On the computation of freely generated modular lattices

By

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Abstract

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Consider subspaces A, B, C of a vector space $\mathbb{V}$. How many subspaces can arise by taking arbitrary "combinations" of A, B, C (such as $(A+B)\cap C$) ? The answer is 28. If there are order relations among A, B, C (e.g. $A \subseteq C$), the corresponding number is smaller than 28. This leads to the concept of a modular lattice $FM(P)$ freely generated by a poset $(P, \leq)$. We compute the cardinality of $FM(P)$ for all P's with at most six elements. For 88 of these P' s the lattice $FM(P)$ is infinite.

Uittreksel

On the computation of freely generated modular lattices

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Gestel drie deelruimtes A, B, C van 'n vektor ruimte $\mathbb{V}$'s gegee. Wat is die maksimum aantal ruimtes wat kan ontstaan deur alle moontlike "kombinasies" van A, B, C te skep (soos bv. $(A + B) \cap C$)? Die antwoord is 28. As daar orde-relasies tussen A, B en C is (bv. $A \subseteq C$), dan is die ooreenkomstige getal kleiner as 28. Dit lei tot die konsep van 'n modulêre tralie $FM(P)$ wat deur 'n parsieelgeordende versameling $(P, \leq)$ vry voortgebring is. Ons bereken die kardinaliteit van $FM(P)$ vir alle P's van grootte hoogstens 6. Vir 88 van hulle die tralie is oneindig.

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- To my son Vanick and my daughter Ange, I miss you so much.

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Dedications

This thesis is dedicated to my wife Kwakep Chanceline, my son Juemo Semegni Vanick Wilfried and my daughter Tchouanang Semegni Ange Gabrielle.

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Chapter 1

Introduction

Lattice Theory offers an important tool for understanding mathematical structures as was stated by G. Birkhoff in the preface of the 1967 edition of his book *Lattice theory* when he wrote : "Lattices and groups provide the most basic tools of universal algebra, and in particular the structure of algebraic systems is usually most clearly revealed through the analysis of appropriate lattices." Birkhoff's book opened the door to intensive research on lattice theory. One problem encountered in studying free lattices is to find an algorithm which decides whether two arbitrary lattice expressions are identical in all lattices. This problem, known as the word problem, has attracted the interest of many researchers. In general the effective computation of free lattices is a difficult problem. G. Birkhoff [1] observed in 1940 that the free lattice on four generators is infinite, and he raised the question of the word problem for free lattices on n generators, which was solved in 1942 by Whitman in a series of two papers [2; 3]. In 1958, Howard L. Rolf [4] gave a description of the free lattices generated by a set of chains and R. Wille [5] in 1977 stated a necessary and sufficient condition under which a lattice freely generated by a poset is finite. Interesting is the word problem for free modular lattices. The free modular lattice on three generators, which is finite and contains 28 elements, was first described by R. Dedekind [6] in 1900. Interest in the word problem for free modular lattices (on n generators) increased after P. Whitman's solution [2; 3] of the word problem for free lattices appeared in the 1940's (see also [7; 8]). In 1973, R. Wille [9] gave a characterization of those posets P such that the modular lattice freely generated by P is finite. The word problem for free modular lattices on $n \geq 5$ generators was shown to be unsolvable by R. Freese [10] in 1982. Based on this result of Freese, C. Herrmann [11] was able to show in 1983 that the word problem for the modular lattice with four generators is unsolvable as well.

In 1994 G. Bartenschläger in his Ph.D. thesis [12] gave a complete list of free distributive lattices for posets up to cardinality five. He used the notion of concept lattices and skeletons to analyse the structure of a free bounded distributive lattice. In my thesis, I will extend his result to posets of cardinality six. More importantly I will generalize the computation to free modular lattices generated by posets of cardinality up to six and for some "good" posets on seven points. Our method to compute the free distributive lattice $F\mathcal{D}(P)$ generated by a poset P is based on the Birkhoff's representation theorem for finite distributive lattices. The computation of the free modular lattices $F\mathcal{M}(P)$ will be based, besides the theory of Wille, on a result by C. Herrmann and M. Wild [13] on the representation of modular lattices by certain closure systems. Another issue is the representation of $F\mathcal{D}(P)$ and $F\mathcal{M}(P)$ in a compact way. Since both $F\mathcal{D}(P)$ and $F\mathcal{M}(P)$ can be represented by closure systems (set of order ideals, and Λ-closed order ideals of some poset respectively), this leads us to find an algorithm that generates all the order ideals and all the Λ-closed order ideals of a given poset.

I will organize the thesis as follows.

Chapter one is the introduction and a brief historical background of the subject.

In chapter two, we will recall some basic notions on posets and lattices.

Chapter three is about closure systems and particularly about the congruence lattice of a lattice. The first and second isomorphism theorems will be discussed and some standard results on transposition and projectivity will be highlighted. We will end this chapter with the proof of the subdirect product decomposition theorem and related results and a construction of subdirect products of lattices via join-homomorphisms.

In chapter four we will first discuss the Birkhoff's representation theorem for distributive lattices, then we will study in more depth the free distributive lattices and discuss two equivalent methods to compute them. An algorithm based on the method of P-labellings will be developed and illustrated by means of examples.

Chapter five will cover modular lattices. We will start this chapter by recalling some preliminary results on modular lattices, namely the Dedekind transposition principle and Dilworth's theorem on the congruence lattice of a lattice. We will outline some results on finite projective geometries and these results will be used to discuss a theory of representing modular lattices which was initiated by C. Herrmann and M. Wild [13].

In chapter six, we will formally introduce the concept of free lattices generated by posets and study in detail the free modular lattice $F\mathcal{M}(P)$ generated by a finite poset P. We will next present an algorithm to illustrate the computational aspect of free modular lattices. A detailed proof of Wille's theorem [9] about the finiteness of $F\mathcal{M}(P)$ will be given at the end of this chapter.

Having represented $F\mathcal{D}(P)$ and $F\mathcal{M}(P)$ as the ideal lattice, respectively Λ-closed ideal lattice of some posets, in chapter seven we will discuss an algorithm called (a, B)-Algorithm, initially developed by M. Wild, to generate all the ideals of a finite poset, and we will apply this algorithm to effectively determine the elements of $F\mathcal{D}(P)$ and $F\mathcal{M}(P)$ and draw their Hasse diagrams.

Some numerical results will be recorded in chapter eight. In section 8.1 we will list for any poset P on up to six points the cardinality of $F\mathcal{D}(P)$, the cardinality of $F\mathcal{M}(P)$, and the number of factors (**2** or M_3) in their subdirect product decompositions respectively. In section 8.2, we are concerned with the good posets [1] on seven points. Thanks to G. Brinkmann and B. D. Mckay [14] who sent me a C^{++} code of their program to generate all posets on up to sixteen points. From this code I extracted all the 2045 posets on seven points, then I wrote a program to select all the 1101 good posets on seven points. The (a, B)-Algorithm was again used to compute $|F\mathcal{M}(P)|$ together with its parameters, for all the good posets on seven points.

The thesis ends with an appendix containing the Hasse diagrams of $F\mathcal{D}(P)$ and $F\mathcal{M}(P)$ for some finite posets of interest. The thesis is self-contained and we have tried as far as possible to illustrate many concepts either by simple examples or by means of pictures.

[1] Good posets are those for which $F\mathcal{M}(P)$ is finite.

Chapter 2

Basic concepts

2.1 Preliminaries on partially ordered sets

2.1.1 Ordered sets

Definition 2.1 *Let P be a non-empty set. A binary relation $\leq$ is said to be an* ***order*** *(or a* ***partial order****) on P if the following properties hold for all $x, y, z \in P$.*

i) ***Reflexivity:*** $x \leq x$.

ii) ***Antisymmetry:*** $x \leq y$ *and* $y \leq x$ *imply* $x = y$.

iii) ***Transitivity:*** $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$.

Definition 2.2 *A* ***partially ordered set*** *(or* ***poset****), denoted $(P, \leq)$, is a non-empty set together with an order relation. Two elements x and y of a poset are said to be* ***comparable*** *if $x \leq y$ or $y \leq x$. Otherwise, they are said to be* ***incomparable****. A* ***chain*** *of a poset $(P, \leq)$ is a set of pairwise comparable elements of P. A chain of n elements will be denoted by* $\boldsymbol{n}$*. A set of pairwise incomparable elements of P is called an* ***antichain****. If P is a poset consisting of two posets P_1 and P_2 such that $P = P_1 \cup P_2$ and for all $a \in P_1$ and $b \in P_2$, a and b are incomparable, then we write $P = P_1 + P_2$. In particular an antichain of n elements is denoted* $\boldsymbol{1}+\boldsymbol{1}+ \cdots + \boldsymbol{1}$ *where there are n terms in the sum.*

Example 2.1

1. *The power set $\mathcal{P}(X)$ of a set X, together with the set inclusion $\subseteq$, is a poset.*

2. *The real line $\mathbb{R}$, the set of integers $\mathbb{Z}$ and the set of nonnegative integers $\mathbb{N}_0$, with their natural order $\leq$, are chains.*

3. *The set of positive integers* $(\mathbb{N}, |)$ *together with the relation of divisibility defined by* $a|b$ *if* $b = na$ *for some* $n \in \mathbb{N}$ *(so* $3|6$ *but* $4 \nmid 6$*), is a poset.*

4. *The vector space* $\mathcal{C}([0,1], \mathbb{R})$ *of continuous functions from* $[0,1]$ *to* $\mathbb{R}$ *ordered by* $f \leq g$ *if and only if* $f(x) \leq g(x)$ *for all* $x \in [0,1]$*, is a poset.*

5. *Let* $(P, \leq)$ *be a poset. The relation* $\geq := \{(a,b) \in P \times P : \ b \leq a\}$ *is an order on* P *and* $(P, \geq)$ *is called the* ***dual*** *of* $(P, \leq)$*.*

Definition 2.3 *Let* $(P, \leq)$ *be a poset and let* X *be a subset of* P*.*

1. *An element* $a \in P$ *is called* ***lower bound*** *of* X *if* $a \leq x$ *for all* $x \in X$*, and it is called* ***upper bound*** *of* X *if* $x \leq a$ *for all* $x \in X$*. We say that* X *is* ***bounded*** *if it has a lower bound and an upper bound.*

2. *The* ***greatest lower bound*** *(or* ***infimum****) of* X*, denoted* $\bigwedge X$ *when it exists, is a lower bound* l *of* X *such that for any other lower bound* m *of* X*,* $m \leq l$*. The* ***least upper bound*** *(or* ***supremum****) of* X*, denoted* $\bigvee X$ *when it exists, is an upper bound* u *of* X *such that for any other upper bound* v *of* X*,* $u \leq v$*.*

3. *The* ***minimum element*** *of* X*, when it exists, is an element* $m \in X$ *such that* $m \leq x$ *for all* $x \in X$*. The* ***maximum element*** *of* X*, when it exists, is an element* $g \in X$ *such that* $x \leq g$ *for all* $x \in X$*.*

4. *An element* $a \in X$ *is said to be* ***maximal*** *in* X *if for any* $x \in X$*,* $a \leq x \Rightarrow a = x$*. Dually an element* $b \in X$ *is said to be* ***minimal*** *in* X *if for any* $x \in X$*,* $b \geq x \Rightarrow b = x$*.*

Remark: Generally $\bigwedge X, \bigvee X \notin X$. But if l is the minimum element of X, then $\bigwedge X = l \in X$ and if g is the maximum element of X, then $\bigvee X = g \in X$. If Φ is a statement about a poset $(P, \leq)$, then the statement Φ^* obtained by replacing any occurrence of $\leq$ by $\geq$ and by switching the infimum and the supremum is called the **dual statement** of Φ. If Φ is true for all posets, then Φ^* is also true for all posets. This fact is known as the **duality principle** and it is very useful in proofs.

Definition 2.4 *Let* $(P, \leq)$ *and* $(Q, \leq)$ *be two posets. An* ***order morphism*** *from* P *to* Q *is a map* $\rho\colon P \longrightarrow Q$ *that preserves the order. That is, for all* $x, y \in P$ *:*

$$x \leq y \quad \Rightarrow \quad \rho(x) \leq \rho(y).$$

An order morphism is sometimes called ***monotone map****. An order morphism is said to be an* ***order isomorphism*** *if it is a bijection and its inverse is an order morphism.*

2.1.2 Graphical representation of posets - Hasse diagram

Let $(P, \leq)$ be a poset and let $x, y \in P$. We write $x < y$ when $x \leq y$ and $x \neq y$. We say that y **covers** x (or y is an **upper cover** of x or x is a **lower cover** of y), and we write $x \prec y$, if $x < y$ and no $a \in P$ satisfies $x < a < y$. Using the covering relation, one can obtain a graphical representation of any finite poset P as follows. Represent each element of P by a dot in such a way that whenever $x \prec y$ then y (i.e. the corresponding dot) is higher than x and the two are connected by a line segment. It is easily seen that for all $x, y \in P$ one has $x < y$ if and only if there is an "increasing path" from x to y. The resulting figure is called a **Hasse diagram** of P. Note that different Hasse diagrams may represent the same poset.

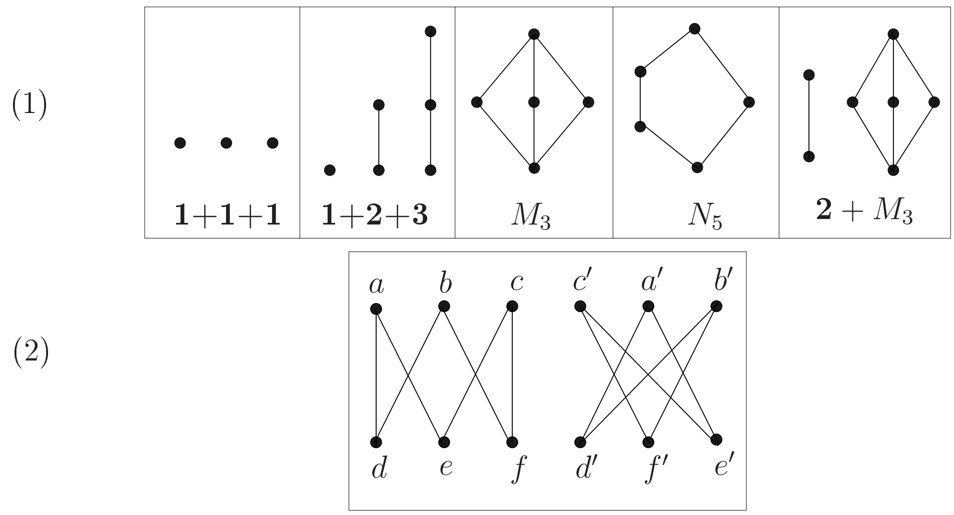
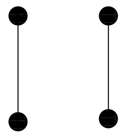
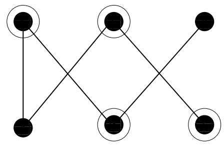


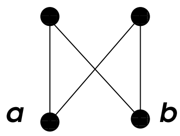
Figure 2.1: (1) The Hasse diagrams of some posets. M_3 is called **Diamond** and N_5 **Pentagon**. (2) Two Hasse diagrams representing isomorphic posets, the isomorphism sends each x to x'.

Definition 2.5 *Each subset X of $(P, \leq)$ yields a* ***subposet*** *of P if X endowed with the induced order is a poset. That is, for all $x, y \in X$, $x \leq y$*

in X if and only if $x \leq y$ in P. For instance X =  *is a subposet of P =:* 

2.2 Basic lattice theoretic concepts

Definition 2.6 *A poset $(L, \leq)$ is said to be a **lattice** if any pair of elements $a, b \in L$ has a least upper bound $a \vee b$ (**join** of a and b), and a greatest lower bound $a \wedge b$ (**meet** of a and b).*

Note that 

is not a lattice since the least upper bound of a and b does not exist.

Proposition 2.1 *If $(L \leq)$ is a lattice, then the binary operations $\vee$ and $\wedge$ satisfy the following properties for all $a, b, c \in L$:*

i) ***Idempotency:*** $a \wedge a = a$ *and* $a \vee a = a$

ii) ***Commutativity:*** $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$

iii) ***Associativity:*** $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ *and* $(a \vee b) \vee c = a \vee (b \vee c)$

iv) ***Absorption:*** $a \wedge (a \vee b) = a$ *and* $a \vee (a \wedge b) = a$

Example 2.2 *1. Any chain is a lattice in which $x \wedge y$ is simply the minimum and $x \vee y$ is the maximum of x and y.*

2. The poset $(\mathcal{P}(X), \subseteq)$ is a lattice in which $A \wedge B = A \cap B$ and $A \vee B = A \cup B$.

3. Let M be a module over a ring and let $Sub(M)$ denote the set of all submodules of M. Then $(Sub(M), \subseteq)$, is a lattice where $S \wedge T = S \cap T$ and $S \vee T = S + T = \{s + t \in M : s \in S \text{ and } t \in T\}$.

4. $(\mathbb{N}, |)$ ordered by divisibility is a lattice in which $a \wedge b = gcd(a, b)$, the greatest common divisor of a and b, and $a \vee b = lcm(a, b)$, the least common multiple of a and b.

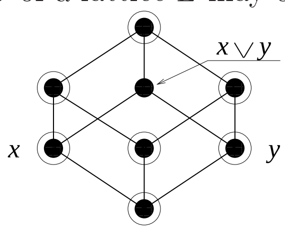
5. *The Pentagon N_5 and the Diamond M_3 are lattices.*

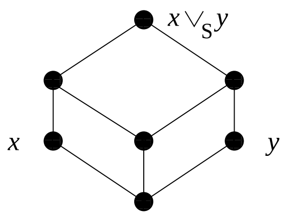
6. *If $(L_1, \leq_1)$ and $(L_2, \leq_2)$ are lattices, then the Cartesian product $L_1 \times L_2$ together with the order $\leq$ defined componentwise is a lattice where the meet and the join are also defined componentwise.*

By associativity, in any lattice L the supremum $\bigvee H$ (respectively infimum $\bigwedge H$) is well defined for every *finite* subset $H \subseteq L$.

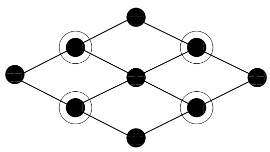
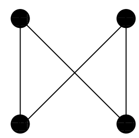
Definition 2.7 *A lattice $(L, \leq)$ is said to be* ***complete*** *if both $\bigwedge X$ and $\bigvee X$ exist for any (not necessarily finite) subset $X \subseteq L$.*

A subposet S of a lattice L may or may not be a lattice:

$L =$


$\longrightarrow$ $S =$


is a lattice, but

$L =$

$\longrightarrow$ $S =$

is not a lattice.

Definition 2.8 *A non-empty subset S of a lattice $(L, \leq)$ is called* ***sublattice*** *of L if $a \wedge b \in S$ and $a \vee b \in S$ for all $a, b \in S$.*

In this case the subposet $(S, \leq)$ not only is a lattice $(S, \wedge_S, \vee_S)$ in its own right; moreover one has $a \wedge_S b = a \wedge b$ and $a \vee_S b = a \vee b$ for all $a, b \in S$.

Remark: A complete lattice is always bounded and any finite lattice is complete. Note that the intersection of any family of sublattices of L is again a sublattice. In particular, if X is a subset of L, then the intersection of all the sublattices containing X is obviously the smallest sublattice that contains X. It is called the **sublattice generated** by X and denoted by $\langle X \rangle$.

Example 2.3 1. *The set $D(n)$ of divisors of an integer $n \in \mathbb{N}$ is a sublattice of the lattice $(\mathbb{N}, |)$.*

2. *If $(L, \leq)$ is a lattice and $a, b \in L$, then the set $\{x \in L : \ a \leq x \leq b\}$ is a sublattice of L called* ***interval*** *and denoted by $[a, b]$. If b covers a, then the interval $[a, b] = \{a, b\}$ is called* ***prime quotient****.*

Definition 2.9 *Let $(L, \leq)$ and $(M, \leq)$ be lattices. A map $\alpha : L \longrightarrow M$ is a **lattice morphism** if it preserves the meet and the join. That is, for all $a, b \in L$*

$$\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b) \quad \textit{and} \quad \alpha(a \vee b) = \alpha(a) \vee \alpha(b).$$

*A lattice morphism is an **isomorphism** if it is a bijection.*

Observe that any lattice morphism is order preserving but that the converse is not always true. If $\alpha : L \to M$ is a surjective morphism, then M is said to be an **epimorphic image** of L.

Proposition 2.2 *Let L be a lattice, P a poset and $\rho : L \longrightarrow P$ a surjective map such that $x \leq y \Longleftrightarrow \rho(x) \leq \rho(y)$ for all $x, y \in L$. Then P is a lattice and ρ is an isomorphism.*

Proof: The reader is e.g. referred to [15] for the proof of this result. ■

Definition 2.10 *(i) Let $(L, \leq)$ be a bounded lattice. An element $a \in L$ is said to be **complemented** if there exists an element $b \in L$, called **complement** of a such that $a \wedge b = 0$ and $a \vee b = 1$. A **complemented lattice** is a lattice in which every element has a complement. L is said to be **relatively complemented** if every interval of L (viewed as a lattice on its own) is complemented.*

*(ii) A lattice $(L, \leq)$ is said to be of **finite height** if there is a finite upper bound to the length of chains in L. The least such upper bound is called **height** of L and denoted by $h(L)$. The height of the interval $[0, a]$ (viewed as a sublattice of L) is simply denoted by $h(a)$ and called height of a.*

*(iii) A bounded lattice is called **graded lattice** if all chains from 0 to 1 have the same length.*

Note that relatively complemented lattices are complemented but the converse is not true, e.g. N_5 is complemented but not relatively complemented.

Definition 2.11 *An element a of a lattice L is called **join-irreducible** (or **$\vee$-irreducible**) if for all $b, c \in L$, $a = b \vee c$ implies $a = b$ or $a = c$ (otherwise a is called join-reducible). The set of nonzero join-irreducible elements of L is denoted by $J(L)$. An element a of L is called **meet-irreducible** (or **$\wedge$-irreducible**) if for all $b, c \in L$, $a = b \wedge c$ implies $a = b$ or $a = c$. Finally, if L is bounded, $a \in L$ is called **atom** if for all $x \in L$, $x \leq a \Rightarrow x = a$ or $x = 0$. Dually a is called **co-atom** if for all $x \in L$, $x \geq a \Rightarrow x = a$ or $x = 1$.*

One easily shows:

Proposition 2.3 *[16] If L is a lattice of finite height, then every element of L is a join of join-irreducible elements of L.*

The decomposition of an element as a join of join-irreducible elements is not necessarily unique as seen below.

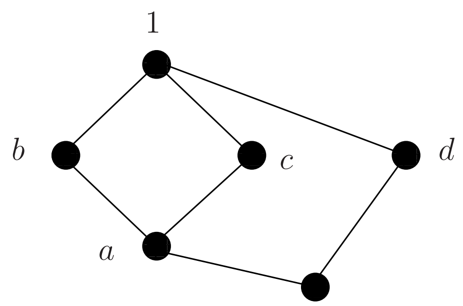


Figure 2.2: $a \vee d = 1 = b \vee c$.

Chapter 3

Congruence relations

3.1 Closure systems

Definition 3.1 *Let A be a set. A map $c : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ is a* ***closure operator*** *on A if c is:*

i) extensive: $X \subseteq c(X)$ *for all* $X \in \mathcal{P}(A)$

ii) monotone: $X \subseteq Y \Rightarrow c(X) \subseteq c(Y)$ *for all* $X, Y \in \mathcal{P}(A)$

iii) idempotent: $c(c(X)) = c(X)$ *for all* $X, Y \in \mathcal{P}(A)$

An element $X \in \mathcal{P}(A)$ is said to be ***closed*** *with respect to c if $X = c(X)$.*

Definition 3.2 *Let A be a set and $\mathcal{F} \subseteq \mathcal{P}(A)$. Then $\mathcal{F}$ is said to be a* ***closure system*** *on A if $A \in \mathcal{F}$ and $\bigcap \mathcal{G} \in \mathcal{F}$ for all non-empty subsets $\mathcal{G}$ of $\mathcal{F}$.*

The following results are well known and have easy proofs.

Proposition 3.1 *Let $\mathcal{F}$ be a closure system on a set A. Then the map*

$$\begin{array}{rccl} c_{\mathcal{F}} : & \mathcal{P}(A) & \longrightarrow & \mathcal{P}(A) \\ & X & \longmapsto & \bigcap\{K \in \mathcal{F} : X \subseteq K\} \end{array}$$

is a closure operator on A. Conversely, let A be a set and c a closure operator on A. Then the set $\mathcal{F}_c = \{c(X) : X \subseteq A\}$ of closed elements is a closure system on A. Moreover if $\mathcal{F}$ is a closure system, then $\mathcal{F} = \mathcal{F}_{c_{\mathcal{F}}}$. This means that any closure system is a complete lattice with the operations given by

$$X \wedge Y = X \cap Y \ \text{ and } \ X \vee Y = c_{\mathcal{F}}(X \cup Y).$$

Let P be a poset. A subset $I \subseteq P$ is called (**order**) **ideal** if for all $x, y \in P$, $x \in I$ and $y \leq x$ imply $y \in I$. The intersection (and trivially the union) of any family of ideals of P is again an ideal of P. Hence the set of ideals of P, denoted $Id(P)$, ordered by the inclusion is a closure system on P, whence a complete lattice in which the meet is the intersection and the join is the union. If S is a subset of P, the **ideal generated** by S, denoted by $\downarrow S$, is the smallest ideal containing S. In particular $\downarrow\{a\}$ is denoted $\downarrow a$ and is called **principal ideal** generated by a. It is straightforward to show that $\downarrow S = \{x \in P : \exists s \in S, x \leq s\}$.
A subset F of a poset P is called (**order**) **filter** if for all $x, y \in P$, $x \in F$ and $x \leq y$ imply $y \in F$. The intersection of any family of filters of P is again a filter of P, hence the set of filters of P, denoted by $Fil(P)$, is a closure system on P. If S is a subset of P, the **filter generated** by S, denoted $\uparrow S$, is the smallest filter of P containing S. If $f \in P$ then $\uparrow\{f\}$ is simply denoted by $\uparrow f$. Note that $\uparrow S = \{x \in P : \exists s \in S, x \geq s\}$. Observe also that $\emptyset$ and P are filters. A filter F of P is called **proper filter** if $\emptyset \neq F \neq P$. We denote by $Fil^*(P)$ the set of proper filters of P. A proper ideal is defined dually.

3.2 Equivalence relations

Let A be a set and $\mathcal{R} \subseteq A \times A$ a binary relation on A. Then $\mathcal{R}$ is an **equivalence relation** on A if $\mathcal{R}$ is reflexive, symmetric and transitive where the symmetry means that $x\mathcal{R}y \iff y\mathcal{R}x$ for all $x, y \in A$. The **equivalence class** of an element $a \in A$, denoted $a_{\mathcal{R}}$ or $a/\mathcal{R}$, is the set of elements $b \in A$ such that $a\mathcal{R}b$. The set of all the equivalence classes of A is denoted by $A/\mathcal{R}$, and the set of all the equivalence relations on A is denoted $Eqv(A)$.

The diagonal of A, written $\Delta A = \{(a, a) : a \in A\}$, and the Cartesian product, $\nabla A = A \times A$, are equivalence relations on A. If A and B are two sets and $f : A \to B$ is a map, then the relation $\mathcal{R}$ defined on A by $x\mathcal{R}y$ if and only if $f(x) = f(y)$ is an equivalence relation called **kernel** of f and denoted $ker(f)$. If $\mathcal{R}$ and $\mathcal{S}$ are equivalence relations on A, then the **composition** of $\mathcal{R}$ and $\mathcal{S}$, denoted $\mathcal{R} \circ \mathcal{S}$, is the binary relation defined on A by

$$x(\mathcal{R} \circ \mathcal{S})y \iff \exists z \in A : \ x\mathcal{R}z \text{ and } z\mathcal{S}y.$$

Proposition 3.2 *Let A be a set. Then $Eqv(A)$ is a closure system on $A \times A$. Hence $Eqv(A)$ is a complete lattice. Further, if $\mathcal{R}, \mathcal{S} \in Eqv(A)$,* *then* $\mathcal{R}\wedge\mathcal{S} = \mathcal{R}\cap\mathcal{S}$ *and* $\mathcal{R}\vee\mathcal{S} = \mathcal{R}\cup(\mathcal{R}\circ\mathcal{S})\cup(\mathcal{R}\circ\mathcal{S}\circ\mathcal{R})\cup(\mathcal{R}\circ\mathcal{S}\circ\mathcal{R}\circ\mathcal{S})\cdots$, *that is,* $a(\mathcal{R}\circ\mathcal{S})b$ *if and only if there is a sequence* $x_0, x_1, \cdots, x_n$ *such that* $a = x_0, b = x_n$ *and* $x_i\mathcal{R}x_{i+1}$ *or* $x_i\mathcal{S}x_{i+1}$ *for all* $i \in \{0, 1, \cdots, n-1\}$.

Proof: This is a standard result, see e.g. [17] for a proof. ■

3.3 Congruences on lattices

Definition 3.3 *Let* L *be a lattice. A binary relation* $\theta \subseteq L \times L$ *is a* ***congruence*** *on* L *if:*

(i) θ *is an equivalence relation on* L *and,*

(ii) for all $a, b, c, d \in L$, $a\theta b$ *and* $c\theta d$ $\Longleftrightarrow$ $(a\wedge c)\theta(b\wedge d)$ *and* $(a\vee c)\theta(b\vee d)$.

The second property is sometimes called **substitution property**. The set of all congruences on L will be denoted by $Con(L)$. The intersection of any family of congruences is again a congruence on L. This implies (Prop.3.1) that $Con(L)$ is a closure system on $L \times L$, and as such, is a complete lattice. One can show that $Con(L)$ is in fact a sublattice of $Eqv(L)$. In other words the join of congruences θ and τ is computed as in Prop.3.2. The smallest congruence containing a subset X of L^2 is called the **congruence generated** by X and it is denoted by $Cg(X)$ or $\langle X\rangle$. The congruence $Cg(\{(a,b)\})$ will be simply denoted by $Cg(a,b)$ or $\langle(a,b)\rangle$, the **principal congruence** collapsing a and b.

Proposition 3.3 *[16] Let* $(L,\leq)$ *be a lattice. An equivalence relation* θ *on* L *is a congruence on* L *if and only if for all* $(a,b)\in\theta$ *and all* $c \in L$, *one has*

$$(a\wedge c, b\wedge c)\in\theta \text{ and } (a\vee c, b\vee c)\in\theta.$$

Example 3.1 *1.* ΔL *and* ∇L *are congruences on the lattice* $(L,\leq)$.

2. If L *and* M *are lattices and* $h : L \to M$ *is a morphism, then the equivalence relation* $ker(h)$ *is a congruence on* L. *One can hence declare on* $L/ker(h)$ *two well defined binary operations* $\wedge$ *and* $\vee$ *by*

$$x_{ker(h)}\wedge y_{ker(h)} = (x\wedge y)_{ker(h)} \quad \text{and} \quad x_{ker(h)}\vee y_{ker(h)} = (x\vee y)_{ker(h)}.$$

These binary operations can be generalised to any quotient lattice L/θ *where* θ *is a congruence on* L.

Proposition 3.4 *[17]**(First isomorphism theorem)** Let L and M be two lattices and $h : L \rightarrow M$ a morphism. Then $L/ker(h) \cong Im(h)$.*

The first (and below the second) isomorphism theorem holds more generally for any algebraic structure. However, the next result is specifically lattice-theoretic.

Proposition 3.5 *[16] Let $(L, \leq)$ be a lattice and θ a congruence on L. Then $a\theta b$ if and only if $(a \wedge b)\theta(a \vee b)$ for all $a, b \in L$. Moreover, any congruence class is a convex sublattice of L, i.e. an interval of L whenever L is finite.*

Proof: In fact, $a\theta b$ implies $(a \wedge b)\theta(b \wedge b) = b$ and $a = (a \vee a)\theta(a \vee b)$. So by symmetry and transitivity of θ, $(a \wedge b)\theta(a \vee b)$. Conversely, if $(a \wedge b)\theta(a \vee b)$ then,

$$\begin{array}{rcll} a & = & a \wedge (a \vee b) & \\ & \theta & a \wedge (a \wedge b) & \text{since} \quad (a \wedge b)\theta(a \vee b) \\ & = & a \wedge b & \\ & \theta & a \vee b & \\ & = & (a \vee b) \vee b & \\ & \theta & (a \wedge b) \vee b & \\ & = & b. & \end{array}$$

The transitivity of θ yields $a\theta b$. Also any congruence class modulo θ is a convex sublattice of L. Indeed, $x \leq z \leq y$ and $x\theta y$ imply $x = (x \wedge z)\theta(y \wedge z) = z$ since θ is a congruence. That is $x\theta z$. ■

We now introduce a kind of special element, called prime element, that yields a congruence on L . The concept of prime element is very important in distributive lattices, in fact we will use this concept to show that any distributive lattice and its congruence lattice have the same height[1]. We will also show that a distributive lattice is completely determined by its prime elements.

Definition 3.4 *Let L be a lattice. An element p of L is said to be **join-prime**[2] if for all $a, b \in L$, $p \leq a \vee b$ implies $p \leq a$ or $p \leq b$.*

It is easy to see that any prime element is join-irreducible but not all join-irreducible elements are necessarily primes as illustrated on the following picture. Note that p and d are primes, b and c are join-irreducibles but not primes.

[1] See definition (2.10)

[2] *We will just say **prime** for short.*

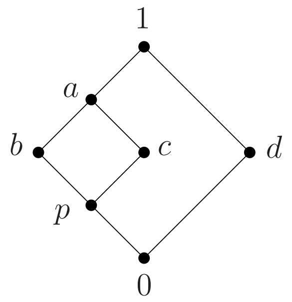


Figure 3.1: Illustration of primality.

Theorem 3.1 *[16] Let L be a lattice and $p \in L$ a prime element. Then the map $\widetilde{p} : L \to \mathbf{2}$ defined by:*

$$\widetilde{p}(a) = \begin{cases} 1 & \textit{if } a \geq p, \\ 0 & \textit{otherwise} \end{cases}$$

is an epimorphism. Conversely suppose L is finite and $g : L \to \mathbf{2}$ is an epimorphism. Then $p := \bigwedge\{a \in L : g(a) = 1\}$ is a prime element of L. ■

The following example illustrates this theorem:

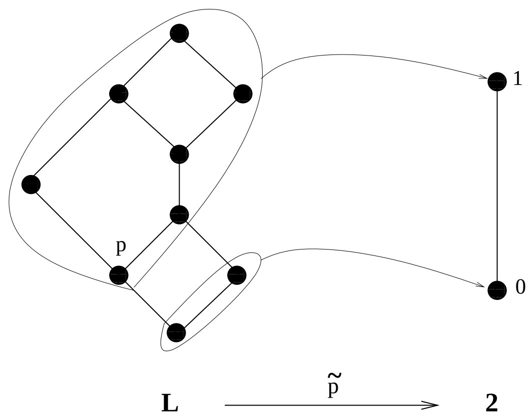


Figure 3.2: Illustration of theorem 3.1.

In general, the prime elements in a finite lattice L correspond to the congruences $\theta \in Con(L)$ with exactly two θ-classes. The latter θ's are co-atoms in $Con(L)$, but the converse need not be true (e.g. take $L = M_3$).

Theorem 3.2 *[18] **(Second isomorphism theorem)***
Let L be a lattice and fix $\theta \in Con(L)$. Every $\phi \in Con(L)$ containing θ

yields a congruence $\phi/\theta \in Con(L/\theta)$ defined by

$$a_\theta(\phi/\theta)b_\theta \quad \text{if and only if} \quad a\phi b. \tag{3.3.1}$$

It follows that $(L/\theta)/(\phi/\theta) \cong L/\phi$, and that $\phi \mapsto \phi/\theta$ yields a lattice isomorphism from the interval $[\theta, \nabla]$ of $Con(L)$ onto $Con(L/\theta)$. ∎

3.4 Transposition and projectivity

Definition 3.5 *Let $(L, \leq)$ be a lattice and let $a, b, c, d \in L$ such that $a \leq b$ and $c \leq d$. We say that the interval $[a, b]$ **transposes up** to the interval $[c, d]$ denoted by $[a, b] \nearrow [c, d]$ if and only if $d = b \vee c$ and $a = b \wedge c$. Similarly we say that the interval $[a, b]$ **transposes down** to the interval $[c, d]$ denoted by $[a, b] \searrow [c, d]$ if and only if $b = a \vee d$ and $c = a \wedge d$. We call $[a, b]$ and $[c, d]$ **transposed** if either $[a, b] \nearrow [c, d]$ or $[a, b] \searrow [c, d]$. Finally, we say that $[a, b]$ and $[c, d]$ are **projective** if there is a finite sequence $[a, b] = [c_0, d_0], [c_1, d_1], \cdots, [c_n, d_n] = [c, d]$ such that $[c_i, d_i]$ and $[c_{i+1}, d_{i+1}]$ are transposed for all $0 \leq i \leq n-1$. For instance in the following figure, $[a, d] \searrow [0, b]$ and $[0, b] \nearrow [c, 1]$, so $[a, d]$ and $[c, 1]$ are projective prime quotients.*

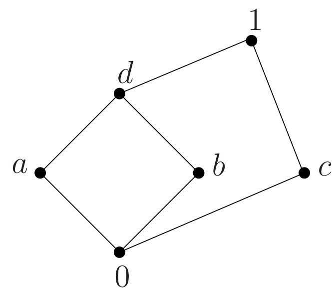


Figure 3.3: Illustration of the projectivity relation.

Theorem 3.3 *[16] Let L be a lattice and let $[a, b]$ and $[c, d]$ be projective intervals of L. Then for all $\theta \in Con(L)$, $a\theta b$ if and only if $c\theta d$.*

Proof: It essentially suffices to observe that from, say, $[a, b] \nearrow [c, d]$ and $a\theta b$, follows $(a \vee c)\theta(b \vee c)$. That is $c\theta d$ since $c = a \vee c$ and $d = b \vee c$ ∎

3.5 Direct and subdirect products

Definition 3.6 *Let $(L, \leq)$ be a lattice. We say that L is **directly indecomposable** if $|L| > 1$ and $L \cong L_1 \times L_2$ implies that either $|L_1| = 1$ or $|L_2| = 1$. We say that L is **simple** if $Con(L)$ has only two elements, i.e. $Con(L) = \{\Delta, \nabla\}$.*

For $L = L_1 \times L_2$ one checks that $\theta_1, \theta_2 \in Con(L)$ if they are defined as follows:

$$\begin{array}{lcl}(x_1, x_2)\theta_1(y_1, y_2) & :\Leftrightarrow & x_1 = y_1 \\ (x_1, x_2)\theta_2(y_1, y_2) & :\Leftrightarrow & x_2 = y_2\end{array}$$

One has $\theta_1 \wedge \theta_2 = \Delta$ (clear) and $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1 = \nabla = \theta_1 \vee \theta_2$. For instance $\theta_1 \circ \theta_2 = \nabla$ since $(x_1, x_2)\theta_1(x_1, y_2)\theta_2(y_1, y_2)$ for all $(x_1, x_2), (y_1, y_2) \in L$. Moreover, if $|L_1|, |L_2| > 1$, then any $\theta_1, \theta_2 \notin \{\Delta, \nabla\}$. Conversely, any lattice L and any $\theta_1, \theta_2 \in Con(L) \backslash \{\Delta, \nabla\}$ with $\theta_1 \wedge \theta_2 = \Delta$ and $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1 = \nabla$ yield a direct decomposition $L \cong L_1 \times L_2$ with $|L_i| > 1$.

An easy induction shows that each finite lattice L is isomorphic to $L_1 \times L_2 \times \cdots \times L_s$ for some directly indecomposable lattices L_i. Interestingly the L_i's are unique up to isomorphism and ordering. Direct products are the special case $S = L_1 \times L_2$ in the definition below.

Definition 3.7 *Let L_1 and L_2 be two lattices. A subset $S \subseteq L_1 \times L_2$ is a **subdirect product** of L_1 and L_2 if*

i) S *is a sublattice of* $L_1 \times L_2$,

ii) $(\forall x \in L_1)(\exists y \in L_2)\ (x, y) \in S$,

iii) $(\forall y \in L_2)(\exists x \in L_1)\ (x, y) \in S$.

*The lattices L_1 and L_2 are called **factors** of the subdirect product S.*

Proposition 3.6 *Let $S \subseteq L_1 \times L_2$ be a subdirect product. Consider the maps $\rho_1 : S \to L_1$ and $\rho_2 : S \to L_2$ defined by $\rho_1(x, y) = x$ and $\rho_2(x, y) = y$. Then ρ_1 and ρ_2 are surjective morphisms and $ker(\rho_1) \cap ker(\rho_2) = \Delta$.*

We omit the easy proof and rather illustrate by the following example where S is the above lattice:

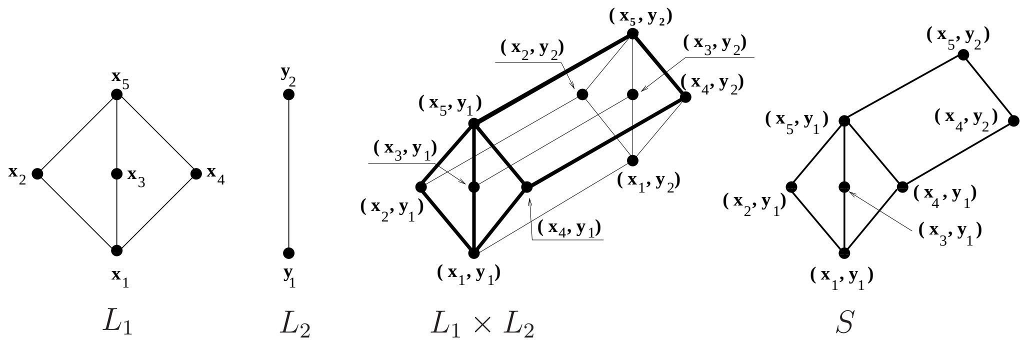


Figure 3.4: S is a subdirect product of factors L_1 and L_2.

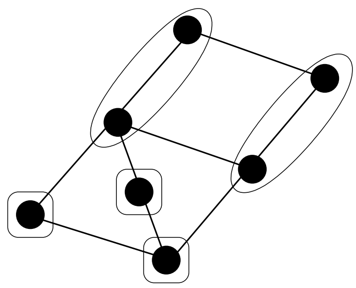


Figure 3.5: Classes of $ker(\rho_1)$.

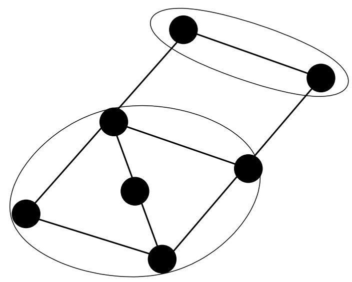
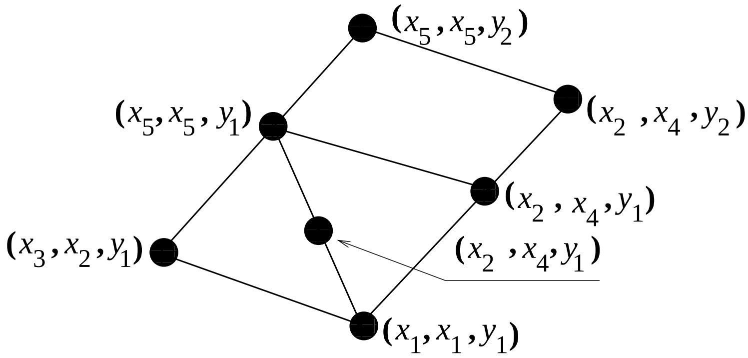


Figure 3.6: Classes of $ker(\rho_2)$.

Consider now this subdirect product $T \subseteq L_1 \times L_1 \times L_2$ where L_1 and L_2 are as in figure 3.4:

$T \quad :=$



Let ρ_1, ρ_2, ρ_3 be the restrictions of the projections of $L_1 \times L_1 \times L_2$ onto T. Although ρ_1, ρ_2 are distinct maps $T \to L_1$, observe that $ker(\rho_1) = ker(\rho_2)$. That means either of the first two subdirect factors is redundant; it could be dropped without changing the isomorphism type of the remaining subdirect product.

Here is a converse of proposition 3.6

Theorem 3.4 *[17](**Subdirect product decomposition theorem**)*
Let T be a lattice and let θ_1, θ_2 be two congruences on T such that $\theta_1 \cap \theta_2 = \Delta$. Put $T' = \{(a_{\theta_1}, a_{\theta_2}) :\ a \in T\}$. Then $T' \cong T$ and T' is a subdirect product of T/θ_1 and T/θ_2.

Proof: Define $\varepsilon : T \to T'$ by letting $\varepsilon(a) = (a_{\theta_1}, a_{\theta_2})$ for all $a \in T$. Then $\varepsilon(a \wedge b) = \big((a \wedge b)_{\theta_1}, (a \wedge b)_{\theta_2}\big) = (a_{\theta_1} \wedge b_{\theta_1}, a_{\theta_2} \wedge b_{\theta_2}) = (a_{\theta_1}, a_{\theta_2}) \wedge (b_{\theta_1}, b_{\theta_2}) = \varepsilon(a) \wedge \varepsilon(b)$. Similarly, one can show that $\varepsilon(a \vee b) = \varepsilon(a) \vee \varepsilon(b)$, so ε is a morphism. For the injection, suppose that $\varepsilon(a) = \varepsilon(b)$, then $(a_{\theta_1}, a_{\theta_2}) = (b_{\theta_1}, b_{\theta_2})$, i.e. $a_{\theta_1} = b_{\theta_1}$ and $a_{\theta_2} = b_{\theta_2}$. So $(a, b) \in \theta_1 \cap \theta_2 = \Delta$, therefore $a = b$. Let us now prove that T' is a subdirect product of T/θ_1 and T/θ_2. Obviously, T' is a sublattice of $T/\theta_1 \times T/\theta_2$. Further if $a_{\theta_1} \in T/\theta_1$, then $a_{\theta_2} \in T/\theta_2$ and $(a_{\theta_1}, a_{\theta_2}) \in T'$. Ditto the other way around. Therefore $T' \subseteq T/\theta_1 \times T/\theta_2$ is a subdirect product. ■

Definition 3.8 *A lattice L is said to be **subdirectly reducible** if there exists a pair of congruences $\theta_1, \theta_2 \in Con(L) \setminus \{\Delta\}$ such that $\theta_1 \cap \theta_2 = \Delta$. L is said to be **subdirectly irreducible** if it is not subdirectly reducible, that is for all pairs of congruences $\theta_1, \theta_2 \in Con(L) \setminus \{\Delta\}$, $\theta_1 \cap \theta_2 \neq \Delta$.*

Remark 3.1 *Note that if L is subdirectly irreducible, then L is directly irreducible. We note also that a finite lattice is subdirectly irreducible if and only if $Con(L)$ has only one atom. Moreover any simple lattice is subdirectly irreducible but the converse does not hold. It is well known (Birkhoff [17]) that every lattice is a subdirect product of subdirectly irreducible lattices.*

3.6 Construction of subdirect products via join-morphisms

For a subdirect product $S \subseteq L_1 \times \cdots \times L_s$ where the L_i's are finite lattices, consider for all $1 \leq i \leq s$, the projections $\rho_i : S \to L_i$, and the "smallest pre-image" map

$$\begin{array}{llll} \sigma_i : & L_i & \longrightarrow & S \\ & x & \longmapsto & \bigwedge\{z \in S : \rho_i(z) = x\} \end{array}$$

One checks that σ_i is $\vee$-preserving, and thus all maps $\rho_{ij} := \rho_j \circ \sigma_i : L_i \to L_j$ are $\vee$-homomorphisms as well. Moreover $\rho_{jk} \circ \rho_{ij} \leq \rho_{ik}$ as is easily seen. This construction can be reversed. More precisely, the following holds.

Theorem 3.5 *[19] Suppose that $L_1, \cdots, L_s$ are lattices and that $\beta(i, j) : L_i \to L_j$ are $\vee$-preserving morphisms such that for all $i, j, k \in \{1, \cdots, s\}$,* (*a*) $\beta(i,i) = id_{L_i}$ *and*

(*b*) $\beta(i,k) \geq \beta(j,k) \circ \beta(i,j)$.

Then there is a subdirect product $L \subseteq L_1 \times \cdots \times L_s$ *such that*

$$\beta(i,j) = \rho_j \circ \sigma_i : \; L_i \xrightarrow{\sigma_i} L \xrightarrow{\rho_j} L_j.$$

Moreover, L *is* $\vee$*-generated by all the*

$$\sigma_i(a) = \big(\beta(i,1)(a), \beta(i,2)(a), \cdots, \beta(i,s)(a)\big),$$

where $a \in J(L_i)$ *and* $1 \leq i \leq s$. ■

Example 3.2 *Let* L_1, L_2, L_3 *and* $\beta(i,j)$ $(1 \leq i,j \leq 3)$ *be as in figure 3.7. We want to compute the subdirect product* $L \subseteq L_1 \times L_2 \times L_3$ *such that* $\beta(i,j) = \rho_j \circ \sigma_i$.

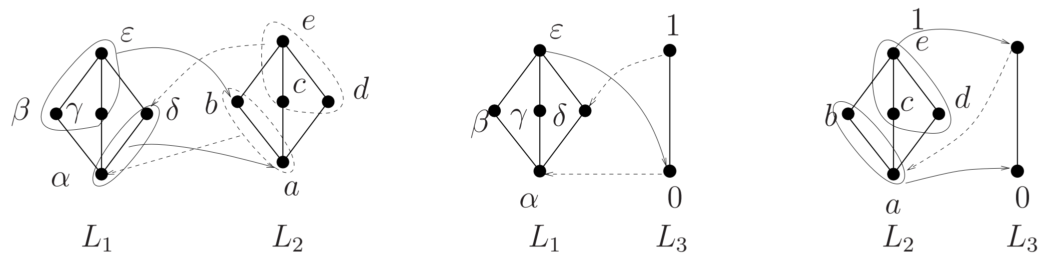


Figure 3.7: For a fixed (i,j), $\beta(i,j)$ is defined with solid lines nd $\beta(j,i)$ is defined with dashed lines.

One can easily check by inspection that the $\beta(i,j)$*'s satisfy* (*a*) *and* (*b*) *of theorem 3.5. We now determine the* $\sigma_i(a)$*'s:*

$$\sigma_1(\alpha) = (\beta(1,1)(\alpha), \beta(1,2)(\alpha), \beta(1,3)(\alpha)) = (\alpha, a, 0) =: \alpha a 0$$
$$\sigma_1(\beta) = (\beta(1,1)(\beta), \beta(1,2)(\beta), \beta(1,3)(\beta)) = (\beta, b, 0) =: \beta b 0$$

In the same manner, one can show that:

$$\sigma_1(\gamma) = \gamma b0, \quad \sigma_1(\delta) = \delta a0, \quad \sigma_1(\varepsilon) = \varepsilon b0, \quad \sigma_2(a) = \alpha a0, \quad \sigma_2(b) = \alpha b0,$$
$$\sigma_2(c) = \delta c1, \quad \sigma_2(d) = \delta d1, \quad \sigma_2(e) = \delta e1, \quad \sigma_3(0) = \alpha a0, \quad \sigma_3(1) = \delta a1.$$

By theorem 3.5, L *is* $\vee$*-generated by* $\mathcal{S} := \{\sigma_1(\beta), \sigma_1(\gamma), \sigma_1(\delta), \sigma_2(b), \sigma_2(c), \sigma_2(d), \sigma_3(1)\}$. *Thus* $\mathcal{S}$ *necessarily contains* $J(L)$ *(plus possibly some more elements) and* L *is obtained by taking all suprema of elements of* $\mathcal{S}$. *The Hasse diagram of* L *(with the elements of* S *circled) is given in figure 3.8. One checks that* $\beta(i,j) = \rho_j \circ \sigma_i$ *for all* $1 \leq i,j \leq 3$.

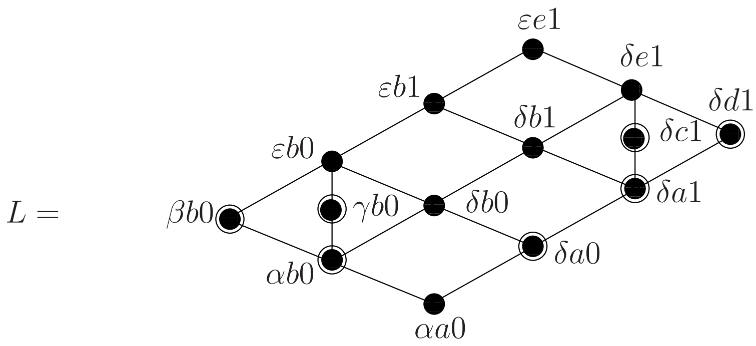


Figure 3.8: Construction of a subdirect product.

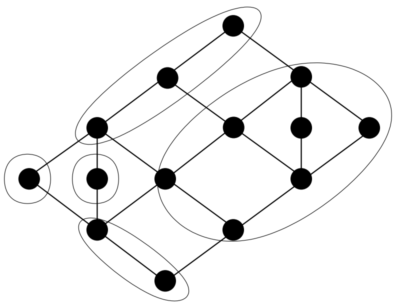


Figure 3.9: $L/ker(\rho_1) \cong L_1$.

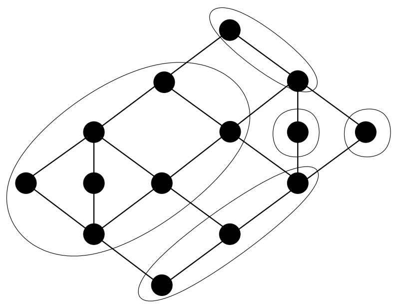


Figure 3.10: $L/ker(\rho_2) \cong L_2$.

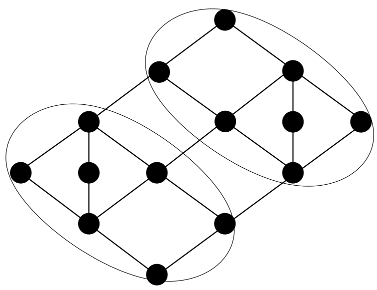


Figure 3.11: $L/ker(\rho_3) \cong L_3$.

Chapter 4

Distributive lattices

4.1 Representation of finite distributive lattices

In the lattice $(\mathcal{P}(X), \subseteq)$ the equality $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ holds for all subsets A, B, C of X. However this equality is not true in all lattices as one can easily check with the Diamond or the Pentagon (see figure 4.1).

Proposition 4.1 *Let $(L, \leq)$ be a lattice. Then the following assertions are equivalent.*

(i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *for all* $x, y, z \in L$.

(ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ *for all* $x, y, z \in L$.

(iii) $(x \vee y) \wedge z \leq x \vee (y \wedge z)$ *for all* $x, y, z \in L$.

Proof: For the proof of this theorem, see [15] or [18]. ■

Definition 4.1 *A lattice is **distributive** if it satisfies one of the equivalent statements of the above proposition.*

Example 4.1 1. *Every chain is a distributive lattice.*

2. $(\mathcal{P}(X), \subseteq)$ *is distributive for any set X. Hence the ideal lattice $(Id(P), \subseteq)$ of any poset P is distributive as a sublattice of the distributive lattice $(\mathcal{P}(P), \subseteq)$.*

3. $(\mathbb{N}, |)$ *is a distributive lattice.*

4. *The lattices M_3 and N_5 are not distributive. In fact, for M_3, $p \vee (q \wedge r) = p \neq t = (p \vee q) \wedge (p \vee r)$, and for N_5, $b \vee (a \wedge c) = b \neq a = (b \vee a) \wedge (b \vee c)$.*

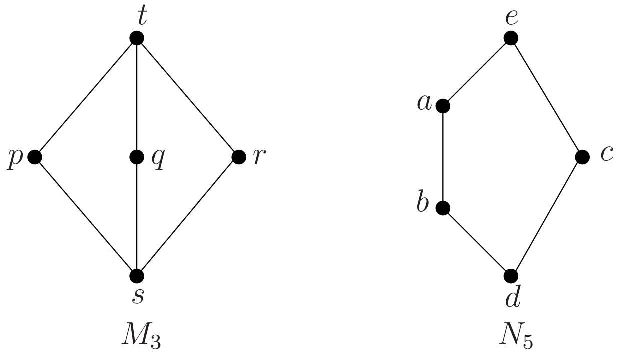


Figure 4.1: The lattices M_3 and N_5.

Since distributivity is inherited by sublattices, M_3 and N_5 cannot appear as sublattices in any distributive lattice. Interestingly, the converse holds as well.

Theorem 4.1 *[16] A lattice is distributive if and only if it contains no sublattice isomorphic either to the Pentagon or the Diamond.*

Theorem 4.2 *[1]**(Birkhoff representation theorem for finite distributive lattices)** A finite lattice is distributive if and only if it is isomorphic to the ideal lattice of some poset.*

Proof: Given any finite lattice L, one verifies that

$$\begin{array}{rcll} J: & L & \longrightarrow & \Big(Id\big(J(L)\big), \subseteq\Big) \\ & a & \longmapsto & J(a) \;=\; \{x \in J(L) : x \leq a\} \;=\; \downarrow a \cap J(L) \end{array}$$

is a $\wedge$-morphism from L into the ideal lattice of its join-irreducible elements. Exactly if L is distributive, J is moreover onto and $\vee$-preserving. In this case the embedding is cover preserving. ■

Example 4.2 *As an example, take the non-distributive lattice $L = N_5$ above with $J(N_5) = \{a, b, c\}$. Then $J: N_5 \to Id(J(N_5), \subseteq)$ is neither $\vee$-preserving nor surjective: $J(b \vee c) = \{a, b, c\} \neq \{b\} \cup \{c\} = J(b) \cup J(c)$ and one checks that $\{b, c\}$ is not in the range of J.*

Proposition 4.2 *[16] Let L be a bounded distributive lattice, then the complement of any element, when it exists, is unique and will be denoted by a'. Further if a, b are complemented, then so are $a \wedge b$ and $a \vee b$ and we have $(a \wedge b)' = a' \vee b'$ and $(a \vee b)' = a' \wedge b'$. The two last equalities are known as the* ***De Morgan's identities.***

Proof: Let b, c be two complements of a. Then $b = b \wedge (a \vee c)$ since $a \vee c = 1$. So $b = (b \wedge a) \vee (b \wedge c)$ since L is distributive. But $b \wedge a = 0$. Hence $b = b \wedge c$ and then $b \leq c$. Similarly $c \leq b$. Therefore $b = c$. Using the distributivity, one shows that $(a' \vee b') \wedge (a \wedge b) = 0$ and $(a' \vee b') \vee (a \wedge b) = 1$. That is $a' \vee b'$ is the complement of $a \wedge b$. ■

Definition 4.2 *A complemented bounded distributive lattice is called* ***Boolean lattice****.*

Note that $(\mathcal{P}(X), \subseteq)$ is a Boolean lattice for any set X. Conversely, if L is a finite Boolean lattice, then $L \cong (\mathcal{P}(X), \subseteq)$ where X is the set of atoms of L.

4.2 Congruences and distributivity

Theorem 4.3 *[18]* ***(Funayama and Nakayama[1940])***
The congruence lattice of any lattice is distributive.

Proof: Let L be a lattice. For $x, y, z \in L$, we set

$$M(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

Let θ, ρ and τ be three congruence relations on L. Then we know that

$$(\theta \wedge \rho) \vee (\theta \wedge \tau) \leq \theta \wedge (\rho \vee \tau).$$

Let us prove the converse inequality. Suppose that $a[\theta \wedge (\rho \vee \tau)]b$. Then $a\theta b$ and $a(\rho \vee \tau)b$. Hence there exists a sequence $x_0, x_1, \cdots, x_n$ such that $x_0 = a$, $x_n = b$ and $x_i \rho x_{i+1}$ or $x_i \tau x_{i+1}$ for $i < n$. By the transitivity of ρ and τ, we can choose this sequence such that

$$\begin{cases} x_i \rho x_{i+1} & \text{for} \quad \text{all} \quad \text{even } i < n \\ x_i \tau x_{i+1} & \text{for} \quad \text{all} \quad \text{odd } i < n. \end{cases}$$

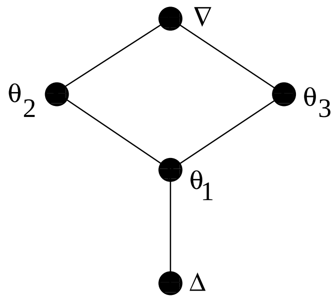
On the other hand $a\theta b$ implies $(a \wedge a)\theta(a \wedge b)$ and $(a \wedge x_i)\theta(b \wedge x_i)$ for all $i \leq n$ since θ is a congruence. Hence for all $i \leq n$

$$[(a \wedge b) \vee (b \wedge x_i) \vee (x_i \wedge a)]\theta[(a \wedge a) \vee (a \wedge x_i) \vee (x_i \wedge a)], \quad i.e. \ M(a, b, x_i)\theta M(a, a, x_i).$$

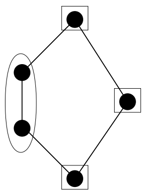
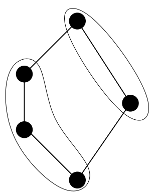
But $M(a, b, x_i)\theta M(a, a, x_i) = a = M(a, a, x_{i+1})\theta M(a, b, x_{i+1})$ implies by transitivity that $M(a, b, x_i)\theta M(a, b, x_{i+1})$. Further for all even $i < n$, $x_i \rho x_{i+1}$ implies that $M(a, b, x_i)\rho M(a, b, x_{i+1})$. Therefore

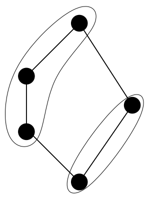
$M(a,b,x_i)(\theta \wedge \rho)M(a,b,x_{i+1})$ for all even $i < n$. Similarly for all odd $i < n$, one proves that $M(a,b,x_i)(\theta \wedge \tau)M(a,b,x_{i+1})$. Since $a = M(a,b,a) = M(a,b,x_0)$ and $b = M(a,b,b) = M(a,b,x_n)$, we can conclude that the sequence $a = M(a,b,x_0), M(a,b,x_1), \cdots, M(a,b,x_n) = b$ satisfies $M(a,b,x_i)(\theta\wedge\rho)M(a,b,x_{i+1})$ or $M(a,b,x_i)(\theta\wedge\tau)M(a,b,x_{i+1})$ for all $i < n$. Hence $a[(\theta\wedge\rho)\vee(\theta\wedge\tau)]b$, which implies that $\theta\wedge(\rho\vee\tau) \leq (\theta\wedge\rho)\vee(\theta\wedge\tau)$. ■

Example 4.3 *For instance $Con(N_5)$ =*



is distributive, where

N_5/θ_1 =  ; N_5/θ_2 =  *and*

N_5/θ_3 = 

Theorem 4.4 *Let L be a finite distributive lattice. Then $p \in L$ is prime if and only if $p \in J(L)$. Further $|J(L)| = h(L)$.*

Proof: We have already shown (see theorem 3.1) that if p is prime then $p \in J(L)$. Conversely, suppose that $p \in J(L)$, if $p \leq a \vee b$, then $p = p \wedge (a \vee b) = (p \wedge a) \vee (p \wedge b)$ by distributivity. Hence $p = p \wedge a$ or $p = p \wedge b$, i.e. $p \leq a$ or $p \leq b$. So p is prime. Now let $p_1, p_2, \cdots, p_n$ be the join-irreducibles of L, then trivially[1] $p_1 \vee p_2 \vee \cdots \vee p_n = 1$. Renumber the p_i's so that $p_i < p_j$ implies $i < j$. If $p_1 \vee p_2 \vee \cdots \vee p_j = p_1 \vee p_2 \vee \cdots \vee p_j \vee p_{j+1}$ for some $j \in \{1, 2, \cdots, n-1\}$, then $p_{j+1} \leq p_1 \vee p_2 \vee \cdots \vee p_j$. Therefore $p_{j+1} \leq p_i$

[1] Since any element of a finite lattice is a join of join-irreducibles by proposition 2.3 on page 10.

for some $i \in \{1, 2, \cdots, j\}$ since p_{j+1} is prime, which is a contradiction. So the chain $0 < p_1 < p_1 \vee p_2 < \cdots < p_1 \vee p_2 \vee \cdots \vee p_n = 1$ is a maximal chain of length n. ■

Theorem 4.5 *Let L be a finite distributive lattice. Then $Con(L)$ is a Boolean lattice with $h\big(Con(L)\big) = h(L)$.*

Proof: For each $p \in J(L)$, set $\theta_p = ker(\widetilde{p})$. Then

$$\begin{aligned} a\Big(\bigwedge_{p\in J(L)} \theta_p\Big)b &\iff (\forall p \in J(L))\ a\theta_p b \\ &\iff (\forall p \in J(L))(a \geq p \iff b \geq p) \\ &\iff J(a) = J(b) \\ &\iff a = b. \end{aligned}$$

So $\bigwedge_{p\in J(L)} \theta_p = \Delta$ is the zero element in $Con(L)$. But $Con(L)$ is distributive by theorem 4.3, so there is a set X with $|X| = d\big(Con(L)\big)$ such that L is cover preserving embedding into $\mathcal{P}(X)$ (theorem 4.2). Therefore each co-atom $\theta_p \in Con(L)$ corresponds to some $X \setminus \{x_p\} \in \mathcal{P}(X)$. From $\bigwedge_{p\in J(L)} \theta_p = \Delta$ follows that $\bigcap_{p\in J(L)} \big(X \setminus \{x_p\}\big) = \varnothing$, i.e. $X = \{x_p | p \in J(L)\}$, i.e. $Con(L) \cong \mathcal{P}(X)$, i.e. $h\big(Con(L)\big) = h\big(\mathcal{P}(X)\big) = |J(L)|$. ■

4.3 Distributive lattices as subdirect products

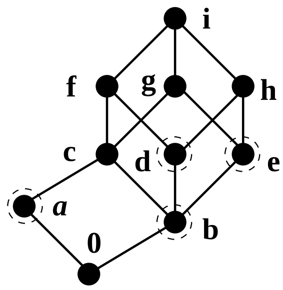
Theorem 4.6 *[1](**Fundamental theorem of Birkhoff**) A distributive lattice is subdirectly irreducible if and only if it is isomorphic to the two-element distributive lattice **2**. Hence each distributive lattice is a subdirect product of two-element lattices.*

Proof: Suppose that D is a distributive lattice and that D contains an element a different from 0 and 1 (i.e. $D \ncong \mathbf{2}$). Define two functions $\omega : D \to D$ and $\sigma : D \to D$ by $\omega(x) = x \wedge a$ and $\sigma(x) = x \vee a$. Then obviously ω and σ are morphisms since D is distributive. Set $\theta_1 = ker(\omega)$ and $\theta_2 = ker(\sigma)$, then $\theta_1, \theta_2 \in Con(L)$. Further if $(x, y) \in \theta_1 \cap \theta_2$, then $x \wedge a = y \wedge a$ and $x \vee a = y \vee a$. Hence

$$\begin{aligned} x &= x \wedge (x \vee a) = x \wedge (y \vee a) = (x \wedge y) \vee (x \wedge a) \\ &= (x \wedge y) \vee (y \wedge a) = y \wedge (x \vee a) = y \wedge (y \vee a) \\ &= y. \end{aligned}$$

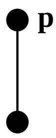
Therefore $\theta_1 \cap \theta_2 = \Delta$. But $(1, a) \in \theta_1$ and $(0, a) \in \theta_2$ imply that $\theta_1, \theta_2 \in Con(L) \setminus \Delta$. So D is subdirectly reducible. ■

Example 4.4 *Consider the distributive lattice D :=*



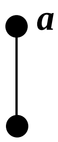
where
$J(D) = \{a,\ b,\ d,\ e\}$. Recall from theorem 4.4 that the co-atoms of $Con(D)$ correspond bijectively to $J(D)$. Namely for $p \in J(D)$, the two congruence classes are $\uparrow p$ and $D \backslash \uparrow p$. A shorthand notation is

. These



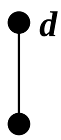
$(p \in J(D))$ "are" the subdirectly irreducible factors of D. In our case, we have:

$D \longrightarrow$

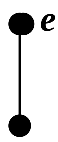


$\times$

$\times$



$\times$



$$\begin{array}{ccccccccccc} \mathrm{c} & \longmapsto & (1 & , & 1 & , & 0 & , & 0) & =: & \vec{c} \\ \mathrm{h} & \longmapsto & (0 & , & 1 & , & 1 & , & 1) & =: & \vec{h} \\ \text{etc.} & & & & & & & & & & \end{array}$$

Notice that $\overrightarrow{J} := \{\vec{a}, \vec{b}, \vec{d}, \vec{e}\}$ is the set of join-irreducibles of D's isomorphic copy $\overrightarrow{D} \subseteq 2^4$, so e.g. $\vec{h} = \vec{d} \vee \vec{e}$.

4.4 Free distributive lattices via filters

Definition 4.3 *Let $(P, \leq)$ be a poset. The* ***free distributive lattice*** *generated by P is the unique (up to isomorphism) distributive lattice denoted by $F\mathcal{D}(P)$ with the following properties.*

(i) There is a generating set $P' \subseteq F\mathcal{D}(P)$ such that P' endowed with the induced order from $F\mathcal{D}(P)$ is isomorphic to P.

(ii) If D is a distributive lattice and $\phi : P' \to D$ is an order preserving map, then ϕ extends to a lattice morphism $\Phi : F\mathcal{D}(P) \to D$.

We shall see in section 6.1 that such a lattice $F\mathcal{D}(P)$ and many other kinds of "free" lattices do in fact exist. The second property (ii) is called **universal mapping property**. Observe that since $F\mathcal{D}(P)$ is distributive and any element x of $F\mathcal{D}(P)$ can be expressed in terms of elements of P, x can be written as $x = \bigvee_{S \in K} \bigwedge S$ for some finite set K of finite antichains of P. Hence the join-irreducibles of $F\mathcal{D}(P)$ must all be of the form $\bigwedge S$ where $\emptyset \neq S \subsetneq P$ is a finite antichain [2]. Conversely (see [20]) every such element is join-irreducible. In particular any element of P is join-irreducible in $F\mathcal{D}(P)$. We conclude that

$$J\big(FD(P)\big) = \{\bigwedge S : \ S \text{ antichain } \text{ of } P \text{ and } \emptyset \neq S \neq P\},$$

is the set of nonzero join-irreducibles of $F\mathcal{D}(P)$. Dually any element of $F\mathcal{D}(P)$ can be expressed as $\bigwedge_{S \in K} \bigvee S$, where K is a finite set of finite antichains of P. Hence the meet-irreducible elements of $F\mathcal{D}(P)$ are precisely the elements of the form $\bigvee S$ where S is a proper antichain of P. In particular any element of P is meet-irreducible in $F\mathcal{D}(P)$. Therefore any element of P is doubly irreducible in $F\mathcal{D}(P)$.

Lemma 4.1 *Let P be a finite poset. Then the map*

$$\begin{array}{cccc} \lambda : & \Big(Fil^*(P), \supseteq\Big) & \longrightarrow & \Big(J\big(F\mathcal{D}(P)\big), \leq\Big) \\ & S & \longmapsto & \bigwedge S \end{array}$$

is a poset isomorphism.

Proof:
Only the injectivity of λ is nontrivial. So consider $R \not\supseteq S$ in the poset $\Big(Fil^*(P), \supseteq\Big)$. Define $\rho : P \longrightarrow \mathbf{2}$ by

$$\rho(a) = \begin{cases} 1 & \text{if } a \in R \\ 0 & \text{if } a \notin R. \end{cases}$$

This clearly order preserving surjective map extends to an epimorphism $\Phi : F\mathcal{D}(P) \longrightarrow \mathbf{2}$ with $\Phi\big(\bigwedge R\big) = 1$ but $\Phi\big(\bigwedge S\big) = 0$ (at least one $a \in S$ is not in R). Hence $\bigwedge R \not\leq \bigwedge S$ in $\Big(J\big(FD(P)\big), \leq\Big)$. ■

[2] Notice that $\bigwedge P = 0$ is not join-irreducible by definition. If $\bigwedge \emptyset = 1$ is join-irreducible, then $1 = \overline{p}$ where $\overline{p}$ is the biggest element of $(P, \leq)$. Therefore $1 = \bigwedge\{\overline{p}\}$, i.e. $S = \emptyset$ is never necessary.

Theorem 4.7 *Let $(P, \leq)$ be a finite poset. Then the free distributive lattice $FD(P)$ is isomorphic to $Id(Fil^*(P), \supseteq)$.*

Proof: By lemma 4.1, $\big(Fil^*(P), \supseteq\ \big) \cong (J, \leq)$ where $J := J\big(FD(P)\big)$. Hence, using Birkhoff's theorem 4.2, $FD(P) \cong Id(J, \leq) \cong Id\big(Fil^*(P)\big), \supseteq)$.
■

Corollary 4.1 *The free distributive lattice $F\mathcal{D}(P)$ is finite if and only if P is finite. In this case $|F\mathcal{D}(P)| \leq 2^{2^{|P|}}$.*

Proof: This is clear by the previous theorem 4.7. ■

4.5 Alternative method for computing $F\mathcal{D}(P)$

In this section, we describe another method to compute $F\mathcal{D}(P)$. As opposed to the method via the proper filters of P studied in the previous section 4.4, it can be generalized (see chapter 6) to the computation of free modular lattices.

Definition 4.4 *Let $(P, \leq)$ be a finite poset and let L be a lattice. A **P-labelling** of L is a couple (λ, L_λ) where $L_\lambda \cong L$ and $\lambda : P \to L_\lambda$ is an order preserving map with the property that $\lambda(P)$ generates L_λ. Two P-labellings (λ_1, L_1) and (λ_2, L_2) are said to be **equivalent** if there exists an isomorphism $\alpha : L_1 \to L_2$ such that $\lambda_2 = \alpha \circ \lambda_1$.*

$$\begin{tikzcd}
P \arrow[r, "\lambda_2"] \arrow[d, "\lambda_1"'] & L_2 \\
L_1 \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

Figure 4.2: Commutative diagram showing two equivalent P-labellings of L.

Definition 4.5 *Let (L_1, λ_1) and (L_2, λ_2) be two P-labellings of a lattice L. A map $\beta : L_1 \to L_2$ is called a **morphism** if*

(i) β is $\vee$-preserving (in particular order preserving) and,

(ii) $\beta\big(\lambda_1(a)\big) \leq \lambda_2(a)$ for all $a \in P$, i.e. β sends labels below labels.

The set of morphisms between two P-labellings (λ_i, L_i) and (λ_j, L_j) of a lattice L, ordered by $\alpha \leq \beta \Leftrightarrow \alpha(x) \leq \beta(x)$ for all $x \in L_i$, clearly contains a greatest element, denoted $\beta_{ij} : L_i \to L_j$.

Lemma 4.2 *Let $\lambda_i : P \to L_i$ $(1 \leq i \leq s)$ be a collection of P-labellings. Then*

(a) $\beta_{ii} = id_{L_i}$ *for all* $i \in \{1, 2, \cdots, s\}$ *and,*

(b) $\beta_{ik} \geq \beta_{jk} \circ \beta_{ij}$ *for all* $i, j, k \in \{1, 2, \cdots, s\}$.

Proof: The proof of (a) is obvious. To prove (b), observe that $\beta_{jk} \circ \beta_{ij}$ is a morphism from L_i to L_k and β_{ik} is the biggest morphism from L_i to L_k, hence $\beta_{ik} \geq \beta_{jk} \circ \beta_{ij}$. ■

We now focus on distributive lattices. Let P be a finite poset and let $D_1, D_2, \cdots, D_s$ be a maximal collection of pairwise non-equivalent P-labellings of **2**. By theorem 3.5 and lemma 4.2, the morphisms β_{ij} $(1 \leq i, j \leq s)$ yield a certain subdirect product $L \subseteq D_1 \times \cdots \times D_s$. We are going to show that $L \cong F\mathcal{D}(P)$. More specifically, denote by 1 the maximum element of D_i and define $\psi_i : D_i \to D_1 \times D_2 \times \cdots \times D_s$ by $\psi_i(x) = \big(\beta_{i1}(x), \beta_{i2}(x), \cdots, \beta_{is}(x)\big)$. Then the set $\mathcal{K} = \{\psi_1(1), \psi_2(1), \cdots, \psi_s(1)\}$ is a poset where the order is defined componentwise. We will show in theorem 4.8 that $F\mathcal{D}(P) \cong Id(\mathcal{K}, \leq)$.

Lemma 4.3 *Let $(P, \leq)$ be a finite poset. Then there is a bijection between the proper filters of P and the P-labellings of the two-element lattice **2**.*

Proof: If $\lambda : P \to D$ is any P-labelling of **2**, then $\lambda(P) \subseteq D$ generates **2** by definition. So $\lambda^{-1}(1) \neq \emptyset$ and $\lambda^{-1}(1) \neq P$. Moreover if $a \in \lambda^{-1}(1)$ and $a \leq b$, then since λ is order preserving, $1 = \lambda(a) \leq \lambda(b)$. It follows that $\lambda(b) = 1$, i.e. $b \in \lambda^{-1}(1)$. So $\lambda^{-1}(1)$ is a proper filter of P. Conversely, every proper filter of P clearly arises that way. ■

Let $Fil^*(P) = \{f_1, f_2, \cdots, f_s\}$ be the set of proper order filters of P and let $\lambda_i : P \to D_i$ $(1 \leq i \leq s)$ be the P-labellings of **2** such that $f_i = \lambda_i^{-1}(1)$, i.e. the labels of the top elements of D_i are precisely the elements of f_i. Then $Fil^*(P)$ with the reverse inclusion is a poset. For $1 \leq i, j \leq s$, we recall that $\beta_{ij} : D_i \to D_j$ is the biggest $\vee$-preserving map such that

$$\beta_{ij}\big(\lambda_i(a)\big) \leq \lambda_j(a) \text{ for all } a \in P. \tag{4.5.1}$$

Lemma 4.4 *For all $1 \leq i, j \leq s$,*

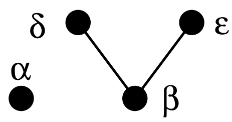
$$\beta_{ij}(1) = 1 \quad if \ and \ only \ if \quad f_i \subseteq f_j.$$

Proof: Suppose that $\beta_{ij}(1) = 1$ and let $a \in f_i = \lambda_i^{-1}(1)$. Then $\lambda_i(a) = 1$. But $\lambda_j(a) \geq \beta_{ij}\big(\lambda_i(a)\big) = \beta_{ij}(1) = 1$ implies that $\lambda_j(a) = 1$. Hence $a \in \lambda_j^{-1}(1) = f_j$. Conversely, suppose that $f_i \subseteq f_j$. Then $\beta : D_i \to D_j$, where $\beta(0) := 0$ and $\beta(1) := 1$ trivially maps labels below corresponding labels, and so $\beta_{ij} \geq \beta$, i.e. $\beta_{ij} = \beta$. ■

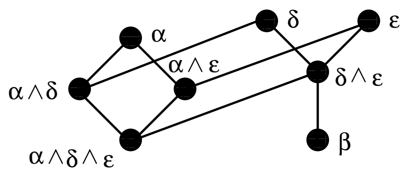
Theorem 4.8 *The free distributive lattice $F\mathcal{D}(P)$ is isomorphic to $Id(\mathcal{K}, \leq)$.*

Proof: Recall that $\mathcal{K} = \{\psi_1(1), \psi_2(1), \cdots, \psi_s(1)\}$ where s is the number of pairwise non-equivalent P-labellings of $\mathbf{2}$ and $\psi_i(1) = \big(\beta_{i1}(1), \beta_{i2}(1), \cdots, \beta_{is}(1)\big)$. Consider the map $\sigma : (\mathcal{K}, \leq) \to (Fil^*(P), \supseteq)$ defined by $\sigma(\psi_i(1)) = f_i$. It is clear that σ is surjective by construction. We will prove that σ is an order isomorphism.
Suppose first that $\psi_i(1) \leq \psi_j(1)$, then $\beta_{ik}(1) \leq \beta_{jk}(1)$ for all $1 \leq k \leq s$. So $\{f_k \in Fil^*(P) | \beta_{ik}(1) = 1\} \subseteq \{f_k \in Fil^*(P) | \beta_{jk}(1) = 1\}$. By lemma 4.4, $\{f_k \in Fil^*(P) | f_i \subseteq f_k\} \subseteq \{f_k \in Fil^*(P) | f_j \subseteq f_k\}$. Therefore $f_i \in \{f_k \in Fil^*(P) | f_j \subseteq f_k\}$, hence $f_i \supseteq f_j$ and so σ is order preserving.
Conversely, suppose that $\psi_i(1) \not\leq \psi_j(1)$. Then $\beta_{ik}(1) \not\leq \beta_{jk}(1)$ for some k, i.e. $\beta_{ik}(1) = 1$, $\beta_{jk}(1) = 0$. By lemma 4.4, it follows that $f_i \subseteq f_k$ and $f_j \not\subseteq f_k$. This implies that $f_i \not\supseteq f_j$. So $(\mathcal{K}, \leq) \cong (Fil^*(P), \supseteq)$, and by theorem 4.7 it follows that $Id(\mathcal{K}, \leq) \cong Id\big(Fil^*(P), \supseteq\big) \cong FD(P)$. ■

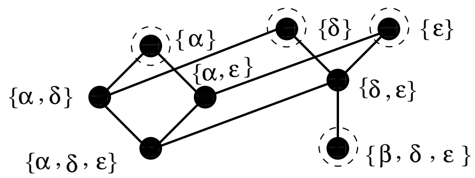
Example 4.5 *Consider for instance the poset* $(P, \leq)$ $:=$



In $F\mathcal{D}(P)$ the join-irreducibles are

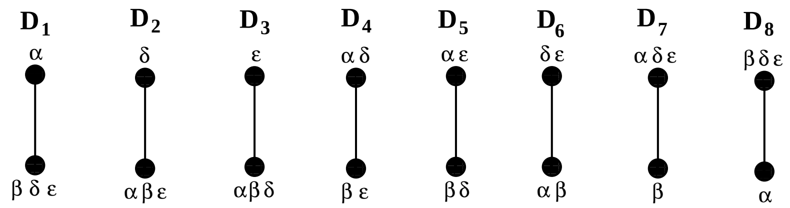


They correspond to the proper filters in reverse ordering:



The principal filters corresponding to the elements of P are indicated, for instance $\uparrow\beta = \{\beta, \delta, \varepsilon\} =: \beta\delta\varepsilon$. Usually $|Fil^*(P)| > |P|$, i.e. $F\mathcal{D}(P)$ has usually more join-irreducibles than doubly irreducibles. Similar to general distributive lattices (section 4.3) let us visualize the subdirect decomposition

$F\mathcal{D}(P) \longrightarrow$



$$\begin{array}{rccccccccc}
\delta\wedge\varepsilon & \longmapsto & (0 & 0 & 0 & 0 & 0 & 1 & 1 & 1) \\
\alpha & \longmapsto & (1 & 0 & 0 & 1 & 1 & 0 & 1 & 0) \\
\varepsilon & \longmapsto & (0 & 0 & 1 & 0 & 1 & 1 & 1 & 1)
\end{array}$$

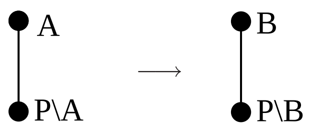
Here, additionally to the join-irreducible A on top (alias $A \in Fil^*(P)$), we write $P \setminus A$ on the bottom of $\mathbf{2}$. In this way we obtain the P-labellings of $\mathbf{2}$ in the same sense of definition 4.4. Once the 8 octuples corresponding to the join-irreducibles of $F\mathcal{D}(P)$ are computed, the whole lattice $F\mathcal{D}(P)$ is determined.

__**Claim:**__ The octuples can be written using the morphisms $\beta_{ij} : D_i \longrightarrow D_j$ as follows:

$$\begin{array}{ccccccc}
\delta\wedge\varepsilon & \longmapsto & (\beta_{61}(1), & \beta_{62}(1), & \beta_{63}(1), & \cdots, & \beta_{68}(1)) \\
\alpha & \longmapsto & (\beta_{11}(1), & \beta_{12}(1), & \beta_{13}(1), & \cdots, & \beta_{18}(1)) \\
\varepsilon & \longmapsto & (\beta_{31}(1), & \beta_{32}(1), & \beta_{33}(1), & \cdots, & \beta_{38}(1))
\end{array}$$

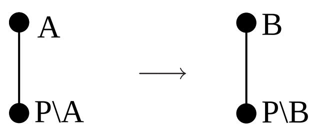
Proof: Recall, for the general P-labellings L_i and L_j the morphism $\beta_{ij} : L_i \to L_j$ is the biggest $\vee$-preserving map that sends labels below corresponding labels. In the case of $L_i = D_i = \mathbf{2}$, there are just two types of β_{ij}:

β_{ij} :



is $\beta_{ij} =$id $\Leftrightarrow$ $\beta_{ij}(1) = 1$ $\Leftrightarrow$ $A \subseteq B$.

β_{ij} :



is $\beta_{ij} \equiv 0$ $\Leftrightarrow$ $\beta_{ij}(1) = 0$ $\Leftrightarrow$ $A \nsubseteq B$.

For instance, $\beta_{65}(1) = 0$ since $\{\delta, \varepsilon\} \nsubseteq \{\alpha, \varepsilon\}$, but $\beta_{67}(1) = 1$ since $\{\delta, \varepsilon\} \subseteq \{\alpha, \delta, \varepsilon\}$. ■

The steps involved in the calculation of $F\mathcal{D}(P)$ via P-labellings can be summarized as follows:

Step 1: Compute all the non-equivalent P-labellings $D_1, D_2, \cdots, D_s$ of the two-element lattice **2** (or equivalently, compute all the proper filters of P).

Step 2: Compute the morphisms $\beta_{ij} : D_i \to D_j$ and set $\mathcal{K} = \{\psi_1(1), \psi_2(1), \cdots, \psi_r(1)\}$, where $\psi_i(1) = (\beta_{i1}(1), \beta_{i2}(1), \cdots, \beta_{is}(1))$. Then $Id(\mathcal{K}, \leq) \cong F\mathcal{D}(P)$ where the order in $\mathcal{K}$ is taken componentwise.

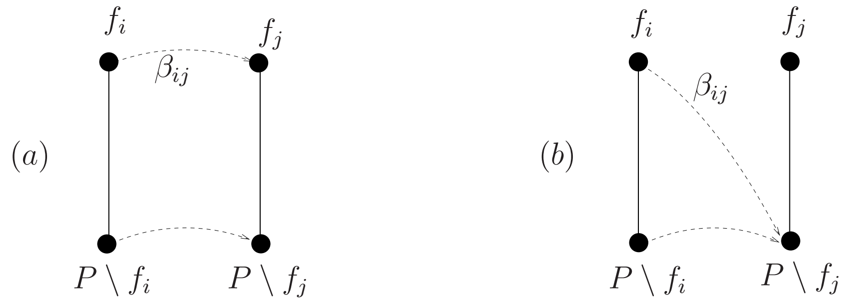


Figure 4.3: (a) $f_i \subseteq f_j$, so $\beta_{ij}(1) = 1$. (b) $f_i \nsubseteq f_j$, so $\beta_{ij} \equiv 0$.

We will later explain how to efficiently compute the elements of $Id(\mathcal{K}, \leq)$ via the algorithm we will introduce in chapter 7.

Example 4.6 *Determine the free distributive lattice $F\mathcal{D}(P)$ and its subdirectly irreducible factors where P is the poset P of figure 4.4(a).*

<u>**Step 1:**</u> The 13 non-equivalent P-labellings of $\mathbf{2}$ are drawn in figure 4.4(b).

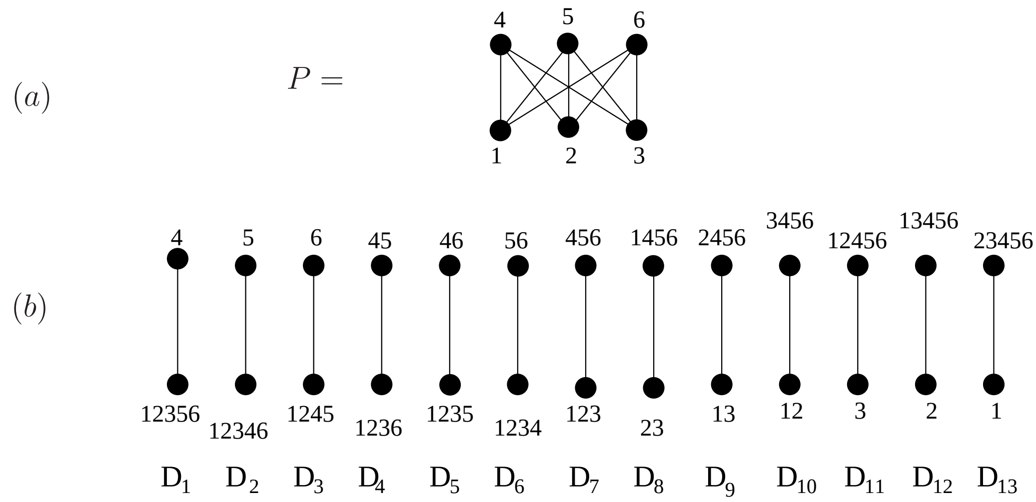


Figure 4.4: (a) The poset P under consideration (b) The 13 P-labellings of $\mathbf{2}$.

<u>**Step 2:**</u> By using lemma 4.4, one can easily compute the β_{ij}'s, for instance, $\beta_{12} = \beta_{13} = \beta_{16} \equiv 0$ and $\beta_{11}(1) = \beta_{14}(1) = \beta_{15}(1) = \beta_{17}(1) = \beta_{1i}(1) = 1$ for all $i \geq 8$. On the other hand, we compute the $\psi_i(1)$ as follows. We list the components of a vector as a string for the sake of simplicity.

$$\begin{array}{rclclcl} \psi_1(1) & = & (\beta_{1i})_{1\leq i\leq 13} & = & (1,0,0,1,1,0,1,1,1,1,1,1,1) & =: & 1001101111111 \\ \psi_2(1) & = & (\beta_{2i})_{1\leq i\leq 13} & = & (0,1,0,1,0,1,1,1,1,1,1,1,1) & =: & 0101011111111. \end{array}$$

In the same manner, we obtain

$$\begin{array}{rclrcl} \psi_4(1) & = & 0010111111111, & \psi_5(1) & = & 0000101111111, \\ \psi_6(1) & = & 0000011111111, & \psi_7(1) & = & 0000001111111, \\ \psi_8(1) & = & 0000000100111, & \psi_9(1) & = & 0000000010101, \\ \psi_{10}(1) & = & 0000000001011, & \psi_{11}(1) & = & 0000000000100, \\ \psi_{12}(1) & = & 0000000000010, & \psi_{13}(1) & = & 0000000000001. \end{array}$$

Letting $\mathcal{K} = \{\psi_i(1)\}_{1\leq i\leq 13}$, we conclude with theorem 4.8 that $F\mathcal{D}(P) \cong Id(\mathcal{K}, \leq)$. The Hasse diagram of $\mathcal{K}$ is shown in figure 4.5.

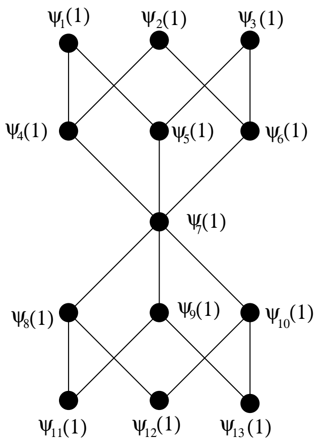


Figure 4.5: The poset $(\mathcal{K}, \leq)$

A computer subroutine called base-of-line.nb of the algorithm described in steps 1 and 2 has been implemented with the Mathematica package. The elements of $Id(\mathcal{K}, \leq)$ will be explicitly computed and its Hasse diagram will be drawn in chapter 7. The 13 subdirectly irreducible factors of $F\mathcal{D}(P)$ are the 13 P-labellings of **2** shown in figure 4.4.

Chapter 5

Modular lattices

5.1 Some preliminary results on modular lattices

Definition 5.1 *A lattice $(L, \leq)$ is said to be **modular** if for all $a, b, c \in L$*

$$a \leq b \quad \Rightarrow \quad a \vee (b \wedge c) = b \wedge (a \vee c). \tag{5.1.1}$$

Note that the inequality $a \vee (b \wedge c) \leq b \wedge (a \vee c)$ is trivial for all $a \leq b$ in any lattice L.

It is not difficult to see that any distributive lattice is modular. It can be shown that M_3 is modular but that N_5 is not. The lattice $Sub(M)$ of the submodules of a module over a ring is modular. Indeed let N_1, N_2 and N_3 be submodules of M such that $N_1 \subseteq N_2$. As seen above, it remains to show that $N_1 + (N_2 \cap N_3) \supseteq N_2 \cap (N_1 + N_3)$. So let $x \in N_2 \cap (N_1 + N_3)$. Then $x \in N_2$ and $x = a + b$ where $a \in N_1$ and $b \in N_3$. So $a \in N_2$ since $N_1 \subseteq N_2$. Therefore $b = x - a \in N_2$ since N_2 is a submodule of M. So $b \in N_2 \cap N_3$ and $x = a + b \in N_1 + (N_2 \cap N_3)$. So $(Sub(M), \cap, +)$ is a modular lattice.

Proposition 5.1 *[6] A lattice is modular if and only if it contains no sublattice isomorphic to N_5.*

In chapter 2, we observed that not every complemented lattice is relatively complemented. However, this is true in modular lattices.

Proposition 5.2 *[16] Every complemented modular lattice L is relatively complemented. In fact one proves that the complement of any element x of an interval $[a, b] \subseteq L$ is $a \vee (x' \wedge b)$.*

Theorem 5.1 *[16](**Dedekind transposition principle**)*
Let $(L, \leq)$ be a modular lattice and let $a, b \in L$. Consider the maps

$$\begin{array}{rccc} \phi_a : & [b, a \vee b] & \longrightarrow & [a \wedge b, a] \\ & x & \longmapsto & a \wedge x \end{array} \quad \text{and} \quad \begin{array}{rccc} \psi_b : & [a \wedge b, a] & \longrightarrow & [b, a \vee b] \\ & x & \longmapsto & b \vee x. \end{array}$$

Then ϕ_a and ψ_b are lattice isomorphisms and $\phi_a^{-1} = \psi_b$. Moreover, the image of a subinterval under either these functions is a transposed of that subinterval. Conversely, if $(L, \leq)$ is a lattice for which the maps ϕ_a and ψ_b are lattice isomorphisms for all $a, b \in L$, then L is a modular lattice.

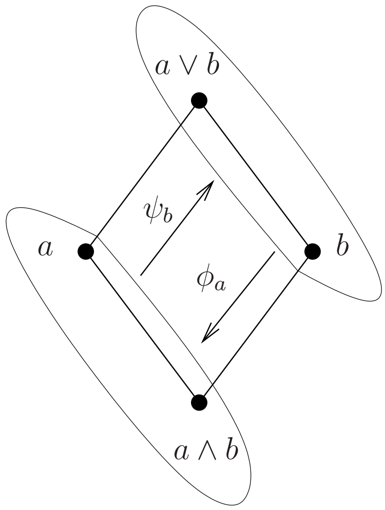


Figure 5.1: Illustration of the Dedekind transposition principle

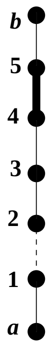
It follows immediately from theorem 5.1 that projective intervals of a modular lattice are isomorphic and that for any modular lattice L and any distinct elements a, b, c :

(i) If both a and b cover c, then $a \vee b$ covers both a and b.

(ii) If c covers both a and b, then a and b both cover $a \wedge b$.

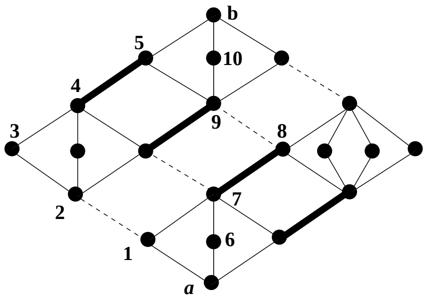
Every finite height modular lattice is graded and

$$h(a) + h(b) = h(a \wedge b) + h(a \vee b) \text{ for all } a, b \in L.$$

Moreover the number of representatives of classes of projective prime quotients within a maximal chain is an invariant. For instance, these two maximal a, b-chains: 

and 

of the modular lattice



each feature **1,1,4** prime quotients of the dotted, fat, and thin projectivity class respectively.

Theorem 5.2 *[16] Every finite height modular lattice L in which 1 is the join of a finite set of atoms is a complemented modular lattice of finite height.*

Proof: Let A be the set of atoms and $x \in L$. Set $d_0 = x$. If $d_i \neq 1$, then pick $a_i \in A$ such that $a_i \nleq d_i$ (such a_i exists otherwise $1 = \bigvee A \leq d_i$ which is a contradiction), and let $d_{i+1} = d_i \vee a_i$. Since A is finite, this construction stops after a finite number of steps, so there exists n such that

$$\begin{aligned} 1 = d_{n+1} &= d_n \vee a_n = d_{n-1} \vee a_{n-1} \vee a_n = \cdots \\ &= d_0 \vee a_0 \vee \cdots \vee a_n = x \vee a_0 \vee \cdots \vee a_n. \end{aligned}$$

Letting $y = a_0 \vee a_1 \vee \cdots \vee a_n$ yields $x \vee y = 1$. Let us prove that $x \wedge y = 0$. For all $i \leq n$, we have $d_i \wedge a_i = 0$ since a_i is an atom and $a_i \nleq d_i$. So by modularity, $h(d_i \vee a_i) = h(d_i) + h(a_i)$. That is,

$$h(x \vee a_0 \vee a_1 \vee \cdots \vee a_{i-1} \vee a_i) = h(x \vee a_0 \vee a_1 \vee \cdots \vee a_{i-1}) + h(a_i).$$

Hence by induction, we have $h(x \vee y) = h(x) + h(a_0) + \cdots + h(a_n)$. Moreover, $h(y) = h(a_0) + h(a_1) + \cdots + h(a_n)$. This implies that $h(x \vee y) = h(x) + h(y)$. Therefore $h(x \wedge y) = 0$, that is $x \wedge y = 0$. Hence y is a complement of x in L. ■

Definition 5.2 *Let $(L, \leq)$ be a lattice. We say that the interval $[a, b] \subseteq L$* ***transposes weakly down*** *into the interval $[c, d]$ and we denote $[a, b] \searrow_w [c, d]$ if $b = a \vee d$ and $c \leq a$. Dually $[a, b]$* ***transposes weakly up*** *into $[c, d]$ denoted $[a, b] \nearrow^w [c, d]$, if $a = b \wedge c$ and $b \leq d$. We say that $[a, b]$* ***transposes weakly*** *into $[c, d]$ if $[a, b]$ transposes either up or down into $[c, d]$. We say that $[a, b]$ is* ***weakly projective*** *into $[c, d]$ if there is a finite sequence*

$$[a_0, b_0] = [a, b], [a_1, b_1], \cdots, [a_n, b_n] = [c, d]$$

such that $[a_i, b_i]$ transposes weakly into $[a_{i+1}, b_{i+1}]$ for all $0 \leq i \leq n-1$. For instance in the following figure, $[a, b] \searrow_w [a_1, b_1] \nearrow_w [a_2, b_2] \searrow_w [c, d]$, so $[a, b]$ is weakly projective into $[c, d]$.

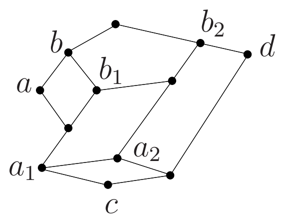


Figure 5.2: Illustration of weak projectivity.

Definition 5.3 *A lattice L is said to satisfy the* ***projectivity property*** *if whenever $[a, b]$ is weakly projective into $[c, d]$, then $[a, b]$ is projective into a subinterval of $[c, d]$.*

Theorem 5.3 *[16] Every modular lattice satisfies the projectivity property.*

Proof: Suppose that $[a, b] \nearrow^w [c, d]$. Then $a = b \wedge c$ and $b \leq d$. So $[a, b] = [b \wedge c, b]$ transposes up into $[c, b \vee c] \subseteq [c, d]$. Dually if $[a, b] \searrow_w [c, d]$, then $[a, b] = [a, a \vee d]$ transposes down into $[a \wedge d, d] \subseteq [c, d]$. So in either case $[a, b]$ transposes into a subinterval of $[c, d]$. Suppose that $[a_0, b_0]$ transposes weakly into $[a_1, b_1]$ which transposes weakly into $[a_2, b_2]$. Then from the previous arguments, $[a_0, b_0]$ transposes into a subinterval $[x_1, y_1] \subseteq [a_1, b_1]$ and $[a_1, b_1]$ transposes into a subinterval $[x_2, y_2] \subseteq [a_2, b_2]$. So $[a_1, b_1] \overset{\rho}{\cong} [x_2, y_2]$ and $[x_1, y_1]$ transposes into $\rho([x_1, y_1]) \subseteq [x_2, y_2]$ by Dedekind transposition principle. That is $[x_1, y_1]$ transposes into a subinterval of $[a_2, b_2]$. So $[a_0, b_0]$ is projective into a subinterval of $[a_2, b_2]$. The proof can now be completed by induction on the length of the chain of weak projectivity. ■

5.2 Congruences and modularity

The following lemma, which is valid in any lattice (see [16]), will be used to prove theorem 5.4, an important result which will allow us to show that $Con(L)$ is Boolean for any modular lattice L.

Lemma 5.1 *[16] **(R.P. Dilworth[1950])** Let L be a lattice with $a, b, c, d \in L$ such that $a \leq b$ and $c \leq d$. Then $(a, b) \in Cg(c, d)$ if and only if there is a sequence*

$$a = e_0 \leq e_1 \leq e_2 \leq \cdots \leq e_n = b$$

such that $[e_i, e_{i+1}]$ is weakly projective into $[c, d]$ for all $i \leq n$.

Theorem 5.4 *Let $(L, \leq)$ be a modular lattice and let $a, b \in L$ with $a \prec b$. Then $Cg(a, b)$ is an atom in $Con(L)$.*

Proof: Let $\theta \leq Cg(a, b)$ and $\langle c, d\rangle \in \theta$ with $c \neq d$ (i.e. $\theta \neq \Delta$). Then $(c \wedge d)\theta(c \vee d)$ by proposition 3.5 on page 14. This implies that $(c \wedge d, c \vee d) \in Cg(a, b)$. Therefore there exists a sequence $c \wedge d = e_0 \leq e_1 \leq \cdots \leq e_n = c \vee d$ such that $[e_i, e_{i+1}]$ is weakly projective into $[a, b]$ for all $0 \leq i \leq n-1$. So $[e_i, e_{i+1}]$ is projective into a subinterval of $[a, b]$ by the projectivity property. But $a \prec b$ implies that $[e_i, e_{i+1}]$ is projective into $[a, b]$. Therefore by theorem 3.3 on page 16, $(a, b) \in Cg(e_i, e_{i+1}) \subseteq Cg(c \wedge d, c \vee d) \subseteq \theta$. So $Cg(a, b) \subseteq \theta$. This implies that $Cg(a, b) = \theta$ and thus $Cg(a, b)$ is an atom. ■

Theorem 5.5 ***(R.P. Dilworth[1950])*** *If $(L, \leq)$ is a modular lattice of finite height, then $Con(L)$ is a Boolean lattice.*

Proof: By theorem 4.3, it suffices to show that $Con(L)$ is complemented. Take a finite maximal chain of L, say

$$0 = a_0 \prec a_1 \prec \cdots \prec a_n = 1.$$

Trivially we have $Cg(a_0, a_1) \vee Cg(a_1, a_2) \vee \cdots \vee Cg(a_{n-1}, a_n) \subseteq Cg(0, 1) = 1_{Con(L)}$. But $(a_0, a_1), (a_1, a_2) \in Cg(a_0, a_1) \vee Cg(a_1, a_2)$ implies by transitivity that $(a_0, a_2) \in Cg(a_0, a_1) \vee Cg(a_1, a_2)$. Also $(a_0, a_2), (a_2, a_3) \in Cg(a_0, a_1) \vee Cg(a_1, a_2) \vee Cg(a_2, a_3)$ implies by transitivity that $(a_0, a_3) \in Cg(a_0, a_1) \vee Cg(a_1, a_2) \vee Cg(a_2, a_3)$. Therefore, continuing this process will give:

$$(0, 1) = (a_0, a_n) \in Cg(a_0, a_1) \vee Cg(a_1, a_2) \vee \cdots \vee Cg(a_{n-1}, a_n).$$

That is:

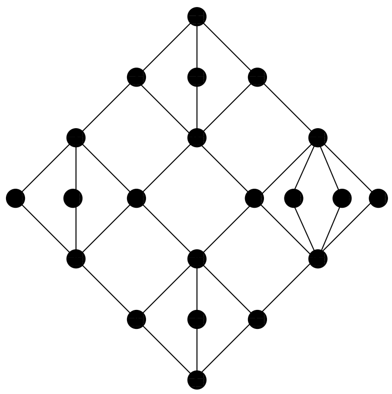
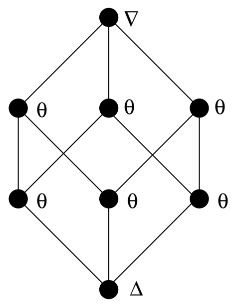
$$1_{Con(L)} \subseteq Cg(a_0, a_1) \vee Cg(a_1, a_2) \vee \cdots \vee Cg(a_{n-1}, a_n).$$

Therefore,

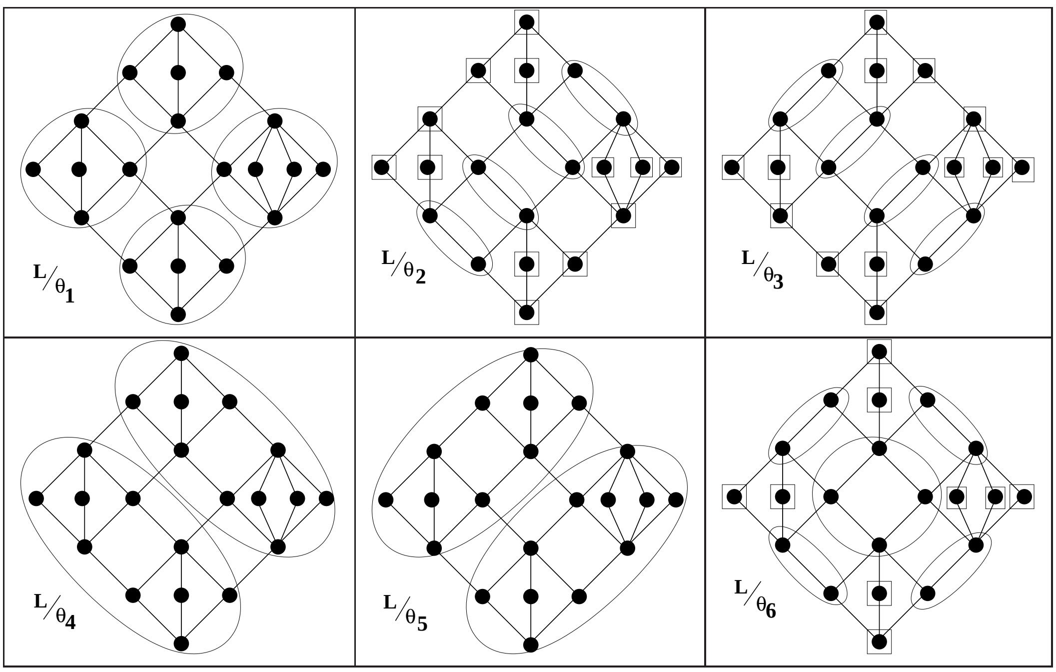
$$1_{Con(L)} = Cg(a_0, a_1) \vee Cg(a_1, a_2) \vee \cdots \vee Cg(a_{n-1}, a_n).$$

Since for all $0 \leq i \leq n-1$, $a_i \prec a_{i+1}$, $Cg(a_i, a_{i+1})$ is an atom in $Con(L)$ by theorem 5.4 and we see that $1_{Con(L)}$ is a join of a finite number of atoms. Therefore by theorem 5.2, $Con(L)$ is a complemented lattice. ∎

Note that if L is a modular lattice of finite height, then $Con(L)$ is of finite height and $h(Con(L)) \leq h(L)$. For instance, considering the modular lattice

$L :=$  , we have $Con(L) =:$ 

where



We see that $Con(L)$ is Boolean and $h\big(Con(L)\big) = 3 < 6 = h(L)$.

Corollary 5.1 *Let L be a finite modular lattice, then L is simple if and only of L is subdirectly irreducible.*

Proof: We have already noticed that if L is any simple lattice then it is subdirectly irreducible (cf. remark 3.1). Conversely if L is subdirectly irreducible, then $Con(L)$ has only one atom. Since $Con(L)$ is Boolean by theorem 5.5, necessarily $Con(L) \cong \mathbf{2}$ since $\mathbf{2}$ is the only Boolean lattice with exactly one atom, therefore L is simple. ■

5.3 Projective geometry and complemented modular lattices.

Definition 5.4 *A* ***projective geometry*** *is a couple (P, Λ) where P is a set of points and $\Lambda \subseteq \mathcal{P}(P)$ is a set of lines satisfying the following properties:*

P_1: *For all distinct points $p, q \in P$ there is exactly one line $l \in \Lambda$ with $p, q \in l$.*

P_2: ***(Pasch Axiom)*** *Each line $l \in \Lambda$ which intersects two sides of a triangle[1] $\Delta := \{l_1, l_2, l_3\}$ also intersects the third side of Δ, in formulas:*

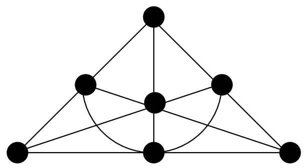
$$\emptyset \neq l \cap {}_1 \neq l \cap l_2 \neq \emptyset \quad \Rightarrow \quad l \cap l_3 \neq \emptyset.$$

There is a finite dimensionality axiom as well, which however is void when P is infinite.

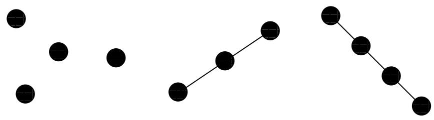
Example 5.1 *1. Let $\mathbf{K}$ be a finite field and $n \geq 2$. Set $P = \{$1-dimensional subspaces of $\mathbf{K}^n\}$ and $\Lambda = \{$2-dimensional subspaces of $\mathbf{K}^n\}$. Then (P, Λ) is a projective geometry.*

2. Suppose that $\Lambda \subseteq \mathbf{2}^P$ satisfies P_1, and any two nontrivial lines[2] intersect. Then P_2 is trivially satisfied. In this case (P, Λ) is called ***projective plane****. The smallest projective plane is*

[1] *A triple of pairwise intersecting lines that do not intersect at the same point.*
[2] *Lines that contain at least three distinct points.*

$(P, \Lambda) =:$ 

3. *Suppose that $\Lambda \subseteq 2^P$ satisfies P_1, and there are no triangles. Then P_2 is trivially satisfied as well. Here is an example (the trivial 2-element lines are not drawn):*



Let (P, Λ) be a projective geometry. A subset $X \subseteq P$ is said to be **Λ-closed** if for any line $l \in \Lambda$, $|l \cap X| \geq 2$ implies $l \subseteq X$. It is clear that the intersection of any family of Λ-closed subsets of (P, Λ) is Λ-closed. Hence the set $C(P, \Lambda)$ of Λ-closed subsets of P is a complete lattice. In fact, it happens to be a complemented modular lattice [21].
Conversely let L be a finite complemented modular lattice. Call $g \subseteq J(L)$ a line if $g = J(a) := \{x \in J(L) | \ x \leq a\}$ for some $a \in L$ with $h(a) = 2$. Let Λ be the family of all lines g. Then the pair $(J(L), \Lambda)$ is a projective geometry and the set $C(J(L), \Lambda)$ of Λ-closed subsets of $J(L)$ is a lattice. Further,

Theorem 5.6 *[22] Let L be a finite height complemented modular lattice, then $L \cong C(J(L), \Lambda)$ with the isomorphism given by $a \mapsto J(a)$*

While each projective geometry (P, Λ) is coupled to a complemented modular lattice, it needs not be **coordinatizable**, i.e. (P, Λ) needs not be associated to a field $\mathbf{K}$ as in example 5.1.1. As an extreme case of theorem 5.6 also notice that the finite height complemented distributive lattices (i.e. Boolean lattices) are precisely the one with an empty set Λ of lines.

Definition 5.5 *A projective geometry (P, Λ) is said to be **non-degenerated** if $C(P, \Lambda)$ is directly irreducible of height ≥ 3. This amounts to say that $|\Lambda| > 1$ and there are no trivial lines.*

It turns out [13; 22] that the subdirectly irreducible factors of $C(P, \Lambda)$ (which here coincide with the directly irreducible factors) correspond bijectively to the connected components of (P, Λ).

5.4 Representation of finite modular lattices.

In this section, we will combine the Birkhoff's representation theorem for finite distributive lattices with ideas from projective geometries to get a representation theorem for finite modular lattices. For starters, we introduce the concept of Λ-closed order ideal.

Definition 5.6 *For any integer $n \geq 3$, we denote by M_n the height two modular lattice with n atoms. An element x of a modular lattice L is said to be a M_n**-element** if there is a height two interval $[x_0, x] \cong M_n$ which contains all the lower covers x_i of x. That is, if $x_1, x_2, \cdots, x_n$ are the lower covers of x, then $x_0 = x_1 \wedge x_2 \wedge \cdots \wedge x_n$ is covered by x_i for all $1 \leq i \leq n$.*

*A **line** corresponding to a fixed M_n-element $x \in L$ is an n-element subset $l_x = \{p_1, p_2, \cdots, p_n\} \subseteq J(L)$ such that $p_i \leq x_i$ and $p_i \not\leq x_0$ $(1 \leq i \leq n)$. A **base of lines** of a finite modular lattice L is a family Λ of lines l_x with exactly one line corresponding to each M_n-element $x \in L$. An order ideal K of $(J(L), \leq)$ is called Λ**-closed** if for all $l \in \Lambda$,*

$$|l \cap K| \geq 2 \quad \Rightarrow \quad l \subseteq K.$$

We denote by $C(J(L), \Lambda)$ the set of Λ-closed order ideals of $J(L)$.

In particular, the projective geometry $\big(J(L), \Lambda\big)$ associated to a finite height complemented modular lattice L, yields a unique base of lines Λ for L (where $p_i = x_i$ throughout). Often bases of lines are not unique though.

Example 5.2 *The following lattice L has two M_n-elements and a base of lines of L is given to the right. We see that b is a M_4-element and d is a M_3-element with $l_b = \{a_1, a_2, a_3, c\}$ and $l_d = \{a_1, a_4, a_5\}$. So $\Lambda = \{l_b, l_d\}$ is a base of lines of L. Observe that $l'_d = \{a_2, a_4, a_5\}$ is also a line corresponding to d, and $\Lambda' = \{l_b, l'_d\}$ is another base of lines.*

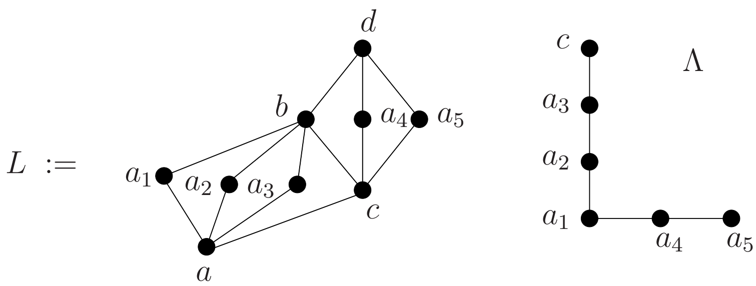


Figure 5.3: A lattice with two M_n-elements and a base of lines of L.

Any order ideal $K \subseteq J(L)$ of type $K = J(a) := \{x \in J(L) : x \leq a\}$ is Λ-closed. In fact let $l_x = \{p_1, p_2, \cdots, p_k\}$ be a line of L and $|l_x \cap K| \geq 2$. Suppose that $p_i, p_j \in K$. Then $p_i \leq a$ and $p_j \leq a$. Therefore $x = p_i \vee p_j \leq a$. But $p_1, p_2, \cdots, p_k \leq x \leq a$ implies that $l_x \subseteq K$. Less obvious is that every Λ-closed order ideal $K \subseteq J(L)$ is of type $K = J(a)$. Thus, generalizing theorem 5.6 we have:

Theorem 5.7 ***(Herrmann-Wild)****[13; 22] Let L be a finite height modular lattice, and let Λ be any base of lines. Then $a \mapsto J(a)$ yields an isomorphism $L \cong C\big(J(L), \Lambda\big)$.*
In particular, the finite distributive lattices L are precisely the ones with empty bases of lines.

Chapter 6

Free modular lattices

6.1 Free lattices within a variety

Definition 6.1 *A class $\mathcal{V}$ of algebras of given type*[1] *is called* ***variety*** *if it is closed under the operations of taking subalgebras, direct products and epimorphic images.*

For instance, the class of all semigroups is a variety, and the class of all commutative semigroups is a subvariety of it. On the other hand, the class of all fields is not a variety since the direct product of two fields is not a field (nonzero elements of type $(x, 0)$ having no inverse). The intersection of any family of varieties of algebras (of the same type) is again a variety. If K is a family of algebras of a given type, the smallest variety containing K is a variety called **variety generated** by K; it is in fact the intersection of all the varieties containing K and denoted by $Var(K)$. By Birkhoff's theorem [17], each $X \in \mathcal{V}$ is a subdirect product of subdirectly irreducible algebras of $\mathcal{V}$.
In the sequel we focus on varieties of *lattices*. For starters, from Birkhoff's theorem and theorem 4.6 follows:

Corollary 6.1 *The variety of lattices generated by $D_2 \cong \mathbf{2}$ is the variety $\mathcal{D}$ of all distributive lattices.*

Theorem 6.1 *The lattice variety $\mathcal{M}_3 := Var(\{M_3\})$ is the class of all subdirect products of M_3 and D_2.*

In fact, for any lattice L, $Var(\{L\})$ is the class of all subdirect products of epimorphic images of sublattices of L [17].

[1] *We refer to [17] for the precise definition of the type of an algebra.*

Definition 6.2 *A* ***lattice polynomial*** *on the variables* $x_1, x_2, \cdots, x_n$ *is defined recursively as follows:*

(i) $x_1, x_2, \cdots, x_n$ *are lattice polynomials.*

(ii) If $p := p(x_1, x_2, \cdots, x_n)$ *and* $q := q(x_1, x_2, \cdots, x_n)$ *are lattice polynomials, then* $(p \vee q)$ *and* $(p \wedge q)$ *are lattice polynomials.*

Example 6.1 $x_1 \vee x_1$, $x_1 \wedge (x_1 \vee x_2)$ *and* $(x_1 \vee x_2) \wedge x_1$ *are distinct lattice polynomials.*

The set of all lattice polynomials on the variables $x_1, x_2, \cdots, x_n$ together with the operations $\vee$ and $\wedge$ is called **term algebra** and it is denoted by $(T_n, \wedge, \vee)$. Note that $(T_n, \wedge, \vee)$ is not a lattice as $x_1 \wedge x_2 \neq x_2 \wedge x_1$. If K is a class of lattices, we define on $(T_n, \wedge, \vee)$ the relation θ_K by:

$p\theta_K q$ if and only if for any lattice $L \in K$ and for any substitution

$$\delta : \{x_1, x_2, \cdots, x_n\} \rightarrow L$$

we have,

$$p(\delta(x_1), \delta(x_2), \cdots, \delta(x_n)) = q(\delta(x_1), \delta(x_2), \cdots, \delta(x_n)) \textit{ in } L.$$

In other words: The "identity" $p \equiv q$ holds in all lattices L of K.

Example 6.2 *For any class* K *of lattices, we have*

$$(x_1 \vee x_1)\, \theta_K\, x_1 \quad \textit{and} \quad (x_1 \wedge x_2)\, \theta_K\, (x_2 \wedge x_1).$$

Moreover if K *is the variety of modular lattices, then*

$$\Big(x_1 \wedge \big(x_2 \vee (x_1 \wedge x_3)\big)\Big)\ \theta_K\ \big((x_1 \wedge x_2) \vee (x_1 \wedge x_3)\big).$$

Theorem 6.2 *The relation* θ_K *is a congruence on the algebra* $(T_n, \wedge, \vee)$ *and*

$$FK(x_1, x_2, \cdots, x_n) \ := \ T_n/\theta_K$$

is the lattice contained in $Var(K)$ *that is generated by* $\overline{x}_j := x_j/\theta_K$ $(1 \leq j \leq n)$ *and satisfies the following* ***universal mapping property:*** *For any* $L \in K$ *and any substitution* $\delta : \{x_1, x_2, \cdots, x_n\} \rightarrow L$*, there is a unique homomorphism* $f : FK(x_1, x_2, \cdots, x_n) \rightarrow L$ *extending* δ*, i.e. the following diagram commutes where* i *is the "inclusion map"* $i(x_j) = \overline{x}_j$*:*

$$\begin{tikzcd}
\{x_1, x_2, \cdots, x_n\} \arrow[r, "\delta"] \arrow[d, "i"'] & L \\
FK(x_1, x_2, \cdots, x_n) \arrow[ur, dashed, "\exists! f"'] &
\end{tikzcd} \qquad f \circ i = \delta$$

Proof: The only candidate for f is

$$f(p(x_1, \cdots, x_n)/\theta_K) \ := \ p(\delta(x_1), \cdots, \delta(x_n)).$$

By the very definition of θ_K, f is well defined. That f is a homomorphism is just as easy. If $K \subseteq K'$, then $\theta_K \supseteq \theta_{K'}$, and so $FK(x_1, \cdots, x_n)$ is an epimorphic image of $FK'(x_1, \cdots, x_n)$. However, if K' is the variety generated by K, then $FK'(x_1, \cdots, x_n) = FK(x_1, \cdots, x_n)$. Essentially this is because an identity $p \equiv q$ that holds in all members of K, also holds in all direct products, sublattices and epimorphic images of such.

Example 6.3 *If $K = \{D_2\}$, then in view of corollary 6.1 and the above we have*

$$FK(x_1, \cdots, x_n) = F\mathcal{D}(x_1, \cdots, x_n).$$

The cardinality $fd(n)$ of $F\mathcal{D}(x_1, \cdots, x_n)$ is only known up to $n = 8$. For instance $fd(3) = 18$ and $fd(4) = 166$. This implies that taking arbitrary unions and intersections of any sets $S_1, \cdots, S_4$ one can obtain at most 166 different sets. Indeed, putting $S = S_1 \cup \cdots \cup S_4$, there is by theorem 6.2 a homomorphism $f : F\mathcal{D}(x_1, \cdots, x_4) \to \mathcal{P}(S)$ with $f(x_i) = S_i$, and so the sublattice of $\mathcal{P}(S)$ generated by $S_1, \cdots, S_4$ has cardinality at most $fd(4)$.

6.1.1 Generators and relations

Let $\mathcal{V}$ be a variety of lattices and put $F := F\mathcal{V}(x_1, \cdots x_n)$. For simplicity we shall henceforth write x_i rather that $\overline{x}_i$ for the generators of F.

Theorem 6.3 *Let $\mathcal{R}$ be a set of pairs $(t_i, s_i) \in F \times F$ $(i \in I)$ interpreted as "relations" $t_i(x_1, \cdots, x_n) = s_i(x_1, \cdots, x_n)$. Put*

$$F\mathcal{V}(x_1, \cdots, x_n; \mathcal{R}) \quad := \quad F/\theta,$$

where θ is the congruence generated by the pairs (t_i, s_i) in $\mathcal{R}$. Then $F\mathcal{V}$ $(x_1, \cdots, x_n; \mathcal{R})$ has another universal mapping property in the sense that for each $L \in \mathcal{V}$ and all $\alpha_1, \cdots, \alpha_n \in L$ which satisfy

$$t_i(\alpha_1, \cdots, \alpha_n) = s_i(\alpha_1, \cdots, \alpha_n),$$

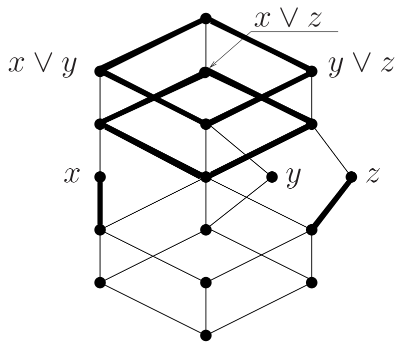
there is a homomorphism $f : F\mathcal{V}(x_1, \cdots, x_n; \mathcal{R}) \to L$ *with* $f(x_j) = \alpha_j$ $(1 \leq j \leq n)$.

Proof: To fix ideas, say $n = 4$, and $t_1 := x_1 \vee x_2$, $s_1 = x_3 \wedge x_4$ (so $|I| = 1$). Thus put $\theta = \langle (x_1 \vee x_2, x_3 \wedge x_4) \rangle \in Con(F\mathcal{V}(x_1, x_2, x_3, x_4))$ and $F\mathcal{V}(x_1, x_2, x_3, x_4; \mathcal{R}) := F\mathcal{V}(x_1, x_2, x_3, x_4)/\theta$. Let $L \in \mathcal{V}$ be such that $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in L$ and $\alpha_1 \vee \alpha_2 = \alpha_3 \wedge \alpha_4$. We look for an homomorphism $f : F\mathcal{V}(x_1, x_2, x_3, x_4; \mathcal{R}) \to L$ with $f(x_j) = \alpha_j$. Consider the map $\delta : \{x_1, x_2, x_3, x_4\} \to L$ defined by $\delta(x_j) = \alpha_j$. By the universal mapping property of theorem 6.2, there is an homomorphism $\rho : F\mathcal{V}(x_1, x_2, x_3, x_4) \to L$ with $\rho(x_j) = \alpha_j$. In particular

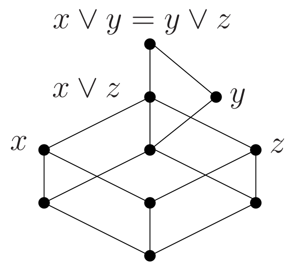
$$\begin{aligned} \rho(x_1 \vee x_2) &= \rho(x_1) \vee \rho(x_2) = \alpha_1 \vee \alpha_2 \\ &= \alpha_3 \wedge \alpha_4 = \rho(x_3) \wedge \rho(x_4) = \rho(x_3 \wedge x_4). \end{aligned}$$

Thus $(x_1 \vee x_2, x_3 \wedge x_4) \in Ker(\rho)$, and so $\theta = \langle (x_1 \vee x_2, x_3 \wedge x_4) \rangle \subseteq Ker(\rho)$. Hence by the second isomorphism theorem 3.2, there is a congruence $Ker(\rho)/\theta$ on $F\mathcal{V}(x_1,\ x_2,x_3,x_4)/\theta$ satisfying $\rho(a) = \rho(b) \iff a_\theta \big(Ker(\rho)/\theta\big) b_\theta$. Thus $f(a_\theta) := \rho(a)$ is a well defined homomorphism from $F\mathcal{V}(x_1, x_2, x_3, x_4)/\theta$ to L which satisfies $f(x_j/\theta) = \rho(x_j) = \alpha_j$. ∎

Example 6.4 *Taking $\mathcal{D}$ to be the variety of distributive lattices, it is well known that the free distributive lattice on three generators $F\mathcal{D}(x, y, z)$ is:*



If $\mathcal{R} = \{(x \vee y, y \vee z)\}$, then $\theta := \langle (x \vee y, y \vee z) \rangle$ collapses the thick prime quotients and so $F\mathcal{D}(x, y, z; \mathcal{R}) := F\mathcal{D}(x, y, z)/\theta$ is given by:



Hence this is the "most general" distributive lattice generated by x, y, z and subject to relation $x \vee y = y \vee z$; any other lattice of that kind is an epimorphic image of it. As opposed to theorem 6.1 the relation $x \vee y = y \vee z$ only holds for the generators; so $a \vee b = b \vee c$ does not hold for all $a, b, c \in F\mathcal{D}(x, y, z; \mathcal{R})$.

6.1.2 $F\mathcal{V}(P, \leq)$ as a special case of $F\mathcal{V}(x_1, \cdots, x_n, \mathcal{R})$

Definition 6.3 *If $(P, \leq)$ is a poset on $P = \{x_1, \cdots, x_n\}$, put*

$$F\mathcal{V}(P, \leq) \quad := \quad F\mathcal{V}(x_1, \cdots, x_n; \mathcal{R})$$

where $\mathcal{R}$ is a set of relations of a very specific type, namely

$$\mathcal{R} \quad := \quad \{(x_i, x_i \wedge x_j) | x_i, x_j \in P, \ x_i < x_j\}.$$

We call $F\mathcal{V}(P, \leq)$ the **lattice freely generated** by the poset P within the variety $\mathcal{V}$.

For $\mathcal{V} = \mathcal{L}$ the variety of all lattices, $F\mathcal{L}(P, \leq)$ has a neat description. Namely, as easy extension of the Whitman test (see [3]) which handles $F\mathcal{L}(x_1, \cdots, x_n)$, it is shown in [23] that $F\mathcal{L}(P, \leq) \cong T_n/\theta$, where $\theta := \{(t_1, t_2) | t_1 \leq' t_2 \text{ and } t_2 \leq' t_1\}$ and $\leq'$ on T_n is defined by induction as follows:

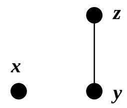
1. $x \leq' y :\iff x \leq y, \quad \text{for all } x, y \in P$
2. $t_1 \vee t_2 \leq' t_3 :\iff t_1 \leq' t_3 \text{ and } t_2 \leq' t_3$
3. $t_3 \leq' t_1 \wedge t_2 :\iff t_3 \leq' t_1 \text{ and } t_3 \leq' t_2$
4. $t_1 \leq' t_3 \text{ or } t_2 \leq' t_3 \Rightarrow t_1 \wedge t_2 \leq' t_3$
5. $t_3 \leq' t_1 \text{ or } t_3 \leq' t_2 \Rightarrow t_3 \leq' t_1 \vee t_2$

Notice that for $\mathcal{R} := \emptyset$ and $P := \overline{\mathbf{n}}$ (the n-element antichain) we get

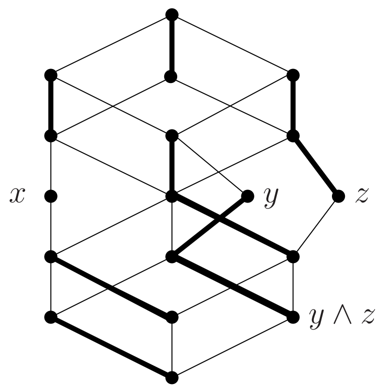
$$F\mathcal{V}(\overline{\mathbf{n}}, \leq) = F\mathcal{V}(x_1, \cdots, x_n; \emptyset) = F\mathcal{V}(x_1, \cdots, x_n).$$

Every n-generated lattice $L \in \mathcal{V}$ is clearly isomorphic to $F\mathcal{V}(x_1, \cdots, x_n; \mathcal{R})$ for a suitable set of relations $\mathcal{R}$, but L needs not be isomorphic to any lattice of type $F\mathcal{V}(P, \leq)$.
When $\mathcal{V}$ is the variety $\mathcal{D}$ of distributive lattices, we are dealing with the lattice $F\mathcal{D}(P)$ introduced in section 4.4.

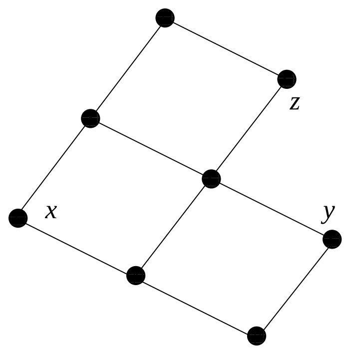
Example 6.5 *If* $(P, \leq) :=$



$=$ ***1+2***, *then* $\theta = \big\langle (y, y \wedge z) \big\rangle$ *collapses the following thick prime quotients of* $F\mathcal{D}(x, y, z)$*:*



Hence $F\mathcal{D}(P, \leq) = F\mathcal{D}(x, y, z)/\theta =$



6.1.3 Specializing the variety $\mathcal{V}$ in $F\mathcal{V}(P, \leq)$

We are now giving an alternative construction of $F\mathcal{V}(P, \leq)$ in the special case where the variety $\mathcal{V}$ is generated by some single finite lattice Y. We first need to know the finitely many subdirectly irreducible members Z_i of

$Var(\{Y\})$. Next take all non-equivalent P-labellings $\lambda_i : P \to L_i$ $(1 \leq i \leq s)$ of these lattices, i.e. each λ_i is monotone and $\lambda_i(P)$ generates L_i (cf. section 4.5). Notice that one Z_i may give rise to many P-labellings $\lambda_j : P \to L_j$ where all $L_j = Z_i$.

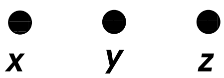
Theorem 6.4 *[24] With notation as above, the lattice $F\mathcal{V}(P)$ is isomorphic to the subdirect product of $L_1, \cdots, L_s$ generated by the s-tuples $(\lambda_1(p), \lambda_2(p), \cdots, \lambda_s(p))$ $\quad (p \in P)$.*

Proof: For notational convenience we *define* $F\mathcal{V}(P)$ as the generated sublattice mentioned above and verify the universal mapping property. It is easy to see that the generating set $\{(\lambda_1(p), \cdots, \lambda_s(p)) | p \in P\}$ of $F\mathcal{V}(P)$ is isomorphic to $(P, \leq)$, and so we identify the two. Take any $X \in \mathcal{V}$ and any monotone map $\alpha : P \to X$. By proposition 6.1, $\langle \alpha(P) \rangle \in \mathcal{V}$ is a subdirect product of some of the Z_i's, say $\langle \alpha(P) \rangle \subseteq Z_1 \times Z_2 \times Z_2$ and correspondingly $\alpha(p) = (\alpha_1(p), \alpha_2(p), \alpha_3(p))$ for all $p \in P$. Thus $\langle \alpha_1(P) \rangle = Z_1$, $\langle \alpha_2(P) \rangle = \langle \alpha_3(P) \rangle = Z_2$. Hence these α_i $(1 \leq i \leq 3)$ must be some of our P-labellings λ_j, w.l.o.g. corresponding to $\lambda_1 : P \to L_1$, $\lambda_2 : P \to L_2$, $\lambda_3 : P \to L_3$. (Note that although $L_2 = L_3 (= Z_2)$, in a nonredundant subdirect product we will have $\lambda_2 \neq \lambda_3$). Hence $F\mathcal{V}(P) \to X$, $(a_1, \cdots, a_s) \mapsto (a_1, a_2, a_3)$ is the sought extension of $\alpha : P \to X$: $(\lambda_1(p), \cdots, \lambda_s(p)) \mapsto (\lambda_1(p), \lambda_2(p), \lambda_3(p))$. ■

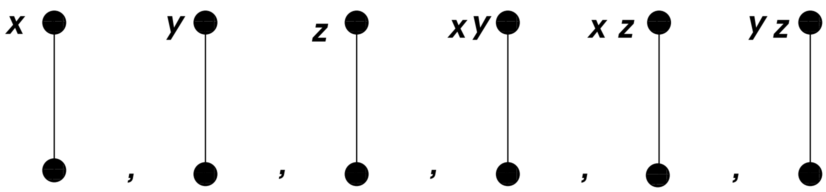
6.2 Free modular lattices $F\mathcal{M}_3(P)$

In this section, $\mathcal{M}$ will denote the variety of all modular lattices and $\mathcal{M}_3$ the variety of modular lattices generated by M_3. Thus by proposition 6.1, $X \in \mathcal{M}_3$ if and only if X is a subdirect product of factors M_3 or D_2. Accordingly, the modular lattice **freely generated by P** within the variety $\mathcal{M}_3$, respectively $\mathcal{M}$ is denoted by $F\mathcal{M}_3(P)$ respectively $F\mathcal{M}(P)$.

Recall that $F\mathcal{M}_3(P)$ is always an epimorphic image of $F\mathcal{M}(P)$. As we will see later (section 6.3), they are actually isomorphic for many types of posets P.

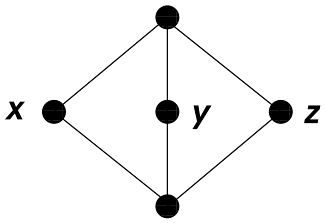
Example 6.6 *For $P :=$* 

, there are 6 non-equivalent P-labellings of $\mathbf{2}$, namely



D_1, D_2, D_3, D_4, D_5, D_6

and a unique P-labelling of M_3, namely



M_3

Therefore $F\mathcal{M}_3(P)$ is a subdirect product of $(D_2)^6 \times M_3$. Since P is isomorphic to the subposet $\{(\lambda_1(p), \cdots, \lambda_7(p)) | p \in P\}$ of $F\mathcal{M}_3(P)$, we can identify any $p \in P$ with the corresponding 7-tuple $(\lambda_1(p), \cdots, \lambda_7(p))$. So

$$\begin{aligned} x &= (\lambda_1(x), \cdots, \lambda_7(x)) = (1,0,0,1,1,0,u) =: 100110u, \\ y &=: 010101v \\ z &=: 001011w \end{aligned}$$

where $\{u, v, w\}$ is the set of atoms of M_3. Thus the Hasse diagram of $F\mathcal{M}_3(x, y, z)$ is given in figure 6.1 where:

$$\begin{array}{ll} p = x \wedge y = 0001000, & q = x \wedge z = 0000100, \\ r = y \wedge z = 0000010, & a = x \wedge (y \vee z) = 000110u, \\ b = y \wedge (x \vee z) = 000101v, & c = z \wedge (x \vee y) = 000011w, \\ u = p \vee q = 0001100, & v = p \vee r = 0001010, \\ w = q \vee r = 00001104, & e = a \vee d (= e^*) = 000111u, \\ f = b \vee d (= f^*) = 000111v, & g = c \vee d (= g^*) = 000111w, \\ 0 = x \wedge y \wedge z = 0000000, & d = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = 0001110, \end{array}$$

and $p^* = x \vee y$ is the dual of p, etc.

For bigger posets P, it becomes computationally cumbersome to compute $F\mathcal{M}_3(P)$ as the sublattice of $L_1 \times \cdots \times L_s$ generated by all s-tuples $(\lambda_1(p), \cdots, \lambda_s(p))$ $(p \in P)$. It would be nice to e.g. first compute all the join-irreducibles and then all joins thereof. However, as opposed to $F\mathcal{D}(P)$ not all join-irreducible of $F\mathcal{M}_3(P)$ are infima of elements of $P \subseteq F\mathcal{M}_3(P)$. For instance, in example 6.6 the join-irreducible a is $x \wedge (y \vee z)$.

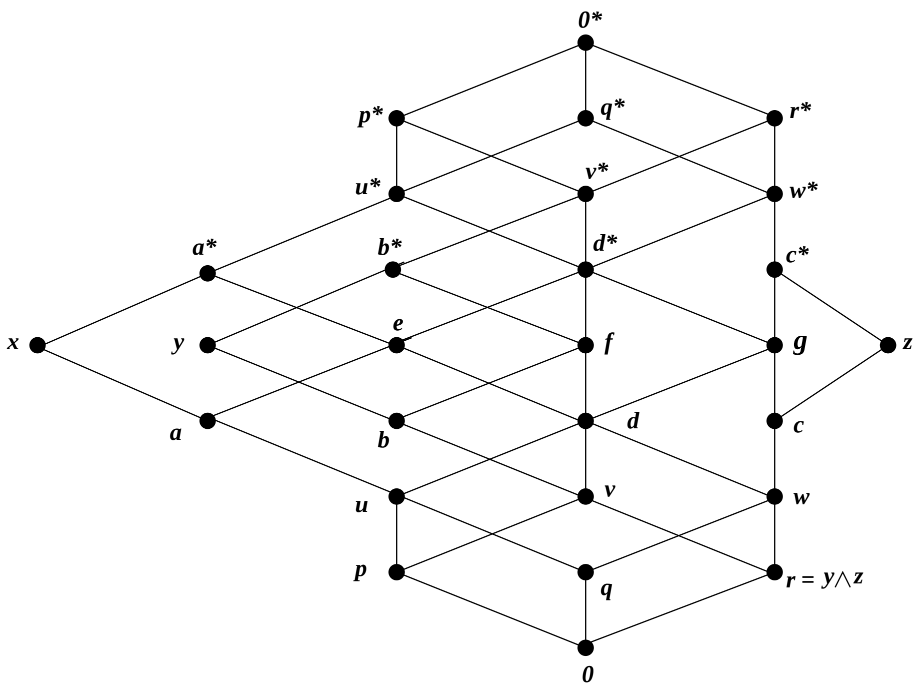


Figure 6.1: Hasse diagram of the free modular lattice on three generators.

In this section we show how one can predict the right $\vee$-morphisms $\beta_{ij} : L_i \to L_j$ among the subdirectly irreducible factors of $F\mathcal{M}_3(P)$ and herewith (section 3.6) get the set J of join-irreducibles of $F\mathcal{M}_3(P)$. Computing one by one all the joins of elements of J is actually infeasible but the fact that only $\vee$ (and not $\wedge$) is involved will allow for some other tricks to get the job done (chapter 7).

The lattice $F\mathcal{M}_3(x, y, z)$ in example 6.6 coincides with the famous 28-element Dedekind lattice $F\mathcal{M}(x, y, z)$. As a preview to section 6.3 we mention that generally, whenever $F\mathcal{M}(P)$ happens to be finite, it coincides with $F\mathcal{M}_3(P)$.

6.2.1 Construction of a base of lines of $F\mathcal{M}_3(P)$

Let $(P, \leq)$ be a finite poset. Suppose that $\lambda_i : P \to L_i$ $(1 \leq i \leq s)$ are the pairwise non-equivalent P-labellings of $\mathbf{2}$, and $\lambda_i : P \to L_i$ $(s{+}1 \leq i \leq s{+}t)$ are the pairwise non-equivalent P-labellings of M_3. As seen in theorem 6.4,

$$F\mathcal{M}_3(P) \subseteq L_1 \times \cdots \times L_s \times L_{s+1} \times \cdots \times L_{s+t}$$

is the subdirect product generated by the tuples $(\lambda_1(p), \cdots, \lambda_{s+t}(p))$ $(p \in P)$. For all $1 \leq i \leq s$, we denote indifferently by 1 the unique nonzero join-irreducible of L_i and for all $s+1 \leq j \leq s+t$, let $(p,j), (q,j), (r,j)$ be the three nonzero join-irreducibles of L_j. Let $\Lambda_j = \{(p,j), (q,j), (r,j)\}$ be the unique base of lines of L_j. We define for all $1 \leq i \leq s+t$, the map $\psi_i : L_i \to L_1 \times \cdots \times L_s \times L_{s+1} \times \cdots \times L_{s+t}$ by

$$\psi_i(x) := (\beta_{i1}(x), \cdots, \beta_{is}(x), \beta_{i(s+1)}(x), \cdots, \beta_{i(s+t)}(x))$$

where as in definition 4.5, we let $\beta_{ij} : L_i \to L_j$ be the biggest $\vee$-preserving morphism between L_i and L_j that maps labels below labels. By theorem 4.2 the β_{ij}'s satisfy $\beta_{ii} = id_{L_i}$ and $\beta_{ik} \geq \beta_{jk} \circ \beta_{ij}$. By theorem 3.5, if we set

$$\begin{aligned} J_1 &= \{\psi_1(1), \psi_2(1), \cdots, \psi_s(1)\} \\ J_2 &= \bigcup_{j=s+1}^{s+t} \{\psi_j(p,j), \psi_j(q,j), \psi_j(r,j)\}, \end{aligned}$$

then the sublattice F of $L_1 \times \cdots \times L_s \times L_{s+1} \times \cdots \times L_{s+t}$ which is $\vee$-generated by $J_1 \cup J_2$ is a subdirect product of $L_1 \times \cdots \times L_s \times L_{s+1} \times \cdots \times L_{s+t}$. In fact by theorem 6.4, $F = F\mathcal{M}_3(P)$. Before we give an example to illustrate this fact, let us clarify the details and the procedure involved in the computation of the base of lines (J, Λ) of $F\mathcal{M}_3(P)$.

6.2.2 Steps to determine a base of lines (J, Λ) of $F\mathcal{M}_3(P)$

Step 1: Determination of the P-labellings of **2** and M_3.
We first compute the P-labellings of **2** and the P-labellings of M_3. The P-labellings of **2** are as in section 4.5, they correspond to the proper filters of P. The computation of the P-labellings of M_3 relies on the number of 3-element antichains of P. In fact any 3-element antichain of P gives rise to at least one P-labelling of M_3.
Step 2: Determination of the morphisms β_{ij}.
The determination of a morphism β_{ij} between two P-labellings of **2** is straightforward since $\beta_{ij}(1) = 1 \Leftrightarrow \lambda_i^{-1}(1) \subseteq \lambda_j^{-1}(1)$. Similarly the calculation of a morphism β_{ij} between a P-labelling of **2** and a P-labelling of M_3 is not difficult. In contrast, the determination of a morphism β between two P-labellings of M_3 needs more attention. By considering the quotient $M_3/ker(\beta)$ and discarding the P-labellings for the moment, we can distinguish five cases.

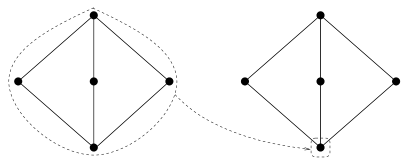


Figure 6.2: $\beta \equiv 0$

1. $|M_3/ker(\beta)| = 1$, i.e. $ker(\beta) = \nabla$, then $\beta(1) = \beta(0) = 0$. So there is only one morphism $\beta \equiv 0$
2. If $|M_3/ker(\beta)| = 2$, then the following four subcases may arise.
   a) $\beta(1) = 1$ and all atoms map to the same image y. Then necessarily $y = 1$ since $\beta(a) = \beta(b) = \beta(c) = y$ implies $y = \beta(a) \vee \beta(b) = \beta(a \vee b) = \beta(1) = 1$.

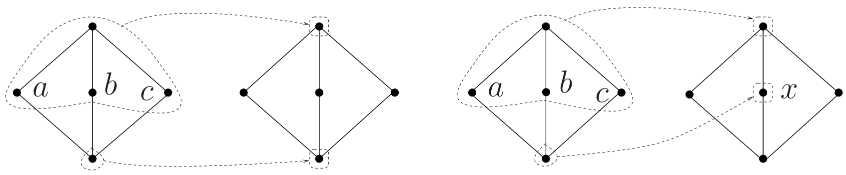


Figure 6.3: Two of the four possible morphisms, x is any of the 3 atoms.

   b) $\beta(1) = 1$ and exactly two of the atoms have the same image y. Then again $y = 1$ and there are exactly 12 possible such morphisms, exactly 3 of which satisfy $\beta(0) = 0$.

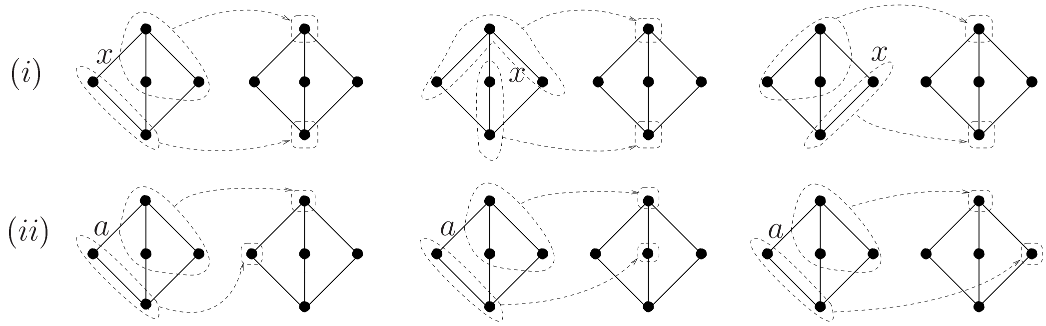


Figure 6.4: (i) The 3 morphisms satisfying $\beta(0) = 0$ and (ii) 3 of the 9 morphisms satisfying $\beta(0) \neq 0$ (each atom, say a for instance yields exactly 3 morphisms).

c) $0 < \beta(1) < 1$ and all atoms have the same image y. Then necessarily, $\beta(1) = y$ and there are exactly 3 possible such morphisms corresponding to 3 choices of y.

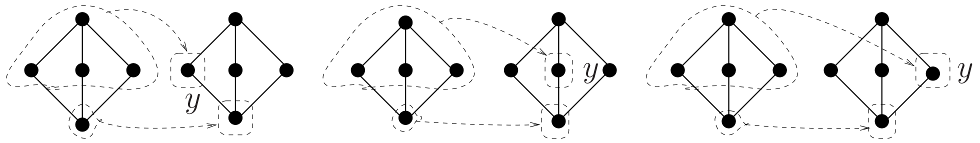


Figure 6.5: 3 possible morphisms corresponding to 3 choices of y.

d) $0 < \beta(1) < 1$ and only two atoms have the same image y. Then necessarily $\beta(1) = y$ and there are exactly 9=3.3 possible such morphisms corresponding to 3 choices for x, each of which comprises 3 choices for y.

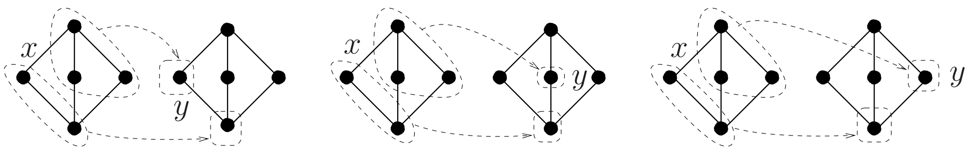


Figure 6.6: 3 choices of y for a fixed x.

3. If $|M_3/ker(\beta)| = 3$, then necessarily two atoms, together with the top element of M_3 must have 1 as image. In this case, there are exactly 9=3.3 possible such morphisms corresponding to 3 choices for x, each of which comprises 3 choices for y.

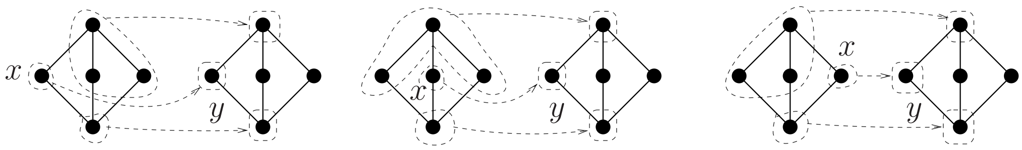
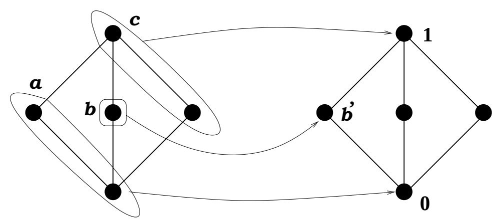


Figure 6.7: 3 choices of x for a fixed y

Note that the map f depicted by



is not a morphism since it is not $\vee$-preserving. In fact $f(a \vee b) = f(c) = 1 \neq b' = 0 \vee b' = f(a) \vee f(b)$.

4. $|M_3/ker(\beta)| = 4$, then exactly one of the atoms, say x, together with the top element of M_3 must have 1 as image. In this case there are exactly $18 = 3.3.2!$ possible morphisms corresponding to 3 choices for x, each of which comprises 3 choices for y and 2! permutations on a and b.

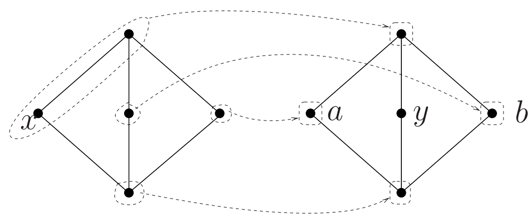
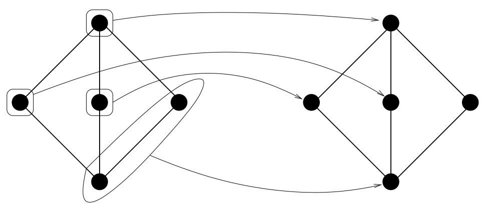


Figure 6.8: A possible such morphism.

Note that the map



is impossible since it is not $\vee$-preserving.

5. $|M_3/ker(\beta)| = 5$, i.e. $\ker(\beta) = \Delta$, there are obviously 6=3! possible morphisms corresponding to the 3! permutations on the images of the 3-element antichains.

Step 3: Determination of the base of lines (J, Λ).
Let $L_1, L_2, \cdots, L_s$ resp. $L_{s+1}, L_{s+2}, \cdots, L_{s+t}$ be maximal sets of pairwise non-equivalent P-labellings of $\mathbf{2}$ resp. M_3. Considering the unique nonzero join-irreducible 1 of L_i $(1 \leq i \leq s)$, and the three atoms $(p, j), (q, j), (r, j)$ of L_j $(s + 1 \leq j \leq s + t)$, we compute

(1) $\psi_i(1)$ for all $i \in \{1, 2, \cdots, s\}$

(2) $l_j = \{\psi_j(p,j), \psi_j(q,j), \psi_j(r,j)\}$ for all $j \in \{s+1, s+2, \cdots, s+t\}$,

and we set

$$\begin{aligned}\Lambda &= \{l_{s+1}, l_{s+2}, \cdots, l_{s+t}\}\\ J &= \{\psi_1(1), \psi_2(1), \cdots, \psi_s(1)\} \cup l_{s+1} \cup l_{s+2} \cup \cdots \cup l_{s+t}.\end{aligned}$$

Then (J, Λ) is a base of lines of $F\mathcal{M}_3(P)$.

Example 6.7 *For the poset P of figure 6.9(a) below, there are 12 pairwise non-equivalent P-labellings of $\mathbf{2}$ and two pairwise non-equivalent P-labellings of M_3 as indicated on figures 6.9(b) and 6.10(a) respectively. In any P-labellings of $\mathbf{2}$, if $f \subseteq P$ is on top, then $P \setminus f$ is on the bottom[2]. We have not indicated the bottom labels for simplification.*

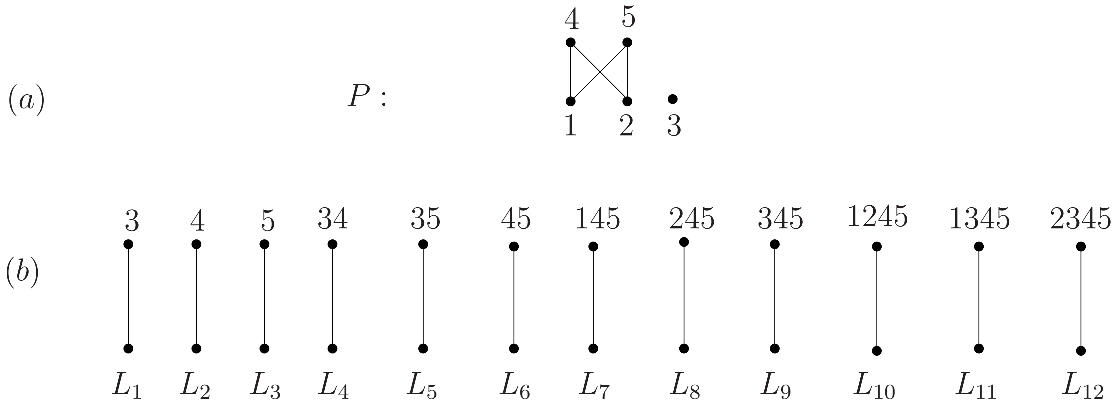


Figure 6.9: (a) A poset P with (b) 12 P-labellings of $\mathbf{2}$.

[2] *We represent a set by listing its elements as a string, for instance 245 represents the set $\{2, 4, 5\}$.*

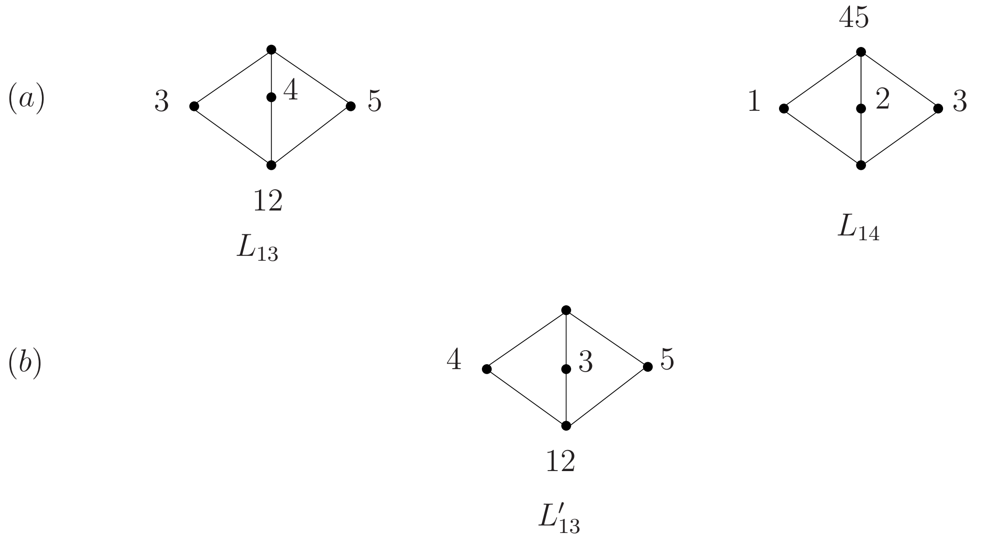


Figure 6.10: (a) Two P-labellings of M_3. (b) L'_{13} and L_{13} are equivalent P-labellings of M_3 (which would trigger redundant subdirect factors).

The morphisms $\beta_{13,14}$ and $\beta_{14,13}$ are given in figure 6.11, while the morphisms $\beta_{13,i}$ and $\beta_{i,13}$ are listed in table 6.1 for all $1 \leq i \leq 12$. Likewise the morphisms $\beta_{14,i}$ and $\beta_{i,14}$ are listed in table 7.3 for all $1 \leq i \leq 12$.

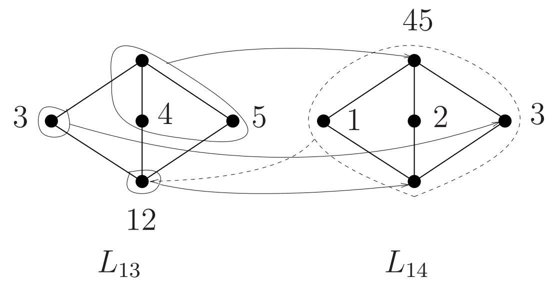
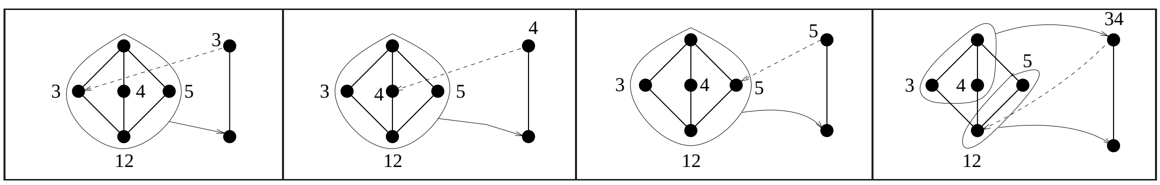


Figure 6.11: The morphisms $\beta_{13,14}$ in thin lines and $\beta_{14,13}$ in dashed lines.



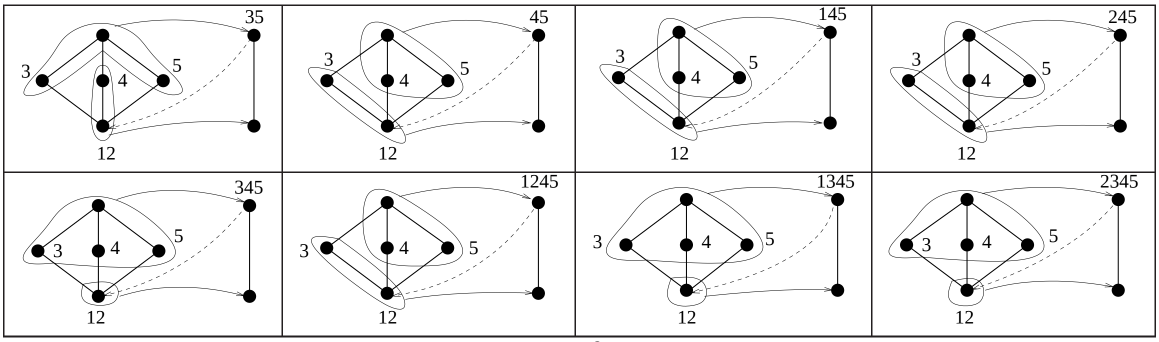


Table 6.1: The morphisms $\beta_{13,i}$, in thin lines and the morphisms $\beta_{i,13}$, in dashed lines ($1 \leq i \leq 12$).

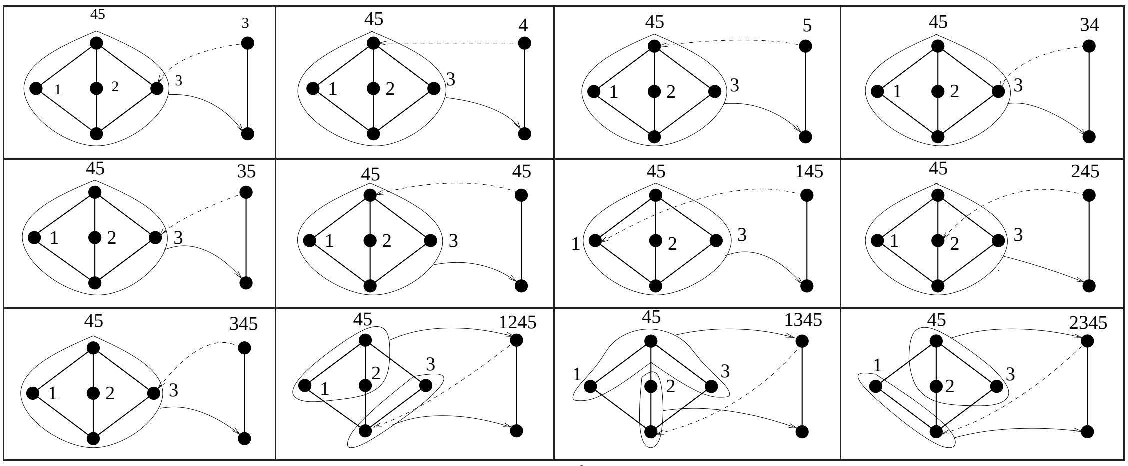
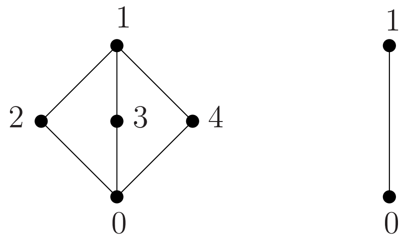


Table 6.2: The morphisms $\beta_{14,i}$, in thin lines and the morphisms $\beta_{i,14}$, in dashed lines ($1 \leq i \leq 12$).

To compute the elements of J, we will identify (by isomorphism) any P-labelling of M_3 to the poset of figure 6.12(a) and any P-labelling of **2** to the poset of figure 6.12(b) i.e. $(p, j) \equiv 2$, $(q, j) \equiv 3$ and $(r, j) \equiv 4$. This does not matter since the order under consideration on the subdirect product $F\mathcal{M}(P) \subseteq L_1 \times \cdots L_s \times L_{s+1} \times \cdots \times L_{s+t}$ is taken componentwise.



(a) (b)

Figure 6.12: (a) Poset identifying the P-labellings of M_3. (b) Poset identifying the P-labellings of **2**.

With this notation, we have

$$\psi_1(1) = (\beta_{1,i}(1))_{1\leq i\leq 14} = (1,0,0,1,1,0,0,0,1,0,1,1,2,4) =: 10011000101124.$$

The other elements of J are computed in the same manner and listed below.

$$\begin{array}{rrr}
\psi_2(1) = 01010111111131, & \psi_3(1) = 00101111111141, & \psi_4(1) = 00010000101104, \\
\psi_5(1) = 00001000101104, & \psi_6(1) = 00000111111101, & \psi_7(1) = 00000010011002, \\
\psi_8(1) = 00000001010103, & \psi_9(1) = 00000000101104, & \psi_{10}(1) = 00000000010000, \\
\psi_{11}(1) = 00000000001000, & \psi_{12}(1) = 00000000000100, & \\
\psi_{13}(2) = 0001100010112, & \psi_{13}(3) = 00010111111131, & \psi_{13}(4) = 00001111111141, \\
\psi_{14}(2) = 00000000011002, & \psi_{14}(3) = 00000000010103, & \psi_{14}(4) = 00000000001104.
\end{array}$$

Put

$$\begin{aligned}
J_1 &= \{\psi_1(1), \psi_2(1), \cdots, \psi_{12}(1)\}, \\
J_2 &= \{\psi_{13}(2), \psi_{13}(3), \psi_{13}(4), \psi_{14}(2), \psi_{14}(3), \psi_{14}(4)\}, \\
\Lambda &= \{\{\psi_{13}(2), \psi_{13}(3), \psi_{13}(4)\}, \{\psi_{14}(2), \psi_{14}(3), \psi_{14}(4)\}\}.
\end{aligned}$$

Then $F\mathcal{M}_3(P)$ has the set of join-irreducibles $J = J_1 \cup J_2$ (the disjoint union of the "distributive" and "modular" parts), and the base of lines (J, Λ) is depicted as:

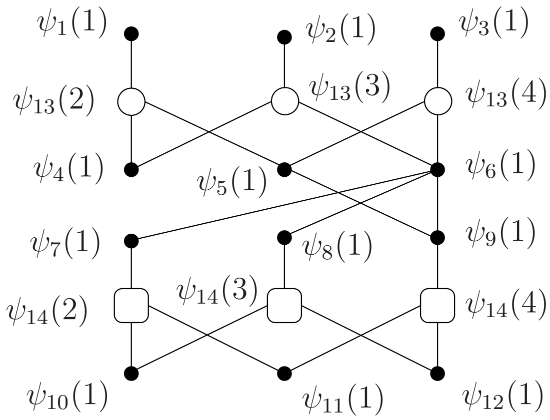


Figure 6.13: The Hasse diagram of the poset of join-irreducibles $(J, \leq)$ of $F\mathcal{M}_3(P)$.

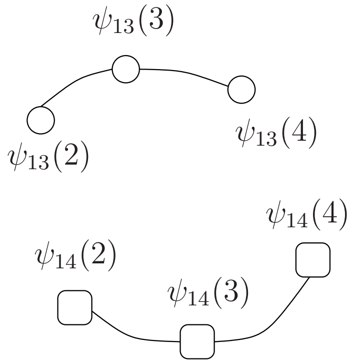
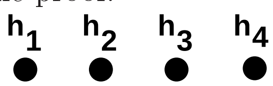


Figure 6.14: A base of lines of $FM_3(P)$.

6.3 A proof of Wille's theorem

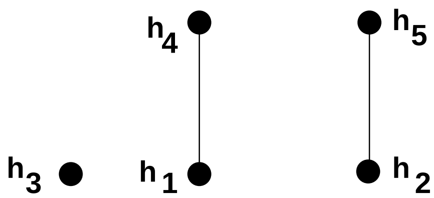
This section is devoted to the proof of Wille's fundamental result. We will study two crucial lemmas on posets (the D_2-lemma and the M_3-lemma) which will be used throughout the proof.

We set $A_4 = \mathbf{1{+}1{+}1{+}1}$:=



, the 4-element antichain and

$H_5 = \mathbf{1{+}2{+}2}$:=



Proposition 6.1 *Let A and B be any two nonempty sets and let $f : A \to B$ be a surjection, then there is an injection $g : B \to A$ with $f \circ g = 1_B$.*

Proof: By the axiom of choice, pick a choice function $\gamma : \mathcal{P}(A) \to A$ and define $g(b) = \gamma(f^{-1}(b))$ for $b \in B$. Then by definition of the choice function, $g(b) = \gamma(f^{-1}(b)) \in f^{-1}(b)$ implies $f(g(b)) = b$. That is $f \circ g = 1_B$. Moreover if $g(b_1) = g(b_2)$ then $f(g(b_1)) = f(g(b_2))$. So $b_1 = b_2$ and therefore g is injective. ∎

Corollary 6.2 *If $f_1 : A \to B$ and $f_2 : B \to A$ are surjective maps, then $A \cong B$.*

Proof: By the previous proposition, there are injective maps $g_1 : B \to A$ and $g_2 : A \to B$. By Cantor-Bernstein's theorem (see [25]) $A \cong B$. ∎

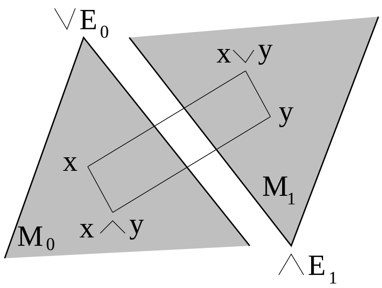
Lemma 6.1 (**D_2-lemma[9]**) *Let M be a subdirectly irreducible modular lattice and let $M = \langle E_0 \cup E_1 \rangle$ where E_0, E_1 are finite and $M \ncong D_2$. Then,*

$$\bigvee E_0 \geq \bigwedge E_1.$$

Proof: Suppose $\bigvee E_0 \ngeq \bigwedge E_1$ and set

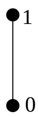
$$M_0 = \{x \in M | x \leq \bigvee E_0\} \quad \text{and} \quad M_1 = \{y \in M | y \geq \bigwedge E_1\}$$

.



Obviously $M_0 \neq \emptyset$ and $M_1 \neq \emptyset$ since $\bigvee E_0 \in M_0$ and $\bigwedge E_1 \in M_1$ by the finiteness of M_0, M_1. If $x \in M_0 \cap M_1$, then $\bigwedge E_1 \leq x \leq \bigvee E_0$ which is a contradiction, whence $M_0 \cap M_1 = \emptyset$. If $x, y \in M_0 \cup M_1$, then $x, y \in M_0$ or $x, y \in M_1$ or $x \in M_0$ and $y \in M_1$. For the first two cases, it is clear that $x \wedge y, x \vee y \in M_0 \cup M_1$. For the later case, $x \wedge y \leq x \leq \bigvee E_0$ implies $x \wedge y \in M_0$ and $x \vee y \geq y \geq \bigwedge E_1$ implies $x \vee y \in M_1$. Therefore $x \wedge y, x \vee y \in M_0 \cup M_1$. Thus $M_0 \cup M_1$ is a sublattice of M. Trivially $E_0 \subseteq M_0$ and $E_1 \subseteq M_1$, so $E_0 \cup E_1 \subseteq M_0 \cup M_1$. Therefore $M = \langle E_0 \cup E_1 \rangle \subseteq M_0 \cup M_1$ since $M_0 \cup M_1$ is a lattice. Hence $M = M_0 \cup M_1$ and the map

$$f : M \longrightarrow$$



$$:= D_2 \quad \text{defined by} \quad f(z) = \begin{cases} 0 & \text{if } z \in M_0, \\ 1 & \text{if } z \in M_1 \end{cases}$$

is well-defined since $M = M_0 \cup M_1$ is a partition. Moreover it is clear that f is an epimorphism. Therefore $M/Ker(f) \cong D_2$. But by assumption $M \ncong D_2$. This implies that $Ker(f) \notin \{\Delta, \nabla\}$, and so M is not simple. By corollary 5.1 on page 42, this is a contradiction since M is subdirectly irreducible by assumption. ∎

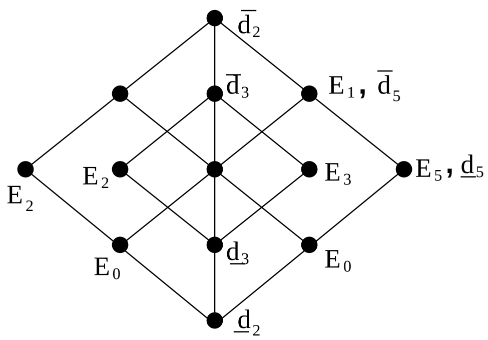
Lemma 6.2 **(M_3-lemma[9])** *Let M be a subdirectly irreducible modular lattice and $M = \langle E_0 \cup E_2 \cup E_3 \cup E_5 \cup E_1 \rangle$ where $E_2 \neq \emptyset$, $E_3 \neq \emptyset$, $E_5 \neq \emptyset$ and all E_i are finite. Set*

$$\overline{d_i} := Sup \bigcup \{E_j | i \ \ divides \ \ j\} \quad and \quad \underline{d_i} := Inf \bigcup \{E_j | j \ \ divides \ \ i\} \quad (i = 2, 3, 5)$$

If $M \not\cong M_3$, then

$$(\overline{d_2} \wedge \overline{d_3}) \vee (\overline{d_2} \wedge \overline{d_5}) \vee (\overline{d_3} \wedge \overline{d_5}) \quad \geq \quad (\underline{d_2} \vee \underline{d_3}) \wedge (\underline{d_2} \vee \underline{d_5}) \wedge (\underline{d_3} \vee \underline{d_5}).$$

To fix ideas take,

$M :=$ 

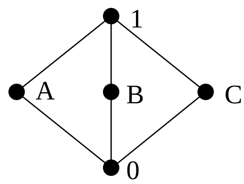
M is clearly subdirectly irreducible since any congruence collapsing a prime quotient collapses the whole M. For simplicity, a point $a \in M$ is labelled E_i if $a \in E_i$. One checks that indeed

$$(\overline{d_2} \wedge \overline{d_3}) \vee (\overline{d_2} \wedge \overline{d_5}) \vee (\overline{d_3} \wedge \overline{d_5}) \quad \geq \quad (\underline{d_2} \vee \underline{d_3}) \wedge (\underline{d_2} \vee \underline{d_5}) \wedge (\underline{d_3} \vee \underline{d_5}),$$

which here boils down to $1 \geq 0$.

Proof of lemma 6.2: Suppose that

$$(\overline{d_2} \wedge \overline{d_3}) \vee (\overline{d_2} \wedge \overline{d_5}) \vee (\overline{d_3} \wedge \overline{d_5}) \quad \not\geq \quad (\underline{d_2} \vee \underline{d_3}) \wedge (\underline{d_2} \vee \underline{d_5}) \wedge (\underline{d_3} \vee \underline{d_5}).$$

We will prove the existence of an epimorphism $M \twoheadrightarrow M_3 :=$ 

This will be a contradiction since M is simple and $M \not\cong M_3$ by hypothesis. Put

$$\begin{array}{ll}
\sigma(0) = \underline{d_0}, & \gamma(0) = (\overline{d_2} \wedge \overline{d_3}) \vee (\overline{d_2} \wedge \overline{d_5}) \vee (\overline{d_3} \wedge \overline{d_5}), \\
\sigma(A) = \underline{d_2} \wedge (\underline{d_3} \vee \underline{d_5}), & \gamma(A) = \overline{d_2} \vee (\overline{d_3} \wedge \overline{d_5}), \\
\sigma(B) = \underline{d_3} \wedge (\underline{d_2} \vee \underline{d_5}), & \gamma(B) = \overline{d_3} \vee (\overline{d_2} \wedge \overline{d_5}), \\
\sigma(C) = \underline{d_5} \wedge (\underline{d_2} \vee \underline{d_3}), & \gamma(C) = \overline{d_5} \vee (\overline{d_2} \wedge \overline{d_3}), \\
\sigma(1) = (\underline{d_2} \vee \underline{d_3}) \wedge (\underline{d_2} \vee \underline{d_5}) \wedge (\underline{d_3} \vee \underline{d_5}), & \gamma(1) = \overline{d_1}.
\end{array}$$

$$\sigma(A) \leq \underline{d_2} := \bigwedge(E_1 \cup E_2) \leq \bigwedge E_2 \leq \bigvee E_2 \leq \bigvee(E_0 \cup E_2) := \overline{d_2} \leq \gamma(A).$$

Similarly $\sigma(P) \leq \gamma(P)$ for all $P \in M_3$. Put

$$S := \bigcup_{p \in M_3} [\sigma(P), \gamma(P)].$$

We show that:

1) $(\forall P, Q \in M_3)\ \ \sigma(P \vee Q) = \sigma(P) \vee \sigma(Q)$ and $\gamma(P \wedge Q) = \gamma(P) \wedge \gamma(Q)$.

2) S is a sublattice of M.

3) $M = S$

4) The five intervals $[\sigma(P), \gamma(P)]$ $(P \in M_3)$ are mutually disjoint.

Proof of 1):

$$\begin{aligned}
\sigma(A) \vee \sigma(B) &= \big(\underline{d_2} \wedge (\underline{d_3} \vee \underline{d_5})\big) \vee \big(\underline{d_3} \wedge (\underline{d_2} \vee \underline{d_5})\big) \\
&= \Big(\big(\underline{d_2} \wedge (\underline{d_3} \vee \underline{d_5})\big) \vee \underline{d_3}\Big) \wedge (\underline{d_2} \vee \underline{d_5}) \\
&\ \text{by modularity since } \underline{d_2} \wedge (\underline{d_3} \vee \underline{d_5}) \leq \underline{d_2} \vee \underline{d_5} \\
&= (\underline{d_2} \vee \underline{d_3}) \wedge (\underline{d_3} \vee \underline{d_5}) \wedge (\underline{d_2} \vee \underline{d_5}) \\
&\ \text{by modularity since } \underline{d_3} \leq \underline{d_3} \vee \underline{d_5} \\
&=: \sigma(1) \\
&= \sigma(A \vee B) \quad \text{since} \quad A \vee B = 1.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\gamma(A) \wedge \gamma(B) &= \big(\overline{d_2} \vee (\overline{d_3} \wedge \overline{d_5})\big) \wedge \big(\overline{d_3} \vee (\overline{d_2} \wedge \overline{d_5})\big) \\
&= (\overline{d_3} \wedge \overline{d_5}) \vee \Big(\overline{d_2} \wedge \big(\overline{d_3} \vee (\overline{d_2} \wedge \overline{d_5})\big)\Big) \\
&\ \text{by modularity since } \overline{d_3} \wedge \overline{d_5} \leq \overline{d_3} \vee (\overline{d_2} \wedge \overline{d_5}) \\
&= (\overline{d_3} \wedge \overline{d_5}) \vee (\overline{d_2} \wedge \overline{d_3}) \vee (\overline{d_2} \wedge \overline{d_5}) \\
&\ \text{by modularity since } \overline{d_2} \wedge \overline{d_5} \leq \overline{d_2} \\
&=: \gamma(0) \\
&= \gamma(A \wedge B) \quad \text{since} \quad A \wedge B = 0.
\end{aligned}$$

Similarly, $\sigma(P \vee Q) = \sigma(P) \vee \sigma(Q)$ and $\gamma(P \wedge Q) = \gamma(P) \wedge \gamma(Q)$ for all $P, Q \in M_3$.

Proof of 2): Take $x, y \in S$. If $x, y \in [\sigma(P), \gamma(P)]$ for some $P \in M_3$, then obviously $x \wedge y, x \vee y \in [\sigma(P), \gamma(P)] \subseteq S$. Suppose say $x \in [\sigma(B), \gamma(B)]$ and $y \in [\sigma(C), \gamma(C)]$. Then

$$x \vee y \geq \sigma(B) \vee \sigma(C) = \sigma(B \vee C) = \sigma(1) \text{ and}$$
$$x \vee y \leq \gamma(B) \vee \gamma(C) \leq \gamma(1) \quad \Rightarrow \quad x \vee y \in [\sigma(1), \gamma(1)] \subseteq S.$$

Also

$$x \wedge y \leq \gamma(B) \wedge \gamma(C) = \gamma(B \wedge C) = \gamma(0) \text{ and}$$
$$x \wedge y \geq \sigma(B) \wedge \sigma(C) \geq \sigma(0) \quad \Rightarrow \quad x \wedge y \in [\sigma(0), \gamma(0)] \subseteq S.$$

Therefore S is a sublattice of M.

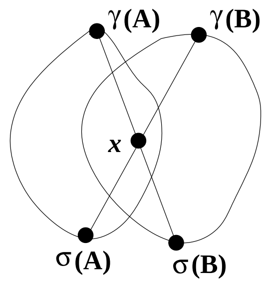
Proof of 3): For each $e \in E_2$,

$$\sigma(A) \leq \underline{d_2} := \bigwedge(E_1 \cup E_2) \leq \bigwedge E_2 \leq e \leq \bigvee E_2 \leq \bigvee(E_0 \cup E_2) := \overline{d_2} \leq \gamma(A).$$

So $E_2 \subseteq [\sigma(A), \gamma(A)] \subseteq S$. Similarly, all $E_i \subseteq S$. So $E_0 \cup E_2 \cup E_3 \cup E_5 \cup E_1 \subseteq S$ implies $M = \langle E_0 \cup E_2 \cup E_3 \cup E_5 \cup E_1 \rangle = S$ since S is a sublattice of M.

Proof of 4): Suppose for instance that $[\sigma(A), \gamma(A)] \cap [\sigma(B), \gamma(B)] \neq \emptyset$. Pick $x \in [\sigma(A), \gamma(A)] \cap [\sigma(B), \gamma(B)]$.

$$\sigma(A) \leq x \leq \gamma(B) \text{ and } \sigma(B) \leq x \leq \gamma(A)$$



Then $\sigma(A) \vee \sigma(B) \leq \gamma(B)$ and $\sigma(A \vee B) = \sigma(A) \vee \sigma(B) \leq \gamma(A)$ imply $\sigma(1) \leq \gamma(A) \wedge \gamma(B) = \gamma(A \wedge B) = \gamma(0)$. This is a contradiction since by assumption

$$\gamma(0) = (\overline{d_2} \wedge \overline{d_3}) \vee (\overline{d_2} \wedge \overline{d_5}) \vee (\overline{d_3} \wedge \overline{d_5}) \not\geq (\underline{d_2} \vee \underline{d_3}) \wedge (\underline{d_2} \vee \underline{d_5}) \wedge (\underline{d_3} \vee \underline{d_5}) = \sigma(1).$$

Similarly other intervals are mutually disjoint.

From (3) and (4) follows that $f : M \longrightarrow M_3$ defined by:

$$f(x) = P :\Longleftrightarrow x \in [\sigma(P), \gamma(P)]$$

is well-defined, and so is obviously an epimorphism by (1). ■

Definition 6.4 *For $h \in (H, \leq)$, denote by $r(h)$ the length of the longest chain between h and a minimal element of H. All sets $\rho_n(H) := \{h \in H | r(h) = n\}, (n \geq 0)$ clearly are antichains.*

Lemma 6.3 *Let $(H, \leq)$ be a poset not containing A_4 as subposet and let $\rho_0(H) := \{m_0, m_1\}$ (i.e. H has exactly two minimal elements). If*

$$U(m_i) := \{p \in H | p \ngeq m_i\} \ (i = 0, 1),$$

then either $U(m_0)$ or $U(m_1)$ is a chain.

Proof: Let $p \in U(m_0)$ and $q \in U(m_1)$. We show that p and q are incomparable. In fact if $p \geq q$, then $q \geq m_0 \Rightarrow p \geq m_0$ which is a contradiction since $p \in U(m_0)$. Therefore $p \ngeq q$. Ditto $q \ngeq p$. Now if neither $U(m_0)$ nor $U(m_1)$ is a chain, then at least two elements of $U(m_0)$ say a, b are incomparable and at least two elements of $U(m_1)$ say c, d are incomparable. By the previous arguments, this implies that $\{a, b, c, d\}$ is a 4-element antichain of H, contradicting the assumption. ■

Lemma 6.4 *If $A_4 \nsubseteq H$ and $H_5 \nsubseteq H$, then each subdirectly irreducible factor of $F\mathcal{M}(H)$ is D_2 or M_3.*

Proof:

Fix any subdirectly irreducible factor M of $F\mathcal{M}(H)$. By proposition 3.6 on page 17, there is an epimorphism $F\mathcal{M}(H) \overset{\rho}{\twoheadrightarrow} M$. Its monotone restriction $\psi : H \longrightarrow M$ is fixed throughout the proof. We emphasize that ψ could be highly non-injective. We assume that $M \ncong D_2$ and $M \ncong M_3$ and we show that $|M| = 1$. Namely, we shall prove by induction on $n := |H|$ that

$$\text{if} \quad M = \langle \psi(H) \rangle, \quad \text{then} \quad |M| = 1.$$

- For $n = 1$, this is trivially true
- For $n > 1$, we may by induction suppose that if $M = \langle \psi(H') \rangle$ for some $H' \subsetneqq H$, then $|M| = 1$.

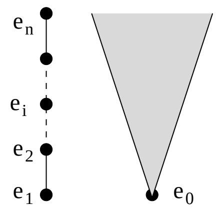
Put $E := \psi(H)$ and $e_i := \psi(h_i)$ for any $h_i \in H$. Let $min(E)$ be the set of minimal elements of E. We proceed now by case distinction according to $|min(H)|$. With $(H, \leq)$ a fortiori $(E, \leq)$ has no subposet A_4. In particular $|min(E)| \leq 3$.

<u>**Case 1**</u>**:** $|min(E)| = 1, \quad \text{say} \quad min(E) = \{e_0\}$.

Since $M = \langle E \rangle = \big\langle \{e_0\} \cup (E \setminus \{e_0\}) \big\rangle$ and $M \not\cong D_2$, $e_0 \geq \bigwedge(E \setminus \{e_0\})$ by the D_2-lemma, i.e. lemma (6.1). Hence since e_0 is the minimum element of E, $e_0 = \bigwedge(E \setminus \{e_0\})$ and thus $M = \big\langle E \setminus \{e_0\} \big\rangle = \big\langle \psi(H') \big\rangle$ for some $H' \subseteq H \setminus \{h_0\}$. So by induction hypothesis $|M| = 1$.

<u>Case 2</u>: $|min(E)| = 2$, say $min(E) = \{e_0, e_1\}$.

Set $U(e_i) = \{e \in H | e \not\geq e_i\}$. By lemma (6.3), either $U(e_0)$ or $U(e_1)$ is a chain. Suppose that $U(e_0)$ is a chain. Then $(E, \leq)$ looks so:



We have $M = \langle E \rangle = \langle E_0 \cup E_1 \rangle$ where $E_0 = \psi(U(e_0)) = \{e_1, \cdots, e_n\}$ and $E_1 = \psi(U(e_1)) = \{e | e \geq e_0\}$. So by the D_2-lemma

$$\bigvee\{e_1, \cdots, e_n\} \;\geq\; \bigwedge\{e | e \geq e_0\}.$$

That is,

$$e_n \;\geq\; e_0 \tag{6.3.1}$$

By the D_2-lemma again,

$$\bigvee\{e_1, \cdots, e_i\} \;\geq\; \bigwedge\{e | e \neq e_1, \cdots, e_i\} \;\; (1 \leq i \leq n-1),$$

hence

$$e_i \;\geq\; e_{i+1} \wedge e_0 \quad (1 \leq i \leq n-1). \tag{6.3.2}$$

Therefore,

$$\begin{aligned} e_1 &\geq e_2 \wedge e_0 && \text{by (6.3.2)} \\ &\geq (e_3 \wedge e_0) \wedge e_0 && \text{by (6.3.2)} \\ &= e_3 \wedge e_0 \\ &\;\vdots \\ &\geq e_n \wedge e_0 \\ &= e_0 && \text{by (6.3.1)} \end{aligned}$$

From $e_1 \geq e_0$ follows at once (see the sketch of $(E, \leq)$) that e_0 is the minimum element of E. As in case 1, one concludes that $e_0 = \bigwedge(E \setminus \{e_0\})$, whence $M = \langle E \setminus \{e_0\} \rangle$, whence $|M| = 1$.

<u>Case 3</u>: $|min(E)| = 3$.

Then necessarily $|\rho_0(H)| = 3$. We will distinguish 3 subcases according to the number $|\rho_1(H)| \leq 3$ of elements of H with length 1.

<u>Case 3.1</u>: $|\rho_1(H)| = 1$, say $\rho_1(H) = \{h_3\}$.

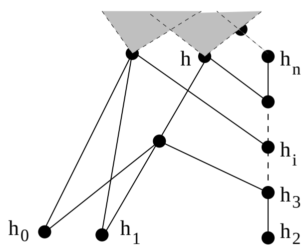
There is at least one element in $\rho_0(H)$, say h_2 with $h_2 \prec h_3$. From $|\rho_0(H)| = 3$ follows $\rho_0(H) = \{h_0, h_1, h_2\}$. If the set $\{h \in H | h \not\geq h_0 \text{ and } h \not\geq h_1\}$ (which at least contains h_2) contains two incomparable elements, say a and b, then $\{a, b, h_0, h_1\}$ would be a 4-element antichain of H which is a contradiction. So

$$\{h \in H | h \not\geq h_0 \ \text{ and } \ h \not\geq h_1\} = \{h_2 \prec h_3 \prec \cdots \prec h_n\} \quad (n \geq 2) \quad (6.3.3)$$

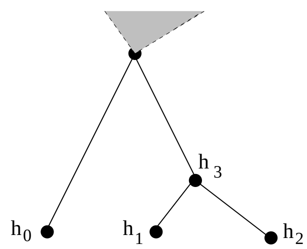
is a chain. Moreover since $\rho_1(H) = \{h_3\}$, we have

$$h_0 < h \quad \text{or} \quad h_1 < h \quad \Rightarrow \quad h \geq h_3 > h_2. \qquad (6.3.4)$$

For instance $(H, \leq)$ is



or



$(n = 2)$

We will show that $\psi(h_0) \geq \psi(h_2)$ which will bring us back to case 2.

By (6.3.4), $h > h_0$ implies $e \geq e_2$. Trivially $h \geq h_i$ implies $e \geq e_i$ $(i = 1, 2)$, and so $\bigwedge(E \setminus \{e_0\}) \geq e_1 \wedge e_2$. But by the D_2-lemma, $e_0 \geq \bigwedge(E \setminus \{e_0\})$. So

$$e_0 \geq e_1 \wedge e_2. \qquad (6.3.5)$$

Next, $M = \langle \{e_0, e_1\} \cup \big(\{e | e > e_0\} \cup \{e | e > e_1\} \cup \{e_i | 2 \leq i \leq n\}\big) \rangle$ by (6.3.3). So by the D_2-lemma

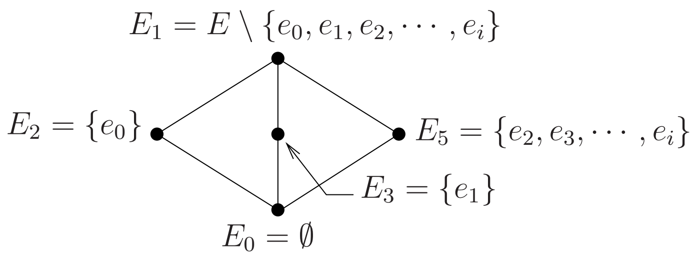
$$e_0 \vee e_1 \ \geq \ (\bigwedge\{e | e > e_0\}) \wedge (\bigwedge\{e | e > e_1\}) \wedge e_2.$$

Subcase A: Either e_0, e_1 or e_0, e_2 are comparable. Then $|min(E)| \leq 2$ and we are back to case 2 or case 1.

Subcase B: Neither e_0, e_1 nor e_0, e_2 are comparable. It then follows from $e = \rho(h) > e_0$ and $\rho_0(H) = \{h_0, h_1, h_2\}$ that $h > h_0$ or $h > h_1$ or $h > h_2$. By (6.3.4) $e \geq e_3$, and so $\bigwedge\{e|e > e_0\} \geq e_3$. Similarly $\bigwedge\{e|e > e_1\} \geq e_3$, and therefore

$$e_0 \vee e_1 \geq e_3 \wedge e_3 \wedge e_2 = e_2. \tag{6.3.6}$$

In the lengthly sequel we are going to strengthen (6.3.6) to $e_0 \geq e_2$. In other words, it will turn out that subcase B is in fact impossible. For starters, consider the partition $E = E_0 \cup E_2 \cup E_3 \cup E_5 \cup E_1$ depicted below for some fixed $i \in \{2, 3, \cdots, n\}$:



Letting

$$e_i' = \bigwedge E_1,$$

we have $e_i' \geq e_2$ by (6.3.4). We define

$$\begin{array}{lll}
\overline{d_2} := \bigvee(E_0 \cup E_2) = e_0, & \overline{d_3} := \bigvee(E_0 \cup E_3) = e_1, & \overline{d_5} := \bigvee(E_0 \cup E_5) = e_i, \\
\underline{d_2} := \bigwedge(E_1 \cup E_2) = e_i' \wedge e_0, & \underline{d_3} := \bigwedge(E_1 \cup E_3) = e_i' \wedge e_1, & \\
\underline{d_5} := \bigwedge(E_1 \cup E_5) = e_i' \wedge e_2 = e_2 \quad \text{since } e_i' \geq e_2. & &
\end{array}$$

By the M_3-lemma, i.e. lemma (6.2),

$$(\overline{d_2} \wedge \overline{d_3}) \vee (\overline{d_2} \wedge \overline{d_5}) \vee (\overline{d_3} \wedge \overline{d_5}) \geq (\underline{d_2} \vee \underline{d_3}) \wedge (\underline{d_2} \vee \underline{d_5}) \wedge (\underline{d_3} \vee \underline{d_5}).$$

So

$$\begin{aligned}
(e_0 \wedge e_1) \vee (e_0 \wedge e_i) \vee (e_1 \wedge e_i) \quad \geq \quad & \big((e_0 \wedge e_i') \vee (e_1 \wedge e_i')\big) \wedge \big((e_0 \wedge e_i') \vee e_2\big) \\
& \wedge \big((e_1 \wedge e_i') \vee e_2\big)
\end{aligned}$$

Since $e_2 \leq e_i'$, it follows by modularity that:

$$\begin{aligned}(e_0 \wedge e_i') \vee e_2 &= (e_0 \vee e_2) \wedge e_i'\\(e_1 \wedge e_i') \vee e_2 &= (e_1 \vee e_2) \wedge e_i'\end{aligned}$$

Taking into account that $(e_0 \wedge e_i') \vee (e_1 \wedge e_i') \leq e_i'$ we get for all $2 \leq i \leq n$

$$\begin{aligned}(e_0 \wedge e_1) \vee (e_0 \wedge e_i) \vee (e_1 \wedge e_i) &\geq \big((e_0 \wedge e_i') \vee (e_1 \wedge e_i')\big) \wedge (e_0 \vee e_2)\\&\quad \wedge (e_1 \vee e_2). \qquad (6.3.7)\end{aligned}$$

Putting

$$\hat{e}_0 := \bigwedge\{e \in E | e > e_0\} \quad \text{and} \quad \hat{e}_1 := \bigwedge\{e \in E | e > e_1\},$$

we get $e_i' = e_{i+1} \wedge \hat{e}_0 \wedge \hat{e}_1$ by (6.3.3). If we set

$$e_{n+1} := \bigwedge\{e \in E | e > e_n\},$$

this includes $e_n' = e_{n+1} \wedge \hat{e}_0 \wedge \hat{e}_1$. We further process the right hand side of (6.3.7):

$\big((e_0 \wedge e_i') \vee (e_1 \wedge e_i')\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2)$
$= \big((e_0 \wedge e_{i+1} \wedge \hat{e}_1) \vee (e_1 \wedge e_{i+1} \wedge \hat{e}_0)\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2)$ since $e_0 \leq \hat{e}_0$ and $e_1 \leq \hat{e}_1$
$= \big((e_0 \wedge e_{i+1} \wedge \hat{e}_1) \vee (e_1 \wedge e_{i+1})\big) \wedge \hat{e}_0 \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2)$ by modularity
$= \big((e_0 \wedge e_{i+1}) \vee (e_1 \wedge e_{i+1})\big) \wedge \hat{e}_0 \wedge \hat{e}_1 \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2)$ by modularity
$= \big((e_0 \wedge e_{i+1}) \vee (e_1 \wedge e_{i+1})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2)$.
The last equality holds since by subcase B and (6.3.4), $e_2 \leq \hat{e}_0$ and $e_2 \leq \hat{e}_1$ imply $e_0 \vee e_2 \leq \hat{e}_0$ and $e_1 \vee e_2 \leq \hat{e}_1$. Coupling the above inequality with inequality (6.3.7) and then taking the meet with $(e_0 \vee e_2) \wedge (e_1 \vee e_2)$, we get

$$\begin{aligned}&\big((e_0 \wedge e_1) \vee (e_0 \wedge e_i) \vee (e_1 \wedge e_i)\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2)\\\geq\ &\big((e_0 \wedge e_{i+1}) \vee (e_1 \wedge e_{i+1})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2), \quad 2 \leq i \leq n \qquad (6.3.8)\end{aligned}$$

Since $e_0 \wedge e_1 \leq (e_0 \vee e_2) \wedge (e_1 \vee e_2)$, modularity applied to the first part of (6.3.8) yields

$$\begin{aligned}&\big((e_0 \wedge e_1) \vee (e_0 \wedge e_i) \vee (e_1 \wedge e_i)\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2)\\=\ &(e_0 \wedge e_1) \vee \Big(\big((e_0 \wedge e_i) \vee (e_1 \wedge e_i)\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2)\Big)\end{aligned}$$

Hence, taking the union of both sides of (6.3.8) with $e_0 \wedge e_1$ leaves invariant the first part of (6.3.8) and yields:

$$\begin{aligned} & (e_0 \wedge e_1) \vee \Big(\big((e_0 \wedge e_i) \vee (e_1 \wedge e_i) \big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \Big) \\ \geq\ & (e_0 \wedge e_1) \vee \Big(\big((e_0 \wedge e_{i+1}) \vee (e_1 \wedge e_{i+1}) \big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \Big) \\ =\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{i+1}) \vee (e_1 \wedge e_{i+1}) \big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\ & \text{by modularity since } e_0 \wedge e_1 \leq (e_0 \vee e_2) \wedge (e_1 \vee e_2). \end{aligned}$$

So we get the following inequality for each $2 \leq i \leq n$:

$$\begin{aligned} & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_i) \vee (e_1 \wedge e_i) \big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\ \geq\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{i+1}) \vee (e_1 \wedge e_{i+1}) \big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \end{aligned} \tag{6.3.9}$$

Iterating (6.3.9) from $i = 2$ to $i = n$, we get:

$$\begin{aligned} & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_2) \vee (e_1 \wedge e_2) \big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\ \geq\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_3) \vee (e_1 \wedge e_3) \big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\ & \vdots \\ \geq\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_n) \vee (e_1 \wedge e_n) \big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\ \geq\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{n+1}) \vee (e_1 \wedge e_{n+1}) \big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \end{aligned}$$

Therefore, since $e_0 \geq e_1 \wedge e_2$ by (6.3.5), we have

$$\begin{aligned} e_0 \ \geq\ & (e_0 \wedge e_1) \vee (e_0 \wedge e_2) \vee (e_1 \wedge e_2) \\ \geq\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_2) \vee (e_1 \wedge e_2) \big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\ \geq\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{n+1}) \vee (e_1 \wedge e_{n+1}) \big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\ \geq\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{n+1}) \vee (e_1 \wedge e_{n+1}) \big) \wedge e_2 \end{aligned}$$

By (6.3.3),

$$\begin{aligned} e_{n+1} &= \Big(\bigwedge \{ e \in E | e > e_n \ \text{ and } \ e > e_0 \} \Big) \wedge \Big(\bigwedge \{ e \in E | e > e_n \ \text{ and } \ e > e_1 \} \Big) \\ &= e^0_{n+1} \wedge e^1_{n+1} \geq e_0 \wedge e_1 \end{aligned}$$

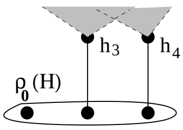
where $e^0_{n+1} := \{ e \in E | e > e_n \ \text{ and } \ e > e_0 \}$ and $e^1_{n+1} := \{ e \in E | e > e_n \ \text{ and } \ e > e_1 \}$. So

$$
\begin{aligned}
e_0 &\geq \big((e_0 \wedge e^1_{n+1}) \vee (e_1 \wedge e^0_{n+1})\big) \wedge e_2 \\
&= \Big(\big(e_0 \vee (e_1 \wedge e^0_{n+1})\big) \wedge e^1_{n+1}\Big) \wedge e_2 \quad \text{by} \quad \text{modularity} \\
&= (e_0 \vee e_1) \wedge e^0_{n+1} \wedge e^1_{n+1} \wedge e_2 \quad \text{by} \quad \text{modularity} \\
&= e^0_{n+1} \wedge e^1_{n+1} \wedge e_2 \quad \text{by} \quad (6.3.6) \\
&= e_2 \quad \text{since} \quad e^0_{n+1},\ e^1_{n+1} \geq e_2.
\end{aligned}
$$

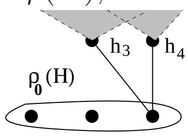
This is the desired contradiction to the hypothesis of subcase B.

Case 3.2: $|\rho_1(H)| = 2$, say $\rho_1(H) = \{h_3, h_4\}$.

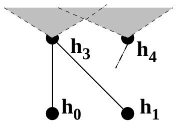
If both h_3 and h_4 cover only one element of $\rho(H)$, then one either has



whence $H_5 \subseteq H$ or one has

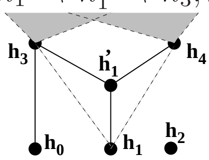


whence $A_4 \subseteq H$. In either case we obtain a contradiction. Hence we may assume that $\rho_0(H) = \{h_0, h_1, h_2\}$ and $h_0, h_1 \prec h_3$:



Case 3.2.1: if $h_1 \prec h_4$ (Case $h_0 \prec h_4$ is analogue).

Consider the new poset $\widehat{H} := H \cup \{h'_1\}$ where $h_1 \prec h'_1 \prec h_3, h_4$ and no other new relation. By transitivity, $\widehat{H}$ looks so:



We claim that $(\widehat{H}, \leq)$ contains no A_4 and no H_5 and $\rho_1(\widehat{H}) = \{h'_1\}$. In fact

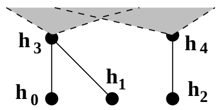
- If $\widehat{H}$ contains an A_4, then necessarily $h'_1 \in A_4$ since $A_4 \nsubseteq H$. So $h_1 \notin A_4$ and $h \notin A_4$ for all $h > h'_1$. Therefore $A_4 \subseteq \{h'_1, h_0, h_2\}$ which is a contradiction.

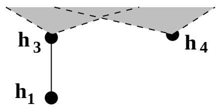
- If $\widehat{H}$ contains a H_4, suppose that there exists $h \in H_5 \backslash \{h', h_0, h_1, h_2, h_3, h_4\}$, then necessarily $h > h_3$ or $h > h_4$ since $\rho_1(H) = \{h_3, h_4\}$. So $\{h'_1 \prec h_3 \prec h\} \subseteq H_5$ or $\{h'_1 \prec h_4 \prec h\} \subseteq H_5$ which, in either case, is a contradiction since  $\nsubseteq H_5$. So $H_5 \subseteq \{h'_1, h_0, h_1, h_2, h_3, h_4\}$. But then each of the 5-element posets $\{h'_1, h_0, h_1, h_2, h_3\}$, $\{h'_1, h_1, h_2, h_3, h_4\}$, $\{h'_1, h_2, h_3, h_4, h_0\}$, $\{h'_1, h_3, h_4, h_0, h_1\}$ and $\{h'_1, h_4, h_0, h_1, h_2\}$ contains

at least 3 relations of type $a < b$ which is a contradiction since H_5 contains only 2 such relations.
So $(\widehat{H}, \leq)$ contains no A_4 and no H_5 as claimed.

It is clear that $\rho_1(\widehat{H}) = \{h_1'\}$ since $h_1 \prec h'$ is the only maximal chain ending at h_1'. Extend $\psi : H \longrightarrow M$ to $\widehat{\psi} : \widehat{H} \longrightarrow M$ with $\widehat{\psi}(h_1') := \psi(h_1)$. Because $M = \langle \widehat{\psi}(\widehat{H}) \rangle$ where $(\widehat{H}, \leq)$ satisfies $\left|\rho_1(\widehat{H})\right| = 1$, case 3.1 implies that $|M| = 1$.

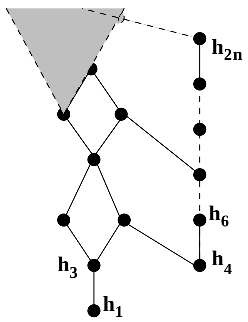
Case 3.2.2: $h_0 \not< h_4, \quad h_1 \not< h_4$

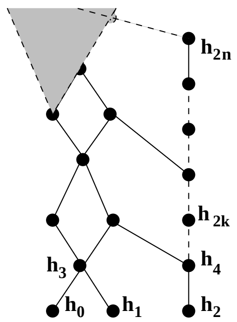
Then necessarily $h_2 \prec h_4$. So H looks so: 

Whence $H \setminus \{h_0, h_2\} := H^- =:$ 

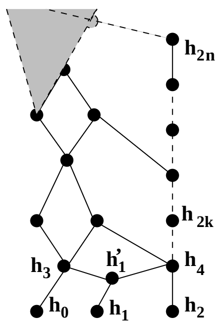
From $\rho_0(H^-) = \{h_1, h_4\}$, applying lemma (6.3) to H^- yields $U(h_1) = \{h \in H^- | h \not\geq h_1\}$ is a chain or $U(h_4) = \{h \in H^- | h \not\geq h_4\}$ is a chain.

Subcase (a): $U(h_1) = \{h \in H^- | h \not\geq h_1\}$ is a chain $h_4 < h_6 < \cdots < h_{2n}$.

For instance $H^- :=$ 

and so $H :=$ 

We would like to extend H to a poset $\widehat{H} := H \cup \{h_1'\}$ where $h_1 \prec h_1' \prec h_3$ and $h_1' \prec h_4$.

So $\widehat{H}$ looks so: 

Correspondingly we would like to extend $\psi : H \longrightarrow M$ to a monotone map $\widehat{\psi} : \widehat{H} \longrightarrow M$ with $\widehat{H}(h'_1) = \psi(h_1)$. If we manage to do that, then we can apply case 3.1 to $\widehat{\psi}$ (since clearly $\rho_1(\widehat{H}) = \{h'_1\}$) and conclude that $|M| = 1$. The problem is that if we want $\widehat{\psi}$ to be monotone, then $\widehat{\psi}(h_1) \leq \widehat{\psi}(h_4)$ (since $h_1 \leq h_4$ in $\widehat{H}$). This works only when $\psi(h_1) \leq \psi(h_4)$. Thus we must show:

$$\psi(h_4) \geq \psi(h_1), \quad \text{i.e. } e_4 \geq e_1. \tag{6.3.10}$$

Proof of (6.3.10):

Let h_{2k} be the biggest element of $U(h_1)$ which is not greater or equal h_0. Put $e_{2n+2} := \bigwedge\{e \in E | e > e_{2n}\}$ and for $2 \leq i \leq k$

$$E_0 = \{e_2, e_4, \cdots, e_{2i}\} \quad \text{and} \quad E_1 = \{e_0, e_1, e_3, \cdots, e_{2i+2}, \cdots, e_{2n}\}$$

Applying the D_2-lemma to $M = \langle E_0 \cup E_1 \rangle$ yields $\bigvee E_0 \geq \bigwedge E_1$, i.e.

$$e_{2i} \geq e_0 \wedge e_1 \wedge e_{2i+2}, \quad 2 \leq i \leq k. \tag{6.3.11}$$

Iterating (6.3.11) from $i = 2$ to $i = k$ yields

$$e_4 \geq e_0 \wedge e_1 \wedge e_6 \geq \cdots \geq e_0 \wedge e_1 \wedge e_{2k+2}$$

Taking into account that $e_{2k+2} \geq e_0$ by definition of h_{2k}, we get

$$e_4 \geq e_0 \wedge e_1. \tag{6.3.12}$$

Also, applying the D_2-lemma to $M = \langle E_0 \cup E_1 \rangle$ where

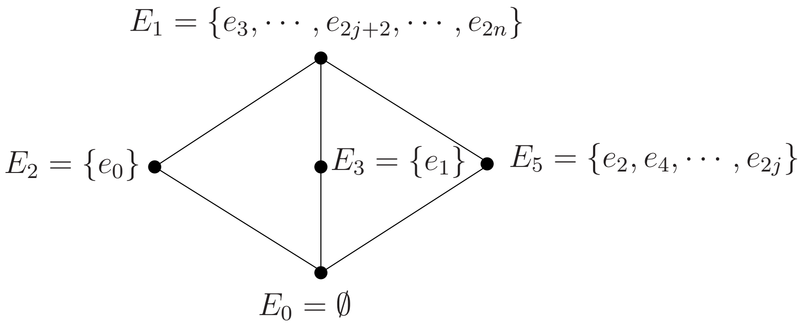
$$E_0 = \{e_0, e_2, e_4, \cdots, e_{2i}\} \quad \text{and} \quad E_1 = \{e_1, e_3, \cdots, e_{2i+2}, \cdots, e_{2n}\},$$

we get

$$e_0 \vee e_{2i} \geq e_1 \wedge e_{2i+2}, \quad 2 \leq i \leq k. \tag{6.3.13}$$

For $2 \leq j \leq k$, $M = \langle E_0 \cup E_2 \cup E_3 \cup E_5 \cup E_1 \rangle$ where

$$\begin{array}{ll}
\bar{d}_2 := \bigvee(E_0 \cup E_2) = e_0, & \bar{d}_3 := \bigvee(E_0 \cup E_3) = e_1, \\
\bar{d}_5 := \bigvee(E_0 \cup E_5) = e_{2j}, & \underline{d_2} := \bigwedge(E_1 \cup E_2) = e_0 \wedge e_{2j+2}, \\
\underline{d_3} := \bigwedge(E_1 \cup E_3) = e_1 \wedge e_{2j+2}, & \underline{d_5} := \bigwedge(E_1 \cup E_5) = e_2 \wedge e_3.
\end{array}$$



By the M_3-lemma, we have

$$\begin{aligned}
& (e_0 \wedge e_1) \vee (e_0 \wedge e_{2j}) \vee (e_1 \wedge e_{2j}) \\
\geq\ & \big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge \big((e_0 \wedge e_{2j+2}) \vee (e_2 \wedge e_3)\big) \wedge \big((e_1 \wedge e_{2j+2}) \vee (e_2 \wedge e_3)\big) \\
=\ & \big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge \big((e_0 \vee (e_2 \wedge e_3)\big) \wedge e_{2j+2} \wedge \big((e_1 \vee (e_2 \wedge e_3)\big) \wedge e_{2j+2} \\
& \text{by modularity since } e_2 \wedge e_3 \leq e_2 \leq e_{2j+2} \\
=\ & \big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_2) \wedge e_3 \wedge (e_1 \vee e_2) \wedge e_3 \wedge e_{2j+2} \\
& \text{by modularity since } e_0, e_1 \leq e_3 \\
=\ & \big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\
& \text{since } e_0, e_1 \leq e_3 \Rightarrow (e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2}) \leq e_3 \wedge e_{2j+2} \\
=\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\
& \text{since } e_0 \wedge e_1 \leq e_4 \leq e_{2j+2} \text{ by (6.3.12)} \Rightarrow e_0 \wedge e_1 \leq e_0 \wedge e_{2j+2}.
\end{aligned}$$

That is for $2 \leq j \leq k$;

$$\begin{aligned}
& (e_0 \wedge e_1) \vee (e_0 \wedge e_{2j}) \vee (e_1 \wedge e_{2j}) \qquad (6.3.14) \\
\geq\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2).
\end{aligned}$$

Taking the intersection of both sides of (6.3.14) with $(e_0 \vee e_2) \wedge (e_1 \vee e_2)$ yields, for $2 \leq j \leq k$,

$$\begin{aligned}
& \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{2j}) \vee (e_1 \wedge e_{2j})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \quad (6.3.15) \\
\geq\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2).
\end{aligned}$$

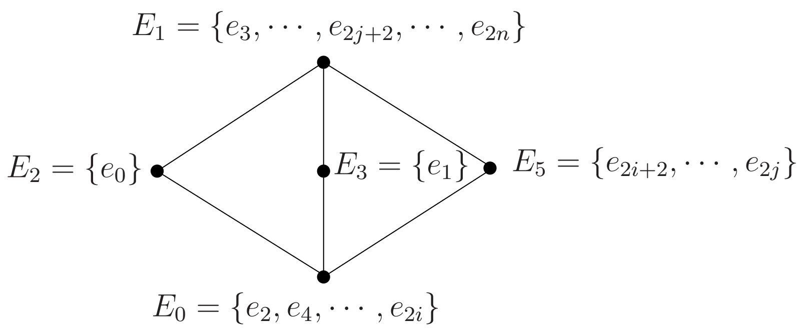
Since $e_4 \geq e_0 \wedge e_1$ by (6.3.12) and $e_4 \geq (e_0 \wedge e_4) \vee (e_1 \wedge e_4)$, we have

$$\begin{aligned}
e_4 &\geq (e_0 \wedge e_1) \vee (e_0 \wedge e_4) \vee (e_1 \wedge e_4) \\
&\geq \big((e_0 \wedge e_1) \vee (e_0 \wedge e_4) \vee (e_1 \wedge e_4)\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\
&\geq \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{2k+2}) \vee (e_1 \wedge e_{2k+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\
&\quad \text{by iterating (6.3.15) from } j = 2 \text{ to } j = k \\
&\geq \big((e_0 \wedge e_{2k+2}) \vee (e_1 \wedge e_{2k+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\
&\quad \text{since } e_0 \wedge e_1 \leq e_4 \leq e_{2k+2} \Rightarrow e_0 \wedge e_1 \leq e_0 \wedge e_{2k+2} \\
&= \big(e_0 \vee (e_1 \wedge e_{2k+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\
&\quad \text{since by definition of } h_{2k},\ e_0 \leq e_{2k+2} \Rightarrow e_0 \wedge e_{2k+2} = e_0 \\
&= (e_0 \vee e_1) \wedge e_{2k+2} \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\
&\quad \text{by modularity since } e_0 \leq e_{2k+2} \\
&= (e_0 \vee e_1) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\
&\quad \text{since } e_0, e_2 \leq e_{2k+2} \Rightarrow e_{2k+2} \wedge (e_0 \vee e_2) = e_0 \vee e_2.
\end{aligned}$$

Therefore

$$e_4 \geq (e_0 \vee e_1) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2). \tag{6.3.16}$$

On the other hand, $M = \langle E_0 \cup E_2 \cup E_3 \cup E_5 \cup E_1 \rangle$ where for $1 \leq i < j \leq k$



$$\begin{aligned}
\bar{d}_2 &:= \bigvee(E_0 \cup E_2) = e_0 \vee e_{2i}, & \bar{d}_3 &:= \bigvee(E_0 \cup E_3) = e_1 \vee e_{2i}, \\
\bar{d}_5 &:= \bigvee(E_0 \cup E_5) = e_{2j}, & \underline{d_2} &:= \bigwedge(E_1 \cup E_2) = e_0 \wedge e_{2j+2}, \\
\underline{d_3} &:= \bigwedge(E_1 \cup E_3) = e_1 \wedge e_{2j+2}, & \underline{d_5} &:= \bigwedge(E_1 \cup E_5) = e_3 \wedge e_{2i+2}.
\end{aligned}$$

By the M_3-lemma, we have

$$\begin{aligned}
&\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee \big((e_0 \vee e_{2i}) \wedge e_{2j}\big) \vee \big((e_1 \vee e_{2i}) \wedge e_{2j}\big) \\
\geq\ &\big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge \big((e_0 \wedge e_{2j+2}) \vee (e_3 \wedge e_{2i+2})\big) \\
&\wedge\big((e_1 \wedge e_{2j+2}) \vee (e_3 \wedge e_{2i+2})\big) \\
=\ &\big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge \big((e_0 \wedge e_{2j+2}) \vee e_{2i+2}\big) \wedge e_3 \\
&\wedge\big((e_1 \wedge e_{2j+2}) \vee e_{2i+2}\big) \wedge e_3 \\
&\text{by modularity since } e_0 \wedge e_{2j+2} \leq e_0 \leq e_3 \text{ and } e_1 \wedge e_{2j+2} \leq e_1 \leq e_3 \\
=\ &\big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_{2i+2}) \wedge e_{2j+2} \wedge (e_1 \vee e_{2i+2}) \wedge e_{2j+2} \wedge e_3 \\
&\text{by modularity since } e_{2i+2} \leq e_{2j+2} \\
=\ &\big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2}) \\
&\text{since } e_0, e_1 \leq e_3 \Rightarrow (e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2}) \leq e_3 \wedge e_{2j+2}.
\end{aligned}$$

Therefore for $1 \leq i < j \leq k$,

$$\begin{aligned}
&\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee \big((e_0 \vee e_{2i}) \wedge e_{2j}\big) \vee \big((e_1 \vee e_{2i}) \wedge e_{2j}\big) \\
\geq\ &\big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2}) \qquad (6.3.17)
\end{aligned}$$

Since for $1 \leq i < j \leq k$, $e_{2i} \leq e_{2j}$, the left hand side of (6.3.17) yields by modularity

$$\begin{aligned}
&\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee \big((e_0 \vee e_{2i}) \wedge e_{2j}\big) \vee \big((e_0 \vee e_{2i}) \wedge e_{2j}\big) \\
=\ &\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2j}) \vee e_{2i} \vee (e_1 \wedge e_{2j}) \vee e_{2i} \\
=\ &\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2j}) \vee (e_1 \wedge e_{2j}) \quad \text{since} \quad e_{2i} \leq (e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})
\end{aligned}$$

That is, by (6.3.17)

$$\begin{aligned}
&\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2j}) \vee (e_1 \wedge e_{2j}) \\
\geq\ &\big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2}).
\end{aligned}$$

Taking the union of both sides of the previous inequality with $(e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})$ yields

$$\begin{aligned}
&\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee \Big(\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2j}) \vee (e_1 \wedge e_{2j})\Big) \\
\geq\ &\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee \Big(\big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2})\Big).
\end{aligned}$$

That is,

$$\begin{aligned}
&\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2j}) \vee (e_1 \wedge e_{2j}) \\
\geq\ &\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee \Big(\big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2})\Big) \\
&\text{since } (e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i}) \leq \big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2j}) \vee (e_1 \wedge e_{2j}).
\end{aligned}$$

Taking the intersection of both sides of the previous inequality with $(e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2})$ yields

$$\begin{aligned}
&\Big(\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2j}) \vee (e_1 \wedge e_{2j})\Big) \wedge (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2}) \\
\geq\ & \Big[\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee \Big(\big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_{2i+2}) \wedge \\
& (e_1 \vee e_{2i+2})\Big)\Big] \wedge (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2}) \\
=\ & \big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee \Big[\big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2}) \\
& \wedge (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2})\Big] \quad \text{by modularity since} \\
& (e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i}) \leq (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2}) \\
=\ & \big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee \Big(\big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2})\Big) \\
=\ & \Big(\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\Big) \wedge (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2}) \\
& \text{by modularity since } (e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i}) \leq (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2}).
\end{aligned}$$

That is for $1 \leq i < j \leq k$, we obtain the following recurrence (with respect to j) inequality:

$$\begin{aligned}
&\Big(\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2j}) \vee (e_1 \wedge e_{2j})\Big) \wedge (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2}) \\
\geq\ & \Big(\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\Big) \wedge (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2})
\end{aligned}$$

Iterating the previous inequality from $j = i+1$ to $j = k$ yields

$$\begin{aligned}
&\Big(\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2i+2}) \vee (e_1 \wedge e_{2i+2})\Big) \wedge (e_0 \vee e_{2i+2}) \\
& \wedge (e_1 \vee e_{2i+2}) \\
\geq\ & \Big(\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2k+2}) \vee (e_1 \wedge e_{2k+2})\Big) \wedge (e_0 \vee e_{2i+2}) \\
& \wedge (e_1 \vee e_{2i+2})
\end{aligned} \tag{6.3.18}$$

On the other hand, we have

$$\begin{aligned}
&\Big(\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2i+2}) \vee (e_1 \wedge e_{2i+2})\Big) \wedge e_1 \\
\geq\ & \Big(\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2i+2}) \vee (e_1 \wedge e_{2i+2})\Big) \wedge (e_0 \vee e_{2i+2}) \\
& \wedge (e_1 \vee e_{2i+2}) \wedge e_1 \\
\geq\ & \Big(\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2k+2}) \vee (e_1 \wedge e_{2k+2})\Big) \wedge (e_0 \vee e_{2i+2}) \\
& \wedge (e_1 \vee e_{2i+2}) \wedge e_1
\end{aligned}$$

That is,

$$\begin{aligned}
&\Big(\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2i+2}) \vee (e_1 \wedge e_{2i+2})\Big) \wedge e_1 \qquad (6.3.19)\\
\geq\ &\Big(\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2k+2}) \vee (e_1 \wedge e_{2k+2})\Big) \wedge (e_0 \vee e_{2i+2})\\
&\wedge(e_1 \vee e_{2i+2}) \wedge e_1
\end{aligned}$$

But

$$\begin{aligned}
&\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2k+2}) \vee (e_1 \wedge e_{2k+2})\\
=\ &\Big((e_1 \vee e_{2i}) \wedge \big((e_0 \vee e_{2i}) \vee (e_1 \wedge e_{2k+2})\big)\Big) \vee (e_0 \wedge e_{2k+2})\\
&\text{by modularity since } e_1 \vee e_{2i} \geq e_1 \geq e_1 \wedge e_{2k+2}\\
=\ &\big((e_1 \vee e_{2i}) \vee (e_0 \wedge e_{2k+2})\big) \wedge \big((e_0 \vee e_{2i}) \vee (e_1 \wedge e_{2k+2})\big)\\
&\text{by modularity since } (e_0 \vee e_{2i}) \vee (e_1 \wedge e_{2k+2}) \geq e_0 \vee e_{2i} \geq e_0 \geq e_0 \wedge e_{2k+2}\\
=\ &(e_1 \vee e_{2i} \vee e_0) \wedge e_{2k+2} \wedge (e_0 \vee e_{2i} \vee e_1) \wedge e_{2k+2}\\
&\text{by modularity since } e_{2i} \leq e_{2k+2}\\
=\ &(e_0 \vee e_1 \vee e_{2i}) \wedge e_{2k+2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Big(\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2k+2}) \vee (e_1 \wedge e_{2k+2})\Big) \wedge (e_0 \vee e_{2i+2})\\
&\wedge(e_1 \vee e_{2i+2}) \wedge e_1\\
=\ &(e_0 \vee e_1 \vee e_{2i}) \wedge e_{2k+2} \wedge (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2}) \wedge e_1\\
=\ &(e_0 \vee e_1 \vee e_{2i}) \wedge (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2}) \wedge e_1\\
&\text{since by definition of } h_{2k},\ e_{2k+2} \geq e_0 \text{ and } e_{2k+2} \geq e_{2i+2} \Rightarrow e_{2k+2} \geq e_0 \vee e_{2i+2}\\
=\ &(e_0 \vee e_{2i+2}) \wedge e_1\\
&\text{since} (e_0 \vee e_1 \vee e_{2i}) \wedge (e_1 \vee e_{2i+2}) \geq e_1
\end{aligned}$$

That is,

$$\begin{aligned}
&\Big(\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i})\big) \vee (e_0 \wedge e_{2k+2}) \vee (e_1 \wedge e_{2k+2})\Big) \wedge (e_0 \vee e_{2i+2})\\
&\wedge(e_1 \vee e_{2i+2}) \wedge e_1\\
=\ &(e_0 \vee e_{2i+2}) \wedge e_1 \qquad (6.3.20)
\end{aligned}$$

On the other hand $e_0 \vee e_{2i+2} \geq e_1 \wedge e_{2i+4}$ by (6.3.13), $e_0 \vee e_{2i+2} \geq e_0 \wedge e_{2i+4}$ and $e_0 \vee e_{2i+2} \geq (e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2})$ imply

$$e_0 \vee e_{2i+2} \ \geq\ \big((e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2})\big) \vee (e_0 \wedge e_{2i+4}) \vee (e_1 \wedge e_{2i+4}).$$

Whence

$$(e_0 \vee e_{2i+2}) \wedge e_1 \quad \geq \quad \Big(\big((e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2}) \big) \vee (e_0 \wedge e_{2i+4}) \vee (e_1 \wedge e_{2i+4}) \Big) \wedge e_1.$$

That is, by (6.3.20)

$$\begin{aligned} & \Big(\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i}) \big) \vee (e_0 \wedge e_{2k+2}) \vee (e_1 \wedge e_{2k+2}) \Big) \wedge (e_0 \vee e_{2i+2}) \\ & \wedge (e_1 \vee e_{2i+2}) \wedge e_1 \qquad (6.3.21) \\ \geq \quad & \Big(\big((e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2}) \big) \vee (e_0 \wedge e_{2i+4}) \vee (e_1 \wedge e_{2i+4}) \Big) \wedge e_1. \end{aligned}$$

That is, by (6.3.19) we have the following recurrence inequality for $1 \leq i \leq k$

$$\begin{aligned} & \Big(\big((e_0 \vee e_{2i}) \wedge (e_1 \vee e_{2i}) \big) \vee (e_0 \wedge e_{2i+2}) \vee (e_1 \wedge e_{2i+2}) \Big) \wedge e_1 \quad (6.3.22) \\ \geq \quad & \Big(\big((e_0 \vee e_{2i+2}) \wedge (e_1 \vee e_{2i+2}) \big) \vee (e_0 \wedge e_{2i+4}) \vee (e_1 \wedge e_{2i+4}) \Big) \wedge e_1. \end{aligned}$$

By Wille [9], page 247:

$$(e_0 \vee e_{2k}) \wedge e_1 \quad \geq \quad e_1 \wedge e_{2k+2}. \tag{6.3.23}$$

But then also (shift indices)

$$(e_0 \vee e_{2k+2}) \wedge e_1 \quad \geq \quad e_1 \wedge e_{2k+4}. \tag{6.3.24}$$

Now $e_0 \leq e_{2k+2}$, and so

$$e_0 \wedge e_{2k+2} = \geq \quad (e_0 \vee e_{2k+2}) \wedge e_1. \tag{6.3.25}$$

Together with (6.3.24) follows

$$e_1 \wedge e_{2k+2} \geq e_1 \wedge e_{2k+4}. \tag{6.3.26}$$

Because $e_0 \leq e_{2k}$ (6.3.23) because

$$e_1 \vee e_{2k} \quad \geq \quad e_1 \wedge e_{2k+2}. \tag{6.3.27}$$

From (6.3.27) and (6.3.26) follows by induction that

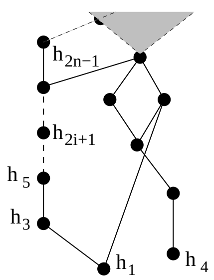
$$e_1 \wedge e_{2k} \geq e_1 \wedge e_{2k+2} \geq \cdots \geq e_1 \wedge e_{2n+2}. \tag{6.3.28}$$

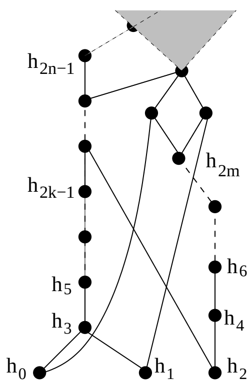
In summary, since $e_4 \geq (e_0 \wedge e_4) \vee (e_1 \wedge e_4)$, one obtains from (6.3.16) that

$$\begin{aligned}
e_4 &\geq \big((e_0 \vee e_1) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2)\big) \vee (e_0 \wedge e_4) \vee (e_1 \wedge e_4) \\
&= \Big(\big((e_0 \vee e_2) \wedge (e_1 \vee e_2)\big) \vee (e_0 \wedge e_4) \vee (e_1 \wedge e_4)\Big) \wedge (e_0 \vee e_1) \\
&\quad \text{by modularity since } e_0 \vee e_1 \geq (e_0 \wedge e_4) \vee (e_1 \wedge e_4) \\
&\geq \Big(\big((e_0 \vee e_2) \wedge (e_1 \vee e_2)\big) \vee (e_0 \wedge e_4) \vee (e_1 \wedge e_4)\Big) \wedge e_1 \\
&\quad \text{since } e_0 \vee e_1 \geq e_1 \\
&\geq \Big(\big((e_0 \vee e_{2k+2}) \wedge (e_1 \vee e_{2k+2})\big) \vee (e_0 \wedge e_{2k+4}) \vee (e_1 \wedge e_{2k+4})\Big) \wedge e_1 \\
&\quad \text{by iterating (6.3.22) from } i = 1 \text{ to } i = k \\
&\geq \big((e_{2k+2} \vee e_0) \vee (e_1 \wedge e_{2k+4})\big) \wedge e_1 \\
&\quad \text{since } e_0 \leq e_{2k+2} \leq e_{2k+2} \Rightarrow (e_0 \vee e_{2k+2}) \wedge (e_1 \vee e_{2k+2}) = e_{2k+2} \\
&\quad \text{and } e_0 \wedge e_{2k+4} = e_0 \\
&= (e_0 \vee e_{2k+2}) \wedge e_1 \\
&\quad \text{since } e_0 \vee e_{2k+2} \geq e_1 \wedge e_{2k+4} \quad \text{by (6.3.13)} \\
&\geq e_1 \wedge e_{2k+2} \quad \text{since} \quad e_0 \leq e_{2k+2} \\
&\ \ \vdots \\
&\geq e_1 \wedge e_{2n+2} \quad \text{by} \quad (6.3.28) \\
&= e_1 \text{ since } \ e_{2n+2} \geq e_1 \quad \text{by definition of } e_{2n+2}
\end{aligned}$$

Therefore $e_4 \geq e_1$, which was to be shown

<u>Subcase</u> (b): $U(h_4)$ is a chain $h_1 < h_3 < \cdots < h_{2n-1}$.

So $H^- := H \setminus \{h_0, h_1\}$ 

and so $H =$ 

If we can show that $\psi(h_1) \geq \psi(h_2)$, then $E = \psi(H)$ has (at most) two minimal elements, whence $|M| = 1$ as in case 2.

<u>Claim</u>: $\psi(h_1) \geq \psi(h_2)$ i.e. $e_1 \geq e_2$.

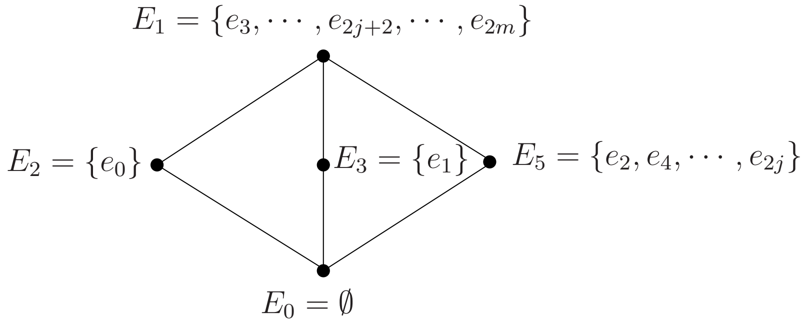
Proof of the Claim: Let h_{2k-1} be the biggest element of $U(h_4)$ which is not $\ngeq h_2$. Moreover let $e_{2n+1} := \{e \in E | e > e_{2n-1}\}$ and let $h_2 \leq h_4 \leq \cdots \leq h_{2m}$ be the elements in the set $\{h \in H | h \ngeq h_0 \text{ and } h \ngeq h_1\}$ (they form a chain since otherwise $A_4 \subseteq H$). Put $e_{2m+2} := \bigwedge\{e \in E | e > e_{2m}\}$. Applying the D_2-lemma to $M = \big\langle \{e_1\} \cup (E \setminus \{e_1\})\big\rangle$ yields

$$e_1 \geq e_0 \wedge e_2. \tag{6.3.29}$$

Also for $1 \leq i \leq k$, we apply the D_2-lemma to $M = \big\langle E_0 \cup E_1 \big\rangle$, where $E_0 = \{e_0, e_1, e_3, \cdots, e_{2i-1}\}$ and $E_1 = \{e_2, e_4, \cdots, e_{2i+1} \cdots, e_{2n-1}\}$ to obtain

$$e_0 \vee e_{2i-1} \geq e_2 \wedge e_{2i+1}, \quad (1 \leq i \leq k). \tag{6.3.30}$$

For $1 \leq j \leq m$, applying the M_3-lemma to $M = \big\langle E_0 \cup E_2 \cup E_3 \cup E_5 \cup E_1 \big\rangle$ where



yields

$$
\begin{aligned}
& (e_0 \wedge e_1) \vee (e_0 \wedge e_{2j}) \vee (e_1 \wedge e_{2j}) \\
\geq\ & \big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge \big((e_0 \wedge e_{2j+2}) \vee (e_2 \wedge e_3)\big) \wedge \big((e_1 \wedge e_{2j+2}) \vee (e_2 \wedge e_3)\big) \\
=\ & \big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge \big(e_0 \vee (e_2 \wedge e_3)\big) \wedge e_{2j+2} \wedge \big(e_1 \vee (e_2 \wedge e_3)\big) \wedge e_{2j+2} \\
& \text{by modularity since } e_{2j+2} \geq e_2 \geq e_2 \wedge e_3 \\
=\ & \big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge \big(e_0 \vee (e_2 \wedge e_3)\big) \wedge \big(e_1 \vee (e_2 \wedge e_3)\big) \\
& \text{since } e_{2j+2} \geq (e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2}) \\
=\ & \big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \wedge e_3 \\
& \text{by modularity since } e_0, e_1 \leq e_3 \\
=\ & \big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\
& \text{since } e_3 \geq (e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})
\end{aligned}
$$

Therefore taking the intersection of both sides with $(e_0 \vee e_2) \wedge (e_1 \vee e_2)$, we get

$$\begin{aligned} & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{2j}) \vee (e_1 \wedge e_{2j})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\ \geq\ & \big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \qquad (6.3.31) \end{aligned}$$

On the other hand, $e_0 \vee e_1 \geq e_0 \geq e_0 \wedge e_1$ and $e_0 \vee e_2 \geq e_1 \geq e_0 \wedge e_1$ imply $(e_0 \vee e_2) \wedge (e_1 \vee e_2) \geq (e_0 \vee e_1)$. Also, $(e_0 \vee e_1) \vee (e_0 \vee e_{2j}) \vee (e_1 \vee e_{2j}) \geq e_0 \vee e_1$. So $\big((e_0 \wedge e_1) \vee (e_0 \wedge e_{2j}) \vee (e_1 \wedge e_{2j})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \geq e_0 \wedge e_1$. Therefore taking into account (6.3.31), we get

$$\begin{aligned} & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{2j}) \vee (e_1 \wedge e_{2j})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\ =\ & (e_0 \wedge e_1) \vee \Big(\big((e_0 \wedge e_1) \vee (e_0 \wedge e_{2j}) \vee (e_1 \wedge e_{2j})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2)\Big) \\ \geq\ & (e_0 \wedge e_1) \vee \Big(\big((e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2)\Big) \\ =\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\ & \text{by modularity since } e_0 \wedge e_1 \leq (e_0 \vee e_2) \wedge (e_1 \vee e_2) \end{aligned}$$

That is for $1 \leq j \leq m$, we obtain the following recurrence inequality

$$\begin{aligned} & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{2j}) \vee (e_1 \wedge e_{2j})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\ & \geq \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{2j+2}) \vee (e_1 \wedge e_{2j+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2). \end{aligned}$$

Iterating the later inequality from $j = 1$ to $j = m$ yields

$$\begin{aligned} & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_2) \vee (e_1 \wedge e_2)\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\ \geq\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_4) \vee (e_1 \wedge e_4)\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\ \vdots\ & \\ \geq\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{2m+2}) \vee (e_1 \wedge e_{2m+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \end{aligned}$$

That is

$$\begin{aligned} & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_2) \vee (e_1 \wedge e_2)\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \qquad (6.3.32) \\ \geq\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{2m+2}) \vee (e_1 \wedge e_{2m+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2). \end{aligned}$$

But $e_1 \geq e_0 \wedge e_1$, $e_1 \geq e_1 \wedge e_2$ and by (6.3.30) $e_1 \geq e_0 \wedge e_2$. So

$$\begin{aligned} e_1 \ \geq\ & (e_0 \wedge e_1) \vee (e_0 \wedge e_2) \vee (e_1 \wedge e_2) \\ \geq\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_2) \vee (e_1 \wedge e_2)\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\ \geq\ & \big((e_0 \wedge e_1) \vee (e_0 \wedge e_{2m+2}) \vee (e_1 \wedge e_{2m+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \quad \text{by (6.3.32)} \\ =\ & \big((e_0 \wedge e_{2m+2}) \vee (e_1 \wedge e_{2m+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\ & \text{since by construction } e_0 \leq e_{2m+2} \text{ or } e_1 \leq e_{2m+2} \text{ implies } e_0 \wedge e_1 \leq e_0 \wedge e_{2m+2} \\ & \text{or } e_0 \wedge e_1 \leq e_1 \wedge e_{2m+2} \end{aligned}$$

But,

$\{e \in E | e > e_{2m}\} = \{e \in E | e > e_{2m}$ and $e > e_0\} \cup \{e \in E | e > e_{2m}$ and $e > e_1\}$. So

$$e_{2m+2} = \bigwedge \{e \in E | e > e_{2m}\} = e^0_{2m+2} \wedge e^1_{2m+2},$$

where $e^0_{2m+2} = \{e \in E | e > e_{2m}$ and $e > e_0\}$ and $e^1_{2m+2} = \{e \in E | e > e_{2m}$ and $e > e_1\}$. So $e_0 \wedge e_{2m+2} = e_0 \wedge e^0_{2m+2} \wedge e^1_{2m+2} = e_0 \wedge e^1_{2m+2}$ since $e_0 \leq e^0_{2m+2}$. Ditto $e_1 \wedge e_{2m+2} = e_1 \wedge e^0_{2m+2}$. Therefore,

$$\begin{aligned}
e_1 &\geq \big((e_0 \wedge e^1_{2m+2}) \vee (e_1 \wedge e^0_{2m+2})\big) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\
&= \big((e_0 \wedge e^1_{2m+2}) \vee e_1\big) \wedge e^0_{2m+2} \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\
&\quad \text{by modularity since } e_0 \wedge e^1_{2m+2} \leq e_0 \leq e^0_{2m+2} \\
&= (e_0 \vee e_1) \wedge e^1_{2m+2} \wedge e^0_{2m+2} \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \\
&\quad \text{by modularity since } e_1 \leq e^1_{2m+2} \\
&= (e_0 \vee e_1) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2) \wedge e_{2m+2} \\
&\quad \text{since } e_{2m+2} = e^0_{2m+2} \wedge e^1_{2m+2} \\
&= (e_0 \vee e_1) \wedge (e_0 \vee e_2) \wedge (e_1 \vee e_2)
\end{aligned}$$

since $e_0, e_2 \leq e_{2m+2}$ or $e_1, e_2 \leq e_{2m+2}$ implies $e_0 \vee e_2 \leq e_{2m+2}$ or $e_1 \vee e_2 \leq e_{2m+2}$

$\geq \ (e_0 \vee e_1) \wedge e_2$ since $(e_0 \vee e_2) \wedge (e_1 \vee e_2) \geq e_2$

This is:

$$e_1 \geq (e_0 \vee e_1) \wedge e_2.$$

Thus it remains to show that $e_0 \vee e_1 \geq e_2$ to get $e_1 \geq e_2$. Taking $i = 1$ in (6.3.30) yields

$$e_0 \vee e_1 \geq e_2 \wedge e_3. \tag{6.3.33}$$

Since $e_{2i-1} \geq e_3 \geq e_0$ for $2 \leq i \leq k$, (6.3.30) implies $e_{2i-1} \geq e_2 \wedge e_{2i+1}$, i.e.

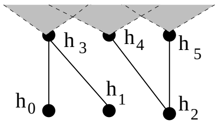
$$e_2 \wedge e_{2i-1} \geq e_2 \wedge e_{2i+1} \qquad 2 \leq i \leq k$$

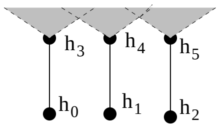
Iterating the later inequality from $i = 2$ to $i = k$ yields

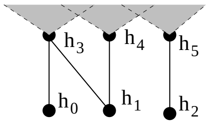
$e_2 \wedge e_3 \geq e_2 \wedge e_5 \geq \cdots \geq e_2 \wedge e_{2k+1} = e_2$ since $e_2 \leq e_{2k+1}$ by definition of h_{2k-1}

Therefore (6.3.33) yields $e_0 \vee e_1 \geq e_2$ which was to be shown.

Case 3.3: $|\rho_1(H)| = 3$ *say* $\rho_1(H) = \{h_3, h_4, h_5\}$.

Since $A_4 \nsubseteq H$, for instance 

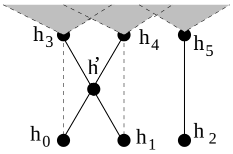
with no other relation is not allow. Thus without lost of generality, if these 

where the only relations among $h_0, h_1, \cdots, h_5$, then we would have $H_5 \subseteq H$. Thus without lost of generality, we have 

with possibly more relations.

In order to avoid that $\{h_0, h_1, \cdots, h_5\} \cong H_5$, one of these cases must occur:

Subcase (a): $h_4 > h_0$.

Extend H to $\widehat{H} := H \cup \{h'\}$ so $\widehat{H}$: 

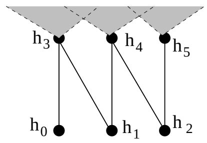
Extend ψ to $\widehat{\psi} : \widehat{H} \longrightarrow M$ by setting $\widehat{\psi}(h') = \psi(h_3) \wedge \psi(h_4)$. We claim that $A_4 \nsubseteq \widehat{H}$ and $H_5 \nsubseteq \widehat{H}$. In fact

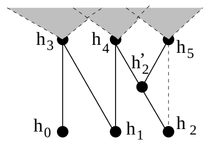
- If $A_4 \subseteq \widehat{H}$, then there are $h_6, h_7 \in \widehat{H} \setminus \{h', h_0, h_1, \cdots, h_5\}$ with $A_4 = \{h', h_5, h_6, h_7\}$ or $A_4 = \{h', h_2, h_6, h_7\}$. But then $h_6 \geq h_3$ or $h_6 \geq h_4$ or $h_6 \geq h_5$ since $\rho_1(H) = \{h_3, h_4, h_5\}$. This implies that $h_6 \geq h'$ or $h_6 \geq h_5$ which is a contradiction.

- If $H_5 \subseteq \widehat{H}$, then there is $h \in \widehat{H} \setminus \{h', h_0, h_1, \cdots, h_5\}$ with $h, h' \in H_5$ since $H_5 \nsubseteq \{h', h_0, h_1, \cdots h_5\}$. But then as before $h \geq h'$ or $h \geq h_5$. If $h \geq h'$, then $\{h' \prec h_3 \prec h\} \subseteq H_5$ or $\{h' \prec h_4 \prec h\} \subseteq H_5$ which is impossible. If $h \geq h_5$, then automatically any other $k \in H_5$ is such that $k \geq h'$ or $k \leq h'$ which is also impossible.

So neither A_4, nor H_5 is contained in $\widehat{H}$. $\widehat{\psi}$ is clearly monotone and since $\rho_1(\widehat{H}) = \{h_5, h'\}$ and $M = \langle \psi(\widehat{H}) \rangle$ where $|\rho_1(\widehat{H})| = 2$, case 3.2 implies $|M| = 1$.

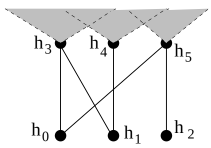
Subcase (b): $h_4 > h_2$.

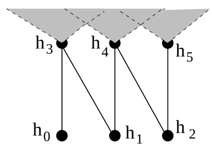
That is 

Extend H to $\widehat{H} := H \cup \{h_2'\}$ so: 

and extend ψ to $\widehat{\psi} : \widehat{H} \longrightarrow M$ via $\widehat{\psi}(h_2') := \psi(h_2)$.
If $A_4 \subseteq \widehat{H}$, then necessarily $h_2' \in A_4$. Since $h_2 \prec h_2' \prec h_4, h_5$, there is at least a point $h \in A_4 \setminus \{h_2', h_0, h_1, h_3\}$. But then $h \geq h_4$ or $h \geq h_5$ or $h \geq h_3$. In the two first cases, $h \geq h_2'$ which is impossible. In the later case $h \geq h_3 \geq h_0, h_2$. So $h_0, h_1, h_3 \notin A_4$. therefore $A_4 = \{h_2', a, b, c\}$ where $a, b, c \prec h_3$. But then $H \supseteq \{h_2, a, b, c\} \cong A_4$ is a contradiction.
If $H_5 \subseteq \widehat{H}$, then necessarily $h_2' \in H_5$. Since $H_5 \not\subseteq \{h_0, h_1, \cdots, h_5\}$, there is a point $h \in H_5 \setminus \{h_2', h_0, \cdots, h_5\}$. But $h \geq h_4$ or $h \geq h_5$ or $h \geq h_3$. The first two cases implies that H_5 contains a chain of type $a \prec b \prec c$, which is impossible. The later case implies that $\{h_3 \prec h\} \subseteq H_5$. Therefore there is a $k \in H_5$ such that $\{h_4 \prec k\} \subseteq H_5$ or $\{h_5 \prec k\} \subseteq H_5$. Either of these cases yields a contradiction.
$A_4 \not\subseteq \widehat{H}$ and $H_5 \not\subseteq \widehat{H}$. Clearly $\widehat{\psi}$ is monotone and $|\rho_1(\widehat{H})| = 2$. So $|M| = 1$ as in case 3.2.

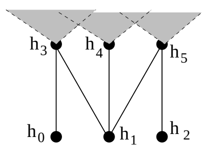
Subcase (c): $h_5 > h_0$.

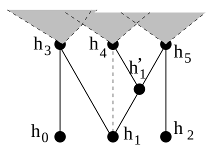
So 

is isomorphic to 

and the conclusion is as in subcase (b).

Subcase (d): $h_5 > h_1$.

So 

extend H to $\widehat{H} := H \cup \{h_1'\}$ so 

and extend ψ to $\widehat{\psi} : \widehat{H} \longrightarrow M$ via $\widehat{\psi}(h_1') = \psi(h_1)$. One can show as in case (b) that $A_4 \not\subseteq \widehat{H}$ and $H_5 \not\subseteq \widehat{H}$. Clearly $\widehat{\psi}$ is monotone and $M = \langle \widehat{\psi}(\widehat{H}) \rangle$ where $|\rho_1(\widehat{H})| = 2$. So $|M| = 1$ as in case 3.2. Because $A_4 \not\subseteq H$, we have $|\rho_0(H)| \leq 3$ and $|\rho_1(H)| \leq 3$, i.e. all cases have been dealt with. $\blacksquare$

We recall Wille's theorem.

Theorem 6.5 *[9](**R. Wille[1973]**) For each finite poset $(H, \leq)$, the following statements are equivalent:*

(i) $\left|F\mathcal{M}(H)\right| < \infty$

(ii) H contains no subposet A_4 *or* H_5.

Proof: $(i) \Rightarrow (ii)$ If H contains A_4, then put

$$\begin{array}{ll} f(h_1) := \left\langle(1,0,0)\right\rangle, & f(h_2) := \left\langle(0,1,0)\right\rangle, \\ f(h_3) := \left\langle(0,0,1)\right\rangle, & f(h_4) := \left\langle(1,1,1)\right\rangle. \end{array}$$

Define $f(h) \in Sub(\mathbb{Q}^3)$ arbitrary for $h \in H \setminus A_4$. Then $\left\langle f(H)\right\rangle \subseteq Sub(\mathbb{Q}^3)$ is known to be infinite. Therefore $\left|F\mathcal{M}(H)\right| = \infty$.
If H contains H_5, then put $f(h_1), f(h_2)$ and $f(h_3)$ as above and

$$f(h_4) = \left\langle(1,0,0),(1,1,1)\right\rangle, \quad f(h_5) = \left\langle(0,1,0),(1,1,1)\right\rangle$$

and take $f(h) \in Sub(\mathbb{Q}^3)$ arbitrary for $h \in H \setminus H_5$. For the same reason as above $\left|F\mathcal{M}(H)\right| = \infty$.

$(ii) \Rightarrow (i)$ Lemma 6.4 implies that $F\mathcal{M}(H) \in \mathcal{M}_3$. Therefore by theorem (6.2), $H \overset{i}{\hookrightarrow} F\mathcal{M}(H)$ extends to an epimorphism $F\mathcal{M}_3(H) \twoheadrightarrow F\mathcal{M}(H)$. Conversely, $F\mathcal{M}(H) \twoheadrightarrow F\mathcal{M}_3(H)$ is an epimorphism since $\mathcal{M}_3 \subseteq \mathcal{M}$. Hence by corollary (6.2), $F\mathcal{M}(H) \cong F\mathcal{M}_3(H)$ which is finite since $\mathcal{M}_3$ is **locally finite** [17] (i.e. any finitely generated free lattice in $\mathcal{M}_3$ is finite). ■

Chapter 7

The (a, B)-Algorithm

In this chapter, we implement an algorithm called (a, B)-Algorithm[1] to compute all the elements of a closure system on any finite set. We give some simple examples and we apply this algorithm to compute $F\mathcal{D}(P)$ and $FM(P)$ for posets of small size.

7.1 The principle of exclusion

Let C be a set and let $P_1, P_2, \cdots, P_n$ be a set of properties that the elements of C may have. In general an element can have zero, one or more than one of these properties. We write $P_i(x)$ to indicate that element x has property P_i and we denote by $N(P_i)$ the number of elements that have property P_i. We want to compute the elements of C that satisfy all the properties P_i . Recall that the **principle of inclusion-exclusion** states that

$$N(P_1 \wedge P_2) = N(P_1) + N(P_2) - N(P_1 \vee P_2).$$

More generally,

$$\begin{aligned} N(P_1 \wedge P_2 \wedge \cdots \wedge P_n) &= \sum_{i=1}^{n} N(P_i) - \sum_{1 \le i < j \le n} N(P_i \vee P_j) \\ &+ \sum_{1 \le i < j < k \le n} N(P_i \vee P_j \vee P_k) + \cdots \\ &\pm N(P_1 \vee P_2 \vee \cdots \vee P_n) \end{aligned}$$

[1] The justification of the name will be apparent later on.

which can be written in a compact form as

$$N(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \;=\; \sum_{I \in \mathcal{P}(\{1,2,\cdots,n\}) \setminus \{\emptyset\}} (-1)^{1+|I|} N\big(\bigvee_{i \in I} P_i\big). \qquad (7.1.1)$$

Note that there are $2^n - 1$ terms on the right hand side of formula (7.1.1) that need to be added or subtracted. So the principle of inclusion-exclusion has exponential time complexity $\mathcal{O}(2^n)$, therefore it will be costly to be implemented.

An alternative solution to this problem was proposed by M. Wild [26], namely the **principle of exclusion**. Basically, the idea behind this principle is very simple. We start with a set $C_0 = C$ and then we recursively exclude all the elements that fail to have property $P_1, P_2, \cdots, P_n$. That is, given C_0, we compute

$$\begin{array}{lcl} C_1 & = & \{x \in C_0 : \; P_1(x)\}, \text{ the set of elements satisfying } P_1. \\ C_2 & = & \{x \in C_1 : \; P_2(x)\}, \text{ the set of elements satisfying } P_1 \textit{ and } P_2. \\ C_n & = & \{x \in C_{n-1} : \; P_n(x)\}, \text{ the set of elements satisfying } P_1, P_2 \textit{ up to } P_n. \end{array}$$

Obviously we have

$$C_0 \supseteq C_1 \supseteq \cdots \supseteq C_n \quad \text{and} \quad N(P_1 \wedge P_2 \wedge \cdots \wedge P_n) = |C_n|.$$

Observe that the principle of exclusion uses only n “steps” to compute $N(P_1 \wedge P_2 \wedge \cdots \wedge P_n)$ as compared to the $2^n - 1$ steps involved in the principle of inclusion-exclusion. The circumstances under which this apparently naive approach is successful are discussed in Wild [26]. Besides the generalities, Wild [26] furthermore focuses on certain properties P_i coupled to so called implications $A \to B$, and introduces the (A, B)-Algorithm. The (a, B)-Algorithm is a special case of the (A, B)-Algorithm in that A becomes $\{a\}$, but it will be further tailored to fit our modular lattices.

7.2 The (a, B)-Algorithm

7.2.1 Preliminaries and notations

Definition 7.1 *Let M be a set. An* ***implication*** *on M is a pair (A, B) of nonempty subsets of M which will be sometimes denoted by $A \longrightarrow B$. In the implication $A \longrightarrow B$, A is called* ***premise*** *and B is called* ***conclusion****. A subset $X \subseteq M$ is said to be (A, B)**-closed** if $A \subseteq X \Rightarrow B \subseteq X$ (equivalently: $A \nsubseteq X$ or $B \subseteq X$). More generally, if Σ is a set of implications on M, a subset $X \subseteq M$ is said to be Σ**-closed** if X is (A, B)-closed for every implication $A \longrightarrow B$ in Σ. The set of Σ-closed subsets of M is denoted by $C(\Sigma)$.*

One easily shows:

Proposition 7.1 *$C(\Sigma)$ is a closure system on M, whence a complete lattice. Conversely any closure system on M is of the form $C(\Sigma)$ for some family Σ of implications on M.*

Proposition 7.2 *Let P be a finite poset. For any non-minimal element a of P, let B_a be the set of lower covers of a and $\Sigma = \{\{a\} \longrightarrow B_a : a$ is non-minimal in $P\}$. Then $C(\Sigma) = Id(P)$.*

Observe that in proposition 7.2, each implication in Σ is of the form $\{a\} \longrightarrow B$ i.e. with singleton premise. This justifies the name of the (a, B)-Algorithm. We will only deal with this kind of implication, but will later see how the (a, B)-Algorithm can be improved in order to take into account the implications that do not have singleton premises.

Definition 7.2 *A **linear extension** of a poset $(P, \leq)$ is a poset $(P, \leq')$ where $\leq'$ is a linear order containing $\leq$.*

As is well known, this amount to iteratively "shelling of" (in any order) the minimal elements of P (refer to [27] for more details about linear extension of a poset).

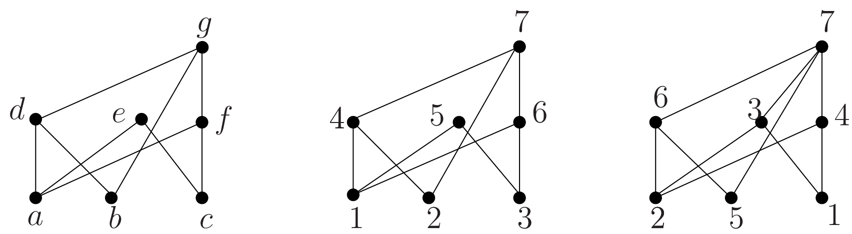


Figure 7.1: A poset P (on the left) and two linear extensions $\{1 < 2 < \cdots < 7\}$ of P.

Before we state the exact formulation of the (a, B)-Algorithm, we begin by introducing an example to illustrate some of its computational details. Let $(P, \leq)$ be a poset with a linear extension $p_1 < p_2 < \cdots < p_n$. Any subset X of P is identified with its characteristic function encoded by the 0,1-vector $(\delta_i)_{1 \leq i \leq n}$, where

$$\delta_i = \begin{cases} 1 & \text{if } p_i \in X \\ 0 & \text{if } p_i \notin X. \end{cases}$$

By definition, a **3-valued row** $r = (r_1, r_2, \cdots, r_n) \in \{0, 1, 2\}^n$ is the family of all subsets X of P that satisfy for all $1 \le i \le n$:

$$\begin{aligned} r_i &= 1 \;\Rightarrow\; p_i \in X \\ r_i &= 0 \;\Rightarrow\; p_i \notin X. \end{aligned}$$

It follows that for $r_i = 2$ there is no restriction on p_i, and so the cardinality of a 3-valued row r is 2^m where m is the number of occurrences of 2 in the vector r. If for example $P = \{p_1, p_2, p_3, p_4, p_5\}$, then $X = \{p_2, p_4, p_5\}$ is encoded by $\delta_X = (0, 1, 0, 1, 1)$. If we consider the 3-valued row $r = (2, 1, 0, 2, 1)$, then $|r| = 2^2$ and r represents the family

$$\{(0, 1, 0, 0, 1), (0, 1, 0, 1, 1), (1, 1, 0, 0, 1), (1, 1, 0, 1, 1)\},$$

which is the same family as

$$\{\{p_2, p_5\}, \{p_2, p_4, p_5\}, \{p_1, p_2, p_5\}, \{p_1, p_2, p_4, p_5\}\}.$$

The power set $\mathcal{P}(P)$ is encoded by $(2, 2, 2, 2, 2)$.

Example 7.1 *Consider the poset P of figure 7.1 and set*

$$\Sigma = \{4 \longrightarrow \{1, 2\},\ 5 \longrightarrow \{1, 3\},\ 6 \longrightarrow \{1, 3\},\ 7 \longrightarrow \{2, 4, 6\}\}.$$

We want to compute $Id(P)$, the lattice of ideals of P. Putting

$$\begin{array}{ll} C_0 = \mathcal{P}(P) = (2,2,2,2,2,2,2), & C_1 = \{X \in C_0 | 4 \in X \Rightarrow \{1,2\} \subseteq X\}, \\ C_2 = \{X \in C_1 | 5 \in X \Rightarrow \{1,3\} \subseteq X\}, & C_3 = \{X \in C_2 | 6 \in X \Rightarrow \{1,3\} \subseteq X\}, \\ C_4 = \{X \in C_3 | 7 \in X \Rightarrow \{2,4,6\} \subseteq X\}, & \end{array}$$

C_1 can be written as $\big(C_1 \cap \{X \in C_0 :\ 4 \in X\}\big) \cup \big(C_1 \cap \{X \in C_0 :\ 4 \notin X\}\big)$. But

$$C_1 \cap \{X \in C_0 :\ 4 \in X\} = \{X \in C_0 :\ 1, 2, 4 \in X\} = (1, 1, 2, 1, 2, 2, 2)$$

and $C_1 \cap \{X \in C_0 : 4 \notin X\} = (2, 2, 2, 0, 2, 2, 2)$. So

$$C_1 = (1, 1, 2, 1, 2, 2, 2) \cup (2, 2, 2, 0, 2, 2, 2).$$

We can compute C_2, C_3 and C_4 in the same manner.

The steps involved in the computation are summarized on the following table 7.2.

i	1	2	3	4	5	6	7	$C_i : a_i \longrightarrow B_i$
0	2	2	2	2	2	2	2	$C_0 = \mathcal{P}(P)$
1	1	1	2	1	2	2	2	$C_1 : 4 \longrightarrow \{1, 2\}$
	2	2	2	0	2	2	2	
	1	1	1	1	1	2	2	
2	1	1	2	1	0	2	2	$C_2 : 5 \longrightarrow \{1, 3\}$
	1	2	1	0	1	2	2	
	2	2	2	0	0	2	2	
	1	1	1	1	1	2	2	
	1	1	1	1	0	1	2	
3	1	1	2	1	0	0	2	$C_3 : 6 \longrightarrow \{1, 3\}$
	1	2	1	0	1	2	2	
	1	2	1	0	0	1	2	
	2	2	2	0	0	0	2	
	1	1	1	1	1	1	1	
	1	1	1	1	1	2	0	
	1	1	1	1	0	1	2	
4	1	1	2	1	0	0	0	$C_4 : 7 \longrightarrow \{2, 4, 6\}$
	1	2	1	0	1	2	0	
	1	2	1	0	0	1	0	
	2	2	2	0	0	0	0	

Table 7.2: Summary of the steps to compute C_4

By the principle of exclusion we know that $C_4 = C(\Sigma)$. From table 7.2 and from proposition 7.2, we deduce that $|C(\Sigma)| = |Id(P)| = 1 + 2 + 2 + 2 + 2^2 + 2 + 2^3 = 21$. We can explicitly compute the elements of $Id(P)$ by considering the 3-valued rows in C_4, for instance, $(1, 1, 1, 1, 1, 1, 1) = \{1, 2, 3, 4, 5, 6, 7\} = P$ and $(1, 1, 1, 1, 1, 2, 0) = \{(1, 1, 1, 1, 1, 1, 0), (1, 1, 1, 1, 1, 0, 0)\} = \{\{1,2,3,4,5,6\}, \{1,2,3,4,5\}\} =: \{123456,\ 12345\}$[2]. If we express the rest of elements of C_4 in the same manner, we obtain

$$\begin{aligned} Id(P) \;\; = \;\; & \{\emptyset, 1, 2, 3, 12, 13, 23, 123, 124, 135, 136, 1234, 1235, 1236, 1356, 12345, \\ & 12346, 12356, 123456, 123467, P\}. \end{aligned}$$

The Hasse diagram of $Id(P)$ is given in figure 7.2

[2]For simplification we represent a set by listing its elements as a string.

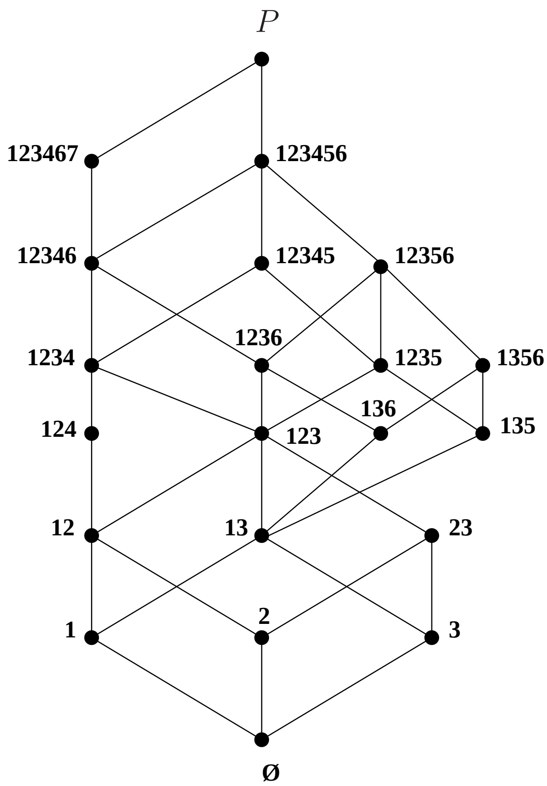


Figure 7.2: The Hasse diagram of the lattice $(Id(P), \subseteq)$

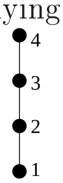
We will now adopt a shorthand notation, for instance $r = (2, b, 0, 1, a)$ to represent the family of subsets $X \in (2, 2, 0, 1, 2)$ satisfying the implication $p_5 \longrightarrow p_2$. Thus $r = (2, b, 0, 1, a)$ is defined as $r = (2, 1, 0, 1, 1) \cup (2, 2, 0, 1, 0)$. Indeed for a fixed $X \in r$, either $p_5 \in X$ or $p_5 \notin X$. If $p_5 \in X$ (whence $a = 1$), then $p_2 \in X$ (whence $b = 1$). This yields the first vector $(2, 1, 0, 1, 1)$. If $p_5 \notin X$ (whence a=0), the implication $p_5 \in X \Rightarrow p_2 \in X$ is always true no matter whether or not $p_2 \in X$ (whence b=2). This yields the second vector $(2, 2, 0, 1, 0)$. Similarly we write for instance $s = (b, 2, b, a, 1)$ to represent the family of subsets $X \in (2, 2, 2, 2, 1)$ such that $p_4 \in X \Rightarrow \{p_1, p_3\} \subseteq X$. Hence $s = (1, 2, 1, 1, 1) \cup (2, 2, 2, 0, 1)$.

From table (7.2), one can observe that the number of rows increases (or remains constant) from one context (or working stack) to the next. This might cause a space problem but in the new variant of the (a, B)-Algorithm, M. Wild [26] exploited the well known technique of LIFO (Last In, First Out) so as to keep the number of rows in the working stack below the number of implications to be imposed. The a, b symbolism we have introduced helps to minimize the number of row splittings necessary and therefore contributes to speed up the program. For instance from the previous example (7.1), replace $C_1 = \{X \in C_0 : \ 4 \longrightarrow \{1,2\} \subseteq X\}$ by the 3-valued row $(b, b, 2, a, 2, 2, 2)$. This yields table 7.3 which is more compact than the previous one.

i	1	2	3	4	5	6	7	$C_i : a_i \longrightarrow B_i$
0	2	2	2	2	2	2	2	$C_0 = \mathcal{P}(P)$
1	b	b	2	a	2	2	2	$C_1 : 4 \longrightarrow \{1,2\}$
2	1	b	1	a	1	2	2	$C_2 : 5 \longrightarrow \{1,3\}$
	b	b	2	a	0	2	2	
3	1	b	1	a	1	2	2	$C_3 : 6 \longrightarrow \{1,3\}$
	1	b	1	a	0	1	2	
	b	b	2	a	0	0	2	
	1	1	1	1	1	1	1	
	1	b	1	a	1	2	0	
4	1	1	1	1	0	1	1	$C_4 : 7 \longrightarrow \{2,4,6\}$
	1	b	1	a	0	1	0	
	b	b	2	a	0	0	0	

Table 7.3: Contracted form of table 7.2

One does not need to cancel[3] any row if the order of the implications in Σ is chosen properly. This order is suggested once a linear extension of the underlying poset is given. For instance, the linear extension of the poset

$P =:$ 

suggests that the premises of the implications should be taken in an increasing order, i.e.

$$\Sigma = \{2 \longrightarrow \{1\}, 3 \longrightarrow \{2\}, 4 \longrightarrow \{3\}\}.$$

In this case, no row needs to be cancelled (see table (a)) as the premises always fall in a label **2**. But if the order in Σ is random, say

$$\Sigma = \{2 \longrightarrow \{1\}, 4 \longrightarrow \{3\}, 3 \longrightarrow \{2\}\},$$

one may have to cancel a row, see table (b) where row 2 0 1 1 is cancelled because of the implication $3 \longrightarrow \{2\}$.

[3] The cancellation operation is costly to the program.

1 2 3 4	
2 2 2 2	C_0
2 0 2 2 1 1 2 2	$C_1: 2 \longrightarrow 1$
2 0 0 2 1 1 2 2	$C_2: 3 \longrightarrow 2$
2 0 0 0 1 1 2 0 1 1 1 1	$C_3: 4 \longrightarrow 3$

Table (a)

1 2 3 4	
2 2 2 2	C_0
2 0 2 2 1 1 2 2	$C_1: 2 \longrightarrow 1$
2 0 2 0 2 0 1 1 1 1 2 0 1 1 1 1	$C_2: 4 \longrightarrow 3$
2 0 0 0 ~~2 0 1 1~~ 1 1 2 0 1 1 1 1	$C_3: 3 \longrightarrow 2$

Table (b)

Having given this example, we are now in a position to state the (a, B)-Algorithm.

7.2.2 Statement of the (a, B)-Algorithm

The (a, B)-Algorithm can be stated as follows:

Input: A poset $(P, \leq)$ with a linear extension $p_1 < p_2 < \cdots < p_n$ and the corresponding set of implications $\Sigma = \{a_1 \longrightarrow B_1, a_2 \longrightarrow B_2, \cdots, a_k \longrightarrow B_k\}$ defined as in proposition 7.2.

Output: $C(\Sigma)$, the set of Σ-closed subsets of P, i.e. the set of order ideals of P.

1. Initialize $i = 0$ and $C_0 = (2, 2, \cdots, 2) = \mathcal{P}(P)$.

2. Suppose that C_i is computed do,

   2.a) $i = i + 1$

   2.b) $C_i = \{X \in C_{i-1} :\ a_i \in X \Rightarrow B_i \subseteq X\}$.

3. If $i < k$, go to step 2. Otherwise output C_k and stop.

It is straightforward by the principle of exclusion to see that $C_k = C(\Sigma)$. Let us now compute the complexity of the (a, B)-Algorithm. Step 1 can be computed in time $\mathcal{O}(1)$. If one imposes the implications in the order listed in Σ, as does the (a, B)-Algorithm, then we will never delete a 3-valued row because the premise of any implication always falls on a label 2. So since the 3-valued rows are mutually disjoint, the list of 3-valued rows at the end comprises at most $N(P)$ rows, where $N(P)$ is the number of order ideals of P. Further any row has been subjected to exactly $k \le P$ iterations (steps 2 and 3), each of which costs $\mathcal{O}\big(\max_{1\le i\le k} |B_{a_i}|\big) \approx \mathcal{O}(|P|)$. So the (a,B)-Algorithm has complexity $\mathcal{O}\big(N(P)|P|^2\big)$.

Theorem 7.1 *Given a finite poset P, the (a,B)-Algorithm computes the $N(P)$ order ideals of P in time $\mathcal{O}\big(N(P)|P|^2\big)$.* ■

Example 7.2 *For the poset of figure 7.3, we apply the (a,B)-Algorithm to determine $Id(P)$.*

$(P,\le) :=$

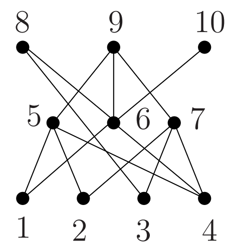


Figure 7.3: P has 4 minimal elements, namely 1, 2, 3 and 4.

By proposition 7.2, $Id(P) = C(\Sigma)$ where $\Sigma = \{5 \longrightarrow \{1,2,4\},\ 6 \longrightarrow \{1,4\},\ 7 \longrightarrow \{2,3,4\},\ 8 \longrightarrow \{3,6\},\ 9 \longrightarrow \{5,6,7\},\ 10 \longrightarrow 6\}$. The (a,B)-Algorithm applied to Σ yields table 7.5.

i	1	2	3	4	5	6	7	8	9	10	$C_i : a_i \longrightarrow B_i$
0	2	2	2	2	2	2	2	2	2	2	$C_0 = \mathcal{P}(P)$
1	b	b	2	b	a	2	2	2	2	2	$C_1 : 5 \longrightarrow \{1,2,4\}$
2	1	b	2	1	a	1	2	2	2	2	$C_2 : 6 \longrightarrow \{1,4\}$
	b	b	2	b	a	0	2	2	2	2	
	1	1	1	1	2	1	1	2	2	2	
3	1	b	2	1	a	1	0	2	2	2	$C_3 : 7 \longrightarrow \{2,3,4\}$
	b	1	1	1	a	0	1	2	2	2	
	b	b	2	b	a	0	0	2	2	2	
	1	1	1	1	2	1	1	2	2	2	
4	1	b	b'	1	a	1	0	a'	2	2	$C_4 : 8 \longrightarrow \{3,6\}$
	b	1	1	1	a	0	1	0	2	2	
	b	b	2	b	a	0	0	0	2	2	
	1	1	1	1	b	1	1	2	a	2	
5	1	b	b'	1	a	1	0	a'	0	2	$C_5 : 9 \longrightarrow \{5,6,7\}$
	b	1	1	1	a	0	1	0	0	2	
	b	b	2	b	a	0	0	0	0	2	

	1	1	1	1	b	1	1	2	a	2	
6	1	b	b'	1	a	1	0	a'	0	2	$C_6 : 10 \longrightarrow 6$
	b	1	1	1	a	0	1	0	0	0	
	b	b	2	b	a	0	0	0	0	0	

Table 7.5: Summary of the (a, B)-Algorithm applied to the poset of figure 7.3

Note that in a 3-valued row r, any occurrence of a (respectively a', a'', $\cdots$) is coupled to $t \geq 1$ occurrence(s) of b (respectively b', b'', $\cdots$) and contributes a factor of $(2^t + 1)$ to $|r|$ whereas m occurrences of 2 account for a factor of 2^m in $|r|$. For instance

$|(1, 1, 1, 1, b, 1, 1, 2, a, 2)| = (2^1 + 1) \cdot 2^2 = 12,$
$|(1, b, b', 1, a, 1, 1.0, a', 2)| = (2^1 + 1)(2^1 + 1) \cdot 2 = 18,$
$|(b, 1, 1, 1, a, 0, 1, 0, 0, 0)| = (2^1 + 1) = 3,$ and
$|(b, b, 2, b, a, 0, 0, 0, 0, 0)| = (2^3 + 1) \cdot 2^1 = 18.$
So $\quad |C_6| = |Id(P)| = 12 + 18 + 3 + 18 = 51.$

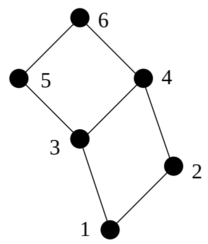
The (a, B)-Algorithm has been implemented with the **Mathematica** 6.0 (refer to [28; 29]), the code of this algorithm will be given in chapter 8. A number of algorithms to compute the set of order ideals of a finite poset exists in the literature (see [30; 31]), most of which have complexity $\mathcal{O}(N(P)|P|^2)$. George Steiner [31] was the first (in 1986) to present an enumeration algorithm with complexity $\mathcal{O}(N(P)|P|)$.
Despite $\mathcal{O}(N(P)|P|^2)$, the (a, B)-Algorithm is usually faster than the Steiner algorithm since one row can encode many ideals.

7.3 Applications of the (a, B)-Algorithm

7.3.1 Explicit computation of free distributive lattices

In chapter 4, we gave the procedure to compute the free distributive lattice $F\mathcal{D}(P)$. In this section, we will discuss two examples to see how the (a, B)-Algorithm comes into play. We saw in step 3 of this procedure that we need to determine $(Id(\mathcal{K}), \leq)$ which can now be done via the (a, B)-Algorithm.

Example 7.3 *(1) Compute $F\mathcal{D}(P, \leq)$ where $(P, \leq) =$*



and sketch its graph.

The eight non-equivalent P-labellings of $\mathbf{2}$ are:

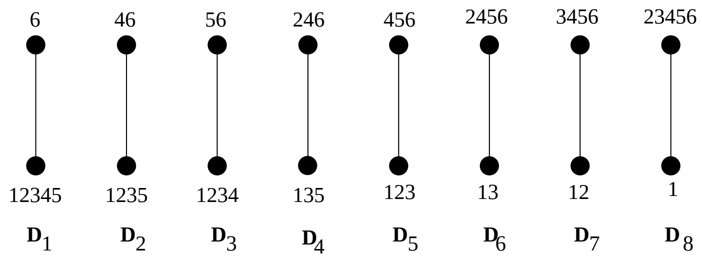


Figure 7.4: The eight non-equivalent P-labellings of $\mathbf{2}$.

Now we compute the morphisms β_{ij}. Obviously $\beta_{1i} = \text{id}$ since $6 \in f_i$ for all proper filters f_i of P. In figure 7.5, we compute β_{2i} and β_{i2} for $1 \leq i \leq 8$. Note that for a fixed i, β_{2i} (in dashed lines) and β_{i2} (in thick lines) are given in the same figure.

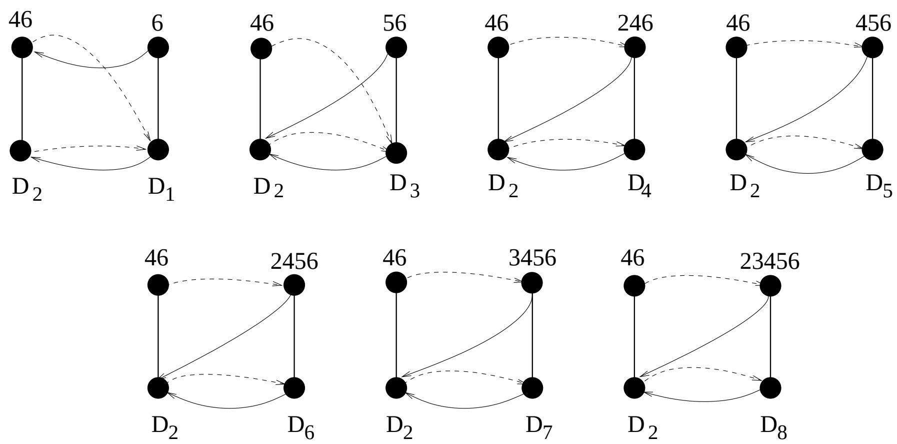


Figure 7.5: $\beta_{21} = \beta_{23} \equiv 0$, $\beta_{22} = \beta_{24} = \beta_{25} = \beta_{26} = \beta_{27} = \beta_{28} = \text{id}$ and $\beta_{12} = \text{id}$, $\beta_{i2} \equiv 0$ for all $i \geq 2$.

One can similarly show that

- $\beta_{31} = \beta_{34} \equiv 0$, and $\beta_{33} = \beta_{35} = \beta_{36} = \beta_{37} = \beta_{38} = \text{id}$.
- $\beta_{41} = \beta_{42} = \beta_{45} = \beta_{47} \equiv id$, and $\beta_{44} = \beta_{46} = \beta_{48} = \text{id}$.
- $\beta_{51} = \beta_{52} = \beta_{53} = \beta_{54}$ and $\beta_{55} = \beta_{56} = \beta_{57} = \beta_{58} = \text{id}$.
- $\beta_{61} = \beta_{62} = \beta_{63} = \beta_{64} = \beta_{65} \equiv 0$ and $\beta_{66} = \beta_{67} = \beta_{68} = \text{id}$.
- $\beta_{7i} \equiv 0$ for $1 \leq i \leq 6$ and $\beta_{77} = \beta_{78} = \text{id}$.
- $\beta_{8i} \equiv 0$ for $1 \leq i \leq 7$ and $\beta_{88} = \text{id}$.

We next compute $\mathcal{K} = \{\psi_1(1), \psi_2(1), \cdots, \psi_8(1)\}$ where

$$\psi_i(1) = (\beta_{i1}(1), \beta_{i2}(1), \cdots, \beta_{i8}(1)).$$

$$\begin{array}{ll} \psi_1(1) = (1,1,1,1,1,1,1,1), & \psi_2(1) = (0,1,0,1,1,1,1,1), \\ \psi_3(1) = (0,0,1,0,1,1,1,1), & \psi_4(1) = (0,0,0,1,0,1,0,1), \\ \psi_5(1) = (0,0,0,0,1,1,1,1), & \psi_6(1) = (0,0,0,0,0,1,0,1), \\ \psi_7(1) = (0,0,0,0,0,0,1,1), & \psi_8(1) = (0,0,0,0,0,0,0,1). \end{array}$$

The Hasse diagram of $(\mathcal{K}, \leq)$ is given below. Note that $(\mathcal{K}, \leq) \cong (Fil^*(P), \supseteq)$, as has already been proved.

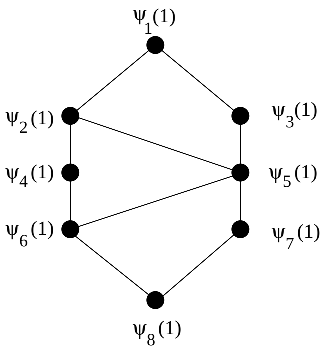


Figure 7.6: The Hasse diagram of $(\mathcal{K}, \leq)$.

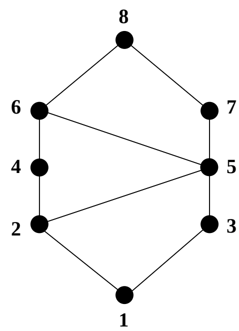


Figure 7.7: A linear extension of $(\mathcal{K}, \leq)$.

We now apply the (a, B)-Algorithm to compute $Id(\mathcal{K}, \leq)$. We consider a linear extension (figure 7.7) of $(\mathcal{K}, \leq)$ and we set $B_2 = \{1\}$, $B_3 = \{1\}$, $B_4 = \{2\}$, $B_5 = \{2, 3\}$, $B_6 = \{4, 5\}$, $B_7 = \{5\}$ and $B_8 = \{6, 7\}$ and $\Sigma = \{2 \longrightarrow B_2, 3 \longrightarrow B_3, \cdots, 8 \longrightarrow B_8\}$. Then by proposition 7.2, $Id(\mathcal{K}, \leq) \cong C(\Sigma)$.

i	1	2	3	4	5	6	7	8	$a_i \longrightarrow B_i$
0	2	2	2	2	2	2	2	2	$C_0 = \mathcal{P}(\mathcal{K})$
1	b	a	2	2	2	2	2	2	$C_1 : 2 \longrightarrow \{1\}$
2	1	2	1	2	2	2	2	2	$C_2 : 3 \longrightarrow \{1\}$
	b	a	0	2	2	2	2	2	
	1	b	1	a	2	2	2	2	
3	1	1	0	1	2	2	2	2	$C_3 : 4 \longrightarrow \{2\}$
	b	a	0	0	2	2	2	2	
	1	1	1	2	1	2	2	2	
4	1	b	1	a	0	2	2	2	$C_4 : 5 \longrightarrow \{2, 3\}$
	1	1	0	1	0	2	2	2	
	b	a	0	0	0	2	2	2	
	1	1	1	b	1	a	2	2	
5	1	b	1	a	0	0	2	2	$C_5 : 6 \longrightarrow \{4, 5\}$
	1	1	0	1	0	0	2	2	
	b	a	0	0	0	0	2	2	
	1	1	1	b	1	a	2	2	
6	1	b	1	a	0	0	0	2	$C_6 : 7 \longrightarrow \{5\}$
	1	1	0	1	0	0	0	2	
	b	a	0	0	0	0	0	2	
	1	1	1	1	1	1	1	1	
	1	1	1	b	1	a	2	0	
7	1	b	1	a	0	0	0	0	$C_7 : 8 \longrightarrow \{6, 7\}$
	1	1	0	1	0	0	0	0	
	b	a	0	0	0	0	0	0	

Table 7.6: Summary of the (a, B)-Algorithm.

From table 7.6 one can see that $|F\mathcal{D}(P, \leq)| = 1 + 6 + 3 + 1 + 3 = 14$. Precisely,

$$
\begin{aligned}
(1,1,1,1,1,1,1,1) &= \{12345678\} \\
(1,1,1,b,1,a,2,0) &= \{1234567, 123456, 123457, 12345, 12357, 1235\} \\
(1,b,1,a,0,0,0,0) &= \{1234, 123, 13\} \\
(1,1,0,1,0,0,0,0) &= \{124\} \\
(b,a,0,0,0,0,0,0) &= \{12, 1, \emptyset\}
\end{aligned}
$$

So,

$$\begin{aligned} F\mathcal{D}(P, \leq) \ = \ & \{12345678, 1234567, 123456, 123457, 12345, 12357, 1235, \\ & 1234, 123, 13, 124, 12, 1, \emptyset\}. \end{aligned}$$

and its Hasse diagram is given below.

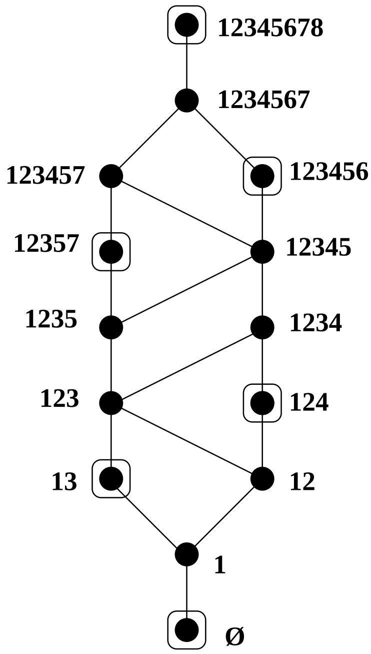
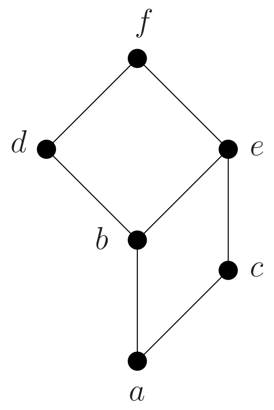
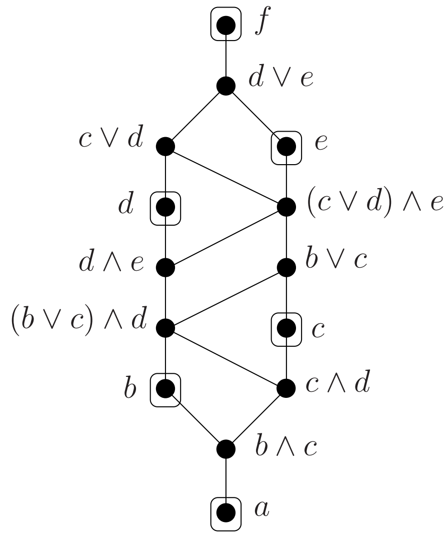


Figure 7.8: The Hasse diagram of the free distributive lattice $F\mathcal{D}(P, \leq)$, the six generators (doubly irreducible elements) are indicated.

We next illustrate with this example the fact that $F\mathcal{D}(P)$ satisfies the definition of a free distributive lattice. It is clear that $F\mathcal{D}(P)$ contains a copy of $(P, \leq)$ which generates $F\mathcal{D}(P)$ as we can see from the following pictures.

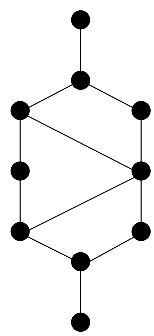


$(P, \leq)$

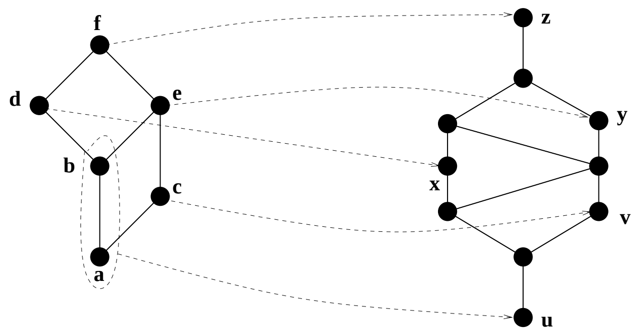


$F\mathcal{D}(P)$

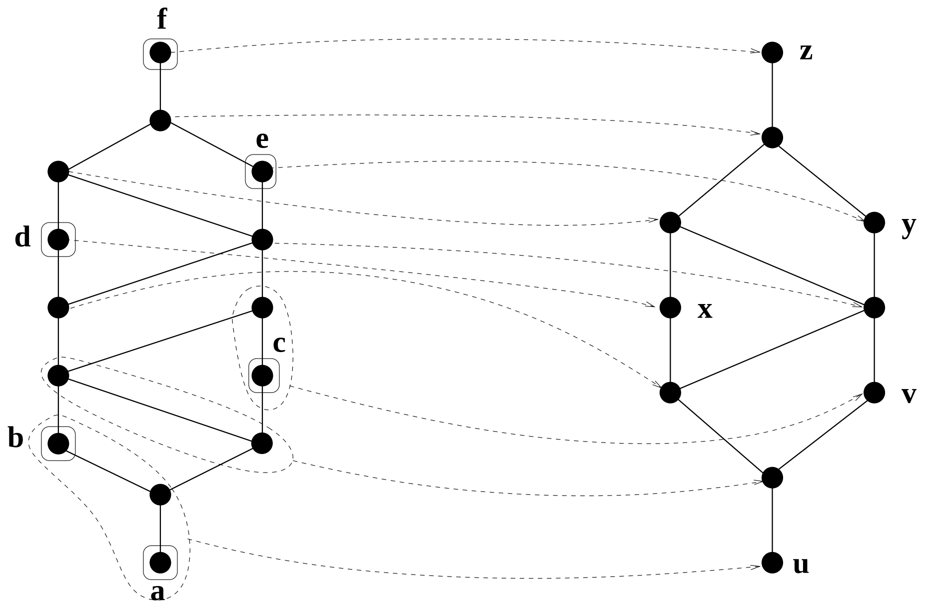
Suppose that D is the distributive lattice $D =$



We pick any order preserving map, say for instance $\phi : P \longrightarrow Q$ depicted as follows:



Consider the map $\Phi : F\mathcal{D}(P) \longrightarrow D$ depicted by the following figure:



One can check by inspection that Φ is a lattice morphism which extends ϕ.

(2) Consider the poset of figure 4.4(a) on page 34. The corresponding poset $(\mathcal{K}, \leq)$ is given in figure 4.5. To apply the (a, B)-Algorithm on $(\mathcal{K}, \leq)$, we first give (figure 7.9) a linear extension of $(\mathcal{K}, \leq)$, then the (a, B)-Algorithm applied to $(\mathcal{K}, \leq)$ yields the next table where the first three steps and the last two steps are summarized.

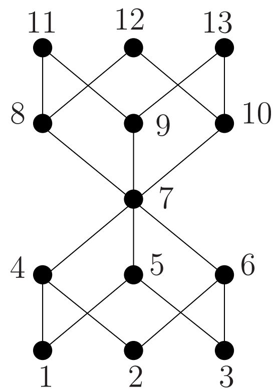


Figure 7.9: A linear extension of the poset $(\mathcal{K}, \leq)$ of figure 4.5.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	$a_i \longrightarrow B_i$
0	2	2	2	2	2	2	2	2	2	2	2	2	2	$C_0 = \mathcal{P}(\mathcal{K})$
1	b	b	2	a	2	2	2	2	2	2	2	2	2	$4 \longrightarrow \{1, 2\}$
2	1	b	1	a	1	2	2	2	2	2	2	2	2	$5 \longrightarrow \{1, 3\}$
	b	b	2	a	0	2	2	2	2	2	2	2	2	
	1	1	1	2	1	1	2	2	2	2	2	2	2	
3	1	b	1	a	1	0	2	2	2	2	2	2	2	$6 \longrightarrow \{2, 3\}$
	b	1	1	a	0	1	2	2	2	2	2	2	2	
	b	b	2	a	0	0	2	2	2	2	2	2	2	
	1	1	1	1	1	1	1	1	b	b'	a	a'	2	
	1	1	1	1	1	1	1	0	1	2	0	0	2	
	1	1	1	1	1	1	1	0	0	1	0	0	2	
9	1	1	1	b	1	1	a	0	0	0	0	0	2	$12 \longrightarrow \{8, 10\}$
	1	b	1	a	1	0	0	0	0	0	0	0	2	
	b	1	1	a	0	1	0	0	0	0	0	0	2	
	b	b	2	a	0	0	0	0	0	0	0	0	2	
	1	1	1	1	1	1	1	1	1	1	2	2	1	
	1	1	1	1	1	1	1	1	b	b'	a	a'	0	
	1	1	1	1	1	1	1	0	1	b	0	0	a	
10	1	1	1	1	1	1	1	0	0	1	0	0	0	$13 \longrightarrow \{9, 10\}$
	1	1	1	b	1	1	a	0	0	0	0	0	0	
	1	b	1	a	1	0	0	0	0	0	0	0	0	
	b	1	1	a	0	1	0	0	0	0	0	0	0	
	b	b	2	a	0	0	0	0	0	0	0	0	0	

From this table we see that

$$|F\mathcal{D}(P)| = 2^2 + (2^1 + 1)(2^1 + 1) + \cdots + (2^2 + 1) \cdot 2^1 = 36.$$

Precisely,

$bb2a000000000 = \{a_1, a_2, \cdots, a_{10}\}$ where

$$\begin{array}{lllll} a_1 = \emptyset, & a_2 = 1, & a_3 = 2, & a_4 = 3, & a_5 = 23, \\ a_6 = 13, & a_7 = 12, & a_8 = 123, & a_8 = 124, & a_{10} = 1234, \end{array}$$

$b11a010000000 = \{a_{11}, a_{12}, a_{13}\}$ where

$$a_{11} = 235, \quad a_{12} = 1236, \quad a_{13} = 12346$$

$1b1a100000000 = \{a_{14}, a_{15}, a_{16}\}$ where

$$a_{14} = 135, \quad a_{15} = 1235, \quad a_{16} = 12345,$$

$111b11a000000 = \{a_{17}, a_{18}, a_{19}\}$ where

$$a_{17} = 12356, \quad a_{18} = 123456, \quad a_{19} = 1234567,$$

$1111111001000 = \{a_{20}\}$ where $a_{20} = 1234567(10)$,
$111111101b00a = \{a_{21}, a_{22}, a_{23}\}$ where

$$a_{21} = 12345679, \quad a_{22} = 12345679(10), \quad a_{23} = 12345679(10)(13),$$

$11111111bb'aa'0 = 11111111b1a10 \cup 11111111b2a00$
$11111111b2a00 = \{a_{24}, a_{25}, a_{26}, a_{27}, a_{28}, a_{29}\}$ where

$$\begin{array}{lll} a_{24} = 12345678, & a_{25} = a_{24} \cup (10), & a_{26} = a_{24} \cup (9), \\ a_{27} = a_{24} \cup 9(10), & a_{28} = a_{24} \cup 9(11), & a_{29} = a_{24} \cup 9(10)(11) \end{array}$$

$11111111b1a10 = \{a_{30}, a_{31}, a_{32}\}$ where

$$a_{30} = a_{24} \cup (10)(12), \quad a_{31} = a_{24} \cup 9(10)(12), \quad a_{32} = a_{24} \cup 9(10)(11)(12),$$

Finally $1111111111221 = \{a_{33}, a_{34}, a_{35}, a_{36}\}$ where

$$\begin{array}{ll} a_{33} = a_{24} \cup 9(10)(13), & a_{34} = a_{24} \cup 9(10)(12)(13), \\ a_{35} = a_{24} \cup 9(10)(11)(13), & a_{26} = \mathcal{K}. \end{array}$$

The Hasse diagram of $F\mathcal{D}(P)$ is given below.

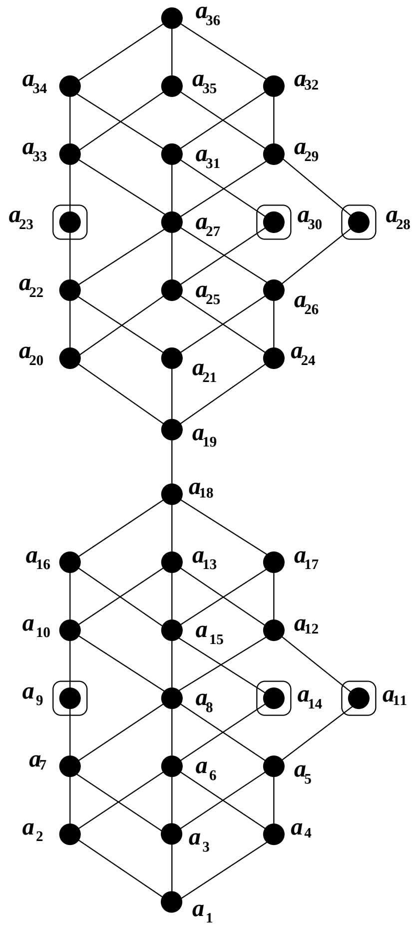
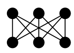


Figure 7.10: The Hasse diagram of $F\mathcal{D}($  $)$, the six generators are indicated.

A complete list is given in chapter 7, where for all posets P with $1 \leq |P| \leq 6$, the cardinality of $F\mathcal{D}(P)$ and the number of factors $\mathbf{2}$ in the subdirect product decomposition of $F\mathcal{D}(P)$ are computed.

7.3.2 Explicit computation of free modular lattices

We have already seen that each finite modular lattice L is isomorphic to $C(J(L), \Lambda)$ where $C(J(L), \Lambda)$ is the set of Λ-closed order ideals of $J(L)$. Since we are only concerned with the variety $\mathcal{M}_3$ of modular lattices having factors $\mathbf{2}$ or M_3 in their subdirect product decomposition, any line $l \in \Lambda$ has exactly three elements, and the lines are mutually disjoint.

Recall that an ideal I of $J(L)$ is Λ-closed if $|l \cap I| \geq 2 \Rightarrow l \subseteq I$ for all $l \in \Lambda$. If $l = \{p, q, r\}$, this means that

$$\begin{cases} \{p,q\} \subseteq I & \Rightarrow \quad \{p,q,r\} \subseteq I \\ \{p,r\} \subseteq I & \Rightarrow \quad \{p,q,r\} \subseteq I \\ \{q,r\} \subseteq I & \Rightarrow \quad \{p,q,r\} \subseteq I. \end{cases}$$

Which can be simplified to

$$\begin{cases} \{p,q\} \subseteq I & \Rightarrow \quad r \in I \\ \{p,r\} \subseteq I & \Rightarrow \quad q \in I \\ \{q,r\} \subseteq I & \Rightarrow \quad p \in I. \end{cases} \tag{7.3.1}$$

Recall that if a is a non-minimal element of the poset $J(L)$, B_a denotes the set of lower covers of a. By proposition 7.2, a subset $I \subseteq J(L)$ is an ideal of $J(L)$ if and only if I satisfies $a \in I \Rightarrow B_a \subseteq I$ for all non-minimal elements $a \in J(L)$. Therefore a subset $I \subseteq J(L)$ is a Λ-closed order ideal of $J(L)$ if and only if I satisfies the implications $a \in I \Rightarrow B_a \subseteq I$ for all non-minimal elements $a \in J(L)$ together with the implications (7.3.1) for all $l = \{p, q, r\} \in \Lambda$. So the (a, B)-Algorithm cannot be applied to $(J(L), \Lambda)$ without modification since the implications (7.3.1) do no longer have singleton premises. In this section, we show how we can upgrade the (a, B)-Algorithm. For convenience we define:

$$333 := \{000, 100, 010, 001, 111\}$$

and we set

$$44 := \{00, 01, 10\} \qquad \text{and} \qquad 55 := \{00, 11\}$$

since then

$$333 = 044 \cup 155.$$

This definition is handy because for each line $l = \{p, q, r\} \subseteq J(L)$, and for each l-closed set X there are exactly five possibilities for $X \cap l$:

$$\varnothing, \quad \{p\}, \quad \{q\}, \quad \{r\}, \quad \text{or} \quad \{p, q, r\}$$

and this is encoded by 333. For instance if $J(L) = \{p_1, p, p_3, p_4, q, p_6, r, p_8\}$ (obvious notation), then $C\big(J(L), \Lambda\big) = (2, 3, 2, 2, 3, 2, 3, 2)$. Furthermore if the implication $p \longrightarrow \{p_3, p_4\}$ is imposed, then $(2, 3, 2, 2, 3, 2, 3, 2)$ shrinks to

$$(2, 0, 2, 2, \mathbf{4}, 2, \mathbf{4}, 2) \cup (2, 1, 1, 1, \mathbf{5}, 2, \mathbf{5}, 2).$$

In fact if $p \notin X$, then it does not matter whether or not p_3 or $p_4 \in X$, and because X is Λ-closed, $q, r \notin X$ or either q or $r \in X$ exclusively. This yields the first 8-tuple $(2, 0, 2, 2, \mathbf{4}, 2, \mathbf{4}, 2)$. If $p \in X$, then $\{p_3, p_4\} \subseteq X$, and because X is Λ-closed, either $\{q, r\} \subseteq X$ or $q, r \notin X$. This yields the second 8-tuple $(2, 1, 1, 1, \mathbf{5}, 2, \mathbf{5}, 2)$. To see how to handle the label 4 and the 5, suppose we further impose the implication $p_1 \longrightarrow \{r\}$, then $(2, 0, 2, 2, \mathbf{4}, 2, \mathbf{4}, 2)$ reduces to

$$(1, 0, 2, 2, 0, 2, 1, 2) \cup (0, 0, 2, 2, 0, 2, 1, 2) \cup (0, 0, 2, 2, 2, 2, 0, 2)$$

and $(2, 1, 1, 1, \mathbf{5}, 2, \mathbf{5}, 2)$ reduces to

$$(1, 1, 1, 1, 1, 2, 1, 2) \cup (0, 1, 1, 1, 1, 2, 1, 2) \cup (0, 1, 1, 1, 0, 2, 0, 2).$$

With the modifications we have just described, we are now able to use the (a, B)-Algorithm to compute the Λ-closed order ideals of $J(L)$. We next give two examples to illustrate the procedure discussed in chapter 5 to explicitly compute $F\mathcal{M}(P)$.

Example 7.4 ***(1)*** *In this example we compute $F\mathcal{M}(\mathbf{1}+\mathbf{1}+\mathbf{1})$, the free modular lattice on three generators.*

There are six non-equivalent P-labellings of $\mathbf{2}$ and only one P-labellings of M_3.

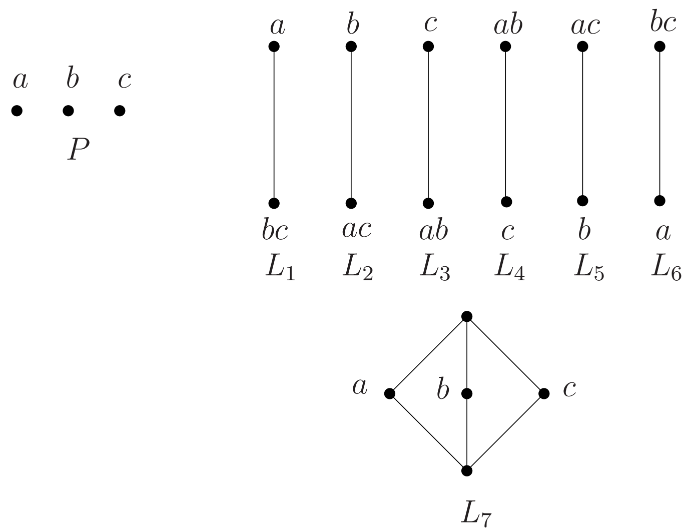


Figure 7.11: The P-labellings of $\mathbf{2}$ and M_3

The morphisms between two P-labellings of $\mathbf{2}$ are computed as in example 7.3(1). The morphisms β_{i7} and β_{7i} are given in figure 7.12 for all $1 \leq i \leq 6$.

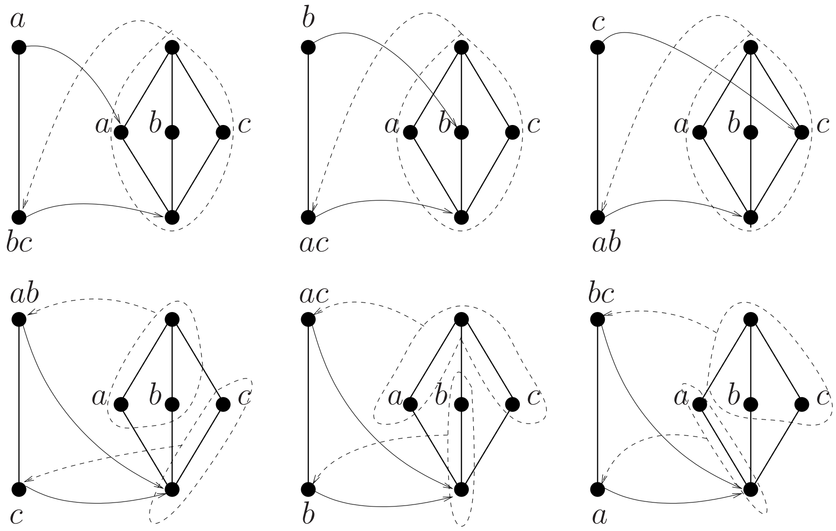


Figure 7.12: The morphisms between the six P-labellings of $\mathbf{2}$ and the unique P-labelling of M_3.

$$
\begin{array}{ll}
\psi_1(1) = (\beta_{1i}(1))_{1\le i\le 7} = (1,0,0,1,1,0,a) & \psi_2(1) = (\beta_{2i}(1))_{1\le i\le 7} = (0,1,0,1,0,1,b) \\
\psi_3(1) = (\beta_{3i}(1))_{1\le i\le 7} = (0,0,1,0,1,1,c) & \psi_4(1) = (\beta_{4i}(1))_{1\le i\le 7} = (0,0,0,1,0,0,0) \\
\psi_5(1) = (\beta_{5i}(1))_{1\le i\le 7} = (0,0,0,0,1,0,0) & \psi_6(1) = (\beta_{6i}(1))_{1\le i\le 7} = (0,0,0,0,0,1,0) \\
\psi_7(a) = (\beta_{7i}(a))_{1\le i\le 7} = (0,0,0,1,1,0,a) & \psi_7(b) = (\beta_{7i}(b))_{1\le i\le 7} = (0,0,0,1,0,1,b) \\
\psi_7(c) = (\beta_{7i}(c))_{1\le i\le 7} = (0,0,0,0,1,1,c) &
\end{array}
$$

$l_7 = \{a, b, c\}$ is a line of L_7, so $\Lambda = \psi(l_7) = \{\psi_7(a), \psi_7(b), \psi_7(c)\}$ is a base of lines of $F\mathcal{M}(P)$ and $J = \{\psi_1(1), \psi_2(1), \psi_3(1), \psi_4(1), \psi_5(1), \psi_6(1), \psi_7(a), \psi_7(b), \psi_7(c)\}$ is the set of nonzero join-irreducibles of $F\mathcal{M}(P)$. The Hasse diagram of $(J, \le)$ and a linear extension of $(J, \le)$ are given below.

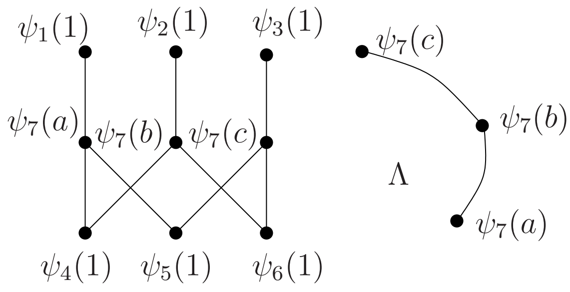


Figure 7.13: A linear space (J, λ).

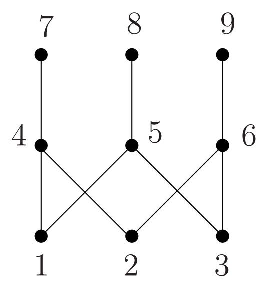


Figure 7.14: A linear extension of $(J, \le)$

We now apply the modified (a, B)-Algorithm to compute $C(J, \Lambda)$. We set $B_4 = \{1, 2\}$, $B_5 = \{1, 3\}$, $B_6 = \{2, 3\}$, $B_7 = \{4\}$, $B_8 = \{5\}$, $B_9 = \{6\}$. Since $\{\psi_7(a), \psi_7(b), \psi_7(c)\}$ is a line, the corresponding elements 4,5,6 in the linear extension will be coded by the sequence 3,3,3 in the (a, B)-Algorithm.

i	1	2	3	4	5	6	7	8	9	$C_i : a_1 \longrightarrow B_i$
0	2	2	2	3	3	3	2	2	2	
1	1	1	2	1	5	5	2	2	2	$C_1 : 4 \longrightarrow \{1, 2\}$
	2	2	2	0	4	4	2	2	2	
	1	1	1	1	1	1	2	2	2	
2	1	1	2	1	0	0	2	2	2	$C_2 : 5 \longrightarrow \{1, 3\}$
	1	2	1	0	1	0	2	2	2	
	2	2	2	0	0	2	2	2	2	
	1	1	1	1	1	1	2	2	2	
3	1	1	2	1	0	0	2	2	2	$C_3 : 6 \longrightarrow \{2, 3\}$
	1	2	1	0	1	0	2	2	2	
	2	b	b	0	0	a	2	2	2	
	1	1	1	1	1	1	2	2	2	
4	1	1	2	1	0	0	2	2	2	$C_4 : 7 \longrightarrow \{4\}$
	1	2	1	0	1	0	0	2	2	
	2	b	b	0	0	a	0	2	2	
	1	1	1	1	1	1	2	2	2	
5	1	1	2	1	0	0	2	0	2	$C_5 : 8 \longrightarrow \{5\}$
	1	2	1	0	1	0	0	2	2	
	2	b	b	0	0	a	0	0	2	
	1	1	1	1	1	1	2	2	2	
6	1	1	2	1	0	0	2	0	0	$C_6 : 9 \longrightarrow \{6\}$
	1	2	1	0	1	0	0	2	0	
	2	1	1	0	0	1	0	0	1	
	2	b	b	0	0	a	0	0	0	

Table 7.8: Summary of the modified (a, B)-Algorithm applied to (J, Λ)

So $|F\mathcal{M}(\mathbf{1}+\mathbf{1}+\mathbf{1})| = |C_6| = 2^3 + 2^2 + 2^2 + 2 + (2^2 + 1).2 = 28$. We explicitly compute $F\mathcal{M}(\mathbf{1}+\mathbf{1}+\mathbf{1})$ as follows.

$$\begin{aligned}(1,1,1,1,1,1,2,2,2) &= \{123456\} \cup \mathcal{P}(789) \\ &= \{123456, 1234567, 1234568, 1234569, 12345678, \\ &\quad 12345679, 12345689, 123456789\}\end{aligned}$$

$$
\begin{aligned}
(1,1,2,1,0,0,2,0,0) &= \{124\} \cup \mathcal{P}(37) \\
&= \{124, 1243, 1247, 12437\} \\
(1,2,1,0,1,0,0,2,0) &= \{135\} \cup \mathcal{P}(28) \\
&= \{135, 1352, 1358, 13528\} \\
(2,1,1,0,0,1,0,0,1) &= \{12369, 2369\}
\end{aligned}
$$

$$
\begin{aligned}
(2,b,b,0,0,a,0,0,0) &= (2,1,1,0,0,1,0,0,0) \cup (2,2,2,0,0,0,0,0,0) \\
(2,1,1,0,0,1,0,0,0) &= \{1236, 236\} \\
(2,2,2,0,0,0,0,0,0) &= \mathcal{P}(123) \\
&= \{\emptyset, 1, 2, 3, 12, 13, 23, 123\}.
\end{aligned}
$$

So $F\mathcal{M}(\mathbf{1}+\mathbf{1}+\mathbf{1}) = \{\emptyset$, J, 1, 2, 3, 12, 13, 23, 123, 124, 135, 236, 1234, 1235, 1236, 1247, 1358, 2369, 12347, 12358, 12369, 123456, 1234567, 1234568, 1234569, 12345678,12345679, 12345689$\}$. The Hasse diagram of $F\mathcal{M}(\mathbf{1}+\mathbf{1}+\mathbf{1})$ is given by the following picture.

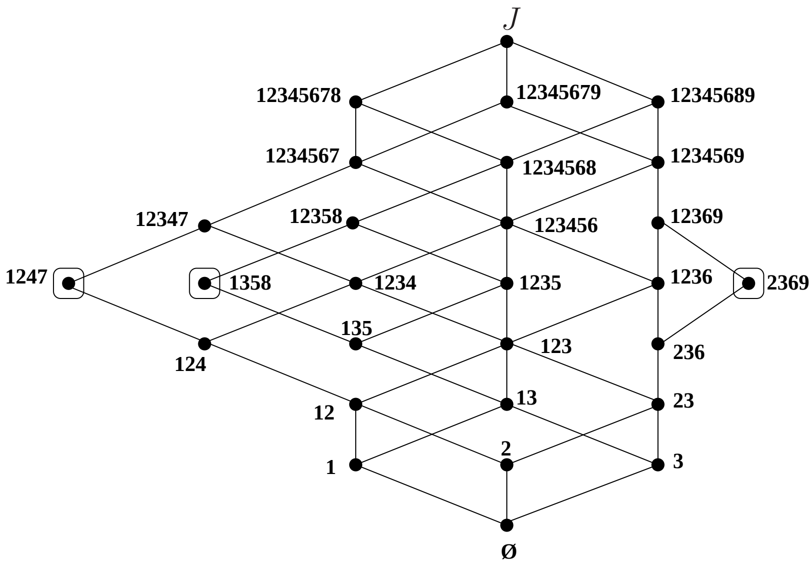


Figure 7.15: The free modular lattice on three generators $F\mathcal{M}(\mathbf{1}+\mathbf{1}+\mathbf{1})$, the generators are indicated.

(2) Applying the modified (a, B)-Algorithm to the poset of figure 6.9, page 59, we obtain $|F\mathcal{M}(P)| = 80$.

Prior to the execution of the (a, B)-Algorithm, a first program called base-of-line.nb is run. It takes input a poset P determined by its covering relation and output the Hasse diagram of $J\big(F\mathcal{M}(P)\big)$ (resp. $J\big(F\mathcal{D}(P)\big)$), a base of lines and a complete set of implications.

Chapter 8

Numerical results

8.1 Cardinalities of the free lattices $F\mathcal{D}(P)$ and $F\mathcal{M}_3(P)$

In this section, we consider all the posets P with $1 \leq |P| \leq 6$. For each one, we compute the cardinality of the free distributive lattice $F\mathcal{D}(P)$ and the cardinality of the free modular lattice $F\mathcal{M}_3(P)$ within the variety $\mathcal{M}_3$ of modular lattices with subdirectly irreducible factors **2** or M_3. The number of subdirrectly irreducible factors is also computed and listed in the form $s + t$, where s is the number of factors **2**, i.e. the number of P-labellings of **2**, and t the number of factors M_3, i.e. the number of P-labellings of M_3. Observe that the height $h(P)$ of $F\mathcal{M}_3(P)$ and the number $j(P)$ of nonzero join-irreducible elements of $F\mathcal{M}_3(P)$ can be determined as $h(P) = s + 2t$ and $j(P) = s + 3t$. Further by theorem 6.5, the modular lattice $F\mathcal{M}(P)$ freely generated by P is finite if and only if $|P| < \infty$ and P contains no subposet isomorphic either to **1+1+1+1** or **1+2+2**. In this case $F\mathcal{M}(P) = F\mathcal{M}_3(P)$, and so the given cardinality of $F\mathcal{M}_3(P)$ also is the cardinality of $F\mathcal{M}(P)$. If the cardinality of $F\mathcal{M}_3(P)$ is boldface, this warns that P contains one of the forbidden subposets, and so $|F\mathcal{M}(P)| = \infty$. These results perfectly match those obtained by Berman and Wolk [32].

$$|P| = 1$$

$(P, \leq)$	$\lvert F\mathcal{D}(P)\rvert$	$\lvert F\mathcal{M}_3(P)\rvert$	$s+t$
	1	1	0+0

Table 8.1: The only one poset of order one.

$$|P| = 2$$

$(P, \leq)$	$\lvert F\mathcal{D}(P)\rvert$	$\lvert F\mathcal{M}_3(P)\rvert$	$s+t$	$(P, \leq)$	$\lvert F\mathcal{D}(P)\rvert$	$\lvert F\mathcal{M}_3(P)\rvert$	$s+t$
	4	4	2+0		2	2	1+0

Table 8.2: The 2 non-isomorphic posets of order 2.

$$|P| = 3$$

$(P, \leq)$	$\lvert F\mathcal{D}(P)\rvert$	$\lvert F\mathcal{M}_3(P)\rvert$	$s+t$	$(P, \leq)$	$\lvert F\mathcal{D}(P)\rvert$	$\lvert F\mathcal{M}_3(P)\rvert$	$s+t$
	3	3	2+0		5	5	3+0
	5	5	3+0		8	8	4+0
	18	28	6+1				

Table 8.3: The 5 non-isomorphic posets of order 3.

$$|P| = 4$$

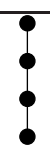
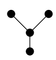
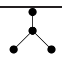
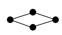
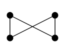
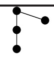
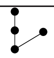
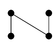
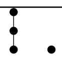
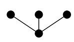
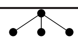
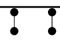
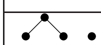
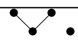
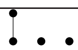
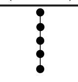
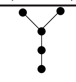
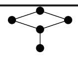
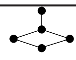
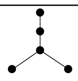
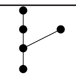
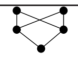
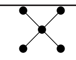
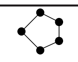
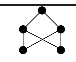
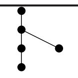
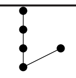
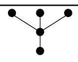
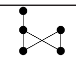
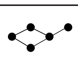
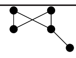
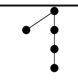
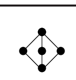
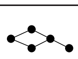
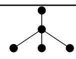
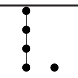
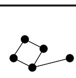
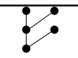
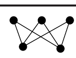
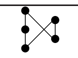
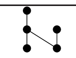
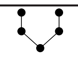
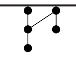
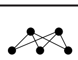
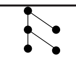
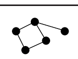
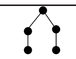
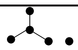
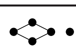
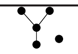
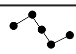
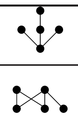
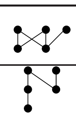
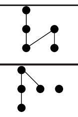
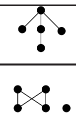
$(P, \leq)$	$\lvert F\mathcal{D}(P)\rvert$	$\lvert F\mathcal{M}_3(P)\rvert$	$s+t$	$(P, \leq)$	$\lvert F\mathcal{D}(P)\rvert$	$\lvert F\mathcal{M}_3(P)\rvert$	$s+t$
	4	4	3+0		6	6	4+0
	6	6	4+0		6	6	4+0
	8	8	5+0		9	9	5+0
	9	9	5+0		12	12	6+0
	13	13	6+0		19	29	7+1
	19	29	7+1		18	18	7+0
	25	36	8+1		25	36	8+1
	48	138	10+3		166	**19982**	14+14

Table 8.4: The 16 non-isomorphic posets of order 4.

$$|P| = 5$$

$(P,\leq)$	$\|F\mathcal{D}(P)\|$	$\|F\mathcal{M}_3(P)\|$	$s+t$	$(P,\leq)$	$\|F\mathcal{D}(P)\|$	$\|F\mathcal{M}_3(P)\|$	$s+t$
	5	5	4+0		7	7	5+0
	7	7	5+0		7	7	5+0
	7	7	5+0		10	10	6+0
	9	9	6+0		9	9	6+0
	10	10	6+0		9	9	6+0
	10	10	6+0		14	14	7+0
	20	30	8+1		12	12	7+0
	13	13	7+0		12	12	7+0
	14	14	7+0		20	30	8+1
	13	13	7+0		20	30	8+1
	19	19	8+0		26	37	8+1
	26	37	9+1		22	32	9+1
	23	23	9+0		17	17	8+0
	19	19	8+0		17	17	8+0
	22	32	9+1		26	37	9+1
	26	37	9+1		19	19	8+0
	33	45	10+1		33	45	10+1
	33	45	10+1		32	44	10+1
	49	139	11+3		29	40	10+1
	29	40	10+1		23	23	9+0
	23	23	9+0		49	139	11+3
	59	154	12+3		51	80	12+2

	59	154	12+3		55	147	12+3
	55	147	12+3		167	**19983**	15+14
	39	51	11+1		39	51	11+1
	33	33	10+0		167	**19983**	15+14
	187	**20180**	16+14		93	352	14+5
	103	629	14+6		187	**20180**	16+14
	75	185	14+3		75	185	13+3
	173	**2603**	16+9		297	**63639**	18+18
	297	**63639**	18+18		885	**160228749**	22+39
	7727		30+125				

Table 8.5: The 63 non-isomorphic posets of order 5.

$|P| = 6$

$(P, \leq)$	$\lvert F\mathcal{D}(P)\rvert$	$\lvert F\mathcal{M}_3(P)\rvert$	$s+t$	$(P, \leq)$	$\lvert F\mathcal{D}(P)\rvert$	$\lvert F\mathcal{M}_3(P)\rvert$	$s+t$
	7828352		62+910		160946		46+333
	22950		38+174		9944		34+133
	7748		32+125		7580		31+125
	22950		38+174		8788		34+108
	2990		30+68		2024		28+51
	1195	**179700889**	26+43		1075	**160667032**	25+39
	906	**179700889**	24+39		3488		30+76
	1326	**296198143**	26+45		936	**160224000**	24+39
	886	**160228750**	23+39		1058	**6306868**	26+37
	670	**434366**	24+26		407	**68915**	22+20
	590	**2472286**	23+22		354	**64461**	21+18
	304	**64461**	21+18		490	**213428**	23+22
	325	**64004**	20+18		298	**63640**	19+18
	255	**20984**	20+15		218	**20392**	19+14
	191	**20184**	18+14		209	**20379**	18+14
	188	**20181**	17+14		170	**19986**	17+14
	168	**19984**	16+14		9944		34+133

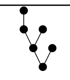
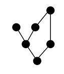
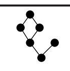
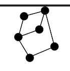
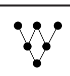
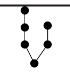
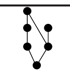
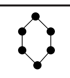
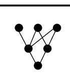
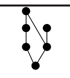
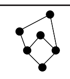
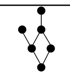
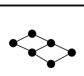
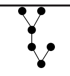
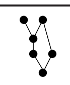
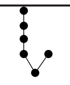
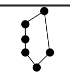
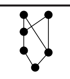
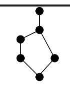
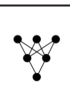
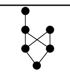
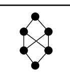
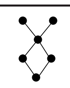
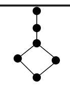
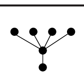
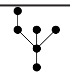
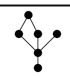
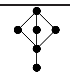
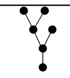
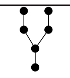
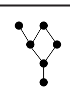
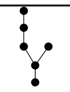
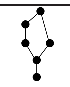
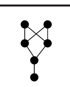
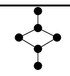
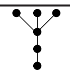
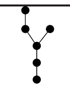
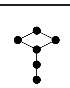
	2024	**2610806855**	28+51		1195	**179700889**	26+43
	596	**153926**	23+22		428	**121130**	22+22
	1326	**296198143**	26+45		472	**138454**	22+22
	318	**63872**	20+18		492	**210044**	22+23
	325	**63943**	20+18		298	**63640**	19+18
	670	**434366**	24+26		987	**1007808**	25+27
	434	**14616**	22+15		243	**3311**	20+11
	488	**60962**	22+18		273	**32449**	20+12
	234	**2895**	19+9		184	**2626**	18+9
	194	**2665**	18+9		174	**2604**	17+9
	194	**2665**	18+9		243	**3311**	20+11
	188	756	18+7		138	584	18+7
	273	**4936**	20+12		154	649	18+7
	127	415	17+5		100	361	16+5
	167	1060	18+8		104	369	16+5
	108	377	16+5		94	353	15+5
	198	622	19+6		108	243	17+4
	180	821	18+6		100	216	16+3
	80	190	15+3		110	242	16+3
	83	195	15+3		76	186	14+3
	81	195	16+4		130	686	17+6
	73	170	15+3		59	151	14+3
	63	157	14+3		56	148	13+3
	490	**213428**	22+23		167	1060	18+8
	110	639	16+6		194	2784	18+10
	119	661	16+6		104	630	15+6
	97	230	16+4		78	178	15+3
	63	158	14+3		71	171	14+3

	60	155	13+3		52	142	13+3
	50	140	12+3		255	**20984**	20+15
	108	243	14+4		81	195	16+4
	97	230	16+4		58	88	14+2
	62	93	14+2		52	81	13+2
	80	106	16+2		60	84	15+2
	58	71	14+1		44	56	13+1
	40	52	12+1		46	68	14+2
	43	55	13+1		33	44	12+1
	34	45	12+1		30	41	11+1
	62	93	14+2		37	49	12+1
	42	55	12+1		34	46	11+1
	29	40	11+1		27	38	10+1
	36	56	13+2		26	36	11+1
	23	33	10+1		23	33	10+1
	21	31	9+1		7748		32+125
	1074	**160667032**	25+39		906	**160229710**	24+39
	936	**160244000**	24+39		318	**63873**	20+18
	325	**64227**	20+18		208	**20378**	18+14
	209	**20379**	18+14		188	**20181**	17+14
	590	**247228**	23+22		354	**64461**	21+18
	234	**2895**	19+9		127	415	17+5
	110	242	16+3		83	195	15+3
	180	873	18+6		130	730	17+6
	86	201	15+3		76	186	14+3
	304	**63651**	20+18		184	**2626**	18+9

	100	361	16+5		83	195	15+3
	62	156	14+3		110	639	16+6
	66	163	14+3		56	278	13+3
	325	**64004**	20+18		104	369	16+5
	194	**2665**	18+9		86	201	15+3
	108	377	16+5		66	163	14+3
	119	661	16+6		70	170	14+3
	71	171	14+3		60	169	13+3
	58	88	14+2		46	59	13+1
	47	60	13+1		36	48	12+1
	40	53	12+1		33	45	11+1
	62	93	14+2		37	49	12+1
	42	55	12+1		34	46	11+1
	29	40	11+1		27	38	10+1
	218	**20392**	19+14		100	216	16+3
	73	170	15+3		46	59	13+1
	78	178	15+3		47	60	13+1
	40	52	12+1		110	242	16+3
	58	71	14+1		68	68	14+0
	48	48	13+0		38	38	12+0
	54	54	13+0		39	39	12+0
	34	34	11+0		43	55	13+1
	34	34	12+0		27	27	11+0
	29	29	11+0		24	24	10+0

	22	22	10+0		20	20	9+0
	191	**20184**	18+14		80	190	15+3
	59	151	14+3		63	158	14+3
	36	48	12+1		37	49	12+1
	30	41	11+1		83	195	15+3
	44	56	13+1		38	38	12+0
	28	28	11+0		39	39	12+0
	28	28	11+0		24	24	10+0
	27	27	11+0		27	27	11+0
	20	20	10+0		21	21	10+0
	17	17	9+0		37	49	12+1
	21	21	11+0		23	23	10+0
	18	18	9+0		16	16	9+0
	14	14	8+0		209	**20379**	18+14
	63	157	14+3		71	171	14+3
	40	53	12+1		42	55	12+1
	34	46	11+1		34	45	12+1
	29	29	11+0		21	21	10+0
	22	22	10+0		18	18	9+0
	42	55	12+1		23	23	10+0
	26	26	10+0		20	20	9+0
	17	17	9+0		15	15	8+0

	26	36	11+1		16	16	9+0
	13	13	8+0		13	13	8+0
	11	11	7+0		170	**20149**	17+14
	52	142	13+3		29	40	11+1
	23	33	10+1		29	40	11+1
	22	22	10+0		16	16	9+0
	17	17	9+0		13	13	8+0
	12	12	8+0		10	10	7+0
	23	33	10+1		13	13	8+0
	10	10	7+0		10	10	7+0
	8	8	6+0		7580		31+125
	886	**160228750**	23+39		295	**63640**	19+18
	188	**20181**	17+14		168	**19984**	16+14
	298	**63640**	19+18		174	**2985**	17+9
	94	354	15+5		76	187	14+3
	56	149	13+3		104	631	15+6
	60	156	13+3		50	141	12+3
	52	82	13+2		40	53	12+1
	30	42	11+1		34	47	11+1
	27	39	10+1		23	34	10+1
	21	32	9+1		188	**20181**	17+14
	76	188	14+3		56	149	13+3

	60	156	13+3		33	46	11+1
	34	47	11+1		27	74	10+1
	40	54	12+1		34	34	11+0
	24	44	10+0		20	20	9+0
	30	42	11+1		24	24	10+0
	17	17	9+0		18	18	9+0
	14	14	8+0		34	47	11+1
	18	18	9+0		20	20	9+0
	15	15	8+0		13	13	8+0
	11	11	7+0		23	34	10+1
	13	13	8+0		10	10	7+0
	10	10	7+0		8	8	6+0
	168	**19984**	16+14		50	278	12+3
	27	74	10+1		21	60	9+1
	27	74	10+1		20	20	9+0
	14	44	8+0		17	15	8+0
	11	11	7+0		10	10	7+0
	8	8	6+0		21	60	9+1
	11	11	7+0		8	8	6+0

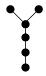
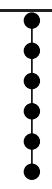
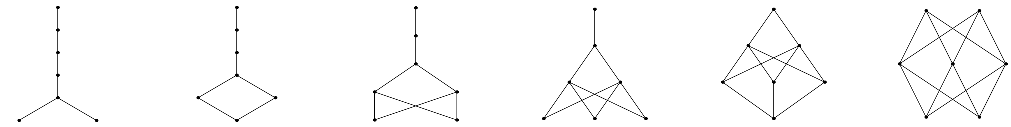
	8	8	6+0		6	6	5+0

Table 8.6: The 318 non-isomorphic posets of order 6.

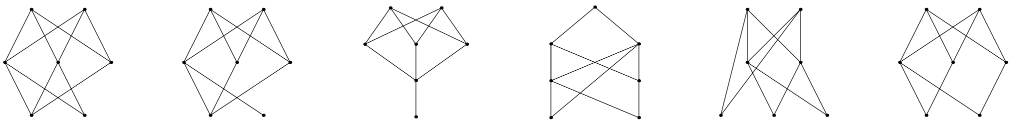
8.2 Cardinalities of $F\mathcal{M}(P)$ for good posets on seven points

In this section, we compute the cardinalities, together with some important parameters, of the free modular lattices generated by good posets on seven points. We call **good** poset any poset that does not contain a 4-element antichain or the poset **1+2+2** as subposet. Recall that $F\mathcal{M}(P)$ is finite if and only if P is a good poset. In this case $F\mathcal{M}(P)$ is a subdirect product of factors M_3 and D_2. There are several algorithms for generating non isomorphic posets on a given number of points ([33; 34; 35]) in the literature, the one we used for this project can be found in [14]. Thanks for Prof G. Brinkmann from Bielefeld University in Germany who sent us the code (written in C^{++}) of this program from which we where able to extract the 2045 non isomorphic posets (their order relations more precisely) on seven points. We then wrote a program that select all the good posets from any finite set of posets on a given number of points. For the 2045 non isomorphic posets on seven points, we obtained 1101 good posets. These good posets are listed below. On top of each good poset P is a list containing $|F\mathcal{M}(P)|$, the number of factors M_3, and the number of factors D_2. With our program we can also compute $F\mathcal{M}(P)$ for good posets of higher order, we are only limited by computing time constraints.

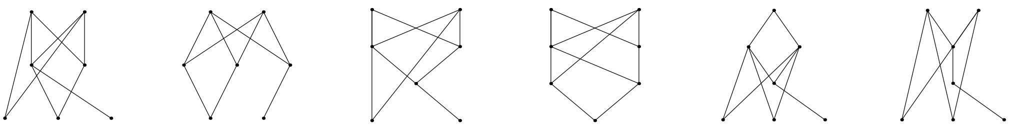
{9, 7, 0} {9, 7, 0} {11, 8, 0} {34, 11, 1} {34, 11, 1} {36, 12, 1}



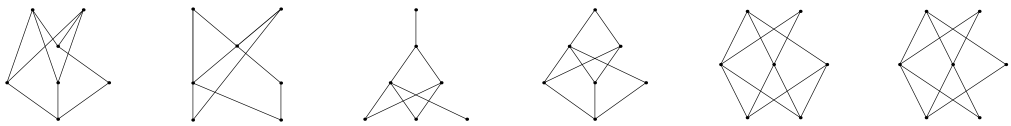
{44, 13, 1} {151, 15, 3} {34, 11, 1} {41, 12, 1} {84, 15, 2} {55, 14, 1}



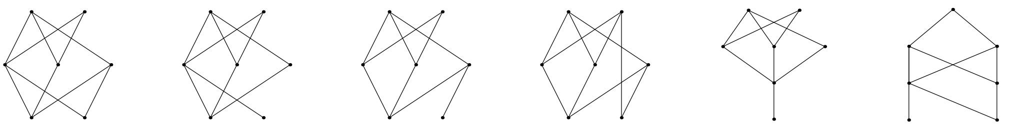
{356, 17, 5} {189, 16, 3} {49, 13, 1} {41, 12, 1} {145, 14, 3} {633, 17, 6}



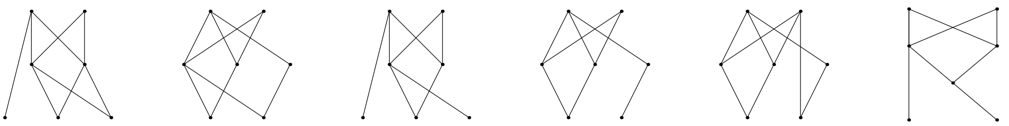
{143, 14, 3} {158, 15, 3} {42, 12, 1} {44, 12, 1} {44, 13, 1} {53, 14, 1}



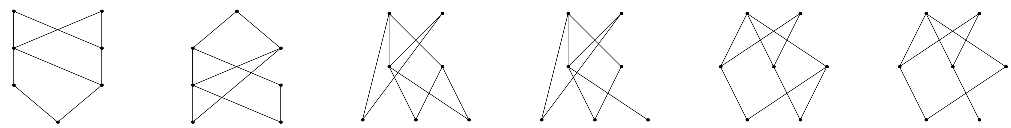
{52, 14, 1} {167, 16, 3} {200, 17, 3} {160, 16, 3} {42, 12, 1} {52, 13, 1}



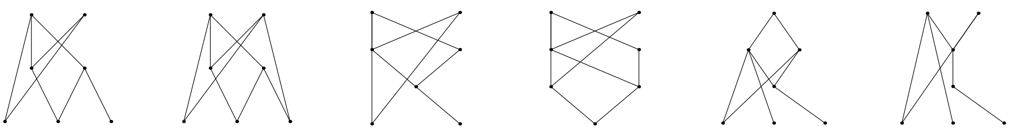
{97, 16, 2} {64, 15, 1} {381, 18, 5} {247, 18, 3} {199, 17, 3} {59, 14, 1}



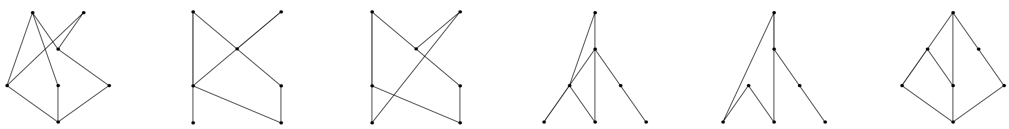
{50, 13, 1} {51, 13, 1} {92, 16, 2} {373, 18, 5} {63, 15, 1} {205, 17, 3}



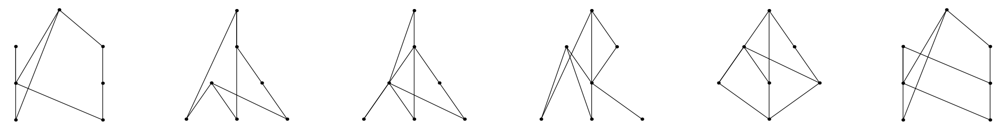
{420, 19, 5} {365, 18, 5} {57, 14, 1} {49, 13, 1} {161, 15, 3} {665, 18, 6}



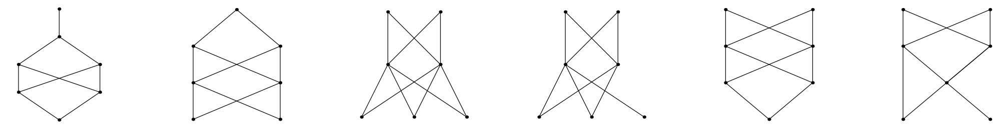
{159, 15, 3} {175, 16, 3} {174, 16, 3} {193, 16, 3} {879, 20, 6} {191, 16, 3}



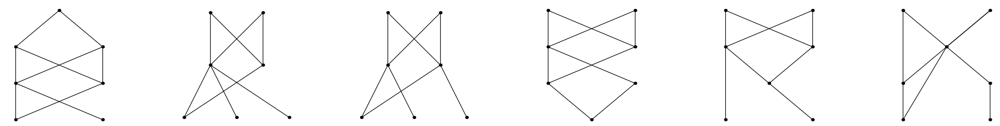
{206, 17, 3} {691, 19, 6} {152, 15, 3} {643, 18, 6} {152, 15, 3} {167, 16, 3}



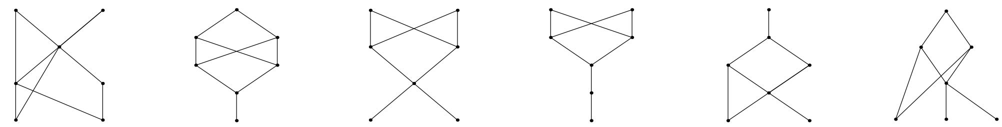
{11, 8, 0} {13, 9, 0} {36, 12, 1} {44, 13, 1} {13, 9, 0} {16, 10, 0}



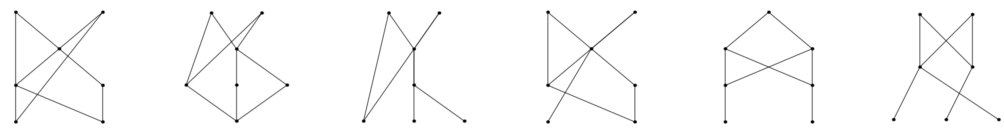
{17, 10, 0} {151, 15, 3} {55, 14, 1} {17, 10, 0} {21, 11, 0} {27, 12, 0}



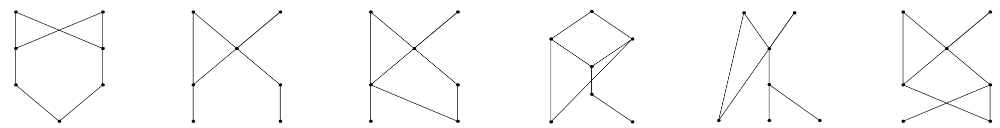
{20, 11, 0} {11, 8, 0} {13, 9, 0} {11, 8, 0} {14, 9, 0} {41, 12, 1}



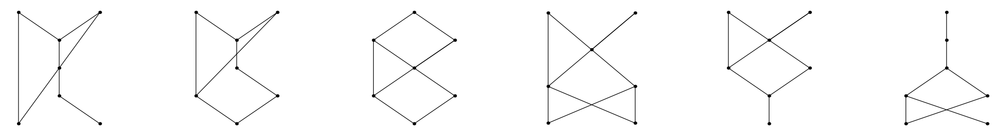
{49, 13, 1} {41, 12, 1} {158, 15, 3} {48, 13, 1} {23, 11, 0} {189, 16, 3}



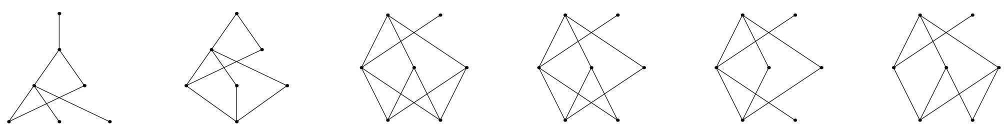
{23, 11, 0} {37, 13, 0} {27, 12, 0} {18, 10, 0} {49, 13, 1} {21, 11, 0}



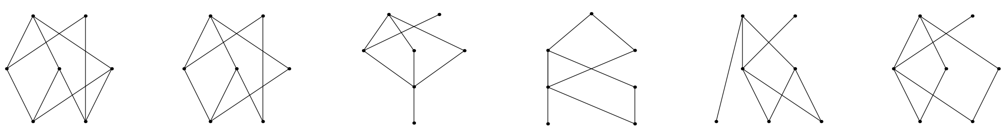
{23, 11, 0} {18, 10, 0} {14, 9, 0} {16, 10, 0} {14, 9, 0} {15, 9, 0}



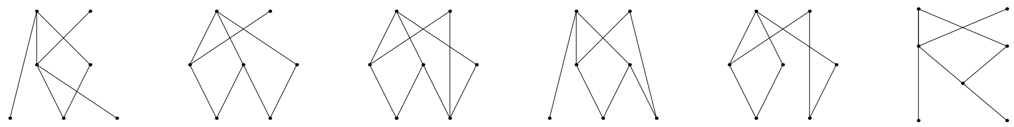
{149, 14, 3} {151, 14, 3} {151, 15, 3} {167, 16, 3} {649, 18, 6} {200, 17, 3}



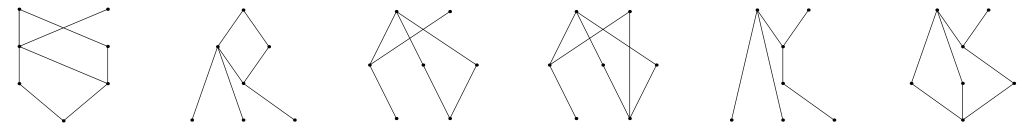
{160, 16, 3} {370, 18, 5} {149, 14, 3} {166, 15, 3} {239, 18, 4} {187, 17, 3}



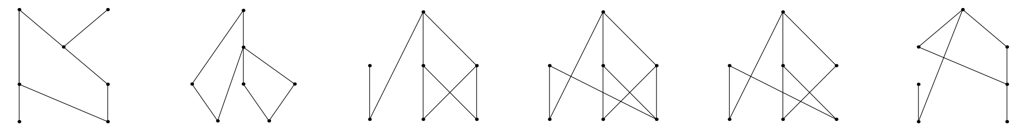
{1070, 20, 8} {226, 18, 3} {179, 17, 3} {658, 20, 7} {424, 19, 5} {180, 16, 3}



{164, 15, 3} {642, 17, 6} {884, 20, 6} {696, 19, 6} {2795, 20, 10} {640, 17, 6}



{671, 18, 6} {198, 16, 3} {637, 21, 6} {253, 19, 4} {772, 21, 7} {253, 18, 3}



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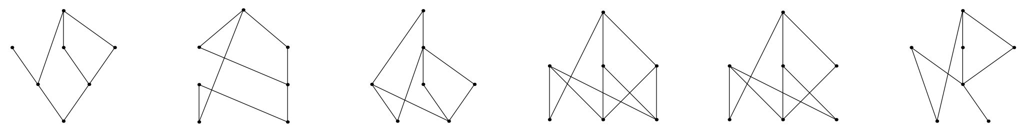

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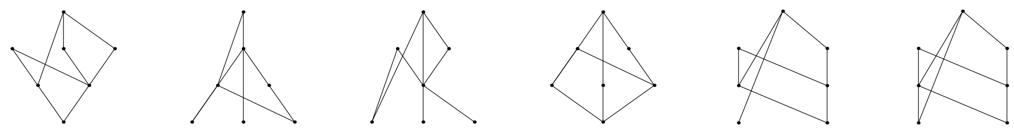

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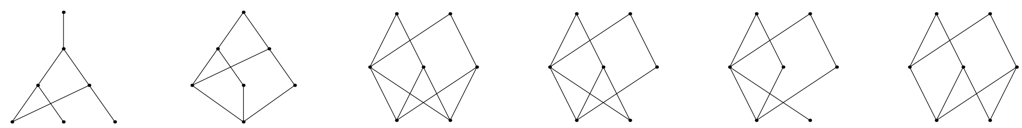

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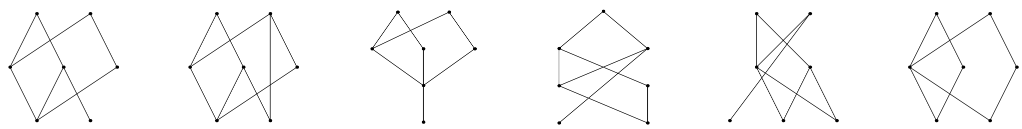

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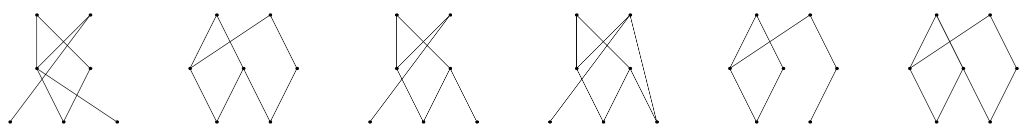

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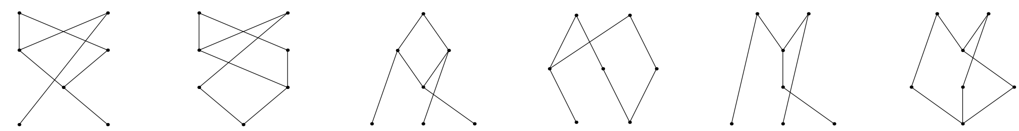

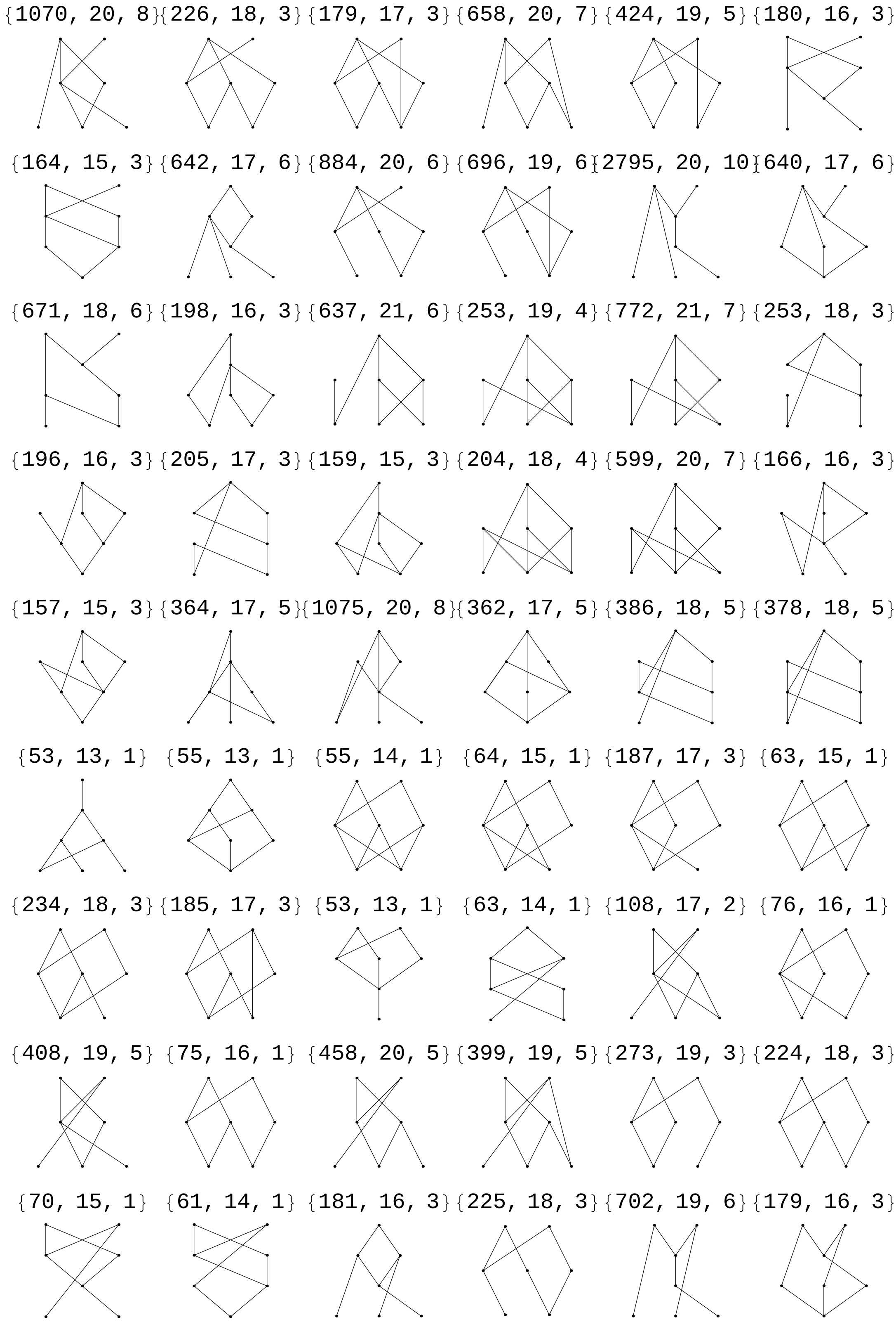
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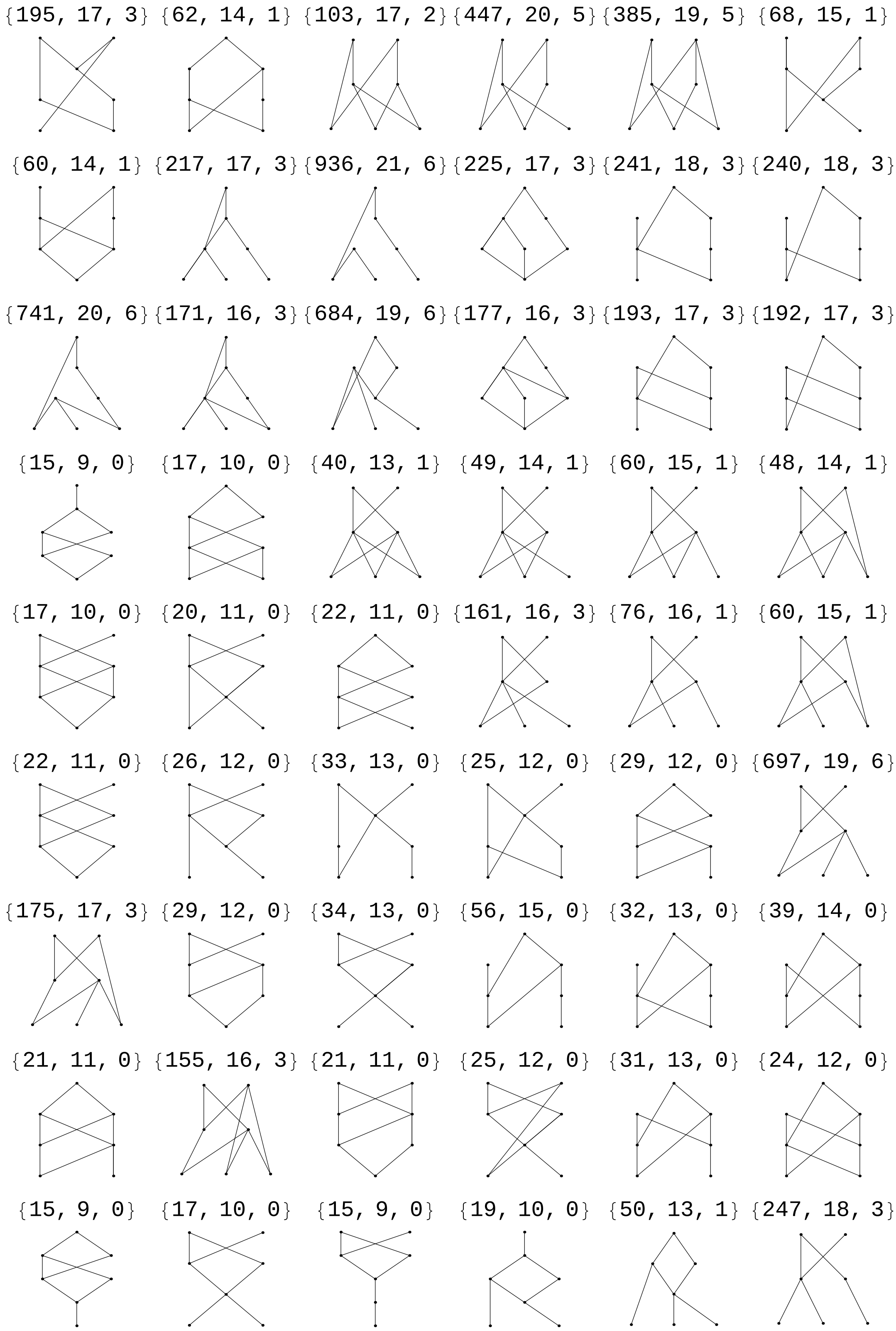
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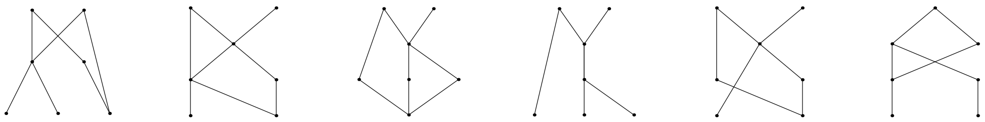

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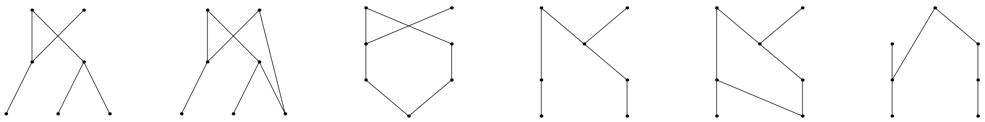

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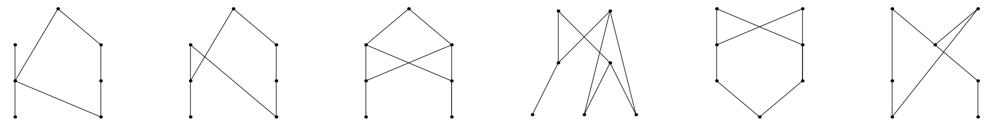

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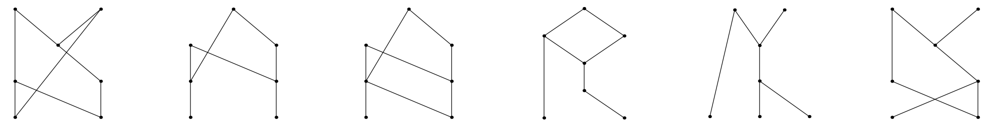

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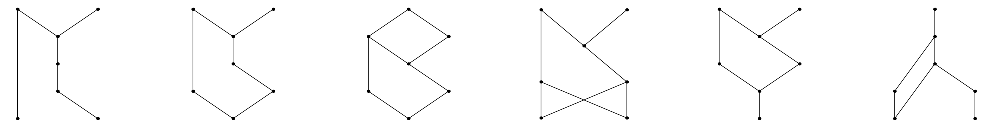

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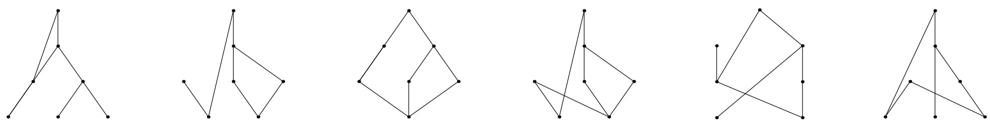

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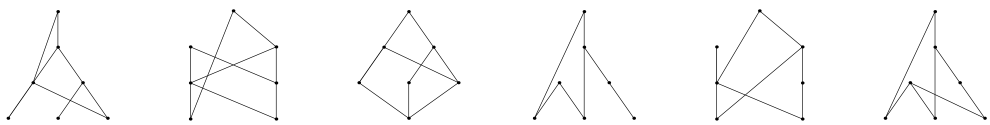

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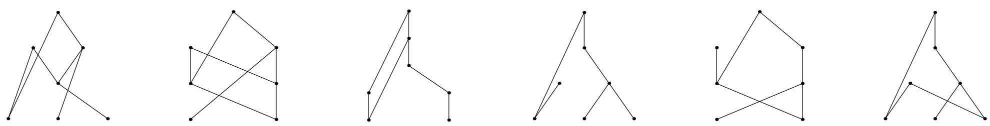

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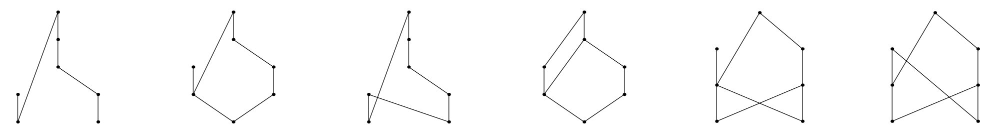

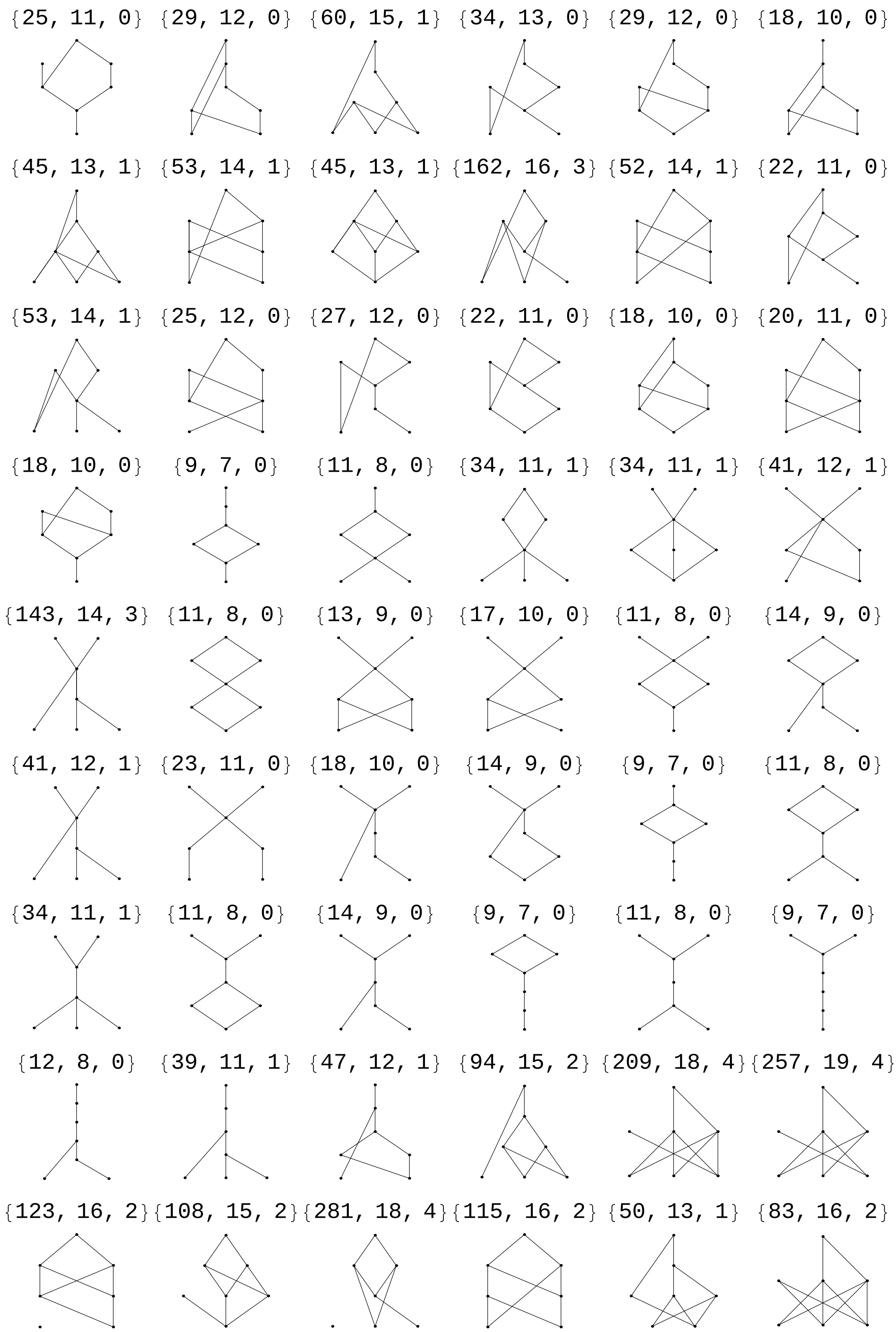
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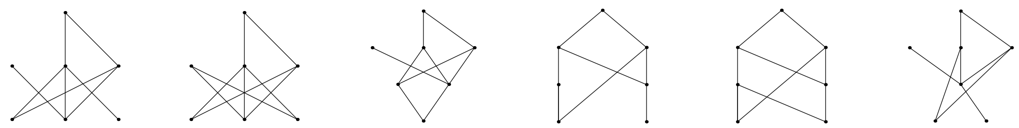

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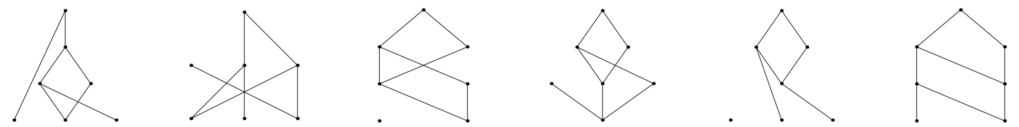

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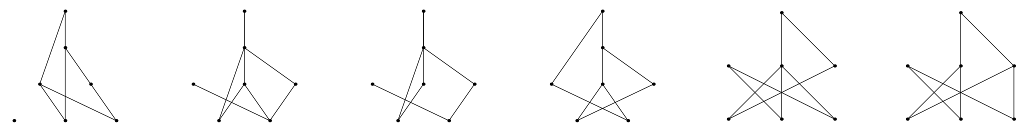

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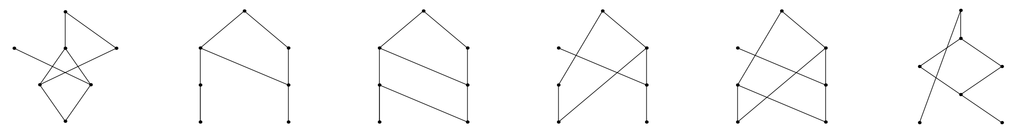

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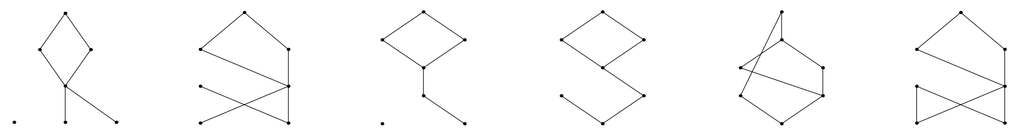

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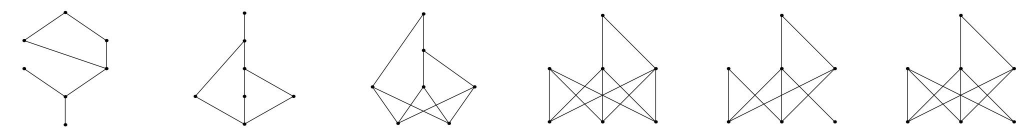

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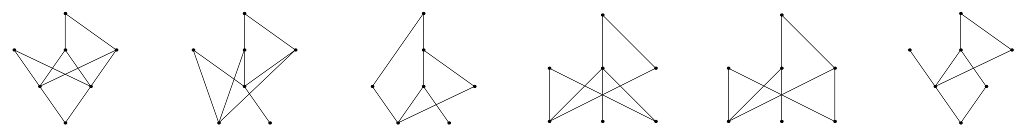

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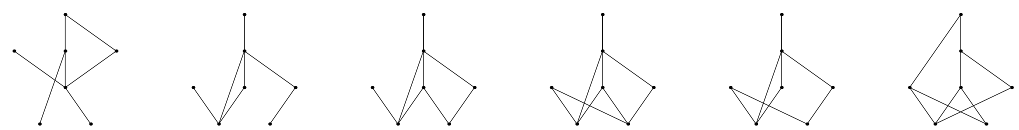

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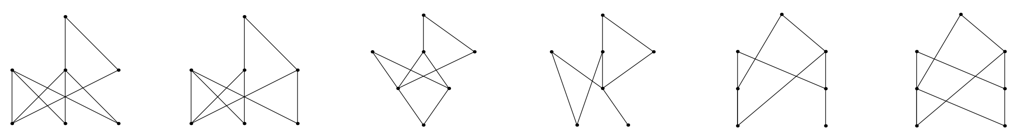

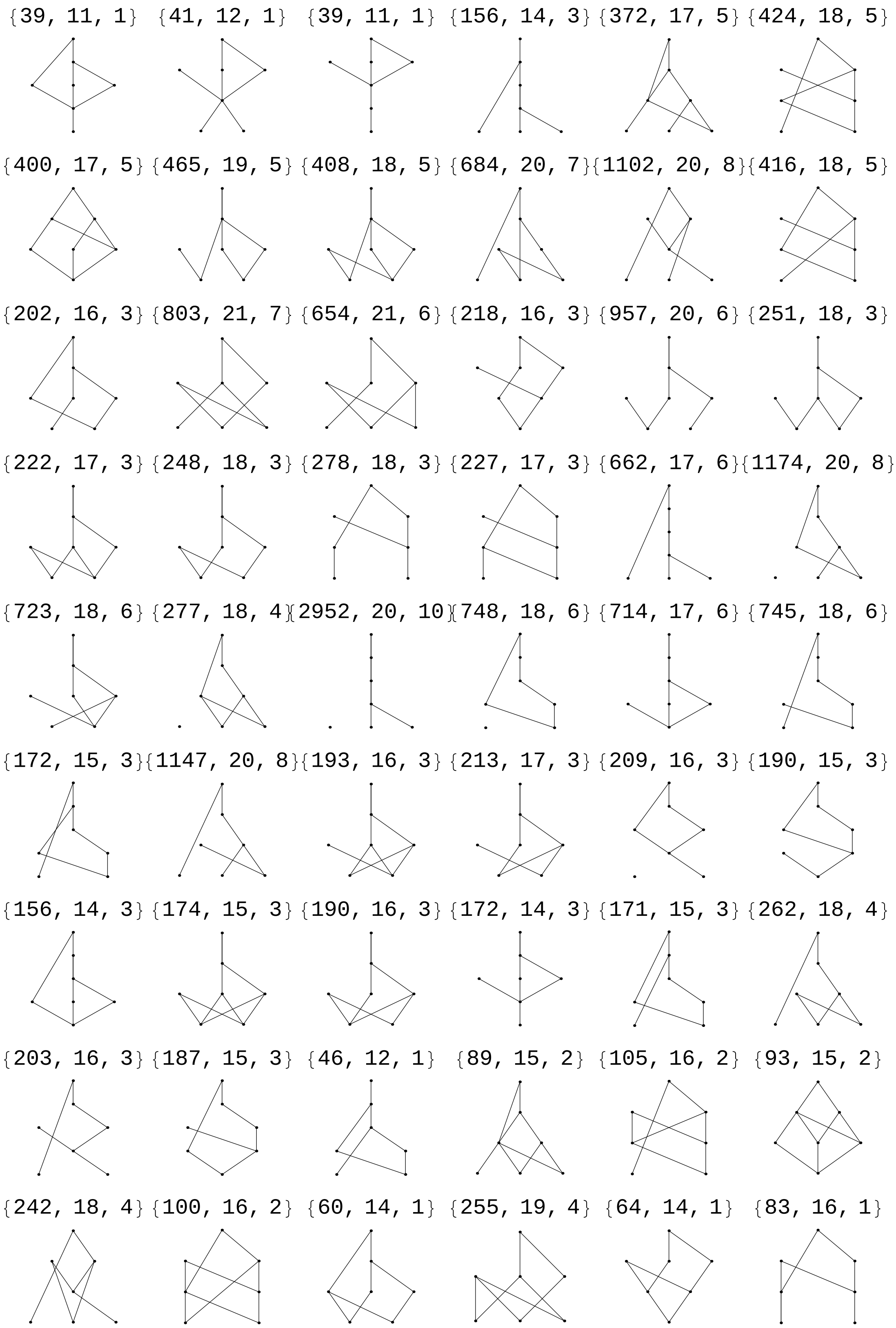
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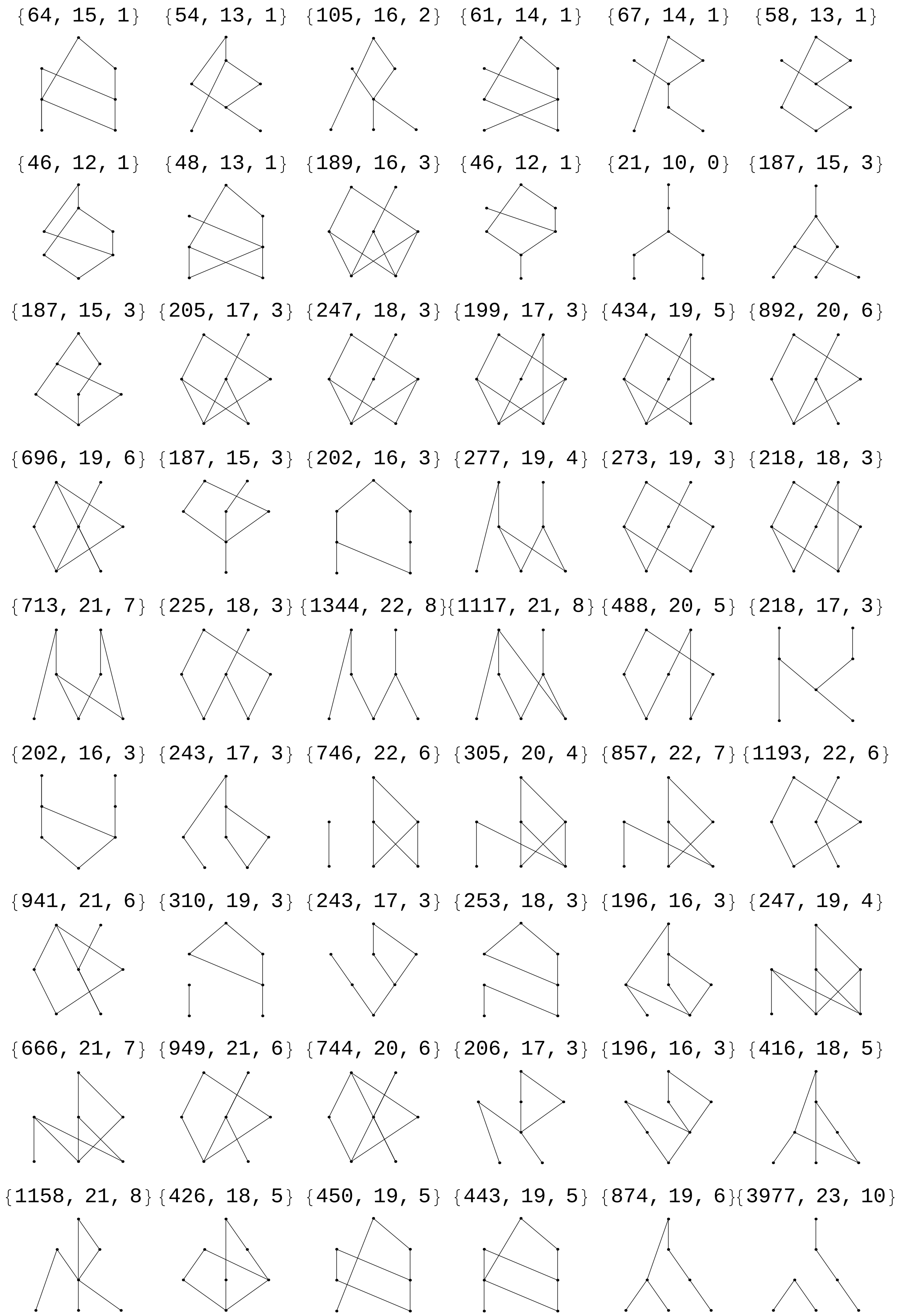
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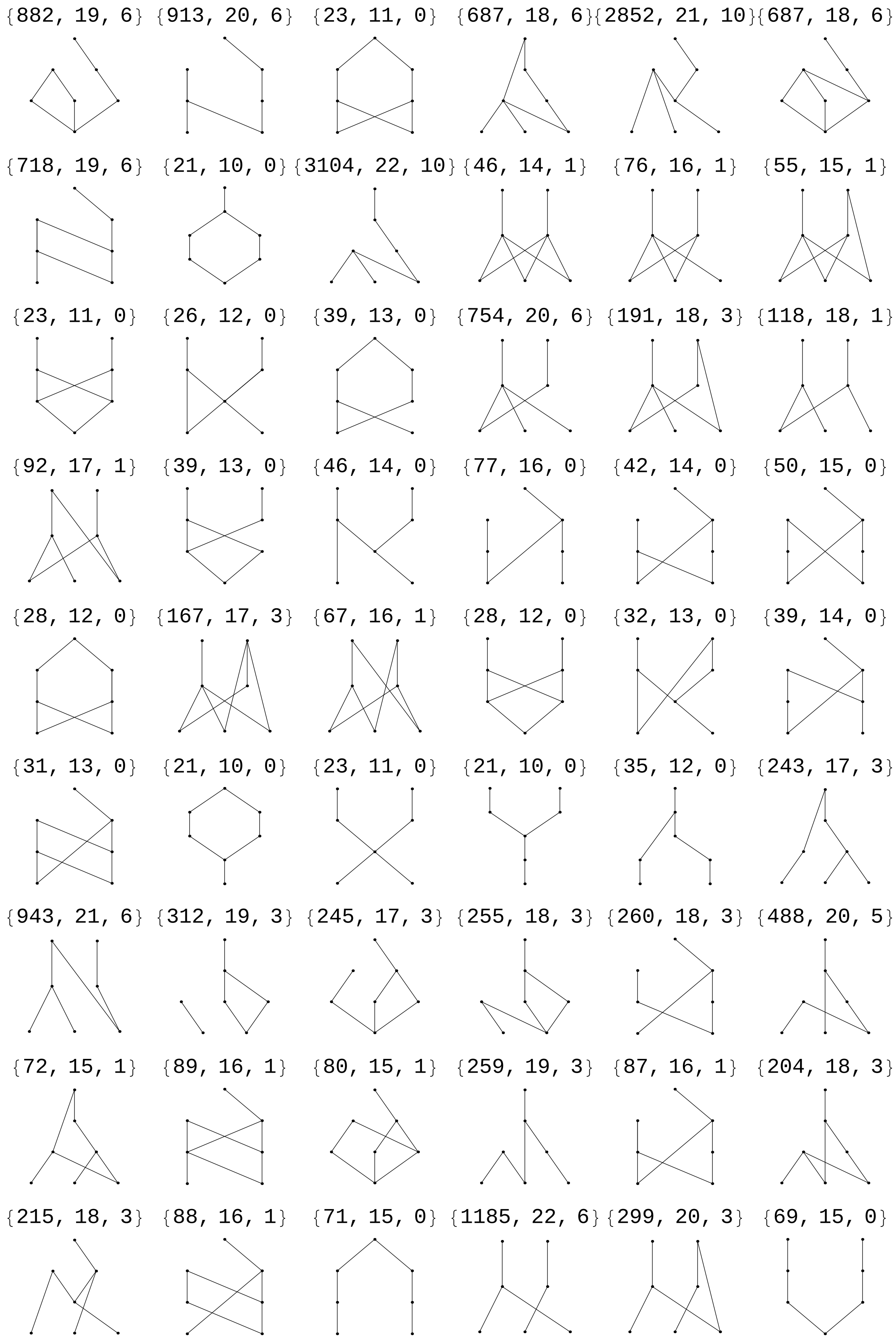
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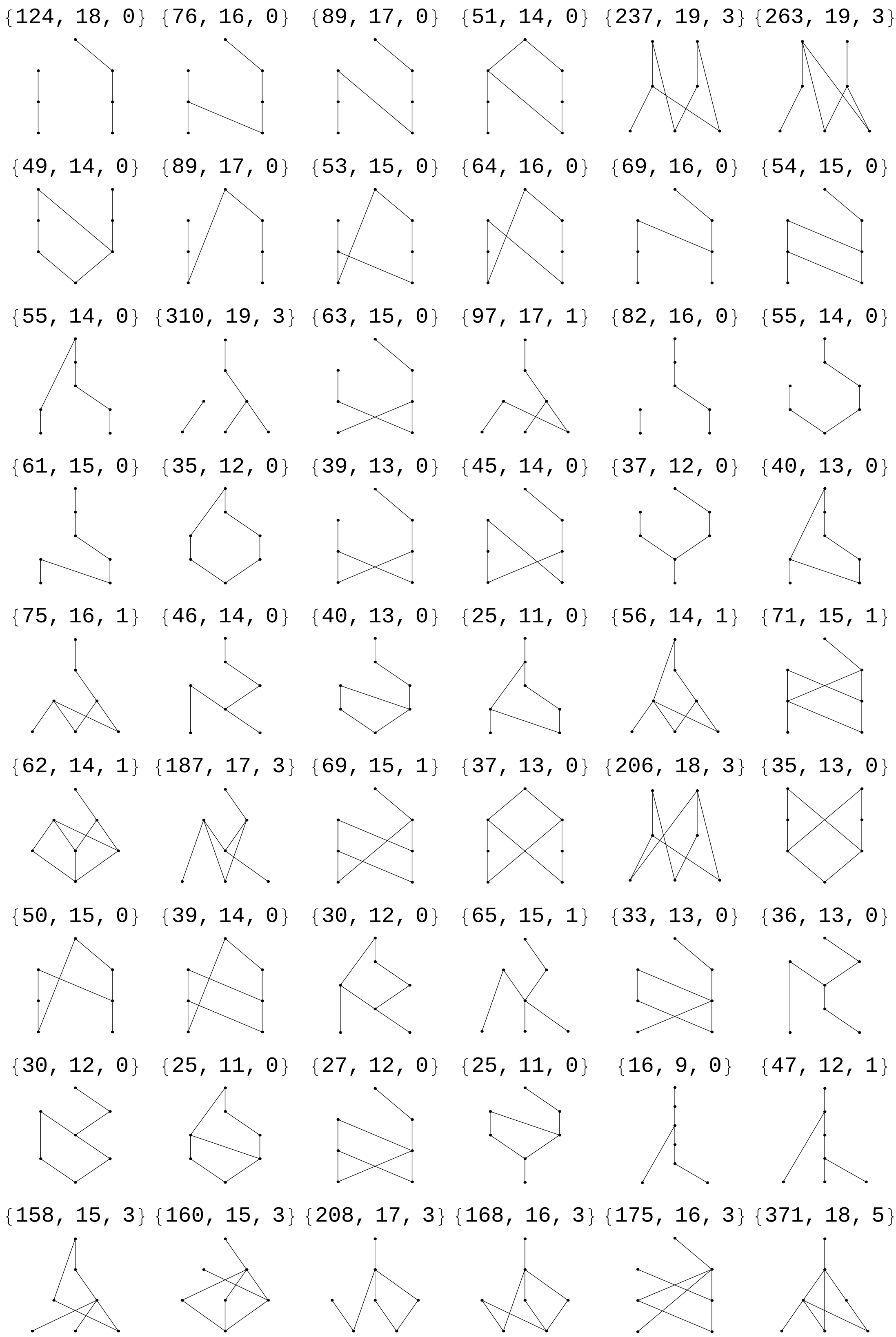
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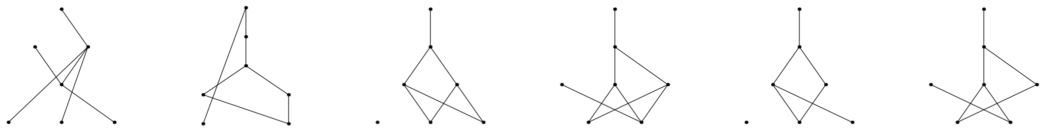

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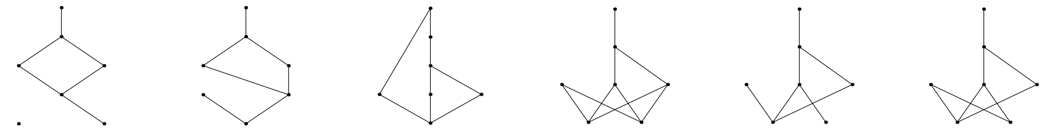

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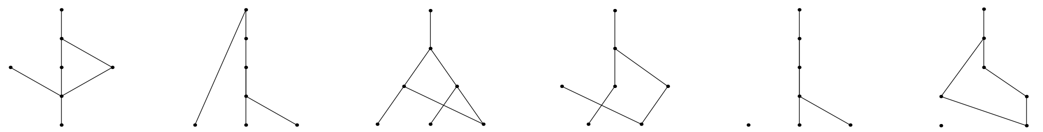

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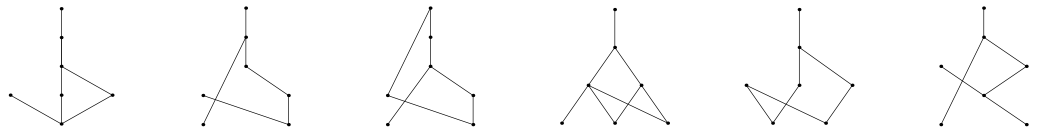

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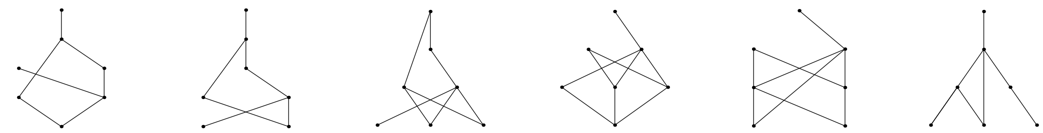

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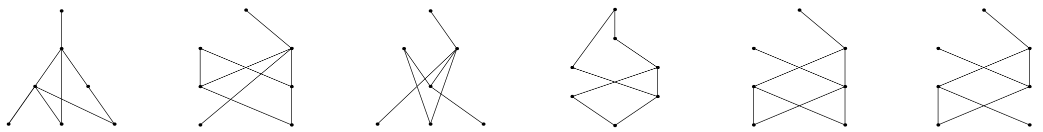

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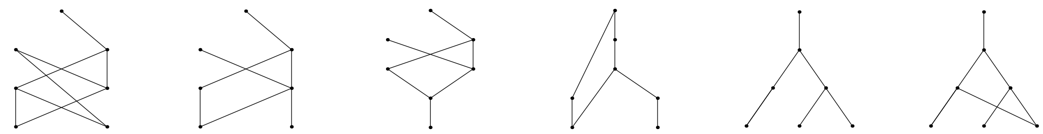

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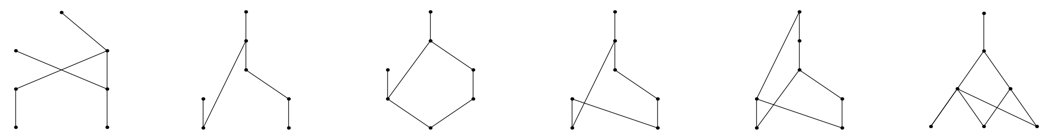

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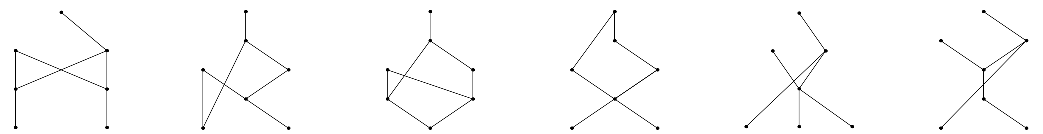

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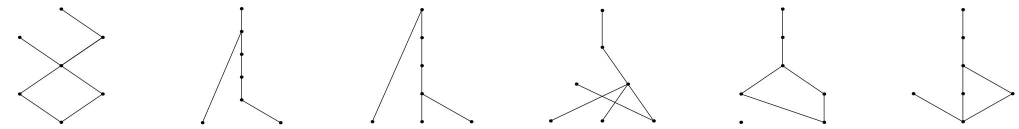

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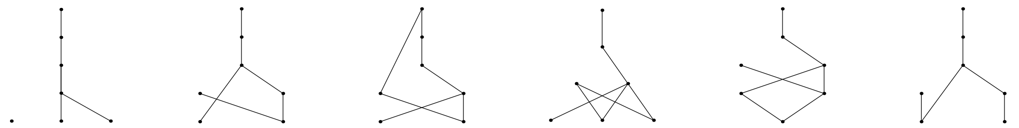

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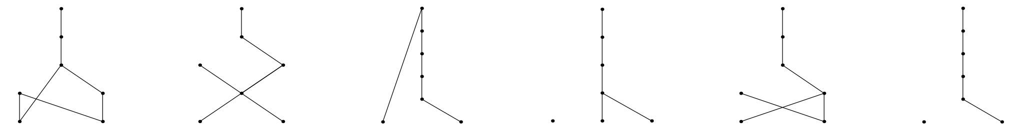

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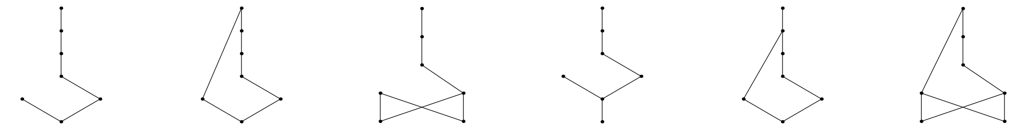

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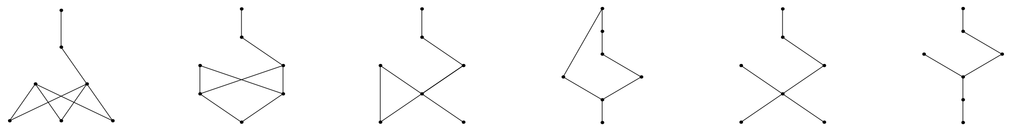

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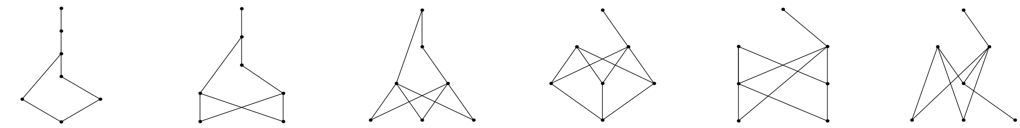

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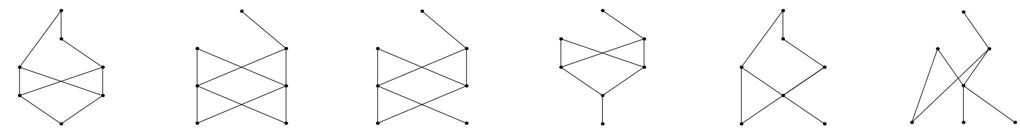

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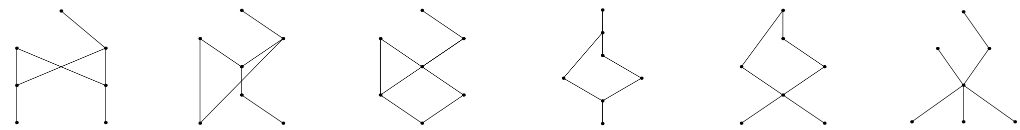

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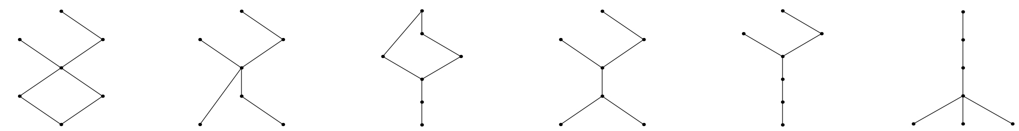

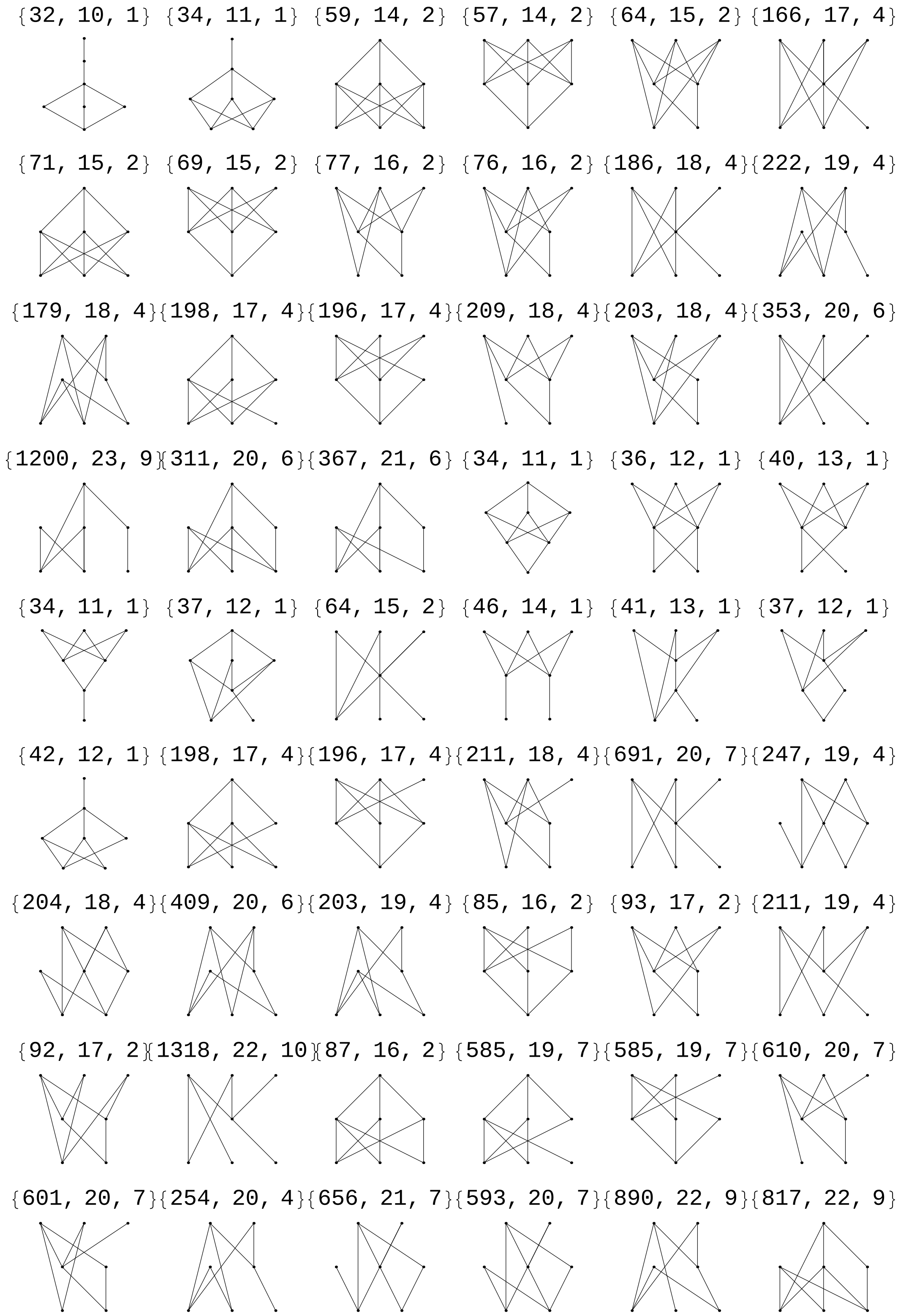
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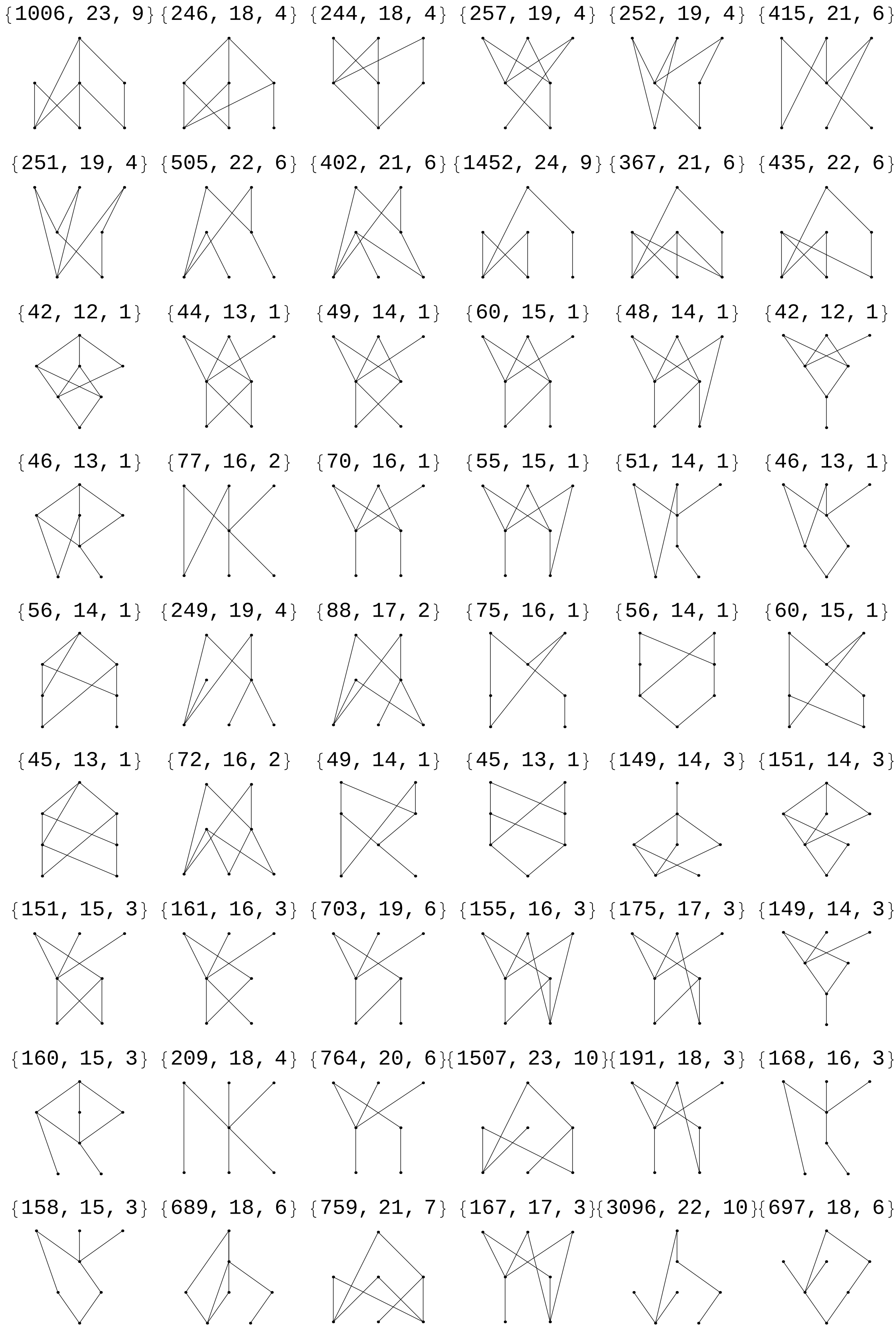
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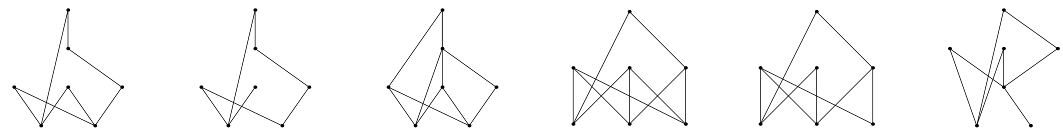

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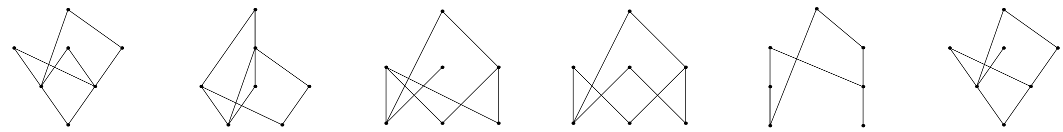

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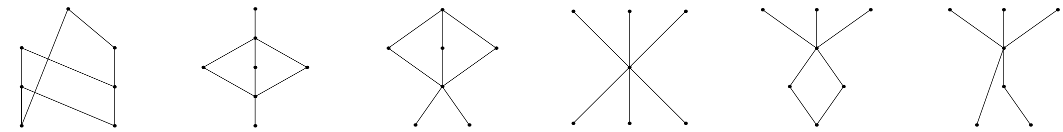

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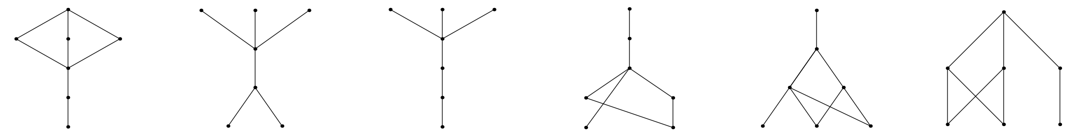

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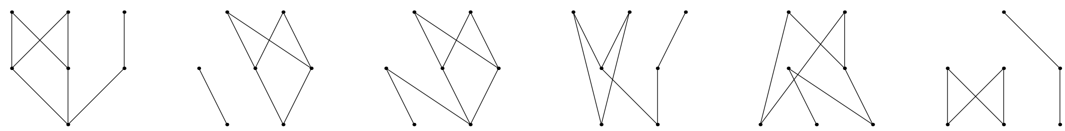

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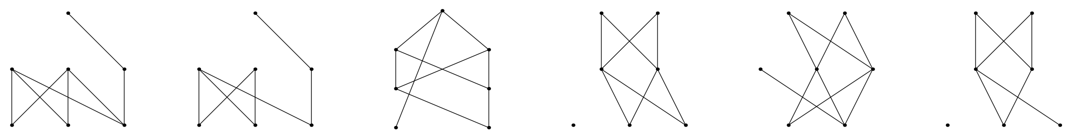

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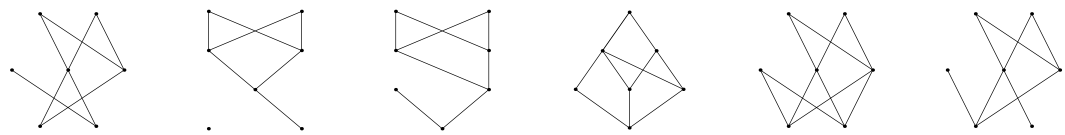

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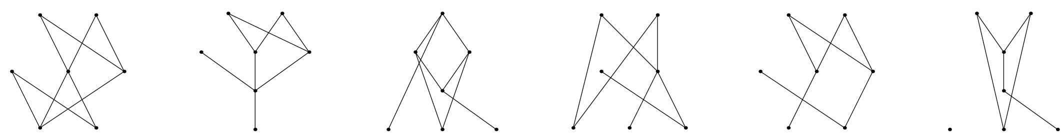

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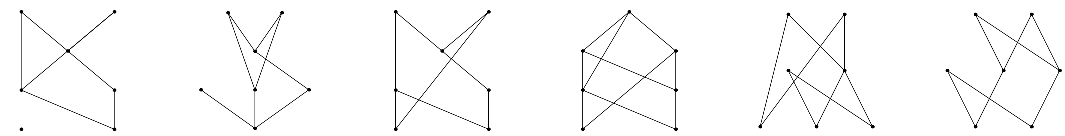

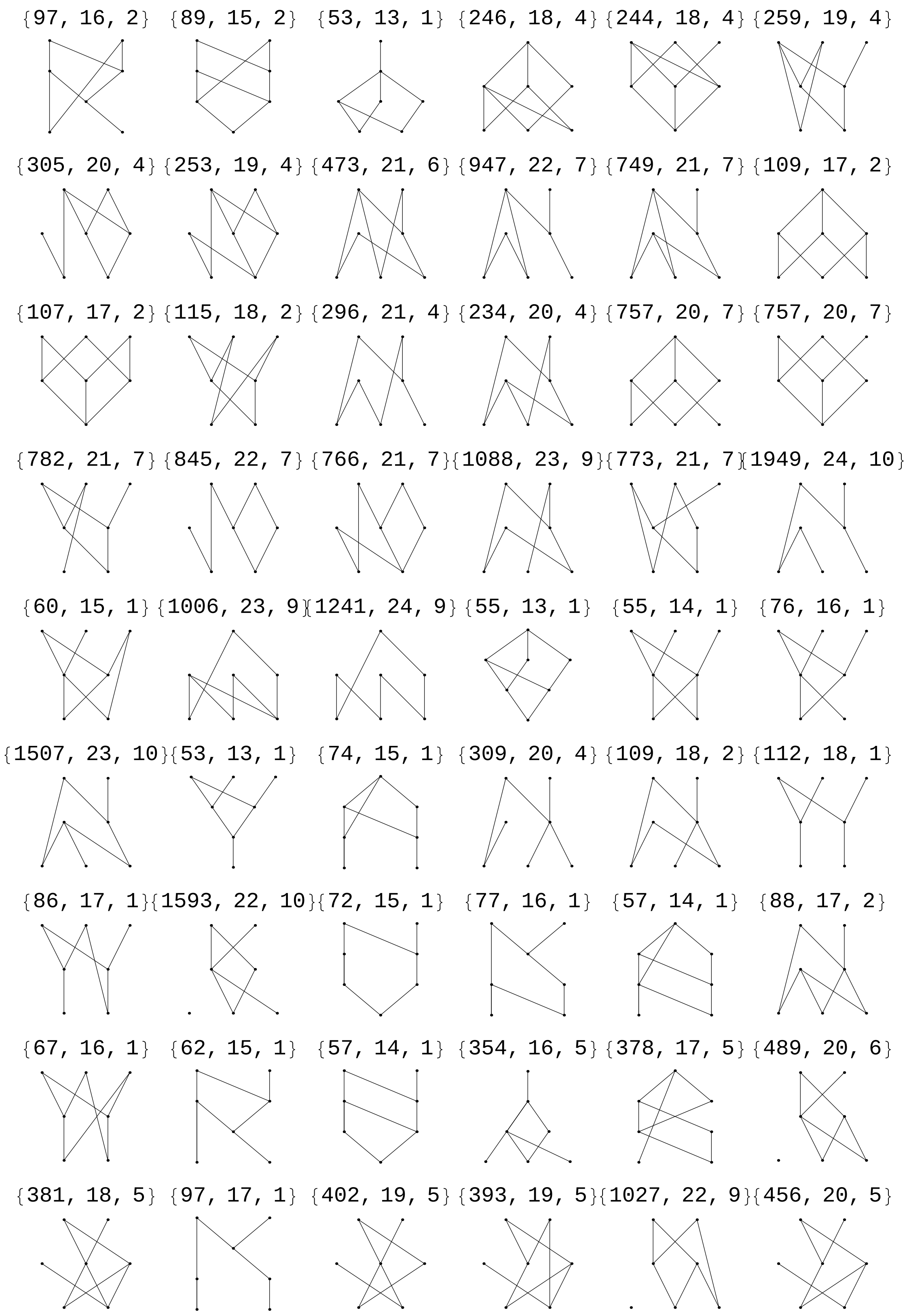
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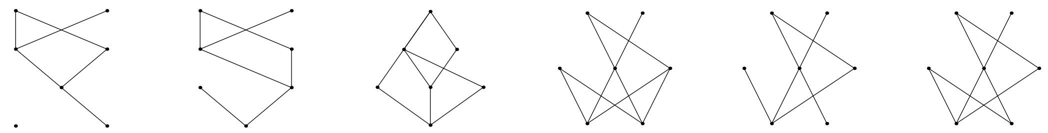

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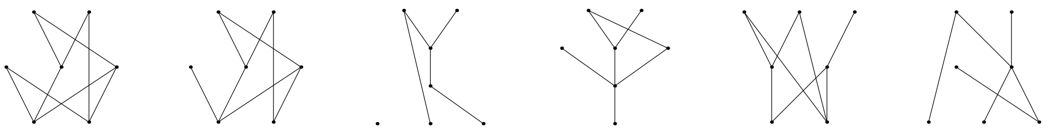

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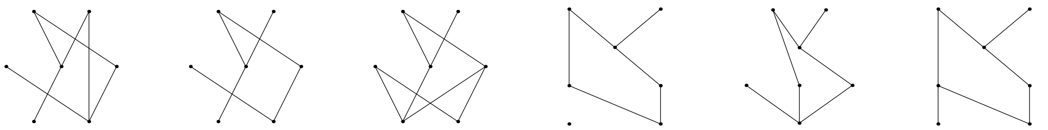

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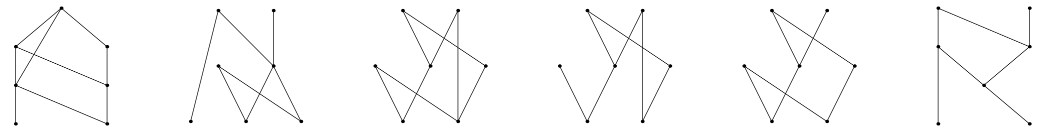

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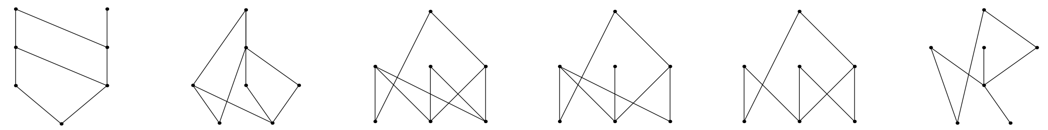

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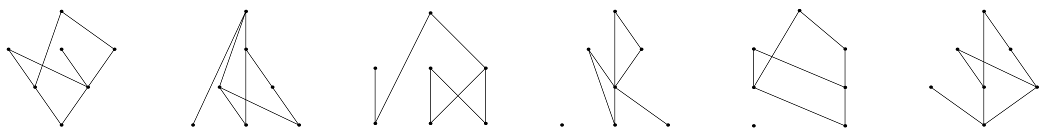

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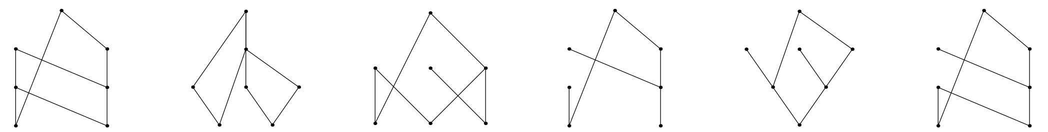

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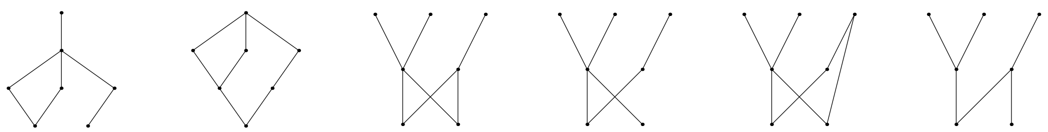

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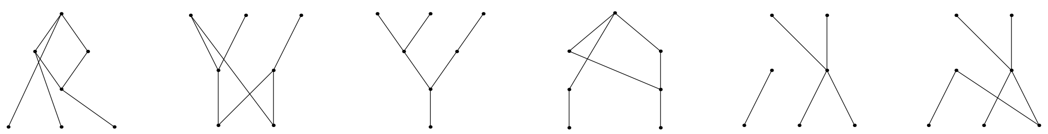

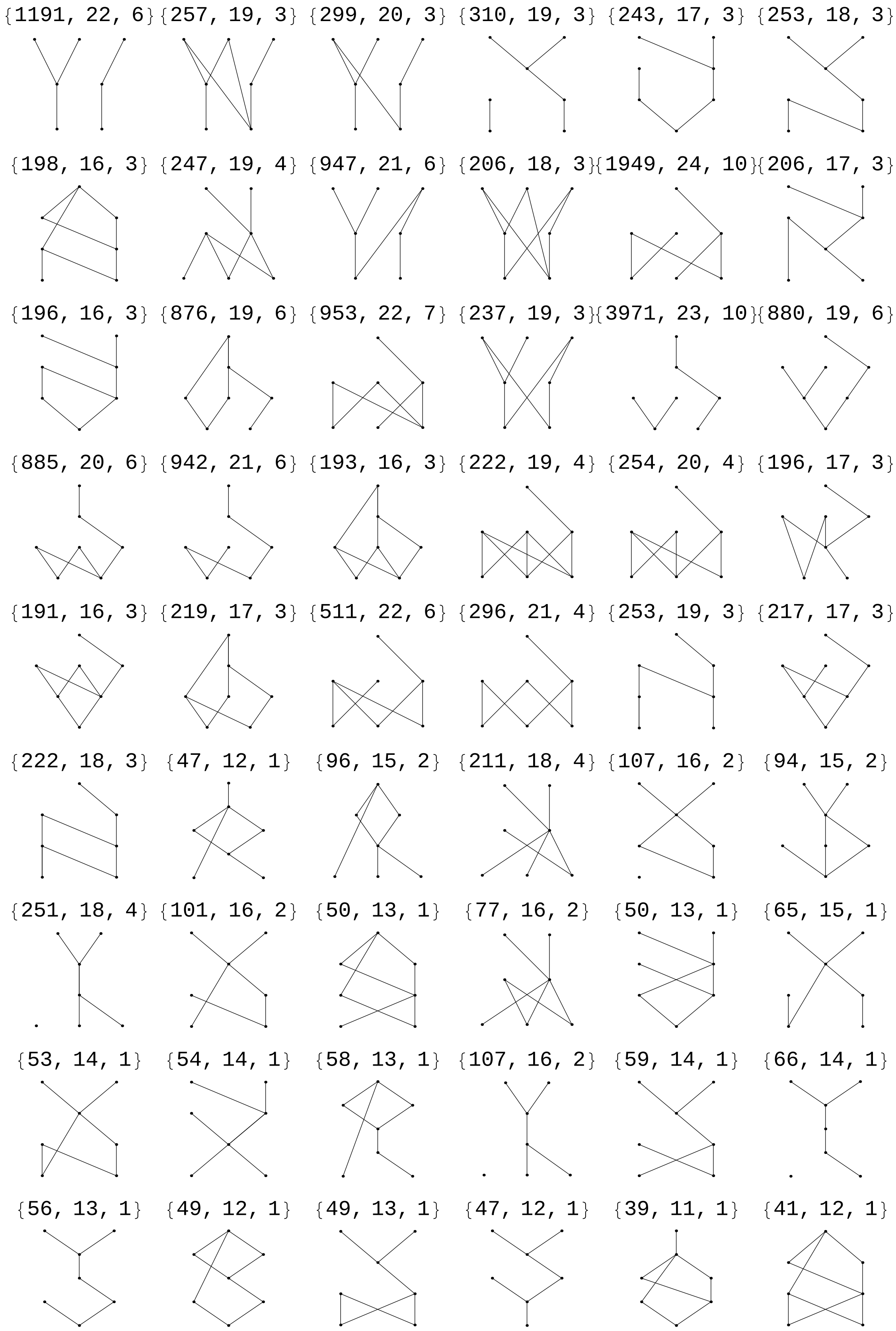
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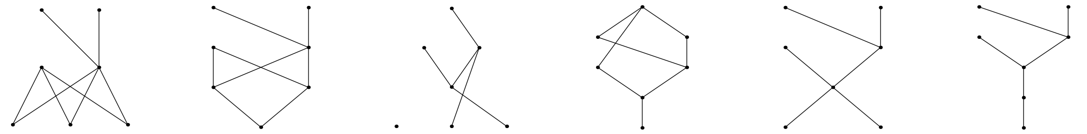

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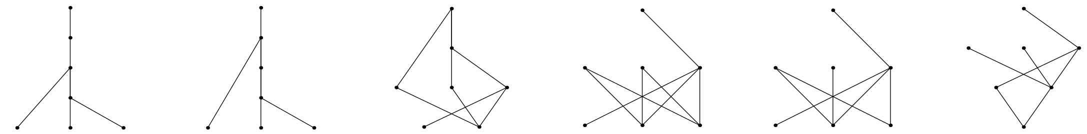

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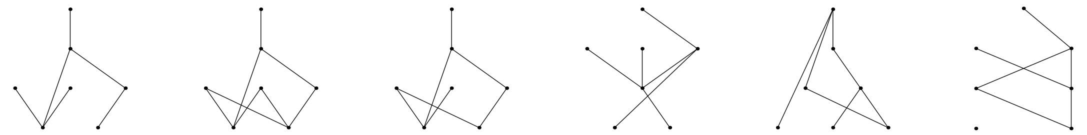

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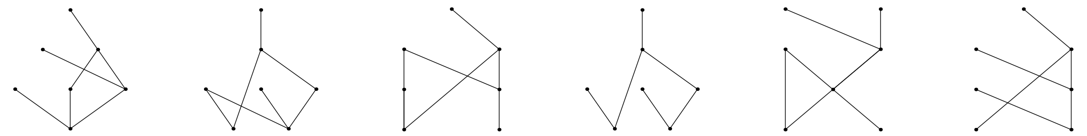

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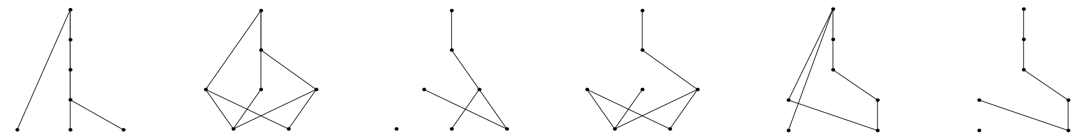

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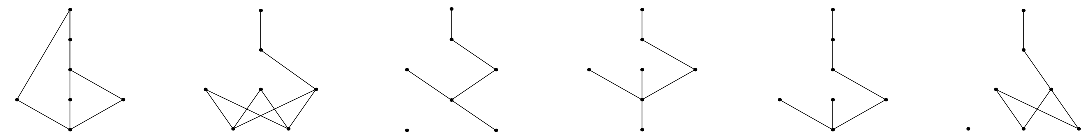

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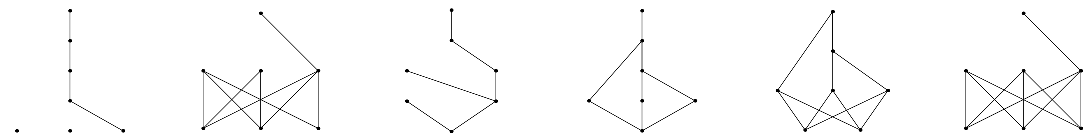

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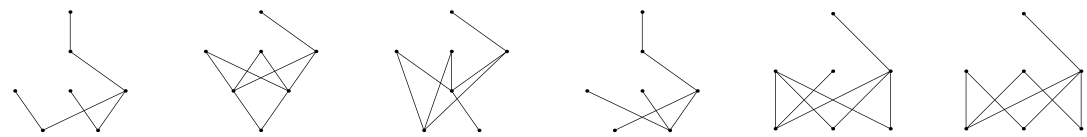
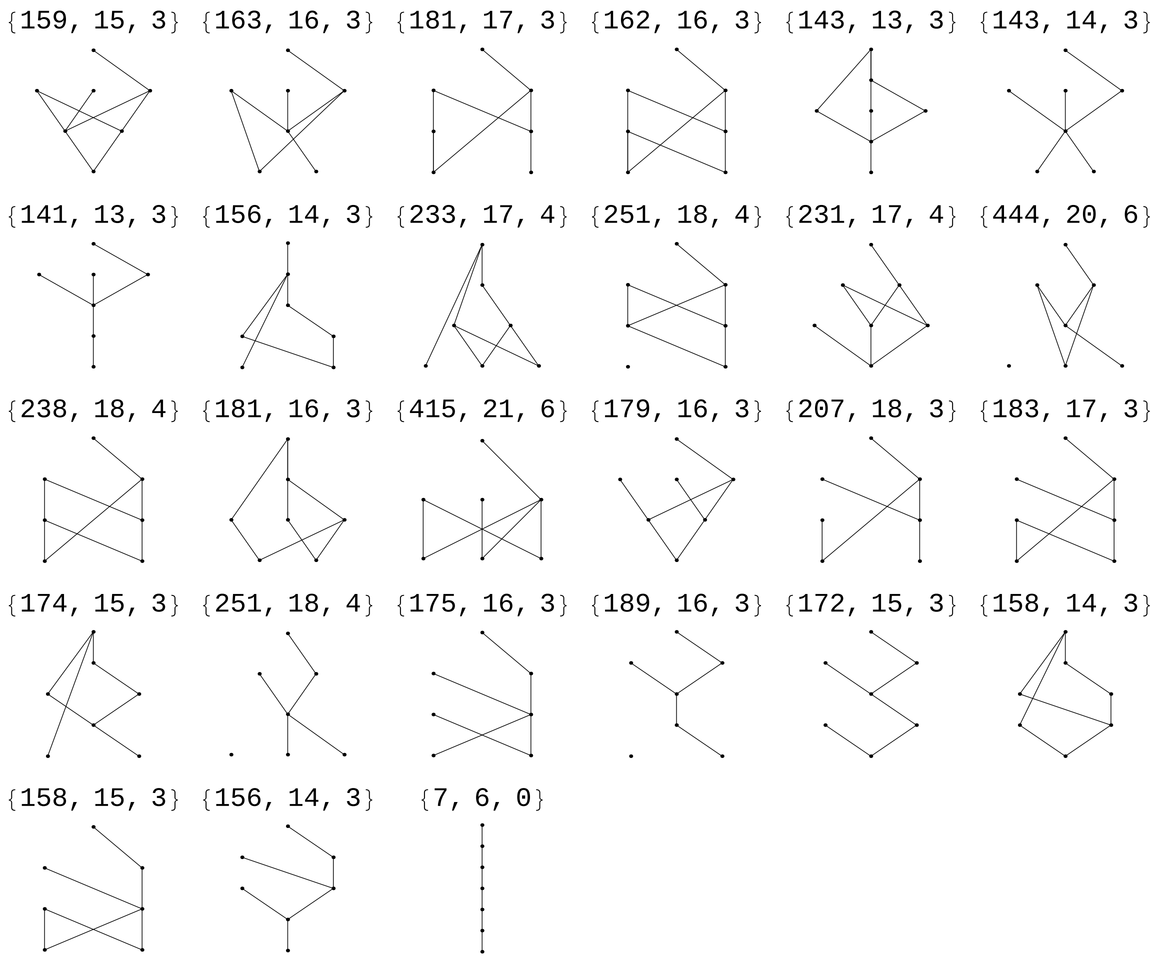




8.3 The code of the (a, B)-Algorithm

A linear extension $1, 2, ..., m, m+1, ..., w$ of the poset P is considered for which m is the number of minimal elements of P. For any non-minimal element k, $m < k \leq w$, the set B_k of lower covers of k is listed in the form $B[k] = B_k$. The algorithm then compute the cardinality of the lattice of order ideals of P.

```
(*Enter the value of m and the value of w where indicated*)

m=value of m;
```

```
w=value of w;

(*Enter the list of conclusions of the implications k --> B_k*)

B[m+1]= B_{m+1} ; B[m+2] =B_{m+2} ; ...; B[w]= B_w;

c=1;  (*the initial length of the many-valued context*)
  zeros[1]= ones[1]={}; twos[1]= Range[w];
  SetOfPremises[1]= {};
  Do[ p= c;
    (* p is a pointer of the current row to be split*)
         While[ p>0,
            BkRest= Complement[ B[k], ones[p] ];
                 If[ BkRest == {}, Goto[Next] ];
             (* leave row p unchanged since B[k] is contained in ones[p] *)
                     If[Intersection[BkRest, zeros[p]]#{} ,
                        zeros[p] = Union[zeros[p],{k}];
                              twos[p]= Complement[twos[p],{k}];
                                     Goto[Next]; ];
 (*the 0 in the conclusion of {k}-->B[k] forces 0 on position k in row p*)
                 SetOfPremises[p] =Union[SetOfPremises[p],{k}] ;
                 con[p, k]= BkRest;
                 twos[p]= Complement[twos[p],{k}, BkRest];
             Goto[Next] ; ];
  (* the implication {k}-->BkRest  falls completely into twos[p] *
      (* there are only 2's and  a's and b's in Bkrest ( and at least one
 a or b does occur ) *)
 (* that means row p is split into a row 0 at position k and an c+1-endrow
 with 1's at the positions in B[k] union {k} : *)
      Nullen = zeros[p]; Einsen = ones[p]; Zweien=twos[p];
      zeros[p]=Union[Nullen,{k}];
      twos[p]=Complement[Zweien,{k}];
      c= c+1;
      zeros[c]=Nullen;
      ones[c]=Union[Einsen, {k},BkRest];
      twos[c]= Complement[Zweien,{k}, BkRest];
      S1= Intersection[SetOfPremises[p], BkRest];
      s1=Length[S1];
      Do[ aa=S1[[i]];
          ones[c] =  Union[ ones[c] ,con[p,aa] ];
         ,{i,s1}  ];
```

```
      S2=Complement[SetOfPremises[p],BkRest];
      s2=Length[S2];
      SetOfPremises[c]={};
      Do[ aa=S2[[i]];
            BB= Complement[ con[p,aa], BkRest];
             If[ BB = {}  ,
          (* so conclusion of {aa}-->con[p,aa] is in B[k] *)
                twos[c]=Union[twos[c],{aa}] ;
          (* con[p,aa] not in B[k] *),
          SetOfPremises[c] =Union[SetOfPremises[c],{aa}];
          con[ c,aa]=BB; ];
        ,{i,s2}  ];
      Label[Next];          p=p-1;                      ];  ,
    {k,m+1,w}   ];
 (* Compute number of order ideals: *)
  card=0;
   Do[ t=Length[twos[p]] ;
      card1=2^t;
          s=Length[SetOfPremises[p]];
          Do[aa=SetOfPremises[p][[i]];
                card1=card1*(2^Length[con[p,aa]]+1) ;
            ,{i,s}];
    card=card+card1;
    ,  {p,c} ] ;
  Print["The context has at the end ",c," three-valued  rows"];
Print["The number of order ideals is ", card] ;
```

8.4 Concluding remarks

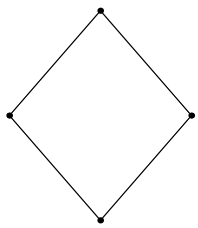
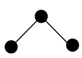
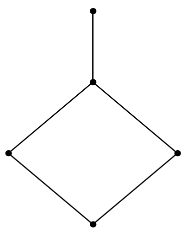
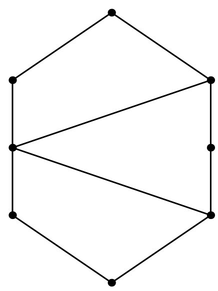
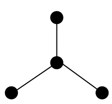
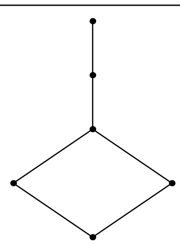
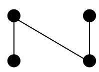
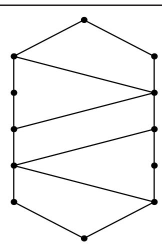
The computation of free lattices in general and free modular lattices in particular are interesting problems. The method often used is that which deals with words on the set of generators (i.e. the poset under consideration). In this thesis, we have proposed another approach based on the representation of modular lattices by closure systems. An algorithm to generate the elements of any finite closure system has been implemented with the Mathematica software suite, enabling us to achieve our main objective which was to effectively compute the elements of a free modular lattice generated by a finite poset and draw its Hasse diagram. Practically, given any finite poset P determined by its covering relation, the first subroutine of the program computes $J\big(F\mathcal{M}(P)\big)$ together with a base of lines and the resulting set

of implications. The second subroutine which is the (a, B)-Algorithm then takes this set of implications and outputs $|F\mathcal{M}(P)|$ and the number of factors ($\mathbf{2}$ or M_3) in the subdirect product decomposition of $F\mathcal{M}(P)$.
Theoretically, our program can compute $|F\mathcal{M}(P)|$ for any finite poset but practically, this is only feasible for small posets. With this program we were able to compute $|F\mathcal{M}(P)|$ for all posets P with $|P| \leq 6$ except for few critical cases. We also computed $|F\mathcal{M}(P)|$ for the 1101 good posets on seven points. It would be interesting to improve the algorithm in order to (a) generate $|F\mathcal{M}(P)|$ for bigger posets, and (b) to develop a similar algorithm to effectively compute the elements of a lattice freely generated by a poset within a fixed locally finite variety. As to (b), I am currently pursuing this in collaboration with Prof. Wild. As to (a), there is hope to handle large posets as long as their structure is symmetric. In fact the exploitation of symmetry is a crucial issue in the general framework [26] of the principle of exclusion.

Appendix A

More pictures of $F\mathcal{D}(P)$ and $F\mathcal{M}(P)$

We start with posets with no 3-element antichain. For these posets, $F\mathcal{D}(P) = F\mathcal{M}(P)$.

P	$F\mathcal{D}(P)= F\mathcal{M}(P)$	P	$F\mathcal{D}(P)= F\mathcal{M}(P)$
n	**n**	**1+1**	
		**1+2**	
			

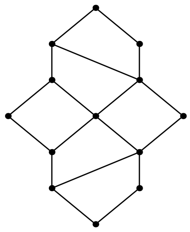
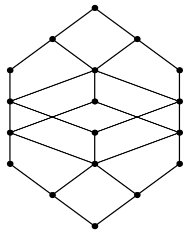
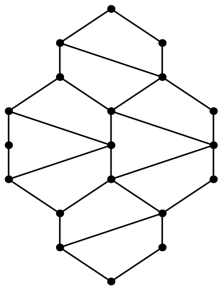
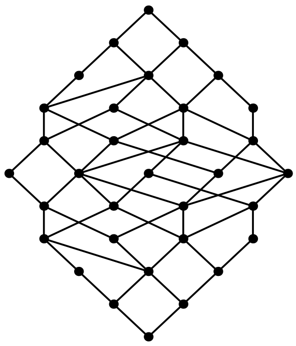
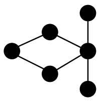
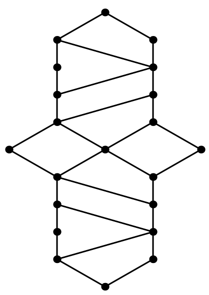
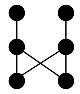
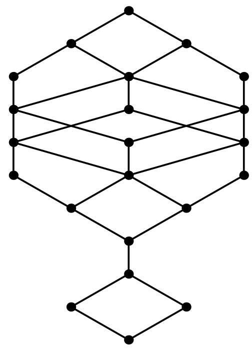
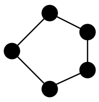
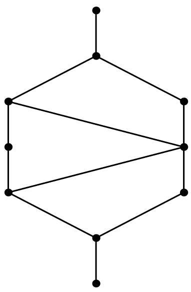
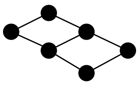
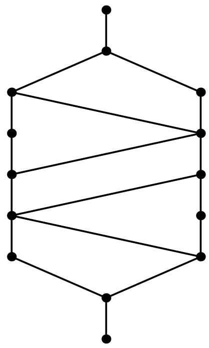
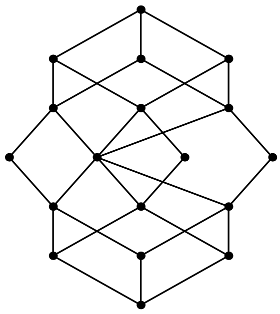
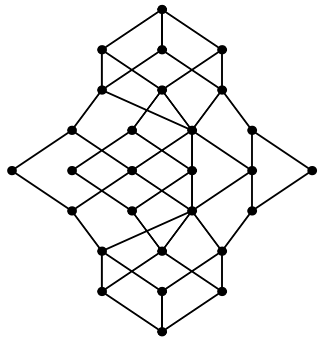
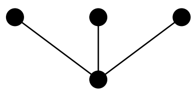
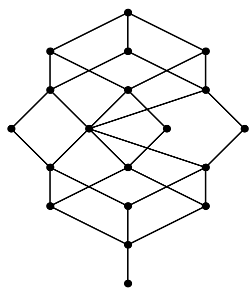
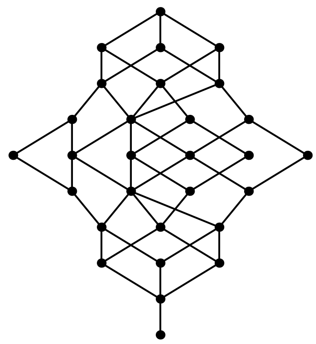
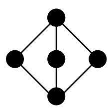
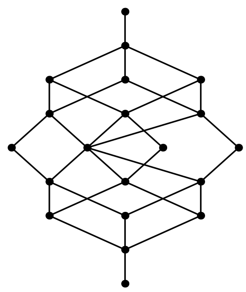
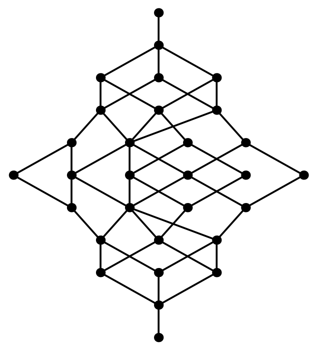
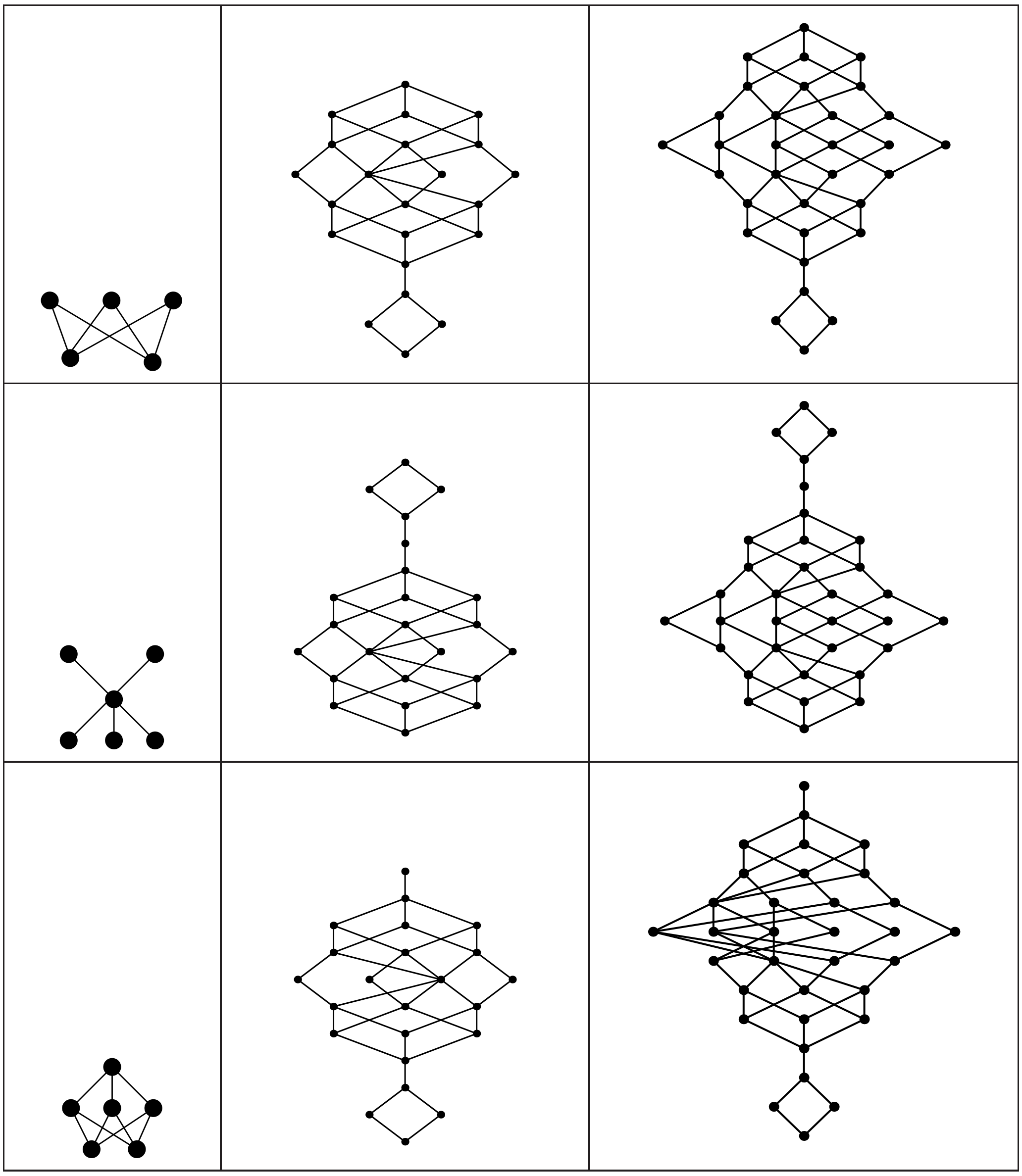
1+3		**2+2**	
1+4		**2+3**	
			
			

Table A.1: Posets with no 3-element antichain.

For poset having a 3-element antichain, $F\mathcal{D}(P) \neq F\mathcal{M}(P)$.

P	$F\mathcal{D}(P)$	$F\mathcal{M}(P)$
1+1+1		
		
		



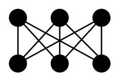
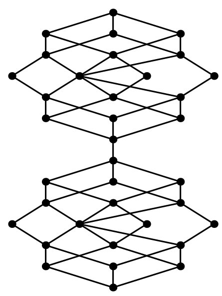
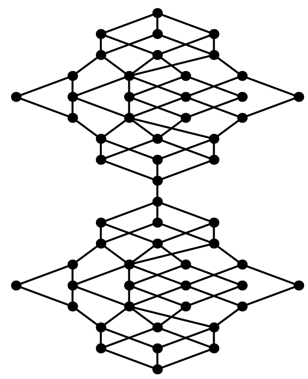
		

Table A.2: Posets with a 3-element antichain.

Bibliography

[1] Birkhoff, G.: *Lattice theory.* Amer. Math. Soc., Providence, 1940.

[2] Whiteman, P.M.: Free lattices 1. *Ann. of Math.*, vol. 42, no. 2, pp. 325–330, 1941.

[3] Whiteman, P.M.: Free lattices 2. *Ann. of Math.*, vol. 43, no. 2, pp. 104–115, 1942.

[4] Rolf, H.L.: The free lattice generated by a set of chains. *Pacific J. of Math.*, vol. 8, no. 3, pp. 585–595, 1958.

[5] Wille, R.: *On lattices freely generated by finite partially ordered sets.* Coll. Math., Budapest, Amsterdam. Janos Bolyai 17 (B.Csakany, E.T. Schmidt eds), 1977.

[6] Dedekind, R.: Über die von drei Moduln erzeugte Dualgruppe. *Math. Ann.*, vol. 53, pp. 371–403, 1900.

[7] Whitman, P.M.: Status of word problems for lattices. *Proc. Sympos. Pure Math., Amer. Math. Soc., Providence*, vol. 2, pp. 17–21, 1961.

[8] Freese, R., Je ˇ zek, J. and Nation, J.B.: *Free lattices, Mathematical Surveys and Monographs.* American Mathematical Society, 1995.

[9] Wille, R.: Über modulare Verbände die von einer endlichen halgeordneten Menge frei erzeugt werden. *Math. Z.*, vol. 131, pp. 241–249, 1973.

[10] Freese, R.: Free modular lattices. *Trans. Amer. Math. Soc.*, vol. 261, pp. 81–91, 1980.

[11] Herrmann, C.: On the word problem for modular lattice with four free generators. *Math. Ann.*, vol. 265, pp. 513–527, 1983.

[12] Bartenschläger, G.: *Free bounded distributive lattices generated by finite ordered sets.* Ph.D. thesis, Technische Hochschule Darmstadt, 1994.

[13] Herrmann, C. and Wild, M.: Acyclic modular lattices and their representations. *Journal of Algebra*, vol. 136, pp. 365–396, 1991.

[14] Brinkmann, G. and Mckay, B.D.: Posets on up to 16 points. *Order*, vol. 19, no. 2, pp. 147–179, 2002.

[15] Semegni, J.Y.: *Some Concepts in Lattice Theory.* African Institute for Mathematical Sciences (AIMS), Jun 2005. Available on the web: www.aims.ac.za/resources/archive/2004.php.

[16] McKenzie, R.N., Taylor, W.F. and McNulty, G.F.: *Algebras, Lattices, Varieties.* Wadsworth & Brooks / Cole, 1987. Advanced Books & Software. Monterey, California.

[17] Burris, S. and Sankappanavar, H.P.: *A course in universal algebra.* Springer Verlag, New York, 1981. Graduate texts in Mathematics.

[18] Grätzer, G.: *General lattice theory.* Birkhäuser Verlag, Basel, 1978.

[19] Wild, M.: Computing various types of lattices freely generated by posets. *Note di matematica e fisica*, vol. 10, pp. 99–128, 1999.

[20] Semegni, J.Y. and Wild, M.: Finite lattices freely generated by posets and partial semilattices, part I, 2008. Submitted.

[21] Jipsen, P. and Rose, H.: *Varieties of lattices.* Springer Verlag, New York, 1992.

[22] Wild, M.: Modular lattices of finite length, 1990. Unpublished manuscript, 28 pages, MIT.

[23] Dilworth, R.P.: Lattices with unique complements. *Trans. Amer. Math. Soc.*, vol. 57, pp. 123–154, 1945.

[24] Semegni, J.Y. and Wild, M.: Finite lattices freely generated by posets and partial semilattices, part II, 2008. Submitted.

[25] Pinter, C.C.: *Set Theory.* Addison - Wesley, 1971.

[26] Wild, M.: Generating all closed sets of a closure system given by Horn formulas, 2008. In preparation.

[27] Schröder, B.S.W.: *Ordered Sets, An Introduction.* Birkhäuser, Boston. Basel, Berlin, 2002.

[28] Pemmaraju, S. and Skiena, S.: *Computational Discrete Mathematics: Combinatorics and Graph Theory with Mathematica.* Cambridge University Press, 2003.

[29] Wolfram, S.: *Mathematica: A System for Doing Mathematics by Computer.* Second Edition. Addison-Wesley Publishing Compagny, Inc, 1991. The Advanced Book Program.

[30] Berman, J. and Köhler, P.: Cardinalities of finite distributive lattices. *Mitteilungen Math. Sem. Giessen*, vol. 121, pp. 103–124, 1976.

[31] Steiner, G.: An algorithm to generate the ideals of a partial order. *Operation Research Letters*, vol. 5, pp. 317–320, 1986.

[32] Berman, J. and Wolk, B.: Free lattices in small varieties. *Algebra Universalis*, vol. 10, pp. 269–289, 1980.

[33] Prömel, H.J.: Counting unlabeled structures. *J. Combinat. Theory*, vol. 44, pp. 83–93, 1987. Ser. A.

[34] Culberson, J.C. and Rawlins, G.J.E.: New results from an algorithm for counting posets. *Order*, vol. 7, no. 3, pp. 361–374, 1991.

[35] Chaunier, C. and Lygeros, N.: The number of orders with thirteen elements. *Order*, vol. 9, no. 3, pp. 203–204, 1991.

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