A SYSTEMATIC ANALYSIS OF THE GENERALISATION CONCEPT IN EARLY ALGEBRA FOR YOUNG LEARNERS – SOME IDEAS FOR THE CLASSROOM

By

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DECLARATION

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ABSTRACT

The importance of introducing algebra concepts and skills in the early years of mathematics education, has become increasingly acknowledged as imperative for algebra success in the secondary grades of mathematics teaching and learning. Research has shown that learners at a young age are able to reason algebraically. Generalisation is described as one of the core aspects of early algebra and should be embedded throughout the mathematics curriculum to form a deep understanding of the underlying structure of mathematics. In South Africa, the field of early algebra remains largely unexplored in the mathematics education research context. The content area, 'Patterns, functions and algebra' which aims to provide guidelines for the teaching of early algebra in South African early years classrooms, seems to be inadequate for the implementation of early algebra in early years classrooms. A lack of a relational approach in the sequencing of curriculum documents and learning and teaching materials, are provided for the teaching of patterns, functions, and algebra in the foundation phase. The purpose of this study was to determine how the generalisation concept can be implemented in early years classrooms to develop early algebra skills and concepts. A systematic literature review was conducted with the aim of extending on current research by designing a higher-order construct from existing literature. A thematic analysis of the literature led to the synthesis of an instructional sequence for the implementation of generalisation in early years classrooms. The instructional sequence was based on the principles of Realistic Mathematics Education from the Netherlands which included guided reinvention and emergent modelling as foundational principles. A historical overview of the development of algebra through the ages indicated three historical stages: the rhetorical stage, the syncopated stage, and the symbolic stage, as well as four conceptual stages: the geometric stage, the static-equation stage, the dynamic function stage, and the abstract stage. The emergence of the main components and big ideas of algebra from these stages provided a valuable insight as to how algebraic thinking developed naturally and informed an instructional sequence for the implementation of generalisation. An in-depth systematic review of the concepts which emerged from history was further conducted to understand the current state of algebra in schools, how algebraic thinking develops, the levels of algebraic thinking and what the main components of early algebra are, with a specific focus on generalisation. The study further explored an appropriate learning approach, namely the problem-centred approach, which ensures that mathematics is learned for understanding. The historical overview and the systematic review of early algebra, generalisation, and structure were used to construct the instructional sequence for the implementation of generalisation in early years classrooms.

OPSOMMING

Dit word algemeen aanvaar dat daar ‘n behoefte bestaan om algebra op ‘n vroeër fase deel te maak van die wiskunde kurrikulum. Vroeë algebra word beskou as noodsaaklik om die sukses van die verstaan van formele algebra in later grade te verseker. Navorsing dui aan dat leerders daartoe in staat is om van ‘n vroeë ouderdom algebraïes te redeneer. Veralgemening word beskryf as een van die kern aspekte van vroeë algebra. Veralgemening moet dwarsdeur die wiskunde kurrikulum integreer word sodat leerders die struktuur van wiskunde gouer en beter kan verstaan. In Suid-Afrika, is die veld van vroeë algebra meestal onontgin in die konteks van vroeë wiskundeonderwys. Die inhoud area ‘Patrone, funksies en algebra’ het die doel om riglyne voor te skryf vir die onderrig en leer van algebra in Suid-Afrikaanse grondslagfase klaskamers, maar blyk onvoldoende te wees vir die effektiewe onderrig en leer van vroeë algebra. ‘n Gebrek aan ‘n verhouding en samehang in die volgorde van kurrikulumdokumente, en onderrig en leer materiaal kan waargeneem word. Die oogmerk van hierdie studie was om te bepaal hoe die veralgemeningskonsep in grondslagfase klaskamers geïmplementeer kan word met die doel om vroeë algebra vaardighede en konsepte in jong leerders te ontwikkel. ‘n Sistematiese literatuurstudie was uitgevoer met die hoop om op huidige literatuur uit te brei deur ‘n hoër-orde konstruksie te ontwerp op grond van bestaande literatuur. ‘n Tematiese analise van die literatuur het gelei tot die sintese van ‘n geordende onderrig patroon of leer-teoretiese model wat die implementering van veralgemening in die grondslagfase klas verduidelik. Die model is gebaseer op die beginsels van die Realistiese Wiskundeonderwysbenadering van Nederland (‘Realistic Mathematics Education’) wat onder andere insluit, gerigte herontdekking (‘guided reinvetion’) en ontluikende modellering (‘emergent modelling’). ‘n Historiese oorsig van die ontwikkeling van algebra deur die eeue het drie ontwikkelingsfases blootgelê: die retoriese fase, die sinkopering fase, en die simboliese fase. Ook is daar vier konseptuele fases aangedui: die geometriese fase, die fase van die oplos van statiese vergelykings, die dinamiese funksie fase, en die abstrakte fase. Die opkoms van die sleutelkonsepte en groot idees van vroeë algebra vanuit hierdie fases, het waardevolle insigte gelewer ten opsigte van die natuurlike ontwikkeling van algebra. Hierdie insigte is gebruik in die ontwerp van die onderrig-volgorde van die implementering van veralgemening. ‘n Verdere in-diepte sistematiese studie van die konsepte wat uit die geskiedenis verskyn het, was uitgevoer om vas te stel wat die huidige situasie van vroeë algebra in skole is, hoe algebraïese denke ontwikkel word, hoe die vlakke van algebraïese denke ontwikkel en wat die kernaspekte van algebra is, met ‘n sterk fokus op veralgemening. Die studie het ook die probleem-gesentreerde benadering gekies as ‘n leerbenadering wat die doel het om die leer van wiskunde-met-begrip te verseker. Die oorsig van die geskiedenis van die ontwikkeling van algebra, sowel as die sistematiese analise van vroeë algebra gefokus op veralgemening en struktuur, het ‘n onderrig-volgorde of leer-teoretiese model vir die implementering van veralgemening in die grondslagfase gelewer.

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CHAPTER 1: INTRODUCTION TO THE STUDY

1.1. INTRODUCTION

Chapter 1 has the objective of describing the context in which the study took place by explaining the motivation, the statement of the problem and the aims and objectives of the study. Furthermore, the research methodology, trustworthiness and delimitation of the study will be explained and justified.

1.2. MOTIVATION

From the earliest days of mathematics, algebra can be regarded as the science of solving equations (Kieran, 2004). Research conducted in the 70s’ and 80’s pointed to difficulties learners experience in this subject and shone light on the need for reform in the views of algebra. At the moment, one of the most important questions in mathematical reform is: “Does early algebra matter?” (Blanton et al., 2019). It is believed that to make algebra more accessible to a larger group of learners, algebra should be introduced in the early grades (Kieran, 2004). Even so, it is necessary to acknowledge that many subareas of algebra are not appropriate for young learners. Instead of questioning whether young learners are prepared for algebra, the question should rather be which algebraic ideas, concepts and methods are within the reach of young learners (Kaput et al., 2008).

Roberts (2012) describes algebra as the study of general properties of numbers, and generalisations, whereas early algebra is seen as the teaching of arithmetic in the early mathematics education with a specific focus on generalising arithmetic, generalising towards the idea of a function, and using modelling as a language of mathematics (Roberts, 2012). To understand which concepts are within the reach of young learners, it is imperative to understand what the literature says about early algebra.

Various characterisations of what early algebra is, can be found in the literature. Five core aspects of algebra has been identified by Kaput (1995): generalisation and formalisation; syntactically guided manipulations; the study of structure; the study of functions, relations and joint variation; and a modelling language. Kieran (1996) developed a model to explain early algebra activity by focusing on the activities typically worked with by learners: generalisation activities, transformational activities and global-meta level activities. These models attempt to clearly map out the focus and components of early algebra and will be discussed in more depth in the literature review. Early algebra differs from algebra that is taught in High School in the sense that it does not focus on the transformational aspects of algebra which involves procedures for solving equations or simplifying equations. Broadly stated, the core of early algebra is in generalising mathematical ideas, representing and justifying generalisations in multiple ways, and arguing with generalisations (Blanton et al., 2015).

Early algebra has the purpose of deepening learners' understanding of the structural form and generality of mathematics, rather than only providing isolated mathematical experiences. This has been proved to ensure better mathematical achievement of learners in later grades (Blanton & Kaput, 2011).

Kaput et al. (2008) provided a framework for in view of the main elements of early algebra with reference to two core aspects which falls within three strands:

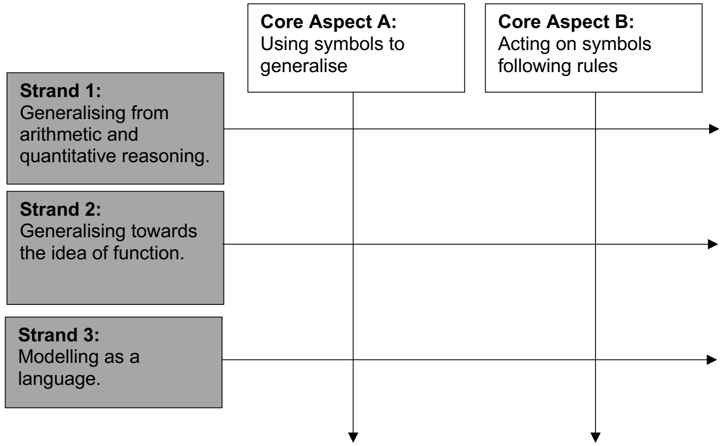


Figure 1.1: Kaput's framework of the main elements of early algebra (Roberts, 2012)

Kaput et al. (2008) characterise algebraic reasoning as symbolising activities that serve purposeful generalisation, and simultaneously as reasoning with symbolised generalisations. This is evident in his two core aspects of early algebra. The first two strands of his framework for early algebra involve the two types of generalising that are at the centre point of algebraic reasoning, namely generalising arithmetic and generalising towards the idea of function. The third strand, modelling as a language, involves the modelling of real-life situations using algebraic reasoning and language (Roberts, 2012). As can be seen in the framework above, generalisation forms an integrated part of early algebra and will serve as the focus of this systematic literature review.

It is critically important to establish firm foundations of algebra in early years education to ensure success in the understanding of complex algebra in the later years. With the aim of teaching algebra for understanding, a problem-centred approach, which focuses on the mathematical process rather than computational skills, may be most effective (Human and Olivier, 1999). In a problem-centred approach, learners are introduced to mathematical concepts by means of engaging problems. A problem-centred approach builds on on learners' existing intuitive and informal knowledge by creating opportunities for them to tackle problems using their common sense (Human & Olivier, 1999, p.3). The problem-centred approach is based on a socio-constructivist learning theory which believes that learning is a social process and the learner is an active participant who constructs his or her own knowledge. Learners learn from each other through talking to each other and the exchangingof ideas.

The National Council of Teachers of Mathematics (NCTM) emphasises the importance of teaching early algebra concepts in the early years and describes algebra as a way of thinking with a set of concepts and skills which aids learners to generalise, model and analyse mathematical situations (Lee, Collins, & Melton, 2016, p. 306). This orientation enables the learner to get a positive feeling about mathematics and what mathematics is.

In the South African CAPS for the Foundation phase mathematics document, “Patterns, Functions and Algebra” is described as one of the five content areas of foundation phase mathematics (Department of Education, 2011). The prescribed activities of this content area focus on the completion of number sequences and the copying and extending of geometrical patterns (Department of Education, 2011, p.506). An absence can be noted of a relational approach to sequencing in the curriculum documents and the order of learning materials provided for teaching (Du Plessis, 2018, p.1). This results in a curriculum which focuses on procedures and is based on memory and recall (Du Plessis, 2018). This approach to patterns, functions and algebra is carried through in all government textbooks and educational materials and thus contributes to a limited exposure to algebra in the early years.

Research shows that the traditional “arithmetic-then-algebra” approach, where an arithmetic curriculum in the early years is followed by formal algebra education in the later years, leads to school failure for many learners (Kaput, 1998, 1999, 2008; Moses & Cobb, 2001; Schoenfeld, 1995) in (Knuth et al., 2014). When algebra is only introduced in the later grades, there is not enough time and space for teachers to develop depth in learners’ algebraic thinking with algebra. This results in limited career opportunities, particularly in STEM-related fields. It is therefore clear that a longitudinal approach to teaching algebra is imperative. This involves including algebra education from pre-school up to Grade 12. Learners should have ample opportunities to experience algebra starting in the early grades. These experiences should build on learners’ natural and informal intuitions of patterns and relationships to develop formal ways of mathematical thinking (NCTM, 2000; NCTM, 2006).

The transition from arithmetic to algebra can be challenging for learners, as many adjustments need to be made. Learners who are proficient at early years arithmetic and who are exposed to arithmetic for extended periods of time, may struggle with the transition to algebra as they are answer-orientated and struggle to focus on the representation of relations (Kieran, 2004). Considerable adjustment is required to develop algebraic thinking, which includes but is not restricted to:

- A focus on relationships and not merely on the calculation of numerical answers;

- A focus on operations as well as their inverses, and on the related idea of doing/undoing;
- A focus on both representing and solving a problem instead of just solving it to find the answer;
- A focus on both numbers and letters rather than just numbers alone. This can include:
    - ❖ Working with letters which may be variables or unknowns
    - ❖ Accepting unclosed literal expressions and responses
    - ❖ Comparing expressions for equivalence based on properties rather than on numerical evaluation
- A refocus of the meaning of the equation sign (Kieran, 2004).

Algebraic thinking can be understood as a way to approach to quantitative situations that emphasises the general relational features with tools that are not automatically letter-symbolic, but which can be used as cognitive support for introducing and for suporting the traditional dialogue of school algebra (Kieran, 1996, p.275).

It has been documented that learners at a young age are able to think algebraically in the following ways:

- Form a **relational understanding** of the equal sign (Carpenter et al., 2003, 2005; Falkner et al., 1999).
- **Generalise mathematical structure** by noticing regularity in arithmetic situations (Schifter, 1999; Bastable and Schifter, 2008; Schifter *et al.*, 2008).
- Use **sophisticated tools** to explore, symbolise and generalise functional relationships (Blanton et al., 2015; Carraher et al., 2006; Cooper & Warren, 2011; Moss et al., 2008).
- Develop **mathematical arguments** that reflect more generalised forms than the empirical, case-based reasoning (Carpenter et al., 2003; Schifter, 2009).
- **Reason about abstract** quantities (volume, length, mass) to represent algebraic relationships (Dougherty, 2003; B. J. Dougherty, 2008).

These ways in which learners are able to think algebraically, informs the types of activities or learning situations which will develop learners’ ability to generalise. It is, therefore, imperative that these ways of thinking is implemented in the design of the instructional sequence in Chapter 5.

Mason, Burton and Stacey (2010) suggests a conceptual framework for the mathematical process which takes place when a learner approaches any problem as seen in the figure below. The framework shows that the mathematical processes involved with tackling a problem is **specialising and generalising**. A learner must first be able to specialise, which refers to understanding and to make a plan to solve a problem. Specialization will lead to the learner being able to solve the problem by carrying out the plan and checking if the solution is appropriate. Here the learner is busy with generalisation.

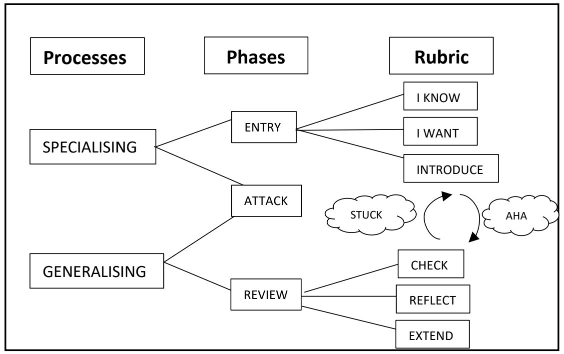


Figure 1.2. Conceptual framework of the mathematical process (Mason, Burton and Stacey, 2010)

Algebraic reasoning is an activity which consists of the generalisation of mathematical ideas, using symbolic representations, which represent functional relationships (Blanton & Kaput, 2011). The publication *Algebra in the Early Grades* (Kaput et al., 2008, p. 77) takes on the perspective that the core of algebraic reasoning is constructed by complex symbolisation processes that underpins purposeful generalisations and reasoning with generalisations. Algebra's broad, rich and natural relationship with naturally occurring human cognitive processes and reasoning is stressed (Kaput et al., 2008).

Certain challenges impair the development of algebraic thinking in the early years. Patterns are often used as the main tool to develop algebraic thinking in the early years, but patterns are only one of the possible avenues to master algebraic thinking. Even so, teachers have limited pedagogical content knowledge in relation to the types, levels and complexity of patterns. Teachers may limit children's development of patterning when only approaching patterns as repeating patterns (Papic & Mulligan, 2007, p. 592).

When teachers have a narrow perspective of algebraic activities, the relation between algebra and early mathematical thinking is obscured (Kaput et al., 2008). The difficulties learners experience further orginate due to an absence of an relevant foundation in arithmetic (Warren, 2004, p.417). The assumption has been that everyday use of the four operations (+, -, x, ÷) (*Back to basic movement* that feeds the traditional teaching approach) will develop a fundamental understanding of the structure of mathematics but research is showing that it is not. Learners should be exposed to and encouraged to engage in patterns that lead to generalised thinking throughout their education (Warren, 2004).

Early algebra has the purpose of deepening learners' understanding of the structural form and generality of mathematics. It aims to produce more than only isolated mathematical experiences. This has been proved to ensure better mathematical achievement of learners in later grades (Cai and Knuth, 2011). "Pattern generalisation is considered one of the prominent routes for introducing students to algebra" (Radford, 2010, p. 1). Developing a feeling for numbers and operations, and the patterns within them, builds a foundation for the development of children's algebraic thinking.

Functions within patterns are rich and may include a wide spectrum of kinds of change. Learners' patterning knowledge shows to have a great influence on the progress of analogical thinking and their ability to generalise, identify and extend patterns through inductive reasoning. The essence of mathematics is grounded in relations and transformations that lead to patterns and generalisations. In research literature, abstracting patterns in mathematical structures is the goal of mathematics and the basis of structural knowledge (Jones, 2018). The **focus of mathematics** should, thus, be on developing fundamental skills in generalising, expressing and systematically justifying generalisations (Warren and Cooper, 2008, p. 171). The focus of **early algebra** should be on a relational approach to learning mathematics, which refers to studying number from a structural perspective (National Research Council, 2001). Structure is drawn out through examining number and space relationally, which sparks reasoning that not only focuses on the object but also on underlying associates the object possesses (Du Plessis, 2018). Learners should be able to see common mathematical structures in representations when solving problems. The emphasis should fall on the structural relationships within problems. These relationships give hints for how problems might be solved. The development of proficient problem solvers involve learners forming mental representations of problems, detecting mathematical relationships and devising new solution methods (National Research Council, 2001).

The levels of thought in the process of developing early algebraic thinking are significant and reveal a trend in the development of thought in algebra. Nixon (2009) explored a theory of the levels of the development of algebraic thinking based on Piaget and Garcia's belief that there are three developmental levels in algebra. Nixon argues for three levels of learning to think algebraically: the perceptual level (precept level) where learners make use of counters or abacuses for counting. In this stage learners need to coordinate their senses and perceptions to form algebraic concepts. In the conceptual level (concept level) a shift from analysing objects to considering the relations of transformations between the objects can be noticed. Learners are able to find interrelationships between properties and start providing definitions and theorems for what they experience. In the abstract level learners use symbols with deep understanding to construct proofs and they can understand the importance of deductions, axioms, postulates and proofs. As learners pass through these levels, it is necessary for them to be guided. They should be motivated to generalise and draw comparisons (Nixon, 2009).

In a study conducted by Carpenter and Levi (2000) learners were introduced to number sentences as a context to engage learners in a discussion about properties of number operations. On the perceptual level learners were introduced to a variety of true-false operations with zero, for example 58 + 0 = 58. Learners could readily say if it was true or false. A mental picture of these types of operations were formed. Learners moved to the conceptual level when they were then challenged with 78 – 49 = 78. Learners had to justify why this could not be true and how it differs from working with a zero. After many examples, learners were challenged to state a rule which they could share with the rest of the class. The learners were able to verbalise that when you add or take away zero from a number, you are always left with the same number.

Learners were thus able to construct their own mathematical concept. When learners are able to 'make a rule' they are starting the process of generalisation.

Algebraic thinking involves two core activities: (a) making generalisations and (b) making use of symbols to represent mathematical ideas and to represent and to solve problems. Generalisation and formalisation involve the articulation and representation of unifying ideas that make clear the mathematical and structural relationships (Carpenter & Levi, 2000).

The focus of algebra teaching in mathematics should be on developing skills in generalising and expressing and justifying generalisations. It seems that some classrooms in the early years focus more on mathematical products than on mathematical processes (Warren & Cooper, 2008, p. 171). Mathematical processes involve communications, connections, mental mathematics and estimations, problem solving, reasoning, technology and visualisation (*Seven Mathematical Processes in Action*, 2018). Over-exposure to result pattern generalisation (regularity in results) can hinder learners' ability to generalise regularities in the process of finding the pattern. It is, therefore, necessary to explore various avenues for the development of algebraic thinking.

Many believe young children are capable of thinking functionally (algebraically). The difficulties that occur may stem from a lack of experiences (with generalisations) in the early years (Warren & Cooper, 2008, p. 172). Algebra in the early years demands a clear comprehension of the mathematical structure of arithmetic expressed by language and gestures making use of concrete materials and representations. The generalisation concept is not integrated into early years mathematics in schools in South Africa.

When examining the emergence of algebra throughout the history of mathematics, important lessons can be learned. The stages through which algebra emerged in history, can be used to inform the sequence of teaching and learning of algebra at a early education of mathematics level. Piaget (1989) claims that advances made in the history of scientific thought from one period to next, do not follow each other in a random manner, but can be organised into sequential stages. Literature (Derbyshire, 2006; Kvasz, 2006; Tabak, 2011; Ferrara and Sinclair, 2016) agrees that the history of algebra occurred over three stages: the rhetorical stage, the syncopated stage and the symbolic stage. When examining these stages in history, it is noticed that the components of early algebra, including the concept of generalisation, emerge in a sequential manner. These stages in the history of algebra and the emergence of early algebra concepts will be used to inform an instructional design sequence.

This systematic literature review study will aim to provide an instructional design sequence for the teaching of the generalisation concept to effectively foster early algebra concepts in young learners.

1.3. STATEMENT OF THE PROBLEM

Generalising can be seen as a mental activity by which one squash multiple instances into a single unitary form (Blanton et al., 2019). The problem is that the generalisation concept is not taught effectively to foster early algebra concepts in young learners.

Mitchelmore (2002) groups generalisation into three categories: Generalisation is often used as a synonym for abstraction (G1) where it is defined as “finding and singling out properties in a whole class of similar objects” (Mitchelmore, 2002). Generalisation as an extention (G2) is formed by at least three aspects: Empirical extension, which applies when one finds other contexts to which a known concept applies; mathematical extention, when one class of mathematical objects is emerged in a larger class based on a different similarity; mathematical invention, when a mathematician deliberately omits a defining property to form a more general concept. Generalisation can also refer to a relationship that holds between all members of a set of objects (G3) (for the purpose of this study, G3 would be most applicable) (Mitchelmore, 2002). For the purpose of this study, generalisation will be seen as learners’ ability to generalise mathematical structure by noticing regularity in arithmetic situations, use sophisticated instruments to explore, generalise, and symbolise functional relationships, construct mathematical arguments that are more generalised forms than the empirical, case-based reasoning often used, and reason about abstract quantities to represent algebraic relationships Knuth et al. (2014).

Teaching mathematics that promote understanding is seen as in important facet of this systematic literature review. School mathematics should be seen as a human acitvity which reflects the work of mathematicians. This involves finding out why certain techniques work, thinking of new techniques and justifying and reasoning about these techniques (Carpenter and Lehrer, 1999). **Learning with understanding** is measured in terms of the outcomes of the process. Learners learn with understanding if they can apply this new knowledge to novel topics to solve unfamiliar topics. For the purpose of this study, the focus will fall on **teaching for understanding.** Three axes of characterisation for instruction for understanding include: tasks, tools and normative practices. Teaching for understanding encourages and integration of problem solving with the learning of basic skills and concepts. **Classrooms that promote understanding** should look like discourse communities where learners are engaged in discussing various techniques to solving problems. Mathematics should become a language for thought rather than a collection of ways to get an answer (Carpenter and Lehrer, 1999). The teaching for robust understanding (TRU) framework described five essential elements for classroom practice: (1) the content; (2) cognitive demand; (3) equitable access; (4) Agency, ownership and identity; and (5) Formative assessment (Schoenfeld, 2019). Teaching for understanding will be discussed in more detail in chapter 3 of the study.

Knuth et al. (2014) identified *five big ideas* around which much of early algebra’s research has matured. These big ideas offer opportunities for the development of deep algebraic reasoning, which includes the practices of generalising, representing, justifying and reasoning with mathematical relationships:

(a) equivalence, expressions, equations and inequalities
(b) generalised arithmetic
(c) functional thinking
(d) variables
(e) proportional reasoning (Knuth *et al.*, 2014)

Kaput et al. (2008) describes algebraic reasoning as symbolising activities that serve purposeful generalisation and at the same time as reasoning with symbolised generalisations. The first two strands of his framework for early algebra involve the two types of generalising that are the focus of algebraic reasoning, namely generalising arithmetic and generalising towards the idea of function. The third strand, modelling as a language, involves the modelling of real-life situations using algebraic reasoning and language (Roberts, 2012). Generalisation forms an integrated part of early algebra and will serve as the focus of this literature review study.

For the purpose of this study, young learners can be taken as Foundation Phase learners in the South African context. This implies that teaching and learning from Grade R, where learners are 5 or 6 years old, to Grade 3, where learners are 8, 9 or 10 years old, will be included in the study.

The purpose of this systematic literature study is to investigate the role of generalisation in the development of early algebra concepts and skills.

To achieve the above stated objective, the following research questions are raised:

RQ 1: What is the role of generalisation in the understanding of early algebra concepts and skills in young children?

RQ 2: How can the historical development of algebra and scholarly trajectories of algebra learning be synthesised to construct an instructional design sequence which focuses on generalisation for early algebra?

From here, the aims and the objectives of the study will be explained.

1.4. AIMS AND OBJECTIVES

To answer the research question stated above, there are some sub-research questions which can be viewed as the aims and objectives for this systematic literature review. These aims and objectives puts the study in focus as it clearly describes the elements and themes which will be researched in the study.

1. What can we learn from the history of the development of algebra for the learning of algebra?

Towards this aim, the researcher will examine the development of algebra from Mesopotamian times to algebra as we know it in schools today. This is an important aspect to investigate as it may provide valuable information and clues for understanding parallels between the development of algebraic thinking in history and the development of learners' algebraic thinking in schools.

2. What is early algebra?

In Chapter 2 and Chapter 3, the researcher will investigate the nature of early algebra by reviewing the literature on the main elements or components that form early algebra. The study will aim to provide a clear perspective on early algebra and the main components thereof, and look at the structure and form of mathematics which is developed through early algebra.

3. What is the role of algebraic thinking and generalisation in the understanding of early algebra?

This study will explore teaching for understanding as a route to the development of algebraic thinking and generalisation of early algebra. The researcher takes on the perspective that a problem-centred approach is necessary in a classroom which promotes understanding. For this objective, ideas for implementing generalisation in a classroom which promotes the understanding of generalisation in early algebra through a problem-centred approach, will be investigated.

4. How is the problem-centred approach implemented in the teaching and learning of early algebra in the search for teaching for understanding?

Here the researcher aims to look deeper into the characteristics of a problem-centred classroom with a specific emphasis on the teaching and learning of early algebra. This will include an investigation of the classroom culture, the types of questions which should be asked and theoretical perspectives on the teaching of mathematics for the role of the teacher.

5. How can the emergence of early algebra concepts from history be used in the construction of an instructional design sequence which shows a possible blueprint for the teaching and learning of generalisation for early algebra?

The aim of this systematic literature review is to extend on existing knowledge in the literature by critically constructing higher-order knowledge to provide a didactical framework for the teaching of generalisation which promotes the understanding of early algebra concepts for young learners. In Chapter 5, an instructional design sequence will be constructed based on the emergence of early algebra concepts from history.

1.5. RESEARCH METHODOLGY

1.5.1 Research approach: This study followed a qualitative research approach, and consists largely of a thematic analysis of data which was generated using a systematic literature review. Dixon-Woods (2016, p. 969) defines the systematic literature review as "a scientific process governed by a set of explicit and demanding rules oriented towards demonstrating comprehensiveness, immunity from bias, and transparency and accountability of technique and execution." Literature review is an essential part of academic research as knowledge advancement must be built on prior research. As existing work is reviewed, we understand the breadth and depth of the existing body of work and can identify gaps in the literature (Xiao and Watson, 2019).

The literature review for the purpose of research, can be categorised in four categories based on the purpose of the review: describe, test, extend and critique. The purpose of this study is to **extend** on existing knowledge with regards to how the development of the generalisation concept improve early algebra skills and concepts in young learners.

An extending review aims to go beyond the summary of data but rather focuses on building on the literature to create new, higher-order constructs. For qualitative research, these techniques include extracting concepts and second-order constructs from the literature and transforming them into third-order constructs (Xiao and Watson, 2019). The study will be elevated by extending on existing literature to designing an instructional design sequence in Chapter 5.

The method of research will be a thematic analysis. Themes were extracted from the literature, clustered, and eventually synthesised into analytical themes. Thomas and Harden (2008, p. 9) in Xiao & Watson (2019) explain that "analytical themes are more appropriate when a specific review question is being addressed (as often occurs when informing policy and practice), and third order constructs should be used when a body of literature is being explored in and of itself, with broader, or emergent, review questions."

1.5.2. Research paradigm: This systematic literature study was conducted within the interpretivism research paradigm. Non-empirical studies often fall in the interpretivism paradigm, taking a philosophical approach to the exploration of the social world with the aim of developing a clear understanding of a specific phenomenon (Gray, 2014, p. 24).

1.5.3. Research design: This study followed the systematic literature review approach as explained by Dixon-Woods (2016) and Xiao & Watson (2019), and is comprised of three major stages:

1. Planning stage: The researcher identifies a need for the review, sets the research questions and develop a review protocol.
2. Conducting the review: The researcher identifies and selects primary studies, extracts, analyses and synthesises data.
3. Reporting the review: The researcher writes a report to disseminate findings from the literature review (Dixon-Woods, 2016; Xiao and Watson, 2019).

Even though procedures in various types of reviews differ, eight steps can be followed when conducting a systematic review. The steps are shown as they follow the three major stages, in the diagram below.

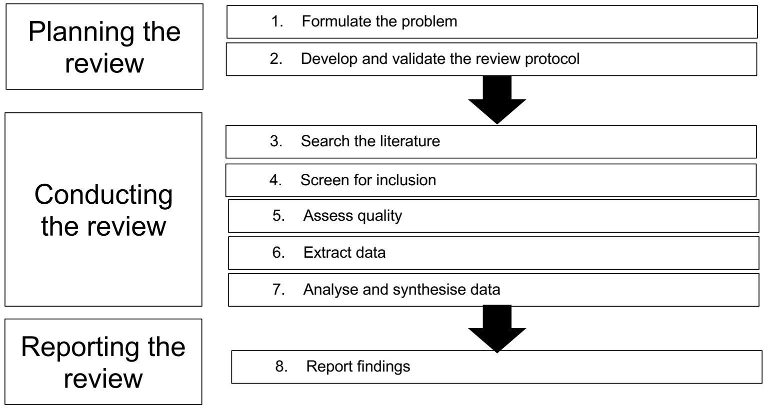


Diagram 3.1. Process of the systematic literature review (Xiao and Watson, 2019)

1.6. TRUSTWORTHINESS

The key principles of good qualitative research is found in the notion of trustworthiness which refers to the neutrality of its findings or decisions (Babbie and Mouton, 2011).
A research approach can be trustworthy if the methodology can be copied by a different researcher to reach the same results and findings. The systematic literature review framework has been used by many researchers in various qualitative studies.

The methodology used in this study was documented meticulously to ensure its replicability. An audit trail was built by carefully documenting online platforms and search terms used to find studies.

Literature was studied thoroughly to ensure that the authors' arguments are correctly and effectively interpreted. Furthermore, emphasis was placed on the accurate citing of all authors and their work to avoid plagiarism. The researcher aimed to interpret all literature in an objective manner.

Validity was achieved through triangulation, i.e., by using multiple sources of data and repeatedly cross-checking data (Merriam, 2009). Data was collected from academic journals, books, websites and various databases. Data from over a long period of time will be included to ensure a valid overview and analysis of the data. This ensures that a full perspective on early algebra and its emergence from history can be formed and analysed.

Triangulation was promoted by involving more than one investigator in the study. Even though this study was conducted by one investigator, work was continuously assessed by the study leader to ensure validity.
Finally, discrepant case analysis was used as a means of triangulation. This involves looking for data which support alternative explanations. This ensures that the researcher is aware of possible bias and take into account contrary explanations and perspectives.

Peer review formed an important part of ensuring validity in this study. Using peer-reviewed data (articles/books/websites) enabled the researcher to include data which were valid and of a high quality.

An audit trail, consisting of a clear methodology and account of how the study was conducted, was kept to ensure the reliability and replicability of the study. The audit trail was continuously monitored by the study leader.

All data extracted in this study is already available in the mathematics education field and no previously unpublished personal information was used. Data used are reviewed by scholars who are experts in the field.

Before the research was conducted, ethical clearance was applied for, and was and granted by Stellenbosch University's human research ethics department (Humanoria).

No external funding was provided for this study, and so no conflicts of interest are introduced through funding relationships.

Since no research participants are involved in this study, the ethical risk of this study can be considered very low.

1.7. DELIMITATION OF THE STUDY

An historical overview of the development of thought in Algebra was included in the study, as it provides a valuable perspective on how algebraic thinking has developed. This developmental trajectory was drawn into relation with how algebraic thinking develops in learners at a school level.

This study only investigated algebraic thinking in young learners. The focus fell on learners between the ages of 6 years and 10 years old. Algebraic thinking in these ages can be referred to as early algebraic thinking.

The Malati project (Linchevski, Kutscher and Olivier, 1999) explained 2 strands of algebra, which includes making generalisations and using symbols to represent mathematical ideas. In this study, the focus fell on generalisations.

Lastly, the findings from this study will not be able to be generalised to the development of other areas of mathematics. It is not necessarily sure to have the same results or findings as this study is solely based on early algebraic concepts.

1.8. CHAPTERING

This thesis consist of six chapters as explained below:

Chapter 1: Introduction. States the purpose of, and motivation for, the study, and provides a study outline.

Chapter 2: The historical review of the development of algebra and the emergence of the big ideas of algebra in history. This chapter aims to explain the core ideas of early algebra and how they emerged in the history of algebra. This chapter greatly informed the instructional design described in Chapter 5.

Chapter 3: Early algebra, generalising and structure. This chapter aims to analyse early algebra, the development of early algebraic thinking, the fundamental components of early algebra and the role generalisation plays in the early years mathematics classroom. This analytical chapter greatly informed the instructional design which is described in Chapter 5.

Chapter 4: The methodology. This chapter aims to explain, justify and evaluate all the methodological choices made in the study.

Chapter 5: Model for implementing the instructional sequence. This chapter uses the findings formed from chapters 2 and 3 to provide a higher order construct in the form of an instructional design sequence for a possible implementation of generalisation in the mathematics classrooms for young learners.

Chapter 6: Conclusion. In this chapter an overview of the findings will be given to conclude the report. Ideas for further research in this field will also be included.

1.9. CONCLUSION

This study aims to explore and construct an instructional sequence to implement early algebra in the classroom with the aim of making algebra in the later grades accessible to more learners. The purpose of chapter 1 is to provide an introductory motivation for why the study is needed. The chapter starts by sketching the background for literature review. It follows by stating the problem which is taken out of the literature. From there a research question and sub-research questions with aims and objectives are explained. The methodology of the systematic literature review is briefly discussed. A detailed explanation and evaluation of the methodology can be seen in Chapter 4. Lastly, the chaptering for the study is set out.

CHAPTER 2: A HISTORICAL REVIEW OF THE DEVELOPMENT OF ALGEBRA AND THE EMERGENCE OF THE BIG IDEAS OF ALGEBRA IN HISTORY

2.1. INTRODUCTION

The history of algebra is a history of the gradual reification of activities and transformation of operations into objects (Kvasz, 2006, p. 291). By examining the development of algebra through history, one can construct an idea of the development of algebraic thought and reasoning through the ages. This trajectory of thought may provide valuable clues as to how algebraic thinking and the generalisation concept should be taught at a school level. Many lessons can be learned from how people from the earliest times have learned algebra.

Spencer (1861) in his work on moral, intellectual and physical education, stated:

> If there be an order in which the human race has mastered its various kinds of knowledge, there will arise in every child an aptitude to acquire these kinds of knowledge in the same order…. Education is a repetition of civilization in little (p.76)

Piaget (1989), who was an outspoken constructivist, agreed:

> The advances made in the course of history of scientific thought from one period to the next, do not, except in rare instances, follow each other in random fashion, but can be seriated, as in psychogenesis, in the form of sequential "stages" (p.28).

Sfard (1995) concurs that it is valid to expect that similarities will be observed when analysing and interpreting phylogeny and ontogeny of mathematical concepts. Due to the fundamental characteristics of knowledge, and to the essence of the relationship between its various levels, similar cyclical phenomena can be traced throughout the historical development of knowledge (Sfard, 1995).

For these reasons, the stages of the development of algebraic thinking in history should be explored and analysed to inform how algebraic thinking is taught today. For this study, these stages will inform the instructional design sequence in Chapter 5.

These "stages" can be drawn into relation with the levels of algebraic thought we see when learners are developing algebraic thinking. It is therefore necessary for this study to take an overview on how algebraic thinking developed in history with the aim of providing a possible instructional trajectory (Stephan, Underwood-Gregg and Yackel, 2016; Gravemeijer, 2020) which teachers should keep in mind in the classroom when teaching and learning of early algebra concepts, especially generalisation, take place. Freudenthal (1973) believes that instructional trajectories should be informed by the conceptual development of mathematical ideas in history. This study aims to use the Real Mathematics Education (RME) design principles and the heuristics of guided reinvention and emerging modelling to construct an instructional framework for the implementation of generalisation in the early algebra classroom (Stephan, Underwood-Gregg and Yackel, 2016; Gravemeijer, 2020). The development of algebra throughout history and its relationship with the fundamental components of algebra which emerge from a systematic literature review, are used as the basis for the design of an instructional sequence and ideas for the classroom setting.

By making reference to the history of algebra, teachers can increase learners' understanding of the material. Very few rectified, current textbooks use a formal, operational instructional design theory specific to mathematics to inform the design of textbooks (Stephan, Underwood-Gregg and Yackel, 2016). Textbooks often follows a logical learning trajectory which is very sterile. The textbooks rarely provide context and reasoning for why the subject matter at hand is important, why people in history were interested in the topic, and why the topic remains interesting today (Katz and Parshall, 2014). RME instructional design and guided reinvention are important in the reform of textbooks and instructional resources (Stephan, Underwood-Gregg and Yackel, 2016). Learners do not understand the usefulness of the subject by simply practicing incoherent examples which follow set out rules. History provides clear reasons for the importance of the subject matter. Knowing these reasons can aid learners in developing the perspective that algebra is a useful tool to solve real life problems. Learners thereby become more involved in the material and form a deep understanding thereof. Based on the ideas of Freudenthal (1973), it is proposed that knowledge of the stages of the historical development of algebra can help a teacher and learner see and understand how the need for generalising and making formalisations emerged as people through the ages engaged in mathematical problems and why we still implement it in all areas of mathematics.

When considering the history of the emergence of algebra, it is important to note the various definitions of algebra. Different understandings of what constitutes algebra result in different accounts of the origins of algebra. Some believe that early signs of algebraic thinking are indicated when an attempt is made to approach computational process in some or other general manner. The search for general solutions of equations was one of the central themes of algebra in history (Sfard, 1995). Others believe that it is necessary to implement algebraic methods to solve problems for the emergence of algebra to be noted. Operational symbolisation is claimed as one of the main features of algebraic thinking. Patterning and the recognition of pattern throughout mathematics, is the core of the development of mathematical reasoning (Papic and Mulligan, 2007; Cooper and Warren, 2008; Mulligan and Mitchelmore, 2009). One of the core goals of mathematics is to find the patterns and structure which emerge from relations and transformations. Abstracting patterns in the total of mathematics is fundamental to the development of structural knowledge and the goal of mathematics (Warren and Miller, 2010). From the literature it can be noticed that generalisation should be the aim of mathematics teaching in the early years and should run through all content areas. It should become a habit of mind of the teacher as well as the learners as they engage in various problems.

The evolution of algebra can be presented as a continuous attempt at transforming computational procedures into mathematical objects (Sfard, 1995). Therefore, algebra evolved as a constant attempt at generalising and making formalisations (see 3.3.1) (Kaput, 1999). An overview of the history of algebra will hopefully provide insights into the nature of developing algebraic thought and generalising mathematics.

In many accounts of the history of algebra, it is stated that algebra went through three predominant stages as it developed through the ages. These stages are the rhetorical, syncopated, and symbolic stages. The rhetorical stage refers to the stage where all statements and arguments were given in words. In the syncopated stage, some arguments are made by abbreviations when treating algebraic expressions. In the final stage, the symbolic stage, there is total symbolisation – all numbers, operations, relationships, and manipulations of symbols transpire under a governed set of well-understood procedures (Katz and Barton, 2007).Sfard (1995) also attempts to provide a brief overview of this history by classifying the development of algebra into three stages, however these differ from those in other literature. Sfard's first stage of the development of algebra is described as the antiquity to renaissance stage. In this stage the focus falls on the science of generalised numerical computations. Stage 2 focuses on Viète's invention where algebra is considered a science of universal computations. In Stage 3, the works from Galois to Bourbaki is studied. Here algebra becomes a science of abstract structures. In all of these stages, generalisation seems to be the underlying factor and goal of mathematics.

Katz and Barton (2007) argue that alongside the development of algebra through these stages of history, four conceptual stages of algebra emerges. These stages are the geometric, static equation solving, dynamic function and abstract stages. None of these stages described can stand independently. They are constantly overlapping with one another. The geometric stage refers to the stage where the bulk of algebra is geometric. The static-equation solving stage aims to find numbers fulfilling certain relationships. Motion is the main idea of the dynamic function stage. Finally, the abstract stage has structure as the main goal (*Mathematics for Teaching*, n.d.).

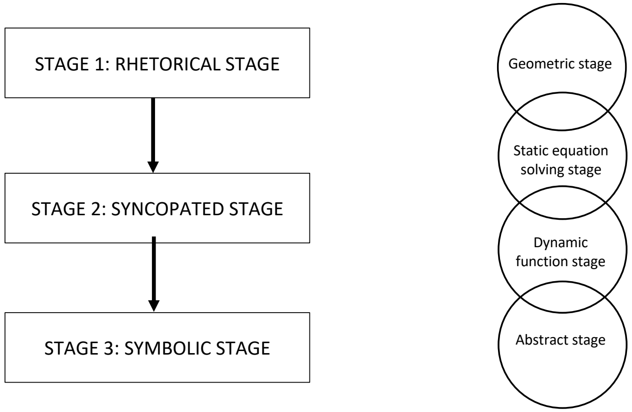


Figure 2.1. Stages in the history of algebra (Katz and Barton, 2007)

In this chapter, the focus will fall on a discussion of the three main stages of algebra through the history and how they are supported by and related to the four conceptual stages. The account of the development of algebra by Sfard (1995) will be considered and integrated into the main three stages of the emergence of algebra through the ages. In this overview of the emergence of algebraic thinking and generalisation, some of the main scholars in its history are explored and discussed. There are many more important scholars who influenced the emergence of algebra and algebraic thinking. Even so, the purpose of this chapter is to sketch an overview of the trajectory by focusing on the main events in the development of algebra to provide a clear view of the stages of development of algebraic thinking and therefore the ability to generalise. These stages of development and their relation to the fundamental components of early algebra will be used to inform the instructional design for the development of generalisation in young learners in Chapter 3.

2.2. EARLY ALGEBRA

2.2.1. Fundamental components of early algebra

In this section of Chapter 2, early algebra as a mathematical domain will be analysed and discussed with reference to its various fundamental components. This is important to the understanding of how these components of early algebra emerged from history. This emergence of the fundamental components of early algebra will inform the instructional design in Chapter 3.

2.2.1.1. Introduction

Algebra emphasises relationships between quantities and how these quantities change in relation to one another. Algebraic thinking involves grasping patterns, relations, and functions; representing and analysing mathematical situations and structures using algebraic symbols; and using mathematical models to represent and understand relationships (Friel, Rachlin and Doyle, 2001). Kaput, Carraher and Blanton (2008) further make a distinction between “algebra” and “algebraic reasoning”. They refer to “Algebra” as an independent body of knowledge, while “algebraic reasoning” is described as a human activity. This is an important distinction to take

note of when inquiring how algebraic reasoning is developed. Even so, when compared to the statement by Freundenthal which states that mathematics is a human activity where learners are provided with a chance to learn in mathematical activities and can extract mathematical ideas or create models of learners’ thinking (Gravemeijer and Terwel, 2000), “Algebra” can also be taken as a human activity in itself. Algebraic reasoning is an activity which consists of the generalisation of mathematical ideas, using symbolic representations of functional relationships (Blanton & Kaput, 2011). So, algebraic reasoning involves the generalisation of the whole of mathematics and communicating these generalities by means of symbolic representations in the form of models.

The publication *Algebra in the Early Grades* (Kaput et al., 2008, p. 77) takes the perspective that the core of algebraic reasoning is constructed by complex symbolisation processes that serve meaningful generalisations and reasoning with generalisations. Algebra’s broad, rich and natural relationship with naturally occurring human cognitive processes and reasoning is stressed (Kaput, Carraher and Blanton, 2008). In other words, algebra serves the purpose of finding generalities in situations, specifically mathematical situations. Kieran (1996) categorised algebra in relation to the activities in which learners engage: generational activities, transformational activities and global meta-level activities. Generational activities of algebra involve the making of expressions or equations that are objects of algebra. This involves activities where geometrical and numerical patterns are generalised in expressions (Kieran, 1996). Friel et al. (2001) elaborates on the fundamental components of algebraic thinking:

2.2.1.2. Understanding patterns, relations, and functions

Understanding algebra starts with understanding patterns and symbols at pre-school level (Lee, Collins and Melton, 2016). Roberts (2012) describes patterns as “regular structure of shapes or numbers which are created by repeating a rule. Young children begin to explore patterns in the world around them through experiences with things such as colour, size, shape, design, words, rhythms, movements and physical objects”(Roberts, 2012b, 2012c). They observe, describe, repeat, extend, compare and create patterns. They can predict what comes next and identify missing components of patterns. They learn to distinguish between different types of patterns (Friel, Rachlin and Doyle, 2001). Du Plessis (2018) agrees that this first experience of patterns through rhythm and rhyme relies on the repeated nature of pattern. This is because of the cyclical structure of patterns. The recognition of rhythm at a young age enables learners to identify and create their own patterns. Rhythm regularity is therefore clarified through learners' implementation of this regularity to express the generality in the sequence of events, far past the perceptual and long before algebraic relationships and structure is formed (Du Plessis, 2018, p. 2). In higher elementary grades, learners develop the representation of patterns numerically, graphically, verbally or symbolically. They learn to look for relationships between numerical and geometrical patterns and analyse how they change or grow. Learners can be encouraged to make generalisations through the use of tables, charts, physical objects and symbols (Friel, Rachlin and Doyle, 2001). As learners move through the grades, it is expected that these processes of generalisation become more and more sophisticated until their understanding expands to include functions of more than one variable, and they learn to do transformations such as composing and inverting commonly used functions (Friel, Rachlin and Doyle, 2001). When extracting patterns from situations, we are not only referring to numerical and geometrical patterns, which form the focus of the CAPS curriculum content strand, "Patterns, functions and algebra' (Department of Basic Education, 2011). Learners must be prompted to see the patterns and structure which underly all mathematical activities (Mulligan and Mitchelmore, 2009). Generalised arithmetic is a content strand which enjoys focus in Kaput's framework for early algebra (see 2.2.4). Generalising arithmetic involves generalising about the properties of numbers and operations and generalising about particular number properties and relationships (Roberts, 2012). When learners are able to generalise arithmetic, they can see patterns and structure within numbers, operations and their relationships.

2.2.1.3. Representing and analysing mathematical situations and structure using algebraic symbols

Young learners can represent mathematical ideas with objects or specific numbers. They use objects, pictures, words or symbols (rather than letters) to represent

mathematical relationships or properties(Ontario Ministry of Education and Training, 2007, p. 31). One of the reasons for this is that children are in the learning stages of reading and perceive letters as sounds rather than symbols (Ontario Ministry of Education and Training, 2007, p. 31). When young learners are prompted to describe and represent quantities in various manners, they learn to recognise equivalent representations and improve their ability to use symbols to communicate their ideas (Friel, Rachlin and Doyle, 2001). A variable is a symbol or a letter which represents an unknown value or quantity, or a generalised number property (Ontario Ministry of Education and Training, 2007, p 31).

Carpenter, Franke and Levi (2003) suggest that early algebra involves the generalisation of properties using every day or symbolic language systems. The symbolising develops as an effective linguistic form of expressions through interactions with mathematical situations. In Chinese elementary school, representation variables represent various numbers simultaneously, they have no number value, and are selected randomly. Variables are used in three distinct ways in Chinese elementary schools. Firstly, variables are implemented as place holders for unknowns, which can be a question mark or box. Secondly, they are used as pattern generalisers. In these cases, words are used rather than letters. The third use is that of representing relationships (Blanton & Kaput, 2011, p. 28).

The use of the variable in early grades can become explicit when children describe steps as "a number plus a number plus 1". When students are able to do this when describing geometrical patterns, Radford (2012) explains that the learners are then able to overcome the spatial meaning of the unknown.

Blanton and Kaput (2011, p. 71) argue that according to Vygotsky's learning theory, the child operates with a concept, to refine conceptual thinking, before he is clearly aware of the characteristics of these operations". This suggests that learning to reason mathematically or algebraically includes the successful understanding of notations which are within the zone of proximal development (ZPD) of the child. This involves a transition of a vague use of symbols to a deliberate use.

2.2.1.4. Using mathematical models to represent and understand quantitative relationships

According to NCTM's (2000, p158) algebra standards, "instructional programs from pre-K through grade 12 should enable all students to use mathematical models to present and understand quantitative relationships". In the early algebra years, concrete activities are recommended to develop this aspect of learners' algebraic thinking and reasoning from an early age (Lee, Collins and Melton, 2016). Young learners initially learn to use objects and pictures to represent stories or model situations. They are later able to informally use symbols. As learners move through the elementary grades, they realise that mathematics can be implemented to model numerical and geometrical patterns, scientific experiments, and other physical situations. Learners discover that mathematical models have the power to predict as well as describe situations. Contextualised problems can be modelled and solved using various representations, such as graphs, tables, and equations (Friel, Rachlin and Doyle, 2001). Ultimately, in high school, learners can use symbolic expressions to represent relationships in various contexts. By using models, learners can make inferences about relationships, formulate and test hypotheses, and draw conclusions about the situations being modelled (Friel, Rachlin and Doyle, 2001). Emergent modelling is a heuristic which will inform the design of the instructional trajectory in Chapter 3. It involves designing instructional activities which prompt learners engaged with models of informal mathematical activity to transition to models of more formal mathematical activity. As learners transition from informal to more formal, the instructional design aids learners' modelling by introducing new tools to communicate learners' reasoning (Stephan, Underwood-Gregg and Yackel, 2016).

2.2.1.5. Analysing change in various contexts

From an early age, learners recognise change in their environments and describe change in qualitative terms. Words such as taller, colder, darker or heavier are used to describe change. When learners are exposed to measuring and comparing quantities, they learn to also describe change quantitatively. During these processes, learners learn that some changes can be predicted, while others cannot. They learn that change can often be represented and described mathematically (Friel, Rachlin and Doyle, 2001). Throughout the early years, learners deepen their understanding of patterns and relationships by investigating patterns and the relationships between numbers and their properties; and how they change or stay the same. As a result of experiences of a variety of patterns using various materials, learners are able to identify, describe and extend progressively complex relationships. The complexity of patterns is based on the number of attributes that are changed in a pattern. This can include the colour, size or shape (Ontario Ministry of Education and Training, 2007, p. 10). By analysing the change in patterns, learners can generalise pattern rules and make predictions. Kaput's (Kaput, Carraher and Blanton, 2008) second content strand in his framework for early algebra involves generalising towards the idea of a function (see 2.2.4). This includes observing regularity in elementary patterns, ideas of change which include linearity, and the representation thereof in graphs, tables and function machines. It includes the investigation of the relationship between two variables which might be fixed or varying (Roberts, 2012).

2.2.1.6. Developing an algebra curriculum

Algebra reform is the focus of mathematical curriculum reform. The potential of early mathematics education teachers to develop algebraic reasoning might be the most important aspect of algebra reform and the reform of the mathematical curriculum in general (Carpenter et al., 2003). An algebra curriculum that effectively develops algebraic thinking must be coherent, focused and well-articulated. It cannot be a series of randomly selected lesson or activities but instead, should be well thought-out and connected. The aim of the trajectory of lessons should be developmental (Friel, Rachlin and Doyle, 2001). In South Africa, the potential of the use of patterns to develop an algebraic habit of mind remains largely unexplored. This can be ascribed to the absence of a relational approach to sequencing of patterning-type activities in the curriculum documents (Du Plessis, 2018, p.1). Mathematical ideas introduced in the early years must deepen and expand. The following instructions should build on that foundation. As learners move through a curriculum, they must continually be challenged to learn and apply progressively more sophisticated algebraic thinking to solve problems in a variety of contexts (Friel et al., 2001, p. 5). Comprehensive algebra instruction is necessary to develop deep algebraic reasoning. This refers to an approach that purposefully integrates early algebraic practices into the early years school curriculum across the different conceptual domains that are seen as important points of entry for algebra (Eric Knuth et al., 2014). The methods and concepts of algebra are a vital component of mathematical literacy in modern life, and the algebra strand of the curriculum is the core of the vision of mathematics education (Friel, Rachlin and Doyle, 2001).

2.2.2. The problem-centred approach for learning algebra

In this part of Chapter 2, I will focus on the learning of early algebra when following the problem-centred approach (PCA). The discussion will focus on the following aspects of the PCA: the theoretical basis, the role of the teacher, the classroom culture, the role of the learner and their reasoning, the kind of problems given and the mathematical structure of problems, and lastly, informing the larger community.

2.2.2.1. Theoretical basis for the Problem-Centred Approach

The problem-centred approach takes on the constructivist perspective that mathematics learning is a process in which learners reorganise their activity to resolve situations that they find problematic (Cobb *et al.*, 2014). A problem-centred approach to learning mathematics accepts the stance that learners are responsible for constructing their own knowledge. The classroom should provide opportunities where learners create individual and social procedures to monitor and improve their own constructions of knowledge (Murray, Olivier and Human, 1998). Social interaction forms a vital part of the PCA classroom. It creates chances for learners to talk about their own thinking and prompts them to reflect. The constructivist perspective on learning emphasises the importance of reflection and verbalising what one is doing. Through classroom social interactions, the teacher and learners construct a consensual domain where mathematical knowledge is to be shared (Murray, Olivier and Human, 1998).

PCA is based on the belief that subjective knowledge should be experienced by the learners as personal constructions and not re-constructed objective knowledge (Murray, Olivier and Human, 1998). In other words, the learner is not seen by the teacher as an empty vessel which should be filled with information and knowledge. Learners must internalise their own understanding and knowledge base.

Problem-solving can be described as a process where one would start with problems or tasks and after working on these problems, learners would be left with a residue of mathematical ideas and concepts (Murray, Olivier and Human, 1998). This process would result in learners having a firm understanding of the mathematical concept at hand.

Hiebert et al.,(1996, p.14) based their problem-centred approach on Dewey's principles of reflective inquiry. Dewey states that reflective inquiry is imperative to move past the distinction between knowing and doing, and provides a novel way of perceiving human behaviour. The fundamental principles of reflective inquiry include: (1) problems are identified, (2) problems are studies through active engagement, (3) conclusions are reached as problems are resolved. When learners solve problems, they should be given chances to corroborate existing knowledge and intuitions, make inventions, make sense, and assign meanings, and interact mathematically. These constructs together embody what it means to solve problems with flexibility and understanding (Biccard and Wessels, 2012).

I take on the perspective that the PCA, which is learning mathematics through problem solving, is an effective way to develop the generalisation concept in young learners. The implementation of this approach to learning early algebra or facilitating early algebraic thinking and generalisation, is used to inform the instructional design based on RME design principles of guided reinvention and emergent modelling (Stephan, Underwood-Gregg and Yackel, 2016).

2.2.2.2. The role of the teacher

The role of the teacher is to facilitate the problem-solving process in which learners are engaged, while not interfering with the learners' thought processes (Murray, Olivier and Human, 1998). Teachers should provide the learners with the necessary context and information regarding the problem they are engaging with. Teachers need to facilitate classroom discussion and prompt interaction among learners. Teachers should also introduce learners to, and facilitate the process of, communicating mathematical thinking and ideas on paper in a generally acceptable manner. This is done, for example, by introducing various uses of the equal sign as the need arises when solving problems. Teachers should also model the use of measuring instruments, calculators, and other mathematical instruments (Murray, Olivier and Human, 1998).

The primary tool for guiding learners' mathematical development is based on the meticulous design of a sequence of instructional activities. The teacher plays a foundational role in the guidance of learners' reinvention. Emphasis on social and socio-mathematical norms which characterise guided reinvention teaching, forms a big portion of the decision making of the teacher in terms of classroom interactions. The instructional design of the sequence of the activities serves the purpose of orchestrating whole-class discussions where certain mathematical practices have been pre-established (Stephan, Underwood-Gregg and Yackel, 2016).

When learners are engaged in focused building of logico-mathematical knowledge, teachers should not interfere and should provide learners with the chance to construct their own knowledge (Murray, Olivier and Human, 1998). The role of the teacher as facilitator is to initiate and guide mathematical negotiations, which is a highly complex activity. Teachers should highlight the conflicts between alternative interpretations and solutions, help learners to develop productive social collaborations, and facilitate mathematical dialogue between learners. Furthermore, teachers should be able to notice certain aspects of contributions in the light of their potential for further mathematical constructions. They should be able to redescribe learners' explanations in more sophisticated terms to guide the development of their understanding (Cobb *et al.*, 2014). From this it is concluded that the learner becomes the active participant in the learning process and his or her thinking develops as he or she engages in mathematical negotiations, interpretations and solutions, and mathematical dialogue between learners. The role of the teacher is to highlight and shed light on important concepts and possible misconceptions.

The role of the teacher in literature about PCA is completely different from that of a teacher in a traditional classroom setting. Instead of focusing on demonstrating, checking, and prescribing, the teacher focuses on setting appropriate problems, organising interaction between learners, and negotiating a style of learning and classroom culture with the learners.

2.2.2.3. The Classroom Culture

The classroom culture and quality of learners' interactions with each other will have a great influence on the mathematical learning that takes place in a classroom. The classroom culture in the PCA should enforce a safe space for consensual social interactions where knowledge between learners and the teacher can be shared (Murray, Olivier and Human, 1998).

Cobb and Yackel (1996) developed three constructs which explain the social dynamics in a mathematics classroom: social norms, socio-mathematical norms and classroom mathematical practices. Social norms indicate the expectations and obligations that the teacher and learners have towards one another during mathematical discussions (Cobb *et al.*, 2014). Four social norms which sustain the classroom culture include (1) explaining and justifying solutions and methods, (2) attempting to make sense of others' explanations, (3) indicating agreement or disagreement, and (4) asking clarifying questions when the need arises. Socio-mathematical norms are normative aspects of mathematical discussions which depend on the mathematical activity. It includes what is perceived as the criteria for an acceptable mathematical explanation. In guided reinvention classrooms, the focus of explanations and justifications should be on descriptions of actions on mathematical objects that are experientially realistic instead of procedural computations. Classroom mathematical practices are the "taken-as-shared" ways of reasoning and arguing mathematically that are content specific. Classroom mathematical processes evolve as discussions of problems, situations, representation and solutions become increasingly sophisticated (Cobb and Yackel, 1996).

The classroom should develop social norms which are imperative for the implementation of an inquiry-based mathematics classroom. One such social norm is for learners to be able to productively engage in small group work without having the teacher constantly monitoring, to ensure a successful collaborative learning environment. Learners should believe in themselves and persist to solve challenging problems, explain personal solutions to their partners, and listen to and make sense of the explanations of other group members. Social norms for whole classroom discussion involve explaining and justifying solutions, making sense of explanations of others, agreeing or disagreeing, and questioning other options in situations where a conflict between interpretations or solutions is clear (Cobb *et al.*, 2014).

A problem-centred classroom differs greatly from a traditional classroom set-up. Instead of a fixed organisation where the teacher is seen as the authoritative source of knowledge, learners work in an individual and co-operative manner to take responsibility for the construction of their own knowledge. The teacher acts as the facilitator or manager to guarantee learners are given sufficient opportunities to learn. The emphasis is placed on empowering learners to make sense of mathematics and deeply understand it, instead of imitating prescribed methods and solutions (Human and Olivier, 1999).

Opportunities to construct mathematical knowledge emerge when learners attempt to resolve conflicts, attempt to reconstruct and verbalise a mathematical idea or solution, and from attempts to create a consensual domain in which one can coordinate their mathematical thinking with that of others (Cobb *et al.*, 2014).

Mathematics classroom cultures should transform to focus on inquiry mathematics. The norms of inquiry mathematics are built on self-evident truths about a taken-to-be-shared mathematical reality by the teacher and students during their classroom interactions (Cobb *et al.*, 2014). A productive classroom culture is characterised by the teacher taking the perspective that the learner's mathematical actions and explanations are reasonable from his or her point of view even if it is not immediately apparent to the teacher what the learner wants to say (Cobb *et al.*, 2014). The teacher should have the perspective that there are learning opportunities in the mathematical reasoning of learners, even if only to make the class aware of certain misconceptions which might emerge from a specific learner's explanation of his or her reasoning.

When creating an effective classroom culture for the development of mathematical ideas, it is important that the teacher does not simply provide learners with a list of rules or social norms which they should follow. Teachers and learners should discuss what is expected from each of them, what their respective roles are, and what it means to do mathematics. By engaging in this process, learners start to view mathematics as an activity where they are expected to solve problematic situations by constructing personally meaningful and justifiable solutions as they contribute to an interactive constitution of an inquiry-based classroom (Cobb *et al.*, 2014).

2.2.2.4. The role of the learner

Learners' learning and thinking is greatly subjective to their beliefs about what mathematics is, about how mathematics is learned, about how mathematics is taught, and by beliefs about what they are capable of (Human and Olivier, 1999). The learner is an active roleplayer in the learning process as the construction of new conceptual knowledge is based on the interaction between new conceptual knowledge and the existing knowledge and ideas which a learner already possesses (Human and Olivier, 1999). So, learners' thinking about new ideas and concepts, and their interpretation thereof, are based on the learners' current knowledge.

Learners share the responsibility for creating a classroom which operates as a community of learners in which they participate. There are two important aspects to take note of when exploring the role of the learner in a PCA classroom. Firstly, learners must take responsibility for sharing the results of their inquiries and for explaining and justifying their thinking and solutions. They provide opportunity for an open culture which is necessary to improve methods and to become a full participant of the classroom community. Secondly, learners need to recognise that learning occurs best when learning occurs from others. Learners should take advantage of the ideas of other learners around them. This asks of learners to listen because of a genuine interest in what a classmate has to say (Hiebert *et al.*, 1996).

Many discussions in mathematics education led to the opportunity of 'shared authority', a type of authority that is non-localised with no definite separation between "the subject and agent of authority" (Solomon, Hough and Gough, 2021). Revised authority highlights the role of whole class discussions, where learners are active role players in shaping the public domain and their personal reflections and deliberations are important in building and developing a community of mathematical thinkers (Solomon, Hough and Gough, 2021).

Working in smaller groups in mathematics classrooms, especially in the early years of mathematics, is beneficial. Learners can easily interact socially with the learners around them, and the teacher can form a good idea of the understanding of mathematical concepts of all learners.

The way in which learners are grouped is often a highly contentious topic. It is important to group learners in various ways and that they are given chances to work with a wide variety of learners. However, there are some instances when it is beneficial to learners to be grouped in ability-matched groups. When the task is focused mainly on the construction of logico-mathematical knowledge, it would be best for learners to be grouped according to ability. When logico-mathematical knowledge is constructed, the learners' thinking should not be interfered with by more advanced ideas for which they are not yet ready. When physical and social knowledge is constructed, it is effective to co-operate with a variety of learners (Murray, Olivier and Human, 1998).

2.2.2.5. Problems and task design

All instructional activities, including arithmetical and numerical computations, should be designed to be potentially problematic to learners at a variety of conceptual levels (Cobb *et al.*, 2014). Problems provided to learners should include scenarios where the development of basic skills is addressed head-on by turning them into problematic situations. Learners should master the basic mathematical skills by seeing patterns and structure within the problems and their solutions (Murray, Olivier and Human, 1998). Conceptual and procedural development should be integrated as learners work through the problems (Cobb *et al.*, 2014).

The mathematical structure of the provided problems have an important part to play. The decision of which problems to use, should be based on comprehensive content analysis and a thorough understanding of how learners develop concepts and misconceptions. Some problems are posed for the purpose of creating the initial classroom culture, whereas other problems might be posed to introduce learners to a certain problem domain, in which case the reflection on the problem and its structure would be more important than the solution itself. Problems should actualise existing, but inexplicit knowledge and intuitions to create new inventions and to mathematical action (Murray, Olivier and Human, 1998). Therefore, teacher should focus on gradually introducing more sophisticated terminology and reasoning as learners become increasingly able to give meanings to these.

It is important to note that 'easier' problems should not be introduced first to develop concepts and skills more easily. This leads to limited constructions which contribute to the types of misconceptions which emerge when learners have limited exposure to a certain concept or experience it only through one type of problem. Problem types should be mixed and not blocked (Murray, Olivier and Human, 1998).

Time plays an necessary part in the development of early algebraic concepts. As a teacher, it is important to remember that at various time learners are at different stages of conceptual development and should not be pushed to operate on levels of abstraction which they are not capable of. Learners should be given enough time to reason with mathematical ideas effectively (Murray, Olivier and Human, 1998).

Access to appropriate notational systems is imperative for the effective development of early algebraic thinking. Appropriate notations should capture the methods learners use to solve problems. The ability to capture one's thoughts on paper is imperative for individual reflection and analysis. In the early years, learners are expected to record their thinking logically and clearly such that others would be able to follow their thinking (Murray, Olivier and Human, 1998). This can be facilitated by implementing the writing of notation and symbols which can be used to represent learners' thinking. The teacher can, for example, model the writing of "I added the 3 to the 6 and then I added another 3 to that answer" as 3 + 6 → 9 + 3. The arrow is a useful symbol to note whenever a learner uses the words "and then".

The PCA does not force formal methods on learners but legitimises and builds on the intuitive and prior knowledge learners possess. Learners are challenged to work with with realistic problems by using their common sense, and compare answers to reach an agreement (Human and Olivier, 1999).

To effectively implement an instructional design trajectory to develop early algebra through generalisation, a problem-centred approach needs to be followed and should inform the design of the sequence of activities as well as the classroom practices, which include the role of the teacher and learner.

2.2.3. Kaput's framework for early algebra

Kaput and Blanton (1999) refer to 'algebrafying' primary mathematics' where algebra is seen to encompass the whole of mathematics in the form of generalised arithmetic. Early algebraic thinking develops when arithmetic are saturated with algebraic meaning, with the aim of the algebraic character coming to the forefront (Knuth *et al.*, 2014).

Kaput (2008) argues that algebra is made up of specific thought practices and content strands. He proposes that algebraic thinking involves (a) "making and expressing generalisations in increasingly formal symbol systems" and (b) "reasoning with symbolic forms". These practices take place over three content strands (Kaput, 2008, p.11):

- Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic and quantitative reasoning (Kaput, 2008, p.11).
- Algebra as the study of functions, relations and joint variation (Kaput, 2008, p.11).
- Algebra as the application of a cluster of modelling languages both inside and outside of mathematics (Kaput, 2008, p.11).

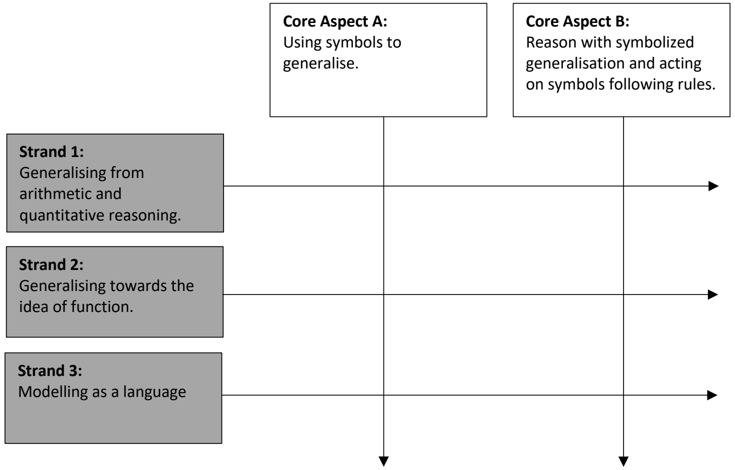


Figure 2.2. Kaput's core aspects and strands of algebra (Roberts, 2012, p.304)

The two core aspects which are important in algebraic thinking are symbolisation activities which serve purposeful generalisation and at the same time reason with symbolised generalisations. The first two strands in this framework for early algebra takes into account the two types of generalising that are central to algebraic thinking: generalising arithmetic and quantitative reasoning; and generalising towards the idea of a function (Kaput, 2008). The third strand focuses on modelling as a process where situations are interpretated using algebraic notation as a language. Kaput (2008) identifies 'algebrafying the arithmetic problem' as an aspect of the modelling strand, where constraints of a particular problem are liberated to explore its more general form (Roberts, 2012). These components of early algebra can be noticed as they emerge in the development of algebra throughout the history of mathematics. This emergence will inform the instructional design which will be constructed in Chapter 3, where a more in-depth discussion of Kaput's framework for early algebra is discussed.

2.2.4. The big ideas of early algebra

The central theory of early algebra education is that it will improve learners' "understanding of algebraic concepts" that will support their conceptual development in largely arithmetic based classrooms in the early mathematics education (Knuth et al., 2014). It will also increase their likelihood of success in later grades when engaging in more complex mathematics (Knuth et al., 2014).

Knuth et al. (2014) identified *five big ideas* out of Kaput's (2008) framework for early algebra on which much of the research on early algebra has developed. These big ideas offer possibilities for the development of deep algebraic reasoning, and the practices of "generalising, representing, justifying and reasoning with mathematical relationships", and include (Knuth *et al.*, 2014, p.43):

1) Equivalence, expressions, equations and inequalities

This big idea involves building a "relational understanding of the equal sign, representing and reasoning with expressions and equations in their symbolic form and describing relationships between generalised quantities". Activities in the early years can include:

- Interpreting equations written in different formats (e.g., other than a + b = c) and evaluating as true or false
- Solving open number sentences (e.g., 8 + 5 = ___ + 4), including by reasoning from the structural relationship in the equation
- Using variable expressions to model linear problem situations
- Identifying the meaning of a variable used to represent an unknown quantity
- Interpreting an algebraic expression in the context of a problem
- Modelling problem situations to produce linear equations of the form x + a = b
- Analysing an equation to determine the value of a variable (Knuth *et al.*, 2014, p.43).

2) Generalised arithmetic

This idea involves generalising "arithmetic relationships, including the fundamental properties of number and operation", and thinking about the structure of arithmetic expressions instead of their computational value. Activities in the early years can include:

- Analysing information to conjecture an arithmetic relationship
- Expressing the conjecture in words and/or variables
- Identifying values or domains of values for which a conjectured generalisation is true
- Describing the meaning of a repeated variable or different variables in the same equation

- Identifying a generalisation in use (e.g., in computational work)
- Justifying an arithmetic generalisation using either empirical arguments or representation-based arguments
- Examining limitations of empirical arguments(Knuth *et al.*, 2014, p.43).

3) Functional thinking

This idea involves generalising "relationships between covarying quantities and representing and reasoning with those relationships through natural language, algebraic notation, tables and graphs". Activities in the early years can include:

- Generating linear data and organising data in a function table
- Identifying the meaning of a variable used to represent a varying quantity
- Identifying a recursive pattern and describing it in words; using the pattern to predict near data
- Identifying a covariational relationship and describing it in words
- Identifying a function rule and describing it in words and variables
- Using a function rule to predict far function values
- Given a value of the dependent variable, determining the value of the independent variable (reversibility)
- Constructing a coordinate graph (Knuth *et al.*, 2014, p.43).

4) Variables

Variables refer to symbolic notation as a language for representing mathematical ideas in concise ways to include the different roles variables play in mathematical contexts. Activities in the early years may include:

- Using variables to represent arithmetic generalisations
- Examining the meaning of a repeated variable or different variables in an equation or rule
- Using variables to represent an unknown quantity (fixed or varying)
- Understanding that a variable represents the measure or amount of an object rather than the object itself
- Interpreting the meaning of a variable within a problem context
- Using variables to represent linear problem situations
- Describing a function rule using variables (Knuth *et al.*, 2014, p.43).

5) Proportional reasoning

Proportional reasoning is "opportunities to reason algebraically about two generalised quantities that are related in such a way that the ratio of one quantity to the other is invariant" (Knuth *et al.*, 2014, p.43).

These *five big ideas* should not be viewed as a mutually exclusive road to algebraic thinking. They should be made available to learners in an integrated and connected manner. These *five big ideas* can be seen as imperative to understanding algebra as they give rich context within which algebraic thinking can take place (Knuth *et al.*, 2014).

As an analysis of the stages of algebra development in history is undertaken, it can be noticed how these fundamental components and big ideas of early algebra emerge. This provides a valuable basis for the design of an instructional sequence based on the levels of algebraic thinking which emerge from the stages of algebra development in history.

2.3 The stages in the history of algebra

2.3.1. The rhetorical stage

2.3.1.1. The Egyptians and Mesopotamians

Different opinions about the origin of algebra are presented in the literature. Some believe that algebra originates from Greek mathematics and others state that the origins lies as far back as the Mesopotamians (Tabak, 2011). The Mesopotamians and Egyptians (2 000 BC) solved various mathematical problems and kept their solutions well-documented. The beginnings of algebra are apparent in the solutions of problems that can be seen in preserved texts of these ancient civilisations (Katz and Parshall, 2014). The challenge in knowing whether they understood algebra lies in the fact that they solved problems in a much different way than it would be done today.

Mathematics in Egypt developed in two major contexts: architecture and government (Katz and Parshall, 2014). In architecture, scale models were drawn by builders to indicate that mathematical techniques and strategies, specifically proportionality, were used in design and construction. In government this included “the collection and distribution of goods, the calculation of the calendar, the levying of taxes and the payment of wages. Evidence for this appears in various papyrus documents” from the period (Katz and Parshall, 2014).

Two important mathematical concepts can be noted from the ancient texts and mathematical papyri dating from Mesopotamian times. This includes the display of arithmetic techniques as well as proportionality (Katz and Parshall, 2014). Proportionality and proportional reasoning play vital roles in the development of

algebraic thinking and generalisation. Proportionality refers to the mathematical structure that models the relationship within situation where two quantities, x and y, change together in such a manner that the rate between the quantities remains constant (Breit-Goodwin, 2015). This idea also underlies the problems introduced in early years classrooms where learners are expected to find the unknown quantity. Therefore, the fact that proportionality can be seen in these early Mesopotamian problems, the simplest and earliest algebra in history is noted (see 2.2.5) (Katz and Parshall, 2014).

Let’s explore an example of the proportionality type problems which were solved by the Mesopotamians (Katz and Parshall, 2014). Below is Problem 75 from the *Rhind Papyrus* named after the Scotsman Alexander H. Rhind (1833-1863) who purchased the Rhind at Luxor in 1858.

> “What is the number of loaves of *pesu* 30 that can be made from the same amount of flour as 155 loaves of *pesu* 20?”
>
> (Katz and Parshall, 2014)

Pesu is an Egyptian measure which can be communicated as the ratio of the number of loaves to the number of *hekats* of grain, where *hekat* is a dry measure approximately equal to $\frac{1}{8}$ of a bushel. The problem is solved by the proportion $x : 30 = 155 : 20$. The scribe solved the problem by first dividing 155 by 20 and then multiplying the result by 30 to get $232\frac{1}{2}$. This problem is an example of the practical use of proportionality to solve problems, but many problems in papyri were more abstract. Even though signs of thinking algebraically emerged in the proportional thinking displayed by Egyptians, little claims of generalisation can be noticed (see 2.2.5).

The Mesopotamian and Egyptian civilisations are roughly the same age and can therefore be grouped together. Mesopotamians showed little interest in generalisations. Their methods of problem solution indicate very little understanding of general theory or equations. Along with that, no algebraic language can be detected in their solutions. The Mesopotamian’s development in mathematics seems to stem from the study of individual problems (Tabak, 2011, p. 3). Mesopotamian mathematics is often named proto-algebra, arithmetic algebra, or numeric algebra. Their work was an important initial step towards the evolution of algebra as we know it today (Tabak, 2011).

The Mesopotamians started with simple arithmetic and advanced to more complex problems (Tabak, 2011). In the same way, young learners start their mathematics journey by working with numbers and engaging in arithmetic. To develop a generalisation habit of mind, a constant seeking for generalisations in arithmetic is imperative in the early years classroom. Learners should be engaged with experiences where they generalise about properties of numbers and operations, and generalise about particular number properties and relationships (see 2.2.4 and 2.2.5) (Roberts, 2012).

Mesopotamian mathematics has two roots, namely accountancy problems and "cut-and-paste" geometry (Katz and Barton, 2007). The accountancy problems were an necessary part of the governmental system of the earliest Mesopotamian dynasties. The "cut-and-paste" geometry was likely used by surveyors as a way to figure out how to understand the division of land. To compare areas of fields, for example, surveyors evidently thought of them as divided into squares and rectangles that they could mentally rearrange. Out of this practice emerged a form of geometrical algebra. This geometrical algebra was a strategy for manipulating areas (cut-and-paste) to determine unknown lengths and widths (Katz and Parshall, 2014). Babylonian algebra developed out of the "cut-and-paste" geometry theories. Old-Babylonian clay tablets show lists of quadratic problems. The aim of these lists was to discover geometric quantities such as the length and width of a rectangle (Katz and Barton, 2007).

The development of geometrical algebra started with the manipulation two-dimensional areas. The methods involve the solving of quadratic problems. Mesopotamian scribes could also deal with linear equations. Like the Egyptians, these problems were solved using proportionality (Katz and Parshall, 2014).

The Mesopotamians did not use modern formulas of geometry, but rather words which described a procedure or algorithm. These algorithms did not contain equal signs or any other conventional notations. They consisted of sentences and some numbers (Derbyshire, 2006). That This lack of formal notation contributed to their difficulty in solving these problems(Tabak, 2011). The lack of generality in their solutions to problems further increased the challenge. The need for some form of generalisation and algebraic thinking becomes clear when considering the work of the Mesopotamians (see 2.2.4 and 2.2.5). Some literature shows that Mesopotamian algebra consisted of more than just a series of blind, trial and error computations to solve problems. The Mesopotamians had to memorise a small quantity of identities, and solving mathematical problems consisted of changing each problem into a standard form and calculating the solution (Sesiano, 2009).

The solving of these geometrical problems was the beginning of algebra, in that it was the first time numerical problems were solved by manipulating original data using set rules (Katz and Barton, 2007). The work of the Egyptians and Mesopotamians can be seen as the beginning of the development of algebraic thinking even though little generalisation was applied. The challenges they evidently experienced in solving discrete problems without the use of symbols or formalisations clearly highlights the value of generalisation.

2.3.1.2. Chinese algebra

The first records of Chinese mathematics can be found during the period of the Han Dynasty, which existed approximately 2000 years later than the Mesopotamian and the Egyptian civilisations. An important record of Chinese mathematics is one of the earliest Chinese mathematical texts called *Nine Chapters on the Mathematical Art* (100 B.C.E – 50 C.E.) (Tabak, 2011, p. 12). The text consists of various problems on taxation, surveying, engineering, and geometry and also solutions and methods. The tone of the text can be described as conversational, which is consistent with the rhetorical stage of the development of algebra (Tabak, 2011). Again, mathematics is approached as a tool for solving various important, real-life problems.

The problem, solution, and the algorithm employed to solve the problem, are communicated only in words and numbers: no use of symbolization can be found in these texts. A clear need for algebraic notation emerges when reading problems such as these found in the Nine Chapters. Extensive writing and vocabulary are needed to explain simple problems if algebraic notation and symbolisation are not used (Tabak, 2011).

Many similarities can be drawn between the works of the Mesopotamian and Chinese mathematicians. It is clear that the aim of these works was not to find general theories or prove that their algorithms worked, but rather was limited to solving various problems (Tabak, 2011). The generality of mathematics was lacking, and therefore, mathematics was perceived to be extremely challenging. Greek mathematics was fundamentally different to that of the Mesopotamians or the Chinese. Greek mathematicians were interested in the nature of number and form (Tabak, 2011). Greek mathematicians started to explore patterns and structure within mathematics and were able to start making generalisations. In the same way, learners' reasoning develops as they move from viewing mathematics as discrete objects and concepts to a more general perspective. And so, the syncopated stage emerged.

2.3.1.3. Critical remark for the teaching of algebra

The rhetorical stage refers to the stage in history where the focus was on solving individual problems with no attempt to generalise. Problems were not grouped or categorized but rather seen in isolation (Katz and Barton, 2007; Tabak, 2011; Katz and Parshall, 2014). Solutions to problems were presented mainly in words where little to no generality can be noted. When problems are presented individually, we notice that little generalisation takes place. In this stage, the reasoning was still answer-orientated, and mathematicians could not relate various mathematical problems and concepts to each other.

This stage is often reflected in school textbooks and classrooms when it comes to teaching algebra. The first step to approaching early algebra type activities, for example, describing patterns, is to express the observed pattern or relationship in words. From learners' observations, certain key aspects can be highlighted to show the mathematical structures observed. This will make it easier for learners to try and

represent patterns and relations in various diagrams (the syncopated stage) and then later using symbols (the symbolic stage). Even so, in the traditional first stage of algebraic development, learners do not make generalisations. The focus often remains on copying or extending patterns with little focus on conjecturing rules (du Plessis, 2018). In other areas of mathematics, the reasoning in this stage is answer-orientated and concepts are perceived discretely instead of in a related manner (Blanton and Kaput, 2011). Generalisation in the rhetorical stage can be achieved by teachers prompting learners to express generalities which they notice in working with mathematics and communicating it informally in their own words.

There are of course learners whose thinking may differ, and who can or want to immediately represent what they observe in a diagram or using symbols. The role of the teacher should be to facilitate a deep understanding of the mathematical processes with which the learner is involved by asking relevant questions. For most learners the rhetorical stage is a necessary one in their development of algebraic thinking. Before learners can be introduced to symbols or abstract mathematics, it is important for the educator to make sure that they have a firm and deep understanding of the mathematics they are engaging with (*History of algebra as framework for teaching it?*, n.d.).

Communication is seen as one of the important mathematical processes. Many learners struggle to communicate their thinking effectively (Ontario Ministry of Education and Training, 2007). When engaged in the rhetorical stage, learners are encouraged to **speak** about their own thinking in their own language. This is not only important when developing algebraic thinking but in all mathematical thinking. Before learners can be expected to write number sentences using the symbols for the operations, learners will **say** what they are doing, for example: "I put the three and the five together. I know I will then have eight in total." The aim of teaching algebra should be to get learners to start making formalisations and generalisations (Kaput, 1995) and in the rhetorical stage they will express these in their own words. Learners can, for example, say: "If I put two numbers together, I know my answer will be bigger than the numbers I started with." From there learners can be encouraged to present their ideas in pictures or diagrams (syncopated stage) and much later learners should be expected to be able to write the number sentence, 3 + 5 = 8 (symbolic stage). Generalising arithmetic (numerical patterns, properties and so forth) begins within a mathematical system, which includes integers and “their properties and operations, where the understanding of the mathematical structures plays the core constraining role” (Kaput, 1995). Another important aspect of generalising is quantitative reasoning which is based in mathematising situations, and this offers a different foundation for generalising and formalising (Kaput, 1995).

2.3.2. The syncopated stage

2.3.2.1. Greek mathematics

The mathematics that began to develop in Ancient Greece around 600 B.C.E. was quite different from the mathematics explored by the Egyptians and Mesopotamians. The Greeks developed a basic political organisation of city-states with governmental units that governed societies of a few thousand individuals. These governments were all controlled by law and adopted a culture of argument and debate (Katz and Parshall, 2014). This context is important to take note of when exploring the history of Greek mathematics. It is probably from this culture of argument and debate that the ethos and necessity of proving reasoning in mathematics stemmed. This need to prove mathematical reasoning led to a need for generalising. Justifying and explaining mathematical reasoning is an important classroom practice in the guided reinvention and problem-centred approach (see 2.2 and 2.3) (Cobb and Yackel, 1996).

When looking at Greek mathematics, one is struck by the emphasis on the large algebraic parts as well as other parts where algebra hides under a geometric cover. From the reports on integrals which were calculated by Archimedes and reports from the numerical astronomy. Greeks must have been in possession of powerful algebraic tools (Freudenthal, 1977). Two different sides of algebra can be seen in the mathematics of Ancient Greece (Sesiano, 2009): archaic algebra which is mostly used in schools now and resembles that seen in Mesopotamian times, and Diophantine algebra, which will be examined in more detail later in the chapter. Even though the problems in archaic Greek algebra are more advanced than those found in

Mesopotamian times, the fundamental form remains the same, being that one finds sequences of unjustified calculations, and only the correctness of the answer suggests an underlying method (Sesiano, 2009). The emphasis is on the answer rather than the solution and little generalisation can be noticed. This corresponds with the historically accepted approach to algebra teaching in schools where the focus is on solving equations rather generalising the structure of mathematics (Smith and Thomson, 2008). This traditional approach has proven to be unsuccessful.

Diophantine algebra is traditionally intended for higher mathematical education. In this algebra the focus shifts to a designated unknown, symbolism, and an explanation and motivation for the given solution method (Sesiano, 2009). The focus here becomes the general structure of mathematics. The belief is that a generalising perspective on the underlying structure of mathematics should be embedded throughout mathematics by requiring that learners provide generalisable explanations and motivations for their thinking from the start of schooling (Roberts, 2012).

Diophantine Greek mathematicians were not interested in mathematics for the sake of solving problems, but in and of itself. They showed interest in questions about number and form. Greek mathematicians started to explore the structure of mathematics. The potential for generalisation emerged as the focus fell on the structure within computations rather than the answer (Kaput, 1999). Ideas were expressed in terms of numbers, points, curves, planes and geometric solids (Tabak, 2011). The Greeks were also the first to make a distinction between exact and approximate results. The Greeks’ fascination with precision influenced the **manner** in which they investigated mathematics as well as **the content** they investigated. It was this focus on precision which lead to the discovery of the Pythagorean theorem (Tabak, 2011, p. 20).

In Greek mathematics, geometry can be seen as the focus and aim of mathematical reasoning. Greek mathematicians wished to combine the two main components (algebra and geometry) of ancient mathematics. Geometric algebra involved analysing and understanding quantities expressed with letters as lengths of line segments. It also consisted of operations performed on these quantities. The reason for this unification of the two mathematical components, is said to be due to geometry's mathematical rigor and consistency (Sfard, 1995). The Greek mathematicians had difficulty solving many geometrical problems. This may be due to the lack of the science of algebra and a generational approach to solving problems (Sfard, 1995). A central figure in the climate of rational debate was Aristotle (384-322 B.C.E.). He codified the principles of logical argument and firmly believed that the development of any domain of knowledge must begin with the definition of terms and with the statement of axioms (Sfard, 1995). His requirement that arguments be proven indicates the need for making generalisations and formalisations. We will now look at other prominent figures who influenced the development of algebra in Ancient Greece.

2.3.2.2. The work of Pythagoras

Pythagoras (572-497 B.C.E.) was an influential Greek mathematician even though no specific discoveries can be attributed to him (Tabak, 2011). Pythagoras made strides in the development of mathematics and greatly influenced the work of Diophantus (Sfard, 1995).

Pythagoras constructed a community in Cortona where he lived with his disciples. Pythagorean exchanged ideas freely and did not take individual credit for any of their work. At the core of the Pythagorean philosophy was the saying: “All is number.” The meaning of this saying is exemplified in the Pythagorean’s understanding of music. The Pythagoreans used an instrument called a monochord to investigate the tones in music , and found that musical tones made by strings can be represented by whole number ratios, which could be used to describe music (Tabak, 2011).

The Pythagoreans were fascinated by numbers. They believed that certain numbers were infused with special, specific properties. For example, 4 was believed to be the number of justice and retribution. Number 1 was the number of reason (Tabak, 2011). It is necessary to note that the Pythagoreans only worked with positive, whole numbers. They did not recognise the number 0, negatives or any fractions as a number (Tabak, 2011).

This Pythagorean reasoning that “all is number” greatly influenced Greek mathematics. to the point where most of the work done before Diophantus, was geometrical. The school of Pythagoras founded their understanding of all mathematics, as well as astronomy and music, on numbers. When irrational numbers were discovered, the Pythagoreans were so disturbed that they turned away from arithmetic. Arithmetic contained numbers which could not be written and geometry with the representation of line segments (Derbyshire, 2006).

The discovery that the idea, “All is number”, was wrong, was one of the most influential discoveries in the history of mathematics. The idea of incommensurability was discovered (Tabak, 2011, p.23). The term incommensurability means ‘to have no common measure’ (Oberheim and Hoyningen-Huene, 2018).

The Greeks easily accepted the proof of incommensurability which shows the early levels of abstract thinking of the Greek mathematicians. They were prepared to accept a mathematical idea which deconstructed their worldviews, if it was a logical result of other, previously accepted mathematical ideas (Tabak, 2011).

Even though the Pythagoreans’ work does not speak of great algebraic discoveries, the way in which they thought about numbers in terms of generalised properties laid the foundation for generalised arithmetic, which focuses on the generalisation of properties of numbers (Mitchelmore, 2002). Furthermore, the Greeks were open to reconstructing their mathematical knowledge based on logical reasoning of previously established mathematical ideas. They, therefore, saw the necessity of having to prove your arguments and reasoning.

2.3.2.3. Geometrical algebra in Euclid’s elements and data

The main source of Greek mathematical ideas can be found in a set of books titled *The Elements* by Euclid of Alexandria (300 B.C.E) (Tabak, 2011). Euclid received his mathematical training in Athens before he settled in Egypt (Derbyshire, 2006). He worked in Alexandria which was the capital of Egypt under the dynasty of the Ptolemies (Katz and Parshall, 2014).

The work Euclid did in the set of books, *The Elements,* was largely based on the principles of Aristotle. His text was a prototype for how mathematics should be expressed and represented. This model included definitions, axioms and logical proofs and still stands today (Katz and Parshall, 2014).

The Elements was written in 13 short books (Tabak, 2011). It contains the elementary consequences of plane geometry which is still part of the secondary mathematics curriculum globally today. It also contains material on elementary number theory and solid geometry. If algebra can be noted in the books is a widely contested subject. The debate centres around the meaning and goal of Book II.

Book II deals with various relationships between rectangles and squares and has no apparent goal. Its propositions are seldomly used elsewhere in *The Elements.* Some explanations state that 'geometrical algebra' can be noted in some propositions from Book I and Book IV. Some representations of algebraic concepts through geometrical figures can be noted (Katz and Parshall, 2014). This includes representing the relationships between the sides of a rectangle as an equation. In doing work that focuses on the relationships between the side of geometrical objects, Euclid must have been engaged in some sort of quantitative reasoning. It can be argued that dealing directly with quantities (the length of the side of a rectangle) and the relationships within them, helps learners to construct an initial comprehension of the concepts of function and functional thinking (Ellis, 2011). This type of quantitative reasoning lies at the core of the growth of algebraic thinking and generalisation (Smith and Thomson, 2008).

Book II lays out the foundation for geometric algebra where geometric thinking was prominent in all Greek mathematics including algebra. The common perception of unknowns like x, y and z are that these variables represent numbers. Euclid had a different approach. Euclid represented unknowns as line segments (Tabak, 2011). In Book II established the rules that permit you to manipulate line segments in the same way that you would manipulate numbers. What we today would represent as equations, Euclid represented using drawings of rectangles, squares, and other geometric forms (Tabak, 2011). He made geometric algebra **visible.** Even though Euclid did not represent the relationships using formal algebra notation as we know it today, he showed the quantitative relationships between geometrical objects through drawings. Quantitative reasoning has the aim of supporting reasoning that is supple and general in character but which does not only count on symbolic expressions (Smith and Thomson, 2008) Carraher and Schlieman (2014) note that quantitative reasoning, along with various manners of representation, can enhance the emergence of algebraic thinking in young learners. When learners in the classroom start to reason quantitatively, they will often start to represent their thinking using informal geometrical drawings as representations. From there, learners will be able to start representing relationships using formal notation. This same trajectory emerges in the historical development of mathematics.

In the work of Euclid and Apollonius, algebraic notions is seen. Numerous propositions show how to directly manipulate rectangles and squares. Some propositions involve Euclid solving algebraic problems for geometric results. Euclid solves these problems by manipulating geometric figures. Development is seen when the manipulations are based on clearly stated axioms (Katz and Barton, 2007).

One of the first algebraic concepts introduced to learners in the classroom is that "multiplication distributes over addition". This idea is called the distributive law: $x(y+z) = xy + xz$ (Tabak, 2011, p.29). The very first proposition that Euclid proves in Book II of *The Elements* is exactly this statement.

> **Proposition: "**If there be two straight lines, and one of them be cut in any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments."
>
> (Tabak, 2011, p.29)

The majority of literature claims that in Book II, Euclid merely aimed to show a fairly logical framework of geometric knowledge that could be applied to other proofs of geometric theorems. Euclid was without a doubt thinking geometrically. This is clear in his initial definition: "Any rectangle is said to be contained by two straight lines forming the right angle." (Katz and Parshall, 2014, p. 35). Consider for example the following:

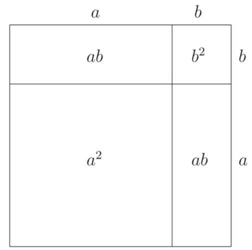


Figure 2.3. Solution to Euclid's proposition (Katz and Parshall, 2014, p. 36)

For Euclid, geometrical results were demonstrated by drawing and comparing the relevant squares and numbers. In Book II, Euclid proves a result concerning "invisible" figures, which are figures indicated in the theorem with respect to an initial line and its segments, by using "visible" figures, which are the actual squares and rectangles which are drawn. By giving lengths a and b to the two segments of line, the results can be translated into a binomial: $(a+b)^2 = a^2 + b^2 + 2ab$ (Katz and Parshall, 2014). This can be taken as an algebraic result.

Euclid only dealt with geometric figures and never really composed rules for them. His formulations of problems, with regards to finding two lengths and satisfying specific conditions, were almost identical to Mesopotamian formulations. He was, however, able to generalise the Mesopotamian problems from rectangles to parallelograms (Katz and Parshall, 2014). Euclid's work greatly influence Islamic mathematics, especially the work of Al-Khwarizmi (Sesiano, 2009), which will be explored in more detail later in this chapter.

2.3.2.4. Diophantus

Diophantus (A.D. 250) was a Greek mathematician who can be described as the father of algebra, as his main focus was the study of algebra separately from the study of geometry (Tabak, 2011). Diophantus is well-known for his work in number theory and algebraic geometry (Sesiano, 2009). In light of Diophantus' work and particularly his knowledge of solving quadratic equations, it is clear that here the development of algebra begins moving towards the equation solving stage (Katz and Barton, 2007). The work of Diophantus started to include some symbols in the form of Greek letters and thus represented syncopated algebra (Sfard, 1995).

The work of Diophantus mainly consisted of two important works: *Arithmetica* (the more famous of the two) and *On Polygonal numbers.* The work *Arithmetica* is quite similar to the Chinese *Nine Chapters*. The main difference is that at the beginning of the work, Diophantus attempts to provide a foundation for algebra. This is a critical moment in the historical development of algebra, as it is the first time that this was attempted (Tabak, 2011).

The remaining part of *Arithmetica* comprises 189 problems in which the goal is to find numbers and families of numbers which satisfy specific circumstances. The focus on generalised arithmetic (see 2.2.4 and 2.2.5) once again emerges from the work of Diophantus and emphasises that the development of algebraic thinking does not start with simplifying equations but rather with a focus on the generality that is evident throughout all areas of mathematics (Roberts, 2012). At the beginning of the work, Diophantus outlines his use of symbolism and methods. From a modern perspective the symbolism may seem simple and primitive, but it was advanced for the time (Derbyshire, 2006). Diophantus used the Greek alphabetic system for writing numbers. Using symbols, even if simple and primitive, show that using variables to model situations in algebraic language emerged here (see 2.2.4 and 2.2.5). This can be seen as a significant point in history in terms of the development of the use of symbols in algebra.

The 27 symbols Diophantus used in his system of writing numbers consisted of the 24 letters of the Greek alphabet, plus three obsolete letters. The first 9 symbols represented the numbers 1-9, the next 9 symbols represented 10-90, and the last 9 symbols represent 100-900. As with other systems of the time, there was no symbol for zero (Derbyshire, 2006).

The following is an example of the work of Diophantus named "Arithmetica" (Sfard, 1995, p.19):

> *To find two numbers such their sum and product are given numbers:*
>
> *"Given sum 20, given product 96. $2x$ the difference of the required. Therefore, the numbers are 10 + x, 10 – x. Hence, 100 - x^2 is 96. Therefore, x is 2 and the required numbers are 12, 8.*
>
> (Sfard, 1995, p.19)

Note that x as a symbol has been used in the example for the sake of clarity. Diophantus used Greek letters as symbols.

It is clear from this example from "Arithmetica", that Diophantus aimed to generalise about specific number properties and relationships. This is the second element of generalisation in early mathematics as explained by Roberts (2010, p. 169).

Two types of analysis and synthesis have evolved in the solving of problems (Katz and Parshall, 2014). One type is characterised in the work of Euclid's *Elements* and the second type can be seen when exploring Diophantus' *Arithmetica.* The characteristic of solving geometric problems is seen in Euclid's work and involves the demonstration of a proposition by first, in the analysis stage, assuming true what is to be proved and reducing that to an identity or other know proposition. In the synthesis stage, the process in reversed. In *Arithmetica* analysis and synthesis provides the problem first in the analysis phase and, supposing the problem to be solved, establishes some relations between the known and unknown quantities. This is then reduced to some relation in terms of the smallest number of unknowns possible. In the synthesis phase, the solution found is checked (Katz and Parshall, 2014).

Diophantus presented conditions for solving problems in the beginning of the work *Arithmetica.* The organisation of each problem is as follows:

1) First, one finds the **statement,** specifying which are the give quantities and which are to be determined.
2) Next, where needed, the **condition** that the given quantities must satisfy for the solution to be rational.
3) The given quantities are then **set** in accordance with the conditions stated.

4) The **solution** then follows by expressing the quantities to be determined in terms of the unknown, after which the problem is solved (Sesiano, 2009, p. 33).

Diophantus had fairly sophisticated algebraic notation at his disposal (Derbyshire, 2006). It is believed that Diophantus was the first to initiate symbols for "unknown quantities, abbreviations for powers of numbers, relationships and operations as used in the syncopated stage" (Nethravati, 2020). Even so, the system he used did not assign specific symbols to operations (Sesiano, 2009).

In pursuit of generality, the computational methods were explained through tangible numerical examples instead of by universal prescriptions. Although the problems were stated using general terms, physical numbers were chosen to explain the solution (Sfard, 1995). Other problems of a similar sort could be solved by means of analogy; in other words, by substituting new numbers instead of using those specific examples. Furthermore, it is necessary to take note that after Diophantus, algebraic symbolism disappeared until the 15th century. Medieval texts, like Latin and Arabic texts, are mostly verbal (Sesiano, 2009).

The decline in Greek mathematics, as well as the decline in the city of Alexandria, which was seen as an academic centre, occurred at the same time as Hypatia's brutal death in 415 C.E. Hypatia was an influential teacher who taught ethics, astronomy, ontology, and mathematics to a number of men who later became leaders in Alexandria. With her death, the city of Alexandria also started to regress and in 645 C.E. was conquered by the Arabs. This marked a cultural and intellectual transition in the medieval world(Katz and Parshall, 2014).. A shift away from the West and toward the East could be noted. A growing demand in the Middle East to rediscover "lost" knowledge occurred (Katz and Parshall, 2014).

2.3.2.5. Al-Khwarizmi

Al-Khwarizmi (A.D. 780) and his colleagues, the Banu Musa, were students at the House of Wisdom in Baghdad. They participated in the translation of Greek scientific manuscripts as well as the study of algebra, geometry and astronomy (O' Connor and Robertson, 1999). Al-Khwarizmi stated that algebra was a discipline of dealing with equations (see 2.2.4). Traditionally, this is one of the focus points of teaching algebra in the secondary years of mathematics education. It has become increasingly clear that the development of algebraic thinking cannot rely solely on the simplification of equations in later grades. Algebraic thinking is a skill that focuses on the relations between numbers and the generalisation of arithmetic. It gives light to reasoning related to the development of mathematical models and frameworks, which includes mental and formal models. These models help to solve algebraic problems, formulate and visualise patterns and construct algebraic language (Ratith Ayu Apsari *et al.*, 2020). This skill should be integrated from the beginning of schooling and mathematics education (Bastable and Schifter, 2008; Schifter *et al.*, 2008; Warren and Cooper, 2008; Warren and Miller, 2010).

The oldest remaining true algebra texts are the works "*Al-Jabr and Al-Muqabala* by Al-Khwarizmi", written in Baghdad around 825 (Katz and Barton, 2007). The meaning of *jabr,* as used in mathematical works, is plussing equal terms to both sides of the equation to eliminate negative numbers. Or, alternatively, multiplying both sides of an equation by the same number to eliminate fractions. The meaning of *muqabala* is to reduce "positive terms by subtracting equal amounts from both sides of an equation" (Waerden, 1985). The word 'equate' can be used here (Waerden, 1985). The combination of the two words *Al-Jabr Al-Muqabala* refers to the science of algebra or performing algebraic operations. *Al-Jabr* is the name from which the word algebra is derived.

The initial part of the text is an instruction book for solving linear and quadratic equations. Al-Khwarizmi classified equations into six types (Waerden, 1985; Katz and Parshall, 2014). For each type he provided an algorithm for the solution. The six types of equations can be reduced as following, where a, b and c are given as positive numbers:

$$ax^2 = bx$$
$$ax^2 = b$$
$$ax = b$$
$$ax^2 + bx = c$$
$$ax^2 + c = bx$$
$$ax^2 = bx + c$$

(Waerden, 1985)

As in Babylonian times, his algorithms were entirely verbal and no use was made of symbols. For example, to solve a quadratic equation he writes:

> "…take half the number of "things", square it, subtract the constant, find the square root, and then add it to or subtract it from the half roots already found."
> (Katz and Barton, 2007, p.190-191)

Functional thinking involves the generalising of relationships between covarying quantities, representing the specific relationships in words, tables, graphs or symbols, and thinking and arguing with the various representations to analyse function behaviour (Blanton et al., 2015). In his work with linear and quadratic equations, Al-Khwarizmi engaged in the earliest stages of functional thinking. He was able to group various equations which represented relationships between various quantities. Even though these equations were represented by means of words, the functions were represented in a general manner.

As in the work of Al-Khwarizmi, the focus start to shift from only finding generalisations in arithmetic or geometrical objects to algebraic thinking which starts to include the focus on relationships between quantities which result in functions. In the same way, learners in the early mathetics classrooms should be exposed to problems based on the relationships between covarying quantities. The aim of functional thinking is to focus on relationships (Smith, 2008).

Al-Khwarizmi's aim was to solve equations. Here the development of algebra moves decisively to the static-equation solving conceptual stage.

2.3.2.6. Critical remark for education

In Katz and Barton's (2007) conceptual stages in the history of algebra, the geometric stage can be seen as the first stage. The syncopated stage retains this geometric thinking. In the Ancient Greek civilisation, most mathematical thinking was geometrical, and the influence of this thinking on the subsequent development of algebra is clear.

This geometric thinking involves representing mathematical thinking by means of geometric figures and forms (Katz and Barton, 2007). This is relevant in the classroom as many learners need to represent the content of algebraic problems by means of diagrams and drawings, these often being geometric. When learners start representing ideas by means of diagrams or drawings, they are starting to make models and mental pictures of the mathematical concepts and real life situations they are working with (Cooper and Warren, 2008). Making models and representing ideas in this way, is an important step towards generalising (see 2.2.4) (Roberts, 2012; Kaput, 2008).

The crucial role played by problem solving in the evolution of algebra and its teaching over the centuries cannot be ignored. Problem solving is an essential component of algebraic thinking which can be seen throughout the whole history if algebra (Bednarz, Kieran and Lee, 1996). It is from the need to solve complex, real-life problems, that algebra developed and emerged through the centuries. The need for generalisations emerged as mathematicians in history realised that problems can be grouped together and can be brought in relation to each other. Doing that can make mathematics more sensible and meaningful. Using problems as a means for conceptual development aligns with the PCA (see 2.2.3).

Examining mathematics in history leads to questions about the idea of problems and the part problems have played within mathematical theories and the building of new theories. The historical works of great mathematicians like Diophantus, Al-Khwarizmi, Cardano and Viète place problem solving in a broader context by illuminating the extent of what is implied by a problem and the progress of the contexts in which these problems have surfaced (Bednarz, Kieran and Lee, 1996).

The variety of words used to represent problems and their use within theory show fundamental conceptions. These conceptions vary from basic application of rules to a particular solutions asking for certain skills, and from a particular solutions to a more general solution for which the set of rules used and the problems solved are extended (Bednarz, Kieran and Lee, 1996). Here we notice how generalisation as algebraic reasoning becomes more prominent and fundamental in the emergence of algebra in history, as thinking in a general manner is central to algebraic thinking (see 2.2.4 and 2.2.5).

The rhetorical and syncopated stages are characterised by the operational outlook of mathematicians in these times. These stages are defined by characteristics which are prolix, and are tediously sequential, which imposes an operational outlook (Sfard, 1995).

The thinking operationally puts a tremendous load on working memory and is much less effective than structural thinking (Sfard, 1995). Structural thinking, which happens when one can generalise mathematics (Cooper and Warren, 2008), is developed and shown by the use of modern notation. Aryabhata (476-550) was an Indian mathematician who wrote 'Aryabhatia'. Aryabhata led the way for the development of variables in algebra. The example below shows his algebraic reasoning (Sfard, 1995, p. 19):

> "Multiply the sum of the progression by eight times the common difference, add the square of the difference between twice the first term, and the common difference, take the square root of this, subtract twice the first term, divide by the common difference, add one, divide by two."

This still forms part of the rhetorical and syncopated stages as the focus is on a verbal description of the problem, but the author here speaks to both the given and the missing numbers by using **general terms.** The expression of the unknown as a variable (even if verbally), is an important advance in the development of algebra. In the classroom, learners should be engaged in expressing their mathematical thoughts in general terms by looking for the underlying structures which can be noticed in mathematics (see 2.2.4 and 2.2.5).

Therefore, until the 16th century, developments in algebra were not marked by changes in the general character of the methods used, as these stayed more or less the same for more than 2 millennia. The changes can be noted in the progressive increase in complexity of the investigated computational processes (Sfard, 1995, p. 20).

When comparing how the knowledge of operational versus structural thinking influences algebra on a school level, some research has shown that learners, even those with some experience in algebraic symbols, fare better when employing verbal rather than symbolic methods (Sfard, 1995).

The precedence of operational over structural thinking in the development of algebra in history is relevant when considering the development of algebra in individual learning. The aim of teaching mathematics in the early years should be to foster structural thinking which leads to generalising (Roberts, 2012).

2.3.3. The symbolic stage

2.3.3.1. Viète's invention – variable as a given

The way Francois Viète (1540-1603) employed symbols in mathematics paved the way for modern symbolism in algebra. Viète was a French lawyer and, in his free time, a geometer who was looking for better techniques for astronomical calculations. Viète developed a 'new algebra' which was first explainedin the work *In Artem Analyticem Isagoge (1591).* This 'new algebra' was completely different from contemporary sixteenth century algebra (Oaks, 2018).

Up to this stage, letters had been implemented in algebra to represent missing numbers (Sfard, 1995). For earlier algebraists, knowns and unknowns in algebra were numbers. They reasoned with positive numbers which could be derived from the unit through addition, subtraction, multiplication, division, and root extraction (Sfard, 1995). Negative and complex numbers were not yet part of the basic presentation of algebra (Oaks, 2018).
Viète understood that the unknown values in equations could represent *types* of objects. He had a much broader view of equations than simply solving for an answer. He stated that if the unknown could represent a type of object, then algebra would be the study of the *relationship* between these types (Tabak, 2011). This higher level of abstract thought led to a breakthrough in the conventional notation of algebra.

Focusing on the relationship between quantities indicates Viète's ability to reason quantitatively as well as in a functional manner (see 3.4.3).

Viète implemented the denotation of symbols for the unknown with vowels. And the given numbers or data were marked with consonants. Because of this convention designed by Viète, entire groups of problems could be solved by using concise algorithms (Sfard, 1995). Viète used two separate series of terms to describe unknowns. He called unknowns "magnitudes", and the rules were presented without the usual connection with arithmetic. Furthermore, his expressions and equations look nothing like those in earlier books, because he used letters for unknowns (Oaks, 2018).

Viète made important strides in the use of variables. Variables are described as one of the five big Ideas of algebraic thinking (see 2.2.4.). Viète initiated the use of symbols as variables in a way that closely relates to modern methods. His use of variables enabled him to think algebraically and to generalise in a way that was not possible before.

It is important to note that also using letters to represent givens precluded finding specific numerical answers. This allowed Viète to see broader patterns in the mathematics. The letters helped him to identify relationships between symbols and the types and classes they represent. This is an important developmental step which emerges from history. A total shift towards generalisation is seen as Viète moved away from a focus on specific numbers. His perspective became process-orientated rather than answer-orientated. This allowed him to see the underlying structures and patterns within mathematics (Tabak, 2011).

Viète's conventions played a significant role in algebra's development from Diophantus's plans and methods for solving various problems, into a genuine science of *general computations*. According to Viète, arithmetic was the science of concrete numbers, whereas algebra was a science of 'species' or types of things rather than the things themselves (Sfard, 1995, p. 24). These 'species' possess dimensions and cannot be identified with numbers (Oaks, 2018). Viète was able to make the shift from simply computing with numbers in arithmetic, to viewing arithmetic in a general manner and implementing generalised arithmetic. Generalised arithmetic also forms one of the Five Big Ideas, and includes being able to analyse information, conjecture an arithmetic relationship, and identify values or groups of values for which a conjectured generalisation is true and justify an arithmetic generalisation employing either empirical arguments or representation-based arguments (Knuth et al., 2014).

Viète's innovation of using letters as known numbers, along with using symbolism for operations, abstracted and reified the total algebraic knowledge in a way that made it much more accessible. Algebra could be used as a user-friendly basis for a completely novel facet of mathematics (Sfard, 1995).

With the emergence of the use of symbols, a new kind of natural science developed. Mathematics could be used to deal with changing amounts and not just constant quantities. Natural scientists employed this invention to represent a variety of natural processes. The concept of function, along with the concept of variable (see 2.2.3. and 2.2.4.), began to emerge (Sfard, 1995). The use of conventional symbolism in mathematics was also seen in geometry.

2.3.3.2. Fermat and Descartes

Geometry is an ancient branch of mathematics. For much of its history it was mainly structural, as it had easily visualisable concrete objects to work with. A transition into operational thinking at a higher level was necessary (Sfard, 1995). In the 17th century, mathematicians initiated the expression of geometric problems and relationships in an algebraic manner (Tabak, 2011). To achieve more generality, the discipline of geometry had to be separated ifrom concrete shapes and instead the focus had to fall on the constructions and transformations by which these shapes are ruled. This would become possible through the employment of algebra to solve geometric problems (Sfard, 1995).

As learners' algebraic thinking develops and becomes more sophisticated, mirroring the historical development of algebra, teachers should guide learners to gradually detach from the use of drawings and concrete objects and start to focus on the constructions and transformations by representing these in a more formal and general symbolic manner. Learners in the early mathematics classroom can use variables to represent functional and quantitative relationships (Knuth et al., 2014). When young learners are prompted to describe and represent quantities and relationships between them in various ways, they learn to recognise equivalent representations and broaden their capability to employ symbols to express their ideas (Friel, Rachlin and Doyle, 2001).

Fermat and Descartes employed symbolism in geometry for the first time. They can be seen as the fathers of analytic geometry. Descartes' main goal was to employ algebra as a problem-solving tool with regards to geometry. Fermat focused on representing curves through algebra (Katz and Barton, 2007). Geometrical figures and transformations were represented by their computational processes (Sfard, 1995). Algebraic descriptions of geometric problems are more concise and easier to manipulate than visual representations. Mathematicians were looking for a way to connect geometric ideas like "curves, lines, and surfaces" with algebraic expressions (Tabak, 2011). This new method was named analytic geometry because the method was based on manipulating algebraic symbols (Sfard, 1995). This has a deep impact on the history of mathematics and science in general. Analytic geometry provided mathematicians with a mechanism for representing motion or movement. Newton used this mechanism as he was developing calculus (Katz and Barton, 2007).

Fermat (1601-1665) took *Arithmetica,* the work of Diophantus, as the starting point for his studies in number theory (Sesiano, 2009). Fermat never formally published his number theory. His results and very limited methods was acknowledged through his comments in the margins of Bachet's translation of *Arithmetica,* as well as through his correspondences with leading scientists of the time (Carcavi, Frenicle, Mersenne) (Kleiner, 2005).

Fermat's most famous work was his Last Theorem. Fermat wrote a note next to problem II.8 in Bachet's translation of *Arithmetica*. The note read:

> It is impossible for a cube to be written as a sum of two cubes or a fourth power to be written as the sum of two fourth powers or, in general, for any number which is greater than the second to be written as the sum of two like powers. I have a truly marvellous demonstration of this proposition, which this margin is too narrow to contain.
>
> (Kleiner, 2005, p. 8)

Symbolically, this theorem can be expressed as follows: $z^n = x^n + y^n$ has no positive integer solutions if n>2. Fermat never published his 'marvellous demonstration'. This theorem was proved 357 years later by Andrew Wiles (Derbyshire, 2006). Even though Fermat's thinking led to the development of representing geometry with algebraic symbols, little of his work has been preserved.

Descartes (1596 – 1650) used the theory of proportions to achieve unification between geometry, arithmetic, and algebra. The relationship between geometry and algebra in Descartes' work can be challenging to comprehend as he almost contradicts himself. On the one hand Descartes attempts to unify geometry and algebra by trying to prove a method of representing a curve via an equation. However, he maintained the logical and epistemological priority of geometry over algebra. This tension can be noted in his constructions of equations for solving geometrical problems (Crippa, 2017).

In *Discours de la méthode* (Discourse on method), and especially in an appendix which discusses his work in geometry, Descartes made the connections between geometry and algebra which led to a novel area of mathematics: Cartesian geometry (Tabak, 2011).

The Cartesian coordinate system is a method for establishing a correspondence between points and numbers. A two-dimensional Cartesian system is formed by identifying a special point, which is called the origin, and a line passing through the origin (x-axis) (Tabak, 2011). A second point on the x-axis is used to establish a direction and distance. The distance from the origin to this second point is taken as one unit (Tabak, 2011). The direction that is travelled from the origin to the second point identifies the directions of increasing x. The line that passes through the origin and is perpendicular to the x-axis is the y-axis (Tabak, 2011).

In his most influential work, *La Geométrie* Descartes introduced the system of algebraic symbolisation used in modern algebra. He also explored the relation between geometry and algebra. His version of analytic geometry was almost the same as that of Fermat. However, Fermat inclined to start with an algebraic equation and interpret it geometrically in terms of a specific curve. Descartes began with a geometrical problem and aimed to solve it and represent it algebraically (Katz and Parshall, 2014).

Descartes also introduced a novel perspective on multiplication. The Greeks formed a relationship amongst the line segments and real numbers by using quantitative reasoning (Smith and Thomson, 2008). A number of magnitude x, can be represented by a line segment with length x. The product of two numbers x and y are represented by a rectangle with line segment with length x forming the one side, and line segment with length y forming the other side (Tabak, 2011, p.88). This works well until you want to represent the product of more than two segments. Descartes' innovation was to use triangles rather than rectangles to represent multiplication. He imagined all products as line segments of the appropriate length as seen in the diagram below (Tabak, 2011, p.88):

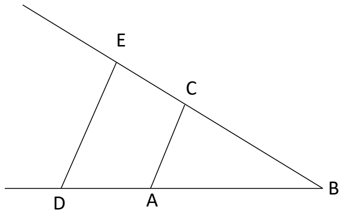


Figure 2.4. Descartes' geometric interpretation of the operation multiplication (Tabak, 2011, p. 88)

With this diagram Descartes provided a more effective and usable representation of the operation of multiplication and therefore, used geometry to represent arithmetic. The use of representations is useful to develop algebraic ideas.

Descartes also examined the relationship between geometric descriptions of conic sections and algebraic equations. Conic sections are curves like hyperbolas, parabolas and the ellipse. He examined the relation between geometry and the equation: $y^2 = ay - bxy + cx - dx^2 + e$. In this equation x and y are the variables and a, b, c, d and e are the coefficients (Tabak, 2011).

Descartes went far in exploring the connections between geometry and algebra and the use of symbols to represent mathematical ideas became more powerful and useful through his discoveries and work. Descartes had founded a method for producing infinite new curves: Write one equation in two variables and the result would be a new curve. The principles of analytic geometry were made clear by Descartes which lead to an enrichment of mathematical vocabulary (Tabak, 2011).

At the core of algebraic reasoning and generalisation in the early years lies a fundamental understanding of the mathematical structure of arithmetic communicated through language and gestures using tangible materials and representations (Warren and Cooper, 2008). Even so, as claimed earlier, learners need to gradually move away from concrete materials and representations and start to represent relationships in more formal manners by using variables (Friel, Rachlin and Doyle, 2001).

2.3.3.3. Peacock and the de-arithmetisation of, and arbitrariness in, algebra

Some mathematicians and prominent thinkers stated their doubts in Viète's invention of the use of symbolism in algebra. Newton claimed that "algebra is the analysis of bunglers in mathematics" (Sfard, 1995, p.27). Many scientists and mathematicians rooted their doubts in the argument that this new discipline of employing letters to symbols, lacked logical reasoning (Sfard, 1995).

The abstract notion of variable was the underlying problem. The notion of variables, which cannot easily be explained by a simple definition, may be one of the most problematic in the whole of mathematics. Doubt in variable still exists in recent professional literature (Sfard, 1995). Bell (1951, p. 101) stated that:

> "… to state fully what a variable is would take a book. And the outcome might be a feeling of discouragement, for our attempts to understand variables would lead us into a morass of doubt concerning the meanings of the fundamental concepts of mathematics" (Bell, 1951, p. 101).

A variable can be seen as something which changes. This aspect of operational change was unacceptable to many 20th century mathematicians. Algebra was viewed as generalised arithmetic with the sole aim of expressing the goals which govern numerical operations in a general manner. The British mathematicians wanted to provide algebra with a solid logical basis, which would eliminate any doubt around the subject (Sfard, 1995).

Peacock (1791-1858) was one of the mathematicians working on the emancipation of algebra. He is seen as one of the earliest and most prominent minds in the development of symbolic algebra. Even so, there is no book of works which can be studied to form a thorough understanding of his work. It is proposed that his work with algebraic symbols stemmed from the aim to resolve problems with imaginary and negative numbers (Pycior, 1981).

Peacock introduced the concept of arbitrariness into algebra. Mathematicians could develop and invent new mathematical ideas without the fear of judgement from others. The laws of logic were the only thing governing their thoughts (Sfard, 1995). Peacock believed that while quantity is the ultimate subject matter of algebra, he adopted a basic symbolic approach to algebra. He viewed it as a science of "arbitrary" or undefined symbols and signs governed by specific laws (Pycior, 1981). After this ontological breakthrough, the introduction of novel mathematical ideas through axiomatic systems developed to be increasingly common. Algebra's connection with numbers and numerical computations was released more and it progressively transformed into a science of abstract structures (Sfard, 1995).

His expression "principle of permanence" stated: "Whatever form is algebraically equivalent to another form expressed in general symbols, must continue to be equivalent whatever the symbols denote" (Novy, 1973, p. 191). Peacock concluded that a variable should no longer be seen as generalised number but must be treated as an object itself, stripped of external sense. From here Peacock arrived at the de-arithmetisation of algebra. The meaning of symbols should not be expected to come from their non-existent value, but must be pursued in a way that formulas are transformed and combined with each other (Sfard, 1995). Peacock was motivated by a concern with the applicability of symbolic algebra and chose to adopt the laws of arithmetic as the laws for symbolic algebra (Pycior, 1981). He did not show the freedom of thought seen in the later algebraists like Hamilton.

2.3.3.4. Hamilton

The freedom, which was created by mathematical thought leaders like Peacock, opened the door for mathematicians like Hamilton to think and reason freely about mathematics. This led to Hamilton's creation of quaternions in the 1850's (Katz and Parshall, 2014).

Hamilton was captivated by the connection between geometry and complex numbers. Complex numbers can be seen as real numbers which are used to deal with everyday life (Shúilleabháin, 2016). Complex numbers could be easily manipulated in two-dimensional geometry. In his earlier works Hamilton presented complex numbers simply as pairs of real numbers governed by formally defined operations (Sfard, 1995). Hamilton wanted to expand the use to three-dimensions but it proved to be impossible (Shúilleabháin, 2016). He began to realise that he only needed consistent axiomatic system to legitimise that existence of an abstract object. His thinking was quite futuristic for the time (Sfard, 1995). This can be noted in the following statement Hamilton made to a friend, John Graves:

> "I have not yet any clear view as to the extent to which we are at liberty to create imaginaries, and to endow them with supernatural properties."
> (Kleiner, 1987, p.233)

He took advantage of this freedom of thought and understood that if he skipped a dimension and operated with numbers in four dimensions instead of three, he would be able to solve the problems. From here the new idea of quaternions developed (Shúilleabháin, 2016). In spite of his freer, axiom-based approach, Hamilton felt it

necessary to prove his idea of quaternions by physical application. Mathematicians who came after him freed themselves from this perspective (Sfard, 1995). The freedom that was seen in the work of Hamilton created new opportunities for mathematicians to think creatively about mathematics, and to explore and push the boundaries of what was known at the time.

Creativity is a necessary aspect of the mathematical process and should be incorporated in mathematics classrooms. Learners should be given ample opportunities to think creatively about mathematics, and to come up with their own original ideas. This supports learners' understanding of the mathematical concepts they are working with.

The historical stages discussed in this section correspond to the stages learners pass through as they develop algebraic thinking. When studying the development of algebraic thinking in young learners, it is therefore imperative to understand how algebraic thinking, and especially generalisation, came about in history, and how this trajectory can be mirrored in the teaching of generalisation in the early mathematics classroom.

2.3.3.5. Critical remarks for education

In the symbolic stage, total symbolisation can be noted. This can be seen as the stage where all numbers, operations and relationships are communicated through the use of a set of simply recognised symbols, and manipulations of the symbols, in accordance with the rules which are well-comprehended (*Mathematics for Teaching*, n.d.).

As stated earlier in this chapter, the stages in the development of the history of algebra can be drawn into relation with the development of algebraic thinking in the classroom. The progression of the stages is also often clear in the ways in which textbooks introduce algebra and algebraic problems. Initially learners are expected to express ideas, patterns, and relationships in words, which resemble the rhetorical stage. From there learners are encouraged to represent ideas using only key words, diagrams, or informal symbols, which reflects the syncopated stage. And lastly the formal use of symbols to represent numbers, operations, and relationships are initiated and introduced to learners. This is the final stage, the symbolic stage (*Mathematics for Teaching*, n.d.).

As seen in the progression of these stages it is mostly necessary to take learners through all the stages of thinking to ensure that they develop a deep understanding of algebra and how it is used. Nixon (2009) argues for three levels of learning to think algebraically: the perceptual level, the conceptual level and the abstract level. These three levels of the development of algebraic thinking relate closely to the stages in the development of algebra throughout history. This progression can be seen as a possible trajectory for the development of algebraic thinking and generalisation. An in-depth discussion of these levels will follow in Chapter 3. An important aspect of these levels to note, is that as learners are guided through these levels, teachers should facilitate situations for them to make generalisations and draw comparisons (Nixon, 2009).

Often, if learners are simply introduced to the formal use of symbols given with a set of complicated rules, learners are expected to memorise mathematics. This is the traditional approach to teaching algebra. This leads to many challenges, as learners do not really understand why they are performing certain actions and do not see the clear need for the use of symbols. Algebraic concepts should be integrated throughout all content areas to ensure that their teaching develops a deep understanding (Schoenfeld, 2019). As an educator, it is important to ensure that learners can clearly explain and communicate their thinking to show they have deep understanding. Learners should realise that mathematics is not answer driven but that all mathematical processes, including communication, are important aspects (Ontario Ministry of Education and Training, 2007). When examining history, an important lesson is learned in terms of visualisation. Greeks found it valuable to use geometric representations to interpret numerical computations. This points to the effectiveness of graphical representations in an abstract subject, such as, algebra (Sfard, 1995). Geometrical objects like squares, triangles and circles can be used as symbols or 'placeholders' in the early years of school mathematics. Even so, when using geometry to assist computations, the necessary precautions should be taken to avoid a literal approach where models might hinder algebraic thinking (Sfard, 1995).

When considering the development of algebra throughout history, one of the first and most obvious challenges and barriers to the progression of algebra was the lack of symbolisation. A convenient symbolism for expressing new ideas and solutions to problems was needed. In this regard, some ideas and concepts were spreading throughout Europe, but not in a conventional and consistent manner. Mathematicians in different geographical and linguistic regions introduced various ways of using symbols (Tabak, 2011). It took time for the symbolisation and notation to become standardised. Much of the algebra in the 16th century resembled the Islamic mathematics of centuries earlier: the focus was still on finding the roots of equations. The equation was seen as a concrete object in the form of a question. At the time, algebra was a set of problem-solving techniques. One of the first mathematicians who understood algebra as more than a problem-solving technique was the French mathematician Francois Viète (Tabak, 2011).

2.3.4. Conceptual stages

Katz and Barton (2007) described four conceptual stages in the development of algebra, which run concurrently with the three historical stages. The four conceptual stages are the geometric, static-equation, dynamic function, and abstract stages. These stages are not separate from each other but overlap. The following systematic analysis of the conceptual stages in the development of algebraic thinking will inform the instructional sequence for developing algebraic thinking presented in Chapter 3.

2.3.4.1. Geometric stage

The geometric stage reflects, for the purpose of this discussion, the beginning of algebra. It first emerged in 4000 years ago in Mesopotamia, as discussed previously (see 2.2.1). Mesopotamians developed the cut and paste geometry used by land surveyors, and solved accounting problems. The Babylonian mathematics (2 000 –

1 700 BCE) can also be included when discussing the geometric stage. The old-Babylonians had clay tablets with extensive lists of solutions to quadratic equations as we know them today. Their goal in developing these lists was to facilitate finding geometric quantities, such as the length and width of a given rectangle. The underlying aim for Babylonian equation solving was to solve geometric problems, but they developed algorithms to solve equations (Katz and Barton, 2007). Some efforts towards generality are evident in the labelling of the sides of geometric object in a manner that resembles modern algebraic symbol systems. However, procedures and algorithms were described in words.

These systems were applied across many similar problems (Katz and Barton, 2007). This can be viewed as the beginning of algebraic thinking and generalisation as it was the first time in recorded history that numerical problems were solved by manipulating original data according to a set of fixed rules with the aim of finding some sort of generality (Tabak, 2011). Similarly, as learners work with numbers and number properties in arithmetic, as well as, solving problems and communicating their ideas and thinking, the generality which learners notice throughout mathematics, are expressed in their own words using informal terminology. As learners become more sophisticated in their ability to express generality, they become able to do so by means of variables (Blanton *et al.*, 2015).

Greek mathematics was mainly based on geometry, but algebraic notions can be observed in many of the works that represent the mathematicians of the time. Euclid's *Elements*, for example, presents methods for manipulating rectangles and squares. In Book II of *The Elements,* there are propositions where Euclid solves problems which seem to be algebraic with geometric results, such as the position of a particular point on a line (Katz and Barton, 2007). His solutions were based on clearly stated axioms and proofs (Katz and Barton, 2007). In the geometric stage, mathematicians developed the sophistication of their algebraic thinking by engaging with algebraic problems through geometric reasoning and representing. The geometric stage is a crucial stage of thinking for learners as they work through algebraic problems and patterning activities in the pre-algebra classroom (Apsari *et al.*, 2020).

Geometric representations make it possible for learners engaged in patterning activities and arithmetic to forge connections between problems, mathematical models, and problem solving strategies, and to become aware of the structure of mathematics (Dekker and Dolk, 2011).

The geometric stage can be compared to the perceptual level of algebraic thinking as described by Nixon (2009) in Chapter 1 and 3 (see 1.1 and 3.4.3.1.) . This level of algebraic thinking involves the coordination of physical senses and perceptions to form algebraic concepts. Learners reason with physical and concrete "objects" like geometric figures, and from there are able to construct informal algebraic concepts. An example of this in the geometric stage would be Euclid's demonstrations of how to manipulate rectangles and squares based on axioms.

Geometric thinking and representation plays a valuable role in students' algebraic thinking. Apsari *et al.* (2020,p.52-53) distinguish various key roles of geometric representation in the pre-algebra classroom:

1) Context

Geometric thinking and representation aids learners in setting the context of the algebraic problem or pattern they are engaged in. Formal algebraic expressions are too sophisticated and abstract for learners at the beginning of their algebraic thinking development. Apsari et al. (2020) found that in a prototype lesson where a problem is presented and analysed merely in words, learners could not find a general relation between numbers and worked fruitlessly to solve the problem. In comparison, when learners investigated patterning activities with geometrical representations, they could see the structure of the pattern in a more realistic way. Geometric representations set the foundation for discovering which aspects of patterns remain the same and which aspects change. Learners can then use geometrical visualisation to elaborate on the changing of mathematical objects and express generalities. (Rivera, 2011) also found that visualisation is imperative to constructing personal inferences when engaging in patterning activities.

2) Model of and model for situation

Models help learners to shift their focus from reality to mathematical objects. Mathematical models aim to provide a visualisation of the actual condition or situation. The Realistic Mathematics Education (RME) approach distinguishes between two types of models: (1) model of situation and (2) model for situation. The model of situation is used to transfer the context or situation to a mathematical object. The model for is used to work with mathematical ideas and concepts. Geometric representation is a bridge used to translate the context of situations and mathematical ideas (Apsari *et al.*, 2020). (Kusumaningsih *et al.*, 2018) also highlighted the need to use numerous representations to develop algebraic thinking and generalisation.

3) Scaffolding

The teacher can use geometric representation as a means of enhancing learners' ability to reason critically. By prompting learners to represent their thinking by means of geometrical objects the teacher can provide support without directly giving learners the answer (Apsari *et al.*, 2020).

4) Learners' mathematical reasoning and proof

Geometry representations are used by learners to show their mathematical reasoning. Illustrations are used to support arguments and help teachers to notice where learners have formed misconceptions in their reasoning (Apsari *et al.*, 2020).

When algorithms start to replace geometry and geometric representations, the conceptual stages start to move from the geometric stage to the static-equation solving stage (Katz and Barton, 2007).

2.3.4.2. Static-Equation solving stage

This stage is characterised by a general concern with numerical problems which involves squares, and can be viewed as algebraic. Algorithms to solve problems (expressed as an equations) are proposed and used to arrive at an answer. The transition from the geometric to the static-equation solving stage becomes evident in the work of Diophantus. Diophantus was able to solve quadratic equations using an algorithm based only on numbers (Katz and Barton, 2007). The first Big Idea of algebra in the early years is working with equivalence, expressions, equations and inequalities (see 2.2.5) (Knuth *et al.*, 2014). Diophantus was able to solve quadratic equations by reasoning from the structural form of the equation. Reasoning with equations is an important skill which should be introduced in the pre-algebra classroom (Knuth *et al.*, 2014). Diophantus studied algebra as a separate field from geometry, even though he also studied geometric algebra. He introduced the use of Greek symbols in equations. His main goal was to provide a foundation for algebra (Tabak, 2011). He used the number system and the fixed rules which govern it, to describe various number properties and the relations between them. For example, rational numbers are

numbers than can be shown as fractions with whole numbers in the numerator and the denominator – and negative numbers (Tabak, 2011). In this way he engaged in the process of generalising arithmetic by providing rules for thinking and finding solutions (Knuth *et al.*, 2014).

As discussed previously, the first authentic algebra text was published by Al-Khwarizmi and was called *Al-Jabr* and *Al-Muabala*. The first section of the book is written as a manual to solve linear and quadratic equations. Al-Khwarizmi was able to classify equations into six types and for each type of equation, an algorithm was presented as a solution. Al-Khwarizmi was able to engage in the first Big Idea of pre-algebra by solving various equations (Knuth *et al.*, 2014). The fact that Al-Khwarizmi aimed to organise equations into categories, shows that he was seeking generality. The solutions here was still entirely verbal. Al-Khwarizmi presented abstract problems as examples for his algorithms. He could interpret the context of a problem by expressing it algebraically, even if it was merely in words (Knuth *et al.*, 2014). He did not stick solely to using geometric objects like width and length (Katz and Barton, 2007). The goal in this era of Islamic mathematics was solving equations. Al-Khwarizmi's approach to and perspective on algebra informed the traditional teaching of algebra in high school. This traditional approach mainly focused on the simplification of equations rather than a relational approach where algebra and generalisation encompasses the whole of mathematics (Ratith Ayu Apsari *et al.*, 2020).

The static-equation solving stage was also introduced to Europe in the twelfth and thirteenth centuries. Even in the sixteenth century the aim of algebra remained the solving of equations (Katz and Barton, 2007). Equations were used to model problem situations and determine the numerical value of unknowns (see 2.2.3.) (Knuth *et al.*, 2014).

Learners involved in the static-equation solving stage are starting to enter the conceptual level of algebraic thinking as described by Nixon (Nixon, 2009) (see 3.4.3.1). In the conceptual level, a shift occurs which moves the focus from analysing **objects** to the transformations of objects and relations between objects. Learners start to find interrelationships between properties and develop definitions and theorems to describe and explain these relationships. In the static-equation solving stage, mathematicians started to move away from primarily analysing geometric objects, and started to represent relationship as equations, which were solved by developing algorithms and proofs. Equivalence, expressions, equations, and inequalities are part of the first Big Idea involved in developing algebraic thinking in the foundational years of mathematics education. The ability to express problem situations and the unknown by means of variables in equations is an important step towards developing algebraic thinking and generalisation (Knuth *et al.*, 2014). In the static-equation solving stage, equations emerge as the next developmental level of thought for learners to engage in (see 3.4.3.1).

2.3.4.3. Dynamic function stage

The dynamic function stage introduced the concepts of motion and movement to mathematics (Katz and Barton, 2007).

Early in their development, algebra and algebraic thinking were seen as difficult and tedious due to the lack of standardised symbols (Tabak, 2011). In the seventeenth century, new notation was introduced by mathematicians like Viète and Descartes (Katz and Barton, 2007). This was quite similar to the algebraic notation we use today (Sfard, 1995). It involved the use of variables (one of the Bid Ideas of pre-algebra) to represent unknown quantities, which could be fixed or varying (see 2.2.5) (Knuth *et al.*, 2014). Viète understood that algebra was more than the developing of techniques to solve various equations. He understood that unknowns in equations represented species of objects, and algebra was about the relationship between these species. Viète's employment of notation in the form of alphabet letters to represent unknowns in equations (Tabak, 2011) showed his understanding of variables and the importance of using variables when working with algebraic problems and equations (Knuth *et al.*, 2014). Viète's introduction of notation and use of symbols made it possible for mathematicians to see broader patterns in mathematics, and to identify relationships between symbols and the classes of objects they represent (Tabak, 2011). Kaput (2018) emphasises that, in the pre-algebra classroom, learners should be able to suspend their conception of what the symbols symbolise, and instead look at the symbols itself. In this way learners are freed to operate on relationships which are more complicated (Kaput, 1999). A syntactic action involves the manipulation of symbols only by looking at the syntax of the symbol system rather than by looking at a reference field for those symbols. A syntactic action is an action on the notational system and not on the representational system. A syntactic manipulation looks at expressions or equations as manipulable object strings which are subject to certain constraints and governed by certain rules (Kaput, 2018).

A change in perspectives on algebra, and the implementation thereof, was also taking place. Mathematicians started to move away from the focus on finding solutions to expressing problems as equations. A growing interest in astronomy and physics further motivated mathematicians to advance their understanding of algebra (Katz and Barton, 2007).

Johan Kepler was fascinated with the paths of planets and Galileo Galilei was interested in the paths of a projectiles. In both these cases the aim was not to find a number, but a curve. Kepler and Galilei drew on the work of Apollonius, whose work was mainly static, to develop the representation of motion. Still, neither of these mathematicians had an effective way to represent motion. They were still using Greek models and not algebra. In 1637, suitable tools for representing motion were developed by Fermat and Descartes. Fermat and Descartes (as discussed in 2.3.3.3) had the goal of representing curves. Descartes wanted to use algebra to solve geometric problems and Fermat was concerned with representing curves using algebra. Both of these mathematicians developed methods for representing curves verbally by using algebra (Katz and Barton, 2007).

Isaac Newton was initially hesitant to use algebra in his work, but when this new algebra (dynamic function) arose, Newton started using it more freely. One of the key goals of the eighteenth century was to transcribe Newton's ideas into algebraic language and prove them using the new calculus. The mathematicians involved were no longer looking for answers expressed as numbers, but as curves. The objective was to see how objects move (Katz and Barton, 2007). As the eighteenth century drew on, algebra developed in such a way that it became easier to represent a curve as the path of motion. The idea of finding curves which solve problems became the central goal of mathematics (Katz and Barton, 2007). From the work on curves in the eighteenth century, the ideas of functional thinking, relations and joint variation emerged (see 2.2.4 and 3.5.4.). Mathematicians and scientists focused on representing change and the motion of objects by means of curves.

In the dynamic function stage, the goal is to represent motion using algebra. Motion can also be viewed as change. This stage still falls in the conceptual level of thinking algebraically as described by Nixon (2009) (see 3.4.3.1.). The conceptual stage focuses on the relations between and transformation of objects. Thus, the aim is to find and represent the interrelationships of properties between objects, for example, the motion of projectile and the curve which it would follow. Learners should be able to represent changes and motion in objects using algebra, even if it is verbally. Here, learners are expected to generalise the relationship between properties. This involves functional thinking, as learners reason about covariational relationships and are expected to represent these in words or by means of various representational tools (Knuth *et al.*, 2014). Functional thinking is perceived as a powerful mathematical idea which should form part of the pre-algebra classroom as it provides opportunity for learners to reason quantitatively about real-world problems and situations and allows learners to study the relationship and change which can be noticed in algebraic problems (Ng, 2018).

2.3.4.4. Abstract stage

In the abstract stage, understanding the structure of mathematics itself becomes the overarching goal (Katz and Barton, 2007). Generalising and abstraction based on computations, where the focus falls on the structure within the computations rather than the process or answer, lead to the emergence of abstract structures (Kaput, 1999). In the nineteenth century, another question started to gain prominence: How could one be certain that the algebraic manipulations made, are correct? A general consensus was reached among mathematicians that if axioms or proofs were in place, it could be assumed that calculations based on them would give the correct results. From there, axioms were formulated for arithmetic and were used to solve equations using algebraic manipulations (Katz and Barton, 2007). Structural knowledge is the skill of recognising all equivalent forms of an expression. It is necessary skill that learners should also be able to justify the structural equivalence which is identified (Liebenberg et al., 1998). In the abstract stage of the development of algebraic thinking in history, this need for justification and proof for argument, emerged (Katz and Barton, 2007).

With the discovery of quaternions by Thomas Hamilton, mathematicians realised that there could be more sets of axioms that provide interesting results. Lagrange conducted a study to determine whether Cardano’s algebraic solutions of polynomials of degrees three and four could be extended to polynomials of a higher degree. He could not find any conclusive results, but he introduced the idea of permutations. Permutations is a mathematical strategy that governs the amount of possible arrangements in a set when the order of the arrangements matter. Galois constructed methods involving what is now called group theory to determine under which conditions polynomial equations are solvable (Katz and Barton, 2007). In all of these events in the history, the aim is shifting from solving equations, but is progression towards finding structure in mathematics.

Group theory continued to develop throughout the nineteenth century, and mathematicians realised that many different mathematical situations had similar or common properties. By the twentieth century algebra became to a lesser extent concerned with finding solutions to equations and more about finding structures in various mathematical objects, with the objects defined by sets of axioms or rules (Katz and Barton, 2007). The study of structure is an important and foundational concept of algebraic thinking (see 2.2.5 and 3.5.3) (Kaput, 1999). The focus of early algebra should be on a relational approach to learning mathematics, which refers to studying number from a structural perspective (Du Plessis, 2018). Structure is obtained when number and space are explored relationally, and this initiates reasoning which focuses not only on the object but also on its fundamental properties (Du Plessis, 2018). Learners should be able to see common mathematical structures in representations when solving problems and working with arithmetic. The emphasis should fall on the structural relationships within problems and numbers and their properties. These relationships provide clues for how problems might be solved (National Research Council, 2001).

In the abstract stage the main goal is to find structure in mathematics, and to use symbols to represent and construct proofs. This aligns with Nixon's abstract level of algebraic thinking. In the abstract level of thought, learners start to use symbols with deep understanding to construct proofs, and they can understand the importance of deductions, axioms, postulates and proofs (Nixon, 2009) (see 3.4.3.1). The abstract stage is the goal of algebra. Here learners generalise mathematics and represent mathematics using symbols, which can be considered the main two aspects of algebra. As learners' algebraic thinking develops, they may move through the levels of algebraic thinking in a sequence which resembles the development of algebraic thought throughout in history. For this reason, having knowledge about and exploring the development of algebra throughout history provides teachers with valuable information about the development of algebraic thinking.

To summarise what can be taken from the conceptual stages in the history of algebra, a swift review of the development of algebraic thinking as seen through the stages, is provided. Learners start out in the (1) geometric stage (Katz and Barton, 2007) or the perceptual level (Nixon, 2009) of algebraic thinking. Here their thoughts about, and work with, mathematical objects are mainly concrete. They are expected to use spatial reasoning to manipulate simple geometric objects (Ratith Ayu Apsari *et al.*, 2020). An example of appropriate learner activities for this stage would be to copy and extend a visual pattern, and use geometric representations to show their understanding of the pattern. As learners progress to the (2) static-equation solving stage (Katz and Barton, 2007), they begin to think conceptually (Nixon, 2009). They start being able to represent their mathematical ideas when analysing the relationship between mathematical objects. These learners, especially in the earky years, will represent their ideas verbally or by means of equations (Knuth *et al.*, 2014). In the example of a geometric pattern, learners at this stage are able to express the rules of the pattern verbally or by means of a verbal equation. In the (3) dynamic function stage, the focus shifts to the motion of objects (Katz and Barton, 2007). Learners understand and can provide proofs for the interrelationships between properties of objects. Learners start

to think and reason with functions by noticing covarying relationships between properties (Knuth *et al.*, 2014) (Kaput, 1999). Learners in this stage are still operating at the conceptual thinking level (Nixon, 2009). When analysing a geometric pattern, these learners are be able to start expressing the changes they see happening in the pattern and explain the relationships between the properties of the pattern. In the last stage, (4) the abstract stage (Katz and Barton, 2007)(Nixon, 2009), learners start to see the structure in the relationships between, and properties of, mathematical objects (Du Plessis, 2018). Learners can use symbols to represent these relationships, and can use mathematical proofs, axioms, and deductions to explain their thinking (Liebenberg et al., 1998). Learners at this stage are able to represent the geometric pattern using mathematical language. They can furthermore prove that their reasoning about the pattern is true, for example by using a table or flow diagram. Learners can also start to express patterns in terms of the terms (n) of the pattern.

Several questions and lessons relevant to pedagogy arise from an examination of the history of algebra. The first question to consider is whether the teaching of algebra should commence with geometry (*Department of Basic Education*, 2018). Most learners first start to reason algebraically in a concrete manner (Katz and Barton, 2007), and need to manipulate concrete objects. The initial focus of algebraic manipulation would therefore be on simple geometric objects like squares, which are more concrete than x. Products of numbers can be presented as rectangles. It is not necessary that a lesson with a deep focus on geometry precedes algebraic thinking, but such a lesson may be valuable in providing a context for the rules of algebra and improving learners' deep understanding thereof (Katz and Barton, 2007). One should further consider whether all the focus should fall on solving problems using equations (Nixon, 2009). In school algebra, courses are often spread widely with no clear central focus or aim. Even when algebra courses have defined aims, these are often not clear to learners. When the aim of a course is the solving of real-world problems with equations, and all manipulations are introduced in this context, it is much easier for learners to see the value of algebra. This would improve their understanding.

The concept of function is seen as a more abstract concept than solving equations (Katz and Barton, 2007), as is evident from the history of algebra. It is therefore, imperative that learners have many experiences with curves in geometry before the abstract concept of function is introduced. Lastly, before the teaching of abstract concepts like rings, group theory, fields, etc., learners need to understand why it useful to generalise and why certain sets of axioms were chosen. They should therefore be exposed to a variety of relevant examples. Abstract concepts should only be introduced much later in the curriculum (Katz and Barton, 2007).

Now that we have a clear understanding of how algebraic thinking and generalisation emerged in history, we are going to zoom in on the history of algebra in schools specifically and what key aspects of algebra in schools arises when reviewing the history. In this way an overview of the history of school algebra can contextualise the current state of school algebra.

2.4. PURPOSES FOR SCHOOL ALGEBRA EMERGING FROM HISTORY

2.4.1. School algebra in the beginning

Algebra as a school subject was introduced much later than arithmetic and Euclidean geometry. Algebra only appeared in the secondary school curriculum in 1673 in London (Ellerton and Clements, 2017). The curriculum was based on a Latin book on algebra written by the swede Johannis Alexandri. This algebra was only taught to boys between the ages of 14 to and years. Before these boys could start studying algebra, they had to study Latin for four and a half years. In addition to the language barrier, the algebra itself was extremely challenging. This was accepted, as it was believed that the study of algebra should be reserved for highly intelligent boys. This remained the prevalent perspective on mathematics education in Europe and North America throughout the seventeenth century. In the eighteenth-century, algebra was introduced into most secondary schools. Teaching was based on textbooks written by highly acclaimed mathematicians, and algebra was mostly presented as generalised arithmetic (Katz and Barton, 2007). John Hodgson (1723), who was teaching algebra at Christ Hospital in London, maintained that the purpose of algebra should be the solution of practical problems. He was an exception at the time (Ellerton and Clements, 2017).

2.4.2. School algebra in the eighteenth and nineteenth centuries

While Hodgson was teaching at Christ's Hospital in London (1709-1755), the Royal Mathematical School (RMS), which was for the most capable boys, became the leader of school mathematics in Europe (Ellerton, Kanbir and Clements, 2017). This led to many other schools introducing algebra into the curriculum. Still, a top-down method was used, and most textbooks were written by professional mathematicians rather than by experienced pedagogues. Teachers were seen as the source of all knowledge and introduced learners to big, abstract ideas from which smaller concepts were derived and explored.

In the eighteenth and nineteenth centuries, colonialization was at the forefront. Mathematics textbooks introduced in colonies, were written and exported from their respective "home" nations (Hans, 1951). These textbooks were written in the language of the home nation by authors based in the home nation. These textbooks were mostly culture-free and therefore, mathematics was viewed as something which is culture-free. In practice, these textbooks proved to be unsuited for the needs of indigenous children both in terms of language and assumed prior learning.

Early in the nineteenth century, the content of school algebra began to be informed by what was expected of learners in tertiary institutions. From the second half of the nineteenth century, the idea emerged that school algebra should be suitable for modelling and solving real-life problems (Ellerton, Kanbir and Clements, 2017). When learners solve real-life problems in context, they start to see the value of mathematics and can from there develop a deeper understanding of the subject and the concepts. At the same time, a movement arose which advocated that school algebra should be help learners to recognise the structure of the real number-system. Lastly, in the twentieth century the concept of variables rose to prominence in algebra classrooms, with a focus on the power of variables to summarise major mathematical ideas and model real-life situations.

2.4.3. Six purposes of algebra

Six purposes for algebra emerge from literature and historical data, as outlined by Ellerton, Kanbir and Clements (2017).

Purpose 1: Knowledge essential for higher mathematics and science

This purpose is evident in mathematics education starting from 1693, particularly in the work of key writers like Descartes, Ditton, Newton, Leibniz and Lacroix. Mathematicians were motivated by the importance of school algebra in enabling learners to understand more challenging concepts and principles in higher education, including conic sections, trigonometry, calculus and mechanics (Ellerton, Kanbir and Clements, 2017). In the traditional curriculum, algebra has been misunderstood and misrepresented as an abstract and challenging subject, which is thought to be taught only to secondary learners with the aim of preparing them for the challenging algebra they will encounter at university. But in recent research (Carraher, Schliemann and Schwartz, 2008; Schifter *et al.*, 2008; Warren and Cooper, 2008; Schifter, 2009; Warren and Miller, 2010; T. Cooper and Warren, 2011; Fonger, Nicole *et al.*, 2015; M. L. Blanton *et al.*, 2015; Kaput, 2018; Blanton, Isler-Baykal, *et al.*, 2019), algebra and algebraic thinking have been found to be fundamental to the basic mathematical education of all learners, beginning in the foundational years (Friel, Rachlin and Doyle, 2001).

In South Africa, algebra is a main subject area in the high school mathematics curriculum and is a pre-requisite subject for post-secondary mathematics, science, and engineering courses. It is therefore seen a gatekeeper course (van Laren and Moore-Russo, 2014). The gatekeeper effect of algebra leads to the marginalisation of some learners, by limiting their opportunities to progress into certain career fields. This disproportionately affects groups which are already underrepresented in STEM (Science, Technology, Engineering and Mathematics) related fields (Blanton, Isler-Baykal, et al., 2019) (Blanton, Stroud, et al., 2019). This effect underlines the need to introduce algebra in a way which learners can understand. Introducing early algebra in the foundational years of education gives learners ample occasions to develop a deeper understanding of the subject. This is discussed further in chapter 3.

Purpose 2: Generalised arithmetic

This purpose is evident in the period starting from 1700. Key writers include Bourdon, Bézout, Euler, Pike and Todhunter. Work motivated by this purpose emphasises the syntax and semantics of early algebra. Solving an equation is the same as finding the unknown value. From this purpose, the idea emerged that secondary-school algebra should not necessarily over-emphasise operating with algebraic symbols and other representations of varying quantities (Ellerton, Kanbir and Clements, 2017). It is known that a “cognitive gap” exists between the traditional arithmetic approach in primary school and transitioning to the learning of formal algebra in the secondary school. The traditional way of teaching arithmetic does not support algebraic thinking in later grades (Roberts, 2012). In early years education, pre-algebra or early algebra is often solely viewed as the manipulation of number and geometric patterns where learners are simply expected to extend, copy, or repeat patterns (Du Plessis, 2018). Very rarely are learners expected to notice patterns in arithmetic. Early algebra should be approached as arithmetic which focuses on the underlying structure and patterns in mathematics (Roberts, 2012). The fundamental purpose of mathematics can be seen as the finding of patterns in numbers and operations. Therefore, generalised arithmetic should become a focus of early years algebra to support learners in seeing the patterns and rules when doing basic operations. Learners should understand why operations have certain results and should be able to communicate the generalisations they notice. Generalised arithmetic involves being deliberate about when something happens and exploring when something happens, as well as when it **always** happens (Roberts, 2012). In other words, the learners are engaged in finding generalities in mathematical situations. When engaged in generalised arithmetic, learners should be prompted to observe patterns in groups of number sentences and sequences of sums (Roberts, 2012). Learners’ ability to generalise should be developed through discussions about how special numbers like zero behave, and which relationships can be noticed between properties. As learners’ algebraic thinking become more sophisticated, they should be able to describe what generalities they notice, and to provide justifications for why some things will always be true (Roberts, 2012). Lastly, learners must be able to talk and reason about equivalence.

Purpose 3: A pre-requisite for entry to higher studies

This purpose is evident in the period starting from 1800 . The key role players in shaping this purpose were university prerequisite developers. In the nineteenth century, algebra as a subject was a prerequisite for many higher education institutions, especially prestigious universities. This has changed over the last 50 years (Ellerton, Kanbir and Clements, 2017). However, a solid background in algebra is generally needed to enter STEM-related fields, and most learners are currently unable to meet this standard. This leads to limited opportunities to study and work in STEM-related fields (Blanton, Isler-Baykal, et al., 2019) (Blanton, Stroud, et al., 2019). Furthermore, being able to think algebraically contributes to success in every avenue of the job market, as being able to see patterns, generalise, and communicate your thoughts clearly are beneficial skills regardless of career path. This purpose further emphasises the need for excellent early years algebra in schools Schoenfeld (1995, p. 11-12).

Purpose 4: A language for modelling real-life problems

In 1870, a new purpose for algebra emerged: using it as a language to model real-life problems. This remains one of the main purposes of school algebra. Key contributors in the literature with regard to this purpose are Hodgson, Lacroix, Perry, Klein and Moore. When doing algebra, learners need to be able to think in a functional manner. This purpose of algebra is to enable learners to solve real-life problems using algebraic objects like tables of value, plotting and interpreting on Cartesian values, and describing sequences recursively and explicitly (Ellerton, Kanbir and Clements, 2017). Modelling situations can be seen as one of the main aims of algebra (Kaput, 1999). Kaput describes algebraic modelling as “algebraifying” an arithmetic problem so that the constraints which govern its context are relaxed, reasoning is liberated, and learners are able to explore the problem in a more general form (Roberts, 2012). Algebraic modelling as a language should be used to appreciate the value of algebra as a means to solve problems (Vermeulen, 2007). It is important that learners realise the value of algebra and algebraic objects in solving real-life problems. Algebra should not be presented as an abstract concept where equations are merely solved without context, as this prevents learners from seeing the value of the subject (Kaput, 1995b).

Purpose 5: An aid for describing basic structural properties

In 1870, algebra started being used as a tool to desribe the fundamental structural properties of mathematics. This is represented in the literature by writers like Klein, Bourbaki, Gattegno and Dienes. Klein employed function concepts, structural ideas, and associated symbolisms from algebra to geometry. This purpose started influencing school curriculums in the 1950’s and 1960’s. Gattegno and Dienes argued that young learners could learn algebra before arithmetic and that the structural properties in mathematics should be emphasised (Ellerton, Kanbir and Clements, 2017), calling into question the traditional approach of arithmetic preceding algebra in early education. It is imperative for learners to be made conscious of the structure underlying everyday mathematics at a young age. Learners can do algebra in the early years, but exactly what this entails should be considered carefully (Roberts, 2012). At the heart of algebraic reasoning in the early years lies a fundamental understanding of the mathematical structure of arithmetic expressed by language and gestures using concrete materials and representations (Warren and Cooper, 2008). Roberts (2012) concurs, arguing that early algebraic teaching should be based around the teaching and learning of arithmetic which focuses on the underlying structures and patterns which emerge from arithmetic. A further consideration of early algebra and the importance of structure will be done in chapter 3.

Purpose 6: A study of variables

In the 1960s, the idea that variables should be introduced to school algebra emerged. This idea was introduced by the School Mathematics Study Group (SMSG) as well as Davis and Chazan. Solving equations was taken to be the finding the value of a variable which would make a given open sentence true or false. Tables of values and Cartesian graphs were seen as representing relationships between variables. Structural properties, like the distributive or commutative properties expressed in algebraic language, were seen to be statements using variables (Ellerton, Kanbir and Clements, 2017). Variables are versatile tools to describe mathematical ideas in concise ways. A variable represents the measure or amount of an object, and not the object itself. Variables can represent discrete or continuous quantities (Blanton *et al.*, 2015). Kaput describes two core aspects of the development of algebraic thinking which run throughout his three strands in the content area: using symbols to generalise, and acting on symbols to follow rules (Kaput, Carraher and Blanton, 2008). In this paradigm, learners solve arithmetic problems where symbols represent variables which themselves represent unknown values. Situations can be modelled by means of variables which express a class of functions. When using modelling as an algebraic language, variables provide parameters to explore effects in pure arithmetic word problems with the aim of “algebrafying” the problem and representing it in multiple ways (Roberts, 2012).

These purposes which emerge from history provides a clear view of what some of the aims of school algebra should be, and closely align with the main components and Big Ideas of early algebra which informs early algebra teaching and learning. It is important that educators and learners are made aware of these aims before they start studying algebra. These purposes for school algebra provide valuable insights which, combined with Kaput’s core aspects and strands of early algebra (Roberts, 2012), and the five Big Ideas of early algebra (Blanton *et al.*, 2015), can inform a framework for the teaching and learning of early algebra in the classroom. Such a framework is described in Chapter 5.

2.5. THE KEY ELEMENTS OF THE DEVELOPMENT OF ALGEBRAIC REASONING FROM AN OVERVIEW OF THE LITERATURE

In a similar manner to the six purposes for algebra arising from the literature about the history of school algebra by (Ellerton, Kanbir and Clements, 2017), Mason and Sutherland (2002) attempted to summarise the key aspects of algebra. To this end, they grouped historical algebraic literature in assemblages according to their time period and location of origin. These five assemblages include (1) a sample of older sources, (2) the work of Bednarz, Kieran and Lee (1996), (3) an Australian assemblage, (4) an American analysis, and (5) an Italian assemblage.

A1: A sample of older sources: J. A. Wright (1906), Sir Percy Nunn (1919), Hans Freudenthal (1973 – 1991)

When examining these works, many of the issues raised in regards to algebra are still relevant today. Wright raised the issue of algebra’s roots in arithmetic (Mason and Sutherland, 2002). He suggested four functions for algebra, which are still considered relevant today:

- To carefully establish and extend the theoretical processes of arithmetic.
- To strengthen learners’ ability to compute by practicing and by developing useful computational devices.
- To develop an equation which can be applied towards the solution of a problem.
- To learn as much about the subject area as is needed for the later study of mathematics or physics.
  (Mason and Sutherland, 2002, p.11)

When engaged in generational thinking in the course of doing arithmetic, learners analyse information to conjecture arithmetic relationships between numbers and their properties. These conjectures can be expressed in words or variables as symbols (Knuth et al., 2014). Learners should not have trouble moving from symbols to the values they represent. They should understand that variables represent a number value associated with an object, rather than object itself (Knuth et al., 2014). Learners should continuously replace symbols with the numbers they symbolise. that the value of symbols lies in their being used to express generality. Even so, learners should be made aware that symbols represent something and are not simply abstract concepts entirely separate from context (Mason and Sutherland, 2002).

Wright further identifies the challenge that algebra’s innate rule-based nature makes it difficult for teachers to teach in a relational rather than an instrumental manner. Often drill work is used a teaching tool when it comes to teaching algebra. On the contrary, an important aim of algebra should be to achieve generality. Wright argues that generality in algebra occurs when problems are classed into groups of similar problems (Mason and Sutherland, 2002). Learners must be led to see the commonality between problems and how to express these generalities.

Nunn makes a clear distinction between arithmetic and algebra. Arithmetic, according to him, is focused on calculations performed to reach an answer, whereas algebra is focused on the **process** of calculating (Wright, 1906). The aim of teaching and learning of arithmetic must be to elucidate the structures and patterns which emerge from mathematical situations (Roberts, 2012). He further makes an important distinction between analysis and generalising. Analysis is the abduction of structure in mathematics, while generalisation is recognising pattern inductively (Mason and Sutherland, 2002). When patterns are analysed for internal structure, n is used to describe the structure of the pattern (Mason and Sutherland, 2002). In many mathematics classrooms, learners are expected to generalise based on an example of a pattern (Du Plessis, 2018), but the structure is rarely analysed. Nunn states that learners cannot see the structure in mathematics by merely looking at one example. Many examples should be examined, and learners should be made aware of the relations between these examples (Mason and Sutherland, 2002).

Nunn (1919) agrees with many authors in mathematics that algebra should be regarded as generalised arithmetic. The traditional practice of teaching specific arithmetic before formal, symbolic algebra, is also questioned by Nunn. Generalised arithmetic is the simplest type of symbolic algebra and should be the steppingstone between arithmetic and formal algebra (Mason and Sutherland, 2002).

Freudenthal (1983, p.467) emphasises the importance of basing instruction on a child's prior experiences. He states that if knowledge called upon in mathematics already exists within the child, learning can be focused on strategies and solution processes. In Freudenthal's China Lectures (Freudenthal, 1991, p.62), he proposes that the question should be raised if negative numbers form part of to arithmetic or algebra. In *Mathematics As An Educational Task* (Freudenthal, 1991, p.224) he continues by claiming that fractions are composed to allow unhindered division, but they emerged from extending the number system to admit solutions to multiplication problems, for example "what times 3 will give 5?" In the same way, negative numbers arose to meet the need to count backwards and answer questions like "what added to 4 gives 2?" Freudenthal calls this the algebraic principle.

For Freudenthal, general number does not exist. There are indeterminates (represented by letters) and unknowns (represented by symbols). When treating algebra as a translation process between languages, certain challenges arise. Freudenthal advocates for distillation of axioms of a field, which is also known as generalised arithmetic (a fundamental component of early algebra (Kaput, Carraher and Blanton, 2008)). The properties of arithmetic emerge from, and are expressed by, the use of symbols to produce rules for manipulating numbers (Mason and Sutherland, 2002). Roberts (2010, p.169) explains that generalising arithmetic is the exploration of the properties of numbers and operations and generalising about particular number properties and relationships which are fundamental for the development of algebraic thinking in the early years. This perspective on the role of generalisation will inform the construction of the framework for implementing Early Algebra in the classroom.

Regarding the difficulty learners experience when working with algebra as a formal school subject, Freudenthal states that 'when calculating starts, the thinking finishes' (Freudenthal, 1973). He blames this difficulty on the didactic method, where learners initially learn by insight and then permanently move on to automatisms (Mason and Sutherland, 2002). The traditional algebra curriculum has over-emphasised the semantics of algebra. When acting on formalisms semantically, one's actions are directed by what one believes the symbols should stand for (Kaput, 1995a). A semantic justification focuses on the numbers in an expression. This results in many learners being unable to see the meaning and value of mathematics. The power of using the form of mathematics as a basis for reasoning is lost when learners are engaged with endless practicing of rules for symbol manipulation (Kaput, 1999). Freudenthal advocates for solving problems in a way which will improve learners' understanding (Mason and Sutherland, 2002). Syntactically guided manipulations on formalisms are the core of algebra but to ensure the effective learning and development of actions on formalisms, a semantic starting perspective should be taken. Formalisms should initially be viewed as representing something which the learner has experienced (Kaput, 1995a). Most actions and manipulations of symbols involve a combination of syntactical and semantic actions (Kaput, 2018). Mathematical activity is the interactions between notational systems and their reference fields. When a new problem is encountered, learners will often reach for something familiar to help them make sense of the situation. This includes concrete objects like blocks, counters, mental pictures, or symbols in the form of numerals or letters. As seen in the geometric stage (Katz and Barton, 2007) of algebraic development, geometry representations are important to provide context for, and mental pictures of, mathematical situations (Ratith Ayu Apsari *et al.*, 2020). Manipulation leads to leaners getting a sense of the problem and how to approach it. Learners can then articulate their thoughts on increasingly sophisticated levels to produce a solution. Understanding algebra means being able to show connections between knowledge of procedures with knowledge of concepts (Kaput, 1999).

In his work *Weeding and Sowing* (1978), Freudenthal challenges the notion that learners learn through the repetition of many examples. It is not certain that when learners are able to see generalisations across a variety of problems, that their ability to generalise stemmed from seeing many cases where generalisation was reached. Freudenthal consistently emphasises the importance of studying structure in mathematics. In his *China Lectures* (Freudenthal, 1991), he furthermore argues for the value of word problems as opportunities to generalise.

When looking at the works from these three authors, many important aspects of algebra in education comes to the fore. Many of these issues raised are still issues today and sheds light on the challenge that educators face in the algebra classroom. A need for reform in the teaching of algebra is imperative. This study will aim to show that the introduction of the teaching of early algebra with the appropriate teaching approach may be one of the possible routes to the improvement of algebra instruction in schools.

A2: Bednarz, Kieran and Lee (1996)

Approaches to Algebra: Perspectives for Research and Teaching (Freudenthal, 1991) provides an overview of four instructional practices in the teaching of algebra. These four practices are generalisation, problem-solving, modelling and function, and are connected with historical perspectives on the teaching of algebra (Freudenthal, 1991).

1. Algebra through generalisation

From the beginning of mathematics, one of its aims was to solve not only problems, but whole classes of problems. Al-Khwarizmi in the syncopated stage aimed to classify the solutions to classes of problems by categorising six types of equations to solve algebraic problems (Katz and Barton, 2007). Often rules to find solutions would be provided. Such rules are expressions of generalities which are expected to be applicable to a whole class of problems (Mason and Sutherland, 2002). Mathematics at school arose through the search for techniques and rules to solve groups of problems. These strategies are isolated and then taught to learners, who are expected to memorise the techniques. Making learners aware of the nature of mathematical generalisation should be at the heart of mathematics. When generalisation underlies all of mathematics the subject area will become less challenging for learners (Bednarz, Kieran and Lee, 1996).

The expression of generality is present from the earliest days in a child's life. Learners can generalise when they come to school, but it is the responsibility of the educator to draw on this and develop the sophistication thereof. Generalisation is an ongoing process of growing sophistication. Below are some of the main aspects of expressing generalisations (Mason and Sutherland, 2002, p. 22-23):

- Awareness of generalisation is present from children's earliest encounters with numbers.
- Noticing generality in specific things, and specific things in generalities, are capabilities which learners bring to school, and should be draw upon.
- Expressing generality should not be seen as a skill which is acquired and then used, but rather one whose sophistication is developed continuously.
- Manipulations of expressions are possible because we recognise that different looking expressions can represent the same result, and that imposing constraints on generality delivers equations and inequalities for which techniques can be developed.
- Generality is present in all mathematics, not only in problem solving.
- For generalising to be taught successfully, it should be emphasised in all content areas of mathematics, not only in patterns or problem solving.
- Learners' capabilities should be drawn upon, rather than teachers doing the work of generalisation for them.

2. Algebra through word problems

When developing algebraic by engaging with problems, learners are expected to by think, imagine, draw on prior experiences, and articulate relationships. From the constructivist perspective, learners interpret what they learn and give it their own meaning based on their existing, but not yet explicit, knowledge (Cobb et al., 2014). Word problems have been present from the beginning of mathematics, and examples can be found in Egyptian, Babylonian, and Chinese mathematics (see 2.2.1). When solving word problems, it is not the problem which is arithmetical or algebraic, but the way in which it is approached. They give rise to structure and symbolic manipulations. Problem-solving is seen as the underlying drive for generalisation and the functional approach (Mason and Sutherland, 2002), and plays a role in number and geometrical or spatial domains. Traditional problems stem from changing an unknown number to get an outcome which is specific. Solving the problem is the process of undoing the transformation (Mason and Sutherland, 2002). Below are some of the main features of word problems in algebra (Mason and Sutherland, 2002, p.27):

- Word problems are often misused at the end of chapters in textbooks to challenge learners. This leads to demotivation and learners who do not have deep understanding of mathematical concepts.
- Problems should be used as the core of mathematics teaching, because they motivate explorations and development in mathematics. Various mathematical techniques should emerge from the problems solved.
- When solving problems, learners use a variety of strategies and methods. These methods are typically arithmetical, unless learners are familiar with expressing generality and confident in using symbols.

3. Algebra through modelling

Mathematical modelling can be described as the construction of mathematical narratives for real-life situations (Mason and Sutherland, 2002). Mathematical models are used to illustrate or explain a mathematical situation or the process of problem solving. Models can be concrete objects, representations of objects, or ideas expressed in words (Roberts, 2012). Learners should use sophisticated mathematical thinking, involving both the particular and the general, when trying to make sense of what they are working with (Mason and Sutherland, 2002). Algebraic modelling provides the opportunity to investigate

any word problem context which is “algebrafied” by liberating the problem from its constraints and allowing for exploration of the patterns which emerge (Roberts, 2012). Below are the main aspects of modelling (Mason and Sutherland, 2002, p. 28):

- Mathematics should be used to solve practical problems concerned with the material world.
- Modelling calls upon the use of many capabilities: mental imagery to move from a specific situation to an abstract imagined world, and then to the world of manipulable mathematical symbols, before going back to the original setting or context.

4. Algebra through functions

Functions are often seen as the most fundamental mathematical objects. Functions can be thought of in various ways: a table of values; a graph showing relationship; a rule expressed using algebraic symbols; and all three manifested in spreadsheets, calculators, and specially designed computer software. A special relationship between symbols x and y is present in most of these functions (Bednarz, Kieran and Lee, 1996). Functional thinking is one of the fundamental components of early algebra (Kaput, Carraher and Blanton, 2008). In the current South African CAPS (Curriculum and Assessment Policy Statement) one content area is patterns, functions and algebra, where functional thinking is emphasised. Functional thinking involves recognising a regularity in elementary patterns, ideas of change including linearity, and representation through tables, graphs and function machines (Roberts, 2012). Below are the main aspects of functions emerging from the work of Bednarz, Kieran and Lee (Mason and Sutherland, 2002, p.30):

- There are multiple representations of functions.
- A functional approach manipulates dynamic imagery possibilities of digital technology.
- A functional approach provides the chance to emphasise the representation and interpretation of relationships, leaving computation to computer software.
- It develops awareness and capability with function, which is a powerful idea in modern mathematics.

A3: Work from the Australian Mathematics Education Community

Research into the teaching of algebra commenced in Australia in the early nineties. The focus of the research was learners' interpretation of letters, understanding of variables, problem solving abilities, and the cognitive and linguistic demands of learning algebra. The research was mainly conducted by means of interviews, questionnaires and classroom observations (Mason and Sutherland, 2002).

Research specifically on learners' understanding of symbols was based on earlier studies by Collis (Mason and Sutherland, 2002), Küchemann (1981) and Booth (1984). More recent Australian research focused on the teaching approaches used to introduce learners to mathematics. Learners' first introduction to the use of letters for unknown or general numbers is in writing formulas for number patterns where two variables are related by a rule (Mason and Sutherland, 2002). The idea of using variables to represent unknowns, linear problem situations, and function rules is one of the Big Ideas of early algebra (Knuth et al., 2014).

MacGregor and Stacey (1997) conducted a test where 2 000 learners aged 7-10 engaged with algebraic problems. They classified learners' interpretation of letters into the following categories of misconceptions:

- The letter is read as an abbreviated word.
- The letter is assigned a numerical value that would be reasonable in context.
- The letter is assigned a numerical value related to its position in the alphabet.
- The letter has the value 1 unless otherwise specified.
- The same letter can represent various quantities (MacGregor and Stacey, 1997).

Following up on their research, MacGregor and Stacey (1997) observed classrooms to find reasons for the common misconceptions learners have when working with algebra. Some of the reasons include:

1) Instinctive assumptions and sensible, logical reasoning about an unfamiliar notation system.
2) Analogies with symbol systems used in everyday life or in other content areas of mathematics or school subjects.
3) Interference from new concepts taught in mathematics.
4) Poorly designed and misleading teaching materials (MacGregor and Stacey, 1997).

MacGregor and Stacey (1997) infer that when the "correct" teaching approach is followed, these misconceptions and misunderstandings can be avoided. Mason and Sutherland (2002) disagree and suggest that all teaching will inevitably lead to constructions of understanding which will be correct in some circumstances and incorrect in other circumstances. This suggests that symbolic algebra can never be taught as a discrete or separate part of the mathematics curriculum. In many curricula, learners are simply introduced to the use of symbolic notation in an isolated module. It is never made relevant to other content areas in mathematics (MacGregor and Stacey, 1997).

Furthermore, the Australian curriculum promotes the development of early algebraic thinking from primary grades. Algebra and algebraic notation are introduced by means of a "pattern-based" approach where algebraic notation is used as a language to indicate the relationship between two variables. The pattern-based approach deals with generality first, which leads to learners' understanding of functional relationships and their algebraic descriptions (MacGregor and Stacey, 1997).

A4: Kilpatrick, Swafford and Findell (2001)

The book *Adding It Up: Helping Children Learn Mathematics* (Mason and Sutherland, 2002) provides an overview of North American and international research.

Algebra in the USA is presented as a separate course from mathematics and is often prerequisite for college or university admissions. Textbooks are predominantly focused on teaching guided transformation of symbols, rather than taking a more balanced approach which focuses both on transforming symbols and expressing generality and abstraction, and includes algebra as generalised arithmetic (Kilpatrick, Swafford and Findell, 2001).

Most courses start by introducing linear functions and then move on to quadratic functions, but it may be beneficial to introduce these levels of functional thinking simultaneously. There is a crucial relationship between functional thinking and the early algebraic thinking practices of generalising, justifying, representing and reasoning with mathematical relationships (Blanton et al., 2015). This relationship could eradicate the misconception that linear and quadratic functions are the only functions which are relevant. In the USA, there is also a disproportionate focus on product over process: arithmetic is predominantly answer orientated and does not focus on the mathematical processes involved. Arithmetic should be approached with attention to the generalities that are present in the methods of calculation, to instil and develop algebraic thinking in learners (Roberts, 2012). This could make algebra less challenging in later grades (Mason and Sutherland, 2002).

Simply introducing algebra as “arithmetic with symbols” has been highly unsuccessful in the history of school algebra (Kilpatrick, Swafford and Findell, 2001). Teachers and researchers have been had difficulty developing a school algebra where learners see the need for using algebraic notation to represent mathematical ideas and solve mathematical problems. Learners struggle greatly to express generality. These challenges may be attributed to the fact that learners are introduced to generality too late in their schooling careers. The concept of generality should be embedded into all aspects of mathematics teaching (Sutherland, 1991).

A5: Work from the Italian Mathematics Education Community

Italian research in mathematics is typified by an analysis of complex problems in small-scale studies, which is followed-up by empirical work in realistic learning situations. The algebra curriculum in Italy highlights rigour, relation between hypotheses expressed and experimental work, and accurate language (Mason and Sutherland, 2002).

Boero (Mason and Sutherland, 2002) focuses on the importance of transformation and anticipation as key processes in algebraic problem solving. He suggests that when transformation happens before formal algebra has been introduced, learners often transform the problem situation by means of arithmetic or geometric or physical manipulations of variables. These transformations can be called pre-algebraic. When transformations occur after the formalisation of algebra, transformations are based on algebraic notations and leads to more possibilities of transformations.

Dettori, Garutti and Lemut (2001) also explores the nature of the patterns and the strategies which learners use to solve problems. They suggest that an arithmetical approach to solving problems makes use of a step-by-step method, whereas an algebraic approach takes a global-synthetic perspective on the problem.

Italian researchers have found that different types of word problems do not necessarily provoke distinct problem-solving approaches. Therefore, presenting a specific type of problem would not necessarily evoke algebraic thinking. The teacher plays a key role in facilitating learners’ thinking and guiding their understanding of mathematical concepts (Mason and Sutherland, 2002).

From Assemblage 1, the importance of the relationship between arithmetic and algebra comes to the fore. Arithmetic should be approached in a general manner where learners predict relationships between numbers and their properties (Mason and Sutherland, 2002, p.11, Knuth et al., 2014). This idea reinforces the need for a constant emphasis on generalisation, especially when engaging learners in arithmetic. This assemblage underpins the generalised arithmetic activities in the instructional design sequence in Chapter 5. Assemblage 2 explores four instructional practices of algebraic reasoning, namely, algebra through generalisation; word problems; algebra through modelling; and algebra through functions (Freudenthal, 1991). These instructional approached were used when designing examples of early algebra activities in the instructional design sequence. In Assemblage 3, it becomes clear that symbolic algebra is an important facet of early algebra teaching and should be made relevant to all other areas of mathematics (MacGregor and Stacey, 1997). Assemblage 4 focusses on research done in North America and specifically mentions the challenges experienced in the teaching of algebra due to a lack of generality in the pedagogical approach (Kilpatrick, Swafford and Findell, 2001, Mason and Sutherland, 2002, Sutherland, 1991). Once again, the idea that generality should be embedded throughout mathematics education comes to the fore and this reinforces the importance of an instructional design sequence which attempts to employ generalisation throughout a variety of mathematical activities. Assemblage 5 is based on research done in Italy and sees transformation and anticipation as the main processes involved in algebraic reasoning. Furthermore, patterns and strategies for problem solving is researched. The role of the teacher as a key role player in the development of algebraic reasoning emerges (Mason and Sutherland, 2002). From this assemblage it is known to be important to accompany the instructional design sequence with a 'blueprint' of the role of the teacher and the classroom setting of an 'ideal' mathematics classroom, which is provided in detail in Chapter 5. The five assemblages provide valuable underpinnings for the construction of the instructional design sequence in Chapter 5.

2.6. CURRENT SITUATION OF EARLY ALGEBRA IN SOUTH AFRICA

The CAPS document in South Africa for Grades 1 to 3 focuses on the teaching of early algebra in the content area "patterns, functions and algebra" (Department of Basic Education, 2011). In this content area, algebra is described as the language for investigating and communicating most of mathematics, and as capable of being extended to the study of functions and relationships between variables. A central focus of this content area in the early years mathematics education is the development of learners' ability to achieve effective manipulative skills in the use of algebra. Other focuses include: the descriptions of patterns and relationships through the use of symbolic expressions, graphs and tables, the identification and analysis of regularities and change in patterns, and relationships that make it possible for learners to make predictions and solve problems (Department of Basic Education, 2011, p.9).

There is an overemphasis on number and geometric pattern-based activities as the primary means of developing algebraic thinking. The CAPS document mentions the following skills which learners should employ when working with patterns (Department of Basic Education, 2011, p.9):

1) Using physical objects, drawings, and symbolic forms to copy, extend, describe, and create patterns.
2) Copying a pattern helps learners to notice the logic of how it is formed.
3) Extending a pattern helps learners to check that they understand the logic of the pattern.
4) Describing patterns develops learners' language skills.
5) Focusing on the logic of patterns lays the foundation for developing algebraic thinking.

6) Working with number patterns supports the development of number concept, while working with geometric patterns supports the development of spatial awareness and shapes.

The first content area of the CAPS document, 'Numbers, operations and relationships', provides ample opportunity for developing learners' knowledge and understanding of algebra (Roberts, 2010). Although early algebra is not explicitly mentioned, attention is paid to noticing relationships between different kinds of numbers and representing numbers in different ways. This content area also allows for the exploration of equivalence and the equal sign (Department of Basic Education, 2011). This appears to be consistent with the aim and fundamental components of early algebra, which include generalised arithmetic and seeing the relationships between numbers and their properties (Kaput, 1995; Roberts, 2012).

Learners should work structurally with patterns in the early years of mathematics education and should be able to notice the underlying structure of patterns which emerges from arithmetic. Almost all of mathematics is based on patterns and structure. A mathematical pattern is any predictable regularity and usually involves number or space. Mathematical structure refers to the way in which a pattern is organised (Mulligan *et al.*, 2008). The CAPS document emphasises the importance of noticing **logic** in patterns (Department of Basic Education, 2011, p.9), but the approach has been neglected in the implementation of early algebra in classrooms, textbooks and teacher support guides (du Plessis, 2018). The potential of patterns to develop algebraic habits of minds by encouraging learners to focus on the underlying structures which emerge from patterns remain unexplored and under emphasised in the South African mathematics education context (du Plessis, 2018). Even though the CAPS document (Department of Basic Education, 2011, p.9) emphasises the importance of logic in patterns, a relational approach to analysing and understanding patterns is missing.

A major challenge in South African mathematics education is the fact that many teachers were themselves not taught the content area of 'Patterns, functions and algebra' in primary school or their teaching training. This results in confusion about the inclusion of this content area and its effective implementation in the classroom (Roberts, 2012).

Research (Mulligan, Mitchelmore and Prescott, 2005; Warren and Miller, 2010; Papic, Mulligan and Mitchelmore, 2011; T. Cooper and Warren, 2011) calls for an early algebra approach where the interconnectedness of pattern, structure and algebraic reasoning, and their power to promote basic numeracy in young learners, are prioritised in classrooms. A lack of coherence is present throughout pedagogic communication and activities (du Plessis, 2018). The random selection and sequencing of activities at a primary level is an obstacle to the development of early algebraic thinking and the ability to generalise. This lack of coherence points to a lack of consciousness in pedagogical practice. In order to effectively develop early algebra, teachers need to reflect and revise their teaching strategies to create a carefully planned sequence of learning activities. This sequence should scaffold the introduction of a structural approach to the mastery of sequencing (du Plessis, 2018) and generalisation throughout the whole of mathematics (Roberts, 2012).

2.7. CONCLUSION

By looking at the history of algebra through the ages, valuable lessons can be learned which should inform the teaching of algebra in classrooms today. From history it becomes clear that there exists a definite need for algebra, algebraic thinking, generalisation and the use of symbolic notation to solve problems we encounter in everyday life. The importance of mathematics education as a whole emerges from the need for problem solving tools. The stages of algebra's emergence through history, closely resembles the levels of thinking that occur as algebraic thinking is developed in schools (Nixon, 2009). Progression in algebraic thinking takes place when the need for algebraic representation and the need to see relationships are present. Learners need to be made aware of the algebraic tools they have access to and the opportunities which algebraic thinking creates for solving various types of problems. The role of the teacher and their perspective on the teaching of algebra becomes very important. Young learners, even with limited exposure to arithmetical thinking, can start to think algebraically and make generalisations. With suitable instructional support and encouragement of the habit of mind of finding generalisations, learners can understand algebraic concepts at a young age (Radford, 2011).

The way in which algebra is currently taught in schools is failing learners. Many experience algebra as a challenging and unattainable content strain of mathematics. They do not see the need for it or its relevance to the material world. This results in many learners either not progressing to university, or opting to study in other than STEM-related fields (Blanton, Isler-Baykal, *et al.*, 2019) (Blanton, Stroud, *et al.*, 2019).

Algebra education in its current form over-emphasises the traditional arithmetic-then-algebra approach (Radford, 2015). Learners are over-exposed to answer-orientated arithmetic and only progress to a more abstract algebra when they are proficient at working with numbers. The focus of algebra in the later grades is then to manipulate symbols and simplify equations. Algebra is taught in an isolated manner which furthermore alienates learners from the possibilities and opportunities it presents. Research has shown that this approach is not working. There is a need for reform in algebra education.

This study aims to explore and construct an instructional sequence to provide ideas for implementing early algebra in the classroom with the aim of making algebra in the later grades accessible to more learners. Learners are able to think algebraically from a very young age. Generality is noticed by very young learners in the world around them. From primary grades the noticing and expression of generality and structure should be embedded in all content areas of mathematics. It cannot be taught as a discrete subject (Blanton, Stroud, *et al.*, 2019). In Chapter 3 early algebra, generality and structure will be explored to provide an instructional design sequence to develop algebraic thinking and generalisation with the aim of closing the gap between arithmetic and formal algebra which occurs in the traditional approach to teaching algebra.

CHAPTER 3: INSTRUCTIONAL SEQUENCES FOR EARLY ALGEBRA, GENERALISATION AND STRUCTURE

3.1. INTRODUCTION

In chapter 2, the history of algebra was explored to see how the fundamental components of algebra emerged. Algebra emerged as a tool to solve problems in everyday life. Solving the problems involve algebraic thinking, generalising and the use of symbolic notation. The stages of the development of algebra throughout history closely relates to the development of level of thought as explained by Nixon (2009). Algebraic thinking develops and progresses when a need for the use of algebraic expressions and seeing regularities emerges. Learners need to be aware of their capabilities in terms of the algebraic tools they have in their possession. Radford (2011) claims that learners are able to think algebraically from a young age. With suitable instructional support habit of mind of finding generalisations, learners are able to understand algebraic concepts. Development of algebraic thinking takes place when a need for algebraic representation and the need to see relationships arises from real-life problem situations. Learners need to be aware of the algebraic tools they can use to model mathematical situations, and the opportunities which algebraic thinking creates for solving various types of problems (Kaput, 2008). The role of the teacher (who will be referred to as he as the general gender in this study) and his perspective on the teaching of algebra is very important. By looking at the history of algebra through the ages, valuable lessons can be learned which should inform the teaching of algebra in classrooms today. From history it becomes clear that there exists a definite need for algebra, algebraic thinking, generalisation and the use of symbolic notation to solve problems we encounter in real life and mathematics (Sfard, 1995; Katz and Barton, 2007; Tabak, 2011).

In this chapter, guided reinvention and emergent modelling as design heuristics based on the principles of Real Mathematics Education (RME) (Gravemeijer, 2007, 2020) will be used as a framework to design an instructional sequence of generalisation activities. The main components of early algebra which emerge from the stages of the historical development of algebra will inform the instructional sequence. A systematic literature review of early algebra, the development of algebraic thinking, and the main components of early algebra will provide the basis for the instructional sequence.

3.2. SETTING THE SCENE FOR EARLY ALGEBRA

In this section of Chapter 3, he emergence of early algebra, the current situation in classrooms, and the importance and scope of early algebra, will be systematically reviewed to inform the construction of the instructional design.

3.2.1. From formal to early algebra

Through the ages, algebra has been a central focus of mathematics. Algebra continues to be an essential component of modern mathematics. Yet, in the school curriculum, algebra has been misunderstood and misrepresented as an abstract and challenging subject, to be taught only to secondary learners. In truth, algebra and algebraic thinking are fundamental to the basic education of all learners, beginning in the foundational years (Friel, Rachlin and Doyle, 2001).

In the book *Algebra in the Early Grades* Kaput (2008) mentions that 'the algebra problem' has become a trend in the 21st century mathematics teaching domain. Traditionally, algebra has been taught only after learners have mastered arithmetic and reasoning with numbers. Arithmetic thinking was seen to be a pre-requisite for moving on to algebraic thinking (Radford, 2015). The curriculum over-emphasised computational work in the early and middle grades followed by a superficial treatment of algebra in the secondary grades (Blanton, Stroud, *et al.*, 2019). In this way, algebra was taught as a discrete and isolated subject area, where learners were expected to merely simplify equations. This approach led to widespread failure in school mathematics and unmotivated learners. Learners did not see the value of working with symbolic mathematics, as it was not made relevant to their everyday lives and no context was provided for algebraic problems. Learners are first exposed to a world of numbers and numerical procedures when working through arithmetic. They are then later introduced to a world of symbols and symbolic procedures in algebra. The connection between arithmetic and algebra, and problems or situations where we would use them, is missing (Smith and Thomson, 2008). It has become a central theme of early algebra that generalising in all aspects of mathematics is central to developing algebraic thinking (Roberts, 2012).

Furthermore, the teaching and learning of functions was historically confined to the secondary grades, as it was believed that learners needed a type of abstract thinking which was only attainable at a certain age. However, functional thinking has recently been argued to be a critical route into the teaching and learning of early algebra (Blanton *et al.*, 2015). Functional thinking is claimed to be a fundamental component of early algebra (Kaput, 2008) (see 3.5.4). There is an important relationship between functional thinking and the early algebraic thinking practices of generalising, justifying, representing, and reasoning with mathematical relationships (Blanton *et al.*, 2015). The approach of teaching computational arithmetic for six to eight years, followed by an isolated and superficial teaching of algebra in the later years, has led to high learner failure. Learner drop outs were especially high among economically and socially disadvantaged groups (Kaput, 2008).

In South Africa, algebra is a key course in the secondary mathematics curriculum. It is a prerequisite subject for post-secondary mathematics, science, and engineering courses. It is therefore considered a gatekeeper course (van Laren and Moore-Russo, 2014). Algebra's gatekeeper effect has led to the marginalisation of learners in algebra education, by depriving learners of opportunities in certain career fields, disproportionately affecting underrepresented groups. This, in turn, has led to underrepresentation of these groups in STEM-related fields especially (Blanton, Isler-Baykal, *et al.*, 2019) (Blanton, *et al.*, 2019). Mathematics results in South Africa, both at primary and secondary levels, continue to indicate failure of the system, teachers, and learners. Evidence suggests that the low levels of mathematics achievement in South African schools can be attributed to the low quality of teaching in primary schools (McAuliffe and Lubben, 2013). Many teachers in the South African context were not taught the content area of 'Patterns, functions and algebra' when they were in school, and thus do not understand why or how to teach this content area (Roberts, 2012). This contributes to the need for teacher development and education, and support for teachers in effectively implementing generalisation and algebraic thinking in the classroom.

It is critically important to establish firm foundations for algebra in the early years, by effectively developing the generalisation concept to ensure success in understanding complex algebra in the later years. The aim should be to teach early algebra through the generalisation concept for understanding. The aim of this chapter is to explore the importance of early algebra, generalising, and the structure of mathematics, to provide an instructional design based on the RME principles, guided reinvention and emergent modelling. The aim of the design is to provide a route for implementing the generalisation concept to develop early algebra in the classroom through a problem-based approach.

3.2.2. The current situation in classrooms

Algebra's status as the gateway to academic and economic success has called for reform in the teaching and learning thereof (Blanton, Stroud, *et al.*, 2019). Current research has identified algebra as a central concern in mathematics education and has emphasised the importance of a longitudinal approach to the teaching of algebra (Blanton *et al.*, 2015). A longitudinal approach involves the teaching of algebraic concepts from the beginning of the schooling years. Problems are evident in the transition from primary to secondary school, where a 'cognitive gap' emerges as learners transition from arithmetic to algebra. There is increasing evidence that the way in which arithmetic is taught in the primary grades is not conducive to the development of algebraic thinking the later grades, when mathematics and algebra become more complex (Roberts, 2012).

In light of the challenges learners experience with algebra in the later grades, it has been widely accepted that a progressive introduction to algebra in the early grades may aid their subsequent understanding of more advanced algebraic concepts, algebraic notation in particular (Radford, 2015). In theory, this approach would allow learners' algebraic thinking to develop more organically by enhancing their natural instincts about structure and relationships from the start of their formal education (Blanton, Stroud, *et al.*, 2019).

The Common Core State Standards in the USA also emphasised the importance of teaching algebra across all grades, with the aim of improving for the likelihood of success in mathematics in the secondary grades (Blanton, Isler-Baykal, *et al.*, 2019). This reform in algebra education represents an important paradigm shift in mathematics education. Even so, there are still questions about the impact that this approach would have on learners' success in school mathematics. Many researchers ask the question: are (young) learners able to successfully engage in a longitudinal, comprehensive approach to algebra that might prepare them for the formal study of algebra in high school? (Blanton, Isler-Baykal, *et al.*, 2019). Transforming the whole of mathematics education is a massive task which will demand financial and intellectual resources. It involves the restructuring of curriculums and changing of classroom practice, as well as assessment of, and changes in, teacher education (Kaput, Carraher and Blanton, 2008).

The focus on generalisation and structure within patterns (and mathematics as a whole) remains largely unexplored in early years mathematics in South Africa. This is because of the lack of a relational approach to the sequencing of curriculum documents and materials which are provided for the teaching of patterns, functions and algebra in the foundation phase (Du Plessis, 2018). It has been proposed that algebraic thinking should be introduced earlier in the schooling of learners, and this has been widely accepted. The South African Curriculum and Assessment Policy (CAPS) (Department of Basic Education, 2011) agrees that learners should understand the logic of patterns, and that this should lay the foundation for algebraic thinking. Even so, in classroom practice little emphasis is placed on the structure which exists in mathematics and patterns. Learners are mostly expected to copy or extend patterns. Learners are not given sufficient opportunities to justify and express generalisations which are noticed within patterns (see 2.6) (Du Plessis, 2018).

Textbooks are perceived as a valuable instructional tool in classrooms (Garner, 1992). According to RME principles, guided reinvention and emergent modelling of the teacher's guidance to learners' development of conceptual knowledge, depends greatly on the instructional tools they use and the sequence in which mathematical activities are organised (Gravemeijer, 2007, 2020; Stephan, Underwood-Gregg and Yackel, 2016). Most textbooks, however, lack coherence and the sequence of activities in a relational approach to make connections between concepts explicit (Velverde & Schmidt, 1998). Sood and Jitendra, (2007) found that traditional textbooks can be criticised for being repetitive, unfocused, and undemanding. They suggest four components for effective textbooks: (1) a clear and meaningful development of mathematics which promotes conceptual understanding through real world experiences, (2) tasks in textbooks that include multiple models, with explicit connections between these models, (3) opportunities in mathematics textbooks for learners to reflect on their performance, and (4) enough opportunities in textbooks for learners to apply new skills by means of scaffolding (Afonso, 2019).

Vermeulen (2016) analysed the development of algebraic thinking in three Grade 4 mathematics textbooks, and found that authors do seem to understand the expectations of the curriculum. Two textbooks offered some opportunity for algebraic thinking to develop, however the third did not develop algebraic thinking. Furthermore, he found the sequencing of activities in all three textbooks to be problematic.

According to Kaput (2008, p.6) 'the algebra problem' needs to be solved to serve four major goals:

- To add a sense of coherence, purpose, depth, and power to the mathematics curriculum from pre-school to high school.
- To amend the late, abrupt, and isolated nature of the teaching of complex algebra in later grades.
- To democratise access and opportunity to powerful ideas by transforming algebra from an exclusive and inequal subject area to a powerful tool for all.
- To build conceptual and institutional capacity and open curricular space for the new 21st century mathematics needed at a high school level.

Even though there is consensus that algebra should be introduced into formal schooling from the start, the implementation of this approach is still questioned and should be further researched. The aim of this chapter is to provide a possible framework for the teaching of early algebra via the generalisation concept in the early

years of mathematics education, based on a systematic review of the research which has already been done.

3.2.3. The importance of early algebra

Young learners, even with limited exposure to arithmetical thinking, have the ability to start thinking algebraically. With suitable instructional support, learners can understand some algebraic concepts, for example pattern generalisation (Radford, 2011). The National Council of Teachers of Mathematics (NCTM) describes algebra as a way of thinking with a set of concepts and skills which aids learners to generalise, model and analyse mathematical situations (Lee et al., 2016, p. 306). Early algebra has the purpose of deepening children's understanding of the structural form and generality of mathematics, rather than only providing isolated mathematical experiences. Algebra has the aim of providing learners with tools to simplify complex, real-life problems and situations (Smith and Thomson, 2008). This has been proved to ensure better mathematical achievement in later grades (Blanton & Kaput, 2011).

In *Does Early Algebra Matter? The Effectiveness of an Early Algebra Intervention in Grades 3 to 5* (Blanton et al., 2019) algebraic thinking is described as a critical way of thinking relevant to virtually any avenue of the job market and every part of schooling. It can, therefore, be seen as one of the most important aspects of mathematics teaching in the early years. Linder, Powers-Costello and Stegelin (2011) argue that the foundation for constructing an understanding of mathematical concepts, such as algebra, starts in the early grades. Meeting the expectation of transitioning from arithmetic to algebra is challenging for learners (see 3.3.2) and many research studies have found that students need earlier opportunities for engaging in activities that encourage algebraic reasoning (Jacobs et al., 2007, p. 259). Fox (2005) agrees that the years prior to formal schooling are a period of profound developmental changes, where many mathematical concepts are formed. Blanton and Kaput (2011, p. 6) explain that algebraic reasoning "is a common thread in the fabric of ideas that constitute mathematical thinking at elementary grades" because algebraic reasoning involves looking for generalities in all of mathematics. The increasingly complex mathematics of the 21st century requires of children to have elementary school experiences which enable a deeper understanding of the underlying structure of mathematics. A recent study by Cai, Ng and Moyer (2011), found that students perform much better when they are able to use abstract strategies. In other words, students will perform better in mathematics when they are able to reason algebraically based on the patterns and structure in mathematics which emerge when looking for generalities.

Algebra can be seen as the crux of mathematics because of its foundational role in all areas of the subject. Algebra provides the mathematical tools to represent and analyse quantitative relationships, to model situations, and to solve problems in every mathematical domain. Algebra, as stated earlier, is a gatekeeper to future educational and employment opportunities (Knuth *et al.*, 2016). Schoenfeld (1995, p. 11-12) said:

> Algebra has become an academic passport for passage into virtually every avenue of the job market and every street of schooling. With too few exceptions, students who do not study algebra are therefore relegated to menial jobs and are unable often to even undertake training programs for jobs in which they might be interested. They are sorted out of the opportunities to become productive citizens in our society.

Traditional learning systems divorce mathematics, and especially algebra, from STEM-related (Science, Technology, Engineering and Mathematics) concepts and their applications in real life (Renganathan *et al.*, 2017). Algebraic thinking is important because it allows one to solve much more complicated problems by acknowledging the unknown and approaching problems in a general manner (Mason, 2008). Problems are thus liberated from the constraints that govern them and can be viewed in a general manner (Roberts, 2012). Early algebra is necessary to bridge the gap between arithmetical thinking in the early years and isolated algebraic thinking in the later years, as early algebra should encompass arithmetic and all areas of mathematics.

3.2.4. The scope of early algebra

Early algebra differs from the algebra which is taught at a high school level; it is not the earlier implementation of a curriculum which is meant for high school learners.

Early algebra is built on the backgrounds and contexts of problems. It only gradually introduces formal notation and is integrated with the other content areas of the mathematics curriculum (Carraher, Schliemann and Schwartz, 2008):

3.2.4.1. Early algebra builds on the backgrounds and contexts of problems.

One could ask why it is necessary to expose learners to deeply nuanced problems if the goal of algebra is to get them to think more abstractly be able to use symbolic notation to represent a problem. The reason for using context rich problems lies in the way in which young learners learn: young learners use a variety of intuition, beliefs, and presumed facts along with principled reasoning and argument (Carraher, Schliemann and Schwartz, 2008). This idea aligns with socio-constructive learning, where the learner is an active participant in the learning process and their thinking and interpretation of new ideas and concepts are based on existing knowledge (Human and Olivier, 1999). It is hoped that by starting from rich contexts and realistic situations, learners will eventually become able to derive conclusions directly from a written system of equations (Carraher, Schliemann and Schwartz, 2008). Conceptual development through real, context based problems is an important principle of RME, and will inform the instructional design sequence for developing generalisation (Stephan, Underwood-Gregg and Yackel, 2016). The role of the teacher becomes very important in facilitating this development of thought in learners. This aspect of early algebra also aligns with the problem-based approach, which is based on the premise that learners should encounter and explore new mathematical concepts by engaging in effective and realistic problems (Human and Olivier, 1999). This ensures that learners understand the value of mathematics, and leads to a deeper understanding of the concepts at hand (Carraher, Schliemann and Schwartz, 2008).

3.2.4.2. In early algebra formal notation is introduced gradually.

Without a certain amount of guidance, it is unlikely that young learners will be able to start with written notation for variables by means of formal symbolism on their own. Variables and the use of symbols are fundamental components of early algebra (see 2.2.4 and 2.2.5) and, even though it might be challenging, learners are able to use variables to represent unknowns in a general manner (Kaput, 2008; Knuth *et al.*, 2014). The use of symbols and notation to represent mathematical situations in a general manner first emerged in the syncopated stage of algebra, where Diophantus started to employ symbols in the form of Greek letters to generalise mathematical situations. His use of symbols was primitive, but was complex and advanced for the time (see 2.3.2.5) (Derbyshire, 2006). In the symbolic stage, Viète employed a use of symbols which closely resembles the algebraic notation we use today (see 2.3.3.2). As algebra developed, a need for representing situations in a more general manner emerged. In response, mathematicians like Diophantus and Viète introduced the use of symbols which enabled them to solve more complex problems (Sfard, 1995).

Algebraic expression as a concept needs to be introduced to learners in a careful manner. Teachers need to introduce unfamiliar terms, representations, and techniques even though learners might not initially understand them as intended. Continuous classroom discussions about algebraic expression would be beneficial to teachers and learners, especially if teachers listen to learners' interpretations and provide them with opportunities to expand and adjust their understanding (Carraher, Schliemann and Schwartz, 2008). This is consistent with the problem-based approach, which treats learning as a social process. Well-planned classroom discussions between learners, with the teacher in a facilitative role, are necessary to ensure the effective development of learners' deep understanding of algebraic expressions (Human and Olivier, 1999). In this way, the teacher guides learners through the reinvention of their conceptual understanding, and learners start to transition from an informal use of models to a more formal use for models (see 3.2.)(Stephan, Underwood-Gregg and Yackel, 2016).

3.2.4.3. Early algebra should be integrated with other content areas in the mathematics curriculum.

Algebraic concepts, and especially generalisation, are embedded in all content areas of the early mathematics curriculum. It is the role of the teacher to facilitate the emergence of learners' algebraic thinking when working with word problems, ration, proportion, measurement, graphs, number lines, tables, and arithmetical notation as

he guides them through the reinvention of concepts based on a carefully designed sequence of activities (Stephan, Underwood-Gregg and Yackel, 2016). Teachers should help to bring the algebraic character of early mathematics to light (Carraher, Schliemann and Schwartz, 2008). Early algebraic thinking involves learners' capability to understand patterns in culturally evolved co-variational ways and use them to engage with questions of remote and unspecified terms. Learners must make use of a coordination of spatial and numeric structures. Their awareness of these structures and coordination involves a complex relationship between inner and outer speech, forms of visualisation and imagination, gesture, and activity on signs (Radford, 2011, p. 23).

3.2.4.4. The role of representation

Mathematical ideas can be represented internally through mental models and cognitive representations of the mathematical ideas which underlie external representations. External representations include concrete materials, diagrams, drawings, spoken words and symbols (T. Cooper and Warren, 2011). Models and representations are related to each other: models are ways of thinking about abstract concepts, while representations refer to the various forms of the models. Mathematical understanding matures when connections form between a learner's internal network of models and representations. Learner proceeds through four states in terms of representation (Cooper and Warren, 2011, p.191): (1) using one representation, (2) using more than one representation in parallel, (3) making links between parallel representations, and lastly (4) integrating representations and moving effortlessly between them.

Aspects of representational forms should be understood and communicated to allow learners to notice commonalities (generalities) across or between representations. Including a variety of forms of representation in instructional sequences, develops learners' ability to switch between forms of representations (Cooper and Warren, 2008b).

Apsari *et al.* (2020) propose the use of geometric objects as representations to help learners see the structure of mathematics and make generalisations. From their research, they concluded that geometric representation supports the development of algebraic thinking. Geometric representation plays the following roles (Apsari *et al.*, 2020) (see 2.3.4.1):

- Geometric representation is used as the context of the problem
- Geometric representation is used as the model of, and model for, situations (see 3.2) (Treffers, 1987)
- The teacher can use geometric representation as a tool for scaffolding critical thinking
- Learners use geometric representations to communicate and represent their mathematical ideas.

RME proposes a guided reinvention process which enables learners to construct their own understanding of mathematical concepts (Apsari *et al.*, 2020). According to Treffers (1987) models are the bridge which helps leaners transition from reality to mathematical ideas or objects. A Mathematical model should give a sense of visualisation to the actual condition in phenomena.

The use of representations can, therefore, be seen as integral to modelling mathematical situations and helping learners to represent and communicate their mathematical thinking.

3.3. THE DEVELOPMENT OF EARLY ALGEBRAIC THINKING

In this section, the development of algebraic thinking will be explored**.** The levels of algebraic thinking and the challenges in developing algebraic thinking will be addressed. This will be followed by an in-depth discussion of the main components of early algebra, which involve: (1) Generalisations and formalization, (2) Syntactically guided manipulation, (3) Study of structure, (4) Functions, relations, and joint variation, and lastly, (5) Modelling as a language (Kaput, Carraher and Blanton, 2008; Roberts, 2012a).

3.3.1. Introduction

The traditional “arithmetic-then-algebra” approach, where an arithmetic curriculum in the early grades is followed by formal algebra in secondary grades, has not proven to be successful, as it does not allow enough time and opportunity for the deep development of algebraic reasoning (see 3.3.1) (Knuth *et al.*, 2014). It is proposed that the artificial separation of arithmetic-then-algebra deprives learners of powerful schemes for thinking mathematically in the foundational years, and makes it harder for them to learn algebra in later years. Understanding takes a long time to develop and should start in the early years of education. The goal of early algebra teaching should not be the skilled use of algebraic procedures, but rather the ability to think algebraically (Carpenter & Levi, 2000). It is, therefore, imperative that algebraic thinking should be developed from an early age and be introduced and integrated through all areas of early mathematics teaching.

In the early grades, algebraic reasoning is developed through the comparison of quantities, observing and making patterns, navigating through different kinds of spaces, and solving problems in playful interactions with objects and peers in the classroom (Linder, Powers-Costello and Stegelin, 2011). Lee et al. (2016, p. 306) further explain that by providing concrete experiences with algebraic concepts in the early years, the foundation is laid for the comprehension of abstract forms of algebra in the later years. It is important that algebraic concepts and skills are introduced in the early grades and embedded through all school years (Lee et al., 2016, p. 306). *The Ontario Ministry of Education and Training guide* states that algebraic reasoning is developed through investigations and discussions of number properties (generalised arithmetic (see 2.2.4 and 3.5.1)) , which aid learners to make generalisations and construct concepts which create the stepping stones for a more formal way of algebraic thinking (Ontario Ministry of Education and Training, 2007, p. 8).

Carpenter and Levi (2000) argue for two central themes at the core of developing algebraic thinking: (1) making generalisations and (2) using symbols to represent mathematical ideas and to represent and solve problems. Learners are encouraged to construct and represent powerful ideas. Examples of generalisations in the primary grade are: “when you add zero to a number, the sum is always that number” or “when you add up three numbers, it does not matter which you add first" (Carpenter & Levi, 2000, p. 2). To develop mastery of generalisation and symbolisation in early algebra, teachers must create a carefully planned order of learning activities that supports the introduction of a structural approach to the learning of sequencing at a foundation phase level (Du Plessis, 2018). RME principles propose implementing guided reinvention and emergent modelling to design an instructional sequence of mathematical activities which guides learners and teachers in the development of conceptual understanding (Gravemeijer, 2007, 2020; Stephan, Underwood-Gregg and Yackel, 2016).

Algebraic thinking is promoted by placing emphasis on ways to represent and analyse underlying structures of numbers, operations, and relationships (Billings, 2017, p. 483). Algebraic thinking incorporates manners of thinking that are in line with the Common Core Standards for Mathematical Practice (CCSMP) (Billings, 2017, p. 483). This framework provides an important overview of the type of robust thinking and reasoning children should be able to engage with. Within and Within (2014) agree with Billings (2017) that the following ways of thinking should be integrated when teaching early algebra:

- Make sense of problems and persist in solving them (SMP 1),
- reason abstractly and quantitatively (SMP 2),
- construct viable arguments and critique the reasoning of others (SMP 3),
- use appropriate tools strategically (SMP 5),
- look for and make sense of structure and patterns (SMP 7), and
- identify and express regularity in repeated reasoning (SMP 8).

Blanton and Kaput (2011) describe these standards as the goals of early algebra teaching.

The CCSMP aim to provide clear and consistent learning goals to help learners prepare for tertiary education and their careers. The following activities to support the development of algebraic thinking in the foundational years are included in the CCSPM (*Common Core States Standards Initiative*, 2021):

- Acting out situations, verbal explanations, expressions or equations.

- Representing the solutions of problems using objects or drawings, and equations with a symbol for the unknown number to represent the problem.
- Understanding and applying properties of operations and the relationships between them.
- Determining unknown numbers in equations.
- Identifying and explaining patterns in arithmetic.

Radford (2012) explains that **structural understanding** (see 2.2.4 and 3.5.3) forms a crucial part of the emergence of algebraic reasoning in the early grades (Ferrara & Sinclair, 2016, p. 3). When introducing algebra in the early years, it is important to move away from the current emphasis on learning the rules of symbol manipulation and instead focus on creating deep algebraic reasoning (Jacobs et al., 2007, p. 259). Du Plessis (2018) emphasises that working structurally in the foundation phase classroom is not only possible, but necessary. At this stage in their development, learners are receptive to an approach that develops habits of mind when dealing with repeating patterns on a relational level. The focus of early algebra is on a relational approach to mathematics.

Kaput (2008) describes a framework for early algebra which includes the following content strands (see 2.2.4 and 3.5) (Roberts, 2012):

- Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic and quantitative reasoning.
- Algebra as the study of functions, relations and joint variation.
- Algebra as a cluster of modelling languages both inside and outside of mathematics.

These thinking practices and content strands should be integrated intentionally across different conceptual domains which are recognised as important entry points for algebraic thinking, so that mathematical connections become accessible to learners at all levels of thinking (Knuth *et al.*, 2014). Mathematical connections can be made when learners are able to represent their mathematical thinking in various ways (see 3.3.5.4). CAPS (DBE, 2011) provides guidance to teachers on investigative work involving problem contexts which can be explored and representing functional relationships that arise, by means of number sentences, input and output tables, and function machines. Graphs can also be used to represent functional relationships (Roberts, 2012). Modelling concepts and encouraging learners to model their understanding and problem-solving processes engages them in mathematical thinking and reasoning. Learners must be aware of different models which can be used to represent problems and contexts (Roberts, 2012).

Patterns can be used as an important tool to develop algebraic thinking. Experiences in describing and extending patterns help learners make generalisations about the composition of different patterns. Learners are motivated to communicate these generalisations as pattern rules, that is, concise descriptions of how a pattern changes or repeats. In the foundational years, learners express these generalisations informally (Ontario Ministry of Education and Training, 2007). Taylor-Cox (2003) agrees that patterns serve as the cornerstone for algebraic thinking. Patterns are visible in everyday life around us. Learners watch the sun set and come up every day, sing songs with rhyme and rhythm, and see how bricks are laid in patterns. Recognising, describing, extending, and translating patterns encourages learners to think in terms of algebraic problem solving. Working with patterns motivates learners to identify relationships and form generalisations. Young learners are able to come up with algebraic formulas to describe patterning activities. According to Radford (2015), this requires a coordination of words, written signs, drawn figures, gestures, perception, and rhythm. Learners engage in interesting gestures of rhythm while trying to make sense of mathematical structure which underlies patterns and sequences.

There is little research systemically comparing children exposed to early algebra and those who experienced a more traditional arithmetic-first approach in terms of the development of their algebraic thinking and their understanding of important algebraic concepts. Even so, a fundamental assumption of early algebra education is that it will increase children's understanding of algebraic concepts which will aid them in later grades (Knuth *et al.*, 2014).

There is a general agreement amongst researchers that early algebra comprises of two core features: (1) generalising, which includes identifying, expressing and justifying mathematical regularities, structures, relationships and properties, and (2) reasoning and actions based on the forms of generalisation (Du Plessis, 2018). In the following section of Chapter 3, a systematic analysis will be given of the views of Radford, Mason and Kaput on the development of algebraic thinking.

3.3.2 Different views of Radford, Mason and Kaput

3.3.2.1. Radford's view on the development of algebraic thinking

Algebraic thinking is a theoretical form which has emerged, evolved and been refined in the course of cultural history (Radford, 2015). Radford (2011) bases his research on the development of early algebraic thinking on the Vygotskian *theory of knowledge objectification*. According to this theory 'thinking' does not happen only in the head, but includes other material and idea-based components including inner and outer speech, objectified forms of sensuous imagination, gestures, tactility, and actions with cultural artifacts. Radford describes thinking as a dynamic unity of material and ideal components. This is a sociable, tangible process in the human body. He describes the development of early algebraic thinking as the appearance of new structuring relationships between the material-ideational components of thinking, and the manner in which these relationships are organised and re-organised. This development does not follow a pre-established path, but is cultural-dependant. Radford (2014) writes about the importance of thinking analytically in algebra. Thinking analytically involves treating indeterminate quantities as though they were known or specific numbers. This way of thinking analytically, where unknown numbers are treated on par with known numbers, distinguishes algebra from arithmetic. Viète characterised algebra as an analytical art (see 2.3.3.2).

Radford proposes three conditions which characterise algebraic thinking (Radford, 2014, p. 260):

1) Indeterminacy: the problem involves unknown numbers in the form of variables.
2) Denotation: The indeterminate numbers in problems need to be named or symbolised. Symbolisation can be achieved in various ways, including natural language, gestures, unconventional signs, or alphanumeric symbols.
3) Analyticity: The indeterminate or unknown quantities should be treated as known numbers. One starts with indeterminate numbers and operates on them. Trial-and-error methods fail the condition of analyticity and therefore cannot be considered algebraic.

3.3.2.2. Mason's view on the development of algebraic thinking

Mason (2008) believes that by the time children start to attend school, they have developed great 'powers' to make sense of the world they live in. The main aim of teaching should be to make use of those 'powers' and to develop them. Algebraic thinking is developed when these 'powers' are used in the context of numbers and relationships. Teachers often underestimate learners and do too much of the work for them. This leads to learner dependency. Mason argues that learners already have the 'powers' necessary to think algebraically in the womb. He argues that one should not see arithmetic as a pre-requisite for algebra, and that algebra and arithmetic can emerge simultaneously from the use of the same powers. Algebraic thinking is needed to make sense of arithmetic. Mason (2008) argues that learner have the following 'powers' which are relevant to the development of early algebraic thinking:

- Imagining and Expressing
- Focusing and De-focusing
- Specialising and Generalising
- Conjecturing and Convincing
- Classifying and Characterising

Even though learners possess these powers innately, it does not mean that they automatically think mathematically. These powers need to be exercised and brought forward through intentional facilitation by the teacher (Mason, 2008).

One of the powers used to induce algebraic thinking is the awareness of and the expression of generality (Mason, 2008). The role of the teacher is to provide ample opportunities for learners to recognise and express the generality that arises in particular mathematical instances. Whenever a teacher notices that learners are able to make generalisations, the opportunity arises to prompt them to specialise by constructing a particular own example. Geometric pattern sequences are one of the possible avenues to get learners to start to generalise and specialise. Learners can be prompted to specify a method for drawing the next term in the pattern. Learners can further be asked to generate their own simpler or more complicated sequences. The mental processes involved when working with patterns are well-described by (Watson, 2000). "Going with the grain" involves making a useful contrast with the obvious. "Going across the grain" means to pause to address what is different and what is the same in each of the statements or terms being explored. When learners are going with the grain, they complete a sequence. The important part of a task takes place when learners go against the grain. It is then when they interpret a generalisation. The basic structural properties in arithmetic can also be generalised. The simple but powerful awareness that adding something to one number and subtracting it from another number leaves the sum invariant is only one example of many possibilities. Learners need to specialise so that they can make sense of generality by reconstructing it to discern what changes and what stays the same. As learners become used to being expected to express generality and justify their thoughts by testing their conjectures with specific cases, they will start to internalise the process and start doing it naturally. Learners will become more efficient and effective at learning mathematics as they take responsibility for their own learning and use their own innate powers (Mason, 2008).

When learners work with problems, Mason (2008) suggests that they should engage their thinking about as-yet-unknown values. After learners have solved a problem, it is valuable for the teacher to turn the question around and ask them to provide problems of the same type which would give the same answer. For example, it is known that 10 + 10 = 20. The teacher can ask learners what other pairs will result in the same sum. When learners engage in such tasks, they are trying particular cases by attending to the structure of the mathematics and finding generalisation. Arithmetic tasks can also be transformed by changing a doing calculation into an undoing calculation. This encourages learners to think and act creatively. It provides them with opportunities to work with the as-yet-unknown instead of focussing on calculating the arithmetical answer.

The notion of a variable emerges in both these contexts for producing algebraic thinking. Variables are present in the world of learners from a very young age and are united by perceiving cases in terms of freedom and constraints. For example, "I am thinking of a number" (freedom emerges here in the concept of number), "the number is between 5 and 7" (here a constraint is provided). Also, a traditional arithmetic sum, like 6 + 3 =? can be reversed to ask 9 =? + ?. Changing the range of permissible change for numbers provides access to a varying degrees of freedom. Most textbook problems can be seen as problems where you start with great freedom and then impose constraints by the choices you make in approaching the problem. Tasks like this provide learners with opportunities to work with the as-yet-unknown as an expression of generality with constraints (Mason, 2008).

Getting learners to express generalities to satisfy certain constraints, leads to **multiple possible expressions** of the generality (Mason, 2008). Each expression states the same thing but looks different. Learning to see situations through the structure of expressions improves learners' sense of there being a variety of ways to express similar things. This further improves their understanding of symbolic expressions and leads to the need to manipulate expressions. The need to manipulate algebraic expressions can emerge naturally when learners use their powers. Simultaneously, learners can improve their understanding of equivalence of symbolic expressions.

Guessing and Testing is described as a good mathematical process for developing early algebraic thinking (Mason, 2008). Over time it can develop into more sophisticated processes, such as trying and improving, where the guess is modified according to a specific principle rather than being random. Learners can also spot and check where an answer is tried and found to be correct. Learners should use structure by using the values which have been tried, to build on the structural features of the problem. Getting learners to use their "powers" is to get them to start thinking mathematically, and therefore, algebraically. By treating algebraic thinking as a natural consequence of the using of learners' powers, algebra's gatekeeper effect fades and it becomes accessible to all.

Across the world, marginalised learners underperform in mathematics, especially with regards to algebra. This leads to limited career opportunities, especially in STEM-related fields. Hunter and Miller (2020) propose using a culturally responsive approach to the development of early algebraic thinking to overcome this worldwide challenge. Mathematics as a subject has long been believed to be value and culture free, but in recent research this position has been refuted. The teaching and learning of mathematics cannot be decontextualised from the learner. Hunter and Miller (2020) emphasises the importance of developing culturally responsive classrooms which set mathematical tasks within the known and lived, social and cultural reality of learners.

3.3.2.3. Kaput's view on developing algebraic thinking

Kaput (2008) takes on the perspective that the heart of algebraic reasoning is comprised of complex symbolisation processes that serve purposeful generalisation and reasoning with generalisations. Kaput (2008) claims that to describe algebra can be challenging, as it is not a static body of knowledge. It evolves as a cultural artifact, as described in Chapter 2, in terms of the symbol systems it embodies. It also evolves as a human activity as learners learn to reason algebraically.

Kaput (2008) describes two core aspects (see 2.2.4) of algebraic reasoning. Firstly, generalisation and the expression of generalisations in increasingly systematic, conventional symbol systems, and, secondly, syntactically guided action on symbols within organised systems of symbols. These core aspect of algebraic reasoning appear across all three strands of algebra, which include algebra as the study of structures and systems abstracted from computations and relations, algebra as the study of functions, relations, and joint variation, and algebra as the application of clusters of modelling languages both inside and outside of mathematics (Kaput, 2008).

These fundamental components as explained by Kaput (2008) will be discussed in more depth in 3.5 of Chapter 3.

3.3.3. The levels of algebraic thinking

Freudenthal's (1973) believes that learners should experience mathematics as a human activity and reinvent it as they are guided by teachers and tasks. His ideas were based on the assertion that the history of mathematics should be the main informant for designing a route along which learners might reinvent mathematics (Gravemeijer, 2020). In this section of Chapter 3, various perspectives on the levels of algebraic thinking will be systematically reviewed and analysed in relation to the stages of the development of algebra in Chapter 2.

3.3.3.1. Nixon's theory on the levels of development of algebraic thinking

The levels of thought that learners pass through in the process of developing early algebraic thinking are significant and reveal a trend in the development of thought in algebra. Nixon (2009) developed a theory of the development of algebraic thinking based on Piaget and Garcia's identification of three developmental levels in algebra. Nixon argues for three levels of learning to think algebraically: the perceptual level, the conceptual level and the abstract level.

Perceptual level: In the initial stages of algebraic learning, learners make use of counters or abacuses for counting. In this stage, learners need to coordinate their senses and perceptions to form algebraic concepts. As learners advance in their use of numbers, they can use symbolisations to represent ideas or events. The perceptual level is regarded as significant as it enables learners to form mental pictures of concepts. This level is crucial for the advancement and development of further algebraic thinking (Nixon, 2009). The perceptual level is consistent with the rhetorical and syncopated historical stages of algebra. In the rhetorical stage (see 2.3.1) of the historical development of algebra, algebraic concepts emerged as mathematicians aimed to solve isolated problems by modelling the problem situations in natural language. Generalisation and algebraic thinking are achieved in the rhetorical stage when learners notice generalities in the solutions of problems, and the generalisation is communicated in words. The aim of teaching should be to get learners to start making formalisations and generalisations (Kaput, 1995) and, in the rhetorical stage, learners will be expressing this in their own words. Learners can, for example, say: "If I put two numbers together, I know my answer will be bigger than the numbers I started with." The syncopated stage is characterised by geometric thinking (see 2.3.2). This involves representing mathematical thinking by means of geometric figures and forms (Katz and Barton, 2007). When learners start to represent ideas by means of diagrams or drawings, they are starting to make models and mental pictures of the mathematical concepts an real life situations they are working with (Cooper and Warren, 2008). Making models and representing ideas in this way is an important step towards generalising (see 2.2.4) (Roberts, 2012; Kaput, 2008).

Alongside the rhetorical stage and syncopated stages, runs the geometric conceptual stage (see 2.3.4.1.), which marks the beginning of algebra in history. In this stage, geometric problems are solved by employing geometric procedures, but an orientation towards generality can be noticed. Through reasoning and engaging in algebraic problems by means of geometric representation, mathematicians developed the sophistication of algebraic thinking they were involved in. The geometric stage is crucial for learners as they work through algebraic problems and patterning activities in the pre-algebra classroom (Apsari *et al.*, 2020). Geometric representations make it possible for learners to make connections between problems, mathematical models, and problem solving strategies, and to notice the structure of mathematics as they engage in patterning activities and arithmetic (Dekker and Dolk, 2011). This is comparable to the perceptual level of algebraic thinking in that it involves the coordination of physical senses and perceptions to form algebraic concepts: learners reason with physical and concrete "objects" in the shape of geometric figures, then construct informal algebraic concepts by representing and modelling the situations in natural language. Modelling as an algebraic language emerges (see 3.5.5). An example of this in the geometric stage would be Euclid's demonstrations of how to manipulate rectangles and squares.

Conceptual level: At this level, a shift from analysing objects to considering the relations of transformations between the objects becomes apparent. A reconstruction of the previous level takes place. When learners at this level learners are able to find interrelationships between properties, and start providing definitions and theorems for what they experience, they are ready to advance to the next level (Nixon, 2009). In the syncopated stage, the focus is on designated unknowns, symbolism, and motivating for the suitability of solution methods (Sesiano, 2009). In Nixon's (2009) conceptual level of thinking, the actions of justifying and proving theorems become central. The focus is on the general structure of mathematics. The belief is that a generalising perspective on the underlying structure of mathematics should be embedded throughout mathematics by expecting of learners to provide explanations and motivations for their thinking in a general manner from the start of schooling (Roberts, 2012). In the conceptual stage, analysis of objects gives way to analysis of the transformations and relations between objects. The aim is for learners to find interrelationships between properties and start to justify what they experience. In the static-equation solving stage, mathematicians started to move away from analysing mainly geometric objects and started to represent relationships as equations. Algorithms and proofs were developed to find the answers to these equations. Equivalence, expressions, equations, and inequalities are part of the first Big Idea to develop in algebraic thinking in the foundational years of mathematics education (see 2.2.4). The ability to express problem situations and the unknown by means of variables in equations is an important step towards developing algebraic thinking and generalisation (Knuth *et al.*, 2014). Using symbols to represent variables and functional thinking as the idea of motion as change, emerge in the work of Viète, Descartes, Kepler and Newton history (symbolic stage (see 2.3.3) which indicates the move into the dynamic function stage of development (see 2.3.4.3.). In the conceptual stage, the aim is to find and represent the interrelationships of properties between objects, for example, the motion of a projectile and the curve which it would follow.

Abstract level: As learners reach the abstract level, they are organising results acquired at the perceptual level. These learners are able to use symbols, with deep understanding, to construct proofs. They can understand the importance of deductions, axioms, postulates and proofs (Nixon, 2009). Learners are thus be able to deduct a rules from patterns, and can understand how symbols can be used to represent these rules. In the abstract stage (2.3.4.4), learning is focused on the underlying structure or mathematics. Structural knowledge is the recognition of all equivalent forms of an expression. It is important that learners can justify the structural equivalence which they identify (Liebenberg et al., 1998). In the abstract stage of the historical development of algebraic thinking, a need for justification and proof for your arguments emerges (Katz and Barton, 2007). The aligns with Nixon's abstract level of algebraic thinking. At the abstract level of thought, learners start to use symbols with deep understanding to construct proofs, and they can understand the importance of deductions, axioms, postulates and proofs (Nixon, 2009).

As learners pass through these levels, it is necessary for them to be guided. They should be motivated to generalise and draw comparisons (Nixon, 2009). These levels of algebraic thinking or learning should be taken into account when designing teaching materials.

3.3.3.2. Mason, Burton and Stacey's conceptual framework for mathematical thinking

Mason, Burton, & Stacey's (2010) conceptual framework for mathematical thinking in a problem-centred classroom (see Fig 3.1 below) , identifies the mathematical processes involved in solving problems as specialising and generalising. A learner must first be able to specialise, meaning to understand, and make a plan to solve, a problem. Specialization enables the learner to solve problems by carrying out plans and checking whether their solutions areis appropriate. This is a generalising activity. The phases of thinking are not crisply distinct. This is due to them being defined by qualities of experiences rather than mechanical activity. Work in one phase may easily lead back to a previous phase or the final phase, without going through all the phases. The RUBRIC words are generally indicative of the phase in which thinking is happening. By specialising, one find out what one *knows*, what one *wants*, and what one might *introduce*. By specializing, one uncovers patterns which lead to generalisation. By generalising, one comes up with conjectures which can be *checked* by further specialising, and one can *extend* the knowledge to wider questions (Mason, Burton and Stacey, 2010). This framework provides valuable sub-levels of the thinking processes learners should engage with as well as a rubric to determine on what level the learner is working. Learners' actions and thinking can be analysed according to this conceptual framework to see if they are able to reach the generalisation level throughout the interventions. This model will be used in the analysis of the meaning of generalisation, and will furthermore provide the main structure for developing an instructional design sequence for the teaching and learning of EA.

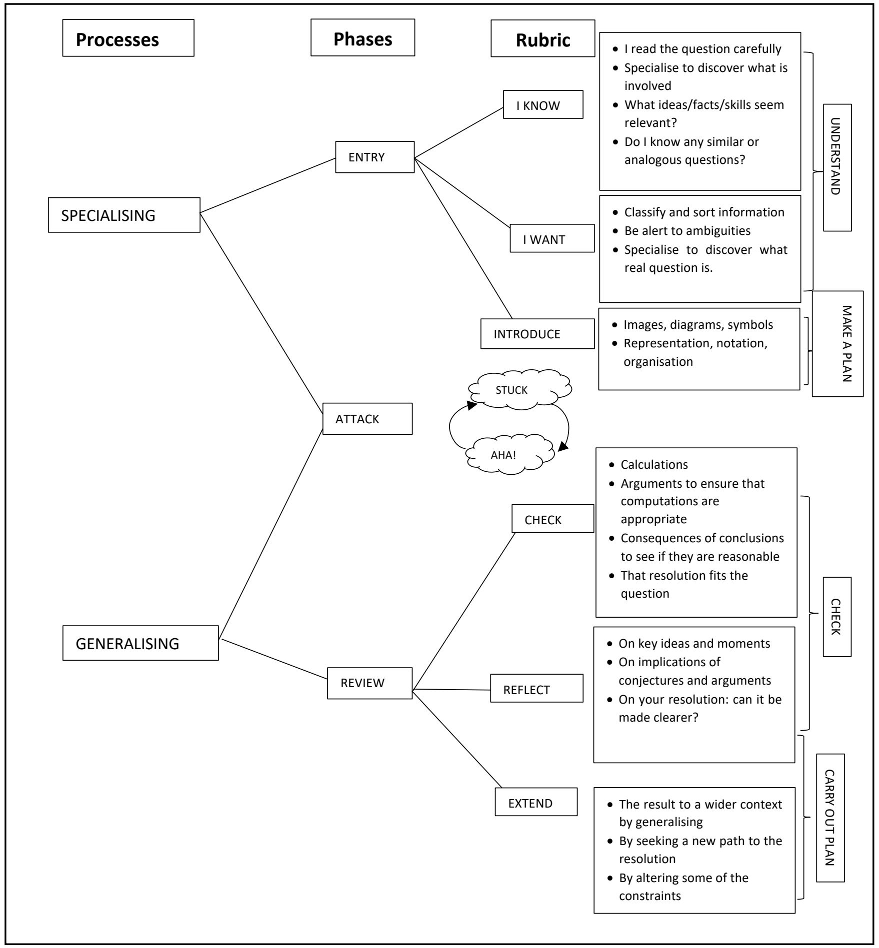


Diagram 3.1: Mathematical processes involved in solving a problem (Mason, Burton and Stacey, 2010)

3.3.3.4. Challenges in developing algebraic thinking

Du Plessis (2018) notes that one of the major barriers to an effective algebra education for learners in South Africa, is the lack of coherence and relational sequencing in the South African CAPS curriculum. Furthermore, a lack in coherence across pedagogic communication and activities is evident. A random selection and sequencing of acitivities is observed in the primary years. This absence of coherence suggests a lack of conciousness in pedagogical practice. A well thought-out sequence of activities is necesarry to create habits of mind through exploration of structure in the early years of mathematics teaching. The further lack of a specific, focused, and research-based teaching and learning approach is apparent in South African foundation phase classrooms. An appropriate and coherent approach, like the problem-centred approach, is necessary to ensure that classrooms promote mathematics which learners can understand.

Any approach to teaching may be affected by obstacles which hinder the efficacy of the teaching and learning program. Most obstacles to the teaching of algebra result from failure to situate generality at the core of the learning experience (Mason, 2008).

Stacey and MacGregor (2000) question the efficacy of basing the teaching of algebraic thinking on the generalisation of picture patterns and tables of consecutive values of function for three reasons:

- There was no research to prove that this approach is more effective than using letters to stand for as-yet-unknown numbers.
- Research on learners' enagement with pattern formulating tasks show low facility.
- Picture patterns emphasise relationships between terms and cause a recursive or inductive specification of rules instead of functional relationships.

Finding research evidence to support 'generality' as an approach, is challenging, as it is not a single or isolated strategy to be used, but rather an holistic perspective on how to approach mathematics. Emphasising the expression of generality is aimed at developing an overall awareness, not just training learners to produce certain behaviours. It should be implemented consistently over time and not merely in isolated lessons (Mason, 2008). This task poses a substantial challenge.

The International Comission on Mathematical Instruction (ICMI) Study Conference on The Future of Teaching and Learning of Algebra has identified the lack of data in relations to professional development as one of the key limitations of research on algebraic reasoning (Jacobs et al., 2007, p. 259). Professional development and the role of the teacher as a facilitator can be described as one of the core factors in the effective development of algebraic reasoning in the classroom. The teacher has an essential role in facilitating children's mathematical patterning activities. It is essential that teachers have sufficient knowledge of mathematical patterning and are able to capatilise on children's interests (Fox, 2005, p. 319).

There is evidence that teachers have limited pedagogical content knowledge in relation to the types, levels and complexity of patterns. A teacher may limit children's development of patterning when only approaching patterns as repeating patterns (Papic & Mulligan, 2007, p. 592). When teachers have a narrow perspective on algebraic activity, the relation between algebra and early mathematical thinking is obscured (Kaput, Carraher and Blanton, 2008). Blanton and Kaput (2011, p. 27) note that most elementary school teachers do not have the experience with algebraic thinking that needs to become the norm in schools.

3.3.4. Quantitative views of early algebra

Algebra can be viewed as the expression, manipulation, and formalisation of mathematical concepts and structures, governed by explicit rule-based notational systems. It is based on the ideas of coherence, representation, generalisation, and abstraction (Smith and Thomson, 2008). Reasoning directly with quantities and relationships within them is a powerful way to help learners build initial understanding of the concepts of functions and functional thinking (Ellis, 2011). Smith and Thomson (2008) proposes that quantitative reasoning lies at the core of algebraic thinking. Algebraic thinking is characterised by its generality and the use of symbols to represent these generalities. These expressions can then be manipulated and compared, and can facilitate numerical evaluations. Quantities can be seen as attributes of objects which can be measured. Whether it is measured or not, these

attributes are referred to as quantities. A quantity is a person's perspective of a quality of an object and the process of assigning a numerical value to this quality. Measurable attributes like length, volume, mass, and area are all quantities. Quantitative reasoning occurs when learners engage with quantities and the relationships between them. Quantitative operations are conceptual operations where one conceives of a new quantity in relation to one or more already conceived quantities (Ellis, 2011).

Research has shown that an awareness of mathematical patterns and structure is imperative for the development of generalisation and abstraction in early mathematics. Mulligan and Mitchelmore (2009) have proposed a framework, Awareness of Mathematical Pattern and Structure (AMPS) which generalises throughout all mathematics content areas, can be effectively measured, and is aligned with the development of structural features of mathematics. The aim of this construct is to provide reliable methods for describing the development of learners' mathematical structures and relationships, and using learners' ideas to develop quantitative reasoning (Mulligan, 2010).

In Kaput's (2008) explanation of algebra in relation to the thinking practices involved and the content strands of these practices (as mentioned in Chapter 1 of this study), the first content strand is stated as:

> Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic and quantitative reasoning.
> (Kaput, 2008, p. 11)

Quantitative reasoning is an integral aspect to consider when exploring the development of algebraic thinking. The role of quantitative reasoning in problem solving is to provide the content for the algebraic expression so that the power of notation can be exploited. Furthermore, quantitative reasoning has the aim of supporting reasoning that is flexible and general in character but which does not only rely on symbolic expressions (Smith and Thomson, 2008).

In *Developing an Essential Understanding of Algebraic Thinking for Teaching Mathematics Grades 3-5,* Blanton *et al* (2015) describe 5 big ideas which are essential to the development of algebraic thinking. Big idea 4 emphasises the importance of quantitative reasoning. Quantitative reasoning extends the relationship between quantities with the aim of expressing generalities among these quantities. Two quantities can relate to each other in the following ways: (1) they can be equal, (2) one can be bigger than the other, and (3) one can be smaller than the other. Known relationships between quantities can be used as a basis for describing relationships with other quantities.

There are various approaches to the teaching of early algebra. One popular approach focuses on reasoning with quantities to develop early algebraic thinking. In this approach, learners are prompted to use letter notation to compare the quantities of measurable objects. It is known that quantitative reasoning along with various forms of representation can support the emergence of algebraic thinking in young learners (Carraher and Schliemann, 2014).

Research conducted by Davydov and fellow mathematicians (1975) stemmed from a need to improve learner achievements in mathematics. Learners needed more support to be able to successfully engage in more complex mathematics when they entered secondary grades. It was agreed by this group of researchers that reform was necessary in the early grades of mathematics. They found that what learners do naturally and instinctively at a young age is to learner compare things. They combined this notion with the works of Piaget and Vygotsky on teaching and learning. The aim of the Russian team was for learners to develop a structural understanding of mathematical systems. They believed that if learners were able to understand structure, they would be able to apply properties and underlying concepts to any other number system (B. Dougherty, 2008).

Vygotsky (1978) identified two ways in which learners learn: by generating spontaneous concepts, or scientific concepts. Spontaneous concepts are developed when learners can abstract properties or concepts from specific situations. This is how traditional mathematics education is set out. Topics are taught in a very specific sequence and are not connected across the different number systems. Scientific concepts develop from experiences that focus on conceptual foundations that then

lead to identifying, applying, and analysing the generalised concepts in specific cases. Learners are here prompted to see the mathematical structures across all number systems. Davydov (1975) believed that a general to specific approach (scientific) was more effective in developing deep understanding than the spontaneous approach to developing mathematical concepts.

The Measure Up (MU) approach as explained by Dougherty (2008) assumes that learners enter the first grade with a view of quantities that centres around comparison. First grades are often concerned with who has the most or the least of a certain object. These young learners have a natural and spontaneous approach to measurement which forms the basis for mathematical development. It emphasises young learners building, recognising, and using properties of real numbers before they deal with whole or natural numbers. This is called the pre-numeric stage. Learners in this stage work with unspecified quantities of length, area, volume, and mass rather than discrete numbers. The measuring of these attributes is developed in an informal manner where learners engage with working definitions which are comprehensible at their developmental stage. As learners work through problems with the comparison of quantities, the teacher needs to facilitate the need to communicate the result of their comparison orally or in writing. Learners find this challenging as they do not have the necessary linguistic capabilities. This creates the need for naming quantities, and the teacher should guide learners towards doing so using letter symbols. This is a precursor to the use of letter symbols to represent variables. When learners can name quantities with letter symbols, the fundamental properties of equality, which are reflexive, symmetric, and transitive, are easily explored. Because learners model these properties with physical quantities, they can develop a clear mental image of how these properties work. The role of the teacher is to guide learners to write the statements as the young learners express them. For example, using the equal sign to express that fact that two quantities are the same amount. The use of the equal sign in these situations helps learners to not perceive the equal sign as an operator.

From the equality properties and comparison experiences, various numbers and operation concepts arise. Quantitative modelling aids learners in seeing different meanings of subtraction, for example, the action of taking away and the action of comparing to find the difference. The comparison model helps learners to see that if two quantities are unequal, the amount by which they are unequal is the difference. Furthermore, learners explore the effects of changing two equal quantities with the constraint that they must contain an equal relationship by adding or subtracting. By modelling the actions of adding and subtracting with physical quantities, learners can make the generalisation that two quantities can be kept equal if you add or subtract the same amount from both. Learners can further engage with keeping an unequal relationship between those two quantities. Lastly, numbers emerge when situations require the quantification of differences in comparisons. Without a unit, it is not possible to quantify the differences or to make definite comparisons. A significant step in learners' development is made when they move from the generalised approach to specific quantities and the counting of units (B. Dougherty, 2008).

The consistent use of letters to label physical quantities provides learners with the confidence to begin manipulating letters in ways characteristic of more sophisticated mathematics learners (B. Dougherty, 2008). When thinking about and exploring different ways in which quantities can be parted, learners start to describe these ways with multiple symbolic representations. Diagrams can also be used to represent relationships. Building on their work with units and relationships within generalised and increasingly specific quantifications, learners begin to use variables to represent unknown quantities. Learners' experiences with part-whole relationships and units enable them to deal with known and unknown quantities. Equations are written to express relationships.

MU promotes the development of properties which are often neglected until later grades, as it introduces all number operations simultaneously. Complex properties are made accessible to learners through generalised arithmetic and the noticing of patterns and structure within mathematics. The multiple presentations used in MU provide structure for solving computational word problems. An approach that focuses on generalised and non-specified quantities is often thought to be too complicated and abstract for early mathematics. Research (Davydov, 1975; B. Dougherty, 2008), however shows that understanding the structure and properties of mathematics opens a way for learners to construct deep understanding of the fundamental concepts of mathematics. This enables them to reason relationally even in unfamiliar problems or situations.

3.3.5. Summary

Defining algebraic levels historically, by blending a cultural artifact and action perspective as described by Kaput (2008) (see 3.4.2.3), focuses on the progress made in equations and the use of equations to model and analyse problem situations in an algebraic manner (see chapter 2). The origins of equations stem from problem situations or assertions about numbers or measurement quantities. Th earliest versions of equation solving appear almost 4 000 years ago in the Rhind Papyrus (see 2.3.1.2). Equations were solved using natural language rather than sophisticated symbolism (rhetorical stage (see 2.3.1). The focus of algebra then turns to the solving of equations in the 16th and 17th centuries, distinguished from their status of models, but merely as mathematical objects of intrinsic intellectual interest. In the 18th century and later, a modern perspective emerges where definitions of algebra include the literal use of symbols as a core feature of algebraic activity (Kaput, 2008). Section 3.4 provided a systematic thematic analysis of the development of algebraic thinking, the views of prominent figures in the domain, the levels of algebraic thinking and how it relates to stages of algebra development in history and lastly a quantitative view on developing algebra. This systematic analysis will be used to inform the instructional sequence based on the principles of RME (Gravemeijer, 2007, 2020). The instructional sequences will especially be grounded in the stages of algebra emerging from history (see Chapter 2), the levels of algebraic thinking as described by Nixon (2009), and the processes of mathematical thinking which learners engage with in a PCA classroom ((Mason, Burton and Stacey, 2010). Kaput (2008) explains his view of algebraic thinking based on a symbolisation perspective which takes into consideration the use of letters and a variety of symbol systems that extend on the traditional systems of the use of symbols.

3.2. GUIDED REINVENTION AND EMERGENT MODELLING AS AN INSTRUCTIONAL DESIGN APPROACH

The aim of this chapter is to systematically review literature on how the generalisation concept can be used to develop early algebra in the early years mathematics classroom. The history of algebra and its relationship to the development of early algebra, will be used as the foundation and basis for the design of a heuristic-based theoretical construct which relies on the pedagogy of guided reinvention and the principles of RME.

There is an internationally documented disparity between the innovative pedagogy advocated by mathematics education researchers, and classroom practice. Shifting the focus from instruction for procedural fluency to the development of learner agency and argumentation requires pedagogic change as well as change in values and beliefs (Solomon, Hough and Gough, 2021). “The goal is to learn more about the complexity of successfully implementing meaningful instructional methods equitably [and] identify ways in which … practices might be problematic for some students … and also the adaptations that successful teachers make to address such problems” (Lubienski, 2002, p.121). This goal can be attained by taking the RME approach to instructional design (Solomon, Hough and Gough, 2021).

RME was constructed as a descriptive theory, whose aim was mainly to distinguish the realistic approach from the structuralist, empirical, and mechanistic approaches (Treffers, 1987)(Gravemeijer, 2020). RME emphasised the role of emergent mathematics, where learners move from models of their informal activity in recognisable contexts to more formal mathematics by engaging in a process of progressive mathematisation of models (Solomon, Hough and Gough, 2021). RME has been characterised by three instructional design heuristics: guided reinvention, didactical phenomenology and emergent modelling (Gravemeijer, 2020). The RME framework was the result of restructuring theories which inform instructional sequences.

Guided reinvention is based on Freudenthal's (1973) ideas that learners should experience mathematics as a human activity and reinvent mathematics as they are guided by teachers and tasks. His ideas were based on the view that the history of mathematics should be the main informant for designing a route along which learners might reinvent mathematics (Gravemeijer, 2020). Guided reinvention requires particular pedagogic practices from the teacher and corresponding modes of participation from the learner (Solomon, Hough and Gough, 2021).

Didactical phenomenology as a heuristic approach was also based on Freudenthal’s (1983) work on the didactical phenomenology of mathematical structures. His perspective was that one of the main characteristics of mathematics and mathematical activity is organising. Organising here refers to organising subject matter from reality or organising mathematical matter on a higher level. This heuristic requires one to analyse which phenomena are organised and how they are organised by the mathematical concepts, rules or procedures which is the aim (Gravemeijer, 2020).

The emergent modelling design heuristic aims to establish an incremental process where models and mathematical conceptions co-evolve. The core of emergent modelling is the use of a series of sub-models which support an overarching model. The overarching model develops the model of informal mathematical activity to a model for more formal mathematical activity (Gravemeijer, 2020).

Stephan, Underwood-Gregg and Yackel (2016) incorporated three heuristics which are used to design mathematical instruction based on the RME design theory.

Heuristic 1: Guided Reinvention

Instructional resources should be designed to encourage learners’ reinvention of key mathematical concepts (Freudenthal, 1973). To initiate the development of an instructional sequence, the designer should visualise a learning route the class and learners might invent themselves. Basic mathematical concepts which are relevant today, took centuries to develop, and learners are expected to develop comprehensive conceptual understandings in a matter of weeks. RME instructional resources help learners to reinvent these concepts and ideas using carefully sequenced problems, tools, and guidance from the teacher. The learning route is designed such that concepts emerge as learners engage in the instructional sequence (Stephan, Underwood-Gregg and Yackel, 2016).

Heuristic 2: Sequences should be experientially real for learners

In an RME approach, the starting point of instructional sequences should be experientially real, to allow learners to engage in personally meaningful activity. Instructional tasks should draw on realistic situations as a semantic foundation for learners' mathematisation. Activities are sequenced so that learners can organise their activity within a realistic context to reinvent important mathematics. Learners start to reason with abstract symbols as their reinventions become more and more sophisticated (Stephan, Underwood-Gregg and Yackel, 2016).

Heuristic 3: Emergent models

This heuristic involves designing instructional activities that motivate learners to engage with models of their informal mathematical activity as they transition to models for more formal mathematical activity. During the transition from informal to formal, the designer supports learners' modelling by introducing new tools or using learner-created tools, including physical devices, inscriptions, and symbols, to explain their mathematical reasoning (Stephan, Underwood-Gregg and Yackel, 2016).

The three heuristics explained above are used to create an instructional sequence while simultaneously envisioning a path that the class might follow as they engage in the tasks. This path of development can be named a hypothetical learning trajectory (HLT). The designer aims to conjecture about the route the mathematical class or community will travel, including the learning goals and tools, as they engage in the instructional tasks. The HLT also analyses the role of the teacher in supporting learners along the developmental route (Stephan, Underwood-Gregg and Yackel, 2016).

RME instructional design and guided reinvention can be seen as crucially important in the development of domain specific mathematical concepts to guide teachers' instructions (Stephan, Underwood-Gregg and Yackel, 2016). RME instructional design principles and the three heuristics described by Stephan, Underwood-Gregg and Yackel (2016) will guide the design of an instructional sequence for the implementation of the generalisation concept in the early years mathematics classroom. The instructional design will aim to provide an analysis of classroom culture by looking at the role of the teacher and learner, as well as proposing an instructional sequence to develop early algebra through generalisation. The early algebra concepts and developmental levels which emerge from the history of algebra, as well as recent literature on the domain of early algebra, will be used to design the sequence.

3.5. MAIN COMPONENTS OF EARLY ALGBRA BASED ON KAPUT' FRAMEWORK

In this section, the main components of early algebra will be reviewed systematically with the aim of informing the types of generalisation activities which should be included in the instructional design (see Chapter 5).

Kaput (2008) proposes a framework outlining the various elements involved in algebra and algebraic thinking (see 2.2.4 and 3.4.2) (Kaput, Carraher and Blanton, 2008). In this section, an in-depth discussion of Kaput's framework is presented to show how these fundamental components emerged from history. These fundamental components, as well as the historical developmental stages of algebra and Nixon's levels of algebraic thinking, are used to construct an instructional sequence based on the principles of guided reinvention and emergent modelling. The fundamental components are organised into two core aspects which are integrated throughout the three content strands. The figure below shows the framework.

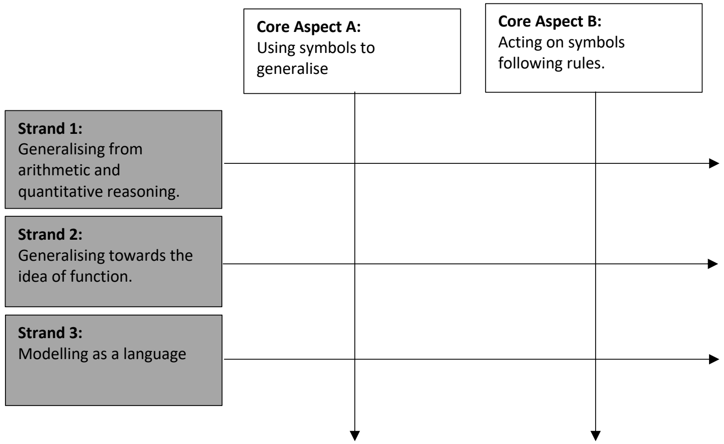


Figure 3.1. Kaput's core aspects and strands of algebra (Roberts, 2012)

The two core aspects are relevant to algebraic reasoning and are (1) symbolisation activities that serve purposeful generalisations and (2) reasoning with symbolised generalisations. In each of the core aspects, generalisation is a focus point of algebraic reasoning. The first two content strands, as explained by Kaput, are relevant to the development of early algebraic thinking. They are concerned with the types of generalisations which are at the core of early algebraic reasoning: generalising arithmetic and generalising towards the idea of a function. The third strand focuses on modelling as a language to represent and interpret algebraic thinking. Roberts (2010) explains that from this framework, early algebra can be perceived as generalising in the early grades of primary school. The instructional sequence of activities in 3.7 suggest generalisation activities for each developmental stage based on the fundamental components of algebra (Kaput, 2008). These generalisation activities are based on the three elements of generalisation as described by Roberts (2010, p.169). These three elements include:

- Generalising arithmetic as the exploration of the properties of numbers and operations.

- Generalising about particular number properties and relationships.
- Generalising towards the idea of a function, which includes recognising regularity in elementary patterns, ideas of change including linearity, and representation through tables, graphs and function machines.

3.5.1. Generalisations and formalisation

The evolution of algebra (see Chapter 2) can be presented as a constant attempt at turning computational procedures into mathematical objects (Sfard, 1995). Therefore, algebra evolved as a constant attempt at generalising and making formalisations (Kaput, 1999). The aim of early algebra is generalisation. When learners are able to find generalisations in mathematics, it creates a deep understanding of the structure of mathematics. Learners are able to translate this understanding to all domains of learning. The power of mathematics lies in relations and transformations that lead to patterns and generalisations. The focus of mathematics teaching should be on developing fundamental skills in generalising, expressing and systematically justifying generalisations (Warren and Cooper, 2008). Fundamental skills in generalising can only be developed when a deep understanding of the structure of mathematics is developed. This should be done through a problem-centred approach where learners encounter non-routine patterning problems which lead to generalisations. When learners are able to make generalisations, they construct their structural knowledge of mathematics and, therefore, develop early algebraic thinking (Roberts, 2012).

Generalising is described as a mental activity by which one compresses multiple instances into a single unitary form (Blanton et al., 2019). Mitchelmore (2002) groups generalisation into three categories:

G1: As a synonym for abstraction
G2: As an extention of an existing concept:
    Empirical extention
    Mathematical extention
    Mathematical invention
G3: A theorem realting existing concepts

Generalisation is often used as a synonym for abstraction (G1) where it is defined as "finding and singling out properties in a whole class of similar objects". Generalisation as an extention (G2) is formed by at least three aspects: Empirical extension, which applies when one finds other contexts to which a known concept applies; mathematical extention, when on class of mathematical objects is embedded in a larger class based on a different similarity; and mathematical invention, when a mathematician deliberately omits a defining property to form a more general concept. Generalisation can also refer to a relationship that holds between all members of a set of objects (G3) (Mitchelmore, 2002). For the purpose of this study, G3 would be most applicable. Radford (2006) classified two components of generalisation: phenomenological and semiotic. The phenomenological component involves understanding the generality through observing the local commonality of all terms, whereas the semiotic component involves expressing generality through gestures, language and algebraic symbols (see 3.3.4.5 on representation and 3.4.2. on Radford's view of developing algebraic thinking).

Roberts (2010) explains that generalisation has two manifestations: generalising from arithmetic and quantitative reasoning; and generalising towards the idea of a function. This is based on the work of Kaput (2008) (see 3.5). In the early years mathematics curriculum, generalising should include the following three elements (Roberts, 2010, p. 169):

- Generalising arithmetic as the exploration of the properties of numbers and operations.
- Generalising about particular number properties and relationships.
- Generalising towards the idea of a function, which includes recognising regularity in elementary patterns, ideas of change including linearity, and representation through tables, graphs and function machines.

These generalisation activities will inform the guided reinvention and emergent modelling instructional sequence (see 3.7). The first two elements differ in the sense that element one looks for properties of numbers and generalisations in general, while element two, relates to properties of and relationships of specific numbers. Arithmetic approaches which encourage "partitioning" or "breaking down" and "building up" numbers draw on these properties of particular numbers and operations (Roberts, 2010).

Algebraic generalisation is developed through identifying a regular pattern based on terms which are known. This is called abduction. Deduction occurs when the observed

regularity is used to produce an expression which is true for any term in the sequence. It is during the deduction stage, that learners develop the type of reasoning which lies at the heart of algebra (Demonty, Vlassis and Fagnant, 2018). Rivera (2013) argues that a continuous interplay between thinking about pattern and analysis of pattern, which is known as abduction and induction, is the essence of algebraic generalisation.

The practices of justifying and reasoning with mathematical structure are forms of generalisation. Justifying entails that one argues about the validity of a generalisation within a certain representational system. When reasoning with a generalisation, one acts on generalisations as mathematical objects in new situations. These practices are socially mediated practices that refine the scope of the generalisation and drive the symbolisation process (Blanton et al., 2019).

Knuth et al. (2014) refer to algebraic activites in which generalisation can be seen as prominent. Research has documented learners' ability to generalise mathematical structure by noticing regularity in arithmetic situations, using sophisticated instruments to explore, generalise, and symbolise functional relationships, building mathematical arguments that reflect more generalised forms than the empirical, case-based reasoning often used, and reasoning about abstract quantities to represent algebraic relationships.

Many elements of constructivist teaching promote generalisation. Elements such as existing knowledge, small-group cooperative learning, the admission of contrasting methods and the reconciliation of conflicting solutions lead to the recognition of commonalities and, therefore, generalisation (Mitchelmore, 2002).

It is necessary to determine certain levels of generalisation which learners are able to achieve to determine wether a specific intervention is effective in developing algebraic thinking and, therefore, the ability to generalise. One should expect that, as learners move through the intervention, they would become able to generalise more and more sophisticatedly.

Warren (2004) described six broad categories in which learners' ability to be generalised can be divided. The categories are presented in descending order in terms of their sophistication:

> *Category 1* – Detailed description is given of the relationship which can be noticed in the activity, with all aspects taken into account and described.
> *Category 2* – Detailed description of the relationship is given, but some aspects are left unmentioned.
> *Category 3* – Less detailed description of the relationship is given. Not all aspects are mentioned.
> *Category 4* – Broad relationship is observed. Learner cannot fully communicate idea.
> *Category 5* – Partial response. Learner can mention one aspect but does not see a relationship.
> *Category 6* – No response.

As learners work on a series of problem-based non-routine patterning activities, one would expect them to progress through the categories, and at the end be able to achieve category one.

The RME movement from the Netherlands is a constructionist curriculum which aims to teach for abstraction and generalisation. Their approach to teaching consists of three stages:

- Develop rules of operation in several specific, familiar, everyday contexts
- Demonstrate that the same structure is present in several such contexts
- Formulate, symbolise and study the common structure.

(Mitchelmore, 2002)

When designing an early algebra intervention, it is important to keep these stages in mind for each lesson, with the aim of developing learners' ability to generalise. The instructional sequence (see 3.7) will be based on the principles of RME.

Number is an abstract concept and represents a quantity that may or may not be clear. Generally, learners begin developing their sense of numbers by counting isolated objects. As they move through different number systems, routines and algorithms, learners struggle to develop a consistent conceptual base that can deal with all numbers as a connected whole (Warren, 2004). Davydov (1975) claimed that learners

should start their mathematics program without number. Rather, they should start by exploring physical attributes that can be compared. He hypothesised that this allows learners to focus more effectively on the underlying concepts of mathematics without the interferences of numbers. For that reason, if learners are able to and motivated to reason quantitatively, their understanding of number systems, routines, and algorithms might be deeper at a later stage.

The traditional approach to teaching patterns, where learners are mainly asked to identify, copy and extend patterns (Department of Education, 2011) is not sufficient to develop early algebraic thinking through generalisations. Studies show that even though patterning activities are present in the classroom, teachers (and the curriculum) fail to fully appreciate their algebraic aspect (Demonty, Vlassis and Fagnant, 2018). Even so, it is not necessary for classroom patterning activities to change completely. The problems used traditionally in arithmetic and algebra can build rich educational environments, but the way in which these activities are understood, should be changed (Demonty et al., 2018, p. 2). Classroom patternings should be adjusted to include early algebraic concepts like generalisation and symbolisations instead of the inappropriate over-emphasis on copying and extending patterns.

Aiding students in generalising is not a simple task. There is a variety of activities which can be used to reach the goal of generalisation. Studies agree that aids such as verbal, figural and numerical representation of patterns, and highlighting the connections between these representations, may help students to generalise (Demonty et al., 2018, p. 5). Warren, Trigueros, & Ursini (2016) consider two approaches that are especially effective for making such connections: looking at invariant relations between pictorial clues provided by visual arrangements, and inviting students to express and also justify their generalisations. Rivera (2013) advocates for motivating learners to think multiplicatively. This can help them to generalise linear patterns, because learners are able to move from arithmetical generalisations to algebraic generalisation, by allowing the iteration of a constant increase between two terms to be generalised.

An activity commonly used to develop generalisation in the Australian context is the exploration of simple repeating patterns using shapes, colours, movement, touch and sound. Students are asked to establish relationships between patterns and their positions, and use this generalisation to form steps in the patterns for other positions. One can distinguish between result pattern generalisation and process pattern generalisation. Result pattern generalisation refers to regularity in the results, whereas process pattern generalisation emphasises the regularity in the process. It is believed that young learners have the ability to think functionally (process pattern generalisation) (Warren & Cooper, 2008, p. 172) and learners should be exposed to patterning problems which lead them to functional thinking. In her research, Warren (2004) proposed that one should ask three types of questions in lessons to elicit learners' ability to reason algebraically (generalise): those which require predicting, justifying and generalising. Learners were asked to record the generalisation they discussed as a class, in their own words. These responses were categorised in terms of sophistication to determine the level of the learner's ability to generalise. Research has shown that the problem-centred approach, elicited by patterning activities involving visual aids, provides a solid foundation for the understanding of algebraic expressions and techniques related to their transformations (Demonty et al., 2018, p.6). Mitchelmore (2002) agrees, and further emphasises that generalisations are part of all aspects of education and thinking. He identifies three methods of teaching generalisation, not all of which are necessarily effective:

> The **ABC method** – Abstract before concrete – generalisations are taught as abstract relations which have to be learned before they can be used in concrete situations. In practice, the ABC-method leads to *abstract-apart* knowledge, which learners cannot apply to any situation or problem.
>
> The exploratory method – tproceeds from the concrete to the abstract. Learners are able to make empirical generalisations. In this method, learners perceive mathematics as more relatable, but the results are still not crystal clear and there is a risk of making false inductions.
> Problem solving – essentialy consists of theoretical generalisations. This method leads to a deep understanding of concepts. This substantively involves using empirical generalisations to make conjectures or even form axioms.

Patterning activities are useful for encouring algebraic thinking if they are focused in a particular direction, such as generalisation and relational thinking (Demonty, Vlassis and Fagnant, 2018). Patterns give learners the chance to transition from arithmetic to algebra by making verbal and symbolic generalisations. A close relationship exists among patterns, algebra and generalisation. Generalisation is the core of algebra and the search for pattern is an essential step towards constructing generalisations (Imre and Akkoc, 2012). Radford (2008) describes the architecture of algebraic pattern generalisation:

- As the first stage of the model, abduction is described as grasping a commonality amongst particulars.
- The next stage is called transforming the abduction, which is defined as extending the generality to all subsequent terms.
- The last stage is deducing P*n* which requires using the generality to represent an expression of any term in the sequence.

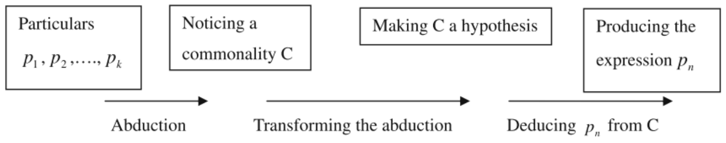


Figure 3.2. The architecture of algebraic pattern generalisation (Radford, 2008)

Generalising and representing are symbiotic processes which are the core aspects of algebraic thinking (Blanton et al., 2019). Generalising occurs when multiple instances are condensed into a single generalisation or rule, which is communicated through representation (symbolisation), and from the resulting unitary form expressed in appropriate notation which can include natural language, variable notation, graphs, tables and pictures (Blanton et al., 2019). The action of representing a generalisation is socially mediated: one's thinking about symbol and referent is constantly refined, leading to a mediation of the generalisation itself (Blanton et al., 2019). Even though there are different ways of representing generalisations, variable notation remains the most recognisable artifact of cultural algebra. Cultural algebra refers to the social context in which algebra is understood. In the early years, when learners are first introduced to the variable, it should be as a fixed unkown quantity associated with missing value problems (Knuth *et al.*, 2014). Many learners struggle to express their generalisations in their own words, even about simple concepts. This may be due to a lack of mathematical vocabulary and, further, their lack of familiarity with the types of conversations that demand generalisation (Warren, 2004).

Research suggests that many learners struggle to extract the underlying structure of generalisation because they have difficulty with language, number and basic mathematics concepts (Warren, 2004). Furthermore, many learners still view mathematics as a series of disconnected objects to be memorised (Mitchelmore, 2002). Generalising patterns is a challenging task for learners. Learners' difficulties with patterns are closely linked to their understanding of patterns and pattern generalisations. The first challenge arises when learners think about variables. Learners does not understand what it means to write the *n*th term. Secondly, learners experience difficulty expressing relationship algebraically. Learners are able to easily extend patterns, but this approach keeps them from seeing the general structure of the elements and being able to express it algebraically. Lastly, learners' use of representations to make generalisation presents further cause for concern, as they struggle to move from verbal representations to more formal representations (Imre and Akkoc, 2012).

3.5.2. Syntactically guided manipulation

The human mind trades in symbols which represent abstract arithmetic, algebraic and logical propositions. These symbols can be manipulated according to internally represented mathematical and logical rules (Landy, Allen and Zednik, 2014). Formalisms are behind the tremendous development we have seen in modern science and technology. When working with formalisms, either traditional algebraic ones or more exotic ones, the focus falls on symbols and the syntactical rules for manipulating symbols in expressions. Manipulation refers to changing of the form of the expression (Kaput, 1995a). The learner should refocus their attention from what the symbols represent to the symbols themselves. In this way, they are liberated from operating on relationships which are more complicated than what would be possible if the focus was on what the symbols represent (Kaput, 1999).

A change in the perspective of algebra and the implementation thereof occurred as algebraic thinking transitions to the dynamic function stage (see 2.3.4.3.). Mathematicians started to ask more questions, and the focus shifted from merely finding solutions to problems expressed as equations. In the seventeenth century, new notation of algebraic ideas was introduced by mathematicians like Viète and Descartes in the symbolic stage (see 2.3.3) (Katz and Barton, 2007). This new notation is similar to the algebraic notation we use today (Sfard, 1995). Variables are used to represent unknown quantities, which may be fixed or varying (see 2.2.5) (Knuth *et al.*, 2014). Viète understood that algebra was more than the developing techniques to solve various equations. He used unknowns in equations to represent species of objects and algebra was about the relationship between the species. Viète's employment of notation in the form of alphabet letters to represent unknowns in equations (Tabak, 2011) showed his understanding of variables and the importance of using variables when working with algebraic problems and equations (Knuth *et al.*, 2014). Viète's introduction of notation and use of symbols made it possible for mathematicians to see broader patterns in mathematics and identify relationships between symbols and the classes of objects they represent (Tabak, 2011). As in the pre-algebra classroom, Kaput (2018) emphasises that learners should be able to suspend their attention on what the symbols stand for, and instead look at the symbols themselves. In this way learners are freed to operate on relationships which are more complicated (Kaput, 1999). To ensure that this process runs smoothly, a well-planned teaching and learning approach and plan is necessary. This study aims to provide such an approach in Chapter 5, by designing an instructional sequence for the implementation of the generalisation concept in the early mathematics classroom.

Actions or manipulations on representations occur in two broad classes (Kaput, 2018):
A syntactic action involves the manipulation of symbols only by looking at the syntax of the symbol system rather than by looking at a reference field for those symbols. A syntactic action is an action on the notational system and not on the representational system. A syntactic manipulation treats expressions or equations as manipulable object strings which are subject to certain constraints.
A semantic action is guided by the referents of the symbols. The elaboration is led by the characteristics of a referent field for the symbol system rather than its syntax. A

semantic action acts on an equation as a comparison of two statements about numbers. It is imperative to note that a specific equation can give rise to various semantically guided actions. This will depend on the different reference fields and how they are linked.

Bruner (1973) explains that syntactic actions are viewed as “opaque” uses of symbols, whereas semantic actions are “transparent” uses of symbols. It is possible that a natural transition from semantic to syntactic actions occur as the symbols and actions associated with them reify into entities that can serve as referents for new symbol systems.

When working with algebraic structure, learners need to eventually work on a purely syntactical level. Liebenberg *et al.* (1998) believe that in order to develop learners’ understanding of structure in numerical and algebraic contexts, they should be encouraged to engage in syntactical *and* semantic discussions for the justification of the equivalence of expressions. When acting on formalisms semantically, one’s actions are guided by what one believes the symbols should stand for (Kaput, 1995a). A semantic justification focuses on the numbers in an expression. A syntactic justification, in contrast, only focuses on the relevant rules (Liebenberg *et al.*, 1998). The syntactically guided approach treats symbols as objective entities and the conceptual system of rules applies to the system of symbols, not what they might stand for. The rules can be thought of as applying to the symbols as physical objects (Kaput, 1995a).

Syntactically guided manipulations on formalisms can be viewed as the core of algebra,. However, to ensure the effective learning and development of actions on formalisms, a semantic starting point should be taken. Formalisms should initially be taken to represent something which the learner has experienced (Kaput, 1995a). Importantly, most actions and manipulations on symbols involve a combination of syntactical and semantic actions (Kaput, 2018). Mathematical activity can be seen as the interactions between the notational systems and its reference field.

The traditional algebra curriculum has over-emphasised the semantics of algebra, which has resulted in many learners being unable to see the meaning and value of mathematics. The power of using the form of mathematics as a basis for reasoning is lost when learners are engaged with endless practicing of rules for symbol manipulation. They lose the connection to the quantitative relationships that the symbols might stand for. This leads to learning without understanding. Learners experience algebra as a challenging subject if they are not provided with opportunities to construct their own knowledge and reflect on what they have learned. For many learners, understanding is remembering which rules to apply in certain situations. This is not a deep understanding of the underlying concepts. Understanding algebra means being able to connect knowledge of procedures with knowledge of concepts (Kaput, 1999).

Young learners are able to perform syntactically guided actions on opaque symbols. We will now consider, as an example, a task for fifth grade learners from “Patterns and Symbols” (Roodhardt *et al.*, 1997). In this example, learners are expected to perform transformations on sequences with letters L and S. These letters represent blocks lying on their sides (L) or standing up (S).

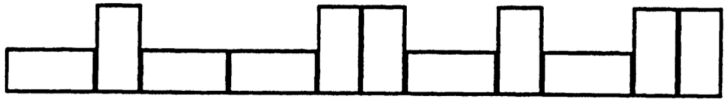


Figure 3.3. Block array presented by letter sequences (LSLLSSLSLSS) (Roodhardt *et al.*, 1997)

Learners must work on various transformation rules to act on these arrays, interpreting their results in terms of string and vice-versa. The learners must make up their own rules and apply them to their own designs. Learners should gradually progress toward more abstract substitution rules which they can apply to new strings or situations. The work on formalisms is necessary throughout mathematics (Kaput, 1999).

Syntactically guided manipulations are an important focus area in algebra instruction, as we want learners to form a deep understanding of the representations they use.

Teachers should encourage learners to reflect intentionally and explicitly on the use of representations and communicate about their mathematical ideas to solve problems (Carpenter and Lehrer, 1999).

3.5.3. Study of structure

Almost all of mathematics is based on patterns and structure. A mathematical pattern is any predictable regularity, usually involving number or space. Mathematical structure refers to the way in which a pattern is organised. This may be in a numerical or spatial manner (Mulligan *et al.*, 2008). Generalising and abstraction, which are focused on the structure within computations rather than the process or answer, lead to the emergence of abstract structures which are associated with traditional algebra (Kaput, 1999). At the heart of algebraic reasoning in the early years lies a deep understanding of the mathematical structure of arithmetic, expressed by language and gestures using conrete materials and representations (Warren and Cooper, 2008). Structural knowledge can be described as the ability to recognise all equivalent forms of an expression. But learners should also be able to justify the structural equivalences they identify (Liebenberg *et al.*, 1998).

In Sfard's (1995) account of the historical development of algebra, she describes the third stage (the works from Galois to Bourbaki) as being concerned with the science of abstract structures. In accordance with the work of Sfard (1995), Katz and Barton (2007) describe the conceptual stages of the development of algebra where the abstract stage involves the emergence of the underlying structure of mathematics (see 2.3.4.4). Generalising and abstraction, which emerge from generalised arithmetic, and where the emphasis is placed on the structure within the computational procedures rather than the process or answer, lead to the emergence of abstract structures (Kaput, 1999). In the 19th century, mathematicians decided that axioms and proofs were needed to justify computations (Katz and Barton, 2007). To develop structural knowledge, learners should learn to recognise equivalent forms of an expression and justify the structural equivalence they've identified (Liebenberg et al., 1998). In the abstract stage, the main goal is to find structure in mathematics and to use symbols to represent and construct proofs. Here learners generalise mathematics and represent mathematics using symbols, which can be taken as the main two aspects of algebra and structural thinking.

Generalising patterns are also thought to be especially critical because the structure of mathematics can be revealed through the search for patterns and relationships (Imre and Akkoc, 2012). In structural-abstract algebra which is taught for understanding, structure emerges from learners' experiences with mathematical ideas, for example, representations of motions on the plane, symmertries of geometric figures, modular arithmetic, and manipulations of letters in words. The aims of abstracting structures in mathematical thinking include (Kaput, 1999, p. 142):

- Articulation of structure in preformal, natural language,
- Enrichment of learners' understanding of the systems from which they are abstracted,
- Providing learners with intrinsically useful structures to compute freely from the particulars and constraints those structures were once tied to, and;
- Giving learners a basis for even more complicated levels of abstraction and formalisation.

Mulligan, Mitchelmore, & Prescott (2005) found that early school achievement in mathematics is closely related to the learner's development and perception of mathematical structure. There is a clear relationship between learners' difficulties in algebra and their lack of understanding of the structural notions of arithmetic (Liebenberg *et al.*, 1998). Learners use various structural strategies to solve nonroutine problems on measures of algebraic knowledge (Blanton et al., 2019, p. 1934). Learners are motivated to justify and reason about the underlying structures of mathematics by reaching generalisations. Learners' structural strategies involve recognising and acting on underlying mathematical relationships which occur when representing a relationship between two quanities using variables, when making a general argument that does not rely on specific values, or when reasoning about equations (Blanton et al., 2019). The aim of an early algebraic intervention would be to help learners develop the skill of extracting the underlying structure of a generalisation (Warren, 2004). Varied algebraic generalisations can be symbolised in different ways, thereby opening the way to comparisons of numerical or algebraic expressions, and subsequently the development of structural understanding of expressions and of equality (Demonty, et al, 2018). Viewing an expression structurally depends on what the chosen emphasis in the structure is. The focus might fall on the numbers in certain instances, in others on only the operations, and sometimes the focus can fall on both. Viewing an expression structurally also means having the ability to see the "surface" structure as well as the "hidden" structure. The ability to see "hidden" structures in complex algebraic expressions and to relate the structure to a simpler equivalent form, is very challenging for most learners. By helping learners to reason with expressions in a syntactical and semantical manner at the same time, this obstacle can be overcome (Liebenberg *et al.*, 1998). At the beginning of algebra, the focus falls on the structural properties of expressions. Liebenberg et al. (1998) follows a pedagogical approach which emphasises numerical expressions whose structures have a possible input on the needed and relevant algebraic expressions which learners deal with initially in algebra. The teaching of algebra and generalisation by finding structure in numbers, are motivated for the following reasons:

1) The structure of the algebraic system is based on the properties of the number system,
2) The numerical context is a familiar context for most learners,
3) It is a meaningful context through situations for the construction of schema, and
4) It provides opportunity for meaningful reflection and verification procedures through calculations (Liebenberg *et al.*, 1998. p. 2).

Mulligan et al. (2005, p. 1) identified four broad stages of structural development which are present in learners' representations as they engage in tasks across a range of mathematical content domains, such as counting, partitioning, patterning, measurement, and space:

- Pre-structural stage: representations lack any evidence of mathematical or spatial structure.
- Emergent (inventive-semiotic) stage: representations show some elements of structure such as use of units. Characerters are first given meaning in relation to previously constructed representations.
- Partial structural stage: some aspects of mathematical notation or symbolism and/or spatial features are present.
- Stage of structural development: representations clearly integrated mathematical and spatial features.

Learners should have a firm understanding of numbers and operations on them before being expected to deal with numerical expressions structurally. As learners engage with numerical expressions, the emphasis should fall on the structure, with calculation used as a way of verifying the equivalence of expressions. If more emphasis is placed on the structural features of numerical expressions, learners may be able to calculate

better and have a better understanding of simple algebraic expressions (Liebenberg *et al.*, 1998).

Mulligan et al. (2005) have found that the more structurally developed a learner's internal representational system is, the more well-organised, coherent and stable in all structural aspects their external representations will be, and the more mathematically competent the child will be. This indicates the enormity of the importance of developing a learner's deep understanding of underlying mathematical structure. In their empirical work, it was further seen that when a learner is able to recognise the spatial structures and features of an object, they are able to make generalisations about the relationship between the features and spatial structure (Mulligan et al., 2005, p. 5). Spatial structure is a critical element in developing structure, because it entails the process of constructing an organisation or form (Mulligan et al., 2005).

Learners' transition from thinking arithmetically to algebraically is influenced by their ability to see structure in patterns. To support learners' development of algebraic thinking, the teacher should choose a helpful context and appropriate visualisation (Apsari *et al.*, 2020). A basic example of extracting structure from patterns can be seen in the grid represented in the figure below (Figure 3.5).

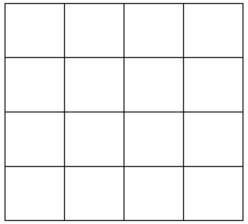


Figure 3.4. Square grid represented by 4 x 4 squares (Apsari *et al.*, 2020)

The figure shows a square which is formed by horizontally and vertically aligning equal-sized squares in 4 rows of 4. Understanding such a grid pattern can lead to the connected understanding of various mathematical ideas. Some examples include the area of a square, counting (which leads to skip counting), multiplication, and division. This understanding enables learners to make various generalisations by finding such properties. This leads to pre-algebraic thinking as learners become able to see structures instead of specific numbers (Mulligan *et al.*, 2008).

Mulligan, Mitchelmore, Marston and Highfield (2008) extensively explored the use of pattern and structure in the development of early algebraic thinking in the Pattern and Structure Assessment (PASA) and the related Pattern and Structure Mathematics Awareness Program (PASMAP). They found that the use of patterns and structural features indicate common characteristics in their mathematical understanding. Their researched showed that young learners can be taught to recognise mathematical pattern and structure**.** This research shows that having learners recognise similarities and differences in mathematical representations develops their ability to recognise pattern and structure in mathematics. Mulligan et al. (2008) conclude their study by emphasising the importance of the development of pattern and structure to the learning of multiplicative concepts, the base ten system, and unitising and portioning in early mathematics teaching and learning. Pattern and structure are at the core of mathematical thinking and should be embedded throughout mathematics teaching.

3.5.4. Functions, relations, and joint variation

Function is the study of relationships (Ng, 2018). The idea of function has for the last 100 years been perceived as a powerful mathematical idea which should fill a central role in the mathematics curriculum (Blanton and Kaput, 2011). It is perceived as one of most important topics because it provides means for thinking quantitatively about real-world problems and situations, and allows learners to study the relationships and change they notice in problems (Ng, 2018). Functional thinking as a content strand should be included in early years algebra teaching to introduce the idea of generality. Functional thinking means to incorporate building and generalising patterns and relationships using diverse linguistic and representational tools. It also involves treating generalised relationships or functions as mathematical objects which are useful in their own right (Blanton and Kaput, 2011). Functional thinking can also be viewed as representational thinking that emphasises the relationship between two or more varying quantities. Specifically, the kinds of thinking which stem from specific relationships in specific instances with the aim of generalising that relationship across a variety of situations. Algebraic reasoning takes place when the learner constructs representational systems to represent a generalisation of a relationship between varying quantities (Smith, 2008).

Functional thinking, as a fundamental component of algebra, emerges throughout the stages of the historical development of algebra (see Chapter 2). Early examples of functional thinking emerged from Al-Khwarizmi’s (see 2.3.2.6) work with linear and quadratic equations. He was able to group various equations which represented relationships between various quantities. Even though these equations were represented by means of words, the functions were represented in a general manner (Katz and Barton, 2007). Al-Khwarizmi’s algebraic thinking introduced the focus on relationships between quantities which result in functions. Learners in the early years work problematic situations which require them to identify relationships between covarying quantities. The foundation of functional thinking is the focus on relationship (Smith, 2008). The dynamic function stage (2.3.4.3) developed the concept of motion and movement in mathematics (Katz and Barton, 2007). Functional thinking is also noticed in the work of Viète (see 2.3.3.2.) as he had the ability to reason quantitatively as well as in a functional manner with variables as unknowns. In Viète’s work, the concept of function as well as variable emerged (Sfard, 1995). Viète’s introduction of notation and use of symbols made it possible for mathematicians to see broader patterns in mathematics and identify relationships between symbols and the classes of objects they represent (Tabak, 2011). As the eighteenth century grew on, algebra developed in such a way that it became easier to represent a curve as the path of motion. The idea of finding curves which solve problems became the central goal of mathematics (Katz and Barton, 2007). From the work on curves in the eighteenth century, the idea of functional thinking, relations and joint variation emerge (see 2.4.5).

Early Algebra is often described as having five “Big Ideas” (see 2.2.4) which described functional thinking (Knuth *et al.*, 2014). Functional thinking involves generalising of relationships between covarying quantities, expressing those relationships in words, tables, graphs or symbols, and reasoning with the various representations to analyse function behaviour (Blanton *et al.*, 2015). Blanton proposed six essential understandings which should be instilled in order to develop functional thinking in the early years classrooms (Blanton *et al.*, 2015, p.13-14):

- A function is a special mathematical relationship between two sets where each element from one set, named a domain, is related uniquely to an element of the second set, which can be called the co-domain.
- Functions should be viewed as tools for expressing covariation between quantities.
- In a functional relationship between quantities, variables can be either dependent or independent, and can represent a discrete or continuous quantity.
- When engaging with functions, various important types of patterns or relationships can be observed between quantities that differ in relation to each other.
- Functions can be represented in multiple ways.
- Various types of functions behave in fundamentally different ways and analysing change in function behaviour is a way to capture the difference.

Smith (2008, p. 143-144) proposes six activities which can be perceived as the constructions of functional thinking, divided across three related themes:

Engaging in a Problematic Within a Functional System
- Engaging in some type of physical or conceptual activity.
- Identifying two or more quantities that differ during the activities and emphasise the relationship between the variables.

Creating a Record
- Making a record of corresponding values of these quantities, often in the form of a table or graph.

Seeking Patterns and Mathematical Certainty
- Identifying patterns in these records.
- Coordinating the identified patterns with the actions involved in carrying out the activity.
- Using this coordination to construct a representation of the pattern in the relationship.

It is not certain that learners will always engage with the activities in this order. These proposed activities correspond with the activities proposed by CCSMP (2021) to develop algebraic thinking in the early years (see 3.4.1)

Blanton and Kaput (2011) use three modes of analysing patterns and relationships as a framework for discussing the learners' functional thinking in classrooms. These modes are adapted from the work of Smith (2008). The three modes are (Blanton and Kaput, 2011, p. 8):

- Recursive patterning, which involves finding variation within a sequence of values.
- Covariational thinking, which is based on analysing how two quantities vary simultaneously and keeping that change as an explicit, dynamic part of the function's description.
- Correspondence relationship, which is based on identifying a correlation between variables.

Recursive strategies are often used by young learners to generalise functional relationships by, for example, describing iterations of adding on to each term in a growing pattern. Recursive strategies help learners to predict the next element in a pattern, but they do not identify the structural relationship between the pattern and the position or support learners in identifying a rule. A covariational strategy involves learners describing a dynamic relationship between quantities in words. For example, the learner can describe the relationship between the pattern and the pattern position. This requires the learner to identify and analyse the underlying structure of the pattern. Lastly, learners need to construct a functioning rule in words or symbols to articulate the underlying structure and identify the relationship between two variables. As learners make increasingly sophisticated generalisations, they move through various levels of thinking (Hunter and Miller, 2020).

Traditionally, the teaching of function only takes place in later grades of the schooling career. However, to ensure success in algebraic thinking, functions should be taught using a longitudinal approach and made accessible to all learners from a young age. Recursive patterning is one aspect of functional thinking which can be found in most early mathematics classrooms, but covariational thinking and correspondence relationship are often omitted. Research has found that learners can use representational tools to reason about functions. They are able to describe recursive, covarying and correspondence relationships in words and symbols, and can use symbolic language to model and solve equations with unknown quantities (Blanton and Kaput, 2011).

Blanton and Kaput (2004) have founded that the types of representations learners use, the progressions of mathematical language in their descriptions of functional relationships, the ways they track and organise data, the mathematical operations they use to interpret functional relationships, and how they express covariation and correspondence between quantities, can be introduced from the earliest grades of formal schooling.

To initiate functional thinking, a teacher should create opportunities for learners to engage with problematic situations that centre around the relationship between varying quantities. Functional thinking emerges when a learner engages in an activity, chooses to pay attention to two or more varying quantities, and then start to focus on the relationship between those quantities. The crux of functional thinking is the focus on relationship (Smith, 2008)**.** When functions are introduced in the lower grades, the focus often falls on covariation between inputs and outputs, and the rules which govern them (Ng, 2018).

When learners engage in solving problems related to functions, the most common starting point is the introduction of a representation, for example, a table (Smith, 2008). It is the role of the teacher to introduce the use of graphs, tables, pictures, words, and symbols in a scaffolded manner to ensure that learners become increasingly more sophisticated in the ways they make sense of data and interpret functional relationships. The use of a T-chart or function table can provide invaluable structure in learners' mathematical reasoning. Blanton and Kaput (2011) advocated for the introduction of the function table even before Grade 1. In the earlier grades it provides learners with the space to re-represent marks made (when counting for example) as numerals, as they learn to understand the correspondence between quantity and number. The introduction of the function table as a tool for organising covarying data, initiates its transformation an opaque to a transparent object in learners' functional thinking as an object which one can "look through". This can further help spread the cognitive load across grades, which would allow learners in the second and third grades to focus on more difficult tasks such as symbolising correspondence and covariational relationships. However, as learners progress from creating a record to reflecting on that record as a representation of a relationship, it is important that they are prompted to use multiple representational tools (Smith, 2008). Confrey (1992, p. 11) claims legitimacy of knowledge in mathematics evolves in relation to the multiple forms in which the idea might be displayed. He takes the position that it is through the interweaving of actions and representations that mathematical meaning is constructed.

If the focus were to fall only on the correspondence between two members of a set, then the criterion for a record, like a table, would be enough to create a function. That would involve placing two values in the same row to indicate that those two values are corresponding members of the two sets. However, functional thinking emphasises the construction of relationships between variables that goes further than merely correspondence. An important part of the process is constructing a certainty in that relationship. Two concepts form the core of this process. Firstly, a distinction should be made between a covariational and correspondence approach to functions. In the covariational approach the emphasis falls on corresponding changes in an individual variable. In the correspondence approach, the focus on the relation between corresponding pairs of variables. This can be seen as the conventional approach, where algebraic expressions are the primary representation, which prevalent is most mathematics classrooms (Smith, 2008).

Smith (2008) proposes that there is an opportunity to provide learners with situations to create linear functions at a very young age based on their construction of conceptual units. Conceptual units emerge from situations involving repeated addition, which develop from cognitive structures related to counting. Linear functions in the elementary years can be seen as an extension of counting. To ensure that functional thinking as a route to early algebra is integrated into classrooms, Blanton and Kaput (2011, p. 16-17) suggest three connected dimensions of change which focus on the role of the teacher in developing learners' functional thinking. These dimensions include (1) transforming teachers' instructional resource base, (2) using learners' thinking to leverage teacher learning, and lastly (3) creating classroom cultures and practices which support algebraic thinking.

3.5.4.1. Transforming teachers' instructional resource base

Very few classroom tasks and materials are currently structured in a manner designed to develop functional thinking. Elementary teachers should focus on transforming their existing tasks to provide opportunities for engagement with covariational and correspondence relationships. Single-numerical-answer arithmetic problems can be transformed to include opportunities for pattern building, conjecturing, generalising, and justifying. Functional thinking can be introduced by varying the parameters of the problems, or by prompting learners to draw diagrams or set up function tables. Varying a problem parameter helps learners to come up with a set of data which has a mathematical relationship. Using increasingly larger quantities for the parameter leads to an algebraic use of number. When teachers engage in the process of transforming their resource base so that arithmetic tasks include mathematical generalisation to become more algebraic, teachers become more confident to overcome the challenges that exist in a school culture where limited resources and pedagogical knowledge is prevalent. Teachers can then start to see algebraic thinking as a fluid domain which encompasses all areas of mathematics. Algebraic thinking becomes a habit of mind (Blanton and Kaput, 2011).

3.5.4.2. Using learners' thinking to leverage teacher learning

Merely transforming instructional tasks to include mathematical generality is not enough to ensure that functional thinking is developed in the early years of formal schooling. Teachers need an 'algebra sense' to constantly be aware of opportunities to extend classroom discussions about arithmetical problems to find mathematical generality. Teachers need to have the necessary skills and internalised understanding of algebraic thinking to interpret what learners are writing and saying. Carpenter and Fennema's (1999) work on Cognitive-Guided Instruction provides valuable knowledge as to how learners' thinking can be brought forward. Focusing on learners' algebraic thinking in professional development builds teachers' ability to identify classroom opportunities for generalisation and to understand and interpret the representational, linguistic and symbolic tools which learners use to reason algebraically (Blanton and Kaput, 2011).

3.5.4.3. Creating classroom culture and practice to support algebraic thinking

Creating the appropriate classroom culture to support functional thinking and ultimately algebraic thinking, is extremely important. to the teacher must construct a culture of conjecturing, arguing, and generalising with purpose. Learners should take arguments seriously to build on their existing knowledge. Robust functional thinking requires that learners engage in complex mathematical ideas, negotiate new notational systems, and use and understand representational tools as objects for mathematical reasoning. These processes should become standard practice on a daily basis in each classroom (Blanton and Kaput, 2011).

3.5.5. Modelling as a language

Quantitative reasoning involves modelling, and it is argued that learning to model situations is the primary aim of studying algebra. Quantitative reasoning involves building mathematical systems through several cycles of improvement and interpretation with the goal of describing phenomena and aiding reasoning about them (Kaput, 1999). The generalised quantitative reasoning aspect can be seen as part of a larger modelling aspect that extends to a wide variety of notational systems used to represent and visualise various phenomena (Kaput, 1995).

Modelling as a language emerged in Greek mathematics, which was part of the syncopated stage of the development of algebra (see 2.3.2). This stage is characterised by geometric thinking and using geometry objects to represent mathematical objects and ideas. The Greeks started to represent various situations using modelling as a language where a wide variety of notational systems (numbers, points, curves, plains and geometric solids) were used to visualise various phenomena (Kaput, 2008). Algebra as a language to model real-life situations emerged in the 19th century as a concept of school algebra (Ellerton, Kanbir and Clements, 2017). Algebraic modelling as a language demonstrates the value of algebra as a means to solve problems ((Vermeulen, 2007). Introducing modelling into school mathematics seems to make the learning of school mathematics more effective. Modelling should be integrated into the curriculum because it enhances learners' involvement in classroom activities, including mathematisation, doing problems, criticising arguments, finding proofs, recognising concepts, and gaining the ability to abstract these from realistic situations (Wessels, 2009).

When modelling, one starts with a specific situation and tries to mathematise it. With the introduction of ever more advanced technology, it is important to think about how we, as teachers, can assist learners to understand the mathematical concepts which represent specific situations or phenomena. Mathematics is used to simulate situations and track data with computers. Computer programming languages are very similar to algebraic languages which can be used to create, extend, and explore mathematical environments (Kaput, 1999). Computer languages amount to algebraic formalisms within which one can construct explorable and extendible mathematical environments (Kaput, 1995). These technological environments change the ways in which we relate to the particular or the general and how we state mathematical conjectures. They even change how we teach and learn mathematics (Kaput, 1999).

Learners initially solve problems by modelling the problem situations with the use of physical objects or drawings. By reflecting on the modelling strategies, learners are able to abstract these strategies so that they no longer need physical materials to solve problems (Carpenter and Levi, 2000b). Young learners are able to model, as the modelling of equations is present when they start representing internal quantitative relationships in word problems and solving word problems with several operations (Hemmi, Bråting and Lepik, 2021).

Models can take on various forms. They can be pictorial-based, coordinate-based, or character-based. Computer can support operations on all notational systems. In traditional modelling, the aim is to mathematise existing phenomena, but computers make it possible to reverse the situation. Computers are able to simulate phenomena before they happen, and make predictions (Kaput, 1995).

Mathematical ideas can be represented externally as well as internally. External representations involve pictures, words, symbols, diagrams, and graphs, while internal

representations involve mental models and cognitive representations of the mathematical ideas underlying the external representations. Models are ways of thinking about mathematical concepts, and representations are the ways in which models are represented. The use of models and their representations is endowed by two fundamental concepts, namely the ability to translate and generalise abstraction (Cooper and Warren, 2011).

RME describes emergent modelling as a design heuristic for constructing an instructional sequence (Gravemeijer, 2007, 2020). Emergent modelling supports an incremental process in which mathematical models and mathematical conceptions co-evolve. As learners engage in an instructional sequence, they should transition from models of informal mathematical activity towards models for more formal mathematical activity. This principle of emergent modelling will inform the design of an instructional sequence in 3.7.

3.6. THE TEACHING AND LEARNING OF EARLY ALGEBRA THROUGH THE PROBLEM-CENTRED APPROACH

Mathematics classrooms should transition away from traditional teaching and learning approaches based on rote learning of decontextualised rules and procedures. These traditional teaching approaches have proven unsuitable for the development of higher order thinking (Biccard and Wessels, 2012).

Murray, Olivier and Human (1998) believe that learning takes place when learners grapple with problems for which they do not have routine problem-solving strategies. The introduction of problems should precede the introduction of solution methods. The teacher should not interfere with the learners as they try to solve problems. From the perspective of learning as a social process, learners should be encouraged to discuss and compare their solutions with each other. Rather than focusing on the mastery and application of prescribed skills, learners should be engaged in solving problems (Hiebert *et al.*, 1996).

Schroeder and Lester (1989) describe three main approaches to problem solving. In the traditional sense, problem solving refers to the solving of word problems as an extension of routine computational tasks. One can describe this approach as teaching for problem solving. It can be taken further to be teaching about problem solving. In this more progressive approach to problem solving, learners are taught to employ various methods or strategies when challenged with a problematic situation. Lastly, learners can be taught through or via problem solving, in which case problems are used as a vehicle to teach learners certain mathematical concepts.

3.7. CONCLUSION

Learners are able to reason algebraically from a very young age (Radford, 2008), and a need exists for the implementation of early algebra concepts and skills in early years mathematics classrooms. In this chapter, a systematic review of the literature on early algebra was presented. This review informed the construction of an instructional design sequence based on the principles of RME (Chapter 5). The purpose of the design sequence is to provide a framework for the implementation of the generalisation concept in early mathematics classrooms. Chapter 3 started by exploring the principles of RME education which include guided reinvention and emergent modelling. From there an in-depth literature study of early algebra, the development of algebraic thinking and the main components of early algebra was undertaken, while constantly keeping the concept of generalisation as the focus. Lastly, the characteristics of problem-centred classroom was explained to ensure that the instructional sequence is based on principles of teaching for understanding. These analytical themes explored in Chapter 3, as well as, the stages of the emergence of the big ideas of early algebra in Chapter 2, was used to construct a higher-order formulation in the format of an instructional design sequence. The sequence and the use thereof is set out in Chapter 5. In the following chapter, the explanation, justification and evaluation of the methodological choices made in this study will be explained.

CHAPTER 4: METHODOLOGY

4.1. INTRODUCTION

The generalisation concept is often not taught effectively in early years education, which deprives learners of its utility in fostering the early algebra skills and early algebraic thinking necessary for successfully engaging in formal algebra in the later grades. The purpose of this systematic literature review is to investigate how the development of the generalisation concept through an instructional sequence designed based on the principles of Realistic Mathematics Education (RME) (Gravemeijer, 2020, 2007), can be used to develop early algebra concepts and skills in early mathematics classrooms. In this chapter, the methodology used, which includes the research approach, research paradigm, research design and research process, is discussed in order to explain, evaluate and justify all methodological choices made in the study.

4.2. QUALITATIVE RESEARCH

Qualitative research means any kind of research that produces findings which does not originate from statistical data which can be quantified. It uses a naturalistic approach which aims to understand phenomena in context-specific settings, such as real-world settings, for example a real classroom (Golafshani, 2003). Qualitative research asks open-ended, exploratory questions which has unlimited, emergent descriptive options. The success of a qualitative study, lies in discovering something new, rather than proving a hypothesis (Elliott and Timulak, 2005). Methods such as interviews and observations are dominant in qualitative research as it leads to results which provide illumination, understanding and extrapolation. In qualitative research, the involvement of the researcher is motivated as the real world is subject to change. A qualitative researcher should be present during the changes to record the events that take place before and after the changes (Golafshani, 2003, p.600). Thematic analysis is a way of analysing primary qualitative research literature. A systematic literature review is useful to synthesise and integrate findings from various qualitative studies (Thomas and Harden, 2008).

4.3. RESEARCH PARADIGM

Ontology is a domain of philosophy involving the assumptions we make in order to believe that something is real and makes sense (Scotland, 2012). Ontology is essential to a paradigm because it helps provide an understanding of the things that constitute the world (Kivunja and Kuyini, 2017). The ontological position of interpretivism is that research is anti-fundamentalist. This assumption rejects the idea that there can only be one truth or that one universal knowledge exists (Guba & Lincoln, 1994). Interpretivist researchers believe in multiple realities and the truth or reality is constructed rather that discovered (Grix, 2004).

In research, epistemology refers to how we to come to know something or how we know the truth or reality. Epistemology is what counts as knowledge in the world. (Kivunja and Kuyini, 2017). The interpretivist epistemology is subjective and external realities are not directly accessible to observers without being influenced by these observers' prior knowledge, ideologies and perspectives (Guba and Lincoln, 1994, p.104).The researcher can be viewed as part of the social reality which is being studied. Single and discrete interpretations should not be viewed as 'correct' but a variety of well-proven interpretations should be included to analyse a certain concept or idea (Grix, 2004). The researcher's interaction with various texts and literature sources is used to describe certain phenomena based on various interpretations with the aim on constructing a new reality.

A paradigm is a collection of understandings on the part of an individual or a group of individuals about the types of things that are done when conducting research in a specific field. It includes the types of questions that one ask , the type of answers that are expected and methods employed to find these answers (Asiala *et al.*, 1996). This systematic literature review will be conducted within an interpretive research paradigm. Non-empirical studies often fall in the interpretivism paradigm as a philosophical perspective on exploring the social world to develop a clear understanding of a specific phenomenon (Gray, 2014, p. 24). A non-empirical study uses research methods to analyse and describe existing data or literature in the domain. It is different from empirical studies because it does not rely on the collection of new data but solely on existing data (Du-Plooy-Cilliers, Davis and Bezuidenhout, 2014, p.68). Layers of understanding are developed as the phenomenon is well-described, instead of reducing events to simplistic interpretations (Scotland, 2012). In the interpretivist paradigm, researchers use hermeneutic and phenomenology, where hermeneutic is the study and analysis of literature and phenomenology refers to the focus on people's subjective interpretations and perceptions on the world around us or a certain phenomenon (Ernest, 1994, p.25). This study aimed to explore a certain phenomenon, which is the emergence of early algebra concepts from the history of mathematics and how it can be used to inform an instructional sequence of generalisation activities. The phenomenon is described based on existing literature in the domain which is interpreted and analysed with the aim of constructing an instructional sequence.

The aim of interpretivist research is not to provide context-free universal knowledge, but rather to analyse and explore various interpretations of certain phenomena to understand the interpretations and interactions of people with the specific phenomena (Creswell, 2007).

4.4. RESEARCH APPROACH

This study followed a qualitative research approach in the interpretivist paradigm. Data was generated by using a systematic literature review. This study followed a qualitative research approach with the aim of enabling deep understand of how early algebraic thinking can be developed through generalisation. The resultant findings were used to design an instructional sequence which may lead to early algebraic thinking through a series of well-planned and pre-determined generalisation activities (Elliott and Timulak, 2005). The primary sources of data which were peer-reviewed articles, books and websites were obtained through a thorough and systematic literature review. Xiao and Watson (2019) explain that literature reviews can be categorised according to their purpose based on the research question. These categories include: describe, test, extend and critique. The purpose of this study was to extend on current literature by designing an instructional sequence for the implementation of generalisation in the early years classroom with the purpose of developing algebraic thinking based on existing literature and concepts in the domain. An extending review aims to go beyond providing a summary for existing literature but aim to construct new, higher order ideas. This type of review lends itself to theory building. For qualitative research studies, this involves extracting concepts and second-order concepts from the literature with the aim of transforming these concepts into third-order (new) concepts (Xiao and Watson, 2019).

4.5. RESEARCH METHODOLOGY AND DESIGN

4.5.1. Research methodology

In this study, a systematic literature review was chosen as the appropriate methodology to generate data to construct an instructional sequence of generalisation activities to develop early algebra in the early years classroom.

4.5.2. Research design

This study was designed as a systematic literature review. The question of which types of generalisation activities can be used to develop early algebraic thinking in the early years classroom was addressed by systematically reviewing the literature with a focus on certain themes which are specified in more detail in this chapter. The findings were applied by synthesising an instructional sequence for the development of generalisation in early years mathematics.

For the purpose of this study, an integration of various conceptualisations of the process of a systematic literature review was implemented.

4.6. SYSTEMATIC LITERATURE REVIEW PROCESS

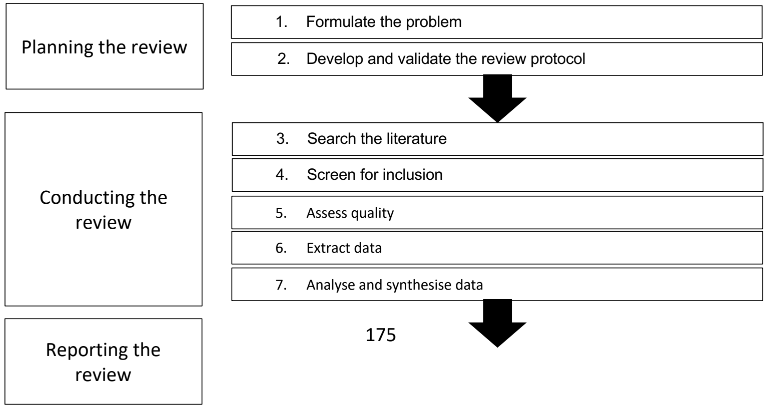
Building research on existing knowledge and relating this existing knowledge is the core of academic research. An effective and well-conducted review as a research method, constructs a solid basis for the advancement of knowledge and can facilitate theory development. By analysing and integrating findings from various empirical studies, a literature review can address research questions with a power that no single study has. A literature review is an effective way to show evidence on a meta-level and shed light on areas which need more research, which is a critical component of constructing theoretical frameworks and building conceptual models. Even so, traditional methods of literature review may lack thoroughness as they are not undertaken in a systematic manner (Snyder, 2019). In this study, conceptualisations are not formed in a traditional (empirical) sense but are formed through the adaption and merging of findings from previous research (Jaakkola, 2020). For that reason, it is proposed that this study is conducted as a systematic literature review which extend theories and concepts which arise in literature based on, mostly, empirical research (Jaakkola, 2020).

A stand-alone systematic literature review aims to make sense of a body of existing literature through aggregation, interpretation, explanation, or integration of existing research. Dixon-Woods explains the characteristics of a systematic review (2016, p. 891):

- Uses an explicit study protocol
- Addresses a formal, pre-specified, highly focused question
- Defines the eligibility criteria for studies to be included in the research in advance
- Is explicit about the methods used for searching studies
- Screens publications for inclusion in the review against pre-specified criteria
- Conducts formalised appraisals to assess scientific quality and otherwise limit the risk of bias
- Use explicit methods to combine findings of studies

Such method of research commonly involves suggest new relationships between existing constructs. The aim is to construct logical arguments and reasons to support these relationships, instead of testing them empirically(Jaakkola, 2020).

Even though procedures in various types of reviews differ, eight steps were followed in conducting a systematic literature review, as illustrated in diagram 4.1.



8. Report findings

Diagram 4.4. Process of the systematic literature review (Xiao and Watson, 2019)

4.6.1. Planning stage

Step 1: Formulate the problem

When planning to conduct a systematic literature review, it is important that the researcher considers why there is a need for the review and what the possible gap in the literature could be . A clear purpose for the review needs to be stated (Snyder, 2019). The research problem for study was the lack of a framework for the effective teaching of the generalisation concept to foster early algebra capacities in young learners. An initial literature review was conducted to review the state of early algebra and the teaching of generalisation in early years classrooms. From the literature, arose the need to further explore how the generalisation concept can be used to teach early algebra in early years classrooms.

From there a purpose for the study was constructed:

> The purpose of this systematic literature study is to investigate the role of generalisation in the development of early algebra concepts and skills.

The analysis of concepts should start by isolating the questions of the emphasised concepts from other concepts (Wilson, 1963). In other words, the research question should narrow the scope of the study and make clear which concepts will be analysed.

The main research question this study aims to answer is as follows: What is the role of generalisation in the understanding of early algebra concepts and skills in young children?

Another research question is further raised: How can the historical development of algebra and scholarly trajectories of algebra learning be synthesised to construct an instructional design sequence which focuses on generalisation for early algebra?

From the main research question, the following sub questions were derived:

- What implications can we derive from the historical development of algebra for early years learning of algebra?
- What is early algebra?
- What is the role of algebraic thinking and generalisation in the understanding of early algebra?
- How can the problem-centred approach be implemented in the teaching and learning of early algebra to foster understanding?

These research questions formed the basis of all decisions and approaches chosen when planning his study. This approach enabled the researcher to answer all the research questions and synthesise the data abstracted in such a way to develop a new, higher-order construct to answer the main research question (Xiao and Watson, 2019).

Step 2: Develop and validate the review protocol

A search strategy was developed to ensure that relevant data sources were found and included. Appropriate search terms, databases, and inclusion and exclusion criteria were selected based on their relevance to the main research question and sub questions. (Snyder, 2019). The review protocol should describe all the elements of the review including the purpose of the study, research questions, inclusion criteria, search strategies, quality assessment criteria, screening procedures, strategies for data extraction, synthesis and reporting (Xiao and Watson, 2019).

Identifying search terms

Search terms are words or phrases used to identify appropriate articles, books and reports. These terms should be related directly to the research question (Snyder, 2019). A keyword search as approach was implemented to find appropriate data sources in this systematic literature review. The following key words (in bold) and phrases, derived from the problem statement and research questions, were used to search for sources:

- **Generalisation** in algebra and early algebra
- **History** of the development of algebra

- Teaching for **understanding** and the **problem-centred approach**

Employing these key words and phrases and derived concepts, ensured that the search was focused, and that the articles, textbooks and reports found were relevant to, and important to include in, the study. The table below shows relevant terms which were searched for under the main key words as described above.

Main key words and search terms		
Generalisation in algebra and early algebra	**History of the development of algebra**	**Teaching for understanding and the problem-centred approach**
Development of algebra *Development of early algebra* *Early algebraic thinking* *Fundamental components of early algebra* *Big Ideas of early algebra*	*Rhetorical stage* *Syncopated stage* *Symbolic stage* *History of school algebra*	*RME approach* *Emergent modelling* *Guided reinvention*

Table 4.1. Main key words and search terms

Only English search terms were used and therefore only English data sources were analysed for the purpose of this study. This choice was based on the study being conducted in English, and the prevalence of sources available in English.

Electronic databases, search engines and other sources

Material for review was found by submitting the selected search terms and key words into electronic databases, search engines and other sources. Only sources published in English were included in this study.

The following electronic databases were employed during the search process:

- Google Scholar
- SAGE Journal Online
- EBSCOhost
- ERIC
- IEEE
- JSTOR
- ScienceDirect
- Taylor & Francis Journals
- ResearchGate

Other sources included in the search process were:

- Bibliographies of reviewed articles – These were scanned to identify further materials for review.
- Conference proceedings – These were reviewed to ensure that recent and relevant themes and studies were included.

Inclusion and exclusion criteria

Initial literature searches yielded many articles and publications, necessitating the application of inclusion and exclusion criteria informed by the research question (Snyder, 2019). The table below explains the inclusion and exclusion criteria used in the systematic literature review.

	Criteria
Literature Included	• All publications relevant up to the year 2021 – because the purpose of the review was to explore the impact of the history of algebra on how algebra is taught today, older sources and newer sources had to be integrated. • Peer reviewed publications – to ensure the quality of publications used. • Literature in English – the study was conducted in English and the researcher understands English. • Full length publications are available – to ensure that misconceptions did not arise from only having part of a publication available • Literature focused on algebra and early algebra • Literature focused on the history of algebra • Literature focused on teaching mathematics for understanding • Literature focussed on the problem-centred approach
Literature Excluded	• Literature only focused on secondary or tertiary education of algebra • Literature which focuses on other content areas of mathematics than algebra • Literature in other languages than English • Literature which does not have full publications available

Table 4.2. Inclusion and exclusion criteria

4.6.2. Conducting the review

Step 3: Search the literature

Literature gathered for review in accordance with the review protocol discussed in 4.6.1 was saved to Mendeley for data management purposes. The Boolean operators (AND, OR, NOT) were used when searching search engines to ensure optimal results (Xiao and Watson, 2019). The terms “early algebra” AND “generalisation” is an example of how the search terms were employed.

Backward searches were conducted to ensure that all relevant literature about the theme could be identified. These backwards searches involve using the list of references at the end of an article to find more relevant articles (Xiao and Watson, 2019).

By consulting experts in the field, including the supervisor of the study, the completeness of the search was continuously monitored and checked. Once candidate materials were identified, this literature was screened for inclusion or exclusion.

Step 4: Screen for inclusion

The researcher had to screen all sources found from submitting search terms to the electronic databases and search engines for inclusion in the study. A two-stage procedure was employed, which involved a coarse sieving process (screening the abstracts of the articles), followed by a refined quality assessment based on a full-text review (Xiao and Watson, 2019).

The researcher started by excluding studies which were duplicates of other sources. After that, the researcher read the abstract of each study to decide whether it was relevant to the research question. In the case of textbooks, the table of contents of each source was studied to determine its relevance. Finally, all sources were read briefly to categorise them according to their relevance to each theme arising from the research questions. These themes included: the development of early algebra, generalisation, history of algebra, teaching for understanding and the problem-centred approach. Mendeley was used to organise and read all the references: once all the references were organised, each was fully read to decide whether it should be included or excluded.

Step 5: Assess quality

For the purpose of this systematic literature review, only peer reviewed studies were included to ensure a high quality of sources used.

Step 6: Extract data

After conducting the literature review and deciding on the final sample, the researcher considered how all the sources would be used to conduct an appropriate analysis. Data was extracted according to the identified themes. All sources were read on Mendeley (data management application), where themes were extracted from each article, textbook or report, and carefully marked and organised according to the related theme. From there, data could easily be analysed and synthesised. Diagram 4.2 shows the process of extracting data which was employed:

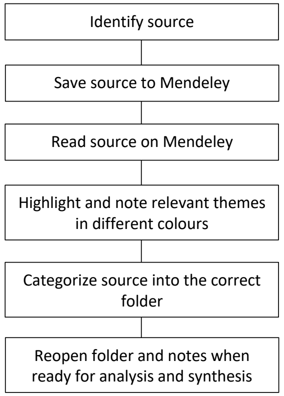


Diagram 4.2. The process of extracting data

Step 7: Analysing and synthesising data

For the purpose of this study, to construct a higher-order instructional sequence for the teaching of generalisation in the early years classroom, a thematic analysis of data was undertaken. Once the data was extracted, the researcher could organise the data in a systematic manner. The researcher generated analytical themes, similar to third-order interpretations, with the aim of answering a specific research question rather than being explanatory (Xiao and Watson, 2019, p.97). In this case, the researcher aimed to answer the question: "What can we learn about the role of generalisation in the understanding of early algebra concepts and skills in young children, from the history of algebra?" To answer the research question, the following analytical themes and sub-themes were generated:

- History of algebra
  - The rhetorical stage
  - The syncopated stage
  - The symbolic stage
  - Conceptual stages
  - Purposes for school algebra emerging from history
  - Algebra situation in South Africa
- Role of generalisation
  - Generalisation and formalisation
  - Generalisation activities
  - Generalising arithmetic
  - Generalising a rule or function
- Teaching for understanding
  - The problem-centred approach to learning algebra
  - Role of the teacher
  - Classroom Culture
  - Role of the learner
  - Problems and task design
  - Real Mathematics Education
  - Emergent modelling
  - Guided Reinvention
- Teaching of early algebra
  - The early algebra curriculum
  - Fundamental components of early algebra
  - The use of mathematical models
  - Kaput's framework for early algebra
  - The Big Ideas of early algebra
  - Developing algebraic thinking

4.6.3. Reporting the review

When reporting the review, the researcher started by explaining clearly the need and the purpose of the study. Review studies can be reported in various ways, but should follow some generalised guidelines. It is necessary to describe transparently how the research process transpired. By clearly explaining the research design, the method for collecting data, how literature was identified, analysed and synthesised, the reader can assess the quality and trustworthiness of the data (Snyder, 2019).

Step 8: Report findings

The report on the findings of this systematic literature review was organised according to the key analytical themes generated during Step 7 of the review process. The chapters of the report were organised in the following manner:

Chapter 1: Introduction of the study, which aims to provide the purpose and motivation for the study, as well as outlining the study.

Chapter 2: The historical review of the development of algebra and the emergence of the big ideas of algebra in history. This chapter aims to explain the core ideas of early algebra and how they emerged in the history of algebra. This chapter greatly informed the instructional design based on the principles of RME in Chapter 5.

Chapter 3: Early algebra, generalising and structure. This chapter aims to analyse early algebra, the development of early algebraic thinking, the fundamental components of early algebra and the role generalisation plays in the early mathematics classroom. This analytical chapter greatly informed the instructional design which is explained in Chapter 5.

Chapter 4: The methodology. This chapter aims to explain, justify and evaluate all the methodological choices made in the study.

Chapter 5: Model for implementation of the instructional sequence. This chapter applies the findings of chapters 2 and 3 by providing a higher order construct in the form of an instructional design sequence for implementing generalisation in the early mathematics classroom.

Chapter 6: Conclusion. In this chapter an overview of the findings will be given to conclude the report.

4.7. TRUSTWORTHINESS OF THE DATA

For the purpose of this study, a systematic literature review was conducted. To achieve trustworthiness, the study methodology was carefully documented, explained and justified in such a manner that it could be duplicated by another researcher to achieve the same results, and an audit trail was built by carefully documenting all online platforms and search terms which were used to find sources and data.

Hirschheim (2008) explains a framework of ensuring the trustworthiness of arguments based on three necessary components. Firstly, (1) claims mention the obvious statements or thesis which the reader of the work needs to understand as true. In this study claims were made based on the overview of the history of algebra and the emergence of early algebra concepts from history. Secondly, (2) grounds are the proofs which underpin the reasoning and aim to reinforce the claims to convince the reader. Evidence is constructed from previous works existent in the literature. Lastly, (3) warrants are the underlying beliefs and assumptions which are taken to be true in a specific research domain. This study is trustworthy because all **claims** are substantiated by **abundant** grounds and can therefore make a noteworthy contribution to knowledge (Hirschheim, 2008).

All sources were read in their entirety to ensure that a complete analysis of the source could be done, and to avoid bias from only reading parts of a source (Snyder, 2019). It was important that authors' arguments were correctly interpreted. The researcher aimed to interpret all literature in an objective manner. To ensure that authors received the necessary credit, all sources were meticulously cited to avoid plagiarism.

Findings and conclusions of literature should be true and valid to readers, practitioners and other researchers. When research is conducted in a rigorous manner, research can effect practice and theory (Merriam, 2009). The validity and reliability of a study depends on the construction of an effective design by clearly stating and justifying the way in which data was collected, analysed, interpreted and the way in which findings are reported. For this study, validity and reliability were achieve by, in this chapter, clearly explaining and justifying all methodological choices. Furthermore, triangulation was achieved by using multiple sources of data and by repeatedly cross-checking data (Merriam, 2009). Data was collected from academic journals, textbooks, articles from various databases and websites. Data from over a long period of time was included to provide a valid overview and analysis of the data. Even though this study was conducted by one researcher, the research was constantly checked by the study leader to further ensure validity through triangulation.

Peer reviewed articles, journals and books were used as the primary sources of data/ When mainly using peer reviewed sources, the validity of the study is enhanced and the study will be of a higher quality (Merriam, 2009).

4.8. ETHICAL CONSIDERATIONS

The systematic literature review makes use of existing data sources which are available on online platforms or in libraries. All data used in the review, is already available in the public domain. No information, which is not yet published, including the personal information of authors, was used in this study.
Ethical clearance was obtained by the Department of Ethics Committee at Stellenbosch University. This study was deemed to be a low risk study as it does not involve research participants to obtain data.

4.9. CONCLUSION

The purpose of this chapter was to explain, justify and evaluate all the methodological choices made in the design of this systematic literature review. The purpose of this systematic review was to see how the concept of generalisation could be implemented in the early mathematics classroom to foster early algebraic concepts and skills. An instructional design sequence was constructed based on the principles of RME and learning trajectories in history, to inform learning sequences for implementation in classrooms. This chapter explained the research approach, the research paradigm, the methodology and design. The systematic review process was explained in detail. This was done to ensure that the study is trustworthy and can be duplicated by other researchers to achieve the same results and findings. Lastly, the ethical considerations and trustworthiness of the study was explained. In chapter 5, the model for implementing generalisation in the early years classroom is set out.

CHAPTER 5: TAKING ALGEBRA TO THE CLASSROOM – IDEAS FOR IMPLEMENTING AN INSTRUCTIONAL SEQUENCE

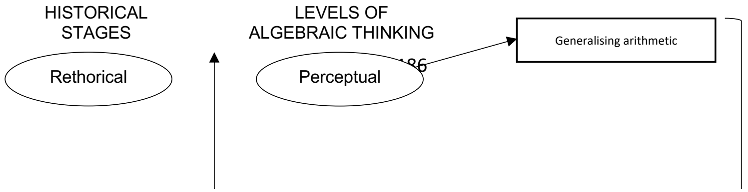
5.1. INTRODUCTION

The aim of this systematic literature review of early algebra, its emergence from history, and the components which develop algebraic thinking, is to inform the design of an instructional sequence for the development of algebra thinking through generalisation activities. This is expected to be an important contribution to the literature, as the traditional approach to algebra, functions and patterns which is currently taught as one the five content strains in the South African CAPS document, is not taught effectively to cultivate learners' deep understanding of early algebraic concepts (du Plessis, 2018). Even though the South African CAPS curriculum mentions and explains the role of early algebra well, the implementation thereof in classrooms is lacking (see 3.2).

The instructional design is constructed by considering the main aspects of the teaching and learning of early algebra which emerged from the systematic review of the literature in terms of the historical development of algebra and the emergence of early algebra concepts from history. The following aspects stood out as underpinnings for a successful instructional design:

1. The stages of the development of algebra throughout history, as well as the conceptual stages (Sfard, 1995; Katz and Barton, 2007; Tabak, 2011; Katz and Parshall, 2014).
2. Levels in the development of algebraic thinking (Nixon, 2009).
3. Thought processes (generalising and specialising) involved in engaging with algebraic problems (Mason, Burton and Stacey, 2010).
4. Key aspects of generalisation (Roberts, 2012c, 2012b).
5. Generalisation activities in a specific sequence.

From there, a diagram was drawn up to indicate how these core aspects of early algebraic thinking are related and can be used to construct an instructional design.



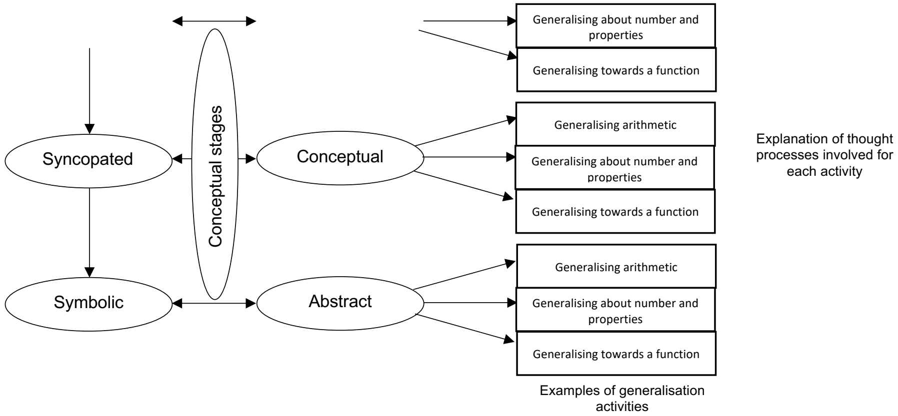


Diagram 5.1. The relationship between aspects involved in the instructional design for generalisation

The instructional design will aim to provide teachers with a roadmap for the implementation and integration of generalising throughout the mathematics curriculum with an emphasis on the patterns and structure which can be noticed in the whole of mathematics. The framework will consist of the following aspects: the theoretical basis of the approach, the role of patterns and structure in the development of generalisation, a possible sequence of activities, the role of the teacher, the role of the learner, and assessing the development of generalisation.

5.2. THEORETICAL PERSPECTIVE ON LEARNING

The design of an instructional sequence based on the principles of emergent modelling and guided reinvention takes on the perspective of the socio-constructivist framework, which holds that the problem-centred approach to learning should be used to develop generalisation in early years classrooms.

Social constructivism is informed by the work of Vygotsky, who argued that learning is a process which occurs within social interactions. Social constructivist theory is grounded in the proposition that children's social and material interactions with their environments are the means through which they learn (Fox, 2005). Human and Olivier agree (1999) that the socio-constructivist learning theory treats learning as a social process whereby learners learn from each other and their teachers. Learning takes places as learners engage in meaningful classroom discussions, where ideas are shared and communicated. In the process of learning, learners should be encouraged to compare and interpret others' ideas, reflect on their own thinking, and try to negotiate a mutual understanding of the concept at hand (Human and Olivier, 1999).

In the socio-constructivist theory, learners are seen as active participants in the learning process. They are responsible for their own construction of knowledge and make meaning of what is being learned. New knowledge is constructed based on learners' prior experiences and natural intuitions. In the constructivist view of learning, learners must interpret what they learn and must give it their own meaning based on their existing, but not yet explicit, knowledge (Cobb *et al.*, 2014).

The constructivist view of how learners learn mathematics with understanding, and construct, interpret, think about, and make sense of mathematical ideas, is determined by the elements and organisations of the relevent mental structures that they use to process their mathematical worlds. To construct new knowledge and make sense of new experiences, learners build on and reflect on their current mental structures through the processes of action, reflection and abstraction (Battista, 2004). For numerous mathematical topics, it is known that learners' development of conceptualisations and reasoning can characterised in terms of levels of sophistication (Battista, 2004). Therefore, development takes place in a progression of levels in a learning trajectory or sequence of activities.

The problem-centred approach (see 2.2.2) is implemented as a vehicle for early algebra which involves engagement with meaningful algebraic problems with the purpose of learning mathematics for understanding. The PCA believes that learners learn through problems rather than implementing and applying discrete skills which wer taught to them to solve problems according to prescribed methods (Biccard and Wessels, 2012).

5.3. REALISTIC MATHEMATICS EDUCATION (RME)

As described in 3.2, the aim of the systematic literature review is to design an instructional sequence based on the principles of RME which include guided reinvention and emergent modelling. RME is based on Freudenthal's (1991) view on mathematics: being that it should be meaningful to learners and should be seen as a human activity. This means that learning takes place as learners engage with problems that are meaningful to them. Treffers (1987) describes five features of RME, which can be summarised as follows (Turmudi and Al Jupri, 2009, p.1):

- Phenomenological exploration or the use of meaningful contexts.
- Using models and symbols for progressive mathematisation.
- Self-reliance, where learners use their own constructions and strategies.
- Interactivity, where learning is part of interactive instruction and classroom discussions.
- Intertwinement, where the importance of an instructional sequence and its relation to other topics emerges.

Gravemeijer (2020) (see 3.2) casts RME as a design theory in terms of three instructional design heuristics:

2) **Guided reinvention** which reflects Freudenthal's (1973) idea that learners should experience mathematics as a human activity and reinvent mathematical ideas and concepts as they are guided by the teacher. The history of mathematics should inform the design of a trajectory for learner reinvention of mathematical concepts. In this study, the history of algebra is used to inform an instructional sequence of generalisation activities to develop early algebra.

3) **Didactical phenomenology** originates from Freudenthal's (1983) view that mathematical activity is based on organising. This heuristic involves analysing what phenomenon is organised and how it is organised by the mathematical concept. Stephan, Underwood-Gregg and Yackel (2016) propose a different view of the second design heuristic in RME; arguing that sequences should be experientially real for learners. Instructional tasks should draw on realistic situations as semantic grounding for learners' mathematisations, and activities should be sequenced so that learners will organise their activity within the realistic context to reinvent important mathematics. For the purpose of this study, the perspective on design heuristics of Stephan, et. al. (2016) is accepted and implemented in the instructional sequence.

4) **Emergent modelling** is a dynamic process of symbolising and modelling, within which the process of symbolising and the development of meaning and understanding are reflexively related. In this model the belief is that learners should start by modeling their own informal

> mathematical activity, and that this model of informal mathematical activity should progressively develop into a model for more formal reasoning. The newly formed formal model should be deeply rooted in learners' prior knowledge and natural instincts (Gravemeijer, 2007). Central to the emergent modelling design heuristic is the use of a series of sub-models, which together constitute the overarching model (Gravemeijer, 2020). During the transitions from informal to formal, the teacher supports learners' modelling by introducing new tools or using learner-created tools to communicate mathematical reasoning (Stephan, Underwood-Gregg and Yackel, 2016).

Gravemeijer (2007, p.3) suggests four levels of mathematical reasoning which should be implemented in an emergent-modelling design:

- Activity in the task setting, in which interpretations and solutions depend on understanding of how to act in the setting
- Referential activity, in which models-of refer to activity in the setting described in instructional activities
- General activity, in which models-for derive their meaning from a framework of mathematical relations
- formal mathematical reasoning, which is no longer dependent on the support of models-for mathematical activity.

These levels of mathematical reasoning will inform the sequence of possible activities chosen to develop generalisation in the early mathematics classroom. Within each proposed activity, the activities themselves build on the levels of mathematical reasoning.

5.4. TEACHING PRACTICE OF A GUIDED REINVENTION TEACHER

Stephan, et. al. (2016) propose five teaching practices for implementing the design principle of guided reinvention in an RME based instructional sequence. These practices are suitable for implementing the instructional sequence in 5.1.6.

5.4.1. Initiating and sustaining social norms

The teacher plays an important role in establishing and sustaining classroom social norms which are conducive to learners' reinvention (Stephan, Underwood-Gregg and Yackel, 2016). The role of the teacher is to implement and create a classroom culture based on inquiry mathematics. Social norms of communication, idea sharing, listening, and own construction of knowledge should be implemented by the teacher (Murray, Olivier and Human, 1998). The teacher serves a foundational role in the guidance of learners' reinvention. Attention to social and socio-mathematical norms which characterise guided reinvention teaching, forms a big portion of the decision making of the teacher regarding classroom interactions. The instructional design of the sequence of the activities serves the purpose of orchestrating whole-class discussions in which certain mathematical practices have been pre-established (Stephan, Underwood-Gregg and Yackel, 2016).

5.4.2. Supporting the development of socio-mathematical norms

The teacher should establish socio-mathematical norms in guided reinvention classrooms whereby, learners' explanations of solutions and strategies are acceptable if they meet the criterion that describe the learners' actions on mathematical objects which are experientially real to them (Stephan, Underwood-Gregg and Yackel, 2016). Listing procedural steps as an explanation for a solution cannot be deemed acceptable unless accompanied by reasons for the calculations. The question: "How do you know?" is instrumental in developing conceptual discourse rather than procedural discourse in the classroom (Cobb and Yackel, 1996).

5.4.3. Capitalising on learners' imagery to create inscriptions and notations

This practice revolves around the teacher's motivation of learners' imagery through notation and tools (Stephan, Underwood-Gregg and Yackel, 2016). Modelling as a language for algebra, which is one of Kaput's (2008) (see 3.5.5) components of algebra, must be incorporated. The teacher can do this by capitalising on the tool development which is part of RME instructional design, and the rich imagery that this design can generate (Stephan, Underwood-Gregg and Yackel, 2016). Encouraging learners to model their understanding and problem-solving processes engages them in mathematical thinking and reasoning. The teacher, as well as the learners, should be aware of the different ways of modelling: talking, showing, drawing, writing number sentences, writing sentences in words, using concrete equipment, using a number line, using a hundred square, fraction walls, tables, graphs, function machines, even acting out or role play (Roberts, 2012). The teacher should model the way in which he or she wants learners to think and speak. It is also the role of the teacher to introduce and model the use of novel mathematical instruments, like measuring instruments.

5.4.4. Developing small groups as communities of learners

The practice involves the teacher establishing productive small groups in the classroom. Teachers should help learners realise the value of working with peers (Stephan, Underwood-Gregg and Yackel, 2016). Working in smaller groups in the mathematics classroom, especially in the early years of mathematics, is beneficial, as learners can easily interact socially with the learners around them and the teacher can form a good idea of the each learner's understanding of mathematical concepts (Murray, Olivier and Human, 1998). How learners should be grouped is often a highly contentious topic. It has been argued that it is important to group learners in various ways, giving them opportunities to work with a variety of peers. (Murray, Olivier and Human, 1998) (see 2.2.2.2). The teacher should establish social norms which are conducive to guided reinvention when working in small groups. These social norms should include (Stephan, Underwood-Gregg and Yackel, 2016, p. 44):

- developing personally meaningful solutions
- explaining one's reasoning to one's partners
- listening to and attempting to understand the explanations of other group members
- persisting with challenging problems
- collaborating to complete activities, including indicating agreement or disagreement with partners
- asking your partner for help before you ask your teacher.

5.4.5. Facilitating genuine mathematical discourse

Genuine mathematical discourse can be facilitated by: (1) introducing mathematical vocabulary and tools to record learners' inventions, (2) asking questions which elicit learners' strategies and scaffold critical thinking, (3) restating learners' solutions in clear or more advanced terms, and (4) using learners' strategies during exploration time to orchestrate meaningful classroom discussions (Stephan, Underwood-Gregg and Yackel, 2016. Teachers should use the framework of genuine discourse to guide learners' thoughts in developing generalisation. Three actions and types of questions are integral to all mathematics lessons which aim to develop early algebraic thinking and generalisation (Warren and Cooper, 2003, p.13):

- Predicting: What do you think will happen?
- Justifying: Why do you think that happened? How do you know?
- Generalising: What is the rule that you notice? Can you represent the rule in words/a diagram/symbols?

The teacher should constantly be sensitive to opportunities to extend learners' mathematical knowledge and thinking when interpreting their explanations of solutions. In this way, the teacher can prevent or correct misconceptions, or prompt further insights.

5.5. PLANNING PRACTICE OF A GUIDED REINVENTION TEACHER

5.5.1. Preparation

When planning the implementation of an instructional sequence, the teacher must consider the goals of the entire unit of study. The teacher must become comfortable with the outline of the instructional sequence, and should understand the mathematical concepts envisioned by the sequence and the preconceptions which learners might have. The teacher should work through the activities to form a hypothetical image of the pathways that can emerge as a result of learners' diverse reasoning (Stephan, Underwood-Gregg and Yackel, 2016).

5.5.2. Anticipation

Before each lesson, the teacher should envision the best ways to introduce tasks, work out problems, and anticipate possible learner reasoning and how it fits with current mathematical practices. The teacher should use predicted learner reasoning to prepare for potential discussion topics. The teacher should uses the anticipate the lesson flow to conjecture which strategies learners will develop, and which are important for progress toward reinventing mathematics ideas (Stephan, Underwood-Gregg and Yackel, 2016).

5.5.3. Reflection

After a lesson has concluded, the teacher should reflect on the learners' discourse and reinvention to determine the status of classroom mathematical practices, including the modelling of learners' thinking via various representations. The teacher should further reflect on the mathematical learning that emerged, and lastly on the status of classroom social and socio-mathematical norms. The reflective analysis is used to make decisions and predictions about subsequent lessons (Stephan, Underwood-Gregg and Yackel, 2016).

5.5.4. Assessment

The guided reinvention teacher creates and implements daily formative assessments to form a picture of the learners' growth individually and as a class (Stephan, Underwood-Gregg and Yackel, 2016). There are three types of formative assessment which can be implemented in an educational framework. These include assessment *as* learning, assessment *of* learning, and lastly, assessment *for* learning. Assessment should be conducted for learning. Assessment for learning requires that teachers observe the knowledge, skills, experience, and interests which learners demonstrate. Teachers then use these observations to tailor instruction to ensure that learners' needs are met. They can also provide valuable direct feedback to students to help improve their learning ('Patterning and Algebra Grades 4 to 6', 2008).

Assessment for learning is different from assessments conducted to serve the purpose of accountability, ranking, or certifying competence (Black & Harrison, 2006). When teachers assess for learning, assessments aim to provide evidence of learners' understanding of the concepts taught. Once a teacher has formed a clear idea of learners' understanding, future learning programs can be accurately modified and adapted to ensure that learners' learning is moving forward (Lee, 2006) (Stephan, Underwood-Gregg and Yackel, 2016).

To ensure that high quality assessment for learning is conducted (Lee, 2006, p. 43):

- Learners must engage in activities or answer questions that fully explore their understanding.
- Learners should have time to think through what they know, understand and can do, and to fully communicate their thoughts.
- Learners must use mathematical language effectively to communicate their understanding and skills in mathematics.

According to Lee (2006, p.44), assessment for learning in mathematics is effective when:

- Learners explore a problem, find they can easily use a mathematical concept to solve the problem, and move on to more complex problems to further explore their understanding.
- Learners mark tests together in groups, and use the results to set out the work that they will have to complete in the following lesson to continue to improve their understanding.
- Teachers asks searching questions and give learners sufficient opportunity to work through them. Teachers should use their knowledge to plan for the next module of work.
- Teachers should observe learners when they are engaged in activites to determine which learners fully comprehend concepts, and which ones need more support.

An assessment can provide opportunity for learning if it provides useful feedback to teachers and learners. This feedback should be used to adapt and modify future learning experiences. Knowledge about learning needs, obtained from assessments, should inform lesson plans (Hodgen and William, 2006).

In the South African CAPS curriculum, there are no assessment standards which guide teachers in the assessment of early algebra (Roberts, 2010). Even so, the CAPS document does provide valuable information on assessment of mathematics. The CAPS document (DBE, 2011a) describes assessment as a continuous planned process of identifying, gathering and interpreting information regarding the performance of learners. It motivates the use of various forms of assessments to ensure assessment for learning. The CAPS documents states that assessing involves four steps:

- Generating and collecting information about achievement;
- Evaluating this evidence;
- Recording the findings; and

- Using information to understand and thereby support the learner's development so that teaching and learning can be improved.

Regular feedback based on assessments is crucial to effective learning. Summative assessments can be used to document learners' cumulative learning (Stephan, Underwood-Gregg and Yackel, 2016).

5.5.5. Revision

Revision happens on two levels: daily revision based on formative assessments, and at the end of an instructional sequence. The latter should identify modifications to be made to the materials, tools, or questions asked (Stephan, Underwood-Gregg and Yackel, 2016.

5.6. GENERALISATION ACTIVITIES AS THE GOAL FOR DEVELOPING EARLY ALGEBRA

Roberts (2010, p.169) describes three generalisation activities which encompass generalisation in mathematics (see 3.5.1).

Roberts (2012) further provides guidelines for primary school teachers to implement early algebra in the classroom via generalisation. Her guidelines are based on the framework for the components of early algebra by Kaput (2008) (see 2.2.3 and 3.5).

5.6.1. Guidelines for generalising arithmetic

Generalisation of arithmetic is located in the 'Numbers, operations and relationships' content area of the CAPS document (DBE, 2011). It can also be applied when working with data handling, shapes and measurement, as patterns underlie all these content areas (Roberts, 2012). Guidelines for generalising arithmetic include (Roberts, 2012, p.308-309):

1. Be deliberate about and explore when something happens, and when it always happens, in mathematics.
2. Look for patterns in groups of number sentences.
3. Look for patterns in sequences of sums.
4. Ask about (observe, describe, talk about) how special numbers behave.

5. Expect and ask for descriptions of what is observed.
6. Ask for explanations to show if something is always true.
7. Explore and talk about equivalence and what the equal sign means.

5.6.2. Guidelines for generalising a rule or function

Generalisation of rules and functions forms the focus of the content area 'Patterns, functions and algebra' in the CAPS document (DBE, 2011). Problem contexts can be explored, along with representations of functions, including families of number sentences, input and output tables, and function machines. Guidelines for generalising a rule or functions include (Roberts, 2012, p.312-315):

1. Expect learners to describe a number pattern in detail.
2. Look at and talk about the operations or functions, not just the numbers.
3. Ask about how operations behave.
4. Set a problem context which requires investigation of a certain function, and then use different representations of the function.
5. Connect work done in 'Patterns, functions and algebra' to work in other content areas.

5.6.3. Guidelines for modelling as a language of mathematics

Modelling concepts, and motivating learners to model their understanding and reasoning, prompts learners' mathematical thinking. Guidelines for generalising a rule or functions include (Roberts, 2012, p.315-317):

1. Model mathematical concepts, problem solving strategies, and calculation techniques. Encourage learners to model their understanding and thinking.
2. Know and make explicit the basic models for the basic operations.
3. Provide opportunities for learners to use concrete objects, draw or imagine objects or processes, and move between representations.
4. 'Algebraify' word problems, and turn them into investigations, to model the process of solving the problem, when one or more of the parameters is relaxed.

These guidelines provide an insight to the types of activities which should be included in the instructional sequence to develop early algebraic thinking in the early mathematics classroom.

5.7. THE INSTRUCTIONAL SEQUENCE TO IMPLEMENT GENERALISATION FOR THE DEVELOPMENT OF EARLY ALGEBRA

The instructional sequence below shows how a variety of generalisation activities can be introduced as an instructional sequence for learners to effectively develop the generalisation concept which ultimately leads to early algebraic reasoning. The design of this instructional sequence is informed by existing research, especially pertaining to the levels of algebraic thinking in which learners are engaged when being challenged with problems. For this design, the historical stages as described in Chapter 2 were used as the developmental stages which learners will work through as they engage in increasingly sophisticated thinking by being introduced to a variety of problems. These stages include the rhetorical, syncopated, and symbolic stages. Alongside these stages run four conceptual stages which are the geometric, static equation solving, dynamic function and abstract stages (see 2.3.4). Furthermore, Nixon's (Nixon, 2009) (see 3.4.3.1.) levels of algebraic thinking were used to inform the sequence in which algebraic problems are introduced. Nixon's levels of algebraic thinking align well with the developmental stages of algebraic thinking, and can be integrated a possible approach for implementing generalisation in early education mathematics.

Three generalisation activities are suggested for each developmental stage. These generalisation activities were based on the three elements of generalisation as described by Roberts (2010, p.169). These three elements include:

- Generalising arithmetic as the exploration of the properties of numbers and operations.
- Generalising about particular number properties and relationships.
- Generalising towards the idea of a function, which includes recognising regularity in elementary patterns, ideas of change including linearity, and representation through tables, graphs and function machines.

It is important to note that these activities can be adapted and adjusted to be more appropriate for each grade in early mathematics education. The sequence of the activities should be taken as the basic structure for the sequence of instruction. It is also important to keep in mind that generalisation should be a habit of mind which the teacher instils in learners. In that sense it should be integrated in all content areas of mathematics.

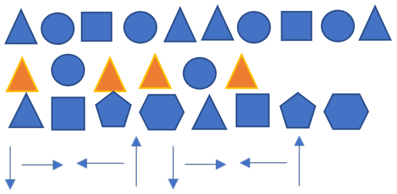
Lastly, the framework will aim to describe the anticipated thought processed which learners engage in as they work through the problems provided to them. These thought processes are taken from Mason, Burton and Stacey's (2010) framework for how mathematical thinking takes place.

The Instructional sequence to implement generalisation for the development of early algebra

Historical stages		Levels of algebraic reasoning	Possible activities according to generalisation elements	Specialising or Generalising (Thought processes)
Rhetorical stage: • Problems solved by looking at individual problems. • Aimed at solving specific problem. • Problems are not categorised. • Solutions to problems are mainly given in words. • Little to no generality can be noticed.	**geometric → static-equation → dynamic function → abstract**	**Perceptual:** • Learners need to coordinate their senses and perceptions. • Learners advance in their use of numbers. • Learners form mental pictures of concepts.	**Generalising arithmetic as the exploration of properties and number operations** **Repeated addition as multiplication** Introduction - Look at the following pictures of children with eyes.  - How many children are there? - How many eyes do they have in total? - How many eyes will two children have? - How many eyes will four children have? - How many eyes will eight children have? - What pattern do you notice? Lesson 1. Look at the picture of hands  2. How many hands do you see? 3. How many fingers does one hand have? 4. How many fingers do two hands have? 5. How many fingers do five hands have? 6. How many fingers do 10 hands have? 7. Do you notice a pattern? 8. Can you describe the pattern in your own words? 9. Write the number of fingers on five hands as an addition sum. 10. Write the number of fingers on five hands as a multiplication sum. Review and discussion - How can you quickly find the answer of a large amount of objects by not counting each object? - What is the relation between addition and multiplication in the sums we did today? - Which one is faster? - How do you know? - Can you always use multiplication in the place of addition? - When can you use multiplication instead of addition?	***Specialising:*** I know: - Learner knows what is asked to do - Specialise to find out how to count in multiples, and the relation to addition and multiplication. - What is relevant and important when repeatedly adding? I want: - Classify and sort the information - Specialise to discover what the real question is - Specialises what is the pattern when counting objects in groups/repeatedly adding Introduce: - Use numbers to show an addition and multiplication sum for repeated addition. ***Generalising:*** Check: - Arguments to check that what has been done so far, is correct. - Whether the rule stands for all amounts of children/hands - Consequences of conclusion to see if they are reasonable Reflect: - On key ideas and moments - On implications of conjectures or arguments - On the resolution: Can it be made clearer? Extend: - Use a given rule to write addition and multiplication sums - Try the result to a wider concept by generalising. - Make a conjecture which is always true.
			Generalising about particular number properties and relationships Introduction - What must be added to 3 to get to 10? - What must be added to 13 to get to 20? - What must be added to 23 to get to 30? - What must be added to 3 to get to 20? - What must be added to 103 to get to 120? Explain. - What must be added to 213 to get to 230? Explain. - What do you notice? - How do you know? - Will this always be true? Lesson 1. Complete the following train sums: 93 + 100 + 120 13 + 30 + 45 53 + 60 + 100 23 + 30 + 35 223 + 240 + 255 2. What do you notice about all the numbers at the beginning of the train sums? 3. What do you notice about all the numbers that you added first? 4. What do you notice about the first answer that you got every time? 5. What is the relationship between 3 and 7? 6. Can you provide a rule in words for the relationship between 3 and 7? 7. Can you think of other number pairs which follow the same rule? Review and discuss? - What did you notice when adding 3 and 7 together every time? - Why is this important to notice? - Does it always work? - Are there other numbers for which this is also true?	***Specialising:*** I know: - Learner knows what is asked to do - Specialise to find out what must be added to 3 to get an answer which is a multiple of ten. - What is relevant and important when adding to 3 every time? I want: - Classify and sort the information - Specialise to discover what the real question is - Specialises what is the relationship between 3 and 7 and how it can be used to complete the ten. Introduce: - Use diagrams as train sums to show how completing the ten makes addition easier. ***Generalising:*** Check: - Arguments to check that what has been done so far, is correct. - Whether the rule stands for all additions of 3 and 7. - Consequences of conclusion to see if they are reasonable Reflect: - On key ideas and moments - On implications of conjectures or arguments - On the resolution: Can it be made clearer? Extend: - Use a given rule to try with other pairs of numbers. - Try the result to a wider concept by generalising.

<table>
<tr><td></td><td></td><td></td><td></td><td>- Make a conjecture which is always true.</td></tr>
<tr><td></td><td></td><td></td><td>Generalising towards the idea of a function
Introduction

- Initial group discussion about pattern
- Identify elements (colour, shape)
- Which elements are changing, and which stays the same?
- Copy and extend the pattern
- Describe in own words the rule of the pattern

Learners investigate a variety of repeating patterns such as:

1. Copy and extend pattern
2. Find elements of the pattern (colour, shape)
3. Name the elements of the patterns in words to find form of the pattern
4. Which elements are changing, and which stay the same?
5. Identify core of the pattern
6. Use core of the pattern to create own pattern that is the same
7. Conjecture a rule of each pattern informally in own words

Review and discuss
- What is the main thing that happens in a repeating pattern?
- What is a pattern made up of?
- What is the core of the pattern?
- Why is it important to know what the core of the pattern is?
- What can one do with the core of the pattern?
- Discuss interesting situations that came to life during the small-group investigation</td><td><i>Specialising:</i>

I know: - Learner knows what is asked to do
- Specialise to find the elements involved
- What is relevant and important in the pattern?

I want: - Classify and sort the information
- Specialise to discover what the real question is
- Specialises how elements change in the pattern

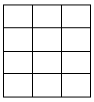
Introduce:- images, diagrams, symbols
- Representation, notation, organisation
- Learner uses alphabet symbols to classify elements of pattern to make clear the form of the pattern

<i>Generalising:</i>

Check: - Arguments to check that what has been done so far, is correct.
- Identify core of the pattern
- Consequences of conclusion to see if they are reasonable
- The core fits the pattern and the question that has been asked
- Informally produce rule for the pattern w.r.t core

Reflect: - On key ideas and moments
- On implications of conjectures or arguments
- On the resolution: can it be made clearer?</td></tr>
</table>

<table>
<tr><td></td><td></td><td></td><td></td><td>Extend: - Learner creates own pattern with the same core, that follows the same rule
- Learner informally describes rule in own words
- Try the result to a wider concept by generalising</td></tr>
<tr><td>Syncopated stage:
• Characterised by geometric thinking.
• Geometric thinking involves representing mathematical thinking by means of geometric figure and forms.
• Learners need to represent the context of algebraic problems by means of diagrams, which involves geometric thinking.</td><td></td><td>Conceptual:
• A shift from analysing objects to the consideration of relations of transformations between objects.
• Learners find interrelationships between properties.
• They start providing definitions and theorems for what they experience.</td><td>Generalising arithmetic as the exploration of properties and number operations
Doubling and halving as inverse relationships

Introduction
- What does it mean to double a number?
- What does it mean to half a number?
- Look at the following picture. First half it and then double it.

- How many blocks do you see? What is half of that amount?
- What happens if you double the answer, you got above?

Lesson

1. Complete the following doubling and halving machines.
<table>
<tr><td>Doubling machine</td><td>IN</td><td>13</td><td>21</td><td>25</td><td>32</td><td>36</td><td>41</td><td>47</td><td>50</td></tr>
<tr><td></td><td>OUT</td><td>26</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Halving machine</td><td>IN</td><td>100</td><td>94</td><td>82</td><td>72</td><td>64</td><td>50</td><td>42</td><td>26</td></tr>
<tr><td></td><td>OUT</td><td>50</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

2. What did you have to do every time to complete the doubling machine?
3. What did you have to do every time to complete the halving machine?
4. Can you use the doubling machine to complete the halving machine?
5. Why and how does it work?
6. What is the relationship you notice between halving and doubling?
7. Describe the relationship between doubling and halving in words.
8. Can you represent the relationship in your own flow diagram?

Review and discussion
- How did you go about completing the halving and doubling machines?
- What does it mean to double and half?</td><td><i>Specialising:</i>

I know: - Learner knows what is asked to do
- Specialise to find out what does doubling and halving mean
- What is relevant and important when doubling and halving diagrams or numbers?

I want: - Classify and sort the information
- Specialise to discover what the real question is
- Specialises what is the relationship between doubling an halving

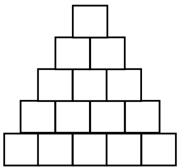
Introduce: - Uses symbols and notation to represent the relationship.
- Uses the relationship to complete doubling and halving machines.

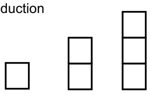
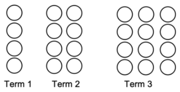
<i>Generalising:</i>

Check: - Arguments to check that what has been done so far, is correct.
- Whether the rule stands for all filled in numbers.
- Consequences of conclusions to see if they are reasonable

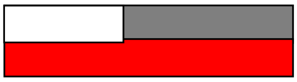
Reflect: - On key ideas and moments
- On implications of conjectures or arguments
- On the resolution: Can it be made clearer?

Extend: - Use relationship to create a flow diagram for doubling and</td></tr>
</table>

			- Is there a relationship between doubling and halving? - How do you know? - Is this always true?	halving. - Try the result to a wider concept by generalising - Make a conjecture that is always true.
			Generalising about particular number properties and relationships Introduction  - Learners complete basic pyramid with rule: add the two numbers that are next to each other to get the number on top. - Learners discuss verbally what the rule is for the pyramid. - Learners create own pyramid with the same rule. Lesson 1. Learners receive pyramids with different rules (add two numbers next to each and double the answer to find the number on top) in order to solve the pyramid. 2. Learners must use the numbers available to work out a rule that is valid for the whole pyramid 3. Learners must complete pyramid according to rule. 4. Learners will then receive blank pyramid and a rule: add numbers next to each other and minus 3 to get answer on top. 5. Learners must design own pyramid with given rule. Review and discussion - How did you know what the rule of each pyramid is? - How can one work it out? - Where did you start when you had to create your own pyramid? - Is there one rule that one must always use for a pyramid? - Discuss interesting situations that came to life during the small-group investigation.	***Specialising:*** I know: - Learner knows what is asked to do - Specialise to find out how does a pyramid work - What is relevant and important when completing the pyramid? I want: - Classify and sort the information - Specialise to discover what the real question is - Specialises what is the rule for completing the pyramid Introduce: - Uses symbols and notation to represent the number patterns. - Use numbers to complete the pyramid according to the rule. ***Generalising:*** Check: - Arguments to check that what has been done so far, is correct. - Whether the rule stands for all filled in numbers - Consequences of conclusion to see if they are reasonable Reflect: - On key ideas and moments - On implications of conjectures or arguments - On the resolution: can it be made clearer? Extend: - Use a given rule to fill in the blank pyramid. - Try the result to a wider concept by generalising - Make a conjecture that is always true.

			Generalising towards the idea of a function Introduction  - Initial group discussion about pattern. - Identify elements in the pattern. - What is the constant change in the pattern? - Copy, extend and describe pattern. - Lesson  1. Identify elements of the pattern in relation to the terms. 2. Copy, extend and describe pattern. 3. Recognise and discuss the constant change. 4. Find a different way to represent the constant change: make a number sequence out of the pattern. 5. Represent pattern in a table form. 6. Find a general rule for the pattern with regards to the relation between the term and the number of circles. Review and discuss - What is the main thing that happens in a growing pattern? - What is this pattern made up of? - What is the core of the pattern? - Why is it important to know what the constant change is? - What can one do with the constant change? - Why is it important to know what the relation between the term and the circles is? - Discuss interesting situations that came to life during the small-group investigation.	***Specialising:*** I know: - Learner knows what is asked to do - Specialise to find the elements involved - What is relevant and important in the pattern? I want: - Classify and sort the information - Specialise to discover what the real question is - Specialises how elements change in the pattern - Identify the constant change in the pattern Introduce: - images, diagrams, symbols - Representation, notation, organisation - Represent pattern in number pattern and in table ***Generalising:*** Check: - arguments to check that what has been done so far, is correct. - Identify constant change of the pattern - Consequences of conclusion to see if they are reasonable - The constant change fits the pattern and the question that has been asked - Extend and describe the pattern - Informally produce rule for the pattern w.r.t constant change - Find a relation between the term and the pattern (functional relationship) Reflect: - On key ideas and moments - On implications of conjectures or arguments - On the resolution: can it be made clearer?

				Extend: - Learner creates own action pattern with the same core, that follows the same rule - Learner informally describes rule in own words - Try the result to a wider concept by generalising
Symbolic stage: • Total symbolisation can be noted. • All numbers, operations and relationships are expressed using symbols. • Manipulations on the symbols are done according to governing rules.		**Abstract:** • Learners use symbols with deep understanding to construct proofs. • They understand the importance of deductions, axioms, postulates, and proofs. • Learners can deduct a rule for patterns. • Understand how symbols can be used to represent the rule.	**Generalising arithmetic as the exploration of properties and number operations** This lesson aims to investigate repeated addition to form a number pattern to see if two patterns can arrive at the same number. Introduction - Number pattern warming up activities - Completing a few basic number patterns: 5: 10: 15 …….. 4; 8; 12 …….. 2; 4; 6 ……. - Which of these number patterns start with even numbers and which start with uneven numbers? How does that influence the numbers in the rest of the pattern. - Provide a rule for each number pattern informally in words. Lesson 1. Learners must investigate two number patterns with regards to their elements and position to see if they both end up at the same number 2. Learners must use the 2's and 3's pattern 3. Start with 2 and repeatedly add 2: $2+2 \rightarrow 4+2 \ \rightarrow 6+2=8+2 \ \rightarrow 10$ ... 4. Start with 3 and repeatedly add 3: $3+3 \rightarrow 6+3 \ \rightarrow 9+3=12+3 \ \rightarrow 15$ ... 5. Will they both arrive at 26? 6. Which one will arrive there first? 7. How can you determine if they will both arrive at 26? 8. Are there number for which 2 and 3 will arrive at the same number? Why/ why not? 9. Provide a rule for your findings using pictures or symbols. Review and discussion - What did we try and find out in this investigation? - Could both number sequences end up at the same number? - Why not? - Is this always true? - How do you know?	***Specialising:*** I know: - Learner knows what is asked to do - Specialise to find out how to complete number patterns - What is relevant and important when completing the number pattern? I want: - Classify and sort the information - Specialise to discover what the real question is - Specialises what is the constant change in the structure of the repeated addition - Recognise that the different repeated addition sums give different answer. Introduce: - Uses symbols and notation to represent the patterns and structure which can be noticed. -Create the 2's and 3's pattern -Find a way to describe the relation between adding even and uneven numbers. ***Generalising:*** Check: - Arguments to check that what has been done so far, is correct. - Follow a rule to create number pattern on certain positions - Identify the rule that arises between the position and element of the pattern - Consequences of conclusion to see if they are reasonable

				- Informally produce rule for the patterns arising - Determine whether the two patterns can ever arrive at the same number? Reflect: - On key ideas and moments - On implications of conjectures or arguments - On the resolution: Can it be made clearer? Extend: - Try another two different numbers to add repeatedly to see if they can arrive at the same number - Learner describes rule using pictures or symbols - Try the result to a wider concept by generalising - Make a conjecture that is always true.
			Generalising about particular number properties and relationships **Arithmetic compensation (If we increase one number by a certain amount, then we must decrease the other number by the same amount for the answer to stay the same.)** (Warren and Cooper, 2003; Cooper and Warren, 2008b) Introduction - Look at the following two models of length.  - Can you provide a name for each of these strips? - How would you describe the length of the white and grey strips in relation to the red strip? - What must I do with the grey strip to keep the total length the same, if I cut a piece of the white strip off? - What must I do with the white strip to keep the total length the same, if I cut a piece of the grey strip off? - How do you know that this will work?	***Specialising:*** I know: - Learner knows what is asked to do - Specialise to find out what the relationship between the strips of paper are. - What is relevant and important when comparing the lengths of the paper. I want: - Classify and sort the information - Specialise to discover what the real question is - Specialises what is the constant change in the structure of arithmetic compensation - Recognise that if one length increases, the other needs to decrease by the same amount to keep the total length constant. Introduce: - Uses symbols and notation to represent the patterns and structure which can be noticed. - Follow the model to show the

*This introduction to the lesson can be repeated a few times with various length models to help consolidate the concept.

- Can you write a rule for what we see happening in your own words?

Lesson (The lesson moves away from quantities to the symbolic world)

1. Look at the sum 7 + 5 = 12.
2. What happens to 7 if we increase 5 by 2 and we want to keep the answer the same? Use your counters.
3. What happens to 5 if we decrease 7 by 4 and we want to keep the answer the same? Use your counters.
4. Examples like this can be repeated a few times.
5. What do you notice?
6. Can you explain a rule for this in words?
7. How do you know that this rule will work?
8. Show your reasoning and calculations in the table below.

7 + 5 =12						7	5
7	+	5	=	12		0	0
	+	7	=	12			↑2
3	+		=	12		↓4	
	+		=	12			
	+		=	12			
	+		=	12			

9. If we can say 7 is represented by ***A*** and 5 is represented by ***B,*** can you write a rule to represent the pattern you notice in the sums above?
10. Apply your rule to the following sum:
    13 + 21 = 34

Review and discussion

- What did you find out in this investigation?
- Will this always work?
- How do you know?
- Does it work the same for subtraction?

relationship between two numbers which are added together.
- Find a way to describe the relation between increasing and decreasing numbers in one sum.

Generalising:

Check:
- Arguments to check that what has been done so far, is correct.
- Follow a rule to complete more examples of increasing and decreasing and represent relationship in a table.
- Consequences of conclusion to see if they are reasonable
- Produce a rule using symbols and words.
- Determine whether the rule can be transferred to higher numbers.

Reflect:
- On key ideas and moments
- On implications of conjectures or arguments
- On the resolution: Can it be made clearer?

Extend:
- Determine whether the rule can be transferred to higher numbers.
- Learner describes rule using pictures or symbols
- Try the result to a wider concept by generalising
- Make a conjecture that is always true.

Generalising towards the idea of a function

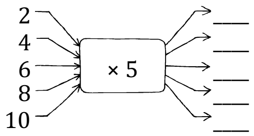
Number patterns in a flow chart

Introduction

Specialising:

I know:
- Learner knows what is asked to do

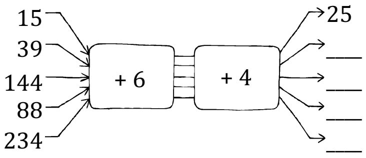
Complete the following flowchart:



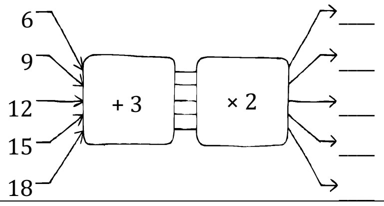
- What do you notice in the flow chart?
- Are there any patterns?

Lesson

1. Complete the following flow chart
2. What do you notice in the flow chart?
3. What is the rule of the flow chart? Give your rule in words.



Complete the following flow chart:



- Specialise to find out how does a flow chart
- What is relevant and important when completing a flow chart?

I want:
- Classify and sort the information
- Specialise to discover what the real question is
- Specialises what is the rule for completing the flow chart

Introduce:
- Uses symbols and notation to represent the number patterns.
- Uses numbers to complete flow chart according to the rule.

Generalising:

Check:
- Arguments to check that what has been done so far, is correct.
- Identify a rule for the number pattern in the input numbers
- Identify a rule for the number pattern in the output numbers
- Consequences of conclusion to see if they are reasonable

Reflect:
- On key ideas and moments
- On implications of conjectures or arguments
- On the resolution: Can it be made clearer?

Extend:
- Provide the rule for a flow chart with input and output numbers by means of symbols.
- Try the result to a wider concept by generalising.
- Create own flow chart with own rules.
- Make a conjecture that is always true.

<table>
<tr>
<td></td>
<td></td>
<td></td>
<td>
1. What do you notice in the flow chart?
2. Are there any patterns in the flow chart?
3. What is the rule of the flow chart?
4. What is the rule of the arising patterns?
5. How can we write the rule using symbols?
6. Organise your numbers in the table below:
<table>
<tr><th>IN</th><th>OUT</th></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
</table>
1. What do you notice now?
2. Can you write a rule using the data table?
3. Try to write a rule using pictures.
</td>
<td></td>
</tr>
</table>

Table 5.1. The Instructional sequence to implement generalisation for the development of early algebra

5.8. CONCLUSION

Freudenthal (1973) opines that the history of the development of mathematics provides valuable inspirations for designing an instructional sequence of generalisation activities based on RME design theories (Gravemeijer, 2020, 2007). To this end, a systematic review of the literature on the historical development of algebra was conducted in Chapter 2.

In Chapter 3, guided reinvention and emergent modelling as design heuristics were explored with the purpose of designing an instructional sequence for the development of early algebra in the foundation phase. A systematics literature review was conducted to set the scene for early algebra (see 3.3). Thematic analysis of the current situation of algebra in the mathematics education classroom was conducted, along with a look at what South African curriculum says about early algebra, the importance of early algebra and its scope.

From there a systematic literature review was conducted to review the development of algebraic thinking (see 3.4) with a specific focus on the levels of algebraic thinking and how they relate to the emergence of algebra from history. Kaput's (2008) main components of early algebra (see 3.5) and how they appear in the history of mathematics was reviewed.

The systematic literature review which was conducted in Chapter 2 as well as Chapter 3, was used to inform the design of an instructional sequence (3.7) to implement generalisation activities in the early years classroom as a possible route to develop early algebraic thinking.

CHAPTER 6: CONCLUSION OF THE STUDY

6.1. INTRODUCTION

Chapter 6 provides a summary, overview and report of the findings of the systematic literature review. The chapter will start by explaining the purpose of the study and how the chapters have been set out to achieve the purpose. From there the findings of the study will briefly be reviewed. Furthermore, this chapter will consider the limitations of the study and mention possible avenues for future research. The chapter will conclude with some recommendations and final remarks.

6.2. PURPOSE AND OVERVIEW OF THE STUDY

The purpose of this systematic literature review study was to investigate the role of generalisation in the development of early algebra concepts and skills. A thorough and systematic literature review of the emergence of early algebra concepts and skills from the historical development of algebra was conducted, and these concepts and skills were used as the basis for the design of an instructional sequence to implement the generalisation concept in the early years classroom.

The main research question of the study was formulated as: What is the role of generalisation in the understanding of early algebra concepts and skills in young children? From there the following aims and objectives were constructed to achieve the purpose of the study:

1. What can we learn from the history of the development of algebra for the learning of algebra?
2. What is early algebra?
3. What is the role of algebraic thinking and generalisation in the understanding of early algebra?
4. How is the problem-centred approach implemented in the teaching and learning of early algebra in the search for teaching for understanding?

Chapter 1 presented a motivation for the study, the problem statement with the research question and sub-research questions, and a brief overview of the study methodology. Chapter 2 is an analytical chapter, involving the use of a systematic literature review to analyse and synthesise the emergence of the main concepts and big ideas of early algebra from in history. The chapter went on to explain how the historical stages could be adopted as the developmental stages of early algebraic thinking. Purposes for school algebra, as they emerged in history, were extracted. A short overview of the history of algebra in South Africa was provided. In Chapter 3, the literature pertaining to early algebra, generalising, and the structure of mathematics was analysed. The study of these analytical themes, together with the concepts emerging from history (Chapter 2), were used to construct an instructional design sequence for the implementation of the generalisation concept in early years classrooms. Chapter 3 also focused on real mathematics education (RME) and the principles of guided reinvention and emergent modelling. Chapter 4 of the study was used to explain, justify, and evaluate the methodology employed. In this chapter the research approach, paradigm, methodology and design of the systematic review process were described. The trustworthiness and ethical dimensions of the study were considered. Chapter 5 applied the findings and synthesis of the analytical chapters to the design of an instructional design sequence based on the principles of RME. Chapter 5 also described the teaching approach and classroom practices and culture which should characterise a early mathematics education classroom.

6.3. FINDINGS OF THE STUDY

In this section of Chapter 6, the findings of the systematic literature review will be briefly summarised according to the analytical themes created for the purposes of the study and research questions.

6.3.1. The emergence of early algebra concepts from history

This analytical theme was mainly explored in Chapter 2. The stages of emergence of algebra in history closely relates to the levels of thought learners go through when learning algebra and developing algebraic thinking (Sfard, 1995; Katz and Barton, 2007; Nixon, 2009).

6.3.1.1. The rhetorical stage

The rhetorical stage is characterised by the need to solve problems, but problems were not yet categorised. Individual solutions for individual problems were described mainly in words. In this stage, reasoning was answer orientated and little generality could be noticed (Katz and Barton, 2007; Tabak, 2011; Katz and Parshall, 2014).

This stage relates to the first step learners take when approaching early algebra type activities. Learners, for example, express patterns or relationships they notice in words. The role of the teacher is to emphasise key aspects in learners observation which will aid learners in alter being able to represent patterns in various representations (syncopated stage) and then later using symbols (the symbolic stage) (Blanton and Kaput, 2011).

6.3.1.2. The syncopated stage

The syncopated stage is characterised by geometric thinking. Geometric thinking involves the representation of mathematical thinking through various representations and especially in the shape of geometric figures and forms (Katz and Barton, 2007).

The geometric thinking stage in the development of algebraic thinking is an important step. Here, learners start to make models and mental pictures of mathematical concepts and real life situations they are working with (Cooper and Warren, 2008). Making models and representing ideas by means of geometric figures, is an important step towards generalising (Roberts, 2012; Kaput, 2008).

Furthermore, the need to solve complex, real-life problems further enhanced the development of algebra in this stage. Mathematicians realised that a need exists for problems to be grouped together and be brought in relation to each other (Bednarz, Kieran and Lee, 1996) Generalisation as algebraic thinking started to become more prominent and fundamental in the emergence of algebra in the syncopated stage.

6.3.1.3. The symbolic stage

The symbolic stage is characterised by total symbolisation. All numbers, operations and relationships are expressed through the use of a set of easily recognised symbols

(*Mathematics for Teaching*, n.d.). The development of convenient symbolisms for expressing new ideas, emancipated mathematicians, and led to great discoveries in this stage.

6.3.1.4. The conceptual stages

Four conceptual stages accompanied the algebra's historical developmental stages, and provide additional insights into the development of algebraic thinking.

Katz and Barton (2007) in their work, refered to four conceptual stages in the development of algebra which runs simultaneously with the three main stages. These four conceptual stages include: the geometric, static-equation, dynamic function and abstract stages. These stages overlap and clearly show a relation to algebraic thinking.

The geometric stage reflects the beginning of algebra where the goal was to find geometric quantities such as the legnth and width of a rectangle. The aim was to solve geometric problems and algorithms was developed to solve equations (Katz and Barton, 2007). In the same way, learners in the early mathematics classroom as learners work with numbers and number properties, as well as, solving problems and communicating their ideas, they should constantly be aiming to find generalities (Blanton *et al.*, 2015). The geometric stage can be brought into relation with the perceptual level of thought explained by Nixon (2009), where the coordination of physical senses and perceptions is used to develop algebraic concepts. Apsari *et al.* (2020,p.52-53) distinguish various major roles of geometric representation in the pre-algebra classroom:

- Context
- Model of, and model for, situation
- Scaffolding
- Learners' mathematical reasoning and proof

When algorithms start to replace geometry and geometry representations, the geometric stage move to the static-equation solving stage (Katz and Barton, 2007).

The static-equation solving stage was characterised by having a general numeric problem which can be viewed as algebraic. An algorithm to solve the problem, expressed as an equation, is then proposed and used to solve the problem to get the answer. The aim of algebra in this stage was the solution of equations (Katz and Barton, 2007). Reasoning with equations is an important idea which should be introduced in the pre-algebra classroom (Knuth *et al.*, 2014). When learners move to the static-equation solving stage, learners also enter the conceptual level of thought explained by Nixon (2009). The focus here shifts from analysing objects to the transformations and relations between objects.

The dynamic function stage developed the concept of movement and motion. Algebra was perceived as a challenging subject due to the lack of standardised symbols (Tabak, 2011). In the 17th century, new notation was introduced by mathematicians like Viètes and Descartes (Katz and Barton, 2007). The introduction of notation and use of symbols made it possible for mathematicians to see broader patterns in mathematics and identify relationships between symbols and classes of objects they represent (Tabak, 2011). When looking at the pre-algebra classroom, Kaput (2018) indicates the importance of learners being able to suspend their attention on what symbols stand for, and rather look at symbols themselves. In so doing, they are liberated to operate on relationships which are more complicated. This stage still falls in the conceptual level of thought explained by Nixon (2009).

The structure of mathematics becomes the underlying goal in the abstract stage (Katz and Barton, 2007). The focus of early algebra should be on a relational approach to learning mathematics, which refers to studying number from a structural perspective. Structure is extracted through exploring number and space relationally (Du Plessis, 2018). This stage aligns with Nixon's (2009) abstract level of algebraic thinking. In this level, learners start to use symbols with deep understanding and construction of proofs. Learners understand the importance of deductions, axioms, postulates, and proofs at this level.

The emergence of algebra throughout history, provides an instructional sequence for the development of algebraic thinking in the early years classroom. There are several purposes for school algebra which emerged from history which will be summarised next.

6.3.1.5. Purpose for school algebra emerging from history

Algebra as a school subject was introduced much later than arithmetic and geometry. Early in the 19th century, school algebra was linked to what was expected of learners at a tertiary level. In the second half of the 19th century, algebra was used to model and solve real-life problems in context which helped learners to form a deeper understanding of the subject and the concepts (Ellerton, Kanbir and Clements, 2017). Six purposes for algebra emerge when studying the literature about the history of algebra. These purposes were outlined by Ellerton, Kanbir and Clements (2017) as:

1. Purpose 1: Knowledge essential for higher mathematics and science
2. Purpose 2: Generalised arithmetic
3. Purpose 3: A pre-requisite for entry to higher studies
4. Purpose 4: A language for modelling real-life problems
5. Purpose 5: An aid for describing basic structural properties
6. Purpose 6: A study of variables

These purposes which emerge from history provides a clear view of what some of the aims of school algebra should be and closely align with the main components of algebra and the Big Ideas of early algebra which inform algebra teaching and learning.

6.3.1.6. The algebra situation in South Africa

In the CAPS document for foundation phase mathematics in South Africa, algebra is described as one of the main content areas (Department of Basic Education, 2011). Algebra is described as the language for investigating and communicating most of mathematics and can be extended to study functions and relationships between variables. The CAPS document mainly focuses on copying and extending patterns. It does not pay attention to the generalisation of mathematics as a whole, even though the first content area, “Numbers, operations and relationships’, provide opportunity to develop the generalisation concept(Roberts, 2010). Early algebra is not explicitly mentioned, but notice is made of observing relationships between different kinds of

numbers, representation of numbers in different ways and allows for the exploration of equivalence and the equal sign (Department of Basic Education, 2011). This correlates with the aim of early algebra which includes generalised arithmetic and seeing the relationship between numbers and their properties (Kaput, 1995; Roberts, 2012).

Learners should notice the underlying structure of patterns which emerge from the whole of mathematics (Mulligan *et al.*, 2008). The very relevant work by Du Plessis (2018) in this field showed that the CAPS document emphasises the importance of noticing logic in patterns, but this approach is not implemented in classrooms. Du Plessis (2018) noticed in his work that a lack of a relational approach to the teaching of early algebra concepts in the foundation phase exists. Another major challenge is the fact that many teachers have not been taught the content area of ‘Patterns, functions and algebra’. It remains an area of confusion regarding why and how to implement this content area of the curriculum effectively. Literature (Mulligan, Mitchelmore and Prescott, 2005; Warren and Miller, 2010; Papic, Mulligan and Mitchelmore, 2011; T. Cooper and Warren, 2011) shows that a need for an early algebra approach where the relation between patterns, structure and algebraic reasoning, and their power to basic numeracy in young learners, is emphasised and implemented in foundation phase classrooms.

6.3.2. The teaching of early algebra

The main components of early algebra which emerged from history were further explored and analysed in Chapter 3, with the purpose of forming a thorough understanding of how early algebra should be implemented in early mathematics classrooms.

6.3.2.1. The early algebra curriculum

The traditional approach to teaching algebra involves an over-emphasis of computational work in arithmetic in the early and middle grades which is followed by a superficial teaching of algebra in the secondary grades (Kaput, 2008) Radford, 2015)(Blanton, Stroud, *et al.*, 2019).

In South Africa, algebra is a key course in the secondary mathematics curriculum and is a pre-requisite for post-secondary mathematics, science, and engineering courses. Algebra can, for that reason, be considered as a gatekeeper course which leads to the marginalisation of learners by sorting learners out of certain career paths, and this deeply affects underrepresented groups (van Laren and Moore-Russo, 2014).

This study was grounded in the proposition that it is crucially important to establish firm foundations for algebra in the early years and that learners are able to think algebraically from a young age. With suitable instructional support, learners can understand some algebraic concepts (Radford, 2011). Early algebra has the purpose of enhancing learners' understanding of the structural form and generality of mathematics and aims to not provide isolated mathematics experiences.

6.3.2.2. Kaput's framework for early algebra

Kaput (2008) states that algebra consists of particular thinking practices and content strand. He proposes that algebraic thinking involves (a) making and expressing generalisations in increasingly formal symbol systems and (b) reasoning with symbolic forms. These practices take place over three content strands (Kaput, 2008, p.11):

Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic and quantitative reasoning.

Algebra as the study of functions, relations and joint variation.

Algebra as the application of a cluster of modelling languages both inside and outside of mathematics.

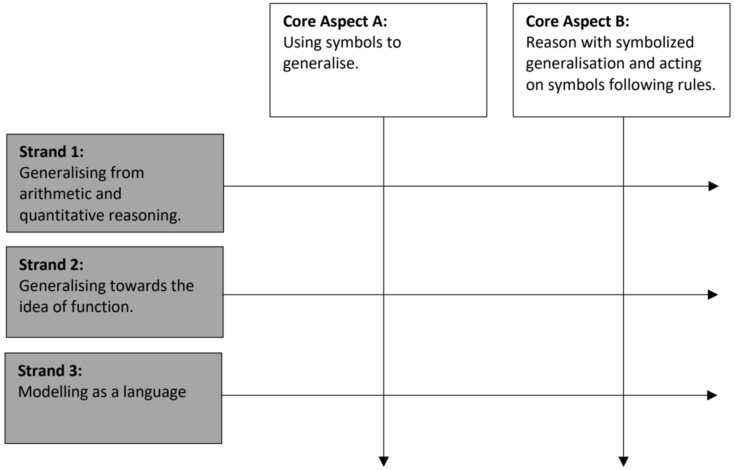


Figure 6.1. Kaput's core aspects and strands of algebra (Roberts, 2012, p.304)

In next section, a summary of these fundamental components will be provided.

6.3.2.3. Fundamental components of early algebra based on Kaput's framework

Kaput's (2008) framework to describe various aspects and strantds involved in algebra and algebraic thinking was used to construct the generalisation activities in the instructional sequence in Chapter 5.

1) Generalisations and formalisation

   Algebra evolved as a constant attempt to make generalisations and formalisations (Kaput, 1999). The aim of early algebra is to find generalisations as it creates an understanding of the underlying structure of mathematics (Warren and Cooper, 2008) (Roberts, 2012).
   Generalisation as an analytical theme will be summarised in 6.3.3.

2) Syntactically guided manipulation

   Symbols are used to represent abstract arithmetic, and algebraic and logical propositions. These symbols are manipulated according to internally represented mathematical and logical rules (Landy, Allen and Zednik, 2014). Manipulation refers to the changing form of the expression. Learners suspend their attention on what the symbols stands for and focus on the symbols itself. The introduction of notation and symbols in the symbolic stage made it possible for mathematicians to see broader patterns and identify relations between symbols and the classes of objects they represent (Tabak, 2011). When this happens in the classrooms, learners are freed to operate on more complicated relationships. To ensure this process runs effectively, well-planned out lessons in a pre-determined sequence needs to be implemented. Manipulations on representations happens in two broad classes (Kaput, 2018): syntactic and semantic. Syntactic action. involves the manipulation of symbols by only looking at the syntax of the symbol system. A semantic action is guided by the referents of the symbol. Syntactically guided manipulations on formalisms is viewed as the core of algebra, but to ensure effective development of actions on formalisms, a semantic starting point should be taken (Kaput, 1995a). Syntactically guided manipulations are an important goal of algebra instruction as learners need to form a deep understanding of representations used (Carpenter and Lehrer, 1999).

3) The study of structure
Generalising and abstraction, where the focus falls on the structure within computations rather than the process or answer, lead to the emergence of abstract structures which are associated with traditional algebra (Kaput, 1999). At the core of early algebra lies a deep understanding of the mathematical structure of arithmetic (Warren and Cooper, 2008). Algebra at the abstract stage of its history is described as the science of structure (Sfard, 1995). Katz and Barton (2007) describe the conceptual stages of the development of algebra, with the abstract stage involving the emergence of the underlying structure of mathematics. To enhance structural knowledge, learners should be able to recognise equivalent forms of an expression and justify the structural equivalence which is identified.

The more developed a learner's internal representational system is structurally, the more well-organised, coherent and stable in all structural aspects their external representations will be, and the more mathematically competent the learner will be. This indicates the importance of developing a learner's structural understanding of mathematics (Mulligan et al., 2005). Patterns and structure are at the core of mathematical thinking and should be embedded throughout mathematics teaching (Mulligan et al. ,2008)

4) Functions, relations and joint variation

Traditionally, the teaching of function only takes place in the secondary grades. However, to ensure success in algebraic thinking, functions should be taught in a longitudinal approach and should be made accessible to all learners from a young age (Blanton and Kaput, 2011). Functional thinking emerges when a learner engages in an activity, chooses to pay attention to two or more varying quantities, and then start to focus on the relationship between those quantities. The crux of functional thinking is the focus on relationship (Smith, 2008).

5) Modelling as a language
It is argued that modelling situations is the primary goal of studying algebra. Modelling involves starting with a specific situation and trying to mathematise it (Kaput, 1999). RME describes emergent modelling as a design heuristic when constructing an instructional sequence (Gravemeijer, 2007, 2020). Emergent modelling supports an incremental process in which mathematical models and mathematical conceptions co-evolve.

6.3.2.4. The big ideas of early algebra

Knuth et al. (2014) propose five big ideas (see 2.2.4) deduced from Kaput's (2008) framework for early algebra. Much of early algebra research has matured around these five big ideas (Knuth *et al.*, 2014, p.43).

In this study, these five big ideas emerged from the systematic analysis of the historical overview of the development of algebra. This review provided valuable support for drawing parallels between this and other perspectives on the progression of levels of algebraic thinking.

6.3.2.5. Developing algebraic thinking.

The traditional "arithmetic-then-algebra" approach has proven to be unsuccessful, as it does not provide ample opportunity for the deep development of algebraic thinking (Knuth *et al.*, 2014). The goal of early algebra should not be skilled use of algebraic procedures but rather algebraic thinking (Carpenter & Levi, 2000). Algebraic thinking should, therefore, be developed from an early age.

In the early years, algebraic thinking is developed through the comparison of quantities, observations and making patterns, navigating through different types of spaces, and solving problems in playful interactions with objects and peers in the classrooms (Linder, Powers-Costello and Stegelin, 2011). Learners should be provided with concrete experiences with algebra concepts (Lee et al., 2016, p. 306). Algebraic thinking is further developed through investigations and discussions of number properties, which help learners to make generalisations and construct concepts to pave the way for formal algebraic thinking in secondary grades (Ontario Ministry of Education and Training, 2007, p. 8). Algebraic thinking can be promoted when the emphasis is placed on finding ways to represent and analyse the underlying structures of numbers, operations and relationships (Billings, 2017, p. 483). This statement shows the importance of including algebra throughout all content areas of the early years curriculum.

The RME principles proposes implementing guided reinvention and emergent modelling to construct an instructional sequence of mathematical activities which guides learners and teachers in the development of the conceptual understanding involved in algebraic thinking (Gravemeijer, 2007, 2020; Stephan, Underwood-Gregg and Yackel, 2016). In the case of this study, these principles were used to develop an instructional sequence of generalisation activities.

6.3.2.6. The levels of algebraic thinking

It is believed that learners should experience mathematics as a human activity and that they should reinvent mathematics as they are guided through mathematical tasks (Freudenthal, 1973). In Chapter 3, various perspectives on the levels of algebraic thinking were systematically reviewed.

Nixon's theory on the levels of the development of algebraic thinking

Nixon (2009) argues for three levels of algebraic thinking (see 3.4.3.1).:
As learners pass through these levels, they need to be guided by a teacher who motivates them to generalise and draw comparisons (Nixon, 2009). These levels of thought were important in the design of the instructional sequence in Chapter 5.

Mason, Burton, & Stacey's (2010) conceptual framework for mathematical thinking see (6.3.2.5 and Figure 6.1) was also an important conceptualisation of algebraic thinking which influenced the instructional design.

6.3.3. The role of generalisation in early algebra teaching

6.3.3.1. Generalisation and formalisation

When learners are able to find generalisations, they develop a deep understanding of the structure of mathematics. The focus of mathematics teaching should be on developing fundamental skills in generalising, expressing, and systematically justifying generalisations (Warren and Cooper, 2008). It is during the deduction stage of generalisaiton, that learners develop the type of reasoning which lies at the heart of algebra (Demonty, Vlassis and Fagnant, 2018)

> The Realistic Mathematics Education (RME) movement from the Netherlands is a constructionist curriculum which aims to teach for abstraction and generalisation.
> (Mitchelmore, 2002)

6.3.3.2. Generalisation activities

A number of generalisation activities are important for the development of the generalisation concept.

Roberts (2010) explains that generalisation has two manifestioans: generlisation from arithmetics and quantitative reasoning; and generalising towards the idea of a function. He identified three elements which should constitute generalisation in the early years (see 3.5.1) (Roberts, 2010, p. 169).

These activities were used to inform the instructional sequence in Chapter 5.

Knuth et al. (2014) refer to algebraic activites in which generalisation is prominent These include the ability to generalise mathematical structure by noticing regularity in arithmetic situations, use sophisticated instruments to explore, generalise, and symbolise functional relationships, build mathematical arguments that reflect more generalised forms than the empirical, case-based reasoning often used, and reason about abstract quantities to represent algebraic relationships.

Generalising arithmetic is an important facet of early algebra as it allows learners to form an understanding of the underlying structure of mathematics. This aspect of early algebra is located in the 'Numbers, operations and relationship' content area of the foundation phase curriculum (DBE, 2011). Guidelines for generalising arithmetic include (Roberts, 2012, p.308-309):

1. Be deliberate about, and explore when something happens and when it always happens in mathematics.

2. Look for patterns in groups of number sentences.
3. Look for patterns in sequences of sums.
4. Ask about (observe, describe, talk about) how special numbers behave.
5. Expect and ask for descriptions of what is observed.
6. Ask for explanations to show if something is always true.
7. Explore and talk about equivalence and what the equal sign means.

Generalising a rule or function is an important aspect of early algebra where problem contexts are explored, as well as the representations of functions. These representations might include number sentences, input and output tables, and function machines. Guidelines for generalising a rule or functions include (Roberts, 2012, p.312-315):

1. Expect learners to describe a number pattern in detail.
2. Look at and talk about the operations or functions, not just the numbers.
3. Ask about how operations behave.
4. Set a problem context which requires investigation of a certain function, and then use different representations of the function.
5. Connect work done in 'Patterns, functions and algebra' to work in other content areas.

Modelling concepts and emphasising the importance of modelling understanding and reasoning, enhances learners' mathematical thinking. Guidelines for generalising a rule or functions include (Roberts, 2012, p.315-317):

1. Model mathematical concepts, problem solving strategies, and calculation techniques. Encourage learners to model their understanding and thinking.
2. Know and make explicit the basic models for the basic operations.
3. Provide opportunities for learners to use concrete objects, draw or imagine objects or processes, and move between the presentations.
4. 'Algebraify' word problems and turn it into an investigation, to model the process of solving the problem, when one or more of the parameters is relaxed.

These guidelines were used to decide the types of activities which was included in the instructional sequence to develop early algebraic thinking in the early mathematics classroom.

From there, the importance of implementing generalisation by means of an effective teaching approach, namely the problem-centred approach was explored. The problem-centred approach should implemented to ensure teaching for understanding.

6.3.4. Teaching for understanding

6.3.4.1. The problem-centred approach for learning algebra

The problem-centred approach takes on the constructionist perspective that mathematics is learned as learners reorganise their activity to resolve solutions they find problematic (Cobb *et al.*, 2014). This approach to learning is based on the belief that subjective knowledge should be experienced by the learner as personal constructions and no re-constructed objective knowledge (Murray, Olivier and Human, 1998). So, the learner is not perceived as an empty vessel in which the teacher 'pours' knowledge. The learner is an active participant in the learning process and is responsible for his own learning. Problem-solving is the process where one starts with problems and after working on these problems, one would be left with a residue of mathematical concepts and ideas (Murray, Olivier and Human, 1998). In this study, PCA as a learning approach, is perceived as an effective way to develop the generalisation concept in young learners. The implementation this approach was used to inform the instructional design sequence in Chapter 5.

6.3.4.2. The role of the teacher

The role of the teacher in this approach is to facilitate the problem-solving process without interrupting learners' mathematical thought processes (Murray, Olivier and Human, 1998). The role of the teacher in PCA-classroom differs from that of the traditional teacher role. Instead of focussing on demonstrating, checking and prescribing, the teacher should focus on setting appropriate problems, organising interaction between learners and negotiating a style of learning and classroom culture with the learners (Cobb *et al.*, 2014, Stephan, Underwood-Gregg and Yackel, 2016,(Murray, Olivier and Human, 1998).

6.3.4.3. Classroom culture

The classroom culture is an important facet of the PCA and will have a great influence of the mathematical learning that takes place in the classroom (Murray, Olivier and Human, 1998).. There are four social norms which should be included when setting a classroom culture: (1) explaining and justifying solutions and methods, (2) attempting to make sense of others' explanations, (3) indicate agreement or disagreement, and (4) ask clarifying question when the need arises (Cobb and Yackel, 1996). Teachers and learners should discuss what is expected of each party in the classrooms, what their own role is, and what is means to do mathematics. By engaging in this process, learners start to perceive mathematics as an activity where they are expected to solve problematic situations by constructing personally meaningful and justifiable solutions as they contribute to an interactive, inquiry-based classroom (Cobb *et al.*, 2014).

6.3.4.4. The role of the learner

The learner is viewed as an active participant in the learning process as the construction of new conceptual knowledge is based on the interaction between the prior knowledge and ideas which a learner already has, and the new conceptual knowledge (Human and Olivier, 1999). Learners share the responsibility to create a classroom which operates as a community of learners. Learners must, firstly, take responsibility for sharing results of their inquiries and for explaining their thinking and solutions. Learners, secondly, need to recognise that learning occurs best when learning from others. This asks of learners to listen because of a genuine interest of what a classmate has to say (Hiebert *et al.*, 1996). The role of the learner is important to take into consideration when implement the instructional design sequence in early years classroom. The sequence is constructed in such a way that the learner is an active participant in his own learning process and the teacher acts as the facilitator.

6.3.4.5. Real Mathematics Education

The aim of the study was to design an instruction sequence based on the principles of RME which include guided reinvention and emergent modelling. Gravemeijer (2020) states that RME is a design theory which can be explained by means of three instructional design heuristics: guided reinvention, didactical phenomenology, and emergent modeling. For the purpose of this study, guided reinvention and emergent modelling was employed as the design principles for the instructional sequence.

Guided reinvention states that learners should experience mathematics as a human activity and that they should reinvent mathematical ideas and concepts as they are guided through tasks by a teacher. The history of mathematics should inform the design of the teaching and learning route (Gravemeijer, 2020). This principle was employed in the design of the instructional sequence as history of algebra was used to inform the sequence of acitivites based on the levels of thought which emerged from history.

Emergent modelling is a dynamic process of symbolising and modelling. The belief is that learners should start with modelling their own informal activity and from there the character of the model should change for the learners. The model of informal mathematical activity should increasingly develop into a model for more formal reasoning (Gravemeijer, 2007). In the instructional design, an increasingly formal use of models by learners can be noticed.

Gravemeijer (2007, p.3) refers to four levels of reasoning levels of mathematical reasoning which should be implemented in an emergent-modelling design:

- Activity in the task setting, in which interpretations and solutions depend on understanding of how to act in the setting
- Referential activity, in which models-of refer to activity in the setting described in instructional activities
- General activity, in which models-for derive their meaning from a framework of mathematical relations
- formal mathematical reasoning, which is no longer dependent on the support of models-for mathematical activity.

These levels of mathematical reasoning will inform the sequence of possible activities chosen to develop generalisation in the early mathematics classroom.

The instructional sequence constructed in Chapter 5, shows how a variety of generalisation activities are introduced as an instructional sequence. The design of the instructional sequence was informed by existing research especially pertaining to the levels of algebraic thinking in which learners engage as they work through problems. For the design, the historical stages have been used at the developmental stages which learners work through as they inform the teaching by looking at how algebraic thinking naturally developed through the ages. Nixon's (2009) levels of algebraic thinking align well with the stage and are integrated in the approach.

The outline below shows how the instructional sequence was designed:

Outline of the instructional sequence to implement generalisation for the development of early algebra

Historical stages		Levels of algebraic reasoning	Possible activities according to generalisation elements	Specialising or Generalising (Thought processes)
Rhetorical stage: • Problems solved by looking at individual problems. • Aimed at solving specific problem. • Problems are not categorised. • Solutions to problems are mainly given in words. • Little to no generality can be noticed.	**geometric → static-equation → dynamic function → abstract**	**Perceptual:** • Learners need to coordinate their senses and perceptions. • Learners advance in their use of numbers. • Learners form mental pictures of concepts.	**Generalising arithmetic as the exploration of properties and number operations**	***Specialising:*** I know I want Introduce ***Generalising:*** Check Reflect Extend
			Generalising about particular number properties and relationships	***Specialising:*** I know I want Introduce ***Generalising:*** Check Reflect Extend
			Generalising towards the idea of a function	***Specialising:*** I know I want Introduce ***Generalising:*** Check Reflect Extend
Syncopated stage: • Characterised by geometric thinking. • Geometric thinking involves representing mathematical thinking by means of geometric figure and forms. • Learners need to represent the context of algebraic problems by means of diagrams, which involves geometric thinking.		**Conceptual:** • A shift from analysing objects to the consideration of relations of transformations between objects. • Learners find interrelationships between properties. • They start providing definitions and theorems for what they experience.	**Generalising arithmetic as the exploration of properties and number operations**	***Specialising:*** I know I want Introduce ***Generalising:*** Check Reflect Extend
			Generalising about particular number properties and relationships	***Specialising:*** I know I want Introduce ***Generalising:*** Check Reflect Extend
			Generalising towards the idea of a function	***Specialising:*** I know I want Introduce ***Generalising:*** Check Reflect Extend
Symbolic stage: • Total symbolisation can be noted. • All numbers, operations and relationships are expressed using symbols. • Manipulations on the symbols are done according to governing rules.		**Abstract:** • Learners use symbols with deep understanding to construct proofs. • They understand the importance of deductions, axioms, postulates, and proofs. • Learners can deduct a rule for patterns. • Understand how symbols can be used to represent the rule.	**Generalising arithmetic as the exploration of properties and number operations**	***Specialising:*** I know I want Introduce ***Generalising:*** Check Reflect Extend
			Generalising about particular number properties and relationships	***Specialising:*** I know I want Introduce ***Generalising:*** Check Reflect Extend
			Generalising towards the idea of a function	***Specialising:*** I know I want Introduce ***Generalising:*** Check Reflect Extend

Table 6.1. Outline of the instructional sequence to implement generalisation for the development of early algebra

6.4. LIMITATIONS OF THE STUDY AND AREAS FOR FUTURE STUDIES

This study was limited to a systematic review of pre-existing literature, and did not include an empirical component. The proposed instructional sequence was not tested experimentally. Furthermore, it is important to note that a majority of sources used originates from the USA or other Western countries, including European countries and Australia. For that reason a critical question can be raised regarding the appropriateness of the findings of this study for implementation in South African foundation phase classrooms. The study can, for that reason, not claim that the implementation of the sequence in South African classrooms will be effective. Lastly, the research was conducted by one primary researcher over a limited period of time, which may have resulted in some valuable sources being excluded from the study. However, through continuous consultation with the study supervisor, the best attempt was made to ensure the completeness of the literature review.

Future research prompted by this study could include:

- An empirical study of the effectiveness of implementing the instructional sequence in South African foundation phase classrooms.
- A deeper analysis of the emergence of the five big ideas in the historical stages of algebra.
- Further study of modelling as a language of early algebra, with the aim of further informing the implementation of early algebra in the foundation phase classroom.
- Exploration of the use of symbolisation in early algebra in the foundation phase classroom.

6.5. A FINAL WORD

The purpose of this systematic literature review was to investigate the role of generalisation in the development of early algebra concepts and skills in the early years classroom. To achieve this purpose an historical overview of the stages of development of algebra was provided. The emergence of the early algebra concepts and skills throughout these stages of development was analysed and further

synthesised into analytical themes. The systematic review and thematic analysis led to construction of an instructional sequence based on the principles of RME, which include guided reinvention and emergent modelling.

This study found overwhelming evidence in the literature that learners are able to reason algebraically from a young age, and that the generalisation concept can be used as route to developing algebraic thinking in early mathematics classrooms. Even so, a carefully planned teaching and learning approach is needed to ensure the effective development of algebraic thinking. The study proposed that the problem-centred approach should be implemented alongside a well-thought-out and planned instructional sequence of generalisation activities based on the systematic literature review.

This study delivers a valuable contribution to the literature of algebra learning as it provides a possible route for the development of algebraic thinking by means of generalisation. As seen in the literature, early algebra remains a largely unexplored domain in the South African early years education context. The findings of this study can be taken as a learning theory model which can be implemented in foundation phase classrooms, as it provides lessons in a pre-planned sequence.

By implementing this instructional sequence in early mathematics classrooms, the hope is that young learners would be empowered to reason algebraically by making generalisations throughout mathematics, and to see the underlying structure of mathematics.

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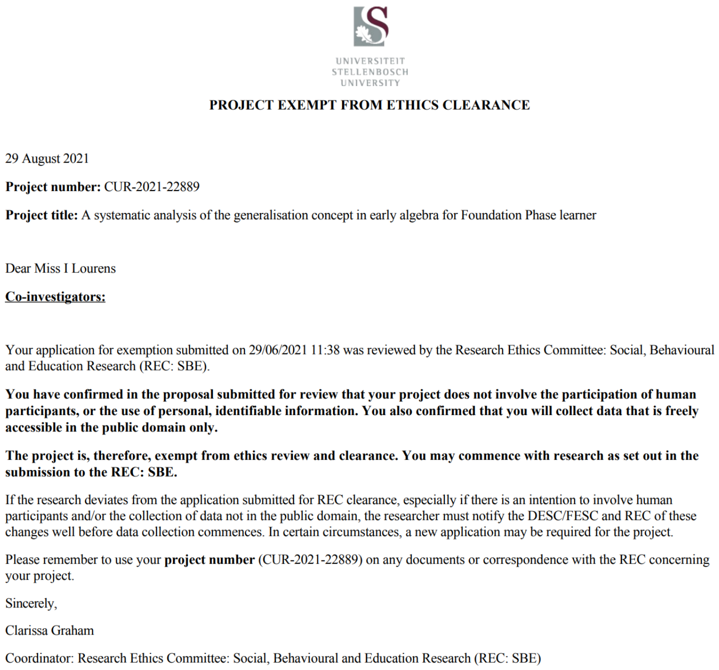
8. ADDENDUM

8.1. LETTER OF EXEMPTION FROM ETHICAL CLEARANCE



UNIVERSITEIT
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UNIVERSITY

PROJECT EXEMPT FROM ETHICS CLEARANCE

29 August 2021

Project number: CUR-2021-22889

Project title: A systematic analysis of the generalisation concept in early algebra for Foundation Phase learner

Dear Miss I Lourens

Co-investigators:

Your application for exemption submitted on 29/06/2021 11:38 was reviewed by the Research Ethics Committee: Social, Behavioural and Education Research (REC: SBE).

You have confirmed in the proposal submitted for review that your project does not involve the participation of human participants, or the use of personal, identifiable information. You also confirmed that you will collect data that is freely accessible in the public domain only.

The project is, therefore, exempt from ethics review and clearance. You may commence with research as set out in the submission to the REC: SBE.

If the research deviates from the application submitted for REC clearance, especially if there is an intention to involve human participants and/or the collection of data not in the public domain, the researcher must notify the DESC/FESC and REC of these changes well before data collection commences. In certain circumstances, a new application may be required for the project.

Please remember to use your **project number** (CUR-2021-22889) on any documents or correspondence with the REC concerning your project.

Sincerely,

Clarissa Graham

Coordinator: Research Ethics Committee: Social, Behavioural and Education Research (REC: SBE)