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Helmut Prodinger*

Some combinatorial matrices and their LU-decomposition

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Abstract: Three combinatorial matrices were considered and their LU-decompositions were found. This is typically done by (creative) guessing, and the proofs are more or less routine calculations.

Keywords: Combinatorial matrix, LU-decomposition, Lehmer's matrix, Fibonacci polynomials

MSC: 05A19; 15B36

1 Introduction

Combinatorial matrices often have beautiful LU-decompositions, which leads also to easy determinant evaluations. It has become a habit of this author to try this decomposition whenever he sees a new such matrix.

The present paper contains three independent ones collected over the last one or two years.

2 A matrix from polynomials with bounded roots

In [11] Kirschenhofer and Thuswaldner evaluated the determinant

$$D_s = \det\Big(\frac{1}{(2l)^2 - t^2(2i-1)^2}\Big)_{1\le i,l\le s}$$

for $t = 1$. Consider the matrix M with entries $1/((2l)^2-t^2(2i-1)^2)$ where s might be a positive integer or infinity. In [11], the transposed matrix was considered, but that is immaterial when it comes to the determinant; we will treat the transposed matrix as well, but the results are slightly uglier.

The aim is to provide a completely elementary evaluation of this determinant which relies on the LU-decomposition $LU = M$, which is obtained by guessing. The additional parameter t helps with guessing and makes the result even more general. We found these results:

$$L_{i,j} = \frac{\prod_{k=1}^{j}\big((2j-1)^2t^2-(2k)^2\big)}{\prod_{k=1}^{j}\big((2i-1)^2t^2-(2k)^2\big)}\frac{(i+j-2)!}{(i-j)!(2j-2)!},$$

$$U_{j,l} = \frac{t^{2j-2}(-1)^j16^{j-1}(2j-2)!}{\prod_{k=1}^{j}\big((2k-1)^2t^2-(2l)^2\big)\prod_{k=1}^{j-1}\big((2j-1)^2t^2-(2k)^2\big)}\frac{(j+l-1)!}{l(l-j)!}.$$

Note that

$$\prod_{k=1}^{j}\big((2i-1)^2t^2-(2k)^2\big) = (-1)^j4^j\frac{\Gamma(j+1-t(i-\frac12))}{\Gamma(1-t(i-\frac12))}\frac{\Gamma(j+1+t(i-\frac12))}{\Gamma(1+t(i-\frac12))}$$

***Corresponding Author: Helmut Prodinger:** Department of Mathematical Sciences, Mathematics Division, Stellenbosch University, Private Bag X1, 7602 Matieland, South Africa, E-mail: hproding@sun.ac.za

and

$$\prod_{k=1}^{j}\big((2k-1)^2t^2-(2l)^2\big)=4^jt^{2j}\frac{\Gamma(j+\frac12+\frac lt)}{\Gamma(\frac12+\frac lt)}\frac{\Gamma(j+\frac12-\frac lt)}{\Gamma(\frac12-\frac lt)};$$

using these formulæ, $L_{i,j}$ resp. $U_{j,l}$ can be written in terms of Gamma functions.

The proof that indeed $\sum_j L_{i,j}U_{j,l} = M_{i,l}$ is within the reach of computer algebra systems (Zeilberger's algorithm). An old version of Maple (without extra packages) provides this summation.

As a bonus, we also state the inverses matrices:

$$L^{-1}_{i,j}=\frac{\prod_{k=1}^{i-1}((2j-1)^2t^2-(2k)^2)}{\prod_{k=1}^{i-1}((2i-1)^2t^2-(2k)^2)}\frac{(-1)^{i+j}(2i-2)!(2j-1)}{(i+j-1)!(i-j)!}$$

and

$$U^{-1}_{j,l}=\prod_{k=1}^{l-1}\big((2k-1)^2t^2-(2j)^2\big)\prod_{k=1}^{l}\big((2l-1)^2t^2-(2k)^2\big)\frac{(-1)^j2j^2}{t^{2l-2}(2l-2)!(j+l)!(l-j)!16^{l-1}};$$

the necessary proofs are again automatic.

Consequently the determinant is

$$D_s=\prod_{j=1}^{s}U_{j,j}.$$

For $t=1$, this may be simplified:

$$\begin{aligned}
D_s&=\frac{1}{s!}\prod_{j=1}^{s}\frac{(-1)^j16^{j-1}(2j-2)!(2j-1)!}{\prod_{k=1}^{j}(2k-2j-1)(2k+2j-1)\prod_{k=1}^{j-1}(2j-2k-1)(2j+2k-1)}\\
&=\frac{1}{s!}\prod_{j=1}^{s}\frac{16^{j-1}(2j-1)!^2}{(4j-1)!!(4j-3)!!}=\frac{4^s}{s!}\prod_{j=1}^{s}\frac{32^{j-1}(2j-1)!^4}{(4j-1)!(4j-2)!}\\
&=\frac{4^s\,16^{s(s-1)}}{s!^2}\Big/\prod_{j=1}^{s}\binom{4j}{2j}\binom{4j-2}{2j-1}=\frac{4^s\,16^{s(s-1)}}{s!^2}\Big/\prod_{j=1}^{2s}\binom{2j}{j}\\
&=\frac{16^{s(s-1)}}{s!^2}\Big/\prod_{j=0}^{2s-1}\binom{2j+1}{j};
\end{aligned}$$

the last expression was given in [11]. We used the notation $(2n-1)!!=1\cdot3\cdot5\cdots(2n-1)$.

Now we briefly mention the equivalent results for the transposed matrix:

$$L_{i,j}=\frac{\prod_{k=1}^{j}((2k-1)^2t^2-(2j)^2)}{\prod_{k=1}^{j}((2k-1)^2t^2-(2i)^2)}\frac{(i+j-1)!j}{(i-j)!(2j-1)!i},$$

$$U_{j,l}=\frac{t^{2j-2}(-1)^j16^{j-1}(2j-1)!}{\prod_{k=1}^{j}((2l-1)^2t^2-(2k)^2)\prod_{k=1}^{j-1}((2k-1)^2t^2-(2j)^2)}\frac{(j+l-2)!}{j(l-j)!},$$

$$L^{-1}_{i,j}=\frac{\prod_{k=1}^{i-1}((2k-1)^2t^2-(2j)^2)}{\prod_{k=1}^{i-1}((2k-1)^2t^2-(2i)^2)}\frac{(-1)^{i+j}(2i)!j^2}{(i-j)!(i+j)!i^2},$$

$$U^{-1}_{j,l}=\prod_{k=1}^{l}\big((2k-1)^2t^2-(2l)^2\big)\prod_{k=1}^{l-1}\big((2j-1)^2t^2-(2k)^2\big)\frac{(2j-1)!l!(-1)^j}{t^{2l-2}16^{l-1}(2l-1)!(l+j-1)!(l-j)!(l-1)!}.$$

3 Lehmer's tridiagonal matrix

Ekhad and Zeilberger [7] have unearthed Lehmer's [12] tridiagonal $n \times n$ matrix $M = M(n)$ with entries (indexed by $1 \le i, j \le n$)

$$M_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ z^{1/2}q^{(i-1)/2} & \text{if } i = j-1, \\ z^{1/2}q^{(i-2)/2} & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

Note the similarity to Schur's determinant

$$\mathsf{Schur}(x) := \begin{vmatrix} 1 & xq^{1+m} & & & & \dots \\ -1 & 1 & xq^{2+m} & & & \dots \\ & -1 & 1 & xq^{3+m} & & \dots \\ & & -1 & 1 & xq^{4+m} & \dots \\ & & & \ddots & \ddots & \ddots \end{vmatrix}$$

that was used to great success in [9]. This success was based on the two recursions

$$\mathsf{Schur}(x) = \mathsf{Schur}(xq) + xq^{1+m}\,\mathsf{Schur}(xq^2)$$

and, with

$$\mathsf{Schur}(x) = \sum_{n \ge 0} a_n x^n,$$

by

$$a_n = q^n a_n + q^{1+m} q^{2n-2} a_{n-1},$$

leading to

$$a_n = \frac{q^{n^2+mn}}{(1-q)(1-q^2)\dots(1-q^n)}.$$

Schur's (and thus Lehmer's) determinant plays an instrumental part in proving the celebrated Rogers-Ramanujan identities and generalizations.

Lehmer [12] has computed the limit for $n \to \infty$ of the determinant of the matrix $M(n)$. Ekhad and Zeilberger [7] have generalized this result by computing the determinant of the finite matrix $M(n)$. Furthermore, a lively account of how modern computer algebra leads to a solution was given. Most prominently, the celebrated q-Zeilberger algorithm [14] and creative guessing were used.

In this section, the determinant in question is obtained by computing the LU-decomposition $LU = M$. This is done with a computer, and the exact form of L and U is obtained by guessing. A proof that this is indeed the LU-decomposition is then a routine calculation. From it, the determinant in question is computed by multiplying the diagonal elements of the matrix U. By telescoping, the final result is then quite attractive, as already stated and proved by Ekhad and Zeilberger [7].

We use standard notation [2]: $(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$, and the Gaussian q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$.

3.1 The LU-decomposition of M

Let

$$\lambda(j) := \sum_{0 \le k \le j/2} \begin{bmatrix} j-k \\ k \end{bmatrix} (-1)^k q^{k(k-1)} z^k.$$

It follows from the basic recursion of the Gaussian q-binomial coefficients [2] that

$$\lambda(j) = \lambda(j-1) - zq^{j-2}\lambda(j-2). \tag{1}$$

Then we have

$$U_{j,j} = \frac{\lambda(j)}{\lambda(j-1)}, \qquad U_{j,j+1} = z^{1/2}q^{(j-1)/2},$$

and all other entries in the U-matrix are zero. Further,

$$L_{j,j} = 1, \qquad L_{j+1,j} = z^{1/2}q^{(j-1)/2}\frac{\lambda(j-1)}{\lambda(j)},$$

and all other entries in the L-matrix are zero.

The typical element of the product $(LU)_{i,j}$, that is

$$\sum_{1\le k\le n} L_{i,k}U_{k,j}$$

is almost always zero; the exceptions are as follows: If $i = j$, then we get

$$L_{j,j}U_{j,j} + L_{j,j-1}U_{j-1,j} = \frac{\lambda(j) + zq^{j-2}\lambda(j-2)}{\lambda(j-1)} = 1,$$

because of the above recursion (1). If $i = j-1$, then we get

$$L_{j-1,j-1}U_{j-1,j} + L_{j-1,j-2}U_{j-2,j} = z^{1/2}q^{(j-2)/2},$$

and if $i = j+1$, then we get

$$L_{j+1,j+1}U_{j+1,j} + L_{j+1,j}U_{j,j} = z^{1/2}q^{(j-1)/2}\frac{\lambda(j-1)}{\lambda(j)}\frac{\lambda(j)}{\lambda(j-1)} = z^{1/2}q^{(j-1)/2}.$$

This proves that indeed $LU = M$. Therefore for the determinant of the Lehmer matrix M we obtain the expression

$$\prod_{j=1}^{n}\frac{\lambda(j)}{\lambda(j-1)} = \frac{\lambda(n)}{\lambda(0)} = \sum_{0\le k\le n/2}\begin{bmatrix} n-k \\ k \end{bmatrix}(-1)^k q^{k(k-1)}z^k.$$

Taking the limit $n \to \infty$, leads to the old result by Lehmer for the determinant of the infinite matrix:

$$\lim_{n\to\infty}\det(M(n)) = \sum_{k\ge 0}\frac{(-1)^k q^{k(k-1)}z^k}{(q;q)_k}.$$

Remarks.

1. For $q = 1$, Lehmer's determinant plays a role when enumerating lattice paths (Dyck paths) of bounded height, or planar trees of bounded height, see [6, 8, 10].
2. Recursions as in (1) have been studied in [3, 4, 13] and are linked to so-called Schur polynomials [15].

4 Matrices for Fibonacci polynomials

Cigler [5] introduced several matrices that have Fibonacci polynomials as determinants; we will only treat two of them as showcases.

The Fibonacci polynomials are

$$F_n(x) = \sum_h \binom{n-h}{h} x^{n-2h};$$

our answers will come out in terms of the related polynomials

$$f_n = \sum_h \binom{n+h}{2h} X^h$$

where we write $X = x^2$ for simplicity. It is easy to check that

$$f_n = (X+2)f_{n-1} - f_{n-2},$$

for instance by comparing coefficients.

The first matrix is

$$M = \left(\binom{i-1}{j}X + \binom{i+1}{j+1}\right)_{0\le i,j<n}$$

and we will determine its LU-decomposition $M = LU$.

We obtained

$$L_{i,j} = \frac{\binom{i+1}{j+1}\sum_h \binom{j+h}{2h}X^h + \binom{i}{j}\sum_h \binom{j+h}{2h-1}X^h}{\sum_h \binom{j+1+h}{2h}X^h} = \binom{i}{j} + \binom{i}{j+1}\frac{f_j}{f_{j+1}}$$

and

$$U_{j,j} = \frac{\sum_h \binom{j+1+h}{2h}X^h}{\sum_h \binom{j+h}{2h}X^h} = \frac{f_{j+1}}{f_j},$$

$$U_{j,l} = (-1)^{j+l}\frac{\sum_h \binom{j+h}{2h-1}X^h}{\sum_h \binom{j+h}{2h}X^h} = (-1)^{j+l}\left(\frac{f_{j+1}}{f_j} - 1\right), \quad j < l.$$

For a proof, we do this computation:

$$\begin{aligned}
\sum_j L_{i,j}U_{j,l} &= L_{i,l}U_{l,l} + \sum_{0\le j<l} L_{i,j}U_{j,l} \\
&= \left[\binom{i}{l} + \binom{i}{l+1}\frac{f_l}{f_{l+1}}\right]\frac{f_{l+1}}{f_l} + \sum_{0\le j<l}\left[\binom{i}{j} + \binom{i}{j+1}\frac{f_j}{f_{j+1}}\right](-1)^{j+l}\left(\frac{f_{j+1}}{f_j} - 1\right) \\
&= \binom{i}{l}\frac{f_{l+1}}{f_l} + \binom{i}{l+1} + \sum_{0\le j<l}\binom{i}{j}\frac{f_{j+1}}{f_j}(-1)^{j+l} + \sum_{0\le j<l}\binom{i}{j+1}(-1)^{j+l} \\
&\quad - \sum_{0\le j<l}\binom{i}{j}(-1)^{j+l} - \sum_{0\le j<l}\binom{i}{j+1}\frac{f_j}{f_{j+1}}(-1)^{j+l} \\
&= \binom{i}{l+1} + \sum_{0\le j\le l}\binom{i}{j}\frac{(X+2)f_j - f_{j-1}}{f_j}(-1)^{j+l} + \sum_{0\le j<l}\binom{i}{j+1}(-1)^{j+l} \\
&\quad - \sum_{0\le j<l}\binom{i}{j}(-1)^{j+l} - \sum_{0\le j<l}\binom{i}{j+1}\frac{f_j}{f_{j+1}}(-1)^{j+l} \\
&= \binom{i}{l+1} + (X+2)\sum_{0\le j\le l}\binom{i}{j}(-1)^{j+l} + \sum_{0\le j<l}\binom{i}{j+1}(-1)^{j+l} - \sum_{0\le j<l}\binom{i}{j}(-1)^{j+l} \\
&\quad - \sum_{0\le j\le l}\binom{i}{j}\frac{f_{j-1}}{f_j}(-1)^{j+l} + \binom{i}{l} + \sum_{1\le j\le l}\binom{i}{j}\frac{f_{j-1}}{f_j}(-1)^{j+l} \\
&= X\binom{i-1}{l} + \binom{i}{l+1} + \binom{i}{l} + \sum_{0\le j\le l}\binom{i}{j}(-1)^{j+l} - \sum_{1\le j\le l}\binom{i}{j}(-1)^{j+l} - (-1)^l \\
&= X\binom{i-1}{l} + \binom{i+1}{l+1}.
\end{aligned}$$

The determinant is then $U_{0,0}U_{1,1}\ldots U_{n-1,n-1}$, and by telescoping

$$\sum_h \binom{n+h}{2h}X^h = \sum_h \binom{2n-h}{h}x^{2n-2h} = F_{2n}(x).$$

For completeness, we also factor the transposed matrix as $LU = M^t$:

$$L_{i,j} = (-1)^{i+j}\frac{\sum_h \binom{j+h}{2h-1}X^h}{\sum_h \binom{j+1+h}{2h-1}X^h}, \quad \text{for } j < i,$$
$$L_{i,i} = 1,$$

and

$$U_{j,l} = \frac{\binom{l}{j}\sum_h \binom{j+h}{2h-1}X^h + \binom{l+1}{j+1}\sum_h \binom{j+h}{2h}X^h}{\sum_h \binom{j+h}{2h}X^h}.$$

Now we move to the second matrix:

$$M = \left(\binom{i}{j}X + \binom{i+2}{j+1}\right)_{0\le i,j<n}.$$

We find

$$L_{i,j} = \frac{\binom{i+1}{j+1}\sum_h \binom{j+1+h}{2h+1}X^h + \binom{i}{j}\sum_h \binom{j+1+h}{2h}X^h}{\sum_h \binom{j+2+h}{2h+1}X^h}$$

and

$$U_{j,j} = \frac{\sum_h \binom{j+2+h}{2h+1}X^h}{\sum_h \binom{j+1+h}{2h+1}X^h},$$
$$U_{j,j+1} = 1, \qquad U_{j,l} = 0 \quad \text{for } l \ge j+2.$$

For a proof, we compute

$$\sum_j L_{i,j}U_{j,l} = \frac{\binom{i+1}{l+1}\sum_h \binom{l+1+h}{2h+1}X^h + \binom{i}{l}\sum_h \binom{l+1+h}{2h}X^h}{\sum_h \binom{l+1+h}{2h+1}X^h} + \frac{\binom{i+1}{l}\sum_h \binom{l+h}{2h+1}X^h + \binom{i}{l-1}\sum_h \binom{l+h}{2h}X^h}{\sum_h \binom{l+1+h}{2h+1}X^h}$$

and

$$\begin{aligned}
\sum_h \binom{l+1+h}{2h+1}X^h \sum_j L_{i,j}U_{j,l} &= \binom{i+2}{l+1}\sum_h \binom{l+1+h}{2h+1}X^h - \binom{i+1}{l}\sum_h \binom{l+1+h}{2h+1}X^h \\
&\quad + \binom{i}{l}\sum_h \binom{l+1+h}{2h}X^h + \binom{i+1}{l}\sum_h \binom{l+h}{2h+1}X^h \\
&\quad + \binom{i+1}{l}\sum_h \binom{l+h}{2h}X^h - \binom{i}{l}\sum_h \binom{l+h}{2h}X^h \\
&= \binom{i+2}{l+1}\sum_h \binom{l+1+h}{2h+1}X^h + \binom{i}{l}\sum_h \binom{l+h}{2h-1}X^h \\
&= \binom{i+2}{l+1}\sum_h \binom{l+1+h}{2h+1}X^h + \binom{i}{l}X\sum_h \binom{l+1+h}{2h+1}X^h
\end{aligned}$$

and therefore

$$\sum_j L_{i,j}U_{j,l} = \binom{i+2}{l+1} + \binom{i}{l}X,$$

as required. The determinant is then

$$\sum_h \binom{n+1+h}{2h+1} X^h = \sum_h \binom{n+1+h}{n-h} X^h = \sum_j \binom{2n+1-j}{j} x^{2n-2j} = x^{-1} F_{2n+1}(x^2).$$

For the transposed matrix $LU = M^t$, we find

$$L_{i,i-1} = \frac{\sum_h \binom{i+h}{2h+1} X^h}{\sum_h \binom{i+1+h}{2h+1} X^h},$$
$$L_{i,i} = 1, \qquad L_{i,j} = 0 \quad \text{for } j < i-1,$$

and

$$U_{j,l} = \frac{\binom{l+1}{j+1} \sum_h \binom{j+1+h}{2h+1} X^h + \binom{l}{j} \sum_h \binom{j+1+h}{2h} X^h}{\sum_h \binom{j+1+h}{2h+1} X^h}.$$

For completeness, we mention another recent paper about matrices and Fibonacci polynomials: [1].

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