A Quantitative Analysis of Investor Over-reaction and Under-reaction in the South African Equity Market: A Mathematical Statistical Approach

by

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Dissertation presented for the degree of Doctor of Philosophy in Statistics in the Faculty of Economic and Management Sciences at Stellenbosch University

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Declaration

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Date: ....April 2022..........

Abstract

A Quantitative Analysis of Investor Over-reaction and Under-reaction in the South African Equity Market: A Mathematical Statistical Approach

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One of the basic foundations of traditional finance is the theory underlying the efficient market hypothesis (EMH). The EMH states that stocks are fairly and accurately priced, making it impossible for investors to use stock selection, technical analysis, or market timing to out-perform the market by earning abnormal returns. Several schools of thought have challenged the EMH by presenting empirical evidence of market anomalies, which seems to contradict the EMH. One such school of thought is behavioural finance, which holds that investors over-react and/or under-react over time, driven by their behavioural biases.

The Barberis *et al.* (1998) theory of conservatism and representativeness heuristics is used to explain investor over-reaction and under-reaction. Investors who exhibit conservatism are slow to update their beliefs in response to recent evidence, and thus under-react to information. Under the influence of the representativeness heuristics, investors tend to produce extreme predictions, and over-react, implying that stocks that under-performed in the past tend to out-perform in the future, and vice-versa (Aguiar *et al.*, 2006).

In this study, it is investigated whether South African investors tend to over-react and/or under-react over time, driven by their behavioural biases. The 100 shares with the largest market capitalisation at the end of every calendar year from 2006 to 2016 were considered for the study. These shares had sufficient liquidity and depth of coverage by analysts and investors to be considered for a study on behavioural finance. In total, a sample of 163 shares had sufficient financial statement data on the Iress and Bloomberg databases to be included in the study. Analyses were done using two mathematical statistical techniques i.e. the more mathematical Fuzzy C-Means model and the Bayesian model, together with formal statistical tests. The Fuzzy C-Means model is based on the technique of pattern recognition, and uses the well-known fuzzy c-means clustering algorithm. The Bayesian model is based on the classical Bayes' theorem, which describes a relationship between the probability of an event conditional upon another event. The stocks in the financials-, industrial- and resources sectors were analysed separately.

Over-reaction and under-reaction were both detected, and differed across the three sectors. No clear patterns of the two biases investigated were visible over time. The results of the Fuzzy C-Means model analysis revealed that the resources sector shows the most under-reaction. In the Bayesian model, under-reaction was observed more than over-reaction in the resources and industrial sectors. In the financial sector, over-reaction was observed more often. The results of this study imply that a momentum and a contrarian investment strategy can lead to over-performance in the South African equity market, but can also generate under-performance in a poorly performing market. Therefore, no trading strategies can be advised based on the results of this study.

Opsomming

A Quantitative Analysis of Investor Over-reaction and Under-reaction in the South African Equity Market: A Mathematical Statistical Approach

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Een van die basiese boustene van tradisionele finansies is die teorie van die effektiewe markhipotese (EMH). Die EMH verklaar dat aandele billik en akkuraat geprys word, wat dit vir beleggers onmoontlik maak om aandeleseleksie, tegniese ontleding of marktydsberekening te gebruik om die mark te oortref deur abnormale opbrengste te verdien. Die EMH is en word uitgedaag deur verskeie denkrigtings wat deur empiriese navorsing ondersteun word en wat die EMH weespreek. Een van die denkrigtings is gedragsfinansies, wat aandui dat beleggers oor tyd oorreageer en/of onderreageer, gedryf deur hul gedragsvooroordele.

Barberis *et al.* (1998) se teorie oor die konserwatiewe en verteenwoordige heuristiek word gebruik om beleggers se oorreaksie en onderreaksie te verklaar.

Beleggers wat konserwatief word, is traag om hul oortuigings aan te pas in reaksie op onlangse getuienis, en reageer dus nie op inligting nie. Onder die invloed van die verteenwoordige heuristiek, is beleggers geneig om ekstreme voorspellings te maak en oor te reageer, wat beteken dat aandele wat in die verlede onderpresteer het, in die toekoms beter presteer en omgekeerd (Aguiar *et al.*, 2006).

In hierdie studie word ondersoek of Suid-Afrikaanse beleggers geneig is om oor tyd te oorreageer / of onderreageer, gedryf deur hul gedragsvooroordele.

Die100 aandele met die grootste markkapitalisasie aan die einde van elke kalenderjaar vanaf 2000 tot 2016 is in hierdie studie gebruik. Hierdie aandele het genoegsame likiditeit en diepte van dekking deur analiste en beleggers gehad om oorweeg te word vir ‘n studie oor gedragsfinansies. ‘n Steekproef van 163 aandele in totaal het genoegsame finansiële-staat-data in die Iress en Bloomberg databasisse gehad sodat dit ingesluit kon word in die studie.

Die analise is gedoen met behulp van twee wiskundige statistiese tegnieke: die meer wiskundige Fuzzy C-Means (FCM) model en die Bayesiaanse model, tesame met formele statistiese toetse. Die FCM model is gebaseer op die tegniek van patroonherkenning en gebruik die bekende fuzzy gemiddelde tros algoritme. Die Bayesiaanse model is gebaseer op die klassieke Bayes-stelling wat ’n verband beskryf tussen die waarskynlikheid van ’n gebeurtenis gegewe ’n ander gebeurtenis. Die aandele in die finansiële, nywerheid en hulpbronne sektore is afsonderlik ontleed.

Beide oorreaksie en onderreaksie is gevind, en het verskil tussen die drie sektore. Geen duidelike patrone van die twee vooroordele was sigbaar nie. Die FCM-ontleding het aan die lig gebring dat die hulpbronsektor die meeste onderreaksie toon. Met die Bayesiaanse model is onderreaksie meer waargeneem as oorreaksie behalwe in die finansiële sektor. Die resultate van hierdie studie impliseer dat momentum en ‘n teenstrydige beleggingstrategie kan lei tot oorprestasie in die Suid-Afrikaanse aandelemark, maar dit kan ook onderprestasie in ‘n swak presterende mark te weeg bring. Daarom kan geen handelstrategieë op grond van die resultate van hierdie studie aanbeveel word nie.

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Special thanks go to anyone who contributed directly and indirectly and whose names are not mentioned here, I thank you wholeheartedly.

Dedications

To my beloved parents, Joseph and Jacqueline Tiekwe, my lovely husband Ridick Roland Takong, and my children, Uriel Johan Takong and Jayden Asher Takong

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List of Abbreviations

EMH . . . . . efficient market hypothesis

FCM . . . . . fuzzy c-means

BBM . . . . . Barberis Bayes Model

JSE . . . . . . Johannesburg Stock Exchange

NYSE . . . . New York Stock Exchange

CAPM . . . . capital asset price model

CAR . . . . . cumulative abnormal return

CR . . . . . . current ratio

ROE . . . . . return on equity

ROA . . . . . return on assets

RR . . . . . . retention ratio

EPS . . . . . . earnings per share

DIV . . . . . . dividend

TAR . . . . . total asset return

NAV . . . . . net asset value

BMT . . . . . book-to-market

DIV . . . . . . dividend yield

EPS . . . . . . earnings per share

SD . . . . . . standardised data

W-L . . . . . winner minus loser

FF . . . . . . Fama and French

FF3FM . . . Fama and French Three-Factor Model

FF5FM . . . Fama and French Five-Factor Model

CH4FM . . . Carhart Four-Factor Model

Chapter 1

Introduction

One of the basic foundations of traditional finance is the theory underlying the efficient market hypothesis (EMH). The EMH states that stocks are fairly and accurately priced, making it impossible for investors to use stock selection, technical analysis, or market timing to out-perform the market by earning abnormal returns. Several schools of thought have challenged the EMH by presenting empirical evidence of market anomalies that seems to contradict the EMH. One such school of thought is behavioural finance, which holds that investors over-react and/or under-react over time, driven by their behavioural biases.

Motivated to provide more insight into investors' behaviour in the financial market, the over-reaction and under-reaction anomalies of the three sectors of the South African equity market from 2006 to 2016, are investigated in this study, using mathematical statistical techniques.

In this chapter, the background of the study is outlined (in Section 1.1). The different models that have been applied to test over- and under-reaction are presented in Section 1.2, followed by the research gaps that the study intended to bridge, in Section 1.3. Next, the research questions and aim and objectives are listed, in Section 1.4. Thereafter, an overview of how the remainder of this dissertation is structured is presented, in Section 1.5. Finally, the conclusion of this chapter is stated in Section 1.6.

1.1 Background

Asset classes are groupings of investments that have similar characteristics. The four asset classes are: equities, bonds, property, and cash (Wilcox and Fabozzi, 2013). Equities, also called stocks or shares, represent a share in a company. Stocks prices can move significantly in the stock market; thus, equities are considered the most risky asset class.

An investor is faced with decisions when buying or selling stocks. The decision process may seem overwhelming, as significant market movements create anxiety for investors (Chaudhary, 2013). Traditional finance assumes that investors always behave rationally; therefore, investors make decisions that will maximise their expected utility. This argument is based on the efficient market hypothesis (EMH), which stipulates that stocks are fairly and accurately priced, making it impossible for investors to use stock selection, technical analysis, or market timing to out-perform the market (Fama, 2021).

In this framework, change in a stock's price is random. Therefore, the future price cannot be predicted based on past prices. This view is built on two arguments. First, when prices differ from their efficient intrinsic value, an arbitrageur will eliminate the market pricing bias by buying in one market and simultaneously selling in another, thereby profiting from a temporary difference in value (Wyart and Bouchaud, 2007). Second, stock prices adjust instantly to any new information entering the market (Malkiel and Fama, 1970).

However, this view does not take into account the psychological aspects of investors' behaviour, whose reaction to new information is not always rational. Studies (Black, 1986; Shleifer and Summers, 1990; Daniel *et al.*, 1998) have shown that investors are prone to behavioural biases that may lead to anomalies in financial markets.

Financial market over-reaction and under-reaction are examples of financial market anomalies, and are caused by investor biases, which pose a challenge to the assumptions of the EMH. Although the EMH supposes that, if prices differ from their fundamental value, an arbitrageur possessing some information not reflected in the price would cancel the market pricing bias, there are limits to arbitrage (Shleifer and Vishny, 1997). Mispricing may persist because rational traders may not be able to bring back the prices to their fundamental value. This is because the arbitrage strategies designed to eliminate mispricing are often costly and risky, or restricted (Barberis and Thaler, 2003). Therefore, understanding investors' behaviour in the financial market is central to understanding price formation in the stock market.

Analyses of over-reaction and under-reaction are related to momentum and contrarian investment strategies respectively (Wouassom, 2017). A momentum investment strategy entails buying stocks with a good record of performance, and selling those that under-performed (Lo and MacKinlay, 1990). A contrarian investment strategy, on the other hand, entails buying stocks with a poor recent record of performance, and selling those that out-performed. This close relationship between over- and under-reaction in investment strategies has led to the development of different models of quantification and prediction. Two models that have specifically made use of interesting quantitative methods to study over- and under-reaction are the Fuzzy C-Means (FCM) model and the Bayesian model.

Aguiar and Sales (2010) introduced the FCM model and suggested that it can be used to analyse over- and under-reaction among investors. The model is based on fuzzy sets theory, which is applied to test for representativeness and anchoring in investors' behaviour. The model comprises two steps: pattern recognition and stocks rating. In the first step, two fundamental profiles (referred to as centres) are formed by applying the FCM algorithm over a model training period. The average performance of the stocks around the two centres is then calculated to identify the "winner" and the "loser" groups according to their relative performance. The winner and loser stocks are defined as such if they belong to the clusters with larger average financial returns and, with smaller average financial returns respectively, and variables such as price-earnings ratios are used when constructing the centres. In the second step, the FCM algorithm is applied again in the subsequent period, to identify two new fundamentally defined centres to test the performance of the previous winner- and loser portfolios. The FCM model has only been applied to the Brazilian (Aguiar, 2012) and USA stock markets (Aguiar and Sales, 2010), and focused only on the petrochemical and textile sectors. In both cases, the FCM model was found to provide a good prediction of both over- and under-reaction. The results indicated that the petrochemical sector presented evidence of over-reaction, while the textile sector presented evidence of under-reaction in both the USA and the Brazilian stock markets (Aguiar and Sales, 2010; Aguiar, 2012).

The Bayesian model proposed by Barberis *et al.* (1998) is also used to analyse over- and under-reaction. In this formulation of over-reaction and under-reaction, earnings follow a random walk over time, and investor beliefs are grouped into two states. The underlying switching process between investor states follows a Markov process. The Bayesian model has only been applied in an artificial financial market, and has not been tested in real market scenarios.

Most studies on market over- and under-reaction used data from developed economies (De Bondt and Thaler, 1985, 1987; Power *et al.*, 1991; Chopra *et al.*, 1992; Clare and Thomas, 1995). It is, however, increasingly important to understand the impact of investor behaviour in developing markets, because of their position in the global economy and the investment interest they attract. The present study's focus is therefore the South African financial market as an example of an emerging market. South Africa holds a unique position among African nations, and has the most developed economy on the African continent (Frisch *et al.*, 2014). However, analysis of investor behaviour in South Africa has thus far been limited. The only enquiries that analysed over-reaction and under-reaction that concentrated on the Johannesburg Stock Exchange (JSE) (Page and Way, 1992; Mun *et al.*, 2001) were based on the approach described by De Bondt and Thaler (1985). This approach does not make use of mathematical statistical methods such as the two methods described above.

The aim of the present research project is therefore to investigate over-reaction and under-reaction in the South African equity market using the mathematical statistical techniques described above. This research will provide a deeper understanding of market over- and under-reaction within the context of a developing economy, and also employ mathematical statistical methods to study these anomalies in general.

1.2 Relevant Methodologies

In this section, more in depth technical detail about the models that have been applied in this dissertation to investigate over- and under-reaction among investors are provided. The two models mentioned in Section 1.1, the FCM model and the Bayesian model are discussed below.

1.2.1 FCM Model

The FCM model is based on the technique of pattern recognition using the well-known FCM algorithm, which is based on the mathematical theory of fuzzy sets (Aguiar *et al.*, 2006). Fuzzy sets are broadly defined as sets or groups containing elements that have varying membership degrees with respect to specific input parameters. The FCM model's algorithm was applied to group or classify the listed companies on the JSE into subsets (clusters) in such a way that the stocks in a specific group are more similar to, or more compatible with, themselves than elements in different groups with respect to certain financial characteristics (features, financial indices, profitability, etc.). The application of the FCM model on South African data is discussed in Chapter 3.

1.2.1.1 Data

The data used for the fuzzy analysis will be the quarterly stock returns data of the listed companies from different sectors of the South African equity market. The data were downloaded from Iress and Bloomberg. The sample period is July 2006 to December 2016. This period includes the global financial crisis of 2008. The JSE's All Share Index will be employed for market benchmarking. The stocks in the financial, industrial, and resources sectors will be analysed separately. A detailed description of the data can be found in Section 3.2.

1.2.2 Bayesian Model

The Bayesian model (Barberis *et al.*, 1998) for decision-making is a well-known statistical model that uses historical data, such as past returns, to update the prior beliefs regarding parameters, such as expected future returns. Investors use information about past events to update the prediction of future events. The model is based on the classical Bayes' theorem (Bolstad and Curran, 2016), which describes a relationship between the probability of an event conditional upon another event. The application of the Bayesian model is discussed in Chapter 4.

1.2.2.1 Data

The 100 shares with the largest market capitalisation at the end of every calendar year from 2006 to 2016 were considered for the study. These shares had sufficient liquidity and depth of coverage by analysts and investors to be considered for a study on behavioural finance. In total, a sample of 163 shares had sufficient financial statement data on the Iress and Bloomberg databases to be included in the study. The stocks in the financial, industrial, and resources sectors will be analysed separately. The variables required for the Bayes analysis of over- and under-reaction are:

- total return index values, which include reinvested dividends for all companies in the sample, downloaded from Bloomberg; and
- earnings per share data, obtained from actual financial statements downloaded from Iress.

1.3 Research Gaps

The following list of research opportunities were identified from the background provided above:

i) the identification of fundamental and financial variables for South African companies from literature; and

ii) the FCM model (Aguiar *et al.*, 2006; Aguiar and Sales, 2010) has only been applied to the Brazilian and USA stock markets in studies focused on a specific industry or sector;

iii) the BBM (Barberis *et al.*, 1998) has only been applied in an artificial financial market, and has not been tested against real market scenarios;

iv) a modification of the Bayesian model proposed by Lam *et al.* (2010), and Lam *et al.* (2012) (referred to as a pseudo Bayesian model) only provides a theoretical approach to explaining over-reaction and under-reaction, and has not been tested using real data;

v) only brief derivations of the above-mentioned models appear in the literature;

vi) to date, studies that analysed over-reaction and under-reaction on the JSE (Page and Way, 1992; Mun *et al.*, 2001) were limited to the approach described by De Bondt and Thaler (1985), which does not make use of the mathematical statistical methods available; and

vii) the use of mathematical statistical methodologies to describe over-reaction and under-reaction are still under-utilised in research.

1.4 Research Questions, Aim, and Objectives

The main research questions (RQs) of this study are as follows:

- RQ1: Does over-reaction and under-reaction really occur in the South African equity market?
- RQ2: Can the application of mathematical statistical models be refined in determining over-reaction and under-reaction in the South African equity market?

Following on the research opportunities outlined in Section 1.3 and the research questions stated above, the aim of this research project is to provide a quantitative investigation of investor over-reaction and under-reaction in the South African equity market, using mathematical statistical techniques.

The research objectives (ROs) of the study are:

i) RO1: To provide detailed theoretical derivations and explanations of the previous methodologies to determine over-reaction and under-reaction;

ii) RO2: To present the two models in a more mathematically and statistically explanatory fashion;

iii) RO3: To identify fundamental and financial variables for South African companies from literature;

iv) RO4: To analyse South African market data by applying the FCM algorithm; and

v) RO5: To further adjust and refine empirical analyses of stock market over-reaction and under-reaction.

1.5 Dissertation Overview

The dissertation is organised as follows:

Chapter 1 is an introductory chapter, where the background of the study, and a brief introduction of the relevant methodologies that had been used to analyse over-reaction and under-reaction in the South African financial market are presented. The research gaps that the present study will attempt to narrow are presented as well, together with the research objectives, research questions, and significance of the study.

In Chapter 2, a theoretical framework of the main theories that are relevant to the discussion of the over-reaction and under-reaction anomalies is provided. Deviations from the expected investor behaviour as predicated by the EMH are discussed, and evidence of irrational investor behaviour across the South African equity market is reviewed. Thereafter, how over-reaction and under-reaction anomalies influence behavioural finance theory is evaluated, reviewed, and explained. Finally, different arguments that have been suggested by researchers as possible causes of over- and under-reaction anomalies in financial markets are provided. In the second part of this chapter, empirical literature is reviewed, and extant research is highlighted. Previous research that provided information about relevant variables in the South African financial market is also discussed. The results of the FCM models in research on the Brazilian and USA markets are then presented. The focus is on identifying methodologies and fundamental factors that have been used in research on price reaction in the South Africa context.

In Chapter 3, mathematical concepts underpinning the FCM model are provided, followed by a detailed explanation of how the model is used to study over- and under-reaction. Data that form the basis of the investigation are described, including the source and processing procedures. All variables are explained and defined. Some data challenges encountered are discussed, and the procedure to address these challenges is discussed. Thereafter, the results of the FCM model analysis of the different sectors of the South African equity market are also presented and discussed.

In Chapter 4, the mathematical formalism of the Bayesian model is presented. In this chapter, the results of the analysis of the over-reaction and under-reaction in the South African financial market, using the Bayesian model, are then presented. This is followed by an interpretation of results and a discussion of the major results.

In Chapter 5, the study is summarised, and conclusions are drawn based on the findings. The contributions of the dissertation, the research challenges, and the limitations of the study are stated. Finally, possible directions for future research are highlighted.

1.6 Conclusion

From the discussions in this chapter, it is clear that it is relevant and important to determine if over-reaction and under-reaction occur in the South African equity market. Using mathematical statistical methodology to determine this can contribute to the body of knowledge in this domain. Hence, in the following chapter, a theoretical framework of the main theories that are relevant to the discussion of over-reaction and under-reaction anomalies is provided.

Chapter 2

Literature Review

In this chapter, a theoretical framework of the main theories that are relevant to the discussion of the over-reaction and under-reaction anomaly is provided. Specifically, key concepts in the literature, such as the EMH and behavioural finance theory are discussed. Deviations from the expected investor behaviour as predicated by the EMH are discussed, together with evidence of irrational investor behaviour across the South African equity market. Second, how over-reaction and under-reaction anomalies influence behavioural finance theory is evaluated, reviewed and explained. Finally, different arguments that have been suggested by researchers as possible causes of over-reaction and under-reaction anomalies in the financial market are provided. In the second part of this chapter, empirical literature and research conducted in this domain are highlighted.

2.1 Theoretical Framework

2.1.1 The Efficient Market Hypothesis

The EMH, introduced by Fama (1965), suggests that stock prices reflect all available information. A direct implication of the theory is that technical and fundamental analyses cannot help an investor to generate returns greater than the market returns.

However, different kinds of information influence stock prices. Consequently, the EMH is subdivided into three levels: weak, semi-strong, and strong forms of market efficiency (Fama, 1970).

- The weak form assumes that market prices incorporate all information on past prices and returns. Such data include historical prices, trading volume, etc. Thus, technical analysis cannot be successfully used to forecast future prices (Mobarek and Keasey, 2002).
- The semi-strong form supposes that market prices incorporate information about historical prices and all publicly available information such as earnings, dividend pay-outs, etc. (Degutis and Novickytė, 2014).
- The strong form of the EMH supposes that private and inside information is available to any market participant (Latif *et al.*, 2011).

The general implication of the EMH is that market prices cannot be beaten by forecasting (return predictability).

2.1.2 Applicable EMH Literature

Evidence in support of over-reaction has been produced by Dreman and Berry (1995), Lobe and Rieks (2011), and Lakonishok *et al.* (1994). Various tests have been used to determine whether equity markets are strongly, semi-strongly, or weakly efficient (Samuelson, 1965; Fama, 1965; Malkiel, 2011).

The weak-form market efficiency tests are correlation tests, used to test return predictability by determining the relation between current and past returns (Elton *et al.*, 2009). A linear correlation between the current and past returns means the returns are predictable; thus, there is no market efficiency. Studies by Grieb and Reyes (1999) and Buguk and Brorsen (2003) are examples of studies on the weak-form market efficiency.

Semi-strong-form market efficiency tests determine whether a change in the value of stocks occurs before, during, or after the announcement of important events (Elton *et al.*, 2009). If the market is efficient, the abnormal stock return should occur around the news release, since prices should adjust quickly and fully to any new public information. Studies by Rose and Selody (1984), and Bacon and McMillan (2007) are examples of studies of semi-strong-form market efficiency.

A test of the strong form of the EMH, also known as a test for private information, determines if an insider-based trader can consistently out-perform the market, indicating superior skill and information processing abilities to the rest of the market participants. Finnerty (1976)'s study is an example of a study of the strong form of the EMH.

Some studies have indicated market inefficiencies (Asamoah, 2010; Zunino *et al.*, 2009; Deshmukh *et al.*, 2008; Lim *et al.*, 2008; Mishkin and Eakins, 2006), while others have supported the EMH and indicated that stock markets are efficient (Gabriela ğiĠan, 2015; Konak and Şeker; Mlonzi *et al.*, 2011; Malkiel, 2011, 2005; Mushidzhi and Ward, 2004). No consensus had been reached on the validity of the EMH. Although there is much empirical evidence in support of the EMH, it has also received criticism for different forms of anomalies that have been identified, which challenge the EMH.

2.1.3 Challenge to the EMH

Frankfurter and McGoun (2001) defined an anomaly as an irregularity or a deviation from the natural order. In traditional finance theory, a financial market anomaly is a situation in which a stock market deviates from the assumptions of the EMH (Latif *et al.*, 2011). The assumptions of the EMH are: stock markets are efficient, investors behave rationally and process available information correctly, and, thus, a stock's price reflects its fundamental value (Sewell, 2011). Fama (1965) investigated whether stock markets are efficient, and noted that, in some cases, there are occurrences of price patterns. Therefore, investors can predict future prices from past prices and form investment strategies to out-perform the market.

Different forms of anomalies have been identified that challenge the EMH. In the next sub-section, some anomalies that are observed in the financial market are presented. The two pertinent to this study, namely over-reaction and under-reaction, are discussed in a separate section.

2.1.4 Calendar Anomalies

Calendar anomalies are linked to a period of the year. Examples of calendar anomalies are the Monday effect, the January effect, and the holiday effect.

1. The January effect is the tendency of stocks' prices to increase in the month of January, compared to any other month.

2. The holiday effect is the tendency of returns and trading volumes to be higher before holidays.

2.1.5 Technical Anomalies

The aim in technical analysis is to use recurring and predictable patterns to generate superior portfolio performance. The following technical anomalies could be considered.

1) Momentum investing is a strategy that capitalises on price continuation. Momentum is a short-term effect of price continuation in stocks' returns (Jegadeesh and Titman, 1993). Evidence of momentum in stock returns suggests that momentum investment will produce excess returns (Jegadeesh and Titman, 1993; Lee and Swaminathan, 2000; Hong *et al.*, 2000; Chan *et al.*, 2000; Balvers and Wu, 2006). Muller and Ward (2013) and Van Rensburg and Robertson (2003) found evidence of momentum in prices in South Africa.

   Many theories are proposed to explain momentum in prices: Barberis *et al.* (1998), Daniel *et al.* (1998), and Hong and Stein (1999) argue that momentum is caused by under-reaction or delayed over-reaction to information. Conrad and Kaul (1998) argue that profits from a momentum strategy are compensation for risk.

   Explanations of momentum anomalies can be broadly split into two:

(i) rational explanations, such as model misspecification (Wang and Wu, 2011), transaction costs (Lesmond *et al.*, 2004)), etc.; and

(ii) irrational behaviour, such as under-reaction (Jegadeesh and Titman, 1993), overconfidence (Daniel *et al.*, 1998), etc..

2) Mean reversion relates to stocks that were mispriced for a period, but eventually reverted to an acceptable level after being over- or under-priced. Contrarian investors base their strategy on negative autocorrelation, which entails buying previous losers, expecting a return reversal, and selling previous winners.

   Some studies (De Bondt and Thaler, 1985, 1987; Chopra *et al.*, 1992) demonstrated that excess returns could be gained by employing both a contrarian- and a momentum investment strategy, which approach contradicts the EMH.

2.1.6 Fundamental Anomalies

Fundamental anomalies are found in trading financial instruments. The following are examples.

(1) Size effect is the tendency of companies with smaller market capitalisation to out-perform those with large market capitalisation over the long-term (Dissanaike, 2002).

(2) Value effect, also termed the book-to-market (BTM) anomaly, is the tendency of value stocks (stocks with high book-value-to-market-value ratios) to out-perform growth stocks (those with low BTM ratios). The BTM anomaly is used to explain stock price over-reaction or deviations of empirical returns from the capital asset price model (CAPM) (Basu, 1977). Hoffman (2012) found evidence of BTM anomalies on the JSE.

Market anomalies arise from the irrationality of investors, a phenomenon analysed in behavioural finance. In the next sub-section, the theory of behavioural finance is discussed.

2.1.7 Behavioural Finance Theory

The existence of market anomalies has been investigated and further explained by introducing the psychology of market participants. Irrational investor behaviour on an individual and group basis is a possible driving force in deviation from the principles that underpin the EMH. In this section, deviations from the expected investor behaviour are discussed. The discussion focuses on specific behavioural concepts that have been cited as the cause of over-reaction and under-reaction.

According to Antonacci (2014), over-reaction is due to a herding effect, representativeness, and overconfidence. Investors' under-reaction comes from anchoring, conservatism, and the slow diffusion of information. According to Barberis *et al.* (1998), investors are negatively influenced by conservatism bias and the use of the representativeness heuristic. Investors are prone to heuristics while processing data and making decisions. Conservatism, disposition effect, and aversion to ambiguity are the most important anomalies that influence investors' decisions (Shleifer, 2000). Behavioural biases can be divided into belief perseverance errors and information processing errors (Institute, 2016). Some belief perseverance biases and information processing errors are discussed below.

2.1.7.1 Belief Perseverance Biases

Belief perseverance bias is a desire to cling to a previous belief, even upon receiving refuting facts. The following are forms of belief perseverance biases.

- Conservatism bias occurs when investors focus on a prior view and fail to consider new information. In Bayesian terminology, investors are slow to update a view because they overweight the initial probabilities and do not adjust probabilities for the new information.

- Representativeness bias is the belief that past events will persist and new information is classified based on past experience. Investors use a similar past experience to assess the probability of an actual event, rather than the underlying probabilities.

2.1.7.2 Information Processing Errors

Examples of information processing biases are the following:

- Anchoring bias occurs when individuals use an initial piece of information to make subsequent judgements (Elton *et al.*, 2009). This bias results in investors selling overvalued stocks and buying undervalued stocks, or holding onto investments that have lost their value.
- Prospect theory explains how investors make decisions based on the potential value of losses and gains. When there is a high probability of gains or a low probability of losses, people are risk-averse. On the other hand, they are risk-seeking when there is a low probability of gains or a high probability of loss (Kahneman and Tversky, 2013).

An inefficient market is a consequence of (heuristic-driven and frame-dependency) biases, which introduce market anomalies (Shefrin, 2002). Financial market over-reaction and under-reaction are examples of financial market anomalies that are caused by investor biases that pose a challenge to the assumptions of the EMH.

The main assumption of the EMH is that stock prices fully reflect all information (Fama, 1965; Malkiel, 1962). However, a conflicting consideration is that stock prices reflect the sentiments of market participants (Daniel *et al.*, 1998; De Bondt, 2000), which is the key to understanding the over-reaction hypothesis (Barberis *et al.*, 1998; De Bondt, 2000). Therefore, prices could deviate from the fundamental value as investors interpret new information differently. The next section focuses on why investors might over-react or under-react in the market, and what drives investors' over-reaction and under-reaction.

2.1.8 Over-reaction and Under-reaction

The term over-reaction means to react above the degree of reaction that is considered normal. The concept originated from psychology, and describes situations in which people over-react to dramatic news. According to Kahneman and Tversky (1982), over-reaction is a result of investors' behavioural biases, which cause them to weigh information asymmetrically. De Bondt and Thaler (1985) found that prior losers (winners) become winners (losers) over the subsequent periods because of investors' over-reaction. "Over-reaction occurs when stock prices rise (fall) too much in response to good (bad) news. When bad news arrives in the stock market regarding certain stocks, investors panic at first and start trading based on this misconception, then prices of these stocks fall and lead to mispricing" (Bassiouny and Ragab, 2014). Under-reaction, in contrast, stems from investors being conservative, i.e. reacting slowly to the new evidence. Therefore, the response of the market to the information is lower than optimal.

According to De Bondt (2000), over-reactions are due to errors in investors' forecasts. Investors assume the continuation of the trend after a sequence of similar news, leading to an over-reaction. Investors extrapolate from random sequences; wherein they expect patterns to continue, resulting in over-reaction (and subsequent reversals), whereas conservatism, creates momentum through under-reaction (Barberis *et al.*, 1998). In the next section, research that has been conducted in the past to test over-reaction or/and under-reaction are highlighted.

2.2 Empirical Literature Review

In this section, an overview of selected empirical tests of over-reaction and under-reaction from prior research is provided. Different methods and statistical procedures have been applied in previous studies of investors' over- and under-reaction:

- multivariate regression, Alagidede (2013),
- event studies, De Bondt and Thaler (1985), Itaka (2014) and Chia *et al.* (2015),
- volatility and variance testing, Heynen *et al.* (1994) and Fang (2013),
- the Bayesian model, Barberis *et al.* (1998) and

- the Fuzzy model, Aguiar *et al.* (2006) and Aguiar and Sales (2010).

Events studies have been conducted to test the EMH, particularly how fast new information is incorporated into stock prices. An event study typically involves the following sequence of steps (Elton *et al.*, 2009) if high-frequency data are available:

(i) collecting a sample of firms that have experienced a surprise announcement or relevant price shock (the event);

(ii) determining the precise day of the event and designating it as Day Zero or Time Zero;

(iii) defining the periods to be studied before and after the event;

(iv) computing various return metrics for each of the firms (e.g., raw total and abnormal returns) and relevant benchmark returns, as required; and

(v) using statistical tests to determine if the abnormal returns are significantly different from zero.

2.2.1 Portfolio-based Event Study

De Bondt and Thaler (1985) formulised the EMH as follows: Consider F_{t-1} the complete set of information available at time t-1. R_{jt}, the return on stock j at time t, and R_{mt}, the return on the market at time t. The residual return u_{jt}, of stock j at time t, is calculated as follows:

$$u_{jt} = R_{jt} - R_{mt}, \tag{2.2.1}$$

and

$$E_m(R_{jt}|F^m_{t-1}). \tag{2.2.2}$$

This is the expectation of R_{jt}, based on the information available at time t-1.

The semi-strong market efficiency hypothesis is:

$$E(R_{jt} - E_m(R_{jt}|F^m_{t-1})|F_{t-1}) = E(u_{jt}|F_{t-1}) = 0 \tag{2.2.3}$$

Theoretically, market information cannot be used to generate excess returns.

If there is efficiency, it implies that $E(u_{Gt}|F_{t-1}) = E(u_{Pt}|F_{t-1}) = 0$, where $E(u_{Gt}|F_{t-1})$ is the expected value of a "Good Performance" portfolio, and $E(u_{Pt}|F_{t-1})$ is the expected value of "Poor Performance" portfolio.

"If the over-reaction hypothesis holds, the expected value of Good Performance will be less than 0, and the expected value of Poor Performance portfolio will be greater than 0, because investors' over-reaction would drive the stock prices to the opposite direction" (Hu, 2012).

Testing over-reaction has been done in many studies, for example those of Itaka (2014), Chia *et al.* (2015), and Hu (2012), and the methodology of De Bondt and Thaler (1985) was followed.

2.2.2 Bayesian Model Application in Behavioural Finance

Barberis *et al.* (1998) proposed a Bayesian model to test over-reaction and under-reaction in investor sentiment. Investors use information about past events to update their prediction of future events. The Bayesian model of Barberis *et al.* (1998) is based on the notion that "people pay too much attention to the strength of the recent evidence they are presented with and too little attention to the statistical weight that it should be assigned while making forecast" (Griffin and Tversky, 1992).

Over-reaction and under-reaction are evident in stock prices after consistent patterns of news. In the forming of over-reaction and under-reaction, the process of the earnings is a random walk over time. Investors do not know the true process of earnings, and hence update their beliefs incorrectly, which is theoretically grouped into two regimes.

The conservative and representative heuristics of investors will lead them to believe that the announcement is either a trending regime or a mean-reverting regime. These two regimes can be formulated as a two-states Markov chain. If investors observe patterns that may continue, for example, consecutive rises or falls in earnings, they reduce the probability of observing a similar outcome in the next earnings announcement (Igboekwu, 2015). The underlying switching process between investor states follows a Markov process.

By modifying the assumptions of the Bayesian model of Barberis *et al.* (1998), Lam *et al.* (2010) developed a pseudo Bayesian approach that incorporates weights on observations that reflect investors' biases. In their formulation, the process of the earnings is random, but earnings shocks are independent, and follow a Gaussian distribution.

The model of Lam *et al.* (2010) was extended using the assumption that the earning shocks follow an exponential distribution. The results indicate that investor behavioural biases create a magnitude effect in the under- and over-reaction phenomena. The momentum/contrarian profit is proportional to the severity of the earnings shocks (Lam *et al.*, 2012).

2.2.3 FCM Model

Aguiar and Sales (2010) introduced the FCM model and suggested that it can be used to analyse over- and under-reaction among investors. The model is based on fuzzy set theory, which is applied to test for representativeness and anchoring in investors' behaviour. It comprises two steps: pattern recognition and stocks rating. In the first step, two fundamental profiles (referred to as centres) are formed by applying the FCM algorithm over a model training period. The average performance of the stocks around the two centres is then calculated to identify the winner- and loser groups according to their relative performance. The winner- and loser stocks are defined as such if they belong to the clusters with larger average financial returns and smaller average financial returns respectively. Evidence of over-reaction was found in the petrochemical sector, while the textile sector presented evidences of under-reaction in the USA and Brazilian stock markets (Aguiar and Sales, 2010; Aguiar, 2012).

Evidence of over-reaction was found in stock markets in the following countries:

- USA, (De Bondt and Thaler, 1985; Aguiar and Sales, 2010),

- UK (Clare and Thomas, 1995),
- China (Fang, 2013),
- Nigeria (Raji, 2015),
- Brazil (Aguiar *et al.*, 2006), and
- India (Choudhary and Sethi, 2014).
- Australia, Belgium, Canada, Denmark, Finland, France, Germany, India, Italy, Japan, the Netherlands, Norway, Pakistan, Spain, Sweden, Switzerland, the UK and the USA (Chan *et al.*, 1997).

Empirical tests of over-reaction and under-reaction hypotheses in the USA stock market, based on the FCM model were conducted by Aguiar and Sales (2010). The methodology thereof, is strongly connected with the representativeness and anchoring heuristics found in behavioural finance (Aguiar, 2012).

Aguiar (2012) investigated over-reaction in the American stock market using portfolios formed from financial ratios of public companies, and found evidence of market over-reaction in the oil and gas sector. The textile sector and the steel and iron sector fluctuated between over-reaction and under-reaction. Results suggested that a contrarian strategy in the USA stock market can generate profit, which confirmed the findings of De Bondt and Thaler (1987).

2.2.4 Results of the FCM Models for the Brazilian Market

The FCM methodology is discussed in details in Section 3.1.2. Aguiar and Sales (2010) applied this methodology and found significant over-reaction and under-reaction in the Brazilian market during a number of trimesters as summarised in Table 2.1. In this dissertation, a similar process will be followed and enhanced. The analysis will be done per quarter and the statistical significant will be tested.

Table 2.1: Average residual returns and t-test for the petrol/petrochemical sector in the Brazilian Market

Trimester	Over/Under-reaction
1 Trim 2001	Under*
2 Trim 2001	Under*
3 Trim 2001	Under*
4 Trim 2001	Over
1 Trim 2002	Over*
2 Trim 2002	Over
3 Trim 2002	Over*
4 Trim 2002	Over*
1 Trim 2003	Over**
2 Trim 2003	Over**
3 Trim 2003	Under
4 Trim 2003	Over
1 Trim 2004	Over
2 Trim 2004	Over
3 Trim 2004	Over*
4 Trim 2004	Under*
1 Trim 2005	Over
2 Trim 2005	Over
3 Trim 2005	Over**

Note: Statistically meaningful at the level of (*) 5% and (**) 10%

Source: Aguiar and Sales (2010)

2.2.5 Prior Methodologies and Conclusions of South African Studies

The first study of over-reaction in the South African market was done by Page and Way (1992) who conducted their research based on monthly data for 204 JSE stocks. Portfolios were formed based on prior return data of companies that had at least 30 trading weeks over the period July 1974 to June 1989. An equally weighted portfolio was used to represent the market index. The results indicated that winner portfolios out-performed loser portfolios by 10% and 20% respectively, which is evidence of investor over-reaction. Mean reversions were also detected for both loser and winner portfolios similar to the findings of De Bondt and Thaler (1987), and De Bondt and Thaler (1985).

Muller (1999) found evidence of investor over-reaction in the JSE using data of the 200 largest stocks for the period 1 January 1985 to 28 February 1998, similar to Page and Way (1992), and De Bondt and Thaler (1985).

Hsieh and Hodnett (2011) investigated investor over-reaction in the JSE over the period 1 January 1993 to 31 March 2009. The results indicated that, in South Africa, under-reaction is cyclical and varies between business cycles. The results also indicated that, when the formation period is longer, mean reversal is stronger. They suggest contrarian investing as appropriate during financial market turmoil.

Hirshleifer *et al.* (2011) found evidence of investor under-reaction on the JSE over the period 1988 to 2009. Portfolio performance measures, correlation analysis, and cumulative spreads methods were used to examine the monthly returns of companies listed on the FTSE/JSE.

Frisch *et al.* (2014) tested for under-reaction and over-reaction in the JSE using data of the FTSE Group JSE Top 40 index from January 2003 to December 2011. Using the cumulative abnormal returns measure and a GARCH (1, 1) model, the results suggested that large price increases and declines are likely to be followed by positive market returns.

Itaka (2014) found evidence of momentum on the JSE by evaluating the cumulative abnormal returns of the winner and loser portfolios formed over the period January 2002 to December 2009. The study indicated that mean reversion is more significant for longer formation portfolios.

The results of the following studies of over-reaction and under-reaction in the JSE were consistent: Page and Way (1992), Muller (1999), Venter (2009) and Hsieh and Hodnett (2011). Conclusions regarding the efficiency of the JSE (Gilbertson and Roux, 1977; Knight, 1985) remain inconsistent. In the next section, previous research that provides information about the relevant variables that have been most often identified as possible contributors to price over-reaction in the South Africa context are discussed.

2.2.6 Variables used in South African Studies

The reaction of the stock price to market information can be allocated to four different categories (Hong and Stein, 1999):

- under-reaction: there is a delay in response to new information (Chan *et al.*, 1996; Bernard, 1993);
- adjustment: there is an immediate response to new information (Hong and Stein, 1999);
- over-reaction: stock prices over-adjust to new information (De Long *et al.*, 1990; Bikhchandani and Sharma, 2000); and
- reversion: stock prices move back towards their fundamental value over time.

Van Rensburg and Robertson (2003) studied the effect of company size, price-to-earnings, and beta on returns on the JSE. The authors found that small-size firms earn higher returns, but have a lower beta. Other studies on size effect on the JSE did not identify small size effect (Bradfield *et al.*, 1988; De Villiers *et al.*, 1986).

Basiewicz and Auret (2010) found evidence of size and value effects on the JSE using the Fama and French Three-Factor model (FF3M) (Fama and French, 1993).

Strugnell *et al.* (2011) found evidence of size and price-to-earnings (P/E) effects on the JSE using a dataset from 1994 to 2007. Contrary to an earlier study by Van Rensburg and Robertson (2003), beta was not statistically significant.

Hoffman (2012) found significant evidence in support of book-to-market, size and momentum effects on the JSE over the period 1985 to 2010 using cross-sectional regression. Auret and Cline (2011) found no evidence of value, size, or January effects on the JSE using data from January 1988 to December 1995 and from January 1996 to December 2006, which results are partially consistent with those of Robins *et al.* (1999). De Villiers *et al.* (1986); Bradfield *et al.*

(1988) and Page and Way (1992) found no evidence of small firm effects on the JSE.

Considering the above-mentioned studies, the variables that have been identified most often as possible contributors to price overreaction in the South Africa context are: size, value (such as the P/E and BTM ratio), and the January effect.

Mahlophe (2015) tested if the Fama and French models; the CAPM, and the FF3FM (Fama and French, 1993), the Carhart four-factor model (C4FM) (Carhart, 1997) and the more recent five-factor model of Fama and French (FF5FM) (Fama and French, 2015) are applicable in estimating expected return on the JSE using data from January 2002 to December 2014. The results indicated that the asset-pricing models explained market anomalies in four of the six sectors examined. The results also suggested that market anomalies depended on the specification of the model.

These methodologies discussed above were not used in the present dissertation. Some of the fundamental concepts that have been mentioned for example the P/E ratio were applied.

2.3 Conclusion

In this chapter, a theoretical framework of the main theories that are relevant to the discussion of the over-reaction and under-reaction anomaly was provided. Deviations from expected investor behaviour as predicated by the EMH were discussed, and evidence of irrational investor behaviour across the South African equity market was reviewed. Different arguments that have been suggested by researchers as possible causes of over-reaction and under-reaction anomalies in the financial markets were provided, together with highlights of extant research. The examples describe how a multi-factor model can effectively capture fundamental patterns to explain returns. There is extensive research on market anomalies in the South African equity market, but it is not known which method is the most appropriate for analysing these market anomalies.

In the current study, the more mathematical FCM model and the Bayesian model were used together with formal statistical tests to investigate over-reaction and under-reaction in the South African equity market. This provides a broader view of different models that can be used to investigate over-reaction and under-reaction. The aim was to determine which sectors are affected more by the over-reaction and under-reaction anomalies. In the following chapter, the FCM model is used to show evidence of over-reaction and under-reaction in the South African equity market.

Chapter 3

Evidence of Over- and Under-reaction using the FCM Model

In this chapter, the FCM model as well as its application to determine over-reaction and under-reaction in the South African equity market are discussed in detail. The methodology employed by Aguiar and Sales (2010), and Aguiar (2012) in analysing over-reaction in Brazilian and USA markets was used. First, the mathematical concepts underpinning the models, and a detailed explanation of how the FCM model is used in this dissertation are provided in Section 3.1. Thereafter, the data used in this study is described in Section 3.2. A brief discussion of the financial ratios used in the FCM model is also provided in Section 3.3. In Section 3.4, the procedure that was followed to process the data is given. The manipulation of the data, the statistical analyses and the interpretation of the results with respect to over-reaction and under-reaction are presented and discussed in Sections 3.5 to 3.13. Finally, the findings are summarised in Section 3.14.

3.1 FCM Model

3.1.1 Mathematical Formalism

The FCM model (Aguiar and Sales, 2010) is based on the technique of pattern recognition using the FCM algorithm (Bezdek *et al.*, 1984) which, is based on mathematical theory of fuzzy sets (Aguiar *et al.*, 2006).

Definition 1. *:(Fuzzy sets) Let* $[\boldsymbol{X} : m \times p]$ *be a* $m \times p$ *matrix where the rows refer to vectors in a Euclidean p-space i.e.*

$$\boldsymbol{X} : m \times p = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1p} \\ x_{21} & x_{22} & \dots & x_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{12} & \dots & x_{mp} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1' \\ \boldsymbol{x_2}' \\ \vdots \\ \boldsymbol{x_m}' \end{bmatrix} \tag{3.1.1}$$

which can be also written as:

$$\boldsymbol{X}' : p \times m = \begin{bmatrix} \boldsymbol{x_1}, & \boldsymbol{x_2}, & \dots, & \boldsymbol{x_m} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{21} & \dots & x_{m1} \\ x_{12} & x_{22} & \dots & x_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1p} & x_{2p} & \dots & x_{mp} \end{bmatrix}. \tag{3.1.2}$$

Suppose the m Euclidean vectors can be grouped in n subsets or clusters where each subset consists of the vectors that are, with respect to their Euclidean distances the nearest to the centre of the subset. Denote the n subsets by

$$C_1,\ C_2,\ \dots,\ C_n$$

with the co-ordinates of the centre of C_i *denoted by:*

$$\boldsymbol{c_i} = \begin{bmatrix} c_{i1} \\ c_{i2} \\ \vdots \\ c_{ip} \end{bmatrix},\ i = 1, 2, \dots, n. \tag{3.1.3}$$

Let

$$0 \leqslant \mu_i(\boldsymbol{x}_j) \leqslant 1,$$

be real numbers $i = 1, 2, \ldots n, \quad j = 1, 2, \ldots, m,$ *such that, for every*

$j = 1, 2, \ldots, m,$

$$\sum_{i=1}^{n} \mu_i(\boldsymbol{x}_j) = 1.$$

The function $\mu_i(\boldsymbol{x}_j)$*, called the membership function, gives the degree of membership of the element* $\boldsymbol{x}_j$ *of the subset* C_i *or the similarity of* $\boldsymbol{x}_j$ *to* $\boldsymbol{c}_i$*. The function ranges on a scale from 0 to* 1*.*

At the end of this section, an optimisation algorithm to determine $\boldsymbol{c}_i$ and $\mu_i(\boldsymbol{x}_j), \; i = 1, 2, \ldots, n, \quad j = 1, \ldots, n$, is introduced.

In this dissertation, two groups were considered $(n = 2)$ because the methodology followed, fundamentally compared two groups. Using two groups made it possible to see how the groups changed over time (Good group moves to Bad group, or Bad group moves to Good group). Grouping the data into three clusters would have created nine scenarios to test. In addition, two groups allowed the testing of a small sample sizes. There would have been more outliers for three groupings of the data of a small sample. The formation of three clusters is listed as an avenue for future investigation.

In the following theorem, expressions for $\boldsymbol{c}_i$ and $\mu_i(\boldsymbol{x}_j), i = 1, 2$ are first derived.

Theorem 3.1.1. *(Bezdek, 2013)*

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$ *be elements of* $\boldsymbol{X}'$ *and consider the problem of grouping these elements in 2 p-dimensional subsets with centres* $\mathbf{c}_1$ *and* $\mathbf{c}_2$*. The FCM algorithm determines the centres of two subsets, i.e.* $\mathbf{c}_1$ *and* $\mathbf{c}_2$ *by minimising the function:*

$$\sum_{i=1}^{2} \sum_{j=1}^{m} [\mu_i(\boldsymbol{x}_j)^2 \|\boldsymbol{x}_j - \boldsymbol{c}_i\|^2] \tag{3.1.4}$$

and the solution of such optimization problem is given by:

$$\boldsymbol{c}_i = \frac{1}{\sum_{i=1}^{2}(\mu_i(\boldsymbol{x}_j))^2}\sum_{j=1}^{m}(\mu_i(\boldsymbol{x}_j))^2\boldsymbol{x}_j \tag{3.1.5}$$

and

$$\mu_i(\boldsymbol{x}_j) = \frac{\frac{1}{\|\boldsymbol{x}_j-\boldsymbol{c}_i\|^2}}{\sum_{k=1}^{2}\frac{1}{\|\boldsymbol{x}_j-\boldsymbol{c}_k\|^2}} \tag{3.1.6}$$

$i = 1, 2,\ j = 1, 2, \ldots, m$. *The vectors* $\boldsymbol{c}_i$, *i=1,2 are called centres, and* $\mu_i(\boldsymbol{x}_j)$ *is the membership degree of the element* $\boldsymbol{x}_j$ *with respect to the fuzzy subset* C_i. *The expression* $\|\boldsymbol{x}_j - \boldsymbol{c}_i\|$ *has the usual meaning i.e.:*

$$\|\boldsymbol{x}_j - \boldsymbol{c}_i\| = \sqrt{(x_{j1} - c_{i1})^2 + \cdots + (x_{jp} - c_{ip})^2}. \tag{3.1.7}$$

Proof.

Consider the objective function:

$$\sum_{i=1}^{2}\sum_{j=1}^{m}\mu_i(\boldsymbol{x}_j)^2\|\boldsymbol{x}_j - \boldsymbol{c}_i\|^2,$$

subject to the constraints

$$\sum_{i=1}^{2}\mu_i(\boldsymbol{x}_j) = 1,\ j = 1, 2, \ldots, m.$$

Introducing the Lagrange multiplier λ, the objective function can be rewritten as:

$$L(\mu_i(\boldsymbol{x}_j), \boldsymbol{c}_i, \lambda) = \sum_{i=1}^{2}\sum_{j=1}^{m}\mu_i(\boldsymbol{x}_j)^2\|\boldsymbol{x}_j - \boldsymbol{c}_i\|^2 - \lambda\left(\sum_{i=1}^{2}\mu_i(\boldsymbol{x}_j) - 1\right).$$

To minimise the objective function with respect to $\boldsymbol{c}_i$, the partial derivative with respect to $\boldsymbol{c}_i$ is determined:

$$\frac{\partial L(\mu_i(\boldsymbol{x}_j), \boldsymbol{c}_i, \lambda))}{\partial \boldsymbol{c}_i} = \frac{\partial}{\partial \boldsymbol{c}_i}\left(\sum_{k=1}^{2}\sum_{j=1}^{m}\mu_k(\boldsymbol{x}_j)^2\|\boldsymbol{x}_j - \boldsymbol{c}_k\|^2\right) - 0$$

Hence,

$$\frac{\partial L(\mu_i(\boldsymbol{x}_j), \boldsymbol{c}_i, \lambda)}{\partial \boldsymbol{c}_i} = \sum_{j=1}^{m} \mu_i(\boldsymbol{x}_j)^2 \frac{\partial}{\partial \boldsymbol{c}_i} \|\boldsymbol{x}_j - \boldsymbol{c}_i\|^2$$

with

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{c}_i} \|\boldsymbol{x}_j - \boldsymbol{c}_i\|^2 &= \begin{bmatrix} \frac{\partial}{\partial ci1} \left[(x_{j1} - c_{i1})^2 + \cdots + (x_{jp} - c_{ip})\right]^2 \\ \frac{\partial}{\partial ci2} \left[(x_{j1} - c_{i1})^2 + \cdots + (x_{jp} - c_{ip})\right]^2 \\ \vdots \qquad \vdots \\ \frac{\partial}{\partial cip} \left[(x_{j1} - c_{i1})^2 + \cdots + (x_{jp} - c_{ip})\right]^2 \end{bmatrix} \\
&= \begin{bmatrix} -2(x_{j1} - c_{i1}) \\ -2(x_{j2} - c_{i2}) \\ \vdots \\ -2(x_{jp} - c_{ip}) \end{bmatrix} \\
&= -2(\boldsymbol{x}_j - \boldsymbol{c}_i)
\end{aligned}$$

so that:

$$\frac{\partial}{\partial \boldsymbol{c}_i} L(\mu_i(\boldsymbol{x}_j), \boldsymbol{c}_i, \lambda) = -2 \sum_{j=1}^{m} \mu_i(\boldsymbol{x}_j)^2 (\boldsymbol{x}_j - \boldsymbol{c}_i). \tag{3.1.8}$$

Setting the partial derivatives to zero, it follows that:

$$\frac{\partial}{\partial \boldsymbol{c}_i} L(\mu_i(\boldsymbol{x}_j), \boldsymbol{c}_i, \lambda) = -2 \sum_{j=1}^{m} \mu_i(\boldsymbol{x}_j)^2 (\boldsymbol{x}_j - \boldsymbol{c}_i) = 0 \tag{3.1.9}$$

which implies that:

$$\sum_{j=1}^{m} (\mu_i(\boldsymbol{x}_j))^2 \boldsymbol{x}_j = \sum_{j=1}^{m} (\mu_i(\boldsymbol{x}_j))^2 \boldsymbol{c}_i.$$

It follows that:

$$\begin{aligned}
\boldsymbol{c}_i &= \begin{bmatrix} c_{i1} \\ c_{i2} \\ \vdots \\ c_{ip} \end{bmatrix}, \; i = 1,2 \\
&= \frac{1}{\sum_{j=1}^{m}(\mu_i(\boldsymbol{x}_j))^2} \sum_{j=1}^{m} (\mu_i(\boldsymbol{x}_j))^2 \boldsymbol{x}_j \\
&= \frac{1}{\sum_{j=1}^{m}(\mu_i(\boldsymbol{x}_j))^2} \left[(\mu_i(\boldsymbol{x}_1)^2 \boldsymbol{x}_1 + (\mu_i(\boldsymbol{x}_2)^2 \boldsymbol{x}_2 + \cdots + (\mu_i(\boldsymbol{x}_m)^2 \boldsymbol{x}_m\right].
\end{aligned}$$

On the other hand, to minimise the objective function with respect to $\mu_i(\boldsymbol{x}_j)$, the partial derivative of $L(\mu_i(\boldsymbol{x}_j), \boldsymbol{c}_i, \boldsymbol{\lambda})$ with respect to $\mu_i(\boldsymbol{x}_j)$ is determined:

$$\begin{aligned}
\frac{\partial}{\partial \mu_i(\boldsymbol{x}_j)} L(\mu_i(\boldsymbol{x}_j), \boldsymbol{c}_i, \lambda) &= \frac{\partial}{\partial \mu_i(\boldsymbol{x}_j)} \left[\sum_{i=1}^{2} \sum_{j=1}^{m} \mu_i(\boldsymbol{x}_j)^2 \|\boldsymbol{x}_j - \boldsymbol{c}_i\|^2 - \lambda \left(\sum_{i=1}^{2} \mu_i(\boldsymbol{x}_j) - 1 \right) \right] \\
&= 2\mu_i(\boldsymbol{x}_j) \|\boldsymbol{x}_j - \boldsymbol{c}_i\|^2 - \lambda.
\end{aligned}$$

Setting this to zero leads to:

$$\mu_i(\boldsymbol{x}_j) = \frac{\lambda}{2\|\boldsymbol{x}_j - \boldsymbol{c}_i\|^2}. \tag{3.1.10}$$

Given that:

$$\sum_{i=1}^{2} \mu_i(\boldsymbol{x}_j) = 1 \qquad \forall j = 1 \ldots m \,,$$

it follows that:

$$\sum_{k=1}^{2} \frac{\lambda}{2\|\boldsymbol{x}_j - \boldsymbol{c}_k\|^2} = 1$$

$\Rightarrow$

$$\sum_{k=1}^{2} \frac{1}{\|\boldsymbol{x}_j - \boldsymbol{c}_k\|^2} = \frac{2}{\lambda}$$

which gives,

$$\frac{\lambda}{2} = \left[\sum_{k=1}^{2} \frac{1}{\|\boldsymbol{x}_j - \boldsymbol{c}_k\|^2} \right]^{-1}.$$

Substituting $\frac{\lambda}{2}$ into (3.1.10), leads to

$$\begin{aligned} \mu_i(\boldsymbol{x}_j) &= \left[\sum_{k=1}^{2} \frac{1}{\|\boldsymbol{x}_j - \boldsymbol{c}_k\|^2} \right]^{-1} \frac{1}{\|\boldsymbol{x}_j - \boldsymbol{c}_i\|^2} \\ &= \frac{\frac{1}{\|\boldsymbol{x}_j - \boldsymbol{c}_i\|^2}}{\sum_{k=1}^{2} \frac{1}{\|\boldsymbol{x}_j - \boldsymbol{c}_k\|^2}}, \end{aligned}$$

which proves the theorem. ■

The steps of the FCM algorithm are summarised below:

- Step 1: Initialise the membership degrees, such that

$$\mu_1(\boldsymbol{x}_j) + \mu_2(\boldsymbol{x}_j) = 1, \; j = 1, 2 \ldots, m, \tag{3.1.11}$$

  and

$$\mu_1(\boldsymbol{x}_j) \geqslant 0 \text{ and } \mu_2(\boldsymbol{x}_j) \geqslant 0, \; j = 1, 2 \ldots, m; \tag{3.1.12}$$

- Step 2: Determine the centres $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$, using (3.1.5);
- Step 3: Use the output centres from Step 2 to update the new membership degrees, via (3.1.6).
- Step 4: Repeat Steps 2 and 3 until the objective function does not decrease further, given the desired precision.

3.1.1.1 Illustration

In this section, it is illustrated with a simple example how the FCM algorithm is applied to the pattern matrix to form two clusters. The results of this classification were obtained using the function cmeans() [in e1071 R package].

Let

$\boldsymbol{x}_1' = (1, 2),\ \boldsymbol{x}_2' = (2, 1),\ \boldsymbol{x}_3' = (2, 3),\ \boldsymbol{x}_4' = (4, 1),\ \boldsymbol{x}_5' = (4, 3)$ and $\boldsymbol{x}_6' = (5, 2)$.

Here $n = 6, \quad p = 2$.

The pattern matrix $\boldsymbol{X}$ follows as:

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}_1' \\ \boldsymbol{x}_2' \\ \boldsymbol{x}_3' \\ \boldsymbol{x}_4' \\ \boldsymbol{x}_5' \\ \boldsymbol{x}_6' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 3 \\ 4 & 1 \\ 4 & 3 \\ 5 & 2 \end{bmatrix}. \tag{3.1.13}$$

The FCM algorithm was applied in the above data set in order to form two clusters. The following results were obtained:

Cluster centres:

$$\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 1.655187 & 2.000045 \\ 4.344813 & 1.999955 \end{bmatrix}$$

Table 3.1: Cluster memberships

j	$\mu_1(x_j)$	$\mu_2(x_j)$	$\sum_{i=1}^{2} \mu_i(x_j)$
1	0.96304821	0.03695179	1
2	0.85311804	0.14690551	1
3	0.85311804	0.14688196	1
4	0.1468819	0.85311804	1
5	0.14690551	0.85309449	1
6	0.03695179	0.96304821	1

In Figure 3.1, the graph of the two clusters with the centres $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ is presented.

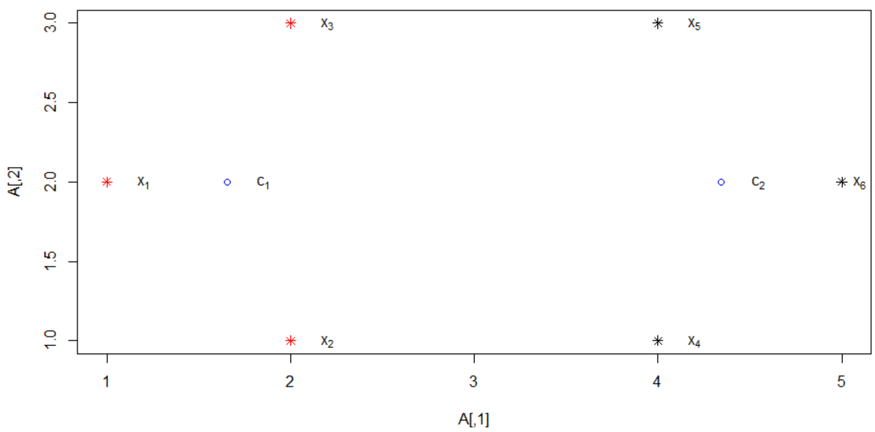


Figure 3.1: Fuzzy C-Means Clustering

From this, it is clear that x_1, x_2 and x_3 belong to $\boldsymbol{c}_1$ and x_4, x_5 and that x_6 belong to $\boldsymbol{c}_2$.

3.1.2 Model Description

In this section, the process of constructing portfolios from the selected stocks is described. In the context of this dissertation, the FCM algorithm was applied to group or classify the sample of listed companies on the JSE into subsets (clusters) in such a way that the stocks of a specific group may be more similar to, or more compatible with, themselves than elements in different groups with respect to certain financial characteristics (features, financial indices, profitability, etc.). The procedure consisted of two steps: pattern recognition (or centre classification) and stock rating (performance measurement). The data were split into two sets for use in the FCM model: (i) the training data set and (ii) the testing data set. The training data set was used for pattern recognition and stock rating, and the testing data set was used to determine over-reaction and under-reaction.

3.1.2.1 Pattern Recognition

In **Step 1**, namely pattern recognition, the FCM algorithm was applied to a set of stocks to produce two subsets. The stocks in a specific subset were more similar, and the two subsets were different.

The analysis was based on a $n \times p$ matrix, $\boldsymbol{X} : n \times p$, named a pattern matrix where n is the number of stocks (elements) and p is the dimension of the vectors, i.e. the characteristics of the stocks. At the end of each quarter of the training data set, the p variables (characteristics) were observed for each stock, to form the $n \times p$ data matrix $\boldsymbol{X} : n \times p$. The j^{th} row vector of the matrix represented the j^{th} stock in the p-dimensional Euclidean space. Let n_1 be the number of stocks in one group (subset, cluster), say Group 1, and n_2 the number of stocks in the second group, say Group 2. The two groups are written as two matrices, $\boldsymbol{X}_1 : n_1 \times p$ and $\boldsymbol{X}_2 : n_2 \times p$ with the vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ the centres of the corresponding groups which were determined by applying the FCM algorithm discussed in Section 3.1.

In the application of the FCM algorithm, the membership degree, on a scale from 0 to 1, of each stock with respect to each group (subset) was determined. This refers to the quantity:

$$\mu_i(\boldsymbol{x}_j), \quad i = 1, 2, \quad j = 1, \ldots, n, \tag{3.1.14}$$

in Equation 3.1.6, and was dependent on the Euclidean distance between the centre of group i i.e. $\boldsymbol{c}_i$ and the j^{th} stock. The membership degree of a stock measured similarity between the stock and a specific group (cluster), and was used to classify a stock as a member of a specific group. The stocks classified in say Group 1 were those with higher membership degrees with respect to Group 1 when compared to the membership degrees of the stocks classified in Group 2. It should be noted that stocks indicating equivalent similarity with regard to each cluster should be discarded.

At the end of period t, the FCM algorithm was applied to the data matrix, $\boldsymbol{X}$, and two groups were obtained. The average financial log return that each cluster produced at the end of the quarter $t+1$ was calculated as follows:

$$r_{t+1} = \frac{1}{n}\sum_{i=1}^{n} \ln \frac{p^i_{t+1}}{p^i_t} \tag{3.1.15}$$

where p^i_t is the value of stock i at the end of quarter t, and n is the number of stocks classified.

The different steps of the application of the FCM algorithm to the matrix of financial ratios to form the two clusters are given in Figure 3.2 as a summary of the method.

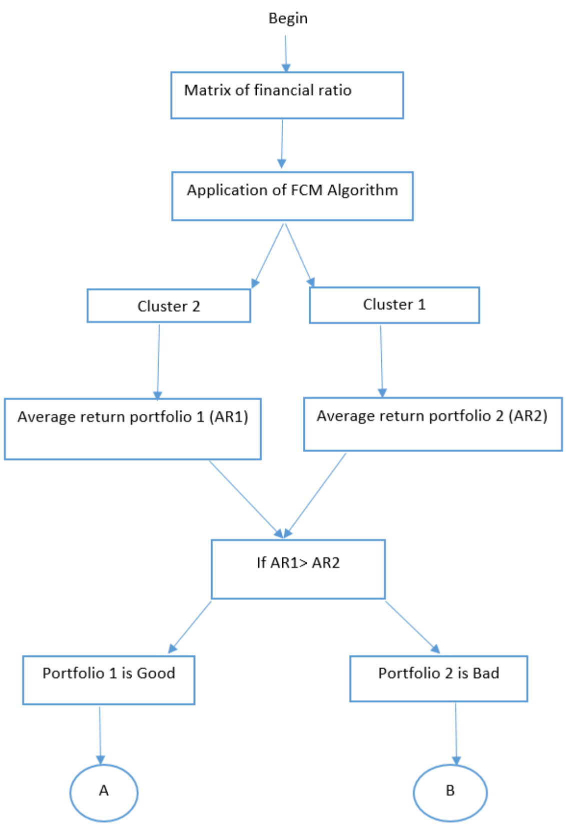


Figure 3.2: Steps of the algorithm

3.1.2.2 Stock Rating

In **Step 2**, the stock rating step, the objective was to identify (classify) at the end of quarter t, $t = 1, \dots, 4$ the stocks whose performance would be good or bad by the end of quarter $t + 1$.

Define for this purpose

$$\boldsymbol{c}_{ij}^{t} : p \times 1 \tag{3.1.16}$$

as the centre of group $i = 1$ for the good group and $i = 2$ for the bad group, after quarter t, $t = 1, \dots, 4$ of year j, $j = 1, \dots n$.

For each quarter t of the n years, the FCM algorithm was applied to the set of good and bad centres' vectors $\boldsymbol{c}_i^t$ obtained in Step 1. For example with respect to quarter t, the FCM algorithm was applied to the following $p \times 2n$ matrix, say $\boldsymbol{X}_t'$:

$$\boldsymbol{X}_t' : \ p \times 2n \ = \begin{bmatrix} \boldsymbol{c}_{11}^t & \boldsymbol{c}_{21}^t & \boldsymbol{c}_{12}^t & \boldsymbol{c}_{22}^t & \dots \boldsymbol{c}_{1n}^t & \boldsymbol{c}_{2n}^t \end{bmatrix}. \tag{3.1.17}$$

In this way, two new reference centres' vectors, which are centres' vectors of sets of centres' vectors were produced for quarter t. The centre around which there was a greater number of good (bad) centres was called the winner (loser) centre and denoted as $\boldsymbol{c}_g^t$ $(\boldsymbol{c}_b^t)$.

After this procedure had been applied, two centres' vectors for each of the four quarters had been determined. For the future data, i.e. the testing data set, at the end of quarter t of a given year, the membership degree of each stock, $\boldsymbol{x}_j$, with respect to the bad and good group, was calculated:

$$\mu_g^t(\boldsymbol{x}_j) = \frac{\frac{1}{\|\boldsymbol{x}_j - \boldsymbol{c}_g^t\|^2}}{\frac{1}{\|\boldsymbol{x}_j - \boldsymbol{c}_g^t\|^2} + \frac{1}{\|\boldsymbol{x}_j - \boldsymbol{c}_b^t\|^2}} \tag{3.1.18}$$

and

$$\mu_b^t(\boldsymbol{x}_j) = \frac{\frac{1}{\|\boldsymbol{x}_j - \boldsymbol{c}_b^t\|^2}}{\frac{1}{\|\boldsymbol{x}_j - \boldsymbol{c}_g^t\|^2} + \frac{1}{\|\boldsymbol{x}_j - \boldsymbol{c}_b^t\|^2}}. \tag{3.1.19}$$

If $\mu_g^t(\boldsymbol{x}_j) > \mu_b^t(\boldsymbol{x}_j)$, stock j with observations, $\boldsymbol{x}_j$, was classified as a promising stock and if $\mu_g^t(\boldsymbol{x}_j) < \mu_b^t(\boldsymbol{x}_j)$, stock j was classified as a non-promising stock. In this way, winner and loser portfolios were constructed by selecting the best- and the worst performing stocks at the end of each quarter.

The different steps of the classification process are given in Figure 3.3 as a summary of the method.

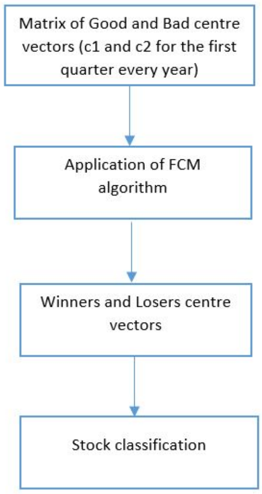


Figure 3.3: Steps of the classification process

3.1.2.3 Computing the Stocks' Residual Returns

Consider the training data set and define:

- p_t^i is the price of stock i of a portfolio (winner or loser) at the end of quarter t,
- $p_{t+1,j}^i$ is the price of stock i, of a portfolio (winner or loser) at the end of week j of quarter $t+1$,
- n is the number of stocks in the portfolio (winner or loser),
- JSE_t is the market index at the end of quarter t,
- $\text{JSE}_{t+1,j}$ is the market index at the end of week j of the quarter $t+1$,

- $r^W_{t+1,j}(r^L_{t+1,j})$ is the average log return of the winner (loser) portfolio from the end of quarter t to the end of week j of quarter $t+1$, i.e.

$$r^W_{t+1,j} = \frac{1}{n}\sum_{i=1}^{n} \ln \frac{p^i_{t+1,j}}{p^i_t}, \tag{3.1.20}$$

- $r^{\text{JSE}}_{t+1,j}$ is the log return on the market index from the end of quarter t to the end of week j of quarter $t+1$, i.e.:

$$r^{\text{JSE}}_{t+1,j} = \ln \frac{\text{JSE}_{t+1,j}}{\text{JSE}_t}. \tag{3.1.21}$$

For each week j of quarter $t+1$, the corresponding residual return $rr^W_{t+1,j}(rr^L_{t+1,j})$ was defined as the return generated by a portfolio from the end of quarter t to the end of week j of quarter $t+1$ minus the corresponding market return i.e.:

$$rr^W_{t+1,j} = r^W_{t+1,j} - r^{JSE}_{t+1,j}, \tag{3.1.22}$$

$$rr^L_{t+1,j} = r^L_{t+1,j} - r^{JSE}_{t+1,j}. \tag{3.1.23}$$

The average residual returns, of the winner portfolio rrm^W_t and similarly rrm^L_t for the loser portfolio in quarter t were calculated as follows:

$$rrm^W_t = \frac{1}{s}\sum_{j=1}^{s} rr^W_{t,j} \tag{3.1.24}$$

$$rrm^L_t = \frac{1}{s}\sum_{j=1}^{s} rr^L_{t,j} \tag{3.1.25}$$

with s the number of weeks in quarter t. The average residual returns were used to test the behavioural hypotheses. In the following section, the hypothesis test is formulated and linked to over-reaction and under-reaction.

3.1.3 Statistical Evidence of Over-reaction and Under-reaction

In this sub-section, the process of testing the significance of the investor over-reaction on the JSE is considered. The statistical significance of the difference between over-reaction and under-reaction was tested by applying the well-known two-sample t-test.

Consider the quantity $rr_{t,j}^{W}$:

$$\begin{aligned} rr_{t,j}^{W} &= \text{log return over first j weeks of quarter t of winner portfolio} \\ &\qquad - \text{log return over first j weeks of JSE market index},\ j = 1 \dots s \\ &= r_{t,j}^{W} - r_{t,j}^{JSE} \\ &= \text{cumulative excess return at week j} \\ &= \text{observation of stochastic variable } RR_{t,j}^{W} \end{aligned}$$

with $RR_{t,1}^{W}, RR_{t,2}^{W}, \dots RR_{t,s}^{W}$ the corresponding sample elements with respect to the stochastic variable RR_{t}^{W}, the cumulative excess return of quarter t. To apply a two-sample t-test to test if the average cumulative excess return of the winner portfolio was significantly different from that of the loser portfolio at the end of quarter t, it is required that RR_{t}^{W} had a normal distribution. Hence, assume as in (Aguiar and Sales, 2010) that RR_{t}^{W} be normally distributed with expected value μ_{t}^{W} and variance $\sigma_{t,W}^{2}$ i.e.

$$RR_{t}^{W} \sim n(\mu_{t}^{W}, \sigma_{t,W}^{2}). \tag{3.1.26}$$

The parameter μ_{t}^{W} can thus be interpreted as the average cumulative excess return of quarter t, estimated by:

$$\widehat{\mu}_{t}^{W} = rrm_{t}^{W} = \frac{1}{s} \sum_{j=1}^{s} rr_{t,j}^{W}. \tag{3.1.27}$$

Similar definitions of the symbols (concepts) follow for the loser portfolio, with the W replaced by an L.

In an efficient market, it is impossible to take advantage of past information to out-perform the market. The future performance of the formed portfolios should not be predictable based on past performance (Fama, 1965). This means, that the expected cumulative excess return of quarter t of the winner portfolio should be equal to the expected cumulative return of quarter t of the loser portfolio, i.e.:

$$H_0 : \mu_t^L = \mu_t^W$$

which gives

$$H_0 : \mu_t^L - \mu_t^W = 0$$

In a case where over-reaction exists, $\mu_t^L - \mu_t^W < 0$, since investors' over-reaction would drive the stock prices in the opposite direction; so,

$$H_{a1} : \mu_t^W - \mu_t^L < 0.$$

Over-reaction occurs if the former winner stocks under-perform loser stocks, in other words, if a winner portfolio has a lower average residual return than a loser portfolio. In this case, there is a reversal effect where stocks that have been losers in a given period subsequently yield higher returns than the corresponding winner stocks.

On the other hand, if the market under-reacts to the arrival of new information,

$$H_{a2} : \mu_t^W - \mu_t^L > 0. \tag{3.1.28}$$

If the winner portfolio again produces superior performance relative to the loser portfolio, it will be classified as under-reaction. This means that there is a momentum effect as stocks with good past performance continue to out-perform in the future.

A parametric two-sample t-test was performed to determine if the difference between average returns in each quarter was significant.

To perform the test it was assumed that:

$$RR_t^W \sim n(\mu_t^W, \sigma_{t,W}^2) \text{ and } RR_t^L \sim n(\mu_t^L, \sigma_{t,L}^2) \tag{3.1.29}$$

under the assumption that $\sigma_{t,W}^2 = \sigma_{t,L}^2 = \sigma_t^2$ (say). Hence, the hypotheses which were:

$$H_0 : \sigma_{t,W}^2 = \sigma_{t,L}^2 = \sigma_t^2$$
$$H_a : \sigma_{t,W}^2 \neq \sigma_{t,L}^2$$

needed to be tested using the test statistic:

$$F = \frac{\widehat{\sigma}_{t,W}^2}{\widehat{\sigma}_{t,L}^2} \sim F_{s-1,s-1}. \tag{3.1.30}$$

If this null hypothesis were not rejected, the pooled estimate of σ_t^2 was given by:

$$\widehat{\sigma_t^2} = s_t^2 = \frac{\left[\sum_{j=1}^{s}(rr_{t,j}^W - rrm_t^W)^2 + \sum_{j=1}^{s}(rr_{t,j}^L - rrm_t^L)^2\right]}{2(s-1)} \tag{3.1.31}$$

and the following hypothesis could be carried out:

$$H_0 : \mu_{t,L}^2 - \mu_{t,W}^2 = 0$$

$$H_a = \begin{cases} \mu_{t,L}^2 - \mu_{t,W}^2 > 0 & \text{over-reaction} \quad (\text{say } H_{a1}) \\ \mu_{t,L}^2 - \mu_{t,W}^2 < 0 & \text{under-reaction} \quad (\text{say } H_{a2}). \end{cases}$$

The two-sample t-statistic used to test this hypothesis was as follows:

$$T_t = \frac{rrm_t^L - rrm_t^W}{\sqrt{2\frac{s_t^2}{s}}} \sim t_{2s-2}. \tag{3.1.32}$$

The null hypothesis proposes that, there is neither over-reaction nor under-reaction. The alternative hypothesis H_{a1} proposes that, there are over-reaction effects because winner portfolios under-perform loser portfolios with lower residual returns. The alternative hypothesis H_{a2} proposes that there are under-reaction effects, as winner portfolios out-perform loser portfolios with higher residual returns.

When the two population variances were not assumed to be equal, Welch's t-test was used to test whether the population means were different. Welch's t-test is defined by the statistic t as follows:

$$t = \frac{rrm_t^L - rrm_t^W}{\sqrt{\frac{s_1^2}{N_1} + \frac{s_2^2}{N_2}}}. \tag{3.1.33}$$

In the following section, data that formed the basis of the investigation are described, including the source, the sample size, and processing procedures. All variables are explained and defined. Some data challenges encountered are discussed, and the detailed procedures for addressing the challenges are presented.

3.2 Data Description

3.2.1 Overview of the South Africa Market

The JSE was formed in 1887 during the first South African gold rush. The JSE joined the World Federation of Exchanges in 1947 and upgraded to an electronic trading system (JET) on 7 June 1996. The JSE Limited listed on its own exchange in 2005 as a result of the bourse demutualising (JSE, 2021).

The JSE is as of 2020 the 19^{th} largest stock exchange in the world with a market capitalisation of USD 894 billion (Finance, 2020). The JSE is the largest of Africa's 29 stock exchanges by market capitalisation.

According to Banerjee and Ghosh (2004), the more liquid the market is, the less it is inefficient. There is evidence showing that under-reaction and over-reaction are linked to trading volume, which was proxied by the turnover ratio (Lee and Swaminathan, 2000). The higher the stock turnover (a measure of stock liquidity), the more liquid a company's stocks are. Following these insights and empirical evidence, the JSE was considered a particularly interesting case study for analysing over-reaction and under-reaction.

3.2.2 Sample Selection

The 100 shares with the largest market capitalisation at the end of every calendar year from 2006 to 2016 were considered for the study. These shares had sufficient liquidity and depth of coverage by analysts and investors to be considered for a study on behavioural finance. In total, a sample of 163 shares had sufficient financial statement data on the Iress and Bloomberg databases to be included in the study.

A list of the companies is provided in Appendix A. The variables that were required for the fuzzy analysis of over- and under-reaction were:

- the quarterly total return index values that included reinvested dividends for all companies in the sample, downloaded from Bloomberg;
- the weekly total return index values for the All Share Index (J203T) for market benchmarking, downloaded from Bloomberg; and
- the data of fundamental variables (current ratio, debt and assets, dividend, earnings per share, net asset value per share, retention rate, total asset return) as obtained from the actual financial statements, downloaded from Iress. Definitions of the variables are provided in Section 3.3.

The companies were divided into three mains sectors according to their SA Sector classification (JSE, 2006), as follows:

- Resources: companies that belong to ICB Industries Oil & Gas (0001) and Basic Materials (1000);

- Financials: companies that belong to ICB Industry Financials (8000); and

- Industrials: companies that do not belong to ICB Industries Financials (8000), Oil & Gas (0001) and Basic Materials (1000).

In Figure 3.4, the percentage of companies allocated to each sector is represented.

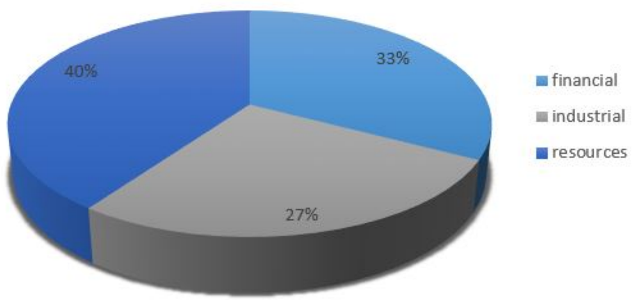


Figure 3.4: Percentage of companies allocated to each sector

In Figure 3.4, it can be seen that the resources sector had the highest representation in the sample: 40%, followed by the financial sector (33%), and the industrial sector (27%). The fundamental variables used in this study were financial ratios of companies. A brief discussion of the financial ratios is provided in the following sub-section.

3.3 Financial Ratios

Financial ratios are used to evaluate the financial statements of a company in order to determine its performance. Ratios ensure that the values are comparable between companies, as the individual line items of financial statements are not always comparable. Financial ratios can be grouped into profitability, liquidity, efficiency ratios, debt ratios, and market ratios (Brigham and Daves, 2014). In the following sub-sections, the various groupings are briefly introduced.

3.3.1 Profitability Ratios

Profitability ratios provide insight into the financial performance of a company, i.e. the return on its investments (Brealey *et al.*, 2018). A higher value means that the business is performing well.

Stock returns and the future profitability of a company can be predicted by the trend in a company's profitability (Akbas *et al.*, 2017). The DuPont system is used for splitting the return on equity (ROE) into its contributing components in order to assess which components contributed towards overall ROE. Some profitability measures can be linked by the Du Pont system relationships (Brealey *et al.*, 2001). Profitability ratios used in the current study were the following:

3.3.1.1 Return on Assets

The return on assets (ROA) ratio measures how efficient a company's management is at using its assets to generate revenues. ROA is calculated as the ratio of net income to total assets (Brealey *et al.*, 2018).

3.3.1.2 Retention Ratio

Retention ratio (RR) is the portion of a company's earnings that is retained and reinvested (Chasan, 2012). RR is calculated as 1 minus the dividend payout ratio. Higher retention rates, combined with profitable reinvestment in assets, mean that the stock is continually appreciating due to company growth; however, the company does not pay out much in dividends.

3.3.2 Liquidity Ratios

Liquidity ratios measure the ease with which a firm can access cash (Brealey *et al.*, 2012). Liquidity ratios used in the current study were as follows:

3.3.2.1 Current Ratio

The current ratio is the ratio of total current assets to total current liabilities (Brealey *et al.*, 2001). It provides some indication as to how financially strong a company is, and also how efficiently it is investing its current assets. A higher value means that the company is capable of paying its short-term obligations by means of its short-term assets. According to Zarb (2018), the current ratio is the most basic and commonly used ratio to measure liquidity.

3.3.2.2 Net Asset Value per Share

The net asset value (NAV) of a single share is calculated by dividing its NAV (total assets less liabilities) by the number of shares that are outstanding (Brigham and Daves, 2014). It is a proxy for the relative size of the company.

3.3.3 Efficiency Ratios

Efficiency ratios or turnover ratios measure how effectively a company is managing its assets and liabilities (Brealey *et al.*, 2001). The efficiency ratio used in the current study was total asset return.

3.3.3.1 Total Asset Return

Total asset return (TAR) is the ratio of net sales to average total assets. Higher total asset return means more revenue for the company.

3.3.4 Debt to Asset Ratio

The debt to asset ratio (DA), also known as the debt ratio indicates how heavily the company is in debt. The debt ratio is the ratio of total debt to total assets (Brigham and Daves, 2014). A high ratio means there is great risk associated with the firm's capital structure.

3.3.5 Market Ratios

Market ratios are used to determine whether a stock is overpriced or underpriced. Market ratios used in the current study were as follows:

3.3.5.1 Dividend Yield

The dividend yield (DIV) is the ratio of the total dividends paid per year to the market price of the stock (Vernimmen *et al.*, 2019). A high dividend yield is attractive to investors.

3.3.5.2 Earnings per Share

Earnings per share (EPS) is the portion of a company's profit that is allocated to each individual share of the stock. EPS is calculated by dividing net income by the average number of common shares (Vernimmen *et al.*, 2019). A higher EPS value means that the company is more profitable and has more profits to distribute to its shareholders.

In the previous section, a brief discussion of the financial ratios which form the foundation of the Fuzzy model was provided. Academic literature does not provide guidance on which ratios are most important (Ou and Penman, 1989). In the current study, seven variables were chosen out of many, because different ratios often convey the same information about a company, thus making other existing financial ratios correlated to these seven ratios. The combination of these seven financial variables gave a more complete picture of different aspects of a company's financial health through insight into the company's liquidity, efficiency, and profitability. One ratio by itself may not give the full picture if not viewed as part of a whole. Thus, the variables formed a suitable framework when viewed together as a seven-dimensional measurement instrument to distinguish between the companies under study.

3.4 Data Processing

The analysis of the FCM model was implemented using R Project for Statistical Computing. The following procedure was followed:

- Data for the seven fundamental variables (CR, DA, DIV, EPS, NAV per share, RR, and TAR), drawn from the companies' financial statements were downloaded from Iress.

- The data were then matched to the quarter in which it were realised.
- The interim financial ratio was calculated using the interim financial statement. For example, to calculate the interim current ratio for stock ACL, total current assets (line 514 on the interim financial statement) was divided by total current liabilities (line 515 on the interim financial statement). The other ratios were calculated according to the definitions provided in Section 3.3. The calculations are provided in Appendix C.
- Data obtained from the interim statement were merged with the year-end financial statement data.
- The merged data for all the stocks were stored in an Excel workbook with eight spreadsheets, with each spreadsheet containing one of the seven variables and the return data for all the quarters and all the years.
- The data were scrutinised for possible errors, for example, any mismatch between columns of the data.
- For each stock and each quarter, data points were selected based on the availability of all seven fundamental variables and return data. For example, the first quarter of 2007 data point for the stock ACL was considered a "GOOD" data point if the CR, DA, DIV, EPS, NAV per share, RR, TAR, and return data were available at that point. If data on one of the variables were unavailable, the data point was considered "INCOMPLETE". Delisted stocks were considered "MISSING" (see illustration in Table 3.2).
- For each stock and each quarter, a boolean function was created to insert the value 1 at every location in the data points matrix where the data frame from the data points' spreadsheet had a value equal to "GOOD!!", and 0 otherwise.
- A matrix was created that contained only 1s and 0s. This was the pattern matrix referred to as $\boldsymbol{X}$ in Section 3.1.2.1, which served as input data for the FCM algorithm.

- For each quarter and each year, the FCM algorithm was applied to the pattern matrix to group or classify stocks.

In Table 3.2, an example of data points as given for five stocks for Quarters 1,2,3 and 4 of 2006, 2007, and 2008 respectively indicates how data were selected to form the entry matrix of the FCM algorithm.

Table 3.2: Data points

CLOSE	ACL	ACP	AEG	AEL	AEN
31/03/2006	MISSING	GOOD!!	MISSING	GOOD!!	MISSING
30/06/2006	MISSING	GOOD!!	GOOD!!	GOOD!!	MISSING
30/09/2006	MISSING	GOOD!!	GOOD!!	GOOD!!	MISSING
31/12/2006	GOOD!!	GOOD!!	GOOD!!	GOOD!!	MISSING
31/03/2007	GOOD!!	GOOD!!	GOOD!!	GOOD!!	MISSING
30/06/2007	GOOD!!	GOOD!!	GOOD!!	GOOD!!	MISSING
30/09/2007	GOOD!!	GOOD!!	GOOD!!	GOOD!!	MISSING
31/12/2007	GOOD!!	GOOD!!	GOOD!!	GOOD!!	MISSING
31/03/2008	GOOD!!	GOOD!!	GOOD!!	GOOD!!	MISSING
30/06/2008	MISSING	GOOD!!	GOOD!!	GOOD!!	MISSING
30/09/2008	MISSING	GOOD!!	GOOD!!	GOOD!!	MISSING
31/12/2008	GOOD!!	GOOD!!	GOOD!!	GOOD!!	MISSING

From Table 3.2, it can be seen that AEN data were missing for 2006, 2007, and 2008, meaning that AEN was delisted during those periods, or have been dropped from the All Shares Index. For AEG, the data for all seven ratios and the return data were available from Quarter 2. There were 163 stocks in total (65 stocks for the resources sector, 54 for the financial sector, and 44 stocks for the industrial sector). Companies that had been delisted for a period were not excluded from the sample, to reduce survivorship bias. New arrays were set, which contained only the data for the companies that had sufficient data at every point. The number of stocks therefore changed from quarter to quarter. In Table 3.3, the number of available data points for the training and the testing data is presented.

Table 3.3: Description of training and testing data

Training data			Testing data		
Year	Quarter	Data points	Year	Quarter	Data points
2006	Q2	99	2012	Q1	128
	Q3	99		Q2	127
	Q4	104		Q3	126
2007	Q1	102		Q4	125
	Q2	104	2013	Q1	126
	Q3	104		Q2	126
	Q4	110		Q3	124
2008	Q1	110		Q4	124
	Q2	112	2014	Q1	127
	Q3	114		Q2	128
	Q4	119		Q3	124
2009	Q1	120		Q4	124
	Q2	121	2015	Q1	125
	Q3	121		Q2	126
	Q4	123		Q3	125
2010	Q1	122		Q4	126
	Q2	122	2016	Q1	127
	Q3	124		Q2	127
	Q4	125		Q3	124
2011	Q1	127		Q4	110
	Q2	126			
	Q3	127			
	Q4	128			

It can be observed from Table 3.3 that the number of stocks (N) varied, and was less than 120 between 2006 and 2009. The FCM algorithm was applied to group or classify the companies listed on the JSE into two distinct clusters at various points in time. In this way, the stocks in a specific group were more similar to each other than the stocks in the other defined groups with respect to the seven financial variables describe above. These variables were: CR, RR, NAV, DA, TAR, DIV, and EPS. In the following section, the results of the over-reaction and under-reaction analysis are presented and discussed.

3.5 Results of Analysis of Over-reaction and Under-reaction

This section contains:

(i) the results of the raw JSE data and the issues encountered because the variables analysed did not have the same order of magnitude;

(ii) a comparison of how the different scaling methods transformed each variable's order of magnitude and how a suitable method was selected;

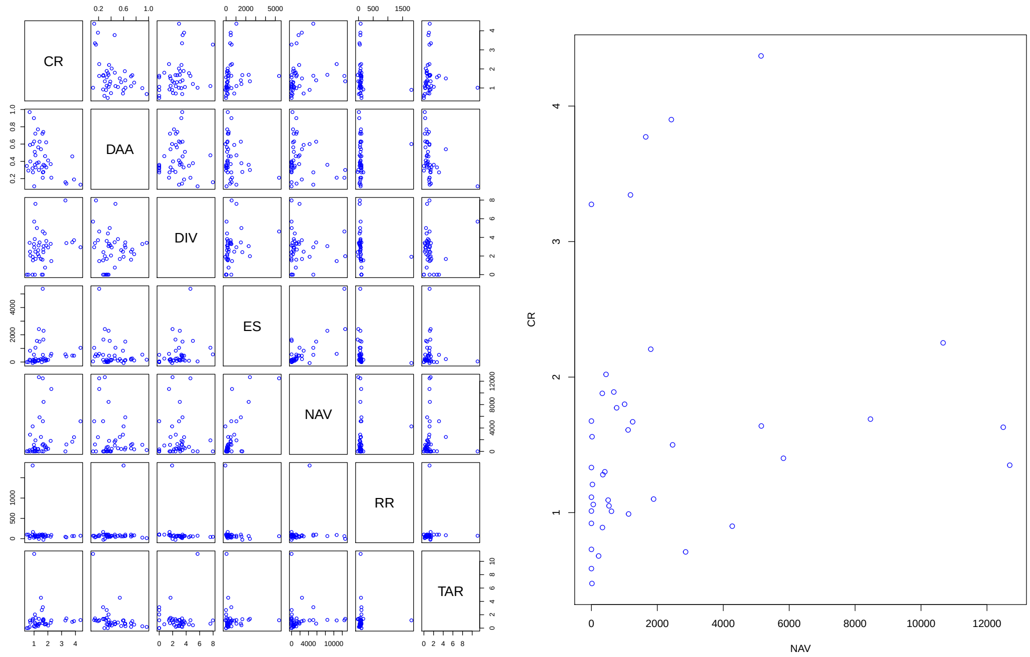
(iii) the impact of winsorizing the raw JSE data before transforming it, because the data contained outliers;

(iv) how the winsorized and re-scaled data were used to identify the historical occurrence of over-reaction and under-reaction in the financial, resources and, industrial sectors, together with a summary of the findings.

The FCM algorithm was applied on both the raw data (original data) and on the standardised data, in order to assess the impact of using transformed data with the FCM model. In this section, a subset of the results using the original data is presented, to show how the FCM performed in terms of the untransformed data.

3.5.1 Classification Results on the Original Data

As an example, and to illustrate the characteristics of the data, a scatter matrix of the original resources sector 2007 Quarter 1 data is depicted in Figure 3.5A. The results are later compared to the transformed data.



(A) Scatter Matrix **(B)** Scatter plot (NAV, CR)

Figure 3.5: Original data

In Figure 3.5A, it is shown that variables were multi-scaled. For example, plotting the original (NAV) data against the (CR) in Figure 3.5B showed that NAV ranged from 0 to 20 000, while CR ranged from 0 to 10. There were significant scaling differences between the two variables, which could cause problems with the Euclidean distance metric. Therefore, the original data of the two variables had to be used together with great caution. The FCM algorithm was applied on the original data at the end of every quarter of the training data period for the different sectors. In Table 3.4, the sizes of the different groups (winners or losers) for every quarter and sector are provided.

Table 3.4: Results of the FCM algorithm on original data

		Resources		Financial		Industrial	
Year	Quarter	Numb Win	Numb Los	Numb Win	Numb Los	Numb Win	Numb Los
2012	Q1	46	7	7	31	12	25
	Q2	5	48	37	0	25	12
	Q3	42	10	32	5	29	8
	Q4	8	43	31	7	29	7
2013	Q1	43	9	9	29	12	24
	Q2	10	41	37	0	26	12
	Q3	40	10	31	6	30	7
	Q4	7	43	30	8	28	8
2014	Q1	42	10	11	28	12	24
	Q2	10	42	39	0	24	13
	Q3	41	10	30	6	29	8
	Q4	7	43	28	9	29	8
2015	Q1	41	10	13	24	12	25
	Q2	8	43	38	0	22	15
	Q3	40	10	30	8	27	10
	Q4	7	43	29	10	27	10
2016	Q1	42	10	13	26	10	26
	Q2	9	43	39	0	25	11
	Q3	40	10	29	10	25	10
	Q4	7	38	22	11	22	10

Note: "Numb Win" and "Numb Los" represent the number of winners and losers respectively.

In Table 3.4, it can be observed that the number of losers in Q2 of years 2012, 2013, 2014, 2015, and 2016 for the financial sector was zero. For certain quarters, the number of stocks in a winner or loser portfolio was very small. For example, the number of losers for the resources sector in Q1 of 2012 was seven out of a total of 53 stocks. The number of losers in the financial sector in Q3 of 2015 was eight out of a total of 38 stocks. The number of winners in the industrial sector in Q1 of 2016 was 10 out of a total of 36 stocks. When using the original data, there was unbalanced classification. The Euclidean distance used by the FCM algorithm was calculated by taking the square root of the sum of the squared differences between observations. Variables with the largest scales were given more importance during dissimilarity calculations, and clustering results were biased. Hence, the data needed to be standardised before applying the FCM algorithm.

The process to determine the effect of the different orders of magnitude in the measurement of the variables on the classification of the stocks in winners and losers involved the following steps:

- After analysing the raw data and calculating the centres of the centres for each quarter, each stock's membership degree to the two centres was calculated, and the stocks were grouped according to their respective membership degrees.
- Next, the variables (ratios) that influenced the classification of the individual stocks as members of the two centres the most were identified to investigate if certain variables dominated the classification. The expectation was that the variables with the largest scale would dominate the classification.
- Thereafter, the percentage contribution of the different ratios to the classification of the stocks was investigated by scrutinising the squared distance of each stock's data from the centre of the centres.

A stock called AGL is used as an example in this section to highlight the issues of order of magnitude. The original data and the detailed results obtained for each step of the calculation are presented in different tables. The membership of AGL to the Quarter 1 centres identified with the FCM algorithm is presented. The raw Quarter 1 data for AGL's seven variables over the testing period are presented in Table 3.5.

Table 3.5: Original matrix of AGL

	CR	**DAA**	**DIV**	**ES**	**NAV**	**RR**	**TAR**
AGL	x_{AGL_1}	x_{AGL_2}	x_{AGL_3}	x_{AGL_4}	x_{AGL_5}	x_{AGL6}	x_{AGL_7}
Q1 2012	2.36	0.29	1.915	358 7.93	274 04.23	137.01	2.631 579
Q1 2013	2.05	0.32	3.088 7	780.22	266 19.96	164.97	2.702 703
Q1 2014	1.94	0.36	3.348 6	984.19	305 11.77	212.17	2.5
Q1 2015	2.13	0.38	3.348 6	130 1.73	289 15.2	140.55	2.439 024
Q1 2016	2.36	0.46	3.435 6	370.49	258 48.28	120.75	2.857 143

The two Quarter 1 training period FCM centres are presented in Table 3.6.

Table 3.6: Centre of centre

k	1	2	3	4	5	6	7
Q1	**CR**	**DAA**	**DIV**	**ES**	**NAV**	**RR**	**TAR**
Winner (c_{wk})	1.72	0.32	2.51	2356.56	13 480.46	94.96	1.55
Loser (c_{lk})	2.01	0.44	3.31	407.69	1602.80	79.14	1.33

The squared differences between observations in Table 3.5 and the winner and loser centre in Table 3.6 were computed to determine the contribution of the different financial ratios to the calculation of the distance of each data point to the centre of the cluster. The calculation was done as follows (the results are presented in Table 3.7:)

$$(x_{jk} - c_{ik})^2; \;\; j = AGL; \;\; i = \text{winner}(W), \; \text{Loser}(L); \;\; k = CR, \ldots, TAR.$$

The total for Quarter 1 of each year was calculated (presented in the last column of Table 3.7) as follows:

$$\sum_k (x_{jk} - c_{ik})^2 = \|\boldsymbol{x}_j - \boldsymbol{c}_i\|^2. \tag{3.5.1}$$

The proportion:

$$\frac{(c_{jk} - c_{ik})^2}{\sum_k (x_{jk} - c_{ik})^2}, \;\; k = 1, \ldots 7 \tag{3.5.2}$$

was calculated for every year, and hence, the average proportion was calculated for every k (variable).

Table 3.7: Squared differences between observations and centre of centre $\sum(x_{jk} - c_{wk})^2$

	CR	DAA	DIV	ES	NAV	RR	TAR	
Winner	$(x_{j1} - c_{w1})^2$	$(x_{j2} - c_{w2})^2$	$(x_{j3} - c_{w3})^2$	$(x_{j4} - c_{w4})^2$	$(x_{j5} - c_{w5})^2$	$(x_{j6} - c_{w6})^2$	$(x_{j7} - c_{w7})^2$	$\sum(x_{jk} - c_{wk})^2$
2012	0.41	0.00	0.36	1 516 270.00	1.94×10^8	1768.21	1.17	195 518 040.20
2013	0.11	8.13×10^{-6}	0.33	2 484 851.00	1.73×10^8	4901.41	1.33	175 186 754.2
2014	0.05	0.00	0.70	1 883 402.00	2.90×10^8	13 738.20	0.90	293 257 223.7
2015	0.17	0.00	0.70	1 112 668.00	2.38×10^8	2078.46	0.79	239 114 748.1
2016	0.41	0.02	0.85	3 944 478.00	1.53×10^8	665.13	1.71	156 941 845.8
Aver	0%	0%	0%	1%	99%	0%	0%	
Loser	$(x_{j1} - c_{l1})^2$	$(x_{j2} - c_{l2})^2$	$(x_{j3} - c_{l3})^2$	$(x_{j4} - c_{l4})^2$	$(x_{j5} - c_{l5})^2$	$(x_{j6} - c_{l6})^2$	$(x_{j7} - c_{l7})^2$	$\sum(x_{jk} - c_{lk})^2$
2012	0.12	0.02	1.94	10 113 904.00	6.66×10^8	3349.43	1.68	676 117 253.9
2013	0.00	0.01	0.05	138 775.90	6.26×10^8	7367.53	1.87	626 1463 30.5
2014	0.00	0.01	0.00	332 348.10	8.36×10^8	17 698.12	1.36	836 035 001 8
2015	0.01	0.00	0.00	799 301.10	7.46×10^8	3771.72	1.22	716 803 074.1
2016	0.12	0.00	0.02	1384.11	5.88×10^8	1731.75	2.32	588 003 348
Aver	0%	0%	0%	0%	100%	0%	0%	

In the analysis above, it is observed that NAV contributed the most to the calculation of $\|x_{AGL} - c_i\|$ and thus had the largest impact on the calculation of $\mu_i(x_{AGL})$, which determined the classification of AGL in the winner or loser group. NAV was thus the biggest influencer in the classification. The same observation and conclusion were made with respect to the NAV of the other stocks. It is thus clear that variables with observed large squared differences $(x_{jk} - c_{ik})^2$ between a specific stock and the winner or loser group had a greater effect on the membership degree of a stock than variables with small squared differences. These larger squared differences followed from the fact that the observations of the NAV variable were of a larger order of magnitude than the other variables.

The FCM algorithm could thus not perform well without a proper standardisation of the dataset. Standardising the observations allowed the respective variables a more equal contribution to the classification of stocks to the two centres. Investigations were performed to determine which standardisation method gave a more reliable classification. In the following subsection, quality clusters are defined.

3.5.2 Transformation of Data: Defining Quality Clusters

The FCM algorithm identified clusters in such a way that the stocks assigned to a specific cluster would be more similar to each other than the stocks assigned to the other cluster(s). For a stock, the membership degree of a strong association was close to 1 for the one cluster centre and close to zero for the other. Conversely, the membership degree of a stock with a weak association to both clusters was expected to be close to 0.5. Therefore, quality clusters were defined by considering the following:

- how close the objects within the same cluster were. Good compactness is characterised by a lower distance between the objects within the same cluster; and
- how well a cluster is separated from other clusters, in other words, larger distances between cluster centres, allowing membership degrees to be close to 1 and zero.

In summary, in a good cluster, the average distance between elements within a cluster is as small as possible, and the average distance between clusters is as large as possible.

3.5.2.1 Compactness of the Clusters

Consider again the function that was minimised by the FCM algorithm. Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \boldsymbol{x}_m$ be elements of $\boldsymbol{X}'$ and consider the problem of grouping these elements in two p-dimensional subsets with centres $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$. The FCM algorithm determines the centres of the two subsets i.e. $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ by minimising the function.

$$SSE = \sum_{i=1}^{2} \sum_{j=1}^{m} [\mu_i(\boldsymbol{x}_j)^2 \|\boldsymbol{x}_j - \boldsymbol{c}_i\|^2]. \tag{3.5.3}$$

In the following sub-section, different standardisation techniques are investigated to address the issue of the different orders of magnitude of the observations of the different variables.

3.5.3 Standardisation Techniques

Seven standardisation techniques were considered in this study, and are discussed below. Let x_{ij} be the i^{th} stock's original value for the j^{th} fundamental factor with $i = 1, 2, \ldots, n$, where n is the number of stocks included in the sample at a specific date, $j = 1, 2, \ldots, m$, where m is the number of fundamental factors and was equal to 7 in this study.

Let $\boldsymbol{s}_j$ be the standard deviation of the j^{th} variable i.e.

$$s_j = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_{ij} - \bar{x_j})^2} \tag{3.5.4}$$

with

$$\bar{x_j} = \frac{1}{n}\sum_{i=1}^{n} x_{ij}. \tag{3.5.5}$$

Define r_j as the range of the j^{th} variable:

$$r_j = \max(x_{ij},\, i = 1, \ldots, n) - \min(x_{ij}, i = 1, \ldots, n).$$

The standardised values z_{ij}, for the different standardised techniques are defined as:

(1)

$$z_{ij} = \frac{(x_{ij} - \bar{x}_j)}{s_j},$$

(2)

$$z_{ij} = \frac{x_{ij}}{s_j},$$

(3)

$$z_{ij} = \frac{(x_{ij} - \bar{x}_j)}{r_j},$$

(4)

$$z_{ij} = \frac{x_{ij}}{\max\{x_{ij}\}},$$

(5)

$$z_{ij} = \frac{(x_{ij} - \min\{x_{ij}\})}{r_j},$$

(6)

$$z_{ij} = \frac{(x_{ij} - \text{median}\{x_{ij}\})}{r_j},$$

(7)

$$z_{ij} = \frac{x_{ij}}{r_j}.$$

The use of these standardisation techniques was aimed at changing the values of the various multi-scaled variables in the dataset to a common and comparable scale, allowing equal contribution of all variables in the fuzzy model classification. The minimum and the maximum value of the standardised data for each standardisation method are presented in Table 3.9. In the next section, the results of the FCM algorithm; obtained using the above standardisation techniques, are described.

3.6 FCM Classification and Standardisation Techniques

The clustering results when using the original data were presented in Section 3.5.1. In this section, the clustering results when standardising the resources sector data by using the seven techniques noted in Section 3.5.3 are presented and compared.

In Table 3.8, the standardisation techniques are simply denoted by (1), (2), etc as defined in Section 3.5.3, while the "win" and "los" headings indicate the number of winner stocks and loser stocks in the respective portfolios formed by the two clusters.

Table 3.8: Results of the FCM classification with different standardisation techniques for the resources sector

		(1)		(2)		(3)		(4)		(5)		(6)		(7)	
Year	Q	win	los	win	los	win	los	win	los	win	los	win	los	win	los
2012	Q1	21	32	44	9	24	29	9	44	52	1	21	32	4	49
	Q2	31	22	32	21	28	25	29	24	31	22	27	26	30	23
	Q3	20	32	19	33	30	22	31	21	35	17	30	22	31	21
	Q4	33	18	42	9	33	18	40	11	8	43	36	15	41	10
2013	Q1	22	30	48	4	23	29	4	48	51	1	22	30	2	50
	Q2	27	24	27	24	29	22	28	23	39	12	24	27	26	25
	Q3	17	33	21	29	29	21	28	22	39	11	26	24	28	22
	Q4	33	17	29	21	33	17	34	16	2	48	34	16	34	16
2014	Q1	20	32	38	14	22	30	15	37	33	19	23	29	13	39
	Q2	30	22	34	18	28	24	32	20	49	3	27	25	32	20
	Q3	19	32	12	39	29	22	37	14	48	3	29	22	41	10
	Q4	31	19	24	26	31	19	24	26	1	49	34	16	22	28
2015	Q1	23	28	9	42	25	26	39	12	50	1	25	26	47	4
	Q2	26	25	35	16	26	25	30	21	48	3	25	26	33	18
	Q3	23	27	14	36	25	25	37	13	47	3	27	23	43	7
	Q4	29	21	26	24	30	20	27	23	3	47	32	18	24	26
2016	Q1	24	28	15	37	25	27	23	29	51	1	23	29	42	10
	Q2	28	24	26	26	26	26	26	26	23	29	26	26	27	25
	Q3	23	27	24	26	26	24	25	25	18	32	26	24	26	24
	Q4	27	18	28	17	27	18	35	10	43	2	32	13	34	11

In Table 3.8, it can be observed that the classification of the number of the stocks in each quarter's portfolio (winner and loser portfolios) depended on the standardisation techniques used. While the FCM classification using some standardisation techniques resulted in more balanced clusters or portfolio sizes overall, the FCM classification using other standardisation techniques resulted in very low numbers of stocks being allocated to portfolios. For example, in Quarter 1 of 2012 (resources sector), with method (5) only one stock was allocated to the loser portfolio and 52 to the winner portfolio. With method(6), the classification changed from 52 stocks in the winner portfolio and one stock in the loser portfolio to 21 stocks in the winner portfolio and 32 in the loser portfolio.

To evaluate the effect of the seven standardisation methods on the procedure, the sum of fuzzy variations of clusters (SSE) was compared to measure the dispersion of individual data points in each clustering.

In Table 3.9, the results of the clusters' evaluation are presented, with the original data used for the top line and the standardised values for lines denoted by (1) through (7).

Table 3.9: Clusters' evaluation

Method		Q1	Q2	Q3	Q4	Min/Max
Original data (x_{ij})	k	8	7	9	9	-24.97
	SSE	373 522	436 944.1	459 003 4	445 004 5	279 00
(1)	k	11	4	4	7	-6.66
	SSE	0.173 596	0.069 108	0.087 05	0.108 071	7.04
(2)	k	70	14	20	11	-0.45
	SSE	0.601 278	0.555 4	0.597 802	0.491 294	7.22
(3)	k	7	6	11	11	-0.45
	SSE	0.011 269	0.005 225	0.009 814	0.009 25	0.88
(4)	k	51	14	16	11	-0.09
	SSE	0.038 158	0.032 17	0.035 99	0.027 265	1
(5)	k	14	14	43	42	0
	SSE	0.039 451	0.040 335	0.052 281	0.053 916	1
(6)	k	11	5	8	15	-0.32
	SSE	0.012 933	0.007 429	0.012 46	0.011 84	0.95
(7)	k	65	13	15	12	-0.71
	SSE	0.039 125	0.031 756	0.041 524	0.029 094	1

Note: k is the number of iterations performed by the Fuzzy C-Means algorithm. Min and Max are the minimum and maximum value for each standardisation method to show scale of the values.

In Table 3.9, it is shown that the FCM algorithm needed varying iterations (k) to identify the two unique centres per standardisation technique and per quarter. The iterations needed to locate the centres are referred to as iterations to convergence. For example, Techniques (2), (4) and (7) needed many iterations for Q1. Method (1) converged in fewer iterations for all the quarters.

The results in Table 3.9 further show that the resultant SSE depended on the scale of the transformed data. The scales of the transformed data were not the same for the different methods (see "Min" and "Max" in Table 3.9); thus, the scales' values could not be used as quality criteria in the resultant cluster identification per method. The standardisation used for the FCM algorithm should aim to eliminate variation in the features.

With Method (1) (z-score), all the fundamental variables were transformed to have a zero mean and unit variance, which, in turn, provided equal contributions among the fundamental variables to the Euclidean similarity measure embedded in the FCM algorithm. The z-score method also had the second-highest SSE of the seven methods, allowing better centre identification between points, because the FCM algorithm struggled with centre identification if the Euclidean distances became too small. This suggested that the z-score, or Method (1), was the best of the seven methods, as it was the most efficient and would give the most accurate results. Because z-score standardisation was central to the analysis, it is defined in detail in Section 3.6.1.

3.6.1 Z-Score Standardisation

In order to define z-score standardisation, let x_{ijt} be the i^{th} stock's value for the j^{th} fundamental factor with $i = 1, 2, \dots, n$, where n is the number of stocks included in the sample for a specific quarter (t), $j = 1, 2, \dots, m$, where m is the number of fundamental factors and is equal to 7 in this study and $t = 1, 2, \dots, T$, where T is the number of quarters in the training period. The mean of fundamental factor j at a specific date was calculated as follows:

$$\bar{x}_{jt} = \frac{1}{n} \sum_{i=1}^{n} x_{ijt}, \text{ with } j = 1, \dots m, \; t = 1, \dots T. \tag{3.6.1}$$

The mean $\bar{x}_{jt}$ was calculated for all the fundamental factors at all quarters. Define s_{jt} as the standard deviation of the values for all the stocks for fundamental factor j on a specific date:

$$s_{jt} = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_{ijt} - \bar{x}_{jt})^2}. \tag{3.6.2}$$

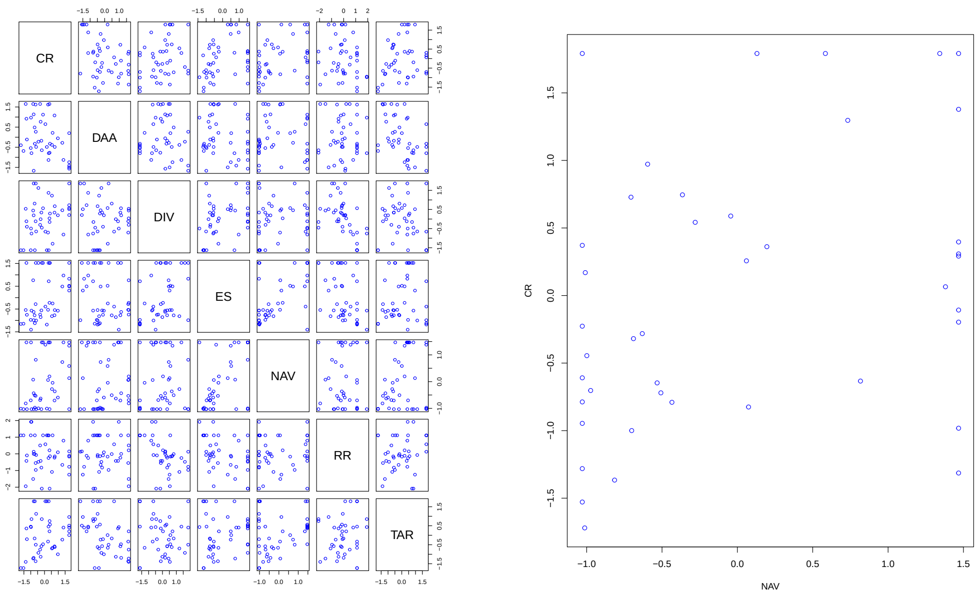
The z-score for the i^{th} stock on j^{th} fundamental factor at quarter t was:

$$z_{ijt} = \frac{x_{ijt} - \bar{x}_{jt}}{s_{jt}}. \tag{3.6.3}$$

In the following section, the FCM classification results with standardised data using the z-score are presented in contrast to prior results, for which the untransformed original data were used.

3.7 FCM Classification Results with Standardised Data (z-scores)

The FCM algorithm used the Euclidean distance metric to determine the centres of the two subsets. The scale of the different variable is thus important in investigating if certain variables influenced the classification. In this section, the scatter matrix of the standardised data for Quarter 1 of 2007 of the resource sector is represented (see Figure 3.6) for the (training period), to compare the scale of the different variables.



(A) Scatter matrix

(B) Scatter plot (NAV, CR)

Figure 3.6: Standardised data

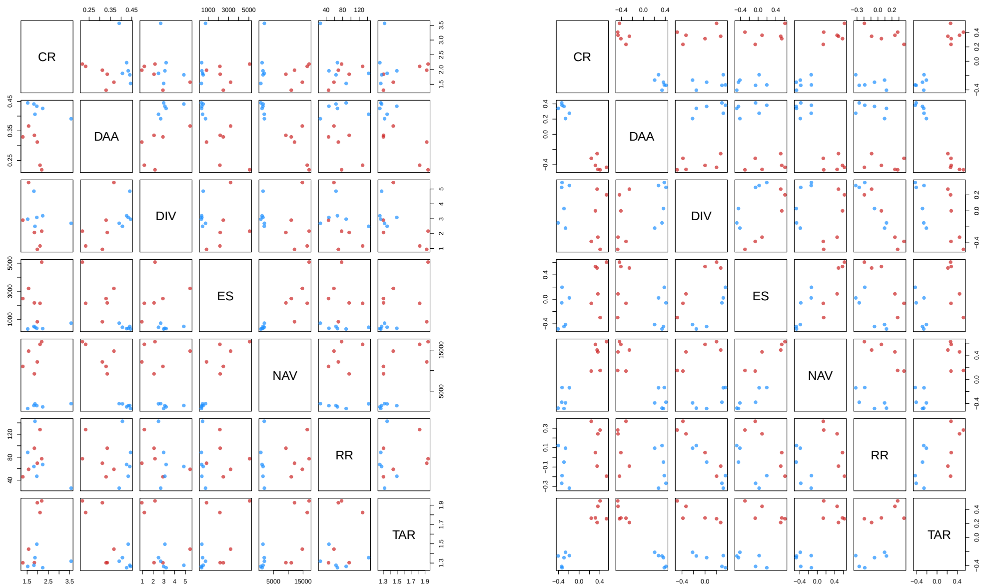
Figure 3.6 shows that variables had equivalent scale. In the next section, the results of the classification using the FCM algorithm on the standardised data are presented. The new results obtained from the FCM classification are presented in Table 3.10. The results of the FCM using original data are presented in brackets alongside the new results.

Table 3.10: Results of the FCM algorithm on normalised data

		Resource		Financial		Industrial	
Year	Quarter	Num_win	Num_los	Num_win	Num_los	Num_win	Num_los
2012	Q1	21 (46)	32 (7)	18 (7)	20 (31)	23 (12)	14 (25)
	Q2	31 (5)	22 (48)	22 (37)	15 (0)	13 (25)	24 (12)
	Q3	32 (42)	20 (10)	17 (32)	20 (5)	25 (29)	12 (8)
	Q4	33 (8)	18 (43)	21 (31)	17 (7)	25 (29)	11 (7)
2013	Q1	22(43)	30(9)	18 (9)	20 (29)	24 (12)	12 (24)
	Q2	27(10)	24 (42)	21 (39)	16(0)	13 (24)	25 (13)
	Q3	33 (40)	17 (10)	17 (31)	20 (6)	25 (30)	12 (7)
	Q4	33 (7)	17 (43)	21 (30)	17 (8)	27 (28)	9 (8)
2014	Q1	20(42)	32 (10)	19 (11)	20 (28)	23 (12)	13 (24)
	Q2	30 (10)	22 (42)	21 (11)	18 (28)	13 (12)	24 (24)
	Q3	32 (41)	19 (10)	15 (13)	21 (24)	26 (12)	11 (25)
	Q4	31(7)	19 (43)	22 (28)	15 (9)	26(29)	11 (8)
2015	Q1	23 (41)	28 (10)	19 (13)	18 (24)	23 (12)	14 (25)
	Q2	26 (8)	25 (43)	18 (38)	20 (0)	13 (22)	24 (15)
	Q3	27 (40)	23 (10)	15 (30)	23(8)	22 (27)	15 (10)
	Q4	29 (7)	21 (43)	23 (29)	16(10)	26 (27)	11 (10)
2016	Q1	24 (42)	28 (10)	25 (13)	14 (26)	24 (10)	12 (26)
	Q2	28 (9)	24(43)	21 (39)	18 (0)	11 (25)	25 (11)
	Q3	27 (40)	23 (10)	17 (29)	22 (10)	24 (25)	11 (10)
	Q4	27 (7)	18 (38)	20 (22)	13 (11)	25(22)	7 (10)

It can be observed that standardising the data changed the classification of the number of the stocks in each quarter's portfolio (winner and loser portfolios), and resolved the prior issues of zero stocks and very low numbers of stocks being allocated to portfolios. The new classification further resulted in more balanced clusters or portfolio sizes overall. In Quarter 1 of 2012 (resources sector), the classification changed from 46 stocks in the winner portfolio and seven stocks in the loser portfolio to 21 stocks in the winner portfolio and 32 in the loser portfolio. In Quarter 2 of 2012, 2013, 2015, and 2016 (financial sector), there were no stocks in the loser portfolio when using the unstandardised data. Also, the FCM algorithm could not identify the two clusters with original data, because the centres were too close to each other. Using standardised data, the FCM algorithm was able to identify the two clusters among the seven variables. To highlight the impact of standardising the data before applying the FCM algorithm, the cluster results of the good and bad centres obtained in Quarter

4 for all the training periods (six years) are presented in Figure 3.7. The two clusters (respectively coloured blue or red) visually show the variable-specific demarcation of clusters.



(A) Before standardisation **(B)** After standardisation

Figure 3.7: Cluster result for Good and Bad Centre Quarter 4 for all the years

From the classification results in Figure 3.7, it can be seen that the classification after standardisation was more balanced than before standardisation. Before standardisation, there was unbalanced grouping. The cluster of blue (or red) dots formed before standardisation (Figure 3.7A) were often mixed with red (or blue) dots, which suggested that the clusters were not well separated. After standardisation (see Figure 3.7B), the two clusters were more separated, making it easy to distinguish them from one another, and lowering the incidence of mixed colour groups. It can therefore be concluded that standardisation before applying the FCM algorithm leads to obtaining a balanced and better grouping. It is also important to select a specific standardisation procedure, according to the nature of the datasets, for the analysis. Once the large scaling inequalities of the seven variables had been resolved, the influence of outliers on the results was investigated.

The data were first winsorized and then standardised before applying the FCM algorithm. In the next section, it is shown how winsorization improved the grouping of the FCM model.

3.8 FCM Classification and Outliers

In order to measure and neutralise the impact of the outliers, the data were firstly winsorized and then standardised (referred to as WSD) before applying the FCM algorithm. The winsorization was done at the 5^{th} percentile, and the 95^{th} percentile implying that values above the 95^{th} percentile and below the 5^{th} percentile were replaced with the 95^{th} and 5^{th} percentile values. In this section, the FCM classification with standardised data (SD) is compared to the FCM classification using WSD. In Table 3.11, the number of stocks that were classified in the resources sector are presented.

Table 3.11: Classification results obtained for the resources sector

		SD			WSD			SD-WSD
Year	Quarter	Numb Win	Numb Los	Win-Los	Numb Win	Numb Los	Win-Los	Win-Los
2012	Q1	21	32	-11	25	28	-3	-8
	Q2	31	22	9	29	24	5	4
	Q3	32	20	12	32	20	12	0
	Q4	33	18	15	29	22	7	8
2013	Q1	22	30	-8	24	28	-4	-4
	Q2	27	24	3	28	23	5	-2
	Q3	33	17	16	29	21	8	8
	Q4	33	17	16	31	19	12	4
2014	Q1	20	32	-12	22	30	-8	-4
	Q2	30	22	8	25	27	-2	10
	Q3	32	19	13	27	24	3	10
	Q4	31	19	12	28	22	6	6
2015	Q1	23	28	-5	23	28	-5	0
	Q2	26	25	1	22	29	-7	8
	Q3	27	23	4	24	26	-2	6
	Q4	29	21	8	26	24	2	6
2016	Q1	24	28	-4	23	29	-6	1
	Q2	28	24	4	28	24	4	0
	Q3	27	23	4	26	24	2	2
	Q4	27	18	9	26	19	7	2

Win–Los numbers per group were calculated for the different methods, to see if the data were skewed to either side, causing the one centre to consistently have a small number of stocks allocated to it, while the other centre had a large group allocated to it. Therefore, the magnitude of the numbers was important. With the SD data, Win–Los < 0 in Q1 every year, because the centres for the first quarter flipped signs for the seven variables. The same pattern was not observed in Q2, Q3, or Q4. With the WSD data, the pattern was different. It can be concluded that winsorization had an effect on cluster formation and stock classification.

When comparing the SD to the WSD, the magnitude of the Win-Los fell, showing that the newly formed centres were better at splitting the data into two distinct groups, in turn indicating that the SD method was influenced by outliers in the data. The results of the Win-Los for the SD-WSD method were positive, except for a few quarters. This implied that WSD created groups of more equal size than SD. Table 3.11 provides only high-level results, and does not show classification on individual stock level. The number of stocks overlapping in the winner and loser portfolio were formed by applying the FCM classification on the SD, and the WSD were counted. A stock overlapped if it was classified as a winner (loser) by both SD and WSD. The results are presented in Table 3.12.

Table 3.12: Overlapping stocks in the Winner and Loser portfolio formed by applying the FCM algorithm on the SD and the WSD

		Resources			Financial			Industrial		
Year	Quarter	Over Win	Over Los	non Over	Over Win	Over Los	non Over	Over Win	Over Los	non Over
2012	Q1	19	26	8	2	3	33	14	22	1
	Q2	26	19	8	20	14	3	13	22	2
	Q3	4	4	44	12	10	17	3	3	31
	Q4	28	17	6	10	14	14	3	22	3
2013	Q1	19	25	8	4	4	30	20	12	4
	Q2	23	19	9	19	14	4	14	20	4
	Q3	3	7	40	9	11	17	4	1	32
	Q4	29	17	4	11	7	20	22	9	5
2014	Q1	19	29	4	4	3	33	20	13	3
	Q2	22	19	12	19	18	2	12	21	4
	Q3	2	5	44	14	9	15	3	4	32
	Q4	27	18	5	10	5	22	22	11	2
2015	Q1	19	24	8	5	4	28	22	14	11
	Q2	20	23	8	16	14	7	13	20	3
	Q3	2	5	43	14	9	10	3	4	30
	Q4	25	20	5	11	6	18	24	11	2
2016	Q1	21	26	5	8	1	20	23	12	1
	Q2	24	20	8	17	14	4	11	23	2
	Q3	2	3	45	9	9	25	5	1	23
	Q4	25	17	9	11	3	18	21	7	4

In Q1, Q2, and Q4, the groupings were similar when the FCM algorithm was applied on the SD and the winsorized data plus the SD. This is evidenced by a large number of stocks in common in the winner- and loser portfolios; for example, 19 stocks out of 21 in the winner portfolio and 26 stocks out of 32 in the loser portfolio for the resources sector in Q1 of 2012. For other quarters, there was a very high number of stocks non-overlapping in Q3 every year in the training period for the resources sector, in Q1 and Q4 for the financial sector, and in Q3 and Q4 for the industrial sector. The number of stocks non-overlapping in 2012 for the resources sector was 44, 33 for the financial sector, and 21 for the industrial sector (see Table 3.12). This meant the groupings derived from using the two methods were different. Some high- or low-performing stocks, in all likelihood, swapped from one cluster to another, thus causing the SD winner portfolio cluster to be a WSD loser portfolio cluster. This happened because the location of the respective centres relative to the seven dimensional points of each stock changed due to the winsorization.

To understand this result better, the centre vectors for the winner and loser portfolios are compared in the next section.

3.8.1 Comparison of the SD and WSD Centre Vectors

In this section, the centre vectors obtained by applying the FCM algorithm (SD and WSD) for the resources sector are considered. The cluster centres based on the data from 2006 to 2016 are shown in Table 3.13.

Table 3.13: Centres vectors obtained with the two methods: resources sector

		SD				WSD				SD - WSD			
		Q1	Q2	Q3	Q4	Q1	Q2	Q3	Q4	Q1	Q2	Q3	Q4
Winners	CR	0.33	- 0.29	0.36	-0.25	0.34	-0.21	-0.20	-0.29	-0.01	-0.08	0.56	0.04
	DAA	-0.49	0.49	-0.55	0.36	-0.36	0.34	0.31	0.32	-0.12	0.16	-0.86	0.04
	DIV	-0.20	0.16	-0.08	0.08	-0.11	0.16	0.08	0.09	-0.09	-0.01	-0.16	-0.01
	ES	0.09	-0.13	0.24	-0.16	0.21	-0.22	-0.27	-0.20	-0.11	0.09	0.51	0.04
	NAV	0.21	-0.17	0.34	-0.27	0.34	-0.30	-0.33	-0.33	-0.13	0.13	0.67	0.06
	RR	0.12	-0.03	0.03	-0.03	0.09	-0.13	-0.08	-0.09	0.03	0.10	0.10	0.06
	TAR	0.18	-0.20	0.20	-0.19	0.33	-0.30	-0.28	-0.30	- 0.14	0.10	0.47	0.11
Losers	CR	-0.25	0.33	-0.29	0.34	-0.28	0.23	0.25	0.36	0.03	0.10	-0.54	-0.03
	DAA	0.38	- 0.55	0.44	-0.49	0.31	-0.36	-0.37	-0.39	0.07	-0.20	0.81	-0.11
	DIV	0.15	-0.21	0.06	-0.10	0.09	-0.17	-0.08	-0.10	0.06	-0.03	0.15	-0.00
	ES	-0.09	0.12	-0.18	0.23	-0.17	0.29	0.36	0.25	0.08	-0.16	-0.54	-0.02
	NAV	-0.19	0.18	-0.25	0.39	-0.29	0.36	0.43	0.42	0.10	-0.19	-0.68	-0.03
	RR	-0.12	0.02	-0.03	0.03	-0.08	0.11	0.07	0.10	- 0.04	-0.10	-0.10	-0.07
	TAR	-0.19	0.20	-0.19	0.22	-0.30	0.31	0.31	0.33	0.12	-0.12	-0.50	-0.11

In Table 3.13, it can be seen that the Q1 and Q2 values for the WSD and the SD method were close to each other. Q3 values were far apart from the SD and the WSD method, except the RR and DIV ratios. It was also observed that Q4 had all the WSD loser points slightly further from the mean than the SD points, and the opposite applied for the winners. Both vectors for each quarter, using SD and WSD, had the same direction for the different quarters, except for Quarter 3. This was to be expected from the inversion of classification seen in Table 3.13. The winner and loser centres obtained using both methods are represented in Figure 3.8.

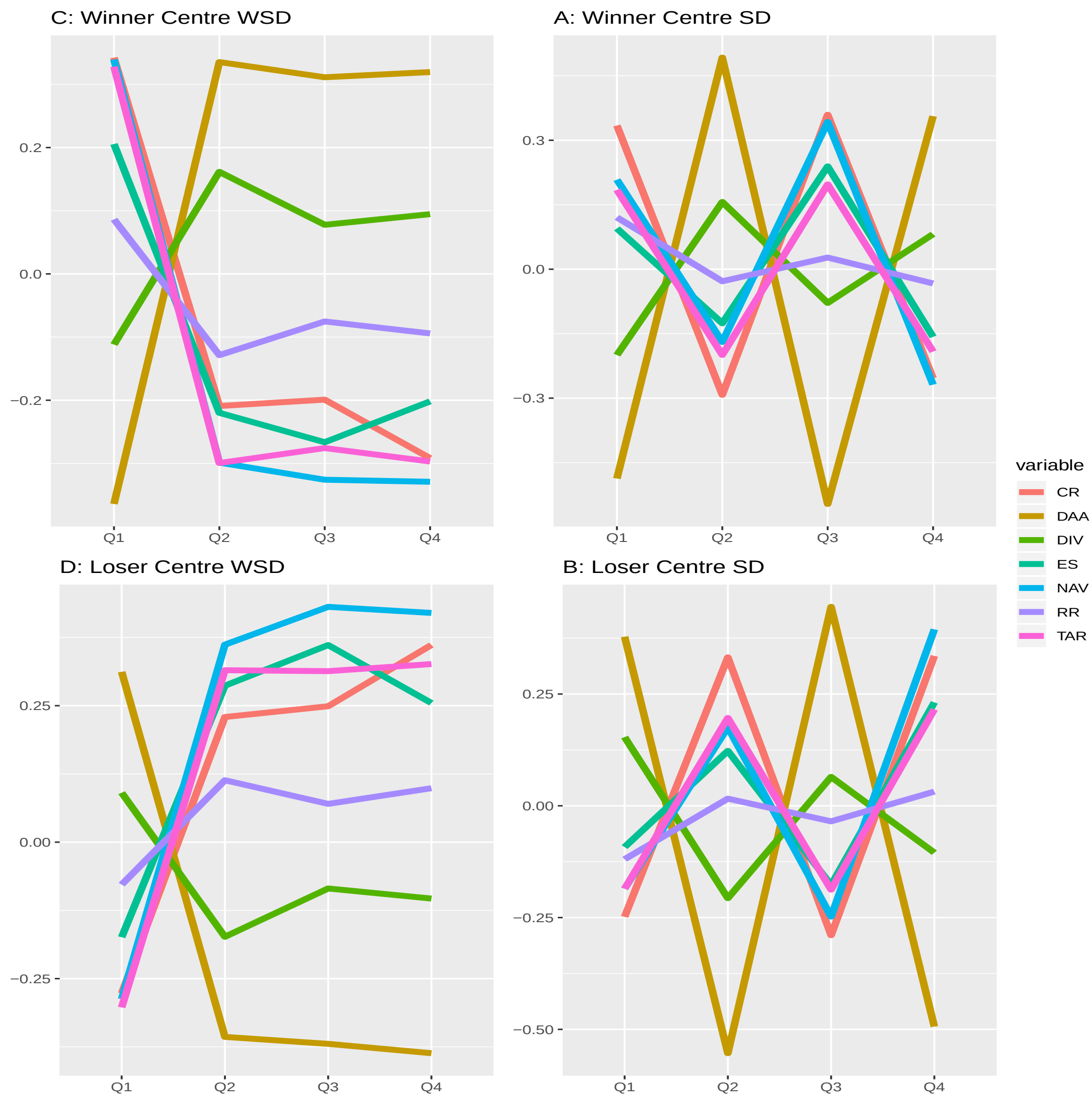


Figure 3.8: Centre vectors obtained with both methods

While the level of the centres remained stable for WSD, the sign of the vector in Quarter 3 for the resources sector changed from positive to negative (see Figure 3.8). The winner (or loser) centre for the standardised data was a loser (or winner) centre for the winsorized data. This happened due to the outliers being winsorised. Therefore, by changing the two centres, stocks with almost equal degrees of membership to the two centres would move from just over 0.5 to just under 0.5, and influence the performance. Outliers are extreme values in the dataset, and their membership was not expected to change much. It was necessary to winsorize the data before doing the analysis. In the following section, all the results were obtained after winsorization.

The statistical procedures used in this analysis were based on the assumption that the data followed a normal distribution. Hence, normality needed to be checked.

3.9 Normality Test

To assess the normality of the data, the sample quantiles of rr_t^W and rr_t^L representing the weekly residual returns of the winner portfolios and loser portfolios, were plotted against theoretical quantiles. The normal Q-Q plots for weekly residual return of the winner portfolio, using the data for the resources sector, are given in Figure 3.9. The letters s and k denote skewness and kurtosis respectively. The kurtosis of a normal distribution equals 3, and excess kurtosis is kurtosis -3. Excess kurtosis is equal to zero for a normal distribution. Normal Q-Q plots for the weekly residual return of the loser portfolios, using data for the resources sector, the weekly residual return of the winners and the losers for the industrial and financial sectors can be found in Appendix B.

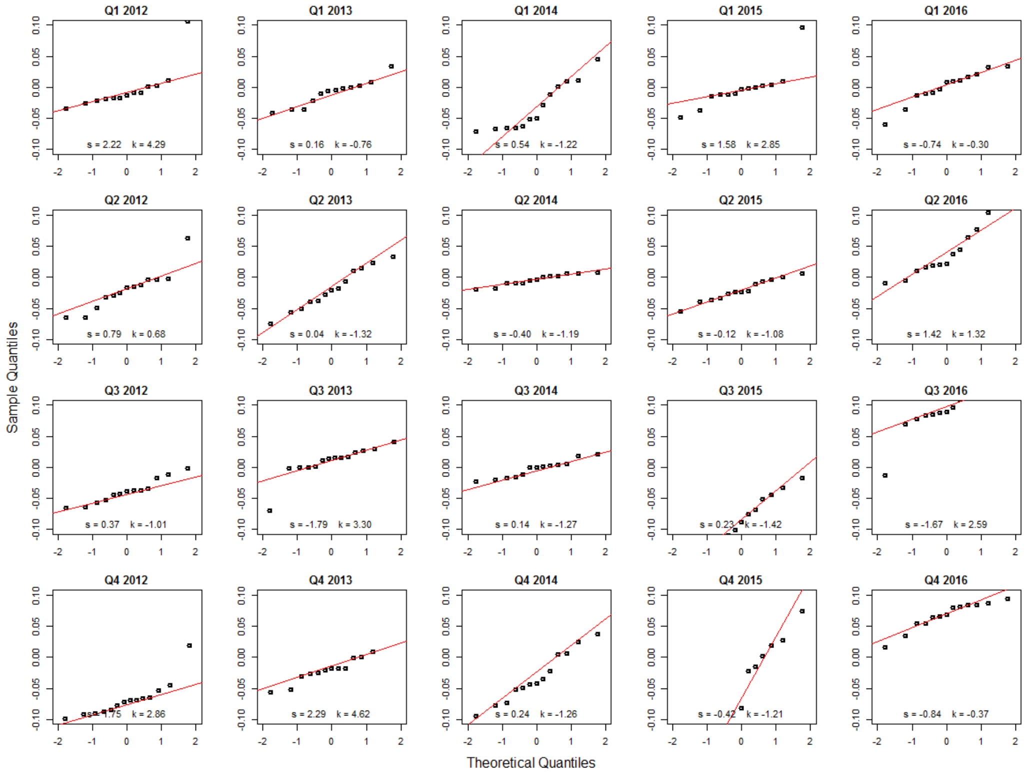


Figure 3.9: Weekly residual return winner

From the Q-Q plots, it seems that normality was not a reasonable assumption to make for the weekly residual return of the winner. In a normal distribution, the kurtosis is equal to 3. If skewness is not close to zero, then the data set is not normally distributed (Elton *et al.*, 2009). The Welch's t-test was applied, it assumes normality but it is more robust to departures from it. The Mann-Whitney U test, the distribution free test was also applied (see Section 3.13) to test for significant under- and over-reaction. In section 3.10, the results of the test of the over-reaction and under-reaction hypothesis using the t-test are presented for the three sectors. The hypothesis of the equality of the variances however, is needed to be tested first, as it was the assumption of the t-test.

3.10 Test of the Hypothesis $H_0 : \sigma^2_{t,W} = \sigma^2_{t,L}$

In this section, the results of the hypothesis testing the equality of the variances of the loser and winner portfolios are presented. In Table 3.14, results of the analysis of the test period for the resource sector are presented.

Table 3.14: Result of the hypothesis testing H_0 for the resources sector

year	quarter	$\hat{\sigma}^2_{t,W}$	$\hat{\sigma}^2_{t,L}$	$\hat{\sigma}^2_{t,W} - \hat{\sigma}^2_{t,L}$	F.test	p.value
2012	Q1	0.07%	0.12%	-0.05%	0.61	0.40
	Q2	0.04%	0.11%	-0.06%	0.40	0.13
	Q3	0.09%	0.04%	0.06%	2.50	0.13
	Q4	0.04%	0.09%	-0.05%	0.44	0.15
2013	Q1	0.05%	0.05%	0.00%	0.99	0.98
	Q2	0.08%	0.11%	-0.03%	0.74	0.61
	Q3	0.04%	0.07%	-0.03%	0.59	0.35
	Q4	0.14%	0.32%	-0.18%	0.45	0.18
2014	Q1	0.09%	0.15%	-0.05%	0.64	0.46
	Q2	0.01%	0.01%	0.00%	1.49	0.50
	Q3	0.01%	0.02%	-0.01%	0.57	0.34
	Q4	0.08%	0.16%	-0.09%	0.46	0.19
2015	Q1	0.14%	0.11%	0.02%	1.20	0.76
	Q2	0.03%	0.03%	0.00%	0.97	0.96
	Q3	0.10%	0.15%	-0.05%	0.66	0.48
	Q4	0.30%	1.45%	-1.15%	0.21	0.01*
2016	Q1	0.98%	0.07%	0.91%	13.74	0.00*
	Q2	0.14%	0.31%	-0.17%	0.44	0.17
	Q3	0.07%	0.12%	-0.05%	0.57	0.34
	Q4	0.05%	0.05%	0.00%	0.99	0.98

Note: * = highly significant

The values in the column titled F-test were calculated using the following formula:

$$F = \frac{\hat{\sigma}^2_{t,W}}{\hat{\sigma}^2_{t,L}}. \tag{3.10.1}$$

The results in the table differ slightly from the normal quotient value, due to approximation.

For example in Q1 of 2012, F was computed as follows:

$\frac{0.0007432264}{0.0001226430} = 0.6060079 \approx 0.61$

on the other hand,

$\frac{0.07\%}{0.12\%} \approx 0.58$

In Table 3.15, results of the analysis of the test period for the industrial sector are presented.

Table 3.15: Result of the hypothesis testing H_0 for the industrial sector

year	quarter	$\hat{\sigma}^2_{t,W}$	$\hat{\sigma}^2_{t,L}$	$\hat{\sigma}^2_{t,W} - \hat{\sigma}^2_{t,L}$	F.test	p.value
2012	Q1	0.02%	0.09%	-0.06%	0.25	0.02*
	Q2	0.07%	0.07%	0.00%	1.02	0.98
	Q3	0.03%	0.05%	-0.02%	0.55	0.31
	Q4	0.08%	0.09%	-0.01%	0.91	0.87
2013	Q1	0.03%	0.07%	-0.04%	0.40	0.15
	Q2	0.08%	0.06%	0.02%	1.31	0.65
	Q3	0.03%	0.04%	-0.01%	0.67	0.48
	Q4	0.13%	0.24%	-0.10%	0.57	0.34
2014	Q1	0.16%	0.04%	0.12%	4.43	0.02*
	Q2	0.01%	0.02%	0.00%	0.78	0.67
	Q3	0.07%	0.08%	-0.02%	0.82	0.74
	Q4	0.07%	0.07%	-0.01%	0.91	0.88
2015	Q1	0.15%	0.02%	0.13%	6.55	0.00*
	Q2	0.05%	0.03%	0.02%	1.74	0.35
	Q3	0.09%	0.13%	-0.05%	0.66	0.49
	Q4	0.08%	0.16%	-0.08%	0.51	0.26
2016	Q1	0.11%	0.10%	0.01%	1.10	0.87
	Q2	0.08%	0.09%	-0.01%	0.89	0.85
	Q3	0.16%	0.12%	0.04%	1.31	0.65
	Q4	0.07%	0.17%	-0.10%	0.42	0.14

Note: * = highly significant

In Table 3.16, results of the analysis of the test period for the financial sector are presented.

Table 3.16: Result of the hypothesis testing H_0 for the financial sector

year	quarter	$\hat{\sigma}^2_{t,W}$	$\hat{\sigma}^2_{t,L}$	$\hat{\sigma}^2_{t,W} - \hat{\sigma}^2_{t,L}$	F.test	p.value
2012	Q1	0.08%	0.09%	-0.01%	0.86	0.80
	Q2	0.09%	0.06%	0.03%	1.61	0.42
	Q3	0.04%	0.02%	0.01%	1.53	0.47
	Q4	0.06%	0.09%	-0.04%	0.59	0.36
2013	Q1	0.03%	0.04%	-0.01%	0.84	0.77
	Q2	0.14%	0.12%	0.02%	1.15	0.82
	Q3	0.11%	0.17%	-0.06%	0.66	0.46
	Q4	0.21%	0.14%	0.07%	1.49	0.50
2014	Q1	0.08%	0.09%	-0.01%	0.94	0.92
	Q2	0.02%	0.02%	-0.01%	0.70	0.55
	Q3	0.04%	0.18%	-0.14%	0.24	0.02*
	Q4	0.08%	0.07%	0.01%	1.16	0.80
2015	Q1	0.13%	0.17%	-0.04%	0.74	0.61
	Q2	0.07%	0.07%	0.00%	0.94	0.92
	Q3	0.09%	0.11%	-0.02%	0.84	0.77
	Q4	0.14%	0.10%	0.04%	1.38	0.58
2016	Q1	0.07%	0.15%	-0.08%	0.47	0.21
	Q2	0.11%	0.08%	0.03%	1.41	0.56
	Q3	0.11%	0.08%	0.02%	1.27	0.69
	Q4	0.06%	0.04%	0.02%	1.53	0.47

Note: * = highly significant

In the present study, Welch's t-test, allowing unequal variances, was used to test if over-reaction and under-reaction on the JSE were statistically significant at the 10% level or lower. Welch's t-test assumes normality but it is more robust to departure from it. The results of Welch's t-test and Student's t-test are the same when the variances are equal (Rasch *et al.*, 2011). In the next section, the results of the t-test are presented for the three sectors to investigate the occurrence of the over-reaction and under-reaction.

3.11 Occurrence of Over-reaction and Under-reaction

3.11.1 Resources Sector

In this subsection, the average residual return and the results of the t-test are presented for each quarter of the testing period (2012 to 2016). The following two rules were applied in interpreting the results: (i) Over-reaction is postulated when the winner under-performs the loser, and (ii) under-reaction is hypothesised when the winner out-performs the loser (see Section 2.1.8 for a discussion of the theoretical links to the observed patterns). The average residual return of the winner minus loser ($RRM_t^W - RRM_t^L$) presented in Table 3.17, is negative in the case of over-reaction and positive in the case of under-reaction. In Table 3.17, results of the analysis for every quarter of the test period for the resource sector are presented.

Table 3.17: Average residual returns and t-test for the resources sector

Year	Quarter	RRM_t^W	RRM_t^L	$RRM_t^W - RRM_t^L$	t-test	p-value	Over/Under
2012	Q1	-0.70 %	-0.33 %	-0.37 %	-0.30	0.76	Over
	Q2	-0.04 %	-1.96 %	1.92 %	1.79	0.09	Under*
	Q3	-3.31 %	-3.89 %	0.58 %	0.57	0.57	Under
	Q4	-4.17 %	-6.78 %	2.62 %	2.79	0.01	Under*
2013	Q1	0.19 %	-0.90 %	1.10 %	1.25	0.22	Under
	Q2	0.61 %	-1.93 %	2.54 %	2.11	0.05	Under*
	Q3	-1.04 %	0.89 %	-1.93 %	-2.21	0.04	Over*
	Q4	0.73 %	-0.65 %	1.38 %	0.74	0.47	Under
2014	Q1	-0.28 %	-3.11 %	2.84 %	2.09	0.05	Under*
	Q2	-0.20 %	-0.38 %	0.17 %	0.44	0.66	Under
	Q3	1.27 %	-0.27 %	1.54 %	3.16	0.00	Under*
	Q4	1.38 %	-3.19 %	4.57 %	3.38	0.00	Under*
2015	Q1	-3.05 %	-0.20 %	-2.85 %	-2.06	0.05	Over*
	Q2	-1.79 %	-2.11 %	0.32 %	0.47	0.64	Under
	Q3	-3.16 %	-8.35 %	5.19 %	3.76	0.00	Under*
	Q4	-1.92 %	-9.09 %	7.17 %	1.95	0.07	Under*
2016	Q1	6.63 %	0.03 %	6.60 %	2.32	0.04	Under*
	Q2	1.78 %	4.61 %	-2.83 %	-1.53	0.14	Over
	Q3	6.22 %	8.91 %	-2.69 %	-2.19	0.04	Over*
	Q4	2.92 %	6.69 %	-3.77 %	-4.28	0.00	Over*

Note: * = statistically significant at the 10% level or lower

In Table 3.17, it is shown that in 2012, 2015 and 2016, the RRM_t^W and RRM_t^L

presented the same pattern. Both were up (2016) or down (2012 and 2015). In 2013, the RRM_t^W and RRM_t^L presented opposite patterns. In 2014, the RRM_t^W and RRM_t^L presented the same pattern for the first two quarters, and the opposite pattern for the last two quarters. The average residual returns for the winner- and loser portfolios showed that the loser portfolio out-performed the average residual return of the winner portfolio by approximately 0.37% in Q1 of 2012, by 1.93% in Q3 of 2013, and by 2.85% in Q1 of 2015. In Q2, Q3, and Q4 of 2012 and in Q1 and Q2 of 2013, the winner portfolio out-performed the loser portfolio consecutively. Over the following short periods: from Q2 of 2012 to Q2 of 2013, from Q4 of 2013 to Q4 of 2014, from Q2 of 2015 to Q1 of 2016, and from Q2 of 2016 to Q4 of 2016, the winner portfolio under-performed the loser portfolio consecutively. Over-reaction was clearly cyclical, and happened after four or five quarters for the resources sector. However, all values were not statistically significant. The patterns could only be inferred from data-points with an asterisk (*). It can also be observed that the magnitude of over-reaction decreased from 2012 to 2015, and increased in 2016.

During the training period, 2012 to 2016, the results of the $RRM_t^W - RRM_t^L$ were negative in the following periods: Q1 of 2012, Q3 of 2013, Q1 of 2015, and Q2, Q3, Q4 of 2016 for the resources sector, consistent with the over-reaction hypothesis. The other periods showed positive results (see Table 3.17) consistent with the under-reaction hypothesis. The analysis did not pinpoint a proper seven-dimensional winner portfolio based on the WSD. The pattern in Figure 3.8 does not reflect the obtained result in Table 3.17.

In the resources sectors, under-reaction occurred more frequently than over-reaction, and once persisted for five consecutive quarters. Out of six cases of observed over-reaction, four were significant, whereas, out of 14 cases of under-reaction, nine were significant (see Table 3.18). The results in Table 3.17 are summarised in the bar chart below.

Table 3.18: Number of significant over- and under-reaction in the resources sector

		Number	Significant
Under	> 0	14	9
Over	< 0	6	4

Figure 3.10 presents the result of over-reaction and under-reaction for the resources sector.

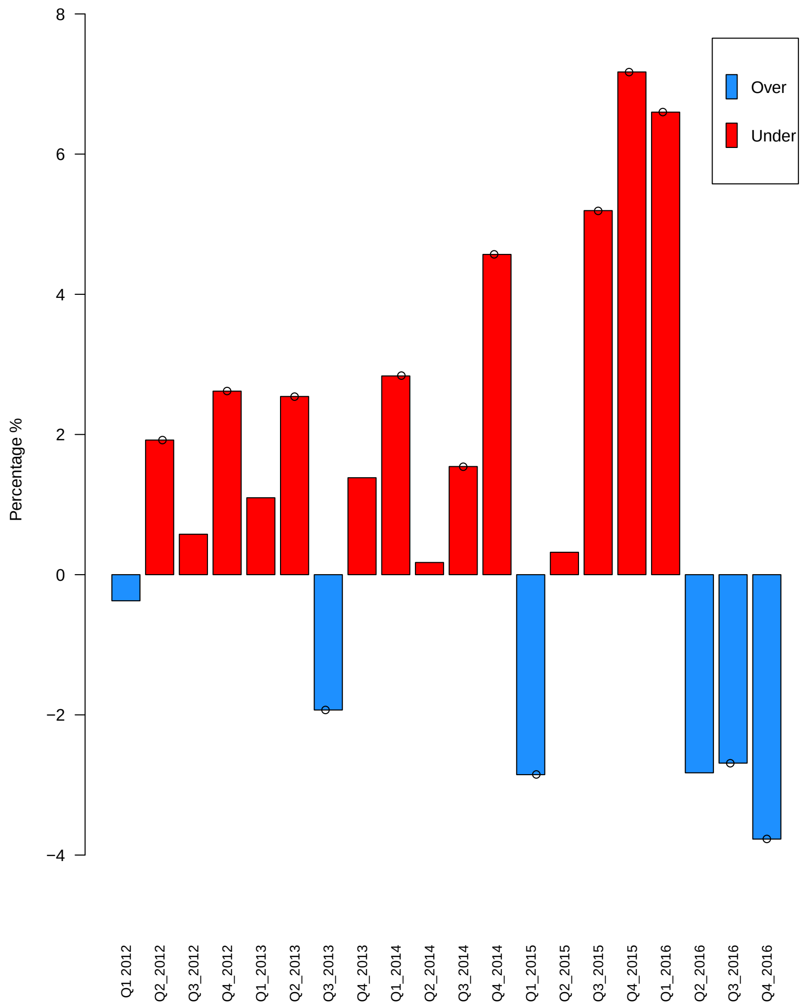


Figure 3.10: Over-reaction and under-reaction: resources sector

The dot in the bar chart indicates quarters where the t-test was significant.

3.11.2 Industrial Sector

In this sub-section, the average residual return and the results of the t-test are presented for each quarter of the testing period (2012 to 2016) for the industrial sector. As previously explained, over-reaction is observed when the winner under-performs the loser, and under-reaction is hypothesised when the winner out-performs the loser.

In Table 3.19, results of the analysis for every quarter of the test period for the industrial sector are presented.

Table 3.19: Average residual returns and t-test for the industrial sector

Year	Quarter	RRM_t^W	RRM_t^L	$RRM_t^W - RRM_t^L$	t-test	p-value	Over/Under
2012	Q1	-0.20 %	0.46 %	-0.66 %	-0.72	0.48	Over
	Q2	1.03 %	0.69 %	0.34 %	0.33	0.74	Under
	Q3	-1.02 %	-2.42 %	0.01	1.78	0.09	Under*
	Q4	-4.35 %	-2.90 %	-1.45 %	-1.32	0.20	Over
2013	Q1	0.72 %	2.07 %	-1.35 %	-1.49	0.15	Over
	Q2	2.46 %	1.54 %	0.93 %	0.88	0.39	Under
	Q3	1.84 %	-0.42 %	2.26 %	3.16	0.00	Under*
	Q4	0.34 %	2.47 %	-2.13 %	-1.26	0.22	Over
2014	Q1	-3.20 %	-1.05 %	-2.15 %	-1.76	0.10	Over*
	Q2	-1.24 %	-0.32 %	-0.92 %	-1.84	0.08	Over*
	Q3	2.17 %	1.89 %	0.29 %	0.27	0.79	Under
	Q4	-0.32 %	0.95 %	-1.27 %	-1.21	0.24	Over
2015	Q1	-2.59 %	0.62 %	-3.21 %	-2.80	0.01	Over*
	Q2	-2.18 %	-3.18 %	1.00 %	1.28	0.21	Under
	Q3	0.01 %	2.61 %	-2.61 %	-1.99	0.06	Over*
	Q4	-6.52 %	-3.78 %	-2.74 %	-2.01	0.06	Over*
2016	Q1	-2.19 %	-3.28 %	1.08 %	0.87	0.39	Under
	Q2	1.65 %	1.15 %	0.50 %	0.43	0.67	Under
	Q3	6.81 %	6.02 %	0.79 %	0.53	0.60	Under
	Q4	3.12 %	1.44 %	1.68 %	1.24	0.237	Under

Note: * = statistically significant at the 10% level or lower

In Table 3.19, it is shown that in 2012, 2013, 2014 and 2015, the RRM_t^W and RRM_t^L presented the same pattern, except in Q1 of 2012, Q3 of 2013, Q4 of 2014, and Q1 of 2015. In 2016, the RRM_t^W and RRM_t^L presented the same pattern. Both went up (in Q2, Q3, and Q4 of 2016) or down (in Q1 of 2016).

The average residual returns for the winner and loser portfolios presented in Table 3.19 shows that the loser portfolio out-performed the average residual return of the winner portfolio by approximately 0.66% in Q1 of 2012, and by 1.45% in Q4 of 2012. Over the following short periods: from Q4 of 2013 to Q2 of 2014, the winner portfolio under-performed the loser portfolio consecutively. From Q1 of 2016 to Q4 of 2016, the winner portfolio out-performed the loser portfolio consecutively. Over-reaction was also cyclical, and occurred after one or two quarters for the industrial sector. However, all values were not statistically significant. The patterns could only be inferred from datapoints with an asterisk (*). It can also be observed that the magnitude of over-reaction varied, without any clear pattern.

During the training period of 2012 to 2016, the results of the $RRM_t^W - RRM_t^L$ were negative in the periods: Q1 and Q4 of 2012, Q1 and Q4 of 2013, Q1, Q2 and Q4 of 2014, and Q1, Q3 and Q4 of 2015 for the industrial sector consistent with the over-reaction hypothesis. The other periods showed positive results (see Table 3.19), consistent with the under-reaction hypothesis. In the industrial sector, the number of under-reaction and over-reaction instances were equal. Under-reaction and over-reaction were also persistent in at least two quarters, except during 2016, where under-reaction occurred in all four quarters. Out of 10 cases of observed over-reaction, five were significant, whereas, out of 10 cases of under-reaction, only two were significant. These results are summarised in Table 3.20.

Table 3.20: Number of significant over- and under-reaction in the industrial sector

		Number	Significant
Under	> 0	10	2
Over	< 0	10	5

In the bar chart presented in Figure 3.11, it is shown that over-reaction and under-reaction were cyclical for the industrial sector. Over-reaction was observed after one or two quarters.

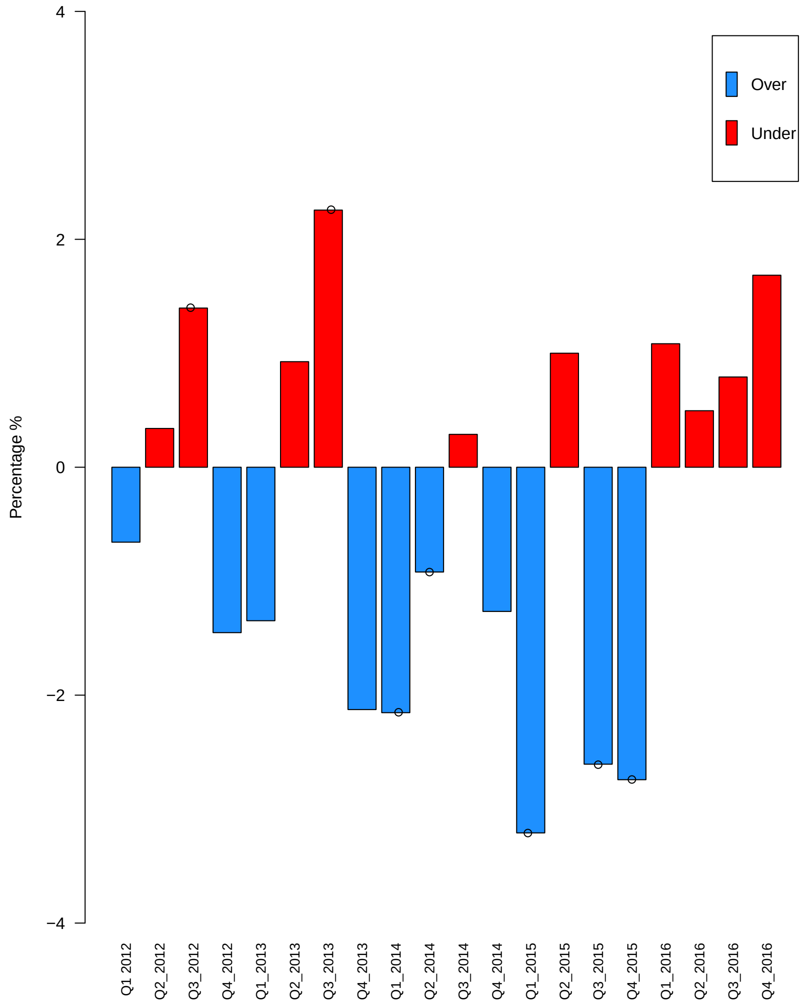


Figure 3.11: Over-reaction and under-reaction: industrial sector

The dot in the bar chart indicates quarters for which the t-test was significant.

3.11.3 Financial Sector

In this sub-section, the average residual return and the result of the t-test are presented for each quarter of the testing period (2012 to 2016) for the financial sector. In Table 3.21, results of the analysis for every quarter of the test period for the financial sector are presented.

Table 3.21: Average residual returns and t-test for the financial sector

Year	Quarter	RRM_t^W	RRM_t^L	$RRM_t^w - RRM_t^L$	t-test	p.value	Over/Under
2012	Q1	0.31 %	1.30 %	-0.99 %	-0.85	0.40	Over
	Q2	1.70 %	0.74 %	0.96 %	0.89	0.38	Under
	Q3	-0.39 %	0.37 %	-0.75 %	-1.10	0.28	Over
	Q4	-2.75 %	-1.98 %	-0.77 %	-0.74	0.46	Over
2013	Q1	-0.75 %	1.30 %	-2.05 %	-2.59	0.02	Over*
	Q2	-1.29 %	0.32 %	-1.61 %	-1.13	0.27	Over
	Q3	-1.17 %	-2.69 %	1.51 %	1.08	0.29	Under
	Q4	-0.67 %	-1.25 %	0.58 %	0.35	0.73	Over
2014	Q1	-3.34 %	-2.40 %	-0.94 %	-0.82	0.42	Over
	Q2	0.47 %	-0.83 %	1.30 %	2.36	0.03	Under*
	Q3	1.67 %	0.22 %	1.45 %	1.09	0.29	Under
	Q4	2.77 %	4.33 %	-1.56 %	-1.44	0.16	Over
2015	Q1	0.47 %	0.99 %	-0.52 %	-0.35	0.73	Over
	Q2	1.89 %	-0.18 %	2.07 %	2.00	0.06	Under*
	Q3	2.79 %	3.91 %	-1.12 %	-0.91	0.37	Over
	Q4	-1.81 %	0.67 %	-2.49 %	-1.83	0.08	Over*
2016	Q1	-1.00 %	-2.38 %	1.38 %	1.04	0.31	Under
	Q2	0.88 %	1.50 %	-0.62 %	-0.52	0.61	Over
	Q3	4.80 %	2.50 %	2.30 %	1.91	0.07	Under*
	Q4	2.05 %	1.72 %	0.33 %	0.39	0.70	Under

Note: * = statistically significant at the 10% level or lower

In Table 3.21, it is shown that in 2016, the RRM_t^W and RRM_t^L presented the same pattern with both up (in Q2, Q3 and Q4 of 2016), and down (Q1 of 2016). In 2012 and 2014, the RRM_t^W and RRM_t^L presented the same pattern except for Q3 of 2012, and Q2 of 2014.

The average residual returns for the winner and loser portfolios presented in Table 3.17 show that the loser portfolio out-performed the average residual return of the winner portfolio by approximately 0.99% in Q1 of 2012 and by 0.75% in Q3 of 2012. In Q2, Q3, and Q4 of 2012, and Q1 and Q2 of 2013, the winner portfolio out-performed the loser portfolio consecutively. Over the short period of Q3 of 2012 to Q3 of 2013, the winner portfolio under-performed the loser portfolio consecutively. From Q4 of 2014 to Q1 of 2015, the winner portfolio consistently out-performed the loser portfolio. For the financial sector, over-reaction was cyclical, and occurred after two quarters. However, not all values were statistically significant. The patterns could only be inferred from datapoints marked with an asterik (*). It can also be observed that the magnitude of over-reaction varied, and did not follow any pattern.

During the training period of 2012 to 2016, the results of the $RRM_t^W - RRM_t^L$ were positive in the periods Q2 of 2012, Q3 of 2013, Q2, Q3 of 2014 and Q1, Q3, Q4 of 2015 for the financial sector, consistent with the under-reaction hypothesis. The other periods showed negative results (see Table 3.19), consistent with the over-reaction hypothesis. In the financial sector, over-reaction occurred more than under-reaction, and was persistent for at least two quarters. In 2016, over-reaction persisted for four quarters. Out of 11 cases of observed over-reaction, two were significant, and out of nine cases of under-reaction, three were significant (see Table 3.22).

Table 3.22: Number of significant over- and under-reaction in the financial sector

		Number	Significant
Under	> 0	9	3
Over	< 0	11	2

Figure 3.12 presents the bar chart of the results of over-reaction and under-reaction for the financial sector.

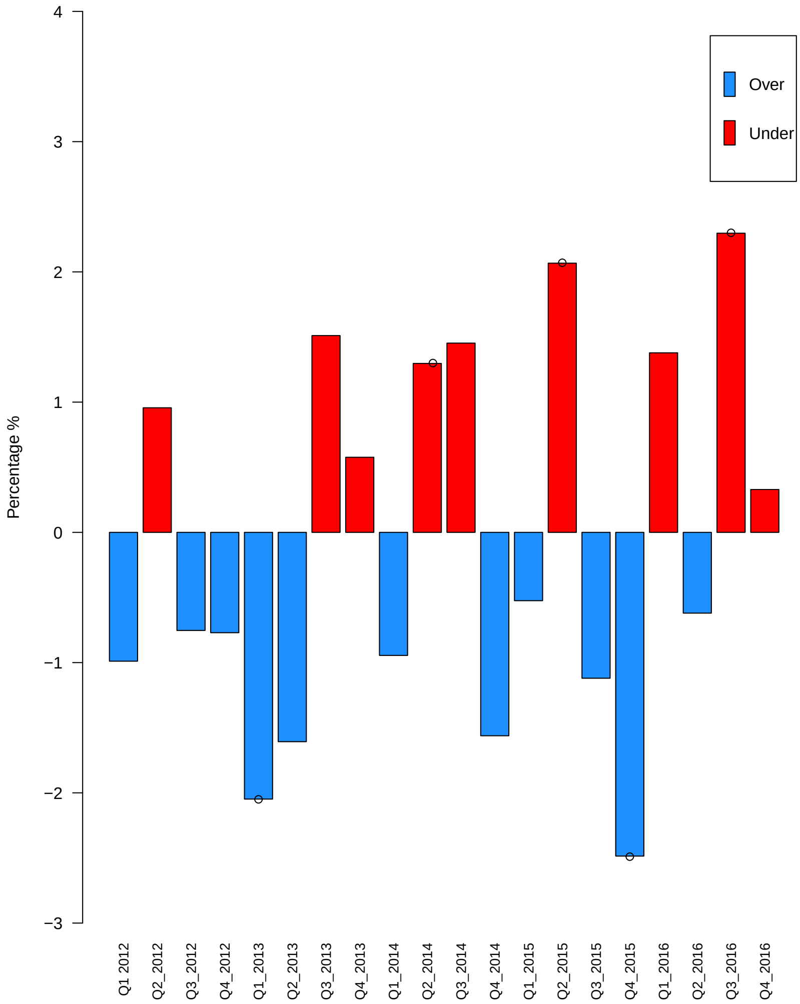


Figure 3.12: Over-reaction and under-reaction: financial Sector

The dot in the bar chart indicates quarters where the t-test was significant.

3.12 Comparison of the Three Sectors

All three sectors were grouped and analysed together, to determine if similar patterns prevailed across the testing period. The results of the occurrence of over-reaction and under-reaction for the different sectors are represented in Figure 3.13. All three sectors were plotted on the same figure to see if there were any similar patterns.

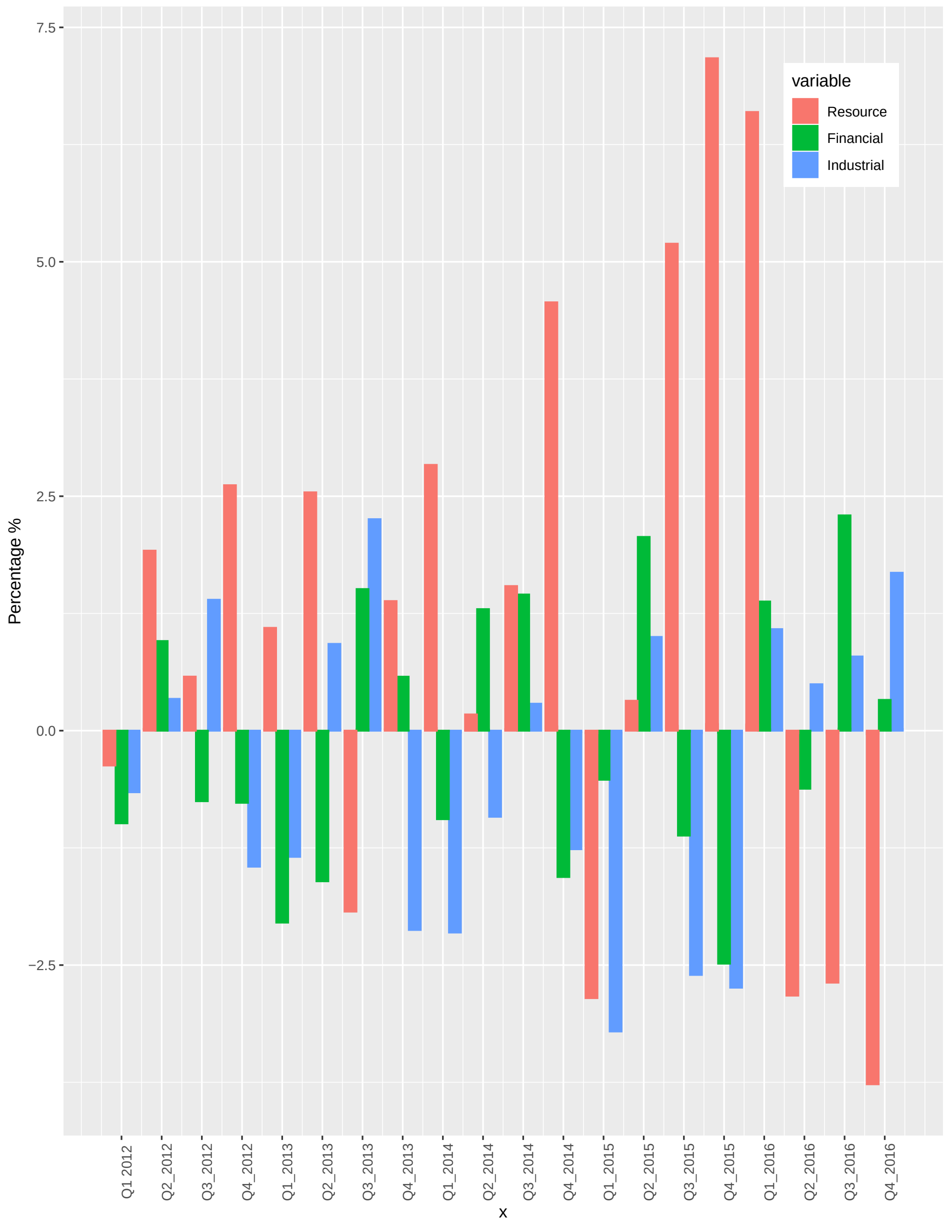


Figure 3.13: Over-reaction and under-reaction per sector

The results reflected in Figure 3.13 indicate that, overall, the two biases investigated depended on the sector considered. Looking at the timing and flow of the occurrence of under-reaction and over-reaction, the analysis revealed that there were periods where all the sectors had similar patterns, namely Q1 of 2012, Q2 of 2012, Q3 of 2014, Q1 of 2015, Q2 of 2015, and Q1 of 2016. Under-reaction was more prevalent in the resources sector, and extremes values for this sector were visible from 2015 to 2016.

3.13 Distribution Free Test

From the analyses to determine if normality could be assumed for the t-tests (see Sub-section 3.9), it can be seen that the residual return of the winner portfolio and the residual return of the loser portfolio were not normally distributed in all cases. Hence, the validity of the t-test was doubtful. Therefore, a distribution free test (Mann-Whitney U test), which does not rely on assumptions that the data are drawn from a normal distribution, was performed.

The Mann-Whitney U test (Elton *et al.*, 2009) was used to test the following pair (H_0, H_1) of hypotheses:

H_0: The distributions of the two populations' are identical.

H_1: The two population distributions are not identical.

This was done using the following u-statistics:

$$U_1 = n_1.n_2 + \frac{n_1.(n_1+1)}{2} - R_1 \tag{3.13.1}$$

$$U_2 = n_1.n_2 + \frac{n_2.(n_2+1)}{2} - R_2 \tag{3.13.2}$$

where, n_1 and n_2 were the sizes of Samples 1 and 2 respectively. R_1 = sum of the ranks for Sample 1, and R_2 = sum of the ranks for Sample 2.

The test statistic is the smallest of these U values. One value of U can be calculated and the other can be found using the following transformation:

$$U_1 = n_1.n_2 - U_2 \tag{3.13.3}$$

There is rejection of H_0 if, by consulting the Mann- Whitney tables, the p corresponding to the $\min(U_1, U_2)$ (the smallest of U both calculated) is smaller than the p or the predetermined α threshold ie.

$$\min(U_1, U_2) < \alpha \text{ threshold.} \tag{3.13.4}$$

If the numbers of observations n_1 and n_2 are larger than eight, a normal approximation, as shown by Mann and Whitney (1947), can be used, that is to say:

$$\mu_u = \frac{n_1.n_2}{2} = \frac{(U_1 + U_2)}{2} \tag{3.13.5}$$

and

$$\sigma_u = \sqrt{\frac{(n_1.n_2)(N+1)}{12}} \tag{3.13.6}$$

where $N = n_1 + n_2$, μ_u corresponds to the average of the U distribution and σ_u corresponds to its standard deviation.

If each group includes more than eight observations, the sample's distribution gradually approaches a normal distribution. If a normal approximation has to be used, the corresponding equation becomes:

$$z = \frac{(U - (n_1.n_2))}{\sigma_u} \tag{3.13.7}$$

and the test statistic becomes, in absolute values:

$$\mid z \mid = \frac{\mid U_1 + U_2 \mid}{\sigma_u} \tag{3.13.8}$$

To test the difference between σ_u and μ_u, the reader can refer to the z-table. If the absolute value of the calculated z is larger or equal to the tabulated $z_{1-\frac{1}{2}\alpha}$ value $(P(z < z_{1-\frac{1}{2}\alpha}) = 1 - \frac{1}{2}\alpha)$ at a α significance level, that is, the null hypothesis is rejected.

Reject H_0 if $\mid z\ \textit{calculated} \mid\ \geqslant \mid z_{1-\frac{1}{2}\alpha}\ \textit{tabulated} \mid$.

The results for the different sectors are presented in the next sub-sections.

3.13.1 Resources Sector

In this sub-section, the average residual return and the results of the Mann-Whitney U test are presented for each quarter of the testing period (2012-2016) for the resources sector. In Table 3.23, results of the analysis for every quarter of the test period for the resources sector are presented together with the results of the t-test.

Table 3.23: Average residual returns and comparison of the Mann-Whitney U test (U test) and t-test for the resources sector

Year	Quarter	$RRM_t^W - RRM_t^L$	Over/Under	p.value (t-test)	p.value (U test)
2012	Q1	-0.37 %	Over	0.76	0.92
	Q2	1.92 %	Under	0.09*	0.03*
	Q3	0.58 %	Under	0.57	0.69
	Q4	2.62 %	Under	0.01*	0.00*
2013	Q1	1.10 %	Under	0.22	0.29
	Q2	2.54 %	Under	0.05*	0.05*
	Q3	-1.93 %	Over	0.04*	0.00*
	Q4	1.38 %	Under	0.47	0.06*
2014	Q1	2.84 %	Under	0.05*	0.04*
	Q2	0.17 %	Under	0.66	0.76
	Q3	1.54 %	Under	0.00*	0.01*
	Q4	4.57 %	Under	0.00*	0.01*
2015	Q1	-2.85 %	Over	0.05*	0.05*
	Q2	0.32 %	Under	0.64	0.88
	Q3	5.19 %	Under	0.00*	0.00*
	Q4	7.17 %	Under	0.07*	0.15
2016	Q1	6.60 %	Under	0.04*	0.05*
	Q2	-2.83 %	Over	0.14	0.11
	Q3	-2.69 %	Over	0.04*	0.01*
	Q4	-3.77 %	Over	0.00*	0.00*

Note: * = statistically significant at the 10% level or lower

In Table 3.23, it can be seen that over-reaction and under-reaction that were significant with the t-test were also significant with the Mann-Whitney U test, except for Q4 of 2013 and Q4 of 2015.

3.13.2 Industrial Sector

In this subsection, the average residual return and the result of the Mann-Whitney U test are presented for each quarter of the testing period (2012-2016) for the industrial sector. In Table 3.24, results of the analysis for every quarter of the test period for the industrial sector are presented together with the results of the t-test.

Table 3.24: Average residual returns and comparison of the Mann-Whitney U test (U test) and t-test for the industrial sector

Year	Quarter	$RRM_t^W - RRM_t^L$	Over/Under	p.value (t-test)	p.value (U test)
2012	Q1	-0.66 %	Over	0.48	0.48
	Q2	0.34 %	Under	0.74	0.58
	Q3	1.40 %	Under	0.09*	0.14
	Q4	-1.45 %	Over	0.20	0.07*
2013	Q1	-1.35 %	Over	0.15	0.20
	Q2	0.93 %	Under	0.39	0.34
	Q3	2.26 %	Under	0.00*	0.00*
	Q4	-2.13 %	Over	0.22	0.03*
2014	Q1	-2.15 %	Over	0.10	0.09*
	Q2	-0.92 %	Over	0.08*	0.02*
	Q3	0.29 %	Under	0.79	0.65
	Q4	-1.27 %	Over	0.24	0.26
2015	Q1	-3.21 %	Over	0.01*	0.01*
	Q2	1.00 %	Under	0.21	0.14
	Q3	-2.61 %	Over	0.06*	0.07*
	Q4	-2.74 %	Over	0.06*	0.02*
2016	Q1	1.08 %	Under	0.39	0.26
	Q2	0.50 %	Under	0.67	0.80
	Q3	0.79 %	Under	0.60	0.48
	Q4	1.68 %	Under	0.23	0.06*

Note: * = statistically significant at the 10% level or lower

In Table 3.24, it can be seen that over-reaction and under-reaction that were significant using the t-test, are also significant when using the Mann-Whitney U test except for Q4 of 2012.

3.13.3 Financial Sector

In this subsection, the average residual return and the result of Mann-Whitney U test are presented for each quarter of the testing period (2012 - 2016) for the financial sector. In Table 3.25, results of the analysis for every quarter of the test period for the financial sector are presented together with the results of the t-test.

Table 3.25: Average residual returns and the Mann-Whitney U test (U test) and the t-test for the financial sector

Year	Quarter	$RRM_t^W - RRM_t^L$	Over/Under	p.value (U test)	p.value (t-test)
2012	Q1	-0.99 %	Over	0.40	0.34*
	Q2	0.96 %	Under	0.38	0.26
	Q3	-0.75 %	Over	0.28	0.17
	Q4	0.77 %	Under	0.46	0.04*
2013	Q1	-2.05 %	Over	0.02*	0.01*
	Q2	-1.61 %	Over	0.27	0.29
	Q3	1.51 %	Under	0.29	0.18
	Q4	-0.58 %	Over	0.73	0.61
2014	Q1	-0.95 %	Over	0.42	0.48
	Q2	1.30 %	Under	0.03*	0.08*
	Q3	1.45 %	Under	0.29	0.42
	Q4	1.56 %	Under	0.16	0.13
2015	Q1	-0.52 %	Over	0.73	1
	Q2	2.07 %	Under	0.06*	0.03*
	Q3	-1.12 %	Over	0.37	0.36
	Q4	2.49 %	under	0.08*	0.1
2016	Q1	1.38 %	Under	0.31	0.22
	Q2	-0.62 %	Over	0.61	0.72
	Q3	2.30 %	Under	0.07*	0.03*
	Q4	-0.33 %	Over	0.70	0.76

Note: * = statistically significant at the 10% level or lower

Except for a few quarters (Q4 of 2013 and 2015 for the resources sector, Q4 of 2012 for the industrial sector, and Q4 of 2012 for the financial sector), for which the results of the t-test and the Mann-Whitney U test did not coincide, over-reaction and under-reaction that were significant using the t-test were also significant when using the Mann-Whitney U test. Hence, the results of the Mann-Whitney U test confirmed the t-test results for the different sectors.

3.14 Summary of Process and Main Findings

In this study, it is investigated whether South African investors tend to over-react and/or under-react over time, driven by their behavioural biases. Barberis *et al.* (1998) proposed a theory that is based on the conservatism and representativeness heuristics to explain investor over-reaction and under-reaction.

Investors who exhibit conservatism will under-react to information because they are slow to react and to update their beliefs in response to recent evidence. "In the context of decision making in economics, the individuals under the influence of the heuristic of representativeness tend to produce extreme predictions, or over-reaction, in which former losers tend to be winners in the future and vice-versa" (Aguiar *et al.*, 2006).

In the present study, weekly and quarterly South African equity market data for the period 2006 to 2016 were used. Stocks in the financial, industrial, and resources sectors were analysed separately and then compared. Because of large data scale differences between the seven variables, seven different standardisation methods were tested, and the normal z-score standardisation method proved to be the optimal transformation method. The data were also winsorized, because of the presence of large outliers that would eventually distort the groups' centres. The data were then split into two sets for use in the FCM model: the training period (2006 to 2011) and the testing period (2012 to 2016).

Using the training set, seven-dimensional winner- and loser centres (in the form of vectors) were created using the seven fundamental variables and stock performance. Using these quarterly vectors in the training period, four vectors representing the winner- and loser groups for each of the four quarters were then compiled as a centre of the centres. For each quarter in the testing period, stocks were then assigned to these centres, in anticipation of these continuing to be winners or losers, depending on the winner- or loser group to which they were matched.

The residual performance per winner and loser group was then calculated per quarter in the testing period, and used to infer behavioural patterns. If the newly formed quarterly winner portfolios under-performed the newly formed loser portfolios, it was hypothesised that over-reaction had occurred. Conversely, if the newly formed winner portfolios maintained their outstanding performance, it was hypothesised that under-reaction had occurred. Welch's t-test, allowing unequal variances, was used to test if over-reaction and under-reaction on the JSE were statistically significant at the 10% level or lower.

The Mann-Whitney U test (the distribution free test) was applied as well, to test for significant under- and over-reaction. In both cases, similar statistically significant over-reaction and under-reaction were found in all three sectors.

Determining if the results varied per sector yielded the following:

- In the resources sectors, under-reaction occurred more frequently than over-reaction, and once persisted for five consecutive quarters. Out of six cases of observed over-reaction, four were significant, whereas, out of 14 cases of under-reaction, nine were significant.

- In the industrial sector, the numbers of instances of under-reaction and over-reaction were equal. Under-reaction and over-reaction were also persistent for at least two quarters, except during 2016, where under-reaction occurred in all four quarters. Out of 10 cases of observed over-reaction, five were significant, whereas out of 10 cases of under-reaction, two were significant.

- In the financial sector, over-reaction occurred more than under-reaction, and was persistent for at least two quarters. In 2016, over-reaction persisted for four quarters. Out of 11 cases of observed over-reaction, two were significant, whereas out of nine cases of under-reaction, three were significant.

The above three bullet points indicate that, overall, the sectors had different occurrences of the two biases investigated. The timing and flow of the under-reaction were then analysed. All three sectors were grouped and analysed together, to determine if similar patterns prevailed across the testing period. The analysis revealed that all the sectors sporadically had similar patterns, but only 30% of the time over the 20 quarters in the test period. Under-reaction was more prevalent in the resources sector, and extreme values for this sector were visible from 2015 to 2016. The result of the normality test of the data showed that the data were not normally distributed. A distribution free test called the Mann-Whitney U test, which does not rely on the assumption that the data were drawn from a normal distribution, was thus used to confirm the results.

Although prior South African studies on over- and under-reaction e.g., (Page and Way, 1992; Muller, 1999; Cubbin *et al.*, 2006; Venter, 2009; Hsieh and Hodnett, 2011) used different methodologies than the one followed in the present study, the results showing over-reaction and under-reaction in the South African market are broadly aligned. The methodologies of earlier studies were based on that of De Bondt and Thaler (1985), who considered the ranking of stocks based on past return. Portfolios of winner- and loser stocks in the prior three or five years were constructed, and their performance in the subsequent three or five years were then measured. The results indicated that the portfolios of most extreme losers had high abnormal returns, whereas the portfolios of winners had negative abnormal returns.

In the current study, the more mathematical FCM algorithm was used, together with formal statistical tests. Significant over- and under-reaction were found in the South African market during the period 2012 to 2016. No clear patterns were visible, and neither did out-performance of one group over another persist over the period under study. It was concluded that the out-performance of winner- and loser groups in the South African market is unpredictable. What is considered a trend according to the tenets of behavioural finance appears to be a random event.

The findings of the current research also indicate that the momentum- and the contrarian investment strategies can lead to excess performance in the South African equity market, but could also generate under-performance relative to the poorly performing market.

The FCM algorithm was accurate and efficient in determining the two unique centres when using South African market data, but proper scaling and winsorization of the dataset before running the FCM algorithm are strongly advised. In the next chapter, the use of the Bayesian model in evaluating investor behaviour in the context of the JSE is discussed.

Chapter 4

Evidence of Over- and Under-reaction using the Bayesian Model

In this chapter, mathematical statistical concepts underlying the Bayesian statistical model are proposed for use in evaluating investor behaviour. Specifically, over- and under-reaction on the JSE are investigated.

Evidence of over-reaction showed that, over a long term horizon, stock prices over-react to consistent patterns of news of the same sign. That is, "stocks that have had a long record of good news tend to become overpriced and then have low average returns afterwards" (De Bondt and Thaler, 1985, 1987).

The evidence of under-reaction showed that stocks prices under-react to news over a short term horizon (Jegadeesh and Titman, 1993). "As a consequence, news is slowly incorporated into prices, which tends to exhibit positive autocorrelations". This implies that prices should have adjusted quicker to good (bad) news, but because investors initially under-reacted, a sequence of positive (negative) returns is evident over time. For the purpose of the current study, it was hypothesised that under-performance but only after prior out-performance events, is evidence of over-reaction among investors. Furthermore, a continuation of out-performance events is evidence of under-reaction among investors.

4.1 Mathematical Formalism

The Bayesian approach provides a mathematical rule that explains how existing beliefs could be updated by new evidence. Compared to classical statistics which ignores prior probabilities, the Bayesian method permits the use of priors. The model proposed by Barberis *et al.* (1998) uses Bayes' s theorem to test how investor sentiment changed in the context of a sequence of prior information flows. It proposes experiments that incorporate data that investors may use to update prior beliefs based on certain parameters, such as earnings surprises (the term earnings shock is used in this dissertation) and valuation levels. Analysts use companies' annual reports and market conditions to predict earnings. An earnings shock occurs when a company's reported profits are different from the estimated profits.

Barberis *et al.* (1998) Bayes model (BBM) is based on the classical Bayes' theorem, which describes a relationship between the probability of an event conditional upon another event.

4.1.1 Properties of Conditional Probability

Definition 2. *(Law of total probability):*

Let events $B_1, B_2, \ldots, B_n$, satisfy the following:

- $S = B_1 \cup B_2 \cup \cdots \cup B_n$
- $B_i \cap B_j = \varnothing$, *for every* $i \neq j$
- $P(B_i) > 0$, *for* $i = 1, \ldots, n$

The events $B_1, B_2, \ldots, B_n$ form a partition of the sample space S. Then for any event A,

$$\mathbb{P}(A) = \mathbb{P}(A/B_1).\mathbb{P}(B_1) + \cdots + \mathbb{P}(A/B_n).\mathbb{P}(B_n). \tag{4.1.1}$$

The Bayes results can be described as follows:

Definition 3. *(Bayes' theorem): Let $B_1, B_2, \ldots, B_n$, partition of the sample space S and let A be an event with $P(A) > 0$. Then, for $j = 1, \ldots, n$, the conditional (posterior) probability of event B_j given that A occurred, is given by Bayes's theorem as:*

$$\begin{aligned}
\mathbb{P}(B_j/A) &= \frac{\mathbb{P}(B_j \cap A)}{\mathbb{P}(A)} \\
&= \frac{\mathbb{P}(B_j)\mathbb{P}(A/B_j)}{\sum_{j=1}^{n} \mathbb{P}(B_j)\mathbb{P}(A/B_j)}.
\end{aligned}$$

As new information is made available, rational investors are expected to use Bayes's rules to form new beliefs. This implies that, at each time period, new information signals should be added to the information set, and that decisions should be made taking all prior signals into consideration. However, the influence of human biases drives investors away from pure Bayesian principles in a variety of ways.

In the model proposed by Barberis *et al.* (1998), the earnings of an asset at time t, N_t is random, and the average investor does not realise that this is the true process of earnings. If an investor believes there are observable patterns that may continue, the investor will thus use a wrong model to update his or her belief which is grouped into two states. The underlying switching process between investor states follows a Markov process in which the change in earnings in period t depends only on the change in earnings in period $t - 1$.

The conservative- and representative heuristics of an investor will, respectively lead him or her to believe that the earnings follow either a trending regime or a mean-reverting regime. It is assumed that the two regimes can be formulated as a two-state Markov chain, adhering to the following principles:

- In the mean-reverting regime, a shock is likely to be followed by another of a different sign in the following period.

- In the trending regime, after an increase (decrease) of a stock price, prices are likely to rise (decrease) further.

The investor adheres to Bayes's rules in this updating process, but uses a wrong model for the earning process. Specifically, periods in which the sign of the shock switches often convince the model investor's belief of a high likelihood of mean reversion in the next period, while a stable sign in consecutive periods increase the model investor's belief that a trending regime is possible.

The model of Barberis *et al.* (1998) is motivated by the fact that, "people pay too much attention to the strength of the recent evidence they are presented with and too little attention to the statistical weight that it should be assigned while making a forecast" (Griffin and Tversky, 1992). Evidence of over-reaction and under-reaction can be identified by the reaction of stock prices after consistent patterns of news. Within the Bayesian framework, over-reaction and under-reaction are explained by two biases in investor behaviour, namely conservatism and representativeness:

- Investors subject to conservatism tend to underweight useful statistical evidence because they are overconfident about their prior information (Barberis *et al.*, 1998).
- Investors subject to a representativeness heuristic see patterns in random sequences.

In the section below, statistical evidence of under-reaction and over-reaction in security returns are summarised.

4.1.2 Statistical Evidence of Under-reaction and Over-reaction

In this sub-section, over-reaction and under-reaction are defined as proposed by Barberis *et al.* (1998). It is assumed that the investor receives news z_t about a particular company in each time period t, where z_t can be good news (G) or bad news (B).

In the BBM (Barberis *et al.*, 1998), good news is implied by earnings that are higher than expected. Following what was described in the introduction of the chapter, and using the same notation as before, over-reaction is deemed to have occurred when the average return following not one but a series of announcements of good news is lower than the average return following a series of bad news announcements.

Mathematically,

$$E(R_{t+1}|z_t = G, z_{t-1} = G, \dots, z_{t-j} = G) < (E(R_{t+1}|z_t = B, z_{t-1} = B, \dots, z_{t-j} = B), \tag{4.1.2}$$

where $j > 1$,

R_{t+1} is the return at time $t + 1$.

After a series of good news, the investor believes that the trend will continue, and therefore, over-reacts, and the price will increase. The stock price will correct (revert to mean) after it was overvalued, and the reaction to good news is then lower than it was before. Under-reaction to news implies that:

$$E(R_{t+1}|z_t = G) > E(R_{t+1}|z_t = B). \tag{4.1.3}$$

where R_{t+1} is the return at time $t + 1$. Investors react lower than optimal to good news. In the following period, the mistake is corrected and the return is higher.

4.1.3 Mathematical Statistical Tools

In the BBM model (Barberis *et al.*, 1998), investors mistakenly believe that earnings come from either of two regimes: mean-reverting and trending. The transition from one regime to the other is a Markov process. In this sub-section, some formal definitions of stochastic processes are given.

Definition 4. *A stochastic process (or random process) is a collection of S-valued stochastic variables $\boldsymbol{X} : \{X(t) : t \in T\}$ defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\boldsymbol{X}$ takes values in a common set S (the state space) which must be measurable with respect to some σ-algebra Σ and indexed by a set T, often either $\mathbb{N}$ or $[0, \infty)$ and thought of as time (discrete or continuous respectively). Recall that Ω is a sample space, $\mathcal{F}$ is a σ-algebra, and $\mathbb{P}$ is a probability measure (Pinsky and Karlin, 2010).*

In the present study, the state space, S, is discrete.

Definition 5. *A finite state Markov process is a stochastic process with the following properties:*

- *The number of possible outcomes or states is finite.*
- *The outcome at any stage depends only on the outcome of the previous stage.*
- *The probabilities are constant over time.*

Definition 6. *(Transition probability): Let $\boldsymbol{X} : \{X(t) : t \in T\}$ be a Markov process. The probability that the process moves from state i to state j is called transition probability, and is defined by:*

$$p_{ij}(s,t) = \mathbb{P}(X(t) = j | X(s) = i), \tag{4.1.4}$$

for all $s, t \in T$ such that $s < t$. If X_t is discrete then it is simply denoted by p_{ij} and given by:

$$p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i), \tag{4.1.5}$$

Equation (4.1.5) is called a one-step transition probability.

Definition 7. *(Transition matrix): The one step transition matrix (given s possible states) is defined as:*

$$\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1s} \\ p_{21} & p_{22} & \dots & p_{2s} \\ \vdots & \vdots & \vdots & \vdots \\ p_{s1} & p_{s2} & \dots & p_{ss} \end{bmatrix}. \tag{4.1.6}$$

Hence, the r-step transition matrix (given s possible states) follows as:

$$P^{[r]} = P^r, \tag{4.1.7}$$

with $(ij)^{th}$ *element:*

$$p_{ij}^{(r)} = \mathbb{P}(X_{t+r} = j/X_t = i). \tag{4.1.8}$$

Define p_r^t *as the state probability:*

$$p_r^t = \mathbb{P}(X_t = r). \tag{4.1.9}$$

and hence let $\mathbf{p}_t$ *be a vector that represents the state probabilities of a system after* t *steps with* $\mathbf{p}_0$ *the vector of initial state probabilities i.e.*

$$\mathbf{p}_0 = \begin{bmatrix} \mathbb{P}(X_0 = 1) \\ \mathbb{P}(X_0 = 2) \\ \vdots \\ \mathbb{P}(X_0 = s) \end{bmatrix} = \begin{bmatrix} p_1^0 \\ p_2^0 \\ \vdots \\ p_s^0 \end{bmatrix}.$$

From this follows:

$$\mathbf{p}_1 = \boldsymbol{P}'\mathbf{p}_0 = \begin{bmatrix} p_{11} & p_{21} & \dots & p_{s1} \\ p_{12} & p_{22} & \dots & p_{s2} \\ \vdots & \vdots & \vdots & \vdots \\ p_{1s} & p_{2s} & \dots & p_{ss} \end{bmatrix} \begin{bmatrix} p_1^0 \\ p_2^0 \\ \vdots \\ p_s^0 \end{bmatrix} = \begin{bmatrix} p_1^1 \\ p_2^1 \\ \vdots \\ p_s^1 \end{bmatrix}$$

so that in general:

$$\mathbf{p}_t = \boldsymbol{P}'\mathbf{p}_{t-1} = \boldsymbol{P}'\boldsymbol{P}'\mathbf{p}_{t-2} = \boldsymbol{P}'^{[2]}\mathbf{p}_{t-2} = \cdots = \boldsymbol{P}'^{[t]}\mathbf{p}_0.$$

Thus, we get a chain of state vectors: $\mathbf{p}_0, \boldsymbol{P}'\mathbf{p}_0, \boldsymbol{P}'^{[2]}\mathbf{p}_0 \ldots$ *where the state probability after* j *iterations is given by* $\boldsymbol{P}'^{[j]}\mathbf{p}_{t-j}$*. Such a chain is called a Markov chain.*

Definition 8. *(Geometric series of matrices): Let* $\mathbf{T}$ *be any square matrix. Then the sequence* $\{\mathbf{S}_n\}_{n\geqslant 0}$*, defined by:*

$$\mathbf{S}_n = \mathbf{I} + \mathbf{T} + \cdots + \mathbf{T}^{n-1},\ \mathbf{S}_0 = \mathbf{I}, \tag{4.1.10}$$

is called the geometric series generated by the matrix $\mathbf{T}$*. The series converges if the sequence* $\{\mathbf{S}_n\}_{n\geqslant 0}$ *converges; this can be written as:*

$$\sum_{n=0}^{\infty} \mathbf{T}^{\mathbf{n}} = \lim_{n\to\infty} \mathbf{S}_n \tag{4.1.11}$$

Theorem 4.1.1. *(Hubbard and Burke Hubbard, 2015) If* $|\mathbf{T}| < 1$*, the geometric series generated by* $\mathbf{T}$ *converges to* $(\mathbf{I} - \mathbf{T})^{-1}$ *as* $n \to \infty$*:*

$$\sum_{k=0}^{\infty} \mathbf{T}^k = (\mathbf{I} - \mathbf{T})^{-1}. \tag{4.1.12}$$

4.1.4 Model Assumptions

The BBM model (Barberis *et al.*, 1998) evaluates investor reaction to public information. The model assumes a representative, risk-neutral investor with discount rate δ. This investor's beliefs are therefore supposed to reflect the "consensus". "There is only one security, which pays out 100% of its earnings as dividends; in this context, the equilibrium price of the security is equal to the net present value of future earnings, as forecasted by the representative investor" (Barberis *et al.*, 1998). In the following paragraph, the assumptions of the model are presented and explained.

Assumption 1. *Y_t is independent and takes discrete values y or $-y$ with equal chance. The earnings of an asset are random. That is, at time t, the earnings are:*

$$N_t = N_{t-1} + Y_t, \tag{4.1.13}$$

where Y_t is an earnings shock a time t.

Investors perceive earnings as following a certain pattern and use the hypothetical patterns to predict future earnings. Investors mistakenly believe that the earnings come from either of two regimes. Regime 1 is mean-reverting where a shock is likely to be followed by another of a different sign in the following period. Regime 2 is trending where after an increase (decrease) of a stock price, prices are likely to rise (decrease) further.

Assumption 2. *The transition probability (from y to $-y$ or from $-y$ to y) is small (smaller than one-half) in the trending regime and is large (larger than one-half) in the mean-reverting regime.*

In the mean-reverting regime, a shock is likely to be followed by another of a different sign in the following period. In the trending regime, after an increase (decrease) of a stock price, the price is likely to rise (decrease) further. For each earnings at time t, y_t is the earning shock, π_t is the probability assigned by the investor at time t to duration of Regime 1 (mean-reverting). Under Regime 1, earnings shocks are likely to be followed by another of a different sign in the following period. At each point in time, the process S_t is in one of two regimes, which is indicated by $S_t = s_t = 1$ and $S_t = s_t = 2$. The state of the world at time t is written as s_t. If for example, $s_t = 1$, the process is in the first regime, and the earnings shock in period t, y_t, is generated by Regime 1. Let λ_1 and λ_2 be the probabilities to move from Regime 1 to Regime 2 (trending) and from Regime 2 to Regime 1 respectively.

Hence:

$$\mathbb{P}(S_{t+1} = 1|s_t = 1) = 1 - \lambda_1, \qquad \mathbb{P}(S_{t+1} = 2|s_t = 2) = 1 - \lambda_2.$$

The transition matrix is given by:

$$\mathbb{P}_{ij} = \mathbb{P}(S_{t+1} = j|s_t = i);$$

where $\mathbb{P}(S_{t+1} = j|s_t = i)$ is the probability of selecting state j next, given that the process is in state i, hence:

$$\mathbf{P} = \begin{bmatrix} \mathbb{P}(S_{t+1} = 1|s_t = 1) & \mathbb{P}(S_{t+1} = 2|s_t = 1) \\ \mathbb{P}(S_{t+1} = 1|s_t = 2) & \mathbb{P}(S_{t+1} = 2|s_t = 2) \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

which can be written as :

	$s_{t+1} = 1$	$s_{t+1} = 2$
$s_t = 1$	$1 - \lambda_1$	λ_1
$s_t = 2$	λ_2	$1 - \lambda_2$

.

Define for Regime 1:

$$\mathbb{P}(Y_{t+1} = y \mid S_t = 1,\ Y_t = y) = \pi_L, \qquad \mathbb{P}(Y_{t+1} = -y \mid S_t = 1,\ Y_t = y) = 1 - \pi_L$$

so that the transition matrix follows as :

$$P_1 = \begin{bmatrix} \mathbb{P}(Y_{t+1} = y \mid S_t = 1,\ Y_t = y) & \mathbb{P}(Y_{t+1} = -y \mid S_t = 1,\ Y_t = y) \\ \mathbb{P}(Y_{t+1} = y \mid S_t = 1,\ Y_t = -y) & \mathbb{P}(Y_{t+1} = -y \mid S_t = 1,\ Y_t = -y) \end{bmatrix}$$

which can be written as:

Regime 1	$y_{t+1} = y$	$y_{t+1} = -y$
$y_t = y$	π_L	$1 - \pi_L$
$y_t = -y$	$1 - \pi_L$	π_L

Under Regime 2, shocks are more likely to switch the sign.

Hence for Regime 2 define:

$$\mathbb{P}(Y_{t+1} = y \mid S_t = 2,\ Y_t = y) = \pi_H, \qquad \mathbb{P}(Y_{t+1} = -y \mid S_t = 2,\ Y_t = y) = 1 - \pi_H$$

so that the transition matrix follows as :

$$P_2 = \begin{bmatrix} \mathbb{P}(Y_{t+1} = y \mid S_t = 2,\ Y_t = y) & \mathbb{P}(Y_{t+1} = -y \mid S_t = 2,\ Y_t = y) \\ \mathbb{P}(Y_{t+1} = y \mid S_t = 2,\ Y_t = -y) & \mathbb{P}(Y_{t+1} = -y \mid S_t = 2,\ Y_t = -y) \end{bmatrix}$$

which can be written as:

Regime 2	$y_{t+1} = y$	$y_{t+1} = -y$
$y_t = y$	π_H	$1 - \pi_H$
$y_t = -y$	$1 - \pi_H$	π_H

If $\pi_H = \pi_L$, the two regimes are identical. Therefore, π_H captures the dissimilarity between, or heterogeneity of the two regimes.

Assumption 3. *The investor updates his or her belief about the probability of the current regime in a Bayesian manner. In other words, the future expected price is determined using an estimation of the current state of the earnings shocks.*

The model discussed here implies that earnings at any time are generated by two regimes. The investor tries to understand which of the two regimes is currently governing earnings. Earnings are observed each period and that information are used to make as good a guess as possible about what regime the process is in.

The regime is the state of the world and this is the parameter of interest in a Bayes context with distribution π. Let π be the apriori probability that the process is in regime 1 given the information available at that time. Then π_t denotes the probability that the process is in regime 1 in period t and the data is considered as the earning shock Y with Y_t and Y_{t+1} the shocks in period t and $t+1$ respectively. The aposteriori probability π_{t+1} needs now to be obtained given the earning shock Y_{t+1} using Bayes' theorem. Regime 2 is treated in a similar fashion.

Note that the observed earning shock y_t during period t can have the same sign or a different sign as during period $t-1$. The notation $Y_{t+1}=y$ and $Y_t=y$ indicates that y_t and y_{t+1} have the same sign. Hence the aposteriori probability that Y_{t+1} was generated by regime 1 given that the observed y_{t+1} has the same sign in period $t+1$ as in period t using fundamental probability laws (leading to the Bayes result) is derived.

$$\begin{aligned}
\pi_{t+1} &= \mathbb{P}(Y_{t+1} \text{ generated by Regime 1 i.e. } S_{t+1}=1 \text{ given that } Y_{t+1}=y, Y_t=y,\ \pi_t) \\
&= \mathbb{P}(S_{t+1}=1 \mid Y_{t+1}=y, Y_t=y,\ \pi_t) \\
&= \frac{\mathbb{P}(S_{t+1}=1 \cap Y_{t+1}=y, Y_t=y,\ \pi_t)}{\mathbb{P}(Y_{t+1}=y, Y_t=y,\ \pi_t)} \\
&= \frac{\mathbb{P}(S_{t+1}=1).\mathbb{P}(Y_{t+1}=y \mid S_{t+1}=1,\ Y_t=y,\ \pi_t)}{\mathbb{P}(Y_{t+1}=y, Y_t=y,\ \pi_t)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\mathbb{P}(S_{t+1}=1)\cdot\mathbb{P}(Y_{t+1}=y \mid S_{t+1}=1,\ Y_t=y,\ \pi_t)}{\mathbb{P}(S_{t+1}=1)\cdot\mathbb{P}(Y_{t+1}=y \mid S_{t+1}=1,\ Y_t=y)+\mathbb{P}(S_{t+1}=2)\cdot\mathbb{P}(Y_{t+1}=y \mid S_{t+1}=2,\ Y_t=y)} \\
&= \frac{A\cdot B}{A\cdot B+C\cdot D}. \qquad (4.1.14)
\end{aligned}$$

with:

$$\begin{aligned}
A &= \mathbb{P}(S_{t+1}=1) \\
&= \mathbb{P}(S_{t+1}=1 \mid S_t=1)\mathbb{P}(S_t=1)+\mathbb{P}(S_{t+1}=1 \mid S_t=2)\mathbb{P}(S_t=2) \\
&= (1-\lambda_1)\pi_t+\lambda_2(1-\pi_t) \qquad (4.1.15)
\end{aligned}$$

using the transition probabalities of the regimes,

$$\begin{aligned} B &= \mathbb{P}(Y_{t+1} = y \mid S_{t+1} = 1,\ Y_t = y) = \pi_L, \\ &\quad \text{using the transition probabilities of } Y \text{ if in regime 1,} \end{aligned} \tag{4.1.16}$$

$$\begin{aligned} C &= \mathbb{P}(S_{t+1} = 2) = \mathbb{P}(S_{t+1} = 2 \mid S_t = 1)\mathbb{P}(S_t = 1) + \mathbb{P}(S_{t+1} = 2 \mid S_t = 2)\mathbb{P}(S_t = 2) \\ &= \lambda_1 \pi_t + (1 - \lambda_2)(1 - \pi_t) \end{aligned} \tag{4.1.17}$$

using the transition probabalities of the regimes,

$$\begin{aligned} D &= \mathbb{P}(Y_{t+1} = y \mid S_{t+1} = 2,\ Y_t = y) = \pi_H \\ &\quad \text{using the transition probabilities of } Y \text{ if in regime 2.} \end{aligned} \tag{4.1.18}$$

If the sign of the shock is stable in two consecutive periods, the investor updates π_{t+1} from π_t by substituting the relevant equations in (4.1.15) to (4.1.18) into (4.1.14) so that:

$$\pi_{t+1} = \frac{[(1-\lambda_1)\pi_t + \lambda_2(1-\pi_t)]\pi_L}{[(1-\lambda_1)\pi_t + \lambda_2(1-\pi_t)]\pi_L + [\lambda_1\pi_t + (1-\lambda_2)(1-\pi_t)]\pi_H}. \tag{4.1.19}$$

Similarly, if the shock in period $t+1$ has the opposite sign to that in period t, π_L and π_H in (4.1.16) and (4.1.18) are replaced by $1 - \pi_L$ and $1 - \pi_H$ respectively i.e.

$$\mathbb{P}(Y_{t+1} = -y \mid S_{t+1} = 1,\ Y_t = y) = 1 - \pi_L, \qquad \mathbb{P}(Y_{t+1} = -y \mid S_{t+1} = 2,\ Y_t = y) = 1 - \pi_H$$

and hence:

$$\begin{aligned} \pi'_{t+1} &= \mathbb{P}(S_{t+1} = 1 \mid Y_{t+1} = -y, Y_t = y,\ \pi_t) \\ &= \frac{[(1-\lambda_1)\pi_t + \lambda_2(1-\pi_t)](1-\pi_L)}{[(1-\lambda_1)\pi_t + \lambda_2(1-\pi_t)](1-\pi_L) + [\lambda_1\pi_t + (1-\lambda_2)(1-\pi_t)](1-\pi_H)} \end{aligned}$$

$$\pi'_{t+1} = \frac{[(1-\lambda_1)\pi_t + \lambda_2(1-\pi_t)](1-\pi_L)}{[(1-\lambda_1)\pi_t + \lambda_2(1-\pi_t)](1-\pi_L) + [\lambda_1\pi_t + (1-\lambda_2)(1-\pi_t)](1-\pi_H)}. \tag{4.1.20}$$

To summarise: assume $\pi_0 = u$ and y_0 is observed, in period 1 y_1 is observed and depending if y_1 has the same sign or a different sign than y_0 either (4.1.19) or (4.1.20) will be used to calculate the updated π_1 and so the process will continue for the periods that follow.

All the probabilities are summarised in the following probability tree.

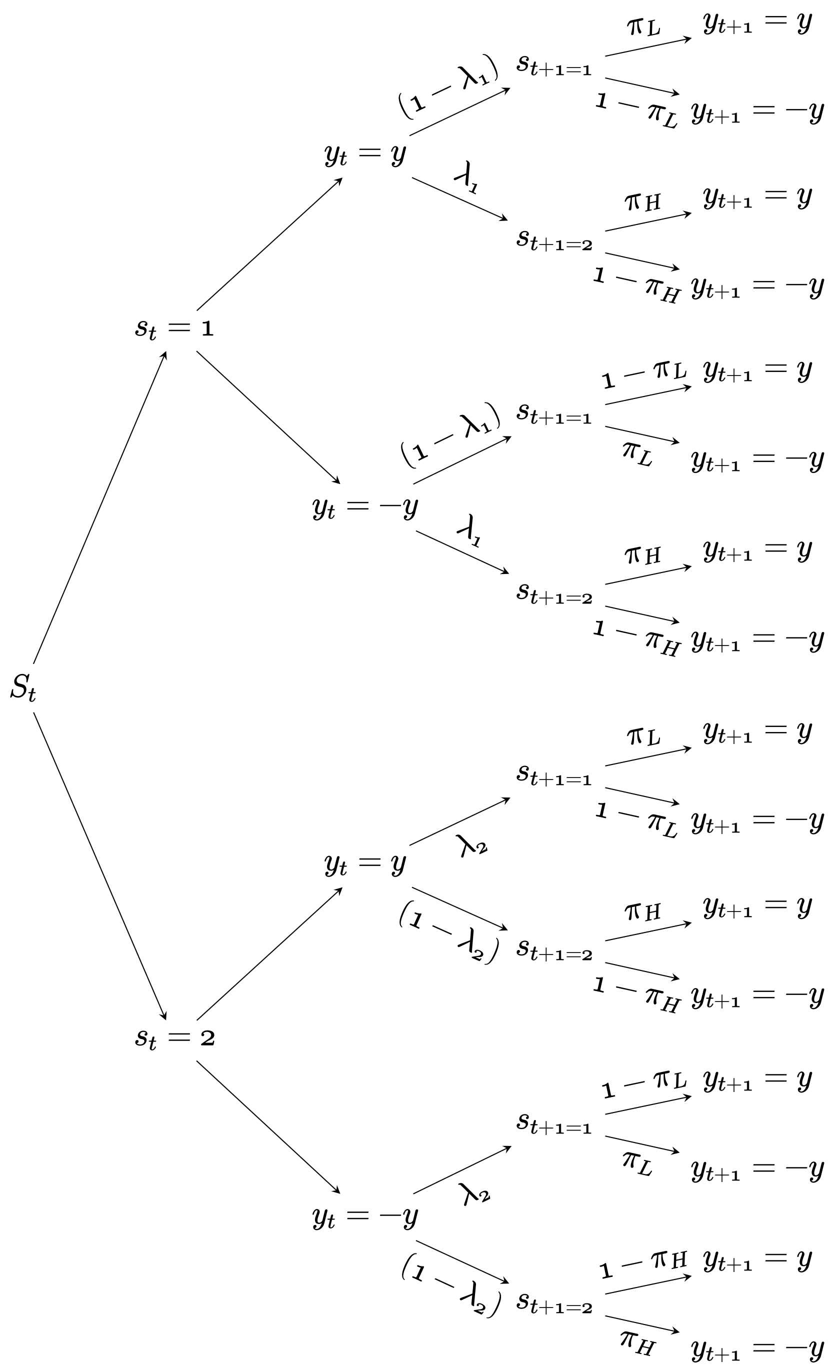


Figure 4.1: Probability Tree

The graph below shows the variation of the function π_{t+1} and π'_{t+1} between [0, 1]. $\pi_L = \frac{1}{3}$, $\pi_L = \frac{3}{4}$ and $\lambda_1 = 0.1$, $\lambda_2 = 0.3$.

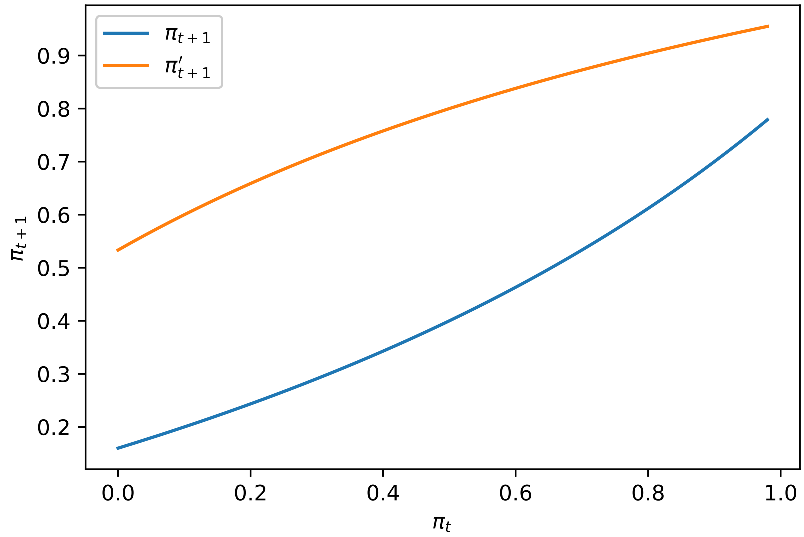


Figure 4.2: Variation of π_{t+1} and π'_{t+1} as function of π_t

The probability π_{t+1} is the probability assigned by an investor at time $t+1$ of being in Regime 1. This depends on the probability to be in Regime 1 at time t, the new earnings observation and the transition probability. In Figure 4.2, it is shown that $\pi'_{t+1} > \pi'_t$ in [0,1]. If, at time t, the earnings shock has the same sign as at time $t+1$, the probability π_t decreases. Therefore, the probability assigned to Regime 2 rises. Similarly, $\pi_{t+1} < \pi_t$ in [0,1]. If the shock in period t has the opposite sign to that in period $t+1$, the probability π_t increases. Therefore, the probability assigned to Regime 1 rises.

To explain the model of Barberis *et al.* (1998), Table 4.1 provides an example of the variation of the probability π_t in the function of time and the shocks y_t. In this example, the shock to earnings y_0 is positive, and the probability assigned to Regime 1 by the investor, i.e., $\pi_0 = 0.5$. Twenty earnings were randomly generated and (see Table 4.1), the investor's belief π_t that the time t shock to earnings was generated by Regime 1 is presented.

Table 4.1: Variation of the probability in function of time t and the shock y_t.

t	y_t	π_t	t	y_t	π_t
0	y	0.50			
1	-y	0.80	11	y	0.74
2	y	0.90	12	y	0.56
3	-y	0.93	13	y	0.44
4	y	0.94	14	y	0.36
5	y	0.74	15	-y	0.74
6	-y	0.89	16	y	0.89
7	-y	0.69	17	y	0.69
8	y	0.87	18	- y	0.87
9	-y	0.92	19	y	0.92
10	y	0.94	20	y	0.72

Table 4.1 is based on an illustrative simulation in Barberis *et al.* (1998) in which $\pi_L = \frac{1}{3}$, $\pi_H = \frac{3}{4}$ and $\lambda_1 = 0.1$, $\lambda_2 = 0.3$. The result of the simulation presented in the table shows that:

- If the shock in period t has a different sign from the shock in period $t-1$, then π_t increases.
- If the shock in period t has the same sign as the shock in period $t-1$, then π_t decreases.

Periods in which the sign of the shock switches often convince the model investor's belief that a high likelihood of mean reversion is to be expected in the next period (a high π_t), while a stable sign in consecutive periods increase the model investor's belief that a trending regime is possible (a low π_t).

In Table 4.1, it is illustrated that, where from period 0 to 1 the sign of y_t changes from positive to negative, the probability π_t increase from 0.5 to 0.8. From period 10 to 14, the sign of y_t remains the same, while the probability π_t decreases from 0.94 to 0.36. When the earnings shock at time t has the same sign as at time $t+1$, the investor puts more weighting on the trending regime and the probability assigned to the trending regime rises. When the earnings shock at time t has the opposite sign as at time $t+1$, the investor puts more weighting on the mean-reverting regime and the probability assigned to the mean-reverting regime rises.

4.1.5 Asset Pricing in the BBM Model

In this section, the implication of the behavioural model for prices is analysed and asset pricing is derived. The BBM model is for an investor who is representative (under the influence of the representativeness heuristics, investors tend to produce extreme predictions, and over-react) and the price of the stock is the value of the stock as perceived by the investor. If the investor did realise that the earnings process is random, the expectation of price is $E_t(N_{t+j}) = N_t$, and price equals

$$
\begin{aligned}
P_t &= E_t\left[\frac{N_{t+1}}{1+\delta} + \frac{N_{t+2}}{(1+\delta)^2} + \ldots\right] \\
&= N_t\left[(1+\delta)^{-1} + (1+\delta)^{-2} + \ldots\right] \\
&= N_t\left[1-(1-\delta)\right]^{-1} \\
&= \frac{N_t}{\delta}.
\end{aligned}
$$

The following proved lemma, summarises the behaviour of prices where investors focus on a combination of Regimes 1 and 2.

Lemma 4.1.1. *(Barberis et al., 1998) Given current period reported earnings y_t; and investors' probability assessment π_t of the current regime being mean-reverting, the stock price is:*

$$P_t = \frac{N_t}{\delta} + y_t(p_1 - p_2\pi_t), \tag{4.1.21}$$

where:

$$p_1 = \frac{1}{\delta}\boldsymbol{\gamma}_0'(1+\delta)[\mathbb{I}(1+\boldsymbol{\gamma}) - \boldsymbol{Q}]^{-1}\boldsymbol{Q}\boldsymbol{\gamma}_1),$$

$$p_2 = -\frac{1}{\delta}\boldsymbol{\gamma}_0'(1+\delta)[\mathbb{I}(1+\boldsymbol{\gamma}) - \boldsymbol{Q}]^{-1}\boldsymbol{Q}\boldsymbol{\gamma}_2),$$

$$\boldsymbol{\gamma}_0' = (1, -1, 1, -1), \quad \boldsymbol{\gamma}_1' = (0, 0, 1, 0), \quad \boldsymbol{\gamma}_2' = (1, 0, -1, 0),$$

$$\boldsymbol{Q}' = \begin{bmatrix} (1-\lambda_1)\pi_L & (1-\lambda_1)(1-\pi_L) & \lambda_1\pi_H & \lambda_1(1-\pi_H) \\ (1-\lambda_1)(1-\pi_L) & (1-\lambda_1)\pi_L & \lambda_1(1-\pi_H) & \lambda_1\pi_H \\ \lambda_2\pi_H & \lambda_2(1-\pi_L) & (1-\lambda_2)\pi_H & (1-\lambda_2)(1-\pi_H) \\ \lambda_2(1-\pi_L) & \lambda_2\pi_L & (1-\lambda_2)(1-\pi_H) & (1-\lambda_2)\pi_H \end{bmatrix}.$$

Proof. The asset price is the expected discounted value of the utility of the earnings over the infinite horizon:

$$P_t = E_t\left(\frac{N_{t+1}}{1+\delta} + \frac{N_{t+2}}{(1+\delta)^2} + \dots\right).$$

Since

$$N_t = N_{t-1} + Y_t$$

It follows that

$$N_{t+1} = N_t + Y_{t+1}, \;\; E_t(N_{t+1}) = N_t + E_t(Y_{t+1}), \;\; E_t(N_{t+2}) = N_t + E_t(Y_{t+1}) + E_t(Y_{t+2})$$

and so on.

Hence

$$
\begin{aligned}
P_t &= \left[\frac{N_t + E_t(Y_{t+1})}{(1+\delta)} + \frac{N_t + E_t(Y_{t+1}) + E_t(Y_{t+2})}{(1+\delta)^2} + \frac{N_t + E_t(Y_{t+1}) + E_t(Y_{t+2}) + E_t(Y_{t+3})}{(1+\delta)^3} + \dots\right] \\
&= \left(\frac{1}{1+\delta} + \frac{1}{(1+\delta)^2} + \frac{1}{(1+\delta)^3} + \dots\right)\left[N_t + E_t(Y_{t+1}) + \frac{E_t(Y_{t+2})}{1+\delta} + \frac{E_t(Y_{t+3})}{(1+\delta)^2} + \dots\right] \\
&= \sum_{n=1}^{\infty} \frac{1}{(1+\delta)^n}\left[N_t + E_t(Y_{t+1}) + \frac{E_t(Y_{t+2})}{1+\delta} + \frac{E_t(Y_{t+3})}{(1+\delta)^2} + \dots\right] \\
&= \left(\frac{1}{1+\delta}\right)\lim_{n\to\infty}\frac{(\frac{1}{1+\delta})^n - 1}{\frac{1}{1+\delta} - 1}\left[N_t + E_t(Y_{t+1}) + \frac{E_t(Y_{t+2})}{1+\delta} + \frac{E_t(Y_{t+3})}{(1+\delta)^2} + \dots\right] \\
&= \frac{1}{\delta}\left[N_t + E_t(Y_{t+1}) + \frac{E_t(Y_{t+2})}{1+\delta} + \frac{E_t(Y_{t+3})}{(1+\delta)^2} + \dots\right] \qquad (4.1.22) \\
&= \frac{1}{\delta}\left(N_t + \sum_{j=1}^{\infty}\left[\frac{E_t(Y_{t+j})}{(1+\delta)^{j-1}}\right]\right).
\end{aligned}
$$

It is thus clear that an expression for $E_t(Y_{t+j})$ is needed.

Let $E_t[.]$ represents the investor's conditional expectation given the information set ϕ_t describing all information available to the investor a time t consisting of the observed earnings series $(y_0, y_1, \dots, y_t)$.

$$E_t(Y_{t+j}|\phi_t) = y_t\mathbb{P}(Y_{t+j} = y_t|\phi_t) + (-y_t)\mathbb{P}(Y_{t+j} = -y_t|\phi_t).$$

The one-step transition matrix for this set-up is:

$$\boldsymbol{Q'} = \begin{bmatrix} q_{11yy} & q_{11y-y} & q_{12yy} & q_{12y-y} \\ q_{11-yy} & q_{11-y-y} & q_{12-yy} & q_{12-y-y} \\ q_{21yy} & q_{21y-y} & q_{22yy} & q_{22y-y} \\ q_{21-yy} & q_{21-y-y} & q_{22-yy} & q_{22-y-y} \end{bmatrix}$$

with :

$$q_{ijab} = \mathbb{P}(S_{t+1} = i/S_t = j).\ \mathbb{P}(Y_{t+1} = a/S_{t+1} = i, Y_t = b),\ \ i, j = 1 \text{ and } 2,\ \ a, b = y \text{ and} -y \qquad (4.1.23)$$

so that:

$$\boldsymbol{Q}' = \begin{bmatrix} (1-\lambda_1)\pi_L & (1-\lambda_1)(1-\pi_L) & \lambda_1\pi_H & \lambda_1(1-\pi_H) \\ (1-\lambda_1)(1-\pi_L) & (1-\lambda_1)\pi_L & \lambda_1(1-\pi_H) & \lambda_1\pi_H \\ \lambda_2\pi_H & \lambda_2(1-\pi_L) & (1-\lambda_2)\pi_H & (1-\lambda_2)(1-\pi_H) \\ \lambda_2(1-\pi_L) & \lambda_2\pi_L & (1-\lambda_2)(1-\pi_H) & (1-\lambda_2)\pi_H \end{bmatrix}.$$

Remember that the unconditional probabilities of being in states at time t are $\mathbb{P}(S_t = 1) = \pi_t$ and similarly $\mathbb{P}(S_t = 2) = 1 - \pi_t$.

Define:

$$\boldsymbol{q}^{t+j} = \begin{bmatrix} q_1{}^{t+j} = \mathbb{P}(S_{t+j} = 1,\ Y_{t+j} = y|\phi_t) \\ q_2{}^{t+j} = \mathbb{P}(S_{t+j} = 1,\ Y_{t+j} = -y|\phi_t) \\ q_3{}^{t+j} = \mathbb{P}(S_{t+j} = 2,\ Y_{t+j} = y|\phi_t) \\ q_4{}^{t+j} = \mathbb{P}(S_{t+j} = 2,\ Y_{t+j} = -y|\phi_t) \end{bmatrix}$$

with initial state probabilities:

$$\boldsymbol{q}^t = \begin{bmatrix} \pi_t \\ 0 \\ 1-\pi_t \\ 0 \end{bmatrix} = \begin{pmatrix} \mathbb{P}(S_t = 1) \\ 0 \\ 1 - \mathbb{P}(S_t = 1) \\ 0 \end{pmatrix}$$

so that

$$\begin{aligned} \boldsymbol{q}^{t+1} &= \boldsymbol{Q}'\boldsymbol{q}^t, \\ \boldsymbol{q}^{t+2} &= \boldsymbol{Q}'\boldsymbol{q}^{t+1} = \boldsymbol{Q}'\boldsymbol{Q}'\boldsymbol{q}^t = \boldsymbol{Q}'^2\boldsymbol{q}^t, \\ &\vdots \\ \boldsymbol{q}^{t+j} &= \boldsymbol{Q}'\boldsymbol{q}^{t+j-1} = \boldsymbol{Q}'\boldsymbol{Q}'\boldsymbol{q}^{t+j-2} = \cdots = \boldsymbol{Q}'^j\boldsymbol{q}^t. \end{aligned}$$

Note that:

$$\begin{aligned}
\mathbb{P}(Y_{t+j} = y_t|\phi_t) &= \mathbb{P}(Y_{t+j} = y_t, S_{t+j} = 1|\phi_t) + \mathbb{P}(Y_{t+j} = y_t, S_{t+j} = 2|\phi_t) \\
&= q_1^{t+j} + q_3^{t+j} \\
&= \boldsymbol{\gamma}'\boldsymbol{q}^{t+j}
\end{aligned}$$

where $\boldsymbol{\gamma}' = (1, 0, 1, 0)$ so that:

$$\mathbb{P}(Y_{t+j} = y_t|\phi_t) = (1, 0, 1, 0) \begin{bmatrix} {q_1}^{t+j} \\ {q_2}^{t+j} \\ {q_3}^{t+j} \\ {q_4}^{t+j} \end{bmatrix}.$$

Also

$$\begin{aligned}
\mathbb{P}(Y_{t+j} = -y_t|\phi_t) &= \mathbb{P}(y_{t+j} = -y_t, s_{t+j} = 1|\phi_t) + \mathbb{P}(y_{t+j} = -y_t, s_{t+j} = 2|\phi_t) \\
&= q_2^{t+j} + q_4^{t+j} \\
&= \underline{\boldsymbol{\gamma}}'\boldsymbol{q}^{t+j}
\end{aligned}$$

where $\underline{\boldsymbol{\gamma}}' = (0, 1, 0, 1)$ so that:

$$\mathbb{P}(Y_{t+j} = -y_t|\phi_t) = (0, 1, 0, 1) \begin{bmatrix} {q_1}^{t+j} \\ {q_2}^{t+j} \\ {q_3}^{t+j} \\ {q_4}^{t+j} \end{bmatrix}.$$

An expression for $E_t(Y_{t+j}|\phi_t)$ follows as:

$$\begin{aligned}
E_t(Y_{t+j}|\phi_t) &= y_t\mathbb{P}(Y_{t+j} = y_t|\phi_t + (-y_t)\mathbb{P}(Y_{t+j} = -y_t|\phi_t \\
&= y_t(\boldsymbol{\gamma}'\boldsymbol{q}^{t+j}) + (-y_t)(\underline{\boldsymbol{\gamma}}'\boldsymbol{q}^{t+j}) \\
&= y_t(\boldsymbol{\gamma}'\boldsymbol{Q}'^j\boldsymbol{q}^t) + (-y_t)(\underline{\boldsymbol{\gamma}}'\boldsymbol{Q}'^j\boldsymbol{q}^t) \\
&= y_t(\bar{\boldsymbol{\gamma}}' - \underline{\boldsymbol{\gamma}}')\boldsymbol{Q}'^j\boldsymbol{q}^t. \qquad (4.1.24)
\end{aligned}$$

Note that:

$$\bar{\gamma}' - \underline{\gamma}' = (1,0,1,0) - (0,1,0,1) = (1,-1,1,-1) = \boldsymbol{\gamma}_0',$$

and:

$$\boldsymbol{q}^t = \begin{bmatrix} \pi_t \\ 0 \\ 1-\pi_t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \pi_t = \boldsymbol{\gamma}_1' + \boldsymbol{\gamma}_2'\pi_t,$$

so that:

$$\begin{aligned} E_t(Y_{t+j}/\phi_t) &= y_t(\bar{\gamma}' - \underline{\gamma}')\boldsymbol{Q}'^j\boldsymbol{q}^t \\ &= y_t\boldsymbol{\gamma}_0'\boldsymbol{Q}'^j(\boldsymbol{\gamma}'_1 - \boldsymbol{\gamma}'_2\pi_t). \end{aligned} \tag{4.1.25}$$

Consider now the second term in the brackets of (4.1.22) i.e.

$$\begin{aligned} \sum_{j=1}^{\infty}\left[\frac{E_t(y_{t+j})}{(1+\delta)^{j-1}}\right] &= \sum_{j=1}^{\infty}\left[\frac{y_t\boldsymbol{\gamma_0}'\boldsymbol{Q}'^j\boldsymbol{\gamma_1}' + y_t\boldsymbol{\gamma}_0'\boldsymbol{Q}'^j\boldsymbol{\gamma_2}'\pi_t}{(1+\delta)^{j-1}}\right] \\ &= y_t\left[\sum_{j=1}^{\infty}\frac{\boldsymbol{\gamma_0}'\boldsymbol{Q}'^j\boldsymbol{\gamma_1}'}{(1+\delta)^{j-1}} + \sum_{j=1}^{\infty}\frac{\boldsymbol{\gamma_0}'\boldsymbol{Q}'^j\boldsymbol{\gamma_2}'}{(1+\delta)^{j-1}}\pi_t\right] \\ &= y_t(1+\delta)\left[\sum_{j=1}^{\infty}\frac{\boldsymbol{\gamma_0}'\boldsymbol{Q}'^j\boldsymbol{\gamma_1}'}{(1+\delta)^{j}} + \sum_{j=1}^{\infty}\frac{\boldsymbol{\gamma_0}'\boldsymbol{Q}'^j\boldsymbol{\gamma_2}'}{(1+\delta)^{j}}\pi_t\right] \end{aligned} \tag{4.1.26}$$

Consider the first term in brackets in (4.1.26) i.e.

$$\begin{aligned}
\sum_{j=1}^{\infty}\frac{\boldsymbol{\gamma_0}'\boldsymbol{Q}'^{j}\boldsymbol{\gamma_1}'}{(1+\delta)^j} &= \left[\sum_{j=1}^{\infty}\boldsymbol{\gamma_0}'\left(\frac{1}{1+\delta}\boldsymbol{Q}'\right)^j\right]\boldsymbol{\gamma_1}' \\
&= \boldsymbol{\gamma_0}'\left[\sum_{j=0}^{\infty}\left(\frac{1}{1+\delta}\boldsymbol{Q}'\right)^j-\left(\frac{1}{1+\delta}\boldsymbol{Q}'\right)^0\right]\boldsymbol{\gamma_1}' \\
&= \boldsymbol{\gamma_0}'\left[(\boldsymbol{I}_n-\frac{1}{1+\delta}\boldsymbol{Q}')^{-1}-\boldsymbol{I}_n\right]\boldsymbol{\gamma_1}',\ldots\text{using }\sum_{j=0}^{\infty}\boldsymbol{X}^j=(\boldsymbol{I}_n-\boldsymbol{X})^{-1}(4.1.1) \\
&= \boldsymbol{\gamma_0}'\left[(\boldsymbol{I}_n-\frac{1}{1+\delta}\boldsymbol{Q}')^{-1}(\boldsymbol{I}_n-(\boldsymbol{I}_n-\frac{1}{1+\delta}\boldsymbol{Q}'))\right]\boldsymbol{\gamma_1}' \\
&= \boldsymbol{\gamma_0}'\left[(\boldsymbol{I}_n-\frac{1}{1+\delta}\boldsymbol{Q}')^{-1}(\frac{1}{1+\delta}\boldsymbol{Q}')\right]\boldsymbol{\gamma_1}' \\
&= \boldsymbol{\gamma_0}'\left[(\boldsymbol{I}_n(1+\delta)-\boldsymbol{Q}')^{-1}\boldsymbol{Q}'\right]\boldsymbol{\gamma_1}'. \qquad (4.1.27)
\end{aligned}$$

In a similar fashion, the second summation in brackets (4.1.26) can be written as:

$$\sum_{j=1}^{\infty}\frac{\boldsymbol{\gamma_0}'\boldsymbol{Q}'^{j}\boldsymbol{\gamma_2}'}{(1+\delta)^j}=\boldsymbol{\gamma_0}'\left[(\boldsymbol{I}_n(1+\delta)-\boldsymbol{Q}')^{-1}\boldsymbol{Q}'\right]\boldsymbol{\gamma_2}'\pi_t. \qquad (4.1.28)$$

Hence, equation (4.1.22) can be written as:

$$
\begin{aligned}
P_t &= \frac{1}{\delta}\left(N_t + \sum_{j=1}^{\infty}\left[\frac{E_t(Y_{t+j})}{(1+\delta)^{j-1}}\right]\right) \\
&= \frac{N_t}{\delta} + \frac{1}{\delta}\sum_{j=1}^{\infty}\left[\frac{E_t(Y_{t+j})}{(1+\delta)^{j-1}}\right] \\
&= \frac{N_t}{\delta} + y_t(1+\delta)\left[\sum_{j=1}^{\infty}\frac{\boldsymbol{\gamma_0}'\boldsymbol{Q}'^j\boldsymbol{\gamma_1}'}{(1+\delta)^j} + \sum_{j=1}^{\infty}\frac{\boldsymbol{\gamma_0}'\boldsymbol{Q}'^j\boldsymbol{\gamma_2}'}{(1+\delta)^j}\right] \ldots \text{ using (4.1.26)} \\
&= \frac{N_t}{\delta} + y_t\frac{1}{\delta}(1+\delta)\left[\boldsymbol{\gamma_0}'\left[(\boldsymbol{I}_n(1+\delta)-\boldsymbol{Q}')^{-1}\boldsymbol{Q}'\right]\boldsymbol{\gamma_1}'\right. \\
&\quad \left. +\boldsymbol{\gamma_0}'\left[(\boldsymbol{I}_n(1+\delta)-\boldsymbol{Q}')^{-1}\boldsymbol{Q}'\right]\boldsymbol{\gamma_2}'\pi_t\right]
\end{aligned}
$$

(using (4.1.27) and (4.1.28))

$$
\begin{aligned}
&= \frac{N_t}{\delta} + y_t\left[\underbrace{\frac{1}{\delta}\boldsymbol{\gamma_0}'(1+\delta)\left[(\boldsymbol{I}_n(1+\delta)-\boldsymbol{Q}')^{-1}\boldsymbol{Q}'\right]\boldsymbol{\gamma_1}'}_{p_1}\right. \\
&\quad \left. +\underbrace{\frac{1}{\delta}\boldsymbol{\gamma_0}'(1+\delta)\left[(\boldsymbol{I}_n(1+\delta)-\boldsymbol{Q}')^{-1}\boldsymbol{Q}'\right]\boldsymbol{\gamma_2}'}_{-p_2}\pi_t\right] \\
&= \frac{N_t}{\delta} + y_t[p_1 - p_2\pi_t].
\end{aligned}
$$

which completes the proof of Lemma 4.1.1.

■

The first term of the expression of the price, $\frac{N_t}{\delta}$ is the value of the asset if the process used by the investor was random.

The second term, $Y_t(p_1 - p_2\pi_t)$ is the deviation of the asset price from its fundamental value. The range of values of π_L, π_H, λ_1, and λ_2 that allow the price function in Lemma 4.1.1 to exhibit both under-reaction and over-reaction to earnings news needs to be determined.

Recalling the definition of over-reaction in Equation (4.1.2), over-reaction can be thought of as meaning that "the expected return following a sufficiently large number of positive shocks should be lower than the expected return following the same number of negative shocks" (Barberis *et al.*, 1998). Mathematically, there exists some number $J \geqslant 1$, such that for all $j \geqslant J$,

$$E_t(p_{t+1}-p_t|y_t = y_{t-1} = \cdots = y_{t-j} = y)-E_t(p_{t+1}-p_t|y_t = y_{t-1} = \cdots = y_{t-j} = -y) < 0.$$

Similarly, under-reaction means that "the expected return following a positive shock should exceed the expected return following a negative shock" see (4.1.3) (Barberis *et al.*, 1998). That is to say,

$$E_t(p_{t+1} - p_t|y_t = +y) - E_t(p_{t+1} - p_t|y_t = -y) > 0.$$

For the case of under-reaction, the stock price does not react sufficiently to the shock. This means that, on average the deviation $y(p_1 - p_2\pi_t)$ must be negative. Let π_{avg} denote an average value of π_t, this implies that one condition must be

$$p_1 < p_2\pi_{avg}. \tag{4.1.29}$$

On the other hand, the price is above the fundamental value in the case of over-reaction. When the sign of the shock remains stable a number of times, π_t is low, indicating a low weighting on Regime 1 and a high weighting on Regime 2. If π_{low} represents a typical low value of π_t, over-reaction then requires that $y(p_1 - p_2\pi_{low})$ be positive, or that

$$p_1 > p_2\pi_{low}. \tag{4.1.30}$$

Putting the two conditions in Equation 4.1.30 and 4.1.29 together leads to:

$$p_2\pi_{low} < p_1 < p_2\pi_{avg}.$$

The following lemma from (Barberis *et al.*, 1998) gives sufficient conditions on p_1 and p_2 to ensure that both over-reaction and under-reaction (inequalities 4.1.3 and 4.1.2) happen. It should be noted that, in the mathematical formulation of over-reaction and under-reaction, there are two simplifications. First, the absolute price change $p_{t+1} - p_t$ is considered a return. Second, the good news is an event $y_t = +y$ i.e. when the change in earnings is positive. Better than expected earnings could also be used.

Lemma 4.1.2. *(Barberis et al., 1998) If the underlying parameters π_L, π_H, λ_1, and λ_2 satisfy*

$$\underline{k}p_2 < p_1 < \bar{k}p_2, \qquad p_2 \geqslant 0,$$

where

$$\underline{k} = \underline{q} + \frac{1}{2}\bar{\Delta}(\underline{q}), \qquad \bar{k} = \underline{q}^e + \frac{1}{2}(c_1 + c_2 q_*),$$

$$c_1 = \frac{\bar{\Delta}(\underline{q})\bar{q} - \underline{\Delta}(\bar{q})\underline{q}}{\bar{q} - \underline{q}}, \qquad c_2 = \frac{\underline{\Delta}(\bar{q}) - \bar{\Delta}(\underline{q})}{\bar{q} - \underline{q}},$$

$$q_* = \begin{cases} \bar{q}^e & \text{if } c_2 < 0, \\ \underline{q}^e & \text{if } c_2 \geqslant 0, \end{cases}$$

q be the probability assigned by the investor.

$$\bar{\Delta}(\underline{q}) = q_{t+1} - q_t | y_{t+1} = -y_t; q_t = q$$

$$\underline{\Delta}(\bar{q}) = q_t - q_{t+1} | y_{t+1} = y_t; q_t = q$$

and $\underline{q}^e$ and $\bar{q}^e$ are bounds on the unconditional mean of the random variable π_t, then the price function in Lemma 4.1.1 exhibits both under-reaction and over-reaction to earnings; and $\bar{k}$ are positive constants that depend on π_L, π_H, λ_1, and λ_2.

The range of the parameters π_L, π_H, λ_1, and λ_2 which the sufficient conditions for both under-reaction and over-reaction held needed to be determined. Let $\lambda_1 = 0.1$ and $\lambda_2 = 0.3$, chosen small to ensure that the regime switches did not occur very often. By fixing λ_1 and λ_2, the range of values of π_H and π_L for which there was over-reaction and under-reaction at the same time, needed to be determined. The model was set up so that in the reversion regime, the chances of earnings shocks to be in the same sign, π_L was believed to be low $0 < \pi_L < 0.5$. In the momentum regime, $0.5 < \pi_H < 1$. The objective was to evaluate the conditions in Lemma 4.1.2 for pairs (π_L, π_H) where π_L ranged from 0.5 at intervals 0.01 and π_H ranged from 0.5 to 1 at intervals of 0.01.

Figure 4.3 presents the conditions for over-reaction and under-reaction to hold are represented.

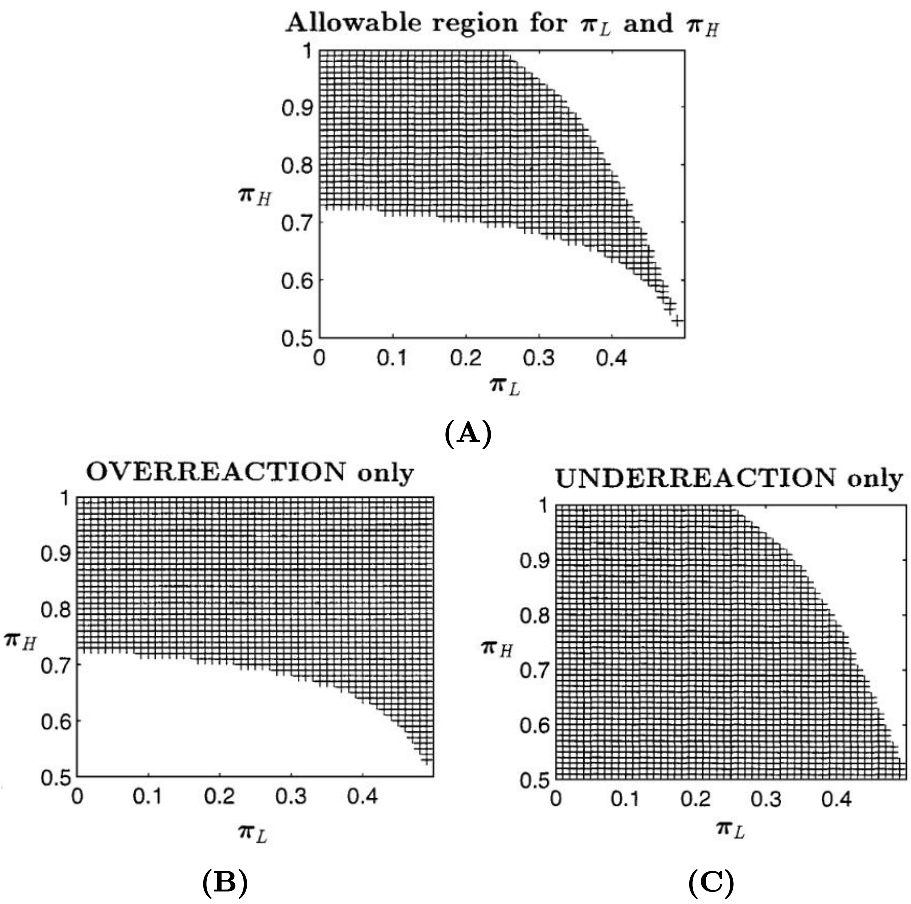


Figure 4.3: Conditions for over-reaction and under-reaction to hold

Source: (Barberis *et al.*, 1998)

In Figure 4.3A, all the pairs for which sufficient conditions for over-reaction and under-reaction held are marked with a $+$. It is observed that over-reaction and under-reaction did not always occur. For some parameter values, at least one of the two phenomena did not occur. For example, if π_H and π_L were near the high end of their feasible ranges, or if both π_H and π_L were near the low end of their ranges, the sufficient conditions did not hold.

There are two cases:

- If π_H and π_L are high, investors always believe in the regime trending whatever the regime. Over-reaction certainly happens, but under-reaction might not. After a positive earnings shock, investors, on average expect another positive earnings shock. Returns will, on average, be negative because the true process is random.
- If π_L and π_H are both at the low end, the investors believe in the reverting regime, regardless of the regime that led to under-reaction but over-reaction might not hold.

Figure 4.3B shows the parameter values for which only over-reaction occurred. In Figure 4.3C, values for which only under-reaction holds are shown. The intersection of the figures for the case where only over-reaction held and the case where only under-reaction held is shown in Figure 4.3A.

In Figure 4.4, the different ranges of (π_L, π_H) pairs that generate both under-reaction and over-reaction for a number of other values of λ_1 and λ_2 are presented.

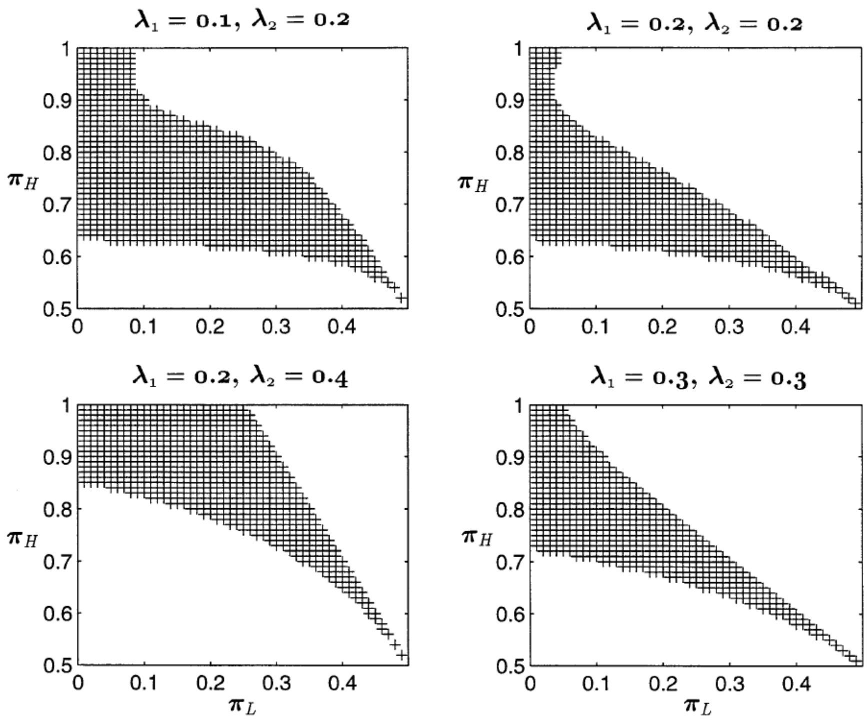


Figure 4.4: Range of (π_L, π_H) that generated both under-reaction and over-reaction

Source: (Barberis *et al.*, 1998)

4.2 Empirical Finding of Over-reaction and Under-reaction

In this section, empirical findings of over-reaction and under-reaction discussed in Chapter 2 and illustrated in (Barberis *et al.*, 1998) using artificial data sets of prices are summarised. The findings were obtained from three experiments. In the following two sub-sections, two of the experiments are described and the results obtained are given.

4.2.1 First Experiment

The aim of the first experiment was to simulate earnings, prices, and returns. The parameter values of the regime-switching model were fixed to $\lambda = 0.1$ and $\lambda = 0.3$. The choice of π_L and π_H was done by referring to Figure 4.3. Setting $\pi_L = \frac{1}{3}$ and $\pi_H = \frac{3}{4}$ enabled one to be in the region for which prices should exhibit both under-reaction and over-reaction. The following process was followed:

- Generate the earnings shocks at time t, y_t for the 2000 realisations.
- Choose an initial N_1 and compute the earnings at time t, $N_t = N_{t-1} + y_t$ for the 2000 realisations.
- Calculate prices from earnings using:

$$P_t = \frac{N_t}{\delta} + y_t(p_1 - p_2\pi_t) \tag{4.2.1}$$

(see lemma 4.1.1).

The simulated data were used to calculate returns following particular realisations of earnings as follows:

- Group the firms to form two portfolios according to the sign of the earnings changes in each of n years, where n ranges from 1 to 4.
- Compute the return for each firm

$$r_{t+1} = \frac{P_{t+1} - P_t}{P_t} \tag{4.2.2}$$

- Calculate the return of each portfolio. The return of the winner portfolio with s stocks in year n is calculated as follows:

$$r_+^n = \frac{1}{s}\sum_{i=1}^{s} r_i + \tag{4.2.3}$$

Table 4.2: Differences between the winner and loser portfolios mean returns

$r_+^1 - r_-^1$	0.0391
$r_+^2 - r_-^2$	0.0131
$r_+^3 - r_-^3$	-0.0072
$r_+^4 - r_-^4$	-0.0309

Similarly, the return of the loser portfolio with s stocks in year n is calculated as follows:

$$r_-^n = \frac{1}{s}\sum_{i=1}^{s} r_i- \tag{4.2.4}$$

where r_i+ and r_i- are returns of stock i in year n in the winner portfolio and loser portfolio respectively.

- Compute the difference $r_+^n - r_-^n$ for each n in the sample.

The quantity $r_+^n - r_-^n$ was expected to be positive, matching the definition of under-reaction to news (see Section 4.1.3).

Table 4.2, below reports the results obtained.

The following are observations from Table 4.2:

- $r_+^n - r_-^n$ is positive, consistent with under-reaction.
- As n increases, $r_+^n - r_-^n$ turns negative, consistent with over-reaction.

4.2.2 Second Experiment

The steps of the second experiment were as follows:

- Form deciles by grouping the different firms using their cumulative return over n years.

- Compute the return of the best- and worst-performing deciles for the year after portfolios formation.
- Compute the mean of the difference between the winner and the loser portfolio, $r_W^n - r_L^n$.

The quantity $r_W^n - r_L^n$ was expected to decreases with n, with $r_W^1 - r_L^1$ positive, as posited by Jegadeesh and Titman (1993) and $r_W^4 - r_L^4$ negative, as posited by De Bondt and Thaler (1987). The results, shown in the following table, were precisely the expected results.

Table 4.3: Differences between the winner and loser portfolios mean returns

$r_W^1 - r_L^1$	0.0280
$r_W^2 - r_L^2$	0.0102
$r_W^3 - r_L^3$	-0.0094
$r_W^4 - r_L^4$	-0.0181

4.3 Results of the Bayesian Model for the South African Financial Market

In this section, firstly, the data used for this part of the analysis are described and the incidence of positive and negative earnings runs across the datasets presented; thereafter the detailed methodology of the Bayes analysis is presented. The chapter concludes with the presentation and discussion of the findings.

4.3.1 Data Description

The 100 shares with the largest market capitalisation at the end of every calendar year from 2006 to 2016 were considered for the study. These shares had sufficient liquidity and depth of coverage by analysts and investors to be considered for a study on behavioural finance. In total, a sample of 163 shares had sufficient financial statement data on the Iress and Bloomberg databases to be included in the study.

A list of the individual companies is provided in Appendix A. The variables that were required for the Bayes analysis of over- and under-reaction are:

- The total return index values, which included reinvested dividends for all companies in the sample downloaded from Bloomberg.
- The EPS data was obtained from financial statements downloaded from Iress.
- The earnings-to-price data were downloaded from Bloomberg.

In Table 4.4, the maximum number of consecutive positive and negative earnings surprise sequences in the dataset is presented. Column (1) lists the length of an earnings run in quarter i.e. how many quarter lasted the consecutive positive or negative earnings? For example, Q = 4 represents four quarters i.e. consecutive positive or negative earnings lasted for four quarters, 5:6 means quarters of length 5 and 6 and similarly for the rest of the columns.

Column (2) lists the number of consecutive positive earnings surprises across stocks in the dataset that only lasted for Q periods. For example, 121 times in the entire history across stocks, the maximum number of consecutive positive earnings surprise runs lasted only four quarters. Column (3) lists the number of consecutive positive earnings surprises across stocks in the dataset that were equal to or longer than the period Q. For example, 240 times in the entire history across stocks, there were four or more consecutive positive earnings surprises.

Table 4.4: Number of the consecutive earnings surprise sequences in the dataset

(1) Length of run in Q	(2) Pos runs = Q	(3) All pos runs >=Q	(4) Neg runs =Q	(5) All neg runs >=Q
4	121	240	116	165
5:6	58	226	36	94
7:8	22	116	11	26
9:10	14	76	2	4
11:12	10	47	0	0
13:16	11	40	0	0
17:26	3	29	0	0
27:42	1	16	0	0

The first pattern for these results that demanded attention was that the maximum run of both positive and negative runs was the highest across all the sample data for Q = 4. This implied that the number of consecutive positive or negative earnings sequences in most companies lasted for only one year, which represented two consecutive reporting periods. The number of positive earnings surprise runs was higher than the number of negative earnings surprise runs, indicating that the sample period was generally a profitable period across the respective companies. With regards to Q above 27, there was only one time in the entire history across stocks when there were consecutive positive earnings surprises. There was only one company with 42 consecutive positive earnings surprises, Capitec Bank Holdings Ltd., which resorts under the financial sector.

4.3.2 First Experiment

For the purpose of the first experiment, positive and negative earnings surprises are defined as new information to which the financial markets could react. Analysts interpret a company's financial reports and current market conditions in order to predict earnings. An earnings shock occurs when a company's reported profits are different from the estimated profits. The method proposed by Barberis *et al.* (1998), that the change in actual earnings over consecutive reporting periods can be used as the earnings surprise proxy, was applied. Deviations in earnings from expected earnings or earnings forecast could also be used (Guerard and Mark, 2021).

In the first experiment, stocks were classified and grouped using earnings surprises (earnings shocks), and the performance of the resulting portfolios were measured. The following procedure was followed to form the portfolios and measure their performance:

- The EPS data from the actual financial statements were downloaded from Iress and matched to the quarter in which the data were released.
- The earnings shocks were calculated using the following formula:

$$y_t = N_t - N_{t-1} \tag{4.3.1}$$

where:

N_t is the earnings at time t.

- It should be noted that new earnings data were generally reported every six months for most companies, either in the interim- or final annual financial statements.

- The earning shocks y_t calculated from Equation 4.3.1 can be positive (positive earnings shock) or negative (negative earning shock). Depending on whether the shocks were positive or negative at the end of quarter t, the stocks were included in either a positive earnings shock portfolio or a negative earnings shock portfolio.

- Compute the six- and 12-months effective returns. The buy-and-hold (defined as a strategy to buy stocks at the closing market price and hold these until the k^{th} anniversary) returns for each stock, with quarterly compounding calculated as follows:

$$r^*_{ik} = \prod_{t=1}^{k}[1 + r_{it}] - 1 \tag{4.3.2}$$

where, r^*_{ik} is the buy-and-hold return for stock i for a holding period of $k = 2, 4$ months and, r_{it} is the raw return on stock i in month t (Drobetz *et al.*, 2005).

- Compute the average return of the winner and loser portfolios over the relevant period after formation at the end of quarter t+1. For example, for a portfolio formed using earnings released at the end of Q1 of 2006, the three-month quarterly performance is measured over the next period, namely Q2 of 2006.

- Compute the difference between the returns of the positive and negative shock portfolios.

- Repeat the procedure for all the quarters to compile a time series of the mean of the differences between the two portfolios' returns.

- Apply the test on a three-, six-, and 12-months basis.

- Count the number of times the positive shock portfolio under-performed the negative shock portfolio.

4.3.3 Illustration

In this section, it is illustrated using data for a single firm how the returns were calculated.

Consider the returns (including dividends) for a single firm (ACL) over five quarters and presented in Table 4.5.

Table 4.5: Return on ACL

Quarter	Returns
30/06/2006	18.09%
30/09/2006	7.03%
31/12/2006	25.16%
31/03/2007	22.36%
30/06/2007	7.98%

The buy-and-hold return is calculated as follows:

From these returns, the six-months returns can be calculated as:

$$30/09/2006 : [(1 + 0.1809)(1 + 0.0703)] - 1 = 26.39\%$$

$$31/12/2006 : [(1 + 0.0703)(1 + 0.2516)] - 1 = 33.96\%$$

And the 12 months return:

$$31/03/2007 : [(1 + 0.1809)(1 + 0.0703)(1 + 0.2516)(1 + 0.2236)] - 1 = 92.52\%$$

$$30/06/2007 : [(1 + 0.0703)(1 + 0.2516)(1 + 0.2236)(1 + 0.0798)] - 1 = 76.99\%$$

Next, it is illustrated how portfolio classification was done:

Consider the earnings shocks data of four companies that are listed below.

To form the positive (winner) and negative (loser) portfolios for each date shown, stocks with positive earnings shocks were used to form the positive

Table 4.6: Earnings shocks data

CLOSE	ACL	ACP	AEG	AEL
31/12/2006	1042	177.3	76.9	-59
31/03/2007	1042	-284.3	76.9	38
30/06/2007	356	-284.3	111.7	-38

(P^+) portfolio whereas stocks with negative earnings shocks were used to form the negative (P^-) portfolio.

For example, for the date 31/12/2006 (portfolio formation, the quarter following portfolios formation was used (31/03/2007) to calculate the average return. The formed portfolio was: $P^+ = \{ACL, ACP, AEG\}$ and $P^- = AEL$.

For the date 31/03/2007 (portfolio formation), the quarter following portfolios formation was used (30/06/2007) to calculate the average return. The formed portfolios were: $P^+ = \{ACL, AEG, AEL\}$ and $P^- = ACP$.

Next, the corresponding three- month returns for the above companies are shown below:

Table 4.7: Three-month returns

CLOSE	ACL	ACP	AEG	AEL
31/12/2006	25.16%	13.63%	23.80%	26.07%
31/03/2007	22.36%	17.11%	36.90%	31.70%
30/06/2007	7.98%	1.16%	8.59%	9.48%

The return of each portfolio formed on 31/12/2006 was calculated by using the returns on 31/03/2007 as follows:

$$r^+ = \text{Average}(22.36\%, 17.11\%, 36.90\%) = 25.46\%$$

$$r^- = 31.70\%$$

For 30/06/2007 the returns for the two portfolios were calculated as follows:

$$r^+ = \text{Average}(7.98\%, 8.59\%, 9.48\%) = 8.68\%$$

$$r^- = 1.16\%$$

Table 4.8 presents the resulting portfolio returns over the three quarters:

Table 4.8: Portfolio's returns

CLOSE	r^-	r^+	$r^+ - r^-$
31/12/2006	26.07%	18.71%	- 7.36%
31/03/2007	31.70%	25.46%	-6.24%
30/06/2007	1.16%	8.68%	7.52%

4.3.4 Results of Experiment 1

The results of the first experiment are presented for the resources, industrial and financial sectors. The objective was to determine if the positive portfolio persistently out-performed the negative portfolio in the period following the date of portfolio formation.

In Table 4.9, the number of times the positive portfolio out-performed (under-performed) the negative portfolio is reported. The sign of the return of the winner minus that of the loser portfolio determined whether there was over-reaction or under-reaction.

Table 4.9: Performance of the portfolios across the sectors.

	Resources		Industrial		Financial	
	Numb Pos	Numb Neg	Numb Pos	Numb Neg	Numb Pos	Numb Neg
3 months	22	20	31	11	24	18
6 months	24	18	32	10	27	15
12 months	22	20	28	14	14	28

The following two rules were applied to interpret the results:

(i) Over-reaction is postulated when the positive portfolio under-performs the negative portfolio, and

(ii) under-reaction is hypothesised when the positive portfolio out-performs the negative portfolio.

Therefore, the number of positive and negative signs in Table 4.9 was the number of observed under- and over-reactions respectively. The results in Table 4.9 reveal an increase in the occurrence of under-reactions over six months and a decrease in under-reactions over 12 months for all the sectors. An opposite pattern was observed for over-reaction across the sectors, i.e. there was a decrease in over-reaction over six months and an increase in over-reaction over 12 months. Across the sectors, more under-reaction than over-reaction was observed, except in the financial sector, where under-reaction was observed more often than over-reaction over 12 months. This was expected, as the number of positive earnings surprise runs was higher than the number of negative earnings surprise runs.

With regard to the three-month period:

- After the first three-month period (Q1 of 2006), stocks with positive returns over the three-month period formed the positive (winner) portfolio, and stocks with negative returns formed the negative (loser) portfolio.
- After the next three-month period (Q2 of 2006), the average quarterly returns (r^+) of the winner and loser portfolios (r^-) as classified at the end of Q1 of 2006 were calculated; $(r^+ - r^-)$ was an observation of the stochastic variable R^*.

Generally, based on the quarterly returns at the end of Q_t, winner and loser portfolios were formed at the end of Q_{t+1}, and the $(r^+ - r^-)$ was observed for these portfolios. A number of observations of R^* were possible. A normality test (the Shapiro-Wilk normality test) was done with the following hypotheses:

H_0: R^* is normally distributed against H_a: R^* is not normally distributed

If the p-value for the t-statistic was greater than the significance level of 0.05, the distribution of the data was not significantly different from a normal distribution, and normality could be assumed. If R^* was found to be not normally distributed, the Mann-Whitney U test was used.

If it was found that R^* was normally distributed, a t-test was performed to test the following hypothesis:

$$H_0 : E(R^*) = \mu = 0 \text{ against } H_a : E(R^*) = \mu \neq 0 \qquad (4.3.3)$$

The same was done for the six-month and one-year periods.

In table 4.10, the p-values for the different tests are reported for the different periods. The Mann-Whitney U test was used when R^* was not normally distributed.

Table 4.10: Results of the Shapiro-Wilk normality test (s-test), t-test and Mann-Whitney U test for the different sectors

Return (W-L)		Resources	Industrial	Financial
3m	Average	0.80%	2.10%	0.80%
	s-test	0.03*	0.37	0.11
	t-test	0.45	0.02*	0.27
	U test	0.21		
6m	Average	0.20%	3.30%	2%
	s-test	0.00*	0.05*	0.51
	t-test	0.91	0.00*	0.03*
	U test	0.35	0.00*	
12m	Average	0.40%	5.20%	- 11.10%
	s-test	0.71	0.87	0.74
	t-test	0.86	0.01*	0.00*

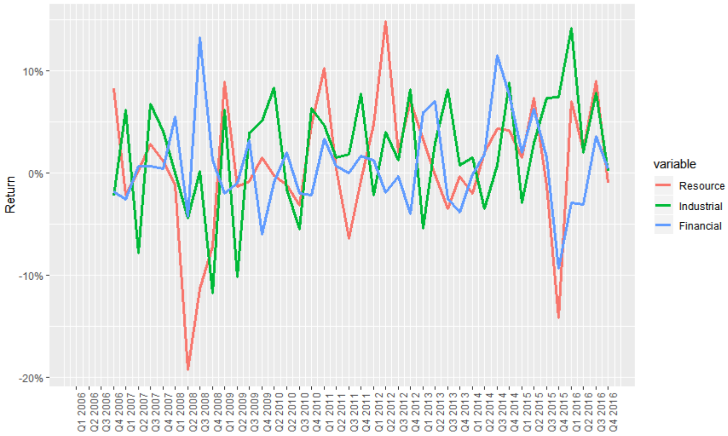
Note: * = statistically significant at the 5% level or lower

In Table 4.10, the average returns of the winner minus loser portfolio for the study period for each of the three sectors are presented. The measurement periods' magnitude could not be compared because the results were not annualised. Only the direction and prolonged movement were of importance. The average return of −11.10% over 12 months in the financial sector was significantly less than zero, which implies a reversal.

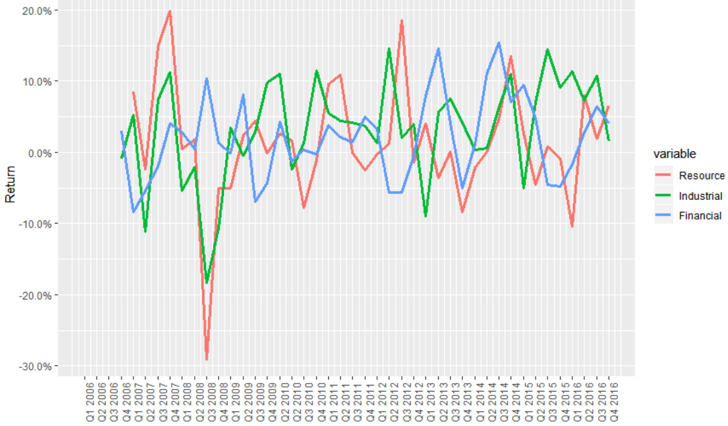
From Table 4.10, it can be seen that there was significant under-reaction in the industrial sector over three months, six months, and 12 months. There was significant over-reaction in the financial sector over 12 months. Further, there was significant under-reaction in the financial sector over six months. The test was applied over three months, six months, and 12 months in order to compare the market reaction to the positive and negative shocks over different time periods, and the results of the over-reaction and under-reaction are presented for the different sectors. All three sectors were grouped and analysed together per analysis period, to determine if similar patterns or prolonged evidence of a certain reaction prevailed. The magnitude of the over-reaction or under-reaction is also shown. The graphs of the winner-minus-loser portfolios for the three periods are presented in the panels of Figure 4.5.

During the period 2006 to 2016, there were periods in which the returns of the winner-minus-loser portfolio were positive, and others in which the returns were negative. The sign of the returns was used to determine whether there was over-reaction or under-reaction. Considering the values reflected in Figure 4.5, both over-reaction and under-reaction were present in the South African equity market during the period 2006 to 2016. Behavioural finance theory argues that investors are led by conservatism, and thus form expectations on the basis of recent news. Investors are, furthermore, slow to react in response to new evidence, which results in under-reaction. Investors' under-reaction to new information allows for a momentum strategy, which entails buying stocks with recent good performance, and selling stocks with recent bad performance, in order to realise a profit (Lo and MacKinlay, 1990). On the other hand, if investors over-react to market information, a contrarian strategy may be profitable.

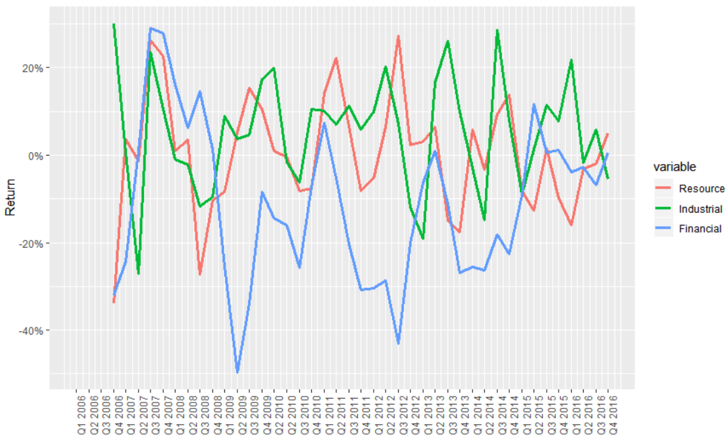
The graphs in Panels A and C of Figure 4.5 show that the difference in the performance of the positive and negative shock portfolios varied greatly over the time periods under study, which could be interpreted as the magnitude of the over-reaction and under-reaction of investors generally not having followed specific patterns. There were, however, periods when all the sectors showed similar patterns. The six-months graph in Panel B is more stable than the three-months and 12-months graphs. The amplitude is small, except for some extreme values in the resources sector (2006, 2007, 2008, and 2012). Some common elements emerged across the three panels: all three sectors were more stable when the returns were measured over six months. The magnitude of the two biases investigated increased over 12 months.



(A) Three-month buy and hold portfolio strategies



(B) Six-month buy and hold portfolio strategies



(C) 12-month buy and hold portfolio strategies

Figure 4.5: Return of the W-L portfolios

4.3.5 Second Experiment

In the second experiment, stocks were grouped according to quartiles using their cumulative returns. The performance of the resulting portfolios was measured in the period after portfolio formation.

In each quarter, stocks were grouped according to quartiles, using their cumulative return over the past three, six, and 12 months respectively. Stocks were assigned a portfolio strength index value ranging from 1 to 4, where 1 indicated the top quartile and 4 indicated the bottom quartile. Stocks in the top quartile were assigned to the winner portfolio, while stocks in the bottom quartile were assigned to the loser portfolio. The return of each portfolio was then measured as an equally weighted average return of the stocks included in the portfolio. The difference between the return of the best- and worst-performing portfolio was computed in the quarter after portfolio formation. The procedure was repeated for all the quarters in the sample, and the time series mean of the difference of the two portfolio returns was computed.

4.3.6 Illustration

In this subsection, it is illustrated how portfolios were classified as either a winner or loser portfolio for experiment 2. For this purpose, the three-month return data of five companies are listed below.

Table 4.11: Three-month returns data of five companies

CLOSE	ACL	ACP	AEG	AEL	AEN
30/06/2006	18.09%	-16.52%	-8.72%	-10.18%	-11.70%
30/09/2006	7.03%	11.45%	28.21%	13.82%	24.14%
31/12/2006	25.16%	13.63%	23.80%	26.07%	19.26%
31/03/2007	22.36%	17.11%	36.90%	31.70%	36.34%
30/06/2007	7.98%	1.16%	8.59%	9.48%	13.75%
30/09/2007	19.01%	2.77%	10.11%	-3.50%	-7.49%
31/12/2007	2.55%	5.55%	12.15%	1.13%	-4.66%
31/03/2008	45.80%	-13.33%	-3.96%	-17.53%	-12.79%
30/06/2008	13.20%	-20.43%	0.00%	-3.70%	1.47%

In each quarter, stocks were grouped according to quartiles, using their cumulative returns (shown in Table 4.11). The top quartile was indicated by 1 and

4 indicated the bottom quartile. Table 4.12 summarises the classification of the five stocks for each quarter.

Table 4.12: Grouping stocks into quartiles

CLOSE	ACL	ACP	AEG	AEL	AEN
30/06/2006	1	4	2	3	4
30/09/2006	3	2	1	2	1
31/12/2006	1	3	2	1	2
31/03/2007	2	3	1	1	1
30/06/2007	4	4	3	3	3
30/09/2007	2	4	3	4	4
31/12/2007	3	3	2	4	4
31/03/2008	1	4	4	4	4
30/06/2008	2	4	3	4	3

To form the positive (winner) and negative (loser) portfolios during each quarter shown, stocks with an index value of 1 as indicated in Table 4.12 were assigned to the winner portfolio (P^{+}), whereas stocks with an index value of 4 were assigned to the loser portfolio (P^{-}).

For example, for the date 30/06/2006, $P^{+} = \{ACL\}$ and $P^{-} = \{ACP, AEN\}$. To calculate the average return of each portfolio, the quarter following portfolio formation was used (30/09/2006).

Specifically, the corresponding three- month returns portfolios constructed based on the above companies are shown below in Table 4.13.

The return of each portfolio formed on 30/09/2006 was calculated using the returns on 30/09/2006 as follows:

$r^{-} = \text{Average}(11.45\%, 24.14\%) = 17.80\% \quad r^{+} = 7.03\%$

Similarly to Experiment 1, based on the quarterly returns at the end of Q_t, winner- and loser portfolios were formed and, at the end of Q_{t+1}, the $(r^{+} - r^{-})$ was observed for these portfolios.

In this way a number of observations of R^{*} was observed as reflected in the final column of Figure 4.13.

Table 4.13: Returns of the formed portfolios

CLOSE	r^-	r^+	$r^+ - r^-$
30/06/2006	17.80%	7.03%	-10.76%
30/09/2006	19.39%	23.80%	4.41%
31/12/2006	26.72%	31.70%	4.98%
31/03/2007	4.57%	8.59%	4.02%
30/06/2007	10.89%	-7.49%	-18.38%
30/09/2007	-1.77%	2.55%	4.31%
31/12/2007	-15.16%	-3.96%	11.20%
31/03/2008	-12.07%	13.20%	25.27%

4.3.7 Results of Experiment 2

In this section, the results of the second experiment is presented for the resources, industrial and financial sectors. In Table 4.14, the number of times the positive portfolio out-performed (under-performed) the negative portfolio is reported.

Table 4.14: Performance of the portfolios across the sectors

	Resources		Industrial		Financial	
	Num pos	Num neg	Num pos	Num neg	Num pos	Num neg
3 months	29	13	27	15	28	14
6 months	41	1	42	0	42	0
12 months	42	0	42	0	42	0

In Table 4.14, the number of occurrences of over-reaction and under-reaction is presented for the different sectors. The sign of the return of the winner-minus-loser portfolio determined whether there was over-reaction or under-reaction. The interpretation of the results was done using the two rules enumerated in Experiment 1. A remarkable feature was an increasing occurrence of under-reaction over six months, and the occurrence of under-reaction remaining stable over 12 months. The over-reaction effect disappeared over six months. Similar to Experiment 1, the normality of the R* distribution was first investigated by means of a Shapiro-Wilk normality test. If the distribution was classified as a normal distribution, a t-test was done; alternatively, a Mann-Whitney U test was conducted in those cases where the distribution was found to be non-normal. The hypotheses of the t-test were as follows:

$$H_0 : E(R^*) = \mu = 0 \text{ against } H_a : E(R^*) = \mu \neq 0 \qquad (4.3.4)$$

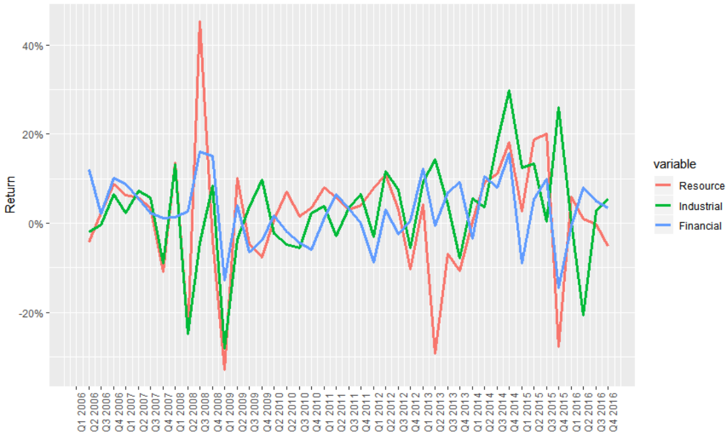
The same was done for the six-month and 12-month periods. In Table 4.15, the p-values for the different tests are reported.

Table 4.15: Shapiro-Wilk normality test (s-test), t-test and Mann-Whitney U test for the different sectors

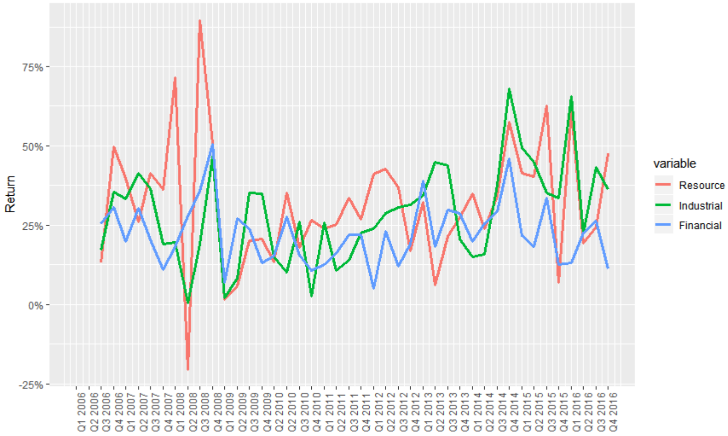
Return (W-L)		Resources	Industrial	Financial
3 months	Average	14.8%	26%	27%
	s-test	0.00*	0.05	0.81
	t-test	0.48	0.13	0.02*
	U test	0.18		
6 months	Average	31.60%	28%	22%
	s-test	0.53	0.43	0.81
	t-test	0.00*	0.00*	0.00*
12 months	Average	70%	64%	50%
	s-test	0.65	0.55	0.61
	t-test	0.00*	0.00*	0.00*

Note: * = statistically significant at the 5% level or lower

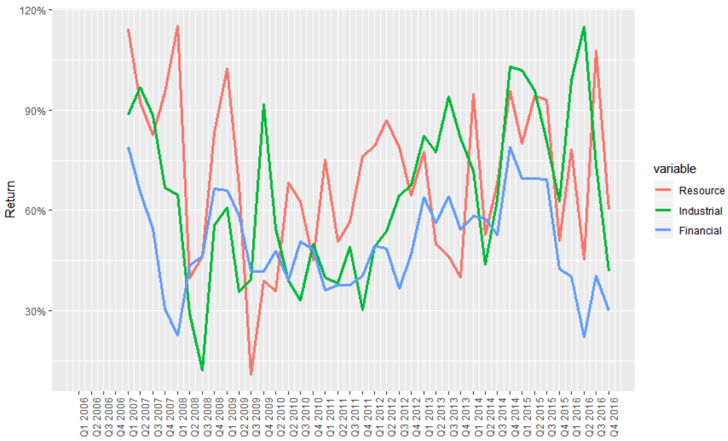
In Table 4.15, it can be seen that there was significant under-reaction in the industrial sector over three months, six months, and 12 months. There was significant under-reaction in the financial sector over six months. The evidence of positive average returns (shown in Table 4.15) indicates that the strategy was profitable. The graphs of the winner-minus-loser portfolios for the three periods are presented in the panels of Figure 4.6.



(A) Three-month buy- and -hold portfolio strategies



(B) Six-month buy- and -hold portfolio strategies



(C) 12-month buy- and - hold portfolio strategies

Figure 4.6: Return of the W-L portfolio

In the graphs in Panels A, B, and C of Figure 4.6, it is shown that the three-month data delivered a different results than the six- and 12-month data. Therefore, the performance of the positive and negative portfolios was different over time. The graphs in Panel B and Panel C have a positive amplitude. The measurement periods' magnitude could not be compared, because the results were not annualised. Only the direction and prolonged movement were of importance.

4.4 Conclusion

Empirical studies in finance have established that investors make mistakes in forming their beliefs, which lead to financial market anomalies such as under-reaction and over-reaction. Behavioural finance holds that under-reaction and over-reaction could be explained by taking into account human behavioural biases such as conservatism and representativeness. Investors who exhibit conservatism will under-react to information, because they react slowly in response to new evidence (Barberis *et al.*, 1998). "The individuals under the influence of the heuristic of representativeness tend to produce extreme predictions, or over-reaction, in which former losers tend to be winners in the future and vice-versa" (Aguiar *et al.*, 2006).

The current study investigated whether South African investors tend to over-react and/or under-react over time due to behavioural biases, using Bayesian model of Barberis *et al.* (1998) to examine investor sentiment in the financial market. The 100 shares with the largest market capitalisation at the end of every calendar year from 2006 to 2016 were considered for the study. These shares had sufficient liquidity and depth of coverage by analysts and investors to be considered for a study on behavioural finance. In total, a sample of 163 shares had sufficient financial statement data on the Iress and Bloomberg databases to be included in the study. The financial, industrial, and resources sectors were analysed separately, and two experiments were conducted. The two experiments proffered by Barberis *et al.* (1998) were replicated using South African equity market data, and yielded the following results:

- In the first experiment, over-reaction and under-reaction were observed for all three sectors, over the entire study period, following earnings shocks. Under-reaction showed a higher incidence than over-reaction in the industrial and resources sectors. In the financial sector, over-reaction had a higher incidence than under-reaction. Significant over-reaction was found in the financial sector, and significant under-reaction was found in the industrial and financial sectors.

- In the second experiment, over-reaction and under-reaction were also observed for all the sectors for the entire study period, based on price momentum. There was significant under-reaction in the industrial and financial sectors. The pattern was more evident over the six- and 12-month periods than over the three-month period.

Chapter 5

Summary, Contribution and Recommendations

Existence of over-reaction and under-reaction in the South African equity market, the largest financial market in Africa, is not clearly evidenced by existing literature. The aim of this research was to investigate whether South African investors over-react and/or under-react over time, driven by their behavioural biases. To this end, two mathematical statistical models were used: the FCM model (Aguiar and Sales, 2010; Aguiar, 2012) and the Bayesian model (Barberis *et al.*, 1998). The FCM model is based on the technique of pattern recognition, and uses the well-known FCM clustering algorithm (Bezdek *et al.*, 1984). The Bayesian model is based on the classical Bayes' theorem, which describes a relationship between the probability of an event conditional upon another event. Using the two models, this dissertation contributes towards addressing the following two main research questions:

- Does over-reaction and under-reaction really occur in the South African equity market?
- Can the application of mathematical statistical models be refined in determining of over-reaction and under-reaction in the South African equity market?

In this chapter, the keys results of the study are revisited and summarised, followed by the research contributions and their implications. The chapter also highlights the practical implications of the results, the research challenges and limitations, and recommendations for future research.

5.1 Summary of Findings

In this study, it was found that South African investors tend to over-react and/or under-react over time, driven by their behavioural biases. Over-reaction and under-reaction differ across sectors, but no clear patterns of the two biases investigated were visible over time. Therefore, no investment strategies can be advised for the South African market based on the results of this study. The occurrence and prevalence of the two biases may therefore be driven by factors not considered in this study.

5.1.1 The FCM Model

The FCM model analysis yielded the following results:

- In the resources sectors, under-reaction occurred more frequently than over-reaction, and once persisted for five consecutive quarters. Out of six cases of observed over-reaction, four were significant. Out of 14 cases of under-reaction, nine were significant.

- In the industrial sector, there were equal numbers of incidences of under-reaction and over-reaction. Under-reaction and over-reaction also persisted for at least two quarters, except during 2016, when under-reaction occurred in all four quarters. Out of 10 cases of observed over-reaction, five were significant. Out of 10 cases of under-reaction, two were significant.

- In the financial sector, over-reaction occurred more than under-reaction, and was persistent for at least two quarters. In 2016, over-reaction persisted for four quarters. Out of 11 cases of observed over-reaction, two were significant. Out of nine cases of under-reaction, three were significant.

The three prior bullet points indicate that, overall, the sectors had different occurrences of the two biases investigated. The timing and flow of under-reaction were then analysed. All three sectors were grouped and analysed together, to determine if similar patterns prevailed across the testing period. The analysis revealed that there were periods when all the sectors showed similar patterns, namely Q1 of 2012, Q2 of 2012, Q3 of 2014, Q1 of 2015, Q2 of 2015, and Q1 of 2016. Under-reaction was more prevalent in the resources sector, and extreme values for this sector were visible from 2015 to 2016. The FCM algorithm was accurate and efficient in determining the two unique centres when using South African market data, but proper scaling and winsorisation of the dataset before running the FCM algorithm are strongly advised.

5.1.2 The Bayesian Model

The model proposed by Barberis *et al.* (1998), uses Bayes's theorem to test how investor sentiment changes in the context of a sequence of prior information flows. According to the theory of behavioural finance, investors over-react and/or under-react over time to market information, driven by their behavioural biases. Investors underweight or overweight market information. Therefore, the financial market does not reflect all available information as predicted by the EMH, but, rather, the sentiment of the market participants.

De Bondt and Thaler (1985) argue that financial markets over-react to information on past earnings. As a result, investors can use trading strategies to out-perform the market. The two experiments using the Bayesian model incorporated data investors use to update their prior beliefs based on earnings surprises and valuation levels, in order to investigate the link between market behaviour and the psychology of investors.

The results of the two experiments using this model revealed the following:

- In the first experiment, over-reaction and under-reaction following earnings shock were observed for all three sectors and the entire study period. Under-reaction was observed more often than over-reaction in the resources and industrial sectors. In the financial sector, over-reaction was observed more often than under-reaction. Significant over-reaction was found in the financial sector, and significant under-reaction was found in the industrial and financial sectors.

- In the second experiment, over-reaction and under-reaction were also observed for all the sectors over the entire study period, based on price momentum indicators. Under-reaction was observed more often than over-reaction. There was significant under-reaction in the industrial and financial sectors.

5.1.3 Overall Results

Overall, the results of this research indicate that, using the two models, over-reaction and under-reaction were detected in the South African equity market during the period 2006 and 2016. The two biases investigated differed across sectors. The results imply that over-reaction and under-reaction related to behavioural biases do exist among investors who trade stocks on the JSE. A further implication of the results of this study is the inefficiency of the South African stock market during the period under study. Market efficiency theory holds that the market adjusts to new information quickly and correctly. Market inefficiency theory holds that the market over-reacts or under-reacts to new information (Grigaliūnienė, 2013). In the current study, both over-reaction and under-reaction were detected. The results are in line with those of previous such studies that focused on the JSE (Page and Way, 1992; Muller, 1999; Cubbin *et al.*, 2006; Venter, 2009; Hsieh and Hodnett, 2011).

5.2 Contribution of the Study

In this section, the contributions of this study are summarised.

5.2.1 Practical Contribution

This research makes the following academic contributions:

- The study focused on the South Africa equity market, whereas most previous studies focused on developed markets.
- The time period of analysis in this study (from 2006 to 2016) included the global financial crisis of 2007 to 2009.
- Data used in this study were obtained from companies' published financial statements, incorporating both their interim and year-end statements, downloaded from Iress. Therefore, the transformation of data using interim statements enhanced the accuracy of the dataset.
- Because of large data scale differences between the seven variables used in this study, seven different standardisation methods were tested, and the normal z-score standardisation method proved to be the optimal transformation method. The data were also winsorized, due to the presence of large outliers that would eventually have distorted the groups' centres. Standardisation and winsorization of the data proved to be reliable and robust transformation techniques that could be used by other researchers in future studies.
- This study differs significantly from prior studies because it includes the analysis of the three largest sectors of the JSE. Evidence and discussion of industry over-reaction and under-reaction in the South African stock market have been neglected in prior studies.

5.2.2 Methodological Contribution

Regarding methodology, the following contributions can be highlighted:

- This study is the first to analyse over-reaction and under-reaction in the South African financial market using rigorous statistical methodologies in the behavioural finance space. Therefore, this study contributes to the statistical analysis of behavioural anomalies in the South African stock market.
- All the mathematical derivations regarding the FCM model and the Bayesian model were explained and presented in a more digestible manner.

- To optimally achieve each research objective, the FCM model and the Bayesian model were refined and applied to South African data.

5.2.3 Policy Implications

The findings of this research are important for investors who trade based on market expectations. The dissertation provides clear evidence of the South African stock market's over-reaction and under-reaction, which are market anomalies. In the financial market, investors, in particular, try to exploit market anomalies and develop investment strategies to out-perform the market. This study sheds light on whether this is possible in the South African equity market. The results of this research clearly imply that opportunities to profit from the data analysis applied in this dissertation are very limited in the South African equity market, because no clear patterns were visible, and no out-performance of one group over another persisted over time. The results further imply that the momentum and the contrarian investment strategies can lead to over- performance in the South Africa equity market, but could also generate under-performance relative to a poorly performing market. Therefore, no trading strategies can be advised based on the results of this study.

5.3 Research Challenges and Limitations

The most influential studies on the over-reaction and under-reaction anomalies in financial markets (De Bondt and Thaler, 1985, 1987; Power *et al.*, 1991; Chopra *et al.*, 1992; Clare and Thomas, 1995) focused on developed markets. The unavailability of data for emerging markets from reliable sources makes conducting research in this domain difficult. No published quarterly data are available for the South African equity market. Availability of such data would enable more frequent analyses and longitudinal studies.

De Bondt and Thaler (1985) used data for 57 years, from January 1926 to December 1982, while the present study used only 10 years' data. This study could therefore be replicated using a longer study period.

Companies' financial statements and the interim financial statements were available from 2006. Therefore, the year 2006 was chosen as the start date in the analysis, as it date from which it was possible to identify the company and gather reliable data from the databases. It should also be noted that the analysis was conducted in 2017. Some of the later data were not included because of the unique data extraction method used, where the results were calculated from the financial statements. It was not a suitable process for all the companies. Therefore, the emphasises was on the data analysis, as data cleaning and processing took time.

5.4 Areas for Further Research

This study identified areas where further research is warranted to continue expanding knowledge about investor over-reaction and under-reaction in the South African equity market. Such areas include the use of new and more modern statistical techniques. The application of machine learning algorithms, artificial intelligence and more advanced mathematical statistical procedures can be investigated to solve the classification task before testing for over-reaction and under-reaction.

In this dissertation, the number of clusters of the FCM algorithm was limited to two. The formation of three clusters could be investigated in a future study.

Finally, this study showed that over- and under-reaction are a reality in the South African equity market. It is, however, also clear that substantial further research would be required to understand investor behaviour with respect to these two anomalies. Specifically, if a causal relationship between predictive variables and over- and under-reaction could be established, a profitable strategy can be devised. Therefore, it is hoped that this dissertation will serve as an introduction to and motivation for further research in this field.

Appendices

Appendix A

List of Stocks

Table A.1: Stock names

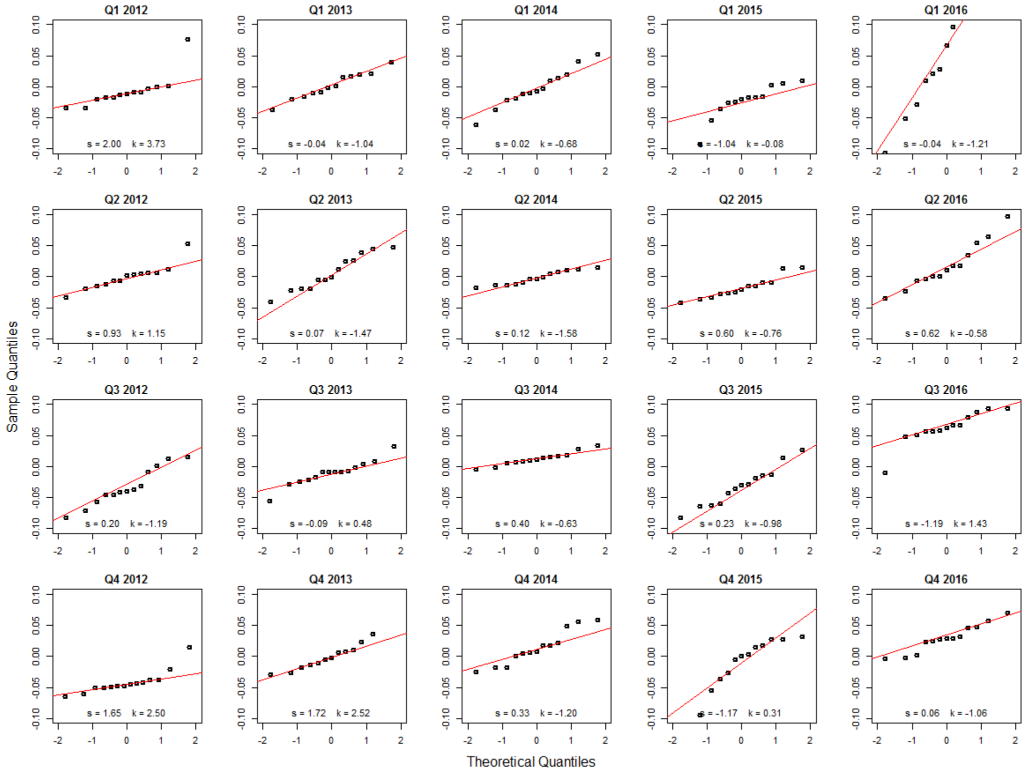
Ticker	Stock Names	Ticker	Stock Names
ACL	ARCELORMITTAL SO	LGL	LIBERTY GROUP
ACP	ACUCAP PROPERTIE	LHC	LIFE HEALTHCARE
AEG	AVENG LTD	LON	LONMIN PLC
AEL	ALLIED ELE-A SHR	MDC	MEDICLINIC INT
AEN	ALLIED ELECTRONICS CORP	MEI	MEDICLINIC INTER
AFE	AECI LTD	MMI	MMI HOLDINGS LTD
AFH	ALEXANDER FORBES GROUP	MND	MONDI LTD
AFX	AFRICAN OXYGEN	MNP	MONDI PLC
AGL	ANGLO AMER PLC	MRF	MERAFE RESOURCES LIMITED
AIP	ADCOCK INGRAM HO	MRP	MR PRICE GROUP
ALT	ALLIED TECHNOLOG	MSM	MASSMART HLDGS
AMS	ANGLO AMERICAN P	MTN	MTN GROUP LTD
ANG	ANGLOGOLD ASHANT	MTX	METOREX LTD
APN	ASPEN PHARMACARE	MUR	MURRAY & ROBERTS
ARI	AFRICAN RAINBOW	MVL	MVELAPHANDA RES
ARL	ASTRAL FOODS LIMITED	NBC	NEW BOND CAPITAL
ASR	ASSORE LTD	NED	NEDBANK GROUP
ATT	EOH HOLDINGS LTD	NEP	RAND MERCHANT IN
AVI	AVIS SOUTHERN AF	NHM	NORTHAM PLATINUM
AXL	AFRICAN PHOENIX	NPK	NAMPAK LTD
BAT	BRAIT SE	NPN	NASPERS LTD-N
BAW	BARLOWORLD LTD	NT1	NET 1 UEPS TECHNOLOGIES INC
BEL	BELL EQUIPMENT LIMITED	NTC	NETCARE LTD
BGA	BARCLAYS AFRICA	OCE	OCEANA GROUP LTD
BID	BID CORP LTD	OML	OLD MUTUAL PLC
BIL	BHP BILLITON PLC	OMN	OMNIA HOLDINGS
BLU	BLUE LABEL TELEC	OPT	OPTIMUM COAL HOL
BTI	BRIT AMER TOBACC	PAM	PALABORA MINING
BVT	BIDVEST GROUP	PAP	PANGBOURNE PPTYS
CAT	CAXTON AND CTP P	PFG	PIONEER FOODS GR
CCO	CAPITAL & COUNTI	PGR	PEREGRINE HOLDINGS LIMITED
CFR	RICHEMONT-DR	PIK	PICK'N PAY STORE
CLH	CITY LODGE HOTELS LIMITED	PMA	PRIMEDIA LTD/SOU
CLS	CLICKS GROUP LTD	PMN	PRIMEDIA-N SHRS
CML	CORONAT	PPC	PPC LTD
COH	CURRO HOLDINGS L	PSG	PSG GROUP LTD
CPF	CAPITAL PROPERTY FUND LIMITED	PTG	PTG Pivot Technology Solutions Inc
CPI	CAPITEC BANK HOL	RBP	ROYAL BAFOKENG P
CSL	Consol Limited	RBX	RAUBEX GROUP LIMITED
DDT	DIMENSION DATA	RCL	RCL Foods Limited
DST	DISTELL GROUP	RDF	REDEFINE PROPERT
DSY	DISCOVERY LTD	REI	REINET INVEST-DR
DTC	DATATEC LTD	REM	REMGRO LTD
ECO	EDGARS CONS STOR	RES	RESILIENT REIT L
EHS	EVRAZ HIGHVELD S	RLO	Reunert Limited
ELE	ELEMENTONE LTD	RMH	RMB HOLDINGS LTD
ELH	Evraz Highveld Steel & Vanadium Ltd	RMI	RAND MERCHANT IN
EMI	EMIRA PROPERTY F	ROC	ROCKCASTLE GLOBA
EOH	EOH HOLDINGS LTD	RPL	REDEFINE INTERNA
EXX	EXXARO RESOURCES	S32	SOUTH32 LTD
FBR	FAMOUS BRANDS LT	SAB	SABMILLER PLC
FFA	FORTRESS-INC-A	SAC	SA CORPORATE REA
FFB	FORTRESS-INC	SAP	SAPPI LTD
FPT	FOUNTAINHEAD PRO	SBK	STANDARD BANK GR
FSR	FIRSTRAND LTD	SGL	SIBANYE GOLD LTD
GFI	GOLD FIELDS LTD	SHP	SHOPRITE HLDGS
GLN	GLENCORE PLC	SIM	SIMMER & JACK
GND	GRINDROD LTD	SLM	SANLAM LTD
GRF	GROUP FIVE LIMITED	SNH	STEINHOFF INT NV
GRT	GROWTHPOINT PROP	SNT	SANTAM LTD
HAR	HARMONY GOLD MINING LTD	SOL	SASOL LTD
HCI	HOSKEN CONS INV	SPG	SUPER GROUP LIMITED
HLM	HULAMIN LIMITED	SPP	SPAR GRP LTD/THE
HMN	HAMMERSON PLC	SUI	SUN INTERNATIONA
HYP	HYPROP INVESTMEN	SYC	SYCOM PROPERTY FUND
ILV	ILLOVO SUGAR LTD	TBS	TIGER BRANDS LTD
IMP	IMPALA PLATINUM	TFG	Foschini Group Ltd
INL	INVESTEC LTD	TKG	TKG SJ Equity
INP	INVESTEC PLC	TON	TONGAAT HULETT
IPF	INVESTEC PROPERTY FUND LIMITED	TRE	TRENCOR LTD
IPL	IMPERIAL HLDGS	TRU	TRUWORTHS INTL
ITE	ITALTILE LIMITED	TSH	TSOGO SUN HOLDIN
ITU	INTU PROPERTIES	UCP	Unicorn Capital Partners Ltd.
IVT	INVICTA HOLDINGS LIMITED	UTR	Ultracharge Ltd
JDG	JD GROUP LTD	VKE	VUKILE PROPERTY
JSE	JSE LTD	VOD	VODACOM GROUP
KAP	KAP INDUSTRIAL	WAR	GOLD FIELDS OPER
KIO	KUMBA IRON ORE L	WBO	WILSON BAYLY HOM
KST	PSG KONSULT LIMITED	WEZ	WESIZWE PLATINUM LIMITED
LBH	LIBERTY HLDGS	WHL	WOOLWORTHS HLDGS
LEW	LEW SJ Equity	ZED	ZEDER INVESTMENT

Appendix B

Normality Test

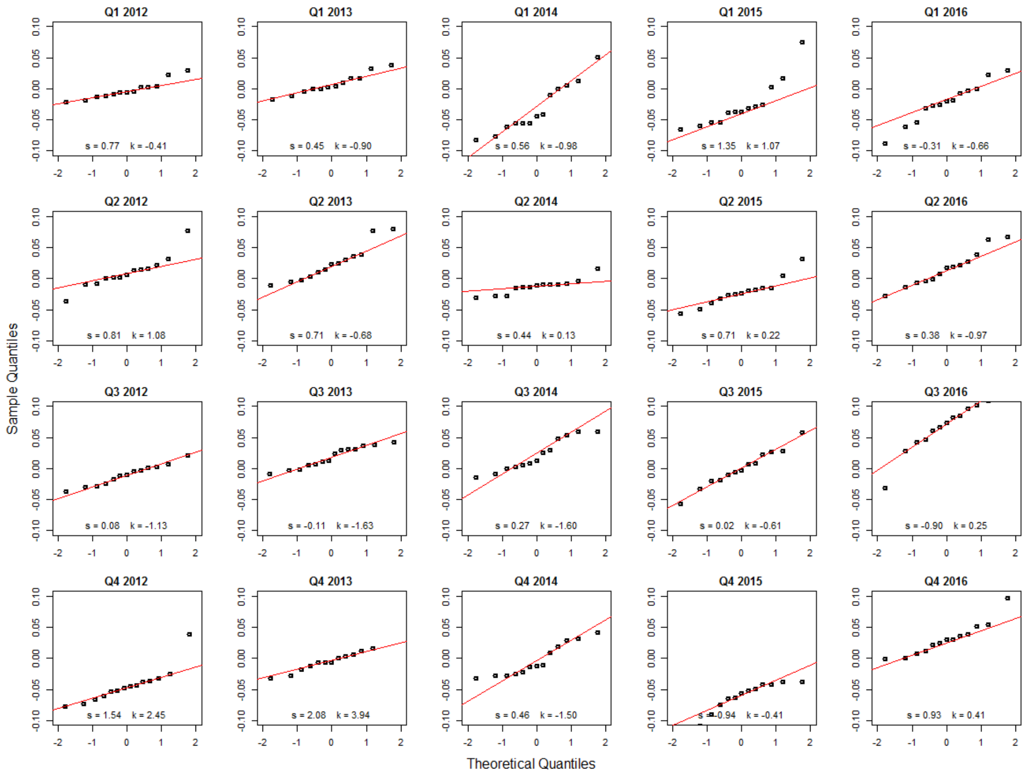
B.1 Resources Sector

B.1.1 Weekly Residual Return Loser

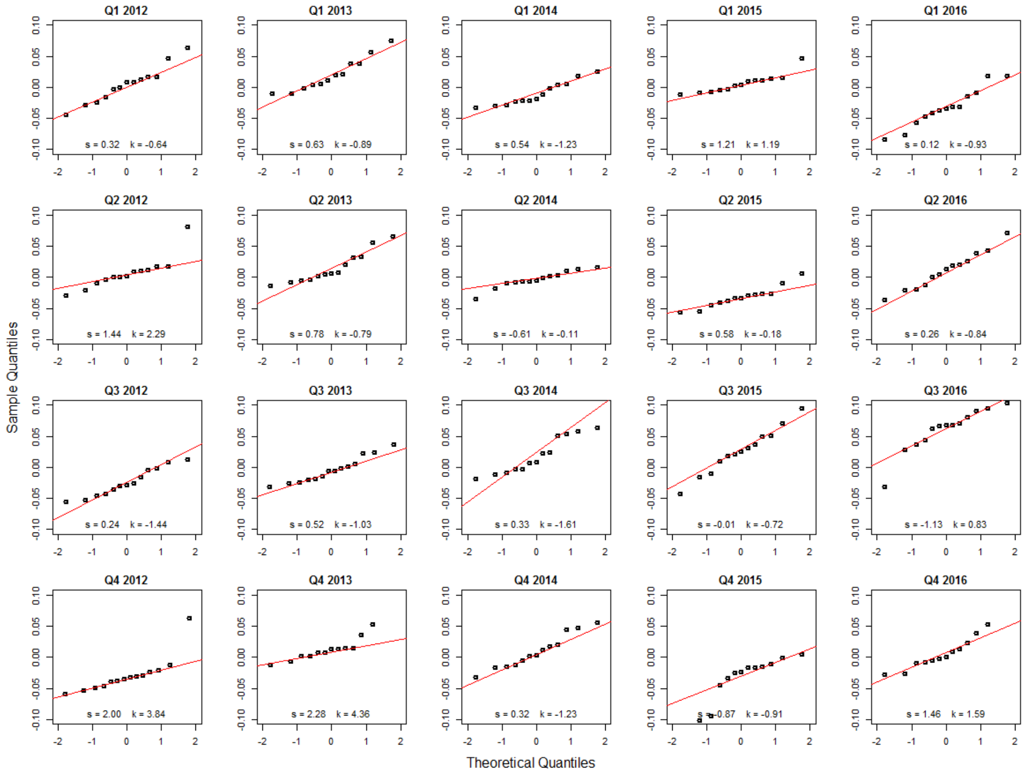


B.2 Industrial Sector

B.2.1 Weekly Residual Return Winner

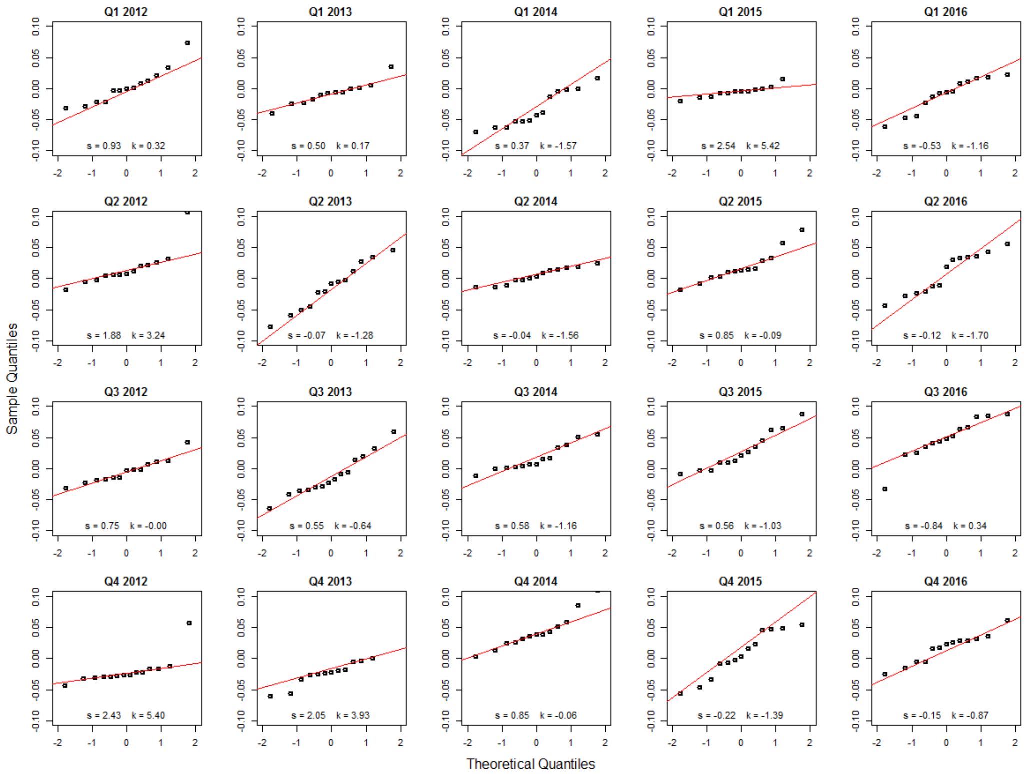


B.2.2 Weekly Residual Return Loser

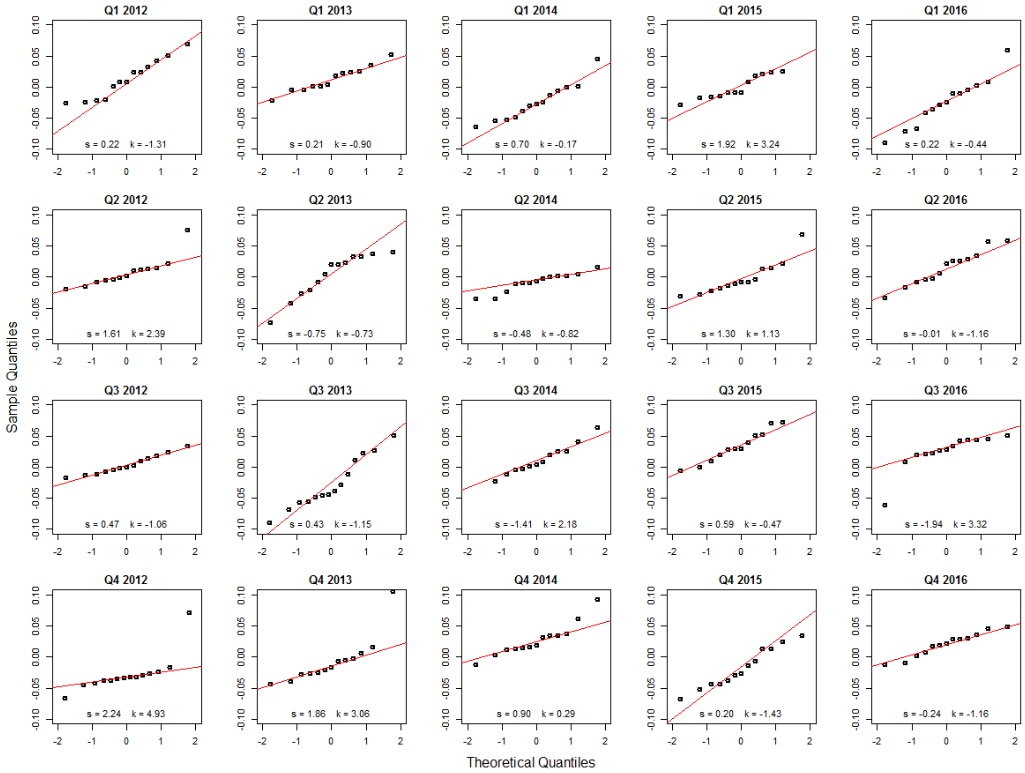


B.3 Financial Sector

B.3.1 Weekly Residual Return Winner



B.3.2 Weekly Residual Return Loser



Appendix C

Calculation of Interim Financial Ratios

Table C.1: Calculation of ratios according to line items in Iress

Ratio	Formula
Current ratio	514/515
Debt to assets	(508+515)/533
Retention rate	Stable
Dividend	Available
EPS	574/10
NAV per share	528
Return on assets	(591 – 586) / (533 * 100)
Total asset return	553/533

The listed number below indicate that these are the line items as provided by Iress.

- 514 Current assets
- 515 Current liabilities
- 508 Long term liabilities
- 533 Total assets
- 574 Total earnings
- 591 Profit before interest and tax
- 586 Extraordinary items
- 553 Gross profit
- 528 NAV per share

List of References

Aguiar, R.A. (2012). Fuzzy logic and behavioral finance: An approach using fuzzy c-means algorithm. *International Journal of Latest Trends in Finance and Economic Sciences*, vol. 2, no. 1, pp. 1–7.

Aguiar, R.A. and Sales, R.M. (2010). Overreact analysis in the American Stock Market: A fuzzy c-means algorithm approach. *International Journal of Trade, Economics and Finance*, vol. 1, no. 4, pp. 325–330.

Aguiar, R.A., Sales, R.M. and De Sousa, L.A. (2006). A Behavioral Fuzzy Model for Analysis of Overreaction and Underreaction in the Brazilian Stock Market. In: *Proceedings of the 9th Joint International Conference on Information Sciences.*, pp. 109–112. Atlantis Press.

Akbas, F., Jiang, C. and Koch, P.D. (2017). The trend in firm profitability and the cross-section of stock returns. *The Accounting Review*, vol. 92, no. 5, pp. 1–32.

Alagidede, P. (2013). Month of the year and pre-holiday effects in African stock markets. *South African Journal of Economic and Management Sciences*, vol. 16, no. 1, pp. 64–74.

Antonacci, G. (2014). *Dual Momentum Investing:: an innovative strategy for higher returns with lower risk.* CITY: TATA McGraw-Hill Education.

Asamoah, G.N. (2010). The impact of dividend announcement on share price behaviour in Ghana. *Journal of Business & Economics Research*, vol. 8, no. 4, pp. 47–58.

Auret, C. and Cline, R. (2011). Do the value, size and January effects exist on the JSE? *Investment Analysts Journal*, vol. 40, no. 74, pp. 29–37.

Bacon, F.W. and McMillan, C.E. (2007). Federal funds rate changes: A test of market efficiency. *Management Review: An International Journal*, vol. 2, no. 2, pp. 37–59.

Balvers, R.J. and Wu, Y. (2006). Momentum and mean reversion across national equity markets. *Journal of Empirical Finance*, vol. 13, no. 1, pp. 24–48.

Banerjee, A. and Ghosh, A. (2004). The measures of the quality of trading in a stock: Empirical evidence from the National Stock Exchange (India). *Available at SSRN 570546*.

Barberis, N., Shleifer, A. and Vishny, R. (1998). A model of investor sentiment. *Journal of Financial Economics*, vol. 49, no. 3, pp. 307–343.

Barberis, N. and Thaler, R. (2003). A survey of behavioral finance. *Handbook of the Economics of Finance*, vol. 1, pp. 1053–1128.

Basiewicz, P. and Auret, C. (2010). Feasibility of the Fama and French three factor model in explaining returns on the JSE. *Investment Analysts Journal*, vol. 39, no. 71, pp. 13–25.

Bassiouny, A. and Ragab, A. (2014). Testing short-term over/underreaction hypothesis: Empirical evidence from the Egyptian Exchange. *Journal of Applied Finance and Banking*, vol. 4, no. 5, pp. 1–7.

Basu, S. (1977). Investment performance of common stocks in relation to their price-earnings ratios: a test of the efficient market hypothesis. *The Journal of Finance*, vol. 32, no. 3, pp. 663–682.

Bernard, V. (1993). Stock price reactions to earnings announcements: a summary of recent anomalous evidence and possible explanations, advances in behavioral finance (R. Thaier, ed.). New York. *Thaler, R., Russell Sage Foundation.*

Bezdek, J.C. (2013). *Pattern recognition with fuzzy objective function algorithms*. Springer Science & Business Media.

Bezdek, J.C., Ehrlich, R. and Full, W. (1984). FCM: The fuzzy c-means clustering algorithm. *Computers & Geosciences*, vol. 10, no. 2-3, pp. 191–203.

Bikhchandani, S. and Sharma, S. (2000). Herd behavior in financial markets. *IMF Staff Papers*, vol. 47, no. 3, pp. 279–310.

Black, F. (1986). Noise. *The Journal of Finance*, vol. 41, no. 3, pp. 528–543.

Bolstad, W.M. and Curran, J.M. (2016). *Introduction to Bayesian Statistics*. CITY: John Wiley & Sons.

Bradfield, D., Barr, G. and Affleck-Graves, J. (1988). Asset pricing in small markets: the South African case. *South African Journal of Business Management*, vol. 19, no. 1, pp. 11–21.

Brealey, R.A., Myers, S.C., Allen, F. and Mohanty, P. (2012). *Principles of corporate finance*. CITY: Tata McGraw-Hill Education.

Brealey, R.A., Myers, S.C., Allen, F. and Mohanty, P. (2018). *Principles of corporate finance, 12/e*, vol. 12. CITY: McGraw-Hill Education.

Brealey, R.A., Myers, S.C. and Marcus, A.J. (2001). *Fundamentals of Corporate Finance*. CITY: McGraw-Hill.

Brigham, E.F. and Daves, P.R. (2014). *Intermediate financial management*. CITY: Cengage Learning.

Buguk, C. and Brorsen, B.W. (2003). Testing weak-form market efficiency: Evidence from the Istanbul Stock Exchange. *International Review of Financial Analysis*, vol. 12, no. 5, pp. 579–590.

Carhart, M.M. (1997). On persistence in mutual fund performance. *The Journal of Finance*, vol. 52, no. 1, pp. 57–82.

Chan, K., Hameed, A. and Tong, W. (2000). Profitability of momentum strategies in the international equity markets. *Journal of Financial and Quantitative Analysis*, vol. 35, no. 02, pp. 153–172.

Chan, K.C., Gup, B.E. and Pan, M.-S. (1997). International stock market efficiency and integration: a study of eighteen nations. *Journal of Business Finance & Accounting*, vol. 24, no. 6, pp. 803–813.

Chan, L.K., Jegadeesh, N. and Lakonishok, J. (1996). Momentum strategies. *The Journal of Finance*, vol. 51, no. 5, pp. 1681–1713.

Chasan, E. (2012). Mid-size firms tap retained earnings to fund growth. *The Wall Street Journal*, vol. 2, no. 5, pp. 23–39.

Chaudhary, A.K. (2013). Impact of behavioral finance in investment decisions and strategies: a fresh approach. *International Journal of Management Research and Business Strategy*, vol. 2, no. 2, pp. 85–92.

Chia, G.S., Wong, A.T.M., Chin, S.T., Lee, J.L. and Wong, K.Y. (2015). *A study of behavioural finance: Industry-based overreaction and underreaction in Malaysia*. Ph.D. thesis, UTAR.

Chopra, N., Lakonishok, J. and Ritter, J.R. (1992). Measuring abnormal performance: Do stocks overreact? *Journal of Financial Economics*, vol. 31, no. 2, pp. 235–268.

Choudhary, K. and Sethi, N. (2014). A study of overreaction hypothesis in the Indian Equity Market. *Asia-Pacific Journal of Management Research and Innovation*, vol. 10, no. 4, pp. 355–366.

Clare, A. and Thomas, S. (1995). The overreaction hypothesis and the UK stock-market. *Journal of Business Finance & Accounting*, vol. 22, no. 7, pp. 961–973.

Conrad, J. and Kaul, G. (1998). An anatomy of trading strategies. *Review of Financial Studies*, vol. 11, no. 3, pp. 489–519.

Cubbin, E., Eidne, M., Firer, C. and Gilbert, E. (2006). Mean reversion on the JSE. *Investment Analysts Journal*, vol. 35, no. 63, pp. 39–47.

Daniel, K., Hirshleifer, D. and Subrahmanyam, A. (1998). Investor psychology and security market under-and overreactions. *The Journal of Finance*, vol. 53, no. 6, pp. 1839–1885.

De Bondt, W. (2000). The psychology of underreaction and overreaction in world equity markets. *Keim, Donald B.(eds., 2000), Security market imperfections in worldwide equity markets, Cambridge*, pp. 65–89.

De Bondt, W.F. and Thaler, R. (1985). Does the stock market overreact? *The Journal of Finance*, vol. 40, no. 3, pp. 793–805.

De Bondt, W.F. and Thaler, R.H. (1987). Further evidence on investor overreaction and stock market seasonality. *Journal of Finance*, pp. 557–581.

De Long, J.B., Shleifer, A., Summers, L.H. and Waldmann, R.J. (1990). Noise trader risk in financial markets. *Journal of Political Economy*, vol. 98, no. 4, pp. 703–738.

De Villiers, P., Lowings, A., Pettit, T. and Affleck-Graves, J. (1986). An investigation into the small firm effect on the Johannesburg Stock Exchange. *South African Journal of Business Management*, vol. 17, no. 4, pp. 191–195.

Degutis, A. and Novickytė, L. (2014). The efficient market hypothesis: A critical review of literature and methodology. *Ekonomika*, vol. 93, pp. 7–23.

Deshmukh, S., Fatemi, A. and Fooladi, I.J. (2008). Complexity of information and trading behavior: The case of dividend increase announcements. *Journal of Economic Psychology*, vol. 29, no. 1, pp. 1–18.

Dissanaike, G. (2002). Does the size effect explain the UK winner-loser effect? *Journal of Business Finance & Accounting*, vol. 29, no. 1-2, pp. 139–154.

Dreman, D.N. and Berry, M.A. (1995). Overreaction, underreaction, and the low-p/e effect. *Financial Analysts Journal*, vol. 51, no. 4, pp. 21–30.

Drobetz, W., Kammermann, M. and Wälchli, U. (2005). Long-run performance of initial public offerings: The evidence for Switzerland. *Schmalenbach Business Review*, vol. 57, no. 3, pp. 253–275.

Elton, E.J., Gruber, M.J., Brown, S.J. and Goetzmann, W.N. (2009). *Modern Portfolio theory and investment analysis*. CITY: John Wiley & Sons.

Fama, E.F. (1965). The behavior of stock-market prices. *The Journal of Business*, vol. 38, no. 1, pp. 34–105.

Fama, E.F. (1970). Efficient capital markets: A review of theory and empirical work. *The Journal of Finance*, vol. 25, no. 2, pp. 383–417.

Fama, E.F. (2021). *Market efficiency, long-term returns, and behavioral finance*. CITY: University of Chicago Press.

Fama, E.F. and French, K.R. (1993). Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics*, vol. 33, no. 1, pp. 3–56.

Fama, E.F. and French, K.R. (2015). A five-factor asset pricing model. *Journal of Financial Economics*, vol. 116, no. 1, pp. 1–22.

Fang, Y. (2013). Empirical study on overreaction and underreaction in Chinese stock market based on anar-tgarch model. *Journal of Financial Risk Management*, vol. 2, no. 4, pp. 71–76.

Finance, Y. (2020). Largest stock exchanges in the world. Available at: https://finance.yahoo.com/news/20-largest-stock-exchanges-world-175549152.html

Finnerty, J.E. (1976). Insiders and market efficiency. *The Journal of Finance*, vol. 31, no. 4, pp. 1141–1148.

Frankfurter, G.M. and McGoun, E.G. (2001). Anomalies in finance: What are they and what are they good for? *International Review of Financial Analysis*, vol. 10, no. 4, pp. 407–429.

Frisch, T., Kolaric, S. and Schiereck, D. (2014). Returns on large stock price declines and increases in the South African stock market: a note on market efficiency. *The International Business & Economics Research Journal (IBER)*, vol. 13, no. 3, pp. 581–592.

Gabriela ğiĠan, A. (2015). The efficient market hypothesis: review of specialized literature and empirical research. *Procedia Economics and Finance*, vol. 32, pp. 442–449.

Gilbertson, B. and Roux, F. (1977). The Johannesburg Stock Exchange as an efficient market. *Investment Analysts Journal*, vol. 6, no. 9, pp. 21–27.

Grieb, T. and Reyes, M.G. (1999). Random walk tests for Latin American equity indexes and individual firms. *Journal of Financial Research*, vol. 22, no. 4, pp. 371–383.

Griffin, D. and Tversky, A. (1992). The weighing of evidence and the determinants of confidence. *Cognitive Psychology*, vol. 24, no. 3, pp. 411–435.

Grigaliūnienė, Ž. (2013). Investor sentiment, overreaction and underreaction in stock market. *Grigaliūnienė, Žana. Studijos šiuolaikinėje visuomenėje. Šiauliai: Šiaurės Lietuvos kolegija, 2013, Nr. 4 (1), p. 32-43. ISSN 2029-431X.*

Guerard, J. and Mark, A. (2021). Earnings forecasts and revisions, price momentum, and fundamental data: Further explorations of financial anomalies. In: *Handbook of financial econometrics, mathematics, statistics, and machine learning*, pp. 1151–1209. CITY: World Scientific.

Heynen, R., Kemna, A. and Vorst, T. (1994). Analysis of the term structure of implied volatilities. *Journal of Financial and Quantitative Analysis*, vol. 29, no. 1, pp. 31–56.

Hirshleifer, D., Lim, S.S. and Teoh, S.H. (2011). Limited investor attention and stock market misreactions to accounting information. *Review of Asset Pricing Studies*, vol. 1, no. 1, pp. 35–73.

Hoffman, A. (2012). Stock return anomalies: evidence from the Johannesburg Stock Exchange. *Investment Analysts Journal*, vol. 41, no. 75, pp. 21–41.

Hong, H., Lim, T. and Stein, J.C. (2000). Bad news travels slowly: size, analyst coverage, and the profitability of momentum strategies. *The Journal of Finance*, vol. 55, no. 1, pp. 265–295.

Hong, H. and Stein, J.C. (1999). A unified theory of underreaction, momentum trading, and overreaction in asset markets. *The Journal of Finance*, vol. 54, no. 6, pp. 2143–2184.

Hsieh, H.-H. and Hodnett, K. (2011). Tests of the overreaction hypothesis and the timing of mean reversals on the JSE Securities Exchange (JSE): The case of South Africa. *Journal of Applied Finance & Banking*, vol. 1, no. 1, pp. 107–130.

Hu, Y. (2012). *Statistical Approach to Stock Market Overreaction and Seasonality (Unpublished)*. Master's thesis, University of California.

Hubbard, J.H. and Burke Hubbard, B. (2015). *Vector calculus, linear algebra, and differential forms: a unified approach (5th edition)*. Matrix Editions.

Igboekwu, A. (2015). *Representative agent earnings momentum models: the impact of sequences of earnings surprises on stock market returns under the influence of the Law of Small Numbers and the Gambler's Fallacy*. Ph.D. thesis, Loughborough University.

Institute, C. (2016). *CFA Program Curriculum 2017 Level III*. No. vol. 1 à 6 in CFA Curriculum 2017. Wiley. ISBN 9781119315964.

Itaka, J.K. (2014). *Test of the overreaction hypothesis in the South African stock market*. Ph.D. thesis, University of the Western Cape.

Jegadeesh, N. and Titman, S. (1993). Returns to buying winners and selling losers: Implications for stock market efficiency. *The Journal of Finance*, vol. 48, no. 1, pp. 65–91.

JSE (2006). SA sector classification.
Available at: https://www.jse.co.za/sa-sector

JSE (2021). Overview JSE.
Available at: https://www.jse.co.za/our-business/history-%26-company-overview

Kahneman, D. and Tversky, A. (1982). Variants of uncertainty. *Cognition*, vol. 11, no. 2, pp. 143–157.

Kahneman, D. and Tversky, A. (2013). Prospect theory: An analysis of decision under risk. In: *Handbook of the fundamentals of financial decision making: Part I*, pp. 99–127. World Scientific.

Knight, R. (1985). An empirical evaluation of the effectiveness of South African investment analysts. *South African Journal of Business Management*, vol. 16, no. 4, pp. 157–160.

Konak, F. and Şeker, Y. (). The efficiency of developed markets: Empirical evidence from FTSE 100. *Journal of Advanced Management Science*, vol. 2, no. 1, pp. 29–32.

Lakonishok, J., Shleifer, A. and Vishny, R.W. (1994). Contrarian investment, extrapolation, and risk. *The Journal of Finance*, vol. 49, no. 5, pp. 1541–1578.

Lam, K., Liu, T. and Wong, W.-K. (2010). A pseudo-Bayesian model in financial decision making with implications to market volatility, under-and overreaction. *European Journal of Operational Research*, vol. 203, no. 1, pp. 166–175.

Lam, K., Liu, T. and Wong, W.-K. (2012). A new pseudo-bayesian model with implications for financial anomalies and investors' behavior. *Journal of Behavioral Finance*, vol. 13, no. 2, pp. 93–107.

Latif, M., Arshad, S., Fatima, M. and Farooq, S. (2011). Market efficiency, market anomalies, causes, evidences, and some behavioral aspects of market anomalies. *Research Journal of Finance and Accounting*, vol. 2, no. 9, pp. 1–13.

Lee, C. and Swaminathan, B. (2000). Price momentum and trading volume. *The Journal of Finance*, vol. 55, no. 5, pp. 2017–2069.

Lesmond, D.A., Schill, M.J. and Zhou, C. (2004). The illusory nature of momentum profits. *Journal of Financial Economics*, vol. 71, no. 2, pp. 349–380.

Lim, K.-P., Brooks, R.D. and Kim, J.H. (2008). Financial crisis and stock market efficiency: Empirical evidence from Asian countries. *International Review of Financial Analysis*, vol. 17, no. 3, pp. 571–591.

Lo, A.W. and MacKinlay, A.C. (1990). When are contrarian profits due to stock market overreaction? *Review of Financial Studies*, vol. 3, no. 2, pp. 175–205.

Lobe, S. and Rieks, J. (2011). Short-term market overreaction on the Frankfurt stock exchange. *The Quarterly Review of Economics and Finance*, vol. 51, no. 2, pp. 113–123.

Mahlophe, M.I. (2015). *Effect of market anomalies on expected returns on the JSE: A cross-sector analysis.* Ph.D. thesis, North-West University.

Malkiel, B.G. (1962). Expectations, bond prices, and the term structure of interest rates. *The Quarterly Journal of Economics*, vol. 76, no. 2, p. 197–218.

Malkiel, B.G. (2005). Reflections on the efficient market hypothesis: 30 years later. *Financial Review*, vol. 40, no. 1, pp. 1–9.

Malkiel, B.G. (2011). The efficient-market hypothesis and the financial crisis. In: *Rethinking finance: perspectives on the crisis (Proceedings of a Conference). CITY: Russel Sage Foundation.*

Malkiel, B.G. and Fama, E.F. (1970). Efficient capital markets: A review of theory and empirical work. *The Journal of Finance*, vol. 25, no. 2, pp. 383–417.

Mann, H.B. and Whitney, D.R. (1947). On a test of whether one of two random variables is stochastically larger than the other. *The annals of mathematical statistics*, pp. 50–60.

Mishkin, F.S. and Eakins, S.G. (2006). *Financial markets and institutions.* CITY: Pearson Education India.

Mlonzi, V., Kruger, J. and Nthoesane, M.G. (2011). Share price reaction to earnings announcement on the JSE-ALtX: a test for market efficiency. *Southern African Business Review*, vol. 15, no. 3, pp. 142–166.

Mobarek, A. and Keasey, K. (2002). Weak-form market efficiency of and emerging market: Evidence from Dhaka Stock Market of Bangladesh.[e-document]. retrieved october 26, 2014, from: http://www. bath. ac. uk/centers/cds/enbs-papers. *Mobarek_ new. htm.*

Muller, C. (1999). Investor overreaction on the Johannesburg stock exchange. *Investment Analysts Journal*, vol. 28, no. 49, pp. 5–17.

Muller, C. and Ward, M. (2013). Style-based effects on the Johannesburg Stock Exchange: a graphical time-series approach. *Investment Analysts Journal*, vol. 42, no. 77, pp. 1–16.

Mun, J.C., Vasconcellos, G.M. and Kish, R. (2001). The contrarian/overreaction hypothesis: an analysis of the US and Canadian stock markets. *Global Finance Journal*, vol. 11, no. 1, pp. 53–72.

Mushidzhi, T. and Ward, M. (2004). Abnormal returns for cash vs share funded acquisitions. *Investment Analysts Journal*, vol. 33, no. 60, pp. 17–31.

Ou, J.A. and Penman, S.H. (1989). Financial statement analysis and the prediction of stock returns. *Journal of Accounting and Economics*, vol. 11, no. 4, pp. 295–329.

Page, M.J. and Way, C. (1992). Stock market over-reaction: The South African evidence. *Investment Analysts Journal*, vol. 21, no. 36, pp. 35–49.

Pinsky, M. and Karlin, S. (2010). *An introduction to stochastic modeling.* CITY: Academic Press.

Power, D.M., Lonie, A. and Lonie, R. (1991). The over-reaction effect—some UK evidence. *The British Accounting Review*, vol. 23, no. 2, pp. 149–170.

Raji, R. (2015). *Investor reaction and seasonality in two of Africa's largest equity markets.* Ph.D. thesis, University of the Witwatersrand, Johannesburg.

Rasch, D., Kubinger, K.D. and Moder, K. (2011). The two-sample t test: pre-testing its assumptions does not pay off. *Statistical Papers*, vol. 52, no. 1, pp. 219–231.

Robins, E.M., Sandler, M. and Durand, F. (1999). Inter-relationships between the January effect, market capitalisation and value investment strategies on the JSE. *Investment Analysts Journal*, vol. 28, no. 50, pp. 53–64.

Rose, A.K. and Selody, J.G. (1984). Exchange market efficiency: A semi-strong test using multiple markets and daily data. *The Review of Economics and Statistics*, vol. 66, no. 4, pp. 669–672.

Samuelson, P.A. (1965). Proof that properly anticipated prices fluctuate randomly. *Industrial Management Review (pre-1986)*, vol. 6, no. 2, p. 41.

Sewell, M. (2011). History of the efficient market hypothesis. *Research Note No. RN/11/04*, vol. 11, no. 4, pp. 4–18.

Shefrin, H. (2002). *Beyond greed and fear: understanding behavioral finance and the psychology of investing.* Oxford University Press on Demand.

Shleifer, A. (2000). *Inefficient markets: an introduction to behavioural finance.* Oxford: Oxford University Press.

Shleifer, A. and Summers, L.H. (1990). The noise trader approach to finance. *The Journal of Economic Perspectives*, vol. 4, no. 2, pp. 19–33.

Shleifer, A. and Vishny, R.W. (1997). The limits of arbitrage. *The Journal of Finance*, vol. 52, no. 1, pp. 35–55.

Strugnell, D., Gilbert, E. and Kruger, R. (2011). Beta, size and value effects on the JSE, 1994–2007. *Investment Analysts Journal*, vol. 40, no. 74, pp. 1–17.

Van Rensburg, P. and Robertson, M. (2003). Explaining the cross-section of returns in South Africa: attributes or factor loadings? *Journal of Asset Management*, vol. 4, no. 5, pp. 334–347.

Venter, J. (2009). Intraday momentum and contrarian effects on the JSE. *Investment Analysts Journal*, vol. 38, no. 70, pp. 47–60.

Vernimmen, P., Quiry, P., Dallocchio, M., Le Fur, Y. and Salvi, A. (2019). *Corporate finance.* 1. Wiley.

Wang, J. and Wu, Y. (2011). Risk adjustment and momentum sources. *Journal of Banking & Finance*, vol. 35, no. 6, pp. 1427–1435.

Wilcox, J.W. and Fabozzi, F.J. (2013). *Financial advice and investment decisions: a manifesto for change*, vol. 195. CITY: John Wiley & Sons.

Wouassom, A. (2017). *Momentum and contrarian trading strategies: implication for risk-sharing and informational efficiency of security markets*. Ph.D. thesis, Queen Mary University of London.

Wyart, M. and Bouchaud, J.-P. (2007). Self-referential behaviour, overreaction and conventions in financial markets. *Journal of Economic Behavior & Organization*, vol. 63, no. 1, pp. 1–24.

Zarb, B.J. (2018). Liquidity, solvency, and financial health: Do they have an impact on US airline companies'profit volatility? *International Journal of Business, Accounting, & Finance*, vol. 12, no. 1, pp. 42–51.

Zunino, L., Zanin, M., Tabak, B.M., Pérez, D.G. and Rosso, O.A. (2009). Forbidden patterns, permutation entropy and stock market inefficiency. *Physica A: Statistical Mechanics and its Applications*, vol. 388, no. 14, pp. 2854–2864.