Investigations on the Wigner derivative and on an integral formula for the quantum 6j symbols

by

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Supervisor: Dr Bruce Bartlett

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Declaration

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Date: April 2022

Abstract

Investigations on the Wigner derivative and on an integral formula for the quantum 6j symbols

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Two separate studies are done in this thesis:

1. The Wigner derivative is the partial derivative of dihedral angle with respect to opposite edge length in a tetrahedron, all other edge lengths remaining fixed. We compute the inverse Wigner derivative for spherical tetrahedra, namely the partial derivative of edge length with respect to opposite dihedral angle, all other dihedral angles remaining fixed. We show that the inverse Wigner derivative is actually equal to the Wigner derivative.

2. We investigate a conjectural integral formula for the quantum 6j symbols suggested by Bruce Bartlett. For that we consider the asymptotics of the integral and compare it with the known formula for the asymptotics of the quantum 6j symbols due to Taylor and Woodward. Taylor and Woodward's formula can be rewritten as a sum of two quantities: ins and bound. The asymptotics of the integral splits into an interior and boundary contribution. We successfully

compute the interior contribution using the stationary phase method. The result is indeed quite similar to although not exactly the same as ins. Though we expect the boundary contribution to be similar to bound, the computation is left for future work.

Uittreksel

Investigations on the Wigner derivative and on an integral formula for the quantum 6j symbols

("Investigations on the Wigner derivative and on an integral formula for the quantum 6j symbols")

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Twee afsonderlike studies word in hiedie tesis gedoen:

1. Die Wigner-afgeleide is die parsiële afgeleide van 'n tweevlakshoek met betrekking tot die teenoorgestelde kandlengte in 'n tetraëder, terwyl alle ander kandlengtes onveranderd bly. Ons bereken die inverse Wigner-afgeleide vir sferiese tetraëders, naamlik die parsiële afgeleide van die kandlengte met betrekking tot teenoortaande tweevlakshoek, terwyl alle ander tweevlakshoeke konstant bly. Ons wys dat die inverse Wigner-afgeleide inderdaad gelyk is aan die Wigner-afgeleide.

2. Ons ondersoek 'n beweerde integralformule vir die kwantum 6j simbole, wat deur Bruce Bartlett as moontlikheid voorgestel is. Daarvoor oorweeg ons die asimptotika van die integraal en vergelyk dit met die bekende formule van die kqantum 6j simbole as gevolg van Taylor en Woodward. Taylor en Woodward se formule kan herskryf word as 'n som van twee hoeveelhede: ins en

bound. Die asimptotika van die integraal verdeel in 'n binne- en grensbydrae. Ons het die interne bydrae suksesvol met behulp van die stilstaande fase metode bereken. Die resultaat is inderdaad baie soortgelyk aan hoewel nie presies dieselfde as ins nie. Alhoewel ons verwag dat die grensbydrae soortgelyk aan bound sal wees, word die berekening gelaat vir toekomstige werk.

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Last but not least, all the glory be to God Who has kept His promises for me to reach this far.

Dedications

To Africa.

Contents

Nomenclature

$\mathbb{R}^n$	The set of all vectors of size n with real entries.
S^3	The unit three sphere.
Tr	Trace of a matrix.
$< a, b >_X$	The inner product between a and b in X.
χ_n	The character of a n-dimensional linear representation.
$SO(4)$	The special orthogonal group of dimension four.
$\sim$	refers to asymptotic.
$\cong$	refers to approximately the same.

Chapter 1

Introduction

Note: Only the normalized classical and quantum 6j symbols are considered throughout the thesis.

Classical 6j symbols, for example

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}$$

with non-negative integer entries, are known to be the coefficients in the change of basis of a certain vector space of morphisms in the category $Rep(SU(2))$ of representations of $SU(2)$. They are real numbers. For instance, $\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}$ are the coefficients for a certain change of basis in

$$Hom_{Rep(SU(2))}(V_{m_{13}}, V_{m_{12}} \otimes V_{m_{02}} \otimes V_{m_{03}})$$

where $V_{m_{ij}}$ denotes the $(m_{ij}+1)-$dimensional irreducible representation of $SU(2)$.

There are at least two ways to compute the classical 6j symbols namely, [3][35][20][12] via Penrose's spin network calculus (which is closely related to the Kauffman bracket) and via integration,

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}^2 = \int_{(SU(2))^4} \prod_{i<j} \chi_{m_{ij}}(g_j g_i^{-1})[\prod_{i=0}^{3} dg_i],$$

where $\chi_{m_{ij}}$ denotes the character of the irreducible representation $V_{m_{ij}}$. This method was introduced by Barrett [5] in 1998 for the evaluation of relativistic spin networks.

Appearing in the diagrammatic/combinatorial way of computing the classical 6j symbols [3][20] and nicely explained in [35], a classical 6j symbol can be geometrically associated to a *non-degenerate Euclidean* tetrahedron where the lengths of its edges are given by the entries of the symbol. For instance, our example is geometrically represented by Figure 1.1.

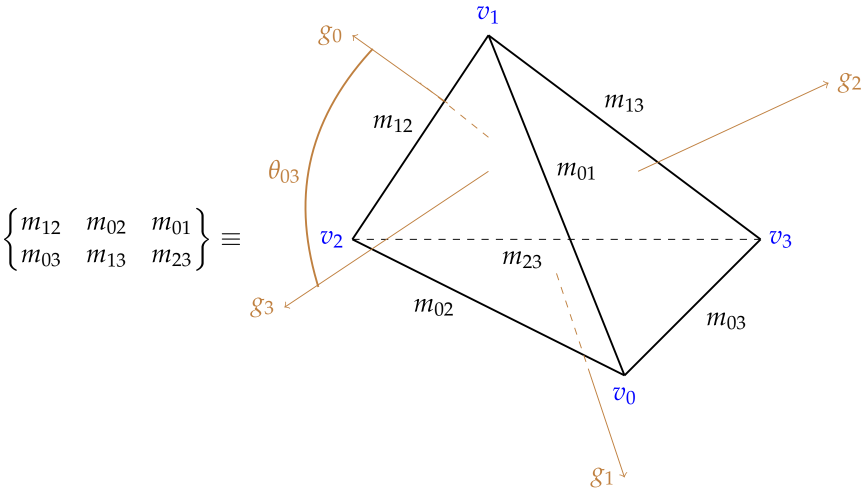


Figure 1.1: Relationship between a non-degenerate Euclidean tetrahedron with edge lengths m_{ij} and the classical 6j symbol with entries m_{ij}.

Definition 1.0.1. *Let f and g be two real-valued functions. The quantities $f(k)$ and $g(k)$, where $g(k)$ is non-zero, are said to be asymptotic, which in our notation*

$$f(k) \sim g(k),$$

when k tends to infinity, if

$$f(k) - g(k) = o(\frac{1}{k})g(k).$$

Initially, Wigner [44] gave an average approximation of the square of a classical 6j symbol with large entries.

In 1968 Ponzano and Regge [32] conjectured and empirically checked a more refined formula for the non-degenerate asymptotics of the classical 6j symbols.

In 1999 this asymptotic formula was proven by Roberts [35] through geometric quantization, and in 2003 Freidel and Louapre re-proved it via the integral formula for the classical 6j symbols with integer entries.

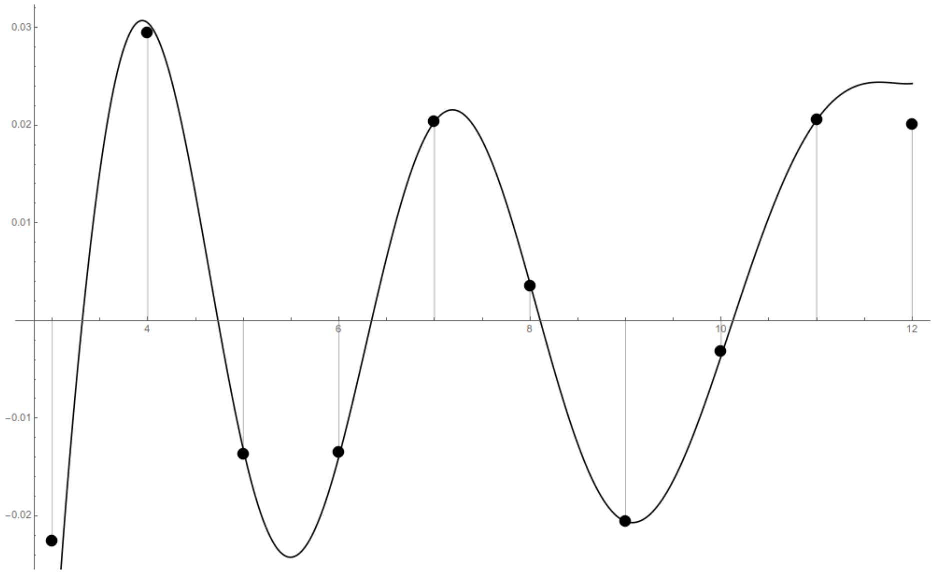


Figure 1.2: Exact values of the classical 6j symbols (dots) with entries $m_{12} = 14$, $m_{02} = 16$, $m_{01} = 18$, $m_{03} = 12$, $m_{13} = 18$ and $m_{23} = j$ where j varies from 6 to 24 in steps of 2 versus its non-degenerate asymptotic approximation (continuous curve). This is generated by the author using Mathematica (see [33]). It corresponds to Ponzano and Regge's original example [32, Figure 5].

The non-degenerate asymptotic formula for the classical 6j symbols is stated as follows:

Theorem 1.0.2 ([35])**.** *Let $m_{01}, m_{02}, m_{03}, m_{12}, m_{13}, m_{23}$ be six non-negative integers such that the triples (m_{12}, m_{01}, m_{02}), (m_{01}, m_{13}, m_{03}), (m_{02}, m_{23}, m_{03}), and (m_{12}, m_{13}, m_{23}) are admissible (see Equation* (2.3)*). Let $k \in \mathbb{N}$, and τ be a non-degenerate Euclidean tetrahedron with edge lengths m_{ij} associated to the classical 6j symbol*

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}.$$

Let θ_{ij} be the exterior *dihedral angle of τ at the edge of length $m_{\overline{ij}}$, which is opposite to the edge of length m_{ij}. Then, the non-degenerate asymptotic formula for the classical 6j symbols is*

$$\begin{Bmatrix} km_{12} & km_{02} & km_{01} \\ km_{03} & km_{13} & km_{23} \end{Bmatrix} \sim \sqrt{\frac{2}{3\pi V k^3}} \cos\{(km_{\overline{ij}} + 1)\frac{\theta_{ij}}{2} + \frac{\pi}{4}\} \tag{1.1}$$

when k tends to infinity, where V is the volume of τ.

As an overview of the proof, we would like to draw the reader's attention to focus on Freidel and Louapre's technique.

In 2003, Freidel and Louapre [17] used the $SO(4)$ symmetry of the integral to rewrite Barrett's twelve-dimensional integral formula for the square of the classical 6j symbols as a six-dimensional one:

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}^2 = \frac{2}{\pi^4} \int_{D_\pi} \frac{\prod_{i<j} \sin\left((m_{ij}+1)l_{ij}\right)}{\sqrt{\det([\cos l_{ij}])}} [\prod_{i<j} \mathrm{d}l_{ij}],$$

where $[\cos l_{ij}]$ is a unitary matrix whose off-diagonal entries are $\cos l_{ij}$ and

$$D_\pi = \{(l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23}) \in [0,\pi]^6, [\cos l_{ij}] \text{ is positive definite }\}.$$

Then, they studied the asymptotic of the integral for large k given by

$$I(k) = \frac{2}{\pi^4} \int_{D_\pi} \frac{\prod_{i<j} \sin\left((km_{ij}+1)l_{ij}\right)}{\sqrt{\det([\cos l_{ij}])}} [\prod_{i<j} \mathrm{d}l_{ij}]$$

via the stationary phase approximation. As a starter, they divided D_π into two components:

$$D^{>}_{\pi,\epsilon} = [\epsilon, \pi - \epsilon]^6 \cap D_\pi$$

and

$$D^{<}_{\pi,\epsilon} = D_\pi - D^{>}_{\pi,\epsilon}$$

where $\epsilon > 0$ is sufficiently small, knowing that this separation is a clear way to explain the approximation contributed by the interior stationary points and those from the boundary critical points.

Here, we call *interior contribution* the asymptotic approximation of the integral obtained by considering the stationary point inside the region $D^{>}_{\pi,\epsilon}$. It was computed [17] to be:

$$c_{\text{in}} = -\frac{1}{3\pi k^3 V} \sin\left(\sum_{i<j} (km_{\overline{ij}}+1)\theta_{ij}\right).$$

Whereas, on the domain $D^{<}_{\pi,\epsilon}$, the asymptotic approximation of the integral obtained by considering the boundary critical point, the *boundary contribution*, is given by

$$c_{\text{bd}} = \frac{1}{3\pi k^3 V}.$$

This way, the sum $c_{\text{in}} + c_{\text{bd}}$ results to the non-degenerate asymptotic formula for the square of the classical 6j symbols $\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}$:

$$\begin{Bmatrix} km_{12} & km_{02} & km_{01} \\ km_{03} & km_{13} & km_{23} \end{Bmatrix}^2 \sim \frac{2}{3\pi V k^3} \cos^2\{(km_{\overline{ij}} + 1)\frac{\theta_{ij}}{2} + \frac{\pi}{4}\} \qquad \text{(from(1.1))}$$

$$= c_{\text{in}} + c_{\text{bd}}. \qquad \left(\text{using} \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)\right)$$

Explaining the painstaking process for the transformation of Barrett's twelve-dimensional integral formula for the square of the classical 6j symbols into a six-dimensional one, of which not all the details are present in [17], is the aim of Chapter 2. A similar transformation is also used in our main calculation in Chapter 4.

1.1 Reciprocity of the Wigner derivative

The first aim of this thesis is to compute the inverse Wigner derivative for spherical tetrahedra.

The Wigner derivative is the partial derivative of dihedral angle with respect to *opposite* edge length, in a spherical tetrahedron, all other lengths remaining fixed. It plays a crucial role in a possible geometric proof, which was later made rigorous by Marché and Paul [27], for the non-degenerate asymptotic formula for the quantum 6j symbols proposed by Taylor and Woodward in [41, 38].

As follows is the formula for the Wigner derivative.

Theorem 1.1.1 (Taylor-Woodward [41])**.** *The Wigner derivative for a spherical tetrahedron is*

$$\frac{\partial \beta(l_{ij})}{\partial l'} = \frac{\sin l \sin l'}{\sqrt{\det G}},$$

where β is the interior dihedral angle at the edge with length l, l' is the length of the opposite edge (see Fig. 3.2), and G is the edge Gram matrix (see Definition 2.3.11) associated to the spherical tetrahedron.

The inverse Wigner derivative is the partial derivative of edge length with respect to *opposite* dihedral angle, all other dihedral angle held constant, in a spherical tetrahedron.

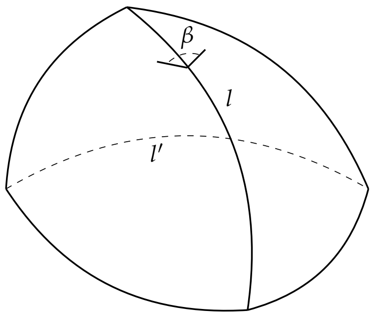


Figure 1.3: Spherical tetrahedron showing the interior dihedral angle β at the edge of length l and the edge of length l', opposite to β.

In 2008 Feng Luo, in [25], showed that any partial derivative of edge length with respect to dihedral angle in a spherical tetrahedron may be expressed in terms of the inverse Wigner derivative. Hence, the result of the inverse Wigner derivative is essential should anyone be interested in the computing the signature of the matrix $H = [\frac{\partial l_{ij}}{\partial \beta_{st}}]$ theoretically. Here, l_{ij} denotes the edge lengths of a spherical tetrahedron and β_{ij} denotes its dihedral angles. We will show that H is the Hessian matrix associated to the phase of the integrals contributing to the stationary phase approximation of the asymptotic of a conjectural integral formula for the square of the quantum 6j symbols. See Chapter 4.

The following is the formula we derive for the inverse Wigner derivative. Although it seem to appear in [41], it is new in the literature. An elaborated discussion on that will be given in Section 3.4.

Theorem 1.1.2. *The inverse Wigner derivative for a spherical tetrahedron is*

$$\frac{\partial l'(\beta_{ij})}{\partial \beta} = \frac{\sin l \sin l'}{\sqrt{\det G}},$$

where β is the interior dihedral angle at the edge with length l, l' is the length of the opposite edge (see Fig. 3.2), and G is the edge Gram matrix (see Definition 2.3.11) associated to the spherical tetrahedron.

As a corollary, by comparing Theorem 1.1.1 and Theorem 1.1.2, we have shown that the Wigner derivative and the inverse Wigner derivative are in fact equal. The details explaining this fact are found in Chapter 3.

1.2 A conjectural integral formula for the quantum 6j symbols

Let $r \geq 3$ and $q = e^{\mathrm{i}\frac{\pi}{r}}$.

Quantum 6j symbols, explicitly defined in Section 4.2, for example

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}_q$$

with non-negative integer entries, are known to be the coefficients in the change of basis of a certain vector space of morphisms in the category $Rep(U_q(sl_2))$ of representations of the quantum group $U_q(sl_2)$. They are real numbers. For instance, $\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}_q$ are the coefficients for a certain change of basis in

$$Hom_{Rep(U_q(sl_2))}(\tilde{V}_{m_{13}}, \tilde{V}_{m_{12}} \otimes \tilde{V}_{m_{02}} \otimes \tilde{V}_{m_{03}})$$

where $\tilde{V}_{m_{ij}}$ denotes the $(m_{ij}+1)-$dimensional irreducible representation of $U_q(sl_2)$.

Analogously to the classical 6j symbols, Taylor and Woodward found a non-degenerate asymptotic formula for the quantum 6j symbols[1], relating them to the geometry of a *non-degenerate spherical tetrahedron* (see Definition 2.3.8). Their formula is equivalent (asymptotically) to a slightly different formula given by Roberts (see [36] and also Appendix C.3). We will adopt a modified version of Roberts' formula in this thesis (see Remark 4.1.5):

Theorem 1.2.1 ([36] [41]). *Let r^0_{ij} be an element of $\mathbb{Q} \cap [0,1]$. Let k be a natural number. Let T^0 be a non-degenerate spherical tetrahedron whose edge lengths are $l^0_{ij} = \pi r^0_{ij}$ and whose* exterior dihedral angle *at the edge $(\overline{ij})$, opposite to the edge (ij), is denoted by θ^0_{ij}. Then,*

$$\begin{Bmatrix} kr^0_{12} & kr^0_{02} & kr^0_{01} \\ kr^0_{03} & kr^0_{13} & kr^0_{23} \end{Bmatrix}^2_{q=e^{\frac{i\pi}{k+2}}} \sim \frac{4\pi^2}{k^3\sqrt{\det([\cos l^0_{ij}])}} \cos^2\{\sum_{i<j}(kr^0_{\overline{ij}}+1)\frac{\theta^0_{\overline{ij}}}{2} + \frac{k+2}{\pi}V(T^0) + \frac{\pi}{4}\} \tag{1.2}$$

when $k \longrightarrow \infty$, where kr^0_{ij} are integers and $[\cos l^0_{ij}]$ is the edge Gram matrix (see Definition 2.3.11) of T^0.

[1] An error has surfaced in the original formula and what is found here is the corrected version. The details on the refinement are found in Chapter 4.

It is a fact that the right hand side of Equation 1.2 may be written as a sum

$$\text{ins} + \text{bound},$$

where

$$\text{ins} = -\frac{2\pi^2}{k^3\sqrt{\det([\cos l_{ij}^0])}} \sin\{\sum_{i<j}(kr_{\overline{ij}}^0 + 1)\theta_{ij}^0 + \frac{2(k+2)}{\pi}V(T^0)\}, \tag{1.3}$$

and

$$\text{bound} = \frac{2\pi^2}{k^3\sqrt{\det([\cos l_{ij}^0])}}. \tag{1.4}$$

We reckon that ins is the *interior* contribution for an asymptotic integral formula for the square of the quantum 6j symbols and bound the *boundary* contribution. This sum is obtained from the fact that $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$.

Taylor and Woodward proved their asymptotic formula by showing that both sides of the equation satisfy a second-order difference equation as one entry of the symbol is varied.

Is it possible to re-prove this asymptotic formula via an integral formula for the quantum 6j symbols? In other words, is there an analogue, for quantum 6j symbols, of the elegant integral formula for the square of the classical 6j symbols

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}^2 = \int_{(SU(2))^4} \prod_{i<j} \chi_{m_{ij}}(g_j g_i^{-1})[\prod_{i=0}^{3} dg_i]?$$

Shouldn't there be a geometric formula for the quantum 6j symbols which directly involves the Lie group $SU(2)$, its irreducible representations V_n and its characters? Since, from the diagrammatic perspective, we can realize the quantum hom-sets as subspaces of the classical hom-sets, i.e.

$$Hom_{Rep_q(SU(2))}(W_i, W_j \otimes W_k) \subseteq Hom_{Vect}(V_i, V_j \otimes V_k)$$

where V_i, V_j, V_k are irreducible representations of $SU(2)$. And, this inclusion suggests that one should be able to express the quantum 6j symbols using the classical irreducible representations of $SU(2)$.

The second aim of this thesis is to investigate the veracity of a conjectural integral formula for the square of the quantum 6j symbols [2]:

[2]The conjecture was introduced by Bartlett in 2019.

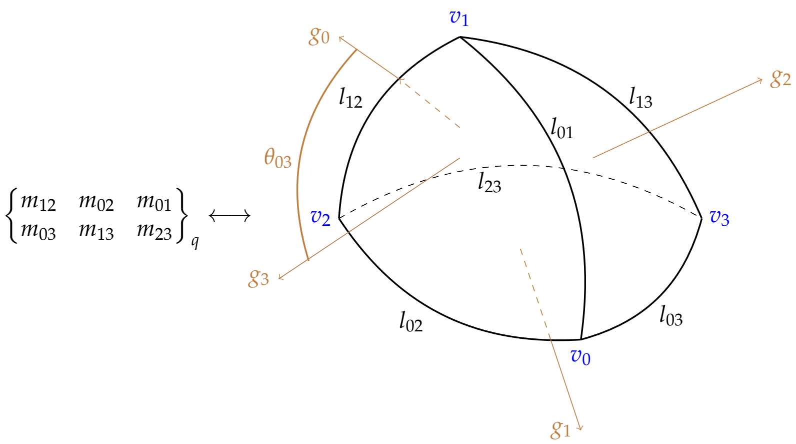


Figure 1.4: A non-degenerate spherical tetrahedron T with edge lengths l_{ij} ($l_{ij} = \frac{\pi m_{ij}}{k}$ in Roberts' formula and $l_{ij} = \frac{\pi(m_{ij}+1)}{k+2}$ in Taylor and Woodward's formula) associated to the quantum 6j symbols with entries m_{ij}.

Conjecture 1.2.2 (Strong form)**.** *Let $m_{01}, m_{02}, m_{03}, m_{12}, m_{13}, m_{23}$ be six natural numbers such that the triples (m_{12}, m_{01}, m_{02}), (m_{01}, m_{13}, m_{03}), (m_{02}, m_{23}, m_{03}) and (m_{12}, m_{13}, m_{23}) are q-admissible (see Equations* (4.7)*,* (4.8)*,* (4.9)*). Let g_0, g_1, g_2, g_3 be four elements in $SU(2)$. Due to the diffeomorphism between $SU(2)$ and S^3 they may be thought as four unit vectors in $\mathbb{R}^4$. Let T be a spherical tetrahedron whose outward normal vectors to each face are the g_i's. Then,*

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}^2_{q=e^{\frac{i\pi}{s+2}}} = \int_{SU(2)^4} \prod_{i<j} \chi_{m_{\overline{ij}}}(g_j g_i^{-1}) e^{\frac{2}{\pi}(s+2)iV(T)} [\prod dg_i],$$

where $V(T)$ denotes the volume of the spherical tetrahedron T and $\overline{ij}$ means neither i nor j features[3] *in the index $\overline{ij}$.*

The idea is to look for an appropriate asymptotic version of the integral in Conjecture 1.2.2, then use the stationary phase approximation to obtain its *interior* and *boundary* contributions and procure the non-degenerate asymptotic formula for the square of the quantum 6j symbols. The expectation is that the asymptotic formula coincides with that of Taylor and Woodward/Roberts.

An asymptotic form of the conjectural integral formula for the square of the quantum 6j symbols in Conjecture 1.2.2 is indeed given by:

[3]Example: $\overline{01} = 23$ when considering the set of indices $\{0, 1, 2, 3\}$

Conjecture 1.2.3 (Asymptotic form). *Let $r^0_{ij} \in \mathbb{Q} \cap [0,1]$ and $k \in \mathbb{N}$ such that kr^0_{ij} is integer. Then as $k \to \infty$,*

$$\begin{Bmatrix} kr^0_{12} & kr^0_{02} & kr^0_{01} \\ kr^0_{03} & kr^0_{13} & kr^0_{23} \end{Bmatrix}^2_{q=e^{\frac{i\pi}{k+2}}} \sim \frac{2}{\pi^4} \int_{D_\pi} [\prod_{i<j} \mathrm{d}\theta_{ij}] \frac{\prod_{i<j} \sin((kr^0_{\overline{ij}}+1)\theta_{ij})}{\sqrt{\det([\cos\theta_{ij}])}} \cos(\frac{2}{\pi}(k+2)V(T)),$$

where $V(T)$ denotes the volume of the spherical tetrahedron with exterior dihedral angles *θ_{ij}, and $\overline{ij}$ means neither i nor j features in the index $\overline{ij}$. Here, the domain of integration is defined by*

$$D_\pi = \{(\theta_{01}, \theta_{02}, \theta_{03}, \theta_{12}, \theta_{13}, \theta_{23}) \in [0,\pi]^6, [\cos\theta_{ij}] \text{ is positive definite } \},$$

where $[\cos\theta_{ij}]$ is the 4×4 unidiagonal symmetric matrix whose off-diagonal elements are $\cos\theta_{ij}$.

Now that a conjectural asymptotic integral formula for the quantum 6j symbols is at hand, let us investigate it and see if it reproduces the non-degenerate asymptotic formula for the quantum 6j symbols by Taylor and Woodward/Roberts. The strategy is to apply the stationary phase approximation on that conjectural asymptotic integral formula to obtain the *interior* and *boundary* contributions. Then, we expect that the non-degenerate asymptotic formula for the square of the quantum 6j symbols would be the sum of the *interior contribution* and the *boundary contribution*, similarly to the asymptotic approximation of the classical 6j symbols by Freidel and Louapre. We were able to procure the interior contribution. It is obtained from the stationary point inside D_π, and its expression is given in the following theorem.

Theorem 1.2.4. *Let $r^0_{ij} \in \mathbb{Q} \cap [0,1]$. Let T^0 be a non-degenerate spherical tetrahedron whose edge lengths $l^0_{ij} = \pi r^0_{ij}$ and whose* exterior dihedral angles *at the edge $(\overline{ij})$, opposite to the edge (ij), is denoted by θ^0_{ij}. Then,*

$$\mathrm{int} = \frac{-\pi^2}{4k^3} \cdot \frac{1}{\sqrt{\det([\cos l^0_{ij}])}} \cos\{\sum_{i<j}(kr^0_{\overline{ij}}+1)\theta^0_{ij} + \frac{2}{\pi}kV(T^0)\} \qquad (1.5)$$

when $k \longrightarrow \infty$, where $[\cos l^0_{ij}]$ is the edge Gram matrix (see Definition 2.3.11) of T^0.

However, after numerical and theoretical computation attempts, it is unfortunate that we could not draw any conclusion about the boundary contribution.

Regarding the interior contribution (1.5), its comparison with the reckoned interior contribution in Equation (1.3) shows that its amplitude is off by a factor of eight (8) and its phase by $\frac{\pi}{2}$. These two shortcomings show that the integral formula for the quantum 6j symbols (Conjecture 1.2.2) is close to being correct and might only have minor correction(s) to be taken into account.

The details on the calculation of the interior contribution (Equation (1.5)), via our conjectural asymptotic integral formula for the quantum 6j symbols, is found in Chapter 4.

1.3 Outline of the thesis

In summary, the thesis is outlined as follows:

- **Chapter 2** contains the painstaking process for the transformation of Barrett's twelve-dimensional integral formula for the square of the classical 6j symbols into a six-dimensional one.
- **Chapter 3** encloses the calculation of the inverse Wigner derivative and the proof of the reciprocity of the Wigner derivative.
- **Chapter 4** presents the investigation on Bartlett's conjectural integral formula for the square of the quantum 6j symbols.
- **Chapter 5** is the conclusion. It contains the summary of the main results in the thesis and some plans for future research.

On another note, quite a number of small mistakes were discovered to appear in different references used during the preparation of the thesis. They will be presented and discussed in each chapter accordingly.

Some necessary proofs that don't appear in the main chapters, plus the MATHEMATICA code for our numerical checking and backing up some arguments in Chapter 4 are presented in the Appendices.

Chapter 2

An integral formula for the classical 6j symbols

2.1 Introduction

The integral formula for the classical 6j symbols first appeared in the literature in 1998. It is a result of Barrett's work on the classical evaluation of relativistic spin networks [5], and is formulated as an integral over four copies of $SU(2)$ shown in the theorem below.

Consider the 6j symbols (see Definition 2.2.4)

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}$$

whose entries are integers. Recall that the 6j symbols are real numbers.

Theorem 2.1.1. *Let $g_0, g_1, g_2, g_3 \in SU(2)$ and $\chi_{m_{ij}}$ denote the character of the $(m_{ij}+1)$-dimensional irreducible representation of $SU(2)$. Then*

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}^2 = \int_{(SU(2))^4} \prod_{i<j} \chi_{m_{ij}}(g_j g_i^{-1})[\prod_{i=0}^{3} dg_i]. \tag{2.1}$$

In 2003, as a tool to generate the asymptotic formula for the classical 6j symbols [17], Freidel and Louapre transformed Barrett's integral in the following way.

Theorem 2.1.2. *The twelve-dimensional integral on the right hand side of Equation* (2.1) *can be re-expressed as a six-dimensional integral over the space of edge lengths of equivalence classes of spherical tetrahedra as follows:*

$$I = \frac{2}{\pi^4} \int_{D_\pi} \frac{\prod_{i<j} \sin((m_{ij}+1)l_{ij})}{\sqrt{\det([\cos l_{ij}])}} [\prod_{i<j} \mathrm{d}l_{ij}]. \tag{2.2}$$

Explicitly, the region of integration is

$$D_\pi = \{(l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23}) \in [0,\pi]^6, [\cos l_{ij}] \text{ is positive definite } \},$$

where $[\cos l_{ij}]$ *is the* 4×4 *unidiagonal symmetric matrix whose off-diagonal elements are* $\cos l_{ij}$.

Remark 2.1.3. *There has been a little confusion about the description of* D_π *in [17]. Further discussion about that will be found in Section 2.9.*

We show that the integral (2.2) is done over the $SO(4)$ equivalence classes (see Definition 2.8.3) of all positively oriented non-degenerate spherical tetrahedra in S^3. Although Freidel and Louapre explained the steps for transforming the integral (2.1) to (2.2) in [17], not all the details were provided in their paper. Therefore, the aim of this chapter is to go through the transformation meticulously. That is because we will use this result in Chapter 4 when investigating a conjectural integral formula for the square of the quantum 6j symbols.

The chapter is outlined as follows: the definition of the classical 6j symbols will be recalled in Section 2.2. Section 2.3 will comprise all the results from spherical triangles and spherical tetrahedra necessary to our calculation. Since the notion of links [25] is crucial to our computation, it will be elaborated in Section 2.4. A set of tetrahedral related variables called here "box variables", used by Freidel and Louapre in [17], will be expounded in Section 2.5. They are employed to transition from the integral (2.1) to (2.2). Section 2.6 is dedicated to the explanation of the Lebesgue measure on S^3. The painstaking process to show the equality between Barrett's integral, and Freidel and Louapre's is explained in Section 2.7. The materials to describe the domain of integration will be provided in Section 2.8. And, the clarification on the domain of integration plus the discussion on Remark 2.1.3 are found in Section 2.9.

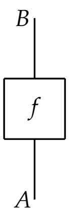
2.2 The classical 6j symbols

For a background, let us recall the definition of a classical 6j symbol.

There are several ways to define the classical 6j symbols (see [35]) but in this section we are going to use the graphical calculus introduced by Penrose in [30]. So, let us remind ourselves on how to use them.

2.2.1 Manual for maps

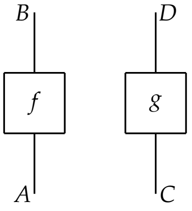
- Here, the maps are pictured vertically and read from bottom to top. For instance, let A and B be two vector spaces and $f : A \longrightarrow B$ a linear map between them. Then, f is depicted as



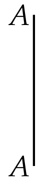
- The tensor product of two linear maps are drawn side by side starting from left to right. For example, let A, B, C and D be four vector spaces and $f : A \longrightarrow B$, $g : C \longrightarrow D$ be two linear maps. Then the tensor product

$$f \otimes g : A \otimes C \longrightarrow B \otimes D$$

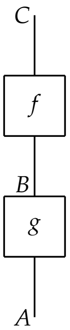
is drawn as



- Let A be a vector space, the identity map $id_A : A \longrightarrow A$ is graphically represented by



- The composition of two linear maps f and g, $f \circ g$ consists of drawing them on top of each other such that g would be at the bottom and f at the top. For instance, let A, B, C be three vector spaces and $f : B \longrightarrow C$, $g : A \longrightarrow B$ be two linear maps. Then, the composite $f \circ g : A \longrightarrow C$ is visualized as



- For linear maps from the base field of the vector space, some examples would explain it better. So, let A be a complex vector space and $f : \mathbb{C} \longrightarrow A$ (resp. $g : \mathbb{C} \longrightarrow A \otimes A$); their graphical representation are respectively given by

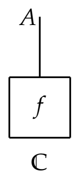


Figure 2.1: The map f

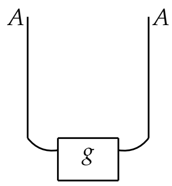


Figure 2.2: The map g

That is done, let us now focus on the representations of $SU(2)$ as the vector spaces.

2.2.2 The 6j symbols

Let V_1 with basis $\{e_1, e_2\}$ be the fundamental representation of $SU(2)$. As mentioned in [35] and [12, page 9, Theorem 2.2.1], the other irreducible representations are given by the $n-$th symmetric power of V_1 (see Section A.2 for a review). Let us denote by $Rep(SU(2))$ the category of all representations of $SU(2)$ and by

$$Hom_{Rep(SU(2))}(V, W)$$

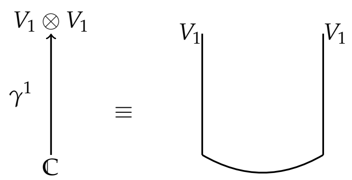
the collection of morphisms from V to W in that category.

The definition of the classical 6j symbols solely depends on the Clebsh-Gordan mapping. So, let us first remind ourselves of its graphical definition.

To start with, let us consider the map

$$\gamma^1 : \mathbb{C} \longrightarrow V_1 \otimes V_1$$

in $Rep(SU(2))$, depicted in the graphical calculus as follows:

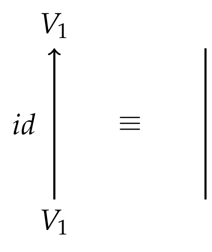


which is defined in such a way that

$$\gamma^1(1) = \mathrm{i}(x \otimes y - y \otimes x)$$

where $\mathrm{i}^2 = -1$.

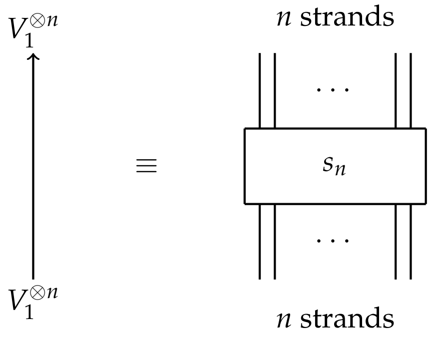
Next, let the identity map from V_1 to V_1 be defined graphically as



Then, let us recall the symmetrizing projector

$$s_n : V_1^{\otimes n} \longrightarrow V_1^{\otimes n}$$

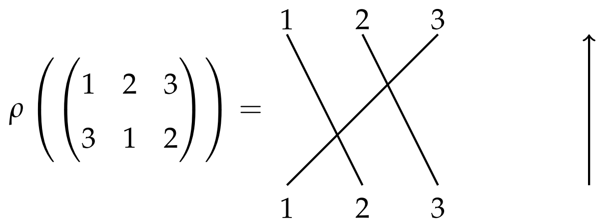
graphically depicted as



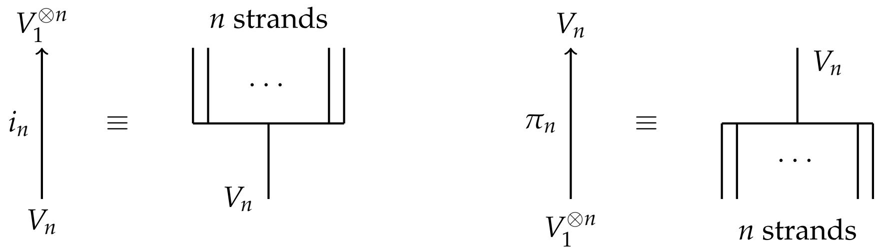
such that

$$s_n := \frac{1}{n!} \sum_{\sigma \in S_n} \rho(\sigma),$$

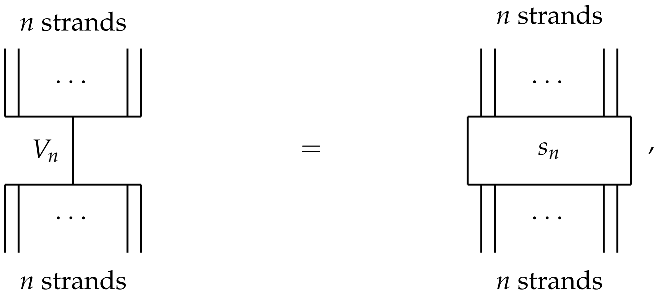
where S_n is the permutation group of n elements and $\rho(\sigma)$ is the map $V_1^{\otimes n} \longrightarrow V_1^{\otimes n}$ implementing the permutation σ by swapping the factors. For example



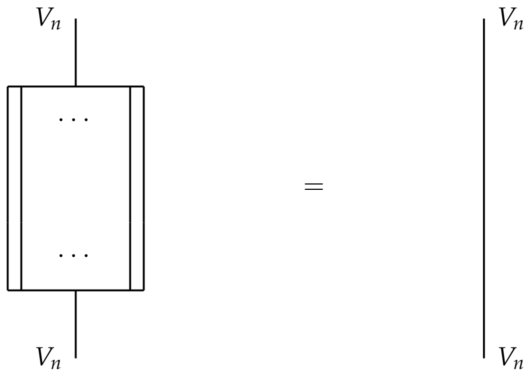
Let $i_n : V^n \hookrightarrow V_1^{\otimes n}$ (resp. $\pi_n : V_1^{\otimes n} \longrightarrow V_n$) be the inclusion of the $n-$th irreducible representation V_n into $V_1^{\otimes n}$ (resp. the projection of $V_1^{\otimes n}$ into V_n). Drawn as follows:



such that the maps i_n and π_n satisfy $i_n \circ \pi_n = s_n$ i.e.



and $\pi_n \circ i_n = id_n$, i.e.

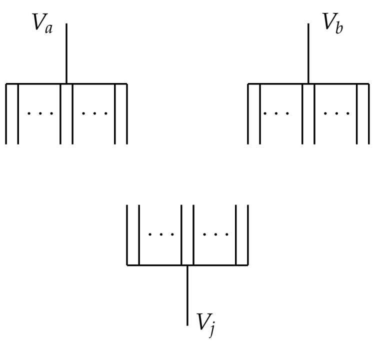


Let $a, b, j \in \mathbb{N}_{>0}$ obeying the conditions

$$a \leq b + j, b \leq a + j, \ j \leq a + b, \tag{2.3}$$

$$a + b + j \in 2\mathbb{Z}, \tag{2.4}$$

and V_a, V_b, V_j objects in $Rep(SU(2))$. Such triplet (a, b, j) is called *admissible*. Then, there exists a unique way [3] to connect the $a + b + j$ strands below using the identity and γ^1 map.

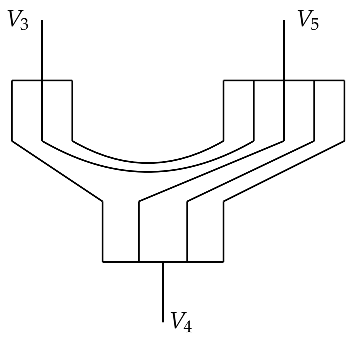


The rules to follow in joining them are as follows:

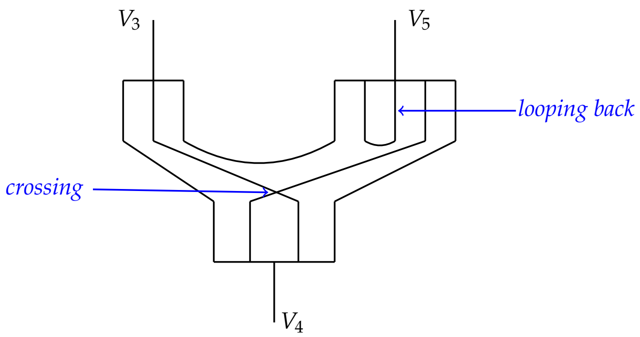
1. no crossing is allowed,
2. looping back is forbidden.

Let us look at two examples where the first one is a "allowed" connection and the second a "forbidden" one.

Example 2.2.1. *Let $a = 3$, $b = 5$ and $j = 4$. They are admissible. Then, the "allowed" connection is shown as:*



Whereas an example of a forbidden connection is depicted by



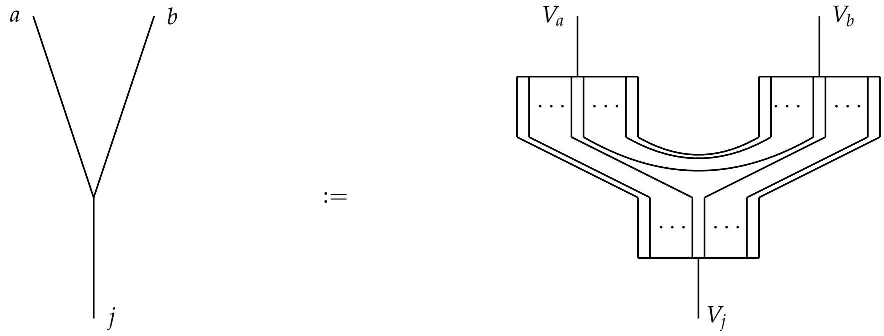
Remark 2.2.2. *The way to have an "allowed" connection is unique. For more details see [3].*

Now that everything is settled, the Clebsch-Gordan map may be defined.

Definition 2.2.3. *Let V_a, V_b, V_j be three representations of $SU(2)$ such that the triplet (a, b, j) is admissible. The $SU(2)-$invariant Clebsch-Gordan map*

$$CG_j^{ab} : V_j \longrightarrow V_a \otimes V_b$$

is defined as $(\pi_a \otimes \pi_b) \circ (id_{n_{aj}} \otimes \gamma^{n_{ab}} \otimes id_{n_{bj}}) \circ i_j$, or in diagrams:

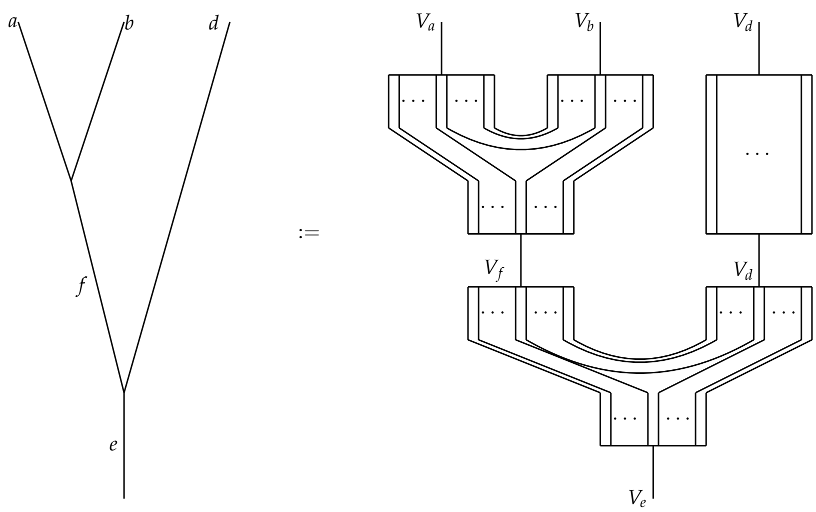


Here id_p *denotes the* $p-$ *fold tensor product of* id, γ^r *is the* $r-$*fold tensor product of* γ^1, *the number of strands connecting* $V_1^{\otimes a}$ *and* $V_1^{\otimes j}$ *is* $n_{aj} = \frac{a+j-b}{2}$, *those linking* $V_1^{\otimes a}$ *with* $V_1^{\otimes b}$ *is* $n_{ab} = \frac{a+b-j}{2}$ *and those joining* $V_1^{\otimes b}$ *and* $V_1^{\otimes j}$ *are of number* $n_{bj} = \frac{b+j-a}{2}$.

Let $V_a, V_b, V_c, V_d, V_e, V_f$ be six objects in $Rep(SU(2))$ such that $(a, b, f), (f, d, e), (b, d, c)$ and (a, c, e) are admissible $(a, b, c, d, e, f$ non-negative integers). Then, depending on the association of the tensor product $V_a \otimes V_b \otimes V_d$ two "fundamental" bases for $Hom_{Rep(SU(2))}(V_e, V_a \otimes V_b \otimes V_d)$ may be considered [41]. Namely,

$$\mathcal{B} = \{b_e^{(ab)d} : V_e \longrightarrow (V_a \otimes V_b) \otimes V_d\}$$

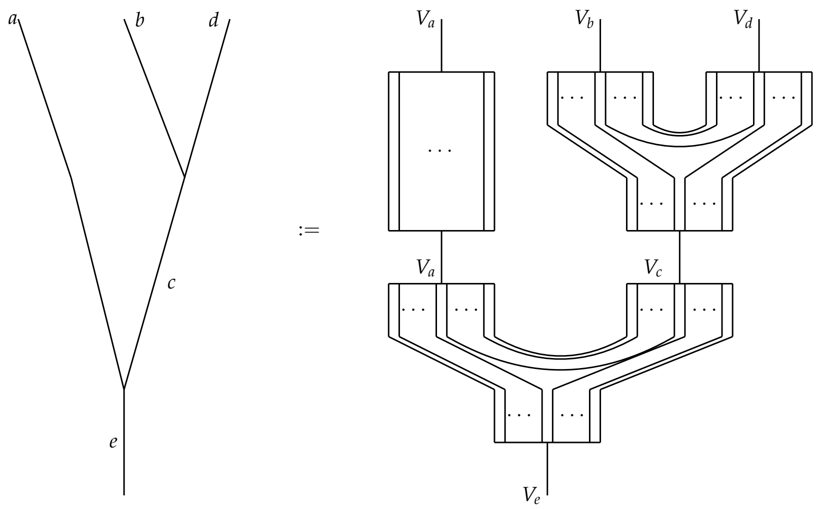
whose element $b_e^{(ab)d}$ is graphically defined as



and

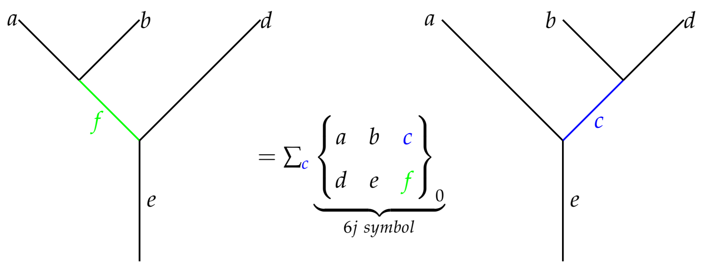
$$\mathcal{B}' = \{b_e^{a(bd)} : V_e \longrightarrow V_a \otimes (V_b \otimes V_d)\}$$

whose element $b_e^{a(bd)}$ is depicted in the graphical calculus as



The classical 6j symbol is known [41][12][20][3] to be the coefficients in the change from the basis $\mathcal{B}$ to $\mathcal{B}'$. It is stated formally as follows:

Definition 2.2.4. *Let a, b, d, e, f be five positive integers such that (a, b, f) and (f, d, e) are admissible. Let V_a denote the $(a+1)-$dimensional irreducible representation of $SU(2)$. Let $\mathcal{B}$ and $\mathcal{B}'$ be the two "fundamental" bases for $Hom_{Rep(SU(2))}(V_e, V_a \otimes V_b \otimes V_d)$. Then,* Classical 6j symbols *are the coefficients while changing from $\mathcal{B}$ to $\mathcal{B}'$ i.e.*



where the sum ranges over all $c \in \mathbb{N}$ such that (b, d, c) and (a, c, e) are admissible. It is a finite sum.

To be completely clear, let us explain this definition in alternative words. An element of a basis in $Hom_{Rep(SU(2))}(V_e, V_a \otimes V_b \otimes V_d)$, the morphism depicted on the left hand side of the equation in Definition 2.2.4, can be written as a linear combination of the elements of the other basis formed by the morphisms depicted on the right hand side of the equation in Definition 2.2.4. Here the sum ranges over all $c \in \mathbb{N}$ such that (b, c, d) and (a, c, e) are admissible. The coefficients in the linear combination are the classical 6j symbols.

For the symbol to represent a Euclidean tetrahedron (that is what we need) which is $SO(4)-$invariant, it has to be normalized. The definition of a normalized classical 6j symbols is given below.

Definition 2.2.5. *Let a, b, c, d, e, f be six non-negative integers such that $(a, b, f), (f, d, e), (b, d, c)$ and (a, c, e) are admissible. Then, the normalized classical 6j symbols is defined as*

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \frac{(-1)^{\frac{a+b+d+e}{2}}}{f+1} \sqrt{\left| \frac{\Theta(a,b,c)\Theta(c,d,e)}{\Theta(a,e,f)\Theta(a,d,f)} \right|} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_0$$

where

$$\Theta(x,y,z) = \frac{n_{xz}!n_{xy}!n_{yz}!(n_{xz}+n_{xy}+n_{yz}+1)!}{(-1)^{n_{xz}+n_{xy}+n_{yz}}x!y!z!}$$

and

$$n_{xz} = \frac{x+z-y}{2},\ n_{xy} = \frac{x+y-z}{2},\ n_{yz} = \frac{y+z-x}{2}$$

for any admissible triples $(x, y, z) \in \mathbb{N}^3$.

For further reading on classical 6j symbols and graphical calculus [20], [12] and [3] are recommended. And for general knowledge on classical 6j symbols see [43].

2.3 Spherical simplices

Given n vectors of unit length in $\mathbb{R}^n$, there are at least two ways of defining an $(n-1)$-dimensional spherical simplex. To wit, by considering the vectors as the vertices of the simplex [21], or by looking at each of them as the unit normal vectors to each face of the simplex [28, page 286, Proof of Lemmas 1 and 2]. When $n = 3$, the simplex is a spherical triangle and for $n = 4$ it is a spherical tetrahedron. Only these two instances are looked at in this section. There are at least two ways to define a non-degenerate spherical simplex, one is given in [21] and the other in [28]. Both of these definitions will be looked at since one ([21]) is more appropriate when dealing with the classical 6j symbol and the other ([28]) when dealing with the quantum 6j symbols. Besides, the notion of cosine and sine laws in a spherical triangle will be reminded since they play an important role in our inner calculations.

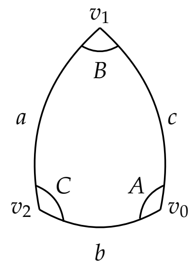


Figure 2.3: Spherical triangle S

2.3.1 Spherical triangle

The aim of this subsection is to recall the definition of a spherical triangle, the cosine law, the dual cosine law and sine law from spherical geometry.

Definition 2.3.1. *Let v_0, v_1, v_2 be three non-zero vectors in $\mathbb{R}^3$ which direct the cone defined by $C = \{\alpha_0 v_0 + \alpha_1 v_1 + \alpha_2 v_2, \alpha_i \geq 0\}$. A* spherical triangle *with vertices v_0, v_1, v_2 is defined by $C \cap S^2$ where S^2 is the unit two sphere.*

Remark 2.3.2. *[45] From spherical geometry, there are at least three results regarding a spherical triangle specifically, the cosine law, the dual cosine law and the sine law. Let's consider the spherical triangle S in Figure 2.3. Then,*

1. *by conceptually taking A as an example, the cosine law reads*

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \tag{2.5}$$

2. *Similarly, the dual cosine law is defined to be*

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}. \tag{2.6}$$

3. *And the sine law is known as*

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \tag{2.7}$$

2.3.2 Spherical tetrahedron

The two ways of defining a non-degenerate spherical tetrahedron from [21] and [28] constitute the content of this subsection.

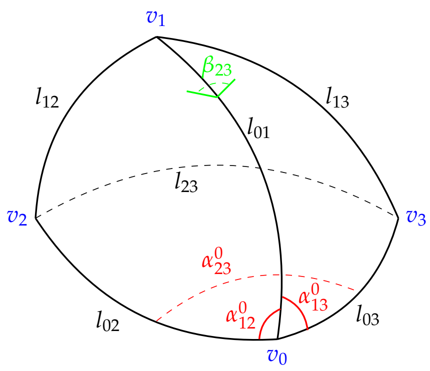


Figure 2.4: Spherical tetrahedron T showing the vertices, lengths of the edges, the interior angles around the vertex v_0 and an example of interior dihedral angle (β_{23}).

Definition 2.3.3 (Definition of a spherical tetrahedron from its vertices)**.** *[21] Let v_0, v_1, v_2, v_3 be four non-zero vectors in $\mathbb{R}^4$ which direct the cone defined by*

$$C = \{\alpha_0 v_0 + \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3, \alpha_i \geq 0\}.$$

A spherical tetrahedron *with vertices v_0, v_1, v_2, v_3 is the intersection of C with S^3 where S^3 is the unit three sphere.*

Remark 2.3.4. *The following terminology will be used throughout this thesis:*

- *The lengths of the edges which here are labeled by l_{ij} are the **edge lengths***
- *The angle between two edges named α^k_{ij} will be called **interior angles***
- *The dihedral angle between two faces of the tetrahedron β_{ij} on the figure are the **interior dihedral angles**.*

A different way to define a spherical tetrahedron is:

Definition 2.3.5 (Definition of a spherical tetrahedron from the outward normal vectors to each of its faces)**.** *[28] Let g_0, g_1, g_2, g_3 be four unit vectors in $\mathbb{R}^4$. Consider the set*

$$H_i = \{x \in \mathbb{R}^4 : x.g_i \leq 0\}.$$

A spherical tetrahedron *T is defined to be the intersection of $\bigcap_{i=0}^{3} H_i$ with the unit sphere S^3.*

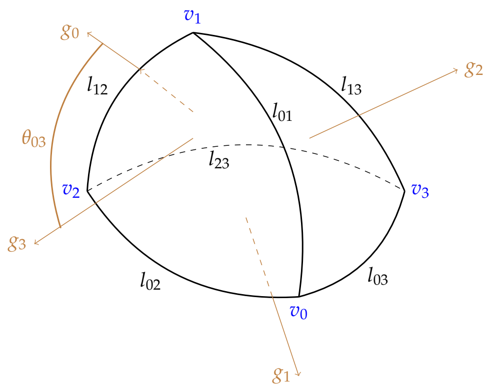


Figure 2.5: Spherical tetrahedron T including the outward normal vectors.

Remark 2.3.6.

- *The vectors g_0, g_1, g_2, g_3 are the outward normal vectors to each face of the spherical tetrahedron.*
- *Let θ_{ij} be the angle between g_i and g_j. By definition θ_{ij} is called the **exterior dihedral angle** of T.*

Remark 2.3.7. *The definitions 2.3.3 and 2.3.5 are equivalent. This equivalence amounts to say that given the vertices of a spherical tetrahedron T, its unit outward normal vectors may be computed and vice versa. So, let us denote the set of vertices of T by*

$$V = \{\textit{vertices } v_0, v_1, v_2, v_3 \textit{ of } T\}$$

and the set of its unit outward normal vectors by

$$O = \{\textit{unit outward normal vectors } g_0, g_1, g_2, g_3 \textit{ of } T\}.$$

Given a vertex v_i of T, then its dual v_i^* is defined by

$$\langle v_i^*, v_j \rangle = \delta_{ij}. \tag{2.8}$$

Then, let us set

$$g_i := -\frac{v_i^*}{\|v_i^*\|}.$$

It follows from (2.8) that g_i is normal to the hyperplane formed by the vectors v_j, v_k, v_l where $j, k, l \neq i$ in the opposite direction to v_i. By repeating that process for $i = 0, \cdots, 3$ one would obtain the four outward normal vectors to each face of the tetrahedron.

Let us consider the four unit outward normal vectors $g_0, g_1, g_2, g_3 \in \mathbb{R}^4$ to T i.e. they define the hyperplanes

$$H_i := \{x \in \mathbb{R}^4 : x.g_i \leq 0\}.$$

Let g_i^* be the dual to g_i, and let us set

$$v_i := -\frac{g_i^*}{\|g_i^*\|}.$$

Therefore, by definition

$$\langle v_i, g_j \rangle = -\frac{1}{\|g_i^*\|}\delta_{ij}$$

which indicates that $v_i \in \bigcap_{i=0}^{3} H_i \bigcap S^3$, with norm one (1) and is indeed the point where the $H_i's$ intersect. Hence, the vector v_i is a vertex of the spherical tetrahedron T. This way, when i varies from 0 to 3, all the vertices of T are recovered.

Definition 2.3.8. *A spherical tetrahedron T is called **non-degenerate** if its vertices, equivalently its outward normal vectors, are linearly independent as unit vectors in $\mathbb{R}^4$. It is otherwise called **degenerate**.*

Definition 2.3.9. *Let T be a spherical tetrahedron with vertices v_0, v_1, v_2, v_3. Let $v_0^*, v_1^*, v_2^*, v_3^*$ be four unit vectors in $\mathbb{R}^4$ such that*

$$\langle v_i, v_j^* \rangle_{\mathbb{R}^4} = \delta_{ij}$$

where δ_{ij} denotes the Kronecker delta. Then the spherical tetrahedron T^* with vertices

$$g_i := -\frac{v_i^*}{\|v_i^*\|}$$

is called the dual tetrahedron *of T.*

Remark 2.3.10. *Geometrically, the four unit outward normal vectors $g_0, g_1, g_2, g_3 \in \mathbb{R}^4$ to the spherical tetrahedron defined by Definition 2.3.5 constitute the vertices of the dual tetrahedron to T. And a spherical tetrahedron T is non-degenerate if and only if its dual is.*

Let us recall that this Chapter aims to reproduce the six-dimensional integral formula for the classical 6j symbols by Freidel and Louapre. From Equation (2.2), the matrix $[\cos l_{ij}]$ in the denominator of the integrand actually corresponds to the edge Gram matrix of a spherical tetrahedron. Hence, let us look at the definition of the edge Gram matrix. Here, Luo's definition found in [24] which is expressed in terms of the dot product between the vertices of the tetrahedron is adopted.

Definition 2.3.11. *Let T be a spherical tetrahedron with vertices v_0, v_1, v_2 and v_3 (see Figure 2.4). The* edge Gram matrix *of T is the 4×4 matrix defined by $G_{ij} = v_i.v_j$, which is equivalent to*

$$G = \begin{pmatrix} 1 & \cos l_{01} & \cos l_{02} & \cos l_{03} \\ \cos l_{01} & 1 & \cos l_{12} & \cos l_{13} \\ \cos l_{02} & \cos l_{12} & 1 & \cos l_{23} \\ \cos l_{03} & \cos l_{13} & \cos l_{23} & 1 \end{pmatrix},$$

where l_{ij} is the length of the edge $(v_i v_j)$.

Besides the edge Gram matrix, there is also the angle Gram matrix which in turn is composed by the negative cosine of the **interior dihedral angles** of the tetrahedron T. Recall its definition:

Definition 2.3.12. *Let T be a spherical tetrahedron with vertices v_0, v_1, v_2, v_3 and* ***interior dihedral angles*** *β_{ij} around the edge $(v_k v_l)$ where $\{i,j\} \neq \{k,l\}$, as shown in Figure 2.4. The* angle Gram matrix *is defined as*

$$G_1 = \begin{pmatrix} 1 & -\cos\beta_{01} & -\cos\beta_{02} & -\cos\beta_{03} \\ -\cos\beta_{01} & 1 & -\cos\beta_{12} & -\cos\beta_{13} \\ -\cos\beta_{02} & -\cos\beta_{12} & 1 & -\cos\beta_{23} \\ -\cos\beta_{03} & -\cos\beta_{13} & -\cos\beta_{23} & 1 \end{pmatrix}.$$

2.4 Links

In his paper [25], Feng Luo presented the beautiful concept of a link which is often used in this Chapter. However, he did not explain why the interior angles of the link are the same as the suitable interior dihedral angles of the spherical tetrahedron. Hence, we give an explanation on that in this section.

Definition 2.4.1. *Let T be a spherical tetrahedron with vertices v_0, v_1, v_2 and v_3 as shown in Figure 2.4. Let $i, j, k \in \{0, 1, 2, 3\}$ and denote by α_{ij}^k the interior angle of T at the vertex v_k and opposite to the edge $(v_i v_j)$. The* link *at vertex v_k, denoted by $Lk(v_k)$, is the spherical*

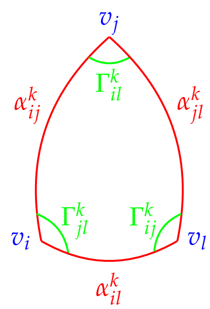


Figure 2.6: Link $Lk(v_k)$

triangle with edge lengths $\alpha_{ij}^k, \alpha_{il}^k, \alpha_{jl}^k$. Let us set $\Gamma_{ij}^k, \Gamma_{il}^k, \Gamma_{jl}^k$ to be its interior angles such that Γ_{ij}^k is opposite to α_{ij}^k as shown in Figure 2.6 .

By looking at the spherical tetrahedron Figure 2.4, the portion of circle representing each interior angle around the vertex v_0 cross at one point of a specific edge. The guess is that the angle between them is the interior dihedral angle along the edge of crossing. And the lemma below, which was not made clear by Luo, formalises that concept.

Lemma 2.4.2. *Let Γ_{il}^k be an interior angle in the link $Lk(v_k)$. Let β_{il} be the interior dihedral angle of T at the edge $(v_j v_k)$. Then,*

$$\Gamma_{il}^k = \beta_{il}.$$

Proof. In this proof, let us pick one example of a link namely $Lk(v_0)$ but the proof may be adapted for any link.

By acting with an appropriate element of $SO(4)$, T may be rotated so that its vertices are in standard position:

$$v_0 = (1, 0, 0, 0),\ v_i = (\cos\theta_i, \sin\theta_i n_i) \quad i = 1, \cdots, 3$$

here $n_i \in S^2$ are the vertices of $Lk(v_0)$ as in Figure 2.7:

$$n_1 = (0, 0, 1), n_2 = (\sin\alpha_{12}^0, 0, \cos\alpha_{12}^0), n_3 = (\sin\alpha_{13}^0 \cos\alpha_{23}^0, \sin\alpha_{13}^0 \sin\alpha_{23}^0, \cos\alpha_{13}^0).$$

By definition,

$$\cos\beta_{23} = -w_2.w_3,$$

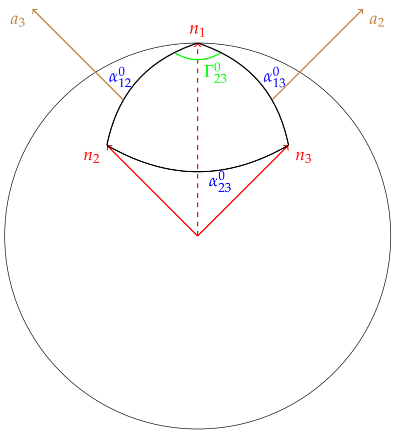


Figure 2.7: The link at v_0

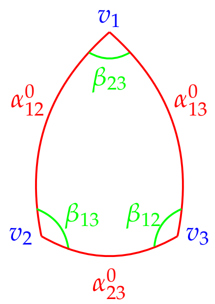


Figure 2.8: Link $Lk(v_0)$

where $w_2, w_3 \in \mathbb{R}^4$ are the outward unit normal vectors to the faces $v_0v_1v_3$ and $v_0v_1v_2$ of T respectively. Clearly we have

$$w_2 = (0, a_2), w_3 = (0, a_3)$$

where $a_2, a_3 \in \mathbb{R}^3$ are the outward unit vectors normal to the edges n_1n_3 and n_1n_2 of $Lk(v_0)$ respectively (see Figure 2.7). But by definition

$$\cos \Gamma^0_{23} = -a_2.a_3$$

which shows that $\Gamma^0_{23} = \beta_{23}$. $\square$

Therefore, from now on the link at v_0 will be represented by the Figure 2.8 and the other links follow accordingly.

2.5 Box variables

The notion of box variables is not new in the literature. They feature in [17] but are not explicitly mentioned as box variables. In this section, we reintroduce the box variables and express the quantities related to a spherical tetrahedron such as edge lengths and interior dihedral angles as functions of them. Although the expression of the determinant of the edge Gram matrix of a spherical tetrahedron in terms of the box variables is present in [17], the detailed calculation was not given. Hence, we provide it in this section as well. The use of the box variables in our integration induces the integration domain to be the entire cube and the integration measure to be a product measure which makes life simpler for numerical calculations. In addition, the determinant of the edge Gram matrix has a "nicer" expression in terms of the box variables. These are some of the reasons why the notion of box variables is re-introduced here.

Definition 2.5.1. *Let T be a spherical tetrahedron with vertices v_0, v_1, v_2, v_3. We call* box variables *the set*

$$B = \{l_{01}, l_{02}, l_{03}, \alpha^0_{12}, \alpha^0_{13}, \beta_{23}\}$$

where l_{01}, l_{02}, l_{03} are the respective lengths of the edges $(v_0v_1), (v_0v_2), (v_0v_3)$, $\alpha^0_{12}, \alpha^0_{13}$ the respective interior angle between the edges $(v_0v_1), (v_0v_2)$ and $(v_0v_1), (v_0v_3)$, and β_{23} the interior dihedral angle around the edge (v_0v_1) as shown in Figure 2.4.

Given four unit vectors v_0, v_1, v_2, v_3 in $\mathbb{R}^4$. They may be expressed in spherical coordinates as:

$$v_i = (\cos x_i, \sin x_i \sin y_i \cos z_i, \sin x_i \sin y_i \sin z_i, \sin x_i \cos y_i).$$

And, formally explained in Definition 2.3.3 v_0, v_1, v_2, v_3 may be considered as the vertices of a spherical tetrahedron T. Let

$$\begin{aligned} L_g : &S^3 \longrightarrow S^3 \\ &v \longmapsto gv \end{aligned}$$

be the restriction to S^3 of the linear action of $SO(4)$ on $\mathbb{R}^4$. Under this action these vectors may be rotated in such a way that the edge lengths and interior dihedral angles of T are preserved. In other words, rotating the vectors is equivalent to rotating T. Let $g \in SO(4)$ such that [4]

$$\begin{aligned}
\tilde{v}_0 &:= gv_0 = (1,0,0,0),\\
\tilde{v}_1 &:= gv_1 = (\cos x_1, 0, 0, \sin x_1),\\
\tilde{v}_2 &:= gv_2 = (\cos x_2, \sin x_2 \sin y_2, 0, \sin x_2 \cos y_2),\\
\tilde{v}_3 &:= gv_3 = (\cos x_3, \sin x_3 \sin y_3 \cos z_3, \sin x_3 \sin y_3 \sin z_3, \sin x_3 \cos y_3),
\end{aligned}$$

where $x_1, x_2, x_3, y_2, y_3 \in [0, \pi]$ and $z_3 \in [0, 2\pi]$. By using these new coordinates, each vector may be expressed in terms of the box variables as follows:

Proposition 2.5.2. *Suppose $z_3 \in [0, \pi]$. Then the variables x_i are equal to the edge lengths l_{0i} of the spherical tetrahedron T (see Figure 2.9), the variables y_i overlap with the interior angles α^0_{1i} at $\tilde{v}_0$ and z_3 with the interior dihedral angle β_{23} i.e.*

$$x_1 = l_{01},\ x_2 = l_{02},\ x_3 = l_{03},$$

$$y_2 = \alpha^0_{12},\ y_3 = \alpha^0_{13},$$

$$z_3 = \beta_{23}.$$

Proof. Let $\overrightarrow{n_i}$ be the unit vector tangent to the edge $(\tilde{v}_0\tilde{v}_i)$ at $\tilde{v}_0$ (see Figure 2.9 below). Here,

$$\begin{aligned}
\overrightarrow{n_1} &= (0,0,1),\\
\overrightarrow{n_2} &= (\sin y_2, 0, \cos y_2),\\
\overrightarrow{n_3} &= (\sin y_3 \cos z_3, \sin y_3 \sin z_3, \cos y_3).
\end{aligned}$$

By definition the cosine of x_i is given by

$$\begin{aligned}
\cos x_i &= \tilde{v}_0.\tilde{v}_i\\
&= \cos l_{0i},
\end{aligned}$$

thus, $x_i = l_{0i}$ when $x_i \in [0, \pi]$.

To understand y_i and z_3, let's use the fact that by definition

$$\overrightarrow{n_i}.\overrightarrow{n_j} = \cos \alpha^0_{ij}.$$

An application of that equality is

$$\begin{aligned}\overrightarrow{n_1}.\overrightarrow{n_2} &= \cos y_2\\ &= \cos \alpha_{12}^0,\end{aligned}$$

which implies that

$$y_2 = \alpha_{12}^0 \tag{2.9}$$

when $y_2 \in [0, \pi]$.

Similarly to the case of y_2,

$$\begin{aligned}\overrightarrow{n_1}.\overrightarrow{n_3} &= \cos y_3\\ &= \cos \alpha_{13}^0.\end{aligned}$$

That implies,

$$y_3 = \alpha_{13}^0 \tag{2.10}$$

when $y_3 \in [0, \pi]$.

For the case of z_3,

$$\begin{aligned}\overrightarrow{n_2}.\overrightarrow{n_3} &= \sin y_2 \sin y_3 \cos z_3 + \cos y_2 \cos y_3\\ &= \cos \alpha_{23}^0.\end{aligned}$$

This equality amounts to say that

$$\sin \alpha_{12}^0 \sin \alpha_{13}^0 \cos z_3 + \cos \alpha_{12}^0 \cos \alpha_{13}^0 = \cos \alpha_{23}^0$$

because of (2.9) and (2.10).

Lastly, by applying Equation (2.5) from Remark 2.3.2 to the link $Lk(\tilde{v}_0)$ one obtains

$$\cos z_3 = \cos \beta_{23}$$

i.e. $z_3 = \beta_{23}$ when $z_3 \in [0, \pi]$. □

Remark 2.5.3. *It is important to keep in mind that only when $z_3 \in [0, \pi]$, then the box variables may be used. This is important while dealing with the integration later.*

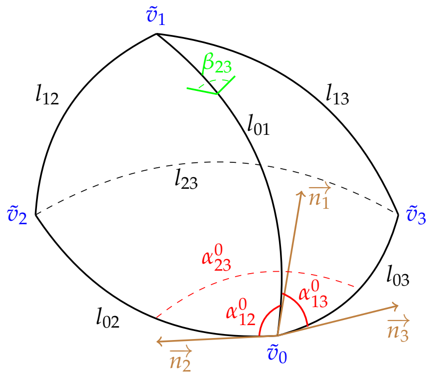


Figure 2.9: Spherical tetrahedron T with the tangent vectors $\overrightarrow{n_i}$.

It is a fact that the edge lengths, the interior angles and interior dihedral angles may be expressed as functions of the box variables. The upcoming results are not only used in the computation of the determinant of the edge Gram matrix, but also in our numerical calculation.

Lemma 2.5.4. *The edge lengths, the interior angles and the interior dihedral angles of a tetrahedron may be expressed as functions of the box variables.*

Proof. By applying the cosine law to all the links and faces of the tetrahedron, the following results, which are supposed to be read consecutively from top to bottom hold:

1. $\cos\alpha^0_{23} = \cos\alpha^0_{12}\cos\alpha^0_{13} + \sin\alpha^0_{12}\sin\alpha^0_{13}\cos\beta_{23}$,
2. $\cos\beta_{13} = \frac{\cos\alpha^0_{13} - \cos\alpha^0_{12}\cos\alpha^0_{23}}{\sin\alpha^0_{12}\sin\alpha^0_{23}}$,
3. $\cos\beta_{12} = \frac{\cos\alpha^0_{12} - \cos\alpha^0_{13}\cos\alpha^0_{23}}{\sin\alpha^0_{13}\sin\alpha^0_{23}}$,
4. $\cos l_{23} = \cos l_{03}\cos l_{02} + \sin l_{03}\sin l_{02}\cos\alpha^0_{23}$,
5. $\cos\alpha^2_{03} = \frac{\cos l_{03} - \cos l_{02}\cos l_{23}}{\sin l_{02}\sin l_{23}}$,
6. $\cos l_{12} = \cos l_{02}\cos l_{01} + \sin l_{02}\sin l_{01}\cos\alpha^0_{12}$,
7. $\cos\alpha^1_{02} = \frac{\cos l_{02} - \cos l_{01}\cos l_{12}}{\sin l_{01}\sin l_{02}}$,
8. $\cos\alpha^2_{01} = \frac{\cos l_{01} - \cos l_{02}\cos l_{12}}{\sin l_{02}\sin l_{12}}$,

9. $\cos\alpha^2_{13} = \cos\alpha^2_{01}\cos\alpha^2_{03} + \sin\alpha^2_{01}\sin\alpha^2_{03}\cos\beta_{13}$,
10. $\cos\beta_{01} = \frac{\cos\alpha^2_{01} - \cos\alpha^2_{03}\cos\alpha^2_{13}}{\sin\alpha^2_{03}\sin\alpha^2_{13}}$,
11. $\cos\beta_{03} = \frac{\cos\alpha^2_{03} - \cos\alpha^2_{01}\cos\alpha^2_{13}}{\sin\alpha^2_{01}\sin\alpha^2_{13}}$,
12. $\cos\beta_{02} = -\cos\beta_{23}\cos\beta_{03} + \sin\beta_{23}\sin\beta_{03}\cos\alpha^1_{02}$,
13. $\cos l_{13} = \cos l_{01}\cos l_{03} + \sin l_{01}\sin l_{03}\cos\alpha^0_{13}$,
14. $\cos\alpha^1_{03} = \frac{\cos l_{03} - \cos l_{01}\cos l_{13}}{\sin l_{01}\sin l_{13}}$,
15. $\cos\alpha^2_{13} = \frac{\cos l_{13} - \cos l_{12}\cos l_{23}}{\sin l_{12}\sin l_{23}}$,
16. $\cos\alpha^2_{03} = \frac{\cos l_{03} - \cos l_{02}\cos l_{23}}{\sin l_{02}\sin l_{23}}$,
17. $\cos\alpha^1_{23} = \frac{\cos\beta_{23} - \cos\alpha^1_{03}\cos\alpha^1_{02}}{\sin\alpha^1_{03}\sin\alpha^1_{02}}$,
18. $\cos\alpha^3_{01} = \frac{\cos l_{01} - \cos l_{03}\cos l_{13}}{\sin l_{03}\sin l_{13}}$,
19. $\cos\alpha^3_{02} = \frac{\cos l_{02} - \cos l_{03}\cos l_{23}}{\sin l_{03}\sin l_{23}}$,
20. $\cos\alpha^3_{12} = \frac{\cos l_{12} - \cos l_{13}\cos l_{23}}{\sin l_{13}\sin l_{23}}$.

□

In particular, as the edge lengths l_{12}, l_{13}, l_{23} are frequently used throughout the thesis, let us emphasize their expression in terms of the box variables.

$$\cos l_{12} = \cos l_{01}\cos l_{02} + \sin l_{01}\sin l_{02}\cos\alpha^0_{12}, \tag{2.11}$$

$$\cos l_{13} = \cos l_{01}\cos l_{03} + \sin l_{01}\sin l_{03}\cos\alpha^0_{13}, \tag{2.12}$$

$$\cos l_{23} = \cos l_{02}\cos l_{03} + \sin l_{02}\sin l_{03}(\cos\alpha^0_{12}\cos\alpha^0_{13} + \sin\alpha^0_{12}\sin\alpha^0_{13}\cos\beta_{23}). \tag{2.13}$$

These were obtained by manipulating the cosine law from the faces $\Delta 012, \Delta 013$, $\Delta 023$ of the spherical tetrahedron and the link $Lk(v_0)$.

If computed directly from the Definition 2.3.11, the expression of the determinant of the edge Gram matrix, which is in terms of the edge lengths of the tetrahedron, is long, heavy and almost impossible to remember. And that may lead to a difficulty in identifying it in the middle of a calculation. However, in terms of the box variables it is short and easy to catch. Therefore, let us express the determinant of the edge Gram matrix in terms of the box variables. For that, one more step needs to be considered.

For the following lemma and proposition, let us consider T to be the spherical tetrahedron with vertices v_0, v_1, v_2, v_3, conform to Figure 2.4.

Lemma 2.5.5. *The following equality holds*

$$\sin^2\alpha^0_{12}\sin^2\alpha^0_{13}\sin^2\beta_{23} = 1-\cos^2\alpha^0_{23}-\cos^2\alpha^0_{12}\cos^2\alpha^0_{13}+2\cos\alpha^0_{12}\cos\alpha^0_{13}\cos\alpha^0_{23}.$$

Proof. By using the equality $\sin^2 x + \cos^2 x = 1$ in the first step of the equalities, and by applying Equation (2.5) to $Lk(v_0)$ in the third equality one obtains

$$\begin{aligned}
\sin^2\alpha^0_{12}\sin^2\alpha^0_{13}\sin^2\beta_{23} &= \sin^2\alpha^0_{12}\sin^2\alpha^0_{13}(1-\cos^2\beta_{23})\\
&= \sin^2\alpha^0_{12}\sin^2\alpha^0_{13} - \sin^2\alpha^0_{12}\sin^2\alpha^0_{13}\cos^2\beta_{23}\\
&= \sin^2\alpha^0_{12}\sin^2\alpha^0_{13} - (\cos\alpha^0_{23}-\cos\alpha^0_{12}\cos\alpha^0_{13})^2\\
&= \sin^2\alpha^0_{12}\sin^2\alpha^0_{13} - \cos^2\alpha^0_{23} + 2\cos\alpha^0_{23}\cos\alpha^0_{12}\cos\alpha^0_{13} + \cos^2\alpha^0_{12}\cos^2\alpha^0_{13}\\
&= \sin^2\alpha^0_{12}(1-\cos^2\alpha^0_{13}) - \cos^2\alpha^0_{23} + 2\cos\alpha^0_{23}\cos\alpha^0_{12}\cos\alpha^0_{13}\\
&+ \cos^2\alpha^0_{12}\cos^2\alpha^0_{13}\\
&= \sin^2\alpha^0_{12} - \sin^2\alpha^0_{12}\cos^2\alpha^0_{13} - \cos^2\alpha^0_{23} + 2\cos\alpha^0_{23}\cos\alpha^0_{12}\cos\alpha^0_{13}\\
&+ \cos^2\alpha^0_{12}\cos^2\alpha^0_{13}\\
&= \sin^2\alpha^0_{12} - \cos^2\alpha^0_{13} - \cos^2\alpha^0_{23} + 2\cos\alpha^0_{23}\cos\alpha^0_{12}\cos\alpha^0_{13}\\
&= 1-\cos^2\alpha^0_{12} - \cos^2\alpha^0_{13} - \cos^2\alpha^0_{23} + 2\cos\alpha^0_{23}\cos\alpha^0_{12}\cos\alpha^0_{13}.
\end{aligned}$$

□

At last, the determinant of the edge Gram matrix of the tetrahedron T as function of the box variables is given by:

Proposition 2.5.6. *The determinant of the edge Gram matrix of T may be expressed in terms of the box variables as*

$$\det[\cos l_{ij}] = \sin^2 l_{01}\sin^2 l_{02}\sin^2 l_{03}\sin^2\alpha^0_{12}\sin^2\alpha^0_{13}\sin^2\beta_{23}.$$

Proof. Let's compute the determinant of $[\cos l_{ij}]$. Note that the Cosine law (2.5) is used while moving from the second to the third equality.

$$\begin{aligned}
\det[\cos l_{ij}] &= \det\begin{pmatrix} 1 & \cos l_{01} & \cos l_{02} & \cos l_{03} \\ \cos l_{01} & 1 & \cos l_{12} & \cos l_{13} \\ \cos l_{02} & \cos l_{12} & 1 & \cos l_{23} \\ \cos l_{03} & \cos l_{13} & \cos l_{23} & 1 \end{pmatrix} \\
&= \det\begin{pmatrix} 1 & \cos l_{01} & \cos l_{02} & \cos l_{03} \\ 0 & 1-\cos^2 l_{01} & \cos l_{12}-\cos l_{01}\cos l_{02} & \cos l_{13}-\cos l_{01}\cos l_{03} \\ 0 & \cos l_{12}-\cos l_{01}\cos l_{02} & 1-\cos^2 l_{02} & \cos l_{23}-\cos l_{02}\cos l_{03} \\ 0 & \cos l_{13}-\cos l_{01}\cos l_{03} & \cos l_{23}-\cos l_{02}\cos l_{03} & 1-\cos^2 l_{03} \end{pmatrix} \\
&= \det\begin{pmatrix} \sin^2 l_{01} & \sin l_{01}\sin l_{02}\cos\alpha^0_{12} & \sin l_{01}\sin l_{03}\cos\alpha^0_{13} \\ \sin l_{01}\sin l_{02}\cos\alpha^0_{12} & \sin^2 l_{02} & \sin l_{02}\sin l_{03}\cos\alpha^0_{23} \\ \sin l_{01}\sin l_{03}\cos\alpha^0_{13} & \sin l_{02}\sin l_{03}\cos\alpha^0_{23} & \sin^2 l_{03} \end{pmatrix} \\
&= \sin^2 l_{01}(\sin^2 l_{02}\sin^2 l_{03} - \sin^2 l_{02}\sin^2 l_{03}\cos^2\alpha^0_{23}) \\
&\quad - \sin l_{01}\sin l_{02}\cos\alpha^0_{12}(\sin^2 l_{03}\sin l_{01}\sin l_{02}\cos\alpha^0_{12} - \sin l_{02}\sin l_{01}\sin^2 l_{03}\cos\alpha^0_{23}\cos\alpha^0_{13}) \\
&\quad + \sin l_{01}\sin l_{03}\cos\alpha^0_{13}(\sin l_{01}\sin l_{03}\sin^2 l_{02}\cos\alpha^0_{12}\cos\alpha^0_{23} - \sin^2 l_{02}\sin l_{01}\sin l_{03}\cos\alpha^0_{13}) \\
&= \sin^2 l_{01}\sin^2 l_{02}\sin^2 l_{03}(1-\cos^2\alpha^0_{23} - \cos^2\alpha^0_{12}\cos^2\alpha^0_{13} + 2\cos\alpha^0_{12}\cos\alpha^0_{13}\cos\alpha^0_{23}) \\
&= \sin^2 l_{01}\sin^2 l_{02}\sin^2 l_{03}\sin^2\alpha^0_{12}\sin^2\alpha^0_{13}\sin^2\beta_{23} \quad \text{(By Lemma 2.5.5).}
\end{aligned}$$

□

2.6 Lebesgue measure on S^3

The integral we would like to transform is over four copies of $SU(2)$, and one of the steps is to translate it to an integration over four copies of S^3. Hence, the intention of this section is to recall the Lebesgue measure on S^3 and show its invariance under the action of $SO(4)$. That is done by the analysis of a volume form on S^3. A general reminder on volume forms and integration on a manifold is provided in Appendix B.

Lemma 2.6.1. *Consider the local parameterization*

$$h^{-1} : (0,\pi)\times(0,\pi)\times(0,2\pi) \longrightarrow S^3$$

such that

$$(\theta, \alpha, \beta) \mapsto g = (\cos\theta, \sin\theta\sin\alpha\cos\beta, \sin\theta\sin\alpha\sin\beta, \sin\theta\cos\alpha).$$

Then,

$$\omega_g = \sin^2\theta\sin\alpha \mathrm{d}\theta \wedge \mathrm{d}\alpha \wedge \mathrm{d}\beta$$

is the volume form on S^3*.*

Proof. As shown in Equation (B.4), the "main" coefficient of a volume form on a manifold is determined by the square root of the determinant of the matrix whose entries are the inner products of the tangent vectors. For S^3 the basis of the tangent space at g is given by

$$\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}\},$$

hence the tangent vectors are

$$g_\theta := \frac{\partial g}{\partial\theta}(\theta, \alpha, \beta) = (-\sin\theta, \cos\theta\sin\alpha\cos\beta, \cos\theta\sin\alpha\sin\beta, \cos\theta\cos\alpha),$$

$$g_\alpha := \frac{\partial g}{\partial\alpha}(\theta, \alpha, \beta) = (0, \sin\theta\cos\alpha\cos\beta, \sin\theta\cos\alpha\sin\beta, -\sin\alpha\sin\beta),$$

$$g_\beta := \frac{\partial g}{\partial\beta}(\theta, \alpha, \beta) = (0, -\sin\theta\sin\alpha\sin\beta, \sin\theta\sin\alpha\cos\beta, 0).$$

Therefore, the matrix composed of the inner products of the tangent vectors reads as follows:

$$G_1 = \begin{pmatrix} < g_\theta, g_\theta > & < g_\theta, g_\alpha > & < g_\theta, g_\beta > \\ < g_\alpha, g_\theta > & < g_\alpha, g_\alpha > & < g_\alpha, g_\beta > \\ < g_\beta, g_\theta > & < g_\beta, g_\alpha > & < g_\beta, g_\beta > \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\theta & 0 \\ 0 & 0 & \sin^2\theta\sin^2\alpha \end{pmatrix}$$

with determinant

$$\det G_1 = \sin^4\theta\sin^2\alpha.$$

Thus, the square root of the determinant is given by

$$\sqrt{\det G_1} = \sin^2\theta\sin\alpha.$$

Which implies that the "main" volume form on S^3 is given by

$$\omega_g = \sin^2\theta \sin\alpha \mathrm{d}\theta \wedge \mathrm{d}\alpha \wedge \mathrm{d}\beta.$$

□

Moreover, this volume form on S^3 is left-invariant under the action of $SO(4)$ on S^3.

Lemma 2.6.2. *The Euclidean volume form $\omega \in \Omega^3(S^3)$ is left-invariant under $SO(4)$ i.e. if $g \in SO(4)$ and*

$$\begin{aligned} L_g : & S^3 \longrightarrow S^3 \\ & v \longmapsto gv \end{aligned}$$

be the restriction to S^3 of the linear action of $SO(4)$ on $\mathbb{R}^4$. Then, $L_g^*\omega = \omega$.

Proof. Let $v \in S^3, g \in SO(4)$ and $v_1, v_2, v_3 \in T_vS^3$.

Therefore, by definition (see Definition B.1.6)

$$\begin{aligned} (L_g^*\omega_v)(v_1, v_2, v_3) &= \omega_{gv}((D_vL_g)v_1, (D_vL_g)v_2, (D_vL_g)v_3) \\ &= \omega_{gv}(L_gv_1, L_gv_2, L_gv_3) \\ &= \omega_{gv}(gv_1, gv_2, gv_3) && L_g \text{ linear} \\ &= \sqrt{\det(< gv_i, gv_j >_{\mathbb{R}^4})} \\ &= \sqrt{\det(< v_i, v_j >_{\mathbb{R}^4})} && \text{as } g \text{ preserves inner products} \\ &= \omega_v(v_1, v_2, v_3). \end{aligned}$$

Therefore $L_g^*\omega = \omega$. □

Based on Proposition B.2.3, a volume form on S^3 induces a measure μ which is finite on S^3 since S^3 is a compact submanifold of $\mathbb{R}^4$. By definition that measure is also positive. It is the Lebesgue measure on S^3 such that

$$\mu(H) := \int_H \omega$$

for a measurable subset H in S^3. And from Lemma 2.6.2, that volume form is invariant under the action of $SO(4)$, hence so is the induced measure.

2.7 Integral formula for the classical 6j symbols

Consider the equality

$$\int_{(SU(2))^4} \prod_{i<j} \chi_{m_{ij}}(g_j g_i^{-1})[\prod_{i=0}^{3} dg_i] = \frac{2}{\pi^4} \int_{D_\pi} \frac{\prod_{i<j} \sin((m_{ij}+1)l_{ij})}{\sqrt{\det([\cos l_{ij}])}} [\prod_{i<j} \mathrm{d}l_{ij}]. \quad (2.14)$$

Although it is proven in [17], a thorough process of the calculations is not provided. So, we have the objective to prove Equation (2.14) meticulously in this section. The strategy is first to convert the left hand side of (2.14) into an integration over the box variables, that is done in Subsection 1, then transform the latter to the right hand side of the equality, see Subsection 2.

Throughout this section, all integration are treated as Riemann integration.

2.7.1 Integral in terms of the box variables

Let us denote the integral on the left hand side of Equation (2.14) by

$$I = \int_{(SU(2))^4} \prod_{i<j} \chi_{m_{ij}}(g_j g_i^{-1})[\prod_{i=0}^{3} dg_i]. \quad (2.15)$$

The aim of this subsection is to rewrite the integral (2.15) in terms of the box variables. Our strategy is to first express it as an integral over $(S^3)^4$, then rewrite it in terms of spherical coordinates.

Let g_0, g_1, g_2, g_3 be four elements in $SU(2)$. Due to the diffeomorphism (A.1)

$$f : S^3 \longrightarrow SU(2),$$

let us set

$$v_0 := f^{-1}(g_0),$$
$$v_1 := f^{-1}(g_1),$$
$$v_2 := f^{-1}(g_2),$$
$$v_3 := f^{-1}(g_3).$$

Let $g \in SO(4)$, recall the action of $SO(4)$ on S^3

$$L_g : S^3 \longrightarrow S^3$$
$$v \longmapsto gv.$$

From (2.15) let us set the product of characters in the integrand of I to be

$$X = \prod_{i<j} \chi_{m_{ij}}(g_j g_i^{-1}).$$

From Proposition A.2.3 the product X becomes

$$X = \prod_{i<j} \frac{\sin((m_{ij}+1)l_{ij})}{\sin l_{ij}}. \tag{2.16}$$

Hence, X may be thought as a function of $\langle v_i, v_j \rangle_{\mathbb{R}^4}$ since $l_{ij} = \arccos(\langle v_i, v_j \rangle_{\mathbb{R}^4})$. In other words, X can be seen as

$$X = F(\langle v_i, v_j \rangle_{\mathbb{R}^4}), \tag{2.17}$$

where $i, j \in \{0, 1, 2, 3\}$. Therefore, the integral (2.15) is equivalent to

$$I = \int_{(S^3)^4} F(\langle v_i, v_j \rangle_{\mathbb{R}^4})[\prod_{i=0}^{3} dv_i]. \tag{2.18}$$

Lemma 2.7.1. *The function F is invariant under the action of $SO(4)$ i.e. for $g \in SO(4)$*

$$F(\langle v_i, v_j \rangle_{\mathbb{R}^4}) = F(\langle gv_i, gv_j \rangle_{\mathbb{R}^4}),$$

where $i, j \in \{0, 1, 2, 3\}$.

Proof. It is obvious since $\langle v_i, v_j \rangle_{\mathbb{R}^4} = \langle gv_i, gv_j \rangle_{\mathbb{R}^4}$ when $g \in SO(4)$. □

As unit vectors in $\mathbb{R}^4$, v_0, v_1, v_2 and v_3 may be written in spherical coordinates as:

$$v_0 = (\cos x_0, \sin x_0 \sin y_0 \cos z_0, \sin x_0 \sin y_0 \sin z_0, \sin x_0 \cos y_0),$$

$$v_1 = (\cos x_1, \sin x_1 \sin y_1 \cos z_1, \sin x_1 \sin y_1 \sin z_1, \sin x_1 \cos y_1),$$

$$v_2 = (\cos x_2, \sin x_2 \sin y_2 \cos z_2, \sin x_2 \sin y_2 \sin z_2, \sin x_2 \cos y_2),$$

$$v_3 = (\cos x_3, \sin x_3 \sin y_3 \cos z_3, \sin x_3 \sin y_3 \sin z_3, \sin x_3 \cos y_3),$$

where $x_i \in [0, \pi]$, $y_i \in [0, \pi]$ and $z_i \in [0, 2\pi]$. And accordingly, F may be expressed in terms of x_i, y_i, z_i, i.e.

$$F \equiv F(x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3, z_0, z_1, z_2, z_3).$$

Furthermore, let us bring back from Lemma 2.6.1 that the normalized volume form on S^3 in spherical coordinates is given by

$$dv_i = \frac{1}{2\pi^2} \sin^2 x_i \sin y_i dx_i dy_i dz_i,$$

since the volume of a three-sphere is $2\pi^2$.

From Definition 2.3.3 v_0, v_1, v_2, v_3 may be considered as the vertices of a spherical tetrahedron T. And by using the same strategy in Section 2.5 they may be rotated in such a way that the action is equivalent to rotating T i.e. [4] there exists $g \in SO(4)$ such that the rotation results in

$$\begin{aligned}
\tilde{v}_0 &:= gv_0 = (1,0,0,0),\\
\tilde{v}_1 &:= gv_1 = (\cos x_1, 0, 0, \sin x_1),\\
\tilde{v}_2 &:= gv_2 = (\cos x_2, \sin x_2 \sin y_2, 0, \sin x_2 \cos y_2),\\
\tilde{v}_3 &:= gv_3 = (\cos x_3, \sin x_3 \sin y_3 \cos z_3, \sin x_3 \sin y_3 \sin z_3, \sin x_3 \cos y_3).
\end{aligned}$$

Based on Lemma 2.6.2 the normalized volume form on S^3 is invariant under the action of $SO(4)$. Additionally, the following lemma holds:

Lemma 2.7.2. *[42] Let ω be an Euclidean volume form on S^3 and f be a continuous compactly supported function on S^3. Let $g \in SO(4)$ and the diffeomorphism*

$$\begin{aligned}
L_g : &S^3 \longrightarrow S^3\\
&m \longmapsto gm.
\end{aligned}$$

Then,

$$\int_{S^3} L_g^*(f)\omega = \int_{S^3} f\omega,$$

where $L_g^* f := f \circ L_g$.

Altogether, Lemma 2.6.2 and Lemma 2.7.2 allow us to say that the twelve-dimensional integral (2.15) may be re-expressed as a six-dimensional Lebesgue integral.

Proposition 2.7.3. *The integral I can be re-written as*

$$I = \frac{2}{\pi^4} \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi F(x_1, x_2, x_3, y_2, y_3, z_3) \sin^2 x_1 \sin^2 x_2 \sin^2 x_3 \sin y_2 \sin y_3$$
$$dz_3 dy_3 dy_2 dx_3 dx_2 dx_1.$$

Proof. By rewriting the integral (2.18) in terms of the spherical coordinates in step one of the equality, and by applying Lemma 2.6.2 to the volume form, and Lemma 2.7.1 and Lemma 2.7.2 to F in the second step, the transformation of the integral in Equation (2.15) reads as follows:

$$
\begin{aligned}
I =& (\frac{1}{2\pi^2})^4 \int_{x_0,x_1,x_2,x_3,y_0,y_1,y_2,y_3=0}^{\pi} \int_{z_0,z_1,z_2,z_3=0}^{2\pi} F(x_0,x_1,x_2,x_3,y_0,y_1,y_2,y_3,z_0,z_1,z_2,z_3) \\
& \prod_{i=0}^{3} \sin^2 x_i \sin y_i \mathrm{d}z_i \mathrm{d}y_i \mathrm{d}x_i \\
=& (\frac{1}{2\pi^2})^4 \int_{x_0,x_1,x_2,x_3,y_0,y_1,y_2,y_3=0}^{\pi} \int_{z_0,z_1,z_2,z_3=0}^{2\pi} F(x_1,x_2,x_3,y_2,y_3,z_3) \prod_{i=0}^{3} \sin^2 x_i \sin y_i \mathrm{d}z_i \mathrm{d}y_i \mathrm{d}x_i \\
=& (\frac{1}{2\pi^2})^4 \int_{x_1,x_2,x_3,y_2,y_3=0}^{\pi} \int_{z_3=0}^{2\pi} F(x_1,x_2,x_3,y_2,y_3,z_3) \sin^2 x_1 \sin^2 x_2 \sin^2 x_3 \sin y_2 \sin y_3 \\
& \mathrm{d}z_3 \mathrm{d}y_3 \mathrm{d}y_2 \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \times \int_{x_0,y_0,y_1=0}^{\pi} \int_{z_0,z_1,z_2=0}^{2\pi} \sin^2 x_0 \sin y_0 \sin y_1 \mathrm{d}z_0 \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}y_0 \mathrm{d}y_1 \mathrm{d}x_0 \\
=& (2\pi)^4 . (\frac{1}{2\pi^2})^4 \int_{x_1,x_2,x_3,y_2,y_3=0}^{\pi} \int_{z_3=0}^{2\pi} F(x_1,x_2,x_3,y_2,y_3,z_3) \sin^2 x_1 \sin^2 x_2 \sin^2 x_3 \sin y_2 \sin y_3 \\
& \mathrm{d}z_3 \mathrm{d}y_3 \mathrm{d}y_2 \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \\
=& (\frac{1}{\pi})^4 \int_{x_1,x_2,x_3,y_2,y_3=0}^{\pi} \int_{z_3=0}^{2\pi} F(x_1,x_2,x_3,y_2,y_3,z_3) \sin^2 x_1 \sin^2 x_2 \sin^2 x_3 \sin y_2 \sin y_3 \\
& \mathrm{d}z_3 \mathrm{d}y_3 \mathrm{d}y_2 \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \\
=& (\frac{1}{\pi})^4 \int_{x_1,x_2,x_3,y_2,y_3=0}^{\pi} \int_{z_3=0}^{\pi} F(x_1,x_2,x_3,y_2,y_3,z_3) \sin^2 x_1 \sin^2 x_2 \sin^2 x_3 \sin y_2 \sin y_3 \\
& \mathrm{d}z_3 \mathrm{d}y_3 \mathrm{d}y_2 \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \\
& + (\frac{1}{\pi})^4 \int_{x_1,x_2,x_3,y_2,y_3=0}^{\pi} \int_{z_3=\pi}^{2\pi} F(x_1,x_2,x_3,y_2,y_3,z_3) \sin^2 x_1 \sin^2 x_2 \sin^2 x_3 \sin y_2 \sin y_3 \\
& \mathrm{d}z_3 \mathrm{d}y_3 \mathrm{d}y_2 \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \\
=& \frac{2}{\pi^4} \int_{x_1,x_2,x_3,y_2,y_3=0}^{\pi} \int_{z_3=0}^{\pi} F(x_1,x_2,x_3,y_2,y_3,z_3) \sin^2 x_1 \sin^2 x_2 \sin^2 x_3 \sin y_2 \sin y_3 \\
& \mathrm{d}z_3 \mathrm{d}y_3 \mathrm{d}y_2 \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1
\end{aligned}
$$

The last equality is obtained by doing a change of variables $u = 2\pi - z_3$ in the second summand in the second last step. □

Now that all the boundaries of the integrals are from 0 to π, the use of the box variables may be incorporated in the calculation. Let us recall from Subsection 2.5 that $B = \{l_{01}, l_{02}, l_{03}, \alpha^0_{12}, \alpha^0_{13}, \beta_{23}\}$, where l_{0i} are the lengths of the edges of the spherical tetrahedron leaving from the vertex v_0, α^0_{1i} are the interior angles around the vertex v_0 and opposite to the edges $(v_1 v_i)$, and β_{23} is the interior dihedral angle along the edge $(v_0 v_1)$. Following Proposition 2.5.2, the angles x_1, x_2, x_3, y_2, y_3 and z_3 in the spherical coordinates may be written in terms of the box variables as $x_1 = l_{01},\ x_2 = l_{02},\ x_3 = l_{03},\ y_2 = \alpha^0_{12},\ y_3 = \alpha^0_{13}$ and $z_3 = \beta_{23}$. Therefore, the integral in Equation (2.15) becomes:

Corollary 2.7.4. *As function of the box variables B, the integral in Theorem 2.1.1 may be expressed as*

$$I = \frac{2}{\pi^4}\int_0^\pi\int_0^\pi\int_0^\pi\int_0^\pi\int_0^\pi\int_0^\pi F(l_{01}, l_{02}, l_{03}, \alpha^0_{12}, \alpha^0_{13}, \beta_{23}) \sin^2 l_{01} \sin^2 l_{02} \sin^2 l_{03} \sin\alpha^0_{12} \sin\alpha^0_{13}$$
$$\mathrm{d}\beta_{23}\mathrm{d}\alpha^0_{13}\mathrm{d}\alpha^0_{12}\mathrm{d}l_{03}\mathrm{d}l_{02}\mathrm{d}l_{01}.$$

2.7.2 Freidel and Louapre's integral

Following from Corollary 2.7.4 and Equation (2.16) the integral I is re-expressed as

$$I = \frac{2}{\pi^4}\int_0^\pi\int_0^\pi\int_0^\pi\int_0^\pi\int_0^\pi\int_0^\pi \prod_{i<j}\frac{\sin((m_{ij}+1)l_{ij})}{\sin l_{ij}} \sin^2 l_{01} \sin^2 l_{02} \sin^2 l_{03} \sin\alpha^0_{12} \sin\alpha^0_{13}\mathrm{d}\beta_{23}\mathrm{d}\alpha^0_{13}\mathrm{d}\alpha^0_{12}\mathrm{d}l_{03}\mathrm{d}l_{02}\mathrm{d}l_{01}. \tag{2.19}$$

Let us be reminded of our goal which is to prove

$$I = \frac{2}{\pi^4}\int_{D_\pi}\frac{\prod_{i<j}\sin((m_{ij}+1)l_{ij})}{\sqrt{\det([\cos l_{ij}])}}[\prod_{i<j}\mathrm{d}l_{ij}].$$

Here, the domain of integration is

$$D_\pi = \{(l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23}) \in [0,\pi]^6, [\cos l_{ij}] \text{ is positive definite }\},$$

where $[\cos l_{ij}]$ is the 4×4 unidiagonal symmetric matrix whose off-diagonal entries are $\cos l_{ij}$. Hence, the variables of the integrand represent the edge lengths of a spherical tetrahedron (Lemma 2.8.5). Therefore, from Lemma 2.5.4, they may be re-written in terms of the box variables. Although, especially in the measure, the "inverse expression" is what's needed i.e. the process requires the expression of the box variables as functions of the edge lengths. But that can easily be obtained from playing around with the equations in Lemma 2.5.4. In other words,

Lemma 2.7.5. *The change in measure, from box variables to edge lengths, in the integral is acquired from*

$$\begin{aligned}
d\alpha_{12}^0 &= \frac{\sin l_{12}}{\sin l_{01} \sin l_{02} \sin \alpha_{12}^0} dl_{12}, \\
d\alpha_{13}^0 &= \frac{\sin l_{13}}{\sin l_{01} \sin l_{03} \sin \alpha_{13}^0} dl_{13}, \\
d\beta_{23} &= \frac{\sin l_{23}}{\sin l_{02} \sin l_{03} \sin \alpha_{12}^0 \sin \alpha_{13}^0 \sin \beta_{23}} dl_{23}.
\end{aligned}$$

Proof. From the proof of Lemma 2.5.4, the following results hold:

$$\begin{aligned}
\cos l_{12} &= \cos l_{01} \cos l_{02} + \sin l_{01} \sin l_{02} \cos \alpha_{12}^0, \\
\cos l_{13} &= \cos l_{01} \cos l_{03} + \sin l_{01} \sin l_{03} \cos \alpha_{13}^0, \\
\cos l_{23} &= \cos l_{02} \cos l_{03} + \sin l_{02} \sin l_{03} (\cos \alpha_{12}^0 \cos \alpha_{13}^0 + \sin \alpha_{12}^0 \sin \alpha_{13}^0 \cos \beta_{23}).
\end{aligned}$$

Therefore, by using the chain rule it follows

$$\begin{aligned}
\frac{d(\cos l_{12})}{d\alpha_{12}^0} &= -\sin l_{12} \frac{dl_{12}}{d\alpha_{12}^0} \\
&= -\sin l_{01} \sin l_{02} \sin \alpha_{12}^0
\end{aligned}$$

which leads us to

$$d\alpha_{12}^0 = \frac{\sin l_{12}}{\sin l_{01} \sin l_{02} \sin \alpha_{12}^0} dl_{12}.$$

Similar calculations produce:

$$\begin{aligned}
d\alpha_{13}^0 &= \frac{\sin l_{13}}{\sin l_{01} \sin l_{03} \sin \alpha_{13}^0} dl_{13}, \\
d\beta_{23} &= \frac{\sin l_{23}}{\sin l_{02} \sin l_{03} \sin \alpha_{21}^0 \sin \alpha_{13}^0 \sin \beta_{23}} dl_{23}.
\end{aligned}$$

□

Eventually, the integral formula for the square of the classical 6j symbols may be rewritten as follows:

Theorem 2.7.6. *The integral formula for the square of the classical 6j symbols may be re-expressed as an integration over the equivalence classes of non-degenerate spherical tetrahedra in the following way:*

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}^2 = \frac{2}{\pi^4} \int_{D_\pi} \frac{\prod_{i<j} \sin((m_{ij}+1) l_{ij})}{\sqrt{\det([\cos l_{ij}])}} [\prod_{i<j} dl_{ij}].$$

Here

$$D_\pi = \{(l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23}) \in [0,\pi]^6, [\cos l_{ij}] \text{ is positive definite } \},$$

where $[\cos l_{ij}]$ *is the* 4×4 *unidiagonal symmetric matrix whose off-diagonal entries are* $\cos l_{ij}$.

Proof. Starting with Equation (2.19), followed by the expansion of the denominator, and an appropriate cancellation in the numerator and the denominator the integral becomes as follows:

$$\begin{aligned}
I =& \frac{2}{\pi^4} \int_{l_{01}, l_{02}, l_{03}, \alpha_{12}^0, \alpha_{13}^0, \beta_{23}=0}^{\pi} \prod_{i<j} \frac{\sin((m_{ij}+1)l_{ij})}{\sin l_{ij}} \sin^2 l_{01} \sin^2 l_{02} \sin^2 l_{03} \sin\alpha_{12}^0 \sin\alpha_{13}^0 \\
& d\beta_{23} d\alpha_{13}^0 d\alpha_{12}^0 dl_{03} dl_{02} dl_{01} \\
=& \frac{2}{\pi^4} \int_{l_{01}, l_{02}, l_{03}, \alpha_{12}^0, \alpha_{13}^0, \beta_{23}=0}^{\pi} \frac{\prod_{i<j} \sin((m_{ij}+1)l_{ij})}{\sin l_{12} \sin l_{13} \sin l_{23}} \sin l_{01} \sin l_{02} \sin l_{03} \sin\alpha_{12}^0 \sin\alpha_{13}^0 \\
& d\beta_{23} d\alpha_{13}^0 d\alpha_{12}^0 dl_{03} dl_{02} dl_{01} \\
=& \frac{2}{\pi^4} \int_{l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23}} \frac{\prod_{i<j} \sin((m_{ij}+1)l_{ij})}{\sin l_{12} \sin l_{13} \sin l_{23}} \\
& \times \frac{\sin l_{01} \sin l_{02} \sin l_{03} \sin\alpha_{12}^0 \sin\alpha_{13}^0 \sin l_{12} \sin l_{13} \sin l_{23}}{\sin l_{01} \sin l_{02} \sin\alpha_{12}^0 \sin l_{01} \sin l_{03} \sin\alpha_{13}^0 \sin l_{02} \sin l_{03} \sin\alpha_{12}^0 \sin\alpha_{13}^0 \sin\beta_{23}} dl_{23} dl_{13} dl_{12} dl_{03} dl_{02} dl_{01} \\
=& \frac{2}{\pi^4} \int_{l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23}} \frac{\prod_{i<j} \sin((m_{ij}+1)l_{ij})}{\sin l_{01} \sin l_{02} \sin l_{03} \sin\alpha_{12}^0 \sin\alpha_{13}^0 \sin\beta_{23}} dl_{23} dl_{13} dl_{12} dl_{03} dl_{02} dl_{01} \\
=& \frac{2}{\pi^4} \int_{D_\pi} \frac{\prod_{i<j} \sin((m_{ij}+1)l_{ij})}{\sqrt{\det([\cos l_{ij}])}} dl_{23} dl_{13} dl_{12} dl_{03} dl_{02} dl_{01} \qquad \text{(Lemma 2.5.6 and Lemma 2.7.5).}
\end{aligned}$$

Here

$$D_\pi = \{(l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23}) \in [0,\pi]^6, [\cos l_{ij}] \text{ is positive definite } \},$$

where $[\cos l_{ij}]$ is the 4×4 unidiagonal symmetric matrix whose off-diagonal entries are $\cos l_{ij}$. □

2.8 Tools to describe D_π

As mentioned in the introduction of this chapter, geometrically the integral domain of (2.2) is the set of all the $SO(4)-$ equivalence classes of positively oriented non-degenerate spherical tedrahedra. That observation is not obvious, hence, requires a bit of explanation. Some materials to understand this geometric description of the domain of integration are provided in this section.

To start with, let us recall the classification of spaces by Milnor in [28].

Lemma 2.8.1. *[28] A non-degenerate simplex must lie either in spherical space, Hyperbolic space, or Euclidean space according to as the determinant of its angle Gram matrix is positive, negative or equal to zero.*

Remark 2.8.2. *Equivalently, Lemma 2.8.1 applies to edge Gram matrices too as referred in [24, Lemma 2.5].*

Let

$$\mathcal{T} = \{T, T \text{ non-degenerate spherical tetrahedron}\}$$

be the set of non-degenerate spherical tetrahedra.

Definition 2.8.3. *Let $T_1, T_2 \in \mathcal{T}$ with respective vertices v_0, v_1, v_2, v_3 and w_0, w_1, w_2, w_3. We say that T_1 and T_2 are $SO(4)-$equivalent i.e.*

$$T_1 \overset{SO(4)}{\sim} T_2$$

if there exists $g \in SO(4)$ such that

$$(w_0, w_1, w_2, w_3) = (gv_0, gv_1, gv_2, gv_3).$$

It turns out that two elements in $\mathcal{T}$ are $SO(4)-$equivalent if and only if their duals are. Details on that are provided in the lemma below.

Lemma 2.8.4. *Let $T_1, T_2 \in \mathcal{T}$ and let us denote by T_1^*, T_2^* their respective duals. Then*

$$T_1 \overset{SO(4)}{\sim} T_2 \Longleftrightarrow T_1^* \overset{SO(4)}{\sim} T_2^*.$$

Proof. ($\Longrightarrow$) Let v_0, v_1, v_2, v_3 be the vertices of T_1 and suppose that

$$T_1 \overset{SO(4)}{\sim} T_2.$$

By the definition of the equivalence relation (Definition 2.8.3), there exists $g \in SO(4)$ such that the vertices of T_2 are gv_0, gv_1, gv_2, gv_3. Hence, the vertices of T_2^*

are $(gv_0)^*, (gv_1)^*, (gv_2)^*, (gv_3)^*$ (see Definition 2.3.9). Let $v_0^*, v_1^*, v_2^*, v_3^*$ be the vertices of T_1^* which are by Definition 2.3.9 the respective dual vectors to v_0, v_1, v_2, v_3. Proving

$$T_1^* \overset{SO(4)}{\sim} T_2^*$$

is equivalent to show that

$$((gv_0)^*, (gv_1)^*, (gv_2)^*, (gv_3)^*) = (gv_0^*, gv_1^*, gv_2^*, gv_3^*).$$

By the definition of dual vectors,

$$\langle gv_i, (gv_j)^* \rangle_{\mathbb{R}^4} = \delta_{ij}, \quad \delta_{ij} \text{ is the Krönecker delta.}$$

Since $SO(4)$ acts linearly on $\mathbb{R}^4$,

$$\begin{aligned} \langle gv_i, gv_j^* \rangle_{\mathbb{R}^4} &= \langle v_i, v_j^* \rangle_{\mathbb{R}^4} \\ &= \delta_{ij}. \end{aligned}$$

Therefore the following equality holds:

$$\langle gv_i, (gv_j)^* \rangle_{\mathbb{R}^4} = \langle gv_i, gv_j^* \rangle_{\mathbb{R}^4},$$

which is equivalent to

$$\langle gv_i, (gv_j)^* - gv_j^* \rangle_{\mathbb{R}^4}.$$

Since $gv_i \neq 0$ and $\langle -, - \rangle_{\mathbb{R}^4}$ is an inner product then

$$(gv_j)^* - gv_j^* = 0$$

i.e.

$$(gv_j)^* = gv_j^*.$$

($\Longleftarrow$) Conversely, suppose that the duals T_1^* and T_2^* are $SO(4)-$equivalent i.e.

$$T_1^* \overset{SO(4)}{\sim} T_2^*.$$

Let v_0^*, v_1^*, v_2^*, v_3^* be the vertices of T_1^* and w_0^*, w_1^*, w_2^*, w_3^* those of T_2^*. Then by Definition 2.8.3, there exists $g \in SO(4)$ such that

$$(w_0^*, w_1^*, w_2^*, w_3^*) = (gv_0^*, gv_1^*, gv_2^*, gv_3^*). \tag{2.20}$$

Let v_0, v_1, v_2, v_3 be the vertices of T_1 and w_0, w_1, w_2, w_3 that of T_2. Showing the equivalence

$$T_1 \overset{SO(4)}{\sim} T_2$$

amounts to prove that

$$w_i = gv_i.$$

On one hand, by definition of a dual vector to w_i and by Equation (2.20), one has

$$\begin{aligned}\langle w_i^*, w_j \rangle_{\mathbb{R}^4} &= \langle gv_i^*, w_j \rangle_{\mathbb{R}^4} \\ &= \delta_{ij}.\end{aligned}$$

On the other hand, by using the linearity of the action of $SO(4)$ on $\mathbb{R}^4$, the following equality holds:

$$\langle gv_i^*, gv_j \rangle_{\mathbb{R}^4} = \langle v_i^*, v_j \rangle_{\mathbb{R}^4} \qquad SO(4) \text{ acts linearly on } \mathbb{R}^4 \tag{2.21}$$

$$= \delta_{ij}. \tag{2.22}$$

By using the expression of w_i^* in (2.20) and the equalities (2.21) and (2.22)

$$\langle gv_i^*, w_j \rangle_{\mathbb{R}^4} = \langle gv_i^*, gv_j \rangle_{\mathbb{R}^4},$$

which is equivalent to say that

$$\langle gv_i^*, w_j \rangle_{\mathbb{R}^4} - \langle gv_i^*, gv_j \rangle_{\mathbb{R}^4} = 0.$$

In other words,

$$\langle gv_i^*, w_j - gv_i \rangle_{\mathbb{R}^4} = 0. \tag{2.23}$$

By definition of an inner product, Equation (2.23) leads to

$$w_j - gv_j = 0.$$

Thus $w_j = gv_j$. □

In the six-dimensional Lebesgue version of the integral formula for the square of the classical 6j symbols (2.2), the condition on the domain of integration is that the four by four matrix $[\cos l_{ij}]$, which is unidiagonal and symmetric should be positive definite. Indeed, that given data describe a non-degenerate spherical tetrahedron with edge lengths l_{ij} up to rotation. That equivalence is carefully checked in the lemma below.

Let

$$\mathcal{U} = \left\{ \begin{array}{c} M, M \ 4\times 4 \text{ unidiagonal symmetric positive definite} \\ \text{whose off-diagonal elements are in the interval } (-1,1) \end{array} \right\}$$

be the set of four by four unidiagonal symmetric positive definite matrices whose off-diagonal elements in the interval $(-1,1)$. Then, the following lemma holds:

Lemma 2.8.5. *There is a one-to-one correspondence between the $SO(4)-$equivalence classes of non-degenerate spherical tetrahedra and the set of unidiagonal four by four symmetric positive definite matrices whose off-diagonal elements are in the interval $(-1,1)$ i.e.*

$$\mathcal{U} \longleftrightarrow \mathcal{T}/\sim .$$

Proof. A full proof of the Lemma may be found in [23], [21] and [28], but since the maps play an important role to understanding the description of D_π, we will write them here. In the one direction, the map from $\mathcal{U}$ to $\mathcal{T}/\sim$ sends a unidiagonal four by four symmetric positive definite matrix with off-diagonal elements in the interval $(-1,1)$, $[a_{ij}]$, to the non-degenerate spherical tetrahedron T with unit outward normal g_0, g_1, g_2, g_3 satisfying $\langle g_i, g_j\rangle_{\mathbb{R}^4} = a_{ij}$ i.e.

$$\begin{aligned} \mathcal{U} &\longrightarrow \mathcal{T}/\sim \\ [a_{ij}] &\longmapsto T \text{ with unit outward normal } g_0,\ g_1,\ g_2,\ g_3 \text{ such that } \langle g_i, g_j\rangle_{\mathbb{R}^4} = a_{ij}. \end{aligned}$$

In the other direction, the map sends a non-degenerate spherical tetahedron to its angle Gram matrix, in other words

$$\begin{aligned} \mathcal{T}/\sim &\longrightarrow \mathcal{U} \\ T &\longmapsto \text{Angle Gram matrix of } T. \end{aligned}$$

Equivalently, since the exterior dihedral angles of a spherical tetrahedron are equal to the edge lengths of its dual, and by taking into account Lemma 2.8.4 the map from $\mathcal{U}$ to $\mathcal{T}/\sim$ may be defined as

$$\begin{aligned} \mathcal{U} &\longrightarrow \mathcal{T}/\sim \\ [a_{ij}] &\longmapsto T \text{ with vertices } v_0, v_1, v_2, v_3 \text{ such that } a_{ij} = \langle v_i, v_j\rangle_{\mathbb{R}^4}, \end{aligned}$$

and that from $\mathcal{T}/\sim$ to $\mathcal{U}$ determined by

$$\mathcal{T}/\sim \longrightarrow \mathcal{U}$$
$$T \longmapsto \text{Edge Gram matrix of } T.$$

□

2.9 Discussion and correction in the literature

The discussion on Remark 2.1.3 is provided in this section.

Let us recall the statement of Theorem 2.1.2 in [17].

Theorem 2.9.1.

$$I = \frac{2}{\pi^4}\int_{D_\pi} \frac{\prod_{i<j}\sin((m_{ij}+1)l_{ij})}{\sqrt{\det([\cos l_{ij}])}}[\prod_{i<j} \mathrm{d}l_{ij}],$$

where D_π is the subset of $[0;\pi]^6$ of angles satisfying the relations:

$$l_{ij} \le l_{ik} + l_{jk}, \tag{2.24}$$

$$l_{ij} + l_{ik} + l_{jk} \le 2\pi \tag{2.25}$$

for any triple (i,j,k) of distinct elements. Geometrically this domain is the set of all possible spherical tetrahedra.

Regarding the domain of integration, we would like to point out that:

- a little confusion occurs in thinking that the conditions (2.24) and (2.25) describe the set of all possible spherical tetrahedra. In fact, some hyperbolic tetrahedra obey to these conditions too. For instance, the tuple

$$(l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23}) = (\frac{\pi}{4}, \frac{\pi}{8}, \frac{\pi}{8}, \frac{\pi}{6}, \frac{\pi}{8}, \frac{\pi}{8})$$

  satisfies the conditions (2.24) and (2.25) but $\det([\cos l_{ij}]) < 0$, of which from Lemma 2.8.1 describes a hyperbolic tetrahedron.

- Next, we understand that what "all possible spherical tetrahedra" is really what we mean by all the non-degenerate spherical tetrahedra. But taking all the non-degenerate spherical tetrahedra as the domain of integration allows the possibility of counting one spherical tetrahedron several times due to its invariance under the $O(4)-$action. Therefore, the geometrical description of the domain in [17] should slightly be modified to be the set of all $O(4)-$equivalence classes of non-degenerate spherical tetrahedra.

These two remarks along with the Lemma 2.8.5 led us to conclude that the domain D_π is algebraically described as follows:

$$D_\pi = \{(l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23}) \in [0, \pi]^6, [\cos l_{ij}] \text{ is positive definite}\},$$

where $[\cos l_{ij}]$ is the 4×4 unidiagonal symmetric matrix whose off-diagonal entries are $\cos l_{ij}$.

Chapter 3

Reciprocity of the Wigner derivative

Note: This Chapter is based on the article [8].

3.1 Introduction

In his seminal book on group theory and quantum mechanics from 1959 ([44], see also [9]), Wigner studied the classical 6j symbols for SU(2) which encode the associator data [35] for the tensor category of representations of SU(2). He related the 6j symbol

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}$$

to a Euclidean tetrahedron with side lengths given by $m_{01}, m_{02}, m_{03}, m_{12}, m_{13}, m_{23}$ and gave a heuristic argument that the square of this 6j symbol should (on average, for large spins) be proportional to the partial derivative $\frac{\partial \theta_{01}}{\partial m_{01}}$ of the dihedral angle θ_{01} at edge of length m_{23} with respect to the length of the opposite edge m_{01}, all other lengths being held fixed (see Figure 3.1).

In 1968 the physicists Ponzano and Regge conjectured a more refined formula for the asymptotics of the classical 6j symbols, which included an oscillatory term. This formula was first proved rigorously by Roberts in 1999, using geometric quan-

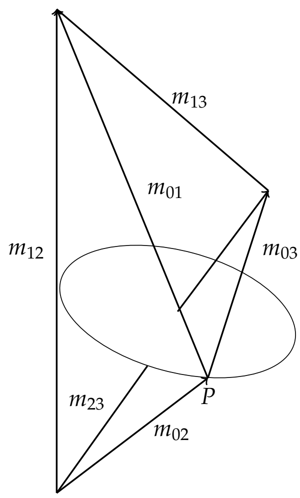


Figure 3.1: If the lengths $m_{03}, m_{02}, m_{13}, m_{12}, m_{23}$ are held constant, then P can still traverse the indicated circle, changing m_{01}. The probability of a given tetrahedron occurring is proportional to $\frac{\partial \theta_{01}(m_{23}, \cdots, m_{01})}{\partial m_{01}}$ where θ_{01} is the dihedral angle at the edge with length m_{23}.

tization techniques [35], and since then a number of other proofs have been given [2][6][16][18].

As mentioned in the introduction, in 2003 Taylor and Woodward gave a corresponding formula for the *quantum* 6j symbols, relating their asymptotics to the geometry of *spherical* tetrahedra [41][38]. In their outline of a possible geometric proof of their formula (this approach was later made rigorous by Marché and Paul [27]), the partial derivative of dihedral angle with respect to opposite edge length (this time for a spherical tetrahedron) again played a crucial role. Following Taylor, we call this the *Wigner derivative* (see Figure 3.2).

Given a spherical tetrahedron with vertices v_0, v_1, v_2, v_3 and edge lengths l_{ij}, let G be the edge Gram matrix, $G_{ij} = \cos(l_{ij})$. Taylor and Woodward's formula for the Wigner derivative is as follows, of which we also give an independent proof in Section 3.3.

Theorem 3.1.1 (Taylor-Woodward [41])**.** *The Wigner derivative for a spherical tetrahedron is*

$$\frac{\partial \beta(l_{ij})}{\partial l'} = \frac{\sin l \sin l'}{\sqrt{\det G}}, \tag{3.1}$$

where β is the interior dihedral angle at the edge with length l and l' is the length of the

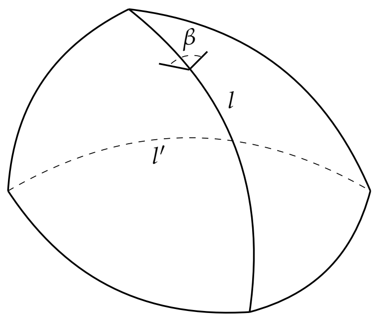


Figure 3.2: The Wigner derivative for a spherical tetrahedron is $\frac{\partial \beta}{\partial l'}$, the partial derivative of dihedral angle with respect to opposite edge length, all other lengths held fixed.

opposite edge (see Fig. 3.2) and G the edge Gram matrix of the tetrahedron.

Unlike a Euclidean tetrahedron, a spherical tetrahedron is determined up to isometry by its six edge lengths as well as by its six dihedral angles. So there is a 1-1 correspondence,

$$(l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23}) \longleftrightarrow (\beta_{01}, \beta_{02}, \beta_{03}, \beta_{12}, \beta_{13}, \beta_{23}),$$

that is due to the machinery of the cosine and dual cosine law. Therefore, it makes sense to ask about the inverse Jacobian matrix $\frac{\partial l_{ij}}{\partial \beta_{kl}}$ and in particular the inverse Wigner derivative $\frac{\partial l'}{\partial \beta}$ in Figure 3.2. Indeed, in our work we were led to consider this inverse Jacobian as it shows up in the stationary phase approximation for a conjectural integral formula for the quantum 6j symbols. The main result of this chapter is as follows:

Theorem 3.1.2. *The inverse Wigner derivative for a spherical tetrahedron is*

$$\frac{\partial l'(\beta_{ij})}{\partial \beta} = \frac{\sin l \sin l'}{\sqrt{\det G}}, \tag{3.2}$$

where β is the interior dihedral angle at the edge with length l and l' is the length of the opposite edge (see Fig. 3.2) and G the edge Gram matrix of the tetrahedron.

Comparing (3.2) with the formula for the Wigner derivative (3.1), the corollary below follows.

Corollary 3.1.3 (Reciprocity of the Wigner derivative)**.** *For spherical tetrahedra, the Wigner derivative and the inverse Wigner derivative are equal:*

$$\frac{\partial\beta(l_{ij})}{\partial l'} = \frac{\partial l'(\beta_{ij})}{\partial\beta}. \tag{3.3}$$

Clarity on contributions Since the chapter is based on a joint paper of the author with Bruce Bartlett, as follows are the contributions made by each of us:

- The author noticed from her numerical calculation that the Wigner derivative is indeed equal to the inverse Wigner derivative. She also checked the veracity of the hypothesis numerically via different methods. Furthermore, she did all the theoretical calculations and the proofs.
- Bruce Bartlett carefully checked Taylor and Woodward's formula in[38, Page 17, Proposition 2.2.0.5] for the Wigner derivative to make sure that it is written in a more explicit way, as given in (3.1). In this way, be clarified which variables are changing and which variables are held constant in Taylor and Woodward's formula, thereby resolving the apparent contradiction with our formula in Equation (3.2). He also noticed that the result is publishable and proposed the idea of checking the Wigner reciprocity for spherical triangles.

Outline of the chapter In Section 3.2 we show, as a warm-up result, that the reciprocity of the Wigner derivative holds for spherical triangles. In Section 3.3 we consider spherical tetrahedra. And in Section 3.4 we will discuss about a confusion in [41, Proposition 4.2.1, (n)].

3.2 Reciprocity of the Wigner derivative for spherical triangles

For sake of precision in the statement of the results, let us choose the spherical triangle Δ as a prototype. However, the results may be applied to any spherical triangle.

Let us consider the spherical triangle Δ with vertices v_0, v_1, v_2, edge lengths a, b, c and interior angles A, B, C as shown in the Figure 3.3. Its edge Gram matrix is given

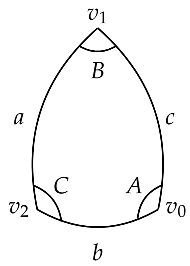


Figure 3.3: spherical triangle Δ

by:

$$G = \begin{pmatrix} 1 & \cos c & \cos b \\ \cos c & 1 & \cos a \\ \cos b & \cos a & 1 \end{pmatrix}.$$

By making G in row echelon form and using the cosine law appropriately, the square root of the determinant of G may be expressed as a function of the edge lengths b and c, and the interior angle A as shown in the lemma below.

Lemma 3.2.1. *The square root of the determinant of the edge Gram matrix of Δ may be written as*

$$\sqrt{\det G} = \sin A \sin b \sin c.$$

Proof. By taking the first row as a pivot, the determinant of G becomes

$$\begin{aligned} \det G = \det & \begin{pmatrix} 1 & \cos c & \cos b \\ 0 & 1 - \cos^2 c & \cos a - \cos b \cos c \\ 0 & \cos a - \cos b \cos c & 1 - \cos^2 b \end{pmatrix} \\ &= \sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2 \\ &= \sin^2 b \sin^2 c - \sin^2 b \sin^2 c \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c} \right)^2 \\ &= \sin^2 b \sin^2 c \, (1 - \cos^2 A) \\ &= \sin^2 b \sin^2 c \; \sin^2 A, \end{aligned}$$

where the cosine law (Equation 2.5 in Remark 2.3.2) was used to obtain the second last step. □

Now, let us focus on one interior angle[1], say A. From the cosine law, A is explicitly written in terms of the edge lengths a, b, c. The partial derivative of A with respect to the edge length opposite to it, a, is given in the lemma below.

Lemma 3.2.2. *For the spherical triangle Δ, the partial derivative of the angle A with respect to its opposite edge length a is given by:*

$$\frac{\partial A(a,b,c)}{\partial a} = \frac{\sin a}{\sin A \sin b \sin c}.$$

Proof. Let us recall the cosine law

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \tag{3.4}$$

Taking the partial derivative of (3.4) with respect to a brings forth the result

$$\frac{\partial \cos A}{\partial a} = \frac{-\sin a}{\sin b \sin c}$$

which by the chain rule is equivalent to say that

$$-\sin A \frac{\partial A}{\partial a} = -\frac{\sin a}{\sin b \sin c}.$$

Hence the equation in Lemma 3.2.2 follows. □

What about the inverse Wigner derivative for spherical triangles i.e. the derivative of edge length[2] with respect to its opposite interior angle? Well, as before, let us focus on an edge length of Δ, say a (a is chosen because it is opposite to A which we chose as the example in Lemma 3.2.2). From the dual cosine law, a can explicitly be expressed as a function of the interior angles A, B, C. Hence, an explicit expression of its partial derivative with respect to its opposite interior angle is given in the lemma below.

Lemma 3.2.3. *For the spherical triangle Δ, the partial derivative of the edge length a with respect to its opposite angle A is given by:*

$$\frac{\partial a(A,B,C)}{\partial A} = \frac{\sin A}{\sin a \sin B \sin C}.$$

[1] Since A is chosen arbitrarily then the "Wigner derivative" formula holds for any interior angle in the spherical triangle with proper adjustment in the notations.

[2] Similarly as before, a is chosen arbitrarily so the formula may be adapted to the partial derivative of any edge length with respect to its opposite interior angle.

Proof. Let us recall the dual cosine law

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}. \tag{3.5}$$

Taking the partial derivative of (3.5) with respect to A results into

$$\frac{\partial \cos a}{\partial A} = -\frac{\sin A}{\sin B \sin C}$$

which by the chain rule is equivalent to say that

$$-\sin a \frac{\partial a}{\partial A} = \frac{-\sin A}{\sin B \sin C}.$$

Hence the equation in Lemma 3.2.3 follows. □

Although $\frac{\partial A(a,b,c)}{\partial a}$ and $\frac{\partial a(A,B,C)}{\partial A}$ seem to not have anything in common in their expression, they are actually equal. And it follows that the partial derivative of any edge length of Δ with respect to its opposite interior angle is equal to the partial derivative of that same interior angle with respect to its opposite edge length.

Theorem 3.2.4 (Wigner reciprocity for spherical triangles)**.** *For the spherical triangle Δ, by focusing on the angle A and its opposite edge length a the partial derivative of A with respect to its opposite edge length a is equal to* $\sin a$ *over the square root of the determinant of the edge Gram matrix of Δ, and is also equal to the partial derivative of the edge length a with respect to its opposite angle A i.e.*

$$\frac{\partial A(a,b,c)}{\partial a} = \frac{\sin a}{\sqrt{\det G}} = \frac{\partial a(A,B,C)}{\partial A}.$$

Proof. The first equation follows from the Lemmas 3.2.1 and 3.2.3. Let us show that $\frac{\partial A}{\partial a} = \frac{\partial a}{\partial A}$.

Let us recall the sine law in the case of our spherical triangle

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

The quotient of $\frac{\partial A}{\partial a}$ with $\frac{\partial a}{\partial A}$ results to

$$\begin{aligned} \frac{\frac{\partial A}{\partial a}}{\frac{\partial a}{\partial A}} &= \frac{\sin^2 a \sin B \sin C}{\sin^2 A \sin b \sin c} && \text{(by the lemmas 3.2.2 and 3.2.3)} \\ &= 1. && \text{(by the sine law)} \end{aligned}$$

Hence, the equality $\frac{\partial A}{\partial a} = \frac{\partial a}{\partial A}$. □

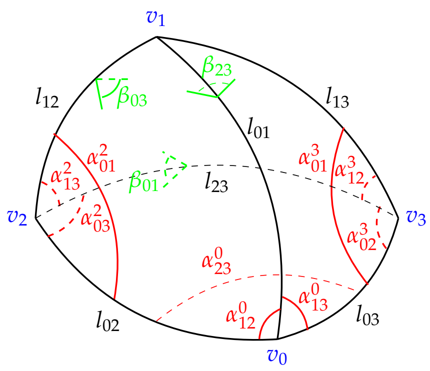


Figure 3.4: Spherical tetrahedron T

3.3 Reciprocity of the Wigner derivative for spherical tetrahedra

Analogously to the strategy in Section 3.2 let us fix a non-degenerate spherical tetrahedron T to work on but all the results are applicable for any non-degenerate spherical tetrahedron.

As a mean to show that the results in this section hold for any edge length of the tetrahedron and its opposite dihedral angle, let $i, j, k, l \in \{0, 1, 2, 3\}$ be all distinct.

The aim of this section is to prove the reciprocity of the Wigner derivative for spherical tetrahedra, i.e. the intention is to show that the partial derivative of a dihedral angle in a spherical tetrahedron with respect to its opposite edge length is equal to the partial derivative of that same edge length with respect to its opposite dihedral angle. Although the results presented here are for **interior dihedral angles**, the results while using **exterior dihedral angles** follow easily.

On one hand, the interior dihedral angles of the spherical tetrahedron T may explicitly be expressed in terms of the edge lengths by appropriate use of the cosine law on its faces. That leads to the statement of the formula for the Wigner derivative in the following lemma.

Lemma 3.3.1. *Let β_{kl} be the interior dihedral angle at the edge (ij) opposite to the edge (kl) of length l_{kl}, then the partial derivative of β_{kl} with respect to l_{kl} is given by*

$$\frac{\partial \beta_{kl}(l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23})}{\partial l_{kl}} = \frac{\sin l_{kl}}{\sin l_{il} \sin l_{ik} \sin \alpha^i_{jl} \sin \alpha^i_{jk} \sin \beta_{kl}}.$$

Proof. For the sake of lighter notations, let us keep in mind that β_{kl} and the links are functions of the edge lengths but their full expression will be omitted. To start with, as follows is the partial derivative of β_{kl} with respect to its opposite edge length:

$$\frac{\partial \beta_{kl}}{\partial l_{kl}} = \frac{\partial \beta_{kl}}{\partial \alpha^i_{kl}} . \frac{\partial \alpha^i_{kl}}{\partial l_{kl}} + \frac{\partial \beta_{kl}}{\partial \alpha^i_{jk}} . \frac{\partial \alpha^i_{jk}}{\partial l_{kl}} + \frac{\partial \beta_{kl}}{\partial \alpha^i_{jl}} . \frac{\partial \alpha^i_{jl}}{\partial l_{kl}}$$

because from the cosine law to the link $Lk(v_i)$(Figure 3.5)

$$\cos \beta_{kl} = \frac{\cos \alpha^i_{kl} - \cos \alpha^i_{jk} \cos \alpha^i_{jl}}{\sin \alpha^i_{jk} \sin \alpha^i_{jl}}. \tag{3.6}$$

However, by applying the cosine law to the spherical triangles Δijl (Figure 3.7) and Δijk (Figure 3.8) the following formulas hold

$$\cos \alpha^i_{jk} = \frac{\cos l_{jk} - \cos l_{ij} \cos l_{ik}}{\sin l_{ij} \sin l_{ik}},$$

$$\cos \alpha^i_{jl} = \frac{\cos l_{jl} - \cos l_{ij} \cos l_{il}}{\sin l_{ij} \sin l_{il}}.$$

These two equalities imply that all the partial derivatives of α^i_{jk} and α^i_{jl} with respect to the length l_{kl} are null. Therefore, the partial derivative of β_{kl} with respect to l_{kl} boils down to

$$\frac{\partial \beta_{kl}}{\partial l_{kl}} = \frac{\partial \beta_{kl}}{\partial \alpha^i_{kl}} . \frac{\partial \alpha^i_{kl}}{\partial l_{kl}}. \tag{3.7}$$

From Equation (3.6), the partial derivative of $\cos \beta_{kl}$ with respect to α^i_{kl} results to

$$\frac{\partial \cos \beta_{kl}}{\partial \alpha^i_{kl}} = \frac{-\sin \alpha^i_{kl}}{\sin \alpha^i_{jk} \sin \alpha^i_{jl}}. \tag{3.8}$$

Hence, by application of the chain rule on the left hand side of (3.8) the partial derivative of β_{kl} with respect to α^i_{kl} is given by

$$\frac{\partial \beta_{kl}}{\partial \alpha^i_{kl}} = \frac{\sin \alpha^i_{kl}}{\sin \alpha^i_{jk} \sin \alpha^i_{jl} \sin \beta_{23}}. \tag{3.9}$$

From the cosine law applied to Δikl (Figure 3.6)

$$\cos \alpha^i_{kl} = \frac{\cos l_{kl} - \cos l_{il} \cos l_{ik}}{\sin l_{il} \sin l_{ik}}. \tag{3.10}$$

By direct computation of a partial derivative of (3.10) with respect to l_{kl}, the following equality is immediate

$$\frac{\partial \cos \alpha^i_{kl}}{\partial l_{kl}} = \frac{-\sin l_{kl}}{\sin l_{il} \sin l_{ik}}. \tag{3.11}$$

Applying the chain rule on the left hand side of (3.11),

$$\frac{\partial \cos \alpha^i_{kl}}{\partial \alpha^i_{kl}} . \frac{\partial \alpha^i_{kl}}{\partial l_{kl}} = \frac{-\sin l_{kl}}{\sin l_{il} \sin l_{ik}}.$$

The latter induces the expression of the partial derivative of α^i_{kl} with respect to l_{kl}, which is

$$\frac{\partial \alpha^i_{kl}}{\partial l_{kl}} = \frac{\sin l_{kl}}{\sin l_{il} \sin l_{ik} \sin \alpha^i_{kl}}. \tag{3.12}$$

By substituting (3.9) and (3.12) in Equation (3.7), the partial derivative of β_{kl} with respect to l_{kl} becomes

$$\frac{\partial \beta_{kl}}{\partial l_{kl}} = \frac{\sin \alpha^i_{kl}}{\sin \alpha^i_{jk} \sin \alpha^i_{jl} \sin \beta_{kl}} . \frac{\sin l_{kl}}{\sin l_{il} \sin l_{ik} \sin \alpha^i_{kl}}.$$

The latter equality implies the final formula for the partial derivative of β_{kl} with respect to l_{kl}, which is

$$\frac{\partial \beta_{kl}}{\partial l_{kl}} = \frac{\sin l_{kl}}{\sin l_{il} \sin l_{ik} \sin \alpha^i_{jk} \sin \alpha^i_{jl} \sin \beta_{kl}}. \tag{3.13}$$

□

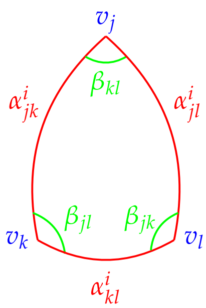


Figure 3.5: $Lk(v_i)$

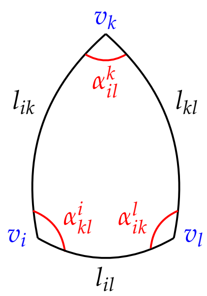


Figure 3.6: Δikl

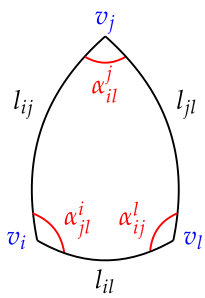


Figure 3.7: Δijl

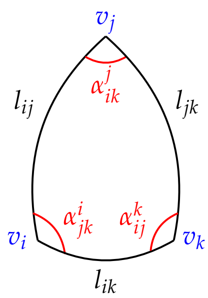


Figure 3.8: Δijk

On the other hand, the edge lengths of the spherical tetrahedron T may be explicitly expressed in terms of the interior dihedral angles by appropriate use of the dual cosine law on its faces. That leads to the statement of the formula for the inverse Wigner derivative provided in the following lemma.

Lemma 3.3.2. *The partial derivative of the edge length l_{kl} with respect to its opposite dihedral angle β_{kl} is given by*

$$\frac{\partial l_{kl}(\beta_{01},\beta_{02},\beta_{03},\beta_{12},\beta_{13},\beta_{23})}{\partial \beta_{kl}} = \frac{\sin \beta_{kl}}{\sin l_{kl} \sin \alpha^l_{ik} \sin \alpha^k_{il} \sin \beta_{jl} \sin \beta_{jk}}.$$

Proof. Similarly to the method in the proof of Lemma 3.3.1, the partial derivative of l_{kl} with respect to β_{kl} may be reduced to

$$\frac{\partial l_{kl}}{\partial \beta_{kl}} = \frac{\partial l_{kl}}{\partial \alpha^i_{kl}} . \frac{\partial \alpha^i_{kl}}{\partial \beta_{kl}}. \tag{3.14}$$

This equality is true from the use of the dual cosine law on Δikl (Figure 3.6)

$$\cos l_{kl} = \frac{\cos \alpha^i_{kl} + \cos \alpha^l_{ik} \cos \alpha^k_{il}}{\sin \alpha^l_{ik} \sin \alpha^k_{il}}, \tag{3.15}$$

and α^l_{ik} and α^k_{il} do not depend on β_{kl} (see the links $Lk(v_k)$ and $Lk(v_l)$).

From Equation (3.15) a direct computation of the partial derivative of $\cos l_{kl}$ with respect to α^i_{kl} results to

$$\frac{\partial \cos l_{kl}}{\partial \alpha^i_{kl}} = \frac{-\sin \alpha^i_{kl}}{\sin \alpha^l_{ik} \sin \alpha^k_{il}}. \tag{3.16}$$

And application of the chain rule to the left hand side of (3.16) implies

$$\frac{\partial l_{kl}}{\partial \alpha^i_{kl}} = \frac{\sin \alpha^i_{kl}}{\sin \alpha^l_{ik} \sin \alpha^k_{il} \sin l_{kl}}. \tag{3.17}$$

The consideration of the dual cosine law applied to $Lk(v_i)$ leads to the expression

$$\cos \alpha^i_{kl} = \frac{\cos \beta_{kl} + \cos \beta_{jl} \cos \beta_{jk}}{\sin \beta_{jl} \sin \beta_{jk}}. \tag{3.18}$$

By applying the chain rule to the left hand side of (3.18) the result below follows:

$$\frac{\partial \cos \alpha^i_{kl}}{\partial \alpha^i_{kl}} . \frac{\partial \alpha^i_{kl}}{\partial \beta_{kl}} = \frac{-\sin \beta_{kl}}{\sin \beta_{jl} \sin \beta_{jk}}.$$

The latter leads to the expression of the partial derivative of α^i_{kl} with respect to β_{kl}:

$$\frac{\partial \alpha^i_{kl}}{\partial \beta_{kl}} = \frac{\sin \beta_{kl}}{\sin \alpha^i_{kl} \sin \beta_{jl} \sin \beta_{jk}}. \tag{3.19}$$

Thus, by substituting (3.17) and (3.19) into Equation (3.14) the partial derivative of l_{kl} with respect to β_{kl} is given by

$$\frac{\partial l_{kl}}{\partial \beta_{kl}} = \frac{\sin \beta_{kl}}{\sin l_{kl} \sin \alpha^l_{ik} \sin \alpha^k_{il} \sin \beta_{jl} \sin \beta_{jk}}. \tag{3.20}$$

□

An expression of the Wigner derivative is introduced in Lemma 3.3.1. However, it can be re-written in a simpler way[3], in terms of the edge Gram matrix of the spherical tetrahedron. Indeed, that simpler expression is given as follows:

Theorem 3.3.3. *The Wigner derivative for a spherical tetrahedron T (see Figure 3.4) is expressed as*

$$\frac{\partial \beta_{kl}}{\partial l_{kl}} = \frac{\sin l_{kl} \sin l_{ij}}{\sqrt{\det G}},$$

where G is the edge Gram matrix of T.

Proof. As seen in Lemma 3.3.1, the partial derivative of β_{kl} with respect to its opposite edge length l_{kl} is given by

$$\begin{aligned}\frac{\partial \beta_{kl}(l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23})}{\partial l_{kl}} &= \frac{\sin l_{kl}}{\sin l_{il} \sin l_{ik} \sin \alpha^i_{jl} \sin \alpha^i_{jk} \sin \beta_{kl}} \\ &= \frac{\sin l_{kl} \sin l_{ij}}{\sin l_{ij} \sin l_{il} \sin l_{ik} \sin \alpha^i_{jl} \sin \alpha^i_{jk} \sin \beta_{kl}}.\end{aligned}$$

And let us remind ourselves that the determinant of the edge Gram matrix of T in terms of the box variables (Lemma 2.5.6) is given by

$$\det G = \det[\cos l_{ij}] = \sin^2 l_{01} \sin^2 l_{02} \sin^2 l_{03} \sin^2 \alpha^0_{12} \sin^2 \alpha^0_{13} \sin^2 \beta_{23}.$$

Notice that the edge Gram matrix is expressed in terms of specific edge lengths, interior angles and an interior dihedral angle, namely $l_{01}, l_{02}, l_{03}, \alpha^0_{12}, \alpha^0_{13}$ and β_{23}. However, in the expression of $\frac{\partial \beta_{kl}}{\partial l_{kl}}$, since it's arbitrary, the variables in the denominator are also arbitrary. Therefore, it remains to be proven that

$$\sin l_{ij} \sin l_{il} \sin l_{ik} \sin \alpha^i_{jl} \sin \alpha^i_{jk} \sin \beta_{kl} = \sqrt{\det G}$$

for some i, j, k, l. Due to the fact that $l_{ab} = l_{ba}$, $\alpha^s_{ab} = \alpha^s_{ba}$ and $\beta_{ab} = \beta_{ba}$ there are six ways to choose i, j, k, l such that they are all distinct and in $\{0, 1, 2, 3\}$. In this proof, let us select one example where $k = 0,\ l = 1,\ i = 2,\ j = 3$, but the method used to prove the other cases is the same. For this particular example, the Wigner derivative is

$$\frac{\beta_{01}(l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23})}{\partial l_{01}} = \frac{\sin l_{01} \sin l_{23}}{\sin l_{23} \sin l_{12} \sin l_{02} \sin \alpha^2_{13} \sin \alpha^2_{03} \sin \beta_{01}}.$$

[3]Taylor and Woodward in [41] initiated the idea of re-writing it in a simpler way by using the determinant of the edge Gram matrix of a spherical tetrahedron.

Comparing its denominator with the square root of the determinant of the edge Gram matrix of T results to

$$\begin{aligned}
\frac{\sin l_{23} \sin l_{12} \sin l_{02} \sin \alpha_{13}^2 \sin \alpha_{03}^2 \sin \beta_{01}}{\sin l_{01} \sin l_{02} \sin l_{03} \sin \alpha_{12}^0 \sin \alpha_{13}^0 \sin \beta_{23}} &= \frac{\sin l_{23} \sin l_{12} \sin \alpha_{03}^2 \sin \alpha_{01}^2 \sin \beta_{13}}{\sin l_{01} \sin l_{03} \sin \alpha_{12}^0 \sin \alpha_{23}^0 \sin \beta_{13}} && \left(\begin{array}{c}\text{sine law on } Lk(v_0), Lk(v_2) \\ \text{Fig. 3.9, 3.10}\end{array}\right) \\
&= \frac{\sin l_{23} \sin l_{12} \sin l_{03} \sin \alpha_{01}^2}{\sin l_{01} \sin l_{03} \sin l_{23} \sin \alpha_{12}^0} && (\text{sine law on } \Delta 023, \text{ Fig. 3.11}) \\
&= 1. && (\text{sine law on } \Delta 012, \text{ Fig. 3.12})
\end{aligned}$$

Thus,

$$\frac{\beta_{01}(l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23})}{\partial l_{01}} = \frac{\sin l_{01} \sin l_{23}}{\sqrt{\det G}}.$$

And, the general result in Theorem 3.3.3 follows for different choices of i, j, k, l. □

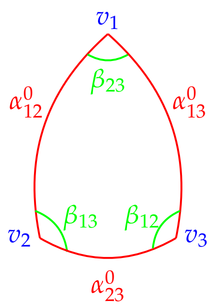


Figure 3.9: $Lk(v_0)$

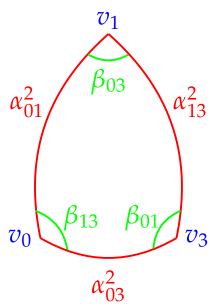


Figure 3.10: $Lk(v_2)$

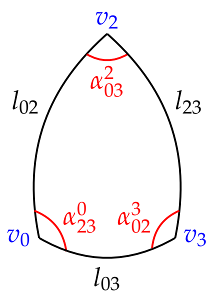


Figure 3.11: $\Delta 023$

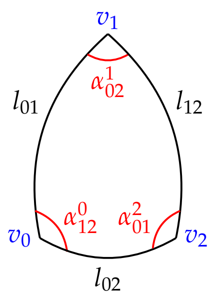


Figure 3.12: $\Delta 012$

Similarly to the Wigner derivative, the expression of the inverse Wigner derivative given in Lemma 3.3.2 may be re-expressed in terms of the determinant of the edge Gram matrix. That is stated in the following theorem.

Theorem 3.3.4. *The inverse Wigner derivative for the spherical tetrahedron T (see Figure 3.4) in terms of the edge Gram matrix is given by*

$$\frac{\partial l_{kl}(\beta_{01}, \beta_{02}, \beta_{03}, \beta_{12}, \beta_{13}, \beta_{23})}{\partial \beta_{kl}} = \frac{\sin l_{kl} \sin l_{ij}}{\sqrt{\det G}}.$$

Proof. Let us look at the sine law applied to the link $Lk(v_i)$,

$$\frac{\sin \alpha_{kl}^i}{\sin \beta_{kl}} = \frac{\sin \alpha_{jk}^i}{\sin \beta_{jk}} = \frac{\sin \alpha_{jl}^i}{\sin \beta_{jl}}.$$

The quotient of the Wigner derivative in Lemma 3.3.1 with the inverse Wigner derivative in Lemma 3.3.2 results to the following:

$$
\begin{aligned}
\frac{\frac{\partial l_{kl}}{\partial \beta_{kl}}}{\frac{\partial \beta_{kl}}{\partial l_{kl}}} &= \frac{\sin \beta_{kl}}{\sin l_{kl} \sin \alpha^l_{ik} \sin \alpha^k_{il} \sin \beta_{jl} \sin \beta_{jk}} . \frac{\sin l_{il} \sin l_{ik} \sin \alpha^i_{jk} \sin \alpha^i_{jl} \sin \beta_{kl}}{\sin l_{kl}} \\
&= \frac{\sin^2 \beta_{kl}}{\sin^2 l_{kl}} . \frac{\sin \alpha^i_{jk}}{\sin \beta_{jk}} . \frac{\sin \alpha^i_{jl}}{\sin \beta_{jl}} . \frac{\sin l_{ik}}{\sin \alpha^l_{ik}} . \frac{\sin l_{il}}{\sin \alpha^k_{il}} \\
&= \frac{\sin^2 \beta_{kl}}{\sin^2 l_{kl}} . \frac{\sin^2 \alpha^i_{jk}}{\sin^2 \beta_{jk}} . \frac{\sin^2 l_{ik}}{\sin^2 \alpha^l_{ik}} && \left(\frac{\sin \alpha^i_{jk}}{\sin \beta_{jk}} = \frac{\sin \alpha^i_{jl}}{\sin \beta_{jl}} \text{ sine law on } Lk(v_i), \text{Fig. 3.5} \right) \\
&= \frac{\sin^2 \beta_{kl} \sin^2 \alpha^i_{jk}}{\sin^2 \beta_{jk} \sin^2 \alpha^l_{ik}} . \frac{\sin^2 \alpha^l_{ik}}{\sin^2 \alpha^i_{kl}} && \left(\frac{\sin l_{ik}}{\sin l_{kl}} = \frac{\sin \alpha^l_{ik}}{\sin \alpha^i_{kl}}, \text{ sine law on } \Delta ikl, \text{Fig. 3.6} \right) \\
&= \frac{\sin^2 \beta_{kl}}{\sin^2 \alpha^i_{kl}} . \frac{\sin^2 \alpha^i_{jk}}{\sin^2 \beta_{jk}} \\
&= 1. && \left(\frac{\sin \alpha^i_{jk}}{sin \beta_{jk}} = \frac{\sin \alpha^i_{kl}}{\sin \beta_{kl}}, \text{ sine law on } Lk(v_i), \text{Fig. 3.5} \right)
\end{aligned}
$$

Hence the expression of the inverse Wigner derivative:

$$
\frac{\partial l_{kl}(\beta_{01}, \beta_{02}, \beta_{03}, \beta_{12}, \beta_{13}, \beta_{23})}{\partial \beta_{kl}} = \frac{\sin l_{kl} \sin l_{ij}}{\sqrt{\det G}}.
$$

□

Remark 3.3.5. *Even though the expression of the inverse Wigner derivative seems to appear in [41][38] but was given a different value, it is new in the literature. Further explanation on that will be provided in Section 3.4.*

From the expressions of the inverse Wigner derivative given in Theorem 3.3.4 and the Wigner derivative in Theorem 3.3.3, we conclude:

Corollary 3.3.6 (Reciprocity of the Wigner derivative)**.** *For spherical tetrahedra, the Wigner derivative is equal to the inverse Wigner derivative.*

3.4 Discussion and correction in the literature

The aim of this section is to compare Taylor and Woodward's Wigner derivative in [38, Page 17, Proposition 2.2.0.5] with our inverse Wigner derivative in Theorem 3.1.2 and clarify a slight confusion between [38, Page 17, Proposition 2.2.0.5], [41, Proposition 2.4.1, (n)] and [40, Theorem 4.0.1, (f)]. Let us look at the comparison first.

3.4.1 The nature of the partial derivative

The term Wigner derivative was referred to by Taylor in [38, Page 17, Proposition 2.2.0.5] and Taylor and Woodward in [41, Proposition 2.4.1, (n)]. Let us recall the exact statement of the formula, taken verbatim from [38, Page 17, Proposition 2.2.0.5].

Proposition 3.4.1 (Wigner derivative)**.**

$$\frac{\partial l_{ab}}{\partial \theta_{cd}} = \frac{\sqrt{\det G}}{\sin l_{cd} \sin l_{ab}}$$

where l_{ab} are the edge length of the spherical tetrahedron $S(1234)$, θ_{ab} is the exterior dihedral angle around the edge e_{ab} for $a, b, c, d \in \{1, 2, 3, 4\}$ and G its edge Gram matrix.

At first glance, one might think that the formula in [41, Prop 2.4.1.(n)] as well as in [38, Prop 2.2.0.5] and [40, Theorem 4.0.1 (f)] which may be rewritten as

$$\frac{\partial l'}{\partial \beta} = \frac{\sqrt{\det G}}{\sin l \sin l'}, \tag{3.21}$$

contradicts our formula for the inverse Wigner derivative presented in Theorem 3.1.2,

$$\frac{\partial l'(\beta_{ij})}{\partial \beta} = \frac{\sin l \sin l'}{\sqrt{\det G}}.$$

However, that is not the case, as these two partial derivatives are completely different. The difference may be understood in knowing what variables are held fixed in (3.21) while performing the partial derivative.

To prove (3.21) Taylor and Woodward argued as follows.

Let $g_a, g_b, g_c, g_d \in SU(2)$ such that

$$g_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, g_b = \begin{pmatrix} e^{l_{ab}} & 0 \\ 0 & e^{-l_{ab}} \end{pmatrix}, g_c = \begin{pmatrix} c_1 + ic_2 & c_3 + ic_4 \\ -c_3 + ic_4 & c_1 - ic_2 \end{pmatrix}, g_d = \begin{pmatrix} d_1 + id_2 & d_3 + id_4 \\ -d_3 + id_4 & d_1 - id_2 \end{pmatrix}$$

where $c_i, d_i \in \mathbb{R}$. Set $v_i = f^{-1}(g_i)$ where f is the diffeomorphism between S^3 and $SU(2)$ described in Section A.1, and consider the spherical tetrahedron with vertices v_a, v_b, v_c, v_d as shown in the Figure 3.13. Let us fix the vertices v_a, v_b, v_d,

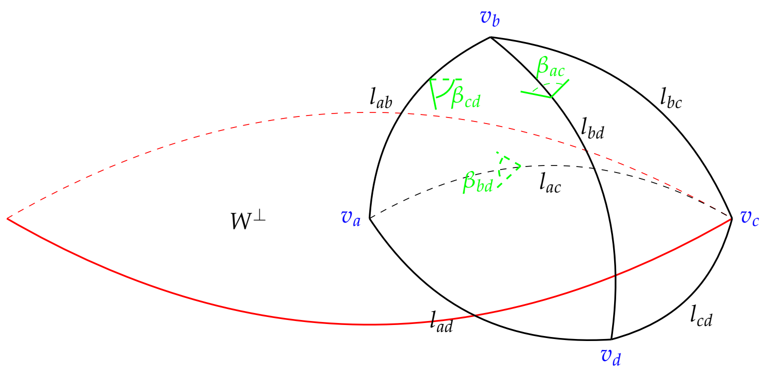


Figure 3.13: The spherical tetrahedron with vertices v_a, v_b, v_c, v_d where the projection of v_c on $W^{\perp} = Span\{a,b\}^{\perp}$ rotates along the red circle.

consider the plane spanned by v_a and v_b, i.e. $W = Span\{v_a, v_b\}$, take the projection of v_c on $W^{\perp}$ and rotate it along the circle in $W^{\perp}$ centered at $(0,0) = pr_{W^{\perp}}(v_a)$ where pr denotes the projection map.

They described that succession of action as the "rotation of v_c around E_{ab} (the edge (ab))". Then the proof carries on as in [41, Prop 2.4.1., Proof (n)].

Let us observe the following feature of their construction. When rotating v_c around the edge (ab), the interior dihedral angles (so are the cosine of the exterior dihedral angles) change accordingly. Let us give an example of why that is the case. By the use of cosine law, the cosine of the interior dihedral angle β_{ac} which is not opposite to the edge (cd) is expressed in terms of the edge length l_{cd} as

$$\cos\beta_{ac} = \frac{\frac{\cos l_{ac} - \cos l_{ad}\cos l_{cd}}{\sin l_{ad}\sin l_{cd}} - \left(\frac{\cos l_{ab} - \cos l_{bd}\cos l_{ad}}{\sin l_{bd}\sin l_{ad}}\right)\left(\frac{\cos l_{bc} - \cos l_{cd}\cos l_{bd}}{\cos l_{cd}\cos l_{bd}}\right)}{\sqrt{1 - \left(\frac{\cos l_{ab} - \cos l_{bd}\cos l_{ad}}{\sin l_{bd}\sin l_{ad}}\right)^2}\sqrt{1 - \left(\frac{\cos l_{bc} - \cos l_{cd}\cos l_{bd}}{\sin l_{cd}\sin l_{bd}}\right)^2}}.$$

Furthermore, the other interior dihedral angles are also expressed as functions of l_{cd} (but whose expressions will not be presented here). Hence, during that process of rotation most of the dihedral angles vary. Therefore, the dihedral angles are not held constant in (3.21), whereas in Theorem 3.1.2 they are. In fact, five of the edge lengths that are fixed in (3.21). Indeed, if we write out the partial derivative on the left hand side of (3.21) explicitly, making clear what the dependent variables are, then it says that

$$\frac{\partial l'(l_{ab}, l_{ac}, l_{ad}, l_{bc}, l_{bd}, \beta)}{\partial\beta} = \frac{\sqrt{\det G}}{\sin l \sin l'} \tag{3.22}$$

where β is the rotation angle of v_c around the edge (ab) i.e. the interior dihedral angle around the edge (ab).

In conclusion, there is no contradiction between the result in Theorem 3.1.2 and (3.21). They are just two entirely different partial derivatives. This also proves that our formula for the inverse Wigner derivative is new in the literature.

3.4.2 Correcting some typos

Let us now clarify the confusion between [38, Page 17, Proposition 2.2.0.5], [41, Proposition 2.4.1, (n)] and [40, Theorem 4.0.1, (f)].

The formula for the Wigner derivative (Proposition 3.4.1) is also present in [41, Proposition 2.4.1, (n)] but seem to have different results. Namely,

Proposition 3.4.2 (Taylor-Woodward [41])**.** *The derivative of dihedral angle with respect to opposite edge length is*

$$\left(\frac{\partial \theta_{cd}}{\partial l_{ab}}\right)^{-1} = -\frac{\sin l_{cd} \sin l_{ab}}{\sqrt{\det G}}. \tag{3.23}$$

Here, $\left(\frac{\partial \theta_{cd}}{\partial l_{ab}}\right)^{-1}$ is understood (see Remark 3.4.3 below) to be $\frac{\partial l_{ab}}{\partial \theta_{cd}}$. Hence, Equation (3.23) really means

$$\frac{\partial l_{ab}}{\partial \theta_{cd}} = -\frac{\sin l_{cd} \sin l_{ab}}{\sqrt{\det G}}. \tag{3.24}$$

However, this appears to contradict Proposition 3.4.1. Some remarks follow immediately.

Remark 3.4.3.

- *On one hand, it is understood that the proof of Proposition 3.4.1 presented in [41, Proposition 2.4.1, (n)] is the same as that of Proposition 3.4.2 presented in [38, Page 17, Proposition 2.2.0.5]. However, as shown in the two propositions, the statement of the results are different. Hence, a conclusion may be drawn that one of them must present an error. But, according to the proof the output given in [38, Page 17, Proposition 2.2.0.5] which is here presented as Proposition 3.4.1 is more convincing. We claim that the error is just a typo which occurs in [41, Proposition 2.4.1, (n)], confirmed by one of the authors [11]. In fact we suggest that the left hand side of Equation* (3.23) *should be* $\frac{\partial \theta_{cd}}{\partial l_{ab}}$ *instead of* $\left(\frac{\partial \theta_{cd}}{\partial l_{ab}}\right)^{-1}$. *This way, Equation* (3.23) *becomes*

$$\frac{\partial \theta_{cd}(l_{ab}, l_{ac}, l_{ad}, l_{bc}, l_{bd}, l_{cd})}{\partial l_{ab}} = -\frac{\sin l_{cd} \sin l_{ab}}{\sqrt{\det G}} \tag{3.25}$$

with all edge lengths other than l_{ab} remaining fixed. Indeed, taking the reciprocal of $\frac{\partial \theta_{cd}}{\partial l_{ab}}$ is the same as

$$\frac{\partial l_{ab}(\theta_{cd}, l_{ac}, l_{ad}, l_{bc}, l_{bd}, l_{cd})}{\partial \theta_{cd}} = -\frac{\sqrt{\det G}}{\sin l_{ab} \sin l_{cd}}. \tag{3.26}$$

- *On the other hand, a little oversight occurred in [38, Page 17, Proposition 2.2.0.5] (see Proposition 3.4.1). In fact, the proposition states that the angle taken into account is the* ***exterior*** *dihedral angle. Although the calculation in its proof is done by considering the* ***interior*** *dihedral angle. That oversight led to a sign error in the final result. In other words, we suggest that*

  $$\frac{\partial l_{ab}(\theta_{cd}, l_{ac}, l_{ad}, l_{bc}, l_{bd}, l_{cd})}{\partial \theta_{cd}} = \frac{\sqrt{\det G}}{\sin l_{ab} \sin l_{cd}} \qquad (\textit{Proposition 3.4.1})$$

  should be

  $$\frac{\partial l_{ab}(\theta_{cd}, l_{ac}, l_{ad}, l_{bc}, l_{bd}, l_{cd})}{\partial \theta_{cd}} = -\frac{\sqrt{\det G}}{\sin l_{ab} \sin l_{cd}},$$

  which is consistent with Equation (3.26).

This same formula in Proposition 3.4.1 appears in [40, Theorem 4.0.1, (f)] which is recalled below.

Theorem 3.4.4 (Taylor-Woodward)**.** *The derivative of an edge length l_{ab} in a spherical tetrahedron τ with respect to its opposite dihedral angle θ_{cd} is given by*

$$\frac{\partial l_{ab}}{\partial \theta_{cd}} = \pm \frac{\sqrt{\det G}}{\sin l_{ab} \sin l_{cd}}.$$

So, following Remark 3.4.3 the formula for the Wigner derivative in Theorem 3.4.4 should be

$$\frac{\partial l_{ab}}{\partial \theta_{cd}} = -\frac{\sqrt{\det G}}{\sin l_{ab} \sin l_{cd}}$$

since θ_{cd} is here understood to the **exterior** dihedral angle opposite to the edge of length l_{ab}.

Chapter 4

A conjectural integral formula for the quantum 6j symbols

Note: Throughout this chapter $\overline{ij}$ means neither i nor j features in the index $\overline{ij}$. For example, $\overline{01} = 23$ while considering the set of indices $\{0, 1, 2, 3\}$.

4.1 Introduction

In 1968 Ponzano and Regge conjectured an asymptotic formula for the classical 6j symbol [32]. The non-degenerate version involves the volume of a *non-degenerate Euclidean tetrahedron*.

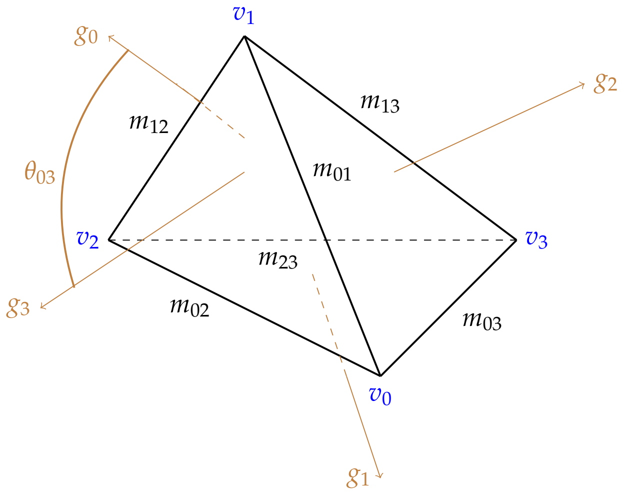
In 1999, Roberts, in [35], reformulated and proved Ponzano and Regge's conjecture.

The non-degenerate version of the asymptotic formula reads as follows:

Theorem 4.1.1 ([35]). *Let $m_{01}, m_{02}, m_{03}, m_{12}, m_{13}, m_{23}$ be six non-negative integers such that the triples (m_{12}, m_{01}, m_{02}), (m_{01}, m_{13}, m_{03}), (m_{02}, m_{23}, m_{03}), and (m_{12}, m_{13}, m_{23}) are admissible. Let $k \in \mathbb{N}$, and τ be a non-degenerate Euclidean tetrahedron with edge lengths m_{ij} associated to the classical 6j symbol*

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}.$$

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix} \equiv$$



Let θ_{ij} be the exterior *dihedral angle of τ at the edge of length $m_{\overline{ij}}$, which is opposite to the edge of length m_{ij}. Then, the non-degenerate asymptotic formula for the classical 6j symbols is*

$$\begin{Bmatrix} km_{12} & km_{02} & km_{01} \\ km_{03} & km_{13} & km_{23} \end{Bmatrix} \sim \sqrt{\frac{2}{3\pi V k^3}} \cos\{(km_{\overline{ij}}+1)\frac{\theta_{ij}}{2}+\frac{\pi}{4}\}$$

when k tends to infinity, where V is the volume of τ.

In 1999 Barrett introduced a twelve-dimensional integral formula for the square of the classical 6j symbols [5]:

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}^2 = \int_{(SU(2))^4} [dg_i] \prod_{i<j} \chi_{m_{ij}}(g_j g_i^{-1}). \tag{4.1}$$

In 2003 Freidel and Louapre, [17], re-wrote Barret's twelve-dimensional integral formula as a six-dimensional integral. Then, they proposed and used an asymptotic version of it to give an alternative proof for Roberts' version of the non-degenerate asymptotic formula for the square of the classical 6j symbols. The stationary phase method was adopted for that. The asymptotic integral formula is given as follows:

$$I(k) = \frac{2}{\pi^4} \int_{D_\pi} [\prod_{i<j} dl_{ij}] \frac{\prod_{i<j} \sin((km_{ij}+1)l_{ij})}{\sqrt{\det([\cos l_{ij}])}}. \tag{4.2}$$

Explicitly, the region of integration is

$$D_\pi = \{(l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23}) \in [0,\pi]^6, [\cos l_{ij}] \text{ is positive definite } \},$$

where $[\cos l_{ij}]$ is the 4×4 unidiagonal symmetric matrix whose off-diagonal elements are $\cos l_{ij}$.

When applying the stationary phase method to an integral, one must consider the critical points of its integrand inside the region of integration and those at the boundary as explained in [15] and [13]. We will call **interior contribution** the asymptotic expansion of the integral contributed by the critical points inside the region and **boundary contribution** that related to the ones on the boundary. The sum of interior and boundary contributions constitute the asymptotic expansion formula for the integral.

From the stationary phase method applied to $I(k)$ (4.2), the non-degenerate asymptotic formula for the square of the classical 6j symbols is presented in the following corollary.

Corollary 4.1.2 ([17]). *Let $m_{01}, m_{02}, m_{03}, m_{12}, m_{13}, m_{23}$ be six natural numbers such that the triples (m_{12}, m_{01}, m_{02}), (m_{01}, m_{13}, m_{03}), (m_{02}, m_{23}, m_{03}), and (m_{12}, m_{13}, m_{23}) are admissible. Let $k \in \mathbb{N}$. And, let τ be a non-degenerate Euclidean tetrahedron with edge lengths m_{ij} associated to the classical 6j symbols*

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}.$$

Let θ_{ij} be the exterior *dihedral angle of τ at the edge of length $m_{\bar{i}\bar{j}}$, opposite to the edge of length m_{ij}. Then, the non-degenerate asymptotic formula for the classical 6j symbols is*

$$\begin{Bmatrix} km_{12} & km_{02} & km_{01} \\ km_{03} & km_{13} & km_{23} \end{Bmatrix}^2 \sim -\frac{1}{3\pi k^3 V} \sin\left(\sum_{i<j}(km_{\bar{i}\bar{j}}+1)\theta_{ij}\right) + \frac{1}{3\pi k^3 V}$$

when k tends to infinity, where V is the volume of τ.

Here, the interior contribution is

$$c_{\text{in}} = -\frac{1}{3\pi k^3 V} \sin\left(\sum_{i<j}(km_{\bar{i}\bar{j}}+1)\theta_{ij}\right),$$

and the boundary contribution is given by

$$c_{\text{bd}} = \frac{1}{3\pi k^3 V}.$$

In 2003 Taylor and Woodward found the asymptotic formulae (based on the possible entries of the symbol) for the quantum 6j symbols , relating them to the geometry of spherical tetrahedra (see [41] and [38]). But in this Chapter, we are only interested in the non-degenerate asymptotic formula which is related to the geometry of a *non-degenerate spherical tetrahedron*. Their formula is asymptotically equivalent to a slightly different formula given by Roberts in [36]. Roberts' version is what we will adopt in this Chapter:

Theorem 4.1.3 ([36] [41]). *Let r^0_{ij} be an element of $\mathbb{Q} \cap [0,1]$. Let k be a natural number. Let T^0 be the non-degenerate spherical tetrahedron with edge lengths $l^0_{ij} = \pi r^0_{ij}$ and whose* exterior dihedral angle *at the edge $(\overline{ij})$, opposite to the edge (ij), is denoted by θ^0_{ij}. Then,*

$$\begin{Bmatrix} kr^0_{12} & kr^0_{02} & kr^0_{01} \\ kr^0_{03} & kr^0_{13} & kr^0_{23} \end{Bmatrix}^2_{q=e^{\frac{i\pi}{k+2}}} \sim \frac{4\pi^2}{k^3\sqrt{\det([\cos l^0_{ij}])}} \cos^2\{\sum_{i<j}(kr^0_{\overline{ij}}+1)\frac{\theta^0_{ij}}{2} + \frac{k+2}{\pi}V(T^0) + \frac{\pi}{4}\} \tag{4.3}$$

when $k \longrightarrow \infty$, where kr^0_{ij} are integers and $[\cos l^0_{ij}]$ is the edge Gram matrix of T^0.

Remark 4.1.4. *The equivalence between Taylor and Woodward's asymptotic formula for the quantum 6j symbols and that of Roberts will be empirically shown in Section C.3.*

Remark 4.1.5. *The asymptotic formula present in Theorem 4.1.3 is slightly different from the one given by Roberts in [36], as well as that of Taylor and Woodward in [41] and [38]. In fact,*

- *a minor slip up has surfaced in the original formula in [36]. Inside the sum, it is written $l_{\overline{ij}}$ when it's expected to be $r^0_{\overline{ij}}$.*
- *In [36], $q = e^{\frac{i2\pi}{k+2}}$ but the formula is only equivalent to that in [41] (as stated in Remark 4.1.4) when $q = e^{\frac{i\pi}{k+2}}$ and the coefficient in front of the volume term is $\frac{k+2}{\pi}$, not $\frac{k}{\pi}$. So, we will consider those as typos as well.*
- *It seems there was a bit of oversight regarding the Schläfli identity in [41, page 550, Proposition 2.4.1 (l)] which led to a sign error in the volume term in the non-degenerate asymptotic formula for the quantum 6j symbols in [41],[36] and later adopted in [27]. More explanation on that will be provided in Section 4.3. That oversight on the Schläfli formula also occurs in [40, Theorem 4.0.1, (f)].*

Remark 4.1.6. *As* $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, *an immediate way of re-writing the right hand side of Equation* (4.3) *as a sum is*

$$\begin{aligned}\begin{Bmatrix} kr^0_{12} & kr^0_{02} & kr^0_{01} \\ kr^0_{03} & kr^0_{13} & kr^0_{23}\end{Bmatrix}^2_{q=e^{\frac{i\pi}{k+2}}} &\sim \frac{1}{2}\frac{4\pi^2}{k^3\sqrt{\det([\cos l^0_{ij}])}}\left(1+\cos\{\sum_{i<j}(kr^0_{ij}+1)\theta^0_{ij}+\frac{2(k+2)}{\pi}V(T^0)+\frac{\pi}{2}\}\right)\\ &= \frac{2\pi^2}{k^3\sqrt{\det([\cos l^0_{ij}])}}\left(1-\sin\{\sum_{i<j}(kr^0_{ij}+1)\theta^0_{ij}+\frac{2(k+2)}{\pi}V(T^0)\}\right)\\ &= -\frac{2\pi^2}{k^3\sqrt{\det([\cos l^0_{ij}])}}\sin\{\sum_{i<j}(kr^0_{ij}+1)\theta^0_{ij}+\frac{2(k+2)}{\pi}V(T^0)\}\\ &\quad+\frac{2\pi^2}{k^3\sqrt{\det([\cos l^0_{ij}])}}.\end{aligned}$$

We guess that if there was an asymptotic integral formula for the square of the quantum 6j symbols, the interior contribution would be

$$\text{ins} = -\frac{2\pi^2}{k^3\sqrt{\det([\cos l^0_{ij}])}}\sin\{\sum_{i<j}(kr^0_{ij}+1)\theta^0_{ij}+\frac{2(k+2)}{\pi}V(T^0)\}$$

and the boundary contribution

$$\text{bound} = \frac{2\pi^2}{k^3\sqrt{\det([\cos l^0_{ij}])}}.$$

Contrarily to the integral formula for the square of the classical 6j symbols, to the best of our knowledge, an analogous integral formula for the square of the quantum 6j symbols is nowhere to be found in the literature.

In early 2019 Bruce Bartlett (the author's supervisor) proposed a conjecture for the square of the quantum 6j symbols and suggested the investigation of that conjecture to be the author's Ph.D. project.

Consider the quantum 6j symbol (see Definition 4.2.1)

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23}\end{Bmatrix}_q,$$

where $q = e^{\mathrm{i}\frac{\pi}{s+2}}$ with $\mathrm{i}^2 = -1$, $s \in \mathbb{N}_{>0}$, and whose entries are non-negative integers. The conjecture is as follows:

Conjecture 4.1.7 (Strong form)**.** *Let* $m_{01}, m_{02}, m_{03}, m_{12}, m_{13}, m_{23}$ *be six natural numbers such that the triples* (m_{12}, m_{01}, m_{02}), (m_{01}, m_{13}, m_{03}), (m_{02}, m_{23}, m_{03}), *and* (m_{12}, m_{13}, m_{23})

are q-admissible (see the Equations (4.7), (4.8), (4.9)*). Let g_0, g_1, g_2, g_3 be four elements in $SU(2)$. Due to the diffeomorphism between $SU(2)$ and S^3 they may be thought as four unit vectors in $\mathbb{R}^4$. Let T be the spherical tetrahedron whose outward normal vectors to each face are the g_i's. Then,*

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}^2_{q=e^{\frac{i\pi}{s+2}}} = \int_{SU(2)^4} \prod_{i<j} \chi_{m_{\overline{ij}}}(g_j g_i^{-1}) e^{\frac{2}{\pi}(s+2)\mathrm{i}V(T)} [\prod dg_i], \tag{4.4}$$

where $V(T)$ denotes the volume of the spherical tetrahedron T.

Our aim is to investigate the veracity of Conjecture 4.1.7. For that, we adopt Freidel and Louapre's strategy:

1. transform the conjectural twelve-dimensional integral formula for the quantum 6j symbols (4.4) to a six-dimensional one,
2. look for an asymptotic analogue of that six-dimensional integral,
3. compute the asymptotic expansion of that asymptotic integral via the stationary phase method and compare it with the known non-degenerate asymptotic formula for the quantum 6j symbols found in [41], [36], and [27].

The integral on the right hand side of Equation (4.4) can be transformed to an integral over a subspace of $\mathbb{R}^6$. The six-dimensional integral is presented in the lemma below.

Lemma 4.1.8. *The twelve-dimensional integral on the right hand side of Equation* (4.4) *can be re-expressed as a six-dimensional integral over the set of $SO(4)-$equivalence classes of non-degenerate spherical tetrahedra as follows:*

$$I(s) = \frac{2}{\pi^4} \int_{D_\pi} [\prod_{i<j} \mathrm{d}\theta_{ij}] \frac{\prod_{i<j} \sin\left((m_{\overline{ij}}+1)\theta_{ij}\right)}{\sqrt{\det\left([\cos\theta_{ij}]\right)}} \cos\left(\frac{2}{\pi}(s+2)V(T)\right),$$

where $V(T)$ denotes the volume of the non-degenerate spherical tetrahedron with exterior dihedral angles θ_{ij}.

Explicitly, the region of integration is

$$D_\pi = \{(\theta_{01}, \theta_{02}, \theta_{03}, \theta_{12}, \theta_{13}, \theta_{23}) \in [0,\pi]^6, [\cos\theta_{ij}] \text{ is positive definite } \},$$

where $[\cos\theta_{ij}]$ *is the* 4×4 *unidiagonal symmetric matrix whose off-diagonal elements are* $\cos\theta_{ij}$.

An asymptotic form of the conjectural integral formula for the quantum 6j symbols (4.4) is stated as follows:

Conjecture 4.1.9 (Asymptotic form)**.** *Let* $r^0_{ij}\in\mathbb{Q}\cap[0,1]$ *and* $k\in\mathbb{N}$ *such that* kr^0_{ij} *is integer. Then as* $k\to\infty$,

$$\begin{Bmatrix} kr^0_{12} & kr^0_{02} & kr^0_{01} \\ kr^0_{03} & kr^0_{13} & kr^0_{23}\end{Bmatrix}^2_{q=e^{\frac{i\pi}{k+2}}} \sim \frac{2}{\pi^4}\int_{D_\pi}\frac{\prod_{i<j}\sin\left((kr^0_{\overline{ij}}+1)\theta_{ij}\right)}{\sqrt{\det([\cos\theta_{ij}])}}\cos\left(\frac{2}{\pi}(k+2)V(T)\right)[\prod_{i<j}\mathrm{d}\theta_{ij}], \tag{4.5}$$

where $V(T)$ *denotes the volume of the non-degenerate spherical tetrahedron with* exterior dihedral angles θ_{ij}. *Here, the domain of integration is defined by*

$$D_\pi=\{(\theta_{01},\theta_{02},\theta_{03},\theta_{12},\theta_{13},\theta_{23})\in[0,\pi]^6,[\cos\theta_{ij}]\ \textit{is positive definite}\ \},$$

where $[\cos\theta_{ij}]$ *is the* 4×4 *unidiagonal symmetric matrix whose off-diagonal elements are* $\cos\theta_{ij}$.

Let us set the integral on the right hand side of Equation (4.5) to be

$$I_c(k)=\frac{2}{\pi^4}\int_{D_\pi}[\prod_{i<j}\mathrm{d}\theta_{ij}]\frac{\prod_{i<j}\sin\left((kr^0_{\overline{ij}}+1)\theta_{ij}\right)}{\sqrt{\det([\cos\theta_{ij}])}}\cos\left(\frac{2}{\pi}(k+2)V(T)\right).$$

The next step is to compute the interior and boundary contributions for $I_c(k)$.

The interior contribution for the integral $I_c(k)$ is given as follows:

Theorem 4.1.10. *Let* $r^0_{ij}\in\mathbb{Q}\cap[0,1]$. *Let* T^0 *be the spherical tetrahedron whose edge lengths* $l^0_{ij}=\pi r^0_{ij}$ *and whose* exterior dihedral angles *at the edge* $(\overline{ij})$, *opposite to the edge* (ij), *is denoted by* θ^0_{ij}. *Then, interior contribution for* $I_c(k)$ *is*

$$\mathrm{int}=\frac{-\pi^2}{4k^3}.\frac{1}{\sqrt{\det([\cos l^0_{ij}])}}\cos\{\sum_{i<j}(kr^0_{\overline{ij}}+1)\theta^0_{ij}+\frac{2}{\pi}(k+2)V(T^0)\} \tag{4.6}$$

when $k\longrightarrow\infty$, *where* $[\cos l^0_{ij}]$ *is the edge Gram matrix of* T^0.

Despite our desire to provide both the interior and boundary contributions for our conjecture, we only managed to explain the interior part. That is due to the observation that if the integral is expressed in terms of the box variables, we are dealing with a critical surface at the boundary, not critical points. That makes the problem more complex. And, if we consider the integral expressed in terms of the exterior dihedral angles, the amplitude tends to infinity at the boundary and the author does not have enough skill to deal with the problem. However, we would like to draw the reader's attention to the following remarks regarding the interior contribution.

Remark 4.1.11. *If we make the case that the decomposition of Equation* (4.3) *as a sum of an "interior" contribution and a "boundary" contribution in Remark 4.1.6 is correct, then the result in Theorem 4.1.10 shows that the interior contribution for the asymptotic form of the conjectural integral formula for the quantum 6j symbols is indeed very close to the interior contribution for the exact asymptotics (see Remark 4.1.6). There are only two shortcomings: the amplitude is short of a factor of eight, and the phase is a cosine, not a sine.*

The aim of this chapter is to prove Theorem 4.1.10.

Here is a brief outline of the chapter: five main sections will be considered, namely, Section 4.2 is about the definition of a quantum 6j symbol, and Section 4.3 contains the elaboration of Remark 4.1.5. The transformation of the integral on the right hand side of Conjecture 4.1.7 to a six-dimensional Lebesgue integral is done in Section 4.4. And, Section 4.5 is dedicated to the calculation of the interior contribution for the asymptotic form of our conjectural integral formula for the quantum 6j symbols.

4.2 The quantum 6j symbols

The aim of this section is to recall the definition of the quantum 6j symbols.

To start with, let us remind ourselves of the definition of a quantum integer.

Let $r \geq 3$ be an integer and $q = e^{\frac{i\pi}{r}}$. Then, for any integer n, the *quantum integer* $[n]$ is defined by

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Its associated *quantum factorial* is known to be

$$[n]! = [n][n-1]\cdots[1],$$

where $[0]! = 1$.

A triple of non-negative integers (a, b, j) is called *q-admissible* if they satisfy the following requirements:

$$a \leq b + j,\ b \leq a + j,\ j \leq a + b, \tag{4.7}$$

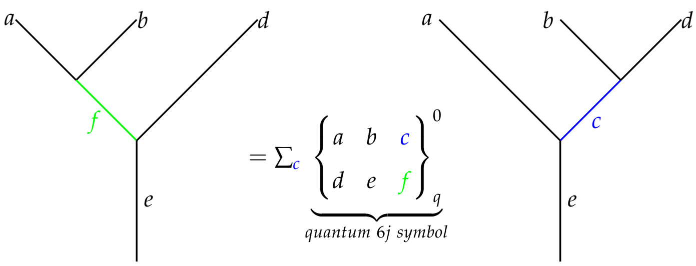
$$a + b + j \in 2\mathbb{Z}, \tag{4.8}$$

$$a + b + j \leq 2r - 4. \tag{4.9}$$

Let a, b, c, d, e, f be non-negative integers such that $(a, b, f), (f, d, e)$ are q-admissible. Let $\tilde{V}_a, \tilde{V}_b, \tilde{V}_c, \tilde{V}_d, \tilde{V}_e, \tilde{V}_f$ be six irreducible representations of the quantum group $U_q(sl_2)$, where $\tilde{V}_n$ is of dimension $(n+1)$. Analogously to the definition of the classical 6j symbols (see Chapter 2), *quantum 6j symbols* are the coefficients for a certain change of basis in

$$Hom_{Rep_q(sl_2)}(\tilde{V}_e, \tilde{V}_a \otimes \tilde{V}_b \otimes \tilde{V}_d),$$

where $Rep_q(sl_2)$ denotes the category of representations of $U_q(sl_2)$. That change may be depicted in the graphical calculus as



where c ranges for all natural numbers making (b, d, c), and (a, c, e) admissible.

Similarly to the classical 6j symbols, there is a normalized version of the quantum 6j symbols. Below is the definition of a normalized quantum 6j symbol.

Definition 4.2.1. *Let a, b, c, d, e, f be non-negative integers such that $(a, b, f), (f, d, e), (b, d, c)$, and (a, c, e) are q-admissible. Then, a normalized quantum 6j symbol is defined as*

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_q = \frac{(-1)^{\frac{a+b+d+e}{2}}}{[f+1]} \sqrt{\left| \frac{\Theta_q(a,b,c)\Theta_q(c,d,e)}{\Theta_q(a,e,f)\Theta_q(a,d,f)} \right|} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_q^0$$

where

$$\Theta_q(x,y,z) = \frac{[n_{xz}]![n_{xy}]![n_{yz}]![n_{xz}+n_{xy}+n_{yz}+1]!}{(-1)^{n_{xz}+n_{xy}+n_{yz}}[x]![y]![z]!}$$

and

$$n_{xz} = \frac{x+z-y}{2},\ n_{xy} = \frac{x+y-z}{2},\ n_{yz} = \frac{y+z-x}{2}$$

for any q-admissible triples $(x, y, z) \in \mathbb{N}^3$.

The normalized version of a quantum 6j symbol presents some symmetries [41][12][20] which enable one to relate the symbol to a spherical tetrahedron. From now on, we will only consider the normalized quantum 6j symbols.

For further reading on the quantum 6j symbols [12] and [20] are recommended.

4.3 Correction in the literature

In this section, let us revise the asymptotic formula for a quantum 6j symbol found in [41], [27] and [36], and expound on the hiccups mentioned in Remark 4.1.5. Note that here, we will focus on the result presented in [41] since those of [27] and [36] are its derivatives.

The statement of the asymptotic formula for a quantum 6j symbol in [41][27] for the non-degenerate case is as follows:

Theorem 4.3.1 (Taylor-Woodward [41])**.** *Let $r > 2$ be an integer. Let $j_{01}, j_{02}, j_{03}, j_{12}, j_{13}, j_{23} \in [0, \frac{(r-2)}{2}] \cap \mathbb{Z}/2$ (half-integers) such that each triple (j_{12}, j_{01}, j_{02}), (j_{01}, j_{13}, j_{03}), (j_{02}, j_{23}, j_{03}), and (j_{12}, j_{13}, j_{23}) satisfies the triangle inequality and their respective sum yields an integer less than $r-2$. Suppose that a non-degenerate spherical tetrahedron exists with edge lengths $l_{ab}(k) = 2\pi \frac{kj_{ab}+\frac{1}{2}}{r(k)}$, and $\theta_{ab}(k)$ denotes the* exterior dihedral angle *at the edge (ab). Then,*

$$\begin{Bmatrix} kj_{12} & kj_{02} & kj_{01} \\ kj_{03} & kj_{13} & kj_{23} \end{Bmatrix}_{q=e^{\frac{\pi i}{r(k)}}} \sim \frac{2\pi}{r(k)^{\frac{3}{2}}[\det(\cos l_{ab})]^{\frac{1}{4}}} \cos\{\frac{r(k)}{2\pi}(\sum_{a<b}\theta_{ab}(k)l_{ab}(k) - 2V(\tau(k))) + \frac{\pi}{4}\} \tag{4.10}$$

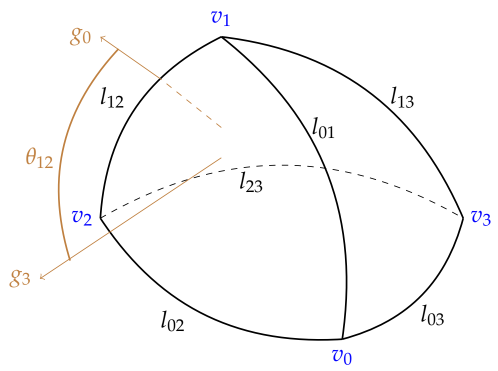


Figure 4.1: Spherical tetrahedron T associated to the classical 6j symbol with entries $j_{01}, j_{02}, j_{03}, j_{12}, j_{13}, j_{23}$ having the exterior dihedral angles labeled according to Taylor and Woodward's notation.

as $k \longrightarrow \infty$. Here, $r(k) = k(r-2)+2$ and $V(\tau(k))$ denotes the volume of the spherical tetrahedron $\tau(k)$.

The spherical tetrahedron associated to $\begin{Bmatrix} j_{12} & j_{02} & j_{01} \\ j_{03} & j_{13} & j_{23} \end{Bmatrix}$, when it exists, is depicted by Figure 4.1. Here, g_0 and g_3 are respectively the outward normal vectors to the faces $\Delta 123$ and $\Delta 012$.

Remark 4.3.2. *As the labeling of the spherical tetrahedron associated to a quantum 6j symbol is delicate and unique up to rotation only, we suggest that the places of l_{14} and l_{23} in [41, Figure 1] should be switched if it's expected to represent the quantum 6j symbol $\begin{Bmatrix} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{Bmatrix}$.*

Although traditionally the word asymptotic refers to scaling the numbers by k and letting k tend to infinity, the behaviour of the asymptotic formula can equally be observed by taking large numbers in the expression of the quantum 6j symbols and varying one number while keeping the five others constant.

When using integers instead of half-integers Equation (4.10) is equivalent to:

Theorem 4.3.3 (A version of Theorem 4.3.1 while using large integers)**.** *Let $k \in \mathbb{N}$. Let $m_{ij} \in \mathbb{N}$ be large such that each triple (m_{12}, m_{01}, m_{02}), (m_{01}, m_{13}, m_{03}), (m_{02}, m_{23}, m_{03}), and (m_{12}, m_{13}, m_{23}) satisfies the triangle inequality and their respective sum yields an even integer less than $2k$. In this case, let us decide to vary m_{23} and keep the rest constant.*

Suppose that the non-degenerate spherical tetrahedron $T(m_{23})$ exists with edge lengths $l_{ab} = \pi\frac{m_{ab}+1}{k+2}$, exterior dihedral angle $\theta_{ab}(m_{23})$ at the edge (ab) and volume $V(m_{23})$. Then the formula in Equation (4.10) *translates into:*

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}_{e^{\frac{\pi i}{k+2}}} \cong \frac{2\pi}{(k+2)^{\frac{3}{2}}[\det G(m_{23})]^{\frac{1}{4}}} \cos\{\sum_{a<b}(m_{ab}+1)\frac{\theta_{ab}(m_{23})}{2} - \frac{k+2}{\pi}V(m_{23}) + \frac{\pi}{4}\}. \tag{4.11}$$

Here, $G(m_{23})$ denotes its edge Gram matrix.

Remark 4.3.4. *Corollary 4.3.3 is more convenient than Theorem 4.3.1 for numerical manipulation.*

4.3.1 Numerical checking

A bit of a challenge occurred when we tried to reproduce Figure 3 in [41]. So, let's re-check the numerical calculation for the example $\begin{Bmatrix} 40 & 48 & 50 \\ 52 & 54 & n \end{Bmatrix}_q$ with $k = 198$ and $q = e^{\frac{i\pi}{k+2}}$.

The software we used is MATHEMATICA and the graph obtained while using the formula in Equation (4.11) is given in Figure 4.2.

As shown on the graph, the plot of the asymptotic formula does not coincide with the plot of the exact values of the quantum 6j symbols. We claim that the sign of the volume in the summand should be changed to plus (+). That is,

Theorem 4.3.5 (Taylor-Woodward (Corrected))**.** *Let k be a positive integer. Let $m_{ij} \in \mathbb{N}$ be large such that each triple (m_{12}, m_{01}, m_{02}), (m_{01}, m_{13}, m_{03}), (m_{02}, m_{23}, m_{03}), and (m_{12}, m_{13}, m_{23}) satisfies the triangle inequality and their respective sum yields an even integer less than $2k$. In this case, let us decide to vary m_{23} and keep the rest constant. Suppose that the non-degenerate spherical tetrahedron $T(m_{23})$ exists with edge lengths $l_{ab} = \pi\frac{m_{ab}+1}{k+2}$, exterior dihedral angle $\theta_{ab}(m_{23})$ at the edge (ab) and volume $V(m_{23})$. Then,*

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}_{e^{\frac{\pi i}{k+2}}} \cong \frac{2\pi}{(k+2)^{\frac{3}{2}}[\det G(m_{23})]^{\frac{1}{4}}} \cos\{\sum_{a<b}(m_{ab}+1)\frac{\theta_{ab}(m_{23})}{2} + \frac{k+2}{\pi}V(m_{23}) + \frac{\pi}{4}\} \tag{4.12}$$

where $G(m_{23})$ denotes the edge Gram matrix of $T(m_{23})$.

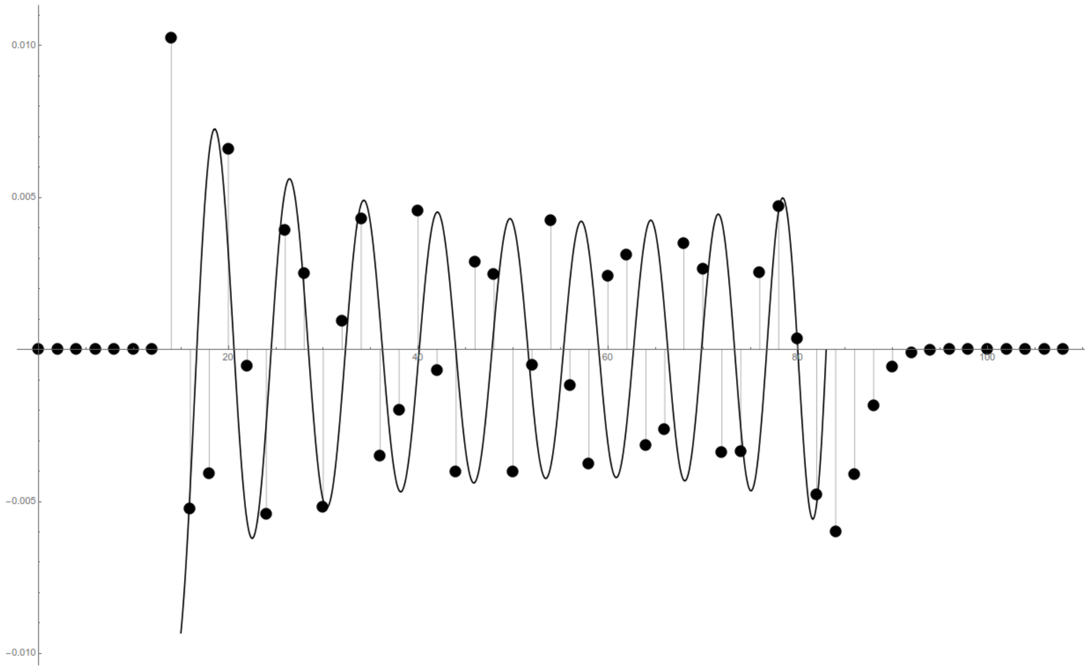


Figure 4.2: Exact values of the quantum 6j symbols vs its asymptotic (Equation (4.11)) . Here, the dots represent the exact values of the quantum 6j symbols and the continuous graph represents the asymptotic. See Appendix C.

When implementing the modified formula (4.12), in [41, Figure 3] is recovered, see Figure 4.3.

Remark 4.3.6.

- *Although the formula for the asymptotics of the quantum 6j symbols in [41, Proposition 2.4.1] The MAPLE code by Taylor and Woodward is not correct, their figure [41, Figure 3] is correct and consistent with Figure 4.3. The reason is that in the MAPLE code available in [39], if one looks under #the prediction for the non-degenerate case, line 84, in the definition of predict, the sign used is plus (+) instead of minus (-). That is why it produces Figure 4.3 and [41, Figure 3].*
- *The MATHEMATICA code generating these two graphs may be found in Appendix C.*

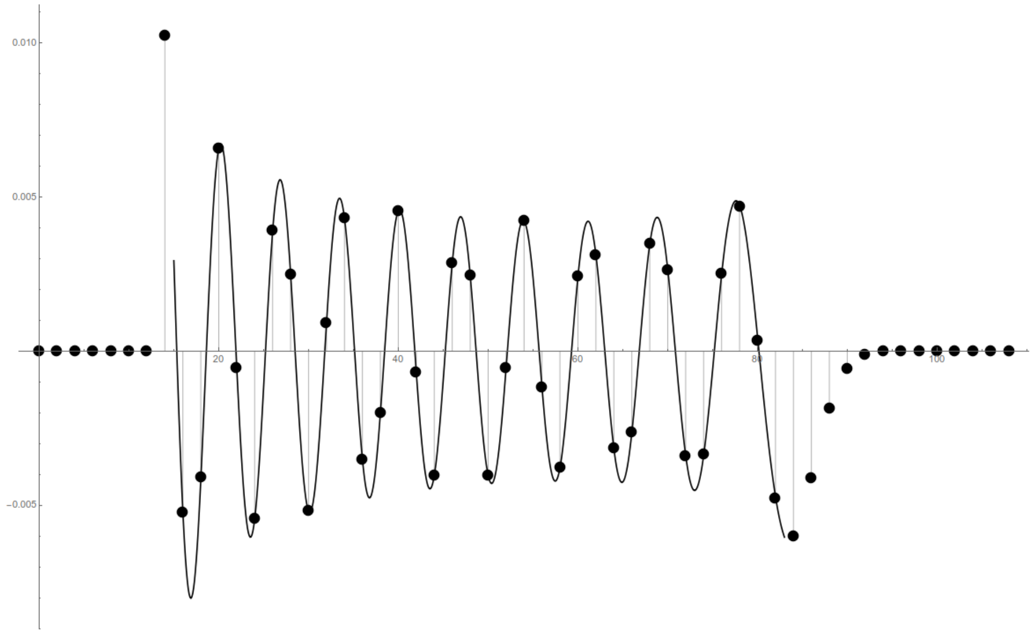


Figure 4.3: Exact values of the quantum 6j symbols vs its asymptotic from Equation (4.12). Here, the dots represent the exact values of the quantum 6j symbols and the continuous graph that of the asymptotic. See Appendix C.

4.3.2 Theoretical argument

This is not a proof of Equation (4.12) but instead an argument on where the error in (4.10) crept in. The full proof is available in [41] as well as in [38, Page 39].

As the area of work is a bit slippery, it can be easy to omit some precision by mistake. In [41, page 550, Proposition 2.4.1, (l) Schläfli formula] as well as in [38, Page 17, Theorem 2.2.0.4] the scenario happened when claiming that θ_F is the **exterior** dihedral angle. This angle should be the **interior** dihedral angle as mentioned in [28, page 281]. That little confusion led to the sign error in [41, page 555], in the last equality in the calculation of $dArea(\gamma)$, and in [38, Page 39], in the last equality in the calculation of $d\phi_{l_{ab}}$.

From now on, we are going to use Equation (4.3) as a reference for our calculations. And for the last two sections we will also use the method in Definition 2.3.5 to construct our spherical tetrahedron.

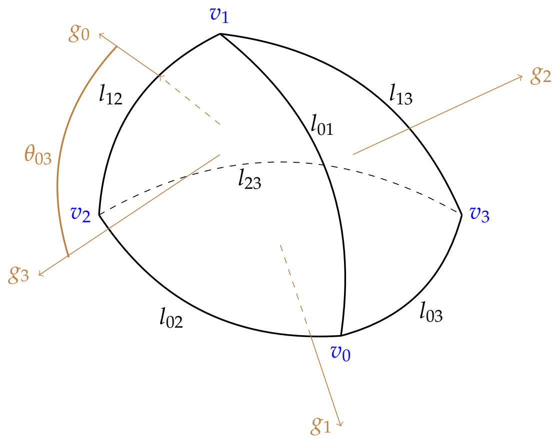


Figure 4.4: Spherical tetrahedron T showing the outward normal vectors the g_i's, in our notation.

4.4 Six-dimensional version of the integral

Let g_0, g_1, g_2, g_3 be four elements in $SU(2)$. Because of the diffeomorphism between $SU(2)$ and S^3 (Section B.1) they may be considered as four points in S^3, hence seen as four unit vectors in $\mathbb{R}^4$. Throughout this section, they will be considered as the unit outward normal vectors to each face of the spherical tetrahedron T (see Figure 4.4).

Let us recall the Conjecture 4.1.7 that we are investigating:

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}^2_{q=e^{\frac{i\pi}{s+2}}} = \int_{SU(2)^4} \prod_{i<j} \chi_{m_{\overline{ij}}}(g_j g_i^{-1}) e^{\frac{2i}{\pi}(s+2)V(T)} [\prod dg_i]. \tag{4.13}$$

The aim here is also to transform the integral on the right hand side of Equation (4.13), which at the moment is a twelve-dimensional integral, into a six-dimensional Lebesgue integral. At first glance, the integral on the right hand side of Equation (4.13) presents strong similarities with the right hand side of Equation (2.1) in Theorem 2.1.1. Therefore, the strategy from Chapter 2 will be adopted verbatim. So, to start with, let us set

$$I_c = \int_{SU(2)^4} \prod_{i<j} \chi_{m_{\overline{ij}}}(g_j g_i^{-1}) e^{\frac{2i}{\pi}(s+2)V(T)} [\prod dg_i].$$

Under the action of $SO(4)$, the unit vectors g_0, g_1, g_2, g_3 transform to [4]:

$$
\begin{aligned}
g_0 &= (1, 0, 0, 0) \\
g_1 &= (\cos x_1, 0, 0, \sin x_1) \\
g_2 &= (\cos x_2, \sin x_2 \sin y_2, 0, \sin x_2 \cos y_2) \\
g_3 &= (\cos x_3, \sin x_3 \sin y_3 \cos z_3, \sin x_3 \sin y_3 \sin z_3, \sin x_3 \cos y_3)
\end{aligned}
$$

where $x_1, x_2, x_3, y_2, y_3 \in [0, \pi]$ and $z_3 \in [0, 2\pi]$.

Since the integral I_c is similar to the right hand side of Equation (2.1) in Theorem 2.1.1, it can be expressed in terms of the spherical coordinates as below.

Lemma 4.4.1. *The integral I_c may be expressed in terms of the spherical coordinates as*

$$
\begin{aligned}
I_c =& \frac{2}{\pi^4} \int_{x_1, x_2, x_3, y_2, y_3, z_3 = 0}^{\pi} F(x_1, x_2, x_3, y_2, y_3, z_3) \cos\left(\frac{2}{\pi}(s+2) V(x_1, x_2, x_3, y_2, y_3, z_3)\right) \\
& \sin^2 x_1 \sin^2 x_2 \sin^2 x_3 \sin y_2 \sin y_3 \mathrm{d}z_3 \mathrm{d}y_3 \mathrm{d}y_2 \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1,
\end{aligned}
$$

where F is exactly the function described in Section 2.7.1.

Proof. Taking from step 5 in the proof of Lemma 2.7.3 along with the argument that the volume of a spherical tetrahedron remains invariant under the action of $SO(4)$, the transformation of the integral I_c reads as follows:

$$\begin{aligned}
I_c =&(\frac{1}{\pi})^4 \int_{x_1,x_2,x_3,y_2,y_3=0}^{\pi} \int_{z_3=0}^{2\pi} F(x_1,x_2,x_3,y_2,y_3,z_3) e^{\mathrm{i}\frac{2}{\pi}(s+2)V(x_1,x_2,x_3,y_2,y_3,z_3)} \\
&\sin^2 x_1 \sin^2 x_2 \sin^2 x_3 \sin y_2 \sin y_3 \mathrm{d}z_3 \mathrm{d}y_3 \mathrm{d}y_2 \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \\
=&(\frac{1}{\pi})^4 \int_{x_1,x_2,x_3,y_2,y_3=0}^{\pi} \int_{z_3=0}^{\pi} F(x_1,x_2,x_3,y_2,y_3,z_3) e^{\mathrm{i}\frac{2}{\pi}(s+2)V(x_1,x_2,x_3,y_2,y_3,z_3)} \\
&\sin^2 x_1 \sin^2 x_2 \sin^2 x_3 \sin y_2 \sin y_3 \mathrm{d}z_3 \mathrm{d}y_3 \mathrm{d}y_2 \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \\
+&(\frac{1}{\pi})^4 \int_{x_1,x_2,x_3,y_2,y_3=0}^{\pi} \int_{z_3=\pi}^{2\pi} F(x_1,x_2,x_3,y_2,y_3,z_3) e^{\mathrm{i}\frac{2}{\pi}(s+2)V(x_1,x_2,x_3,y_2,y_3,z_3)} \\
&\sin^2 x_1 \sin^2 x_2 \sin^2 x_3 \sin y_2 \sin y_3 \mathrm{d}z_3 \mathrm{d}y_3 \mathrm{d}y_2 \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \\
=&(\frac{1}{\pi})^4 \int_{x_1,x_2,x_3,y_2,y_3=0}^{\pi} \int_{z_3=0}^{\pi} F(x_1,x_2,x_3,y_2,y_3,z_3) e^{\mathrm{i}\frac{2}{\pi}(s+2)V(x_1,x_2,x_3,y_2,y_3,z_3)} \\
&\sin^2 x_1 \sin^2 x_2 \sin^2 x_3 \sin y_2 \sin y_3 \mathrm{d}z_3 \mathrm{d}y_3 \mathrm{d}y_2 \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \\
+&(\frac{1}{\pi})^4 \int_{x_1,x_2,x_3,y_2,y_3=0}^{\pi} \int_{z_3=0}^{\pi} F(x_1,x_2,x_3,y_2,y_3,z_3) e^{\mathrm{i}\frac{2}{\pi}(s+2)V(x_1,x_2,x_3,y_2,y_3,2\pi-z_3)} \\
&\sin^2 x_1 \sin^2 x_2 \sin^2 x_3 \sin y_2 \sin y_3 \mathrm{d}z_3 \mathrm{d}y_3 \mathrm{d}y_2 \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \\
=&(\frac{1}{\pi})^4 \int_{x_1,x_2,x_3,y_2,y_3=0}^{\pi} \int_{z_3=0}^{\pi} F(x_1,x_2,x_3,y_2,y_3,z_3) e^{\mathrm{i}\frac{2}{\pi}(s+2)V(x_1,x_2,x_3,y_2,y_3,z_3)} \\
&\sin^2 x_1 \sin^2 x_2 \sin^2 x_3 \sin y_2 \sin y_3 \mathrm{d}z_3 \mathrm{d}y_3 \mathrm{d}y_2 \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \\
+&(\frac{1}{\pi})^4 \int_{x_1,x_2,x_3,y_2,y_3=0}^{\pi} \int_{z_3=0}^{\pi} F(x_1,x_2,x_3,y_2,y_3,z_3) e^{-\mathrm{i}\frac{2}{\pi}(s+2)V(x_1,x_2,x_3,y_2,y_3,z_3)} \\
&\sin^2 x_1 \sin^2 x_2 \sin^2 x_3 \sin y_2 \sin y_3 \mathrm{d}z_3 \mathrm{d}y_3 \mathrm{d}y_2 \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1 \\
=&\frac{2}{\pi^4} \int_{x_1,x_2,x_3,y_2,y_3,z_3=0}^{\pi} F(x_1,x_2,x_3,y_2,y_3,z_3) \cos\left(\frac{2}{\pi}(s+2)V(x_1,x_2,x_3,y_2,y_3,z_3)\right) \\
&\sin^2 x_1 \sin^2 x_2 \sin^2 x_3 \sin y_2 \sin y_3 \mathrm{d}z_3 \mathrm{d}y_3 \mathrm{d}y_2 \mathrm{d}x_3 \mathrm{d}x_2 \mathrm{d}x_1.
\end{aligned}$$

Let (x,y,z,t) denote the positive orientation of $\mathbb{R}^4$. Let us represent the spherical tetrahedron T with outward normal vectors g_0, g_1, g_2, g_3 by the tuple (g_0, g_1, g_2, g_3). Changing z_3 to $2\pi - z_3$ is equivalent to taking into account the mirror image of g_3, hence that of (g_0, g_1, g_2, g_3), with respect to the $(xyt)-$hyperplane. That amounts to say that the spherical tetrahedron changes its orientation. Therefore, the sign of the volume of the spherical tetrahedron T changes i.e.

$$V(x_1,x_2,x_3,y_2,y_3,2\pi - z_3) = -V(x_1,x_2,x_3,y_2,y_3,z_3).$$

Thus, the second last equality holds. And the last equality follows from the Euler's formula $\cos x = \frac{e^{\mathrm{i}x}+e^{-\mathrm{i}x}}{2}$. □

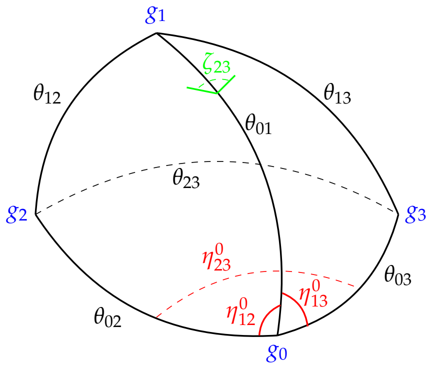


Figure 4.5: The dual spherical tetrahedron T^* of T.

Let us denote by T^* the dual to the spherical tetrahedron T. As stated in Remark 2.3.10 its vertices are composed by the vectors g_0, g_1, g_2, g_3. Let the Figure 4.5 illustrate T^*.

From the fact that T^* itself is a spherical tetrahedron, one may identify its box variables. For the case of Figure 4.5, the box variable is given by

$$B^* = \{\theta_{01}, \theta_{02}, \theta_{03}, \eta^0_{12}, \eta^0_{13}, \zeta_{23}\}$$

where $\theta_{01}, \theta_{02}, \theta_{03}$ indicates the edge lengths of the dual tetrahedron T^* with common vertex g_0, the variables η^0_{12}, η^0_{13} its interior angles around g_0, and the interior dihedral angle ζ_{23} of T^*.

The integral I_c may be re-expressed in terms of the box variables for T^* as follows:

Corollary 4.4.2. *Let $\theta_{01}, \theta_{02}, \theta_{03}, \eta^0_{12}, \eta^0_{13}, \zeta_{23} \in [0, \pi]$. The integral in Conjecture 4.1.7 may be re-written as a function of the box variables in the following way:*

$$I_c = \frac{2}{\pi^4} \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi f(\theta_{01}, \theta_{02}, \theta_{03}, \eta^0_{12}, \eta^0_{13}, \zeta_{23}) \cos\left(\frac{2}{\pi}(s+2)V(\theta_{01}, \theta_{02}, \theta_{03}, \eta^0_{12}, \eta^0_{13}, \zeta_{23})\right)$$
$$\sin^2\theta_{01} \sin^2\theta_{02} \sin^2\theta_{03} \sin\eta^0_{12} \sin\eta^0_{13} d\zeta_{23} d\eta^0_{13} d\eta^0_{12} d\theta_{03} d\theta_{02} d\theta_{01}.$$

Furthermore, the twelve-dimensional conjectural integral formula for the square of a quantum 6j symbol in Conjecture 4.4 may be re-written as a six-dimensional integral. That is shown in the lemma below.

Lemma 4.4.3. *The twelve-dimensional integral on the right hand side of Equation* (4.4) *can be re-expressed as a six-dimensional integral over the space of* exterior dihedral angles *of* $SO(4)-$*equivalence classes of non-degenerate spherical tetrahedra as follows:*

$$I(s) = \frac{2}{\pi^4} \int_{D_\pi} \frac{\prod_{i<j} \sin((m_{\overline{ij}}+1)\theta_{ij})}{\sqrt{\det([\cos\theta_{ij}])}} \cos\left(\frac{2}{\pi}(s+2)V(\theta_{01},\theta_{02},\theta_{03},\theta_{12},\theta_{13},\theta_{23})\right) [\prod_{i<j} \mathrm{d}\theta_{ij}]. \tag{4.14}$$

Explicitly, the region of integration is

$$D_\pi = \{(\theta_{01},\theta_{02},\theta_{03},\theta_{12},\theta_{13},\theta_{23}) \in [0,\pi]^6, [\cos\theta_{ij}] \textit{ is positive definite }\},$$

where $[\cos\theta_{ij}]$ *is the* 4×4 *unidiagonal symmetric matrix whose off-diagonal elements are* $\cos\theta_{ij}$.

Proof. The proof goes hand in hand with that of Theorem 2.7.6. □

4.5 Interior contribution for the asymptotic

Let $r_{ij}^0 \in \mathbb{Q} \cap [0,1]$ and $k \in \mathbb{N}$ such that kr_{ij}^0 is integer.

Let's consider the integral

$$I_c(k) = \frac{2}{\pi^4} \int_{D_\pi} \frac{\prod_{i<j} \sin\left((kr_{\overline{ij}}^0+1)\theta_{ij}\right)}{\sqrt{\det([\cos\theta_{ij}])}} \cos\left(\frac{2}{\pi}(k+2)V(T)\right) [\prod_{i<j} \mathrm{d}\theta_{ij}]. \tag{4.15}$$

We are investigating the conjecture (Conjecture 4.1.9) which states that

$$\begin{Bmatrix} kr_{12}^0 & kr_{02}^0 & kr_{01}^0 \\ kr_{03}^0 & kr_{13}^0 & kr_{23}^0 \end{Bmatrix}^2_{q=e^{\frac{i\pi}{k+2}}} \sim I_c(k). \tag{4.16}$$

The aim of this section is to generate the interior contribution for the integral in Equation (4.15). So, the critical points taken into account are within the domain of integration not at the boundary. The interior contribution is computed by using the stationary phase approximation.

4.5.1 Stationary phase method

The stationary phase method is often used to approximate $n-$dimensional integrals of the form

$$\int_D a(x)e^{\mathrm{i}kp(x)}\,dx. \tag{4.17}$$

It has been studied by several authors including Nicholas Chako [13] and Wong [46]. For handy examples, Cooke's paper [15] is recommended. The approximation mainly depends on the nature of either the critical points of the phase $p(x)$ or that of the amplitude $a(x)$. But in this subsection, we are only interested in the contribution from the stationary points of the phase. For more practicality to our case, we will use the approximation of an $n-$dimensional integral from [4] which reads:

$$\int_D a(x)e^{\mathrm{i}kp(x)}\,dx \sim (2\pi)^{\frac{n}{2}} \sum_i \frac{a(x_i)}{|\det H(x_i)|^{\frac{1}{2}}} e^{\mathrm{i}S(x_i)+\frac{\mathrm{i}\pi}{4}\sigma(H(x_i))}, \tag{4.18}$$

where the sum is done over the stationary points x_i of $S = kp$, $H(x_i)$ is the matrix whose entries are the second partial derivatives of S at the point x_i and $\sigma(H(x_i))$ its signature.

For instance, in our case we have a six dimensional integral, so Equation (4.18) becomes

$$\int_D a(x)e^{\mathrm{i}S(x)}\,dx \sim (2\pi)^3 \sum_i \frac{a(x_i)}{|\det H(x_i)|^{\frac{1}{2}}} e^{\mathrm{i}S(x_i)+\frac{\mathrm{i}\pi}{4}\sigma(H(x_i))}. \tag{4.19}$$

4.5.2 Tools for the approximation

4.5.2.1 The Schläfli formula

As mentioned in the beginning of this section, the stationary phase method will be observed. Therefore, the knowledge of the interior stationary points of the phase is required, in our case the Schläfli formula plays an important role in their computation. Hence, let us recall the Schläfli identity formula.

Lemma 4.5.1 (Schläfli identity, see [28])**.** *Let T be a spherical tetrahedron with edge lengths l_{ij} and* interior *dihedral angles β_{ij} at the edge $(\overline{ij})$. Then*

$$dV = \frac{1}{2}\sum l_{\overline{ij}}\,d\beta_{ij}$$

where V is the volume of T.

4.5.2.2 Relationship between the Jacobian and the Gram matrices

While looking at the denominator $\sqrt{\det([\cos\theta_{ij}])}$ of the integrand in (4.15), one would expect that the final result of the asymptotic approximation of the integral will remain a function of the exterior dihedral angles. But the aim is to reproduce the right hand side of Equation (4.3), whose denominator is the determinant of the edge Gram matrix of a specific spherical tetrahedron. To convert from one expression of the denominator to the other, Kaminski and Steinhaus, in [19], provided a formula which is recalled in the following lemma.

Lemma 4.5.2. *[19] Let T be a spherical tetrahedron with edge lengths l_{ij} and* interior *dihedral angle $\beta_{\overline{ij}}$ around the edge (ij). Then*

$$\det([\frac{\partial\beta_{ij}}{\partial l_{st}}]) = -\frac{\det([-\cos\beta_{ij}])}{\det([\cos l_{ij}])}, \tag{4.20}$$

where $[\frac{\partial\beta_{ij}}{\partial l_{st}}]$ is the matrix whose entries are the derivatives of the interior dihedral angles with respect to the edge lengths, the symbol $[\cos l_{ij}]$ (resp $[\cos\beta_{ij}]$) denotes the 4×4 matrix which has 1 on the diagonal and $\cos l_{ij}$ (resp $\cos\beta_{ij}$) elsewhere.

By adapting Equation (4.20) to our situation, Lemma 4.5.2 is equivalent to:

Corollary 4.5.3. *Let T be a spherical tetrahedron with edge lengths l_{ij} and* exterior *dihedral angle $\theta_{\overline{ij}}$ around the edge (ij) opposite to the edge $(\overline{ij})$. Then*

$$\det([\frac{\partial l_{ij}}{\partial\theta_{st}}]) = -\frac{\det([\cos l_{ij}])}{\det([\cos\theta_{ij}])},$$

where $[\frac{\partial l_{ij}}{\partial\theta_{st}}]$ is the matrix whose entries are the derivatives of the edge lengths with respect to the exterior dihedral angles, and $[\cos l_{ij}]$ (resp $[\cos\theta_{ij}]$) denotes the 4×4 matrix which has 1 on the diagonal and $\cos l_{ij}$ (resp $\cos\theta_{ij}$) elsewhere.

Proof. By considering the dual tetrahedron T^* of T, the edge lengths are θ_{ij} and $\pi - l_{\overline{ij}}$ are the interior dihedral angles at the edge (ij). Then Equation (4.20) becomes

$$\det([\frac{-\partial l_{ij}}{\partial\theta_{st}}]) = -\frac{\det([\cos l_{ij}])}{\det([\cos\theta_{ij}])}.$$

But since we are in dimension four $[\frac{-\partial l_{ij}}{\partial\theta_{st}}]$ is a six dimensional matrix, therefore $\det[\frac{-\partial l_{ij}}{\partial\theta_{st}}] = \det[\frac{\partial l_{ij}}{\partial\theta_{st}}]$. □

4.5.3 Approximation of the integral

Notice that the integral $I_c(k)$ in Equation (4.15) may be rewritten as

$$\begin{aligned} I_c(k) =& \frac{1}{\pi^4}\int_{D_\pi} \frac{\prod_{i<j}\sin\left((kr^0_{ij}+1)\theta_{ij}\right)}{\sqrt{\det([\cos\theta_{ij}])}} e^{\mathrm{i}\frac{2}{\pi}(k+2)V(\theta_{01},\theta_{02},\theta_{03},\theta_{12},\theta_{13},\theta_{23})}[\prod_{i<j}\mathrm{d}\theta_{ij}] \\ &+ \frac{1}{\pi^4}\int_{D_\pi} \frac{\prod_{i<j}\sin\left((kr^0_{ij}+1)\theta_{ij}\right)}{\sqrt{\det([\cos\theta_{ij}])}} e^{-\mathrm{i}\frac{2}{\pi}(k+2)V(\theta_{01},\theta_{02},\theta_{03},\theta_{12},\theta_{13},\theta_{23})}[\prod_{i<j}\mathrm{d}\theta_{ij}] \\ =& I_{c1}(k) + I_{c2}(k). \end{aligned}$$

In this subsection, the intention is to provide asymptotic expansions for $I_{c1}(k)$ and $I_{c2}(k)$, then add them up to obtain the asymptotic expansion for $I_c(k)$. That method makes sense since the sum is not null.

4.5.3.1 Asymptotic expansion of $I_{c1}(k)$

The strategy here is to transform the product of sinuses in the integrand of $I_{c1}(k)$ into a product of exponential functions. By doing so, $I_{c1}(k)$ becomes a sum of 64 integrals. But, upon analysis only one integral contributes to the stationary phase approximation. That is given in the proposition below.

Proposition 4.5.4. *The only integral contributing to the stationary phase approximation of $I_{c1}(k)$ is*

$$I_{c1,\epsilon_{ij}=1}(k) = \frac{1}{(2\mathrm{i})^6}\frac{1}{\pi^4}\int_{D_\pi}\frac{e^{\mathrm{i}(\sum_{i<j}\theta_{ij}+\frac{4}{\pi}V(\theta_{ij}))}}{\sqrt{\det([\cos\theta_{ij}])}}e^{\mathrm{i}k(\sum_{i<j}r^0_{ij}\theta_{ij}+\frac{2}{\pi}V(\theta_{ij}))}[\prod_{i<j}d\theta_{ij}].$$

Proof. When expanding $\prod_{i<j}\sin((kr^0_{ij}+1)\theta_{ij})$ to an exponential form, it produces a sum of 64 integrals of the form

$$\frac{1}{(2\mathrm{i})^6}\frac{1}{\pi^4}\int_{D_\pi}\frac{e^{\mathrm{i}(\sum_{i<j}\epsilon_{ij}\theta_{ij}+\frac{4}{\pi}V(\theta_{ij}))}}{\sqrt{\det[\cos\theta_{ij}]}}e^{\mathrm{i}k(\sum_{i<j}\epsilon_{ij}r^0_{ij}\theta_{ij}+\frac{2}{\pi}V(\theta_{ij}))}[\prod_{i<j}d\theta_{ij}],$$

where $\epsilon_{ij} = \pm 1$.

Let us set

$$S(\theta_{01},\theta_{02},\theta_{03},\theta_{12},\theta_{13},\theta_{23}) = k(\sum_{ij}\epsilon_{ij}r^0_{kl}\theta_{ij} + \frac{2}{\pi}V(\theta_{ij})).$$

Then, from the Schläfli identity in Lemma 4.5.1 the partial derivatives of S with respect to θ_{ij} are given by

$$\frac{\partial S}{\partial \theta_{ij}} = \epsilon_{ij} r^0_{\overline{ij}} - \frac{2}{\pi}.\frac{1}{2} l_{\overline{ij}}. \qquad (\theta_{ij} = \pi - \beta_{ij})$$

To procure the stationary points of S, one needs to equate $\frac{\partial S}{\partial \theta_{ij}}$ to zero. That is,

$$\frac{\partial S}{\partial \theta_{ij}} = 0 \Longleftrightarrow \epsilon_{ij} r^0_{\overline{ij}} - \frac{1}{\pi} l_{\overline{ij}} = 0.$$

The latter equality implies

$$l_{\overline{ij}} = \pi \epsilon_{ij} r^0_{\overline{ij}}.$$

This argument means that the stationary points are spherical tetrahedra with edge lengths $\pi\epsilon_{ij} r^0_{\overline{ij}}$. However, our region is $D_\pi \subset [0, \pi]^6$. Therefore, the only allowed length for a tetrahedron is $l_{\overline{ij}} = \pi r^0_{\overline{ij}}$ i.e. all $\epsilon_{ij} = 1$. Hence, the only integral which contributes to the stationary phase method for the inside contribution of $I_{c1}(k)$ is that given in Proposition 4.5.4. □

By taking the result in Proposition 4.5.4 into consideration, the asymptotic expansion of the integral $I_{c1}(k)$ may now be computed. And the following proposition shows the output.

Proposition 4.5.5. *Let $r^0_{ij} \in \mathbb{Q} \cap [0, 1]$. Let T^0 be the non-degenerate spherical tetrahedron whose edge lengths $l^0_{ij} = \pi r^0_{ij}$ and whose exterior dihedral angles at the edge $(\overline{ij})$ opposite to the edge (ij) is given by θ^0_{ij}. Let the phase function of $I_{c1,\epsilon_{ij}=1}(k)$ be denoted by*

$$S_1(\theta_{01}, \theta_{02}, \theta_{03}, \theta_{12}, \theta_{13}, \theta_{23}) = k(\sum_{i,j} r^0_{\overline{ij}} \theta_{ij} + \frac{2}{\pi} V(\theta_{ij})).$$

Then the asymptotic expansion of $I_{c1}(k)$ is

$$I_{c1}(k) \sim \frac{-\pi^2}{8k^3} . \frac{e^{\mathrm{i}\frac{\pi}{4}\sigma(H_1(\theta^0_{ij}))}}{\sqrt{\det([\cos l^0_{ij}])}} e^{\mathrm{i}(\sum_{i<j}(k r^0_{\overline{ij}}+1)\theta^0_{ij} + \frac{2}{\pi}(k+2)V(T^0))},$$

where $[\cos l^0_{ij}]$ denotes the edge Gram matrix of T^0, H_1 is the Hessian matrix of the phase S_1.

Proof. From Proposition 4.5.4 the stationary phase approximation of $I_{c1,\epsilon_{ij}=1}(k)$ is the same as that of $I_{c1}(k)$. So, this proof is all about the asymptotic expansion of $I_{c1,\epsilon_{ij}=1}(k)$. Let the phase function be denoted by

$$S_1(\theta_{01},\theta_{02},\theta_{03},\theta_{12},\theta_{13},\theta_{23}) = k(\sum_{i,j} r^0_{\overline{ij}}\theta_{ij} + \frac{2}{\pi}V(\theta_{ij})).$$

Then, from the Schläfli identity in Lemma 4.5.1,

$$\frac{\partial S_1}{\partial \theta_{ij}} = k(r^0_{\overline{ij}} - \frac{2}{\pi}.\frac{1}{2}l_{\overline{ij}}) \qquad\qquad (\theta_{ij} = \pi - \beta_{ij}).$$

Therefore, the partial derivative of S_1 with respect to θ_{ij} is given by

$$\frac{\partial S_1}{\partial \theta_{ij}} = 0 \Longleftrightarrow r^0_{\overline{ij}} - \frac{1}{\pi}l_{\overline{ij}} = 0.$$

That is equivalent to say

$$l_{\overline{ij}} = \pi r^0_{\overline{ij}},$$

which means that our critical point is the non-degenerate spherical tetrahedron T^0.

Since the Hessian matrix of S_1 is required, the derivative of S_1 with respect to the θ_{ij}'s at the point

$$E := (\theta_{01},\theta_{02},\theta_{03},\theta_{12},\theta_{13},\theta_{23})$$

is needed. That is given by:

$$(\tfrac{\partial S_1}{\partial \theta_{01}}, \tfrac{\partial S_1}{\partial \theta_{02}}, \tfrac{\partial S_1}{\partial \theta_{03}}, \tfrac{\partial S_1}{\partial \theta_{12}}, \tfrac{\partial S_1}{\partial \theta_{13}}, \tfrac{\partial S_1}{\partial \theta_{23}})_{|E} = (kr^0_{23} - \tfrac{k}{\pi}l_{23},\ kr^0_{13} - \tfrac{k}{\pi}l_{13},\ kr^0_{12} - \tfrac{k}{\pi}l_{12},\ kr^0_{03} - \tfrac{k}{\pi}l_{03},\ kr^0_{02} - \tfrac{k}{\pi}l_{02},\ kr^0_{01} - \tfrac{k}{\pi}l_{01}).$$

Notice that l_{ij} are functions of θ_{ij}. Hence the Hessian matrix H_1 of S_1 is

$$H_1(\theta_{ij}) = (\frac{-k}{\pi})^6 \begin{pmatrix} \frac{\partial l_{23}}{\partial \theta_{01}} & \frac{\partial l_{23}}{\partial \theta_{02}} & \frac{\partial l_{23}}{\partial \theta_{03}} & \frac{\partial l_{23}}{\partial \theta_{12}} & \frac{\partial l_{23}}{\partial \theta_{13}} & \frac{\partial l_{23}}{\partial \theta_{23}} \\ \frac{\partial l_{13}}{\partial \theta_{01}} & \frac{\partial l_{13}}{\partial \theta_{02}} & \frac{\partial l_{13}}{\partial \theta_{03}} & \frac{\partial l_{13}}{\partial \theta_{12}} & \frac{\partial l_{13}}{\partial \theta_{13}} & \frac{\partial l_{13}}{\partial \theta_{23}} \\ \frac{\partial l_{12}}{\partial \theta_{01}} & \frac{\partial l_{12}}{\partial \theta_{02}} & \frac{\partial l_{12}}{\partial \theta_{03}} & \frac{\partial l_{12}}{\partial \theta_{12}} & \frac{\partial l_{12}}{\partial \theta_{13}} & \frac{\partial l_{12}}{\partial \theta_{23}} \\ \frac{\partial l_{03}}{\partial \theta_{01}} & \frac{\partial l_{03}}{\partial \theta_{02}} & \frac{\partial l_{03}}{\partial \theta_{03}} & \frac{\partial l_{03}}{\partial \theta_{12}} & \frac{\partial l_{03}}{\partial \theta_{13}} & \frac{\partial l_{03}}{\partial \theta_{23}} \\ \frac{\partial l_{02}}{\partial \theta_{01}} & \frac{\partial l_{02}}{\partial \theta_{02}} & \frac{\partial l_{02}}{\partial \theta_{03}} & \frac{\partial l_{02}}{\partial \theta_{12}} & \frac{\partial l_{02}}{\partial \theta_{13}} & \frac{\partial l_{02}}{\partial \theta_{23}} \\ \frac{\partial l_{01}}{\partial \theta_{01}} & \frac{\partial l_{01}}{\partial \theta_{02}} & \frac{\partial l_{01}}{\partial \theta_{03}} & \frac{\partial l_{01}}{\partial \theta_{12}} & \frac{\partial l_{01}}{\partial \theta_{13}} & \frac{\partial l_{01}}{\partial \theta_{23}} \end{pmatrix}.$$

Therefore, from Equation (4.19) the asymptotic expansion of the integral $I_{c1,\epsilon_{ij}=1}(k)$ arises as

$$\begin{aligned}
I_{c1,\epsilon_{ij}=1}(k) &\sim \frac{(2\pi)^3}{\pi^4(2\mathrm{i})^6}.\frac{e^{\mathrm{i}(\sum_{i,j}\theta^0_{ij}+\frac{4}{\pi}V(\theta^0_{ij}))}}{\sqrt{\det([\cos\theta^0_{ij}])|\det H_1(\theta^0_{ij})|^{\frac{1}{2}}}}e^{\mathrm{i}k(\sum_{i,j}r^0_{ij}\theta^0_{ij}+\frac{2}{\pi}V(T^0))+\mathrm{i}\frac{\pi}{4}\sigma(H_1(\theta^0_{ij}))} \\
&= \frac{2^3}{\pi(2\mathrm{i})^6}\frac{e^{\mathrm{i}(\sum_{i<j}\theta^0_{ij}+\frac{4}{\pi}V(\theta^0_{ij}))}}{\sqrt{\det([\cos\theta^0_{ij}])\frac{(k^3)}{\pi^3}|\det([\frac{\partial l_{ij}}{\partial\theta_{st}}]_{|\theta^0_{ij}})|^{\frac{1}{2}}}}e^{\mathrm{i}k(\sum_{i,j}r^0_{ij}\theta^0_{ij}+\frac{2}{\pi}V(T^0))+\mathrm{i}\frac{\pi}{4}\sigma(H_1(\theta^0_{ij}))} \\
&= \frac{2^3\pi^2}{k^3(2\mathrm{i})^6}\frac{e^{\mathrm{i}(\sum_{i<j}\theta^0_{ij}+\frac{4}{\pi}V(\theta^0_{ij}))}}{\sqrt{\det([\cos\theta^0_{ij}])|\det([\frac{\partial l_{ij}}{\partial\theta_{st}}]_{|\theta^0_{ij}})|^{\frac{1}{2}}}}e^{\mathrm{i}k(\sum_{i,j}r^0_{ij}\theta^0_{ij}+\frac{2}{\pi}V(T^0))+\mathrm{i}\frac{\pi}{4}\sigma(H_1(\theta^0_{ij}))} \\
&= \frac{-\pi^2}{8k^3}\frac{e^{\mathrm{i}(\sum_{i<j}\theta^0_{ij}+\frac{4}{\pi}V(\theta^0_{ij}))}}{\sqrt{\det([\cos\theta^0_{ij}])|\det([\frac{\partial l_{ij}}{\partial\theta_{st}}]_{|\theta^0_{ij}})|^{\frac{1}{2}}}}e^{\mathrm{i}k(\sum_{i,j}r^0_{ij}\theta^0_{ij}+\frac{2}{\pi}V(T^0))+\mathrm{i}\frac{\pi}{4}\sigma(H_1(\theta^0_{ij}))} \\
&= \frac{-\pi^2}{8k^3}\frac{e^{\mathrm{i}(\sum_{i<j}\theta^0_{ij}+\frac{4}{\pi}V(\theta^0_{ij}))}}{\sqrt{\det([\cos l^0_{ij}])}}e^{\mathrm{i}k(\sum_{i<j}r^0_{ij}\theta^0_{ij}+\frac{2}{\pi}V(T^0))+\mathrm{i}\frac{\pi}{4}\sigma(H_1(\theta^0_{ij}))} \qquad \text{(by Lemma 4.5.3)} \\
&= \frac{-\pi^2}{8k^3}.\frac{e^{\mathrm{i}\frac{\pi}{4}\sigma(H_1(\theta^0_{ij}))}}{\sqrt{\det([\cos l^0_{ij}])}}e^{\mathrm{i}(\sum_{i<j}(kr^0_{ij}+1)\theta^0_{ij}+\frac{2}{\pi}(k+2)V(T^0))}.
\end{aligned}$$

□

Remark 4.5.6. *The stationary phase approximation to the integral* $I_{c1}(k)$ *produces a Jacobian determinant* $\det([\frac{\partial l_{ij}}{\partial\theta_{st}}])$ *in the denominator, while the asymptotics of the quantum 6j symbols have a Gram determinant* $\det([\cos l^0_{ij}])$ *in the denominator. It is therefore not obvious that the one has anything to do with the other. Hence, Lemma 4.5.3 is a nontrivial step and is perhaps evidence that the conjecture is on the right track.*

4.5.3.2 Asymptotic expansion of $I_{c2}(k)$

Similarly to the argument for $I_{c1}(k)$, the integral $I_{c2}(k)$ is equal to the sum of 64 integrals when changing the sine terms in its integrand into their exponential expression. However, only one of these sixty four integrals contributes to its asymptotic expansion. That is,

Proposition 4.5.7. *The only integral contributing to the stationary phase approximation of* $I_{c2}(k)$ *is*

$$I_{c2,\epsilon_{ij}=-1}(k) = \frac{1}{(2\mathrm{i})^6}\frac{1}{\pi^4}\int_{D_\pi}\frac{e^{-\mathrm{i}(\sum_{i<j}\theta_{ij}+\frac{4}{\pi}V(\theta_{ij}))}}{\sqrt{\det[\cos\theta_{ij}]}}e^{-\mathrm{i}k(\sum_{i<j}r^0_{ij}\theta_{ij}+\frac{2}{\pi}V(\theta_{ij}))}[\prod_{i<j}d\theta_{ij}].$$

Proof. The statement of the proof goes hand in hand with that of Proposition 4.5.4. The only difference is that in this case all $\epsilon_{ij} = -1$. □

By taking the result in Proposition 4.5.7 into consideration, let us now derive the asymptotic expansion of the integral $I_{c2}(k)$. The latter is provided in the following proposition.

Proposition 4.5.8. *Let $r^0_{ij} \in \mathbb{Q} \cap [0,1]$. Let T^0 be the non-degenerate spherical tetrahedron whose edge lengths $l^0_{ij} = \pi r^0_{ij}$ and whose exterior dihedral angle at the edge $(\overline{ij})$ opposite to the edge (ij) is given by θ^0_{ij}. Let the phase function of $I_{c2,\epsilon_{ij}=-1}(k)$ be denoted by*

$$S_2(\theta_{01}, \theta_{02}, \theta_{03}, \theta_{12}, \theta_{13}, \theta_{23}) = -k(\sum_{i,j} r^0_{\overline{ij}}\theta_{ij} + \frac{2}{\pi}V(\theta_{ij})).$$

Then the asymptotic expansion of the integral $I_{c2}(k)$ is

$$I_{c2}(k) \sim \frac{-\pi^2}{8k^3}.\frac{e^{\mathrm{i}\frac{\pi}{4}\sigma(H_2(\theta^0_{ij}))}}{\sqrt{\det([\cos l^0_{ij}])}}e^{-\mathrm{i}(\sum_{i<j}(kr^0_{\overline{ij}}+1)\theta^0_{ij}+\frac{2}{\pi}(k+2)V(T^0))},$$

where $[\cos l^0_{ij}]$ denotes the edge Gram matrix of T^0, H_2 the Hessian matrix related to S_2.

Proof. The proof of Proposition 4.5.8 can be considered to go hand in hand with that of Proposition 4.5.5. However, to be careful in the result let us do it separately.

From Proposition 4.5.7 the stationary phase approximation of $I_{c2,\epsilon_{ij}=-1}(k)$ is the same as that of $I_{c2}(k)$. So, this proof is all about the approximation of $I_{c2,\epsilon_{ij}=-1}(k)$.

Let us consider the phase function of $I_{c2,\epsilon_{ij}=-1}(k)$,

$$S_2(\theta_{01}, \theta_{02}, \theta_{03}, \theta_{12}, \theta_{13}, \theta_{23}) = -k(\sum_{i,j} r^0_{\overline{ij}}\theta_{ij} + \frac{2}{\pi}V(\theta_{ij})).$$

Then, from the Schläfli identity in Lemma 4.5.1 the partial derivative of S_2 with respect to θ_{ij} is given by

$$\frac{\partial S_2}{\partial \theta_{ij}} = -k(r^0_{\overline{ij}} - \frac{2}{\pi}.\frac{1}{2}l_{\overline{ij}}) \qquad (\theta_{ij} = \pi - \beta_{ij}). \tag{4.21}$$

As the aim is to find the stationary point of S_2, equating the partial derivative to zero (0) gives

$$\frac{\partial S_2}{\partial \theta_{ij}} = 0 \Longleftrightarrow r^0_{\overline{ij}} - \frac{1}{\pi}l_{\overline{ij}} = 0.$$

This latter equality amounts to say that

$$l_{\overline{ij}} = \pi r^0_{\overline{ij}},$$

which means our critical point is the spherical tetrahedron T^0. The next thing to do is look for the Hessian matrix of S_2 at the point $E := (\theta_{01}, \theta_{02}, \theta_{03}, \theta_{12}, \theta_{13}, \theta_{23})$. For that, let us first recall from (4.21) the derivative of S_2 with respect to the θ_{ij}'s

$$(\tfrac{\partial S_2}{\partial\theta_{01}}, \tfrac{\partial S_2}{\partial\theta_{02}}, \tfrac{\partial S_2}{\partial\theta_{03}}, \tfrac{\partial S_2}{\partial\theta_{12}}, \tfrac{\partial S_2}{\partial\theta_{13}}, \tfrac{\partial S_2}{\partial\theta_{23}})_{|E} = (-kr^0_{23} + \tfrac{k}{\pi} l_{23},\ -kr^0_{13} + \tfrac{k}{\pi} l_{13},\ -kr^0_{12} + \tfrac{k}{\pi} l_{12},\ -kr^0_{03} + \tfrac{k}{\pi} l_{03},\ -kr^0_{02} + \tfrac{k}{\pi} l_{02},\ -kr^0_{01} + \tfrac{k}{\pi} l_{01}).$$

Notice that l_{ij} are functions of θ_{ij}. Therefore, the Hessian matrix H_2 of S_2 is

$$H_2(\theta_{ij}) = (\frac{k}{\pi})^6 \begin{pmatrix} \frac{\partial l_{23}}{\partial\theta_{01}} & \frac{\partial l_{23}}{\partial\theta_{02}} & \frac{\partial l_{23}}{\partial\theta_{03}} & \frac{\partial l_{23}}{\partial\theta_{12}} & \frac{\partial l_{23}}{\partial\theta_{13}} & \frac{\partial l_{23}}{\partial\theta_{23}} \\ \frac{\partial l_{13}}{\partial\theta_{01}} & \frac{\partial l_{13}}{\partial\theta_{02}} & \frac{\partial l_{13}}{\partial\theta_{03}} & \frac{\partial l_{13}}{\partial\theta_{12}} & \frac{\partial l_{13}}{\partial\theta_{13}} & \frac{\partial l_{13}}{\partial\theta_{23}} \\ \frac{\partial l_{12}}{\partial\theta_{01}} & \frac{\partial l_{12}}{\partial\theta_{02}} & \frac{\partial l_{12}}{\partial\theta_{03}} & \frac{\partial l_{12}}{\partial\theta_{12}} & \frac{\partial l_{12}}{\partial\theta_{13}} & \frac{\partial l_{12}}{\partial\theta_{23}} \\ \frac{\partial l_{03}}{\partial\theta_{01}} & \frac{\partial l_{03}}{\partial\theta_{02}} & \frac{\partial l_{03}}{\partial\theta_{03}} & \frac{\partial l_{03}}{\partial\theta_{12}} & \frac{\partial l_{03}}{\partial\theta_{13}} & \frac{\partial l_{03}}{\partial\theta_{23}} \\ \frac{\partial l_{02}}{\partial\theta_{01}} & \frac{\partial l_{02}}{\partial\theta_{02}} & \frac{\partial l_{02}}{\partial\theta_{03}} & \frac{\partial l_{02}}{\partial\theta_{12}} & \frac{\partial l_{02}}{\partial\theta_{13}} & \frac{\partial l_{02}}{\partial\theta_{23}} \\ \frac{\partial l_{01}}{\partial\theta_{01}} & \frac{\partial l_{01}}{\partial\theta_{02}} & \frac{\partial l_{01}}{\partial\theta_{03}} & \frac{\partial l_{01}}{\partial\theta_{12}} & \frac{\partial l_{01}}{\partial\theta_{13}} & \frac{\partial l_{01}}{\partial\theta_{23}} \end{pmatrix}.$$

Thus, from Equation (4.19) the asymptotic expansion of the integral $I_{c2}(k)$ reads

$$\begin{aligned}
I_{c2,\epsilon_{ij}=-1}(k) &\sim \frac{(2\pi)^3}{\pi^4(2\mathrm{i})^6}\cdot\frac{e^{-\mathrm{i}(\sum_{ij}\theta^0_{ij}+\frac{4}{\pi}V(\theta^0_{ij}))}}{\sqrt{\det([\cos\theta^0_{ij}])}|\det H_2(\theta^0_{ij})|^{\frac{1}{2}}} e^{-\mathrm{i}k(\sum_{ij} r^0_{\overline{ij}}\theta^0_{ij}+\frac{2}{\pi}V(T^0))+\mathrm{i}\frac{\pi}{4}\sigma(H_2(\theta^0_{ij}))} \\
&= \frac{2^3}{\pi(2\mathrm{i})^6}\frac{e^{-\mathrm{i}(\sum_{i<j}\theta^0_{ij}+\frac{4}{\pi}V(\theta^0_{ij}))}}{\sqrt{\det([\cos\theta^0_{ij}])\frac{(k^3)}{\pi^3}}|\det([\frac{\partial l_{ij}}{\partial\theta_{kl}}]_{|\theta^0_{ij}})|^{\frac{1}{2}}} e^{-\mathrm{i}k(\sum_{ij} r^0_{\overline{ij}}\theta^0_{ij}+\frac{2}{\pi}V(T^0))+\mathrm{i}\frac{\pi}{4}\sigma(H_2(\theta^0_{ij}))} \\
&= \frac{2^3\pi^2}{k^3(2\mathrm{i})^6}\frac{e^{-i(\sum_{i<j}\theta^0_{ij}+\frac{4}{\pi}V(\theta^0_{ij}))}}{\sqrt{\det([\cos\theta^0_{ij}])}|\det([\frac{\partial l_{ij}}{\partial\theta_{kl}}]_{|\theta^0_{ij}})|^{\frac{1}{2}}} e^{-\mathrm{i}k(\sum_{ij} r^0_{\overline{ij}}\theta^0_{ij}+\frac{2}{\pi}V(T^0))+\mathrm{i}\frac{\pi}{4}\sigma(H_2(\theta^0_{ij}))} \\
&= \frac{-\pi^2}{8k^3}\frac{e^{-\mathrm{i}(\sum_{i<j}\theta^0_{ij}+\frac{4}{\pi}V(\theta^0_{ij}))}}{\sqrt{\det([\cos\theta^0_{ij}])}|\det([\frac{\partial l_{ij}}{\partial\theta_{kl}}]_{|\theta^0_{ij}})|^{\frac{1}{2}}} e^{-\mathrm{i}k(\sum_{ij} r^0_{\overline{ij}}\theta^0_{ij}+\frac{2}{\pi}V(T^0))+\mathrm{i}\frac{\pi}{4}\sigma(H_2(\theta^0_{ij}))} \\
&= \frac{-\pi^2}{8k^3}\frac{e^{-\mathrm{i}(\sum_{i<j}\theta^0_{ij}+\frac{4}{\pi}V(\theta^0_{ij}))}}{\sqrt{\det([\cos l^0_{ij}])}} e^{-\mathrm{i}k(\sum_{i<j} r^0_{\overline{ij}}\theta^0_{ij}+\frac{2}{\pi}V(T^0))+\mathrm{i}\frac{\pi}{4}\sigma(H_2(\theta^0_{ij}))} \qquad \text{(by Lemma 4.5.3)} \\
&= \frac{-\pi^2}{8k^3}\cdot\frac{e^{\mathrm{i}\frac{\pi}{4}\sigma(H_2(\theta^0_{ij}))}}{\sqrt{\det([\cos l^0_{ij}])}} e^{-\mathrm{i}(\sum_{i<j}(kr^0_{\overline{ij}}+1)\theta^0_{ij}+\frac{2}{\pi}(k+2)V(T^0))}.
\end{aligned}$$

□

4.5.3.3 Asymptotic expansion of $I_c(k)$

Eventually, since the asymptotic expansion of $I_{c1}(k)$ added up with that of $I_{c2}(k)$ is non-zero then, the asymptotic expansion of $I_c(k)$ can be considered as this sum. Hence, we are now ready to prove Theorem 4.1.10.

Remark 4.5.9. *We empirically computed the signature of $\sigma(H_1(\theta^0_{ij}))$ of H_1 which is the same as that of H_2 and it gave us zero (0). The MATHEMATICA code may be found in Appendix C.*

Proof of Theorem 4.1.10. As we set from the beginning, the integral $I_c(k)$ may be written as the sum of two integrals $I_{c1}(k)$ and $I_{c2}(k)$ i.e.

$$I_c(k) = I_{c1}(k) + I_{c2}(k).$$

Since the sum of the asymptotics of $I_{c1}(k)$ and $I_{c2}(k)$ is non-zero, it will constitute the asymptotic approximation of $I_c(k)$. In other words,

$$\begin{aligned}
I_c(k) \sim & \frac{-\pi^2}{8k^3}\cdot\frac{e^{i\frac{\pi}{4}\sigma(H_1(\theta^0_{ij}))}}{\sqrt{\det([\cos l^0_{ij}])}} e^{i(\sum_{i<j}(kr^0_{ij}+1)\theta^0_{ij}+\frac{2}{\pi}(k+2)V(T^0))} \\
& -\frac{\pi^2}{8k^3}\cdot\frac{e^{i\frac{\pi}{4}\sigma(H_2(\theta^0_{ij}))}}{\sqrt{\det([\cos l^0_{ij}])}} e^{-i(\sum_{i<j}(kr^0_{ij}+1)\theta^0_{ij}+\frac{2}{\pi}(k+2)V(T^0))} \\
= & \frac{-\pi^2}{8k^3}\cdot\frac{1}{\sqrt{\det([\cos l^0_{ij}])}} e^{i(\sum_{i<j}(kr^0_{ij}+1)\theta^0_{ij}+\frac{2}{\pi}(k+2)V(T^0))} \\
& -\frac{\pi^2}{8k^3}\cdot\frac{1}{\sqrt{\det([\cos l^0_{ij}])}} e^{-i(\sum_{i<j}(kr^0_{ij}+1)\theta^0_{ij}+\frac{2}{\pi}(k+2)V(T^0))} \\
= & \frac{-\pi^2}{4k^3}\cdot\frac{1}{\sqrt{\det([\cos l^0_{ij}])}} \left(\frac{e^{i(\sum_{i<j}(kr^0_{ij}+1)\theta^0_{ij}+\frac{2}{\pi}(k+2)V(T^0))}+e^{-i(\sum_{i<j}(kr^0_{ij}+1)\theta^0_{ij}+\frac{2}{\pi}(k+2)V(T^0))}}{2}\right) \\
= & \frac{-\pi^2}{4k^3}\cdot\frac{1}{\sqrt{\det([\cos l^0_{ij}])}} \cos\{\sum_{i<j}(kr^0_{ij}+1)\theta^0_{ij}+\frac{2}{\pi}(k+2)V(T^0)\}.
\end{aligned}$$

Note that the first equality results from Remark 4.5.9 and the last equality follows from $\cos x = 2\cos^2(\frac{x}{2}) - 1$. □

Remark 4.5.10. *The integral in Lemma 4.1.8 has been approximated numerically using a form of Gaussian quadrature [7]. For the sequence of 6j symbols*

$$I_k = \begin{Bmatrix} \frac{k}{3} & \frac{k}{3} & \frac{k}{6} \\ \frac{k}{6} & \frac{k}{6} & \frac{k}{3} \end{Bmatrix}_{q=e^{\frac{\pi i}{k+2}}} \qquad k = 12, 16, 36, \cdots$$

the following results were obtained:

k	I_k
12	0.0038215
24	0.0005396
36	0.00041512

However, the numerical calculation proved to be very expensive for larger k and it was not possible to reliably compute the boundary contribution using this approach.

Chapter 5

Conclusion

5.1 Summary

The computation of the partial derivative of an angle with respect to the length of its opposite edge and the partial derivative of the length of an edge with respect to its opposite angle in a spherical triangle, provided in Theorem 3.2.4, are presented as warm up calculations leading to the first main result in this thesis.

For spherical tetrahedra, Taylor and Woodward in [41] [38] called Wigner derivative the partial derivative of a dihedral angle with respect to the length of its opposite edge, all other edge lengths remaining fixed. They gave a formula for it which is also found in Theorem 3.1.1, of which the author calculated independently in Lemma 3.3.1. The first main theorem in this thesis is the formula for the inverse Wigner derivative, which is the partial derivative of the length of an edge with respect to its opposite dihedral angle, all other dihedral angles remaining fixed. It is found in Theorem 3.1.2 This result plays an important role in a the calculation of an asymptotic expansion of a possible integral formula for the square of the quantum 6j symbols. As a corollary, see Corollary 3.1.3, the equality between the Wigner derivative and the inverse Wigner derivative is concluded.

The second main theorem in this thesis arises from a conjectural integral formula for the square of the quantum 6j symbols by Bruce Bartlett, found in Conjecture 4.1.7. The aim was to compute the asymptotic expansion of the conjectural integral formula and compare it with the corrected version of the known asymptotic formula for the quantum 6j symbols by Taylor and Woodward [41][38], which is asymptotically equivalent to the corrected version of Roberts' in [36], found in Theorem 4.1.3. The strategy is to use the stationary phase method to asymptotically approximate the integral. A partial result is obtained, which is the asymptotic expansion of the integral from the critical point inside the domain of integration. That is given in Theorem 4.1.10.

5.2 Future plans

1. As the reciprocity of the Wigner derivative is correct for spherical triangles and spherical tetrahedra, it is natural to ask if it stands for an $n-$dimensional spherical simplex as well as for hyperbolic simplices. These areas may be explored in the future.

2. Regarding the investigation on the conjectural integral formula for the square of the quantum 6j symbols, the asymptotic expansion of the integral contributed by the critical points at the boundary of the domain of integration remains a mystery. So, this is still open for investigation.

3. The numerical computation of the integral also remains incomplete. So, that may be carried on as a part of future research.

Appendices

Appendix A

The characters of $SU(2)$

The aim of Chapter 2 was to transform the integral over $SU(2)^4$ in Theorem 2.1.1 into an integral over the set of a 6-tuple $\in [0, \pi]^6$ representing the edge lengths of non-degenerate spherical tetrahedra as described in Theorem 2.1.2. This action requires the transition from one integrand to the other. Recall the equation,

$$\int_{(SU(2))^4} \prod_{i<j} \chi_{m_{ij}}(g_j g_i^{-1})[\prod_{i=0}^{3} dg_i] = \frac{2}{\pi^4} \int_{D_\pi} \frac{\prod_{i<j} \sin((m_{ij}+1)l_{ij})}{\sqrt{\det([\cos l_{ij}])}} [\prod_{i<j} \mathrm{d}l_{ij}].$$

While the left hand side involves the product of characters $\chi_n(gh^{-1})$ $(g, h \in SU(2))$, the right hand side involves sines. The link between those two is suggested in [22], [42] and [17] by

$$\chi_n(gh^{-1}) = \frac{\sin((n+1)\phi)}{\sin\phi}, \tag{A.1}$$

where ϕ denotes the angle between the vectors $f(g)$ and $f(h)$ of S^3. Here, f is the diffeomorphism between $SU(2)$ and S^3, and n a positive integer. Since this equality is not obvious the aim of this Appendix is to provide a rigorous proof for it.

Should the reader be interested in more details on the materials present in this section, they may be found in [37], [22], [42] and [35]. And, assume that the reader is familiar with the notion of representation theory of groups.

A.1 The Lie group $SU(2)$

The aim of this section is to show that there is an equivalence between the inner product in S^3 and that of $SU(2)$. This is crucial for the identification of the angle ϕ

in

$$\frac{\sin((n+1)\phi)}{\sin\phi}.$$

So, to start with let us recall the definitions of the group $SU(2)$, Lie group and eventually look at the relationship between the inner products.

By definition, $SU(2)$ is the group of complex matrices U such that $U^{\dagger}.U = 1$ with $\det U = 1$, where $U^{\dagger}$ is the transpose of the complex conjugate of U. It can explicitly be written as

$$SU(2) = \{\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1\},$$

where the bar denotes complex conjugation. And a Lie group is a smooth manifold G equipped with a group structure so that the maps

$$\mu : G \times G \longrightarrow G$$
$$(x, y) \longmapsto xy$$

and

$$i : G \longrightarrow G$$
$$x \longmapsto x^{-1}$$

are smooth. Van den Ban, in [42, page 7, Example 2.10 and page 9-10, Example 2.19], explicitly shows how $SU(2)$ is a Lie group. In addition, there is a diffeomorphism between $SU(2)$ and S^3 as manifolds. It is defined by

$$f : S^3 \longrightarrow SU(2)$$

such that

$$(a_1, b_2, b_1, a_2) \mapsto \begin{pmatrix} a_1 + ia_2 & b_1 + ib_2 \\ -b_1 + ib_2 & a_1 - ia_2 \end{pmatrix},$$

whose inverse is given by:

$$f^{-1} : SU(2) \longrightarrow S^3$$

such that

$$\begin{pmatrix} a_1 + ia_2 & b_1 + ib_2 \\ -b_1 + ib_2 & a_1 - ia_2 \end{pmatrix} \mapsto (a_1, b_2, b_1, a_2).$$

Let $a, b \in \mathbb{R}^4$ and denote by $< a, b >_{\mathbb{R}^4}$ their dot product. Let $A, B \in SU(2)$; their inner product is defined by

$$< A, B >_{SU(2)} := Tr(AB^\dagger).$$

Then, f induces an equivalence between the inner product in S^3 and $SU(2)$ by considering S^3 as a subset of $\mathbb{R}^4$. That is shown in the lemma below.

Lemma A.1.1. *Let g and h be elements of $SU(2)$. Then the cosine of the angle between g and h as elements of S^3 is equal to the trace of the matrix gh^{-1} i.e.*

$$\cos\theta = \frac{1}{2} Tr(gh^{-1}).$$

Proof. Let $g = \begin{pmatrix} g_1 + ig_2 & g_3 + ig_4 \\ -g_3 + ig_4 & g_1 - ig_2 \end{pmatrix}$, and $h = \begin{pmatrix} h_1 + ih_2 & h_3 + ih_4 \\ -h_3 + ih_4 & h_1 - ih_2 \end{pmatrix}$. It follows

$$\begin{aligned} Tr(gh^{-1}) &= 2Re[(h_1 + ih_2)(g_1 - ig_2) + (h_3 + ih_4)(g_3 - ig_4)] \\ &= 2(h_1 g_1 + h_2 g_2 + h_3 g_3 + h_4 g_4) \\ &= 2 < f^{-1}(g), f^{-1}(h) >_{\mathbb{R}^4} \\ &= 2\cos\theta. \end{aligned}$$

□

A.2 The irreducible representations of $SU(2)$

Let $g, h \in SU(2)$ and recall the diffeomorphism $f : S^3 \longrightarrow SU(2)$ from Section A.1. It is not obvious that the character $\chi_n(gh^{-1})$ may be written in terms of ϕ (angle between $f^{-1}(g)$ and $f^{-1}(h)$ in S^3) as

$$\chi_n(gh^{-1}) = \frac{\sin((n+1)\phi)}{\sin\phi}.$$

This expression is key for the transformation of the integral over $SU(2)^4$ from Theorem 2.1.1 into its six-dimensional version in Theorem 2.1.2. Hence, the aim of this subsection is to prove this equality. We assume that the reader is familiar with the notion of representation theory of groups. However, below is a lemma which plays a crucial role in our proof.

Lemma A.2.1. *Let G be a group and V, V' be two representations of G such that V and V' are isomorphic. Suppose V is irreducible. Then, V' is an irreducible representation of G.*

Proof. Let

$$\rho : G \longrightarrow GL(V)$$

$$\pi : G \longrightarrow GL(V')$$

be two respective linear representations of G and consider the isomorphism

$$i : V \longrightarrow V'.$$

By definition, V' is irreducible if and only if its only stable subspace under G are 0 and V'. So, let W' be a subspace of V' such that $\pi_s(W') \subseteq W'$ for all $s \in G$, and let us prove that either $W' = 0$ or $W' = V'$. The proof will be conducted in a direct way. It is a fact that $i^{-1}(W')$ is a subspace of V, and every element in W' is of the form $i(w)$ where

$$w \in i^{-1}(W').$$

Let $s \in G$. Therefore,

$$\pi_s(i(w)) \in W'.$$

Hence,

$$i^{-1}(\pi_s i(w)) \in i^{-1}(W').$$

Since i is an isomorphism,

$$\rho_s = i^{-1}\pi_s i$$

which implies

$$\rho_s(w) \in i^{-1}(W')$$

i.e. $i^{-1}(W')$ is stable. But V is irreducible so, $i^{-1}(W') = 0$ or $i^{-1}(W') = V$. Thus, $W' = 0$ or $W' = V'$ since i is an isomorphism. □

Let

$$\rho : SU(2) \longrightarrow End(V_1)$$

be the fundamental representation of $SU(2)$ on $V_1 = \mathbb{C}^2$. Its dual is defined as the vector space

$$V_1^* := \{f : V_1 \longrightarrow \mathbb{C}, f \text{ linear}\}$$

which may be identified as the vector space of homogeneous polynomial of degree one, $P_1(\mathbb{C}^2)$, with basis $\{z, w\}$. Let $\{e_1, e_2\}$ be a basis for V_1 and by respectively sending e_1 to z, e_2 to w, V_1 is isomorphic to V_1^*.

Let us define the vector space

$$\overset{n}{\otimes} V_1 := \underbrace{V_1 \otimes V_1 \otimes \cdots \otimes V_1}_{n \text{ times}},$$

which is the tensor product of n copies of V_1. Recall that an element of the basis for $\overset{n}{\otimes} V_1$ is of the form

$$e_{i_1} \otimes \cdots \otimes e_{i_n}$$

where $i_j \in \{1, 2\}$.

Note that $\overset{n}{\otimes} V_1$ is also a representation of $SU(2)$ by the tensor product action (see [37]) defined by

$$\rho^n : SU(2) \longrightarrow \overset{n}{\otimes} V_1$$

such that

$$\rho_g^n(v_1 \otimes \cdots \otimes v_n) = \rho_g(v_1) \otimes \cdots \otimes \rho_g(v_n),$$

for $g \in SU(2)$.

Let us consider the projection

$$s_n : \overset{n}{\otimes} V_1 \longrightarrow \overset{n}{\otimes} V_1$$

defined on tensor product of vectors in V_1 by:

$$v_1 \otimes \cdots \otimes v_n \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

The image of s_n is called the n-**Symmetric power** of dimension $n + 1$ which we call here V_n, i.e.

$$V_n := \mathrm{Im}(s_n).$$

Note that V_n is an invariant subspace, i.e. it is a representation of $SU(2)$ in its own right.

Lemma A.2.2 ([35][12]). *Every irreducible representation of $SU(2)$ is isomorphic to some V_n.*

Consider V_n^* to be the dual vector space of V_n. Similarly to the case for V_1^*, V_n^* may be identified with the vector space of homogeneous polynomials of degree n, $P_n(\mathbb{C}^2)$, with basis

$$\{z^k w^{n-k}; 0 \leq k \leq n\}.$$

Let us set the isomorphism

$$i_n : P_n(\mathbb{C}^2) \longrightarrow V_n$$

such that

$$z^k w^{n-k} \longmapsto \sum_{\sigma \in S_n} e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_n)},$$

where S_n denotes the symmetric group of n elements, $i_j \in \{1,2\}$ and the number of e_1 occurring in the expression $e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_n)}$ is k whereas that of e_2 is $n-k$.

The knowledge of these results is required for the proof of the equality

$$\chi_n(gh^{-1}) = \frac{\sin((n+1)\phi)}{\sin \phi}.$$

Now that everything is settled, let us state Equation A.1 as a proposition, then proceed to its proof.

Proposition A.2.3. *For $n > 0$ and $g, h \in SU(2)$, the character of the n-th representation of $SU(2)$ at the element gh^{-1} is the same as the quotient of the sine of the angle between $f^{-1}(g)$ and $f^{-1}(h)$ scaled by $n+1$ with the sine of the angle between $f^{-1}(g)$ and $f^{-1}(h)$, i.e.*

$$\chi_n(gh^{-1}) = \frac{\sin((n+1)\phi)}{\sin \phi},$$

where $f : S^3 \longrightarrow SU(2)$, ϕ is the angle between $f^{-1}(g)$ and $f^{-1}(h)$ and $\phi \in [0, \pi]$.

Proof. Let

$$W = \{w_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \theta \in \mathbb{R}\}.$$

It is obvious that W is a subgroup of $SU(2)$.

Let $p_k = z^k w^{n-k} \in P_n(\mathbb{C}^2)$, and consider the representation of W given by:

$$\begin{aligned} \rho : W &\longrightarrow End(P_n(\mathbb{C}^2)) \\ w_\theta &\longmapsto \rho_{w_\theta} : P_n(\mathbb{C}^2) \longrightarrow P_n(\mathbb{C}^2) \end{aligned}$$

such that

$$\rho_{w_\theta}(p_k) = e^{(2k-n)\theta} p_k.$$

Then, two cases occur:

Case 1: n is even

The matrix representing ρ_{w_θ} is given by

$$\rho_{w_\theta} = Diag(e^{in\theta}, e^{i(n-2)\theta}, e^{i(n-4)\theta}, \cdots, e^{i2\theta}, 1, e^{-i2\theta}, \cdots, e^{-i(n-4)\theta}, e^{-i(n-2)\theta}, e^{-in\theta}).$$

Hence the trace of ρ_{w_θ} is given by,

$$\begin{aligned}
Tr(\rho_{w_\theta}) =& 2[\frac{e^{in\theta} + e^{-in\theta}}{2} + \frac{e^{i(n-2)\theta} + e^{-i(n-2)\theta}}{2} + \frac{e^{i(n-4)\theta} + e^{-i(n-4)\theta}}{2} + \cdots + \frac{e^{i2\theta} + e^{-i2\theta}}{2}] + 1 \\
=& 2\cos n\theta + 2\cos(n-2)\theta + 2\cos(n-4)\theta + \cdots + 2\cos 2\theta + 1 \\
=& \frac{2\sin\theta\cos n\theta + 2\sin\theta\cos(n-2)\theta + 2\sin\theta\cos(n-4)\theta + \cdots + 2\sin\theta\cos 2\theta + \sin\theta}{\sin\theta} \\
=& \frac{\sin(n+1)\theta}{\sin\theta} \quad (\text{since } \sin(a+b) + \sin(a-b) = 2\sin a\cos b).
\end{aligned}$$

Case 2: n is odd

The endomorphism ρ_{w_θ} may be identified as

$$\rho_{w_\theta} = Diag(e^{in\theta}, e^{i(n-2)\theta}, e^{i(n-4)\theta}, \cdots, e^{i\theta}, e^{-i\theta}, \cdots, e^{-i(n-4)\theta}, e^{-i(n-2)\theta}, e^{-in\theta}).$$

Therefore the trace of ρ_{w_θ} is computed as,

$$\begin{aligned}
Tr(\rho_{w_\theta}) =& 2[\frac{e^{in\theta} + e^{-in\theta}}{2} + \frac{e^{i(n-2)\theta} + e^{-i(n-2)\theta}}{2} + \cdots + \frac{e^{i3\theta} + e^{-i3\theta}}{2} + \frac{e^{i\theta} + e^{-i\theta}}{2}] \\
=& 2\cos n\theta + 2\cos(n-2)\theta + \cdots + 2\cos 3\theta + 2\cos\theta \\
=& \frac{2\sin\theta\cos n\theta + 2\sin\theta\cos(n-2)\theta + \cdots + 2\sin\theta\cos 3\theta + 2\sin\theta\cos\theta}{\sin\theta} \\
=& \frac{\sin(n+1)\theta}{\sin\theta} \quad (\text{since } \sin(a+b) + \sin(a-b) = 2\sin a\cos b).
\end{aligned}$$

So, in all cases

$$Tr(\rho_{w_\theta}) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

As explicitly explained in [22, page 76, Section 2.2] every element of $SU(2)$ is conjugate to an element in W i.e. $\forall A \in SU(2), \exists w \in W$ such that $A = SwS^{-1}$, where $S \in SU(2)$. Hence, there exists $S \in SU(2)$ and $\theta \in \mathbb{R}$ such that

$$gh^{-1} = S \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} S^{-1}.$$

On one hand,

$$Tr(gh^{-1}) = 2\cos\theta.$$

On the other hand from Lemma A.1.1,

$$Tr(gh^{-1}) = 2\cos\phi.$$

So, if $\theta \in [0, \pi]$ then $\theta = \phi$. Which is the case considered here. Therefore,

$$gh^{-1} = S \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} S^{-1},$$

where ϕ is the angle between $f^{-1}(g)$ and $f^{-1}(h)$ in S^3.

And from the property of characters, $\chi(xyx^{-1}) = \chi(y)$. Thus,

$$\chi_n(gh^{-1}) = \frac{\sin((n+1)\phi)}{\sin\phi}.$$

□

Appendix B

Volume form and integration on a manifold

Note: Some of the materials presented here were extracted from [34]. And for further reading [26], [1] are recommended.

Let us remind ourselves that the target of Chapter 2 was to transform the integral formula for the square of classical 6j symbols given by

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}^2 = \int_{(SU(2))^4} [\prod_{i=0}^{3} dg_i] \prod_{i<j} \chi_{m_{ij}}(g_j g_i^{-1}), \tag{B.1}$$

which is an integration over $SU(2)^4$ into a six-dimensional Lebesgue integral over the region $D_\pi \subset [0,\pi]^6$,

$$I = \frac{2}{\pi^4} \int_{D_\pi} [\prod_{i<j} \mathrm{d}l_{ij}] \frac{\prod_{i<j} \sin((m_{ij}+1)l_{ij})}{\sqrt{\det([\cos l_{ij}])}}. \tag{B.2}$$

In order to reach that goal, one of the steps is to compute the Euclidean volume form on S^3 via the spherical coordinates. In addition, the integration of differential forms will be needed in Chapter 4. Hence, this appendix is set as a reminder on the volume forms on a manifold and the integration of differential forms on manifolds.

B.1 Volume form on a manifold

We suppose that the reader is familiar with the notion of manifolds and charts.

Let M be an n-dimensional oriented Riemannian manifold. From [26, page 86] the volume form on M determines a measure μ_M on M analogous to the Riemannian measure on $\mathbb{R}^n$. Therefore, this idea may be exploited to generate a Lebesgue measure on S^3. The process requires the knowledge of differential forms on a manifold, volume form and orientation on a manifold, hence the notion on those will be reminded before proceeding to the computation of the volume form on S^3. More details on the materials present here may be found in [26] and [1].

As the definition of a differential form depends solely on the knowledge of an alternating form on the tangent space of M at a point p, let us recall the definition of alternating forms on a vector space, and that of the tangent space T_pM of M at p.

Definition B.1.1. *Let V be a vector space over $\mathbb{R}$. A k-linear map $\omega : V^k \longrightarrow \mathbb{R}$ is said to be* alternating *if $\omega(v_1, \cdots, v_k) = 0$ whenever $v_i = v_j$ for some pair $i \neq j$. The vector space of alternating $k-$linear maps is denoted by $Alt^k(V)$.*

Notice that $Alt^0(V) = \mathbb{R}$.

Let $p \in M$ and (U, h) a smooth chart in the neighbourhood of p. Let

$$\mathcal{C} = \{\gamma : I \longrightarrow M, \gamma(0) = p, I \text{ open interval in } \mathbb{R} \text{ around } 0\}$$

be the vector space of smooth curves on M passing through p. An equivalence relation may be defined in $\mathcal{C}$ as follows:

$$\gamma_1 \sim \gamma_2 \Longleftrightarrow \frac{dh(\gamma_1(t))}{dt}_{|t=0} = \frac{dh(\gamma_2(t))}{dt}_{|t=0}. \tag{B.3}$$

Remark B.1.2. *The equivalence relation is independent of the choice of (U, h).*

Definition B.1.3. *The set of equivalence classes with respect to the equivalence relation (B.3) is called the* tangent space *T_pM at the point p in M.*

Remark B.1.4.

- *The tangent space of a manifold at a point p is a vector space of dimension equal to the dimension of the manifold.*
- *Given a smooth chart (U, h) around $p \in M$, a basis for T_pM is given by*

$$(\frac{\partial}{\partial x_1})_p, \cdots, (\frac{\partial}{\partial x_n})_p,$$

  where $(\frac{\partial}{\partial x_i})_p$ is the image under $D_{h(p)}h^{-1} : \mathbb{R}^n \longrightarrow T_pM$ of the i-th standard basis vector $e_i \in \mathbb{R}^n$.

Let us now define a differential form on a manifold.

Let M be a smooth manifold of dimension n. A local parameterization is the inverse of a smooth chart. Let $\omega = \{\omega_p\}_{p\in M}$ be a family of alternating $k-$forms on T_pM i.e. $\omega_p \in Alt^k(T_pM)$. Let $g : W \longrightarrow M$ be a local parameterization, and $x \in W$.

The map

$$D_xg : \mathbb{R}^n \longrightarrow T_{g(x)}M$$

is an isomorphism, therefore it induces an isomorphism

$$Alt^k(D_xg) : Alt^k(T_{g(x)}M) \longrightarrow Alt^k(\mathbb{R}^n)$$

defined as $Alt^k(D_xg)(\omega)(\zeta_1, \cdots, \zeta_k) = \omega(D_xg(\zeta_1), \cdots, D_xg(\zeta_k)), \zeta_i \in \mathbb{R}^n$.

Consider the map $g^*(\omega) : W \longrightarrow Alt^k(\mathbb{R}^n)$ whose value at x is

$$g^*(\omega)_x = Alt^k(D_xg)(\omega_{g(x)}).$$

Definition B.1.5. *Let M be a smooth manifold of dimension n, and W be an open subset of $\mathbb{R}^n$. A family $\omega = \{\omega_p\}_{p\in M}$ of alternating $k-$forms on T_pM is said to be smooth if $g^*(\omega) : W \longrightarrow Alt^k(\mathbb{R}^n)$ is a smooth function for every local parameterization (W, g). The set of such smooth families is a real vector space $\Omega^k(M)$, it is the* vector space of differential forms *on M.*

Being used later to procure the volume form on S^3, as follows is the definition of a "pullback" of a smooth map between two smooth manifolds.

Definition B.1.6. *Let M, N be smooth manifolds and $\phi : M \longrightarrow N$ be a smooth map. It induces a map*

$$\phi^* : \Omega^k(N) \longrightarrow \Omega^k(M)$$

such that if $\tau \in \Omega^k(N)$, and $p \in M$,

$$\phi^*(\tau)_p = Alt^k(D_p\phi)(\tau_{\phi(p)})$$

where $\tau_{\phi(p)}$ lives in $Alt^k(T_{\phi(p)}N)$.

To generate a volume form on a manifold, an orientation on that manifold is required. And as this notion is key in our calculation, let us recall its definition.

Definition B.1.7. 1. *A smooth manifold M of dimension n is called* orientable, *if there exists a differential form $\omega \in \Omega^n(M)$ with $\omega_p \neq 0$ for all $p \in M$. Such form is called an* orientation form *on M.*

2. *An equivalence relation on the set of orientation forms on M may be defined as: two orientation forms ω, τ on M are equivalent if $\tau = s.\omega$, for some $s \in \Omega^0(M)$ with $s(p) > 0$ for all $p \in M$. An* orientation *of M is an equivalence class of orientation forms on M.*

Remark B.1.8.

- *On the Euclidean space $\mathbb{R}^n$, the form*

$$dx_1 \wedge \cdots \wedge dx_n$$

represents the standard orientation of $\mathbb{R}^n$.

- *If M is connected, then there are precisely two orientations on M.*

Later, when dealing with the integration on a manifold, the concept of oriented charts needs to be understood. Hence, below is its definition.

Definition B.1.9. *Let M be an $n-$dimensional oriented smooth manifold, $p \in M$ and (U, h) a chart around p such that $h(U) = U'$. Let ω be the orientation form on U'. Then (U, h) is called* an oriented chart *if the orientation determined by $h^*(\omega)$ is the same as that determined by ω.*

Definition B.1.10. *Let M be a manifold oriented by the orientation form $\omega \in \Omega^n(M)$, $p \in M$ and suppose $\{b_1, \cdots, b_n\}$ is a basis for T_pM. Then the basis is positively (resp. negatively) oriented with respect to ω if $\omega_p(b_1, \cdots, b_n)$ is positive (resp. negative). If for every $p \in M$ and an oriented basis $(b_1, b_2, \cdots, b_n)$ for T_pM, $\omega_p(b_1, b_2, \cdots, b_n) > 0$, the form ω is called a* volume form.

As a handy method to compute a volume form, let S be a $k-$dimensional submanifold of $\mathbb{R}^n$ and B be a $k-$dimensional box spanned by $v_1, v_2, \cdots, v_k \in T_pS$.

Let

$$Gr = \begin{pmatrix} < v_1, v_1 > & < v_1, v_2 > & \cdots & < v_1, v_k > \\ \vdots & \vdots & \cdots & \vdots \\ < v_k, v_1 > & < v_k, v_2 > & \cdots & < v_k, v_k > \end{pmatrix},$$

where $< v_i, v_j >$ denotes the inner product of v_i and v_j in $\mathbb{R}^n$.

A volume form ω on S may be obtained by:

$$\begin{aligned} \omega_p(v_1, v_2, \cdots, v_k) &= Vol(B) \\ &= \sqrt{\det(Gr)}. \end{aligned}$$

Which in local coordinates $x_1, x_2, \cdots, x_k$ for S near p translates into

$$\Phi^* \omega_p = \sqrt{\det A}\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_k, \tag{B.4}$$

where

$$A_{ij} =< \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} >,$$

$\Phi : W \longrightarrow S$ is the local parameterization at p and $\Phi^* : \Omega^k(S) \longrightarrow \Omega^k(\mathbb{R}^k)$.

B.2 Integration on a manifold

The whole point of differential forms is to use them for integration. When forgetting about orientation, the Riemann integral has been used in calculus. Now, when orientation is taken into account, i.e. when dealing with integration of differential forms, the Riemann integral may still be adapted but with a slight change. The integration of differential forms are recalled in this subsection. The materials present here are fully covered in [26] and [10] should the reader would like to explore.

As integration on manifolds requires a compactly supported differential form, let us remind ourselves of its definition. But before that, let us look at the definition of the support of a differential form.

Definition B.2.1. *Let M be a smooth manifold. Let $\omega \in \Omega^p(M)$. The support of ω is defined by $supp(\omega) = \overline{\{p \in M, \omega_p \neq 0\}}$.*

Definition B.2.2. *Let M be a smooth manifold and $\omega \in \Omega^k(M)$. The differential form ω is called compactly supported if its support is compact in M. The vector space of compactly supported differential $k-$forms on M is denoted $\Omega^k_c(M)$.*

We are mostly working with charts; hence, the following proposition from [26] is what we will consider when integrating over a manifold.

Proposition B.2.3. *[26] Let M be an n-dimensional oriented smooth manifold and (U, h) be a positively oriented C^∞ chart on M. Then there exists a unique linear map*

$$\int_M : \Omega_c^n(M) \longrightarrow \mathbb{R}$$

such that if $\omega \in \Omega_c^n(M)$ has support contained in U, then

$$\int_M \omega = \int_{h(U)} (h^{-1})^* \omega.$$

Since a change of variables is often encountered while dealing with integrals, and as we will mostly work on charts, the following lemma explains how the integration of differential forms on a manifold in $\mathbb{R}^4$ transforms under a change of variables.

Lemma B.2.4. *Let V, W be open subsets of $\mathbb{R}^n$ and $\phi : V \longrightarrow W$ be a diffeomorphism. Let $s \in V$ and suppose that the determinant of $D_x\phi$ is of constant sign $\delta = \pm 1$. Let $\omega \in \Omega_c^n(W)$. Then,*

$$\int_V \phi^*(\omega) = \delta \int_W \omega.$$

Remark B.2.5. *As stated at the beginning of this subsection, there is a close relationship between the Riemann integral and the integration of a differential forms. In fact, the integration of a differential form is exactly the same as the Riemann integral when ignoring orientations. For instance, if $\omega \in \Omega_c^n(\mathbb{R}^n)$, then ω is of the form*

$$\omega = f(x_1, \cdots, x_n) dx_1 \wedge \cdots \wedge dx_n,$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, infinitely differentiable and with compact support, and $dx_1 \wedge \cdots \wedge dx_n$ is the standartd orientation form on $\mathbb{R}^n$. Let us set $d\mu_n$ to be the usual Lebesgue measure on $\mathbb{R}^n$, then

- $\int_{\mathbb{R}^n} f(x_1, \cdots, x_n)\, dx_1 \wedge \cdots \wedge dx_n = \int_{\mathbb{R}^n} f(x_1, \cdots, x_n)\, d\mu_n.$

- *If σ is a permutation of n elements, then*

$$\int_{\mathbb{R}^n} f(x_1, \cdots, x_n)\, dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(n)} = Sign(\sigma) \int_{\mathbb{R}^n} f(x_1, \cdots, x_n)\, dx_1 \wedge \cdots \wedge dx_n.$$

Appendix C

Numerical calculations

This Appendix contains the numerical computation generating the two graphs in Section 4.3.1, the code comparing Taylor and Woodward's formula for the asymptotic of the quantum 6j symbols with that of Roberts' mentioned in Remark 4.3.6 and the program computing the signature of the Hessian matrix of the phase function as pointed in Remark 4.5.9.

A link to the actual code is present in [33].

C.1 Quantum 6j symbols

To start with, as follows is the code generating the graphs in Subsection 4.3.1. Let us be reminded that the example taken into account throughout this section is

$$\begin{Bmatrix} 40 & 48 & 50 \\ 52 & 54 & n \end{Bmatrix}_q,$$

where $n \in \{0, 2, 4, ..., 108\}$ and $k = 198$.

C.1.1 Code to compute the exact values

Let k be a positive integer. Here, $q = e^{\frac{i\pi}{k+2}}$ and the quantum integer $[n]$ is [14]

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Then the quantum factorial associated to the quantum integer $[n]$ is the real number defined by $[n]! = [n][n-1]...[1]$, where $[0]! = 1$.

Let us recall that the triple (a,b,c) is q-admissible if

1. $a+b-c \geq 0,\ c+b-a \geq 0,\ a+c-b \geq 0,$
2. $a+b+c \in 2\mathbb{Z},$
3. $a+b+c \leq 2(k+2)-4.$

The quantum 6j symbols

$$\left\{\begin{array}{ccc} a & b & i \\ c & d & j \end{array}\right\}_q,$$

where $a,b,c,d,i,j \in I_k = \{0,1,...,(k+2)-2\}$ exists if the triples $(a,b,i),(d,c,i),(c,b,j)$ and (a,d,j) are q-admissible. And its formula is given by

$$\left\{\begin{array}{ccc} a & b & i \\ c & d & j \end{array}\right\}_q = \Delta(a,b,i)\Delta(b,c,j)\Delta(c,d,i)\Delta(a,d,j) \sum_{m\leq t\leq M} \frac{(-1)^t[t+1]!}{\Pi_l\,[t-a_l]!\Pi_p\,[b_p-t]!},$$

where

$$\begin{aligned}
a_1 &= \frac{a+b+i}{2},\\
a_2 &= \frac{d+c+i}{2},\\
a_3 &= \frac{c+b+j}{2},\\
a_4 &= \frac{a+d+j}{2},\\
b_1 &= \frac{a+b+c+d}{2},\\
b_2 &= \frac{b+i+d+j}{2},\\
b_3 &= \frac{a+i+c+j}{2},\\
m &= \mathrm{Max}\{a_1,a_2,a_3,a_4\},\\
M &= \mathrm{Min}\{b_1,b_2,b_3\},\\
\Delta(a,b,i) &= \frac{\left[\frac{a+b-i}{2}\right]!\left[\frac{a-b+i}{2}\right]!\left[\frac{-a+b+i}{2}\right]!}{\left[\frac{a+b+i+1}{2}\right]!}.
\end{aligned}$$

Then, the exact values of the quantum 6j symbols may be computed by using the following code.

```
TWQuantum6jsymbol[a_, b_, i_, c_, d_, j_, k_Integer] :=
Module[{QuantumSymbol, qF, qf},
(*quantum integer*)

qint[n_Integer] :=
If[n == 0,
1, (Exp[Pi*I/(k + 2)]^n -
Exp[Pi*I/(k + 2)]^(-n))/(Exp[Pi*I/(k + 2)] -
Exp[Pi*I/(k + 2)]^(-1))
];

(*quantum factorial*)

qfact[n_Integer] := (lqint = Table[qint[p], {p, 1, n}];
If[n == 0, 1, qfactorial = Times @@ lqint]);

(*quantum triangle*)

qTriangle[x_, y_, z_] :=
Sqrt[(qfact[(x + y - z)/2]*qfact[(x - y + z)/2]*
qfact[(-x + y + z)/2])/qfact[(x + y + z)/2 + 1]];

(*the denominator*)

qf = qfact[t - (a + b + i)/2]*qfact[t - (d + c + i)/2]*
qfact[t - (a + d + j)/2]*qfact[t - (c + b + j)/2]*
qfact[(b + d + i + j)/2 - t]*qfact[(a + c + i + j)/2 - t]*
qfact[(a + b + c + d)/2 - t];

(*the sum*)
qF =
```

```
Sum[(-1)^t*qfact[t + 1]/qf, {t,
Max[(a + b + i)/2, (d + c + i)/2, (c + b + j)/2, (a + d + j)/2],
Min[(b + d + i + j)/2, (a + c + i + j)/2, (a + b + c + d)/2]}];

(*quantum 6j symbol by Taylor and Woodward*)

QuantumSymbol =
qTriangle[a, b, i]*qTriangle[c, b, j]*qTriangle[c, d, i]*
qTriangle[a, d, j]*qF
]
```

For the example

$$\begin{Bmatrix} 40 & 48 & 50 \\ 52 & 54 & n \end{Bmatrix}_q,$$

the code to plot the values of the symbols, in Figures 4.2 and 4.3, is given by:

```
(*the step must be equal to 2 since we are working with integers not
half-integers*)
plotQuantum6jsymbol =
ListPlot[Table[{n, N[TWQuantum6jsymbol[40, 48, 50, 52, 54, n, 198]]},
{n, 0, 108, 2}], PlotStyle -> Black]
```

C.1.2 Data needed for the asymptotic

The asymptotic formula for the quantum 6j symbols involves the edge lengths, the exterior dihedral angles, the edge Gram matrix and the volume of its associated spherical tetrahedron. Hence, the code computing these quantities are provided in this subsection.

So, let us firstly start with the definition of the list of the numbers constituting the quantum 6j symbol.

```
Na[n_] := {40, 48, 50, 52, 54, n}
```

```
(*this is the list of the six numbers {m12,m02,m01,m03,m13,m23}
when n varies*)
```

Secondly, the functions generating the lengths $l_{\text{ij}} = \pi\frac{m_{\text{ij}}+1}{k+2}$ of the tetrahedron are:

```
l01[n_, k_] := (Pi*(Na[n][[3]] + 1))/(k + 2)
l02[n_, k_] := (Pi*(Na[n][[2]] + 1))/(k + 2)
l03[n_, k_] := (Pi*(Na[n][[4]] + 1))/(k + 2)
l23[n_, k_] := (Pi*(Na[n][[6]] + 1))/(k + 2)
l13[n_, k_] := (Pi*(Na[n][[5]] + 1))/(k + 2)
l12[n_, k_] := (Pi*(Na[n][[1]] + 1))/(k + 2)
```

Thirdly, the functions defining the edge Gram matrix

$$G = \begin{pmatrix} 1 & \cos\text{l}_{01} & \cos l_{02} & \cos\text{l}_{03} \\ \cos\text{l}_{01} & 1 & \cos\text{l}_{12} & \cos\text{l}_{13} \\ \cos\text{l}_{02} & \cos\text{l}_{12} & 1 & \cos\text{l}_{23} \\ \cos\text{l}_{03} & \cos\text{l}_{13} & \cos\text{l}_{23} & 1 \end{pmatrix}$$

of the tetrahedron and its determinant are

```
G[n_, k_] := {{1, Cos[l01[n, k]], Cos[l02[n, k]],
Cos[l03[n, k]]}, {Cos[l01[n, k]], 1, Cos[l12[n, k]],
Cos[l13[n, k]]}, {Cos[l02[n, k]], Cos[l12[n, k]], 1,
Cos[l23[n, k]]}, {Cos[l03[n, k]], Cos[l13[n, k]], Cos[l23[n, k]], 1}}
det[n_, k_] := Det[G[n, k]]
```

Fourthly, due to some complications that may occur we are going to generate the functions computing the **interior dihedral** angles of T one by one.

To start with, let us define two general functions which will be used throughout. Namely, $f(x) = \sqrt{1 - x^2}$ and that computing the cosine law:

```
f[x_] := Sqrt[1 - x^2]
lcosinelaw[x_, y_, z_] := (Cos[x] - Cos[y]*Cos[z])/(f[Cos[y]]*f[Cos[z]])
```

By direct computation from the cosine law in a tetrahedron, the cosine of the interior dihedral angle

- around the edge (01):

$$\cos \beta_{23} = \frac{\frac{\cos l_{23} - \cos l_{02} \cos l_{03}}{\sin l_{02} \sin l_{03}} - \left(\frac{\cos l_{12} - \cos l_{01} \cos l_{02}}{\sin l_{01} \sin l_{02}}\right)\left(\frac{\cos l_{13} - \cos l_{03} \cos l_{01}}{\sin l_{03} \sin l_{01}}\right)}{\sqrt{1 - \left(\frac{\cos l_{12} - \cos l_{01} \cos l_{02}}{\sin l_{01} \sin l_{02}}\right)^2}\sqrt{1 - \left(\frac{\cos l_{13} - \cos l_{03} \cos l_{01}}{\sin l_{03} \sin l_{01}}\right)^2}}.$$

```
lcosinebeta23[n_, k_] := (lcosinelaw[l23[n, k], l02[n, k], l03[n, k]] -
lcosinelaw[l12[n, k], l01[n, k], l02[n, k]]*
lcosinelaw[l13[n, k], l03[n, k], l01[n, k]])/(f[
lcosinelaw[l12[n, k], l01[n, k], l02[n, k]]]*
f[lcosinelaw[l13[n, k], l03[n, k], l01[n, k]]])
```

Hence, the interior dihedral angle β_{23} is computed by

```
beta23[n_, k_] := ArcCos[lcosinebeta23[n, k]]
```

- Around the edge (03):

$$\cos \beta_{12} = \frac{\frac{\cos l_{12} - \cos l_{02} \cos l_{01}}{\sin l_{02} \sin l_{01}} - \left(\frac{\cos l_{23} - \cos l_{03} \cos l_{02}}{\sin l_{03} \sin l_{02}}\right)\left(\frac{\cos l_{13} - \cos l_{03} \cos l_{01}}{\sin l_{03} \sin l_{01}}\right)}{\sqrt{1 - \left(\frac{\cos l_{23} - \cos l_{03} \cos l_{02}}{\sin l_{03} \sin l_{02}}\right)^2}\sqrt{1 - \left(\frac{\cos l_{13} - \cos l_{03} \cos l_{01}}{\sin l_{03} \sin l_{01}}\right)^2}}.$$

```
lcosinebeta12[n_, k_] := (lcosinelaw[l12[n, k], l02[n, k], l01[n, k]] -
lcosinelaw[l23[n, k], l03[n, k], l02[n, k]]*
lcosinelaw[l13[n, k], l01[n, k], l03[n, k]])/(f[
lcosinelaw[l23[n, k], l03[n, k], l02[n, k]]]*
f[lcosinelaw[l13[n, k], l01[n, k], l03[n, k]]])
```

Hence, the interior dihedral angle β_{12}

```
beta12[n_, k_] := ArcCos[lcosinebeta12[n, k]]
```

- Around the edge (02):

$$\cos\beta_{13} = \frac{\frac{\cos l_{13} - \cos l_{03}\cos l_{01}}{\sin l_{03}\sin l_{01}} - \left(\frac{\cos l_{23} - \cos l_{03}\cos l_{02}}{\sin l_{03}\sin l_{02}}\right)\left(\frac{\cos l_{12} - \cos l_{02}\cos l_{01}}{\sin l_{02}\sin l_{01}}\right)}{\sqrt{1 - \left(\frac{\cos l_{23} - \cos l_{03}\cos l_{02}}{\sin l_{03}\sin l_{02}}\right)^2}\sqrt{1 - \left(\frac{\cos l_{12} - \cos l_{02}\cos l_{01}}{\sin l_{02}\sin l_{01}}\right)^2}}.$$

```
lcosinebeta13[n_, k_] := (lcosinelaw[l13[n, k], l01[n, k], l03[n, k]] -
lcosinelaw[l12[n, k], l02[n, k], l01[n, k]]*
lcosinelaw[l23[n, k], l03[n, k], l02[n, k]])/(f[
lcosinelaw[l12[n, k], l02[n, k], l01[n, k]]]*
f[lcosinelaw[l23[n, k], l03[n, k], l02[n, k]]])
```

Hence, the interior dihedral angle β_{13}

```
beta13[n_, k_] := ArcCos[lcosinebeta13[n, k]]
```

- Around the edge (12):

$$\cos\beta_{03} = \frac{\frac{\cos l_{03} - \cos l_{13}\cos l_{01}}{\sin l_{13}\sin l_{01}} - \left(\frac{\cos l_{23} - \cos l_{13}\cos l_{12}}{\sin l_{13}\sin l_{12}}\right)\left(\frac{\cos l_{02} - \cos l_{12}\cos l_{01}}{\sin l_{12}\sin l_{01}}\right)}{\sqrt{1 - \left(\frac{\cos l_{23} - \cos l_{13}\cos l_{12}}{\sin l_{13}\sin l_{12}}\right)^2}\sqrt{1 - \left(\frac{\cos l_{02} - \cos l_{12}\cos l_{01}}{\sin l_{12}\sin l_{01}}\right)^2}}.$$

```
lcosinebeta03[n_, k_] := (lcosinelaw[l03[n, k], l13[n, k], l01[n, k]] -
lcosinelaw[l02[n, k], l01[n, k], l12[n, k]]*
lcosinelaw[l23[n, k], l12[n, k], l13[n, k]])/(f[
lcosinelaw[l02[n, k], l01[n, k], l12[n, k]]]*
f[lcosinelaw[l23[n, k], l12[n, k], l13[n, k]]])
```

Hence, the interior dihedral angle β_{03}

```
beta03[n_, k_] := ArcCos[lcosinebeta03[n, k]]
```

- Around the edge (13):

$$\cos\beta_{02} = \frac{\frac{\cos l_{02}-\cos l_{12}\cos l_{01}}{\sin l_{12}\sin l_{01}} - \left(\frac{\cos l_{03}-\cos l_{13}\cos l_{01}}{\sin l_{13}\sin l_{01}}\right)\left(\frac{\cos l_{23}-\cos l_{12}\cos l_{13}}{\sin l_{12}\sin l_{13}}\right)}{\sqrt{1-\left(\frac{\cos l_{03}-\cos l_{13}\cos l_{01}}{\sin l_{13}\sin l_{01}}\right)^2}\sqrt{1-\left(\frac{\cos l_{23}-\cos l_{12}\cos l_{13}}{\sin l_{12}\sin l_{13}}\right)^2}}.$$

```
lcosinebeta02[n_, k_] := (lcosinelaw[l02[n, k], l01[n, k], l12[n, k]] -
lcosinelaw[l03[n, k], l13[n, k], l01[n, k]]*
lcosinelaw[l23[n, k], l12[n, k], l13[n, k]])/(f[
lcosinelaw[l03[n, k], l13[n, k], l01[n, k]]]*
f[lcosinelaw[l23[n, k], l12[n, k], l13[n, k]]])
```

Hence, the interior dihedral angle β_{02} is

```
beta02[n_, k_] := ArcCos[lcosinebeta02[n, k]]
```

- Around the edge (23):

$$\cos\beta_{01} = \frac{\frac{\cos l_{01}-\cos l_{13}\cos l_{03}}{\sin l_{13}\sin l_{03}} - \left(\frac{\cos l_{02}-\cos l_{23}\cos l_{03}}{\sin l_{23}\sin l_{03}}\right)\left(\frac{\cos l_{12}-\cos l_{13}\cos l_{23}}{\sin l_{23}\sin l_{13}}\right)}{\sqrt{1-\left(\frac{\cos l_{02}-\cos l_{23}\cos l_{03}}{\sin l_{23}\sin l_{03}}\right)^2}\sqrt{1-\left(\frac{\cos l_{12}-\cos l_{13}\cos l_{23}}{\sin l_{23}\sin l_{13}}\right)^2}}.$$

```
lcosinebeta01[n_, k_] := (lcosinelaw[l01[n, k], l03[n, k], l13[n, k]] -
lcosinelaw[l02[n, k], l03[n, k], l23[n, k]]*
lcosinelaw[l12[n, k], l23[n, k], l13[n, k]])/(f[
lcosinelaw[l02[n, k], l03[n, k], l23[n, k]]]*
f[lcosinelaw[l12[n, k], l23[n, k], l13[n, k]]])
```

Hence, the interior dihedral angle β_{01}

```
beta01[n_, k_] := ArcCos[lcosinebeta01[n, k]]
```

Fifthly, as the angles used in the asymptotic are the **exterior dihedral angles**, as follows are the functions computing them.

```
extbeta01[n_, k_] := Pi - beta01[n, k]
extbeta02[n_, k_] := Pi - beta02[n, k]
extbeta03[n_, k_] := Pi - beta03[n, k]
extbeta12[n_, k_] := Pi - beta12[n, k]
extbeta13[n_, k_] := Pi - beta13[n, k]
extbeta23[n_, k_] := Pi - beta23[n, k]
```

Sixthly, we used formula in [24] to compute the volume of the spherical tetrahedra. That is given by

$$V(\sigma^3) = \mu_3^{-1}\sqrt{|\det(G)|}\int_{(\mathbb{R}^4)_{\geq 0}} e^{-y^tGy}\,dy, \tag{C.1}$$

where $\mu_3 = \int_0^\infty x^3 e^{-x^2}\mathrm{dx} = \frac{1}{2}$ and G is the edge Gram matrix. Here is the function for that

```
t = {t1, t2, t3, t4};
V[n_, k_] := 2*Sqrt[Det[G[n, k]]]*
NIntegrate[Exp[-t.G[n, k].t], {t1, 0, Infinity}, {t2, 0, Infinity}, {t3, 0,
Infinity}, {t4, 0, Infinity}]
```

However, since this computation is a bit costly, we listed all the needed values of the volume at once. The code generating them is given below.

```
(*this was used to generate the values of the volume given below,
however running it several times was costly in time. So, the strategy
was to run it once and keep the values*)

TWvolume=Table[V[n,198],{n,15,86,0.1}]
(*List of volumes with step 0.1*)
```

Then, the values of the volume were copied in a list called `volumeTW`.

Seventhly, the sum

$$\sum_{i<j} \frac{\theta_{\text{ij}}}{2} \left(m_{\text{ij}} + 1\right)$$

is generated by

```
sum[n_, k_] := 1/2*((Na[n][[1]] + 1)*extbeta03[n, k] + (Na[n][[2]] + 1)*
extbeta13[n, k] + (Na[n][[3]] + 1)*extbeta23[n, k] + (Na[n][[4]] + 1)*
extbeta12[n, k] + (Na[n][[5]] + 1)*extbeta02[n, k] + (Na[n][[6]] + 1)*
extbeta01[n, k])
(*list of the values taken by the sum when i varies from 15 to 83 in
steps of 0.1*)
ListSum = Table[sum[i, 198], {i, 15, 83, 0.1}];
```

Eighthly, the list of all the values of the determinant of the edge Gram matrix is provided by:

```
Listdet = Table[det[i, 198], {i, 15, 83, 0.1}];
```

C.1.3 Code to compute the asymptotic of the quantum 6j symbols by directly using Taylor and Woodard's formula

This code generates the asymptotic for the quantum 6j symbol while using the formula

$$\frac{2\pi}{(k+2)^{\frac{3}{2}} \left(\det\left(\cos l_{\text{ij}}\right)\right)^{\frac{1}{4}}} \cos\left(\sum_{i<j} \frac{\theta_{\text{ij}}}{2} \left(m_{\text{ij}} + 1\right) - \frac{k+2}{\pi} V + \frac{\pi}{4}\right). \tag{C.2}$$

```
(*i here is the range for ListSum,  volumeTW and Listdet*)
TWAsymptoticQuantum6jSymbol[i_, k_] := 2*Pi/((k + 2)^(3/2)*
(Listdet[[i]])^(1/4))* Cos[ListSum[[i]] - (k + 2)/Pi*volumeTW[[i]] + Pi/4]
(*the code generating the list of the data points constituting the
asymptotics of the quantum 6j symbols*)
Table[{15 + (i - 1)*0.1, TWAsymptoticQuantum6jSymbol[i, 198]}, {i, 1, 681}];
```

Its plot against the exact values of the quantum 6j symbols is generated by

```
(*The code generating the plot of the asymptotic of the quantum 6j symbols*)
plotTW = ListLinePlot[
Table[{15 + (i - 1)*0.1, TWAsymptoticQuantum6jSymbol[i, 198]}, {i, 1, 681}],
 PlotStyle -> Black];
(*The code generating the plot of the quantum 6j symbols and its
asymptotic in the same plane*)
Show[plotQuantum6jsymbol, plotTW, PlotRange -> All]
```

C.1.4 Code to compute the asymptotic of the quantum 6j symbols by using the corrected version of Taylor and Woodard's formula

This code generates the asymptotic of the quantum 6j symbols while using the formula

$$\frac{2\pi}{(k+2)^{\frac{3}{2}}\left(\det\left(\cos l_{\mathrm{ij}}\right)\right)^{\frac{1}{4}}}\cos\left(\sum_{i<j}\frac{\theta_{\mathrm{ij}}}{2}\left(m_{\mathrm{ij}}+1\right)+\frac{k+2}{\pi}V+\frac{\pi}{4}\right).$$

```
(*i here is the label of ListSum, volumeTW and Listdet*)
meTWAsymptoticQuantum6jSymbol[i_, k_] :=
2*Pi/((k + 2)^(3/2)*(Listdet[[i]])^(1/4))*
Cos[ListSum[[i]] + (k + 2)/Pi*volumeTW[[i]] + Pi/4]
(*the code generating the list of the data points constituting the
 asymptotics of the quantum 6j symbols*)
Table[{n, meTWAsymptoticQuantum6jSymbol[n, 198]}, {n,1, 681}];
```

Its plot against the exact values of the quantum 6j symbols is generated by

```
(*The plot of the corrected asymptotic formula for the quantum 6j symbols*)
meplotTW = ListLinePlot[
Table[{15 + (i - 1)*0.1, meTWAsymptoticQuantum6jSymbol[i, 198]}, {i, 1, 681}],
PlotStyle -> Black];
(*The code to plot the exact values of the quantum 6j symbols
vs the asymptotic formula*)
```

```
Show[plotQuantum6jsymbol, meplotTW, PlotRange -> All]
```

C.1.5 General remarks

C.1.5.1 Volume

As a double check on our volume calculations, we also implemented Murakami's volume formula [29]. Note that this is faster in compiling. To the best of the author's knowledge, this is the first time Murakami's formula was implemented in actual code.

The statement of the formula will follow after the presentation of all the needed ingredients.

Let T be a spherical tetrahedron and β_{ij} be its dihedral angles at edges $l_{\overline{ij}}$ where $(\overline{ij})$ is the direct opposite edge to (ij).

Consider the quantities $a_1 = e^{i\beta_{03}}$, $a_2 = e^{i\beta_{23}}$, $a_3 = e^{i\beta_{02}}$, $a_4 = e^{i\beta_{12}}$, $a_5 = e^{i\beta_{01}}$, $a_6 = e^{i\beta_{13}}$. These allows to define the following variables:

$$
\begin{aligned}
q_0 =& a_1a_2 + a_2a_5 + a_3a_6 + a_1a_2a_6 + a_1a_3a_5 + a_2a_3a_4 + a_4a_5a_6 + a_1a_2a_3a_4a_5a_6, \\
q_1 =& -(a_1 - a_1^{-1})(a_4 - a_4^{-1}) - (a_2 - a_2^{-1})(a_5 - a_5^{-1}) - (a_3 - a_3^{-1})(a_6 - a_6^{-1}), \\
q_2 =& a_1^{-1}a_4^{-1} + a_2^{-1}a_5^{-1} + a_3^{-1}a_6^{-1} + a_1^{-1}a_2^{-1}a_6^{-1} + a_1^{-1}a_3^{-1}a_5^{-1} + a_2^{-1}a_3^{-1}a_4^{-1} \\
&+ a_4^{-1}a_5^{-1}a_6^{-1} + a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1}a_5^{-1}a_6^{-1}.
\end{aligned}
$$

Which enable the expression

$$
z_0 = \frac{-q_1 + \sqrt{q_1^2 - 4q_0q_2}}{2q_2}.
$$

Furthermore, let us consider the integral

$$
\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\, dt
$$

for a real number $x < 1$. Then, let us recall the dilogarithm function defined by analytic continuation of $\mathrm{Li}_2(x)$ and the function

$$\begin{aligned} L(a_1,a_2,a_3,a_4,a_5,a_6,z) =& \frac{1}{2}(\mathrm{Li}_2(z)+\mathrm{Li}_2(a_1^{-1}a_2^{-1}a_4^{-1}a_5^{-1}z)+\mathrm{Li}_2(a_1^{-1}a_3^{-1}a_4^{-1}a_6^{-1}z) \\ &+\mathrm{Li}_2(a_2^{-1}a_3^{-1}a_5^{-1}a_6^{-1}z)-\mathrm{Li}(-a_1^{-1}a_2^{-1}a_3^{-1}z)-\mathrm{Li}_2(-a_1^{-1}a_5^{-1}a_6^{-1}z) \\ &-\mathrm{Li}_2(-a_2^{-1}a_4^{-1}a_6^{-1}z)-\mathrm{Li}_2(-a_3^{-1}a_4^{-1}a_5^{-1}z)+\sum_{j=1}^{3}\log a_j \log a_{j+3}). \end{aligned}$$

Now, the volume of a spherical tetrahedron is given as follows:

Theorem C.1.1 ([29]). *Let T be a spherical tetrahedron and β_{ij} be its dihedral angles at edges $l_{\overline{ij}}$ where $(\overline{ij})$ is the direct opposite edge to (ij). Consider the quantities $a_1 = e^{i\beta_{03}}$, $a_2 = e^{i\beta_{23}}$, $a_3 = e^{i\beta_{02}}$, $a_4 = e^{i\beta_{12}}$, $a_5 = e^{i\beta_{01}}$, $a_6 = e^{i\beta_{13}}$ and $Vol(T)$ be the volume of T. Then,*

$$Vol(T) = -\Re(L(a_1,a_2,a_3,a_4,a_5,a_6,z)) + \pi\left(\arg(-q_2)+\frac{1}{2}\sum_{0=i<j}^{3}\beta_{ij}\right) - \frac{3}{2}\pi^2 \mod 2\pi^2,$$

where $\Re(z)$ is the real part of z and q_2 defined earlier.

The program to compute the volume of a spherical tetrahedron via Murakami's formula is provided below.

```
Vol[l01_, l02_, l03_, l12_, l13_, l23_] :=
Block[{cosinebeta01, cosinebeta02, cosinebeta03, cosinebeta12,
cosinebeta13, cosinebeta23, V, a1, a2, a3, a4, a5, a6, q0, q1, q2, z0, L},
f[x_] := Sqrt[1 - x^2];
lcosinelaw[x_, y_, z_] := (Cos[x] - Cos[y]*Cos[z])/(f[Cos[y]]*f[Cos[z]]);

(*cosine of the dihedral angles in terms of the edge lengths*)

cosinebeta01 = (lcosinelaw[l01, l03, l13] -
lcosinelaw[l02, l03, l23]*lcosinelaw[l12, l23, l13])/
(f[lcosinelaw[l02, l03, l23]]*f[lcosinelaw[l12, l23, l13]]);

cosinebeta02 = (lcosinelaw[l02, l01, l12] -
lcosinelaw[l03, l13, l01]*lcosinelaw[l23, l12, l13])/
```

```
(f[lcosinelaw[l03, l13, l01]]*f[lcosinelaw[l23, l12, l13]]);

cosinebeta03 = (lcosinelaw[l03, l13, l01] -
lcosinelaw[l02, l01, l12]*lcosinelaw[l23, l12, l13])/
(f[lcosinelaw[l02, l01, l12]]*f[lcosinelaw[l23, l12, l13]]);

cosinebeta13 = (lcosinelaw[l13, l01, l03] -
lcosinelaw[l12, l02, l01]*lcosinelaw[l23, l03, l02])/
(f[lcosinelaw[l12, l02, l01]]*f[lcosinelaw[l23, l03, l02]]);

cosinebeta12 = (lcosinelaw[l12, l02, l01] -
lcosinelaw[l23, l03, l02]*lcosinelaw[l13, l01, l03])/
(f[lcosinelaw[l23, l03, l02]]*f[lcosinelaw[l13, l01, l03]]);

cosinebeta23 = (lcosinelaw[l23, l02, l03] -
lcosinelaw[l12, l01, l02]*lcosinelaw[l13, l03, l01])/
(f[lcosinelaw[l12, l01, l02]]*f[lcosinelaw[l13, l03, l01]]);
a1 = cosinebeta03 + I*f[cosinebeta03];
a2 = cosinebeta23 + I*f[cosinebeta23];
a3 = cosinebeta02 + I*f[cosinebeta02];
a4 = cosinebeta12 + I*f[cosinebeta12];
a5 = cosinebeta01 + I*f[cosinebeta01];
a6 = cosinebeta13 + I*f[cosinebeta13];

q0 = a1*a4 + a2*a5 + a3*a6 + a1*a2*a6 + a1*a3*a5 + a2*a3*a4 +
a4*a5*a6 + a1*a2*a3*a4*a5*a6;
q1 = -(a1 - a1^(-1))*(a4 - a4^(-1)) - (a2 - a2^(-1))*(a5 -
a5^(-1)) - (a3 - a3^(-1))*(a6 - a6^(-1));
q2 = a1^(-1)*a4^(-1) + a2^(-1)*a5^(-1) + a3^(-1)*a6^(-1) +
a1^(-1)*a2^(-1)*a6^(-1) + a1^(-1)*a3^(-1)*a5^(-1) +
a2^(-1)*a3^(-1)*a4^(-1) + a4^(-1)*a5^(-1)*a6^(-1) +
```

```
a1^(-1)*a2^(-1)*a3^(-1)*a4^(-1)*a5^(-1)*a6^(-1);

z0 = (-q1 + Sqrt[q1^2 - 4*q0*q2])/(2*q2);

L = 1/2*(PolyLog[2, z0] + PolyLog[2, a1^(-1)*a2^(-1)*a4^(-1)*a5^(-1)*z0] +
PolyLog[2, a1^(-1)*a3^(-1)*a4^(-1)*a6^(-1)*z0] +
PolyLog[2, a2^(-1)*a3^(-1)*a5^(-1)*a6^(-1)*z0] -
PolyLog[2, -a1^(-1)*a2^(-1)*a3^(-1)*z0] -
PolyLog[2, -a1^(-1)*a5^(-1)*a6^(-1)*z0] -
PolyLog[2, -a2^(-1)*a4^(-1)*a6^(-1)*z0] -
PolyLog[2, -a3^(-1)*a4^(-1)*a5^(-1)*z0] + Log[a1]*Log[a4] +
Log[a2]*Log[a5] + Log[a3]*Log[a6]);

V = -Re[L] + Pi*(Arg[-q2] + 1/2*(ArcCos[cosinebeta01] + ArcCos[cosinebeta02] +
ArcCos[cosinebeta03] + ArcCos[cosinebeta12] +
ArcCos[cosinebeta13] + ArcCos[cosinebeta23])) - 3/2*Pi^2]
```

The volume computed by using Luo's formula (C.1) and that of Murakami in Theorem C.1.1 agree. The intention is to plot them in the same plane. So, to start with, let us procure the list of the volumes obtained from using Murakami's method by considering the same edge lengths as those present in Luo's method.

```
Table[Vol[l01[n, k], l02[n, k], l03[n, k], l12[n, k], l13[n, k],
l23[n, k]], {n, 15, 86, 0.1}]
```

Hence, the figure comparing the two volumes is given by:

```
ListLinePlot[{volumeTW, Murakamivolume}, PlotLabels -> Automatic,
PlotLegends -> {Luo, Murakami}]
```

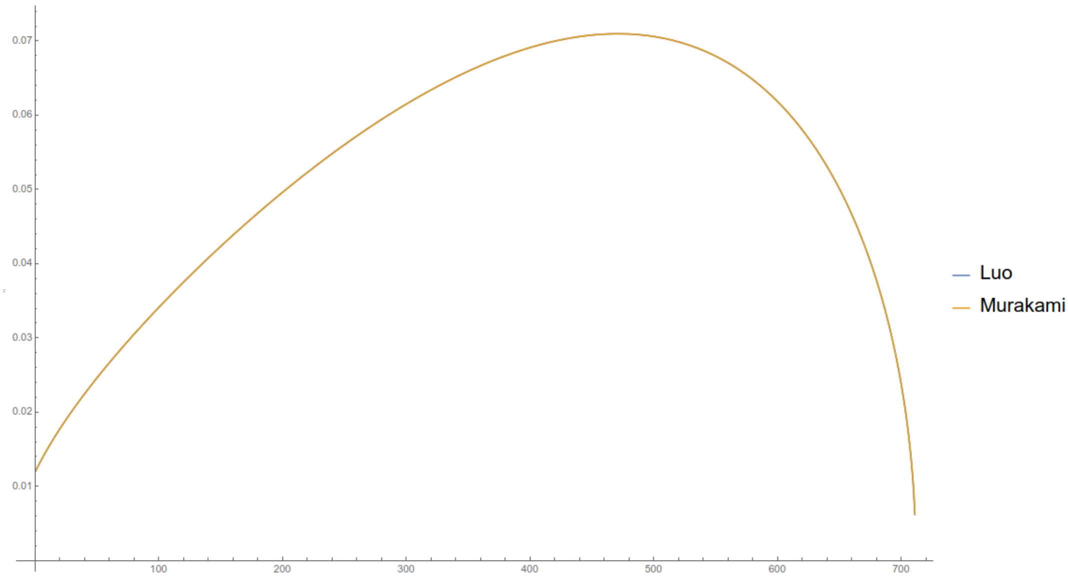


Figure C.1: Graphs of the volume via Luo vs Murakami's method. The two graphs are identical.

So, as presented here the two methods coincide.

C.1.5.2 Methods

Our process of implementing the asymptotic for the quantum 6j symbols differs from that of Taylor and Woodward at https://sites.math.rutgers.edu/ ctw/6j.html

In fact, they used the Schläfli formula to approximate the volume while we are computing its exact value each time.

C.2 Asymptotic of the quantum 6j symbols via Roberts

From Roberts' paper [36, Section 6], a second way of stating the asymptotic formula for the quantum 6j symbols is presented and it reads as follows:

Let r_{ij} be elements of $\mathbb{Q} \cap [0,1]$ where $i, j \in \{0,1,2,3\}$. Let T be a spherical tetrahedron with edge lengths $l_{ij} = \pi r_{ij}$ and with associated exterior dihedral angles θ_{ij} at the edge $(\overline{ij})$, direct opposite to the edge (ij). Let V be the volume of T and G its edge Gram matrix. Then, Roberts' corrected (recall from Section 4.3 the sign error for the volume term) formula is given by

$$\begin{Bmatrix} kr_{12} & kr_{02} & kr_{01} \\ kr_{03} & kr_{13} & kr_{23} \end{Bmatrix}_{q=e^{\frac{\pi i}{k+2}}} \sim \sqrt{\frac{4\pi^2}{k^3\sqrt{\det G}}} \cos\left(\sum_{i<j}(kr_{ij}+1)\frac{\theta_{\overline{ij}}}{2} + \frac{(k+2)}{\pi}V + \frac{\pi}{4}\right).$$

When using large numbers, the asymptotic formula for the quantum 6j symbols by Roberts translates into the approximation

$$\begin{Bmatrix} m_{12} & m_{02} & m_{01} \\ m_{03} & m_{13} & m_{23} \end{Bmatrix}_{q=e^{\frac{\pi i}{k+2}}} \cong \sqrt{\frac{4\pi^2}{k^3\sqrt{\det G}}} \cos\left(\sum_{i<j}(m_{ij}+1)\frac{\theta_{\overline{ij}}}{2} + \frac{(k+2)}{\pi}V + \frac{\pi}{4}\right). \tag{C.3}$$

For example, consider

$$\begin{Bmatrix} 40 & 48 & 50 \\ 52 & 54 & n \end{Bmatrix}_{q=e^{\frac{\pi i}{k+2}}},$$

where $k = 198$. Let us plot its asymptotic.

Let us recall

```
Na[n_] := {40, 48, 50, 52, 54, n}
```

To start with, the functions generating the edge lengths $l_{ij} = \pi\frac{m_{ij}}{k}$ of the tetrahedron are:

```
rl01[n_, k_] := (Pi*Na[n][[3]])/k
rl02[n_, k_] := (Pi*Na[n][[2]])/k
rl03[n_, k_] := (Pi*Na[n][[4]])/k
rl23[n_, k_] := (Pi*Na[n][[6]])/k
rl13[n_, k_] := (Pi*Na[n][[5]])/k
rl12[n_, k_] := (Pi*Na[n][[1]])/k
```

Secondly, the edge Gram matrix and its determinant are provided by

```
rG[n_, k_] := {{1, Cos[rl01[n, k]], Cos[rl02[n, k]], Cos[rl03[n, k]]},
 {Cos[rl01[n, k]], 1, Cos[rl12[n, k]], Cos[rl13[n, k]]},
{Cos[rl02[n, k]], Cos[rl12[n, k]], 1, Cos[rl23[n, k]]},
{Cos[rl03[n, k]], Cos[rl13[n, k]], Cos[rl23[n, k]], 1}}
rdet[n_, k_] := Det[rG[n, k]]
```

Thirdly, similarly to the case of the computation via Taylor and Woodward's method, we will compute the interior dihedral angles one by one. The functions `f` and `cosinelaw` are the same as in Subsection C.1.2.

- Around the edge (01):

```
rcosinebeta23[n_, k_] := (lcosinelaw[rl23[n, k], rl02[n, k], rl03[n, k]] -
lcosinelaw[rl12[n, k], rl01[n, k], rl02[n, k]]*
lcosinelaw[rl13[n, k], rl03[n, k], rl01[n, k]])/(f[
lcosinelaw[rl12[n, k], rl01[n, k], rl02[n, k]]]*
f[lcosinelaw[rl13[n, k], rl03[n, k], rl01[n, k]]])
```

  Hence, the interior dihedral angle β_{23} is computed by

```
rbeta23[n_, k_] := ArcCos[rcosinebeta23[n, k]]
```

- Around the edge (03):

```
rcosinebeta12[n_, k_] := (lcosinelaw[rl12[n, k], rl02[n, k], rl01[n, k]] -
lcosinelaw[rl23[n, k], rl03[n, k], rl02[n, k]]*
lcosinelaw[rl13[n, k], rl01[n, k], rl03[n, k]])/(f[
lcosinelaw[rl23[n, k], rl03[n, k], rl02[n, k]]]*
f[lcosinelaw[rl13[n, k], rl01[n, k], rl03[n, k]]])
```

  Hence, the interior dihedral angle β_{12} is given by

```
rbeta12[n_, k_] := ArcCos[rcosinebeta12[n, k]]
```

- Around the edge (02)

```
rcosinebeta13[n_, k_] := (lcosinelaw[rl13[n, k], rl01[n, k], rl03[n, k]] -
lcosinelaw[rl12[n, k], rl02[n, k], rl01[n, k]]*
lcosinelaw[rl23[n, k], rl03[n, k], rl02[n, k]])/(f[
lcosinelaw[rl12[n, k], rl02[n, k], rl01[n, k]]]*
f[lcosinelaw[rl23[n, k], rl03[n, k], rl02[n, k]]])
```

Hence, the interior dihedral angle β_{13} is computed by

```
rbeta13[n_, k_] := ArcCos[rcosinebeta13[n, k]]
```

- Around the edge (12):

```
rcosinebeta03[n_, k_] := (lcosinelaw[rl03[n, k], rl13[n, k], rl01[n, k]] -
lcosinelaw[rl02[n, k], rl01[n, k], rl12[n, k]]*
lcosinelaw[rl23[n, k], rl12[n, k], rl13[n, k]])/(f[
lcosinelaw[rl02[n, k], rl01[n, k], rl12[n, k]]]*
f[lcosinelaw[rl23[n, k], rl12[n, k], rl13[n, k]]])
```

Hence, the interior dihedral angle β_{03} is provided by

```
rbeta03[n_, k_] := ArcCos[rcosinebeta03[n, k]]
```

- Around the edge (13):

```
rcosinebeta02[n_, k_] := (lcosinelaw[rl02[n, k], rl01[n, k], rl12[n, k]] -
lcosinelaw[rl03[n, k], rl13[n, k], rl01[n, k]]*
lcosinelaw[rl23[n, k], rl12[n, k], rl13[n, k]])/(f[
lcosinelaw[rl03[n, k], rl13[n, k], rl01[n, k]]]*
f[lcosinelaw[rl23[n, k], rl12[n, k], rl13[n, k]]])
```

Hence, the interior dihedral angle β_{02} is given by

```
rbeta02[n_, k_] := ArcCos[rcosinebeta02[n, k]]
```

- Around the edge (23):

```
rcosinebeta01[n_, k_] := (lcosinelaw[rl01[n, k], rl03[n, k], rl13[n, k]] -
lcosinelaw[rl02[n, k], rl03[n, k], rl23[n, k]]*
lcosinelaw[rl12[n, k], rl23[n, k], rl13[n, k]])/(f[
lcosinelaw[rl02[n, k], rl03[n, k], rl23[n, k]]]*
f[lcosinelaw[rl12[n, k], rl23[n, k], rl13[n, k]]])
```

Hence, the interior dihedral angle β_{01} is given by

```
rbeta01[n_, k_] := ArcCos[rcosinebeta01[n, k]]
```

Fourthly, the exterior dihedral angles are computed by

```
rextbeta01[n_, k_] := Pi - rbeta01[n, k]
rextbeta02[n_, k_] := Pi - rbeta02[n, k]
rextbeta03[n_, k_] := Pi - rbeta03[n, k]
rextbeta12[n_, k_] := Pi - rbeta12[n, k]
rextbeta13[n_, k_] := Pi - rbeta13[n, k]
rextbeta23[n_, k_] := Pi - rbeta23[n, k]
```

Fifthly, the volume of the spherical tetrahedron via Luo's method is calculated from

```
t = {t1, t2, t3, t4};
rV[n_, k_] := 2*Sqrt[Det[rG[n, k]]]*
NIntegrate[Exp[-t.rG[n, k].t], {t1, 0, Infinity}, {t2, 0, Infinity}, {t3, 0,
Infinity}, {t4, 0, Infinity}]
```

Since the calculation of the volume is costly in time, we computed all the volumes at once and put it as a list called `volumeR`. The code generating the volumes is given by

```
Rvolume=Table[rV[n,198],{n,15,86,0.1}]
```

Sixthly, the sum

$$\sum_{i<j}(m_{ij}+1)\frac{\theta_{\overline{ij}}}{2}$$

is implemented from

```
rsum[n_, k_] :=
1/2*((Na[n][[1]] + 1)*rextbeta03[n, k] + (Na[n][[2]] + 1)*rextbeta13[n, k]
+ (Na[n][[3]] + 1)*rextbeta23[n, k]+ (Na[n][[4]] + 1)*rextbeta12[n, k]
+ (Na[n][[5]] + 1)*rextbeta02[n, k] + (Na[n][[6]] + 1)*rextbeta01[n, k])
```

```
Listrsum = Table[rsum[n, 198], {n, 15, 86, 0.1}];
```

Seventhly, the list of all the values of the determinant of the edge Gram matrix when n varies from 15 to 86 in steps of 0.1 is procured from

```
Listrdet = Table[rdet[n, 198], {n, 15, 86, 0.1}];
```

Lastly, the asymptotic of the quantum 6j symbol is given by:

```
RAsymptoticQuantum6jSymbol[i_, k_] := 2*Pi/((k)^(3/2)*(Listrdet[[i]])^(1/4))*
Cos[Listrsum[[i]] + (k+2)/Pi*volumeR[[i]] + Pi/4]
(*here i is the label of the tables Listrdet, Listrsum and volumeR*)
```

The code generating the plot of the asymptotic quantum 6j symbols via Roberts' formula is

```
(*List of the data points for the asymptotic of the quantum 6j symbols*)
meRAsymptoticTable =
Table[{15 + (i - 1)*0.1, meRAsymptoticQuantum6jSymbol[i, 198]}, {i, 1, 681}]
(*The graph of the asymptotic of the quantum 6j symbols*)
meplotRoberts =
ListLinePlot[meRAsymptoticTable, PlotLabels -> "Roberts", PlotStyle -> Red]
```

Furthermore, the code implementing the graphs of the exact values of the quantum 6j symbols compared to the asymptotic via Roberts' statement of the formula is

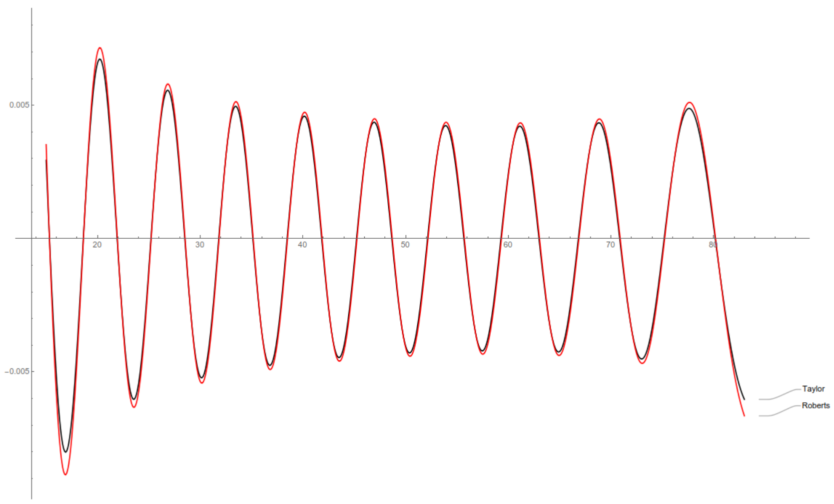
```
Show[plotQuantum6jsymbol, meplotRoberts, PlotRange -> All]
```

C.3 Comparison between Taylor and Woodward's asymptotic formula with that of Roberts'

In this section will be shown the comparison between the graphs of the asymptotic formula for the quantum 6j symbols by using Taylor and Woodward's versus by using Roberts'. That plot is obtained by the following code.

```
Show[meplotTW, meplotRoberts, PlotRange -> All, PlotLabels -> Automatic]
```

Which generates the graph:



From this, we see that the non-degenerate asymptotic formula for the classical 6j symbols by Roberts (C.3) is asymptotically equivalent to that of Taylor and Woodward (C.2).

C.4 The signature of the Hessian matrix

C.4.1 The Hessian matrix

A general program generating the Hessian matrix which appears in the stationary phase approximation of the integral $I_c(k)$ in Section 4.5 is provided below.

```
(*This Hessian matrix is in function of theta01, theta02, theta03,
theta12, theta13, theta23 where thetaij = Pi-betaij, betaij are the
interior dihedral angles at the edge opposite to ij*)
(*I am not making it as a
function since it contains several derivatives which I would like to
evaluate later and which can be done by assigning a value in the end*)


NewHessian =
Block[{cosinel01, cosinel02, cosinel03, cosinel12, cosinel13,
cosinel23, listderivative, dcosinel12beta01, dcosinel12beta02,
dcosinel12beta03, dcosinel12beta12, dcosinel12beta13,
dcosinel12beta23, dcosinel13beta01, dcosinel13beta02,
dcosinel13beta03, dcosinel13beta12, dcosinel13beta13,
dcosinel13beta23, dcosinel23beta01, dcosinel23beta02,
dcosinel23beta03, dcosinel23beta12, dcosinel23beta13,
dcosinel23beta23, dcosinel03beta01, dcosinel03beta02,
dcosinel03beta03, dcosinel03beta12, dcosinel03beta13,
dcosinel03beta23, dcosinel02beta01, dcosinel02beta02,
dcosinel02beta03, dcosinel02beta12, dcosinel02beta13,
dcosinel02beta23, dcosinel01beta01, dcosinel01beta02,
dcosinel01beta03, dcosinel01beta12, dcosinel01beta13,
dcosinel01beta23, H, dl12beta01, dl12beta02, dl12beta03,
dl12beta12, dl12beta13, dl12beta23, dl13beta01, dl13beta02,
dl13beta03, dl13beta12, dl13beta13, dl13beta23, dl23beta01,
dl23beta02, dl23beta03, dl23beta12, dl23beta13, dl23beta23,
dl03beta01, dl03beta02, dl03beta03, dl03beta12, dl03beta13,
dl03beta23, dl02beta01, dl02beta02, dl02beta03, dl02beta12,
dl02beta13, dl02beta23, dl01beta01, dl01beta02, dl01beta03,
dl01beta12, dl01beta13, dl01beta23},
f[x_] := Sqrt[1 - x^2];
```

```
cosinelaw[a_, b_, c_] := (Cos[a] + Cos[b] *Cos[c])/(f[Cos[b]]*f[Cos[c]]);

(*the cosines of the edge lengths of the tetrahedron as functions
of the interior dihedral angles,
their respective derivatives with respect to the interior dihedral
angles and the derivatives of edge lengths with respect to the
interior dihedral angles*)

cosinel12 = (cosinelaw[betajk, betajl, betakl] +
cosinelaw[betaik, betail, betakl]*cosinelaw[betaij, betajl, betail])/(f[
cosinelaw[betaik, betail, betakl]]*f[cosinelaw[betaij, betajl, betail]]);
dcosinel12beta01 = D[cosinel12, betaij];
dcosinel12beta02 = D[cosinel12, betaik];
dcosinel12beta03 = D[cosinel12, betail];
dcosinel12beta12 = D[cosinel12, betajk];
dcosinel12beta13 = D[cosinel12, betajl];
dcosinel12beta23 = D[cosinel12, betakl];
dl12beta01 = -1/f[cosinel12]*dcosinel12beta01;
dl12beta02 = -1/f[cosinel12]*dcosinel12beta02;
dl12beta03 = -1/f[cosinel12]*dcosinel12beta03;
dl12beta12 = -1/f[cosinel12]*dcosinel12beta12;
dl12beta13 = -1/f[cosinel12]*dcosinel12beta13;
dl12beta23 = -1/f[cosinel12]*dcosinel12beta23;

cosinel02 = (cosinelaw[betaik, betakl, betail] +
cosinelaw[betajk, betajl, betakl]*cosinelaw[betaij, betail, betajl])/(f[
cosinelaw[betajk, betajl, betakl]]*f[cosinelaw[betaij, betail, betajl]]);
dcosinel02beta01 = D[cosinel02, betaij];
dcosinel02beta02 = D[cosinel02, betaik];
dcosinel02beta03 = D[cosinel02, betail];
dcosinel02beta12 = D[cosinel02, betajk];
```

```
dcosinel02beta13 = D[cosinel02, betajl];
dcosinel02beta23 = D[cosinel02, betakl];
dl02beta01 = -1/f[cosinel02]*dcosinel02beta01;
dl02beta02 = -1/f[cosinel02]*dcosinel02beta02;
dl02beta03 = -1/f[cosinel02]*dcosinel02beta03;
dl02beta12 = -1/f[cosinel02]*dcosinel02beta12;
dl02beta13 = -1/f[cosinel02]*dcosinel02beta13;
dl02beta23 = -1/f[cosinel02]*dcosinel02beta23;


cosinel01 = (cosinelaw[betaij, betail, betajl] +
cosinelaw[betajk, betajl, betakl]*cosinelaw[betaik, betail, betakl])/(f[
cosinelaw[betajk, betajl, betakl]]*f[cosinelaw[betaik, betail, betakl]]);
dcosinel01beta01 = D[cosinel01, betaij];
dcosinel01beta02 = D[cosinel01, betaik];
dcosinel01beta03 = D[cosinel01, betail];
dcosinel01beta12 = D[cosinel01, betajk];
dcosinel01beta13 = D[cosinel01, betajl];
dcosinel01beta23 = D[cosinel01, betakl];
dl01beta01 = -1/f[cosinel01]*dcosinel01beta01;
dl01beta02 = -1/f[cosinel01]*dcosinel01beta02;
dl01beta03 = -1/f[cosinel01]*dcosinel01beta03;
dl01beta12 = -1/f[cosinel01]*dcosinel01beta12;
dl01beta13 = -1/f[cosinel01]*dcosinel01beta13;
dl01beta23 = -1/f[cosinel01]*dcosinel01beta23;


cosinel13 = (cosinelaw[betajl, betajk, betakl] +
cosinelaw[betail, betaik, betakl]*cosinelaw[betaij, betaik, betajk])/(f[
cosinelaw[betail, betaik, betakl]]*f[cosinelaw[betaij, betaik, betajk]]);
dcosinel13beta01 = D[cosinel13, betaij];
```

```
dcosinel13beta02 = D[cosinel13, betaik];
dcosinel13beta03 = D[cosinel13, betail];
dcosinel13beta12 = D[cosinel13, betajk];
dcosinel13beta13 = D[cosinel13, betajl];
dcosinel13beta23 = D[cosinel13, betakl];
dl13beta01 = -1/f[cosinel13]*dcosinel13beta01;
dl13beta02 = -1/f[cosinel13]*dcosinel13beta02;
dl13beta03 = -1/f[cosinel13]*dcosinel13beta03;
dl13beta12 = -1/f[cosinel13]*dcosinel13beta12;
dl13beta13 = -1/f[cosinel13]*dcosinel13beta13;
dl13beta23 = -1/f[cosinel13]*dcosinel13beta23;


cosinel03 = (cosinelaw[betail, betaik, betakl] +
cosinelaw[betajl, betajk, betakl]*cosinelaw[betaij, betaik, betajk])/(f[
cosinelaw[betajl, betajk, betakl]]*f[cosinelaw[betaij, betaik, betajk]]);
dcosinel03beta01 = D[cosinel03, betaij];
dcosinel03beta02 = D[cosinel03, betaik];
dcosinel03beta03 = D[cosinel03, betail];
dcosinel03beta12 = D[cosinel03, betajk];
dcosinel03beta13 = D[cosinel03, betajl];
dcosinel03beta23 = D[cosinel03, betakl];
dl03beta01 = -1/f[cosinel03]*dcosinel03beta01;
dl03beta02 = -1/f[cosinel03]*dcosinel03beta02;
dl03beta03 = -1/f[cosinel03]*dcosinel03beta03;
dl03beta12 = -1/f[cosinel03]*dcosinel03beta12;
dl03beta13 = -1/f[cosinel03]*dcosinel03beta13;
dl03beta23 = -1/f[cosinel03]*dcosinel03beta23;


cosinel23 = (cosinelaw[betakl, betajk, betajl] +
```

```
cosinelaw[betail, betaij, betajl]*cosinelaw[betaik, betaij, betajk])/(f[
cosinelaw[betail, betaij, betajl]]*f[cosinelaw[betaik, betaij, betajk]]);
dcosinel23beta01 = D[cosinel23, betaij];
dcosinel23beta02 = D[cosinel23, betaik];
dcosinel23beta03 = D[cosinel23, betail];
dcosinel23beta12 = D[cosinel23, betajk];
dcosinel23beta13 = D[cosinel23, betajl];
dcosinel23beta23 = D[cosinel23, betakl];
dl23beta01 = -1/f[cosinel23]*dcosinel23beta01;
dl23beta02 = -1/f[cosinel23]*dcosinel23beta02;
dl23beta03 = -1/f[cosinel23]*dcosinel23beta03;
dl23beta12 = -1/f[cosinel23]*dcosinel23beta12;
dl23beta13 = -1/f[cosinel23]*dcosinel23beta13;
dl23beta23 = -1/f[cosinel23]*dcosinel23beta23;


(*the Hessian matrix*)
(*what we are looking for is the
derivative of edge lengths with respect to thetaij which is Pi-
betaij. So, the Hessian matrix is given by:*)

H = {{-dl23beta01, -dl23beta02, -dl23beta03, -dl23beta12,
-dl23beta13, -dl23beta23}, {-dl13beta01, -dl13beta02, -dl13beta03,
-dl13beta12, -dl13beta13, -dl13beta23}, {-dl12beta01, -dl12beta02,
-dl12beta03, -dl12beta12, -dl12beta13, -dl12beta23}, {-dl03beta01,
-dl03beta02, -dl03beta03, -dl03beta12, -dl03beta13, -dl03beta23},
{-dl02beta01, -dl02beta02, -dl02beta03, -dl02beta12, -dl02beta13,
-dl02beta23}, {-dl01beta01, -dl01beta02, -dl01beta03, -dl01beta12,
-dl01beta13, -dl01beta23}}
]
```

Remark C.4.1.
- *Here, all the necessary derivatives are computed one by one to be careful. However, since the matrix is symmetric the number of implementation may be reduced to 21 derivatives instead of 36.*
- *One may as well compute the Hessian matrix by using the results from [25] as well as those presented in [31].*

C.4.2 The signature

The signature of a matrix is known to be the number of positive eigenvalues subtracted with the number of negative eigenvalues of the matrix. In this subsection, let us look at one example: the signature of the Hessian matrix associated to

$$\begin{Bmatrix} 40 & 48 & 50 \\ 52 & 54 & 20 \end{Bmatrix}_{q=e^{\frac{i\pi}{k+2}}}$$

where $k = 198$.

Let $m_{12} = 40$, $m_{02} = 48$, $m_{01} = 50$, $m_{03} = 52$, $m_{13} = 54$, $m_{23} = 20$. The edge lengths of the tetrahedron associated to the quantum 6j symbols are computed by

```
(*the edge lengths of the tetrahedron T*)
l01 = Pi*m01/k;
l02 = Pi*m02/k;
l03 = Pi*m03/k;
l12 = Pi*m12/k;
l13 = Pi*m13/k;
l23 = Pi*m23/k;
```

Since the Hessian matrix is a function of the interior dihedral angles, the following code computes the interior dihedral angles as functions of the edge lengths.

```
(*computation of the interior dihedral angles as functions of the edge
lengths,
it returns the dihedral angles {beta01, beta02, beta03, beta12,
beta13, beta23}
```

```
where eg beta03 is the interior dihedral angle at the edge 12*)

TDihedralAngle[l01_, l02_, l03_, l12_, l13_, l23_] :=
Block[{cosinebeta01, cosinebeta02, cosinebeta03, cosinebeta12,
cosinebeta13, cosinebeta23, listdihedralangle},
f[x_] := Sqrt[1 - x^2];
lcosinelaw[x_, y_, z_] := (Cos[x] - Cos[y]*Cos[z])/(f[Cos[y]]*f[Cos[z]]);

(*cosine of the dihedral angles in terms of the edge lengths*)

cosinebeta01 = (lcosinelaw[l01, l03, l13] -
lcosinelaw[l02, l03, l23]*lcosinelaw[l12, l23, l13])/(f[
lcosinelaw[l02, l03, l23]]*f[lcosinelaw[l12, l23, l13]]);

cosinebeta02 = (lcosinelaw[l02, l01, l12] -
lcosinelaw[l03, l13, l01]*lcosinelaw[l23, l12, l13])/(f[
lcosinelaw[l03, l13, l01]]*f[lcosinelaw[l23, l12, l13]]);

cosinebeta03 = (lcosinelaw[l03, l13, l01] -
lcosinelaw[l02, l01, l12]*lcosinelaw[l23, l12, l13])/(f[
lcosinelaw[l02, l01, l12]]*f[lcosinelaw[l23, l12, l13]]);

cosinebeta13 = (lcosinelaw[l13, l01, l03] -
lcosinelaw[l12, l02, l01]*lcosinelaw[l23, l03, l02])/(f[
lcosinelaw[l12, l02, l01]]*f[lcosinelaw[l23, l03, l02]]);

cosinebeta12 = (lcosinelaw[l12, l02, l01] -
lcosinelaw[l23, l03, l02]*lcosinelaw[l13, l01, l03])/(f[
lcosinelaw[l23, l03, l02]]*f[lcosinelaw[l13, l01, l03]]);

cosinebeta23 = (lcosinelaw[l23, l02, l03] -
```

```
lcosinelaw[l12, l01, l02]*lcosinelaw[l13, l03, l01])/(f[
lcosinelaw[l12, l01, l02]]*f[lcosinelaw[l13, l03, l01]]);
listdihedralangle = {N[ArcCos[cosinebeta01]],N[ArcCos[cosinebeta02]],
 N[ArcCos[cosinebeta03]], N[ArcCos[cosinebeta12]],
  N[ArcCos[cosinebeta13]], N[ArcCos[cosinebeta23]]}
]
```

For our example, the interior dihedral angles are given by

```
In[15]:= TDihedralAngle[l01, l02, l03, l12, l13, l23]


Out[15]= {1.03678, 1.83613, 1.18779, 0.67861, 2.29937, 0.340975}
```

In other words,

```
beta01 = 1.0367806072139365;
beta02 = 1.836132438028897;
beta03 = 1.187786192293169;
beta12 = 0.6786100199178243;
beta13 = 2.299365644070834;
beta23 = 0.34097497062735044;
```

Therefore, the Hessian matrix is computed by

```
H1 = NewHessian /. {betaij -> beta01, betaik -> beta02,
betail -> beta03, betajk -> beta12, betajl -> beta13,
betakl -> beta23};
MatrixForm[H1]
```

which results to

Out[26]//MatrixForm=

$$\begin{pmatrix} -0.515599 & -1.75775 & -1.42568 & -1.64899 & -1.96729 & -2.36549 \\ -1.75775 & -4.7258 & -3.56259 & -5.79431 & -5.91701 & -4.6136 \\ -1.42568 & -3.56259 & -2.51174 & -4.7847 & -4.4103 & -2.94345 \\ -1.64899 & -5.79431 & -4.7847 & -4.17842 & -4.16245 & -3.90641 \\ -1.96729 & -5.91701 & -4.4103 & -4.16245 & -3.87617 & -3.66589 \\ -2.36549 & -4.6136 & -2.94345 & -3.90641 & -3.66589 & -3.0719 \end{pmatrix}$$

And its eigenvalues are procured from

```
In[9]:= Eigenvalues[H1]
```

```
Out[9]= {-22.2373, 2.81259, 1.24052, -0.808616, 0.166714, -0.053502}
```

This shows that the signature of the Hessian matrix is null.

List of References

[1] S. Alexander. Michael Spivak, A comprehensive introduction to differential geometry. *Bulletin of the American Mathematical Society*, 84(1):27–32, 1978.

[2] V. Aquilanti, H. M. Haggard, A. Hedeman, N. Jeevanjee, R. G. Littlejohn, and L. Yu. Semiclassical mechanics of the Wigner 6j-symbol. *Journal of Physics A: Mathematical and Theoretical*, 45(6):065209, 2012.

[3] J. Baez and T. Bartels. Quantum gravity seminar- fall 2000. *available at math. ucr. edu/home/baez/qg-fall2000*.

[4] J. C. Baez, J. D. Christensen, and G. Egan. Asymptotics of 10j symbols. *Classical and Quantum Gravity*, 19(24):6489, 2002.

[5] J. W. Barrett. The Classical evaluation of relativistic spin networks. *Adv. Theor. Math. Phys.*, 2:593–600, 1998.

[6] J. W. Barrett and C. M. Steele. Asymptotics of relativistic spin networks. *Classical and Quantum Gravity*, 20(7):1341, 2003.

[7] B. Bartlett. Numerical approximation of possible integral formula for quantum 6j symbols. *available at math.sun.ac.za/bartlett/assets/napprox.zip*.

[8] B. Bartlett and V. H. Ranaivomanana. Reciprocity of the Wigner derivative for spherical tetrahedra. *arXiv preprint arXiv:2012.10609*, 2020.

[9] L. C. Biedenharn and J. D. Louck. *The Racah-Wigner algebra in quantum theory*. Addison-Wesley, 1981.

[10] R. Bott, L. W. Tu, et al. *Differential forms in algebraic topology*, volume 82. Springer, 1982.

[11] C. Woodward. Personal communication.

[12] J. S. Carter, D. E. Flath, M. Saito, J. S. Stein, and M. Griffiths. *The classical and quantum 6j-symbols*, volume 43. Princeton University Press, 1995.

[13] N. Chako. Asymptotic expansions of double and multiple integrals occurring in diffraction theory. *IMA Journal of Applied Mathematics*, 1(4):372–422, 1965.

[14] Q. Chen and T. Yang. Volume conjectures for the Reshetikhin–Turaev and the Turaev–Viro invariants. *Quantum Topology*, 9(3):419–460, 2018.

[15] J. Cooke. Stationary phase in two dimensions. *IMA Journal of Applied Mathematics*, 29(1):25–37, 1982.

[16] M. J. Costantino, Francesco. Generating series and asymptotics of classical spin networks. *Journal of the European Mathematical Society*, 017(10):2417–2452, 2015.

[17] L. Freidel and D. Louapre. Asymptotics of 6j and 10j symbols. *Classical and Quantum Gravity*, 20(7):1267, 2003.

[18] S. Garoufalidis and R. Van Der Veen. Asymptotics of classical spin networks. *Geometry & Topology*, 17(1):1–37, 2013.

[19] W. Kamiński and S. Steinhaus. Coherent states, 6 j symbols and properties of the next to leading order asymptotic expansions. *Journal of Mathematical Physics*, 54(12):121703, 2013.

[20] L. H. Kauffman and S. Lins. *Temperley-Lieb Recoupling Theory and Invariants of 3-Manifolds (AM-134), Volume 134*. Princeton University Press, 2016.

[21] A. Kolpakov, A. Mednykh, and M. Pashkevich. Volume formula for a Z 2-symmetric spherical tetrahedron through its edge lengths. *Arkiv för Matematik*, 51(1):99–123, 2013.

[22] Y. Kosmann-Schwarzbach et al. *Groups and symmetries*. Springer, 2010.

[23] F. Luo. On a problem of Fenchel. *Geometriae Dedicata*, 64(3):277–282, 1997.

[24] F. Luo. Continuity of the volume of simplices in classical geometry. *Communications in Contemporary Mathematics*, 8(03):411–431, 2006.

[25] F. Luo. 3-dimensional Schläfli formula and its generalization. *Communications in Contemporary Mathematics*, 10(01):835–842, 2008.

[26] I. H. Madsen, J. Tornehave, et al. *From calculus to cohomology: de Rham cohomology and characteristic classes*. Cambridge University Press, 1997.

[27] J. Marché and T. Paul. Toeplitz operators in tqft via skein theory. *Transactions of the American Mathematical Society*, 367(5):3669–3704, 2015.

[28] J. W. Milnor. *Collected Papers of John Milnor*. American Mathematical Soc., 1994.

[29] J. Murakami. Volume formulas for a spherical tetrahedron. *Proceedings of the American Mathematical Society*, 140(9):3289–3295, 2012.

[30] R. Penrose. Applications of negative dimensional tensors. *Combinatorial mathematics and its applications*, 1:221–244, 1971.

[31] M. Petrera and Y. B. Suris. Spherical geometry and integrable systems. *Geometriae Dedicata*, 169(1):83–98, 2014.

[32] G. Ponzano and T. Regge. Semiclassical limit of Racah coefficients. Technical report, Princeton Univ., NJ, 1969.

[33] V. H. Ranaivomanana. Mathematica code related to classical and quantum 6j symbols. *available at https://sites.google.com/aims.ac.za/hosana/project-page*.

[34] V. H. Ranaivomanana. Euler classes and Frobenius algebras. Master's thesis, Stellenbosch: Stellenbosch University, 2019.

[35] J. Roberts. Classical 6j-symbols and the tetrahedron. *Geometry & Topology*, 3(1):21–66, 1999.

[36] J. Roberts. Asymptotics and 6j-symbols. *Geometry & Topology Monographs*, 4:245–261, 2002.

[37] J.-P. Serre. *Linear representations of finite groups*, volume 42. Springer, 1977.

[38] Y. U. Taylor. *Quantum 6j symbols and a semiclassical invariant of three-manifolds*. Rutgers The State University of New Jersey-New Brunswick, 2003.

[39] Y. U. Taylor and C. Woodward. Numerical approximation of the quantum 6j symbols. *available at https://sites.math.rutgers.edu/ ctw/6j.html*.

[40] Y. U. Taylor and C. T. Woodward. Spherical tetrahedra and invariants of 3-manifolds. *arXiv preprint math/0406228*, 2004.

[41] Y. U. Taylor and C. T. Woodward. 6 j symbols for $U\_q(sl\_2)$ and non-Euclidean tetrahedra. *Selecta Mathematica*, 11(3):539–571, 2006.

[42] E. P. van den Ban. Lie groups. *Lecture Notes in Mathematics, MRI, University of Utrecht, Holland*, 2003.

[43] A. Varshalovich, Dmitrij, A. N. Moskalev, and V. K. Khersonskii. *Quantum theory of angular momentum*. World Scientific, 1988.

[44] E. Wigner. *Group theory: and its application to the quantum mechanics of atomic spectra*, volume 5. Elsevier, 2012.

[45] Wikipedia contributors. Spherical law of cosines.— Wikipedia, the free encyclopedia, 2020. Online, accessed 18-December-2020.

[46] R. Wong. *Asymptotic approximations of integrals*. SIAM, 2001.