQ on how to edit your application and follow the steps carefully: HOW TO RESPOND TO REC FEEDBACK.

Where revision to supporting documents is required, please ensure that you replace all outdated documents on your application form with the revised versions.

INVESTIGATOR RESPONSIBILITIES

Please take note of the General Investigator Responsibilities attached to this letter. You may commence with your research after complying fully with these guidelines.

If the researcher deviates in any way from the proposal approved by the REC: SBE, the researcher must notify the REC of these changes.

Please use your SU project number (19017) on any documents or correspondence with the REC concerning your project.

Please note that the REC has the prerogative and authority to ask further questions, seek additional information, require further modifications, or monitor the conduct of your research and the consent process.

CONTINUATION OF PROJECTS AFTER REC APPROVAL PERIOD

Please note that a progress report should be submitted to the REC: SBE before the approval period has expired if a continuation of ethics approval is required. The Committee will then consider the continuation of the project for a further year (if necessary)

Included Documents:

Document Type	File Name	Date	Version
Research Protocol/Proposal	UvileA_Proposal_16314239	18/10/2020	1
Data collection tool	Presentation on AR & CT1	18/10/2020	1
Data collection tool	Rubric for task assessment	18/10/2020	1
Request for permission	Letter to principal	18/10/2020	1
Data collection tool	Interview questions	18/10/2020	1
Data collection tool	Observation tool	18/10/2020	1
Data collection tool	Tasks 1to4 16314239	01/03/2021	2
Data collection tool	Post-reflective questionnaire	01/03/2021	2
Default	RESPONSE LETTER 16314239	02/03/2021	1
Assent form	Assent Form	09/07/2021	2
Parental consent form	Parent-Legal guardian Consent form	09/07/2021	2
Informed Consent Form	Consent form	09/07/2021	2
Default	Data collection - covid mitigation stratergy	09/07/2021	2
Default	RESPONSE LETTER 16314239	09/07/2021	2

If you have any questions regarding this application or the conditions set, please contact the REC Secretariat at

cgraham@sun.ac.za. Sincerely,

Clarissa Graham

Secretariat: Research Ethics Committee: Social, Behavioural and Education Research (REC: SBE)

National Health Research Ethics Committee (NHREC) registration number: REC-050411-032.

The Research Ethics Committee: Social, Behavioural and Education Research complies with the SA National Health Act No.61 2003 as it pertains to health research. In addition, this committee abides by the ethical norms and principles for research established by the Declaration of Helsinki (2013) and the Department of Health Guidelines for Ethical Research: Principles Structures and Processes (2nd Ed.) 2015. Annually a number of projects may be selected randomly for an external audit.

Principal Investigator Responsibilities

Protection of Human Research Participants

As soon as Research Ethics Committee approval is confirmed by the REC, the principal investigator (PI) is responsible for the following:

Conducting the Research: The PI is responsible for making sure that the research is conducted according to the REC-approved research protocol. The PI is jointly responsible for the conduct of co-investigators and any research staff involved with this research. The PI must ensure that the research is conducted according to the recognised standards of their research field/discipline and according to the principles and standards of ethical research and responsible research conduct.

Participant Enrolment: The PI may not recruit or enrol participants unless the protocol for recruitment is approved by the REC. Recruitment and data collection activities must cease after the expiration date of REC approval. All recruitment materials must be approved by the REC prior to their use.

Informed Consent: The PI is responsible for obtaining and documenting affirmative informed consent using **only** the REC-approved consent documents/process, and for ensuring that no participants are involved in research prior to obtaining their affirmative informed consent. The PI must give all participants copies of the signed informed consent documents, where required. The PI must keep the originals in a secured, REC-approved location for at least five (5) years after the research is complete.

Continuing Review: The REC must review and approve all REC-approved research proposals at intervals appropriate to the degree of risk but not less than once per year. There is **no grace period.** Prior to the date on which the REC approval of the research expires, **it is the PI's responsibility to submit the progress report in a timely fashion to ensure a lapse in REC approval does not occur.** Once REC approval of your research lapses, all research activities must cease, and contact must be made with the REC immediately.

Amendments and Changes: Any planned changes to any aspect of the research (such as research design, procedures, participant population, informed consent document, instruments, surveys or recruiting material, etc.), must be submitted to the REC for review and approval before implementation. Amendments may not be initiated without first obtaining written REC approval. The **only exception** is when it is necessary to eliminate apparent immediate hazards to participants and the REC should be immediately informed of this necessity.

Adverse or Unanticipated Events: Any serious adverse events, participant complaints, and all unanticipated problems that involve risks to participants or others, as well as any research-related injuries, occurring at this institution or at other performance sites must be reported to the REC within **five (5) days** of discovery of the incident. The PI must also report any instances of serious or continuing problems, or non-compliance with the RECs requirements for protecting human research participants.

Research Record Keeping: The PI must keep the following research-related records, at a minimum, in a secure location for a minimum of five years: the REC approved research proposal and all amendments; all informed consent documents; recruiting materials; continuing review reports; adverse or unanticipated events; and all correspondence and approvals from the REC.

Provision of Counselling or emergency support: When a dedicated counsellor or a psychologist provides support to a participant without prior REC review and approval, to the extent permitted by law, such activities will not be recognised as research nor the data used in support of research. Such cases should be indicated in the progress report or final report.

Final reports: When the research is completed (no further participant enrolment, interactions or interventions), the PI must submit a Final Report to the REC to close the study.

On-Site Evaluations, Inspections, or Audits: If the researcher is notified that the research will be reviewed or audited by the sponsor or any other external agency or any internal group, the PI must inform the REC immediately of the impending audit/evaluation.

APPENDIX H: RESEARCH APPROVAL LETTER_WCED

Western Cape Government
Education

Directorate: Research

meshack.kanzi@westerncape.gov.za
Tel: +27 021 467 2350
Fax: 086 590 2282
Private Bag x9114, Cape Town, 8000
wced.wcape.gov.za

REFERENCE: 20211001-6322

ENQUIRIES: Mr M Kanzi

Mrs Uvile Asekun
Pinot Mews 1
Nuutgevonden Road Stellenbosch
7600

Mrs Uvile Asekun

RESEARCH PROPOSAL: DEVELOPING ALGEBRAIC REASONING IN THE INTERMEDIATE PHASE TO ENCOURAGE CRITICAL THINKING: A CASE STUDY OF TEACHERS.

Your application to conduct the above-mentioned research in schools in the Western Cape has been approved subject to the following conditions:

1. Principals, educators and learners are under no obligation to assist you in your investigation.
2. Principals, educators, learners and schools should not be identifiable in any way from the results of the investigation.
3. You make all the arrangements concerning your investigation.
4. Educators' programmes are not to be interrupted.
5. The Study is to be conducted from **11 January 2022 till 30 June 2022.**
6. No research can be conducted during the fourth term as schools are preparing and finalizing syllabi for examinations (October to December).
7. Should you wish to extend the period of your survey, please contact Mr M Kanzi at the contact numbers above quoting the reference number.
8. A photocopy of this letter is submitted to the principal where the intended research is to be conducted.
9. Your research will be limited to the list of schools as forwarded to the Western Cape Education Department.
10. A brief summary of the content, findings and recommendations is provided to the Director: Research Services.
11. The Department receives a copy of the completed report/dissertation/thesis addressed to:

    The Director: Research Services Western
    Cape Education Department Private Bag
    X9114
    CAPE TOWN
    8000

We wish you success in your research.
Kind regards,
Meshack Kanzi
Directorate: Research DATE:
11 January 2022

1 North Wharf Square, 2 Lower Loop Street,
8000 Foreshore, Cape Town 8001
tel: +27 21 467 2531

Private Bag X 9114, Cape Town,
Safe Schools: 0800 45 46 47
wcedonline.westerncape.gov.za

APPENDIX I: PERMISSION LETTER FOR SCHOOL PRINCIPAL

UNIVERSITEIT·STELLENBOSCH·UNIVERSITY
jou kennisvennoot • your knowledge partner

Dear Principal

Request permission for your teachers & learners to participate in a research study

I am currently busy with my master's degree in Curriculum Studies at the Faculty of Education, Stellenbosch University. The title of my research is: "Developing algebraic reasoning in the intermediate phase to encourage critical thinking: a case study of teachers".

The research is focused on the intermediate phase. Therefore, teachers from Grade 4 – 6 and learners in Grade 5 and 6 will be required for the study. The teachers will be required to participate in presentations and interviews while the learners will be observed. The teachers will engage in the presentation by assessing tasks for critical thinking using CAPS cognitive levels and assess if the task encourages the development of algebraic reasoning. An interview with the teachers across all the grades in the intermediate phase will help the researcher understand how teachers prepare lessons that stimulate critical thinking to develop learners' algebraic reasoning skills. The data gathered will assist educators in gaining insight to improve programmes and instructional practices in the field of Mathematics.

Grades 5 and 6 learners will be required for focus group interviews and participate in engaging tasks during school hours. The learners will be observed as the teacher guides learners by asking critical thinking questions. No names of schools or participants will appear in the research study.

I would therefore like to ask for permission from the administration of Mbekweni Primary School to carry out my research in your intermediate classrooms. I am positive the outputs from this research will benefit your school because it has the potential to increase the mathematical outcomes of the intermediate phase. If permission is given, please sign the slip below to confirm the participation of your teachers and learners in this research study. Once completed, the research study will be available for your perusal.

Please feel free to contact me should you require further information.

I, ………………………………………….., give permission for a group of learners and teachers (selected by the researcher) to participate in the virtual Mathematics lesson and presentation.

Sign:……………………….………

Date:…………………………

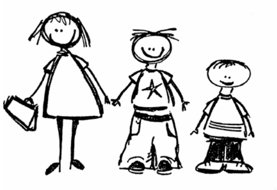
… Yours sincerely,

Uvile Asekun
M.Ed. Student
Department of Curriculum Studies
Faculty of Education
Stellenbosch University
e: 16314239@sun.ac.za

APPENDIX J: ASSENT FORM

STELLENBOSCH UNIVERSITY

ASSENT FORM FOR MINORS



TITLE OF THE RESEARCH PROJECT: Developing algebraic reasoning in the intermediate phase to encourage critical thinking: a case study of teachers.

RESEARCHERS' NAME(S): Mrs Uvile Oluwadarasimi Asekun
RESEARCHER'S CONTACT NUMBER: uvileasekun@gmail.com.
What is RESEARCH?
Research is something we do find NEW KNOWLEDGE about the way things (and people) work. We use research projects or studies to help us find out more about children and teenagers and the things that affect their lives, their schools, their families and their health. We do this to try and make the world a better place!

What is this research project all about

The study is to find out if particular questions can help learners in the intermediate phase to explain in detail their answers when they learn math.

Why have I been invited to take part in this research project?

You are invited to take part in the study because the grade you are in has been chosen for the study.

Who is doing the research?

I am a teacher by profession and currently a student at Stellenbosch University. The research will help to understand how learners learn math in your grade.

What will happen to me in this study?

If you agree to be in the study and your parents give permission, we will ask you to:

- Answer pattern tasks questions

You will be asked to complete a worksheet and will be expected to answer fully so to understand how you solve the problems.

- Be observed

If you agree to be part of the study, you will be observed during the regular class once or twice for two weeks in class. During the lesson you will be observed how you respond to the questions your teacher is asking you and so to understand your way of learning. If you say it is okay but then feel uncomfortable or change your mind, I can or stop the observation at any time. Just let me know.

Assent template. REC: Humanities (Stellenbosch University) 2017

Can anything bad happen to me?

There is absolutely nothing bad that will happen to you because your information will be strictly confidential.

Can anything good happen to me?

You may learn new ways of solving math problems.

Will anyone know I am in the study?

Your identity and willingness of participating in the study will be strictly kept confidential. Besides you and your parents, the researcher is the only one who will know the details of your study participation. If we publish reports or give talks about this research, we will only discuss group results. We will not use your name or any other personal information that would identify you.

Who can I talk to about the study?

You may withdraw your consent at any time and discontinue participation without penalty. You are not waiving any legal claims, rights or remedies because of your participation in this research study. If you have questions regarding your rights as a research participant, contact Ms Maléne Fouché [mfouche@sun.ac.za; 021 808 4622] at the Division for Research Development.

What if I do not want to do this?

Research is something you do only if you want to. No one will get mad at you if you don't want to be in the study. And whether you decide to participate or not, either way, will have no effect on your grades at school. However, you permit the researcher to use the data already collected in the case of withdrawal from the study.

.Do you understand this research study and are you willing to take part in it?

YES NO

Has the researcher answered all your questions?

YES NO

Do you understand that you can STOP being in the study at any time?

YES NO

Signature of Child

Date

Assent template. REC: Humanities (Stellenbosch University) 2017

APPENDIX K: PARENT -LEGAL GUARDIAN CONSENT FORM

UNIVERSITEIT•STELLENBOSCH•UNIVERSITY
jou kennisvennoot • your knowledge partner

STELLENBOSCH UNIVERSITY

PARENT/LEGAL GUARDIAN CONSENT FOR CHILD TO PARTICIPATE IN RESEARCH

I would like to invite your child to take part in a study conducted by Mrs Uvile O. Asekun from the Department of Curriculum Studies at Stellenbosch University. Your child will be invited as a possible participant because the research is done in your child's grade.

1. **PURPOSE OF THE STUDY**

The study aims to find out if questions can help learners in the intermediate phase explain in detail their answers when they learn math.

2. **WHAT WILL BE ASKED OF MY CHILD?**

If you consent to your child taking part in this study, the researcher will then approach the child for their assent to take part in the study. If the child agrees to participate in the study, he/she will be asked to complete four activities in their classroom during math period. Participation in the study will not exceed 1 hour for each activity.

3. **POSSIBLE RISKS AND DISCOMFORTS**

To avoid any inconvenience, the participant will be requested to partake in the study only during their math period. There are no physical or psychological risks associated with this study.

4. **POSSIBLE BENEFITS TO THE CHILD OR TO THE SOCIETY**

The possible benefits are that their thinking skills will be sharpened as they engage with the visualization of the mathematical concepts. They will explore and visualize patterns in ways beyond the scope of paper- and – pencil activities.

5. **PAYMENT FOR PARTICIPATION**

Participants will not receive payment for taking part in the research.

6. PROTECTION OF YOUR AND YOUR CHILD'S INFORMATION, CONFIDENTIALITY AND IDENTITY

Any information you or your child will share with me during this study can identify you, or your child will be protected. Besides you and your child, the researcher is the only one who will know the details of your study participation. If we publish reports or give talks about this research, we will only discuss group results. We will not use the name or any other personal information that would identify your child.

7. PARTICIPATION AND WITHDRAWAL

You and your child can choose whether to be part of this study or not. If you consent to your child taking part in the study, please note that your child may choose to withdraw or decline participation at any time without any consequence. Your child may also refuse to answer any questions they don't want to answer and remain in the study. The researcher may withdraw your child from this study if circumstances arise that warrant doing so.

8. RESEARCHERS' CONTACT INFORMATION

If you have any questions or concerns about the research, please feel free to contact Uvile Asekun, 16314239@sun.ac.za

9. RIGHTS OF RESEARCH PARTICIPANTS

Your child may withdraw their consent at any time and discontinue participation without penalty. Neither you nor your child are waiving any legal claims, rights or remedies because of your participation in this research study. If you have questions regarding your or your child's rights as a research participant, contact Ms Maléne Fouché [mfouche@sun.ac.za; 021 808 4622] at the Division for Research Development.

DECLARATION OF CONSENT BY THE PARENT/ LEGAL GUARDIAN OF THE CHILD-PARTICIPANT

As the parent/legal guardian of the child I confirm that:

- I have read the above information and it is written in a language that I am comfortable with.
- I have had a chance to ask questions and all my questions have been answered.
- All issues related to privacy, and the confidentiality and use of the information have been explained.

By signing below, I ____________________________ *(name of parent)* agree that the researcher may approach my child to take part in this research study, as conducted by Mrs Uvile O. Asekun.

____________________________________ ____________________

Signature of Parent/Legal Guardian **Date**

DECLARATION BY THE PRINCIPAL INVESTIGATOR

As the principal investigator, I hereby declare that the information contained in this document has been thoroughly explained to the parent/legal guardian. I also declare that the parent/legal guardian was encouraged and given ample time to ask any questions.

____________________________________ ____________________

Signature of Principal Investigator **Date**

APPENDIX L: CONSENT FORM

UNIVERSITEIT•STELLENBOSCH•UNIVERSITY
jou kennisvennoot • your knowledge partner

STELLENBOSCH UNIVERSITY CONSENT TO PARTICIPATE IN RESEARCH

Developing algebraic reasoning in the intermediate phase to encourage critical thinking: a case study of teachers.

You are asked to take part in a study conducted by Mrs Uvile Oluwadarasimi Asekun, from the Department of Curriculum Studies at Stellenbosch University. You have been approached as a possible participant because the research is specifically aimed at the Intermediate Phase educators.

1. PURPOSE OF THE STUDY

The research seeks to understand how teachers stimulate IP learners' critical thinking to develop algebraic reasoning in their teaching. The study is a mixed approach study and will be used to describe the impact of critical thinking has on the development of algebraic reasoning. It also aims to help teachers develop strategies and processes to ensure that their planned lessons can develop learners' algebraic proficiency.

2. WHAT WILL BE ASKED OF ME?

If you volunteer to participate in this study, we will ask you to do the following things:

- Avail yourself to for the presentation that will be done by the researcher
- Prepare for a lesson observation that will be based on what was discussed presentation.
- Be prepared to be interviewed, one-on-one about the use of critical thinking.

Due to the pandemic, to avoid physical contact participation on via electronic media will be considered. Which means the interviews and presentation will be completed during contact sessions, via electronic media such as WhatsApp, Google Hangouts, Zoom or via e-mail as a last resort.

3. POSSIBLE RISKS AND DISCOMFORTS

The only foreseeable inconvenience is the presentation and observations that need to be done without affecting any teachers' lessons. To prevent the inconveniences stated above, the participants will be requested to suggest a suitable time and location. Since time and date is going to be communicated there is no other foreseen risks or discomfort.

4. POSSIBLE BENEFITS TO PARTICIPANTS AND/OR TO THE SOCIETY

The findings of this study may:

- imply for teachers to improve teachers' key questions in a lesson to be more critical to encourage learners critical thinking.
- Also help teachers to improve learner's ability to transfer learned content skills to new applications.
- also, add to teachers' skills to be able to identify and analyse learners' responses to critical questions.
- the learners will improve understanding of their thinking as they engage in critical thinking tasks

5. PAYMENT FOR PARTICIPATION

Participants will not receive payment from taking part in the research.

6. PROTECTION OF YOUR INFORMATION, CONFIDENTIALITY AND IDENTITY

All the information which is received in connection with this study and that can be identified with you will remain confidential and will be disclosed only with your permission or as required by law. Confidentiality will be kept through pseudonyms for all teacher names and schools. Other materials such as images will be anonymised by blurring face and recognisable features such as school badges if used in the dissertation or resulting papers.

This interview will be recorded and transcribed. The identity of participants will remain confidential with only the interviewer. Individuals are welcome to review the transcriptions and can dictate which parts they want to be utilised.

7. PARTICIPATION AND WITHDRAWAL

You can choose whether to be in this study or not. If you agree to take part in this study, you may withdraw at any time without any consequence. You may also refuse to answer any questions you don't want to answer and still remain in the study. The researcher may withdraw you from this study if circumstances arise which warrant in doing so.

8. RESEARCHERS' CONTACT INFORMATION

If you have any questions or concerns about the research, please feel free to contact Uvile Asekun, uvileasekun@gmail.com.

9. RIGHTS OF RESEARCH PARTICIPANTS

You may withdraw your consent at any time and discontinue participation without penalty. You are not waiving any legal claims, rights or remedies because of your participation in this research study. If you have questions regarding your rights as a research participant, contact Ms Maléne Fouché [mfouche@sun.ac.za; 021 808 4622] at the Division for Research Development.

DECLARATION OF CONSENT BY THE PARTICIPANT

By signing below, I ________________________________ *(name of participant)* agree to take part in this research study. The information above was described to (*the participant)* by _____________ in ____ *[Afrikaans/English/Xhosa/other]* and I am in command of this language or it was satisfactorily translated to me. I was given the opportunity to ask questions and these questions were answered to my satisfaction.

I hereby consent voluntarily to participate in this study. I have been given a copy of this form.

____

Name of Subject/Participant

____

Name of Legal Representative (if applicable)

________________________________ ___________

Signature of Subject/Participant or Legal Representative Date

DECLARATION BY THE PRINCIPAL INVESTIGATOR

The information above was described to me by Uvile O. Asekun in English/IsiXhosa and I, am in command of this language. I was given the opportunity to ask questions and these questions were answered to my satisfaction.

____ ____________________

Signature of Principal Investigator **Date**

Written consent template. REC: Humanities (Stellenbosch University) 2017