The definable (p, q)-theorem for dense pairs of certain geometric structures

by

Tsinjo Odilon Rakotonarivo

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Supervisor:

Dr. G. Boxall

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Abstract

The definable (p, q)-theorem for dense pairs of certain geometric structures

T.O. Rakotonarivo

Department of Mathematical Sciences,
University of Stellenbosch,
Private Bag X1, Matieland 7602, South Africa.

Dissertation: PhD

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The definable (p, q)-conjecture is a model-theoretic version of a (p, q)-theorem in combinatorics, which was expressed in the form of a question by A. Chernikov and P. Simon in 2015. Researchers have proved that the property holds for certain classes of structures. Based on those existing results, the main objective of the present thesis is to show that the definable (p, q)-conjecture holds for a dense pair of geometric distal structures that satisfies the following condition: algebraic closure and definable closure are the same in sense of the original geometric structure. Independently, we also explore a different approach to prove that under some conditions, the definable (p, q)-conjecture holds in certain cases for dense pairs of real closed ordered fields.

Uittreksel

Die definieerbare (p, q)-stelling vir digte pare van sekere meetkundige strukture

("The definable (p, q)-theorem for dense pairs of certain geometric structures")

T.O. Rakotonarivo

Department van Wiskundige Wetenskappe,
Universiteit van Stellenbosch,
Privaatsak X1, Matieland 7602, Suid Afrika.

Proefskrif: PhD

Desember 2021

Die definieerbare (p, q)-vermoede is 'n model-teoretiese weergawe van 'n (p, q)-stelling in kombinatorika, wat in 2015 deur A. Chernikov en P. Simon in die vorm van 'n vraag uitgedruk is. Navorsers het bewys dat die eienskap vir seker klasse van strukture geldig is. Op grond van daardie bestaande resultate is die hoofdoel van hierdie tesis om aan te toon dat die definieerbare (p, q)-vermoede geldig is vir 'n digte paar meetkundige distale strukture wat die volgende voorwaarde bevredig: algebraïese afsluiting en definieerbare afsluiting is dieselfde in die sin van die oorspronklike meetkundige struktuur. Onafhanklik hiervan ondersoek ons 'n ander benadering om te bewys dat, onder seker voorwaardes, die definieerbare (p, q)-vermoede geldig is vir seker gevalle vir digte pare van reële geslote geordende liggame.

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Dedications

Ho an'i Dada sy i Neny, izay tsy nitsahatra niteny ahy hoe mianara!

Contents

Notations

Throughout this thesis, we use the following notations.

dim	dimension
acl	algebraic closure
dcl	definable closure
scl	small closure
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{R}$	set of the real numbers
$\mathbb{Q}_p$	p-adic field
$v_p(x)$	p-adic valuation on $\mathbb{Q}_p$
$\mathbb{R}_{>0}$	The positive elements of the set of real numbers
max	maximum
$\varphi, \theta, \gamma, \psi, \tau$	formulas
x, y, z	variables
a, b, c, d	parameters
$(\bar{a}_i)_{i\in\mathbb{N}}, (\bar{b}_i)_{i\in\mathbb{N}}, (\bar{c}_i)_{i\in\mathbb{N}}, (\bar{d}_i)_{i\in\mathbb{N}}$	sequences or strict Morley sequences

$p(x), q(y), r(z)$ types (complete or partial)

$\models$ realize or models

$\mathsf{tp}_L(a/M)$ L-type of a over M

$\mathsf{qftp}_L(a/M)$ quantifier-free L-type of a over M

N, M structures

$\subseteq, \subset$ Subset and proper subset

P Predicate

L, L', L'', L_P languages

T, T_P theories

$(N, P(N)); (M, P(M))$ pairs of structures

$|x|$ length of x

∂ differential

f, g, g_i definable functions

σ, σ_i automorphisms

$b \equiv_M b'$ b and b' have the same type over M

$M \prec N$ M is an elementary substructure of N

I, J Set of indices (linearly ordered set)

$S_n(A)$ Space of n-types over A

EM-type Ehrenfeucht-Mostowski type

VC Vapnik-Chervonenkis

NIP Negation of the independence property

CODF Closed ordered differential field

RCF Real closed field

$M_\partial\langle c\rangle, M_\partial\langle d\rangle$ sub-differential ring of N generated resp. by c and d over M

$M_\partial[X]$ differential polynomial ring

$M[c]$, $M[d]$ ring generated respectively by c and d over M

Introduction

Motivation

The independence property was first introduced by S. Shelah in 1971 [She71]. From this notion, the author defines the negation of the independence property known as NIP, which generalizes the concept of stable theories. Later B. Poizat [Poi81] extended the work of S. Shelah on theories without the independence property. Since then, many theories have been classified to be NIP.

Several authors have published important results with regard to the aforementioned classification. Examples of such results include the work of [Dri98b], [Adl08] and [Sim15a] in which the authors proved that o-minimal structures are NIP, and the work of [JS20] where F. Jahnke and P. Simon proved that the theory of a tame henselian valued field is NIP under certain conditions. The p-adic fields were proved to be NIP by L. Bélair [Bél99].

Characterization of NIP also became a center of research. In this regard, let us mention the work of P. Estevan and I. Kaplan about non-forking and preservation of NIP [EK21]. L. Scow characterized NIP theories using ordered graph-indiscernibles in [Sco12].

NIP has been thoroughly studied over the last few decades. Among the most important results and references concerning NIP is the work of A. Chernikov and P. Simon [CS15] in which the authors asked about the definable (p, q)-conjecture. In the more general NTP_2 setting, A. Chernikov and I. Kaplan established some important properties of NIP theories which are used in this thesis.

One of the topics in the study of NIP theories involves the investigation of the properties of stable theories and whether such properties hold for all NIP theories. The definable (p, q)-conjecture is, indeed, an example of such investigation. It was known to hold for stable theories long before it was formulated into a conjecture. However the question of whether this conjecture holds for NIP is still unanswered.

The discovery of the link between a combinatorics property, namely the (p, q)-theorem, and model theory enlarged the study of NIP.

The (p, q)-theorem is a combinatorial property and the model theoretic version of it about forking for certain formulas was conjectured for the class of NIP theories in 2015, see [CS15]. It should be noted that the authors expressed the so-called conjecture in the form of a question. Below is the conjecture stated in a first order setting.

Conjecture 0.1. *Let T be a complete NIP theory and $M \models T$. Let $M \prec N$ sufficiently saturated. Let $b \in N$. Suppose we have an L-formula $\varphi(x, b)$ which does not fork over M. Then there exists $\psi(y) \in \mathtt{tp}(b/M)$ such that $\{\varphi(x, b') : b' \in N \text{ and } N \models \psi(b')\}$ is finitely consistent.*

It has been known that the definable (p, q)-conjecture holds for stable theories since around the early 80's when it was not conjectured yet. Notions of directionality were established by I. Kaplan and S. Shelah to classify NIP theories according to the number of global extensions of a complete type, see [KS14]. Later in 2014, the definable (p, q)-conjecture was proved to hold for dp-minimal theories with small or medium directionality by P. Simon [Sim14]. Then it was asked for NIP in 2015 by A. Chernikov and P. Simon in [CS15]. In the same year, P. Simon [Sim15b] proved that the conjecture stands for theories with small or medium directionality. This was a generalisation of the results in [Sim14] where the assumption about dp-minimality is removed. In [SS14], P. Simon and S. Starchenko also proved that the definable (p, q)-conjecture holds for some other dp-minimal theories.

Recently, G. Boxall and C. Kestner demonstrated that the conjecture holds for distal theories [BK18]. Distal theories are known to be a subclass of NIP theories which are totally unstable. The notion of distality was first introduced by P. Simon [Sim13].

In this thesis, we aim to extend the results obtained in the latter study, *i.e.* we aim to show that the conjecture holds for theories beyond distality. So we start with a distal geometric structure, then we form a dense pair of distal geometric structures in which the notion of distality is not preserved. This gives an opportunity to prove that the definable (p, q)-conjecture holds for dense pairs of distal geometric structures. Towards the process, we add the condition that algebraic closure must be the same as definable closure in the underlying geometric structure.

Chapters overview

Besides the introduction, this thesis is divided into 4 main chapters.

Chapter 1 contains useful notions, properties and an introduction to some basic model theory. Definitions, facts, and well-known theorems that are vital in understanding the results of this dissertation are also stated. This chapter includes some fundamentals of NIP theories. The famous (p, q)-theorem is also presented at the end of the chapter.

Chapter 2 concerns the definable (p, q)-theorem for distal geometric theories. We extend the result from [BK18], in which the authors proved that the definable (p, q)-conjecture holds for distal, by adding the notion of dimension so as to obtain a stronger result which applies to geometric distal.

Chapter 3 is the main work of this dissertation. We prove that the definable (p, q)-conjecture holds for dense pairs of distal geometric structures, under only one condition: the algebraic closure is equal to the definable closure for the distal geometric theories. Real closed fields, p-adic fields are interesting examples of distal geometric structures where the aforementioned equality holds. Then obviously the results obtained in this chapter apply to dense pairs of o-minimal structures, dense pairs of p-adics. We also make use of the result obtained in Chapter 2 to complete the aim in this chapter.

Chapter 4 is an attempt to prove the definable (p, q)-theorem for a reduct of distal, mainly for the special structures: dense pairs of real closed fields. It should be noted that the case of dense pairs of real closed fields is already covered by Chapter 3. However, this chapter is independent from the others. It is a very interesting idea as this may be a way of proving the definable (p, q)-conjecture for any reduct of a distal theory. Unfortunately, we were not able to successfully complete all the cases, although we managed to cover some cases under some assumptions. We also have the opportunity to open new doors for further study.

Chapter 1

Preliminaries

We study dense pairs in a general setting where the original structures are geometric. Namely, we want to investigate if the definable (p, q)-conjecture holds for dense pairs of distal geometric structures.

1.1 Basic model theory

In this section, we present some basic definitions of model theory. We assume that readers are familiar with the very basic model theory. We suggest the following references for more information [Hod93], [Mar02], [Poi12], [Sac09] and [TZ12].

Definition 1.1.1 (Types). *Let L be a language, M a model of an L-theory T, $A \subseteq M$. A type over A is a set $p(\bar{x})$ of formulas in the language $L(A)$ such that for any finite subset $p_0(\bar{x})$ of $p(\bar{x})$, there is some $\bar{b} \in M$ such that $M \models p_0(\bar{b})$.*

The type of an element b over A denoted by $\mathtt{tp}(b/A)$, is the set of all formulas $\phi(x, a)$ with $a \in A$ and $M \models \phi(b, a)$.

Two basic and useful techniques in model theory are the use of saturation and the compactness theorem. The following definitions and theorem relate to these techniques.

Definition 1.1.2. *Let κ be a cardinal. An L-structure M is said to be κ-saturated if all types over a subset $A \subseteq M$ with $|A| < \kappa$, are realised in M.*

It is known that homogeneity is a weaker notion than saturation. In fact, if a model is κ-saturated then it is κ-homogeneous. By combining the concept of automorphism with κ-homogeneity, we have a stronger property named strongly κ-homogeneous. We define the latter property as follows.

Definition 1.1.3. *We say that an L-structure M is strongly κ-homogeneous if for any two tuples from M of length less than κ having the same types, then there is an automorphism of M which sends one tuple to the other tuple.*

Fact 1.1.4. *For every cardinal κ, every structure has a κ-saturated and strongly κ-homogeneous elementary extension.*

From now on, we say that a structure is sufficiently saturated if it is κ-saturated and strongly κ-homogeneous for some large enough cardinal κ.

The compactness theorem is amongst the most used property in model theory as it allows to relate any theory to its finite subsets. This is also a property used to show that sufficiently saturated elementary extensions exist. It is worth mentioning that this is a topological property, expressed in a first order setting. Below is the aforementioned property.

Theorem 1.1.5 (Compactness theorem)**.** *A theory T is finitely consistent if and only if it is consistent.*

Mostly, model theory is studying definable set. Sometimes model theorists characterize definable sets for some special kind of structures. For instance, let M be an o-minimal structure, then definable subsets of $X \subset M$ are finite unions of points and intervals.

Definition 1.1.6. *Let M be a structure, $X \subseteq M^n$ is said to be definable if and only if there is a formula $\psi(\bar{x}, \bar{y})$ where $|\bar{x}| = n$ and $\bar{b} \in M^m$ such that $X = \{\bar{a} \in M^n : M \models \psi(\bar{a}, \bar{b})\}$. A definable set over a set A is a definable set with the parameters taken from A^k.*

1.2 Indiscernible sequences

Indiscernible sequences and their underlying properties play a crucial role in proving the results of this work. Here we introduce some useful definitions and properties related to these sequences.

Let us define $\models \psi(\bar{a})$ to denote that $\psi(\bar{a})$ is true in a monster model where a monster model is a model which is sufficiently saturated and strongly homogeneous. Generally we are working inside a monster model.

Definition 1.2.1. *Let I be a linearly ordered set. A sequence $(b_i)_{i\in I}$ is indiscernible if for every formula $\varphi(x_1, \ldots, x_n)$, we have*

$$i_1 < \cdots < i_n \text{ and } j_1 < \cdots < j_n$$

$$\Longrightarrow$$

$$(\models \varphi(b_{i_1}, \ldots, b_{i_n}) \iff \models \varphi(b_{j_1}, \ldots, b_{j_n})).$$

In general, a sequence $(b_i)_{i\in I}$ is indiscernible over a set of parameters $A \subseteq M$, if $\mathrm{tp}(b_{i_1}, \ldots, b_{i_n}/A) = \mathrm{tp}(b_{j_1}, \ldots, b_{j_n}/A)$ for any $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$.

An indiscernible sequence can be totally indiscernible if it satisfies a certain condition stated as follows.

Definition 1.2.2. *A sequence $(b_i)_{i\in I}$ is totally indiscernible over A if for any $n \in \mathbb{N}$ and any $i_1, \ldots, i_n$ in I and $j_1, \ldots, j_n$ in I such that $i_k \neq i_{k'}$ and $j_k \neq j_{k'}$ for $k \neq k'$, we have $\mathrm{tp}(b_{i_1}, \ldots, b_{i_n}/A) = \mathrm{tp}(b_{j_1}, \ldots, b_{j_n}/A)$.*

Later, new sequences were proposed, namely the Morley sequences. Generally, those sequences are indiscernible and also entail the notion of forking. Before giving a formal definition of Morley sequence, we first need to define forking and dividing.

Definition 1.2.3. *Let A be a set and b a tuple. The formula $\varphi(x, b)$ divides over A if and only if there is an indiscernible sequence over A, say $(b_i)_{i<\omega}$ such that $b = b_0$ and $\{\varphi(x, b_i) : i = 1, 2, \ldots\}$ is inconsistent.*

Definition 1.2.4. *Let A be a set and b a tuple. The formula $\varphi(x, b)$ forks over A if there are formulas $\psi_i(x, a_i)$ for $i < n$ such that $\varphi(x, a) \vdash \bigvee_{i<n} \psi_i(x, a_i)$ and $\psi_i(x, a_i)$ divides over A for every $i < n$.*

It should be noted that a type does not fork over A if it does not imply any forking formula over A.

We now define a Morley sequence as follows.

Definition 1.2.5. *A sequence $(b_i)_{i\in I}$ is said to be a Morley sequence over A if it is A-indiscernible and A-independent in the sense of forking, which means for all $i \in I$, $\mathrm{tp}(b_i/Ab_1 \ldots b_{i-1})$ does not fork over A.*

Before we state another definition, recall that the type $\mathrm{tp}(a/Ab)$ strictly does not fork over A if there exists a global extension type, say p, of $\mathrm{tp}(a/Ab)$ which does not fork over A and for any set B containing Ab, if c realizes the restriction of p to B then $\mathrm{tp}(B/Ac)$ does not fork over A.

A Morley sequence can be a strict Morley sequence if the following condition holds.

Definition 1.2.6 (Strict Morley sequence)**.** *A sequence $(b_i)_{i\in I}$ is said to be a strict Morley sequence over A if it is A-indiscernible and it is strictly A-independent in the sense of forking. Which means for all $i \in I$, $\mathrm{tp}(b_i/Ab_1, \ldots b_{i-1})$ strictly does not fork over A.*

Assuming NIP, the following two properties apply for strictly Morley sequences, see [CK12] and [Sim15a] for proofs.

i (Existence) Let $q(y)$ be a complete type over a model, then there exists a strict Morley sequence $(b_i)_{i\in\mathbb{N}}$ for $q(y)$.

ii (Witnessing property) Assume a formula $\varphi(x, b)$ forks over a model M, then for any strict Morley sequence $(b_i)_{i\in\mathbb{N}}$ from $\mathrm{tp}(b/M)$ where $b_1 = b$, $\{\varphi(x, b_i) : i = 1, 2, \cdots\}$ is inconsistent.

1.3 NIP

In this section, we present some basic facts about NIP which stands for non independence property. Interested readers are referred to [Sim15a] and [Adl08] for detailed information about NIP.

Let us start with definitions and examples.

Definition 1.3.1. *A theory T has the independence property if there are a formula $\varphi(\bar{x}, \bar{y})$ and tuples $(\bar{a}_i : i \in \omega)$ and $(\bar{b}_I : I \subset \omega)$ such that $\models \varphi(\bar{a}_i, \bar{b}_I)$ if and only if $i \in I$.*

Definition 1.3.2. *A theory T is said to be dependent if it does not have the independence property. Dependent theories are also called as NIP theories.*

Example 1.3.3. *Real closed ordered fields, algebraically closed fields, p-adic fields are NIP.*

The following fact is unique to NIP. It shows how forking and dividing are formulated under NIP.

Fact 1.3.4. *[Sim15a] Assume $\varphi(\bar{x}, \bar{b})$ is NIP. If $\varphi(\bar{x}, \bar{b})$ does not divide over M, then for any finitely many realisations of $\mathtt{tp}(\bar{b}/M)$, say $\bar{b}_1, \cdots, \bar{b}_k$, the set of realisations of $\{\varphi(\bar{x}, \bar{b}_i) : i = 1, \cdots, k\}$ is non-empty.*

Remark 1.3.5. *The class of NIP contains o-minimal, dp-minimal, strongly minimal, distal, and all kinds of stable theories.*

Now, we give a definition of distal structures. In fact there are different ways to define distal and the one stated below is the most appropriate to our study.

Definition 1.3.6. *[CS15] A structure M is said to be distal if it is NIP and, for all $M \prec N$, indiscernible sequence $(b_i)_{i \in \mathbb{Z}}$ in $N^{|b_0|}$ and $A \subset N$, if $(\ldots, b_{-1}, b_1 \ldots)$ is indiscernible over A then $(\ldots, b_{-1}, b_0, b_1 \ldots)$ is indiscernible over A.*

A theory T is distal if all models of T are distal.

In [HN17] P. Hieronymi and T. Nell proved that the real field $(\mathbb{R}, +, \times, 0, 1, <)$ expanded by the cyclic multiplicative subgroup $2^{\mathbb{Z}}$ of the $\mathbb{R}_{>0}$ is distal. In fact, we have the language of $\mathbb{R}$ including one unary predicate P such that $P(x)$ means x is in $2^{\mathbb{Z}}$. In their paper, they also proved that theories of dense pairs are not always distal, when the original structure is NIP and the substructure is small and dense in the original structure.

Example 1.3.7.

1. *o-minimal structures are distal.*
2. *Let p be a prime number. Let $v_p(x)$ be the p-adic valuation on $\mathbb{Q}_p$. In the language $(+, \cdot, v_p(x) \geq v_p(y))$, the p-adic fields $\mathbb{Q}_p$ are distal.*

There are a lot of facts about NIP but in this dissertation we only use few of them. In fact, the following property is one of those and is amongst the main NIP results we are using most. It was established by A. Chernikov and I. Kaplan [CK12] for more general context, but for our study it is enough to state it using NIP.

Theorem 1.3.8. *[CK12] Let T be NIP and $M \models T$. Then forking is equal to dividing over the model M i.e a formula forks over M if and only if it divides over M.*

Definition 1.3.9. *Let N be a κ-saturated and strongly κ-homogeneous model. Let $A \subset N$ be small,* i.e. *A has cardinality less than κ. Let p be a complete type over N. The type p is said to be A-invariant if any automorphism σ of N which fixes A pointwise, fixes the type p.*

Recall that a global type is a type over a saturated and strongly homogeneous model. We now state a result which relates forking to invariance.

Theorem 1.3.10. *[Sim15a] Let T be NIP and $M \prec N \models T$ where N is a monster model. A global type does not fork over M if and only if it is M-invariant.*

The following fact is an important result about NIP, it was introduced by A. Chernikov and I. Kaplan in [CK12] and P. Simon discussed it in [Sim15b].

Fact 1.3.11. *Let T be NIP. Let $M \models T$. A partial type does not fork over M if and only if it does not divide over M, if and only if it extends to a global M-invariant type.*

The following theorem is a direct consequence of Fact 1.3.11, as one can think of a restriction of the global type.

Theorem 1.3.12. *[Sim15a] Let T be NIP, let $M \models T$. Let $M \prec N$ be sufficiently saturated and assume $\mathrm{tp}(\bar{b}/M)$ does not fork over M. Then there is a small model N' ($|N'|$ less than the degree of saturation) such that $M \prec N' \prec N$ and $\mathrm{tp}(\bar{b}/N')$ does not fork over M.*

1.4 Geometric structures

Let L be a language, M an L-structure. Let $a, b \in M$ and $A \subseteq M$. The element a is in the algebraic closure of the set A in the language L, denoted by $a \in \mathrm{acl}_L(A)$, if for some formula $\psi(x)$ over A, ψ has finitely many solutions in M and a is among the solutions.

Recall that M has the exchange property if: $a \in \mathrm{acl}(A \cup \{b\}) \setminus \mathrm{acl}(A) \implies b \in \mathrm{acl}(A \cup \{a\})$. This closure property typically defines a pregeometry and the latter gives a notion of independence. Explicitly, for a model M of a theory T, $\bar{a} \in M^{|n|}$, $A \subset B \subset M$, $\bar{a}$ is independent from B over A if $\dim_{\mathrm{acl}}(\bar{a}/A) = \dim_{\mathrm{acl}}(\bar{a}/B)$.

Analogously, a is in the definable closure of the set A in the language L, denoted by $a \in \mathrm{dcl}_L(A)$, if for some formula $\psi(x)$ over A, a is the unique solution of ψ in M.

Definition 1.4.1. *Let T be a complete theory, assume T has infinite models. We say that T is geometric if for any model M of T, the two following properties are satisfied:*

(i) algebraic closure satisfies the exchange property and

(ii) T eliminates the quantifier $\exists^\infty$: for any formula $\varphi(x, \bar{y})$, there is $n < \omega$ such that for every $M \models T$ and $\bar{b} \in M^{|y|}$, if $M \models \exists^{>n} x \varphi(x, \bar{b})$ then $M \models \exists^\infty x \varphi(x, \bar{b})$.

The above definition is the one used by E. Hrushovski and A. Pillay [HP94].

Example 1.4.2. *o-minimal structures, strongly minimal structures are geometric. A structure M is strongly minimal if the definable subsets of M and every elementary extension of M are finite or co-finite. By definable, we mean definable over a set of parameters.*

Recall that a geometric structure is a model of a geometric theory.

Definition 1.4.3. *Let $A \subset B$ and $q \in S_n(B)$.*

1. *q is free over A if for any realization a of q, a is independent from B over A.*

2. *q is said to be algebraic if there exists an integer k such that q has at most k realisations. We say q is non-algebraic if there is no such k.*

Let us state one more definition before we move to the next section.

Definition 1.4.4. *We say that a set $A \subseteq M$ is algebraically closed and finite dimensional if $A = \mathrm{acl}(A)$ and A is the algebraic closure of a finite set.*

1.5 Dense pairs

As dense pairs are the main subject of this thesis, definitions and theorems about them would be given advantage and some concepts will be given in details. Several authors have done work on dense pairs, including A. Berenstein and E. Vassiliev [BV10], I. Ben-Yaacov, A. Pillay and E. Vassiliev [BYPV03], G. Boxall [Box11], L. van den Dries [Dri98a], A. Fornaserio [For11], A. Macintyre [Mac75], A. Robinson [Rob59] and many more.

Let us start by introducing lovely pairs as defined by A. Berenstein and E. Vassiliev [BV10].

Let T be a geometric theory for a language L, assume T has quantifier elimination. Let P be a unary predicate and define $L_P = L \cup \{P\}$. Let T' be the L_P-theory of all structures $(M, P(M))$ where $M \models T$ and $P(M) \prec M$.

Definition 1.5.1. *A structure $(M, P(M))$ is said to be a lovely pair if*

a) $(M, P(M)) \models T'$

b) *(Coheir property/ denseness property) If $A \subset M$ is algebraically closed and finite dimensional and $q \in S_1(A)$ is non-algebraic, there is $a \in P(M)$ such that $a \models q$.*

c) *(Extension property) If $A \subset M$ is algebraically closed and finite dimensional and $q \in S_1(A)$ is non-algebraic, there is $a \in M$ such that $a \models q$ and $a \notin \mathrm{acl}(A \cup P(M))$.*

In a general context, the authors of [BV10] proved the following result.

Lemma 1.5.2. *Let $(M, P(M)) \models T'$, then $(M, P(M))$ is a lovely pair of models of T if and only if*

1) *(Generalised coheir property) If $A \subset M$ is algebraically closed and finite dimensional and $q \in S_n(A)$ is free over $P(A)$, then there is $\bar{a} \in P(M)^n$ such that $\bar{a} \models q$.*

2) *(Generalised extension property) If $A \subset M$ is algebraically closed and finite dimensional and $q \in S_n(A)$, then there is $\bar{a} \in M^n$ realising q such that $\mathrm{tp}(\bar{a}/A \cup P(M))$ is free over A.*

Definition 1.5.3. *Let $(M, P(M))$ be a lovely pair. Let A be a subset of $(M, P(M))$, A is said to be P-independent if A is independent from $P(M)$ over $P(M) \cap A$.*

As a matter of fact, M itself is P-independent.

Dense pairs are particular cases of lovely pairs but generally we say dense pairs to mean lovely pairs. A dense pair of real closed fields is the motivating example for this terminology, where the substructure is dense with respect to the usual topology. So we use dense pairs as another way of saying lovely pairs.

Lemma 1.5.4. *[BV10] For any two lovely pairs $(M, P(M))$ and $(N, P(N))$, we let $\bar{a} \in M$ and $\bar{b} \in N$ be finite tuples having the same length. Assume that the tuples $\bar{a}, \bar{b}$ are P-independent and have the same quantifier free L_P-type. Then $\bar{a}, \bar{b}$ have the same L_P-type.*

There are a lot of results about dense pairs in the literature but we are only interested in the following results which concern definable sets with regard to dense pairs of geometric structures.

Theorem 1.5.5. *If $X \subseteq P(M)^n$ is definable in the pair $(M, P(M))$ then there exists $Y \subseteq M^n$, definable in the original language L such that $X = Y \cap P(M)^n$.*

Let T_P be the L_P-theory of lovely pairs of models of M. It is known that T_P is complete.

Theorem 1.5.6. *Each L_P-formula $\psi(y_1, \ldots, y_n)$ is equivalent in T_P to a boolean combination of formulas of the form*

$$\exists x_1 \ldots \exists x_m (P(x_1) \wedge \cdots \wedge P(x_m) \wedge \phi(x, y))$$

where $\phi(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is an L-formula.

We refer to [Dri98a] for the proofs of the above two theorems for dense pairs of o-minimal ordered groups and to [BV10] for the general results about dense pairs of geometric structures.

Fact 1.5.7. *It has been proved that dense pairs are NIP when the original geometric theory is NIP. It was initially proved for dense pairs of o-minimal structures by A. Günaydin and P. Hieronimy in 2011, see [GH11]. Similar results which cover all lovely pairs of geometric structures have been proved by A. Berenstein, A. Dolich and A. Onshuus [BDO11]. See also [Box11].*

1.6 The (p, q)-theorem

This section contains notions and definitions that are essential in understanding the (p, q)-theorem. We start by presenting the original concepts, i.e. in combinatorics. Thereafter we present the model theoretic version of the theorem. We will also discuss the relation between the two.

We should note that most of the notation and contents in this section are taken from [Sim15a], mainly the first four definitions.

Definition 1.6.1. *Let X be a set, X can be finite or infinite. Let $\mathscr{S}$ be a family of subsets of X. We call $(X, \mathscr{S})$ a set system.*

Usually X is defined to be the union of $\mathscr{S}$.

Definition 1.6.2. *Let $A \subseteq X$. The family of subsets of X, denoted by $\mathscr{S}$, shatters A if for every subset A' of A, there is a set $S \in \mathscr{S}$ such that $S \cap A = A'$.*

The above definitions are essential to understand the notion of VC-dimension, where VC stands for Vapnik-Chervonenkis. The property was named after the two people who discovered it, namely Vapnik and Chervonenkis. We define it as follows.

Definition 1.6.3. *The family $\mathscr{S}$ has VC-dimension at most n, denoted by $VS(\mathscr{S}) \leq n$, if there is no subset A of X of cardinality $n+1$ such that $\mathscr{S}$ shatters A.*

We also say that $\mathscr{S}$ has VC-dimension n if it has VC-dimension at most n and it shatters some subset of X which has size n.

Moreover $\mathscr{S}$ is a VC-class if $VS(\mathscr{S}) < \infty$.

Suppose that for each n, $\mathscr{S}$ shatters a subset of X having cardinality n. We say that S has infinite VC-dimension.

It should be mentioned here that there exists a notion of duality about the set system and so for VC-dimension.

Definition 1.6.4. *Let $(X, \mathscr{S})$ be a set system. The dual set system of $(X, \mathscr{S})$ denoted by $(X^*, \mathscr{S}^*)$, is defined by $X^* = \mathscr{S}$ and $\mathscr{S}^* = \{\mathscr{S}_a : a \in X\}$ where $\mathscr{S}_a = \{S \in \mathscr{S} : a \in S\}$.*

The dual VC-dimension is defined accordingly and is denoted by $VC^*(S)$.

There have been several works about the VC-dimension, the early results were done by R. Dudley [Dud85], D. Pollard [Pol84] and P. Assouad [Ass83]. Then in 1992, M. Laskowski in [Las92] discovered that it has analogy in logic by showing that a class of subsets of a structure uniformly definable by a first-order formula is a Vapnik-Chervonenkis class if and only if the formula itself does not have the independence property.

With this information, we are now ready state the so called (p, q)-theorem in a combinatorics version and it will be followed by the model theoretic concept.

Theorem 1.6.5. *[The (p, q)-theorem by Alon-Kleitman and Matousek] [AK92], [Mat04] Let $p \geq q$ be integers. Then there is an $N \in \mathbb{N}$ such that the following holds:*

Let $(X, \mathscr{S})$ be a set system where every $S \in \mathscr{S}$ is non-empty.

Assume

- $VC^*(\mathscr{S}) \leq q$;

- *For every p many sets of $\mathscr{S}$, some q many sets have non-empty intersection.*

Then there is a subset of X of size N which intersects every element of $\mathscr{S}$.

Conjecture 1.7. *[CS15] Let T be a complete NIP theory and $M \models T$. Let $M \prec N$ sufficiently saturated. Let $b \in N$. Suppose we have an L-formula $\varphi(x, y)$ such that $\varphi(x, b)$ does not fork over M. Then there exists $\psi(y)$ such that $N \models \psi(b)$ and $\{\varphi(x, b') : b' \in N$ and $N \models \psi(b')\}$ is finitely consistent.*

We should note that the definable (p, q)-conjecture was stated in the form of a question [CS15].

One obvious question is that how does the theorem relate to the conjecture?

Suppose $\varphi(x, b)$ does not divide over M. Since our theory is NIP and by the compactness theorem, there is $\theta(y) \in \mathrm{tp}(b/M)$ and such that $\varphi(x, b')$ does not divide over M for any $b' \in \theta(N)$, [Sim15b].

Let our set system be the family of sets defined by $\{\varphi(x, b') : b' \in \theta(N)\}$. The (p, q)-theorem gives a finite set A_θ such that for each $b' \in \theta(N)$, there is $a \in A_\theta$ such that $\models \varphi(a, b')$. For each $a \in A_\theta$, we have $\varphi(a, N) \cap \theta(N)$. If any of the sets $\varphi(a, N) \cap \theta(N)$ is also defined by a formula in $\mathrm{tp}(b/M)$, or contains a set defined by such formula, say $\psi(y)$, then we obtain the conclusion of the definable (p, q)-conjecture when replacing $\theta(y)$ in the set system by the stronger formula $\psi(y)$ and A_θ by A_ψ such that $\{\varphi(x, b') : b' \in \psi(N)\}$ is consistent.

Chapter 2

The improved definable (p, q)-theorem for distal geometric theories

This chapter concerns the definable (p, q)-theorem for distal geometric theories. The definable (p, q)-conjecture has been proved by G. Boxall and C. Kestner in [BK18] to hold for the class of distal structures. In this chapter, we add to this result for distal geometric theories by introducing notions of dimension.

2.1 Properties of geometric structures

In this section we give properties of geometric theories. Let T be a geometric theory and $M \models T$.

Theorem 2.1.1. *[Uniform finiteness] Let $\{X_{\bar{a}} : \bar{a} \in M^k\}$ be a uniformly definable family of subsets of M^n,* i.e. *there is a definable set $X \subseteq M^k \times M^n$ such that for each $\bar{a} \in M^k$ we have $X_{\bar{a}} = \{\bar{x} \in M^n : \langle \bar{a}, \bar{x} \rangle \in X\}$. Then there exists $d \in \mathbb{N}$ such that for all $\bar{a} \in M^k$ we have*

$$|X_{\bar{a}}| > d \iff X_{\bar{a}} \text{ is infinite.}$$

We refer to [GKL$^+$07, Theorem 5.6.7] and [KPS86] for details about the theorem.

Fact 2.1.2. *Elimination of quantifier $\exists^{\infty}$ is equivalent to the uniform finiteness (also known as uniform boundedness). See [Gag05].*

Let M be a geometric structure. Then we define dimension as follows.

Definition 2.1.3. *Let $X \subseteq M^n$ be a set definable over $B \subseteq M$. Assume M is κ-saturated for some cardinal κ and $|B| < \kappa$. We define dimension of the set X, denoted by $\dim(X)$, by $\dim(X) = max\{\dim(\bar{a}/B) : \bar{a} \in X\}$. We call such $\bar{a}$ a generic point of X over B.*

Moreover, if $\varphi(\bar{x}, \bar{b})$ defines a set X, then $\dim(\varphi(\bar{x}, \bar{b})) = \dim(X)$.

The notion of dimension is definable in geometric structures, see [Mac00], [Mat95] and [Sta09]. In the next proposition, we will state the aforementioned property.

Proposition 2.1.4 (Definability of dimension)**.** *Let $X \subseteq M^{m+n}$ be definable, and for each $\bar{a} \in M^m$, let $X_{\bar{a}} := \{\bar{y} \in M^n : (\bar{a}, \bar{y}) \in X\}$. For each $i = 0, 1, \ldots, n$ let $X(i) = \{\bar{x} \in M^m : \dim(X_{\bar{x}}) = i\}$. Then each set $X(i)$ is definable, and for each i,*

$$\dim(\{(\bar{x}, \bar{y}) \in X : \bar{x} \in X(i)\}) = \dim(X(i)) + i.$$

The proof of Proposition 2.1.4 is done by induction on n and using the uniform finiteness theorem. Readers are referred to [Dri89] and [HPS00] for more details.

2.2 Dimension version of the definable (p, q)-theorem

In this section we start by proving a proposition which is crucial to the proof of the main theorem in this chapter. A similar proposition was proved by G. Boxall and C. Kestner in [BK18], and we generally use the ideas from that paper, adapting them to match our case.

Before we state the proposition, let us introduce some results that were taken from [BK18]. They will be used to finalize the proof of Proposition 2.2.3. The first came from a characterization of distality for NIP structures which was introduced in [CS15]. The second is about the existence of some indiscernible sequence which is required to have the same EM-types as a strict Morley sequence.

Proposition 2.2.1. *For any distal L-structure N and any L_N-formula $\varphi(\bar{x},\bar{y})$, there exist an L_N-formula $\theta(\bar{x},\bar{z})$ and a natural number k such that $|z| = k|y|$ and for all $B \subseteq C \subseteq N^{|y|}$, where B is finite, and $a \in N^{|x|}$, if*

1. $|C| \geq 2$,
2. $N \models \varphi(\bar{a},\bar{b})$ *for all* $\bar{b} \in B$

then there is some $\bar{c} \in C^k$ such that $N \models \theta(\bar{a},\bar{c})$ and, for all $\bar{b} \in B$, $N \models \forall\bar{x}\left(\theta(\bar{x},\bar{c}) \rightarrow \varphi(\bar{x},\bar{b})\right)$.

Lemma 2.2.2. *Let M and N be NIP such that $M \prec N$. Assume N is sufficiently saturated. Let $(\bar{b}_i)_{i\in\mathbb{N}}$ be a strict Morley sequence for $q(\bar{y})$ in $N^{|y|}$, where $q(\bar{y})$ is a complete type over M. Then for any set B, a finite set of realizations of q, there is a sequence $(\bar{d}_i)_{i\in\mathbb{Z}}$ such that for each $\bar{d} \in B$, $(\cdots,\bar{d}_{-1},\bar{d},\bar{d}_1,\cdots)$ is indiscernible over M and has the same EM-types as $(\bar{b}_i)_{i\in\mathbb{N}}$.*

The authors of [BK18] proved the above Lemma using the initial strict Morley sequences, and apply the compactness theorem.

Let $M \prec N$ be distal geometric structures where N is sufficiently saturated. We will use this notation for remaining parts of this section.

Let $q(\bar{y}) = \mathrm{tp}(\bar{b}/M)$ and assume $|\bar{b}| = m$. Let $(\bar{b}'_i)_{i\in\mathbb{N}}$ be a strict Morley sequence in $N^{|\bar{y}|}$ for $q(\bar{y})$. Let $\dim(\varphi(N,\bar{b})) = l$. Assume that for any $t \in \mathbb{N}$, $\dim\left(\bigwedge_{i=1,\dots,t}\varphi(N,\bar{b}'_i)\right) = l$, *i.e.* there is some $\bar{a} \in N^{|\bar{x}|}$ such that $\bar{a}$ realizes $\bigcap_{i=1,\dots,t}\varphi(N,\bar{b}'_i)$ and $\dim(\bar{a}/M\bar{b}'_1\cdots\bar{b}'_t) = l$. In addition, assume that $\mathrm{tp}(\bar{a}/M\bar{b}'_1\cdots\bar{b}'_t)$ does not divide over M.

Proposition 2.2.3. *There exist an L_M-formula $\theta(\bar{x},\bar{z})$, a natural number k such that $|\bar{z}| = k|\bar{y}|$ and a complete type $r(\bar{z})$ over M such that, for each finite set $B \subseteq N^{|\bar{y}|}$ of realisations of $q(\bar{y})$, there is some $\bar{c} \in N^{|\bar{z}|}$ such that:*

(1) $N \models r(\bar{c})$

(2) $\theta(\bar{x},\bar{c})$ does not divide over M

(3) for all $\bar{d} \in B, \varphi(N,\bar{d})$ contains $\theta(N,\bar{c})$ such that $\dim(\theta(N,\bar{c})) = l$ and $\dim\left(\bigwedge_{\bar{d}\in B}\varphi(N,\bar{d})\right) = l$.

Proof. It should be noted that all the proofs used in [BK18] still work in our case. We only need to include notions of dimension into the proofs.

Given that N is distal, then by Proposition 2.2.1, for any formula $\varphi(\bar{x},\bar{y})$ there exist a natural number k and formula $\theta(\bar{x},\bar{z})$ with $|\bar{z}| = k|\bar{y}|$, and for all finite sets $B \subseteq C \subseteq N^{|\bar{y}|}$ where $|C| \geq 2$ and $\bar{a} \in N^{|\bar{x}|}$, if $N \models \varphi(\bar{a},\bar{b})$ for all $b \in B$ then there exists some $\bar{c} \in C^k$ such that $\theta(\bar{a},\bar{c})$ holds in N and for all $\bar{b} \in B$, we have the following implication: $N \models \forall\bar{x}\left(\theta(\bar{x},\bar{c}) \to \varphi(\bar{x},\bar{b})\right)$.

Given the strict Morley Sequence $(\bar{b}'_i)_{i\in N}$ and by assumption, we have

$$\dim\left(\bigwedge_{i=1,\ldots,k+1}\varphi(N,\bar{b}'_i)\right) = l.$$

Then there exists an element $\bar{a} \in N^n$ such that for any $i = 1,\cdots,k+1$, $N \models \varphi(\bar{a},\bar{b}'_i)$ and $\dim\left(\bar{a}/M\bar{b}'_1\ldots\bar{b}'_{k+1}\right) = l$; moreover $\mathrm{tp}(\bar{a}/M\bar{b}'_{i_1}\cdots\bar{b}'_{i_k}\bar{b}'_{i_{k+1}})$ does not divide over M.

Therefore for any tuple $i_1\ldots i_k$ from $\{1,\ldots,k+1\}$, $\dim\left(\bar{a}/M\bar{b}'_{i_1}\cdots\bar{b}'_{i_k}\right) = l$. Thus there is a tuple $i_1\ldots i_k$ from $\{1,\ldots,k+1\}$ such that $N \models \theta(\bar{a},\bar{b}'_{i_1}\cdots\bar{b}'_{i_k})$, $\theta(\bar{x},\bar{b}'_{i_1}\cdots\bar{b}'_{i_k})$ has dimension l and such that $\mathrm{tp}(\bar{a}/M\bar{b}'_{i_1}\cdots\bar{b}'_{i_k})$ does not divide over M and for all $\bar{a}' \in N^n$, if $N \models \theta(\bar{a}',\bar{b}'_{i_1}\cdots\bar{b}'_{i_k})$, then for all $i \in \{1,\cdots,k+1\}, N \models \varphi(\bar{a}',\bar{b}'_i)$.

Since $\mathrm{tp}(\bar{a}/M\bar{b}'_{i_1}\cdots\bar{b}'_{i_k})$ does not divide over M and $\theta(\bar{x},\bar{b}'_{i_1}\ldots\bar{b}'_{i_k}) \in \mathrm{tp}(\bar{a}/M\bar{b}'_{i_1}\cdots\bar{b}'_{i_k})$ then $\theta(\bar{x},\bar{b}'_{i_1}\ldots\bar{b}'_{i_k})$ does not divide over M.

Let $i \in \{1,\cdots,k+1\}\setminus\{i_1,\cdots,i_k\}$. Let $B \subseteq N^m$ be a finite set of realizations of $\mathrm{tp}_L(\bar{b}/M)$. Using Lemma 2.2.2, let $(\bar{d}_i)_{i\in\mathbb{Z}}$ be an indiscernible sequence over M with the same EM-types over M as the strict Morley sequence $(\bar{b}'_i)_{i\in\mathbb{N}}$ satisfying the following property: for all $\bar{d} \in B$, the sequence $(\bar{d}_i)_{i\in\mathbb{Z}}$ is still indiscernible over M if we replace $\bar{d}_0$ by $\bar{d}$, *i.e* the sequence $(\cdots,\bar{d}_{-2},\bar{d}_{-1},d,\bar{d}_1,\bar{d}_2,\cdots)$ is indiscernible over M. One can take $r(\bar{z}) = \mathrm{tp}_L(\bar{b}'_{i_1}\cdots\bar{b}'_{i_k}/M)$, a complete type which contains $\theta(\bar{x},\bar{b}'_{i_1}\ldots\bar{b}'_{i_k})$.

For any $\bar{c}$ realizing $r(\bar{z})$, $\theta(\bar{x}, \bar{c})$ does not divide over M because $\bar{c}$ has the same type as $\bar{b}'_{i_1} \cdots \bar{b}'_{i_k}$ over M. We let $\bar{c} = \bar{d}_{i_1 - i} \cdots \bar{d}_{i_k - i}$. Then for any $\bar{a}$ realizing $\theta(\bar{x}, \bar{c})$, $\bar{a}$ realizes $\varphi(\bar{x}, \bar{d})$ for any $\bar{d} \in B$. Therefore $\bar{a}$ realises $\bigwedge_{\bar{d} \in B} \varphi(\bar{x}, \bar{d})$. Moreover $\theta(\bar{x}, \bar{c})$ has dimension l, therefore $\bigwedge_{\bar{d} \in B} \varphi(\bar{x}, \bar{d})$ has dimension l. □

Before we tackle the main theorem of this chapter, let us state a result which is essential in the process of the theorem's proof. Recall the assumptions made before Proposition 2.2.3.

Corollary 2.2.4. *For any finitely many $\bar{b}_1, \cdots, \bar{b}_s$ from $\mathrm{tp}_L(\bar{b}/M)$, we have*

$$\dim(\varphi(N, \bar{b})) = \dim\left(\bigwedge_{i=1,\ldots,s} \varphi(N, \bar{b}_i)\right).$$

Proof. By assumption, $\dim(\varphi(N, \bar{b})) = \dim\left(\bigwedge_{i=1,\ldots,t} \varphi(N, \bar{b}'_i)\right)$ for any $t \in \mathbb{N}$; then there is $\bar{a}$ such that $\dim(\bar{a}/M\bar{b}'_1 \cdots \bar{b}'_t) = l$ and $\mathrm{tp}(\bar{a}/M\bar{b}'_1 \cdots \bar{b}'_t)$ does not divide over M. By compactness there is $\bar{a}'$ such that $\dim(\bar{a}'/M \cup \{\bar{b}'_i : i \in \mathbb{N}\}) = l$ and $\mathrm{tp}(\bar{a}/M \cup \{\bar{b}'_i : i \in \mathbb{N}\})$ does not divide over M.

By Proposition 2.2.3, there are $\theta(\bar{x}, \bar{z})$ and $k \in \mathbb{N}$ such that $|z| = k|y|$; and for any finite set B from $\mathrm{tp}(\bar{b}/M)$ there is some $\bar{c}$, composed of k elements from $(\bar{d}_i)_{i \in \mathbb{Z}}$ in which $\bar{d}_0$ is replaced by any $\bar{d} \in B$ such that for any $\bar{a} \models \theta(\bar{a}, \bar{c})$, we have $\models \bigwedge_{d \in B} \varphi(\bar{x}, \bar{d})$. Moreover, $\theta(\bar{x}, \bar{c})$ does not divide over M and $\dim(\theta(\bar{x}, \bar{c})) = l$, and so there is $\bar{a}'$ such that $\dim(\bar{a}'/M\bar{c}) = l$ and $\models \theta(\bar{a}', \bar{c})$. Therefore $\models \bigwedge_{d \in B} \varphi(\bar{a}', \bar{d})$ and $\dim(\bar{a}'/M) = l$. □

Proposition 2.2.5. *There are a formula $\psi(\bar{y}) \in \mathrm{tp}_{L_M}(\bar{b}/M)$ and a finite set A_ψ, depending on ψ, such that for any $\bar{a} \in A_\psi$, $\dim(\bar{a}/M) = l$ and for any $\bar{b}' \in \psi(M)$, there exists $\bar{a} \in A_\psi$ such that $N \models \varphi(\bar{a}, \bar{b}')$.*

Proof. A variant form of this Proposition is presented in [BK18], which refers to [CS15]. The proof of this proposition is based on the proof made by P. Simon [Sim15b] of his Proposition 2.5.

Since $\varphi(\bar{x}, \bar{y})$ is NIP, then its dual VC-dimension is finite.

Let us fix some large number $s > VC^*(\varphi)$. By Corollary 2.2.4 and compactness, we can choose $\psi(\bar{y}) \in \mathrm{tp}_{L_M}(\bar{b}/M)$ such that for any s realisations

$\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_s$ of $\psi(M)$, we have $\dim\left(\bigwedge_{i=1,\ldots,s} \varphi(N, \bar{b}_i)\right) = l$. Then we can choose some element $\bar{a}$ such that $\models \bigwedge_{i=1,\ldots,s} \varphi(\bar{a}, \bar{b}_i)$ and $\dim(\bar{a}/M) = l$ with the property that $\mathrm{tp}(\bar{a}/M\bar{b}_1 \cdots \bar{b}_s)$ does not divide over M.

We apply the same process for any subset of size s from $\psi(M)$. We let X be the set of all $\bar{a}$, realising s conjunctions of formulas and such that $\dim(\bar{a}/M) = l$.

Let $\mathscr{S} = \{\varphi(N, \bar{b}') \cap X : \bar{b}' \in \psi(M)\}$. We have a system $(X, \mathscr{S})$. Let p and q be two integers such that $p \geq q = s$, then the set system $(X, \mathscr{S})$ has the (p,q)-property, *i.e.* any p sets of $\mathscr{S}$, q sets have non-empty intersection.

We apply Theorem 1.6.5, *i.e.* the (p,q)-theorem to get a subset of X of size N, say A_ψ which intersects every element of $\mathscr{S}$. $\square$

With these results, we are now ready to prove the main result of this chapter. Below the theorem is stated and its proof follows.

Theorem 2.2.6. *Let $M \prec N$ be models of a distal geometric theory, with N sufficiently saturated. Let $\varphi(\bar{x}, \bar{b})$, with $\bar{b} \in N$, be non-forking over M such that $\dim(\varphi(\bar{x}, \bar{b})) = l$. Assume that for some strict Morley sequence $(\bar{b}'_i)_{i \in \mathbb{N}}$ for $q(\bar{y})$ in $N^{|\bar{y}|}$, there is some $\bar{a} \in N^{|\bar{x}|}$ such that $\bar{a}$ realizes $\bigwedge_{i=1,\ldots,t} \varphi(N, \bar{b}'_i)$ and $\dim(\bar{a}/M\bar{b}'_1 \cdots \bar{b}'_t) = l$ for all $t \in \mathbb{N}$. Moreover, assume that $\mathrm{tp}(\bar{a}/M\bar{b}'_1 \cdots \bar{b}'_t)$ does not divide over M for all $t \in \mathbb{N}$.*
Then there exists $\psi(\bar{y}) \in \mathrm{tp}(\bar{b}/M)$ such that, for all $s \in \mathbb{N}$, if $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_s \in \psi(N)$ then $\dim\left(\bigwedge_{i=1,\ldots,s} \varphi(N, \bar{b}_i)\right) = l$.

Proof. We let $r(\bar{z})$ be as in Proposition 2.2.3 and $\bar{c} \in N^{|\bar{z}|}$ be such that $N \models r(\bar{z})$. We know that $\theta(\bar{x}, \bar{c})$ does not divide over M. We apply Proposition 2.2.5 to $\theta(\bar{x}, \bar{c})$, then there exist $\tau(\bar{z}) \in r(\bar{z})$ and a finite set $A_\tau \subseteq N^{|\bar{x}|}$ such that, for all $\bar{c}' \in \tau(M)$, there is some $\bar{a} \in A_\tau$ such that $N \models \theta(\bar{a}, \bar{c}')$. It should also be noted that for any $\bar{a} \in A_\tau$, $\dim(\bar{a}/M) = l$.

Now let j be the cardinality of A_τ. By Proposition 2.2.3 and the compactness theorem, there is some $\psi(\bar{y}) \in \mathrm{tp}(\bar{b}/M)$ such that for all $B \subseteq \psi(M)$ with $|B| \leq j$, there is some $\bar{c}' \in \tau(M)$ such that $\dim\left(\theta(\bar{x}, \bar{c}')\right) = l$ and $N \models \forall \bar{x}\left(\theta(\bar{x}, \bar{c}') \rightarrow \varphi(\bar{x}, \bar{d})\right)$ for all $\bar{d} \in B$.

Proof of Theorem 2.2.6 is complete if we can choose some $\bar{a} \in A_\tau$ such that $N \models \varphi(\bar{a}, \bar{b}')$ for all $\bar{b}' \in \psi(M)$.

The rest of the proof is the same as the last part of the proof of Theorem 1.2 in [BK18]. In their paper the authors use contradiction in their proof. In fact, we assume that there is no such $\bar{a}$. Given that A_τ is finite, one can enumerate all of its elements, naming the elements as $\bar{a}_1, \cdots, \bar{a}_j$. Thus for any $1 \leq i \leq j$, there exists $\bar{b}_i \in \psi(M)$ such that $N \models \neg\varphi(\bar{a}_i, \bar{b}_i)$. Let $B = \{\bar{b}_1, \cdots, \bar{b}_j\} \subseteq \psi(M)$. The cardinality of B is j, then we obtain $\bar{c}' \in \tau(M)$ such that $N \models \forall \bar{x}(\theta(\bar{x}, \bar{c}') \rightarrow \varphi(\bar{x}, \bar{b}_i))$ for all $\bar{b}_i \in B$. Besides, for some $\bar{a}_i \in A_\tau$, we have $N \models \theta(\bar{a}_i, \bar{c}')$. Hence $N \models \varphi(\bar{a}_i, \bar{b}_k)$ for some $\bar{a}_i \in A_\tau$ and for all $\bar{b}_k \in B$. We then get a contradiction, which confirms the existence of such realization. $\square$

Chapter 3

The definable (p, q)-theorem for dense pairs of distal geometric structures

In this chapter, we prove the definable (p, q)-theorem for dense pairs of distal geometric structures with the condition that algebraic closure is the same as definable closure. More precisely, we aim to show the following theorem.

Theorem 3.0.1. *Let $(M, P(M)) \prec (N, P(N))$ be two dense pairs of geometric structures, in which the geometric structures are distal and have the property that algebraic closure is equal to definable closure. Let $\varphi(\bar{x}, \bar{y})$ be an L_P-formula. Let $\bar{b} \in N^m$ and suppose that $\varphi(\bar{x}, \bar{b})$ does not divide over M. Then there is an L_P-formula $\theta(\bar{y}) \in \mathrm{tp}_{L_P}(\bar{b}/M)$ such that $\{\varphi(\bar{x}, \bar{b}') : \bar{b}' \in \theta(N)\}$ is finitely consistent.*

To prove this theorem, *i.e.* the definable (p, q)-conjecture holds for an L_P-formula $\varphi(\bar{x}, \bar{b})$, we need to construct an L-formula for which the definable (p, q)-conjecture holds and then work backward. To obtain such L-formula, we need to derive a sequence of L_P-formulas from $\varphi(\bar{x}, \bar{b})$ by reducing the length of its variables. Once the definable (p, q)-conjecture holds for the L-formula, we work backward by proving that the definable (p, q)-conjecture holds for each of the constructed L_P-formulas including $\varphi(\bar{x}, \bar{b})$.

It is noteworthy to mention that this chapter contains the main findings of this study. We assume readers are familiar with forking and dividing, dense pairs of geometric structures. Interested readers are referred to important references respectively including [Sim15a], [CK12] and [BV10], [Dri98a]. Additionally, we also present proofs for some known results that are used frequently but only stated as facts or exercises.

Let $M \prec N$ be NIP. Assume $\varphi(\bar{x}, \bar{b})$ does not divide over M. Recall that a formula $\varphi(\bar{x}, \bar{b})$ satisfies the definable (p, q)-conjecture if there exists a formula $\psi(\bar{y}) \in \mathrm{tp}(b/M)$ such that $\{\varphi(\bar{x}, \bar{b}') : \bar{b}' \in \psi(M)\}$ is finitely consistent.

From now on, let $(M, P(M)) \prec (N, P(N))$ be two dense pairs of geometric structures where $(N, P(N))$ is sufficiently saturated and in which the geometric structures are distal and have the property that algebraic closure is equal to definable closure. Let $\varphi(\bar{x}, \bar{y})$ be an L_P-formula and $\bar{b} \in N^m$. Suppose that $\varphi(\bar{x}, \bar{b})$ does not divide over M.

3.1 Useful Lemmas

This section contains important facts and lemmas that will be used to prove the theorem later in this chapter. We also present proofs of known results stated as fact or exercises in the literature.

The following lemma illustrates the first step of the proof as stated at the beginning of this chapter. That is more precisely, we show and explain in the following how to choose a new L_P-formula from an original one by selecting an appropriate basis for its parameters over M.

Lemma 3.1.1. *In order to prove that $\varphi(\bar{x}, \bar{b})$ satisfies the definable (p, q)-conjecture, it is sufficient to show that the conjecture holds for any L_P-formula $\varphi'''(\bar{x}, \bar{b}')$ in which $\bar{b}' = (b_1, \cdots, b_j) \in N^j$ for some $j \leq m$ with $b_1, \cdots, b_l$ being acl_L-independent over $M \cup P(N)$ (taken from $\bar{b}$) and $b_{l+1}, \cdots, b_j \in P(N)$ being acl_L-independent over M (not necessarily from $\bar{b}$), and such that $\bar{b} \in \mathrm{acl}_L(M\bar{b}')$.*

Proof. First recall the definition of small closure over a set $A \subset N$ to be the algebraic closure of $A \cup P(N)$.

For an arbitrary $\bar{b} \in N^m$, we can choose a tuple $(b_1, \cdots, b_l) \in N^l$ from $\bar{b}$, for some $l \leq m$ such that $(b_1, \cdots, b_l)$ is an scl_L-basis over M for $\bar{b}$. From the definition of small closure, we can choose a tuple $(b_{l+1}, \cdots, b_j)$ (not necessarily from the tuple $\bar{b}$) from $P(N)$ such that $\bar{b} \in \mathrm{acl}_L(M\bar{b}')$, where

$\bar{b}' = (b_1, \cdots, b_j) \in N^j$. Note that we can choose $(b_{l+1}, \cdots, b_j)$ to be acl_L-independent over M. Recall that $\mathrm{acl}_L = \mathrm{dcl}_L$ and $\bar{b} \in \mathrm{dcl}_L(M\bar{b}')$. Then we have a definable function f over M, from N^j to N^n such that $f(\bar{b}') = \bar{b}$.

For such choice of f, we let $\varphi'''(\bar{x}, \bar{b}')$ be a formula equivalent to $\exists \bar{b}(\varphi(\bar{x}, \bar{b}) \wedge f(\bar{b}') = \bar{b})$. Assume the definable (p,q)-conjecture holds for the L_P-formula $\varphi'''(\bar{x}, \bar{b}')$. We want to show that the definable (p,q)-conjecture holds for $\varphi(\bar{x}, \bar{b})$.

By assumption there are two possibilities. If $\varphi'''(\bar{x}, \bar{b}')$ does not divide over M, then there is an L_P-formula $\theta(\bar{z}) \in \mathrm{tp}_{L_P}(\bar{b}'/M)$ such that $\{\varphi'''(\bar{x}, \bar{b}'') : \bar{b}'' \models \theta(M)\}$ is finitely consistent. Then $\{\exists \bar{b}(\varphi(\bar{x}, \bar{b}) \wedge f(\bar{b}'') = \bar{b}) : \bar{b}'' \models \theta(M)\}$ is finitely consistent. Since f is L_P-definable, then $\{\varphi(\bar{x}, \bar{b}''') : \bar{b}''' \in \phi(M)\}$ is finitely consistent where $\phi(\bar{y}) = \exists \bar{z}(\theta(\bar{z}) \wedge f(\bar{z}) = \bar{y})$. It is clear that $\phi(\bar{y}) \in \mathrm{tp}_{L_P}(\bar{b}/M)$. Therefore the definable (p,q)-conjecture holds for $\varphi(\bar{x}, \bar{b})$.

Assuming now that $\varphi'''(\bar{x}, \bar{b}')$ divides over M, we need to prove that $\varphi(\bar{x}, \bar{b})$ divides over M. For some $\bar{b}'_1, \cdots, \bar{b}'_k \in \mathrm{tp}_{L_P}(\bar{b}'/M)$, there is no tuple $\bar{a}$ such that $(N, P(N)) \models \varphi'''(\bar{a}, \bar{b}'_1) \wedge \cdots \wedge \varphi'''(\bar{a}, \bar{b}'_k)$. Let $\bar{b}_i = f(\bar{b}'_i)$ for all $i = 1, \cdots, k$, then there is no tuple $\bar{a}$ such that $(N, P(N)) \models \varphi(\bar{a}, \bar{b}_1) \wedge \cdots \wedge \varphi(\bar{a}, \bar{b}_k)$. If $\bar{b}_i \in \mathrm{tp}_{L_P}(\bar{b}/M)$ then by the NIP Fact 1.3.4, $\varphi(\bar{x}, \bar{b})$ divides over M and we have what we desired. Let us prove that $\bar{b}_i \in \mathrm{tp}_{L_P}(\bar{b}/M)$.

Given the fact that $\mathrm{tp}_{L_P}(\bar{b}'_i/M) = \mathrm{tp}_{L_P}(\bar{b}'/M)$, we let σ be an automorphism of $(N, P(N))$ which fixes M, such that $\sigma(\bar{b}'_i) = \bar{b}'$. Since f is definable over M, we have $f(\sigma(\bar{b}'_i)) = f(\bar{b}') \iff \sigma(f(\bar{b}'_i)) = \bar{b} \iff \sigma(\bar{b}_i) = \bar{b}$. Therefore $\bar{b}_i \models \mathrm{tp}_{L_P}(\bar{b}/M)$, which means $\varphi(\bar{x}, \bar{b})$ divides over M and the definable (p,q)-conjecture holds for $\varphi(\bar{x}, \bar{b})$. □

The following fact is a well known result which relates definable sets and automorphisms.

Fact 3.1.2. *If Y is a definable set and every automorphism of N which fixes M pointwise fixes Y setwise, then Y is definable over M.*

From now on, we may assume $\bar{b}$ has the nice property stated in Lemma 3.1.1, *i.e.* $\bar{b} = (b_1, \cdots, b_k, b_{k+1}, \cdots, b_m)$ is acl_L-independent over M and

$b_1, \cdots, b_k$ are acl_L-independent over $M \cup P(N)$; $b_{k+1}, \cdots, b_m$ from $P(N)$ are acl_L-independent over M.

We are now going to construct sequences with certain properties. These properties are necessary to complete the full proof of the theorem in this chapter, therefore we provide solid evidence.

Lemma 3.1.3. *We can choose an independent sequence $(\bar{b}_i = (b_{i_1}, \cdots, b_{i_k}, b_{i_{(k+1)}}, \cdots, b_{i_m}))_{i \in \mathbb{N}}$ from $\mathrm{tp}_{L_P}(\bar{b}/M)$ such that for any $i \in \mathbb{N}$ and $1 \leq j \leq k$,*

$$b_{i_j} \notin \mathrm{acl}_L \left(M \cup P(N) \cup \{\bar{b}_1 \cdots \bar{b}_{i-1} b_{i_1} \cdots b_{i_{(j-1)}}\} \right),$$

and for any $k+1 \leq j \leq m$, $\bar{b}_j$ is from $P(N)$ such that

$$b_{i_j} \notin \mathrm{acl}_L \left(M \bar{b}_1 \cdots \bar{b}_{i-1} b_{i_1} \cdots b_{i_{(j-1)}} \right).$$

Proof. Let $i = 1$.

We have $b_1 \notin \mathrm{acl}_L(M \cup P(N))$. Then by the extension property, we can choose $b_{1_1} \notin \mathrm{acl}_L(M \cup P(N))$ such that $\mathrm{tp}_L(b_1/M) = \mathrm{tp}_L(b_{1_1}/M)$. Since b_{1_1} and b_1 have the same length and are both scl-independent over M then by Lemma 1.5.4

$$\mathrm{tp}_{L_P}(b_1/M) = \mathrm{tp}_{L_P}(b_{1_1}/M).$$

Moreover, $b_2 \notin \mathrm{acl}_L(M \cup P(N) \cup \{b_1\})$. Then by the extension property, we can choose $b_{1_2} \notin \mathrm{acl}_L(M \cup P(N) \cup \{b_{1_1}\})$ such that the type $\mathrm{tp}_L(b_{1_2}/M b_{1_1})$ corresponds to the type $\mathrm{tp}_L(b_2/M b_1)$. In addition, $\mathrm{tp}_L(b_1/M) = \mathrm{tp}_L(b_{1_1}/M)$, then $\mathrm{tp}_L(b_1 b_2/M) = \mathrm{tp}_L(b_{1_1} b_{1_2}/M)$. These properties combined with the fact that $b_1 b_2$ and $b_{1_1} b_{1_2}$ are scl-independent over M impliy that

$$\mathrm{tp}_{L_P}(b_1 b_2/M) = \mathrm{tp}_{L_P}(b_{1_1} b_{1_2}/M).$$

Following the same fashion, we can reach b_j. Since $b_j \notin \mathrm{acl}_L(M \cup P(N)) \cup \{b_1, \cdots b_{j-1}\}$, we can choose $b_{1_j} \notin \mathrm{acl}_L \left(M \cup P(N) \cup \{b_{1_1} \cdots b_{1_{(j-1)}}\} \right)$ such that $\mathrm{tp}_L(b_{1_j}/M\, b_{1_1} \cdots b_{1_{(j-1)}})$ corresponds to $\mathrm{tp}_L(b_j/M b_1 \cdots b_{(j-1)})$. Furthermore

$$\mathrm{tp}_L \left(b_1 \cdots b_{(j-1)}/M \right) = \mathrm{tp}_L \left(b_{1_1} \cdots b_{1_{(j-1)}}/M \right),$$

then $\mathrm{tp}_L\left(b_1\cdots b_j/M\right)=\mathrm{tp}_L\left(b_{1_1}\cdots b_{1_j}/M\right)$. Since $b_1,\cdots,b_j$ and $b_{1_1}\cdots b_{1_j}$ are scl-independent over M, then

$$\mathrm{tp}_{L_P}\left(b_1\cdots b_j/M\right)=\mathrm{tp}_{L_P}\left(b_{1_1}\cdots b_{1_j}/M\right).$$

Now we are dealing with the parts that are in $P(N)$ and acl_L-independent over M. Given that $b_{j+1}\in P(N)$ and is acl_L-independent over M, we use the denseness property to choose $b_{1_{(j+1)}}\in P(N)$ such that $b_{1_{(j+1)}}\notin \mathrm{acl}_L(M)$ and $\mathrm{tp}_L(b_{1_{(j+1)}}/Mb_{1_1}\cdots b_{1_j})$ corresponds $\mathrm{tp}_L(b_{j+1}/Mb_1\cdots b_j)$. The property of corresponding types combined with the fact that

$$\mathrm{tp}_L(b_1\cdots b_j/M)=\mathrm{tp}_L(b_{1_1}\cdots b_{1_j}/M),$$

implies

$$\mathrm{tp}_L(b_1\cdots b_{j+1}/M)=\mathrm{tp}_L(b_{1_1}\cdots b_{1_{j+1}}/M).$$

In addition $b_1\cdots b_{j+1}$ and $b_{1_1}\cdots b_{1_{j+1}}$ are acl_L-independent over M, b_{j+1} and $b_{1_{j+1}}$ are in $P(N)$. Then by Lemma 1.5.4

$$\mathrm{tp}_{L_P}(b_1\cdots b_{j+1}/M)=\mathrm{tp}_{L_P}(b_{1_1}\cdots b_{1_{j+1}}/M).$$

Furthermore, $b_{1_{(j+1)}}$ is chosen in a very independent way from $b_{1_1}\cdots b_{1_j}$, hence

$$b_{1_{(j+1)}}\notin \mathrm{acl}_L(Mb_{1_1}\cdots b_{1_j}).$$

We proceed with the same argument until we reach b_m. Knowing that $b_m\in P(N)$, $b_m\notin \mathrm{acl}_L(Mb_1\cdots b_{m-1})$, we use the denseness property to choose $b_{i_m}\in P(N)$ such that $b_{i_m}\notin \mathrm{acl}_L\left(Mb_{1_1}\cdots b_{1_{(m-1)}}\right)$ and $\mathrm{tp}_L(b_{1_m}/Mb_{1_1}\cdots b_{1_{m-1}})$ corresponds to the type $\mathrm{tp}_L(b_m/Mb_1,\cdots,b_{m-1})$. The property of corresponding types combined with the fact that

$$\mathrm{tp}_L(b_1\cdots b_{m-1}/M)=\mathrm{tp}_L(b_{1_1}\cdots b_{1_{(m-1)}}/M),$$

implies

$$\mathrm{tp}_L(b_1\cdots b_m/M)=\mathrm{tp}_L(b_{1_1}\cdots b_{1_m}/M).$$

By Lemma 1.5.4 , then

$$\mathrm{tp}_{L_P}(b_1\cdots b_m/M)=\mathrm{tp}_{L_P}(b_{1_1}\cdots b_{1_m}/M).$$

Let $i=2$.

We follow the same reasoning as previously starting with $b_1 \notin \mathrm{acl}_L(M \cup P(N))$, but when choosing the next element we add all the previous elements to the right side with the algebraic closure, *i.e.* we have

$$b_{2_1} \notin acl_L \left(M \cup P(N) \cup \{b_{1_1} \cdots b_{1_m}\}\right)$$

such that $\mathrm{tp}_L(b_1/M) = \mathrm{tp}_L(b_{2_1}/M)$. Since b_{2_1} and b_1 are both scl-independent over M then

$$\mathrm{tp}_{L_P}(b_1/M) = \mathrm{tp}_{L_P}(b_{2_1}/M).$$

Moreover, $b_2 \notin \mathrm{acl}_L(M \cup P(N) \cup \{b_1\})$. Then by the extension property, we can choose $b_{2_2} \notin \mathrm{acl}_L(M \cup P(N) \cup \{\bar{b}_1 b_{2_1}\})$ such that the type $\mathrm{tp}_L(b_{2_2}/Mb_{2_1})$ corresponds to the type $\mathrm{tp}_L(b_2/Mb_1)$. Besides this we have $\mathrm{tp}_L(b_1/M) = \mathrm{tp}_L(b_{2_1}/M)$, then $\mathrm{tp}_L(b_1 b_2/M) = \mathrm{tp}_L(b_{2_1} b_{2_2}/M)$. These properties combined with the fact that $b_1 b_2$ and $b_{2_1} b_{2_2}$ are scl-independent over M imply that

$$\mathrm{tp}_{L_P}(b_1 b_2/M) = \mathrm{tp}_{L_P}(b_{2_1} b_{2_2}/M).$$

Following the same fashion, we can reach b_j. Since $b_j \notin \mathrm{acl}_L(M \cup P(N) \cup \{b_1 \cdots b_{j-1}\})$, we can choose $b_{2_j} \notin \mathrm{acl}_L \left(M \cup P(N) \cup \{\bar{b}_1 b_{1_1} \cdots b_{1_{(j-1)}}\}\right)$ such that the type $\mathrm{tp}_L(b_{2_j}/Mb_{2_1} \cdots b_{2_{(j-1)}})$ corresponds to $\mathrm{tp}_L(b_j/Mb_1 \cdots b_{j-1})$. On the other side, we have

$$\mathrm{tp}_L \left(b_1 \cdots b_{j-1}/M\right) = \mathrm{tp}_L \left(b_{2_1} \cdots b_{2_{(j-1)}}/M\right),$$

then $\mathrm{tp}_L \left(b_1, \cdots, b_j/M\right) = \mathrm{tp}_L \left(b_{2_1} \cdots b_{2_j}/M\right)$. These properties together with the fact that $b_1, \cdots, b_j$ and $b_{2_1} \cdots b_{2_j}$ are scl-independent over M imply

$$\mathrm{tp}_{L_P}(b_1 \cdots b_j/M) = \mathrm{tp}_{L_P}(b_{2_1} \cdots b_{2_j}/M).$$

Now we are dealing with the parts that are in $P(N)$ and acl_L-independent over M. Given that $b_{j+1} \in P(N)$ and is acl_L-independent over M, we use the denseness property to choose $b_{2_{(j+1)}} \in P(N)$ such that $b_{2_{(j+1)}} \notin \mathrm{acl}_L(M)$ and $\mathrm{tp}_L(b_{2_{(j+1)}}/Mb_{2_1} \cdots b_{2_j})$ corresponds $\mathrm{tp}_L(b_{j+1}/Mb_1 \cdots b_j)$. The property of corresponding types combined with the fact that

$$\mathrm{tp}_L(b_1 \cdots b_j/M) = \mathrm{tp}_L(b_{2_1} \cdots b_{2_j}/M),$$

implies

$$\mathsf{tp}_L(b_1 \cdots b_{j+1}/M) = \mathsf{tp}_L(b_{2_1} \cdots b_{2_{j+1}}/M).$$

Note that $b_1, \cdots, b_{j+1}$ and $b_{2_1}, \cdots, b_{2_{j+1}}$ are acl_L-independent over M, b_{j+1} and $b_{2_{j+1}}$ are in $P(N)$. Then

$$\mathsf{tp}_{L_P}(b_1 \cdots b_{j+1}/M) = \mathsf{tp}_{L_P}(b_{2_1} \cdots b_{2_{j+1}}/M).$$

Furthermore, $b_{2_{(j+1)}}$ is chosen in a very independent way from $b_{2_1} \cdots b_{2_j}$, hence

$$b_{2_{(j+1)}} \notin \mathrm{acl}_L(M\bar{b}_1 b_{2_1} \cdots b_{2_j}).$$

We proceed with the same argument until we reach b_m. Knowing that $b_m \in P(N)$, $b_m \notin \mathrm{acl}_L(M)$, we use the denseness property to choose $b_{2_m} \in P(N)$ such that $b_{2_m} \notin \mathrm{acl}_L\left(M\bar{b}_1 b_{2_1} \cdots b_{2_{(m-1)}}\right)$ and $\mathsf{tp}_L(b_{2_m}/Mb_{2_1} \cdots b_{2_{(m-1)}})$ corresponds to the type $\mathsf{tp}_L(b_m/Mb_1 \cdots b_{m-1})$. The property of corresponding types combined with the fact that

$$\mathsf{tp}_L(b_1 \cdots b_{m-1}/M) = \mathsf{tp}_L(b_{2_1} \cdots b_{2_{(m-1)}}/M),$$

implies

$$\mathsf{tp}_L(b_1 \cdots b_m/M) = \mathsf{tp}_L(b_{2_1} \cdots b_{2_m}/M).$$

Since $b_1 \cdots b_m$ and $b_{2_1} \cdots b_{2_m}$ are scl-independent, then

$$\mathsf{tp}_L(b_1 \cdots b_m/M) = \mathsf{tp}_L(b_{2_1} \cdots b_{2_m}/M).$$

Now let $i > 2$.

For the components of $\bar{b}_i$ where $i \geq 2$, we repeat all of the above process and use saturation with the extension property to choose

$$b_{i_j} \notin \mathrm{acl}_L(M \cup P(N) \cup \{\bar{b}_1 \cdots \bar{b}_{i-1} b_{i_1} \cdots b_{i_{(j-1)}}\}),$$

if b_{i_j} corresponds to the acl_L-independent over $M \cup P(N)$ part of $\bar{b}$ and such that $\mathsf{tp}_L(b_{i_j}/Mb_{i_1} \cdots b_{i_{(j-1)}})$ corresponds to the type $\mathsf{tp}_L(b_j/Mb_1 \cdots b_{j-1})$ and $\mathsf{tp}_L(b_{i_1} \cdots b_{i_{(j-1)}} b_{i_j})/M = \mathsf{tp}_L(b_1 \cdots b_{j-1} b_j/M)$. Given that $b_{i_1} \cdots b_{i_j}$ and $b_1 \cdots b_j$ are scl-independent over M, we obtain

$$\mathsf{tp}_{L_P}(b_1 \cdots b_j/M) = \mathsf{tp}_{L_P}(b_{i_1} \cdots b_{i_j}/M).$$

Similarly we use saturation with the denseness property to choose

$$b_{i_j} \notin \mathrm{acl}_L(M\bar{b}_1 \cdots \bar{b}_{i-1} b_{i_1} \cdots b_{i_{(j-1)}}),$$

if b_{i_j} corresponds to the acl_L-independent over M part of $\bar{b}$ which is from $P(N)$ such that $\mathrm{tp}_L(b_{i_j}/Mb_{i_1} \cdots b_{i_{(j-1)}})$ corresponds to $\mathrm{tp}_L(b_j/Mb_1 \cdots b_{j-1})$ and $\mathrm{tp}_L(b_{i_1} \cdots b_{i_{j-1}} b_{i_j})/M = \mathrm{tp}_L(b_1 \cdots b_{j-1} b_j/M)$. Given that $b_{i_1} \cdots b_{i_j}$ and $b_1 \cdots b_j$ are acl_L-independent over M, we then get

$$\mathrm{tp}_{L_P}(b_1 \cdots b_j/M) = \mathrm{tp}_{L_P}(b_{i_1} \cdots b_{i_j}/M).$$

By saturation, we are able to repeat this process infinitely many times to get an independent sequence with part scl-independent over M and part from $P(N)$ which is acl_L-independent over M. □

As we use the definition of dimension in the sense of small closure, we recall that $\dim_{\mathrm{scl}}(\bar{b}/M) = \dim_{\mathrm{acl}_L}(\bar{b}/M \cup P(N))$.

Lemma 3.1.4. *Any strict Morley sequence $(\bar{b}_i)_{i\in\mathbb{N}}$ from $\mathrm{tp}_{L_P}(\bar{b}/M)$ is acl_L-independent over M, moreover for any $i \in \mathbb{N}$, $\bar{b}_i = (b_{i_1}, \cdots, b_{i_m})$ where $b_{i_1}, \cdots, b_{i_k}$ are acl_L-independent over $M \cup P(N)$ together with all the previous b_{i_k}'s and $b_{i_{k+1}}$, $\cdots, b_{i_m}$ are elements of $P(N)$ and are acl_L-independent over M together with all the previous b_{i_k}'s.*

Proof. We consider two cases.

Case 1. Let us start with $\bar{b} = b$ assumed to be one element. This case will be proved using sub-cases. Note that this is covered by Case 2 but since the proof of subcase 1 is slightly different from the other cases, it is worth including it in the proof. Besides, it can be seen as an overview of the general case.

subcase 1.

Suppose $b \in P(N)$ and is acl_L-independent over M. Then it is enough to prove that for each $i \in \mathbb{N}$, $b_i \in P(N)$, $b_i \notin \mathrm{acl}_L(M)$ and for any $n \in \mathbb{N}$, we have $b_{n+1} \notin \mathrm{acl}_L(Mb_1 \cdots b_n)$. We note that we use a similar argument as used in [BDO11] for this subcase.

Since $b \notin M = \mathrm{acl}_L(M)$ and $\mathrm{tp}_{L_P}(b_i/M) = \mathrm{tp}_{L_P}(b/M)$ then for each $i \in \mathbb{N}$, $b_i \in P(N)$ and $b_i \notin M = \mathrm{acl}_L(M)$. Assume for contradiction that $b_{n+1} \in \mathrm{acl}_L(Mb_1 \cdots b_n)$, then there is a definable function over M, $f : N^n \longrightarrow N$ such that $f(b_1 \cdots b_n) = b_{n+1}$. Using the fact that $b_1, b_2, \cdots$ is a strict Morley sequence, then $\mathrm{tp}_{L_P}(b_{n+1}/Mb_1 \cdots b_n)$ does not fork over M. So for any $\theta(y, b_1 \cdots b_n) \in \mathrm{tp}_{L_P}(b_{n+1}/Mb_1 \cdots b_n)$, we have the following:

$$\{\theta(y, b_1' \cdots b_n') : b_1' \cdots b_n' \models \mathrm{tp}_{L_P}(b_1 \cdots b_n/M)\} \text{ is finitely consistent.}$$

Now, let $b_1'' \cdots b_n'' \models \mathrm{tp}_{L_P}(b_1 \cdots b_n/M)$. Then using finite consistency, there is $a \in N$ such that $\theta(a, b_1' \cdots b_n') \wedge \theta(a, b_1'' \cdots b_n'')$ is realised. Now let

$$\theta(y, x_1 \cdots x_n) \equiv f(x_1 \cdots x_n) = y,$$

then

$$f(b_1' \cdots b_n') = f(b_1'' \cdots b_n'') = a.$$

Therefore, f is constant when restricted to the realizations of $\mathrm{tp}_{L_P}(b_1 \cdots b_n/M)$ and $a = b_{n+1}$.

Let σ be an automorphism of N which fixes M. Then $\sigma(f(b_1' \cdots b_n')) = f(\sigma(b_1' \cdots b_n')) = b_{n+1}$. So σ fixes b_{n+1}. Therefore $b_{n+1} \in \mathrm{dcl}_L(M) = M$. We have a contradiction with the fact that b_{n+1} is not in M. Therefore, the strict Morley sequence is acl_L-independent over M.

subcase 2.

For the case where b is acl_L-independent over $M \cup P(N)$, the b_i's are acl_L-independent over $M \cup P(N)$ for all $i \in \mathbb{N}$. This is because $\mathrm{tp}_{L_P}(b_i/M) = \mathrm{tp}_{L_P}(b/M)$ for each $i \in \mathbb{N}$. We want $b_{n+1} \notin \mathrm{acl}_L(M \cup P(N) \cup \{b_1 \cdots b_n\})$ for any $n \in \mathbb{N}$.

Assume for a contradiction that n is the smallest integer such that

$$\dim_{\mathrm{scl}}(b_n/M) > \dim_{\mathrm{scl}}(b_n/Mb_1 \cdots b_{n-1})$$

and $\dim_{\mathrm{scl}}(b_1 \cdots b_{n-1}/M) = n - 1$. Then b_n also reduces the scl-dimension of $b_1 \cdots b_{n-1}$, *i.e.* $\dim_{\mathrm{scl}}(b_1 \cdots b_{n-1}/Mb_n) \leq n - 2$. Let $\gamma(b_n, y_1, \cdots, y_{n-1})$ be a formula witnessing this dimension.

Now let us replace b_n by the variable y_n and $y_1, \cdots, y_{n-1}$ by the tuples $b_1 \cdots b_{n-1}$. We have $\gamma(y_n, b_1 \cdots b_{n-1})$. We need to show that $\gamma(y_n, b_1 \cdots b_{n-1})$ divides over M.

Assume for a contradiction that $\gamma(y_n, b_1 \cdots b_{n-1})$ does not divide over M, *i.e.* $\{\gamma(y_n, b'_1 \cdots b'_{n-1}) : b'_1 \cdots b'_{n-1} \models \mathrm{tp}_{L_P}(b_1 \cdots b_{n-1}/M)\}$ is finitely consistent.

Let us rename the tuples as, $\bar{b} = (b_1 \cdots b_{n-1})$ and $\bar{b}'_i = (b'_{i_1} \cdots b'_{i_{(n-1)}})$ for any $i \in \mathbb{N}$ such that $\bar{b}'_i \models \mathrm{tp}_{L_P}(\bar{b}/M)$. By Lemma 3.1.3, we may choose $(\bar{b}'_i)_{i \in \mathbb{N}}$ to be acl_L-independent over M with part scl-independent over M and part from $P(N)$ which is acl_L-independent over M. Then by finite consistency, for some finite $l \in \mathbb{N}$ there is some $d \in N$ such that $\models \bigwedge_{i=1}^{l} \varphi(d, \bar{b}'_i)$. Besides

$$\dim_{\mathrm{scl}}(\bar{b}'_1 \cdots \bar{b}'_l/M) = (n-1) \times l.$$

Moreover,

$$\begin{aligned}\dim_{\mathrm{scl}}(\bar{b}'_1 \cdots \bar{b}'_l d/M) &= \dim_{\mathrm{scl}}(\bar{b}'_1 \cdots \bar{b}'_l/M) + \dim_{\mathrm{scl}}(d/M\bar{b}'_1 \cdots \bar{b}'_l) \\ &= \dim_{scl}(d/M) + \dim_{\mathrm{scl}}(\bar{b}'_1 \cdots \bar{b}'_l/Md).\end{aligned}$$

We then get

$$\begin{aligned}\dim_{\mathrm{scl}}(\bar{b}'_1 \cdots \bar{b}'_l/Md) &= \dim_{\mathrm{scl}}(\bar{b}'_1 \cdots \bar{b}'_l/M) + \dim_{\mathrm{scl}}(d/M\bar{b}'_1 \cdots \bar{b}'_l) \\ &\quad - \dim_{scl}(d/M) \\ &= (n-1) \times l + \dim_{\mathrm{scl}}(d/M\bar{b}'_1 \cdots \bar{b}'_l) - \dim_{scl}(d/M) \\ &\geq (n-1) \times l - 1. \qquad (3.1.1)\end{aligned}$$

We also have

$$\begin{aligned}\dim_{\mathrm{scl}}(\bar{b}'_1 \cdots \bar{b}'_l/Md) &= \dim_{\mathrm{scl}}(\bar{b}'_1 \cdots \bar{b}'_{l-1}/Md) + \dim_{\mathrm{scl}}(\bar{b}'_l/M\bar{b}'_1 \cdots \bar{b}'_{l-1}d) \\ &= \dim_{\mathrm{scl}}(\bar{b}'_1/Md) + \dim_{\mathrm{scl}}(\bar{b}'_2/M\bar{b}'_1 d) \\ &\quad + \dim_{\mathrm{scl}}(\bar{b}'_l/M\bar{b}'_1 \cdots \bar{b}'_{l-1}d) \\ &\leq l(n-2). \qquad (3.1.2)\end{aligned}$$

By Equations (3.1.1) and (3.1.2), $(n-1) \times l - 1 \leq l(n-2)$, which is impossible when $l \geq 2$. Therefore we cannot have such d, so $\gamma(y_n, b_1 \cdots b_{n-1})$ divides over M.

Since $\gamma(y_n, b_1 \cdots b_{n-1}) \in \mathrm{tp}_{L_P}(b_n/Mb_1 \cdots b_{n-1})$ then $\gamma(y_n, b_1 \cdots b_{n-1})$ should not divide over M. We have a contradiction and consequently the strict Morley sequence $(b_i)_{i\in\mathbb{N}}$ is scl-independent over M.

Case 2.

Recall that $\bar{b}$ is a tuple with part acl_L-independent over $M \cup P(N)$ and part from $P(N)$ which is acl_L-independent over M. Then overall, the tuple $\bar{b}$ is acl_L-independent over M. Moreover, the $\bar{b}_i$'s are acl_L-independent over M and have parts acl_L-independent over $M \cup P(N)$ and parts from $P(N)$ which are acl_L-independent over M.

We want to prove that for any $j \in \mathbb{N}$ we have

$$\dim_{\mathrm{acl}_L}(\bar{b}_j/M) = \dim_{\mathrm{acl}_L}(\bar{b}_j/M\bar{b}_1 \cdots \bar{b}_{j-1})$$

and

$$\dim_{\mathrm{scl}}(\bar{b}_j/M) = \dim_{\mathrm{scl}}(\bar{b}_j/M\bar{b}_1 \cdots \bar{b}_{j-1}).$$

The proofs are almost equal for the different dimensions. We showed in subcase 2 how we deal with scl-dimension. We do exactly the same process with tuples and we get what we desire in this case. We consider acl_L-dimension for the remaining part of the proof.

Assume for a contradiction that for some j, we have the following

$$\dim_{\mathrm{acl}_L}(\bar{b}_j/M) = m > \dim_{\mathrm{acl}_L}(\bar{b}_j/M\bar{b}_1 \cdots \bar{b}_{j-1}) = l.$$

Then there is a formula $\psi(\bar{y}_j, \bar{b}_1 \cdots \bar{b}_{j-1})$ witnessing the above dimension, *i.e.* we have $\dim_{\mathrm{acl}_L}(\psi(\bar{y}_j, \bar{b}_1 \cdots \bar{b}_{j-1})) = l$. Then by the exchange property $\bar{b}_j$ is lowering the dimension of $\bar{b}_1 \cdots \bar{b}_{j-1}$ and after some calculations, we have

$$\dim_{\mathrm{acl}_L}(\bar{b}_1 \cdots \bar{b}_{j-1}/M\bar{b}_j) \leq (j-2) \times m + l. \tag{3.1.3}$$

Hence, there is also a formula $\gamma(\bar{b}_j, \bar{y}_1 \cdots \bar{y}_{j-1})$ witnessing this dimension.

Now we replace the parameter $\bar{b}_j$ by the variable $\bar{y}_j$ and the variables $(\bar{y}_1 \cdots \bar{y}_{j-1})$ by the parameters $(\bar{b}_1 \cdots \bar{b}_{j-1})$ in the formula $\gamma(\bar{b}_j, \bar{y}_1 \cdots \bar{y}_{j-1})$, to get the formula $\gamma(\bar{y}_j, \bar{b}_1 \cdots \bar{b}_{j-1})$. We aim to show that $\gamma(\bar{y}_j, \bar{b}_1 \cdots \bar{b}_{j-1})$ divides over M. If this statement holds, we could get a contradiction with the fact that $\text{tp}_{L_P}(\bar{b}_j/M\bar{b}_1\cdots\bar{b}_{j-1})$ does not divide over M. Which means we cannot have such a formula, therefore the strict Morley sequence $(\bar{b})_{i\in\mathbb{N}}$ is acl_L-independent over M and for any $j \in \mathbb{N}$, we have

$$\dim_{\text{acl}_L}(\bar{b}_j/M) = \dim_{\text{acl}_L}(\bar{b}_j/M\bar{b}_1\cdots\bar{b}_{j-1}).$$

In order to achieve this aim we assume for a contradiction that the formula $\gamma(\bar{y}_j, \bar{b}_1\cdots\bar{b}_{j-1})$ does not divide over M. Then the set

$$\{\gamma(\bar{y}_j, \bar{b}'_1\cdots\bar{b}'_{j-1}) : (\bar{b}'_1\cdots\bar{b}'_{j-1}) \models \text{tp}_{L_P}(\bar{b}_1\cdots\bar{b}_{j-1}/M)\}$$

is finitely consistent.

In the remainder of this proof, we adopt new notation for the purpose of simplicity. Consider $\bar{b}'' = (\bar{b}_1\cdots\bar{b}_{j-1})$. Since $\bar{b}''$ has length $(j-1)\times m$, then we may rename its components to be $(b_1,\cdots,b_{(j-1)m})$. Let $\bar{b}''_1,\cdots,\bar{b}''_n$ be realizations of $\text{tp}_{L_P}(\bar{b}''/M)$ and let $\bar{b}''_i = (b''_{i_1}\cdots b''_{i_{(j-1)m}})$ for any $i = 1,\cdots,n$.

By Lemma 3.1.3, we can choose an independent sequence $(\bar{b}''_i)_{i\in\mathbb{N}}$ from $\text{tp}_{L_P}(\bar{b}''/M)$ such that

$$b''_{i_k} \notin \text{acl}_L(M\cup P(N)\bar{b}''_1\cdots\bar{b}''_{i-1}b''_{i_1}\cdots b''_{i_{(k-1)}}),$$

if b''_{i_k} corresponds to the acl_L-independent over $M\cup P(N)$ part of $\bar{b}$, and

$$b''_{i_k} \notin \text{acl}_L(M\bar{b}''_1\cdots\bar{b}''_{i-1}b''_{i_1}\cdots b''_{i_{(k-1)}}),$$

if b''_{i_k} corresponds to the acl_L-independent over M part of $\bar{b}$ which is from $P(N)$.

Now we have some finite and independent tuples $\bar{b}''_1,\cdots,\bar{b}''_n$ from $\text{tp}_{L_P}(\bar{b}''/M)$. We let $\bar{d}$ be a realization of $\bigwedge_{i=1}^n \gamma(\bar{y}_j, \bar{b}''_i)$, we are using properties of dimension to obtain the contradiction stated earlier.

We know that $\dim_{\text{acl}_L}(\bar{b}''_1,\cdots,\bar{b}''_n/M) = (j-1)\times m\times n$. We also know that

$$\begin{aligned}\dim_{\text{acl}_L}(\bar{b}''_1,\cdots,\bar{b}''_n\bar{d}/M) &= \dim_{\text{acl}_L}(\bar{b}''_1,\cdots,\bar{b}''_n/M) + \dim_{\text{acl}_L}(\bar{d}/M\bar{b}''_1,\cdots,\bar{b}''_n)\\ &= \dim_{\text{acl}_L}(\bar{d}/M) + \dim_{\text{acl}_L}(\bar{b}''_1,\cdots,\bar{b}''_n/M\bar{d}).\end{aligned}$$

Then we get

$$\begin{aligned}\dim_{\mathrm{acl}_L}(\bar{b}''_1,\cdots,\bar{b}''_n/M\bar{d}) &= \dim_{\mathrm{acl}_L}(\bar{b}''_1,\cdots,\bar{b}''_n/M)+\dim_{\mathrm{acl}_L}(\bar{d}/M\bar{b}''_1,\cdots,\bar{b}''_n)\\ &\quad -\dim_{\mathrm{acl}_L}(\bar{d}/M)\\ &\geq (j-1)\times m\times n-m+\dim_{\mathrm{acl}_L}(\bar{d}/M\bar{b}''_1,\cdots,\bar{b}''_n)\\ &\geq (j-1)\times m\times n-m+l\\ &= [((j-1)\times n-1)\times m]+l. \qquad (3.1.4)\end{aligned}$$

On the other side, we have

$$\begin{aligned}\dim_{\mathrm{acl}_L}(\bar{b}''_1,\cdots,\bar{b}''_n/M\bar{d}) &= \dim_{\mathrm{acl}_L}(\bar{b}''_1,\cdots,\bar{b}''_{n-1}/M\bar{d})\\ &\quad+\dim_{\mathrm{acl}_L}(\bar{b}''_l/M\bar{b}''_1,\cdots,\bar{b}''_{n-1}\bar{d})\\ &= \dim_{\mathrm{acl}_L}(\bar{b}''_1,\cdots,\bar{b}''_{n-2}/M\bar{d})\\ &\quad+\dim_{\mathrm{acl}_L}(\bar{b}''_{n-1}/M\bar{b}''_1,\cdots,\bar{b}''_{n-2}\bar{d})\\ &\quad+\dim_{\mathrm{acl}_L}(\bar{b}''_n/M\bar{b}''_1,\cdots,\bar{b}''_{n-1}\bar{d})\\ &= \dim_{\mathrm{acl}_L}(\bar{b}''_1/M\bar{d})+\dim_{\mathrm{acl}_L}(\bar{b}''_2/M\bar{b}''_1\bar{d})+\cdots\\ &\quad+\dim_{\mathrm{acl}_L}(\bar{b}''_n/M\bar{b}''_1,\cdots,\bar{b}''_{n-1}\bar{d}).\end{aligned}$$

On the right side of the above equality, we have n terms where each term is less than or equal to $(j-2)\times m+l$, by Equation (3.1.3). Therefore

$$\dim_{\mathrm{acl}_L}(\bar{b}''_1,\cdots,\bar{b}''_n/M\bar{d}) \leq [(j-2)\times m+l]\times n. \qquad (3.1.5)$$

By Equations (3.1.4) and (3.1.5), we have

$$[(j-1)\times n-1]\times m+l \leq [(j-2)\times m+l]\times n.$$

After simplification, we obtain $m(n-1)+l \leq l\times n$ which is impossible when $n\geq 2$, since $l<m$. Hence $\gamma(\bar{y}_j,\bar{b}_1\cdots\bar{b}_{j-1})$ divides over M. We get another contradiction since $\mathrm{tp}_{L_P}(\bar{b}_j/M\bar{b}_1\cdots\bar{b}_{j-1})$ does not divides over M and $\gamma(\bar{y}_j,\bar{b}_1\cdots\bar{b}_{j-1})\in\mathrm{tp}_{L_P}(\bar{b}_j/M\bar{b}_1\cdots\bar{b}_{j-1})$. Hence we conclude that the strict Morley sequence $(\bar{b}_i)_{i\in\mathbb{N}}$ is acl_L-independent over M, moreover the $\bar{b}_i$'s are independent from each other. □

Remark 3.1.5. *Suppose $\bar{b}=(b_1,\cdots,b_m)$ is an acl_L-independent tuple over M. Moreover assume $b_1,\cdots,b_k$ are acl_L-independent over $M\cup P(N)$; $b_{k+1},\cdots,b_m$ are elements of $P(N)$ and are acl_L-independent over M. Then there is a strict*

Morley sequence $(\bar{b}_i)_{i\in\mathbb{N}}$ *for* $\mathrm{tp}_L(\bar{b}/M)$ *which is* acl_L*-independent over* M *and moreover, for any* $i \in \mathbb{N}$, $\bar{b}_i = (b_{i_1}, \cdots, b_{i_m})$, $b_{i_1}, \cdots, b_{i_k}$ *are* acl_L*-independent over* $M \cup P(N)$ *together with all the previous* b_{i_k}*'s and* $b_{i_{k+1}}, \cdots, b_{i_m}$ *are elements of* $P(N)$ *which are* acl_L*-independent over* M *together with all the previous* b_{i_k}*'s.*

This remark follows from a simplified version of Lemma 3.1.4.

From now on let $(\bar{b}_i)_{i\in\mathbb{N}}$ be a strict Morley sequence for $\mathrm{tp}_{L_P}(\bar{b}/M)$.

As we deal with two languages, we also want to know if the property of a sequence being a strict Morley sequence in sense the of L_P is preserved in the sense of L. Interestingly, we get positive results as shown below. Furthermore, the algebraic independence property is also preserved in the sense of L.

Proposition 3.1.6. *The sequence* $(\bar{b}_i)_{i\in\mathbb{N}}$ *is a strict Morley sequence in the sense of* L.

Proof. In fact, from the definition of strict Morley sequence, $(\bar{b}_i)_{i\in\mathbb{N}}$ is indiscernible over M. The only concern is about the strict non-forking of $\mathrm{tp}_L(\bar{b}_n/M\bar{b}_1\cdots\bar{b}_{n-1})$ for any $n > 0$. Let p be a global extension L_P-type of $\mathrm{tp}_{L_P}(\bar{b}_n/M\bar{b}_1\cdots\bar{b}_{n-1})$, which does not fork over M. The L_P-type $p(\bar{y})$ is determined by the L-type together with certain information about the $\bar{b}_i'$'s. It should also be noted that the L-type part of $p(\bar{y})$, say $q(\bar{y})$, is a global extension L-type of the type $\mathrm{tp}_L(\bar{b}_n/M\bar{b}_1\cdots\bar{b}_{n-1})$ and obviously it does not fork over M in sense of L_P.

Let $\varphi(\bar{y},\bar{c})$ be an L-formula from $q(\bar{y})$ and assume it divides over M in the sense of L. For an arbitrary parameter $\bar{c} \in N^m$, we can choose a tuple $(c_1,\cdots,c_l) \in N^l$ from $\bar{c}$, for some $l \leq m$ such that $(c_1,\cdots,c_l)$ is an scl_L-basis over M for $\bar{c}$. We may choose $c_1,\cdots,c_l$ to be the l first components of $\bar{c}$, otherwise we can always rearrange the components of $\bar{c}$ to make it work. From the definition of small closure, we can choose a tuple $(c_{l+1},\cdots,c_j)$ from $P(N)$ such that $\bar{c} \in \mathrm{acl}_L(M\bar{c}')$, where $\bar{c}' = (c_1,\cdots,c_j) \in N^j$. Note that we can choose $(c_{l+1},\cdots,c_j)$ to be acl_L-independent over M. Since $\mathrm{acl}_L = \mathrm{dcl}_L$ and $\bar{c} \in \mathrm{dcl}_L(M\bar{c}')$ then we have a definable function h over M, from N^j to N^n such that $h(\bar{c}') = \bar{c}$.

For such choice of h, we let $\varphi'(\bar{y}, \bar{c}')$ be a formula equivalent to $\exists \bar{c}(\varphi(\bar{y}, \bar{c}) \wedge h(\bar{c}') = \bar{c})$. Now we need to show that $\varphi'(\bar{y}, \bar{c}')$ divides over M, in the sense of L. Assume for contradiction that $\varphi'(\bar{y}, \bar{c}')$ does not divide over M. Then for any $\bar{c}'_1, \cdots, \bar{c}'_j$ from $\mathrm{tp}_L(\bar{c}'/M)$ there exists $\bar{b}$ such that $\bar{b} \models \bigwedge_{i=1}^{j} \varphi'(\bar{y}, \bar{c}'_i)$. Set $\bar{c}_i = h(\bar{c}'_i)$ for all $i = 1, \cdots j$, then if we assume $\bar{c}_i \in \mathrm{tp}_L(\bar{c}/M)$, we contradict the fact that $\varphi(\bar{y}, \bar{c})$ divides.

We are left to prove that the $\bar{c}_i$'s are from $\mathrm{tp}_L(\bar{c}/M)$. We start with the idea that $\mathrm{tp}_L(\bar{c}'_i/M) = \mathrm{tp}_L(\bar{c}'/M)$, then for all i, we let σ_i be an automorphisms of N which fixes M, such that $\sigma_i(\bar{c}'_i) = \bar{c}'$. Knowing that h is a definable function over M, we have $h(\sigma_i(\bar{c}'_i)) = h(\bar{c}')$. Thus $\sigma_i(h(\bar{c}'_i)) = \bar{c}$, which implies that $\sigma_i(\bar{c}_i) = \bar{c}$. So the $\bar{c}_i$'s are from $\mathrm{tp}_L(\bar{c}/M)$ and then $\varphi'(\bar{y}, \bar{c}')$ divides over M, in the sense of L.

Then there is an indiscernible sequence $(\bar{a}_i)_{i \in N}$ witnessing dividing of $\varphi'(\bar{y}, \bar{c}')$ in sense of L. Moreover $\bar{c}' = c_1, \cdots, c_j$ is of the nice form where $c_1, \cdots, c_l$ are acl_L-independent over $M \cup P(N)$ and $c_{l+1}, \cdots, c_j \in P(N)$ are acl_L-independent over M.

Then by Remark 3.1.5, the sequence $(\bar{a}_i)_{i \in N}$ can be chosen to have a nice form such as $\bar{c}'$ which implies that for all $i \in \mathbb{N}$ we have $\mathrm{tp}_{L_P}(\bar{a}_i/M) = \mathrm{tp}_{L_P}(\bar{c}'/M)$.

Since $(\bar{a}_i)_{i \in \mathbb{N}}$ witnesses dividing of $\varphi'(\bar{y}, \bar{c}')$ as an L_P-formula we have a contradiction with $q(\bar{y})$ not dividing over M in the sense of L_P. Taking everything into account, $q(\bar{y})$ does not divide over M in the sense of L and so $\mathrm{tp}_L(\bar{b}_n/M\bar{b}_1 \cdots \bar{b}_{n-1})$ does not divide over M.

We use the same argument to prove that $\mathrm{tp}_L(\bar{b}_1 \cdots \bar{b}_{n-1}/M\bar{b}_n)$ does not divide over M in the sense of L. Since the two types $\mathrm{tp}_L(\bar{b}_n/M\bar{b}_1 \cdots \bar{b}_{n-1})$ and $\mathrm{tp}_L(\bar{b}_1 \cdots \bar{b}_{n-1}/M\bar{b}_n)$ do not divide over M in the sense of L, then the type $\mathrm{tp}_L(\bar{b}_n/M\bar{b}_1 \cdots \bar{b}_{n-1})$ strictly does not fork over M. Therefore the sequence $(\bar{b}_i)_{i \in \mathbb{N}}$ is a strict Morley sequence in the sense of L. □

The next lemma establishes the existence of a tuple $\bar{a} \in N^{|x|}$ such that for all $i \in \mathbb{N}$, $N \models \varphi(\bar{a}, \bar{b}_i)$ and $\mathrm{tp}_{L_P}(\bar{a}/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$ does not divide over M.

Recall first that $\varphi(\bar{x}, \bar{b})$ does not divide over M and that $(\bar{b}_i)_{i\in\mathbb{N}}$ is a strict Morley sequence for $\mathrm{tp}_{L_P}(\bar{b}/M)$.

Lemma 3.1.7. *There exists $\bar{a} \in N^n$ such that for all $i \in \mathbb{N}$ we have $(N, P(N)) \models \varphi(\bar{a}, \bar{b}_i)$ and such that $\mathrm{tp}_{L_P}(\bar{a}/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$ does not divide over M.*

Proof. Since $\varphi(\bar{x}, \bar{b})$ does not divide over M, then the fact that $\{\bar{b}_i : i \in \mathbb{N}\}$ is a strict Morley sequence is enough to guarantee the existence of a tuple $\bar{a} \in N^n$ such that for all $n \in \mathbb{N}$ we have $(N, P(N)) \models \varphi(\bar{a}, \bar{b}_n)$.

We first need to show that there is some $\bar{a}$ such that $\mathrm{tp}_{L_P}(\bar{a}/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$ does not divide over M. For that, we use the fact that $\varphi(\bar{x}, \bar{b}_i)$ does not divide over M for any $i \in \mathbb{N}$.

We consider $\varphi(\bar{x}, \bar{b}_1)$ as a partial type over N. By Fact 1.3.11, the partial type $\varphi(\bar{x}, \bar{b}_1)$ extends to a complete type over N, say $p(\bar{x})$, which is invariant over M and does not divide over M.

Since any automorphism σ of N which fixes M fixes the type p, then $\varphi(\bar{x}, \bar{b}_i) \in p(\bar{x}) \iff \varphi(\bar{x}, \sigma(\bar{b}_i)) \in p(\bar{x})$ for any $i \neq j$. Moreover $\bar{b}_i \equiv_M \bar{b}_j$, so by strong homogeneity there is an automorphism σ of N which fixes M and sends b_i to b_j. Therefore $\varphi(\bar{x}, \bar{b}_i) \in p(\bar{x})$ for all $i \in \mathbb{N}$.

We also know that by M-invariance, $p(\bar{x})$ does not divide over M. Let us fix $\bar{a}''$, a realization of p. Then clearly $\mathrm{tp}_{L_P}(\bar{a}''/N)$ does not divide over M.

Now let us consider the complete type $\mathrm{tp}_{L_P}(\bar{a}''/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$ and for a contradiction assume it divides over M. It is a partial type over N, and has a complete extension type over N which is $p(\bar{x})$. Recall that a partial type divides over M if it implies some dividing formula over M and so $p(\bar{x})$ divides. This shows a contradiction with the fact that $p(\bar{x})$ does not divide over M. Therefore $\mathrm{tp}_{L_P}(\bar{a}''/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$ does not divide over M.

Finally, we need to choose some $\bar{a}'$ which is in N such that $\bar{a}'$ realizes $\mathrm{tp}_{L_P}(\bar{a}''/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$. Since $M \cup \{\bar{b}_i : i \in \mathbb{N}\}$ is a small set and contained in N where N is assumed to be saturated enough, then by saturation we can choose some tuple $\bar{a}'$ from N such that $\mathrm{tp}_{L_P}(\bar{a}''/M \cup \{\bar{b}_i : i \in \mathbb{N}\}) = \mathrm{tp}_{L_P}(\bar{a}'/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$. □

From now, we may assume that $\bar{a}$ exists and has the property shown in Lemma 3.1.7.

Given the realization $\bar{a}$, we want to choose $\bar{\alpha}$ to be an acl_L-basis for $\bar{a}$. We claim the following.

Claim 3.1.8. *There exist a tuple $\bar{\alpha} = (\alpha_1, \cdots, \alpha_j) \in N^j$ and a function g, from some $Z \subseteq N^j$ to N^n, such that g and Z are L-definable over M, $g(\bar{\alpha}) = \bar{a}$ and there exists $l \leq j$ such that $\bar{\alpha} = (\alpha_1, \cdots, \alpha_j)$ with $\alpha_1, \cdots, \alpha_l$ being acl_L-independent over $M \cup P(N)$ and $\alpha_{l+1}, \cdots, \alpha_j \in P(N)$ being acl_L-independent over M.*

Proof. Since $\bar{a}$ is a tuple from N^m, then we can choose a tuple $(\alpha_1, \cdots, \alpha_l) \in N^l$ from $\bar{a}$, say the l first components of $\bar{a}$, for some $l \leq m$ such that $(\alpha_1, \cdots, \alpha_l)$ forms an scl_L-basis for $\bar{a}$ over M. It should be noted that we can rearrange the components of the tuple $\bar{a}$ to make it happen.

Moreover, we can choose an acl_L-independent tuple over M from $P(N)$, say $(\alpha_{l+1}, \cdots, \alpha_j)$, such that $\bar{a} \in \mathrm{acl}_L(M\bar{\alpha})$, where $\bar{\alpha} = (\alpha_1, \cdots, \alpha_j) \in N^j$. We should also note that $\alpha_{l+1}, \cdots, \alpha_j$ may not be components of $\bar{a}$.

Since $\mathrm{acl}_L = \mathrm{dcl}_L$ and $\bar{a} \in \mathrm{dcl}_L(M\bar{\alpha})$ then we have a definable function over M, g from N^j to N^n such that $g(\bar{\alpha}) = \bar{a}$. Let $Z \subseteq N^j$ be the domain of the definable function g, then Z is also definable over M. □

Now we proceed with the next step of the proof of the main Theorem 3.0.1. We fix such choice of $\bar{\alpha}, g$ and Z. Let $\varphi'(\bar{z}, \bar{y})$ be a formula equivalent to $(z \in Z) \wedge \varphi(g(\bar{z}), \bar{y})$. Clearly, $\varphi'(\bar{z}, \bar{y})$ is an L_P-formula.

Corollary 3.1.9. *The tuple $\bar{\alpha}$ realizes $\{\varphi'(\bar{z}, \bar{b}_i) : i \in \mathbb{N}\}$.*

Proof. Since $\bar{a}$ realizes $\{\varphi(\bar{x}, \bar{b}_i) : i \in \mathbb{N}\}$ then obviously $\bar{\alpha}$ realizes $\{\varphi'(\bar{z}, \bar{b}_i) : i \in \mathbb{N}\}$. □

Recall that $\varphi(\bar{x}, \bar{b})$ does not divide over M.

Lemma 3.1.10. *The L_P-formula $\varphi'(\bar{z}, \bar{b})$ does not divide over M.*

Proof. We know that $\bar{a}$ realizes $\varphi(\bar{x}, \bar{b}_i)$ for all $i \in \mathbb{N}$. By Corollary 3.1.9, we have $\bar{\alpha}$ realizing $\varphi'(\bar{z}, \bar{b}_i)$ and $g(\bar{\alpha}) = \bar{a}$. Then the sequence $(\bar{b}_i)_{i \in \mathbb{N}}$ does not witness dividing of the formula $\varphi'(\bar{z},\bar{b})$. According to the witnessing property of strict Morley sequences — if $\varphi'(\bar{z},\bar{b})$ divides over M then the strict Morley sequence $(b_i)_{i\in\mathbb{N}}$ would witness dividing [Sim15b], therefore $\varphi'(\bar{z},\bar{b})$ does not divide over M. □

We would like to have that $\mathrm{tp}_{L_P}(\bar{\alpha}/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$ does not divide over M. But if it is not the case then we can rechoose $\bar{\alpha}$ to have this property. The first part of the proof is similar to the one for the realization $\bar{a}$, whereas in the second part the choice of $\bar{\alpha}$ is done differently as $\bar{\alpha}$ must satisfy more properties.

Lemma 3.1.11. *We can choose $\bar{\alpha}'$ such that $|\bar{\alpha}| = |\bar{\alpha}'|$, $\mathrm{tp}_{L_P}(\bar{\alpha}/M \cup \{\bar{b}\}) = \mathrm{tp}_{L_P}(\bar{\alpha}'/M \cup \{\bar{b}\})$ and $\mathrm{tp}_{L_P}(\bar{\alpha}'/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$ does not divide over M. Moreover $\bar{\alpha}' \in Z$ corresponds to some $\bar{a}' = g(\bar{\alpha}')$, where $\models \varphi(\bar{a}',\bar{b})$ and $\mathrm{tp}_{L_P}(\bar{a}'/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$ does not divide over M.*

Proof. This is clear when g is injective. In fact, we do not need to make any choice as $\bar{\alpha}$ itself satisfies the property.

If g is not injective, a suitable tuple $\bar{\alpha}'$ must be chosen so that the type $\mathrm{tp}_{L_P}(\bar{\alpha}'/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$ does not divide over M.

Assume g is not injective. We use the same idea as in Lemma 3.1.7, where $\varphi'(\bar{z},\bar{b})$ takes the role of $\varphi(\bar{x},\bar{b})$ and $\bar{\alpha}$ takes the place of $\bar{a}$. We let $q(x)$ be a complete extension type over N of $\mathrm{tp}_{L_P}(\bar{\alpha}/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$, which does not fork over M. Then for a realization $\bar{\alpha}''$ of $q(x)$, we can choose some tuple $\bar{\alpha}'$ from N such that $\bar{\alpha}''$ and $\bar{\alpha}'$ have the same type over $M \cup \{\bar{b}_i : i \in \mathbb{N}\}$ and with the property that $\mathrm{tp}_{L_P}(\bar{\alpha}'/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$ does not divide over M. We may choose $\bar{\alpha}'$ such that for some $\bar{a}' \in N^n$, we have $g(\bar{\alpha}') = \bar{a}'$ and $\mathrm{tp}_{L_P}(\bar{a}'/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$ does not divide over M. Such choice can be done using saturation. □

From now on, we may assume that $\bar{\alpha} = (\alpha_1, \cdots, \alpha_j)$ satisfies the following property: $\alpha_1, \cdots, \alpha_l$ are acl_L-independent over $M \cup P(N)$ and $\alpha_{l+1}, \cdots, \alpha_j \in P(N)$ are acl_L-independent over M. Moreover $\bar{\alpha}$ realizes $\varphi'(\bar{z},\bar{b}_i)$ for any $i \in \mathbb{N}$ and $\mathrm{tp}_{L_P}(\bar{\alpha}/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$ does not divide over M.

We give a well known fact about dimension with its proof.

Lemma 3.1.12. *Let $\bar{b}$ be an acl_L-independent tuple over M such that $\dim(\bar{b}/M) = |b| = m$. Let α_1 be acl_L-independent over M where $\dim(\alpha_1/M) = |\alpha_1| = 1$. Assume that α_1 reduces the dimension of $\bar{b}$ over M, then*

$$\dim(\bar{b}/M\alpha_1) = m - 1.$$

Proof. We use the following property of dimension: $\dim(ab/M) = \dim(a/M) + \dim(b/Ma)$, to get

$$\dim(\bar{b}\alpha_1/M) \quad = \dim(\bar{b}/M) + \dim(\alpha_1/M\bar{b}). \tag{3.1.6}$$

By the exchange property, $\bar{b}$ reduces the dimension of α_1. Since $\dim(\alpha_1/M) = 1$, then $\dim(\alpha_1/M\bar{b}) = 0$. It implies $\dim(\bar{b}\alpha_1/M) = \dim(\bar{b}/M) = m$.

Besides,

$$\dim(\bar{b}\alpha_1/M) \quad = \dim(\alpha_1/M) + \dim(\bar{b}/M\alpha_1). \tag{3.1.7}$$

We bring Equations (3.1.6) and (3.1.7) together to yield

$$\dim(\alpha_1/M) + \dim(\bar{b}/M\alpha_1) = m.$$

Therefore, $\dim(\bar{b}/M\alpha_1) = m - \dim(\alpha_1/M) = m - 1$. □

It should be underlined that this property is true for both acl_L-dimension and scl-dimension. Generally, if $\bar{\alpha}$ reduces the dimension of $\bar{b}$ over M, then the dimension of $\bar{b}$ over $M\bar{\alpha}$ can only be reduced by at most the dimension of $\bar{\alpha}$ over M.

We now prove that the tuple $\bar{\alpha}$ satisfies the following property.

Claim 3.1.13. *For all $i \in \mathbb{N}$, $\alpha_1, \cdots, \alpha_l$ are acl_L-independent over $M \cup P(N) \cup \{\bar{b}_i : i \in \mathbb{N}\}$ and $\alpha_{l+1}, \cdots, \alpha_j \in P(N)$ are acl_L-independent over $M \cup \{\bar{b}_i : i \in \mathbb{N}\}$.*

Proof. Recall that $\{\bar{b}_i : i \in \mathbb{N}\}$ is a strict Morley sequence. So for any tuples $\bar{b}_{i_1}, \cdots, \bar{b}_{i_m}$ and $\bar{b}_{j_1}, \cdots, \bar{b}_{j_m}$ from $\{\bar{b}_i : i \in \mathbb{N}\}$ such that $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_m$, we have

$$\mathrm{tp}_{L_P}(\bar{b}_{i_1}, \cdots, \bar{b}_{i_m}/M) = \mathrm{tp}_{L_P}(\bar{b}_{j_1}, \cdots, \bar{b}_{j_m}/M).$$

We use a dimension version of the exchange property in the next step and split the proof into two cases: first by considering the scl-dimension and second by using acl_L-dimension.

scl **part**

Recall that $\bar{b} = (b_1, \cdots, b_m)$ with $b_1, \cdots, b_k$ being acl_L-independent over $M \cup P(N)$ and $b_{k+1}, \cdots, b_m \in P(N)$ being acl_L-independent over M. Note that $\dim_{\mathrm{scl}}(\bar{b}/M) = k$.

Assume α_1 is not acl_L-independent over $M \cup P(N) \cup \{\bar{b}_i : i \in \mathbb{N}\}$. Then we let r be smallest integer such that α_1 is acl_L-independent over $M \cup P(N) \cup \{\bar{b}_1, \cdots, \bar{b}_{r-1}\}$ but not acl_L-independent over $M \cup P(N) \cup \{\bar{b}_1, \cdots, \bar{b}_{r-1}, \bar{b}_r\}$. In fact

$$\dim_{\mathrm{scl}}(\bar{b}_1, \cdots, \bar{b}_{r-1}/M\alpha_1) = k \times (r-1)$$

and by the exchange property

$$\dim_{\mathrm{scl}}(\bar{b}_1, \cdots, \bar{b}_{r-1}, \bar{b}_r/M\alpha_1) < k \times r.$$

By Lemma 3.1.12, we have

$$\dim_{\mathrm{scl}}(\bar{b}_1, \cdots, \bar{b}_{r-1}, \bar{b}_r/M\alpha_1) = k \times r - 1.$$

Let us introduce different naming to shorten the notation. Let

$$\begin{aligned}
\bar{b}''_1 &= \bar{b}_1, \cdots, \bar{b}_r; \\
\bar{b}''_2 &= \bar{b}_{r+1}, \cdots, \bar{b}_{2r}; \\
&\cdots \\
\bar{b}''_n &= \bar{b}_{(n-1)r+1}, \cdots, \bar{b}_{nr}.
\end{aligned}$$

By M-invariance, $\dim_{\mathrm{scl}}(\bar{b}''_i/M\alpha_1) = k \times r - 1$ for any $i \in \mathbb{N}$. Then we have the following,

$$\begin{aligned}
\dim_{\mathrm{scl}}(\bar{b}''_1, \cdots, \bar{b}''_n/M\alpha_1) &= \dim_{\mathrm{scl}}(\bar{b}''_1, \cdots, \bar{b}''_{n-1}/M\alpha_1) \\
&\quad + \dim_{\mathrm{scl}}(\bar{b}''_n/M\bar{b}''_1, \cdots, \bar{b}''_{n-1}\alpha_1) \\
&= \dim_{\mathrm{scl}}(\bar{b}''_1, \cdots, \bar{b}''_{n-2}/M\alpha_1) \\
&\quad + \dim_{\mathrm{scl}}(\bar{b}''_{n-1}/M\bar{b}''_1, \cdots, \bar{b}''_{n-2}\alpha_1)
\end{aligned}$$

$$
\begin{aligned}
&+ \dim_{\mathrm{scl}}(\bar{b}''_n / M\bar{b}''_1, \cdots, \bar{b}''_{n-1}\alpha_1) \\
&= \dim_{\mathrm{scl}}(\bar{b}''_1 / M\alpha_1) + \dim_{\mathrm{scl}}(\bar{b}''_2 / M\bar{b}''_1\alpha_1) + \cdots \\
&+ \dim_{\mathrm{scl}}(\bar{b}''_n / M\bar{b}''_1, \cdots, \bar{b}''_{n-1}\alpha_1).
\end{aligned}
$$

We compute the left and the right sides of the equation to compare them later. On the left side, we use Lemma 3.1.12 to obtain that

$$\dim_{\mathrm{scl}}(\bar{b}''_1, \cdots, \bar{b}''_n / M\alpha_1) = k \times r \times n - 1. \tag{3.1.8}$$

On the right side, we sum n terms and apply Lemma 3.1.12 for each term to get

$$\dim_{\mathrm{scl}}(\bar{b}''_1, \cdots, \bar{b}''_n / M\alpha_1) = (k \times r - 1) \times n. \tag{3.1.9}$$

By equalizing Equations (3.1.8) and (3.1.9), we then have $k \times r \times n - 1 = (k \times r - 1) \times n$, which is not possible when n is large enough, more precisely when $n \geq 2$. It implies that α_1 is acl_L-independent over $M \cup P(N) \cup \{\bar{b}_1, \cdots, \bar{b}_{r-1}, \bar{b}_r\}$.

In the next step, we include α_1 in the set of parameters. We assume for a contradiction that α_2 is not acl_L-independent over the set $M \cup P(N) \cup \{\bar{b}_1, \cdots, \bar{b}_{r-1}, \bar{b}_r, \alpha_1\}$. By the exchange property α_2 lowers the scl-dimension of $\bar{b}_1, \cdots, \bar{b}_{r-1}, \bar{b}_r$, *i.e.*

$$\dim_{\mathrm{scl}}(\bar{b}_1, \cdots, \bar{b}_{r-1}, \bar{b}_r / M\alpha_1\alpha_2) = k \times r - 1.$$

Note that the α_i's are independent of each others so when using the same notation as before and by M-invariance, we get

$$\dim_{\mathrm{scl}}(\bar{b}''_i / M\alpha_1\alpha_2) = k \times r - 1.$$

By computing $\dim_{\mathrm{scl}}(\bar{b}''_1, \cdots, \bar{b}''_n / M\alpha_1\alpha_2)$, we get a contradiction as we did for α_1.

By continuing with the same argument we reach α_l, and end up with α_l being acl_L-independent over $M \cup P(N) \cup \{\bar{b}_i : i \in \mathbb{N}\}$.

acl_L part

Recall that $\bar{b} = (b_1, \cdots, b_m)$ with $b_1, \cdots, b_k$ being acl_L-independent over $M \cup P(N)$ and $b_{k+1}, \cdots, b_m \in P(N)$ being acl_L-independent over M. These sequences were chosen in a very independent way, and when combined they are still acl_L-independent over M, *i.e.* $\dim_{\mathrm{acl}_L}(\bar{b}/M) = m$.

We already know that $\alpha_{l+1}, \cdots, \alpha_j$ are in $P(N)$ and are acl_L-independent over M. But we want them to be acl_L-independent over $M \cup \{b_i : i \in \mathbb{N}\}$. Clearly $\alpha_{l+1}, \cdots, \alpha_j$ are acl_L-independent over $M\alpha_1, \cdots, \alpha_l$ since they were chosen in such an independent way.

Assume for a contradiction that α_{l+1} is not acl_L-independent over the set $M\bar{b}_1 \cdots \bar{b}_r\alpha_1 \cdots \alpha_l$. Then by the exchange property, α_{l+1} reduces the acl_L-dimension of $\bar{b}_1 \cdots \bar{b}_r$ over $M\alpha_{l+1}$, and by Lemma 3.1.12 it can only be reduced by 1, *i.e.*

$$\dim_{\mathrm{acl}_L}(\bar{b}_1, \cdots, \bar{b}_r/M\alpha_1, \cdots, \alpha_l\alpha_{l+1}) = m \times r - 1.$$

Note that

$$\dim_{\mathrm{acl}_L}(\bar{b}_1, \cdots, \bar{b}_r/M\alpha_1, \cdots, \alpha_l) = m \times r.$$

Keeping previous notation used in the scl part and applying M-invariance to the $\bar{b}_i''$'s, we do the same argument as before but the difference lies in the set of parameters where in the present case, $\alpha_1, \cdots, \alpha_l$ is included in the set of parameters. There will be a contradiction when we compute $\dim_{\mathrm{acl}_L}(\bar{b}_1'', \cdots, \bar{b}_n''/M\alpha_1, \cdots, \alpha_l\alpha_{l+1})$. Therefore α_{l+1} is acl_L-independent over $M\bar{b}_1 \cdots \bar{b}_r\alpha_1, \cdots, \alpha_l$.

While showing that α_{l+2} is acl_L-independent over $M\bar{b}_1 \cdots \bar{b}_r\alpha_1, \cdots, \alpha_l$, we also take this opportunity to prove that it is acl_L-independent over the set $M\bar{b}_1 \cdots \bar{b}_r\alpha_1, \cdots, \alpha_l, \alpha_{l+1}$, *i.e.* we add α_{l+1} to the parameters. We proceed by contradiction. Assume α_{l+2} is not acl_L-independent over $M\bar{b}_1 \cdots \bar{b}_r\alpha_1, \cdots, \alpha_l, \alpha_{l+1}$. By the exchange property α_{l+2} reduces the acl_L-dimension of $\bar{b}_1 \cdots \bar{b}_r$ over M (or $M\alpha_1, \cdots, \alpha_l, \alpha_{l+1}$ since the α_i's are independent of each other). Based on Lemma 3.1.12, we have

$$\dim_{\mathrm{acl}_L}(\bar{b}_1, \cdots, \bar{b}_r/M\alpha_1, \cdots, \alpha_l\alpha_{l+1}\alpha_{l+2}) = m \times r - 1.$$

Using M-invariance a contradiction arises when we compute the dimension $\dim_{\mathrm{acl}_L}(\bar{b}_1'', \cdots, \bar{b}_n''/M\alpha_1, \cdots, \alpha_l\alpha_{l+1}\alpha_{l+2})$. Therefore α_{l+2} is acl_L-independent over $M\bar{b}_1 \cdots \bar{b}_r\alpha_1, \cdots, \alpha_l, \alpha_{l+1}$.

We repeat the same process until we reach α_j where $\alpha_1, \cdots, \alpha_{j-1}$ become parameters. We then get α_j being acl_L-independent over $M\bar{b}_1 \cdots \bar{b}_r\alpha_1, \cdots, \alpha_l$, $\cdots \alpha_{j-1}$. This concludes the proof of Claim 3.1.13. □

Before we go further, let us mention a result about lovely pairs derived from Lemma 1.5.4 which turns out to be a key point in the later proof.

Corollary 3.1.14. *The type $\mathrm{tp}_{L_P}(\bar{b}/M)$ is implied by $\mathrm{tp}_L(\bar{b}/M)$ together with the information that $\bar{b} = (b_1, \cdots, b_m)$ with $b_1, \cdots, b_k$ being acl_L-independent over $M \cup P(N)$ and $b_{k+1}, \cdots, b_m \in P(N)$ being acl_L-independent over M.*

A similar result was first done in 2003 by I. Ben-Yaacov, A. Pillay and E. Vassiliev about simple theories in [BYPV03]. A. Berenstein and E. Vassiliev [BV10] generalize the result for lovely pairs of geometric structures.

A case of Corollary 3.1.14 is used by L. v.d.Dries in [Dri98a], his paper about dense pairs of o-minimal structures, to define an L_P-type.

Remark 3.1.15. *For tuples, being acl_L-independent over $M \cup P(N)$ implies P-independent in the sense of Definition 1.5.3. The same applies for tuples in $P(N)$ which are being acl_L-independent over M. Moreover, if we have a tuple composed only with elements that are acl_L-independent over $M \cup P(N)$ and elements that are from $P(N)$ and acl_L-independent over M, then the tuple is P-independent.*

Now we proceed with another step in the proof of the main Theorem 3.0.1.

From the L_P-formula $\varphi'(\bar{z}, \bar{b})$, we construct a new L-formula $\varphi''(\bar{z}, \bar{b})$, which agrees with $\varphi'(\bar{z}, \bar{b})$ for certain tuples. We then prove that the L-formula $\varphi''(\bar{z}, \bar{b})$ does not divide over M.

Claim 3.1.16. *There is an L-formula $\varphi''(\bar{z}, \bar{y})$ over M such that, for all $\bar{\alpha}' = (\alpha'_1, \cdots, \alpha'_j)$ and $\bar{b}' = b'_1, \cdots, b'_m$ with $\alpha'_1, \cdots, \alpha'_l b'_1, \cdots, b'_k$ being acl_L-independent over $M \cup P(N)$ and $\alpha'_{l+1}, \cdots, \alpha'_j b'_{k+1}, \cdots, b'_m$ in $P(N)$ being acl_L-independent over $M \cup \bar{b}_i$, $(N, P(N)) \models \varphi'(\bar{\alpha}', \bar{b}'_i)$ if and only if $(N, P(N)) \models \varphi''(\bar{\alpha}', \bar{b}'_i)$.*

Proof. We know that $\bar{\alpha}\bar{b}$ is acl_L-independent over M. By Corollary 3.1.14, $\mathrm{tp}_{L_P}(\bar{\alpha}\bar{b}/M)$ is determined by $\mathrm{tp}_L(\bar{\alpha}\bar{b}/M)$ in conjunction with the information that $\bar{\alpha}\bar{b}$ is P-independent together with the fact that $\bar{\alpha}$ and $\bar{b}$ are composed by elements which are scl-independent over M and elements from $P(N)$ which are acl_L-independent over M.

Let $q(\bar{z}\bar{y})$ be a partial L_P-type which expresses the independence nature of the tuple $\bar{\alpha}\bar{b}$. We know that $q(\bar{z}\bar{y})$ is contained in $\mathrm{tp}_{L_P}(\bar{\alpha}\bar{b}/M)$. Then by the compactness theorem we shall obtain an L-formula $\varphi''(\bar{z},\bar{y})$ such that $\varphi''(\bar{z},\bar{y}) \wedge q(\bar{z}\bar{y})$ implies $\varphi'(\bar{z},\bar{y})$ and clearly $\varphi''(\bar{z},\bar{y}) \wedge q(\bar{z}\bar{y})$ implies $\varphi'(\bar{z},\bar{y}) \wedge q(\bar{z}\bar{y})$.

For each realization $\bar{\alpha}_i''\bar{b}_i''$ of $\varphi'(\bar{z},\bar{y}) \wedge q(\bar{z}\bar{y})$, we can choose an L-formula $\varphi_i''(\bar{z},\bar{y})$ such that $\varphi_i''(\bar{z},\bar{y}) \wedge q(\bar{z}\bar{y})$ implies $\varphi'(\bar{z},\bar{y}) \wedge q(\bar{z}\bar{y})$. Since we have infinitely many realizations of $\varphi'(\bar{z},\bar{y}) \wedge q(\bar{z}\bar{y})$, then we also have infinitely many L-formulas $\varphi_i''(\bar{z},\bar{y})$. Thus $\varphi'(\bar{z},\bar{y}) \wedge q(\bar{z}\bar{y})$ is implied by the conjunction of the formulas $\varphi_i''(\bar{z},\bar{y}) \wedge q(\bar{z}\bar{y})$, and by the compactness theorem we get an L-formula $\varphi''(\bar{z},\bar{y})$ such that $\forall\bar{z}\bar{y}(\varphi'(\bar{z},\bar{y}) \wedge q(\bar{z}\bar{y}) \longrightarrow \varphi''(\bar{z},\bar{y}) \wedge q(\bar{z}\bar{y}))$.

We then get the desired equivalence. □

Recall that $\bar{\alpha} = (\alpha_1, \cdots, \alpha_j)$ is such that $\alpha_1, \cdots, \alpha_l$ are acl_L-independent over $M \cup P(N) \cup \{\bar{b}_i : i \in \mathbb{N}\}$ and $\alpha_{l+1}, \cdots, \alpha_j \in P(N)$ are acl_L-independent over $M \cup \{\bar{b}_i : i \in \mathbb{N}\}$. Furthermore, $\bar{\alpha}$ realizes $\varphi'(\bar{z},\bar{b}_i)$ for any $i \in \mathbb{N}$ and $\mathrm{tp}_{L_P}(\bar{\alpha}/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$ does not divide over M.

Let $\varphi''(\bar{z},\bar{y})$ be the L-formula as in Claim 3.1.16.

Lemma 3.1.17. *The formula $\varphi''(\bar{z},\bar{b})$ does not divide over M.*

Proof. The proof of this Lemma is similar to the proof used for Lemma 3.1.10. We know that $\varphi'(\bar{z},\bar{b})$ does not divide over M. Consider $(\bar{b}_i)_{i\in\mathbb{N}}$, the strict Morley sequence from $\mathrm{tp}_{L_P}(\bar{b}/M)$ witnessing the non-dividing of $\varphi'(\bar{z},\bar{b})$. We have $\bar{\alpha}$ realizing $\varphi'(\bar{z},\bar{b}_i)$ for any $i \in \mathbb{N}$, such that $\alpha_1, \cdots, \alpha_l$ are acl_L-independent over $M \cup P(N) \cup \{\bar{b}_i : i \in \mathbb{N}\}$ and $\alpha_{l+1}, \cdots, \alpha_j \in P(N)$ are acl_L-independent over $M \cup \{\bar{b}_i : i \in \mathbb{N}\}$.

Based on Claim 3.1.16, $\bar{\alpha}$ realizes $\varphi''(\bar{z},\bar{b}_i)$ for any $i \in \mathbb{N}$ and by Proposition 3.1.6, $(\bar{b}_i)_{i\in\mathbb{N}}$ is a strict Morley sequence in the sense of L. Then by the witnessing property of strict Morley sequences, $\varphi''(\bar{z}, \bar{b})$ does not divide over M. $\square$

3.2 The definable (p, q)-theorem

In this section, we prove the main theorem stated at the beginning of this chapter by making use of the results provided in the previous section. From this point forward, most of the properties are taking the form of the definable (p, q)-conjecture.

Recall $\bar{b} = (b_1, \cdots, b_m)$ is an acl_L-independent tuple over M. Moreover, $b_1, \cdots, b_k$ are acl_L-independent over $M \cup P(N)$; $b_{k+1}, \cdots, b_m$ are elements of $P(N)$ and are acl_L-independent over M. The formula $\varphi''(\bar{z}, \bar{b})$ is as in Claim 3.1.16. We make use of results from Chapter 2 to get a stronger version of the definable (p, q)-theorem for $\varphi''(\bar{z}, \bar{b})$.

Lemma 3.2.1. *There is an L-formula $\psi'(\bar{y}) \in \mathrm{tp}_L(\bar{b}/M)$ such that for any finite $\bar{b}'_1, \cdots, \bar{b}'_k \in \psi'(M)$ we have*

$$\dim_{\mathrm{acl}_L}\left(\bigwedge_{i=1}^{k} \varphi''(\bar{z}, \bar{b}'_i)\right) = |\bar{z}|.$$

Proof. By Lemma 3.1.17, $\varphi''(\bar{z}, \bar{b})$ does not divide over M in the sense of L. Moreover we already have $\bar{\alpha}$, a realisation of $\varphi''(\bar{z}, \bar{b})$ which is acl_L-independent over $M\bar{b}$. Then the acl_L-dimension of $\varphi''(\bar{z}, \bar{b})$ over M is the length of $\bar{z}$, *i.e.*

$$\dim_{\mathrm{acl}_L}\left(\varphi''(\bar{z}, \bar{b})\right) = |\bar{z}|.$$

We also have a strict Morley sequence $(\bar{b}_i)_{i \in \mathbb{N}}$ such that for any $t \in \mathbb{N}$, $\dim_{\mathrm{acl}_L}\left(\bigwedge_{i=1}^{t} \varphi''(\bar{z}, \bar{b}_i)\right) = |\bar{z}|$. Moreover, we have $\bar{\alpha}$ with $\dim_{\mathrm{acl}_L}(\bar{\alpha}/M) = l$ and such that $\forall i \in \mathbb{N} \models \varphi''(\bar{\alpha}, \bar{b}_i)$ and $\mathrm{tp}(\bar{a}/M \cup \{\bar{b}_i : i \in \mathbb{N}\})$ does not divide over M. All of the assumptions of Theorem 2.2.6 are satisfied. Therefore by Theorem 2.2.6 of Chapter 2, there exists an L-formula $\psi'(\bar{y}) \in \mathrm{tp}_L(\bar{b}/M)$ such that for any $s \in \mathbb{N}$ if $\bar{b}'_1, \bar{b}'_2, \cdots, \bar{b}'_s \in \psi'(M)$ then

$$\dim_{\mathrm{acl}_L}\left(\bigwedge_{i=1}^{s} \varphi''(\bar{z}, \bar{b}'_i)\right) = |\bar{z}|.$$

$\square$

We now prove that the definable (p,q)-conjecture holds for the L_P-formula $\varphi'(\bar{z},\bar{y})$. We use this result to prove the main theorem later in this section.

Proposition 3.2.2. *There is an L_P-formula $\theta'(y) \in \mathrm{tp}_{L_P}(\bar{b}/M)$ such that $\{\varphi'(\bar{z},\bar{b}') : \bar{b}' \in \theta'(N)\}$ is finitely consistent.*

Proof. By Lemma 3.2.1, we have an L-formula $\psi'(\bar{y}) \in \mathrm{tp}_L(\bar{b}/M)$ such that for any finite set of elements of $\psi'(M)$, $\dim_{\mathrm{acl}_L}\left(\bigwedge_{i=1}^{k}\varphi''(\bar{z},\bar{b}'_i)\right) = |\bar{z}|$. Then the set $\{\varphi''(\bar{z},\bar{b}') : \bar{b}' \in \psi'(N)\}$ is finitely consistent.

Recall $q(\bar{z}\bar{y})$ to be the partial type defined in the proof of Claim 3.1.16. Let $p(\bar{y})$ be a partial type that covers the part of $q(\bar{z}\bar{y})$ which does not involve the variable $\bar{z}$, *i.e.* the set of all formulas in $q(\bar{z}\bar{y})$ which do not involve $\bar{z}$. Clearly $p(\bar{y})$ is an L_P-type. By the compactness theorem, there is a formula $\gamma(\bar{y}) \in p(\bar{y})$ such that for all $\bar{b}' \in \gamma(M)$ and $\bar{\alpha}'$, if $\alpha'_1,\cdots,\alpha'_l$ are acl_L-independent over $M \cup P(N) \cup \{\bar{b}'\}$ and $\alpha'_{l+1},\cdots,\alpha'_j \in P(N)$ are acl_L-independent over $M \cup \{\bar{b}'\}$, then $\models \varphi'(\bar{\alpha}',\bar{b}')$ if and only if $\models \varphi''(\bar{\alpha}',\bar{b}')$.

Let $\theta'(\bar{y}) = \psi'(\bar{y}) \wedge \gamma(\bar{y})$, then $\{\varphi''(\bar{z},\bar{b}') : \bar{b}' \in \theta'(N)\}$ is finitely consistent. It should be noted that $\theta'(N)$ is non-empty since $\bar{b}$ is one of its element and by construction $\theta'(\bar{y})$ is an L_P-formula. Recall that for any tuples $\bar{b}'_1,\cdots,\bar{b}'_s \in \theta'(N)$, $\dim_{\mathrm{acl}_L}\left(\bigwedge_{i=1}^{s}\varphi''(N,\bar{b}'_i)\right) = |\bar{z}|$.

Therefore we have $\bar{\alpha}'$ which is acl_L-independent over $M\bar{b}'_1\cdots\bar{b}'_s$ such that $\bar{\alpha}' \models \bigwedge_{i=1}^{s}\varphi''(\bar{z},\bar{b}'_i)$. Moreover, $\bar{\alpha}' = (\alpha'_1,\cdots,\alpha'_j)$ can be chosen such that $\alpha'_1,\cdots,\alpha'_l$ are acl_L-independent over $M \cup P(N) \cup \{\bar{b}'_1\cdots\bar{b}'_s\}$ and $\alpha'_{l+1},\cdots,\alpha'_j \in P(N)$ are acl_L-independent over $M \cup \{\bar{b}'_1\cdots\bar{b}'_s\}$. Since $\theta'(N) \subseteq \gamma(N)$, then $\models \varphi'(\bar{\alpha}',\bar{b}'_i)$ for any $i = 1,\cdots,s$. Therefore $\{\varphi'(\bar{z},\bar{b}') : \bar{b}' \in \theta'(N)\}$ is finitely consistent. □

We now state the main theorem of this chapter again, which is followed by its proof.

Theorem 3.2.3. *Let $(M,P(M)) \prec (N,P(N))$ be two dense pairs of geometric structures, in which the geometric structures are distal and have the property that algebraic closure is equal to definable closure. Let $\varphi(\bar{x},\bar{y})$ be an L_P-formula. Let $\bar{b} \in N^m$ and suppose that $\varphi(\bar{x},\bar{b})$ does not divide over M. Then there is an L_P-formula $\theta(\bar{y}) \in \mathrm{tp}_{L_P}(\bar{b}/M)$ such that $\{\varphi(\bar{x},\bar{b}') : \bar{b}' \in \theta(N)\}$ is finitely consistent.*

Proof. For any $\bar{b} \in N^m$, we can choose $\bar{b}' = b'_1, \cdots, b'_k, b'_{k+1}, \cdots, b'_m$ and a definable function f over M such that $f(\bar{b}') = \bar{b}$; $b'_1, \cdots, b'_k$ are acl_L-independent over $M \cup P(N)$ and taken from the tuple $\bar{b}$, and $b'_{k+1}, \cdots, b'_m$ are from $P(N)$ and are acl_L-independent over M. In Lemma 3.1.1, we show that in order to prove that the definable (p,q)-conjecture holds for $\varphi(\bar{x}, \bar{b})$, it is sufficient to prove that the definable (p,q)-conjecture holds for $\varphi'''(\bar{x}, \bar{b}')$ where $\varphi'''(\bar{x}, \bar{y}') \equiv \exists \bar{y}(\varphi(\bar{x}, \bar{y}) \wedge f(\bar{y}') = \bar{y})$.

When proving the definable (p,q)-conjecture for $\varphi'''(\bar{x}, \bar{b}')$, defined in Section 3.1, we managed to reduce it to the case where for some realization $\bar{a}$ of $\varphi'''(\bar{x}, \bar{b}')$ we obtain a tuple $\bar{\alpha} = \alpha_1, \cdots \alpha_l, \alpha_{l+1}, \cdots \alpha_j$ and a definable function g over M such that $g(\bar{\alpha}) = \bar{a}$ and $\mathrm{tp}(\bar{\alpha}/M \cup \bar{b}_i : i \in \mathbb{N})$ does not divide over M. It should also be noted that $\alpha_1, \cdots, \alpha_l$ are acl_L-independent over $M \cup P(N) \cup \{\bar{b}_i : i \in \mathbb{N}\}$ and $\alpha_{l+1}, \cdots, \alpha_j \in P(N)$ are acl_L-independent over $M \cup \{\bar{b}_i : i \in \mathbb{N}\}$. For such choice of g, we get a new L_P-formula $\varphi'(\bar{z}, \bar{y}')$ which is equivalent to $\exists \bar{x}(\varphi'''(\bar{x}, \bar{y}') \wedge g(\bar{z}) = \bar{x})$.

We also proved in Proposition 3.2.2 that there exists an L_P-formula $\theta'(\bar{y}) \in \mathrm{tp}_{L_P}(\bar{b}'/M)$ such that $\{\varphi'(\bar{z}, \bar{b}') : \bar{b}' \in \theta'(N)\}$ is finitely consistent. Then $\{\exists \bar{x}(\varphi'''(\bar{x}, \bar{b}') \wedge g(\bar{z}) = \bar{x}) : \bar{b}' \in \theta'(N)\}$ is finitely consistent. Since g is L-definable then $\{\varphi'''(\bar{x}, \bar{b}') : \bar{b}' \in \theta'(N)\}$ is finitely consistent, therefore the definable (p,q)-conjecture holds for $\varphi'''(\bar{x}, \bar{b}')$.

Finally, by assumption $\varphi(\bar{x}, \bar{b})$ does not divide over M and Lemma 3.1.1 indicates that there is an L_P-formula $\theta(\bar{y}) \in \mathrm{tp}_{L_P}(\bar{b}/M)$, where $\theta(\bar{y}) = \exists \bar{y}'(\theta'(\bar{y}') \wedge f(\bar{y}') = \bar{y})$, such that $\{\varphi(\bar{x}, \bar{b}'') : \bar{b}'' \in \theta(N)\}$ is finitely consistent. $\square$

We are also interested in the definable (p,q)-conjecture for dense pairs of distal geometric structures where algebraic closure is not the same as the definable closure. This is considered in a future work.

Chapter 4

The definable (p, q)-theorem for reducts of distal theories

In this chapter, we consider to prove the definable (p, q)-conjecture for reducts of distal theories. More precisely we are trying to prove that the definable (p, q)-conjecture holds for dense pairs of real closed ordered fields by considering closed ordered differential fields. This chapter is very independent and the main result is a particular case of the main result presented in Chapter 3. Although, a different and interesting approach is introduced here to prove the result, which might be generalised to prove that the definable (p, q)-conjecture holds for any reduct of a distal theory.

Let T be a complete distal NIP theory, with a language L which we assume to be one-sorted. Let $K \models T$. Let K' be the reduct of K with respect to some language $L' \subseteq L$. Let $T' = Th(K')$. Let $M \prec N \models T'$ with N assumed to be $|M|^+$-saturated. We aim to show that the definable (p, q)-conjecture holds for some case of non-dividing L'-formula over M in the sense of L'.

4.1 Theory of reducts

This section contains notions of reducts of structures. Definitions and properties that relate original structures with their reducts are provided. This section is essential as it provides a clear idea of the steps to follow toward the goal of this chapter.

Definition 4.1.1. *Let L be a one-sorted language and T a complete L-theory. Let*

$M \models T$ and $L' \subset L$. We say that M' is an L'-reduct of M if M and M' have the same underlying set but M' is an L'-structure in which the interpretation of all symbols in M' agrees with the interpretation in M restricted to L'.

The following example illustrates the above definition. It is presented in [Kun07].

Example 4.1.2. *Let L be the language $\{0, 1, +, \times, <\}$ and consider the L-structure $\mathscr{R} = \{\mathbb{R}, 0, 1, +, \times, <\}$. $\mathscr{R}$ is known to be a real closed ordered field. Now let $L' = \{0, +\}$ and $\mathscr{R}' = \{\mathbb{R}, 0, +\}$. We have that $\mathscr{R}'$ is an L'-reduct of $\mathscr{R}$. In fact, $\mathscr{R}'$ and $\mathscr{R}$ share the same underlying set (which is $\mathbb{R}$) but $\mathscr{R}'$ is an L'-structure. The L'-structure $\mathscr{R}'$ is an abelian group.*

In fact, the notion of reducts consists of a reduction of the language to its a sublanguage. Note that this notion is similar to the idea of a forgetful functor in category theory. For instance, by neglecting the multiplication in a ring we obtain an Abelian group.

While reduction generally preserves tameness properties of the original structure, it has been proved by P. Simon in [Sim15a] that reduction does not always preserve distality. Example 4.1.2 gives evidence of this finding. In fact, real closed ordered fields are o-minimal and belong the class of distal theories, but abelian groups without the ordering are not distal. Interestingly, the ordering is very important to preserve distality. This gives us the opportunity to investigate if the definable (p, q)-conjecture is preserved under for some reduction of language.

Let us give some known properties that we use later in the proofs. For the rest of this section. L, L', T, T', K, K, M and N are as in the paragraph before Section 4.1.

Lemma 4.1.3. *The L'-structure M has an $|M|^+$-saturated elementary extension U' which is a reduct of a model of T.*

Proof. Let U be an $|M|^+$-saturated elementary extension of K and U' its reduct in the language $L' \subseteq L$. We are left to show that U' is also saturated, then use a very well-known result about saturation to embed M into it.

Let $p'(x)$ be an L'-type over $A \subseteq U'$, with $|A| < |M|^+$. By the definition of reducts, $p'(x)$ is a type over A in the sense of L. Therefore we can expand $p'(x)$ to a complete L-type $p(x)$ over A such that $p'(x) \subseteq p(x)$. Since U is saturated, then $p(x)$ is realized in U and so is $p'(x)$.

Due to the fact that $p'(x)$ is an arbitrary L'-type over A, and is realized in U', we conclude that U' is also $|M|^+$-saturated. Therefore, one can elementarily embed M into U'. □

From now on we assume the elementary extension N of M is a reduct of a model of T. The next proposition considers dividing in a structure and its reducts.

Proposition 4.1.4. *Let $\varphi(x, y)$ be an L'-formula and b a tuple in N. Suppose $\varphi(x, b)$ does not divide over M in the sense of L', then it does not divide over M in the sense of L.*

Proof. Clearly $\varphi(x, b)$ is an L-formula. We just need to verify that it does not divide over M in the sense of L. For a contradiction, assume $\varphi(x, b)$ divides over M in the sense of L. Then there is an indiscernible (over M) sequence $(b_i)_{i\in\mathbb{N}}$ such that $\{\varphi(x, b_i) : i = 1, 2, \dots\}$ is inconsistent.

In the language L', the sequence $(b_i)_{i\in\mathbb{N}}$ is still an indiscernible sequence. By the definition of dividing, $\varphi(\bar{x}, \bar{b})$ divides over M in the sense of L'. This fact contradicts the fact that $\varphi(x, b)$ does not divide over M in the sense of L'. Hence $\varphi(x, b)$ does not divide over M in the sense of L. □

4.2 Closed ordered differential fields

In this section we show properties of closed ordered differential fields. It is known that dense pairs arise as reducts of closed ordered differential fields. Here we give an explanation of this process. We also present a few properties.

The following definitions are essential in understanding the proofs presented later in this section.

Definition 4.2.1. *Let R be a ring. We call R a differential ring if it is equipped with a map ∂ from R to itself such that for any $a, b \in R$, the next following properties are satisfied:*

(1) $\partial(a + b) = \partial(a) + \partial(b)$.

(2) $\partial(a - b) = \partial(a) - \partial(b)$.

(3) $\partial(ab) = a\partial(b) + b\partial(a)$ *(Leibniz product rule).*

Moreover, if R is a field then it is a differential field.

Definition 4.2.2. *Let R, S be two differential rings. A homomorphism of differential rings from R to S is a homomorphism of rings $\sigma : R \longrightarrow S$ such that $\partial_S \circ \sigma = \sigma \circ \partial_R$.*

We recall the differential polynomial ring over R in n indeterminates to be the ring $R_\partial[X] = R[X_1, \partial X_1, \partial^2 X_1, \ldots, X_2, \partial X_2, \partial^2 X_2, \ldots, X_n, \partial X_n, \partial^2 X_n, \ldots]$, where $X = (X_1, \ldots, X_n)$, $\partial^0 X = X$ and $\partial^{n+1} = \partial(\partial^n)$.

Let $f \in R_\partial[X]$, we define the order of f to be the largest $n \in \mathbb{N}$ such that ∂^n appears in f. Given n, the order of f, we define the degree of f to be the highest power of any of $\partial^i X_j$ for $i = 0, 1, 2, \cdots, n$.

The following definition can be found in [Asl17], [MM14].

Definition 4.2.3. *A differential field R is said to be differentially closed if for any $f, g \in R_\partial[X]$ such that g is nonzero and has order less than the order of f, we can find a tuple a from R which verifies $f(a) = 0$ and $g(a) \neq 0$.*

An ordered differential field is a differential ring which is also an ordered field. This notion was introduced by A. Robinson [Rob73] in 1973. Five years later, M. Singer [Sin78] proved that ordered differential fields have a model completion. The model completion is known as the theory of closed ordered differential fields (CODF).

Let the language $L'' = \{+, -, \cdot, 0, 1, <, \partial\}$. We define the theory of CODF as follows.

Definition 4.2.4. *Let (R,∂) be an ordered differential field. We say that (R,∂) is a CODF if it satisfies the following two properties.*

(1) R is a real closed field in the language $\{+,-,\cdot,0,1,<\}$.

(2) For any $f \in R_\partial[X]$ where the order of f is n, and $g_1,\cdots,g_m \in R_\partial[X]$ where the order of g_i, $i=1,\cdots,m$, is less than or equal to n, by considering f and g_i $(i=1,\cdots,m)$ as polynomials of $(n+1)$-variables say $\hat{f}$ and $\hat{g}_i$, i.e. *$\hat{f}$ and $\hat{g}_i$ have $X,\partial X,\cdots,\partial^n X$ as variables, if there exists $\bar{c}=(c_0,c_1,\cdots,c_n)$ a solution to the system*

$$\hat{f}(\bar{c})=0\wedge\left(\frac{\partial\hat{f}}{\partial^n X}\right)(\bar{c})\neq 0\wedge\bigwedge_{i=1}^{m}\hat{g}_i(\bar{c})>0,$$

then there exists a solution to the differential polynomial system

$$f(x)=0\wedge\bigwedge_{i=1}^{m}g_i(x)>0.$$

As a consequence of Definition 4.2.4, the set of constants $C=\{a\in R:\partial(a)=0\}$ is a dense, codense real closed subfield of R, see [Asl17], [KP19].

Fact 4.2.5. *The theory of CODF is complete and has quantifier elimination, see [Sin78] for a detailed proof.*

As a matter of fact, those nice properties of CODF lead us to make steps towards the study of its reducts.

We attempted to address the question of whether the definable (p,q)-theorem holds for a reduct of a distal structure. We intended to prove it for a general reduct of any CODF. However, we found that the proofs require lots of assumptions and consequently, we only consider the conjecture for a particular reduct of CODF, which is the case of dense pairs of real-closed ordered fields. It is worth noting that we had to consider various assumptions in proving the latter case and were only able to complete the proof that the definable (p,q)-conjecture holds for that particular reduct in one case.

Since C is defined by a first-order L''-formula with no quantifiers and no additional parameters, we could include a predicate with no significant resulting change in the language and get $L = \{+, -, \cdot, 0, 1, <, \partial, P\}$ where $P(a)$ means $a \in C$. We can now use L as our language for closed ordered differential fields.

Given a CODF in the language L, its reduct to the language $L' = \{+, -, \cdot, 0, 1, <, P\}$, obtained by forgetting ∂, is a dense pair of real closed ordered fields, see [KP19].

It should be noted that the theory of CODF belongs to the class of distal theories. This result is due by P. Kovacsics and F. Point in [KP19] and we want to use it to prove that the definable (p, q)-conjecture holds for the theory of dense pairs of real closed ordered fields.

Let (M, ∂) be a closed ordered differential field. Let $(M, P(M))$ be the dense pair of real-closed ordered fields obtained from the CODF (M, ∂) where $P(M)$ is the set of all $a \in M$ such that $\partial(a) = 0$. By Fact 4.2.5, $(M, P(M), \partial)$ has quantifier elimination.

Let $(N, P(N))$ be an elementary extension of $(M, P(M))$, which is κ-saturated and strongly κ-homogeneous for some cardinal κ. By Lemma 4.1.3, $(N, P(N))$ can be chosen to be a reduct of a CODF.

Let b be a tuple from $P(N)$ and let $q(y) = \mathrm{tp}_{L'}(b/M)$. Let $\varphi(x, y)$ be an L'-formula and assume that $\varphi(x, b)$ does not divide over M in the sense of L'. Our goal is to show that the L'-type $q(y)$ has a unique extension to an L-type over M. To do so, we let c and d be two realizations of $q(y)$ in $P(N)$ and prove that c and d satisfy the same quantifier-free L-formulas over M.

Let $M[c]$ and $M[d]$ be the subrings of N generated by c and d over M, respectively. Let $M_\partial\langle c\rangle$ and $M_\partial\langle d\rangle$ be the sub-differential-rings of N generated by c and d over M, respectively. Recall $M_\partial[X]$ to be the differential polynomial ring over M and let $M_\partial\langle X\rangle$ be the free differential ring generated by X over M.

Lemma 4.2.6. $M_\partial\langle X\rangle = M_\partial[X]$.

Proof. It is clear that $M_\partial[X] \subseteq M_\partial\langle X\rangle$.

Let $f(X) \in M_\partial\langle X\rangle$. Then there are some $a_1, \cdots, a_n$ from M such that

$f(X)$ is formed from a combination of $a_1, \cdots, a_n$ and X together with the operations $+, -, \cdot$ and ∂. By Definition 4.2.1, the derivative of products, sums and subtractions can be written in the form of sums, products and subtractions of derivatives. Moreover, we can group all the terms using the family $\{(\partial X)^{i_1}(\partial^2 X)^{i_2} \cdots (\partial^n X)^{i_n}\}$ as a basis for some $n \in \mathbb{N}$. Since M is closed under the aforementioned operations, the derivatives of sums, products and subtractions of the a_i's are in M, which implies that the coefficients are from M. Therefore $f(X) \in M_\partial[X]$. $\square$

Recall that c and d are two realizations of $q(y)$ in $P(N)$.

Claim 4.2.7. *$M_\partial\langle c\rangle = M[c]$ and $M_\partial\langle d\rangle = M[d]$.*

Proof. Obviously, we have $M[c] \subseteq M_\partial\langle c\rangle$. We are left to prove that $M_\partial\langle c\rangle \subseteq M[c]$. We proceed by induction on the order of elements of $M_\partial\langle c\rangle$. Recall from Lemma 4.2.6 that elements of $M_\partial\langle c\rangle$ are polynomials composed of $X, \partial X, \partial^2 X \cdots$ and with coefficients from M where we replace X by c.

Let $f(X) \in M_\partial\langle X\rangle$, assume that it has order 1 and has degree $k \geq 1$. Then by Lemma 4.2.6, $f(X)$ is of the form of $a_0 + a_1 Q_0(X) + a_2 Q_1(X)\partial(X) + a_3 Q_2(X)(\partial(X))^2 + \cdots + a_{k+1} Q_k(X)(\partial(X))^k$ where $Q_i(X)$ are just polynomials without any form of differential with coefficients from M and the a_i's are also from M. By replacing X by c, we get $f(c) = a_0 + a_1 Q_0(c)$ which is an element of $M[c]$.

Assume $f(X) \in M_\partial\langle X\rangle$ has order 2 and with degree m. By Lemma 4.2.6,

$$f(X) = \sum_{i_1;i_2=0,1,\cdots,m} Q_{i_1;i_2}(X)(\partial X)^{i_1}(\partial^2 X)^{i_2}.$$

Since $\partial c = 0$ and $\partial^2 c = \partial(\partial c)$, then when we replace X by c in $f(X)$, we obtain a simple expression for $f(c)$ as $f(c) = Q_{0;0}(c)$, which is an element of $M[c]$.

Assume $f(X) \in M_\partial\langle X\rangle$ has order n and degree m. By Lemma 4.2.6,

$$f(X) = \sum_{i_1;\cdots;i_n=0,1,\cdots,m} Q_{i_1;\cdots;i_n}(X)(\partial X)^{i_1}(\partial^2 X)^{i_2} \cdots (\partial^n X)^{i_n}.$$

Since $\partial c = 0$, $\partial^2 c = \partial(\partial c), \cdots, \partial^{n+1}(c) = \partial(\partial^n c)$, then when we replace X by c in $f(X)$, we get a simple expression for $f(c)$ as $f(c) = Q_{0;\cdots;0}(c)$, which is an element of $M[c]$.

Analogously, we have $M_\partial\langle d\rangle = M[d]$. □

Claim 4.2.8. *Any σ, a ring isomorphism from $M[c]$ to $M[d]$ which fixes M and sends c to d, is a differential ring isomorphism from $M_\partial\langle c\rangle$ to $M_\partial\langle d\rangle$.*

Proof. Let σ be a ring isomorphism from $M[c]$ to $M[d]$ which fixes M and assume that $\sigma(c) = d$. Clearly we have $\partial(\sigma(c)) = \partial(d) = 0$ and $\sigma(\partial(c)) = 0$ so $\partial(\sigma(c)) = \sigma(\partial(c))$.

Let $f(X) \in M[X]$. Let $a = f(c)$ and $\partial(a) = \partial(f(c))$. We have $a = a_0 + a_1 c + \cdots + a_n c^n$ and $\partial(a) = \partial(a_0) + \partial(a_1)\cdot c + \cdots + \partial(a_n)\cdot c^n$. It follows that

$$\begin{aligned}
\sigma(\partial(a)) &= \sigma(\partial(a_0) + \partial(a_1)\cdot c + \cdots + \partial(a_n)\cdot c^n) \\
&= \sigma(\partial(a_0)) + \sigma(\partial(a_1)\cdot c) + \cdots + \sigma(\partial(a_n)\cdot c^n) \\
&= \sigma(\partial(a_0)) + \sigma(\partial(a_1))\cdot\sigma(c) + \cdots + \sigma(\partial(a_n))\cdot\sigma(c^n) \\
&= \sigma(\partial(a_0)) + \sigma(\partial(a_1))\cdot\sigma(c) + \cdots + \sigma(\partial(a_n))\cdot(\sigma(c))^n \\
&= \sigma(\partial(a_0)) + \sigma(\partial(a_1))\cdot d + \cdots + \sigma(\partial(a_n))\cdot d^n. \qquad (4.2.1)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\sigma(a) &= \sigma(a_0 + a_1 c + \cdots + a_n c^n) \\
&= \sigma(a_0) + \sigma(a_1 c) + \cdots + \sigma(a_n c^n) \\
&= \sigma(a_0) + \sigma(a_1)d + \cdots + \sigma(a_n)d^n.
\end{aligned}$$

Thus

$$\begin{aligned}
\partial(\sigma(a)) &= \partial(\sigma(a_0)) + \partial(\sigma(a_1)d) + \cdots + \partial(\sigma(a_n)d^n) \\
&= \partial(\sigma(a_0)) + \partial(\sigma(a_1))\cdot d + \cdots + \partial(\sigma(a_n))\cdot d^n. \qquad (4.2.2)
\end{aligned}$$

The results we obtain in Equations (4.2.1) and (4.2.2) clearly indicate that $\partial(\sigma(a))$ and $\sigma(\partial(a))$ are both elements of $M_\partial\langle d\rangle$.

Moreover M and (M, ∂) are closed respectively as a ring and a differential ring; σ is an isomorphism which fixes M. Therefore $\sigma(a_i) = a_i$ and

$\sigma(\partial(a_i)) = \partial(a_i)$ for $i = 1, \cdots, n$. Hence $\partial(\sigma(a_i)) = \sigma(\partial(a_i))$ for all $i = 1, 2 \cdots, n$.

We then have the equality: Equation (4.2.1) = Equation (4.2.2). This means that for any $a \in M[c]$, $\partial(\sigma(a)) = \sigma(\partial(a))$. Consequently, σ is a differential ring isomorphism from $M_\partial\langle c\rangle$ to $M_\partial\langle d\rangle$. □

The following result is a very important finding about dense pairs of real closed fields in the language L'.

Fact 4.2.9. *The theory of dense pairs of real closed fields is complete and is near-model complete in the language L'. Due to [Rob59].*

4.3 The definable (p, q)-theorem

In this section, we prove the main result of this chapter by making use of the various properties stated in the previous sections. At the end, we also present some open problems.

Now let $L = \{+, -, \cdot, 0, 1, <, \partial, P\}$ where $P(a)$ means $\partial(a) = 0$. Let (M, ∂) be a closed ordered differential field in the language L. Let $L' = \{+, -, \cdot, 0, 1, <, P\}$ be the sublanguage of L obtained by forgetting ∂. Let $(M, P(M))$ be the L'-reduct of (M, ∂). $(M, P(M))$ is also known as a dense pair of real closed fields in the language L'. Assume $(M, \partial) \prec (N, \partial)$ is sufficiently saturated. Let b be a tuple from $P(N)$, and $\varphi(x, y)$ an L'-formula such that $\varphi(x, b)$ does not divide over M. We want to prove that there exists an L'-formula $\psi(y) \in \mathrm{tp}_{L'}(b/M)$ such that $\{\varphi(x, b_i) : b_i \in \psi(M)\}$ is consistent.

In order to achieve the aim we just stated, we start by proving the following lemma. Let $q(y) = \mathrm{tp}_{L'}(b/M)$.

Lemma 4.3.1. *There is only one complete L-type over M containing $q(y)$.*

Proof. Let c, d be two realizations of $q(y)$, then we have

$$\mathrm{qftp}_{L'}(c/M) = \mathrm{qftp}_{L'}(d/M). \tag{4.3.1}$$

We would like to extend this property while we introduce the differential ∂ into Equation 4.3.1. To be exact, we want the following:

$$\mathsf{qftp}_L(c/M) = \mathsf{qftp}_L(d/M). \tag{4.3.2}$$

By Claim 4.2.8, the differential subrings of N generated by c and d over M are isomorphic over M. Then we obtain what we desired. Since $(M, P(M), \partial)$ has quantifier elimination in the language L, thus c and d realize the same L-type over M. □

By Lemma 4.3.1, there is a unique extension of $q(y)$, say $p(y)$, to a complete L-type over M. Note that $p(y)$ is also realized in $P(N)$.

Theorem 4.3.2. *Assume $(M, P(M))$ is a dense pair of real-closed ordered fields obtained from the CODF (M, ∂) in the language $L' \subset L$. Assume $(M, P(M)) \prec (N, P(N))$ is sufficiently saturated. Let b be a tuple in $P(N)$ and suppose $\varphi(x, b)$ does not divide over M in the sense of L'. Then there is an L'-formula $\psi(y) \in \mathsf{tp}_{L'}(b/M)$ such that $\{\varphi(x, b') : b' \in \psi(M)\}$ is finitely consistent.*

Proof. In this process, we are handling $\varphi(x, b)$ as an L-formula and we expect to end up with a set of realizations defined by an L'-formula.

By Proposition 4.1 of [BK18], there are an L-formula $\theta(x, z)$, a natural number k such that $|z| = k|y|$, and a complete L-type $r(z)$ over M such that, for any finite set $B \subseteq P(N)^{|y|}$ of realizations of $p(y)$, there is some $c \in P(N)^{|z|}$ such that $N \models r(c)$, $\theta(x, c)$ does not divide over M and for all $d \in B$ we have $N \models \forall x\big(\theta(x, c) \longrightarrow \varphi(x, d)\big)$.

Using Proposition 2.2 of [BK18], there exist $\tau(z) \in r(z)$ and a finite set $A_\tau \subseteq N^{|x|}$ such that, for all $c' \in \tau(M)$, there is some $a \in A_\tau$ such that $N \models \theta(a, c')$.

Since A_τ is finite, then let j be its cardinality. By the compactness theorem, there is a formula $\psi'(y)$ from $\mathsf{tp}_L(b/M)$ such that, for any B of cardinality less than or equal to j and $B \subseteq \psi'(M)$, there exists $c' \in \tau(M)$ such that $N \models \forall x\big(\theta(x, c') \longrightarrow \varphi(x, d)\big)$ for all $d \in B$.

Furthermore, since $\mathsf{tp}_L(b/M)$ is implied by $\mathsf{tp}_{L'}(b/M)$, $\psi'(y)$ is implied by a conjunction of formulas from $\mathsf{tp}_{L'}(b/M)$. Moreover by the compactness theorem we obtain an L'-formula $\psi(y) \in \mathrm{tp}_{L'}(b/M)$ such that $\forall y(\psi(y) \longrightarrow \psi'(y))$; and for all $B \subseteq \psi(M)$ with $|B| \leq j$, there exists $c' \in \tau(M)$ such that $N \models \forall x\big(\theta(x, c') \longrightarrow \varphi(x, d)\big)$ for all $d \in B$.

The last step is to prove that there exists $a \in A_\tau$ such that $N \models \varphi(a, d')$ for all $d' \in \psi(M)$. The rest of the proof of Theorem 4.3.2 is exactly the same as the last paragraph in the proof of Theorem 2.2.6. $\square$

Theorem 4.3.2 was proved assuming that $(M, P(M))$ is an L'-reduct of some L-structure, and this is only one case among many. Other cases exist, including the case where b is totally far from $P(N)$, known as acl-independent tuple over $M \cup P(N)$, another case is when b is just any tuple from N. This is subject to future research.

Finally, there are many other questions related to this concept that need to be addressed, an example of which is what if there are finitely many extension L-types containing $q(y)$? This is already a strong assumption. A more challenging question is what if there are infinitely many extension L-types containing $q(y)$.

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