Some Statistical Aspects of LULU smoothers

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I, the undersigned, hereby declare that the work contained in this dissertation is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

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Summary

The smoothing of time series plays a very important role in various practical applications. Estimating the signal and removing the noise is the main goal of smoothing. Traditionally linear smoothers were used, but nonlinear smoothers became more popular through the years.

From the family of nonlinear smoothers, the class of median smoothers, based on order statistics, is the most popular. A new class of nonlinear smoothers, called LULU smoothers, was developed by using the minimum and maximum selectors. These smoothers have very attractive mathematical properties. In this thesis their statistical properties are investigated and compared to that of the class of median smoothers.

Smoothing, together with related concepts, are discussed in general. Thereafter, the class of median smoothers, from the literature is discussed. The class of LULU smoothers is defined, their properties are explained and new contributions are made. The compound LULU smoother is introduced and its property of variation decomposition is discussed. The probability distributions of some LULU smoothers with independent data are derived. LULU smoothers and median smoothers are compared according to the properties of monotonicity, idempotency, co-idempotency, stability, edge preservation, output distributions and variation decomposition. A comparison is made of their respective abilities for signal recovery by means of simulations. The success of the smoothers in recovering the signal is measured by the integrated mean square error and the regression coefficient calculated from the least squares regression of the smoothed sequence on the signal. Finally, LULU smoothers are practically applied.

Opsomming

Die gladstryking van tydreekse speel 'n baie belangrike rol in verskeie praktiese toepassings. Die beraming van die sein en die verwydering van ruis is die hoofdoel van gladstryking. Tradisioneel is lineêre gladstrykers gebruik, maar nielineêre gladstrykers het deur die jare meer gewild geword.

Mediaangladstrykers, gebaseer op orde statistieke, is die mees gewilde klas uit die familie van nielineêre gladstrykers. LULU-gladstrykers, 'n nuwe klas van nielineêre gladstrykers, is ontwikkel deur van die minimum en maksimum gebruik te maak. Hierdie gladstrykers besit baie aantreklike wiskundige eienskappe. In hierdie tesis word hul statistiese eienskappe ondersoek en met dié van die mediaanklas vergelyk.

Gladstryking, tesame met verwante konsepte, word in die algemeen bespreek. Daarna word die klas van mediaangladstrykers vanuit die literatuur bespreek. Die klas van LULU-gladstrykers word gedefinieer, hul eienskappe word verduidelik en nuwe bydraes word gemaak. Die saamgestelde LULU-gladstryker word voorgestel en sy eienskap van die opbreek van variasie word bespreek. Die waarskynlikheidsverdelings van 'n aantal LULU-gladstrykers met onafhanklike data word afgelei. LULU-gladstrykers en mediaangladstrykers word vergelyk ooreenkomstig die eienskappe van monotonisiteit, idempotensie, ko-idempotensie, stabiliteit, die minimering van randsteurings, die verdelings van uitsette en opbreking van die variasie. 'n Vergelyking van hulle onderskeie vermoëns om sein te herwin is met behulp van simulasies gedoen. Die sukses van die gladstrykers in die herwinning van die sein is met die geïntegreerde gemiddelde-kwadraatfout en die regressiekoëffisiënt van die kleinste kwadrate regressie van die gladgestrykte reeks op die sein, gemeet. Laastens is die LULU-gladstrykers prakties toegepas.

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Chapter 1

Introduction

Smoothing of a time series may be defined as the process of obtaining from an irregular sequence of data values of a dependent random variable, obtained over time, a corresponding smooth set of values consistent in a general way with the observed values. The ultimate goal of smoothing a time series is that this smooth set of values produced by the smoothing process estimates the **signal** of the time series as effectively as possible. Here a signal is defined as ***"the most useful relationship between the time varying random variable, generating the time series, and time"*** (Anderson, 1971). In this thesis the signal of a time series can be a **smooth continuous curve** (also referred to as a pattern) that does not jiggle too rapidly, i.e. each data point is similar to, or well supported by, the points in its vicinity. It can, however, also be a curve with blockpulses or shifts caused by jumps in the underlying ***"useful relationship"*** implying steep edges that need to be preserved. In contrast, with curve fitting, Brown (1963) states that in the first place a smoothing model for time series data should fit the current data well, while it is not important that it fits data obtained a long time ago. Secondly, the computational procedure in the case of smoothing must be fast and simple, for the process is essentially iterative, thus repeated with each new observation. In this regard Simonoff (1996) gives a very apt description of smoothing methods as methods that provide a bridge between making no assumptions on formal structure (a purely nonparametric approach) and making very strong assumptions (a parametric approach).

Traditionally, linear smoothers were used for smoothing, especially in engineering applications where digital signal processing based on linear smoothers often forms part of a hardware system. However, in contrast to their behaviour for data containing well-behaved Gaussian noise, linear smoothers do not respond well to data containing impulsive noise, outliers or noise from heavy tailed distributions (Pitas & Venetsanopoulos, 1990). In image processing applications, linear smoothers tend to blur the edges, do not remove impulsive noise effectively, and do not perform well in the presence of signal dependent noise. For these reasons nonlinear smoothers have been studied extensively in the literature and are regularly used in practice. They have been designed to meet criteria such as robustness, adaptability to noise probability distributions, preservation of edge information, and preservation of image details.

Based on his experience in the field of robustness, Tukey introduced median-based smoothers in the 1970s as a more robust smoothing technique (Tukey, 1971, 1977). Since then many extensions and modifications of median smoothers have been proposed by Wendt *et al.* (1986), Pitas & Venetsanopoulos (1990), Shmulevich & Arce (2001), Choi *et al.* (2001) and references contained therein. They have also been generalised to more general nonlinear ones based on a number of central order statistics, for example L-smoothers (Arce *et al.* , 1998; Barner & Arce, 1998). These median and other order statistic-based smoothers have been successful in a wide ranging spectrum of applications. See also the more general results on nonlinear smoothers by Mallows (1980) and Velleman (1980).

In the late 1980s and early 1990s a new class of nonlinear smoothers was introduced in the literature by Rohwer (1989). These smoothers are based on the extreme order statistics rather than on the central ones, and Rohwer named them LULU (Lower-Upper-Lower-Upper) smoothers. Rohwer (1989, 1999, 2002a,b, 2004a,b, 2005, 2006), Rohwer & Toerien (1991) and Rohwer & Wild (2002, 2007) in a series of papers showed that they have very attractive mathematical properties. Since their introduction they have been studied and applied fairly extensively, including in the engineering literature (Marquardt *et al.* , 1991; Kao, 2001). However, to date, their properties have only been studied in a deterministic setting and distribution theory based on random sequences has been lacking. Furthermore, they have not been applied in the traditional statistical world where analysing time series is one of the most applied statistical practices.

With this as background, the purpose of the current study is

1. to introduce LULU smoothers to statisticians
2. to propose and study some new LULU smoothers, called compound LULU smoothers
3. to derive and study distribution theory for LULU smoothers based on random sequences
4. to evaluate LULU smoothers and compare their performance with that of other well-known non-linear smoothers, especially the class of median smoothers
5. to illustrate the attractive properties of LULU smoothers by applying them to practical data sets
6. to indicate some open questions for further research

In order to conduct a meaningful evaluation of LULU smoothers and to compare them with other well-known linear smoothers, it is essential that smoothing and smoothers in general are understood.

Chapter 2 is reserved for this purpose. Terms used in smoothing will be defined and explained. Certain criteria used to evaluate smoothers, namely efficiency, monotonicity, idempotency, co-idempotency and stability are defined and discussed. Concepts such as an impulse, a blockpulse and an edge will be defined and illustrated. Statistical and distributional properties of smoothers are discussed. A new type of Winsorised smoother, the compound Winsorised smoother, will be introduced. Variation reduction, a valuable property of the compound smoother, is explained together with the shape preservation of a smoother. Finally, the handling of end-values and the classes of different smoothers, as found in the literature, will be summarised.

The issues regarding smoothing in general that are highlighted in Chapter 2 will be addressed throughout Chapters 3, 4, 5 and 6. These issues are:

- How is it determined whether a specific smoother is the "best" estimate of the unknown signal?
- Given (pure) signal as input, does a specific smoother change it, or does it recognise it as signal and preserve it?

- If a sequence of smoothers is applied consecutively, what is a reasonable stopping rule for the process?
- What is meant by ”smoothness” and how is it measured?
- What is the influence of different smoothers with a window of size ν operating on an n-monotone sequence?
- Does monotonicity play a role in producing a root of a smoother?
- How do different smoothers deal with outliers, blockpulses and edges?

In **Chapter 3** the class of median smoothers, the most well-known class of nonlinear smoothers found in the literature, is introduced. The most important theoretical and practical properties are given and discussed. The deterministic and statistical properties of this class, as found in the literature, are explained. The efficiency, monotonicity, idempotency, co-idempotency and stability of the median smoothers, and also the way median smoothers deal with impulses, blockpulses and edges, will be investigated. Finally, the modifications and extensions of the median smoother proposed in the literature are listed.

In **Chapter 4** the class of LULU smoothers is introduced. The most important theoretical results and properties will be given. The efficiency, monotonicity, idempotency, co-idempotency and stability of the LULU smoothers, and also the way LULU smoothers deal with impulses, blockpulses and edges are investigated. New smoothers based on the LULU smoothers will be suggested and some new distributional results are derived. The variation decomposition of the compound LULU smoothers is illustrated.

In **Chapter 5** LULU smoothers are compared with median smoothers with respect to

- monotonicity
- idempotency and co-idempotency
- stability
- statistical properties
- edge preservation properties
- variation reduction properties

In **Chapter 6** the success of LULU smoothers in recovering signal is compared to the nonlinear smoothers based on the median suggested by Tukey (1971) and Velleman (1975) by means of simulation studies. The first series that is studied is formed by constructing a sinusoidal curve for different frequencies and adding noise to it. A second study is where a blockpulse of length three is added to this sinusoidal curve. Thereafter, the influence of a positive and negative blockpulse on a negative and positive slope on the smoothers is investigated. In addition to this, LULU smoothers are also applied to a number of other interesting examples of series found in the literature, namely Blocks, Bumps, HeaviSine and Doppler. The sequence smoothed by the nonlinear smoothers is further smoothed by a linear smoother. Finally, the success of the procedure in recovering the signal is measured using different statistical measures.

In **Chapter 7** LULU smoothers are applied to two types of data. The first application uses financial data, while the second application comes from the medical field. The output of the median smoothers and the compound LULU smoothers applied to these time series will be compared and discussed. Decisions based on the calculated variation decomposition of the compound LULU smoothers will be illustrated.

In **Chapter 8** the work carried out in this thesis is summarised; the main contributions are highlighted and some areas for further research are suggested.

Chapter 2

Smoothing and smoothers in general

2.1 Introduction

As was stated in Chapter 1, one of the focuses of this study is to evaluate and study LULU smoothers and compare their performance with those of other well-known smoothers. In order to conduct a meaningful comparison of smoothers, it is essential to understand the terms ”smoothing” and ”smoother”, as well as relevant concepts related to smoothing. This chapter is hence reserved for smoothing in general, starting with a discussion of ”the art of smoothing” in Section 2.2.

As background, smoothing is explained by defining and describing concepts involved in the process of smoothing. These concepts have a direct influence on the process of smoothing a sequence of data, and are thus used to develop criteria to evaluate smoothers. In Section 2.3 these aspects are clarified. Definitions of time series, signal, noise, and a smoother are given. The concepts of effectiveness, consistency, idempotency, co-idempotency, stability and efficiency, which are criteria for the design and comparison of smoothers, are explained in detail with an emphasis on their role in the comparison of smoothers (Rohwer, 2005, p. 3). In Chapter 5 the behaviour of the smoothers dealt with in this thesis will be compared according to these properties. Monotone sequences and their behaviour in the process of smoothing are explained. The root of a smoother, as a sequence that is invariant to repeated smoothing, is defined. An edge, impulse, blockpulse and oscillation are defined, and the resistance and robustness of a smoother are defined and discussed. Some statistical and distributional properties of the smoother output are discussed as possible measures of the performance of a smoother. Total variation, which is used to illustrate variation reduction, is defined. The relevant definitions concerning trend preservation are given. Methods for the treatment of end-values of the sequences being smoothed are described.

Winsorised and compound smoothers are defined in general in Section 2.4. The theory given here is used to develop compound smoothers in Chapter 4. The latter have the very useful property of variation decomposition. In Section 2.5 classes of smoothers, e.g. linear and nonlinear smoothers, are summarised.

2.2 The art of smoothing

Smoothing is a process similar to curve fitting where there is a set of data to which some appropriate curve is to be fitted.

Data can generally be decomposed into a smooth component (fitted curve, trend) and a rough component (noise, residuals), i.e.

$$\textit{data} = \textit{smooth component} + \textit{rough component}.$$

Smoothing techniques attempt to separate the smooth component of data from the rough component. This reducing of the distracting effects of the noise leads to insight about the process (smooth component) that generates the data. Hence, some more specific goals of smoothing are

1. to find a functional model that describes the relationship between response and explanatory variables
2. to detect the underlying trend
3. to reduce outliers
4. to examine patterns in the residuals
5. to minimise the effect of aggregated values

One of the most challenging aspects of a smoothing process is its evaluation. In this regard, Anderson (1971) remarks that the variance of the smoother sequence should be small relative to the variance of the original sequence, which is an indication of reduced random error. In addition to this, Goodall (1990) discusses the following three aspects for evaluating different smoothers:

1. The trade-off between the effectiveness of a smoother in reducing noise and its faithfulness to an underlying pattern in the data (variance vs bias)
2. The smoother's response to outliers and abrupt changes of level (resistance properties at spikes and edges)
3. A smoother's ability to weaken rapidly varying noise components in the data while leaving slowly varying trend components unchanged (frequency domain analysis)

In Section 2.3 criteria for evaluating smoothers will be defined and explained with more detailed discussions in later chapters. To conclude, smoothing can be considered a very important tool in statistics because:

1. Smoothing is valuable since it leads to reduced noise and gives insight into functional relationships.
2. Smoothing suggests functional forms which should be incorporated into any fitting strategy.

2.3 Criteria for evaluating smoothers

2.3.1 Definitions

Definition 2.1. *A* ***time series*** *x is defined as a doubly-infinite numerical sequence:*

$$x = \{\ldots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \ldots\}$$

with x_i the observation at time i of a time varying random variable or phenomenon X.

When a time series x is observed, the assumption is made that the observed value is the sum of **signal** (measurable response to a stimulus) or **smooth**, and **noise** (components of measurement that interfere with the detection of signal) or **rough**. The signal and noise of a time series are defined below.

Definition 2.2. *The* ***signal*** *of a time series is defined as the most useful relationship between the time varying random variable generating the time series, and time (Anderson, 1971).*

In the smoothing of a time series it is assumed that the signal is a **smooth continuous curve** (also referred to as a pattern) that does not jiggle too rapidly, i.e. each data point is similar to or well supported by the points in its vicinity. This can happen in several ways: A data point can be about the same level as its neighbours or it can be consistent with changes at a steadily changing rate, the points on either side of a **jump** can be supported on one side and **peaks** can be supported by a consistent trend up and down from the top. Usually the curve is obscured by measurement error and other **noise**. There will be distinguished between Gaussian noise or well-behaved noise, and non-Gaussian noise.

Definition 2.3. ***Gaussian noise*** *is noise obtained by generating a sequence of independent, identical random observations from a Gaussian distribution.*

Definition 2.4. ***Non-Gaussian noise*** *is any noise not generated by a normal distribution, for example by long tail (extreme) distributions, and is observed in the form of extreme data points or groups of consecutive extreme data points or blockpulses. The extreme data points are not supported by the points in their vicinity and the groups of consecutive extreme data points are not supported by the points in their vicinity on either side; hence they are not considered part of the signal or well-behaved noise.*

In order to uncover the signal of a time series, i.e. to highlight the patterns that are present in the data, the distracting effects of the noise have to be reduced. This is done by transforming the observed series with a **smoother**, defined in general below.

Definition 2.5. *In general a* ***smoother*** $\mathbf{P}$ *is an algorithm that operates on x to produce a new series $\mathbf{P}x$ with $(\mathbf{P}x)_i$ the resulting smoothed value at time i.*

In this thesis most of the smoothers are algorithms that operate on a running window. These smoothers have an index n which determines the window size and they are defined as follows:

Definition 2.6. *The* ***smoother*** $\mathbf{P}_n$ *with index $n = 1, 2, 3, \ldots$ is defined as an algorithm that computes its output at index i from the running window, $W_i = \{x_{i-n}, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{i+n}\}$, of size $\nu = 2n + 1$ with output $(\mathbf{P}_n x)_i$. When $(\mathbf{P}_n x)_i \in W_i$, $\mathbf{P}_n$ is called an* ***order selector.***

This leads to the following definition of a **sequence of smoothers:**

Definition 2.7. *The sequence of smoothers $\{\mathbf{P}_n\}$ is defined as the smoothers $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \ldots$ operating on window sizes ν with increasing n.*

When a **time series** x is smoothed with a **smoother** $\mathbf{P}$ on x, a number of fundamental questions arises regarding signal, noise, smoothness of a smoothed series and the properties of smoothers. These issues follow from the fact that when smoothing is used to estimate signal, there is no strict set of mathematical or statistical rules and assumptions. A further complication is that some of the smoothing concepts and terminology are rather vague. In the remainder of this chapter the existing literature on these issues will be discussed and clarified. Where relevant, new insights and views will be added.

2.3.2 Effectiveness

Definition 2.8. *A smoother,* $\mathbf{P}$*, is* ***effective*** *if for each* x*,* $\mathbf{P}x$ *is signal and* $(\mathbf{I}-\mathbf{P})x = x - \mathbf{P}x$ *is noise (Rohwer, 2005, p. 3).*

Here I is the identity operator. The ultimate goal of developing smoothers is to produce an **effective smoother**. In practice, of course, it is unknown whether a smoother is effective, since the signal is unknown. A smoother can thus only be considered as an estimate of the signal. The goal is to find a good/best estimate of the signal. Given the signal, the noise can be obtained. In order to develop an effective smoother the following issues need to be addressed:

- How is it known that $\mathbf{P}x$ is in fact the "best" estimate of the unknown signal?
- Given (pure) signal as input, does a specific smoother change it, or does it recognise it as signal and preserve it?
- If a sequence of smoothers is applied consecutively, what is a reasonable stopping rule for this process?
- What is meant by "smoothness" and how is it measured?

These issues will be addressed in Chapter 6 where smoothers are applied to a known signal with noise added. The smoothed sequence is compared to the original signal using regression coefficients and the integrated mean square error.

2.3.3 Consistency

Definition 2.9. *A smoother is* ***consistent*** *if it preserves signal and maps noise onto 0 (Rohwer, 2005, p. 3).*

Consistency is closely related to idempotency and co-idempotency, defined as follows:

Definition 2.10. *A smoother,* $\mathbf{P}$*, is* ***idempotent*** *if for each* x*,* $\mathbf{P}(\mathbf{P}x) = \mathbf{P}x$ *and* ***co-idempotent*** *if* $(\mathbf{I}-\mathbf{P})^2x = (\mathbf{I}-\mathbf{P})x$*, i.e.* $(\mathbf{I}-\mathbf{P})x - \mathbf{P}(\mathbf{I}-\mathbf{P})x = (\mathbf{I}-\mathbf{P})x$ *and hence* $\mathbf{P}(\mathbf{I}-\mathbf{P})x = 0$*.*

Rohwer (2005, p. 4) calls an idempotent smoother **signal-consistent** and a co-idempotent smoother **noise-consistent**. When a smoother is thus idempotent and also co-idempotent it means that when the output of the smoother is passed through the smoother a second time, no further smoothing takes place and it could be argued that according to the specific smoother, the signal has been identified. **An idempotent smoother is thus consistent in the sense that it defines its own output as the signal, recognises it as the signal and does not smooth it further, but preserves it.** Furthermore, if the smoother is also co-idempotent, the output is zero when the noise of the smoother is passed through the smoother a second time, i.e. $\mathbf{P}(\mathbf{I}-\mathbf{P})x = 0$. However, the question of whether the output of an idempotent smoother, $\mathbf{P}$, should be considered as the signal if some other idempotent smoother (say) $\mathbf{Q}$, smooths the output of $\mathbf{P}$, remains unanswered here and needs to be addressed.

As already mentioned above, most of the smoothers in this thesis are algorithms that operate on a running window. It will also be seen that the larger the window size, the smoother the output, resulting in $\mathbf{P}_{n+1}(\mathbf{P}_n x) \neq \mathbf{P}_n x$. Thus, when sequences of smoothers are evaluated, two issues are at stake:

1. The smoothing effect of the increasing window size, i.e. $\mathbf{P}_n$ vs $\mathbf{P}_{n+k}$, for $n, k = 1, 2, 3, \ldots$.
2. The difference between the smoothing effects of different sequences of smoothers $\{\mathbf{P}_n^m\}$ with $\mathbf{P}_n^m$ the n-th smoother of the m-th sequence operating on the window $W_i = \{x_{i-n}, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{i+n}\}$, of size $\nu = 2n + 1$ with output $(\mathbf{P}_n^m x)_i$.

This further complicates the issue that $\mathbf{P}_n^m x$ is defined as the signal by the idempotent smoother $\mathbf{P}_n^m$, but it is not necessarily defined as the signal by $\mathbf{P}_{n+k}^m$, or by $\mathbf{P}_n^{m+s}$, or $\mathbf{P}_{n+k}^{m+s}$. So an idempotent and co-idempotent smoother is not necessarily consistent just from the fact that $\mathbf{P}(\mathbf{P}x) = \mathbf{P}x$ and $(\mathbf{I} - \mathbf{P})^2 x = (\mathbf{I} - \mathbf{P})x$. It is necessary to distinguish between the idempotency and co-idempotency of a single smoother $\mathbf{P}$ and the idempotency and co-idempotency of a sequence of smoothers $\{\mathbf{P}_n\}$. This leads to the following definitions:

Definition 2.11. *A sequence $\{\mathbf{P}_n\}$ is signal-consistent if there is a k_P such that $\mathbf{P}_{k_P}\mathbf{P}_{k_P-1}\ldots\mathbf{P}_1 x = signal$ and $\mathbf{P}_{k_P+1} signal = \mathbf{P}_{k_P+1}\mathbf{P}_{k_P}\mathbf{P}_{k_P-1}\ldots\mathbf{P}_1 x = signal$.*

If $\{\mathbf{P}_n\}$ and $\{\mathbf{Q}_n\}$ are two consistent sequences, $\mathbf{P}_n$ is better than $\mathbf{Q}_n$ at x if $k_P \leq k_Q$. Note that k depends on the particular series x .

Definition 2.12. *A sequence of smoothers $\{\mathbf{P}_n\}$ is idempotent if each individual smoother of the sequence is idempotent, i.e. if for each x, $\mathbf{P}_n(\mathbf{P}_n x) = \mathbf{P}_n x$ for all n and co-idempotent if each individual smoother of the sequence is co-idempotent, i.e. if for each x, $(\mathbf{I} - \mathbf{P}_n)^2 x = (\mathbf{I} - \mathbf{P}_n)x$ for all n.*

When selecting a smoother the modus operandi should be to consider sequences of idempotent and co-idempotent smoothers. However, to select the "best" sequence and hence the "best" individual smoother given the best sequence, some additional criteria and theory to evaluate the effectiveness and consistency are needed. This suggests that a measure that measures the amount of roughness that is smoothed away at each increase of the window size of a **smoothing sequence** and, in addition to this, some sort of stopping rule for n are needed. This issue will be addressed in Section 2.3.10, where the variation of smoothers and the reduction of the total variation in a series are studied.

2.3.4 Monotonicity and the root of a smoother

Definition 2.13. *An n-**monotone sequence** (also called a LOMO($n + 2$) sequence) is defined as a sequence such that the subset $(x_i, x_{i+1}, \ldots, x_{i+n+1})$ is monotone increasing or monotone decreasing for each i. The set of sequences that are n-monotone is denoted by $\mathcal{M}_n$ (Rohwer, 2005, pp. 22, 24).*

Note that for any sequence x, $x \in \mathcal{M}_0$, and a sequence of nested subsets is formed by $\mathcal{M}_0 \supset \mathcal{M}_1 \supset \cdots \supset \mathcal{M}_n \supset \cdots$.

Definition 2.14. *A **root** of a smoother $\mathbf{P}$ is a sequence x such that $\mathbf{P}x = x$ (Arce* et al. *, 1986).*

From Definitions 2.10 and 2.13 it follows that if a smoother $\mathbf{P}$ is idempotent, it produces its own root after just one pass, since $\mathbf{P}(\mathbf{P}x) = \mathbf{P}x$. Thus, if a smoother $\mathbf{P}$ is idempotent and a smoother $\mathbf{Q}$ needs more than one pass to produce its root, i.e. $\mathbf{Q}$ is not idempotent, $\mathbf{P}$ may be considered a "better" smoother than $\mathbf{Q}$. It may be concluded that an idempotent smoother **converges** to its root within one pass.

From a smoothing perspective it is important to study the monotonicity of a sequence since increased monotonicity means increased lengths of subsets of consecutive points that are either increasing and/or remain constant, or that are decreasing and/or remain constant. **Monotonicity is thus a measure of smoothness of a sequence in the sense that it gives an indication of the minimum length of consecutive monotone subsets of the sequence**. It thus follows that when different smoothers are considered, the way in which they deal with the monotonicity of sequences needs to be understood. In this regard the following questions arise:

- What is the influence of different smoothers operating on a window of size ν on an m-monotone sequence? If it is known, for example, that the passing of any sequence through a smoother with window size ν produces a sequence that is $f(\nu)$-monotone, it is known to what extent the smoother smoothes the original sequence.
- Does monotonicity play a role in producing a root of a smoother? In other words, is there a relationship between the window size ν of a smoother $\mathbf{P}$ and the monotonicity of the sequence x if $\mathbf{P}x = x$? If this relationship is known, the circumstances under which a smoother has no smoothing effect are known.

It is thus clear that knowing the monotonicity of a sequence before and after passing through a smoother is very important in the evaluation of a smoother. In addition to this, the rate of convergence of a smoother to its root is of further importance.

These questions are addressed in Chapters 3, 4 and 5 in the sections where the influence of the monotonicity of a sequence on the different smoothers is discussed.

2.3.5 Stability

Definition 2.15. *A smoother is **stable** or **resistant** when small proportions of large input perturbations do not distort the output excessively (Rohwer, 2005, p. 3).*

The property of stability is of particular importance when a series is subjected to non-Gaussian noise. It is well known that linear smoothers perform poorly, especially when the non-Gaussian noise is in the form of impulsive noise. Hence, it can be said that linear smoothers are not stable. Nonlinear smoothers, on the other hand, have been designed in particular to meet criteria such as, *inter alia*, stability. The stability property of the nonlinear smoothers in this thesis will receive attention in several discussions throughout the thesis.

The concept of robustness is closely related to the stability of a smoother and its definition is as follows:

Definition 2.16. *A smoother is considered **robust** if the signal is recovered well through smoothing of a sequence where noise, e.g. Gaussian and/or non-Gaussian (long-tailed, or spiky), was added to the signal (Velleman, 1980).*

A good smoother should be robust to noise.

The stability of a smoother also depends on the way it deals with an impulse, blockpulse and oscillation, and thus these concepts are discussed in the following section.

2.3.6 An impulse and blockpulse

Definition 2.17. *A **constant neighbourhood** is a region consisting of a number of consecutive points, all of which have identical values (Pitas & Venetsanopoulos, 1990).*

Definition 2.18. *An **impulse** is a constant neighbourhood followed by at least one point, but no more than a finite number of points, that has another value to the constant neighbourhood, which then is followed by another constant neighbourhood having the same value as the first constant neighbourhood (Pitas & Venetsanopoulos, 1990).*

The goal of smoothing is to eliminate impulses (or spikes) of short length, and thus a smoother can be evaluated by the way it treats impulses. It is thus important to know how a specific smoother deals with impulses.

Impulses of longer length are called blockpulses, which can be defined as follows (Rohwer, 2005, p. 15):

Definition 2.19. *An n-**blockpulse** is a sequence x such that*

$$x = \{\ldots\ 0, b_1, b_2,\ \ldots, b_n, 0,\ \ldots\}$$

with $b_1 = b_2 = \ \ldots\ = b_n = b$ and infinitely many zeros on both sides.

It is called upward if it is nonnegative and downward if it is negative.

In practice a blockpulse appears in a time series when an n-blockpulse is added to a general monotone sequence. This blockpulse in a time series is usually the result of an important event in the sequence, and a good smoother should deal with it in a consistent and known way.

2.3.7 Efficiency

Definition 2.20. *A smoother is **efficient** if the computations are economical (Rohwer, 2005, p. 3).*

Given the powerful hardware and software available today, this property might not be of great importance, but is crucial in real-time applications.

2.3.8 An edge

Definition 2.21. *An **edge** is a monotone increasing or decreasing set of points surrounded on both sides by constant neighbourhoods (Pitas & Venetsanopoulos, 1990).*

Two types of edges, as illustrated in Figure 2.1, can be identified:

- A **step edge** of height h is present if the observations have values a and $a + h$ on the two edge sides respectively.
- A **ramp edge** is present if the observations increase monotonically from level a to level $a + h$.

When an edge is mentioned throughout the thesis, it is being referred to a step edge. A good smoother should preserve an edge.

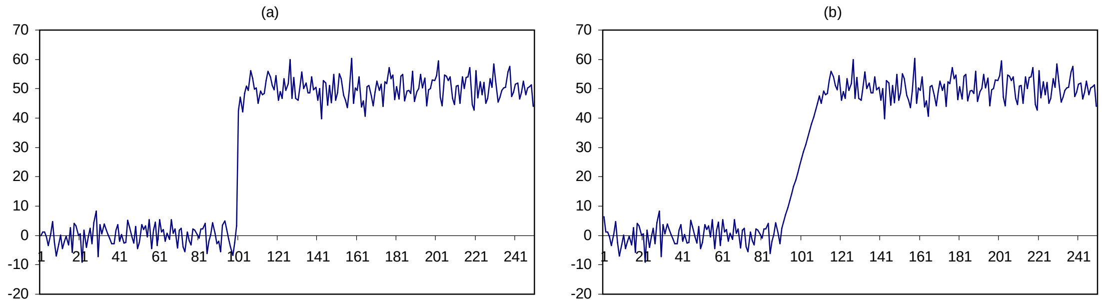

Figure 2.1: (a) Step edge with added Gaussian noise and (b) Ramp edge with added Gaussian noise

2.3.9 Statistical and distributional properties

In order to understand the statistical behaviour of smoothers, and to compare different smoothers, it is neccessary to derive and study the distributions and statistical properties of their output, given certain types of input. Of particular interest is the variation of these distributions as measured by the variance or standard deviation, and the total variation (see Section 2.3.10), since these measures are one way of determining how a smoother reduces the distracting effects of the noise. When the distributions of the output of smoothers are derived and the statistical properties are studied, the following aspects need to be considered:

- **Statistical dependence or independence of the input sequence.** The most simple case to consider is the case where the input sequence x consists of independent, identically distributed (iid) random variables, each with distribution function F_X. The distribution of the median smoother in this case is well known and it is given in Chapter 3. In Chapter 4 the distributions of some LULU smoothers are derived. The distributions of the more complex LULU smoothers are complicated and are very messy to derive. The case of independent, non-identically distributed components is also manageable. In the case of dependent observations, the derivation of the distributions becomes analytically very demanding, if at all possible. Much work has still to be done in this regard.
- **Distribution of input sequence.** In Chapters 3 and 4 distributions of the output of the median smoother and LULU smoothers are given and derived respectively for chosen distributions of input sequences. In this study, the distributions of LULU smoothers' output, when the distributions of the input sequences are long-tailed distributions, are of special interest, since nonlinear smoothers, and in particular LULU smoothers, deal with these types of input sequence in a very constructive and measurable way.

The derivation of the statistical and distributional properties of smoothers is thus not just a theoretical exercise, but is part of a process to understand the behaviour of smoothers better, in order to make an informed practical decision when different smoothers are compared.

2.3.10 Variation reduction and shape preservation

In this section the underlying definitions and results building up to the very important decomposing property of fully trend preserving smoothers are given, explained and discussed.

Definition 2.22. *The* ***total variation*** *of a sequence x is defined as*

$$T(x) = \| \vartriangle x\|_1 = \sum_{i=-\infty}^{\infty} |x_{i+1} - x_i|. \tag{2.1}$$

For

$$x \in l_1, \text{ i.e. } \|x\|_1 = \sum_{i=-\infty}^{\infty} |x_i| < \infty,$$

it follows that

$$\sum_{i=-N}^{N} |x_{i+1} - x_i| = \sum_{i=-N}^{N} |(\vartriangle x)_i| \leq \sum_{i=-N}^{N} (|x_{i+1}| + |x_i|) \leq 2\|x\|_1 < \infty$$

and hence $\vartriangle x \in l_1$.

In Rohwer (2002b) and Rohwer (2005, pp. 52, 57, 58) a number of very useful results, given in the following theorems and definitions, regarding $T(x)$ and shape preservation, are proved.

Theorem 2.1.

1. *$T(Ex) = T(x)$ where E is the forward shift operator ($Ex_i = x_{i+1}$).*
2. *$T(x + y) \leq T(x) + T(y)$.*
3. *$T(\alpha x) = |\alpha| T(x)$ and $T(x) = 0 \Leftrightarrow x = 0$.*

Definition 2.23. *A smoother $\mathbf{P}_n$ is* ***trend preserving*** *if $(\mathbf{P}_n x)_i = x_i$ whenever $x_{i-r}, \ldots, x_i, \ldots, x_{i+r}$ is monotone and $n \geq r$.*

Definition 2.24. *A smoother $\mathbf{P}_n$ is* ***global sequence preserving*** *if $(\mathbf{P}_n x) = x$ whenever x is n-monotone.*

From Definition 2.14 it follows that this definition implies that, if a smoother is global sequence preserving, the input sequence is a root of the smoother.

Definition 2.25. *A smoother $\mathbf{P}$ is called* ***neighbour trend preserving*** *if, for each sequence x, it follows that $x_{i+1} \leq x_i \Longrightarrow (\mathbf{P}x)_{i+1} \leq (\mathbf{P}x)_i$ and $x_{i+1} \geq x_i \Longrightarrow (\mathbf{P}x)_{i+1} \geq \mathbf{P}x_i$ for each index i.*

A neighbour trend preserving smoother carries the ordering of neighbours in the input sequence over to neighbours in the output sequence, hence the name **neighbour trend preserving**.

Definition 2.26. *A smoother $\mathbf{P}$ is* ***difference reducing*** *if $|(\mathbf{P}x)_{i+1} - (\mathbf{P}x)_i| \leq |x_{i+1} - x_i|$ for each sequence x and each index i.*

A difference reducing smoother is thus also a total variation decreasing smoother.

Definition 2.27. *A smoother* $\mathbf{P}$ *is* ***fully trend preserving*** *if it is neighbour trend preserving and difference reducing.*

A fully trend preserving smoother preserves the ordering of the input sequence but reduces the consecutive differences of the input sequence and hence reduces the total variation. The following theorems, proved in Rohwer (2002b), are given:

Theorem 2.2. *All* ***compositions*** *of fully trend preserving operators are fully trend preserving.*

Theorem 2.3. *Let* $\mathbf{P}$ *be a fully trend preserving smoother, then* $T(x) = T(\mathbf{P}x) + T(x - \mathbf{P}x)$.

The last result implies that for a fully trend preserving smoother the total variation can be decomposed into a part explained by the smoother (signal as defined by the smoother) and a part resulting from the rough (noise defined by the smoother) that is removed from the sequence. Variation decomposition is a very important result since it provides an elegant smoothing algorithm that can be applied as follows:

Let $\mathbf{P}_n$ be a fully trend preserving smoother, for each $n = 1, 2, 3, \ldots$, then

$$\begin{aligned}
T(x) &= T(\mathbf{P}_1x) + T(x - \mathbf{P}_1x) \\
&= T(\mathbf{P}_2\mathbf{P}_1x) + T(\mathbf{P}_1x - \mathbf{P}_2\mathbf{P}_1x) + T(x - \mathbf{P}_1x) \\
&= T(\mathbf{P}_3\mathbf{P}_2\mathbf{P}_1x) + T(\mathbf{P}_2\mathbf{P}_1x - \mathbf{P}_3\mathbf{P}_2\mathbf{P}_1x) + T(\mathbf{P}_1x - \mathbf{P}_2\mathbf{P}_1x) + T(x - \mathbf{P}_1x) \\
&= \ldots\ldots\ldots\ldots\ldots \\
&= T(\mathbf{P}_k\mathbf{P}_{k-1}\ldots\mathbf{P}_3\mathbf{P}_2\mathbf{P}_1x) + T(\mathbf{P}_{k-1}\ldots\mathbf{P}_3\mathbf{P}_2\mathbf{P}_1x - \mathbf{P}_k\mathbf{P}_{k-1}\ldots\mathbf{P}_3\mathbf{P}_2\mathbf{P}_1x) + \ \ldots \\
&\quad + T(\mathbf{P}_2\mathbf{P}_1x - \mathbf{P}_3\mathbf{P}_2\mathbf{P}_1x) + T(\mathbf{P}_1x - \mathbf{P}_2\mathbf{P}_1x) + T(x - \mathbf{P}_1x) \\
&= T(\mathbf{R}_kx) + T(\mathbf{R}_{k-1}x - \mathbf{R}_kx) + \ldots + T(\mathbf{R}_1x - \mathbf{R}_2x) + T(x - \mathbf{R}_1x)
\end{aligned} \tag{2.2}$$

with

$$\mathbf{R}_i = \mathbf{P}_i\mathbf{P}_{i-1}\ldots\mathbf{P}_3\mathbf{P}_2\mathbf{P}_1.$$

Smoothing successively by $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_k$ yields a monotone reduction of $T(x)$. As soon as large reductions in variation stop, it could be argued that the noise has been adequately removed. At each window size the variation that is removed is calculable, and therefore indicates the contribution by noise at that window size. It is clear that the successive application of $\mathbf{P}_1, \mathbf{P}_2, \ldots$ implies a successive trimming (peeling off) of the series that can be monitored by means of the variation that is removed at each level. **If a smoother is thus fully trend preserving, this algorithm provides a very informative smoothing procedure to estimate the signal of a given sequence, since it gives a well-based theoretical method to measure the additional noise, as measured by total variation, that is removed at each window size**. This coincides with the argument that when no further noise, as measured by the total variation, can be removed, the signal should have been reached. It also addresses to some extent the issues discussed in Section 2.3.3.

This procedure will be further explored in Chapters 4 and 5, since Rohwer (2002b) has shown that LULU smoothers are fully trend preserving. This property is not shared by the median smoothers or other well-known nonlinear smoothers.

2.3.11 End-values

Handling end-values is a problem for all smoothing methods using a running window. Methods like the moving average (mean), running median, and LULU have to make use of a rule to treat end-values, because the values at the beginning and end of the input sequence are not preceded or succeeded by enough other values to make up the window size. If nothing is done to extend the original sequence, values are lost at the end points of the smoothed sequence. The number of values lost at the beginning and end of the sequence depends on the window size.

The choice of the way to treat the end-values may influence the smoothed output sequence, especially for short sequences. There are no guidelines for which method to choose.

The different approaches found in the literature for dealing with the ends, with x the original sequence and y the smoothed sequence, are summarised as follows:

1. Replicate end-value rule

   This is the most common approach where the first observation, x_1, and the last observation, x_L, of the original series are copied n times for a window size of $v = 2n+1$. The original sequence is thus extended, and $2n$ values added. For a sequence of finite length (L), the extended sequence has by definition two monotonic subsequences of length $n+1$, namely $x_{-n+1}, \ldots, x_1$ and $x_L, \ldots, x_{L+n}$ where $x_{1-i} = x_1$, $i = 1, \ldots, n$ and $x_{L+i} = x_L$, $i = 1, \ldots, n$. A smoothed value can be obtained for each of the original observations $x_1, \ldots, x_L$, namely $y_1, \ldots, y_L$. There is no loss of values in the smoothed sequence.

2. Copy-on end-value rule

   This method starts smoothing at those original observations for which the full window size, $v = 2n+1$, applies. The smoothed sequence will result in a loss of values, namely n values at the beginning of the sequence and n values at the end of the sequence. The end-values which were not smoothed are just copied from the nearest observation in the original sequence. The smoothed sequence will be $x_1, \ldots, x_n, y_{n+1}, \ldots, y_{L-n}, x_{L-n+1}, \ldots, x_L$.

3. Extrapolation end-value rule

   This is a rule introduced by Tukey called Tukey's extrapolation end-value rule (Tukey, 1977). The method uses current smoothed values in a linear extrapolation to construct an additional observation beyond the end of the sequence. The second and third smoothed values are extrapolated linearly to construct a 0-th observation $\hat{y}_0$. If y_2 and y_3 are the already smoothed values, and for equally spaced data with t-spacing Δt, the line at the lower end has slope

   $$\frac{y_3 - y_2}{\Delta t}. \tag{2.3}$$

   The extrapolation is two t-intervals beyond y_2, and the estimated value is

   $$\begin{aligned} \hat{y}_0 &= y_2 - 2\Delta t(y_3 - y_2)/\Delta t \\ &= 3y_2 - 2y_3. \end{aligned} \tag{2.4}$$

Similarly, for the last value of a sequence of L observations, a succeeding point is estimated as

$$\hat{y}_{L+1} = 3y_{L-1} - 2y_{L-2}. \tag{2.5}$$

The median of the extrapolated point, the observed endpoint, and the smoothed point next to the end will be used as the smoothed end-value.

Thus, the first smoothed value is

$$y_1 = med(\hat{y}_0, x_1, y_2), \tag{2.6}$$

and the last smoothed value is

$$y_L = med(\hat{y}_{L+1}, x_L, y_{L-1}). \tag{2.7}$$

4. Cyclic end-value rule

   Sometimes the data seem to follow a natural cyclical pattern where the original values are measured around a *clockface*, so that the first and last values are adjacent. This phenomenon appears in seasonal data where a seasonal adjustment, e.g. the month to month changes in the data averaged across several years after removing the long-term trend, has to be computed. To find a smoothed value for the first observation, values from the end of the sequence are used to match the window size. For example, using a window size of five, the smoothed value of the first observation will be calculated using the observations x_{L-1}, x_L, x_1, x_2, x_3. The data can be visualised in a circle with the first and last values adjacent.

5. Step-down end-value rule

   For this method the window size is decreased near the end of the sequence. This method starts smoothing the values where the full window size applies. Moving towards the ends, the following or previous value can be smoothed by applying the same smoother for a smaller window size. This process continues towards the ends until the last values at each end remain. For the first and last values just copy the original end-values.

   In the literature it was found that the step-down rule is used in conjunction with the extrapolation end-value rule discussed in 3.

6. Omit end-value rule

   The smoothed sequence is shorter than the original sequence due to the window size. The endpoints do not have corresponding smoothed output values. These endpoints are omitted and the original sequence is shortened according to the smoothed sequence.

From the literature it is found that the replicate end-value rule is most commonly used in signal processing. The step-down rule, followed by Tukey's extrapolation end-value rule, is commonly used in exploratory data analysis (Goodall, 1990).

2.4 Winsorised and compound smoothers

In this section, Winsorised and compound smoothers, both of which are types of trimmed smoothers, are defined and discussed.

In this thesis **lower smoothers** and **upper smoothers**, defined below, will be studied frequently.

Definition 2.28. ***Lower*** *and* ***upper*** *smoothers, denoted respectively as* $\mathbf{P}_{lj}$ *and* $\mathbf{P}_{uj}$*, are such that for any sequence* x, $\mathbf{P}_{lj}x \leq \mathbf{P}_{uj}x$.

Mathematical morphology is used in image processing. The morphological representation of images is used for the description of the geometrical properties of the image objects (Pitas & Venetsanopoulos, 1990). The definition of the morphological centre will be given in order to understand the concept of a Winsorised smoother (Rohwer, 2005, p. 62).

Definition 2.29. *The morphological centre of two smoothers* $\boldsymbol{G}$ *and* $\boldsymbol{F}$ *with* $\mathbf{G}x \leq \mathbf{F}x$ *is defined as*

$$\mathbf{B} = (\mathbf{F} \wedge \mathbf{I}) \vee \mathbf{G} = (\mathbf{F} \vee \mathbf{G}) \wedge (\mathbf{I} \vee \mathbf{G}) = \mathbf{F} \wedge (\mathbf{I} \vee \mathbf{G})$$

with

$$((\mathbf{F} \vee \mathbf{G})\, x)_i = (\mathbf{F}x)_i \vee (\mathbf{G}x)_i = \max\{(\mathbf{F}x)_i\,,\ (\mathbf{G}x)_i\}$$

and

$$((\mathbf{F} \wedge \mathbf{G})\, x)_i = (\mathbf{F}x)_i \wedge (\mathbf{G}x)_i = \min\{(\mathbf{F}x)_i\,,\ (\mathbf{G}x)_i\}.$$

In Theorem 2.4 below it will be proved that the Winsorised smoother, defined below, is in fact the morphological centre of two smoothers $\mathbf{P}_{lj}$ and $\mathbf{P}_{uj}$.

Definition 2.30. *A* ***Winsorised smoother,*** $\mathbf{W}_j$*, based on the upper and lower smoothers* $\mathbf{P}_{lj}$ *and* $\mathbf{P}_{uj}$*, has output:*

$$(\mathbf{W}_j x)_i = \begin{cases} x_i & \text{if } x_i \in \left[\left(\mathbf{P}_{lj}x\right)_i, \left(\mathbf{P}_{uj}x\right)_i\right], \\ \left(\mathbf{P}_{lj}x\right)_i & \text{if } x_i < \left(\mathbf{P}_{lj}x\right)_i, \\ \left(\mathbf{P}_{uj}x\right)_i & \text{if } x_i > \left(\mathbf{P}_{uj}x\right)_i. \end{cases} \tag{2.8}$$

The following result holds for a Winsorised smoother.

Theorem 2.4. *The Winsorised smoother* $\mathbf{W}_j$ *is the morphological centre of the smoothers* $\mathbf{P}_{lj}$ *and* $\mathbf{P}_{uj}$.

Proof

Consider

$$\begin{aligned} \left(\left(\mathbf{P}_{uj} \wedge \mathbf{I}\right) x\right)_i &= \left(\mathbf{P}_{uj}x\right)_i \wedge (\mathbf{I}x)_i \\ &= \min\{\left(\mathbf{P}_{uj}x\right)_i,\ (\mathbf{I}x)_i\} \\ &= \begin{cases} \left(\mathbf{P}_{uj}x\right)_i & \text{if } x_i > \left(\mathbf{P}_{uj}x\right)_i \\ x_i & \text{if } x_i < \left(\mathbf{P}_{uj}x\right)_i \end{cases} \\ &= (\mathbf{H}x)_i \ \text{(say)}. \end{aligned}$$

Next

$$
\begin{aligned}
\left(\left(\left(\mathbf{P}_{uj}\wedge\mathbf{I}\right)\vee\mathbf{P}_{lj}\right)x\right)_i &= \left(\left(\mathbf{H}\vee\mathbf{P}_{lj}\right)x\right)_i \\
&= \max\{(\mathbf{H}x)_i\,,\ \left(\mathbf{P}_{lj}x\right)_i\} \\
&= \begin{cases} \left(\mathbf{P}_{lj}x\right)_i & \text{if } \left(\mathbf{P}_{lj}x\right)_i > (\mathbf{H}x)_i \\ (\mathbf{H}x)_i & \text{if } \left(\mathbf{P}_{lj}x\right)_i < (\mathbf{H}x)_i \end{cases} \\
&= \begin{cases} \left(\mathbf{P}_{lj}x\right)_i & \text{if } x_i < \left(\mathbf{P}_{lj}x\right)_i \\ x_i & \text{if } \left(\mathbf{P}_{lj}x\right)_i \leq x_i \leq \left(\mathbf{P}_{uj}x\right)_i \\ \left(\mathbf{P}_{uj}x\right)_i & \text{if } x_i > \left(\mathbf{P}_{uj}x\right)_i \end{cases} \\
&= (\mathbf{W}_j x)_i.
\end{aligned}
$$

Hence, $\mathbf{W}_j \equiv \left(\mathbf{P}_{uj}\wedge\mathbf{I}\right)\vee\mathbf{P}_{lj}$, is the morphological centre of $\mathbf{P}_{lj}$ and $\mathbf{P}_{uj}$.

The following new general smoother, the compound smoother, is defined. In Chapter 4 it will be extended to form a new LULU smoother defined for the first time, the compound LULU smoother.

Definition 2.31. *A **compound smoother** is defined as:*

$$
(\mathbf{V}_j x)_i = \begin{cases} (\mathbf{V}_{j-1}x)_i & \textit{if } (\mathbf{V}_{j-1}x)_i \in \left[\left(\mathbf{R}_{lj}x\right)_i, \left(\mathbf{R}_{uj}x\right)_i\right], \\ \left(\mathbf{R}_{lj}x\right)_i & \textit{if } (\mathbf{V}_{j-1}x)_i < \left(\mathbf{R}_{lj}x\right)_i, \\ \left(\mathbf{R}_{uj}x\right)_i & \textit{if } (\mathbf{V}_{j-1}x)_i > \left(\mathbf{R}_{uj}x\right)_i, \end{cases} \tag{2.9}
$$

with

$$
\begin{aligned}
\mathbf{R}_{lj} &= \mathbf{P}_{lj}\mathbf{P}_{l(j-1)}\ldots\mathbf{P}_{l3}\mathbf{P}_{l2}\mathbf{P}_{l1} \qquad \textit{and} \\
\mathbf{R}_{uj} &= \mathbf{P}_{uj}\mathbf{P}_{u(j-1)}\ldots\mathbf{P}_{u3}\mathbf{P}_{u2}\mathbf{P}_{u1}.
\end{aligned}
$$

Hence, $\mathbf{V}_0 = \mathbf{W}_0 = \mathbf{I}$, $\mathbf{V}_1 = \mathbf{W}_1$, $\mathbf{V}_2 = \mathbf{W}_2\mathbf{W}_1$, $\mathbf{V}_3 = \mathbf{W}_3\mathbf{W}_2\mathbf{W}_1\ldots$, $\mathbf{V}_k = \mathbf{W}_k\ldots\mathbf{W}_3\mathbf{W}_2\mathbf{W}_1$. It is clear that if $\mathbf{P}_{lj}$ and $\mathbf{P}_{lj}$ are idempotent and co-idempotent, $\mathbf{W}_j$ and $\mathbf{V}_j$ are also idempotent and co-idempotent. It is further trivial to prove that if $\mathbf{P}_{lj}$ and $\mathbf{P}_{uj}$ are fully trend preserving, then $\mathbf{W}_j$ and $\mathbf{V}_j$ are also fully trend preserving and hence:

$$
\begin{aligned}
T(x) &= T(\mathbf{W}_j x) + T(x - \mathbf{W}_j x) \qquad \text{and} \\
T(x) &= T(\mathbf{V}_j x) + T(x - \mathbf{V}_j x),
\end{aligned}
$$

so that from (2.2) follows for the compound smoother that

$$
T(x) = T(\mathbf{V}_k x) + T(\mathbf{V}_{k-1}x - \mathbf{V}_k x) + \ldots + T(\mathbf{V}_1 x - \mathbf{V}_2 x) - T(x - \mathbf{V}_1 x). \tag{2.10}
$$

The smoother $\mathbf{V}_j$ is a type of trimmed smoother where the part of the original series lying between $(\mathbf{R}_{lj}x)_i$ and $(\mathbf{R}_{uj}x)_i$ that was not smoothed by $(\mathbf{V}_r x)$, for $r < j$, is preserved. As in (2.2), the successive application of $\mathbf{V}_1$, $\mathbf{V}_2$, ... implies a successive trimming (peeling off) of the series that can be monitored by means of the variation that is removed at each step, while a certain part of the original series is preserved. The variation removed at each level can be calculated. The question is to what extent $\mathbf{V}_j$ preserves ”well-behaved” noise plus signal and removes ”non-well-behaved” noise. As soon as large reductions in variation stop, it could be argued that the non-well-behaved noise has been adequately removed and then a linear smoother can be applied to remove the remaining well-behaved noise. Although the terms *large reductions* and *adequately removed* remain vague, the above algorithm gives a procedure that can be explored further. Note that the Winsorised smoother does not share this property of the decomposition of the variation.

In Chapters 4 and 5 these issues will again be addressed when LULU smoothers are studied.

2.5 Classes of smoothers

Smoothers fall into one of two basic classes (categories), **linear** and **nonlinear** smoothers. **Linear** smoothers can be expressed as linear functions of the data and include the moving averages, kernel smoothers, local polynomial smoothers, *loess*, splines and kriging. **Nonlinear** smoothers include running medians, other median-based smoothers and LULU smoothers introduced by Rohwer (1989).

2.5.1 Linear smoothers

Linear smoothers can be defined as a convolution of the observations with a specific constant sequence α (Rohwer, 2005, p. 109):

Definition 2.32. *A sequence y is the (discrete) convolute of sequences x and α if and only if*

$$y_i = \sum_{j=-\infty}^{\infty} \alpha_j x_{i-j}. \tag{2.11}$$

For a window size $\nu = 2n + 1$, (2.11) becomes

$$y_i = \sum_{j=-n}^{n} \alpha_j x_{i-j}.$$

These are called nonrecursive smoothers and appear under the names ”finite response (FIR) smoothers”, ”transversal smoothers”, ”tapped delay line smoothers”, or ”MA-smoothers” in the literature.

Linear smoothers can also be defined by means of a recursion where already smoothed values are used in the window. With α, β constant sequences, the recursive smoothers are defined as

$$y_i = \sum_{j=-\infty}^{\infty} \alpha_j x_{i-j} + \sum_{j=-\infty}^{\infty} \beta_j y_{i-j}. \tag{2.12}$$

This is an implicit definition of y, and the recursively calculable version, where β_j is chosen 0 for j nonpositive, is generally used. These recursive smoothers are known as ”infinite impulse response (IIR) smoothers”, ”ladder smoothers”, ”lattice smoothers”, ”ARMA-smoothers”, and ”ARIMA-smoothers”.

Smoothers can be designed to possess various properties. *Low-pass* smoothers are designed to remove high frequency noise, while *high-pass* smoothers remove low frequency noise, and *bandpass* smoothers remove noise outside a certain range of frequencies. Each of these smoothers is used according to the purpose of the smoothing process, e.g. if it were required to eliminate high frequency components, a low-pass smoother would apply.

There are several shortcomings when using linear smoothers to smooth sequences with extreme data values present such as spikes, or impulses. Linear smoothers are not resistant, for example an outlier is smeared across several surrounding data values. Linear smoothers perform poorly on data with changing levels (edges) and corrupting noise that is either heavy tailed or signal dependent, as well as where the underlying processes are non-Gaussian or when system nonlinearities are present (Pitas & Venetsanopoulos, 1990). The latter authors also mention that in image processing applications, linear smoothers tend to blur the edges, do not remove impulsive noise effectively, and do not perform well in the presence of signal dependent noise.

2.5.2 Nonlinear smoothers

Nonlinear smoothers have been considered in literature for the reasons discussed in Section 2.5.1. They can be designed to meet criteria such as resistance, robustness, adaptivity to noise probability distributions, preservation of edge information, and preservation of image details. Nonlinear smoothers differ from linear smoothers in two important ways: they are more insensitive or resistant to the presence of occasional outliers in the data, but they are much less tractable analytically (Mallows, 1980).

Families of nonlinear smoothers are illustrated in Figure 2.2 (Pitas & Venetsanopoulos, 1990).

Most of the approaches attempted to achieve smoothing make use of geometric rather than analytic features of signals. Pitas & Venetsanopoulos (1990) discuss each of the families of morphological smoothers, homomorphic smoothers and polynomial smoothers in a separate chapter. Since each of these smoothers is based on specialised mathematical knowledge, such as mathematical morphology, homomorphic system and Volterra series not discussed here, they will not be discussed further in this thesis.

One of the most popular families of nonlinear smoothers is the order statistic smoothers. Order statistic smoothers use the theory of robust statistics as a theoretical basis. This family includes the median smoother which was first introduced by Tukey (1977). Since LULU smoothers use the maximum and minimum operators, they are included in this family of order statistic smoothers.

2.6 Summary

This chapter provided a philosophical and broad background to smoothing and smoothers in general. The reasons for smoothing, goals of smoothing and criteria for evaluating smoothers were discussed.

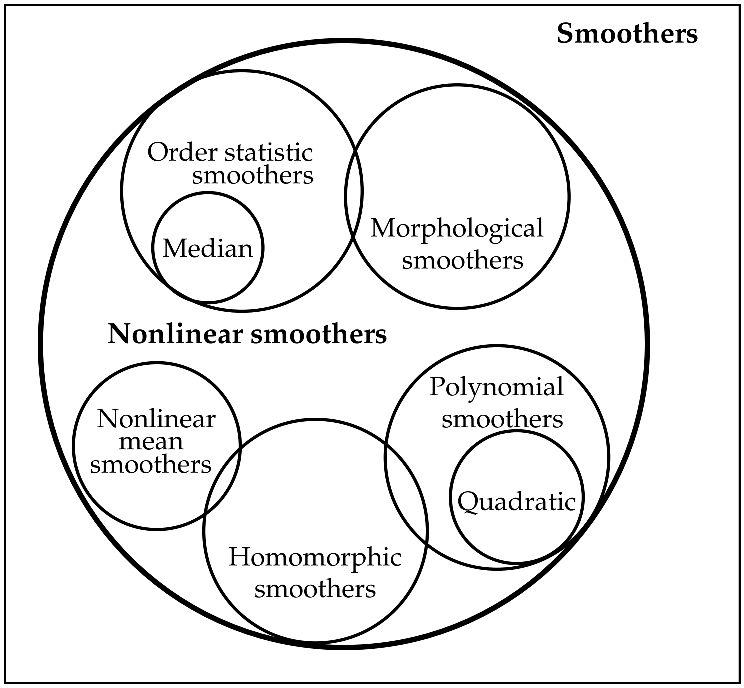

Figure 2.2: Families of nonlinear smoothers

Important concepts involved in the process of smoothing a time series were defined and explained. These definitions provide a framework for understanding the process of smoothing. Criteria for evaluating smoothers such as effectiveness, consistency, stability and efficiency were defined. Other concepts that were explained are idempotency and co-idempotency, monotonicity, root, edge, impulse and blockpulse. Statistical and distributional properties of smoothers such as dependence and measures of variation were discussed. Aspects involved in variation reduction and shape preservation were defined and derived. Approaches for dealing with end-values also received attention.

Winsorised and compound smoothers were defined in general. These are important and are applied in the following chapters of this thesis.

The classes of smoothers of time series were discussed. The smoothers of interest in this study fall into the family of order statistic based nonlinear smoothers since the median smoother and the LULU smoothers are included here. The concepts discussed in this chapter should be used as a reference for the following chapters.

In the next chapter the class of median smoothers is discussed.

Chapter 3

The class of median smoothers

3.1 Introduction

One of the most popular families of nonlinear smoothers is that of order statistic smoothers, which includes the median smoother. Order statistics smoothers use the theory of robust statistics as their theoretical basis (David, 1980).

The class of median smoothers was first introduced by Tukey (1977). Median smoothers perform well where the noise is of an impulsive type. In linear smoothing methods, an outlying data point will dominate the averages in which it participates. Median smoothers, on the other hand, provide protection from such noise spikes (Velleman, 1980).

The median smoothers and their modifications have found numerous applications in digital image processing, digital image analysis, digital TV applications, speech processing and coding, spectral analysis and various other applications (Pitas & Venetsanopoulos, 1990). The reason for their success is their good performance and computational simplicity. The theoretical analysis of their deterministic and statistical properties started at the end of the seventies and the classical work is described in Justusson (1981), Tyan (1981) and Arce *et al.* (1986).

The purpose of this chapter is to introduce median smoothers and provide an understanding of their properties. In Section 3.2 the median smoother is defined and in Sections 3.3, 3.4, 3.5, 3.6 and 3.7 investigated for possible properties, such as monotonicity, idempotency, co-idempotency, stability, statistical and edge preservation. Modifications and extensions of the median smoother, such as compound median smoothers, are discussed in Section 3.8. A summary is given in Section 3.9.

3.2 Definitions

The standard median smoother in a window is obtained when an odd number of successive observations in a sequence is sorted, and the middle or median value is used as the output. This median smoother is also called the moving median, running median, rank-ordered median or nonrecursive median. Formally, the median smoother is defined as:

Definition 3.1. *The **median output value** on a window of v observations $\{x_1, x_2, \ldots, x_v\}$, denoted by med, is*

$$med = \begin{cases} x_{(n+1)} & v = 2n+1, \\ \frac{1}{2}(x_{(n)} + x_{(n+1)}) & v = 2n. \end{cases} \tag{3.1}$$

where $x_{(i)}$ denotes the i-th order statistic.

The definition for an odd v is mainly used:

Definition 3.2. *The **median smoother** of the i-th value of the sequence x, using window size $v = 2n + 1$, is defined as*

$$(M_n x)_i \;=\; med(x_{i-n}, \ldots, x_i, \ldots, x_{i+n}), \tag{3.2}$$

with $\{x_{i-n}, \ldots, x_i, \ldots, x_{i+n}\}$ a running window.

The previous definitions refer to one-dimensional observations. Although one-dimensional smoothers are the focus of this study, the definition of the two-dimensional median smoother is also given for the sake of completeness.

Definition 3.3. *A two-dimensional median smoother can be defined as*

$$med(x_{i+r,\ j+s}), \tag{3.3}$$

where (r, s) are elements in the smoother window A represented by sets of numbers on a square lattice x_{ij} where (i, j) runs over $\mathbb{Z}^2$.

In Section 2.3 a number of important concepts that have a direct influence on the process of smoothing were defined and explained. The way in which median smoothers relate to these concepts will be discussed further on in this chapter.

3.3 Monotonicity and the root

Monotonicity, as defined in Definition 2.13, is a very important concept in evaluating a smoother. The number of values in a subset of a sequence that are monotone increasing or monotone decreasing, together with the window size of the smoother applied, play a substantial role in the pattern of the output sequence. As was stated in Section 2.3.4 monotonicity can also be considered a measure of smoothness.

A sequence that is invariant under median smoothing is called a *root* of the median smoother (the root of a smoother was defined in Definition 2.14). The important question is what type of sequence, if any, passes through the median smoother unchanged?

The following theorems, proved in Tyan (1981), describe the monotonicity and the root sequences for the median smoothers:

Theorem 3.1. *If a sequence x is **monotonic**, i.e. monotone increasing ($x_i < x_j$ for every $i < j$), or monotone decreasing ($x_i > x_j$ for every $i < j$) throughout the sequence, it is invariant w.r.t median smoothing, and thus a root under median smoothing of window size $(2n + 1)$ for any n.*

The following theorem illustrates that the median smoother is invariant with respect to monotonic data transformations. It also implies that scale is irrelevant as far as medians are concerned.

Theorem 3.2. *If $g(x)$ is monotonic, then*

$$med(g(x_1), \dots, g(x_{2n+1})) = g(med(x_1, \dots, x_{2n+1})).$$

A sufficient condition for a sequence to be a root of the median smoother is given by the following theorem:

Theorem 3.3. *An m-monotone sequence (LOMO($m+2$)) is invariant to median smoothing of window size $(2n+1)$, for all n, $n \leq m$.*

Two types of roots can be characterised by the following two theorems which give the necessary conditions for a sequence to be a root of the median smoother.

Theorem 3.4. *(Type I roots)*
If a sequence x is a root of the median smoother M_n of length $(2n + 1)$ and if there exists a monotonic segment $(x_p, x_{p+1}, \dots, x_{p+n})$ of length $(n + 1)$, then x is LOMO($n + 2$), or n-monotone.

The following definition will clarify Theorem 3.5:

Definition 3.4. *A sequence is nowhere n-monotone (nowhere LOMO($n + 2$)) if it does not contain any monotonic segment of length $(n + 2)$.*

Theorem 3.5. *(Type II roots)*
If a sequence x is a root of the median smoother M_n of length $(2n + 1)$ and it is nowhere $(n - 1)$-monotone (nowhere LOMO($n + 1$)), then x is a bi-valued sequence, i.e. x can take on only two values.

There are no Type II roots of median smoothers of window size three, but they exist for median smoothers of window length greater than three. The reason is that median smoothers of length three do not possess oscillatory roots because any segment of two observations is monotonic and therefore any sequence is LOMO(3). By using mathematical induction this property is generalised as follows:

Theorem 3.6. *If a sequence x is invariant to median smoothers M_k of length $(2k + 1)$ for every $k = 1, 2, \dots, n$, then x is LOMO($n + 2$).*

From these theorems, the monotonicity properties of the median smoother can be summarised as follows:

Remark 3.1.

(1) If a sequence is monotonic, then $M_n x = x$ for any window size $(2n+1)$.

(2) If a sequence x is n-monotone (i.e. $x \in \mathcal{M}_n$), then $M_k x = x$ for $k \leq n$.

(3) If $M_k x = x$ for all $k \leq n$, then x is n-monotone.

(4) Remarks 3.1(2) and 3.1(3) imply that $M_k x = x$ for all $k \leq n$ if and only if x is n-monotone.

(5) If $M_n x = x$ and if there exists a monotonic segment $(x_p, x_{p+1}, \dots, x_{p+n})$ of length $(n+1)$, then x is n-monotone.

(6) If $M_n x = x$ and x is nowhere $(n-1)$-monotone, then x is a bi-valued sequence, i.e. x can take on only two values.

Remark 3.2. *The median smoothers M_n are trend preserving, in the sense that if $x_{i-n}, \dots, x_i, \dots, x_{i+n}$ is monotone, then $(M_n x)_i = x_i$.*

3.4 Idempotency and co-idempotency

Consistency was defined in Section 2.3.3, together with the concepts of idempotency and co-idempotency, which are closely related to consistency. A smoother is idempotent if it reaches the root of the sequence in one pass. Median smoothers do not reach the root of a sequence in one pass, and are thus neither idempotent nor co-idempotent. This will be illustrated by an example in Chapter 5.

A deterministic property of median smoothers is that the rate of convergence to a root sequence can be determined. It can be shown that any nonroot sequence of finite length L converges to a root sequence after repeated median smoothing.

To provide a smoothed sequence of the same length as the original sequence for the window of length $(2n+1)$, it is typical to append constant values to the front and back of the L-length sequence. Thus, the first and last sequence points are invariant to smoothing. The rate of convergence to a root sequence depends on the length of the nonroot sections of the input sequence. An upper bound on the number of passes to convergence is given by the following theorem:

Theorem 3.7. *Upon successive median smoothing window passes of any size, any nonroot sequence of length L will become a root after a maximum of*

$$\tfrac{1}{2}(L-2) \tag{3.4}$$

successive passes.

Proof

See Gallagher & Wise (1981).

Theorem 3.8 below provides a tighter bound on the number of passes for any sequence to converge to a root:

Theorem 3.8. *A nonroot sequence of length L will converge to a root in at most*

$$3\left[\frac{(L-2)}{2(n+2)}\right] \tag{3.5}$$

passes of a median with window size $v = 2n + 1$.

Proof

See Wendt *et al.* (1986).

It is clear that the rate of convergence depends on both the length of the sequence, L, and the size of the window, $(2n + 1)$. For larger window sizes, the rate of convergence will be faster, and thus the root sequence is obtained in fewer passes of the median smoother than for smaller window sizes. However, care must be taken in using too large window sizes, for some important detail can be lost because large window sizes provide a ″smoother″ smoothed sequence than smaller window sizes (oversmoothing).

An important point of difference between the median smoother and the LULU smoothers concerning the concepts of idempotency and co-idempotency will be illustrated in Section 5.3 .

3.5 Stability

One of the main advantages of median smoothers is that they are very useful for smoothing impulse (spiky) noise. Moving medians suppress impulse noise, i.e. a signal disturbed by very large positive or negative values of short duration, quite well. The window size of the median smoother is important in deleting the impulse. The size of the window chosen must be at least twice the length of the impulse to delete the impulse (Justusson, 1981). Figure 3.1 illustrates the application of M_1 and M_2 on a signal with two impulses, a positive impulse and a negative impulse, of length one. The window size of M_1 is three, and that of M_2 is five, both of which are greater than twice the length of the impulse. Thus, both these median smoothers delete the impulses of length one. In Figure 3.2 a signal with a positive impulse of length three (blockpulse) and a negative impulse of length three (blockpulse) is illustrated. It can be seen that the median smoothers M_1 and M_2 preserve the blockpulses. The median smoother M_3 with window size seven, which is greater than twice the length of the blockpulses, deletes the blockpulses of length three.

Median smoothers can thus be considered stable for impulses since they do not distort the output excessively. Blockpulses can be preserved or deleted, depending on the window size of the chosen median smoother. If a blockpulse in a time series is due to an important event, it should be preserved. Median smoothers possess the ability to preserve an important blockpulse.

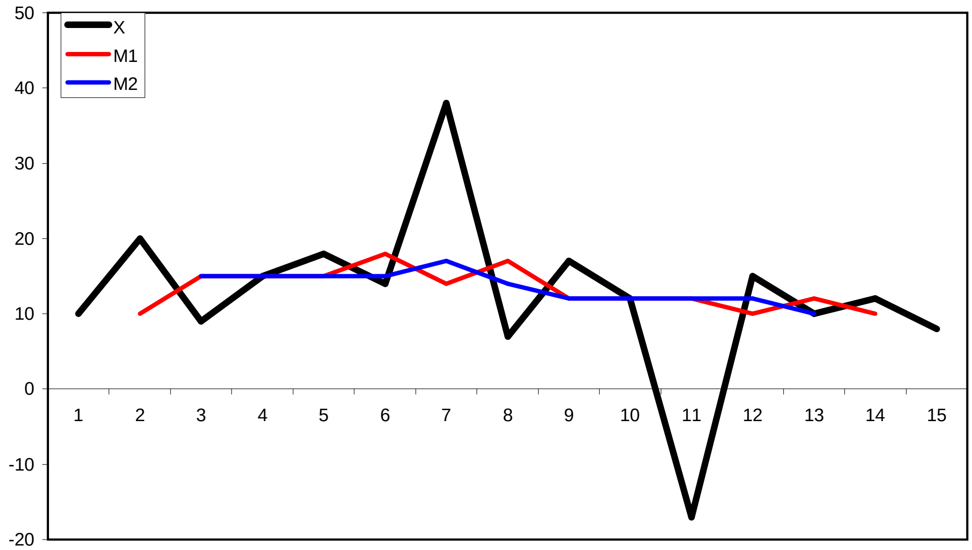

Figure 3.1: Median smoothers on signal with single value impulses

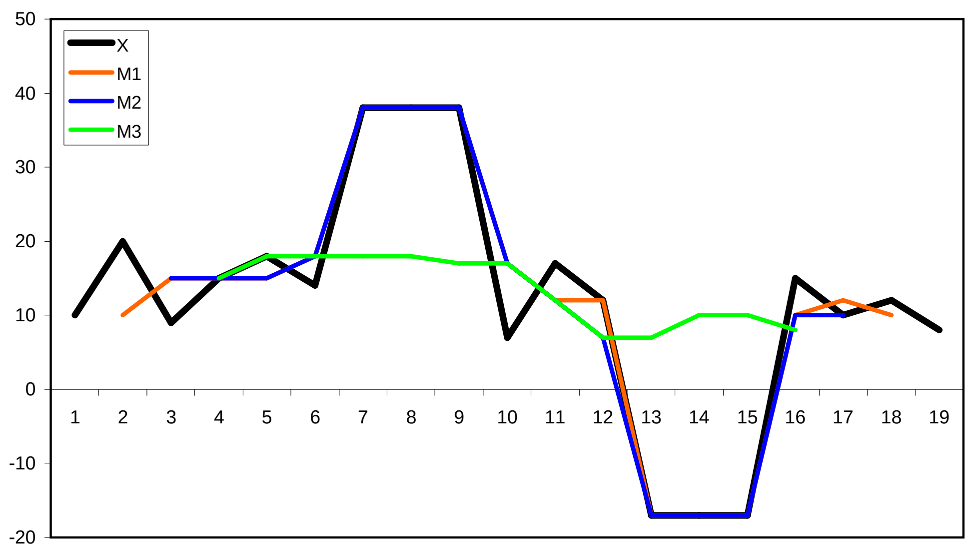

Figure 3.2: Median smoothers on signal with impulses of length three

3.6 Statistical properties

The statistical properties of median smoothers can be investigated through measures such as output variances in cases where the output distributions can be derived. An assumption is that the input sequence is a constant signal with noise added. If the output distribution of the median smoother is calculated over a constant region, the noise smoothing capabilities of the median can be measured by the output variance (Barner & Arce, 1998).

3.6.1 White noise

If the input time series consists of independent identically distributed (iid) observations x_i, $i = 1, \dots, N$, of X, with probability density function (pdf) $f_X(x)$ and cumulative distribution function $F_X(x)$, then the cumulative distribution function (cdf) of the median of window size $\nu = 2n + 1$ is given by (David, 1980):

$$F_{n+1}(x) = \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} F_X(x)^i (1 - F_X(x))^{2n+1-i}. \tag{3.6}$$

Its probability density function is

$$f_{n+1}(x) = (2n+1) \binom{2n}{n} f_X(x) F_X^n(x) (1 - F_X(x))^n. \tag{3.7}$$

Calculating the output mean and variance from equations (3.6) and (3.7) is often quite difficult.

The asymptotic behaviour of the median smoother can be investigated to gain insight into its smoothing characteristics. The distribution of the median for large window size ν is approximately normal $N(\tilde{m}, \sigma^2_{med(large)})$, where $\tilde{m}$ is the theoretical median which is determined by $F_X(\tilde{m}) = 0{,}5$. The asymptotic variance of the median for window size ν is

$$\sigma^2_{\mathrm{med(large)}} = \frac{1}{4\nu f_X^2(\tilde{m})}. \tag{3.8}$$

Thus, irrespective of the input distribution, the median produces a consistent ($\lim_{\nu\to\infty} \sigma^2_{\mathrm{med(large)}} = 0$) estimate of the input distribution median (Barner & Arce, 1998). Interestingly, the output variance is not proportional to the input variance, but to $\frac{1}{f_X^2(\tilde{m})}$.

For small values of ν, a better approximation of the variance is (Justusson, 1981):

$$\sigma^2_{\mathrm{med(small)}} = \frac{1}{4(\nu + b) f_X^2(\tilde{m})}, \tag{3.9}$$

where

$$b = \frac{1}{4 f_X^2(\tilde{m}) \sigma_x^2} - 1. \tag{3.10}$$

This coefficient b is chosen in such a way that $\sigma^2_{\mathrm{med(small)}}$ becomes 1 for $\nu = 1$ (Pitas & Venetsanopoulos, 1990).

A few distributions, which vary in tail length, were chosen to illustrate statistical properties throughout the thesis. The chosen distributions are the uniform distribution with very short tails, the normal (Gaussian) distribution, the logistic distribution with longer tails than the Gaussian distribution, the

Laplace (double exponential) distribution with even longer tails, the t-distribution with the length of tails varying according to the degrees of freedom, and the contaminated normal distribution where the length of tails vary according to the percentage of outliers with their variance.

The asymptotic variance of the median will be illustrated with the following examples:

1. **Uniform distribution**

(1) A sequence of points iid uniformly distributed on [0, 1] has cdf and pdf respectively

$$F(x) = x,$$

and

$$f(x) = 1. \tag{3.11}$$

The mean of this distribution is $\mu_x = 0{,}5$ with variance $\sigma_x^2 = \frac{1}{12}$.

From (3.8) the asymptotic variance of the median smoother for window size ν is

$$\sigma^2_{\text{med(large)}} = \frac{1}{4\nu(1)^2} = \frac{1}{4\nu}. \tag{3.12}$$

From (3.9) and (3.10) an approximation of the variance for small ν is

$$\sigma^2_{\text{med(small)}} = \frac{1}{4(\nu+2)(1)^2} = \frac{1}{4(\nu+2)}, \tag{3.13}$$

with

$$b = \frac{1}{4(1)^2\frac{1}{12}} - 1 = 3 - 1 = 2.$$

Consider a set of observations $\{x_1, \ldots, x_\nu\}$ and let $X_{(r)}$ be the r-th order statistic with $x_{(r)}$ its observed value, then a well-known result is that the r-th order statistic is beta distributed,

$$X_{(r)} \sim beta(r, \nu - r + 1)$$

with

$$E(X_{(r)}) = \frac{r}{\nu+1} \qquad \text{and} \qquad Var(X_{(r)}) = \frac{1}{\nu+2} \cdot \frac{r}{\nu+1}\left(1 - \frac{r}{\nu+1}\right). \tag{3.14}$$

The exact value of the variance of the median, where $\nu = 2n+1$ and $r = n+1$, is

$$\begin{aligned} Var(X_{(n+1)}) &= \tfrac{1}{\nu+2} \cdot \tfrac{n+1}{\nu+1}\left(1 - \tfrac{n+1}{\nu+1}\right) \\ &= \tfrac{n+1}{(\nu+2)(\nu+1)}\left(\tfrac{\nu+1-(n+1)}{\nu+1}\right) \\ &= \tfrac{(n+1)(2n+2-(n+1))}{(\nu+2)(2n+2)^2} \\ &= \tfrac{(n+1)(n+1)}{(\nu+2)(2(n+1))^2} \\ &= \tfrac{(n+1)^2}{(\nu+2)4(n+1)^2} \\ &= \tfrac{1}{4(\nu+2)}. \end{aligned}$$

This result confirms the approximation in (3.13) as exact in this case.

An example of the uniform distribution as in Barner & Arce (1998) follows:

(2) For the uniform distribution on $\left[-\sqrt{3\sigma_x^2};\ \sqrt{3\sigma_x^2}\right]$,

$$f(x) = \frac{1}{\sqrt{12\sigma_x^2}}. \tag{3.15}$$

Using (3.8) the asymptotic variance of the median is computed as

$$\sigma^2_{\text{med(large)}} = \frac{1}{4\nu\left(\frac{1}{\sqrt{12\sigma_x^2}}\right)^2} = \frac{3\sigma_x^2}{\nu}. \tag{3.16}$$

In this case (3.10) gives

$$b = \frac{1}{4(\frac{1}{\sqrt{12\sigma_x^2}})^2\sigma_x^2} - 1 = 3 - 1 = 2,$$

and the approximation (3.9) becomes

$$\sigma^2_{\text{med(small)}} = \frac{1}{4(\nu+2)(\frac{1}{\sqrt{12\sigma_x^2}})^2} = \frac{3\sigma_x^2}{\nu+2}. \tag{3.17}$$

2. Normal distribution

(1) The Gaussian distribution $N(0, 1)$ on $(-\infty;\ +\infty)$ with cdf $\Phi(x)$ has mean $\mu_x = 0$ and variance $\sigma_x^2 = 1$. The pdf of the iid Gaussian distributed sequence is

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}. \tag{3.18}$$

From (3.8) the asymptotic variance of the median smoother for window size ν is

$$\sigma^2_{\text{med(large)}} = \frac{1}{4\nu(\frac{1}{\sqrt{2\pi}})^2} = \frac{\pi}{2\nu}. \tag{3.19}$$

From (3.9) and (3.10) an approximation of the variance for small ν is

$$\sigma^2_{\text{med(small)}} = \frac{1}{4(\frac{1}{\sqrt{2\pi}})^2(\nu+\frac{\pi}{2}-1)} = \frac{\pi}{2(\nu+\frac{\pi}{2}-1)}, \tag{3.20}$$

with

$$b = \frac{1}{4(\frac{1}{\sqrt{2\pi}})^2(1)^2} - 1 = \frac{\pi}{2} - 1.$$

(2) If the x observations are normally distributed $N(m, \sigma_x^2)$ where $\tilde{m} = m$ on $(-\infty; \ +\infty)$, the pdf of the iid distributed sequence is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-m)^2}. \tag{3.21}$$

The asymptotic variance of the median, using (3.8), is computed as

$$\sigma^2_{\text{med(large)}} = \frac{1}{4\nu\left(\frac{1}{\sqrt{2\pi\sigma_x^2}}\right)^2} = \frac{\pi\sigma_x^2}{2\nu}. \tag{3.22}$$

From (3.9) and (3.10) for small ν the variance of the median is approximated by

$$\sigma^2_{\text{med(small)}} = \frac{1}{4(\nu + \frac{\pi}{2} - 1)\left(\frac{1}{\sqrt{2\pi\sigma_x^2}}\right)^2} = \frac{\pi\sigma_x^2}{2(\nu + \frac{\pi}{2} - 1)}, \tag{3.23}$$

with

$$b = \frac{1}{4(\frac{1}{\sqrt{2\pi\sigma_x^2}})^2\sigma_x^2} - 1 = \frac{\pi}{2} - 1.$$

3. Logistic distribution

The logistic distribution $L(0, 1)$ on $(-\infty; \ +\infty)$ has cdf and pdf respectively

$$F(x) = \frac{1}{1 + e^{-x}},$$

$$f(x) = \frac{e^{-x}}{(1 + e^{-x})^2}. \tag{3.24}$$

The mean of this distribution is $\mu_x = 0$ with variance $\sigma_x^2 = \frac{\pi^2}{3}$.

From (3.8) the asymptotic variance of the median smoother for window size ν is

$$\sigma^2_{\text{med(large)}} = \frac{1}{4\nu(\frac{1}{4})^2} = \frac{4}{\nu}. \tag{3.25}$$

From (3.9) and (3.10) an approximation of the variance for small ν is

$$\sigma^2_{\text{med(small)}} = \frac{1}{4(\frac{1}{4})^2(\nu + \frac{12}{\pi^2} - 1)} = \frac{4}{(\nu + \frac{12}{\pi^2} - 1)}, \tag{3.26}$$

with

$$b = \frac{1}{4(\frac{1}{4})^2\frac{\pi^2}{3}} - 1 = \frac{12}{\pi^2} - 1.$$

4. **Laplacian distribution**

The Laplace (double exponential) distribution with parameters α (mean)= 0 and $\beta = 1$ on $(-\infty;\ +\infty)$ has pdf

$$f(x) = \frac{1}{2}e^{-|x|}. \tag{3.27}$$

The mean of this distribution is $\mu_x = 0$ with variance $\sigma_x^2 = 2$.

From (3.8) the asymptotic variance of the median smoother for window size ν is

$$\sigma^2_{\text{med(large)}} = \frac{1}{4\nu(\frac{1}{2})^2} = \frac{1}{\nu}. \tag{3.28}$$

From (3.9) and (3.10) an approximation of the variance for small ν is

$$\sigma^2_{\text{med(small)}} = \frac{1}{4(\frac{1}{2})^2(\nu - \frac{3}{4})} = \frac{1}{\nu - \frac{3}{4}}, \tag{3.29}$$

with

$$b = \frac{1}{4(\frac{1}{2})^2(2)^2} - 1 = \frac{1}{4} - 1 = -\frac{3}{4}.$$

5. ***t*-distribution**

The t-distribution with m degrees of freedom on $(-\infty;\ +\infty)$ has pdf

$$f(x) = \frac{\Gamma(m+1)/2}{\sqrt{m\pi}\Gamma(m/2)} \frac{1}{(1 + x^2/m)^{(m+1)/2}}. \tag{3.30}$$

The mean of this distribution is $\mu_x = 0$ with variance $\sigma_x^2 = \frac{m}{m-2}$.
The Cauchy distribution is obtained for $m = 1$. It is a long-tailed distribution, but the expected value and higher moments do not exist. For $m = 2$ the expected value exists, but the variance is infinite. For $m \geq 3$ the variance is finite. As $m \to \infty$ the t-distribution tends to the normal distribution.

For the case where $m = 3$, the variance $\sigma_x^2 = 3$.

From (3.8) the asymptotic variance of the median smoother for window size ν is

$$\sigma^2_{\text{med(large)}} = \frac{1}{4\nu\left(\frac{2}{\pi\sqrt{3}}\right)^2} = \frac{3\pi^2}{16\nu}. \tag{3.31}$$

From (3.9) and (3.10) an approximation of the variance for small ν is

$$\sigma^2_{\text{med(small)}} = \frac{1}{4\left(\frac{2}{\pi\sqrt{3}}\right)^2\left(\nu + \frac{\pi^2}{16}\right)} = \frac{3\pi^2}{16\nu + \pi^2}, \tag{3.32}$$

with

$$b = \frac{1}{4\left(\frac{2}{\pi\sqrt{3}}\right)^2 3} - 1 = \frac{\pi^2}{16}.$$

6. **Contaminated normal distribution**

The cdf of the contaminated normal distribution on $(-\infty;\ +\infty)$ is

$$F(x) = (1-\varepsilon)\Phi(x) + \varepsilon\Phi(\frac{x}{\tau}), \tag{3.33}$$

where ε represents the proportion of contamination (outliers) and τ^2 is the variance of the outliers. The mean of this distribution is $\mu_x = 0$ with variance $\sigma_x^2 = (1-\varepsilon) + \varepsilon\tau^2$.

The pdf of the contaminated normal distribution is

$$f(x) = (1-\varepsilon)\phi(x) + \frac{\varepsilon}{\tau}\phi(\frac{x}{\tau}). \tag{3.34}$$

The approximate variance for the median smoother varies according to the combinations of ε and τ, and is computed using (3.8) and (3.9).

3.6.2 Nonwhite noise

If input sequences are random processes with statistically dependent random variables it is very difficult to obtain simple exact formulas for the probability distribution and for the variance of the median smoother. However, there are theorems which give approximations of the variance of the median analogous to (3.8) (Justusson, 1981). The conditions needed for the limit theorems are that the input process x is stationary and mixing. The mixing condition means that process variables lying far apart are almost independent.

For a stationary, mixing, normal input process x with covariance function (Justusson, 1981)

$$\mathrm{Cov}(x_i, x_{i+\tau}) = \sigma_x^2\, r_x(\tau), \quad \tau = 0, \pm 1, \ldots \tag{3.35}$$

where $r_x(\tau) = \mathrm{Corr}(x_i, x_{i+\tau})$, the following expression gives the approximate variance of a median for window size $\nu = 2n+1$

$$\sigma_{med}^2 \simeq \frac{\sigma_x^2}{\nu + \pi/2 - 1} \sum_{j=-\nu+1}^{\nu-1} \left(1 - \frac{|j|}{\nu}\right) \arcsin[r_x(j)]. \tag{3.36}$$

3.7 Edge preservation

Edge information is very important for human perception and thus in image processing (Pitas & Venetsanopoulos, 1990). An edge is defined in Section 2.3.8, where a distinction is also made between a step edge and a ramp edge.

The median smoother has very good edge preservation properties and it preserves any step edge (Justusson, 1981). Linear low-pass smoothers, such as the moving average smoother, destroy the edges because the high frequency content is decreased and blurred. The step edges are transformed to ramp edges and the ramp edges are widened by average smoothing. This is not the case with the median smoother (Pitas & Venetsanopoulos, 1990).

The edge preservation properties of the median smoother will be investigated on sequences with a step edge with added white noise. The model is (Pitas & Venetsanopoulos, 1990):

$$x_i = s_i + z_i, \qquad s_i = \begin{cases} h & i \leq k, \\ 0 & i > k, \end{cases} \tag{3.37}$$

where z_i is white Gaussian noise having distribution $N(0, \sigma^2)$.

This model is also valid if the values h and 0 interchange.

The probability density function of the median with window size $\nu = 2n+1$, where the random variables have different distributions, can be given by the following:

Let k independent observations, $x_1, \ldots, x_k$, have distribution function $F_1(x)$, and $(\nu - k)$ independent observations, $x_{k+1}, \ldots, x_\nu$, have distribution function $F_2(x)$. The pdf of the median with window size $\nu = 2n + 1$ is (Justusson, 1981):

$$f_{n+1}(x) = g_1(x) + g_2(x), \tag{3.38}$$

where

$$g_1(x) = \sum_j k\binom{k-1}{j}\binom{\nu-k}{n-j} f_1(x)F_1(x)^j F_2(x)^{n-j}[1-F_1(x)]^{k-j-1}[1-F_2(x)]^{\nu-k-n+j}, \tag{3.39}$$

$$g_2(x) = \sum_j (\nu-k)\binom{k}{j}\binom{\nu-k-1}{n-j} f_2(x)F_1(x)^j F_2(x)^{n-j}[1-F_1(x)]^{k-j}[1-F_2(x)]^{\nu-k-n+j-1}. \tag{3.40}$$

The summations are carried over all natural numbers j for which all involved binomial coefficients $\binom{p}{q}$ satisfy $p \geq q \geq 0$. The means and standard deviations can be computed by numerical integration of (3.38).

The behaviour of the median smoother is investigated as the edge height h varies. This will be illustrated by calculating the mean and standard deviation of the median smoother for different window sizes ν, for different numbers of observations in the step k and varying edge height h. The probability density function, $f_{n+1}(x)$, of a median smoother for window size $\nu = 2n + 1$ with k observations x_i being from a $N(h, \sigma^2)$ distribution and $(\nu - k)$ observations x_i being from a $N(0, \sigma^2)$ distribution can be obtained from (3.38), (3.39) and (3.40). For $\sigma^2 = 1$ and some arbitrarily chosen values of n and k, at $h = 0, 1, 2, 3, 4, \geq 5$ the expected values and standard deviations of the median smoothers M_n are tabulated in Table 3.1 (Justusson, 1981).

For example, for a window size of five, with one observation coming from a $N(5, 1)$ distribution, the expected value of the median is equal to 0,297, and the standard deviation of the median is 0,600.

All the cases listed in Table 3.1 are for $k < n$ and the conclusions are valid for this condition. It may be concluded that the expected value and standard deviation increase as the edge height h increases. Also,

Table 3.1: Expected value and standard deviation of median on edge-plus-noise observations

			h					
$2n+1$	k		0	1	2	3	4	≥ 5
3	1	$E(M_1)$ $\sigma(M_1)$	0,000 0,670	0,305 0,697	0,486 0,760	0,549 0,806	0,563 0,822	0,564 0,826
9	3	$E(M_4)$ $\sigma(M_4)$	0,000 0,408	0,318 0,424	0,540 0,471	0,626 0,513	0,641 0,527	0,642 0,529
5	1	$E(M_2)$ $\sigma(M_2)$	0,000 0,536	0,179 0,551	0,270 0,580	0,294 0,596	0,297 0,600	0,297 0,600
5	2	$E(M_2)$ $\sigma(M_2)$	0,000 0,536	0,386 0,560	0,676 0,631	0,808 0,705	0,841 0,740	0,846 0,748
25	5	$E(M_{12})$ $\sigma(M_{12})$	0,000 0,248	0,184 0,256	0,286 0,271	0,312 0,280	0,315 0,282	0,315 0,282
25	10	$E(M_{12})$ $\sigma(M_{12})$	0,000 0,248	0,391 0,260	0,719 0,295	0,900 0,346	0,944 0,371	0,948 0,375

for the same window size, a larger number of observations at the edge height h delivers a larger expected value, and slightly larger standard deviation. For example, at the window size of five, the expected values over the values of h for $k = 2$ are much larger than for $k = 1$.

The estimated mean square error (MSE) will be used as a goodness-of-fit measure to compare the efficiencies of smoothers on edge-plus-noise sequences (Justusson, 1981). At any N points close to the edge, the estimated MSE is

$$\frac{1}{N}\sum_{i=1}^{N}(y_i - s_i)^2, \tag{3.41}$$

where y_i denotes the output values of the smoother.

Calculated estimated MSE values will be used in Chapter 5 to compare the performance of various median smoothers and LULU smoothers on edge-plus-noise data.

3.8 Other median smoothers

Modifications and extensions of the standard median smoother have been developed to improve the performance of the median smoother (Barner & Arce, 1998).

Various authors have proposed modifications to the median smoother. It is important to be familiar with compound median smoothers, which will be used in Chapter 6. These are discussed in Section 3.8.1. The other modifications found in the literature are not used in this thesis and are only briefly mentioned in Section 3.8.2.

3.8.1 Compound smoothers

Tukey (1971) proposed smoothers that are combinations of running median smoothers and linear smoothers to improve the performance of the running medians. Over the years Tukey's idea has resulted in smoothers such as 53H, (53H,twice), 3RSSH, (3RSSH,twice), (4253H,twice) and (43RSR2H,twice).

The notation used is as follows:

- **R** denotes repeated *resmoothing* where one smoother is applied to the results of a previous smoother.
- **S** denotes *splitting* where the data sequence is divided into two separate pieces in the middle of a width-2 peak or trough. Each end is smoothed separately with the end-value rule and then the parts are glued together.
- **H** denotes *hanning*, after Julius von Hann, who advocated its use (Blackman & Tukey, 1958). This is a *running weighted average* with the requirement that the weights must sum to one. The three-point moving average with weights $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$ is used for most data exploration.
- **twice** involves the operation *reroughing* where the rough sequence is obtained by subtracting the smoothed sequence from the data. To recover patterns from the residuals, this rough sequence is smoothed and the result added to the smoothed sequence. If the same smoother is used in both smoothing and *reroughing*, it is called **twice**.

The compound smoothers, (**53H,twice**) and (**4253H,twice**), used in the simulation studies in Chapter 6, are now defined.

The compound smoother (**53H,twice**) consists of a median of window size five (M_2), followed by a median of window size three (M_1). Then *hanning*, **H**, follows using the three-point moving average

$$z_i = \tfrac{1}{4}y_{i-1} + \tfrac{1}{2}y_i + \tfrac{1}{4}y_{i+1}. \tag{3.42}$$

Reroughing is applied and this smoother is denoted by (**53H,twice**) (Tukey, 1971).

The compound smoother (**4253H,twice**) was introduced by Velleman (1975). He proposed even-window medians to reduce some difficulties found in odd-window medians. (**4253H,twice**) starts with a median of window four, followed by a median of window two. This averages the two middle y values in its window and falls naturally between two middle values. Hereafter, the same operation used to obtain (**53H,twice**) is followed to obtain the smoother (**4253H,twice**).

3.8.2 Modifications and extensions

A large number of modifications and extensions of median smoothers have been proposed in the literature. The following are some of these:

1. recursive median smoother (Pitas & Venetsanopoulos, 1990)
2. weighted median smoother (Justusson, 1981):

- linear combination of weighted medians (LCWM) (Choi *et al.* , 2001)
- recursive weighted medians (RWM) (Yli-Harja *et al.* , 1991)

3. centre-weighted median smoother (CWM) (Barner & Arce, 1998):
   - adaptive centre-weighted median smoother (ACWM) (Ko & Lee, 1991)
   - lower-upper-middle smoother (LUM) (Hardie & Boncelet, 1993)
4. weighted order statistic smoother (Yli-Harja *et al.* , 1991)
5. stack smoother (Fitch *et al.* , 1984)
6. max/median smoother (Arce & McLoughlin, 1987)
7. multistage median smoother (Pitas & Venetsanopoulos, 1990)
8. median hybrid smoother (Heinonen & Neuvo, 1987)
9. α-trimmed mean smoother (Bednar & Watt, 1984)
10. L-smoothers, M-smoothers, and R-smoothers (Bovik *et al.* , 1983)

Since these smoothers will not be used in the subsequent analyses, they are not discussed further.

3.9 Summary

In this chapter the class of median smoothers was defined. The behaviour of the standard median smoother on monotone sequences of different degrees and root sequences was discussed. The rate of convergence of the median smoother depends both on the length of the nonroot sequence and on the window size. Median smoothers are considered stable, for impulses do not change the output excessively. A blockpulse can be preserved or deleted by a median smoother depending on the window size. The statistical properties were described for white noise and nonwhite noise. The variance of the median was illustrated for several input distributions.

The median has good edge preservation properties. Expected values and standard deviations of the median smoother for different window sizes, different numbers of observations in the step and varying edge height were investigated.

Other median smoothers, such as the compound median smoothers, were discussed. Variations and extensions of the standard median smoother were merely mentioned. From these modifications it is seen that a large variety of nonlinear smoothers based on the median exist. The successful application of these smoothers depends heavily on the characteristics of the problem and the experience of the scientist.

Chapter 4

The class of LULU smoothers

4.1 Introduction

The extreme selectors, minimum and maximum, are used as basic constituents of composite smoothers for the removal of impulsive noise. Rohwer (1989) used these extreme selectors to define his class of so-called LULU smoothers, which are nonlinear smoothers. He has since studied and applied them extensively, see for example Rohwer & Toerien (1991), Rohwer (1999), Rohwer (2002a) and Rohwer (2002b). These publications resulted in a monograph (Rohwer, 2005). In these papers it was shown that LULU smoothers have a very attractive mathematical structure and satisfy important criteria for smoothers compared to other nonlinear smoothers such as the median smoother. The most useful of these properties, for the purpose of this thesis, are discussed in the subsequent sections.

In this chapter LULU smoothers will be introduced and their mathematical and statistical properties will be discussed. A number of new LULU-based smoothers will also be defined which, in later studies, were found to have appealing properties. In Section 4.2 definitions are given and remarks made on the forward and backward operators, L_n, U_n, and LULU smoothers based on L_n and U_n. The way in which LULU smoothers behave on monotone sequences of different degrees is discussed in Section 4.3. In Section 4.4 the idempotency and co-idempotency of LULU smoothers are investigated. The stability of LULU smoothers is discussed in Section 4.5, and their behaviour on blockpulses in Section 4.5.1. Some distribution theory of L_nU_n and U_nL_n, and their asymptotic distributions are derived and discussed in Section 4.6. These are new results and were published recently (Conradie *et al.* , 2006). Edge preservation properties of LULU smoothers are investigated in Section 4.7. In Section 4.8 variation reduction and the shape preservation properties of LULU smoothers are discussed together with the property of variation decomposition in Section 4.9. Section 4.10 contains a summary of this chapter.

4.2 Definitions

The definitions and remarks in this section are based on data in the form of a doubly infinite numerical sequence (cf. Definition 2.1). The definitions and remarks, as well as the theorems with their proofs, can be found in Rohwer (2005).

4.2.1 Basic operators

Definition 4.1. *The maximum $\vee^n$ and minimum operators $\wedge^n$ are defined as*

$$(\vee^n x)_i = \max\{ x_i, \ldots, x_{i+n}\}, \quad \textit{and} \tag{4.1}$$
$$(\wedge^n x)_i = \min\{ x_{i-n}, \ldots, x_i\}. \tag{4.2}$$

Note that $(\vee^n x)$ is a forward operator and $(\wedge^n x)$ is a backward operator. Clearly a single upward impulse (point) will be removed by $\wedge x$ and a single downward impulse (point) will be removed by $\vee x$.

Some important properties of $\vee^n$ and $\wedge^n$ are given by the following theorems with proofs in Rohwer (2005, p.10):

Theorem 4.1.

(1) Let $x > y$, that is $x_i > y_i$ for all i. For $n \geq 1$ follows

$$\vee^n x \geq \vee^n y \text{ and } \wedge^n x \geq \wedge^n y.$$

(2) Let I be the identity operator, then

$$\vee\wedge \leq I \leq \wedge\vee .$$

(3) $\vee^n\wedge^n$ is monotonically non-increasing, that is, for $n = 1, 2, \ldots$

$$\vee^{n+1}\wedge^{n+1} \leq \vee^n\wedge^n$$

and
$\wedge^n\vee^n$ is monotonically non-decreasing, that is, for $n = 1, 2, \ldots$

$$\wedge^{n+1}\vee^{n+1} \geq \wedge^n \vee^n .$$

Remark 4.1.

(1) Theorem 4.1(1) implies that both $\vee^n$ and $\wedge^n$ are order preserving . This means that the order of the output of the sequence after the operator has been applied will be the same as the order of the original sequence. Operators are called syntone if this order preserving property holds.

(2) Theorem 4.1(2) indicates that the original sequence will always be bounded by $\vee\wedge$ from below and by $\wedge\vee$ from above.

(3) Combining Theorems 4.1(2) and 4.1(3) results in

$$\ldots \leq \vee^3\wedge^3 \leq \vee^2\wedge^2 \leq \vee\wedge \leq I \leq \wedge\vee \leq \wedge^2\vee^2 \leq \wedge^3\vee^3 \leq \ldots$$

(4) The size of the window on which the operators $\vee^n$ and $\wedge^n$ operate is $(n + 1)$.

(5) To remove both upward and downward isolated single impulses, the compositions $\wedge\vee$ and $\vee\wedge$ are needed. Similarly, for some appropriate n, $\wedge^n\vee^n$ and $\vee^n\wedge^n$ will remove consecutive upward and downward impulses in the sequence.

An example to illustrate Theorem 4.1(2) is given in Figure 4.1.

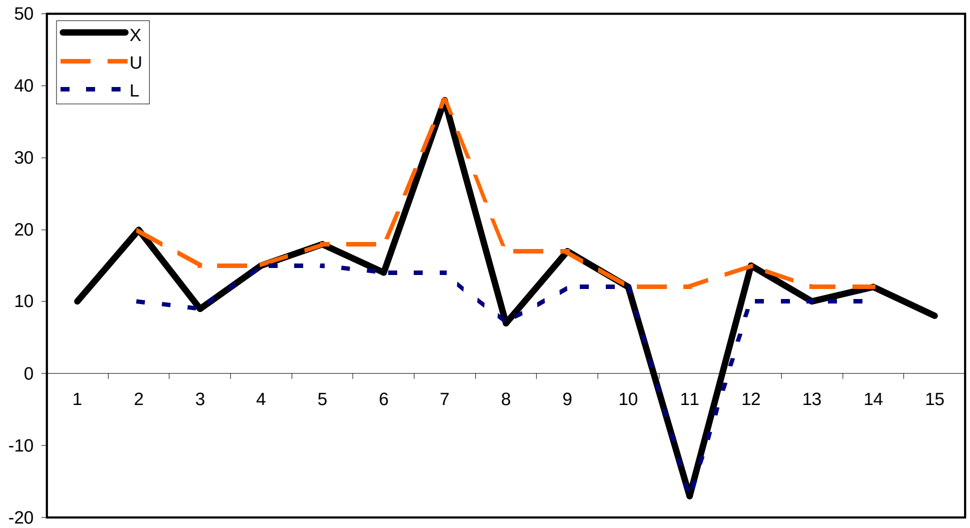

Figure 4.1: Example of $L = \vee\wedge < I < \wedge\vee = U$

4.2.2 Half smoothers L_n and U_n

Definition 4.2. *U_n and L_n are defined as the half smoothers $\wedge^n\vee^n$ and $\vee^n\wedge^n$ respectively, that is*

$$(U_n x)_i \equiv (\wedge^n \vee^n x)_i = \min\{\max(x_{i-n},\ldots,x_i),\ldots,\max(x_i,\ldots,x_{i+n})\} \quad \textit{and} \tag{4.3}$$
$$(L_n x)_i \equiv (\vee^n \wedge^n x)_i = \max\{\min(x_{i-n},\ldots,x_i),\ldots,\min(x_i,\ldots,x_{i+n})\}. \tag{4.4}$$

U_n and L_n are called half smoothers because they only smooth from one side, either from above or from below. This is a typical example of upper and lower smoothers, which were defined in general in Definition 2.28. It is clear that $U_n = \wedge^n\vee^n$ ”pulls up” downward lying points (smoothes from below) and $L_n = \vee^n\wedge^n$ ”pulls down” upward lying points (smoothes from above).

Compositions of the half smoothers L_n and U_n will be used to define the class of LULU half and full smoothers which smooth from above and below simultaneously.

Corollary 4.1. *From Theorems 4.1(1) and 4.1(2) it follows in terms of U_n and L_n that*

$$L_{n+1} \leq L_n \leq L_1 \leq I \leq U_1 \leq U_n \leq U_{n+1}.$$

The following theorems, with proofs found in Rohwer (2005, pp. 13, 35, 15), illustrate some important properties of L_n and U_n:

Theorem 4.2.

(1) If $m \geq n$

$$L_n L_m = L_m \quad \textit{and} \quad U_n U_m = U_m.$$

(2) If $(M_m x)_i$ is the median smoother defined in Definition 3.2, then for $m \leq n$

$$L_n \leq M_m \leq U_n.$$

(3) Let N denote the negative of a sequence, that is $N\{x\} = \{-x\}$, then, for all $n > 0$

$$L_n N = N U_n \quad \text{and} \quad U_n N = N L_n.$$

Remark 4.2.

(1) Theorem 4.2(1), also called the first swallowing theorem, indicates that lower order smoothing is "swallowed" by higher order smoothing.

(2) From Theorem 4.2(1) and $m = n$

$$(L_n)^2 = L_n \quad \text{and} \quad (U_n)^2 = U_n.$$

This implies that L_n and U_n are idempotent (cf. Definition 2.10).

(3) Since all the smoothers are syntone (a smoother P is syntone if $x \geq y \Rightarrow Px \geq Py$, (Rohwer, 2005, p. 3)) and L_n and U_n are idempotent, any power of M_m in Theorem 4.2(2) will also be bounded by L_n and U_n.

(4) From Theorem 4.2(3) it follows that L_n and U_n are duals (a smoother S is the dual of P if $PN = NS$, (Rohwer, 2005, p. 15)) of each other and all compositions of smoothers of these two types are again dual to the smoother formed by interchanging L_n and U_n for any n.

(5) The size of the window of the half smoothers L_n and U_n is $(2n + 1)$.

LULU half smoothers will mainly be used throughout this thesis and they will be called smoothers. When full smoothers are used, they will be explicitly referred to as such.

4.2.3 The smoothers $L_n U_n$ and $U_n L_n$

From the discussion above, it is clear that any outlier in the upward direction will be removed by L_n and any pulse in the downward direction will be removed by U_n. This fact that L_n removes only pulses in the upward direction and U_n only those in the downward direction is a serious defect. There are two obvious problems, even if outliers are expected on one side. Additional Gaussian noise will "pull down" the average of L_n compared to the original sequence, and if an occasional outlier occurs in the wrong direction, the usual problem still arises. Thus, the half smoothers L_n and U_n can be concatenated. Since they are not commutative, the two smoothers $L_n U_n$ and $U_n L_n$ will be studied.

Some important properties, proved in Rohwer (2005, pp. 13, 14, 15), of the smoothers $L_n U_n$ and $U_n L_n$ follow.

Theorem 4.3.

$$L_n \leq L_n U_n L_n \leq L_n U_n \quad \text{and} \quad U_n L_n \leq U_n L_n U_n \leq U_n.$$

Remark 4.3.

(1) *The smoothers are idempotent .*

$$(L_nU_n)^2 = L_nU_n \quad \textit{and} \quad (U_nL_n)^2 = U_nL_n,$$

and

$$(L_nU_nL_n)^2 = L_nU_nL_n \quad \textit{and} \quad (U_nL_nU_n)^2 = U_nL_nU_n.$$

This means that nothing is gained by a further application of the same smoother.

(2) *For $m \geq n$,*

$$L_nU_mL_n = U_mL_n \quad \textit{and} \quad U_nL_mU_n = L_mU_n,$$

and for $m = n$

$$L_nU_nL_n = U_nL_n \quad \textit{and} \quad U_nL_nU_n = L_nU_n.$$

(3) *For any integer $n \geq 0$*

$$L_nU_nL_n \leq U_nL_nU_n$$

and from Remark 4.3(2) it follows that

$$U_nL_n \leq L_nU_n.$$

(4) *Thus*

$$L_n \leq L_nU_nL_n \leq L_nU_n \quad \textit{and} \quad U_nL_n \leq U_nL_nU_n \leq U_n.$$

(5) *The two smoothers U_nL_n and L_nU_n remove impulse noise in both directions, are idempotent, and bound the median M_n*

$$U_nL_n \leq M_n \leq L_nU_n.$$

Unlike the case with the inclusion of any power of M_m for U_n and L_n in Remark 4.2(3), the statement above is not true for $m < n$ in general.

(6) *For $N\{x\} = \{-x\}$ and $n > 0$*

$$L_nU_nN = NU_nL_n \quad \textit{and} \quad U_nL_nN = NL_nU_n.$$

Thus, these smoothers are duals of each other.

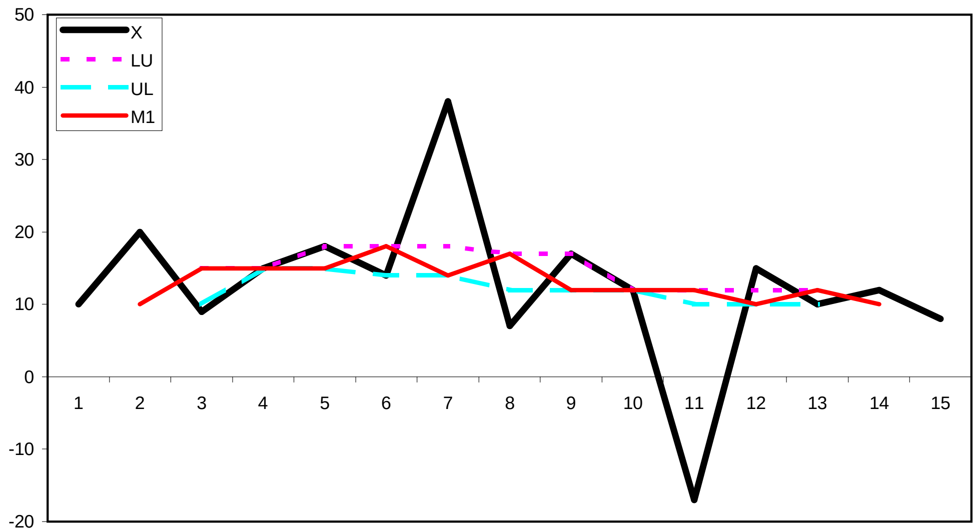

Figure 4.2: Example of $U_1L_1 < M_1 < L_1U_1$

Table 4.1: Composition table of LULU structure

	L	***U***	***UL***	***LU***
L	L	LU	UL	LU
U	UL	U	UL	LU
UL	UL	LU	UL	LU
LU	UL	LU	UL	LU

See Figure 4.2 for an example of Remark 4.3(5) with $n = 1$.

Table 4.1 illustrates all possible combinations where the operators $L = L_n$ and $U = U_n$. If L in the first row is applied to UL in the third column, the result is $LUL = UL$ in the cell of the table where the first row and third column cross each other. The LU in the lower right-hand corner of the table implies that $LULU = LU$. This is what suggested the name "LULU smoothers".

Since L_nU_n and U_nL_n are basic smoothers, combinations of them can be used to define full smoothers such as Q_n, G_n, W_n^*, W_n, A_n, B_n^* and B_n, which are given in the following sections.

4.2.4 The smoother Q_n

Definition 4.3. *The full smoother Q_n is defined as*

$$Q_n = U_n + L_n - I. \tag{4.5}$$

Heuristically, the following argument seems sound. Since U_n removes downward pulses and L_n upward pulses, the sum should be twice the signal plus both impulses. Subtracting the signal plus both impulses, namely the sequence x, should leave virtually only the signal.

For $n \geq 0$, remarks on Q_n follow from Rohwer (2005, pp. 27, 18, 19):

Remark 4.4.

(1) $L_n \leq Q_n \leq U_n$.

(2) $Q_n L_n = U_n L_n \quad and \quad Q_n U_n = L_n U_n$.

(3) $(L_n Q_n)^2 = (L_n Q_n)^3 \quad and \quad (U_n Q_n)^2 = (U_n Q_n)^3$.

(4) Q_n *is selfdual, since* $Q_n N = U_n N + L_n N - N = I - U_n - L_n = NQ_n$.

(5) Q_n *is not idempotent, since if:* $x = \langle(-1)^i\rangle$, $U_n x = 1$ *and* $L_n x = -1$. *Therefore* $Q_n x = -x$, *and similarly* $(Q_n x)^2 = x$.

(6) Q_n *is not necessarily syntone, and not necessarily a selector. However,* Q_1 *is a syntone selector. It was shown that* $Q_1 = M_1$, *but this identity only holds for* $n = 1$.

The above remarks have a remarkable similarity to those of the median smoother M_n. This is extremely odd, since Q_n is generally not a selector.

4.2.5 The smoother G_n

Definition 4.4. *The full smoother* G_n *is defined as the average of* $L_n U_n$ *and* $U_n L_n$,

$$G_n = \tfrac{1}{2}[L_n U_n + U_n L_n]. \tag{4.6}$$

The following remarks are made by Rohwer (2005, p. 27) on G_n:

Remark 4.5.

(1) $L_n \leq G_n \leq U_n$.

(2) $G_n L_n = U_n L_n \quad and \quad G_n U_n = L_n U_n$.

(3) $(L_n G_n)^2 = (L_n G_n)^3 \quad and \quad (U_n G_n)^2 = (U_n G_n)^3$.

4.2.6 The full smoothers W_n^*, W_n and A_n

In Chapter 2 Winsorised smoothers were defined in general in Definition 2.30. The Winsorised smoother, in terms of the smoothers $L_n U_n$ and $U_n L_n$, is defined as follows (Rohwer, 2005, p. 36):

Definition 4.5. *The smoother W_n^* is defined as*

$$W_n^*(x)_i = \begin{cases} x_i & \text{if} \quad x_i \in [(U_nL_nx)_i;\ (L_nU_nx)_i], \\ (U_nL_nx)_i & \text{if} \quad x_i < (U_nL_nx)_i, \\ (L_nU_nx)_i & \text{if} \quad x_i > (L_nU_nx)_i. \end{cases} \tag{4.7}$$

Clearly $W_n^* \in [U_nL_n;\ L_nU_n]$.

An extension of the Winsorised smoother of Rohwer (2005), the compound smoother in terms of L_nU_n and U_nL_n, is defined as follows (cf. Definition 2.31):

Definition 4.6. *The compound smoother W_n is defined by*

$$W_n(x)_i = \begin{cases} (W_{n-1}x)_i & \text{if} \quad (W_{n-1}x)_i \in [(U_nL_nx)_i;\ (L_nU_nx)_i], \\ (U_nL_nx)_i & \text{if} \quad (W_{n-1}x)_i < (U_nL_nx)_i, \\ (L_nU_nx)_i & \text{if} \quad (W_{n-1}x)_i > (L_nU_nx)_i, \end{cases} \tag{4.8}$$

with $W_0 = I$.

This smoother is a new smoother and has some appealing properties which will be highlighted in Section 4.8. An alternative to W_n^* is the smoother A_n defined below (Rohwer, 2005, p. 37).

Definition 4.7. *The smoother A_n is defined as*

$$A_n(x)_i = \begin{cases} x_i & \text{if} \quad x_i \in [(U_nL_nx)_i;\ (L_nU_nx)_i], \\ G_n = \frac{1}{2}[(L_nU_nx)_i + (U_nL_nx)_i] & \text{elsewhere}. \end{cases} \tag{4.9}$$

A_n is often useful for it was found to behave somewhat better than W_n^* (Rohwer, 2005, p. 37).

4.2.7 The smoothers C_n, F_n and the full smoothers B_n^*, B_n

Definition 4.8. *The smoothers C_n and F_n are recursively defined as follows (Rohwer, 2005, p. 40):*

$$C_1 = L_1U_1 \quad \text{and} \quad C_n = L_nU_nC_{n-1}, \tag{4.10}$$

and

$$F_1 = U_1L_1 \quad \text{and} \quad F_n = U_nL_nF_{n-1}, \tag{4.11}$$

with $C_0 = F_0 = I$.

Remarks on C_n and F_n found in Rohwer (2005, p. 40, 64) follow:

Remark 4.6.

(1) *It is clear that* $C_n = L_nU_nL_{n-1}U_{n-1}\dots\dots\dots L_2U_2L_1U_1$ *and* $F_n = U_nL_nU_{n-1}L_{n-1}\dots\dots\dots U_2L_2U_1L_1$*. They are respectively called the* ***ceiling*** *smoother and the* ***flooring*** *smoother, since* C_n *smoothes from the bottom and is an upper smoother, while* F_n *smoothes from the top and is a lower smoother.*

(2) $U_nL_n \leq F_n \leq C_n \leq L_nU_n$*. This property shows that the interval* $[F_n;\ C_n]$ *narrows the interval* $[U_nL_n;\ L_nU_n]$*.*

(3) *For* $n = \max(k, m)$*,* $C_mC_k = C_n$ *and* $F_mF_k = F_n$*. Thus* C_n *and* F_n *are idempotent, that is* $(C_n)^2 = C_n$ *and* $(F_n)^2 = F_n$*.*

Replacing L_nU_n and U_nL_n in W_n^* (cf. Definition 4.5) by C_n and F_n, leads to the following Winsorised smoother:

Definition 4.9. *The smoother* B_n^* *is defined as*

$$B_n^*(x)_i = \begin{cases} x_i & \text{if} \quad x_i \in [(F_nx)_i;\ (C_nx)_i], \\ (F_nx)_i & \text{if} \quad x_i < (F_nx)_i, \\ (C_nx)_i & \text{if} \quad x_i > (C_nx)_i. \end{cases} \tag{4.12}$$

B_n^* can be extended to the compound smoother B_n, defined below:

Definition 4.10. *The compound smoother* B_n *is defined as*

$$B_n(x)_i = \begin{cases} (B_{n-1}x)_i & \text{if} \quad (B_{n-1}x)_i \in [(F_nx)_i;\ (C_nx)_i], \\ (F_nx)_i & \text{if} \quad (B_{n-1}x)_i < (F_nx)_i, \\ (C_nx)_i & \text{if} \quad (B_{n-1}x)_i > (C_nx)_i, \end{cases} \tag{4.13}$$

with $B_0 = I$*.*

This is also a new smoother with some appealing properties which will be highlighted in Section 4.8.

This will be studied in Chapters 5 and 6, and applied in Chapter 7.

Remark 4.7.

(1) *It is clear from Definition 4.10 that* $F_n \leq B_n \leq C_n$*.*

(2) *From Remark 4.6(2) it follows that*

$$U_nL_n \leq F_n \leq B_n \leq C_n \leq L_nU_n$$

.

(3) *These results are used throughout this thesis to compare the performance of the LULU smoothers to the median smoother.*

4.3 Monotonicity

Monotonicity has been discussed in Section 2.3.4 with Definition 2.13 defining an n-monotone sequence (LOMO($n+2$) sequence).

Rohwer (2005, p. 22) proves the following theorems in this section on the monotonicity of LULU smoothers:

Theorem 4.4.

(1) For each n, $L_nx = U_nx = x$ if and only if x is n-monotone.

(2) For each n, $L_nU_nx = U_nL_nx = x$ if and only if x is n-monotone.

This means that the smoothers L_nx, U_nx, L_nU_nx and U_nL_nx will result in the original series x if the series is n-monotone. The LULU smoothers result in a root signal as defined in Definition 2.14 if the series is n-monotone.

Remark 4.8. *$U_nL_nx = L_nU_nx$ does not imply that x is n-monotone.*

Remark 4.8 is in contrast with the case where $U_nx = L_nx$. Here $L_n \leq I \leq U_n$ proves that $U_nx = L_nx = x$. Since generally it is not true that x lies between U_nL_nx and L_nU_nx, $U_nL_nx = L_nU_nx$ generally does not imply that $x = L_nU_nx = U_nL_nx$. The following theorem is proved in Rohwer (2005, p. 23).

Theorem 4.5. *For each integer $n \geq 0$, $U_nL_n \leq M_n \leq L_nU_n$.*

Corollary 4.2. *$U_nL_nx = x$ if and only if x is n-monotone, and $U_nL_xx = x$ if and only if $x = L_nU_nx$.*

Theorem 4.6. *For each n, $W_n^*x = x$ if and only if x is n-monotone.*

Proof

Theorem 4.4(2) states that if $x \in \mathcal{M}_n$, then $U_nL_nx = x = L_nU_nx$.
Since $U_nL_n \leq L_nU_n$ and from Corollary 4.2 it follows that if $x \in \mathcal{M}_n$ then $U_nL_nx = L_nU_nx = x$.
Since $U_nL_n \leq W_n^* \leq L_nU_n$, it follows that $W_n^*x = x$.

Conversely, let $W_n^*x = x$. Then $U_nL_nx = x = L_nU_nx$.

From Theorem 4.4(2) this implies that x is n-monotone.

Corollary 4.3. *For each n, $W_nx = x$ if and only if x is n-monotone.*

Theorem 4.7. *For each n, $C_n x = F_n x = x$ if and only if x is n-monotone.*

Proof

Theorem 4.4(2) states that if $x \in \mathcal{M}_n$, then $U_n L_n x = x = L_n U_n x$.
From Remark 4.6(2) it follows that if $x \in \mathcal{M}_n$, then

$$U_n L_n x = F_n x = C_n x = L_n U_n x = x,$$

i.e. if $x \in \mathcal{M}_n$, then $C_n x = x$ and $F_n x = x$.

Conversely, let $C_n x = F_n x = x$. Then

$$\begin{aligned} U_n L_n x &= U_n L_n (F_n x) \\ &= U_n L_n (U_n L_n F_{n-1} x) \\ &= U_n L_n F_{n-1} x \\ &= F_n x \\ &= x. \end{aligned}$$

Also

$$\begin{aligned} L_n U_n x &= L_n U_n (C_n x \\ &= L_n U_n (L_n U_n C_{n-1} x) \\ &= L_n U_n C_{n-1} x \\ &= C_n x \\ &= x. \end{aligned}$$

From Theorem 4.4(2) this implies that x is n-monotone.

Theorem 4.8. *For each n, $B_n x = x$ if and only if x is n-monotone.*

Proof

Theorem 4.4(2) states that if $x \in \mathcal{M}_n$, then $U_n L_n x = x = L_n U_n x$.
From Remarks 4.6(2) and 4.7(1) it follows that if $x \in \mathcal{M}_n$ then $C_n x = F_n x = x$.
Since $F_n \leq B_n \leq C_n$, it follows that $B_n x = x$.

Conversely, let $B_n x = x$. Then $F_n x = x = C_n x$.

From Theorem 4.4(2) this implies that x is n-monotone.

Corollary 4.4. *For each n, $B_n^* x = x$ if and only if x is monotone.*

The following two theorems are proved in Rohwer (2005, p. 24):

Theorem 4.9. *For each sequence $x \in \mathcal{M}_0$ (i.e. x is 0-monotone), U_nL_nx and L_nU_nx are n-monotone.*

Theorem 4.10. *For $x = U_nx$ or $x = L_nx$, $M_nx = L_nU_nx = U_nL_nx$.*

The following theorem will be proved.

Theorem 4.11. *For each sequence $x \in \mathcal{M}_0$, C_nx and F_nx are n-monotone.*

Proof

Since $C_nx = L_nU_n(C_{n-1}x)$, the result for C_n follows immediately from Theorem 4.9.
Similarly for F_n, since $F_nx = U_nL_n(F_{n-1}x)$.

The theorems in this section can be summarised as follows:

1. For each n,
    - $L_nx = U_nx = x$
    - $L_nU_nx = U_nL_nx = x$
    - $W_n^*x = x$
    - $W_nx = x$
    - $C_nx = F_nx = x$
    - $B_n^*x = x$
    - $B_nx = x$

    if and only if x is n-monotone.
2. For each sequence $x \in \mathcal{M}_0$,
    - L_nU_n and U_nL_n
    - C_n and F_n

    are n-monotone.
3. For $x = U_nx$ or $x = L_nx$, it follows that $M_nx = L_nU_nx = U_nL_nx$.

4.4 Idempotency and co-idempotency

Consistency is defined in Section 2.3.3. This leads to the definitions of idempotency and co-idempotency of a smoother in the same section.

One of the properties that most LULU smoothers have is idempotency. This property means that no further smoothing occurs when the same LULU smoother is applied for a second time. Thus the sequence is invariant under the second or any further application of the same LULU smoother. This means that a root signal has been constructed after one pass of that particular LULU smoother.

From Remark 4.2(2) it follows that L_n and U_n are both idempotent and from Remark 4.3(1) it follows that that both L_nU_n and U_nL_n are also idempotent. From Remark 4.6(3) it follows that C_n and F_n are idempotent. The following theorems prove that the compound Winsorised smoother W_n and compound LULU smoother B_n are idempotent.

Theorem 4.12. *For each nonnegative integer n, $W_n(W_n(x))_i = W_n(x)_i$, i.e. W_n is idempotent.*

Proof

From Definition 4.6 it follows that $W_n(x)_i = \begin{cases} (W_{n-1}x)_i & \text{if} \quad (W_{n-1}x)_i \in [(U_nL_nx)_i;\ (L_nU_nx)_i], \\ (U_nL_nx)_i & \text{if} \quad (W_{n-1}x)_i < (U_nL_nx)_i, \\ (L_nU_nx)_i & \text{if} \quad (W_{n-1}x)_i > (L_nU_nx)_i. \end{cases}$

This means that $W_n(x)_i$ is equal to $(U_nL_nx)_i$, or $(L_nU_nx)_i$ for $(W_{n-1}x)_i$-values outside $[(U_nL_nx)_i;\ (L_nU_nx)_i]$ and equal to $(W_{n-1}x)_i$ for $(W_{n-1}x)_i$-values inside $[(U_nL_nx)_i;\ (L_nU_nx)_i]$.

(1) Consider $W_n(x)_i = (U_nL_nx)_i$:
$U_nL_nW_n(x)_i = U_nL_n(U_nL_nx)_i = (U_nL_nx)_i$
and
$L_nU_nW_n(x)_i = L_nU_n(U_nL_nx)_i = (L_nU_nL_nx)_i = (U_nL_nx)_i$.
Thus for $W(x)_i = (U_nL_nx)_i$ the application of U_nL_n or L_nU_n on $W_n(x)_i = (U_nL_nx)_i$ does not change $W_n(x)_i$.

(2) Consider $W_n(x)_i = (L_nU_nx)_i$:
$U_nL_nW_n(x)_i = U_nL_n(L_nU_nx)_i = (U_nL_nU_nx)_i = (L_nU_nx)_i$
and
$L_nU_nW_n(x)_i = L_nU_n(L_nU_nx)_i = (L_nU_nx)_i$.
Thus for $W_n(x)_i = (L_nU_nx)_i$ the application of U_nL_n or L_nU_n on $W_n(x)_i = (L_nU_nx)_i$ does not change $W_n(x)_i$.

(3) Consider $W_n(x)_i = (W_{n-1}x)_i$ for $(U_nL_nx)_i < (W_{n-1}x)_i < (L_nU_nx)_i$:

Since

- $U_nL_n(U_nL_nx)_i = (U_nL_nx)_i$, $(W_{n-1}x)_i > U_nL_n(U_nL_nx)_i$ if $(W_{n-1}x)_i > (U_nL_nx)_i$,
- $L_nU_n(U_nL_nx)_i = (L_nU_nL_nx)_i = (U_nL_nx)_i$, $(W_{n-1}x)_i > L_nU_n(U_nL_nx)_i$ if $(W_{n-1}x)_i > (U_nL_nx)_i$,
- $L_nU_n(L_nU_nx)_i = (L_nU_nx)_i$, $(W_{n-1}x)_i < L_nU_n(L_nU_nx)_i$ if $(W_{n-1}x)_i < (L_nU_nx)_i$,
- $U_nL_n(L_nU_nx)_i = (U_nL_nU_nx)_i = (L_nU_nx)_i$, $(W_{n-1}x)_i < U_nL_n(L_nU_nx)_i$ if $(W_{n-1}x)_i < (L_nU_nx)_i$,

it follows that if $(U_nL_nx)_i < W_n(x)_i = (W_{n-1}x)_i < (L_nU_nx)_i$ the application of U_nL_n or L_nU_n on $W_n(x)_i$ does not change the inequality $(U_nL_nx)_i < W_n(x)_i = (W_{n-1}x)_i < (L_nU_nx)_i$. In other words, the $(W_{n-1}x)_i$-values inside $[(U_nL_nx)_i;\ (L_nU_nx)_i]$ will still be inside the upper and lower limits.

Thus combining (1), (2) and (3) implies that $W_n(W_n(x))_i = W_n(x)_i$. The smoother W_n is thus idempotent.

Along the same lines it can be proved that the Winsorised smoother W_n^* is idempotent.

Theorem 4.13. *For each nonnegative integer n, $B_n(B_n(x))_i = B_n(x)_i$, i.e. B_n is idempotent.*

Proof

From Definition 4.10 it follows that $B_n(x)_i = \begin{cases} (B_{n-1}x)_i & \text{if } (B_{n-1}x)_i \in [(F_nx)_i;\ (C_nx)_i], \\ (F_nx)_i & \text{if } (B_{n-1}x)_i < (F_nx)_i, \\ (C_nx)_i & \text{if } (B_{n-1}x)_i > (C_nx)_i. \end{cases}$

This means that $B_n(x)_i$ is equal to $(F_nx)_i$, or $(C_nx)_i$ for $(B_{n-1}x)_i$-values outside $[(F_nx)_i;\ (C_nx)_i]$ and equal to $(B_{n-1}x)_i$ for $(B_{n-1}x)_i$-values inside $[(F_nx)_i;\ (C_nx)_i]$.

(1) Consider $B_n(x)_i = (F_nx)_i$:
From Remark 4.6(3) it follows that
$F_nB_n(x)_i = F_n(F_nx)_i = (F_nx)_i$.
From Theorem 4.11 it follows that F_nx is n-monotone and hence from Theorem 4.7 it follows that
$C_nB_n(x)_i = C_n(F_nx)_i = (F_nx)_i$.
Thus for $B(x)_i = (F_nx)_i$ the application of F_n or C_n on $B_n(x)_i = (F_nx)_i$ does not change $B_n(x)_i$.

(2) Consider $B_n(x)_i = (C_nx)_i$:
$F_nB_n(x)_i = F_n(C_nx)_i = (C_nx)_i$.
From Theorem 4.11 it follows that C_nx is n-monotone and hence from Theorem 4.7 it follows that
$C_nB_n(x)_i = C_n(C_nx)_i = (C_nx)_i$.
Thus for $B_n(x)_i = (C_nx)_i$ the application of F_n or C_n on $B_n(x)_i = (C_nx)_i$ does not change $B_n(x)_i$.

(3) Consider $B_n(x)_i = (B_{n-1}x)_i$ for $(F_nx)_i < (B_{n-1}x)_i < (C_nx)_i$:

Since

- $F_n(F_nx)_i = (F_nx)_i,\ (B_{n-1}x)_i > F_n(F_nx)_i$ if $(B_{n-1}x)_i > (F_nx)_i$,
- $C_n(F_nx)_i = (F_nx)_i,\ (B_{n-1}x)_i > C_n(F_nx)_i$ if $(B_{n-1}x)_i > (F_nx)_i$,
- $C_n(C_nx)_i = (C_nx)_i,\ (B_{n-1}x)_i < C_n(C_nx)_i$ if $(B_{n-1}x)_i < (C_nx)_i$,
- $F_n(C_nx)_i = (C_nx)_i,\ (B_{n-1}x)_i < F_n(C_nx)_i$ if $(B_{n-1}x)_i < (C_nx)_i$,

it follows that if $(F_nx)_i < B_n(x)_i = (B_{n-1}x)_i < (C_nx)_i$ the application of F_n or C_n on $B_n(x)_i$ does not change the inequality $(F_nx)_i < B_n(x)_i = (B_{n-1}x)_i < (C_nx)_i$. Thus, the $(B_{n-1}x)_i$-values inside $[(F_nx)_i;\ (C_nx)_i]$ will still be inside the upper and lower limits.

Thus combining (1), (2) and (3) implies that $B_n(B_n(x))_i = B_n(x)_i$. The smoother B_n is idempotent.

A smoother P is co-idempotent if $(I-P)^2 = (I-P)$. This means that the residual of a sequence is invariant under any further smoothing by P and is noise-consistent, i.e. there is no signal left in the residual. Idempotent nonlinear smoothers are not always co-idempotent. This means that a signal-consistent (no noise left in the signal) smoother is not necessarily noise-consistent, but may still contain signal in the residual. An easy way to test for co-idempotency is whether $P(I-P) = 0$, where 0 is the zero operator (Rohwer, 2005, p. 5).

The following theorem and corollary on co-idempotency of LULU smoothers are proved by Rohwer (2005, p. 38):

Theorem 4.14. $U_nL_n(I - U_nL_n) = 0$ *and* $L_nU_n(I - L_nU_n) = 0$.

Corollary 4.5.

(1) $L_n(I - U_nL_n) \leq 0$ *and* $U_n(I - L_nU_n) \geq 0$.

(2) $L_nU_n(U_nL_n - I) = 0$ *and* $U_nL_n(L_nU_n - I) = 0$.

The following theorem and corollary prove that C_n and F_n are co-idempotent (Rohwer, 2005, pp. 66, 67).

Theorem 4.15. $C_j(I - C_n) = C_j - C_n$, *for* $j \leq n$ *and* $F_j(I - F_n) = F_j - F_n$, *for* $j \leq n$.

Corollary 4.6. *For* $j = n :\ C_n(I - C_n) = C_n - C_n = 0$ *and* $F_n(I - F_n) = F_n - F_n = 0$.

From a number of practical applications it seems that W_n^*, W_n, B_n^* and B_n are all co-idempotent. This leads to the following conjecture.

Conjecture 4.1. *The Winsorised smoothers* W_n^* *and* B_n^* *and the compound smoothers* W_n *and* B_n *are all co-idempotent, i.e.* $(I - W_n^*)^2 = (I - W_n^*)$, $(I - B_n^*)^2 = (I - B_n^*)$, $(I - W_n)^2 = (I - W_n)$ *and* $(I - B_n)^2 = (I - B_n)$.

The relation between the monotonicity of a sequence, and idempotency and co-idempotency of a smoother, is summarised in Tables 4.2, 4.3 and 4.4. In Table 4.2 P_n denotes one of the LULU smoothers $L_n, U_n, L_nU_n, U_nL_n, W_n^*, W_n, C_n, F_n, B_n^*$ and B_n, with s_{ij} the j-th smoothed series which is i-monotone given that the input series is j-monotone.

In Table 4.3 the output of stepwise smoothing with smoothers C_n and F_n is tabulated for different degrees of monotonicity of the input sequence x. Let s_{ij} be the j-th smoothed series which is i-monotone, and $s_{ij} = L_iU_i(s_{(i-1)\ j})$, and $t_{ij} = U_iL_i(s_{(i-1)\ j})$.

Since for $n > 1$, $F_n = U_nL_n(F_{n-1})$ with $F_1 = U_1L_1$ and $C_n = L_nU_n(C_{n-1})$ with $C_1 = L_1U_1$, the smoothers F_n and C_n smooth a sequence in a stepwise manner by "peeling off" variation systematically. This leads to the compound LULU smoother B_n which is a type of trimmed smoother where that part of the sequence between F_n and C_n, which was not smoothed by B_m, $m < n$, is preserved. The amount of variation "peeled off" at each level of applying B_n can be calculated. Based on this, a decision can be made if any further smoothing is useful.

The output of stepwise smoothing by the compound smoother B_n is tabulated in Table 4.4 for different degrees of monotonicity of the input sequence x. Here s_{ij} represents the j-th smoothed series which is i-monotone and $s_{ij} = B_i(s_{(i-1)\ j})$.

Table 4.2: Output of idempotent and co-idempotent smoother P_n given different degrees of monotonicity of the input sequence x

	Monotonicity of input sequence x							
P_n	1	2	3	.	$n-1$	n	$n+1$	$n+2$
P_1x	x	x	x	x	x	x	x	x
$P_1(P_1x)$	x	x	x	x	x	x	x	x
$P_1(I-P_1)x$	0	0	0	0	0	0	0	0
P_2x	s_{21}	x	x	x	x	x	x	x
$P_2(P_2x)$	s_{21}	x	x	x	x	x	x	x
$P_2(I-P_2)x$	0	0	0	0	0	0	0	0
P_3x	s_{31}	s_{32}	x	x	x	x	x	x
$P_3(P_3x)$	s_{31}	s_{32}	x	x	x	x	x	x
$P_3(I-P_3)x$	0	0	0	0	0	0	0	0
	.	.	.	.	.	.	.	.
P_nx	s_{n1}	s_{n2}	s_{n3}	.	$s_{n\ (n-1)}$	x	x	x
$P_n(P_nx)$	s_{n1}	s_{n2}	s_{n3}	.	$s_{n\ (n-1)}$	x	x	x
$P_n(I-P_n)x$	0	0	0	.	0	0	0	0

Table 4.3: Stepwise smoothing with C_n and F_n given different degrees of monotonicity of the input sequence x

	Monotonicity of input sequence x							
P_n	1	2	3	.	$n-1$	n	$n+1$	$n+2$
$C_1x = L_1U_1x$	x	x	x	x	x	x	x	x
$F_1x = U_1L_1x$	x	x	x	x	x	x	x	x
$C_2x = L_2U_2C_1x$	s_{21}	x	x	x	x	x	x	x
$F_2x = U_2L_2F_1x$	t_{21}	x	x	x	x	x	x	x
$C_3x = L_3U_3C_2x$	s_{31}	s_{32}	x	x	x	x	x	x
$F_3x = U_3L_3F_2x$	t_{31}	t_{32}	x	x	x	x	x	x
	.	.	.	.	.	.	.	.
$C_nx = L_nU_nC_{n-1}x$	s_{n1}	s_{n2}	s_{n3}	.	$s_{n\ (n-1)}$	x	x	x
$F_nx = U_nL_nF_{n-1}x$	t_{n1}	t_{n2}	t_{n3}	.	$t_{n\ (n-1)}$	x	x	x

Table 4.4: Stepwise smoothing by B_n given different degrees of monotonicity of the input sequence x

	Monotonicity of input sequence x							
B_n	1	2	3	.	$n-1$	n	$n+1$	$n+2$
B_1x	x	x	x	x	x	x	x	x
B_2x	s_{21}	x	x	x	x	x	x	x
B_3x	s_{31}	s_{32}	x	x	x	x	x	x
	.	.	.	.	.	.	.	.
B_nx	s_{n1}	s_{n2}	s_{n3}	.	$s_{n\ (n-1)}$	x	x	x

4.5 Stability

Stability as defined in Section 2.3.5 is an important criterion for the design and comparison of smoothers. The concept of robustness complements the stability of a smoother.

The stability of LULU smoothers will be investigated by studying the way they treat a single impulse (spike) value, or impulses of length greater than one (blockpulses). The way LULU smoothers treat blockpulses will be discussed in Section 4.5.1.

4.5.1 Blockpulses

An n-blockpulse is defined in Definition 2.19.

In terms of the LULU operators, remarks from Rohwer (2005, pp. 15, 16, 17) on blockpulses follow:

Remark 4.9.

(1) *Let x be an upward n-blockpulse and z a downward n-blockpulse. For each integer $k \geq 0$ the following are true:*

- $\wedge^k x$ *is an upward blockpulse of length* $\max\{0;\ n-k\}$
- $\vee^k x$ *is an upward blockpulse of length* $n+k$
- $\wedge^k z$ *is a downward blockpulse of length* $n+k$
- $\vee^k z$ *is a downward blockpulse of length* $\max\{0;\ n-k\}$

Thus upward blockpulses are narrowed by $\wedge$ and widened by $\vee$, while downward blockpulses are widened by $\wedge$ and narrowed by $\vee$.

(2) *If x is an upward blockpulse of length n, then*

$$\begin{aligned} L_m x &= \begin{cases} 0 & \text{if} \quad m \geq n, \\ x & \text{if} \quad 0 \leq m \leq n, \quad \text{and} \end{cases} \\ U_m x &= \;\; x \quad \text{if} \quad m \geq 0. \end{aligned}$$

(3) *Blockpulses are dual, since if x is an upward n-blockpulse, Nx is a downward n-blockpulse and $L_m N = NU_m$, since L_m and U_m are dual smoothers.*

(4) *If blockpulses of length $k \leq n$ are considered as noise, then L_n and U_n are smoothers for the upward and downward blockpulses respectively.*

(5) *$L_n U_n$ would remove both upward and downward blockpulses of length $\leq n$, since U_n removes the downward blockpulse and preserves the upward pulses, which are then removed by L_n. Similarly $U_n L_n$ removes both types.*

(6) *If the length of a blockpulse is greater than n, it will be considered as signal and will be perfectly preserved by both $L_n U_n$ and $U_n L_n$.*

An illustration of how LULU operators treat blockpulses is given in Figure 4.3. A signal with a positive impulse of length three and a negative impulse of length three is used as an example. The smoothers L_1U_1, U_1L_1, L_2U_2 and U_2L_2 all preserve the blockpulses of length three, and thus only L_2U_2 and U_2L_2 are illustrated in the graph. From the graph it can be seen that the smoothers L_3U_3 and U_3L_3 remove the blockpulses of length three.

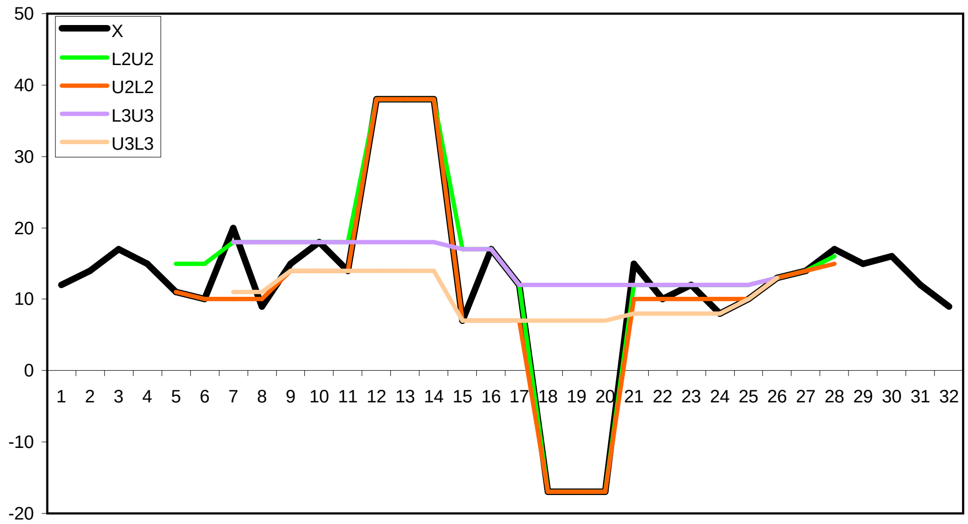

Figure 4.3: LULU smoothers on a signal with impulses of length three

4.6 Probability distributions of LULU smoothers

As in the case with most nonlinear smoothers, it is difficult to analytically derive the distributions of LULU smoothers in general. However, a fair amount of work has been done for the cases of independent identically distributed data and non-identically distributed data (Conradie *et al.* , 2006). The exact and asymptotic distributions in these cases will be derived in Sections 4.6.1, 4.6.2 and 4.6.3.

4.6.1 Exact distributions of L_nU_n and U_nL_n in the independent identically distributed case

Let $X = (\ldots, X_{-3}, X_{-2}, X_{-1}, X_0, X_1, X_2, X_3, \ldots)$ be a double infinite sequence of independent identically distributed random variables (iid's) with a cumulative distribution function (cdf) F_X. Write

$$\begin{aligned} Y_i &= \vee(X_i, X_{i+1}, \ldots, X_{i+n}) = (\vee^n X)_i \\ Z_i &= \wedge(Y_i, Y_{i-1}, \ldots, Y_{i-n}) = (\wedge^n Y)_i \\ A_i &= \wedge(Z_i, Z_{i-1}, \ldots, Z_{i-n}) = (\wedge^n Z)_i \\ B_i &= \vee(A_i, A_{i+1}, \ldots, A_{i+n}) = (\vee^n A)_i \end{aligned}$$

then L_nU_n can be written as

$$(L_nU_nX)_i = (\vee^n \wedge^n \wedge^n \vee^n X)_i = B_i.$$

The $(L_nU_nX)_i$ are again iid's with cdf denoted by

$$F_{(L_nU_nX)_i}(x) \equiv F_{L_nU_n(X)}(x) = P(B_i \leq x) = P(A_i \leq x,\ \ldots, A_{i+n} \leq x).$$

$L_nU_n(X)$ is written for an element $(L_nU_nX)_i$.

Since clearly $L_nU_n(-X) = -U_nL_n(X)$, it follows that

$$\begin{aligned} F_{U_nL_n(X)}(x) &= P(U_nL_n(X) \leq x) \\ &= P(-L_nU_n(-X) \leq x) \\ &= P(L_nU_n(-X) > -x) \\ &= 1 - F_{L_nU_n(-X)}(-x). \end{aligned} \tag{4.14}$$

The cdf of $L_nU_n(X)$ is derived first and then (4.14) is applied to obtain the cdf of $U_nL_n(X)$.

To derive $F_{L_nU_n}(x)$ the following two lemmas are needed.

Lemma 4.1.

Write

$$P(Y_1 > x, Y_2 > x,\ \ldots, Y_{r-1} > x, Y_r \leq x) = m_r^Y(x)$$

then

$$m_r^Y(x) = \begin{cases} d_n(x) & \textit{if} \quad r = 2,\ \ldots, n+2, \\ q^Y_{r-(n+2)}(x)d_n(x) & \textit{if} \quad r \geq n+3, \end{cases} \tag{4.15}$$

where $d_n(x) = (1 - F_X(x))F_X^{n+1}$ *and* $q_r^Y = P(Y_1 > x, Y_2 > x,\ \ldots, Y_r > x)$.

Proof

Note that

$$\{Y_r \leq x\} \cap \{Y_{r-1} > x\} = \{X_{r-1} > x, Y_r \leq x\} = \{X_{r-1} > x, X_r \leq x, X_{r+1} \leq x,\ \ldots, X_{r+n} \leq x\}.$$

Since

$$\{X_{r-1} > x\} \subset \{Y_k > x\} \quad \text{if} \quad k = r-n-1,\ \ldots, r-1$$

then

$$\{Y_{r-n-1} > x,\ \ldots, Y_{r-2} > x\} \cap \{X_{r-1} > x\} = \{X_{r-1} > x\}.$$

Thus

$$\begin{aligned} &\{Y_1 > x, Y_2 > x,\ \ldots, Y_{r-1} > x, Y_r \leq x\} \\ &= \{Y_1 > x,\ \ldots, Y_{r-n-2} > x\} \cap \{Y_{r-n-1} > x,\ \ldots, Y_{r-2} > x\} \cap \{Y_{r-1} > x, Y_r \leq x\} \\ &= \{Y_1 > x,\ \ldots, Y_{r-n-2} > x\} \cap \{Y_{r-n-1} > x,\ \ldots, Y_{r-2} > x\} \cap \{X_{r-1} > x\} \cap \{Y_r \leq x\} \\ &= \{Y_1 > x,\ \ldots, Y_{r-n-2} > x\} \cap \{X_{r-1} > x\} \cap \{Y_r \leq x\} \\ &= \{Y_1 > x,\ \ldots, Y_{r-n-2} > x\} \cap \{X_{r-1} > x\ \ X_r \leq x,\ \ldots, X_{r+n} \leq x\}. \end{aligned}$$

Note that $\{Y_1 > x, \ldots, Y_{r-n-2} > x\}$ is a function only of X_k's, with $k \leq r-2$. This gives

$$m_r^Y(x) = q_{r-(n+2)}^Y(x)(1 - F_X(x))F_X^{n+1}(x).$$

Here $r - n - 2 \geq 1$, which gives $r \geq n+3$.

(Note: $Y_{r-n-2} = \vee(X_{r-n-2}, \ldots, X_{r-2})$ and $Y_1 = \vee(X_1, \ldots, X_{n+1})$ which shows that $r-n-2$ must be greater or equal to 1, i.e. $r \geq n+3$.)

This proves the second expression in (4.15).

Further, for $r \leq n+2$, it follows that

$$\{X_{r-1} > x\} \subset \{Y_k > x\} \quad \text{if} \quad k = 1, \ldots, r-2.$$

(Note:$Y_1 = \vee(X_1, \ldots, X_{n+1})$ and since $n+2 \geq r$, then $n+1 \geq r-1$.)

It follows that for such r

$$\begin{aligned} & \{Y_1 > x, \ldots, Y_{r-2} > x\} \cap \{Y_{r-1} > x\} \cap \{Y_r \leq x\} \\ = & \{Y_1 > x, \ldots, Y_{r-2} > x\} \cap \{X_{r-1} > x\} \cap \{Y_r \leq x\} \\ = & \{X_{r-1} > x\} \cap \{Y_r \leq x\}. \end{aligned}$$

Thus

$$\begin{aligned} m_r^Y &= P(X_{r-1} > x, X_r \leq x, \ldots, X_{r+n} \leq x) \\ &= (1 - F_X(x))F_X^{n+1}(x). \end{aligned}$$

This proves the first expression in (4.15) and thus the lemma.

Lemma 4.2.

For $n = 1, 2, \ldots,$

$$\begin{aligned} q_r^Y(x) &= P(Y_1 > x, Y_2 > x, \ldots, Y_r > x) \\ &= \begin{cases} 1 - F_X^{n+1}(x) & \text{if} \quad r = 1, \\ q_1^Y(x) - (r-1)d_n(x) & \text{if} \quad r = 2, \ldots, n+2, \\ q_{n+2}^Y(x) - d_n(x) \displaystyle\sum_{k=1}^{r-(n+2)} q_k^Y(x) & \text{if} \quad r \geq n+3. \end{cases} \end{aligned} \tag{4.16}$$

Proof

For $r = 1$ the result is trivial.

Consider $r \geq 2$.

Note that $q_r^Y(x)$ can always be written as

$$q_r^Y(x) = q_{r-1}^Y(x) - m_r^Y(x).$$

Iterating this expression gives, for $s \leq r-1$

$$q_r^Y(x) = q_s^Y(x) - \sum_{k=s+1}^{r} m_k^Y(x).$$

Choosing successively $s = 1$ and $s = n+2$ in (4.15) results in the second and third expressions in (4.16) respectively. This proves the lemma.

Theorem 4.16.

For $n = 1, 2, \ldots$, the cdf of L_nU_n is

$$\begin{aligned} F_{L_nU_n(X)}(x) &= F_X^{n+1}(x) + n(1-F_X(x))F_X^{n+1}(x) + (1-F_X(x))F_X^{2(n+1)}(x) \\ &\quad + \tfrac{1}{2}(n-1)(n+2)(1-F_X(x))^2F_X^{2(n+1)}(x). \end{aligned} \tag{4.17}$$

Proof

$$\begin{aligned} F_{(L_nU_nX)_i}(x) &= P(B_i \leq x) \\ &= P(A_i \leq x, A_{i+1} \leq x, \ldots, A_{i+n} \leq x) \\ &= p_{n+1}^A(x) \text{ (say)}, \end{aligned}$$

for each $i = 0, \pm 1, \pm 2, \ldots$.

Now

$$\begin{aligned} p_{n+1}^A(x) &= P(A_1 \leq x, A_2 \leq x, \ldots, A_{n+1} \leq x) \\ &= P(A_1 \leq x, A_2 \leq x, \ldots, A_n \leq x) - P(A_1 \leq x, A_2 \leq x, \ldots, A_n \leq x, A_{n+1} > x) \\ &= p_n^A(x) - P(A_2 \leq x, A_3 \leq x, \ldots, A_n \leq x, A_{n+1} > x) \\ &\quad + P(A_1 > x, A_2 \leq x, \ldots, A_n \leq x, A_{n+1} > x). \end{aligned}$$

Clearly $P(A_1 > x, A_2 \leq x, \ldots, A_n \leq x, A_{n+1} > x) = 0$, and iterating the above, it follows

$$\begin{aligned} p_{n+1}^A(x) &= p_n^A(x) - P(A_n \leq x, A_{n+1} > x) \\ &= p_{n-1}^A(x) - P(A_{n-1} \leq x, A_n > x) - P(A_n \leq x, A_{n+1} > x) \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots \\ &= p_1^A(x) - \sum_{k=1}^{n} P(A_k \leq x, A_{k+1} > x) \\ &= 1 - P(A_1 > x) - \sum_{k=1}^{n} P(Z_{k-n} \leq x, Z_{k-n+1} > x, \ldots, Z_{k+1} > x) \\ &= 1 - P(Z_{1-n} > x, \ldots, Z_1 > x) - \sum_{k=1}^{n} P(Z_1 \leq x, Z_2 > x, \ldots, Z_{n+2} > x) \\ &= 1 - q_{n+1}^Z(x) - nP(Z_1 \leq x, Z_2 > x, \ldots, Z_{n+2} > x) \\ &= 1 - q_{n+1}^Z(x) - nP(Z_2 > x, Z_3 > x, \ldots, Z_{n+2} > x) + nP(Z_1 > x, Z_2 > x, \ldots, Z_{n+2} > x) \\ &= 1 - (n+1)q_{n+1}^Z(x) + nq_{n+2}^Z(x). \end{aligned}$$

Substituting in terms of the Y's, it follows that

$$p_{n+1}^A(x) = 1 - (n+1)q_{2n+1}^Y(x) + nq_{2n+2}^Y(x).$$

Substituting $q_{2n+1}^Y(x)$ and $q_{2n+2}^Y(x)$ using (4.16), gives

$$\begin{aligned} p_{n+1}^A(x) &= 1 - (n+1)\left\{q_{n+2}^Y(x) - d_n(x)\sum_{k=1}^{n-1} q_k^Y(x)\right\} + n\left\{q_{n+2}^Y(x) - d_n(x)\sum_{k=1}^{n} q_k^Y(x)\right\} \\ &= 1 - q_{n+2}^Y(x) - nd_n(x)q_n^Y(x) + d_n(x)\sum_{k=1}^{n-1} q_k^Y(x). \end{aligned}$$

Substituting $q_{n+2}^Y(x)$ and $q_n^Y(x)$ using (4.16), gives

$$\begin{aligned} p_{n+1}^A(x) &= 1 - \left\{q_1^Y(x) - (n+1)d_n(x)\right\} - nd_n(x)\left\{q_1^Y(x) - (n-1)d_n(x)\right\} + d_n(x)q_1^Y(x) + d_n(x)\sum_{k=2}^{n-1}\left\{q_1^Y(x) - (k-1)d_n(x)\right\} \\ &= 1 - q_1^Y(x) + (n+1)d_n(x) - q_1^Y(x)d_n(x) + d_n^2(x)\left\{n(n-1) - \sum_{k=2}^{n-1}(k-1)\right\} \\ &= 1 - q_1^Y(x) + (n+1)d_n(x) - q_1^Y(x)d_n(x) + \tfrac{1}{2}(n-1)(n+2)d_n^2(x). \end{aligned}$$

Substituting $d_n(x)$ and q_1^Y leads to

$$\begin{aligned} p_{n+1}^A(x) &= F_X^{n+1}(x) + (n+1)(1-F_X(x))F_X^{n+1}(x) - (1 - F_X^{n+1}(x))(1-F_X(x))F_X^{n+1}(x) \\ &\quad + \tfrac{1}{2}(n-1)(n+2)\left\{(1-F_X(x))F_X^{n+1}\right\}^2 \\ &= F_X^{n+1}(x) + n(1-F_X(x))F_X^{n+1}(x) + (1-F_X(x))F_X^{2(n+1)}(x) \\ &\quad + \tfrac{1}{2}(n-1)(n+2)(1-F_X(x))^2F_X^{2(n+1)}(x) \end{aligned}$$

which proves the theorem.

Theorem 4.17.

For $n = 1, 2, \ldots$, the cdf of U_nL_n is

$$\begin{aligned} F_{U_nL_n(X)}(x) =& 1 - (1-F_X(x))^{n+1} - nF_X(x)(1-F_X(x))^{n+1} - F_X(x)(1-F_X(x))^{2(n+1)} \\ & - \tfrac{1}{2}(n-1)(n+2)F_X^2(x)(1-F_X(x))^{2(n+1)}. \end{aligned} \tag{4.18}$$

Proof

Write the cdf of $L_nU_n(X)$ in (4.17) as a function of g of $F_X(x)$, say

$$F_{L_nU_n(X)}(x) = g(F_X(x)).$$

From (4.14) follows

$$\begin{aligned} F_{L_nU_n(X)}(x) &= 1 - F_{L_nU_n(-X)}(-x) \\ &= 1 - g(F_{-X}(-x)). \end{aligned}$$

Remembering that $F_{-X}(-x) = P(-X \le -x) = P(X > x) = 1 - F_X(x)$, it follows that

$$F_{U_nL_n(X)}(x) = 1 - g(1 - F_X(x)).$$

Thus replacing $F_X(x)$ in (4.17) with $1 - F_X(x)$ and subtracting from one leads to (4.18).

Corollary 4.7.

For $n = 1, 2, \ldots$

$$F_{U_n(X)}(x) = F_X^{n+1}(x) + n(1 - F_X(x))F_X^{n+1}(x) \tag{4.19}$$

and

$$F_{L_n(X)}(x) = 1 - (1 - F_X(x))^{n+1} - nF_X(x)(1 - F_X(x))^{n+1}. \tag{4.20}$$

Proof

Using Lemma 4.2 it follows that

$$\begin{aligned} F_{U_n(X)}(x) &= P(Z_i \le x) \\ &= 1 - q_1^Z(x) \\ &= 1 - q_{n+1}^Y(x) \\ &= 1 - q_1^Y(x) + nd_n(x). \end{aligned}$$

This proves (4.19). The proof of (4.20) follows in a similar way.

Remark 4.10. *For the ceiling and flooring smoothers, C_n and F_n, the exact distributions were derived for $n = 2$. For any $n > 2$ the exact distributions of these smoothers become very messy (Conradie & de Wet, 2003).*

Let $F_{C_2(X)}(x) = P(C_2(X) \le x)$, and $F_{F_2(X)}(x) = P(F_2(X) \le x)$, then

$$\begin{aligned} F_{C_2(X)}(x) = {} & 2F_X^{12}(x) - 8F_X^{11}(x) + 8F_X^{10}(x) + 4F_X^9(x) - 10F_X^8(x) + 4F_X^7(x) + 4F_X^6(x) \\ & -9F_X^5(x) + 3F_X^4(x) + 3F_X^3(x). \end{aligned} \tag{4.21}$$

The proof of (4.21) follows in the same fashion as the derivation of the distribution of $L_nU_n(X)$ in Theorem 4.16.

Write $F_{C_2(X)}(x) = W(F_X(x))$, then the distribution of $F_2(X)$ follows by using (4.14) as

$$F_{F_2(X)}(x) = 1 - W(1 - F_X(x)).$$

4.6.2 Exact distributions of L_nU_n and U_nL_n in the non-identically distributed case

Let $\{X_i\}$ be independent with $X_i \sim F_i$ $i = 0, \pm 1, \pm 2, \ldots$ The same notation is used as defined in Section 4.6.1 in terms of Y_i, Z_i, A_i, B_i. Finding the distribution function of $(L_nU_nX)_i \equiv B_i$, or $P(B_i \le x)$ is of interest here. Note that B_i had the running window $\{X_{i-2n}, \ldots, X_i, \ldots, X_{i+2n}\}$ as support.

Notation

For indices used as subscripts, running from l to m, the shorthand l, m will be used to denote the sequence $l, l+1, \ldots, m$.

As before, p and q with superscripts will be used, but now the specific random variables involved have to be indicated more explicitly. This will be done in the subscripts. Thus, for example

$$P(B_i \leq x) = P(A_i \leq x, \ldots, A_{i+n} \leq x) \equiv p^A_{i,\, i+n}(x).$$

As before the process is built upwards,

$$B \to A \to Y \to Z \to X,$$

and then the independence of the X_j's is used to obtain the final expression.

Remark 4.11.

(1) From its definition it follows immediately that

$$q^Y_k(x) = 1 - \prod_{j=k}^{k+n} F_j(x).$$

The following lemma is needed in proving the theorem.

Lemma 4.3. *Let l, m be integers with $l \leq m$.*

(i) If $n + 3 \leq m - l \leq 2n + 2$, then

$$q^Y_{l+1,\, m}(x) = q^Y_{l+1,\, m-1}(x) - q^Y_{l+1,\, m-n-2}(x).d_{m-1,\, n}(x). \tag{4.22}$$

(ii) If $m - l \leq n + 2$, then

$$q^Y_{l+1,\, m}(x) = q^Y_{l+1}(x) - \sum_{r=l+1}^{m-1} d_{r,\, n}(x), \tag{4.23}$$

with

$$q^Y_{l+1}(x) = 1 - \prod_{j=l+1}^{l+n+1} F_j(x).$$

Proof

The proof of both parts follows a similar pattern, using the technique employed previously.

(i) Write

$$q^Y_{l+1,\, m}(x) = q^Y_{l+1,\, m-1}(x) - P(Y_{l+1} > x, \ldots, Y_{m-1} > x, Y_m \leq x).$$

Consider the event in the latter probability:

$$\{Y_m \leq x\} \Rightarrow X_m \leq x, \ldots, X_{m+n} \leq x$$

and if also

$$\{Y_{m-1} > x\} \Rightarrow X_{m-1} > x.$$

If these two inequalities hold, then

$$\{Y_{m-2} > x, \ldots, Y_{m-n-1} > x\}$$

holds, and $Y_{m-n-2}, \ldots, Y_{l+1}$ are independent of $X_{m-1}, \ldots, X_{m+n}$.

These give

$$\begin{aligned} P(Y_{l+1} > x, \ldots, Y_{m-1} > x, Y_m \leq x) &= q^Y_{l+1,\ m-n-2}(x).\,(1 - F_{m-1}(x))\,.\prod_{j=m}^{m+n} F_j(x) \\ &= q^Y_{l+1,\ m-n-2}(x).d_{m-1,\ n}(x). \end{aligned}$$

Substituting in the above gives the desired result.

(ii) Again

$$q^Y_{l+1,\ m}(x) = q^Y_{l+1,\ m-1}(x) - P(Y_{l+1} > x, \ldots, Y_{m-1} > x, Y_m \leq x).$$

Arguing as in (i), but now remembering that $m - l \leq n + 2$ immediately gives

$$q^Y_{l+1,\ m}(x) = q^Y_{l+1,\ m-1}(x) - d_{m-1,\ n}(x).$$

Iterating this gives

$$q^Y_{l+1,\ m}(x) = q^Y_{l+1}(x) - \sum_{r=l+1}^{m-1} d_{r,\ n}(x),$$

and the desired result. The expression for $q^Y_{l+1}(x)$ follows trivially.

This concludes the proof of the lemma.

The following is the main result.

Theorem 4.18. *Let $\{X_i\}$ be independent random variables with $X_i \sim F_i$, $i = 0, \pm 1, \pm 2, \ldots$ With $(L_n U_n X)_i \equiv B_i$, it follows that*

$$\begin{aligned} P(B_i \leq x) = 1 - q^Y_{i-n}(x) + \sum_{r=i-n}^{i} d_{r,\ n}(x) + \sum_{r=i+1}^{i+n-1} d_{r,\ n}(x)\left[q^Y_{i-n}(x) - \sum_{s=i-n}^{r-n-2} d_{s,\ n}(x)\right] \\ - \sum_{k=i}^{i+n-1} d_{k,\ n}(x)\left[q^Y_{k-2n}(x) - \sum_{r=k-2n}^{k-n-2} d_{r,\ n}(x)\right], \end{aligned} \tag{4.24}$$

with

$$d_{r,\ n}(x) = (1 - F_r(x))\,.\prod_{j=r+1}^{r+n+1} F_j(x).$$

Proof

Now,

$$\begin{aligned} P(B_i \le x) &= P(A_i \le x, \ldots, A_{i+n} \le x) \\ &= p^A_{i,\, i+n}(x) \\ &= p^A_{i+1,\, i+n}(x) - P(A_i > x, A_{i+1} \le x, \ldots, A_{i+n} \le x) \\ &= p^A_{i+1,\, i+n}(x) - P(A_i > x, A_{i+1} \le x, \ldots, A_{i+n-1} \le x) + P(A_i > x, A_{i+1} \le x, \ldots, A_{i+n-1} \le x, A_{i+n} > x). \end{aligned}$$

It may easily be seen that the latter probability is zero, giving

$$p^A_{i,\, i+n}(x) = p^A_{i+1,\, i+n}(x) - P(A_i > x, A_{i+1} \le x, \ldots, A_{i+n-1} \le x).$$

Iterating this gives

$$\begin{aligned} p^A_{i,\, i+n}(x) &= p^A_{i+n,\, i+n}(x) - \sum_{k=i}^{i+n-1} P(A_k > x, A_{k+1} \le x) \\ &= p^A_{i+n}(x) - \sum_{k=i}^{i+n-1} [P(A_k > x) - P(A_k > x, A_{k+1} > x)] \\ &= 1 - q^A_{i+n}(x) - \sum_{k=i}^{i+n-1} q^A_k(x) + \sum_{k=i}^{i+n-1} q^A_{k,\, k+1}(x) \\ &= 1 - \sum_{k=i}^{i+n} q^A_k(x) + \sum_{k=i}^{i+n-1} q^A_{k,\, k+1}(x), \end{aligned}$$

where the double subscript notation $k, k \equiv k$ is used.

Thus

$$P(B_i \le x) = 1 - \sum_{k=i}^{i+n} q^A_k(x) + \sum_{k=i}^{i+n-1} q^A_{k,\, k+1}(x).$$

Now

$$\begin{aligned} q^A_k(x) &= P(Z_k > x, \ldots, Z_{k-n} > x) \\ &= q^Z_{k-n,\, k}(x). \end{aligned}$$

This gives

$$P(B_i \le x) = 1 - \sum_{k=i}^{i+n} q^Z_{k-n,\, k}(x) + \sum_{k=i}^{i+n-1} q^Z_{k-n,\, k+1}(x).$$

Then

$$\begin{aligned} q^Z_{k-n,\, k}(x) &= P(\wedge(Y_{k-n}, \ldots, Y_{k-2n}) > x, \ldots, \wedge(Y_k, \ldots, Y_{k-n}) > x) \\ &= P(Y_{k-2n} > x, \ldots, Y_k > x) \\ &= q^Y_{k-2n,\, k}(x) \end{aligned}$$

and

$$q^Z_{k-n,\ k+1}(x) = q^Y_{k-2n,\ k+1}(x).$$

Thus

$$P(B_i \leq x) = 1 - \sum_{k=i}^{i+n} q^Y_{k-2n,\ k}(x) + \sum_{k=i}^{i+n-1} q^Y_{k-2n,\ k+1}(x).$$

Note that

$$\begin{aligned} q^Y_{k-2n,\ k+1}(x) &= P(Y_{k-2n} > x, \ldots, Y_{k+1} > x) \\ &= P(Y_{k-2n} > x, \ldots, Y_k > x) - P(Y_{k-2n} > x, \ldots, Y_k > x, Y_{k+1} \leq x) \\ &= q^Y_{k-2n,\ k}(x) - P(Y_{k-2n} > x, \ldots, Y_k > x, Y_{k+1} \leq x). \end{aligned}$$

Substituting gives

$$P(B_i \leq x) = 1 - q^Y_{i-n,\ i+n}(x) - \sum_{k=i}^{i+n-1} P(Y_{k-2n} > x, \ldots, Y_k > x, Y_{k+1} \leq x).$$

Consider the event

$$\{Y_{k-2n} > x, \ldots, Y_k > x, Y_{k+1} \leq x\}.$$

From the definition of the Y's it follows:

$$\{Y_{k+1} \leq x\} \Rightarrow X_{k+1} \leq x, \ldots, X_{k+n+1} \leq x,$$

and with this,

$$\{Y_k > x\} \Rightarrow X_k > x.$$

If the above two inequalities hold, then

$$Y_{k-1} > x, \ldots, Y_{k-n} > x$$

will hold.

Furthermore $Y_{k-2n}, \ldots, Y_{k-n-1}$ are independent of $X_k, \ldots, X_{k+n+1}$.

Taken together, this gives

$$\begin{aligned} &P(Y_{k-2n} > x, \ldots, Y_{k-n-1} > x, Y_{k-n} > x, \ldots, Y_{k-1} > x, Y_k > x, Y_{k+1} \leq x) \\ &= P(Y_{k-2n} > x, \ldots, Y_{k-n-1} > x).P(X_k > x).P(X_{k+1} \leq x, \ldots, X_{k+n+1} \leq x) \\ &= q^Y_{k-2n,\ k-n-1}(x).\,(1 - F_k(x))\,.\prod_{j=k+1}^{k+n+1} F_j(x) \\ &= q^Y_{k-2n,\ k-n-1}(x).d_{k,\ n}(x), \end{aligned}$$

where

$$d_{r,\ n}(x) = (1 - F_r(x)) \,.\prod_{j=r+1}^{r+n+1} F_j(x).$$

This gives

$$P(B_i \le x) = 1 - q^Y_{i-n,\ i+n}(x) - \sum_{k=i}^{i+n+1} q^Y_{k-2n,\ k-n-1}(x).d_{k,\ n}(x).$$

Apply the lemma to the expressions on the right-hand side of this.

Applying (i) twice gives

$$\begin{aligned} q^Y_{i-n,\ i+n}(x) &= q^Y_{i-n,\ i+n-1}(x) - q^Y_{i-n,\ i-2}(x).d_{i+n-1,\ n}(x) \\ &= q^Y_{i-n,\ i+n-2}(x) - q^Y_{i-n,\ i-3}(x).d_{i+n-2,\ n}(x) - q^Y_{i-n,\ i-2}(x).d_{i+n-1,\ n}(x), \end{aligned}$$

and iterating this, gives

$$q^Y_{i-n,\ i+n}(x) = q^Y_{i-n,i+1}(x) - \sum_{r=i+1}^{i+n-1} q^Y_{i-n,\ r-n-1}(x).d_{r,\ n}(x).$$

Note that in this the subscripts of the q^Y's have a length of at most $n+2$, so (ii) of the lemma can be applied. This gives

$$\begin{aligned} q^Y_{i-n,i+n}(x) &= q^Y_{i-n,\ i+1}(x) - \sum_{r=i+1}^{i+n-1} q^Y_{i-n,\ r-n-1}(x).d_{r,\ n}(x) \\ &= q^Y_{i-n}(x) - \sum_{r=i-n}^{i} d_{r,\ n}(x) - \sum_{r=i+1}^{i+n-1} d_{r,\ n}(x)\left[q^Y_{i-n}(x) - \sum_{s=i-n}^{r-n-2} d_{s,\ n}(x)\right]. \end{aligned}$$

Also

$$q^Y_{k-2n,\ k-n-1}(x) = q^Y_{k-2n}(x) - \sum_{r=k-2n}^{k-n-2} d_{r,\ n}(x).$$

Putting this together gives

$$\begin{aligned} P(B_i \le x) &= 1 - q^Y_{i-n,\ i+n}(x) - \sum_{k=i}^{i+n-1} q^Y_{k-2n,\ k-n-1}(x).d_{k,\ n}(x) \\ &= 1 - q^Y_{i-n}(x) + \sum_{r=i-n}^{i} d_{r,\ n}(x) + \sum_{r=i+1}^{i+n-1} d_{r,\ n}(x)\left[q^Y_{i-n}(x) - \sum_{s=i-n}^{r-n-2} d_{s,\ n}(x)\right] \\ &\quad - \sum_{k=i}^{i+n-1} d_{k,\ n}(x)\left[q^Y_{k-2n}(x) - \sum_{r=k-2n}^{k-n-2} d_{r,\ n}(x)\right]. \end{aligned}$$

This concludes the theorem.

Remark 4.12.

(1) *Note that the distribution function of B_i is a function of $F_{i-2n}, \ldots, F_{i+2n}$.*

(2) *A special case of interest is that of a "jump", i.e. where the underlying distribution function is the same up to time $(T-1)$, and then changes to a new distribution function at time T. Thus*

$$F_k = \begin{cases} F_1 & \text{if } k \leq T-1 \\ F_2 & \text{if } k \geq T, \end{cases}$$

with F_1 and F_2 being two given distribution functions.

Corollary 4.8. *Write $F_{L_nU_n(X)}(x) = G(F_X(x))$, then the distribution of $U_nL_n(X)$ follows by using (4.14) as*

$$\begin{aligned} P((U_nL_nX)_i \leq x) = {} & 1 - \prod_{j=i-n}^{i}\Big(1-F_j(x)\Big) - \sum_{r=i-n}^{i}\left(F_r(x)\prod_{j=r+1}^{r+n+1}\Big(1-F_j(x)\Big)\right) \\ & - \sum_{r=i+1}^{i+n-1}\left(F_r(x)\prod_{j=r+1}^{r+n+1}\Big(1-F_j(x)\Big)\right)\left[1-\prod_{j=i-n}^{i}\Big(1-F_j(x)\Big)-\sum_{s=i-n}^{r-n-2}\left(F_s(x)\prod_{j=r+1}^{r+n+1}\Big(1-F_j(x)\Big)\right)\right] \\ & + \sum_{k=i}^{i+n-1}\left(F_k(x)\prod_{j=k+1}^{k+n+1}\Big(1-F_j(x)\Big)\right) \\ & \times\left[1-\prod_{j=k-2n}^{k-n}\Big(1-F_j(x)\Big)-\sum_{r=k-2n}^{k-n-2}\left(F_r(x)\prod_{j=r+1}^{r+n+1}\Big(1-F_j(x)\Big)\right)\right]. \end{aligned} \tag{4.25}$$

The results in Remark 4.12(2) and Corollary 4.8 will be further used in Section 4.7. Here they are applied in the calculation of the distribution function of edges, where a number of consecutive observations come from a different distribution to the underlying distribution of the remaining observations.

4.6.3 Asymptotic distribution of L_nU_n

The distributional behaviour of the LULU smoothers is considered when the window size tends to infinity. It is expected that the results from Extreme Value Theory (EVT) will apply since the quantities are based on extreme order statistics.

Let $X_1, X_2, \ldots, X_n$ be a sample of iid random variables on X, with cdf F_X. Let

$$X_{(n)} = \max(X_1, X_2, \ldots, X_n),$$

define the maximum of the sample.

The well-known Fisher-Tippett theorem states that if there are sequences of constants $a_n > 0$ and $\{b_n\}$, such that $a_n^{-1}(X_{(n)} - b_n)$ converges in distribution to some distribution function H, as $n \to \infty$, then H has a generalised extreme value distribution. This distribution has one of three possible forms, called after Frechet, Gumbel or Weibull (Embrechts *et al.* , 1997). In this case it can be said that F_X lies in the maximum domain of attraction of H, written $F_X \in MDA(H)$. Using this result, the following theorem is proved.

Theorem 4.19.

Suppose $F_X \in MDA(H)$, with sequences of constants $a_n > 0$ and $\{b_n\}$.

As $n \to \infty$ it follows that

$$J_n(x) \equiv F_{U_n}(a_n x + b_n) \xrightarrow{D} H(x) - H(x)\log H(x) \equiv J(x), \tag{4.26}$$

and

$$G_n(x) \equiv F_{L_n U_n}(a_n x + b_n) \xrightarrow{D} H(x) - H(x)\log H(x) + \frac{1}{2}\left[H(x)\log H(x)\right]^2 \equiv G(x). \tag{4.27}$$

Proof

From (Embrechts *et al.* , 1997) it follows that if $F_X \in MDA(H)$ and

$$H_n(x) \equiv P\left(\frac{(X_{(n)} - b_n)}{a_n} \leq x\right) = F_X^n(a_n x + b_n),$$

then

$$H_n(x) \xrightarrow{n\to\infty} H(x). \tag{4.28}$$

Further, suppose that $a_n,\ \{b_n\}$ and x are such that

$$a_n x + b_n \xrightarrow{n\to\infty} \infty$$

then

$$F_X(a_n x + b_n) \xrightarrow{n\to\infty} 1 \tag{4.29}$$

and

$$1 - F_X(a_n x + b_n) \xrightarrow{n\to\infty} 0. \tag{4.30}$$

Consider a first order Taylor expansion of $\log H_n(x)$:

$$\begin{aligned}
\log H_n(x) &= n\log F_X(a_n x + b_n) \\
&= n\log[1 - (1 - F_X(a_n x + b_n))] \\
&= -n(1 - F_X(a_n + b_n)) - \tfrac{1}{2}n(1 - F_X(a_n x + b_n))^2 + \ \ldots \\
&= -n(1 - F_X(a_n + b_n)) + o(1).
\end{aligned}$$

Since $H_n(x) \xrightarrow{n\to\infty} H(x)$, it follows that

$$\log H_n(x) \xrightarrow{n\to\infty} \log H(x)$$

and hence

$$n(1 - F_X(a_n x + b_n)) \xrightarrow{n\to\infty} -\log H(x) \tag{4.31}$$

For equation (4.26) it follows from (4.19) that

$$J_n(x) = F_X^{n+1}(a_n x + b_n) + n\,(1 - F_X(a_n x + b_n))\,F_X^{n+1}(a_n x + b_n).$$

Thus, applying (4.28), (4.29) and (4.31), for $n \to \infty$,

$$J_n(x) \equiv F_{U_n}(a_n x + b_n) \to H(x) - H(x)\log H(x).$$

From (4.17) it follows that

$$\begin{aligned} G_n(x) \equiv F_{L_n U_n(X)}(a_n + b_n) = {} & F_X^{n+1}(a_n x + b_n) + n(1 - F_X(a_n x + b_n))F_X^{n+1}(a_n x + b_n) \\ & +(1 - F_X(a_n x + b_n))F_X^{2(n+1)}(a_n x + b_n) \\ & +\tfrac{1}{2}\tfrac{(n-1)(n+2)}{n^2}[n(1 - F_X(a_n x + b_n))]^2 F_X^{2(n+1)}(a_n x + b_n). \end{aligned}$$

Equation (4.27) follows using this and applying (4.28), (4.29), (4.30) and (4.31).

Figure 4.4 illustrates $G(x)$ and the asymptotic behaviour of $L_n U_n$ for $n = 2, 5, 10, 100$ when X has the standard normal distribution.

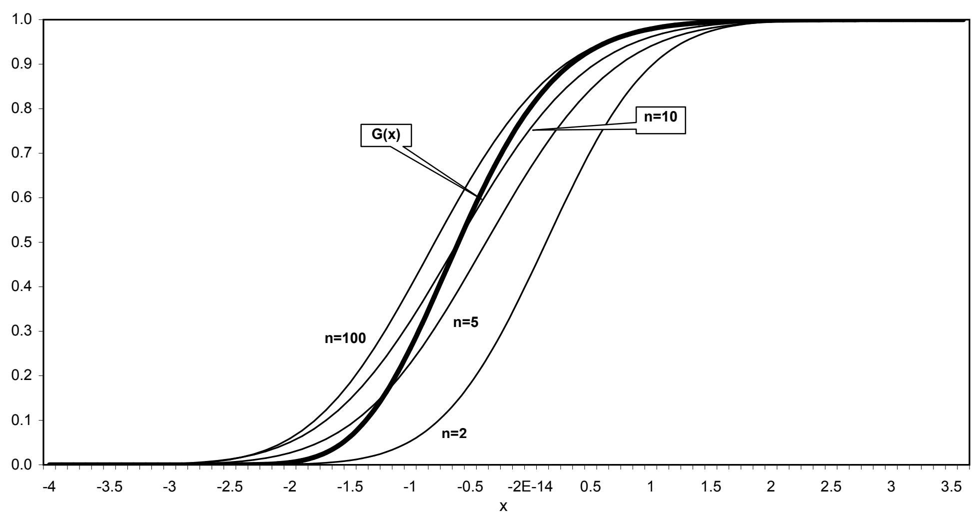

Figure 4.4: $G(x)$ and $F_{L_n U_n}(a_n x + b_n)$ for $n = 2, 5, 10, 100$

Remark 4.13.

(1) The normalising constants a_n and b_n are determined by F_X. Consider the following two cases:

(a) If $F_X(x) = \Phi(x)$, the standard normal cdf, then $H(x)$ is in the Gumbel class (Embrechts et al. *, 1997), i.e.*

$$H(x) = \exp(-\exp(-x))$$

with

$$a_n = (2 \ln n)^{-\frac{1}{2}}$$

and

$$b_n = (2 \ln n)^{\frac{1}{2}} - \frac{\ln(\ln n) + \ln 4\pi}{2(2 \ln n)^{\frac{1}{2}}}.$$

(b) If $F_X(x) = U(0, 1)$, then $H(x)$ is in the Weibull class (Embrechts et al. *, 1997), i.e.*

$$H(x) = \begin{cases} 1 & x \geq 0, \\ 0 & x < 0. \end{cases}$$

with normalising constants $a_n = n - 1$ and $b_n = 1$.

(2) The results for U_nL_n and L_n follow in the same way in terms of the asymptotic distribution of sample minima. Here

$$P\left(\frac{L_n(X) - b_n}{a_n} \leq x\right) \xrightarrow{D} 1 - (1 - H(x))[1 - \log(1 - H(x))].$$

(3) Consider the relationship between the limiting distributions in Theorem 4.19 and those for the upper order statistics. The following result is known for the latter (see e.g. Reiss (1989)).

Let $X_1, X_2, \ldots, X_n$ be as above and let $X_{(n-k+1)}$ be the k-th largest order statistic. Then for k fixed, and $a_n > 0$, $\{b_n\}$ as before,

$$P\left(\frac{(X_{(n-k+1)} - b_n)}{a_n} \leq x\right) \rightarrow H_k(x),$$

where

$$H_k(x) = H(x) \sum_{s=0}^{k-1} \frac{(-\log H(x))^s}{s!}.$$

For example

$$H_1(x) = H(x)$$

and

$$H_2(x) = H(x)(1 - \log H(x)).$$

From the last expression and (4.26) it follows that U_n has the same asymptotic distribution as the second largest order statistic.

For L_nU_n, from (4.27) the asymptotic distribution $G(x)$ can be written as

$$G(x) = H_2(x) + c_1(x) \quad \textit{and} \quad G(x) = H_3(x) - c_2(x) \tag{4.32}$$

where

$$c_1(x) = \frac{1}{2}[H(x) \log H(x)]^2 \geq 0 \tag{4.33}$$

and

$$c_2(x) = \frac{1}{2}[\log H(x)]^2 H(x)(1 - H(x)) \geq 0. \tag{4.34}$$

Thus

$$H_2(x) \leq G(x) \leq H_3(x). \tag{4.35}$$

It may be concluded that asymptotically L_nU_n is stochastically bounded from below by $X_{(n-2)}$ (the third largest order statistic) and from above $X_{(n-1)}$ (the second largest order statistic).

From equations (4.33) and (4.34) it follows that

$$c_1(x) - c_2(x) = \frac{1}{2}(\log H(x))^2 H(x)(2H(x) - 1).$$

This implies that $c_1(x) \geq c_2(x)$ (G is closer to H_3) or $c_1(x) \leq c_2(x)$ (G is closer to H_2) depending on whether

$$x \geq \textit{median}(H) \quad \textit{or} \quad x \leq \textit{median}(H).$$

It may be that L_nU_n has the same asymptotic distribution as some linear combination of the second and third largest order statistics, but a result of this kind has not yet been found.

4.7 Edge preservation

An edge was defined and illustrated in Section 2.3.8. The edge preservation properties of a median smoother were discussed in Section 3.7. A model of a step edge was given in (3.37).

To investigate the edge preservation properties of LULU smoothers, (4.24) and (4.25) in Section 4.6.2 are used to find the cdfs of L_nU_n and U_nL_n, respectively. This will be illustrated for a window size of five. Now $X_1 \sim F_1(x)$, $X_2 \sim F_2(x)$, $X_3 \sim F_3(x)$, $X_4 \sim F_4(x)$ and $X_5 \sim F_5(x)$. An edge is the result of a jump with two different underlying distribution functions, namely $F_1(x)$ and $F_2(x)$ (cf. Remark 4.12(2)).

Use the five observations $\{x_1, x_2, x_3, x_4, x_5\}$ as an example, but any five consecutive observations can be used. Let k observations be from a $N(h, \sigma^2)$ distribution and $(5-k)$ observations be from a $N(0, \sigma^2)$ distribution. For a window of size five, consider $(L_1U_1X)_3$. From (4.24) it follows that

$$P((L_1U_1X)_3 \leq x) = 1 - q_2^Y(x) + d_{2,\,1}(x) + d_{3,\,1}(x) - d_{3,\,1}(x)q_1^Y(x)$$

with

$$\begin{aligned}
q_1^Y(x) &= 1 - \textstyle\prod_{j=1}^{2} F_j(x) = 1 - F_1(x)F_2(x) \\
q_2^Y(x) &= 1 - \textstyle\prod_{j=2}^{3} F_j(x) = 1 - F_2(x)F_3(x) \\
d_{2,\,1}(x) &= (1 - F_2(x)) \textstyle\prod_{j=3}^{4} F_j(x) = (1 - F_2(x))F_3(x)F_4(x) \\
d_{3,\,1}(x) &= (1 - F_3(x)) \textstyle\prod_{j=4}^{5} F_j(x) = (1 - F_3(x))F_4(x)F_5(x).
\end{aligned}$$

$P((L_1U_1X)_3 \leq x)$ is different for each different number of observations, k, from the first distribution, $F_1(x)$.

If $k = 0$, all the observations will be from the same distribution $F_2(x)$ $(N(0, \sigma^2))$ with mean zero.

If $k = 1$, the first observation x_1 comes from $F_1(x)$ $(N(h, \sigma^2))$ and the other observations from $F_2(x)$. Thus $F_2(x) = F_3(x) = F_4(x) = F_5(x) \equiv F_2(x)$ and

$$P((L_1U_1X)_3 \leq x) = 2F_2^2(x) - F_2^3(x) + F_1(x)F_2^3(x) - F_1(x)F_2^4(x).$$

If $k = 2$, the first and second observations come from $F_1(x)$, therefore $F_1(x) = F_2(x) \equiv F_1(x)$. The other observations are from $F_2(x)$ and thus $F_3(x) = F_4(x) = F_5(x) \equiv F_2(x)$. Now

$$P((L_1U_1X)_3 \leq x) = F_2^2(x) + F_1(x)F_2(x) - F_1(x)F_2^2(x) + F_1^2(x)F_2^2(x) - F_1^2(x)F_2^3(x).$$

If $k = 3$, the first three observations are from $F_1(x)$ and thus $F_1(x) = F_2(x) = F_3(x) \equiv F_1(x)$. The other two observations are from $F_2(x)$ and $F_4(x) = F_5(x) \equiv F_2(x)$. Now

$$P((L_1U_1X)_3 \leq x) = F_1^2(x) + F_1(x)F_2(x) - F_1^2(x)F_2(x) + F_1^2(x)F_2^2(x) - F_1^3(x)F_2^2(x).$$

If $k = 4$, the first four observations are from $F_1(x)$ and thus $F_1(x) = F_2(x) = F_3(x) = F_4(x) \equiv F_1(x)$. The fifth observation is from $F_2(x)$ and thus $F_5(x) \equiv F_2(x)$. Now

$$P((L_1U_1X)_3 \leq x) = 2F_1^2(x) - F_1^3(x) + F_1^3(x)F_2(x) - F_1^4(x)F_2(x).$$

If $k = 5$, all the observations come from the same distribution $F_1(x)$ with mean h.

From (4.25), for this example with five observations $\{x_1, x_2, x_3, x_4, x_5\}$, it follows that

$$P((U_1L_1X)_3 \leq x) = 1 - \prod_{j=2}^{3}\left(1 - F_j(x)\right) - \sum_{r=2}^{3}\left(F_r(x)\prod_{j=r+1}^{r+2}\left(1 - F_j(x)\right)\right) + F_3(x)\prod_{j=4}^{5}\left(1 - F_j(x)\right)\left[1 - \prod_{j=1}^{2}\left(1 - F_j(x)\right)\right].$$

If $k = 1$, it follows that

$$P((U_1L_1X)_3 \leq x) = 4F_2^2(x) - 4F_2^3(x) + F_2^4(x) + F_1(x)F_2(x) - 3F_1(x)F_2^2(x) + 3F_1(x)F_2^3(x) - F_1(x)F_2^4(x).$$

If $k = 2$, then

$$P((U_1L_1X)_3 \leq x) = 3F_1(x)F_2(x) - 5F_1(x)F_2^2(x) + 2F_1(x)F_2^3(x) - F_1^2(x)F_2(x) + 2F_1^2(x)F_2^2(x) - F_1^2(x)F_2^3(x) + 2F_2^2(x) - F_2^3(x).$$

If $k = 3$, then

$$P((U_1L_1X)_3 \leq x) = 3F_1(x)F_2(x) - 5F_1^2(x)F_2(x) + 2F_1^3(x)F_2(x) - F_1(x)F_2^2(x) + 2F_1^2(x)F_2^2(x) - F_1^3(x)F_2^2(x) + 2F_1^2(x) - F_1^3(x).$$

If $k = 4$, it follows that

$$P((U_1L_1X)_3 \leq x) = 4F_1^2(x) - 4F_1^3(x) + F_1^4(x) + F_1(x)F_2(x) - 3F_1^2(x)F_2(x) + 3F_1^3(x)F_2(x) - F_1^4(x)F_2(x).$$

The pdfs of L_1U_1 and U_1L_1 are found by differentiation. The mean and standard deviation are then calculated by numerical integration. These results will be used in Chapter 5 where the edge preservation property of LULU smoothers will be compared to that of median smoothers.

4.8 Variation reduction and shape preservation

Definitions and results on variation and shape preservation were explained and discussed in Section 2.3.10.

The application of variation reduction results to LULU smoothers from Rohwer (2002b) is summarised as follows:

Remark 4.14.

(1) Any operator O that is a composition of $\vee$ and $\wedge$ is variation diminishing.

(2) $T(\wedge \vee x) = T(\vee x)$ and $T(\vee \wedge x) = T(\wedge x)$ for each $x \in l_1$, which indicates that $\wedge$ does not reduce variation on any output of $\vee$ and vice versa.

(3) $T(\vee Ux) = T(Ux)$ and $T(\wedge Lx) = T(Lx)$.

(4) $T(x) = T(\vee x) + 2\|Ux - x\|_1$ and $T(x) = T(\wedge x) + 2\|Lx - x\|_1$.

(5) $T(x) = T(Ux) + T(Ux - x)$ and $T(x) = T(Lx) + T(Lx - x)$.

(6) $T(x) = T(LUx) + T(LUx - x)$ and $T(x) = T(ULx) + T(ULx - x)$.

(7) $T(x) = T(F_nx) + T(F_{n-1}x - F_nx) + \ \ldots + T(F_1x - F_2x) + T(x - F_1x)$ and
$T(x) = T(C_nx) + T(C_{n-1}x - C_nx) + \ \ldots + T(C_1x - C_2x) + T(x - C_1x)$.

(8) $T(x) = T(W_nx) + T(W_{n-1}x - W_nx) + \ \cdots + T(x - W_1x)$.

(9) $T(x) = T(B_nx) + T(B_{n-1}x - B_nx) + \ \cdots + T(x - B_1x)$.

The application of shape preservation results to LULU smoothers from Rohwer (2002b) is summarised as follows:

Remark 4.15.

(1) The operators $\vee\wedge$ and $\wedge\vee$, as well as all compositions of these, are neighbour trend preserving (cf. Definition 2.25).

(2) L_n and U_n and all compositions of them are neighbour trend preserving, which implies that L_nU_n and U_nL_n are neighbour trend preserving (cf. Definition 2.25).

(3) L_n and U_n are difference reducing, and since they are also neighbour trend preserving, they are fully trend preserving (cf. Definition 2.27).

(4) L_nU_n and U_nL_n are also fully trend preserving (cf. Definition 2.27).

4.9 Variation decomposition

Variation decomposition was illustrated in (2.2). Smoothing successively by $F_1, F_2, \ldots, F_n$ and $C_1, C_2, \ldots, C_n$ yields a monotone reduction of $T(x)$. From Definition 4.8, where the flooring and ceiling smoothers were defined, it is clear that the successive application of $F_1, F_2, \ldots$ and $C_1, C_2, \ldots$ implies a successive trimming (peeling off) of the sequence that can be monitored by means of the variation that is removed at each level.

The compound smoother $B_n x$ provides a smoothed sequence, which is a type of trimmed sequence, where the trimming follows from

$$(B_n x)_i = (F_n x)_i \; if \; (B_{n-1}x)_i < (F_n x)_i,$$
$$(B_n x)_i = (C_n x)_i \; if \; (B_{n-1}x)_i > (C_n x)_i.$$

If $(F_n x)_i \leq (B_{n-1}x)_i \leq (C_n x)_i$ it is considered as signal plus well-behaved noise. The advantage of $(B_n x)_i$ is that it preserves $(B_{n-1}x)_i$ between $(F_n x)_i$ and $(C_n x)_i$.

At each level the compound smoother B_n can be applied and $T(x) = T(B_n x) + T(B_{n-1}x - B_n x) + \ldots + T(x - B_1 x)$ (cf. Remark 4.14(9)) can be calculated and studied.

Table 4.5 illustrates the decomposition of the variation in an example. The data used in the example are the daily closing prices of the Standard and Poor's 500 stock index for the period 1999/01/04 to 2000/10/03, tabulated in Appendix D. This series is analysed in detail in Chapter 7. The amount of variation reduced at each level of applying F_n, C_n and B_n was calculated. The corresponding percentages for each $n = 1, 2, \ldots, 5$ are given in Table 4.5.

From Table 4.5 the total variation of the sequence is calculated as $T(x) = 5\,085{,}45$. F_1 removes 2 210,31 of the total variation at the first smoothing step, i.e. 43,46%. F_2 removes a further 13,67%, F_3 a further 7,71%, F_4 a further 6,82% and F_5 a further 3,10%. In total F_5 removes 74,76% of the total variation and hence preserves 25,46% of the total variation. Similarly it follows that C_1 removes 44,25% of the total variation and C_5 removes 72,65% of the total variation and preserves 24,35% of the total variation. B_1 removes 32,09% of the total variation, and after the application of B_5, 62,62% of the total variation is removed, preserving 37,38% of the total variation. The total variation removed by B_5 is less than that removed by F_5, or C_5. This is because B_n preserves the part of the original series that was not smoothed by B_r, $r < n$. The amount of variation removed at each level can be used to monitor the impulsive noise removed at each level. The additional percentage of the total variation removed at each level by F_n, C_n and B_n is illustrated in Figure 4.5.

It is thus clear that the variation decomposition property of the LULU smoothers is very useful in monitoring the smoothing process.

Table 4.5: Reduction of total variation

n	$T(x)$	$T(F_n)$	$T(F_{n-1} - F_n)$	$T(F_{n-2} - F_{n-1})$	$T(F_{n-3} - F_{n-2})$	$T(F_{n-4} - F_{n-3})$	$T(F_{n-5} - F_{n-4})$
1	5 085,45	2 875,14	2 210,31	0	0	0	0
%	100%	56,54%	43,46%	0	0	0	0
2	5 085,45	2 179,76	695,38	2 210,31	0	0	0
%	100%	42,86%	13,67%	43,46%	0	0	0
3	5 085,45	1 787,8	391,96	695,38	2 210,31	0	0
%	100%	35,16%	7,71%	13,67%	43,46%	0	0
4	5 085,45	1 440,92	346,88	391,96	695,38	2 210,31	0
%	100%	28,33%	6,82%	7,71%	13,67%	43,46%	0
5	5 085,45	1 283,32	157,60	346,88	391,96	695,38	2 210,31
%	100%	25,24%	3,10%	6,82%	7,71%	13,67%	43,46%
	$T(x)$	$T(C_n)$	$T(C_{n-1} - C_n)$	$T(C_{n-2} - C_{n-1})$	$T(C_{n-3} - C_{n-2})$	$T(C_{n-4} - C_{n-3})$	$T(C_{n-5} - C_{n-4})$
1	5 085,45	2 835,22	2 250,23	0	0	0	0
%	100%	55,75%	44,25%	0	0	0	0
2	5 085,45	2 079,8	755,42	2 250,23	0	0	0
%	100%	40,90%	14,85%	44,25%	0	0	0
3	5 085,45	1 726,41	353,39	755,42	2 250,23	0	0
%	100%	33,95%	6,95%	14,85%	44,25%	0	0
4	5 085,45	1 407,75	318,66	353,39	755,42	2 250,23	0
%	100%	27,68%	6,27%	6,95%	14,85%	44,25%	0
5	5 085,45	1 238,07	169,68	318,66	353,39	755,42	2 250,23
%	100%	24,35%	3,34%	6,27%	6,95%	14,85%	44,25%
	$T(x)$	$T(B_n)$	$T(B_{n-1} - B_n)$	$T(B_{n-2} - B_{n-1})$	$T(B_{n-3} - B_{n-2})$	$T(B_{n-4} - B_{n-3})$	$T(B_{n-5} - B_{n-4})$
1	5 085,45	3 453,33	1 632,12	0	0	0	0
%	100%	67,91%	32,09%	0	0	0	0
2	5 085,45	2 932,65	520,68	1 632,12	0	0	0
%	100%	57,67%	10,24%	32,09%	0	0	0
3	5 085,45	2 555,15	377,5	520,68	1 632,12	0	0
%	100%	50,24%	7,42%	10,24%	32,09%	0	0
4	5 085,45	2 133,57	421,58	377,5	520,68	1 632,12	0
%	100%	41,95%	8,29%	7,42%	10,24%	32,09%	0
5	5 085,45	1 901,11	232,46	421,58	377,5	520,68	1 632,12
%	100%	37,38%	4,57%	8,29%	7,42%	10,24%	32,09%

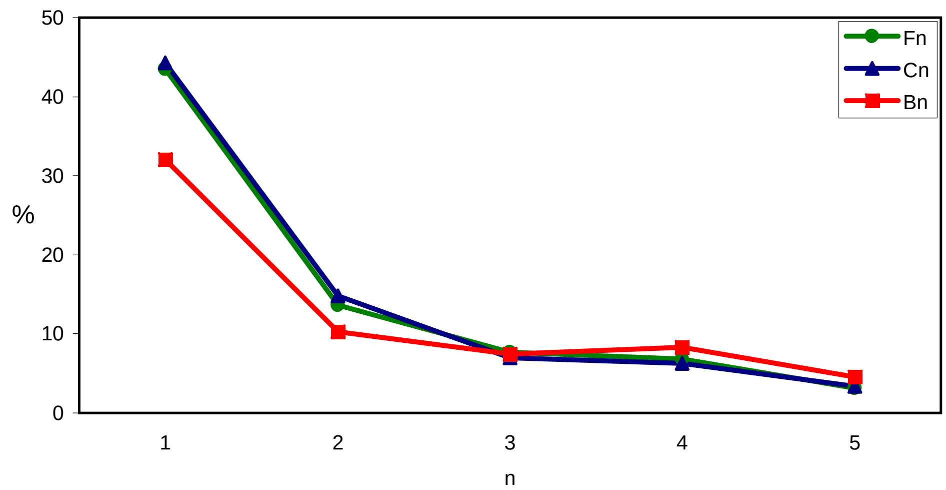

Figure 4.5: Additional percentage variation removed at each level

4.10 Summary

In this chapter the definitions and mathematical properties of the operators which form the building blocks of LULU smoothers as well as the smoothers were given. It is clear that U_n smoothes from below and L_n smoothes from above. Smoothers L_nU_n, U_nL_n, C_n and F_n are formed if L_n and U_n are concatenated. Combinations of these smoothers lead to full smoothers such as Q_n, G_n, W_n^*, W_n, A_n, B_n^* and B_n. These smoothers have useful properties, especially in terms of variation reduction.

The way LULU smoothers treat n-monotone sequences was investigated. If a monotone sequence is smoothed by a LULU smoother, the output is the original sequence. They also treat blockpulses effectively. The exact distributions of L_nU_n and U_nL_n in the independent identically distributed case and the non-identically distributed case were derived. The asymptotic behaviour of L_nU_n was derived using results from Extreme Value Theory (EVT). The distributional results were used in an example to find the distribution of L_nU_n and U_nL_n in the case of an edge. This will be further investigated in Chapter 5. Variation reduction and shape preservation results on LULU smoothers were listed and illustrated in an application to Standard and Poor 500 data.

The following chapter will compare median smoothers and LULU smoothers with respect to some of these properties.

Chapter 5

A comparison of median and LULU smoothers

5.1 Introduction

The properties that are important in the smoothing process have been discussed in Chapter 2. In Chapter 3 the median smoother and in Chapter 4 the LULU smoothers and their properties were discussed. In this chapter some of the issues mentioned in Chapter 2 will be addressed and illustrated using the results of Chapters 3 and 4. In the following sections median smoothers and LULU smoothers are compared with respect to monotonicity, idempotency, co-idempotency, stability, edge preservation, output distributions and variation decomposition.

5.2 Monotonicity

Monotonicity was discussed in Section 2.3.4 with Definition 2.13 defining an n-monotone sequence (LOMO($n+2$) sequence).

In Chapter 2 the argument was made that from a smoothing perspective it is important to study the monotonicity of a smoother output sequence given the monotonicity of the input sequence, since increased monotonicity means increased lengths of subsets of consecutive points that are either increasing and/or remain constant or that are decreasing and/or remain constant. Monotonicity is thus a measure of smoothness of a sequence in the sense that it gives an indication of the minimum length of consecutive monotone subsets of the sequence. It thus follows that when the class of LULU smoothers is compared with the class of median smoothers, the way they deal with the monotonicity of sequences needs to be understood. In this regard the following remark on similarities and differences is made:

Remark 5.1.

From Theorems 3.1, 4.4(1) and 4.4(2) it follows that a series x is a root of the median smoothers M_k as well as of the LULU smoothers L_k, U_k, L_kU_k, U_kL_k, C_k, F_k, W_k^*, W_k, B_k^* and B_k for all $k \leq n$ if and only if x is n-monotone. For both classes the window size relative to the monotonicity of the input series determines the monotonicity of the *output series. An n-monotone sequence is thus not smoothed by the smoothers above if the window size is smaller or equal to*

- $(2n+1)$ *in the case of* M_n, L_n *and* U_n,
- $(4n+1)$ *in the case of* L_nU_n, U_nL_n, W_n^* *and* W_n,
- $(2n(n+1)+1)$ *in the case of* C_n, F_n, B_n^* *and* B_n.

For example, if a 2-monotone sequence is smoothed by median smoother M_1 or M_2 it will result in the original root sequence x. Median smoothers $M_3, M_4, \ldots$ with window size greater than five will change a 2-monotone sequence x and thus do not produce the original root sequence. Compound median smoothers discussed in Section 3.8.1 such as M_2M_1, known as (53), will produce the original root sequence x for a window size less than that of the largest median smoother, which is M_2 in the case of a 2-monotone sequence.

For this example, the LULU half smoothers, L_1, U_1 (window size = 3), or L_2, U_2 (window size = 5), will also produce the original root sequence x if the sequence is 2-monotone, but the smoothers with window size greater than five will change the sequence. A 2-monotone sequence smoothed by L_1U_1, U_1L_1, W_1^* (window size = 5), or L_2U_2, U_2L_2, W_2^* (window size= 9) will produce the original root sequence, while a window size greater than nine for these smoothers will change the sequence. For the ceiling, flooring and compound LULU smoothers, C_n, F_n, and B_n, the original root sequence x will result if C_1, F_1, B_1 (window size= 5), or C_2, F_2, B_2 (window size = 13) is applied to a 2-monotone sequence, while a window size greater than thirteen will change the sequence.

A practical illustration of the behaviour of the median and LULU smoothers on a 2-monotone sequence, i.e. $x \in \mathcal{M}_2$ with four successive elements monotone nondecreasing (non-increasing), is given in Example 5.1. The sequence is plotted in Figure 5.1. In the case of a 2-monotone sequence the result of smoothing by the smoothers for $n \leq 2$, is the original sequence. Only the smoothed sequences which were changed by the smoothers are illustrated in Figures 5.2 and 5.3.

Example 5.1. x : 3 2 1 1 1 2 2 2 3 4 4 4 0 0 0 1 2 3 3 3 2 2 2 1 1

The value 3 *is appended to the beginning of the sequence and the value* 1 *to the end of the sequence.*

Table 5.1 contains the output sequences if LULU smoothers and median smoothers are applied to the 2-monotone sequence x.

It is clear that $L_1x = U_1x = M_1x = L_2x = U_2x = M_2x = x$. The output sequence changed after being smoothed by L_3, U_3 and M_3. This illustrates the fact that a 2-monotone sequence will change if smoothed by L_n, U_n and M_n if $n > 2$.

For $n = 1$, it follows that $L_1U_1x = U_1L_1x = W_1^*x = W_1x = B_1^*x = B_1x = x$ if applied to this 2-monotone sequence. Furthermore, it follows that $L_2U_2x = U_2L_2x = x$ and thus $W_2^*x = W_2x = x$. Also $F_2x = C_2x = B_2^*x = B_2x = x$ $(= L_2U_2x = U_2L_2x = W_2^*x = W_2x)$.

Although $L_3U_3x = U_3L_3x = W_3^*x = W_3^*x = F_3x = C_3x = B_3^*x = B_3x$, they are not equal to the original sequence x. This illustrates that $L_nU_nx = U_nL_nx$ does not imply that the sequence x is n-monotone. In

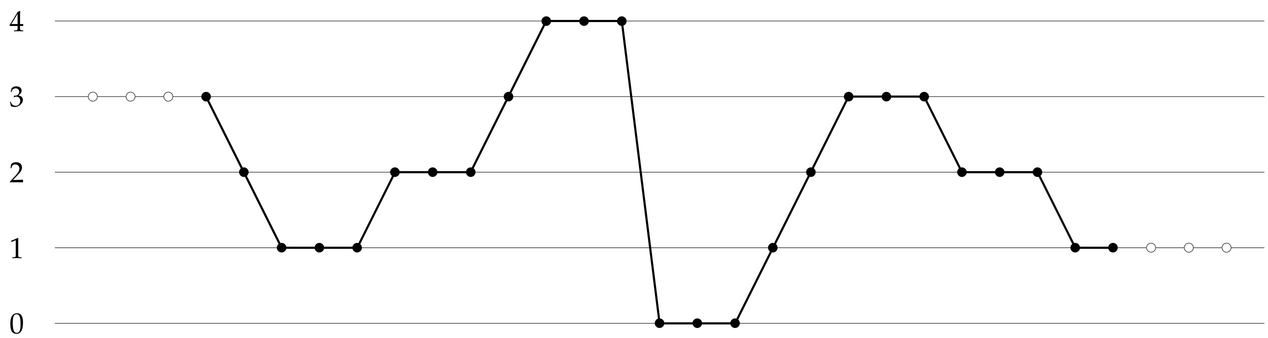

Figure 5.1: Example of 2-monotone sequence x

Table 5.1: Output sequences of LULU and median smoothers for Example 5.1

x	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
L_1	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
U_1	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
M_1	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
L_2	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
U_2	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
M_2	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
L_3	3	2	1	1	1	2	2	2	3	3	3	3	0	0	0	1	2	2	2	2	2	2	2	1	1
U_3	3	2	2	2	2	2	2	2	3	4	4	4	1	1	1	1	2	3	3	3	2	2	2	1	1
M_3	3	2	2	2	2	2	2	2	3	3	3	3	1	1	1	1	2	2	2	2	2	2	2	1	1
L_1U_1	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
U_1L_1	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
$W_1^* = W_1 = B_1^* = B_1$	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
L_2U_2	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
U_2L_2	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
W_2^*	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
W_2	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
F_2	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
C_2	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
B_2^*	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
B_2	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
(53) = M_2M_1	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
L_3U_3	3	2	2	2	2	2	2	2	3	3	3	3	1	1	1	1	2	2	2	2	2	2	2	1	1
U_3L_3	3	2	2	2	2	2	2	2	3	3	3	3	1	1	1	1	2	2	2	2	2	2	2	1	1
W_3^*	3	2	2	2	2	2	2	2	3	3	3	3	1	1	1	1	2	2	2	2	2	2	2	1	1
W_3	3	2	2	2	2	2	2	2	3	3	3	3	1	1	1	1	2	2	2	2	2	2	2	1	1
F_3	3	2	2	2	2	2	2	2	3	3	3	3	1	1	1	1	2	2	2	2	2	2	2	1	1
C_3	3	2	2	2	2	2	2	2	3	3	3	3	1	1	1	1	2	2	2	2	2	2	2	1	1
B_3^*	3	2	2	2	2	2	2	2	3	3	3	3	1	1	1	1	2	2	2	2	2	2	2	1	1
B_3	3	2	2	2	2	2	2	2	3	3	3	3	1	1	1	1	2	2	2	2	2	2	2	1	1

general it is not true that x lies between L_nU_nx and U_nL_nx, and thus $L_nU_nx = U_nL_nx$ does not imply $L_nU_nx = U_nL_nx = x$. This contrasts with the case where $L_nx = U_nx$, which implies that $L_nx = U_nx = x$, from the fact that $L_n \leq I \leq U_n$ (Rohwer, 2005, p. 23).

For $n = 2$, $M_2x = x = U_2L_2x = L_2U_2x$ and thus $U_2L_2 \leq M_2 \leq L_2U_2$. The same result holds for $n = 1$. For

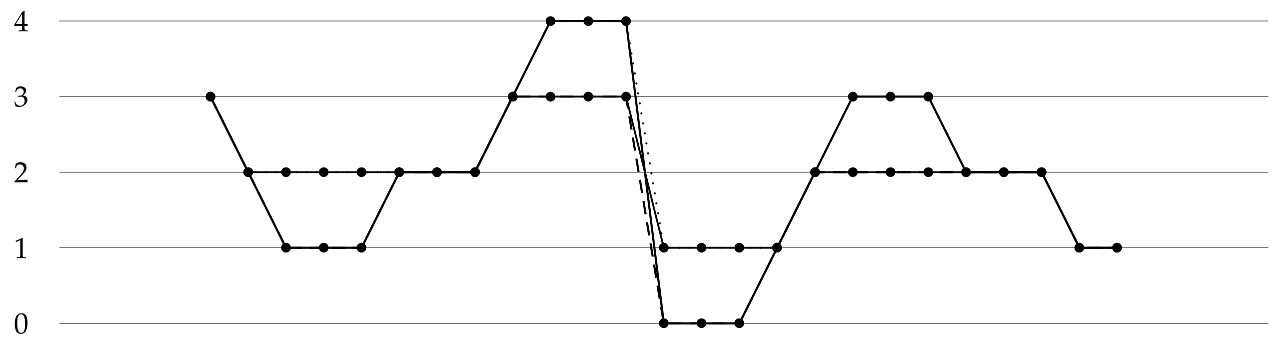

Figure 5.2: $L_3(- - -)$, $U_3(\cdots)$ and M_3(solid thick line) on a 2-monotone sequence x

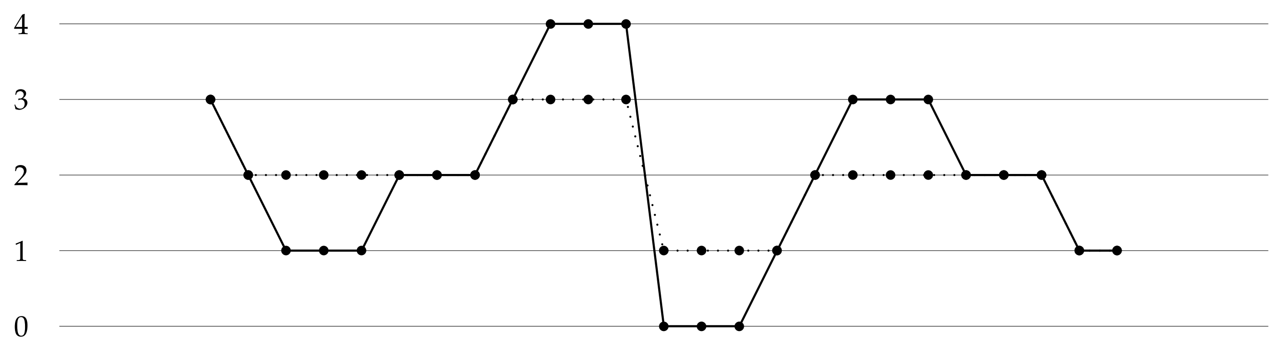

Figure 5.3: $L_3U_3(\cdots)$, $U_3L_3(\cdots)$, $W_3^*(\cdots)$ of a 2-monotone sequence x

$n = 3$, $M_3x = U_3L_3x = L_3U_3x$, but is not equal to the original sequence x. This is an illustration of the fact that for an n-monotone sequence, smoothing by the median and LULU smoothers results in the original sequence x for any integer less than or equal to n. For an integer greater than n, the corresponding median will always fall in the interval $[U_nL_n;\ L_nU_n]$.

The applications of a median smoother on a LULU smoother and of a LULU smoother on a median smoother are illustrated in Table 5.2.

It follows that $M_1U_1x = L_1U_1x = M_2U_2x = L_2U_2x = x$ and $M_1L_1x = U_1L_1x = M_2L_2x = U_2L_2x = x$. This illustrates the result that the median smoothers M_1 or M_2 do not modify the sequence that has been smoothed by LULU smoothers.

For $n = 3$, it follows that $M_3U_3x = L_3U_3x$ and $M_3L_3x = U_3L_3x$, but they are not equal to the sequence x. If the median smoother is applied to a sequence already smoothed by U_n or L_n, it results in the same sequences L_nU_n and U_nL_n respectively. This corresponds with the results that, for any n, the median smoother M_n cannot smooth L_nU_n and U_nL_n any further.

Applying LULU smoothers to the sequence smoothed by median smoothers also does not modify the sequence, since $U_1L_1M_1x = M_1x$ and $L_1U_1M_1x = M_1x$. This is also true for $n = 2$ and $n = 3$.

Table 5.2: Output sequences of median smoothers on LULU smoothers and vice versa for Example 5.1

x	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
M_1U_1	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
L_1U_1	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
M_1L_1	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
U_1L_1	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
$U_1L_1M_1$	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
$L_1U_1M_1$	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
M_2U_2	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
L_2U_2	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
M_2L_2	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
U_2L_2	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
$U_2L_2M_2$	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
$L_2U_2M_2$	3	2	1	1	1	2	2	2	3	4	4	4	0	0	0	1	2	3	3	3	2	2	2	1	1
M_3U_3	3	2	2	2	2	2	2	2	3	3	3	3	1	1	1	1	2	2	2	2	2	2	2	1	1
L_3U_3	3	2	2	2	2	2	2	2	3	3	3	3	1	1	1	1	2	2	2	2	2	2	2	1	1
M_3L_3	3	2	2	2	2	2	2	2	3	3	3	3	1	1	1	1	2	2	2	2	2	2	2	1	1
U_3L_3	3	2	2	2	2	2	2	2	3	3	3	3	1	1	1	1	2	2	2	2	2	2	2	1	1
$U_3L_3M_3$	3	2	2	2	2	2	2	2	3	3	3	3	1	1	1	1	2	2	2	2	2	2	2	1	1
$L_3U_3M_3$	3	2	2	2	2	2	2	2	3	3	3	3	1	1	1	1	2	2	2	2	2	2	2	1	1

The behaviour of the median and LULU smoothers on any sequence with no specific degree of monotonicity will now be illustrated with 25 observations extracted from the middle of a large randomly generated sequence in Example 5.2.

Example 5.2. $y: 11\ -5\ -2\ -20\ -4\ 4\ 26\ 0\ -11\ 19\ 0\ -2\ -7\ 0\ -2\ -2\ 8\ -8\ 2\ -9\ 7\ -13\ 5\ 12\ 14$

The sequence is illustrated in Figure 5.4.

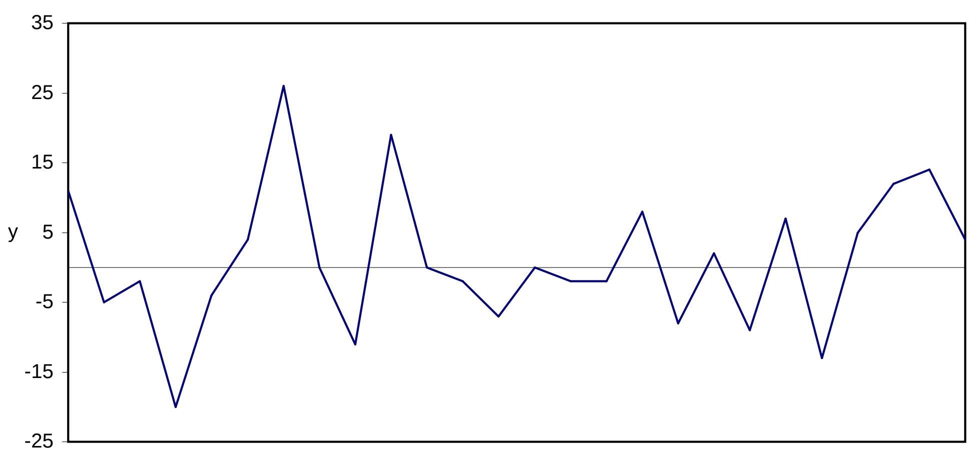

Figure 5.4: The sequence of Example 5.2

The output of the smoothers applied to the observations in Example 5.2 is given in Table 5.3. No pattern is observed for any of the median or LULU smoothers for $n = 1, 2, 3$. None of the smoothers produced the root sequence in these three steps. However, there are small differences between the output of B_1 and B_2, and even less between the output of B_2 and B_3. The root will most probably be reached in the following steps, which are not illustrated here.

Table 5.3: Output sequences of LULU and median smoothers for Example 5.2

y	11	-5	-2	-20	-4	4	26	0	-11	19	0	-2	-7	0	-2	-2	8	-8	2	-9	7	-13	5	12	14
L_1	-5	-5	-5	-20	-4	4	4	0	-11	0	0	-2	-7	-2	-2	-2	-2	-8	-8	-9	-9	-13	5	12	12
U_1	11	-2	-2	-4	-4	4	26	0	0	19	0	-2	-2	0	-2	-2	8	2	2	2	7	5	5	12	14
M_1	-5	-2	-5	-4	-4	4	4	0	0	0	0	-2	-2	-2	-2	-2	-2	2	-8	2	-9	5	5	12	12
L_2	-5	-5	-5	-20	-4	0	0	0	-11	-2	-2	-2	-7	-2	-2	-2	-2	-8	-8	-9	-9	-13	5	5	5
U_2	11	-2	-2	-2	-2	4	26	19	19	19	0	0	0	0	0	0	8	2	2	2	7	7	7	12	14
M_2	-2	-5	-4	-4	-2	0	0	4	0	0	-2	0	-2	-2	-2	-2	-2	-2	2	-8	2	5	7	5	5
L_3	-9	-9	-9	-20	-4	-4	-4	-4	-11	-7	-7	-7	-7	-2	-2	-2	-2	-8	-8	-9	-9	-13	4	4	4
U_3	11	-2	-2	-2	-2	4	26	19	19	19	0	0	0	0	0	0	8	7	7	7	7	7	7	12	14
M_3	-2	-4	-4	-2	-2	-2	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2	2	2	5	5	5	5
U_1L_1	-5	-5	-5	-5	-4	4	4	0	0	0	0	-2	-2	-2	-2	-2	-2	-8	-8	-9	-9	-9	5	12	12
L_1U_1	2	-2	-2	-4	-4	4	4	0	0	0	0	-2	-2	-2	-2	-2	2	2	2	2	5	5	5	12	12
$W_1^* = W_1 = B_1^* = B_1$	2	-5	-2	-5	-4	4	4	0	0	0	0	-2	-2	-2	-2	-2	2	-8	2	-9	5	-9	5	12	12
U_2L_2	-5	-5	-5	-5	-4	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2	-2	-8	-8	-9	-9	-9	5	5	5
L_2U_2	11	-2	-2	-2	-2	4	19	19	19	19	0	0	0	0	0	0	2	2	2	2	7	7	7	10	10
W_2^*	11	-5	-2	-5	-4	4	19	0	-2	19	0	-2	-2	0	-2	-2	2	-8	2	-9	7	-9	5	10	10
W_2	11	-2	-2	-2	-2	4	19	19	19	19	0	0	0	0	0	0	2	2	2	2	7	7	7	10	10
F_2	-5	-5	-5	-5	-4	0	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-8	-8	-9	-9	-9	5	5	5
C_2	2	-2	-2	-2	-2	0	0	0	0	0	0	-2	-2	-2	-2	-2	2	2	2	2	5	5	5	10	10
B_2^*	2	-5	-2	-5	-4	0	0	0	0	0	0	-2	-2	-2	-2	-2	2	-8	2	-9	5	-9	5	10	10
B_2	2	-5	-2	-5	-4	0	0	0	0	0	0	-2	-2	-2	-2	-2	2	-8	2	-9	5	-9	5	10	10
U_3L_3	-9	-9	-9	-9	-4	-4	-4	-4	-7	-7	-7	-7	-7	-2	-2	-2	-2	-8	-8	-8	-8	-8	4	4	4
L_3U_3	11	-2	-2	-2	-2	4	19	19	19	19	0	0	0	0	0	0	7	7	7	7	7	7	7	12	14
W_3^*	-5	-5	-5	-9	-4	0	0	0	-7	-2	-2	-2	-7	-2	-2	-2	-2	-8	-8	-8	-8	-8	5	5	5
W_3	11	-2	-5	-2	-2	4	19	19	19	19	-2	0	0	0	0	0	7	7	7	7	7	7	7	12	14
F_3	-5	-5	-5	-5	-4	0	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-8	-8	-8	-8	-8	5	5	5
C_3	2	-2	-2	-2	-2	0	0	0	0	0	0	-2	-2	-2	-2	-2	2	2	2	2	5	5	5	10	10
B_3^*	2	-5	-2	-5	-4	0	0	0	0	0	0	-2	-2	-2	-2	-2	2	-8	2	-8	5	-8	5	10	10
B_3	2	-5	-2	-5	-4	0	0	0	0	0	0	-2	-2	-2	-2	-2	2	-8	2	-8	5	-8	5	10	10

Example 5.1 illustrates that both the median and LULU smoothers reach the root sequence after one pass if $n \leq 2$ for a 2-monotone sequence. Applying the median smoother to the LULU smoother and vice versa did not change the smoothed sequence. For any sequence with no monotonicity as illustrated by Example 5.2, no specific pattern was observed.

5.3 Idempotency and co-idempotency

In Section 2.3.3 it was explained that if a smoother is idempotent and also co-idempotent it means that when the output of the smoother is passed through the smoother a second time, no further smoothing takes place. It could be argued that according to the specific smoother, signal has been identified. An idempotent smoother is thus consistent in the sense that it defines its own output as signal, recognises it as signal and does not smooth it further, but preserves it. Furthermore, if the smoother is also co-idempotent, the output is zero when the noise of the smoother is passed through the smoother a second time, i.e. $\mathbf{P}(\mathbf{I} - \mathbf{P})x = 0$.

From Section 3.4 it follows that any nonroot sequence of length L has to be smoothed successively by the median of any window size for at least $\frac{1}{2}(L-2)$ times before it will produce a sequence that will not change a root by any further smoothing. If the window size is $(2n+1)$, the nonroot sequence of length L will become a root series in at most $3\left[\frac{L-2}{2(n+2)}\right]$ passes of the median. **Comparing this with the LULU smoothers it is clear from the idempotent and co-idempotent properties of the LULU smoothers that they are in that regard preferable to the median smoothers**. The LULU smoothers are only applied once to obtain a root signal, which saves time and calculations. This makes the LULU smoothers more economical than the median smoothers. However, in some applications in this thesis, $P_1, P_2, \ldots$ (sequences of LULU smoothers) are used successively to see how the variation reduction decreases, thus losing the idempotency.

The idempotency and co-idempotency of the LULU smoothers at different degrees of monotonicity were tabulated in Section 4.4. Knowing the degree of monotonicity of a sequence before passing it through a smoother of a chosen window size is valuable. For an i-monotone input sequence, the output sequence is the original sequence x if passed through a LULU smoother P_n for $n \leq i$. According to the idempotency property of a smoother, the root sequence is produced after a first pass, meaning that a second application of that same smoother changes nothing. Thus, for an i-monotone input sequence passed through a LULU smoother P_n for $n > i$, the output sequence will be a root sequence. The question of whether the output of the idempotent and co-idempotent LULU smoothers should be considered as signal, in other words, whether they are consistent, is not of course answered by this. The class of LULU smoothers consists of a number of different smoothers; some of them intuitively better than others, for example the compound smoothers relative to the ceiling and flooring smoothers. However, the idempotent and co-idempotent properties help us to understand the smoothing process of these smoothers better. This is not the case for the median smoothers.

Some additional criteria and theory to evaluate the effectiveness and consistency of smoothers are still needed. This suggests that a measure that measures the amount of roughness that is smoothed away at each increase of the window size of a smoothing sequence is needed and, in addition to this, some sort of stopping rule for n. In Section 5.7 this issue will be addressed again.

The idempotency and co-idempotency of the median and LULU smoothers were verified by testing them on a randomly generated sequence from the $N(0,1)$ distribution. Tables 5.4 and 5.5 illustrate the influence of different smoothers on ten values taken from the generated sequence. The LULU smoothers that are idempotent and co-idempotent are U_n, L_n, L_nU_n, U_nL_n, C_n, F_n, W_n^*, W_n, B_n^* and B_n.

Smoothers that are neither idempotent nor co-idempotent are the median smoother M_n, and the LULU smoothers G_n and Q_n.

Note that from Table 5.5 it follows that $Q_1x = M_1x$, which confirms Remark 4.4(6).

5.4 Stability

The stability property of a smoother is not measurable. Conclusions about the stability of a smoother can be made by inspecting the way it treats an impulse (spike, outlier).

Table 5.4: Idempotent and co-idempotent smoothers

x	-0,007	-0,212	-0,215	0,833	-0,766	0,180	-0,896	0,697	-1,284	0,537
U_1	-0,007	-0,212	-0,212	0,833	0,180	0,180	0,180	0,697	0,537	0,537
$(U_1)^2$	-0,007	-0,212	-0,212	0,833	0,180	0,180	0,180	0,697	0,537	0,537
$I - U_1$	0	0	-0,003	0	-0,946	0	-1,076	0	-1,821	0
$U_1(I - U_1)$	0	0	0	0	0	0	0	0	0	0
L_1	-0,212	-0,212	-0,215	-0,215	-0,766	-0,766	-0,896	-0,896	-1,284	0,537
$(L_1)^2$	-0,212	-0,212	-0,215	-0,215	-0,766	-0,766	-0,896	-0,896	-1,284	0,537
$I - L_1$	0,205	0	0	1,048	0	0,946	0	1,593	0	0
$L_1(I - L_1)$	0	0	0	0	0	0	0	0	0	0
L_1U_1	-0,212	-0,212	-0,212	0,180	0,180	0,180	0,180	0,537	0,537	0,537
$(L_1U_1)^2$	-0,212	-0,212	-0,212	0,180	0,180	0,180	0,180	0,537	0,537	0,537
$I - L_1U_1$	0,205	0	-0,003	0,653	-0,946	0	-1,076	0,160	-1,821	0
$L_1U_1(I - L_1U_1)$	0	0	0	0	0	0	0	0	0	0
U_1L_1	-0,212	-0,212	-0,215	-0,215	-0,766	-0,766	-0,896	-0,896	-0,896	0,537
$(U_1L_1)^2$	-0,212	-0,212	-0,215	-0,215	-0,766	-0,766	-0,896	-0,896	-0,896	0,537
$I - U_1L_1$	0,205	0	0	1,048	0	0,946	0	1,593	-0,388	0
$U_1L_1(I - U_1L_1)$	0	0	0	0	0	0	0	0	0	0
C_2	-0,212	-0,212	-0,212	0,180	0,180	0,180	0,180	0,537	0,537	0,537
$(C_2)^2$	-0,212	-0,212	-0,212	0,180	0,180	0,180	0,180	0,537	0,537	0,537
$I - C_2$	0,205	0	-0,003	0,653	-0,946	0	-1,076	0,160	-1,821	0
$C_2(I - C_2)$	0	0	0	0	0	0	0	0	0	0
F_2	-0,215	-0,215	-0,215	-0,215	-0,766	-0,766	-0,896	-0,896	-0,896	0,537
$(F_2)^2$	-0,215	-0,215	-0,215	-0,215	-0,766	-0,766	-0,896	-0,896	-0,896	0,537
$I - F_2$	0,208	0,003	0	1,048	0	0,946	0	1,593	-0,388	0
$F_2(I - F_2)$	0	0	0	0	0	0	0	0	0	0
$W_1^* = B_1$	-0,212	-0,212	-0,215	0,180	-0,766	0,180	-0,896	0,537	-0,896	0,537
$(W_1^*)^2$	-0,212	-0,212	-0,215	0,180	-0,766	0,180	-0,896	0,537	-0,896	0,537
$I - W_1^*$	0,205	0	0	0,653	0	0	0	0,160	-0,388	0
$W_1^*(I - W_1^*)$	0	0	0	0	0	0	0	0	0	0
B_2	-0,212	-0,212	-0,215	0,180	-0,766	0,180	-0,896	0,537	-0,896	0,537
$(B_2)^2$	-0,212	-0,212	-0,215	0,180	-0,766	0,180	-0,896	0,537	-0,896	0,537
$I - B_2$	0,205	0	0	0,653	0	0	0	0,160	-0,388	0
$B_2(I - B_2)$	0	0	0	0	0	0	0	0	0	0

In Section 3.5 practical examples were used to illustrate the behaviour of the median smoother to impulses (spikes) of different lengths. The same was done for LULU smoothers in Section 4.5. From the illustrations and remarks in those sections, it may be concluded that both the class of median smoothers and the class of LULU smoothers can either preserve or remove impulses. The length of the impulse and the size of the window play an important role in whether the smoother preserves or removes the impulse.

The window size of the median smoother must be at least twice the length of the impulse to remove

Table 5.5: Smoothers neither idempotent nor co-idempotent

x	-0,007	-0,212	-0,215	0,833	-0,766	0,180	-0,896	0,697	-1,284	0,537
G_1	-0,212	-0,212	-0,214	-0,017	-0,293	-0,293	-0,358	-0,179	-0,179	0,537
$(G_1)^2$	-0,212	-0,212	-0,213	-0,213	-0,293	-0,293	-0,293	-0,179	-0,179	0,537
$I - G_1$	0,205	0	-0,001	0,850	-0,473	0,473	-0,538	0,876	-1,105	0
$G_1(I - G_1)$	0	0	-0,001	0,236	0	0	-0,032	-0,032	-0,269	0
Q_1	-0,212	-0,212	-0,212	-0,215	0,180	-0,766	0,180	-0,896	0,537	0,537
$(Q_1)^2$	-0,212	-0,212	-0,212	-0,212	-0,215	0,180	-0,766	0,180	0,537	0,537
$I - Q_1$	0,205	0	-0,003	1,048	-0,946	0,946	-1,076	1,594	-1,821	0
$Q_1(I - Q_1)$	0	0	0	-0,003	0,946	-0,946	0,946	-1,076	0	0
M_1	-0,212	-0,212	-0,212	-0,215	0,180	-0,766	0,180	-0,896	0,537	0,537
$(M_1)^2$	-0,212	-0,212	-0,212	-0,212	-0,215	0,180	-0,766	0,180	0,537	0,537
$I - M_1$	0,205	0	-0,003	1,048	-0,946	0,946	-1,076	1,594	-1,821	0
$M_1(I - M_1)$	0	0	0	-0,003	0,946	-0,946	0,946	-1,076	0	0
M_2	-0,215	-0,212	-0,212	-0,212	-0,215	0,180	-0,766	0,180	0,537	0,697
$(M_2)^2$	-0,212	-0,212	-0,212	-0,212	-0,212	-0,212	0,180	0,180	0,537	0,537
$I - M_2$	0,208	0	-0,003	1,045	-0,551	0	-0,130	0,517	-1,821	-0,160
$M_2(I - M_2)$	0	0	-0,003	0	-0,130	-0,130	-0,130	0,517	-0,160	-0,160

the impulse, otherwise the impulse (blockpulse) will be preserved. Thus, for an impulse of length k (blockpulse), the median smoother M_n will remove the impulse only if $(2n+1) > 2k$, i.e. when $k - \frac{1}{2} < n$. This condition was illustrated in Figure 3.2 where M_1, M_2 and M_3 were applied to a signal with a positive 3-blockpulse and a negative 3-blockpulse.

For LULU smoothers it follows that a blockpulse of length $k \le n$ will be removed by L_nU_n or U_nL_n. Figure 4.3 illustrates the removal and preservation of blockpulses by applying LULU smoothers L_2U_2, U_2L_2, L_3U_3 and U_3L_3 to a signal with a positive 3-blockpulse and a negative 3-blockpulse.

In comparison, there is a difference of $\frac{1}{2}$ on the length of the impulse k for the median smoother M_n and the LULU smoother L_nU_n or U_nL_n to remove an impulse of length k. For M_n the condition to remove an impulse of length k is $k - \frac{1}{2} < n$, and for L_nU_n it is $k \le n$. These two conditions are equivalent.

5.5 Edge preservation

One of the main advantages of nonlinear median smoothing is that it preserves sharp edges (step edges), while linear smoothers like the moving average, blur such edges. The way that LULU smoothers handle sequences with sharp edges compared to the median smoother is of interest. The definition of an edge was given in Section 2.3.8, while a model of an edge was given in Section 3.7.

Table 3.1 summarised the mean and standard deviation of the median smoother at different window sizes for different edge heights. For the median smoother of window size five, the mean and standard deviation were tabulated for $k = 1$ and $k = 2$ only. The mean and standard deviation for $k = 3$ and $k = 4$ were calculated in order to compare them with those of the LULU smoothers shown in Table 5.6.

The distributions of $L_1U_1(x)$ and $U_1L_1(x)$ were derived in Section 4.7 for the case of five observations from two underlying distributions $F_1(x)$ and $F_2(x)$.

Let k observations be from a $N(h, \sigma^2)$ distribution with cdf $F_1(x)$, and $(5-k)$ observations be from a $N(0, \sigma^2)$ distribution with cdf $F_2(x)$. As an example, an edge height of $h = 5$ and $\sigma^2 = 1$ were chosen arbitrarily. The pdfs of L_1U_1 and U_1L_1, are found by differentiation. The means and standard deviations for U_1L_1, L_1U_1, and the median smoother M_2 were calculated for $k = 1, 2, 3, 4$ by numerical integration.

Table 5.6: Expected value and standard deviation of M_2, U_1L_1 and L_1U_1 on edge-plus-noise observations for window size five and edge height five

	Expected value			**Standard deviation**		
k	M_2	U_1L_1	L_1U_1	M_2	U_1L_1	L_1U_1
1	0,297	-0,103	0,282	0,600	0,662	0,803
2	0,846	0,282	0,564	0,747	0,803	0,826
3	4,154	4,436	4,718	0,747	0,825	0,803
4	4,703	4,718	5,099	0,600	0,803	0,662

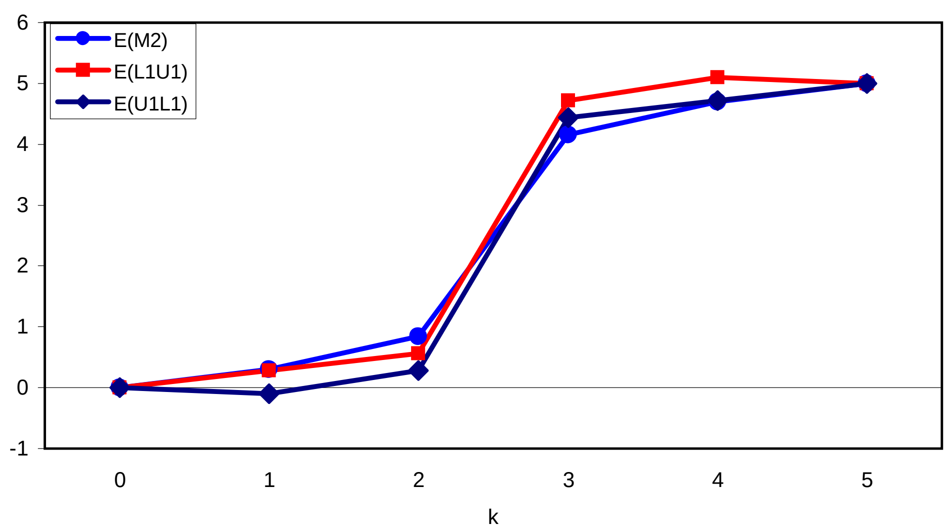

Figure 5.5: Expected values of M_2, U_1L_1 and L_1U_1 for window size five, $h = 5$ and $\sigma = 1$

The expected values of M_2, U_1L_1 and L_1U_1 are given in Figure 5.5. The expected values of U_1L_1 are the closest to zero for $k = 1, 2,$ where only one or two of the five observations are from the distribution with edge height five. For $k = 3, 4,$ where three or four observations are from the distribution with edge height five, the expected values of L_1U_1 are the closest to five. Thus, the expected values of U_1L_1 and L_1U_1 are closest to the mean of the dominating distribution around the edge. If the expected value of smoothers of

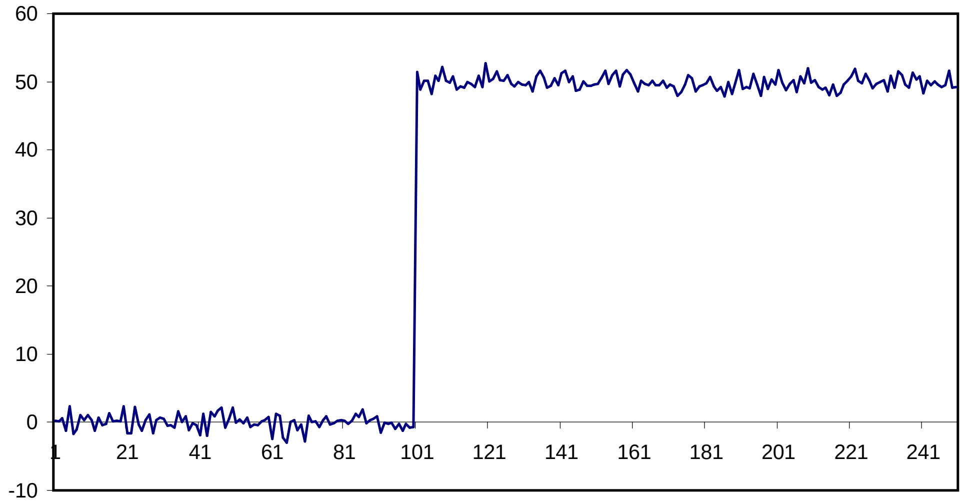

Figure 5.6: Example of step edge-plus-noise sequence

the same window size is used as a measure of comparison, then U_1L_1 and L_1U_1 preserve the edge better than the median smoother.

The standard deviation of the median smoother follows a symmetrical pattern around the values of k. The standard deviation of U_1L_1 and L_1U_1 are the same in a reverse order for the different values of k. The values of the standard deviation of the median for all values of k are less than the standard deviations of U_1L_1 and L_1U_1.

The MSE defined in (3.41) will be used as a goodness-of-fit measure in the following example.

An example of a *step* edge-plus-Gaussian noise sequence with the step $h = 50$ in the second distribution, and $\sigma = 1$ is shown in Figure 5.6. The first 100 observations were generated from a normal distribution with mean zero, and the next 150 observations were generated from a normal distribution with mean 50.

The median smoothers M_1, M_2, M_3, M_5, M_7 and the compound median smoothers (53H,twice) and (4253H,twice) were used to smooth this sequence. The compound LULU smoothers B_1, B_2, B_3, B_4 and B_5 were applied to this edge-plus-noise sequence. The MSE for 20 observations, 10 on each side of the step, were calculated for each of these smoothers and are tabulated with their window sizes in Table 5.7.

For the 20 points close to the edge, M_2 with window size 5, preserves the edge best of all the median smoothers. The compound median smoothers have very large MSEs probably due to the linear component in them, and thus do not perform well for edge preservation compared to the other smoothers. The MSEs of the LULU smoothers have the same value for B_1 and B_2, and the same value for B_3 and B_4, the latter two being slightly smaller. The compound LULU smoother B_5 seems to preserve the edge best amongst the LULU smoothers. Considering the 20 points close to the edge, the MSE of B_5 falls between the MSEs of M_2 and M_3.

Table 5.7: Estimated mean square errors of smoothers on edge-plus-noise observations

Smoother	**Window size**	**MSE(N = 20)**
M_1	3	0,2463
M_2	5	0,1673
M_3	7	0,1974
M_5	11	0,2402
M_7	15	0,2310
(53H,twice)	17	15,7718
(4253H,twice)	25	24,0511
B_1	5	0,3004
B_2	13	0,3004
B_3	25	0,2066
B_4	41	0,2066
B_5	61	0,1876

5.6 Distributions

In Section 3.6 the statistical properties of the median smoother were investigated for input that is a constant signal with additive white noise. The noise smoothing performance of a smoother can be measured using the smoother output variance. Comparing the performance of LULU smoothers with median smoothers for various identified input distributions by computing the output variances is of interest. This is possible since the output distribution of the median has been derived, as given in (3.6) and (3.7). The exact output distributions of the LULU smoothers L_nU_n and U_nL_n have been derived in Section 4.6.1 and result in (4.17) and (4.18) respectively. These are the only exact distributions of LULU smoothers that are available and hence the ones that are considered.

Time series sequences with known input distributions are considered. The cdf and pdf together with the mean and variance of each of these distributions were given in Section 3.6. The same input distributions were used here.

5.6.1 Output distributions

Graphs of the LULU output distributions (n = 2, 5, 10) for each of these input distributions are shown in Figures 5.7, 5.9, 5.11, 5.13, 5.15 and 5.17. The running median smoothers, of which some have the same window sizes as the LULU smoothers (n = 1, 2, 4, 10, 20), for these input distributions are illustrated in Figures 5.8, 5.10, 5.12, 5.14, 5.16 and 5.18. The t-distribution is used with three degrees of freedom. The contaminated normal distribution is used with ε = 0,1 and τ = 3. The output of the median smoothers and LULU smoothers will be discussed for each of the chosen input distributions.

1. Uniform(0, 1) distribution

As L_nU_n and U_nL_n are upper and lower smoothers, the output distributions of these smoothers as shown in Figure 5.7, where the uniform input distribution is used, have long-tailed distributions which are symmetrical to each other around the mean value of 0,5. Figure 5.8 shows the output distributions of the median smoothers for this input distribution. The output distributions of L_nU_n are skewed to the left, while those of U_nL_n are skewed to the right. The median smoothers have a symmetrical shape around the mean of 0,5 with less variation as the window size increases.

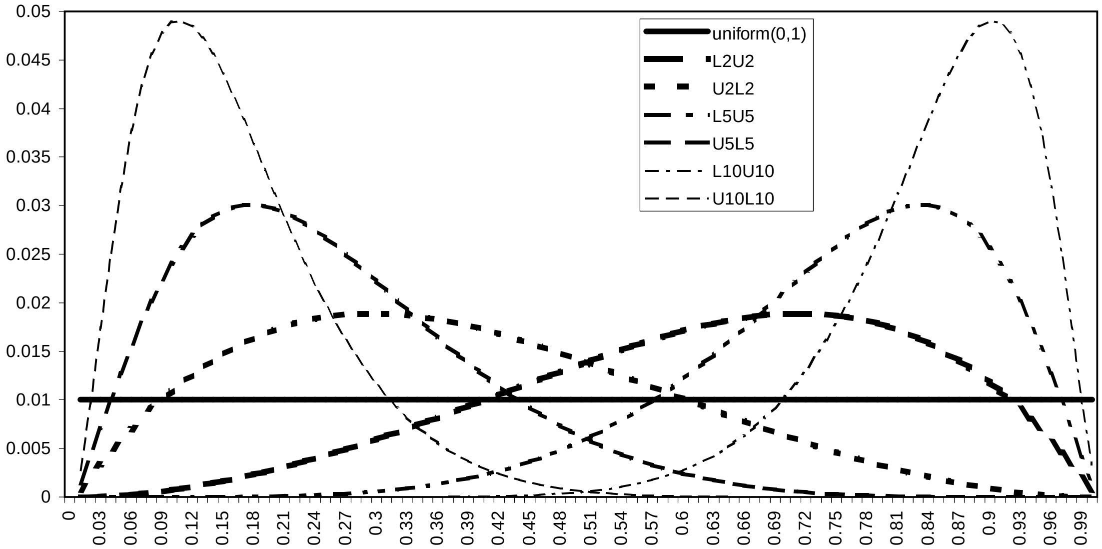

Figure 5.7: LULU output distributions for the uniform input distribution

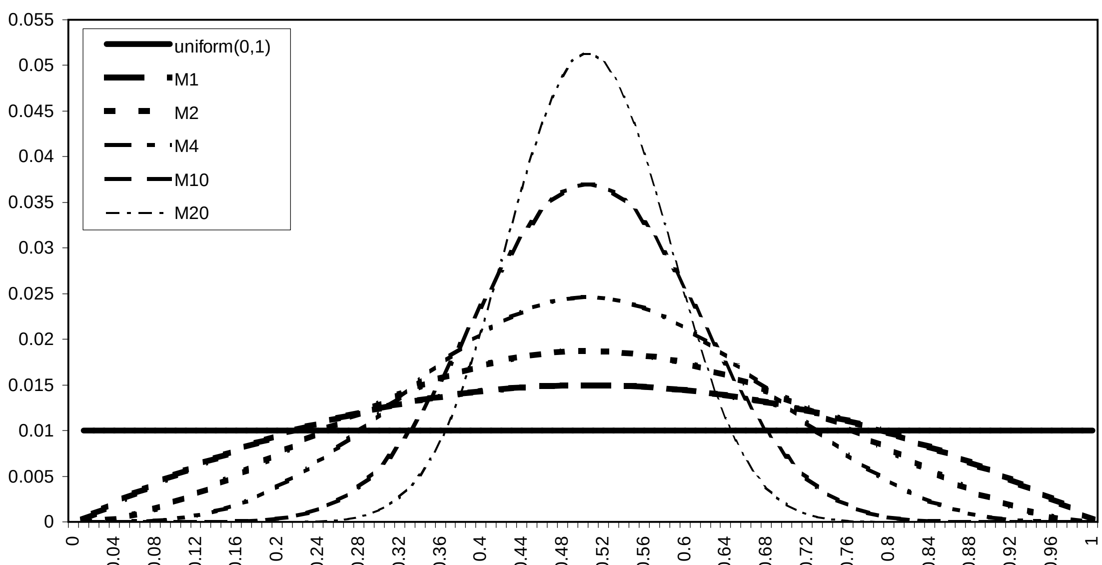

Figure 5.8: Median output distributions for the uniform input distribution

2. Normal(0, 1) distribution

The output distributions of the LULU smoothers for the standard normal input distribution, illustrated in Figure 5.9, have the shape of a normal distribution, but with different means and variances for different window sizes. Note that $L_nU_n \xrightarrow{P} +\infty$ and $U_nL_n \xrightarrow{P} -\infty$ as $n \to \infty$, and thus the distributions of L_nU_n and U_nL_n move further apart as n increases. Figure 5.10 illustrates the output distributions of the median smoothers which are approximately normal with mean zero.

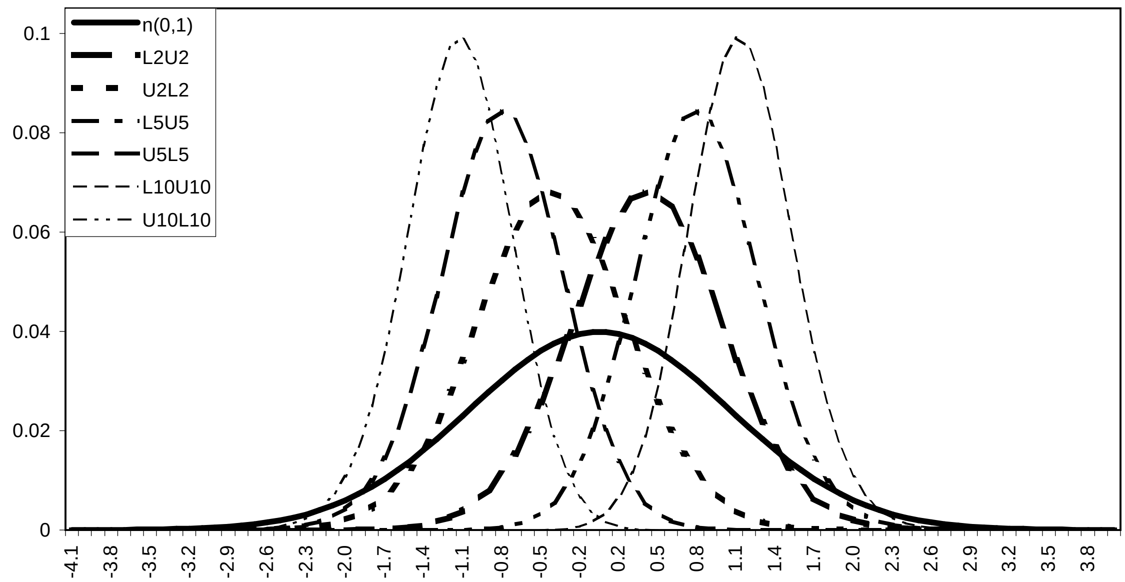

Figure 5.9: LULU output distributions for the standard normal input distribution

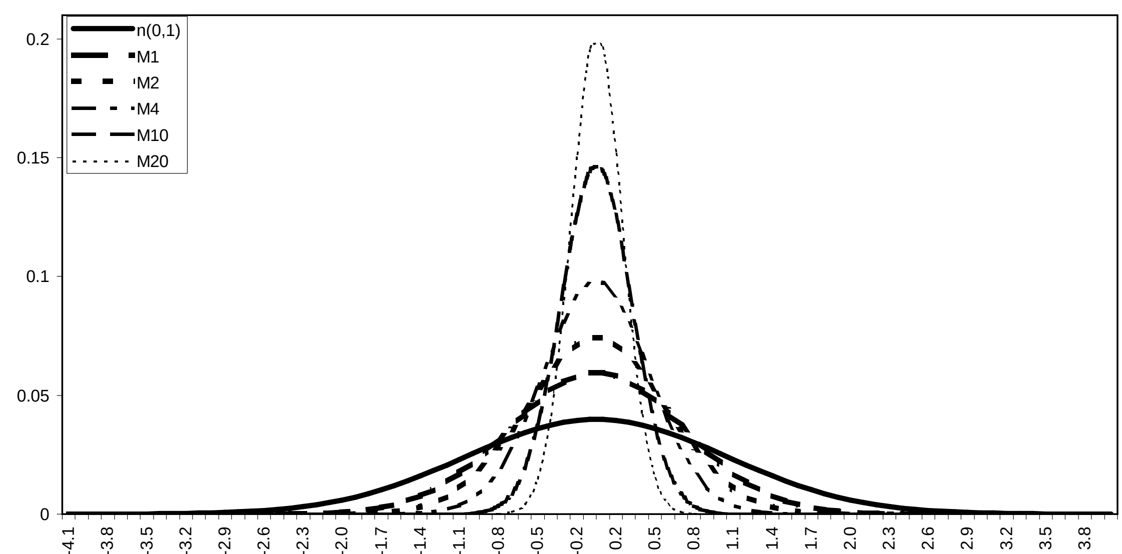

Figure 5.10: Median output distributions for the standard normal input distribution

3. Logistic(0, 1) distribution

From Figure 5.11 it follows that the output distributions of the LULU smoothers for the logistic input distribution are similar to their output for the standard normal input distribution. The logistic distribution is bell-shaped with a mean value of 0 and variance $\frac{\pi^2}{3}$. This variance is larger than that of the standard normal distribution which results in longer tails. The output distributions of the median smoothers illustrated in Figure 5.12 are also approximately normal with mean zero. The variances of the output distributions are all less than the variance of the logistic distribution, and become smaller as the window size increases.

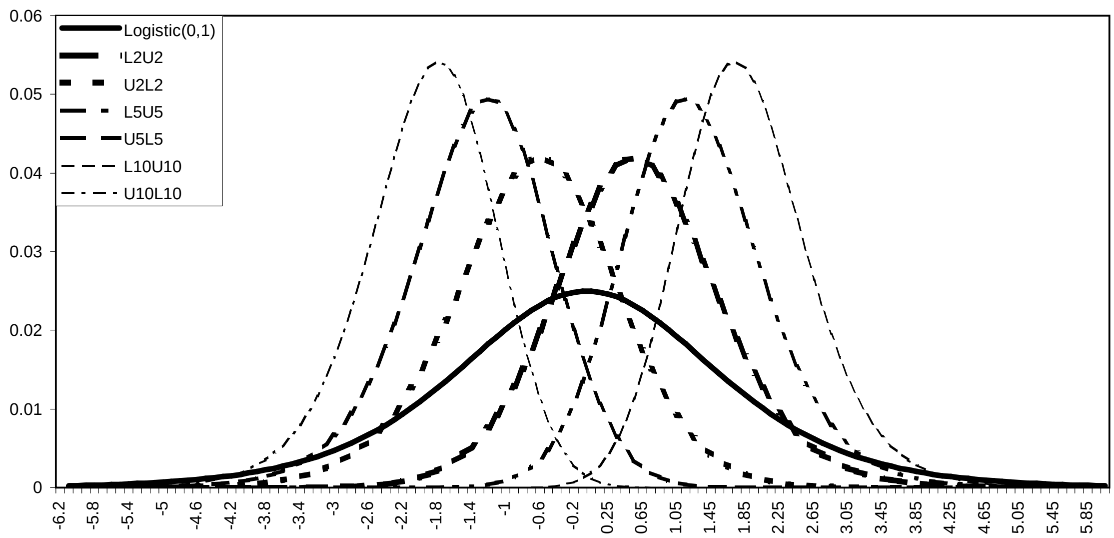

Figure 5.11: LULU output distributions for the logistic input distribution

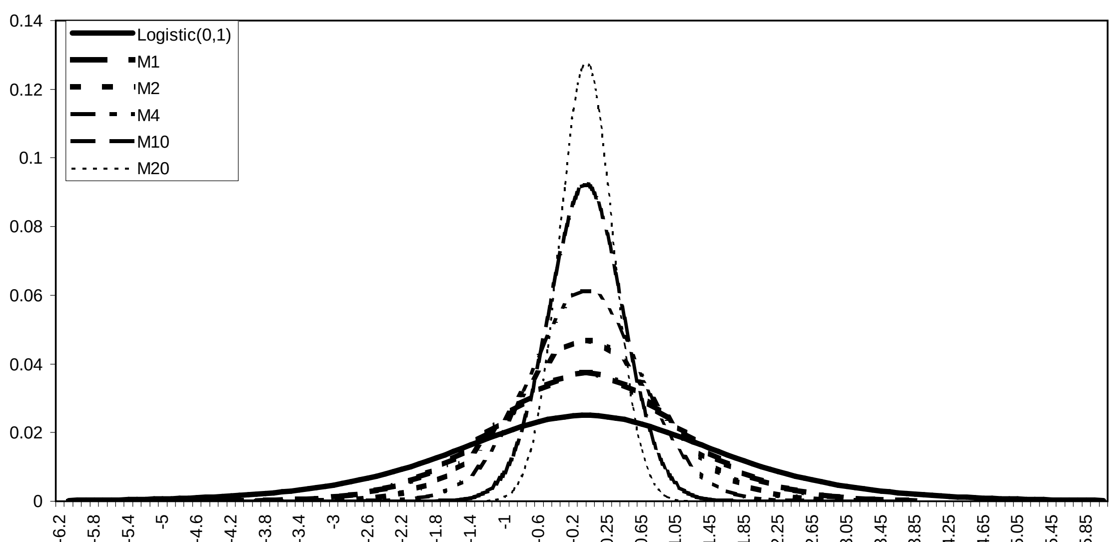

Figure 5.12: Median output distributions for the logistic input distribution

4. Laplace(0, 1) distribution

From Section 3.6 it follows that the Laplace distribution with parameters $\alpha = 0$ and $\beta = 1$ has a mean value of 0 and a variance of two. The LULU output distributions for the Laplace input distribution are illustrated in Figure 5.13. From Figure 5.14 the median output distributions follow the shape of the Laplace input distribution with mean zero and decreasing variances as the window size increases and, as expected, the familiar form is observed.

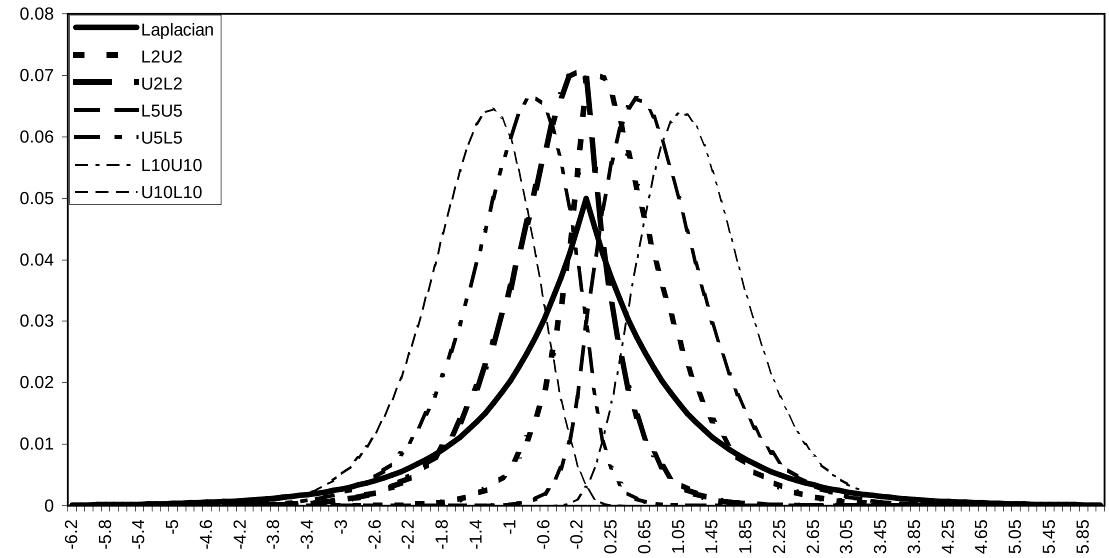

Figure 5.13: LULU output distributions for the Laplace input distribution

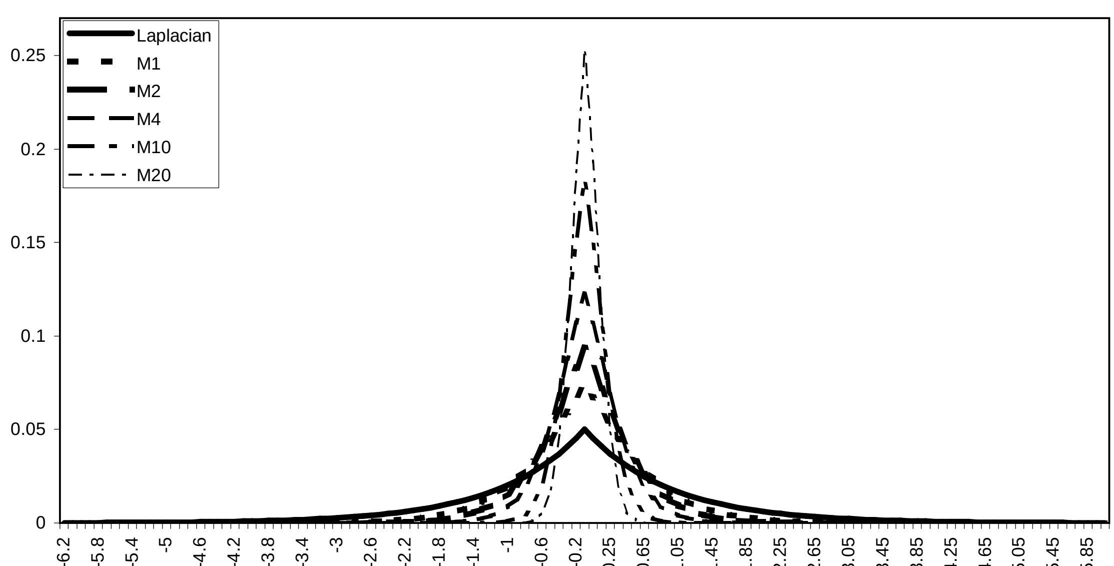

Figure 5.14: Median output distributions for the Laplace input distribution

5. t-distribution with 3 d.f.

From Section 3.6 the mean of the *t*-distribution is zero and for three degrees of freedom, the variance is three, which is an indication of long tails. From Figures 5.15 and 5.16 it follows that the output distributions of the LULU and median smoothers for the *t*-input distribution with three degrees of freedom are similar to the output of the normal distribution (cf. Figures 5.9 and 5.10).

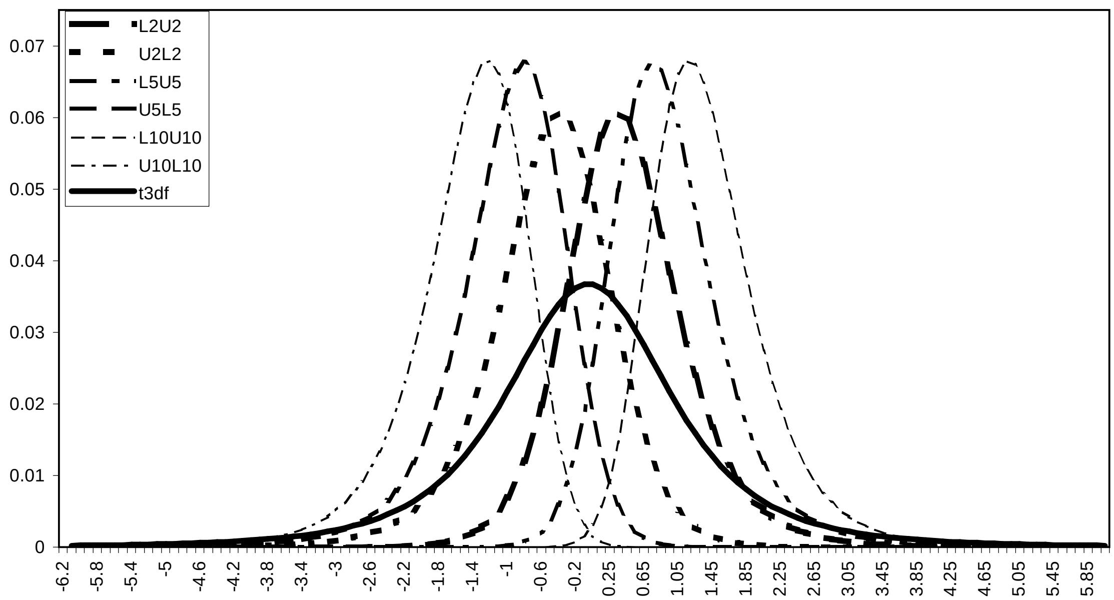

Figure 5.15: LULU output distributions for the *t*-input distribution with 3 degrees of freedom

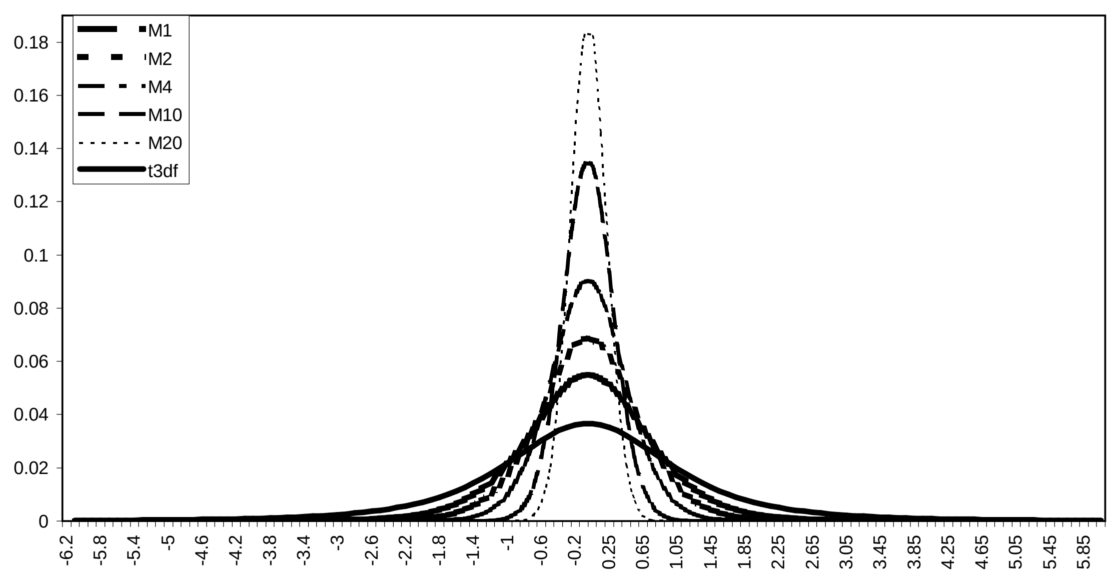

Figure 5.16: Median output distributions for the *t*-input distribution with 3 degrees of freedom

6. Contaminated normal distribution (ε = 0,1, τ = 3)

The output distributions of the LULU and median smoothers for the contaminated normal (ε = 0,1, τ = 3) input distribution are given in Figures 5.17 and 5.18.

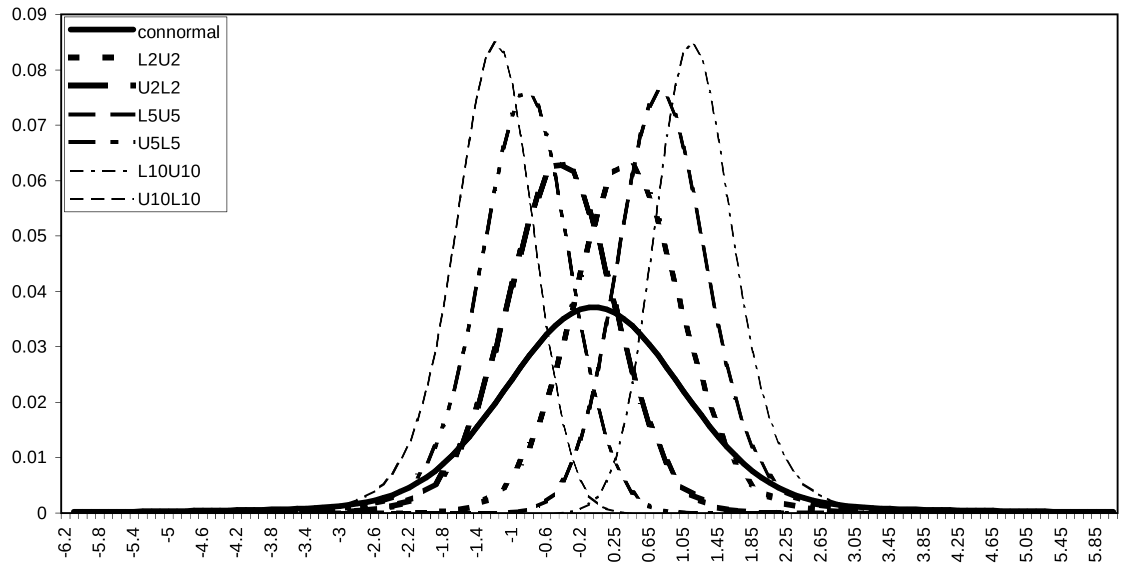

Figure 5.17: LULU output distributions for the contaminated normal (ε = 0,1, τ = 3) input distribution

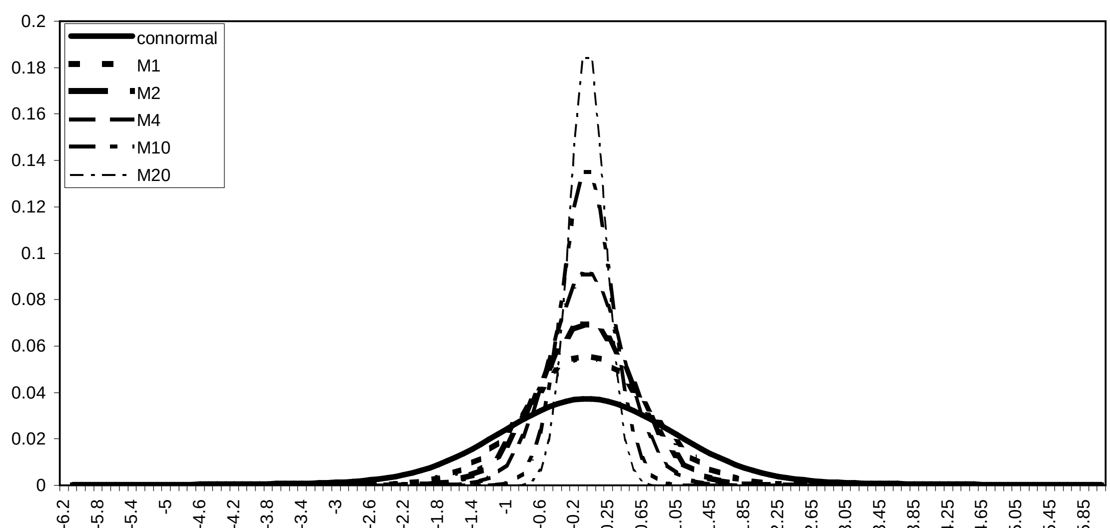

Figure 5.18: Median output distributions for the contaminated normal (ε = 0,1, τ = 3) input distribution

5.6.2 Output expected values and variances

Simple exact formulæ for the mean and variance of the LULU smoothers cannot be easily derived theoretically. The cdf of the LULU smoothers consists of the sum of powers of the cdf of the input distribution, involving n, which makes it quite complicated. The moments can, however, be calculated by numerical integration.

The output expected values and variances of the median smoother and LULU smoothers are computed and compared. These were calculated for the LULU smoothers, L_nU_n and U_nL_n for $n = 1, 2, 5$. Note that the window sizes for the LULU smoothers are larger than that for the median smoothers at the same value of n. The results, in the columns named "Exact Values", are given in Tables 5.8 and 5.9.

Simulation was also used and 500 repetitions of 500 values from each input distribution were simulated. The expected values and variances of M_n, U_nL_n, L_nU_n and B_n that were calculated from these simulated values are tabulated in the columns under "Simulated values" in the same tables. In all cases the exact and simulated values are in fairly close agreement.

Table 5.8: Expected values of M_n, U_nL_n U_nL_n and B_n

Input distribution	**Exact values**			**Simulated values**			
	M_n	U_nL_n	L_nU_n	M_n	U_nL_n	L_nU_n	B_n
1. Uniform(0, 1)				Mean= 0,49955			
$n = 1$	0,500	0,450	0,550	0,499	0,449	0,549	0,499
$n = 2$	0,500	0,376	0,624	0,499	0,376	0,623	0,499
$n = 5$	0,500	0,248	0,752	0,499	0,249	0,752	0,499
2. Normal(0, 1)				Mean=-0,00281			
$n = 1$	0	-0,149	0,149	-0,002	-0,151	0,147	-0,002
$n = 2$	0	-0,368	0,368	-0,003	-0,372	0,365	-0,004
$n = 5$	0	-0,755	0,756	-0,004	-0,756	0,757	-0,005
3. Logistic(0, 1)				Mean=-0,00013			
$n = 1$	0	-0,250	0,250	-0,002	-0,254	0,252	-0,001
$n = 2$	0	-0,619	0,619	-0,001	-0,621	0,621	-0,001
$n = 5$	0	-1,277	1,277	-0,002	-1,279	1,282	-0,005
4. Laplace(0, 1)				Mean=-0,00276			
$n = 1$	0	-0,172	0,172	-0,003	-0,174	0,169	-0,003
$n = 2$	0	-0,425	0,425	-0,002	-0,425	0,424	-0,002
$n = 5$	0	-0,886	0,886	-0,001	-0,885	0,881	-0,002
5. t with 3 d.f.				Mean=-0,00035			
$n = 1$	0	-0,183	0,183	-0,005	-0,183	0,174	-0,005
$n = 2$	0	-0,453	0,453	-0,004	-0,444	0,442	-0,001
$n = 5$	0	-0,939	0,939	-0,010	-0,960	0,922	-0,000
6. Contaminated normal($\varepsilon = 0{,}1$, $\tau = 3$)				Mean=0,00142			
$n = 1$	0	-0,166	0,166	0,001	-0,165	0,167	0,001
$n = 2$	0	-0,410	0,410	0,002	-0,410	0,411	0,002
$n = 5$	0	-0,845	0,845	0,000	-0,835	0,847	-0,001

For the Uniform(0, 1) distribution, the expected values of M_n and B_n for different window sizes are all very close to 0,5, the mean of the input distribution. For the expected values of U_nL_n and L_nU_n it follows that $\mathrm{E}(L_nU_n) = 1 - \mathrm{E}(U_nL_n)$ and they are thus symmetrically located around the mean of 0,5. As the window size increases, the expected values of L_nU_n increase to the value one, while that of U_nL_n decrease

Table 5.9: Variances of M_n, U_nL_n, L_nU_n and B_n

Input distribution	**Exact values**			**Simulated values**			
	M_n	U_nL_n	L_nU_n	M_n	U_nL_n	L_nU_n	B_n
1. Uniform(0, 1)				Variance=0,083 s.e.=0,0129			
$n = 1$	0,050	0,050	0,050	0,050	0,050	0,050	0,055
$n = 2$	0,036	0,038	0,038	0,035	0,038	0,038	0,046
$n = 5$	0,019	0,019	0,019	0,019	0,019	0,019	0,030
2. Normal(0, 1)				Variance=1,0013 s.e.=0,0447			
$n = 1$	0,440	0,454	0,454	0,447	0,452	0,452	0,502
$n = 2$	0,282	0,352	0,352	0,285	0,350	0,351	0,403
$n = 5$	0,136	0,232	0,232	0,135	0,228	0,229	0,229
3. Logistic(0, 1)				Variance=3,320 s.e.=0,0814			
$n = 1$	1,244	1,311	1,311	1,295	1,310	1,319	1,464
$n = 2$	0,767	1,023	1,023	0,788	1,017	1,024	1,151
$n = 5$	0,357	0,738	0,738	0,356	0,718	0,732	0,621
4. Laplace(0, 1)				Variance=2,0079 s.e.=0,0633			
$n = 1$	0,444	0,687	0,687	0,632	0,650	0,651	0,729
$n = 2$	0,235	0,524	0,524	0,347	0,519	0,521	0,550
$n = 5$	0,098	0,458	0,458	0,137	0,448	0,446	0,255
5. t **with** 3 **d.f.**				Variance=3,0578 s.e.=0,0763			
$n = 1$	0,512	0,739	0,739	0,720	0,739	0,736	0,820
$n = 2$	0,329	0,586	0,586	0,407	0,582	0,580	0,622
$n = 5$	0,159	0,490	0,490	0,181	0,477	0,495	0,321
6. Contaminated normal($\varepsilon = 0{,}1$, $\tau = 3$)				Variance=1,80613 s.e.=0,060			
$n = 1$	0,601	0,583	0,583	0,572	0,579	0,583	0,647
$n = 2$	0,361	0,456	0,456	0,344	0,448	0,452	0,505
$n = 5$	0,164	0,338	0,338	0,158	0,333	0,335	0,271

to zero. This is also clear from Figure 5.7, where the distributions of L_nU_n move towards the value one, and those of U_nL_n move towards the value zero as the window size increases.

The expected values of M_n and B_n are very close to the mean of the normal input distribution (zero) over all window sizes. Also $\mathrm{E}(U_nL_n) = -\mathrm{E}(L_nU_n)$, since the normal input distribution is symmetrical. As the window size increases, the expected values of L_nU_n increase, while those of U_nL_n decrease.

Since all the other input distributions are symmetrical with mean zero, the expected values of M_n and B_n are close to zero for all window sizes. Also, from Table 5.8 it is clear that the expected values of L_nU_n and U_nL_n are symmetrical around the value zero.

The variances for different window sizes of M_n and the LULU smoothers, U_nL_n, L_nU_n and B_n, for the uniform, standard normal, logistic, Laplace, t- and contaminated normal input distribution are tabulated in Table 5.9. The output variance of a smoother can be used to measure the noise smoothing performance of a smoother. The smaller the output variance, the better the smoother smoothed the noise. Since small window sizes are used, the variance of the median smoother is approximated by (3.9). The variances for U_nL_n and L_nU_n are equal. For all the smoothers the variances decrease as the window size increases, since more smoothing takes place at larger window sizes.

For the Uniform(0, 1) distribution the variances of M_n and U_nL_n (L_nU_n) are of the same order, although the window sizes of the LULU smoothers are larger. The variance of B_n is a little larger than the variances of the other smoothers for the same n. Thus, M_n, U_nL_n and L_nU_n perform similarly in this case.

The other input distributions are all symmetrical around the mean value of zero, and the output distributions of the median and LULU smoothers at different window sizes are bell-shaped with different means and variances. The median smoother, M_n, has smaller variances than the LULU smoothers in all cases, which mean they smooth noise better. In this regard it is important to remember that L_nU_n and U_nL_n are half smoothers. B_n, on the other hand, is a full smoother that preserves the original series between C_n and F_n, and hence its variances are higher as expected. The only exception is for the contaminated normal ($\varepsilon = 0{,}1$, $\tau = 3$) distribution (exact values) with $n = 1$ where LULU does marginally better. There are considerable differences between the exact and simulated variances of M_n for the Laplace and t-distributions which does not occur for the other distributions. The smaller the window size, the greater the difference. It seems that the approximation of the variance of the median for small n using (3.9) is not so accurate for these two distributions.

5.7 Variation decomposition

The median smoothers M_n are *trend preserving* in the sense that, if $\{x_{i-n}, \ldots, x_i, \ldots, x_{i+n}\}$ is monotone, then $(M_nx)_i = x_i$. The operators L_nU_n and U_nL_n share the trend preserving property with median smoothers. Another stronger result is the global sequence preservation $L_nU_nx = U_nL_nx = x$ if x is n-monotone which M_n shares.

Successive smoothing by LULU smoothers yields a monotone reduction of the total variation. This property is not shared by the median smoothers or other well-known nonlinear smoothers. The variation that is removed at each level is calculable, and it indicates the contribution of noise at that resolution level as illustrated in Section 4.9. As soon as large reductions in variation stop, it indicates that the impulsive noise has been adequately removed.

No stopping rule has been formulated, and the decision at which level to stop is currently intuitive. The amount of variation removed at two successive levels is compared, and if the difference seems to be a fair amount less than the difference of the variation removed between the previous two levels, it can be decided to stop. This variation decomposition is one of the most attractive properties of LULU smoothers not shared by the currently known nonlinear smoothers and makes them more competitive relative to other nonlinear smoothers. It will be further illustrated in Chapter 7.

5.8 Conclusions

The performance of median and LULU smoothers with respect to properties important for the process of smoothing was compared. For an n-monotone input sequence, the output sequence for a smoother P_k, $k \leq n$, is the original sequence and thus the root sequence. The median and LULU smoothers give the same result in the case of an n-monotone sequence. Most LULU smoothers are idempotent and co-idempotent and thus are consistent. This means that if the output of the smoother is passed through the smoother a second time, the output remains unchanged. The output is thus defined as signal and is not smoothed any further. The median smoother is neither idempotent nor co-idempotent. Convergence to a root signal after a number of passes has been derived mathematically. Since LULU smoothers are only applied once to obtain a root signal, they are more economical and efficient than the median smoothers.

Impulses (blockpulses) can be removed or preserved by median and LULU smoothers. The window size of the chosen smoother determines whether the impulse is removed or preserved. Median and LULU smoothers are considered stable in the sense that they deal well with impulses. Median and LULU smoothers preserve sharp step edges compared to linear smoothers, such as the moving average, which blur such edges. Considering the estimated mean square errors of some points close to the edge, the LULU smoothers perform just as well as the median smoothers.

The output distributions for the median and LULU smoothers at different window sizes were illustrated for various chosen input distributions. The input distributions had different tail lengths. The median smoothers resulted in bell-shaped distributions with varying variances around the mean of the input distribution. The output distributions of L_nU_n and U_nL_n resulted in skewed or bell-shaped distributions symmetrical to each other on either side of the mean of the input distribution with varying variances. They were more spread out than the median output distributions due to slower convergence. Both smoothers performed well for long-tailed input distributions.

The amount of variation removed (peeled off) at each level can be monitored for compound LULU smoothers. Although this can also be calculated for median smoothers, it is not additive. Compound LULU smoothers provide a structured mechanism for the smoothing process which is a very attractive property not shared by other currently known nonlinear smoothers.

The comparison of median and LULU smoothers is continued in the next chapter where a comparison is made of their respective abilities for signal recovery. The latter is measured by integrated mean square errors and regression coefficients calculated from least squares regression of the smoothed sequence on the signal sequence.

Chapter 6

Signal recovery with LULU smoothers

6.1 Introduction

In this chapter the success of LULU smoothers in the recovering of signal is measured and compared with the nonlinear smoothers suggested by Tukey (1971) and Velleman (1975, 1977). In his evaluation of the latter smoothers, Velleman (1980) used the idea that the success of a smoother to recover a smooth signal from a signal with added noise can be measured by the means of the least squares regression of the smoothed sequence on the signal sequence. This procedure, as well as estimates of the integrated mean square error, will be used in this chapter to compare LULU smoothers with some other nonlinear smoothers.

The first series that is studied here is formed by constructing a sinusoidal curve as signal and adding noise to it. The noise is simulated from a contaminated normal distribution. The percentage of outliers (spikes) in the noise as well as their magnitude can be controlled by tuning the parameters of the contaminated distribution. Different sine curves were generated for different frequencies. Multiples of sixteenths were chosen for the frequencies. The nonlinear smoothers that were applied to these sequences to remove the non-Gaussian noise (spikes, outliers) are the compound LULU smoothers, and the compound median smoothers of Velleman and Tukey. For each smoother, the regression coefficients were calculated for the regression of the smoothed sequence on the signal sequence for each of the sequences formed by the 16 frequencies. An estimate of the expected value of the integrated mean square error to measure how well each smoother recovers the signal was also obtained in each case. This process was repeated 200 times for each smoother and each frequency. The means of the 200 calculated regression coefficients and the 200 integrated mean square errors were used as estimates of the regression coefficient and the integrated mean square error in each case.

The sequences smoothed by the nonlinear smoothers were further smoothed by a linear smoother, the moving average, to remove the Gaussian noise. The regression coefficients of the resulting smoothed sequence on the original signal sequence were calculated, as well as the estimated integrated mean square error. A regression coefficient close to one indicates that the signal has been recovered relatively successfully, while a coefficient close to zero indicates poor recovery of the signal. Regarding the integrated mean square error, the smaller the value, the better the smoother.

A successful smoother can thus be defined as one that is capable of recovering a noise-contaminated signal as well as if no noise were present, that is, one whose estimated regression coefficient changes little when noise is added to the signal and also one whose integrated mean square error is the smallest.

The procedure used to generate the noise-contaminated signal will be described in Section 6.2. The calculated results with graphs are illustrated and discussed in Section 6.3. In Section 6.4 some other interesting examples of series found in the literature are considered. The conclusions follow in Section 6.5.

6.2 Simulation procedure for sine signal

6.2.1 Simulation model

In this model, the simulated data is the sum of signal and noise:

$$Data_t = Signal_t + Noise_t = X_t \tag{6.1}$$

with the noise simulated from a contaminated normal distribution denoted by D_t and the signal generated by the sine curve

$$Signal_t = \mu_t = \eta t + A \sin B(t - C) \tag{6.2}$$

with t = index, η = slope of trend, $|A|$ = amplitude, $B = \frac{2\pi}{d}$ where d is the period and the frequency is $\frac{1}{d}$, and C = displacement. Hence

$$\begin{aligned} X_t &= \mu_t + D_t \\ &= \eta t + A \sin B(t - C) + D_t. \end{aligned}$$

Contaminated normal observations were generated as follows: Let Z be a standard normal random variable and define the random variable X as

$$X = \begin{cases} \alpha Z & \text{if } Y = 1 \\ \beta Z & \text{if } Y = 0 \end{cases} \tag{6.3}$$

with Y a Bernoulli(p) random variable independent from Z. Thus $P(Y = 1) = p$ and $P(Y = 0) = 1 - p$ so that

$$\begin{aligned} P(X \le x) &= P(\alpha Z \le x | Y = 1).P(Y = 1) + P(\beta Z \le x | Y = 0).P(Y = 0) \\ &= p\Phi\left(\frac{x}{\alpha}\right) + (1 - p)\Phi\left(\frac{x}{\beta}\right) \end{aligned} \tag{6.4}$$

with Φ the standard normal cumulative distribution function.

The variance of this distribution follows as

$$\begin{aligned} Var(X) &= Var(\alpha Z | Y = 1).P(Y = 1) + Var(\beta Z | Y = 0).P(Y = 0) \\ &= p\alpha^2 + (1 - p)\beta^2 \end{aligned} \tag{6.5}$$

and $E(X^4)$ as

$$\begin{aligned} E(X^4) &= E((\alpha Z)^4|Y=1).P(Y=1) + E((\beta Z)^4|Y=0).P(Y=0) \\ &= 3(p\alpha^4 + (1-p)\beta^4) \end{aligned} \quad (6.6)$$

since $E(Z^4) = 3$.

Hence the kurtosis follows as

$$\frac{E(X^4)}{\{E(X^2)\}^2} = \frac{3(p\alpha^4 + (1-p)\beta^4)}{\{p\alpha^2 + (1-p)\beta^2\}^2}.$$

In this study the value of $\beta = 1$ was chosen. Then (6.4) becomes

$$P(X \leq x) = p\Phi\left(\frac{x}{\alpha}\right) + (1-p)\Phi(x), \quad (6.7)$$

i.e. the contaminated normal distribution considered before.

To obtain a contaminated normal distribution with a high kurtosis, the value of α is set as 5,06 in this study so that for $p = 0{,}1$, $E(X^4) = 199{,}36$, $Var(X) = (0{,}1)(5{,}06)^2 + 0{,}9 = 3{,}46$ and kurtosis $= \frac{199{,}36}{(3{,}46)^2} = 16{,}65$. In the simulation of the contaminated normal distribution approximately 10% of the values come from a $N(0, (5{,}06)^2)$ distribution and approximately 90% from a $N(0, 1)$ distribution. Figure 6.1 displays the probability density functions of the standard normal and the contaminated normal distributions for $\alpha = 5{,}06$ and $p = 0{,}10$ on the left graph. On the right graph the cumulative distribution functions of the standard normal and contaminated normal distributions are displayed. Note the heavy tails of the contaminated normal distribution compared to the standard normal distribution.

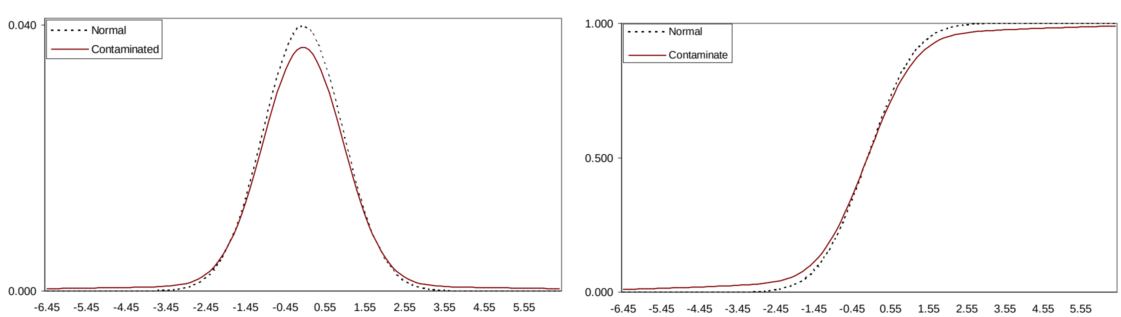

Figure 6.1: Probability density functions (left) and cumulative distribution functions (right) of the standard normal and the contaminated normal distribution

To illustrate the high kurtosis of the contaminated normal distribution versus the standard normal distribution, the probability of X greater than 3 was calculated for each distribution. For the standard normal distribution P($X > 3$) = 0,00138, while for the contaminated normal distribution P($X > 3$) = 0,02888. The probability in the case of the contaminated normal distribution is approximately 21 times greater than the probability of the standard normal distribution.

6.2.2 Simulation and smoothing process

The procedure to form the sequence containing the signal and added noise can be summarised in the following steps:

- Calculate m signal values using the sine curve $\mu_j = \eta t_j + A \sin B(t_j - C)$, $j = 1, \ldots, m$. This is repeated for frequencies of $\frac{1}{16}, \frac{2}{16}, \ldots, \frac{15}{16}, 1$.
- Generate noise D_{ij}, $j = 1, \ldots, m$; $i = 1, \ldots, N$, i.e. N independent sets of noise values of size m using the contaminated normal distribution.
- Form N new sequences $X_{ij} = \mu_j + D_{ij}$, $j = 1, \ldots, m$ for each $i = 1, \ldots, N$.
- Smooth each sequence $\{X_{ij}, j = 1, \ldots, m\}$ for each of the N independent sets with a nonlinear smoother P_n, giving $Y_{ij_r} = (P_n X)_{ij_r}$, $r = 1, \ldots, k < m$. $\{Y_{ij_r}\}$ can be considered an estimator of signal + Gaussian noise.
- Smooth each sequence $\{Y_{ij_r}\}$ with a linear smoother Q_n, giving $Z_{iq_r} = (Q_n Y)_{iq_r}$, $r = 1, \ldots, l < k < m$, for each $i = 1, \ldots, N$.

The simulation notation is summarised in the following table.

Table 6.1: Simulation notation

	Simulation 1	Simulation 2	......	Simulation N
μ_j	$\mu_1, \mu_2, \ldots, \mu_m$	$\mu_1, \mu_2, \ldots, \mu_m$	......	$\mu_1, \mu_2, \ldots, \mu_m$
X_{ij}	$X_{11}, X_{12}, \ldots, X_{1m}$	$X_{21}, X_{22}, \ldots, X_{2m}$	......	$X_{N1}, X_{N2}, \ldots, X_{Nm}$
Y_{ij_r}	$Y_{1j_1}, Y_{1j_2}, \ldots, Y_{1j_k}$	$Y_{2j_1}, Y_{2j_2}, \ldots, Y_{2j_k}$	......	$Y_{Nj_1}, Y_{Nj_2}, \ldots, Y_{Nj_k}$
Z_{iq_r}	$Z_{1q_1}, Z_{1q_2}, \ldots, Z_{1q_l}$	$Z_{2q_1}, Z_{2q_2}, \ldots, Z_{2q_l}$	......	$Z_{Nq_1}, Z_{Nq_2}, \ldots, Z_{Nq_l}$

6.2.3 Measures of signal recovery

Consider a signal plus noise process $\mu_t + D_t$ and let Y_t be a smoothed version of it. The following is used as a measure of closeness of Y_t to μ_t:

Definition 6.1. *The integrated mean square error (IMS_Y) is defined as a measure (per time unit) of the quality of the smoother Y:*

$$IMS_Y = \frac{1}{T} \int_0^T (Y_t - \mu_t)^2 dt. \tag{6.8}$$

Since IMS_Y is random, the expected value is taken to obtain

$$EIMS = E\left(\frac{1}{T} \int_0^T (Y_t - \mu_t)^2 dt \right). \tag{6.9}$$

EIMS can be is estimated through simulation by

$$\widehat{EIMS} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{k}\sum_{r=1}^{k}(Y_{ij_r} - \mu_{j_r})^2. \tag{6.10}$$

The smaller the $\widehat{EIMS}$, the better the smoother recovers the signal.

The variance of *IMS* is

$$Var(IMS) = E(IMS - E(IMS))^2 \tag{6.11}$$

and can be estimated through simulation by

$$\widehat{Var}(IMS) = \frac{1}{N-1}\sum_{i=1}^{N}\left(\frac{1}{k}\sum_{r=1}^{k}(Y_{ij_r} - \mu_{j_r})^2 - \frac{1}{N}\sum_{i=1}^{N}\frac{1}{k}\sum_{r=1}^{k}(Y_{ij_r} - \mu_{j_r})^2\right)^2 \tag{6.12}$$

for P_n and similarly for Q_n.

The smaller the $\widehat{Var}(IMS)$, the more stable the process.

As an alternative, according to Velleman (1980), the success of a smoother in recovering a smooth signal from contaminated noise can be measured by the least squares regression of the smoothed sequence on the signal sequence. If the regression coefficient is close to one, it indicates that the signal has been well recovered. If the value of the regression coefficient is close to zero, it indicates poor recovery.

For this, the following linear models are considered

$$\begin{aligned} Y_{ij_r} &= \beta_i\mu_{j_r} + \beta_i^* + \epsilon_{ij_r},\ r = 1,\ldots,k,\ i = 1,\ldots,N, \\ Z_{iq_r} &= \gamma_i\mu_{q_r} + \gamma_i^* + \epsilon_{iq_r}^*,\ r = 1,\ldots,l < k,\ i = 1,\ldots,N. \end{aligned} \tag{6.13}$$

with Y_{ij_r}, Z_{iq_r} and μ_{j_r} defined as earlier. The least squares estimates of the regression parameters are denoted by $\hat{\beta}_i$, $\hat{\beta}_i^*$, $\hat{\gamma}_i$ and $\hat{\gamma}_i^*$ for the i-th simulation run, $i = 1,\ldots,N$.

For each smoother applied to a sequence of signal with added noise, the following measures were calculated from the simulation:

- $\widehat{EIMS}$ as in (6.10).
- $\widehat{Var}(IMS)$ as in (6.12).
- $\hat{\beta} = \frac{1}{N}\sum_{i=1}^{N}\hat{\beta}_i$, $\hat{\beta}^* = \frac{1}{N}\sum_{i=1}^{N}\hat{\beta}_i{}^*$, $\hat{\gamma} = \frac{1}{N}\sum_{i=1}^{N}\hat{\gamma}_i$ and $\hat{\gamma}^* = \frac{1}{N}\sum_{i=1}^{N}\hat{\gamma}_i{}^*$.

6.2.4 Simulation of the signal

The sine function was considered for the choices $\eta = 0{,}7$, the amplitude $|A| = 3$ and the displacement $C = 1$. The choices of η, $|A|$ and C were to some degree arbitrary. The purpose was to produce a "nice"

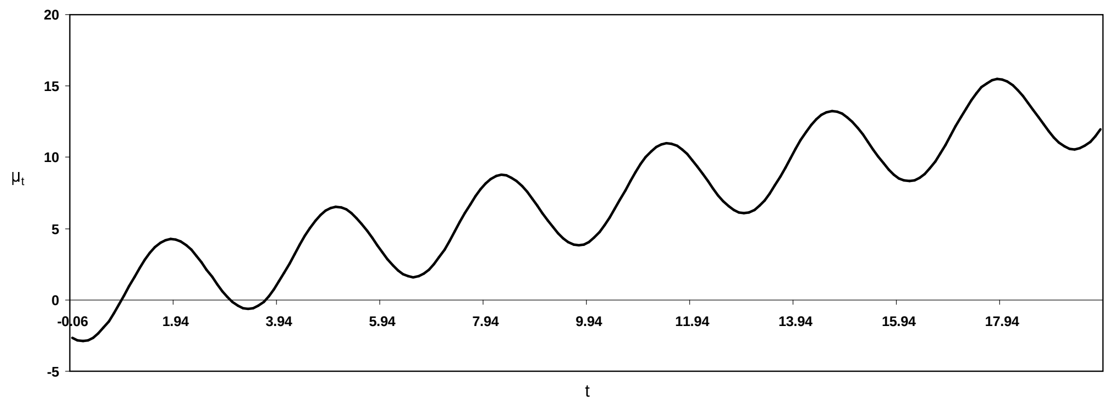

Figure 6.2: Example of a sine signal of frequency $\frac{5}{16}$ (0,3125)

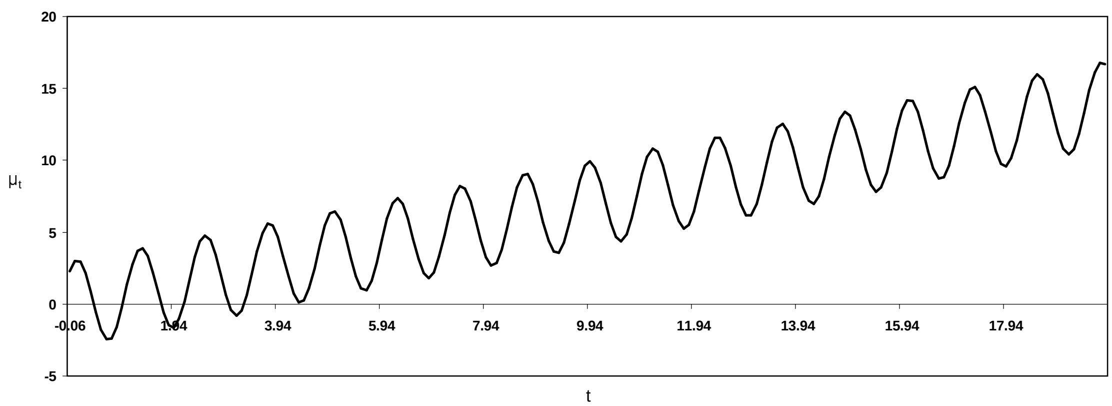

Figure 6.3: Example of a sine signal of frequency $\frac{13}{16}$ (0,8125)

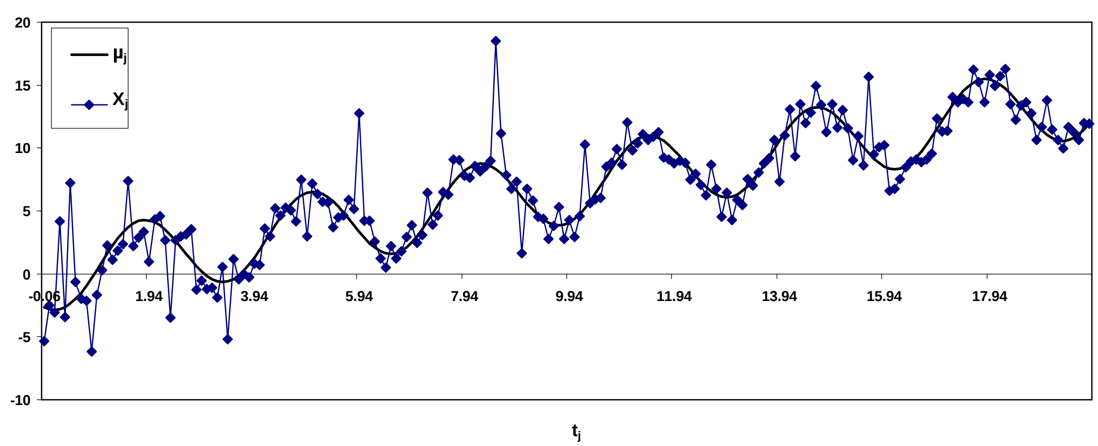

Figure 6.4: Signal of frequency $\frac{5}{16}$ (0,3125) with contaminated normal noise added

curve as for example in Figures 6.2 and 6.3. The effect of the larger frequency on the oscillation of the signal is evident.

For the function $\mu_j = \eta t_j + A \sin B(t_j - C)$, 200 values were calculated for t_j between $-0{,}0584$ and 19,8416 with increments of 0,1 at each of 16 different frequencies, viz $\frac{1}{16}, \frac{2}{16}, \ldots, \frac{16}{16}$. These series then form the 16 signal series.

Figure 6.4 is an illustration of a sine signal of frequency $\frac{5}{16}$ (0,3125), and the same signal series with noise added. The noise was generated from the contaminated normal distribution with $\beta = 1$, $p = 0{,}1$ and $\alpha = 5{,}06$ implying a kurtosis of 16,65. Since approximately 10% of the values were from a $N(0, (5{,}06)^2)$ distribution and approximately 90% from a $N(0, 1)$ distribution, the noise thus includes a number of randomly spread large outliers which form spikes (non-Gaussian noise) on the graph.

6.3 Simulation results for sine signal

All the conclusions in this section are summarised at the end of the section in Table 6.3.

6.3.1 Nonlinear smoothing on sine signal

The steps of the procedure described in Section 6.2.2 were executed $N = 200$ times independently. The measures of signal recovery as described in Section 6.2.3 were calculated. To remove the spikes the nonlinear compound LULU smoothers B_1, B_2, B_3, B_4, B_5, and the median smoothers M_1, M_2, (53H,twice) and (4253H,twice) (see Section 3.8.1) were applied.

Table 6.2 summarises the window size of each nonlinear smoother used in this study.

Table 6.2: Window sizes of nonlinear smoothers

Smoother	B_1	B_2	B_3	B_4	B_5	M_1	M_2	(53H,twice)	(4253H,twice)
Window size	5	13	25	41	61	3	5	17	25

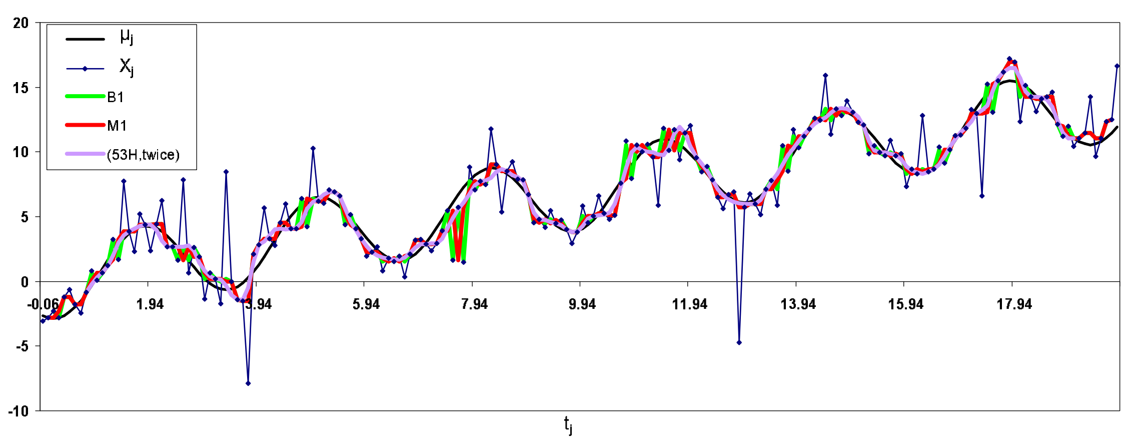

Figure 6.5: Example of nonlinear smoothing on sine signal of frequency $\frac{5}{16}$ (0,3125)

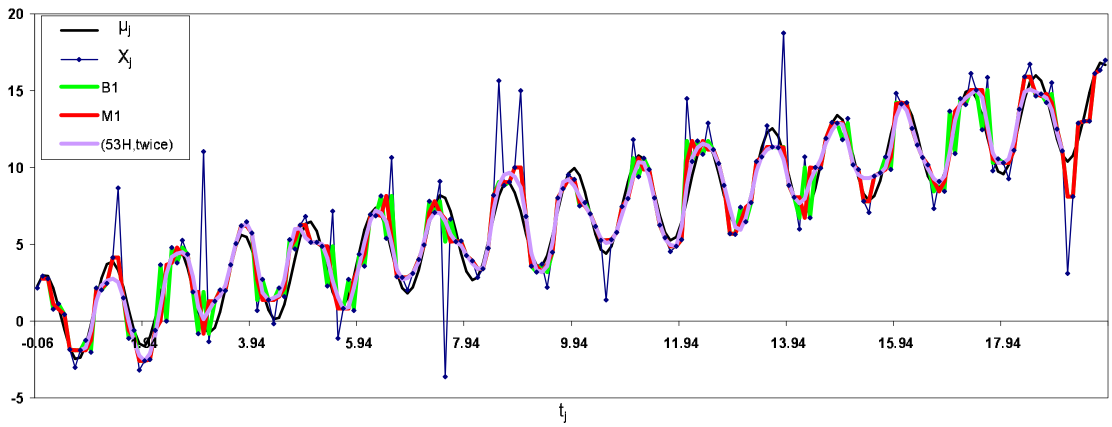

Figure 6.6: Example of nonlinear smoothing on sine signal of frequency $\frac{13}{16}$ (0,8125)

Figure 6.5 illustrates the nonlinear smoothers B_1, M_1 and (53H,twice) on a sine signal of frequency $\frac{5}{16}$ (0,3125) and Figure 6.6 displays the same nonlinear smoothers on a sine signal of frequency $\frac{13}{16}$ (0,8125). Note how the non-Gaussian noise (spikes, outliers) are mostly removed by these smoothers at both frequencies.

For each nonlinear smoother the estimated regression coefficients and their variances, as well as the estimated expected integrated mean square error and its variance at the different frequencies, are tabulated in Appendix A.

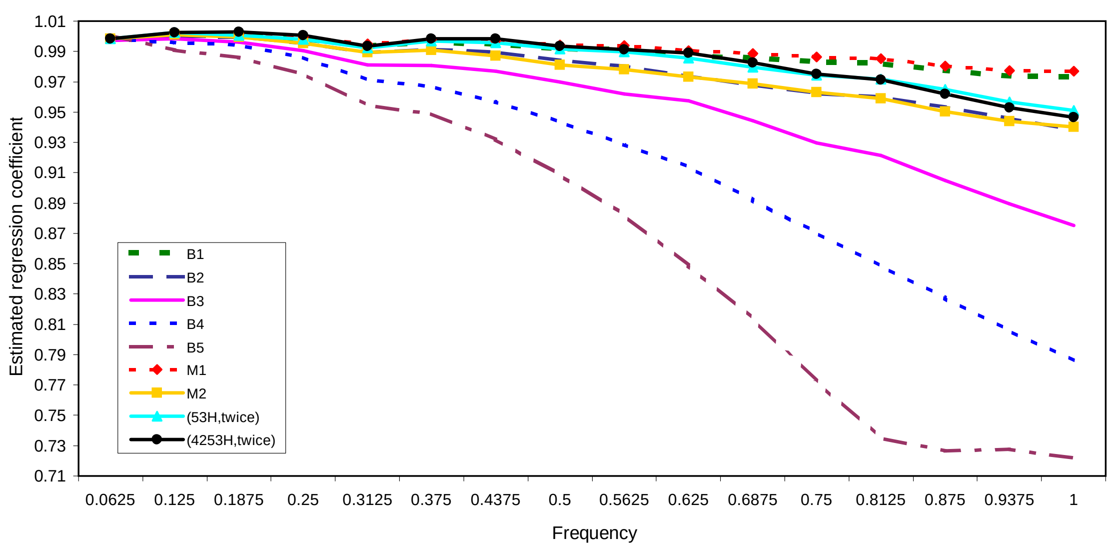

Figure 6.7: Estimated regression coefficients of nonlinear smoothers

Figure 6.7 displays the estimated regression coefficients of nonlinear smoothers applied to signal plus noise sequences at different frequencies. For small frequencies all the smoothers perform the same, but as the frequencies increase, two basic groups emerge. The smoothers B_3, B_4, and B_5 perform poorly relative to the other smoothers especially when the frequencies are high. The other group consisting of the smoothers B_1, B_2, M_1, M_2, (53H,twice) and (4253H,twice) perform very well with estimated regression coefficients ranging from 1,0029 (4253H,twice) for a frequency of $\frac{3}{16}$ (0,1875) to 0,9384 (B_2) for a frequency of 1. From the latter smoothers, B_1 and M_1 are the best in the sense that their regression coefficients remain the closest to 1 for all frequencies. It is thus important to note that the two less sophisticated smoothers from the LULU and median classes, i.e. B_1 (window size = 5) and M_1 (window size = 3), perform the best in recovering the signal as measured by the estimated regression coefficients.

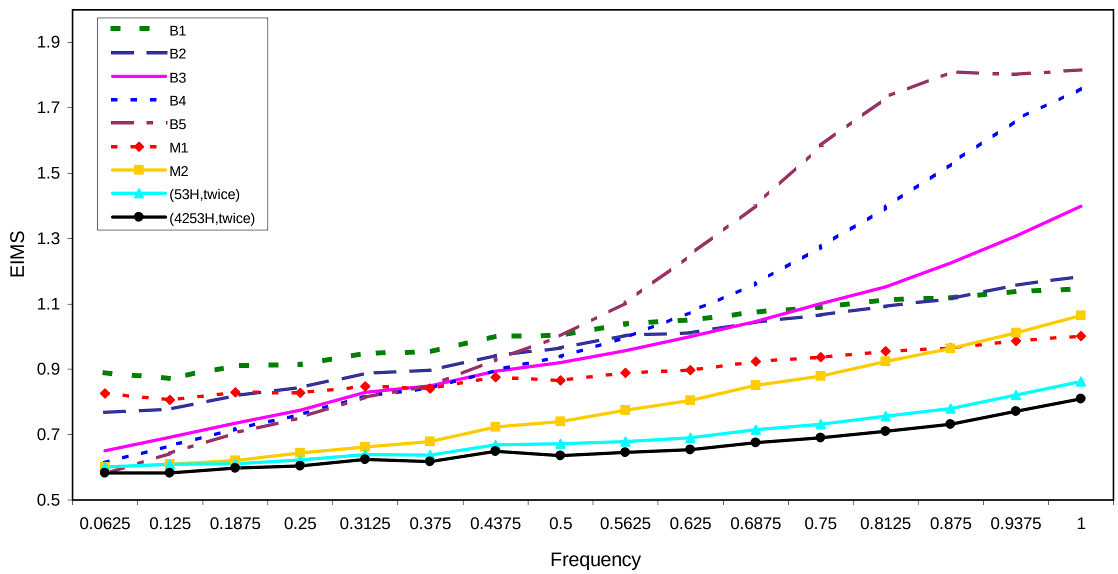

Figure 6.8: $\widehat{EIMS}$ of nonlinear smoothers

The estimates of the expected value of the integrated mean square error, $\widehat{EIMS}$ of the nonlinear smoothers applied to signal plus noise sequences at different frequencies are displayed in Figure 6.8. The smoothers B_1, B_2, M_1, M_2, (53H,twice) and (4253H,twice) form a broad group. The smoother (4253H,twice) has the smallest value of all the different frequencies, ranging from 0,58353 to 0,80929. The other compound median smoother (53H,twice) is the second best with $\widehat{EIMS}$ values very close to those of (4253H,twice). The $\widehat{EIMS}$ values of the median smoother M_2 increase a little faster from 0,60161 to 1,06386 as the frequency increases. The $\widehat{EIMS}$ values of M_1 stays relatively constant ranging from 0,82572 to 1,00152. The $\widehat{EIMS}$ values of the compound LULU smoother B_1 range from 0,88995 to 1,14768. The compound LULU smoothers B_3, B_4 and B_5 have small $\widehat{EIMS}$ values for small frequencies, but they increase rapidly as the frequency increases and thus perform worse for high frequencies. This is expected since, as a result of their large window sizes, they remove an increasing part of the signal. Regarding the six better smoothers, B_1, B_2, M_1, M_2, (53H,twice) and (4253H,twice), the $\widehat{EIMS}$ values range from 0,58196 (the smallest) for (4253H,twice) at a frequency of $\frac{2}{16}$ (0,125) to 1,1833 (the largest) for B_2 at a frequency of 1.

From the discussion above it is clear that when the estimated regression coefficients and the $\widehat{EIMS}$ values are used to measure how successful a smoother is in recovering signal, the six smoothers, B_1, B_2, M_1, M_2, (53H,twice) and (4253H,twice) are similar. If a small window size is added as criterion, B_1 and M_1 have an edge on the other nonlinear smoothers. This can also be explained by the fact that the length of the randomly spread spikes is one and they are successfully removed by M_1 and B_1. The other smoothers, such as (4253H,twice), have a window size of 25, which is much larger than the window sizes of the other smoothers. Further, if the total variation property of the LULU smoothers is taken into consideration, B_1 has this further advantage.

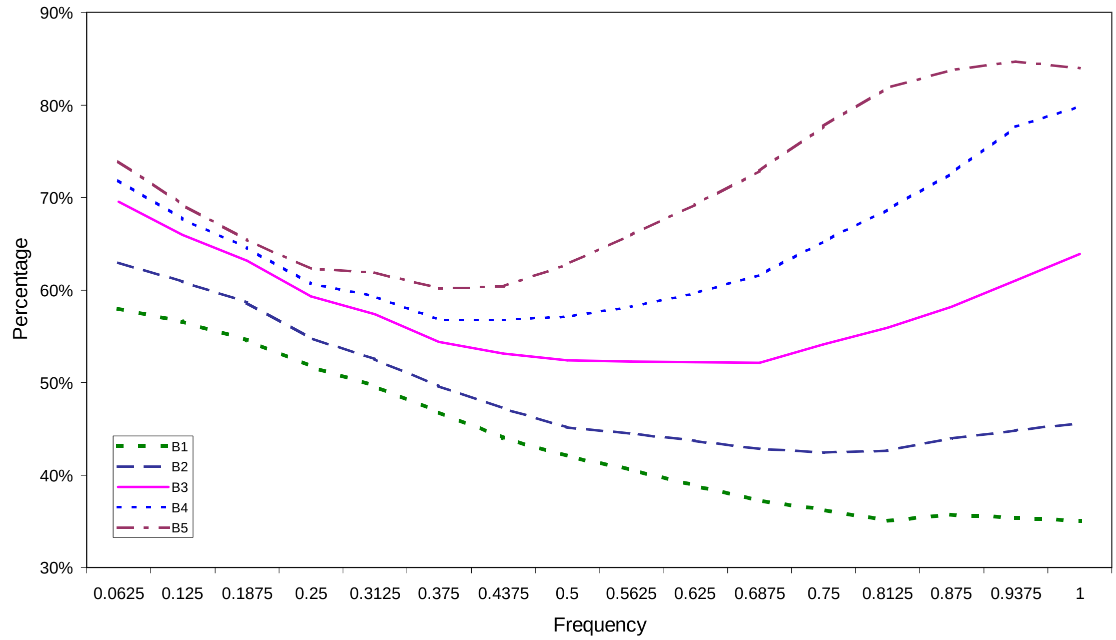

Figure 6.9: Percentage total variation removed by LULU smoothers

A feature of the compound LULU smoothers is that the percentage variation removed at each recursive step of applying the smoother can be calculated. These percentages are displayed in Figure 6.9 for the sine signal of different frequencies. For smoothers B_1 and B_2 the percentage variation removed starts in the order of 60% for sine signals with small frequencies and the percentages decrease as the frequency increases. The percentage of variation removed by the smoothers B_3, B_4 and B_5 decreases as the frequency increases to $\frac{7}{16}$ (0,4375) and thereafter increases sharply as the frequency increases. Again this emphasises the fact that in the case of the smoothers B_3, B_4 and B_5, the fast oscillation sine signal is not recognised as signal and they remove a substantial portion of this signal. In contrast, the greatest proportion of variation removed by the smoothers B_1 and B_2 is at the smaller frequencies from where the proportion decreases as the frequency increases, indicating that these smoothers do not remove the faster oscillation sine signal. Oversmoothing is thus clearly a potential problem in the case of B_3, B_4 and B_5 for this type of data.

6.3.2 Linear on nonlinear on sine signal

Figure 6.10 displays the estimated regression coefficients after the moving average, MA_2, of window size five, has been applied to the sequences smoothed by the respective nonlinear smoothers. For small frequencies the MA_2 smoother on all the nonlinear smoothed signals performs very similarly to the nonlinear smoothers, but as the frequency increases, the smoothers M_1 and B_1 perform best. The same nonlinear smoothers as in Figure 6.7 seem to group together after the linear smoother has been applied. When Figures 6.7 and 6.10 and the tables for each smoother in Appendix A are compared, it is clear that the estimated regression coefficients have slightly smaller values after the linear smoother has been applied.

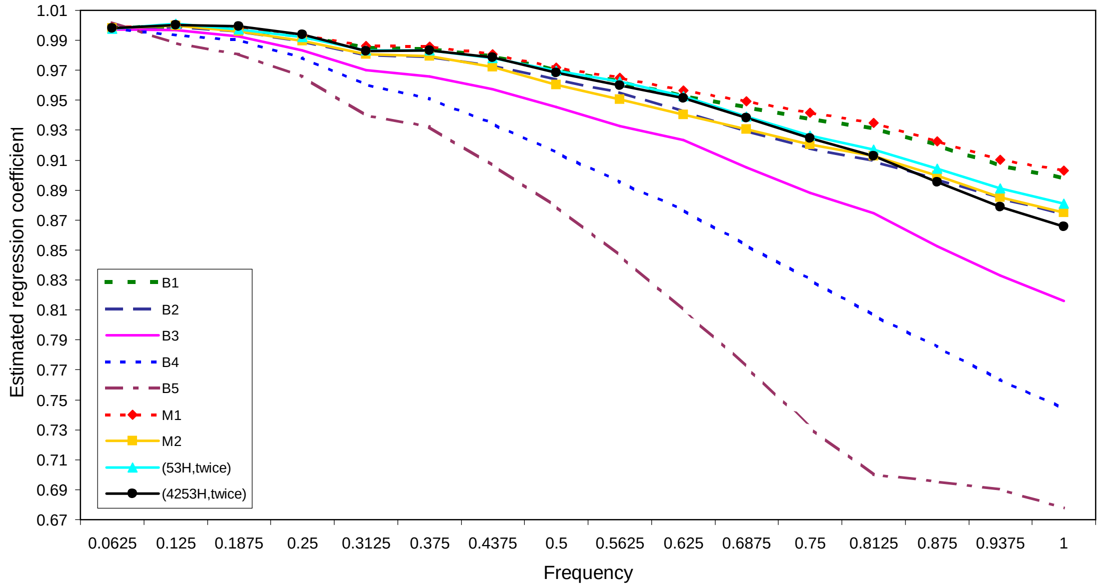

Figure 6.10: Estimated regression coefficients of MA_2 on nonlinear smoothers

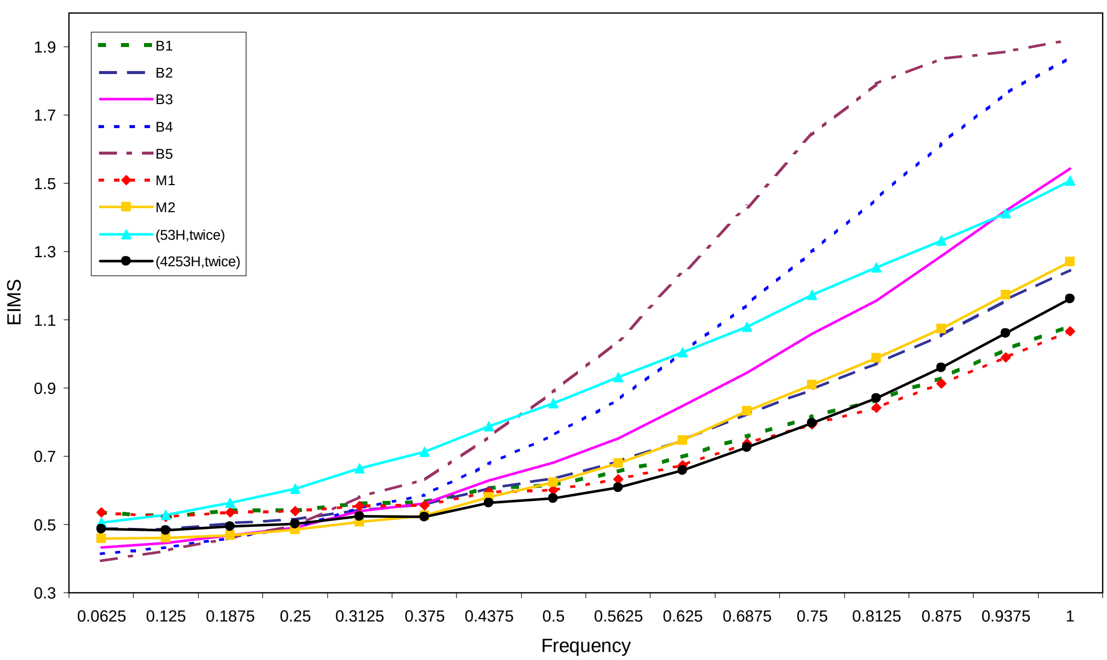

Figure 6.11: $\widehat{EIMS}$ of MA_2 on nonlinear smoothers

Figure 6.11 displays the $\widehat{EIMS}$ values of the linear smoother MA_2 applied to the different nonlinear smoothers. The $\widehat{EIMS}$ values of the compound LULU smoothers B_3, B_4 and B_5 are the smallest for small frequencies, but the values increase sharply for larger frequencies. All the smoothers are very close together at the small frequencies up to about $\frac{7}{16}$ (0,4375). For frequencies greater than $\frac{7}{16}$ (0,4375) the smoothers M_1, B_1 and (4253H,twice) have the smallest $\widehat{EIMS}$ values.

Taking moving averages of the nonlinear smoothers leads to smaller $\widehat{EIMS}$ values but also smaller estimated regression coefficients. This is an apparent contradiction since smaller $\widehat{EIMS}$ values indicate better signal recovery while smaller regression coefficients indicate poorer signal recovery. Therefore performance as measured by the two criteria has to be interpreted very carefully. The decrease in the $\widehat{EIMS}$ values when moving averages are applied to the nonlinear smoothers is, however, relatively more than the decrease in the regression coefficients suggesting that a better signal is indeed accomplished. From Table A.1 the smoother B_1 at frequency $\frac{8}{16}$ (0,5) is used as an example. The $\widehat{EIMS}$ value for nonlinear smoothing is 1,00518, while for linear on nonlinear smoothing, it is 0,61568, which is a decrease of 0,3895 or 38,7%. The estimated regression coefficient of B_1 for nonlinear smoothing is 0,99181, and for linear on nonlinear smoothing 0,97007, indicating a decrease of 0,02174 or 2,19% which is much less than the percentage decrease in the $\widehat{EIMS}$ values.

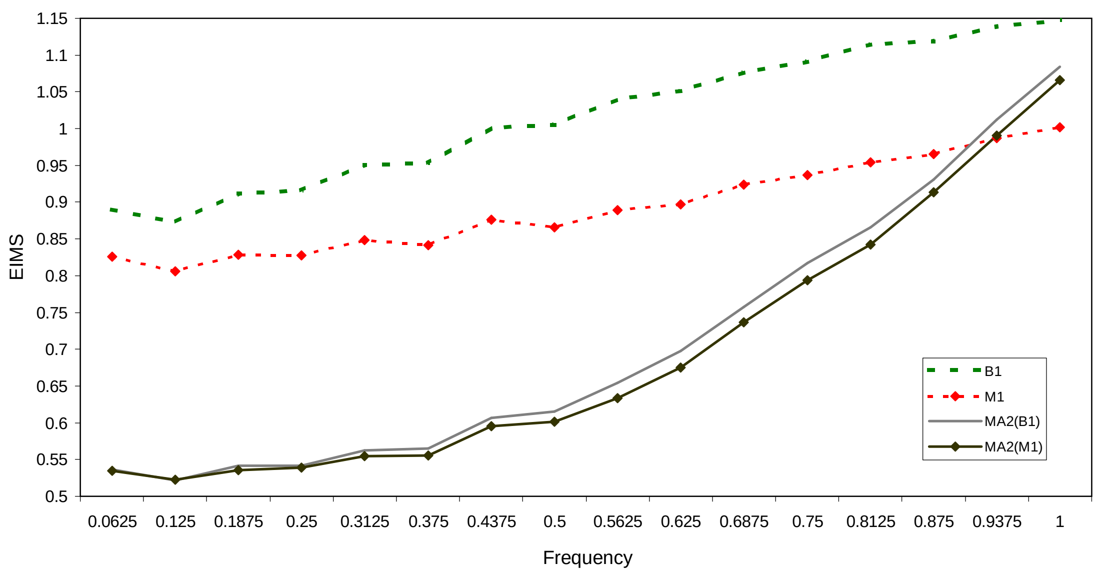

Figure 6.12: $\widehat{EIMS}$ of B_1, M_1, $MA_2(B_1)$ and $MA_2(M_1)$

The results of this section correspond with those of the previous section, i.e. if a small window size is added as criterion, B_1 and M_1 are the better smoothers in recovering signal, with B_1 possessing the further advantage of decomposing the total variation property. Figure 6.12 illustrates the $\widehat{EIMS}$ values for nonlinear smoothers B_1 and M_1, and the $\widehat{EIMS}$ values when the moving average, MA_2, was applied on these nonlinear smoothers. The $\widehat{EIMS}$ values of the linear on nonlinear smoothers are much less than the $\widehat{EIMS}$ values for the nonlinear smoothers for small frequencies. For frequencies greater than $\frac{7}{16}$ (0,4375) the $\widehat{EIMS}$ values of MA_2 on the nonlinear smoothers increase sharply but still remain less than those of the nonlinear smoothers for frequencies less than $\frac{15}{16}$.

To illustrate how the smoothers perform compared to the original signal some figures at different frequencies are included. Figure 6.13 illustrates the nonlinear smoothers B_1 and B_5 on a sine signal of frequency $\frac{5}{16}$ (0,3125), while Figure 6.14 gives $MA_2(B_1)$ and $MA_2(B_5)$ at the same frequency. The success of these two smoothers at this frequency is evident. Figure 6.15 displays the same nonlinear smoothers on a sine signal of frequency $\frac{13}{16}$ (0,8125) and Figure 6.16 $MA_2(B_1)$ and $MA_2(B_5)$ on the same frequency. It can be seen that B_5 smoothes the curves more than B_1. This is due to the large window size of B_5 where the high oscillation is treated as noise and a part of the signal is removed and hence oversmoothing takes place.

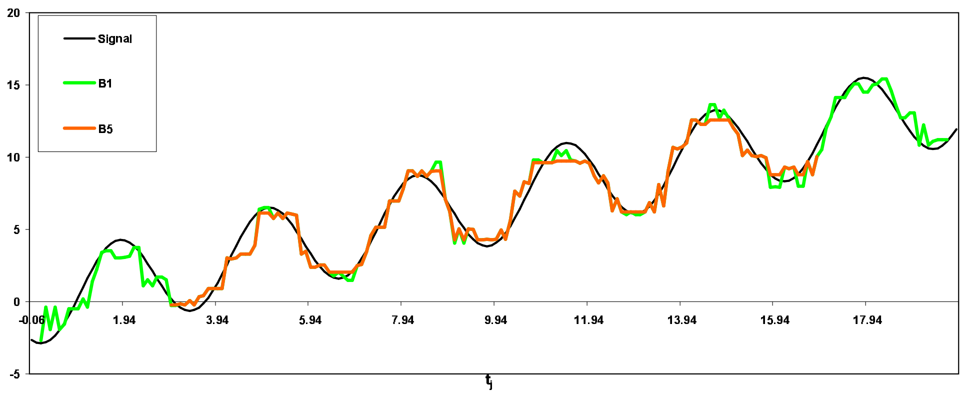

Figure 6.13: Nonlinear smoothers B_1 and B_5 on sine signal of frequency $\frac{5}{16}$ (0,3125)

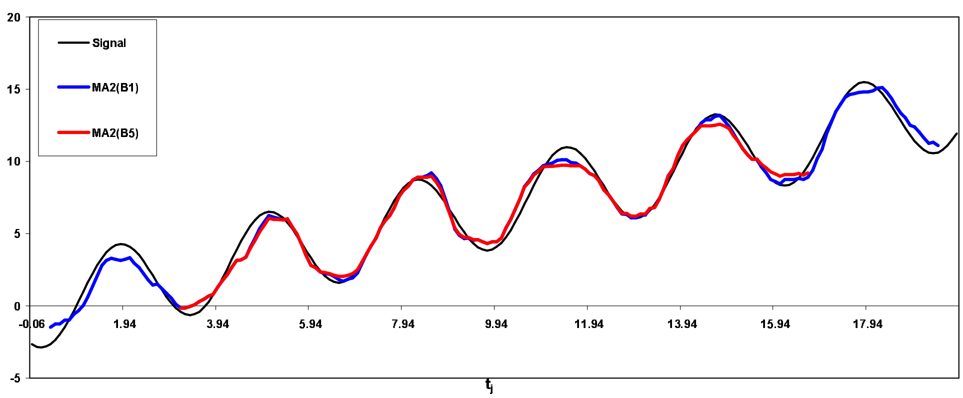

Figure 6.14: $MA_2(B_1)$ and $MA_2(B_5)$ on sine signal of frequency $\frac{5}{16}$ (0,3125)

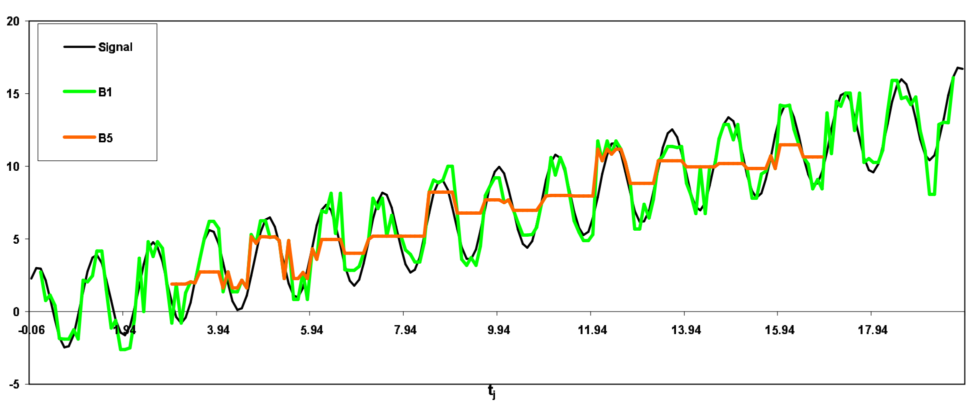

Figure 6.15: Nonlinear smoothers B_1 and B_5 on sine signal of frequency $\frac{13}{16}$ (0,8125)

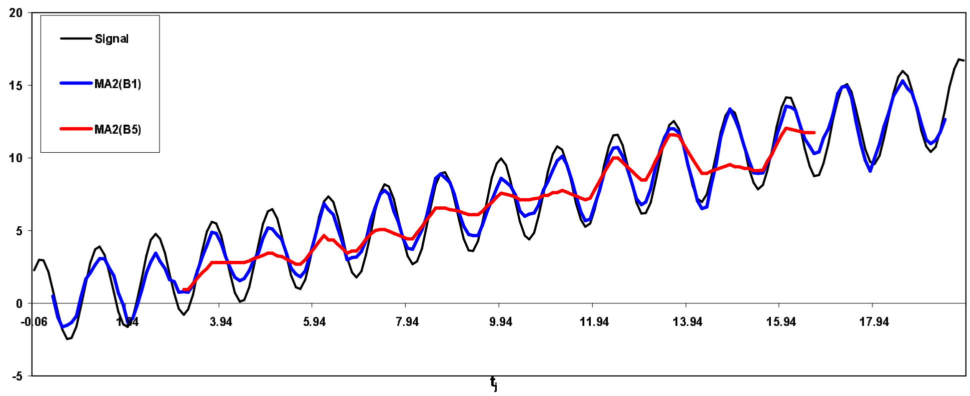

Figure 6.16: $MA_2(B_1)$ and $MA_2(B_5)$ on sine signal of frequency $\frac{13}{16}$ (0,8125)

6.3.3 Nonlinear on sine signal with impulse

One of the attractive properties of LULU smoothers is their way of treating blockpulses. To investigate how the different smoothers deal with these, an impulse was constructed on the original sine signal by adding three values of size ten to the signal. These three values were kept constant at the same index independent of the change in frequency. The effect of the position of the impulse on the slope of the sine signal will be discussed in Section 6.3.5.

Figure 6.17 illustrates a blockpulse of 3 values of 10 each added at t_j = 9,5416, 9,6416 and 9,7416 on a sine signal of frequency $\frac{4}{16}$ (0,25). Notice the blockpulse of length 3 on the upward slope at the positions where 10 was added. Figure 6.18 illustrates a blockpulse of 3 values of 10 added at the same positions on a sine signal of frequency $\frac{5}{16}$ (0,3125). The position of the blockpulse shifts from being on an upward slope to being on a downward slope when the frequency changes by $\frac{1}{16}$.

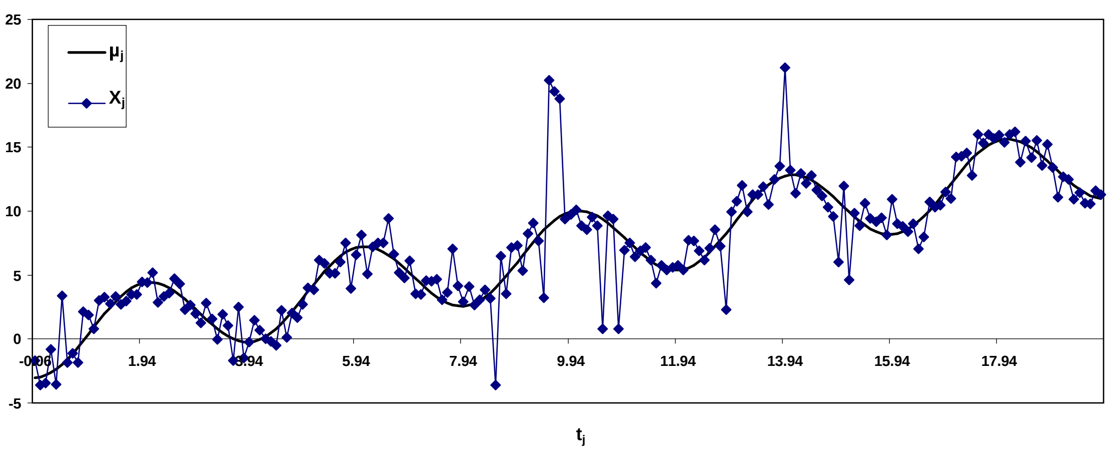

Figure 6.17: Blockpulse on sine signal of frequency $\frac{4}{16}$ (0,25)

Figure 6.19 displays the estimated regression coefficients of the nonlinear smoothers applied to the impulse signals for different frequencies. All the nonlinear smoothers seem to follow a zigzag pattern

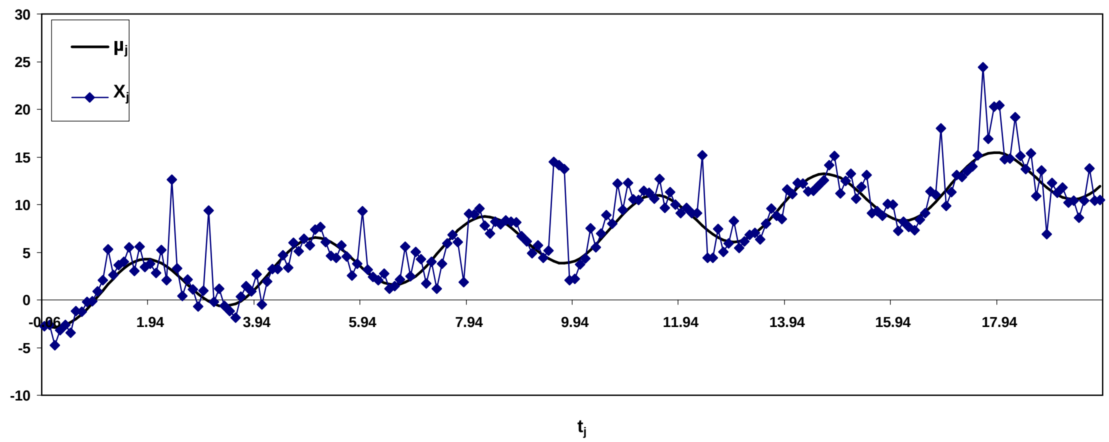

Figure 6.18: Blockpulse on sine signal of frequency $\frac{5}{16}$ (0,3125)

which becomes weaker as the frequency increases. This zigzag pattern resulted from the fact that the blockpulse of 3 values moves around on the sequence as the frequency increases. Sometimes the blockpulse is on the upward slope and sometimes on the downward slope.

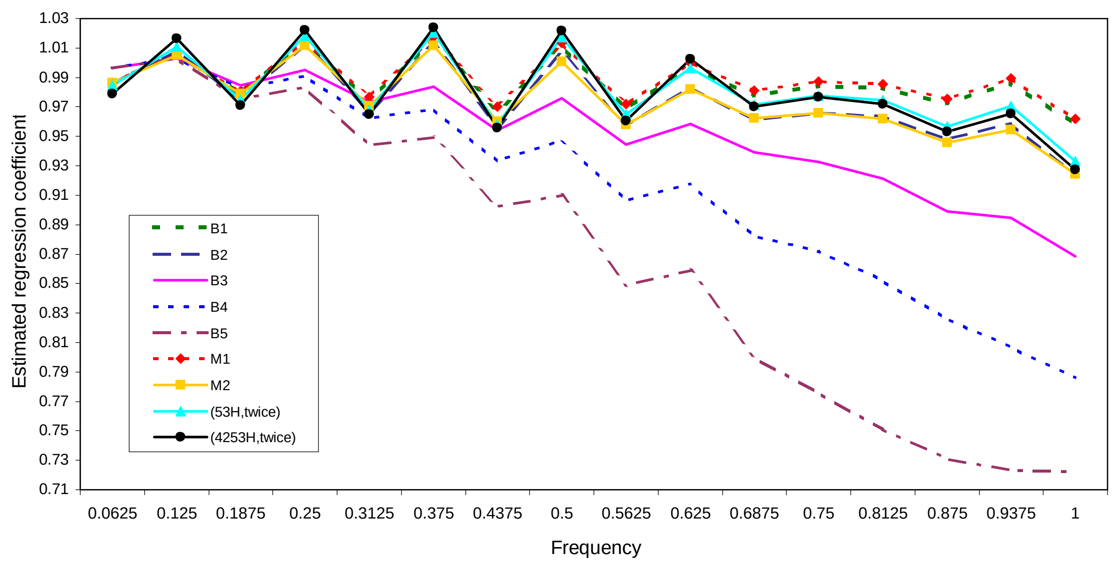

Figure 6.19: Estimated regression coefficient of nonlinear smoothers on signal with impulse

For slower oscillations at small frequencies the estimated regression coefficients show great variation. The zigzag pattern fades as the oscillations become faster at larger frequencies indicating that the position of the blockpulse on the slope of the sequence has a lesser effect for larger frequencies, i.e. faster oscillations. It is clear that the same pattern regarding the success of the smoothers emerges as for the sequences without the blockpulse. It seems as if the smoothers M_1 and B_1 perform best, but from the theory it is known that M_1 and B_1 do not remove the blockpulse of length 3. The 3 values where the smoothed sequence and the signal differ most form 3 outliers on the regression scatter plot and these 3 outliers are found to be not enough or large enough to be influential in the estimation of the regression coefficients. As an example Figure 6.20 gives the regression scatter plots with the fitted regression lines of

B_1 on the signal for frequency with and without the outliers for an arbitrary chosen simulation run. The two regression coefficients are for all practical purposes the same, the difference being only 0,0006 in this case indicating that the three outliers are not influential in the estimation of the regression coefficients. These three relatively large differences have however a large effect on the $\widehat{EIMS}$ value. Therefore, the regression coefficient is less discriminant than the $\widehat{EIMS}$ in evaluating the smoothers in this situation. Hence, from now on, the $\widehat{EIMS}$ will be used as the primary criterion for evaluating the recovery of signal when blockpulses are present. In this case the $\widehat{EIMS}$ is 2,3938 for the case with outliers and 0,9527 for the case without the outliers. The first $\widehat{EIMS}$ is 2,5126 times the second indicating the sensitivity of the $\widehat{EIMS}$ for outliers.

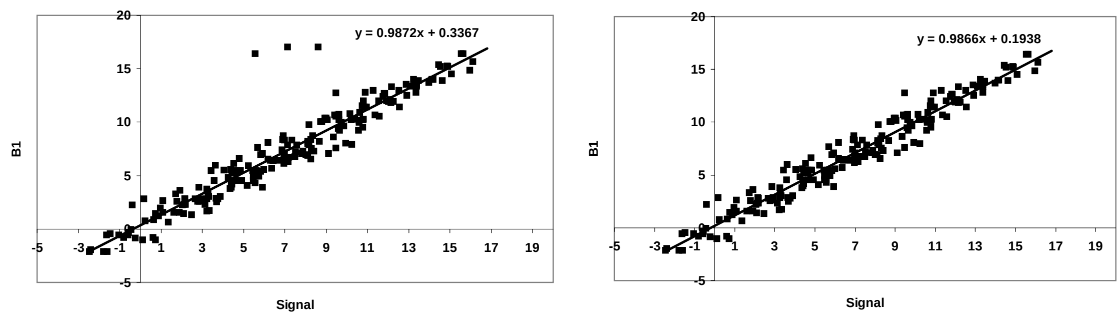

Figure 6.20: Regression of B_1 on signal with impulse and without impulse

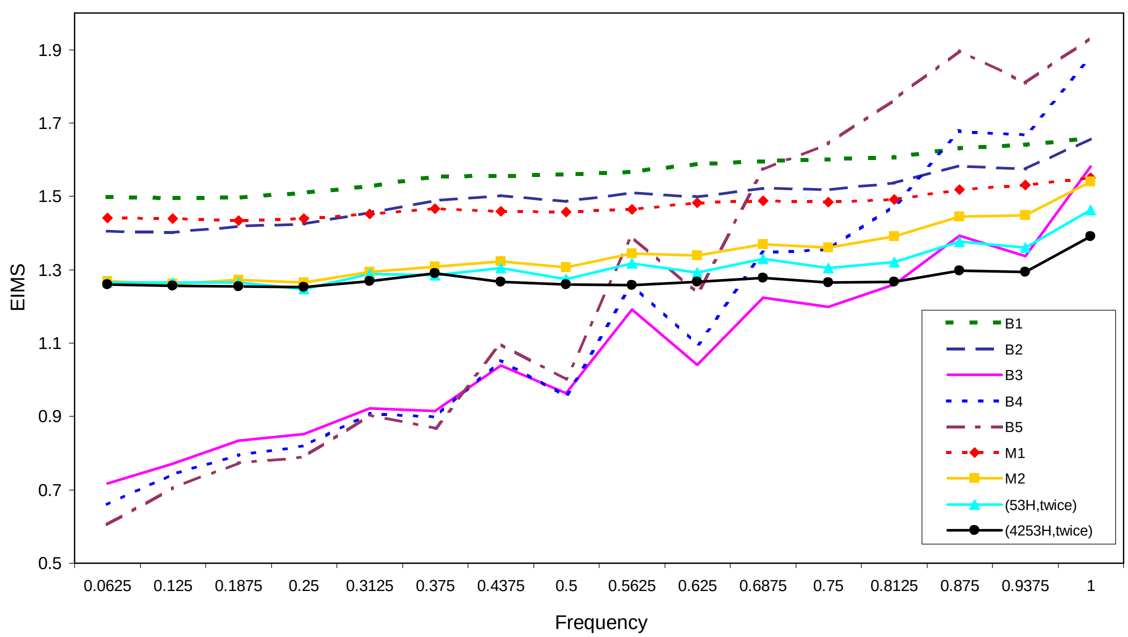

Figure 6.21: $\widehat{EIMS}$ of nonlinear smoothers on signal with impulse

The $\widehat{EIMS}$ values of the nonlinear smoothers applied to the signal with an impulse added are displayed in Figure 6.21. It can be concluded that for

- frequencies $\leq \frac{8}{16}$ (0,5), the smoothers B_3, B_4 and B_5 are very similar and perform best
- $\frac{9}{16}$ (0,5625) $\leq$ frequencies $\leq \frac{13}{16}$ (0,8125), B_3 performs best
- frequencies $\geq \frac{14}{16}$ (0,875), (4253H,twice) performs best

The success of B_3, B_4 and B_5 corresponds with the theory that they should remove a blockpulse of length three at lower frequencies. The poor performance of B_4 and B_5 at high frequencies is again evidence of their removal of signal at high frequencies.

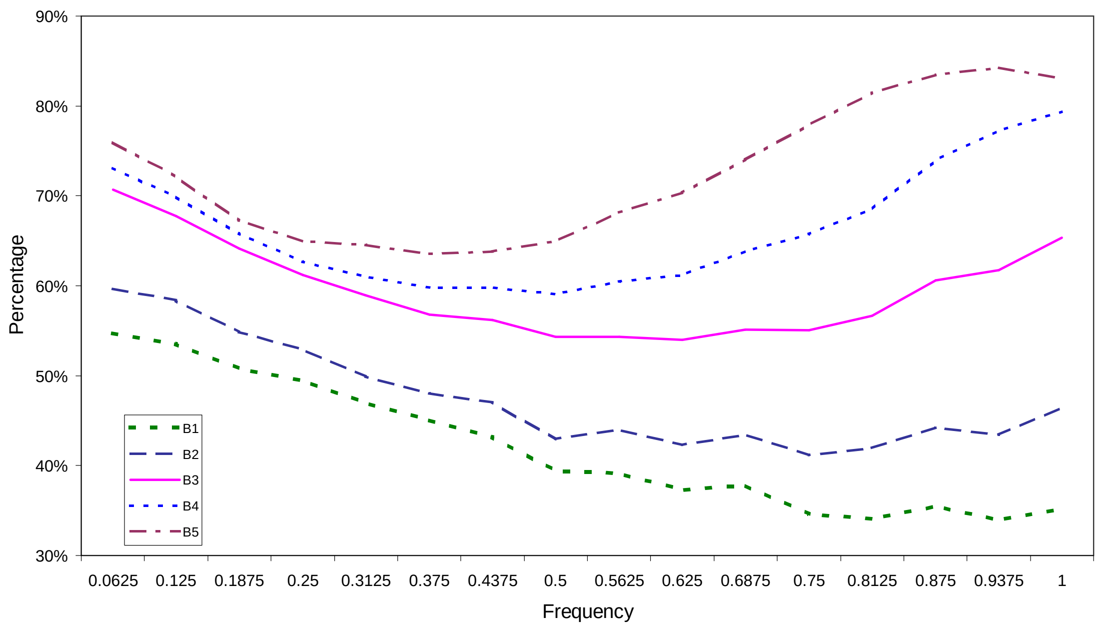

Figure 6.22: Percentage total variation removed by LULU smoothers on signal with impulse

Figure 6.22 displays the percentage variation removed if an impulse of 3 values of size 10 has been added to the sine signal for different frequencies. The percentage variation removed by the compound LULU smoothers follows the same pattern as in the case of the signal without an impulse added, illustrated in Figure 6.9. The percentage variation of the signal removed with the impulse added are slightly smaller, less than 60%, for B_1 and B_2, than in the case with no impulse added. For these smoothers a small zigzag pattern can be observed for frequencies greater than $\frac{6}{16}$ (0,375). The zigzag pattern is less obvious for B_3, B_4 and B_5 and their percentage variation removed is of the same order as in the case if no impulse was added. This phenomenon can be explained by the fact that the latter smoothers remove a blockpulse of length 3.

6.3.4 Linear on nonlinear on sine signal with impulse

In the case where the linear smoother, the moving average MA_2, has been applied to the already smoothed sequences by nonlinear smoothers, the zigzag pattern continues with the same features, as illustrated in Figure 6.23. If patterns are compared to those depicted in Figure 6.19 it can be seen that they are the

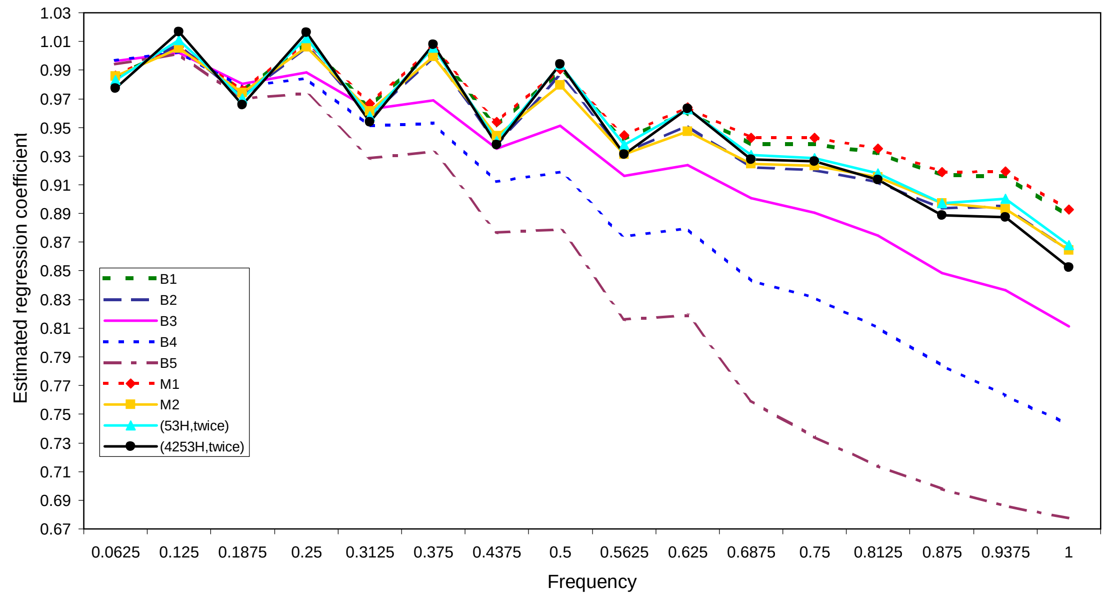

Figure 6.23: Estimated regression coefficient of MA_2 on nonlinear smoothers on signal with impulse

same, although the values of the estimated regression coefficients are slightly smaller. These smaller values may be due to the resmoothing, or smoothing of already smoothed sequences.

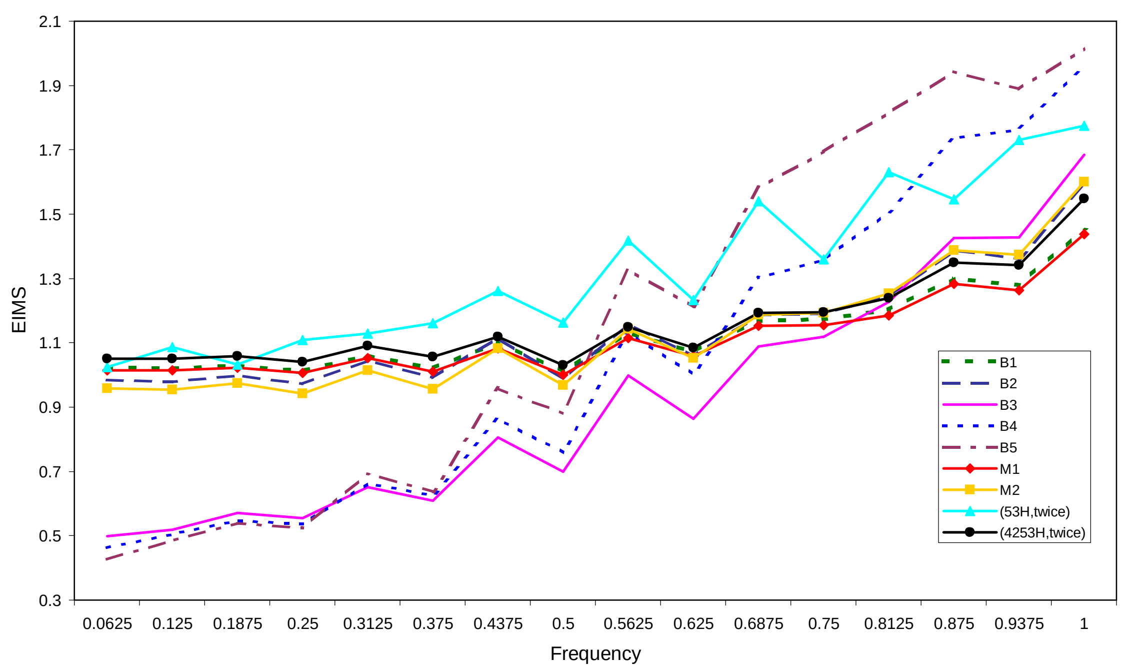

Figure 6.24: $\widehat{EIMS}$ of MA_2 on nonlinear smoothers on signal with impulse

Figure 6.24 displays the $\widehat{EIMS}$ values of the linear smoother, the moving average MA_2, applied to the smoothed impulse signal. Along the same lines as in Section 6.3.3 it can be concluded from the $\widehat{EIMS}$

values that for

- frequencies $\leq \frac{4}{16}$ (0,25), the smoothers B_3, B_4 and B_5 are very similar
- $\frac{5}{16}$ (0,3125) $\leq$ frequencies $\leq \frac{12}{16}$ (0,75), the smoother B_3 performs best
- frequencies $\geq \frac{13}{16}$ (0,8125), the smoothers M_1 and B_1 perform best

The estimated regression coefficients of all the nonlinear smoothers follow a zigzag pattern which becomes weaker as the frequency increases. This zigzag pattern results from the fact that the blockpulse of 3 values moves around on the sequence as the frequency increases. Sometimes the blockpulse is on the upward slope and sometimes on the downward slope. The estimated regression coefficient for each smoother is thus not only influenced by the frequency and the blockpulse, but also by the position of the blockpulse on the sequence. Hence, in the next section results will be given where the blockpulse is kept in the same position on an up or down slope for different frequencies.

6.3.5 Impulse and slope

The zigzag pattern of the mean regression coefficients and $\widehat{EIMS}$ values of the smoothers in Sections 6.3.3 and 6.3.4 is a result of the position and direction of the impulse on the sine signal with added noise at different frequencies. In this section the influence of the position and direction of the impulse on the sine signal is investigated.

An impulse of 3 values of size 10 was added to the noisy sine signal at different positions for different frequencies to keep the impulse on a slope of the same direction. This resulted in four cases, 1. slope negative, impulse positive, 2. slope positive, impulse positive, 3. slope negative, impulse negative, and 4. slope positive, impulse positive. Each of these cases was investigated at each of the 16 frequencies previously used. For each frequency the signal was simulated 200 times. The nonlinear smoothers used in the previous sections were applied, and the estimated regression coefficients and $\widehat{EIMS}$ values were calculated for each smoother at each frequency. Thereafter, the linear smoother MA_2 was applied to the smoothed series and the estimated regression coefficients and $\widehat{EIMS}$ values were calculated. All these values are tabulated in Appendix B for the four different cases.

1. Slope negative, impulse positive

For this case the figures are as follows:

- Figure 6.25 illustrates the estimated regression coefficients of the nonlinear smoothers.
- Figure 6.26 illustrates the $\widehat{EIMS}$ values of the nonlinear smoothers.
- Figure 6.27 illustrates the estimated regression coefficients of MA_2 on the nonlinear smoothers.
- Figure 6.28 illustrates the $\widehat{EIMS}$ values of MA_2 on the nonlinear smoothers.

From Figure 6.26 conclusions from the $\widehat{EIMS}$ values for a negative slope and positive impulse are the same as those made for the $\widehat{EIMS}$ values in Section 6.3.3 and are as follows:

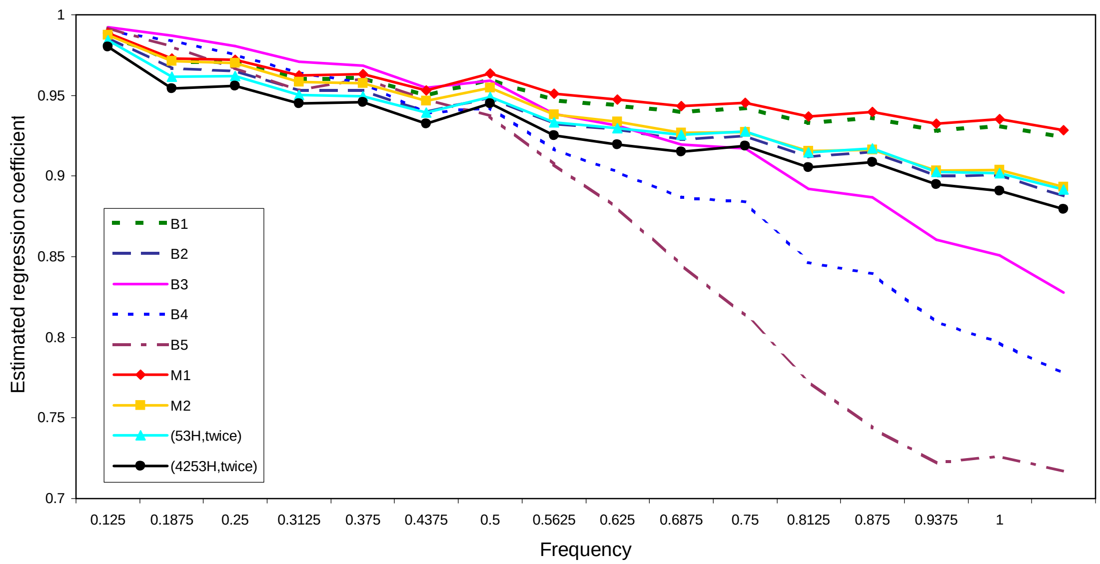

Figure 6.25: Estimated regression coefficients of nonlinear smoothers for negative slope and positive impulse

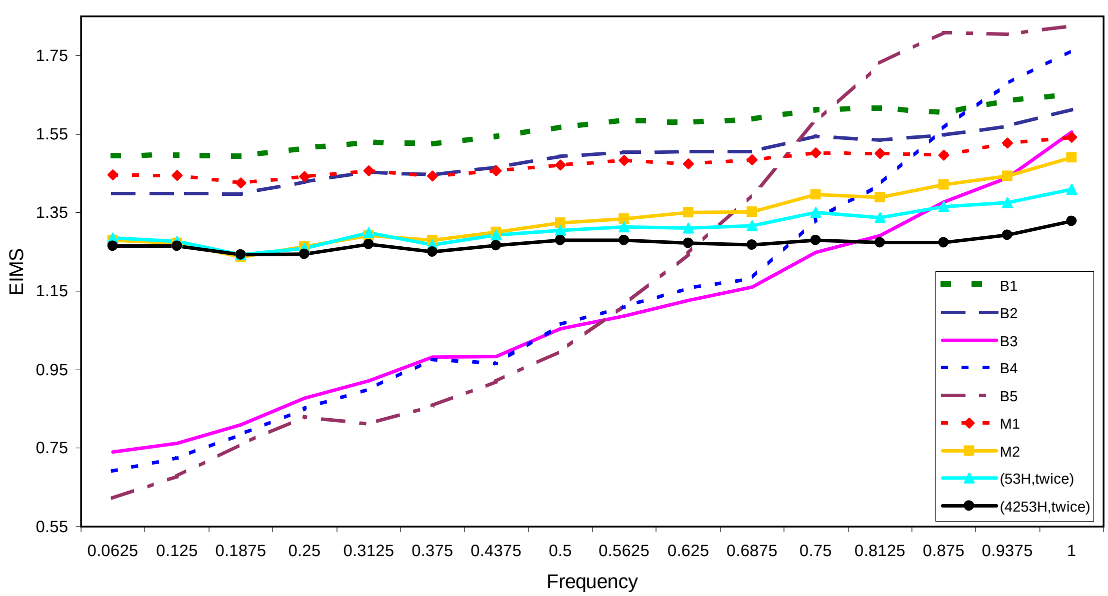

Figure 6.26: $\widehat{EIMS}$ of nonlinear smoothers for negative slope and positive impulse

- frequencies $\leq \frac{8}{16}$ (0,5), the smoothers B_3, B_4 and B_5 are very similar and B_5 performs best
- $\frac{9}{16}$ (0,5625) $\leq$ frequencies $\leq \frac{12}{16}$ (0,75), B_3 performs best
- frequencies $\geq \frac{13}{16}$ (0,8125), (4253H,twice) performs best

From Figure 6.25 it is clear that the estimated regression coefficients for the smoothers above are all close to one in the specific frequency intervals and thus support the conclusions based on the $\widehat{EIMS}$ values.

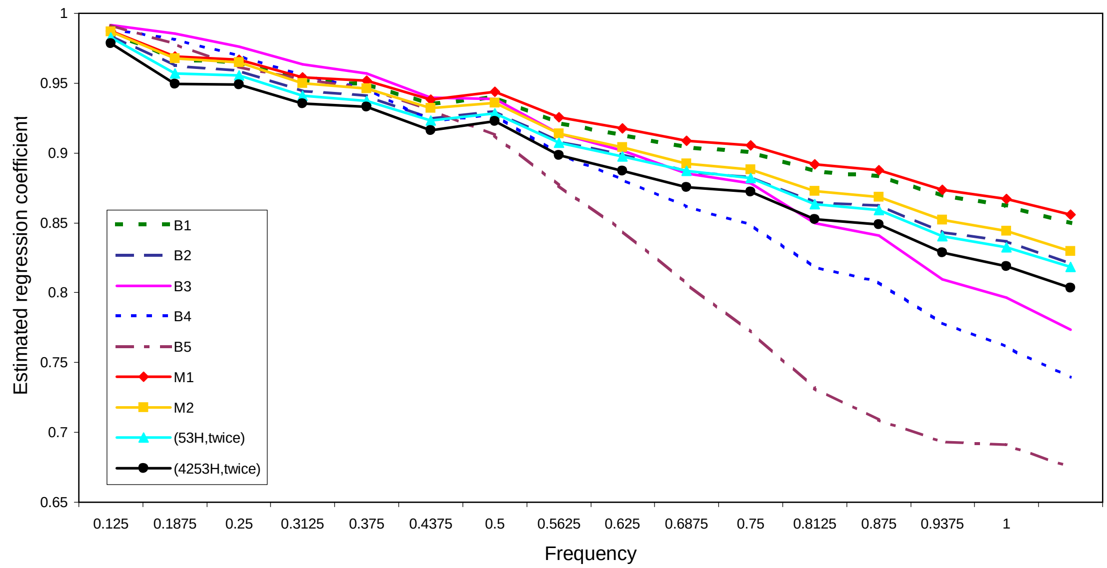

Figure 6.27: Estimated regression coefficients of MA_2 on nonlinear smoothers for negative slope and positive impulse

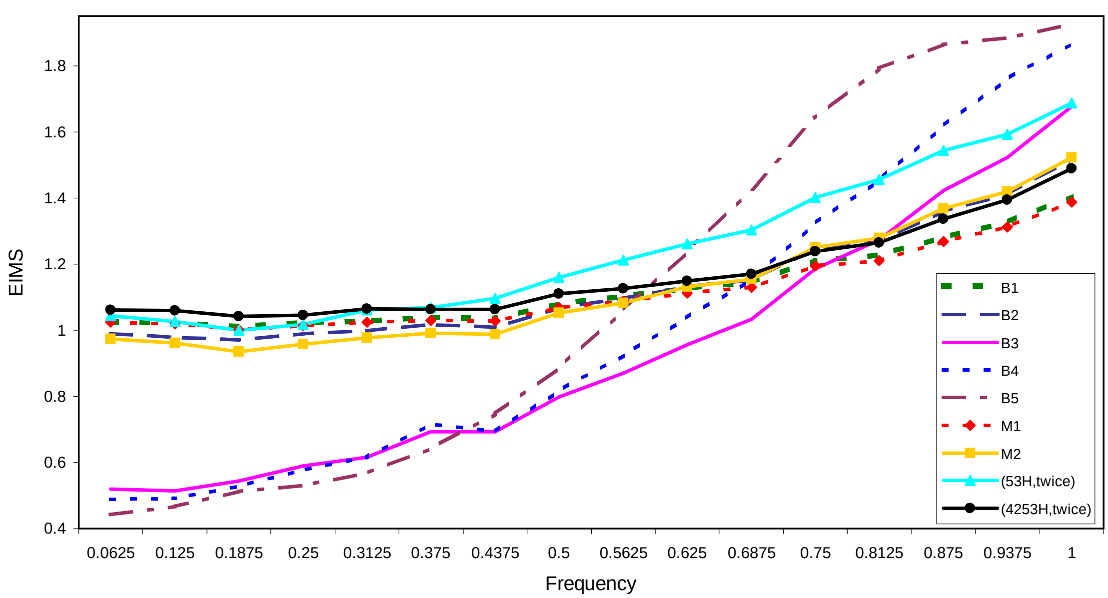

Figure 6.28: $\widehat{EIMS}$ of linear MA_2 on nonlinear smoothers for negative slope and positive impulse

The $\widehat{EIMS}$ values for the linear on nonlinear smoothers for negative slope and positive impulse as displayed in Figure 6.28 form three groups with the conclusions as follows:

- frequencies $\leq \frac{7}{16}$ (0,4375), the smoothers B_3, B_4 and B_5 are very similar and B_5 performs best
- $\frac{8}{16}$ (0,5) $\leq$ frequencies $\leq \frac{12}{16}$ (0,75), B_3 performs best
- frequencies $\geq \frac{13}{16}$ (0,8125), B_1 and M_1 perform best

From Figure 6.27 it is clear that the estimated regression coefficients for the smoothers above are all close to one in the specific frequency intervals and thus support the conclusions based on the $\widehat{EIMS}$ values. It is important to note that the LULU smoothers B_3, B_4 and B_5 perform substantially better than the other smoothers for frequencies less or equal to $\frac{12}{16}$ (0,75). As expected according to the theory, these smoothers should remove the blockpulse more effectively than the other smoothers. It is also informative to see how the application of the linear smoother to the nonlinear smoothers causes the $\widehat{EIMS}$ values to decrease, indicating better signal recovery (cf. Figures 6.26 and 6.28).

2. Slope positive, impulse positive

For this case the figures are as follows:

- Figure 6.29 illustrates the estimated regression coefficients of the nonlinear smoothers.
- Figure 6.30 illustrates the $\widehat{EIMS}$ values of the nonlinear smoothers.
- Figure 6.31 illustrates the estimated regression coefficients of MA_2 on the nonlinear smoothers.
- Figure 6.32 illustrates the $\widehat{EIMS}$ values of MA_2 on the nonlinear smoothers.

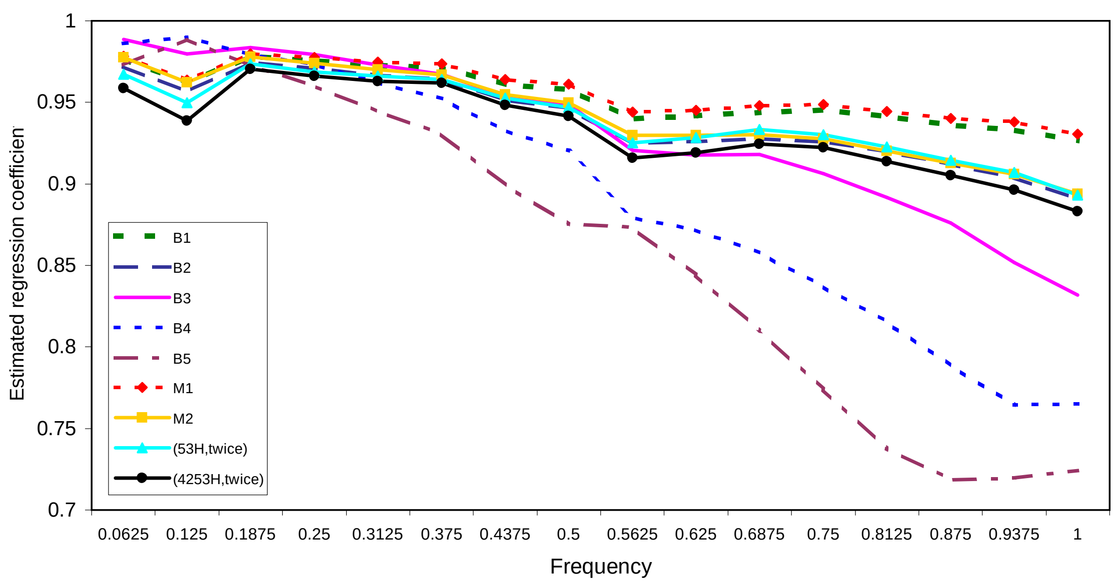

Figure 6.29: Estimated regression coefficients of nonlinear smoothers for positive slope and positive impulse

From Figure 6.30 conclusions from the $\widehat{EIMS}$ values for a positive slope and positive impulse are

- frequencies $\leq \frac{10}{16}$ (0,625), the smoothers B_3, B_4 and B_5 are very similar and perform best
- $\frac{11}{16}$ (0,6875) $\leq$ frequencies $\leq \frac{13}{16}$ (0,8125), B_3 performs best
- frequencies $\geq \frac{14}{16}$ (0,875), (4253H,twice) performs best

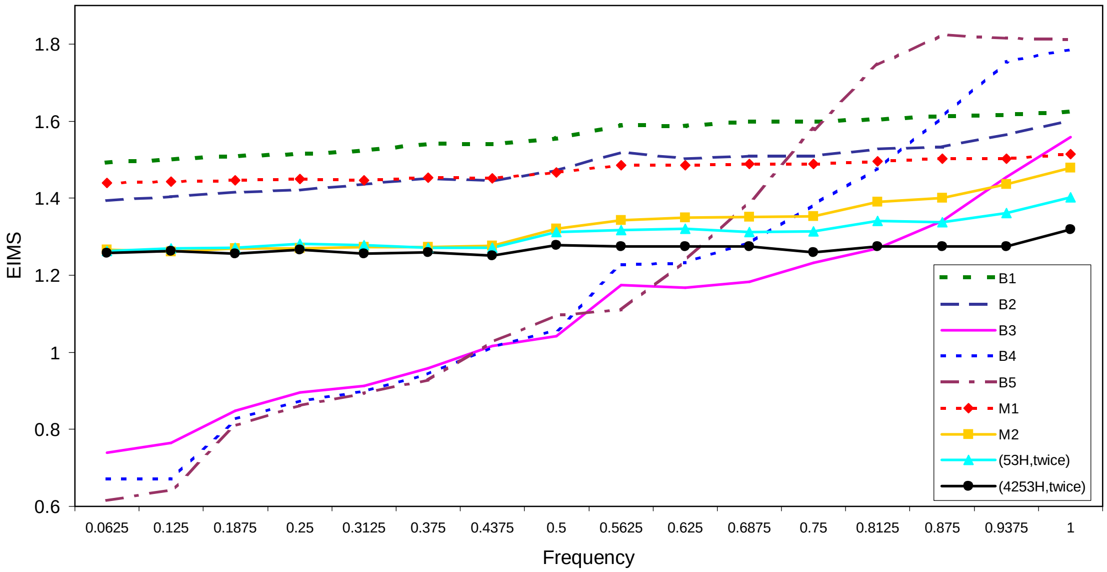

Figure 6.30: $\widehat{EIMS}$ of nonlinear smoothers for positive slope and positive impulse

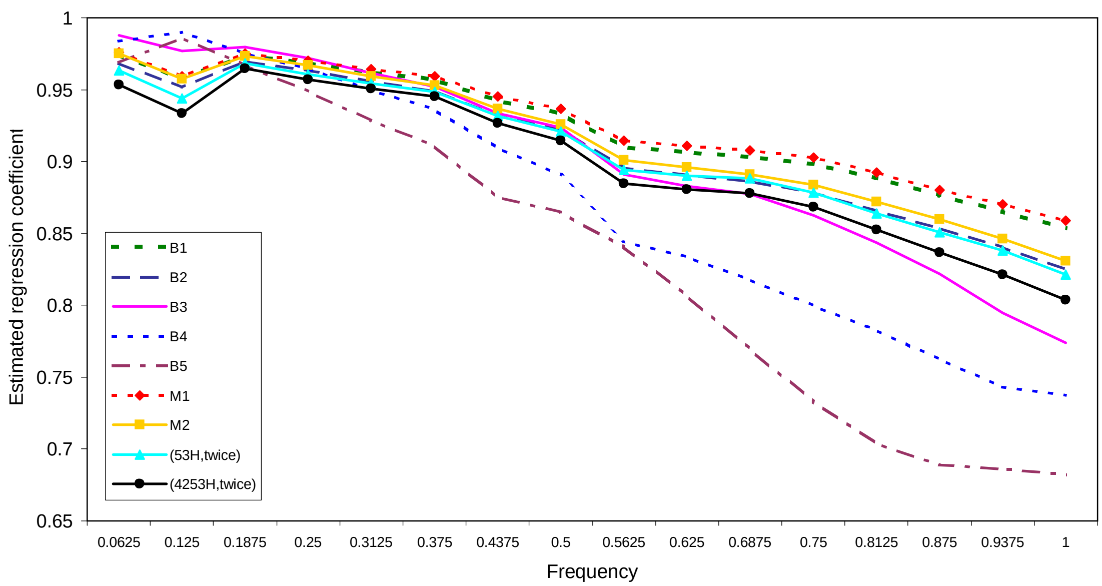

Figure 6.31: Estimated regression coefficients of MA_2 on nonlinear smoothers for positive slope and positive impulse

From Figure 6.29 it is clear that the estimated regression coefficients for the smoothers above are all close to one in the specific frequency intervals and thus support the conclusions based on the $\widehat{EIMS}$ values.

The $\widehat{EIMS}$ values for the linear on nonlinear smoothers for positive slope and positive impulse as displayed in Figure 6.32 are

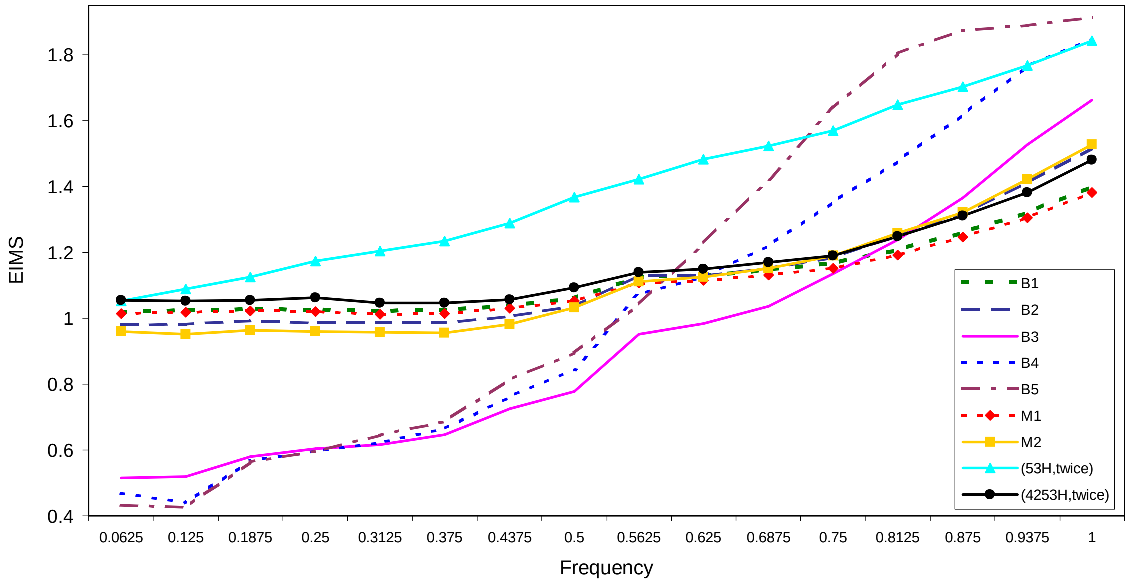

Figure 6.32: $\widehat{EIMS}$ of MA_2 on nonlinear smoothers for positive slope and positive impulse

- frequencies $\leq \frac{4}{16}$ (0,25), the smoothers B_3, B_4 and B_5 are very similar and perform best
- $\frac{5}{16}$ (0,3125) $\leq$ frequencies $\leq \frac{12}{16}$ (0,75), B_3 performs best
- frequencies $\geq \frac{13}{16}$ (0,8125), B_1 and M_1 perform best

From Figure 6.31 it is clear that the estimated regression coefficients for the smoothers above are all close to one in the specific frequency intervals and thus support the conclusions based on the $\widehat{EIMS}$ values. Here also the LULU smoothers B_3, B_4 and B_5 perform better than the other smoothers for frequencies less or equal to $\frac{12}{16}$ (0,75), and for frequencies greater than $\frac{12}{16}$ (0,75), B_1 and M_1 perform better.

3. Slope negative, impulse negative

For this case the figures are as follows:

- Figure 6.33 illustrates the estimated regression coefficients of the nonlinear smoothers.
- Figure 6.34 illustrates the $\widehat{EIMS}$ values of the nonlinear smoothers.
- Figure 6.35 illustrates the estimated regression coefficients of MA_2 on the nonlinear smoothers.
- Figure 6.36 illustrates the $\widehat{EIMS}$ values of MA_2 on the nonlinear smoothers.

From Figure 6.34 the conclusions from the $\widehat{EIMS}$ values for a negative slope and negative impulse can be divided into four categories according to frequencies

- frequencies $\leq \frac{8}{16}$ (0,5), B_5 performs best
- $\frac{9}{16}$ (0,5625) $\leq$ frequencies $\leq \frac{10}{16}$ (0,625), B_4 performs best

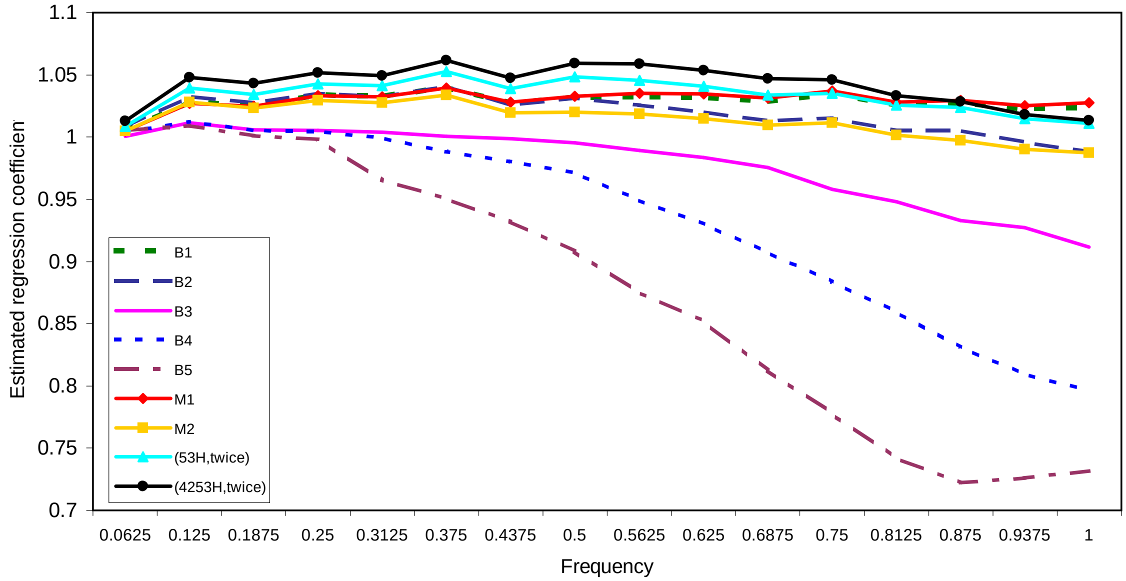

Figure 6.33: Estimated regression coefficients of nonlinear smoothers for negative slope and negative impulse

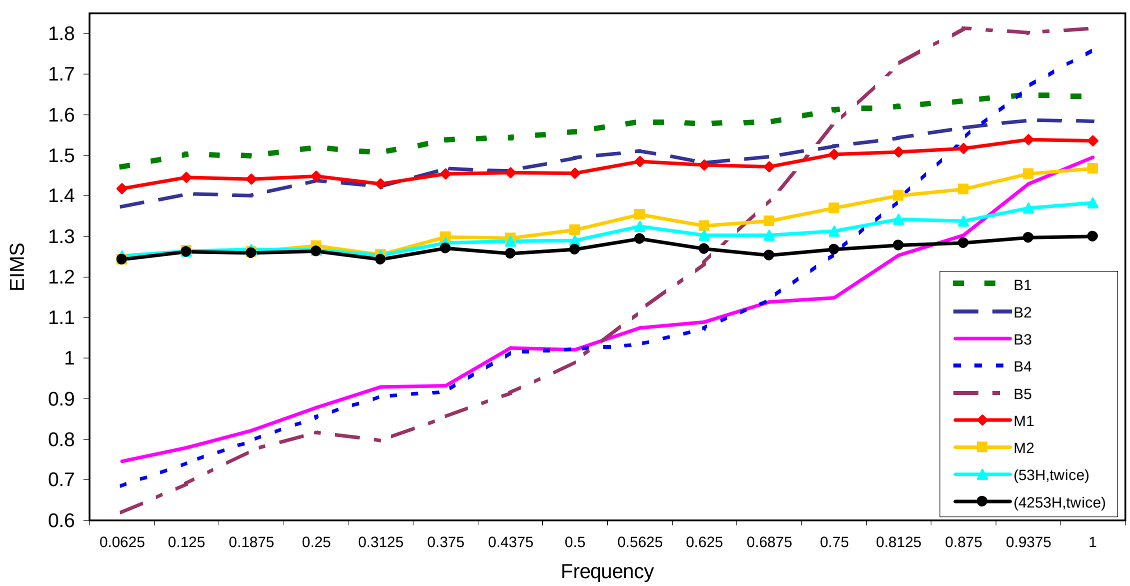

Figure 6.34: $\widehat{EIMS}$ of nonlinear smoothers for negative slope and negative impulse

- $\frac{11}{16}$ (0,6875) $\leq$ frequencies $\leq$ $\frac{13}{16}$ (0,8125), B_3 performs best
- frequencies $\geq$ $\frac{14}{16}$ (0,875), (4253H,twice) performs best

From Figure 6.33 it is clear that the estimated regression coefficients for the smoothers above are all close to one in the specific frequency intervals and thus support the conclusions based on the $\widehat{EIMS}$ values.

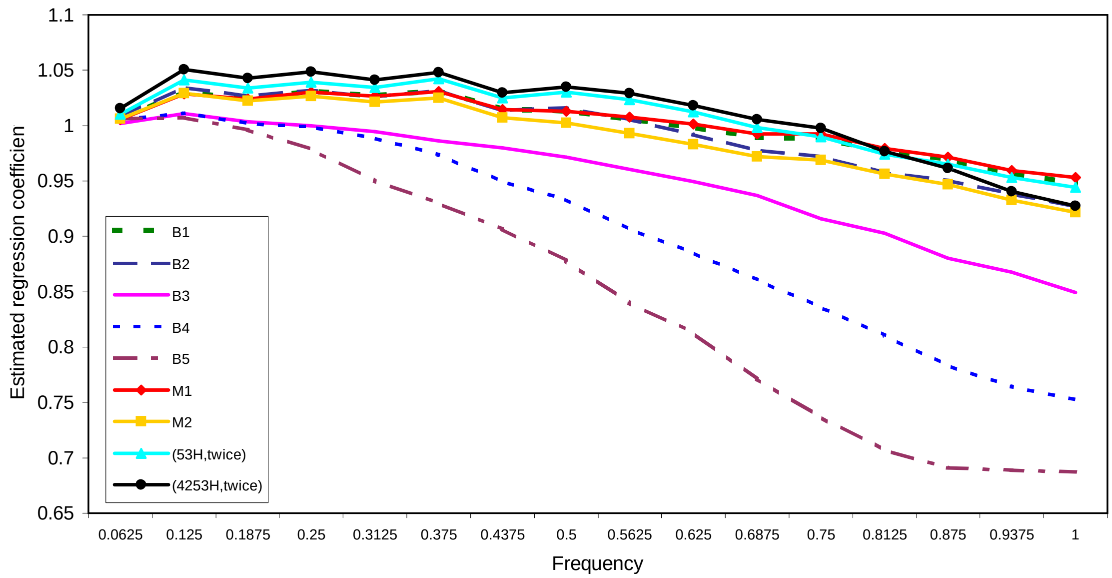

Figure 6.35: Estimated regression coefficients of MA_2 on nonlinear smoothers for negative slope and negative impulse

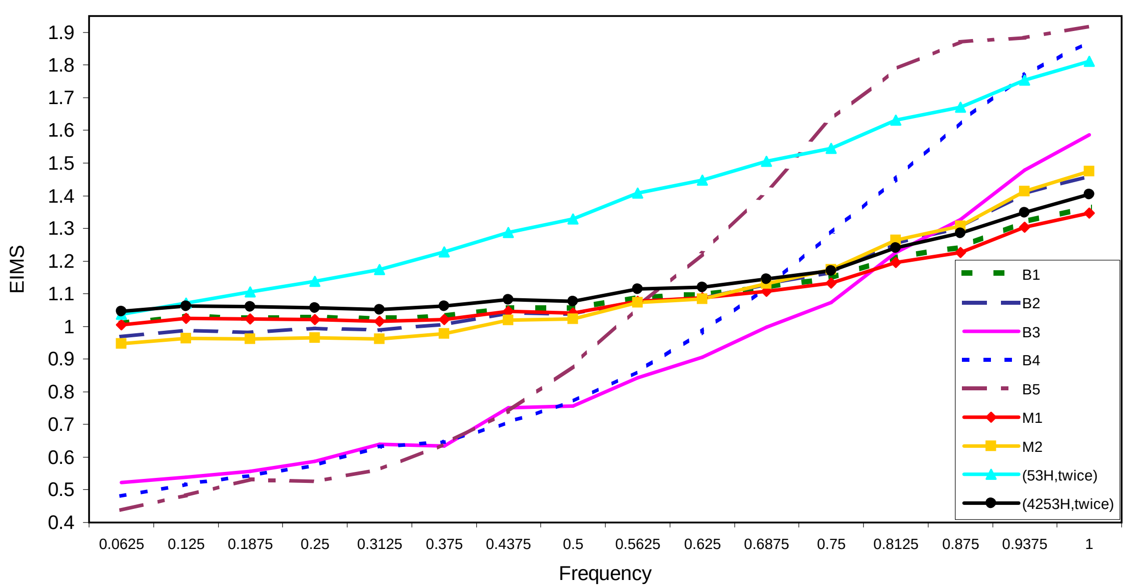

Figure 6.36: $\widehat{EIMS}$ of MA_2 on nonlinear smoothers for negative slope and negative impulse

The conclusions from the $\widehat{EIMS}$ values for the linear on nonlinear smoothers for negative slope and negative impulse as displayed in Figure 6.36 can also be divided into four categories as for the nonlinear smoothers in Figure 6.34. The frequency boundaries and the smoother of the last category differ from the nonlinear smoothers as follows:

- frequencies $\leq \frac{6}{16}$ (0,375), B_5 performs best
- $\frac{7}{16}$ (0,4375) $\leq$ frequencies $\leq \frac{8}{16}$ (0,5), B_4 performs best

- $\frac{9}{16}$ (0,5625) $\leq$ frequencies $\leq$ $\frac{12}{16}$ (0,75), B_3 performs best
- frequencies $\geq$ $\frac{13}{16}$ (0,8125), M_1 performs best

From Figure 6.35 it is clear that the estimated regression coefficients for the smoothers above are all close to one in the specific frequency intervals and thus support the conclusions based on the $\widehat{EIMS}$ values. Again the LULU smoothers perform better for frequencies less than or equal to $\frac{12}{16}$ (0,75).

4. Slope positive, impulse negative

For this case the figures are as follows:

- Figure 6.37 illustrates the estimated regression coefficients of the nonlinear smoothers.
- Figure 6.38 illustrates the $\widehat{EIMS}$ values of the nonlinear smoothers.
- Figure 6.39 illustrates the estimated regression coefficients of MA_2 on the nonlinear smoothers.
- Figure 6.40 illustrates the $\widehat{EIMS}$ values of MA_2 on the nonlinear smoothers.

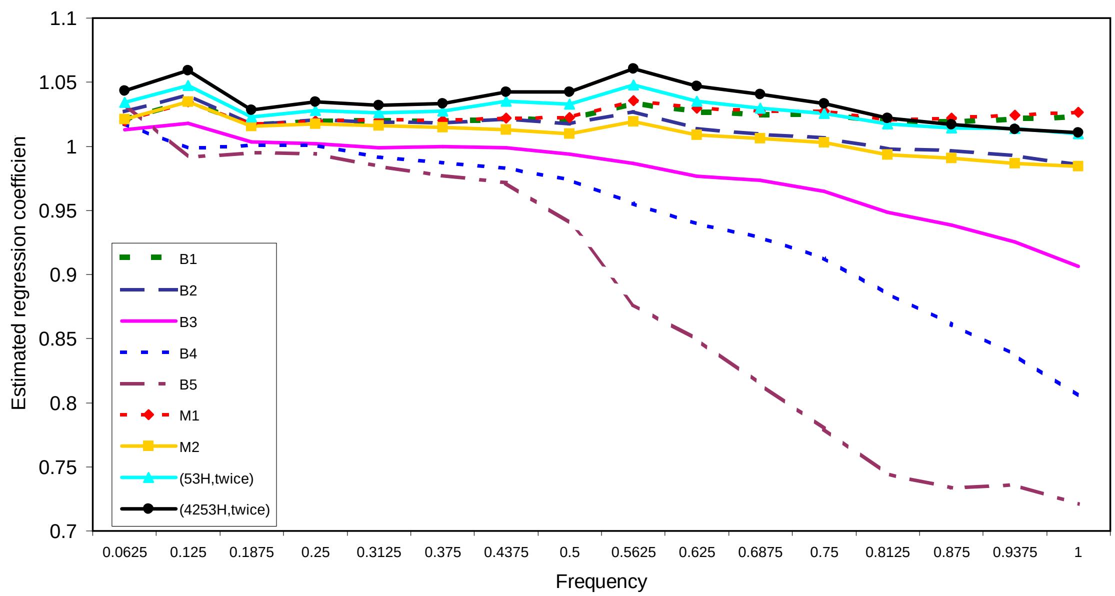

Figure 6.37: Estimated regression coefficients of nonlinear smoothers for positive slope and negative impulse

From Figure 6.38 the conclusions from the $\widehat{EIMS}$ values for a positive slope and negative impulse can be divided into three categories according to frequencies

- frequencies $\leq$ $\frac{6}{16}$ (0,375), B_5 performs best with B_4 very similar for frequencies $\geq$ $\frac{3}{16}$ (0,1875)
- $\frac{7}{16}$ (0,4375) $\leq$ frequencies $\leq$ $\frac{12}{16}$ (0,75), B_3 performs best
- frequencies $\geq$ $\frac{13}{16}$ (0,8125), (4253H,twice) performs best

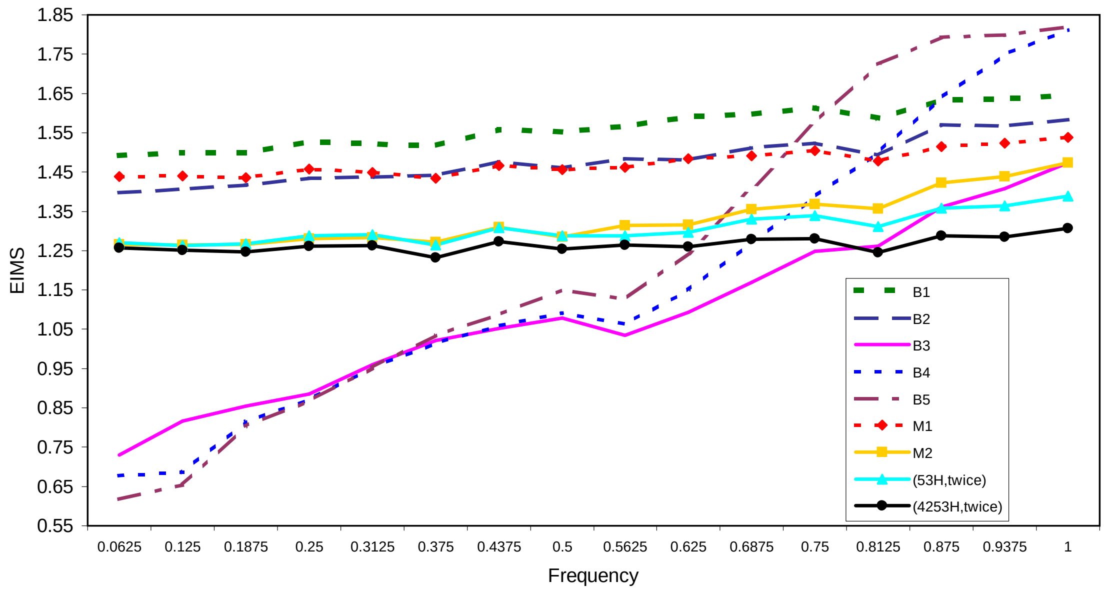

Figure 6.38: $\widehat{EIMS}$ of nonlinear smoothers for positive slope and negative impulse

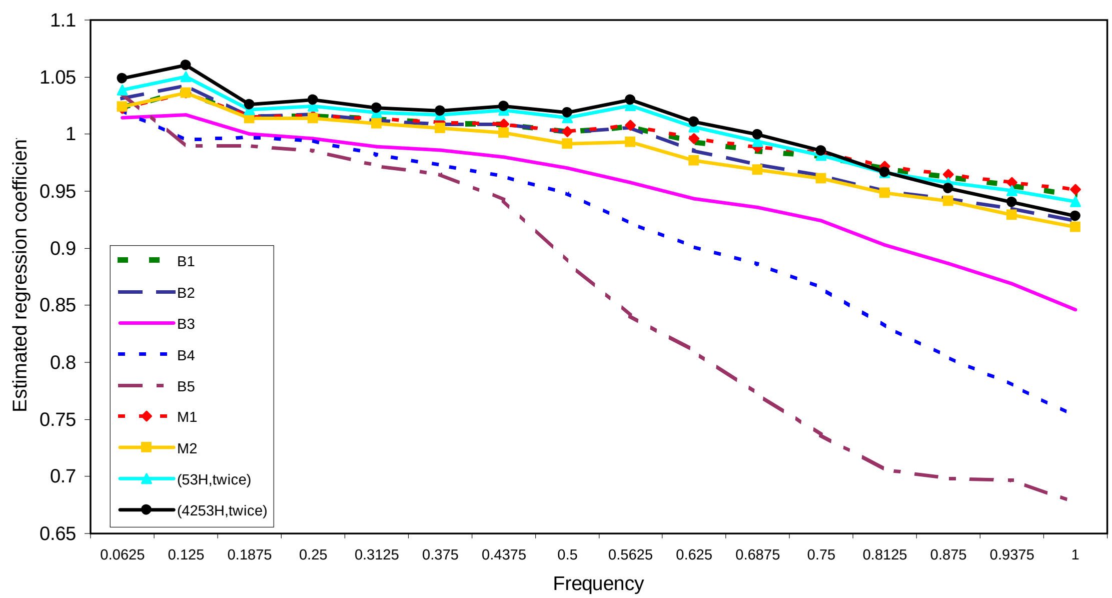

Figure 6.39: Estimated regression coefficients of MA_2 on nonlinear smoothers for positive slope and negative impulse

From Figure 6.37 it is clear that the estimated regression coefficients for the smoothers above are all close to one in the specific frequency intervals and thus support the conclusions based on the $\widehat{EIMS}$ values.

The conclusions from the $\widehat{EIMS}$ values for the linear on nonlinear smoothers for positive slope and negative impulse displayed in Figure 6.40 can be divided into three categories according to frequencies

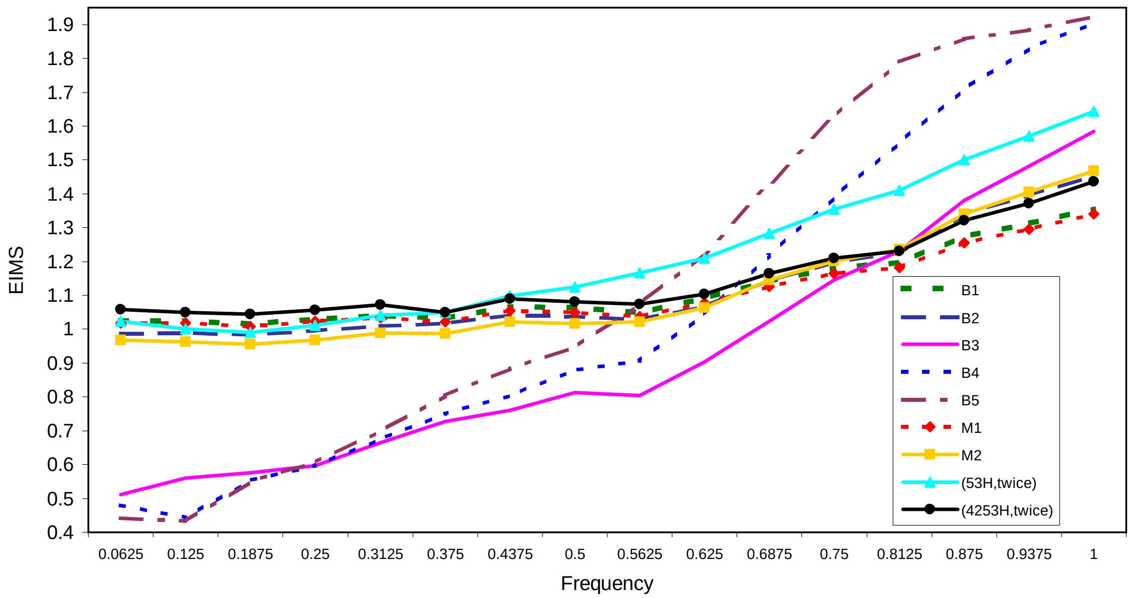

Figure 6.40: $\widehat{EIMS}$ of MA_2 on nonlinear smoothers for positive slope and negative impulse

- frequencies $\leq \frac{4}{16}$ (0,25), B_5 performs best with B_4 very similar for frequencies $\geq \frac{2}{16}$ (0,125)
- $\frac{5}{16}$ (0,3125) $\leq$ frequencies $\leq \frac{12}{16}$ (0,75), B_3 performs best
- frequencies $\geq \frac{13}{16}$ (0,8125), B_1 and M_1 perform best

From Figure 6.39 it is clear that the estimated regression coefficients for the smoothers above are all close to one in the specific frequency intervals and thus support the conclusions based on the $\widehat{EIMS}$ values.

It may be concluded here that, based on the $\widehat{EIMS}$ values for the boundaries of the intervals, the smoother that performs best varies from case to case. Roughly, B_5 performs best for the first few frequencies which lie between $\frac{4}{16}$ (0,25) and $\frac{8}{16}$ (0,5). This may be due to the large window size which smoothes the slower oscillations well. Then B_3, which removes the impulse of length 3, performs best overall for the middle part of the frequencies which range between $\frac{12}{16}$ (0,75) and $\frac{14}{16}$ (0,875). For the last frequencies, (4253H,twice) performs best when the nonlinear smoothers only are applied. For linear on nonlinear, B_1 and M_1, the nonlinear smoothers with the small window sizes perform best on the frequencies with fast oscillations.

The estimated regression coefficients for the all the smoothers are close to one and thus support the conclusions based on the $\widehat{EIMS}$ values. These findings correspond with the theory of how LULU smoothers deal with blockpulses as discussed in Section 4.5.1.

The most important conclusions, however, are that the LULU smoothers B_3, B_4 and B_5 perform best for frequencies less than or equal to $\frac{12}{16}$ (0,75). It is also informative to observe how the application of the linear smoother to the nonlinear smoothers causes the $\widehat{EIMS}$ values to decrease. This means that the application of the linear smoother resulted in better signal recovery. These results support Mallows' idea that it would be sensible to design a robust smoother to achieve the desired insensitivity to outliers and then follow it with a linear smoother to achieve a desired transfer shape. The way that LULU smoothers deal with blockpulses, the unique feature of the LULU smoothers that the percentage variation removed at each recursive step can be calculated, as well as the success of applying the simple moving average on the LULU smoothers as illustrated with the simulation studies, makes this procedure a worthy candidate for satisfying Mallows' idea. Table 6.3 summarises the conclusions of the entire section.

Table 6.3: Summary of conclusions on sine signal

Type of smoothing	**Regression**		**EIMS**	
	Frequency	**Better smoothers**	**Frequency**	**Better smoothers**
Nonlinear on sine signal	[0; 0,5] [0,5625; 1]	B_1, M_1,(53H,twice), (4253H,twice) B_1, M_1	[0; 1]	(53H,twice), (4253H,twice)
Linear on nonlinear on sine signal	[0; 0,5] [0,5625; 1]	B_1, M_1,(53H,twice), (4253H,twice) B_1, M_1	[0; 0,4375] [0,5; 1]	All B_1, M_1, (4253H,twice)
Nonlinear on sine signal with impulse	[0; 0,625] [0,6875; 1]	B_1, M_1,(53H,twice), (4253H,twice) B_1, M_1	[0; 0,5] [0,5625; 0,8125] [0,875; 1]	B_3, B_4, B_5 B_3 (4253H,twice)
Linear on nonlinear on sine signal with impulse	[0; 0,625] [0,6875; 1]	B_1, M_1,(53H,twice), (4253H,twice) B_1, M_1	[0; 0,25] [0,3125; 0,75] [0,8125; 1]	B_3, B_4, B_5 B_3 B_1, M_1
Nonlinear on sine signal with positive impulse on negative slope			[0; 0,5] [0,5625; 0,75] [0,8125; 1]	B_3, B_4, B_5 B_3 (4253H,twice)
Linear on nonlinear on sine signal with positive impulse on negative slope			[0; 0,4375] [0,5; 0,75] [0,8125; 1]	B_3, B_4, B_5 B_3 B_1, M_1
Nonlinear on sine signal with positive impulse on positive slope			[0; 0,5625] [0,625; 0,8125] [0,875; 1]	B_3, B_4, B_5 B_3 (4253H,twice)
Linear on nonlinear on sine signal with positive impulse on positive slope			[0; 0,25] [0,3125; 0,75] [0,8125; 1]	B_3, B_4, B_5 B_3 B_1, M_1
Nonlinear on sine signal with negative impulse on negative slope			[0; 0,5] [0,5625; 0,625] [0,6875; 0,8125] [0,875; 1]	B_5 B_4 B_3 (4253H,twice)
Linear on nonlinear on sine signal with negative impulse on negative slope			[0; 0,375] [0,4375; 0,5] [0,5625; 0,75] [0,8125; 1]	B_5 B_4 B_3 M_1
Nonlinear on sine signal with negative impulse on positive slope			[0; 0,375] [0,4375; 0,75] [0,8125; 1]	B_4, B_5 B_3 (4253H,twice)
Linear on nonlinear on sine signal with negative impulse on positive slope			[0; 0,25] [0,3125; 0,75] [0,8125; 1]	B_4, B_5 B_3 B_1, M_1

6.4 Other examples

In this section the nonlinear smoothers are applied to some special functions well known in the literature.

Donoho & Johnstone (1994) simulated data for four special functions, Blocks, Bumps, HeaviSine and Doppler. These functions have been chosen because they display the spatially variable functions that arise in imaging, spectroscopy and other scientific processing. They used the functions to visually compare a wavelet oracle with a Fourier domain oracle.

Figure 6.41 displays these four functions which have also been analysed by various authors in the literature. Donoho *et al.* (1995) used them to illustrate the reconstruction of the noisy functions by wavelet shrinkage, spline smoothing and the windowed Fourier series method. Fan & Gijbels (1995) and Fan & Gijbels (1996) illustrate a data-driven bandwidth selection procedure in local polynomial fitting via the four functions. The results were also compared visually with wavelet thresholding techniques. Davies & Kovac (2001) compared the taut-string method with local squeezing on these functions visually with wavelet thresholding.

The formulæ for the functions are given in Table 6.4.

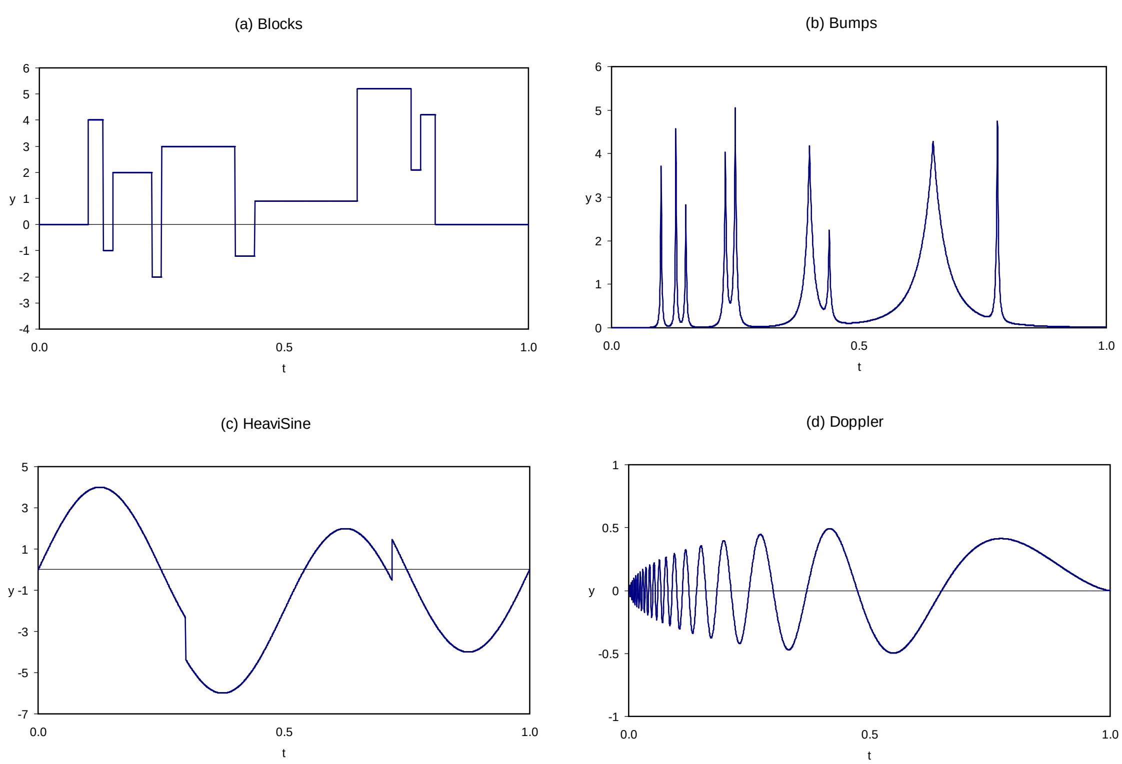

Figure 6.41: Four spatially variable functions

Gaussian white noise with $\sigma = 1$ was added to each function. Thereafter, as in Donoho & Johnstone (1994), each function was rescaled to have a signal-to-noise ratio, standard deviation of the function f to σ, equal to 7. Data from these rescaled functions are displayed in Figure 6.42.

Two hundred data sets of size $n = 2048$, as in Donoho & Johnstone (1994), were simulated from each of the functions. The nonlinear smoothers used in the previous sections were applied to these rescaled noisy functions. The estimated regression coefficients of the regression of the smoothed function on

(a) Blocks	$f(t) = \Sigma h_j K(t - t_j),\ K(t) = \{1 + \text{sgn}(t)\}/2$ $t_j = (0{,}1;\ 0{,}13;\ 0{,}15;\ 0{,}23;\ 0{,}25;\ 0{,}40;\ 0{,}44;\ 0{,}65;\ 0{,}76;\ 0{,}78;\ 0{,}81)$ $(h_j) = (4;\ -5;\ 3;\ -4;\ 5;\ -4{,}2;\ 2{,}1;\ 4{,}3;\ -3{,}1;\ 2{,}1;\ -4.2)$
(b) Bumps	$f(t) = \Sigma h_j K(t - t_j/w_j),\ K(t) = (1 + \|t\|)^{-4}$ $t_j = t_{Blocks}$ $(w_j) = (0{,}005;\ 0{,}005;\ 0{,}006;\ 0{,}01;\ 0{,}01;\ 0{,}03;\ 0{,}01;\ 0{,}01;\ 0{,}005;\ 0{,}008;\ 0{,}005)$
(c) HeaviSine	$f(t) = 4\sin 4\pi t - \text{sgn}(t - 0{,}3) - \text{sgn}(0{,}72 - t)$
(d) Doppler	$f(t) = \{t(1-t)\}^{\frac{1}{2}} \sin\{2\pi(1+\varepsilon)/(t+\varepsilon)\},\ \varepsilon = 0{,}05$

Table 6.4: Formulæ for test functions

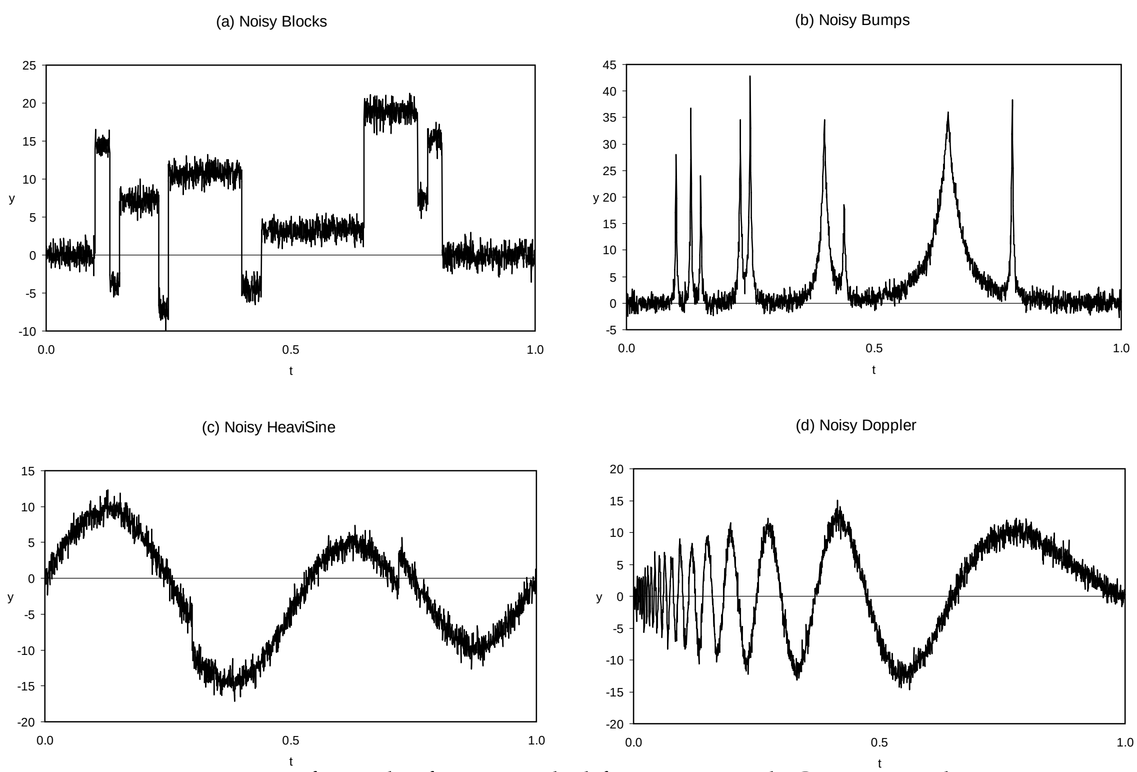

Figure 6.42: Data from the four rescaled functions with Gaussian white noise

the signal function and the $\widehat{EIMS}$ values were calculated for each nonlinear smoother. The values are displayed in Figure 6.46 and Figure 6.47. Thereafter the linear smoother, the moving average MA_2, was applied to each of these smoothed sequences. The estimated regression coefficients for the regression of the resmoothed function on the signal function and the $\widehat{EIMS}$ values were calculated. Figure 6.48 and Figure 6.49 illustrate the calculated values. All the values calculated in this section are tabulated in Appendix C.

Examples to illustrate the behaviour of the nonlinear and linear on nonlinear smoothers are included. Figure 6.43 displays B_1 and $MA_2(B_1)$ applied to the noisy rescaled functions. Figure 6.44 displays B_5 and $MA_2(B_5)$ applied to the noisy rescaled functions and Figure 6.45 displays M_1 and $MA_2(M_1)$ applied to the noisy rescaled functions.

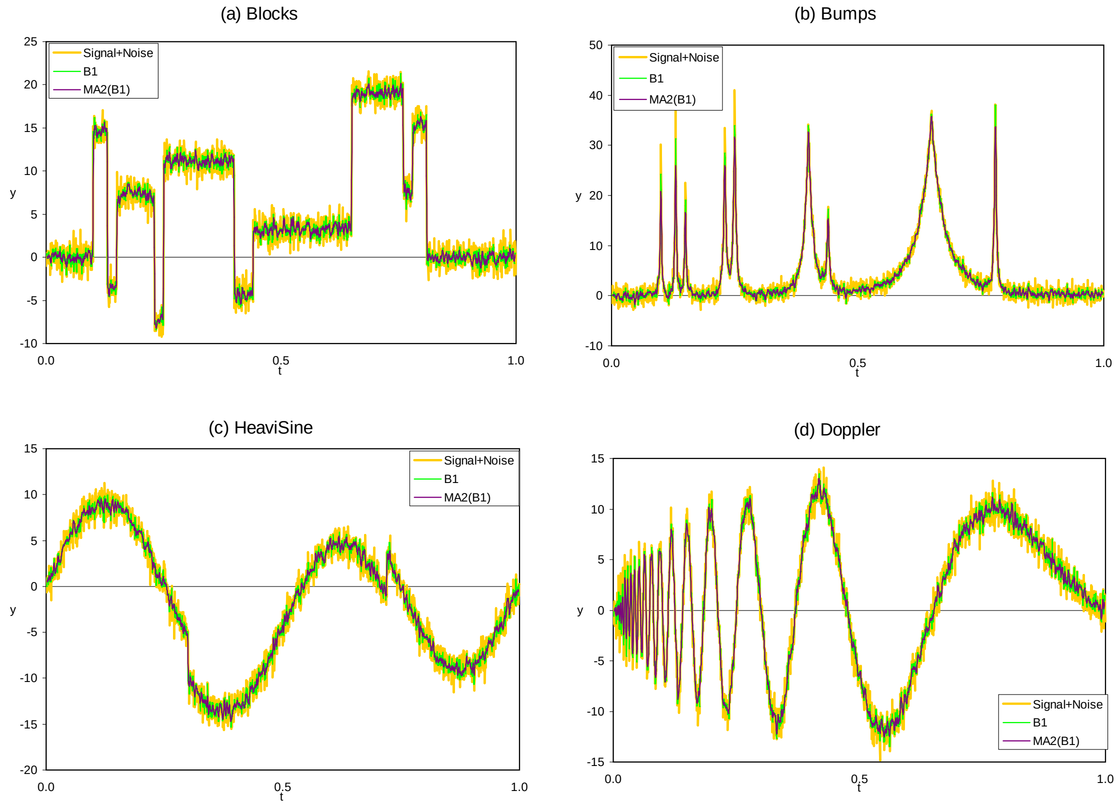

Figure 6.43: B_1 and $MA_2(B_1)$

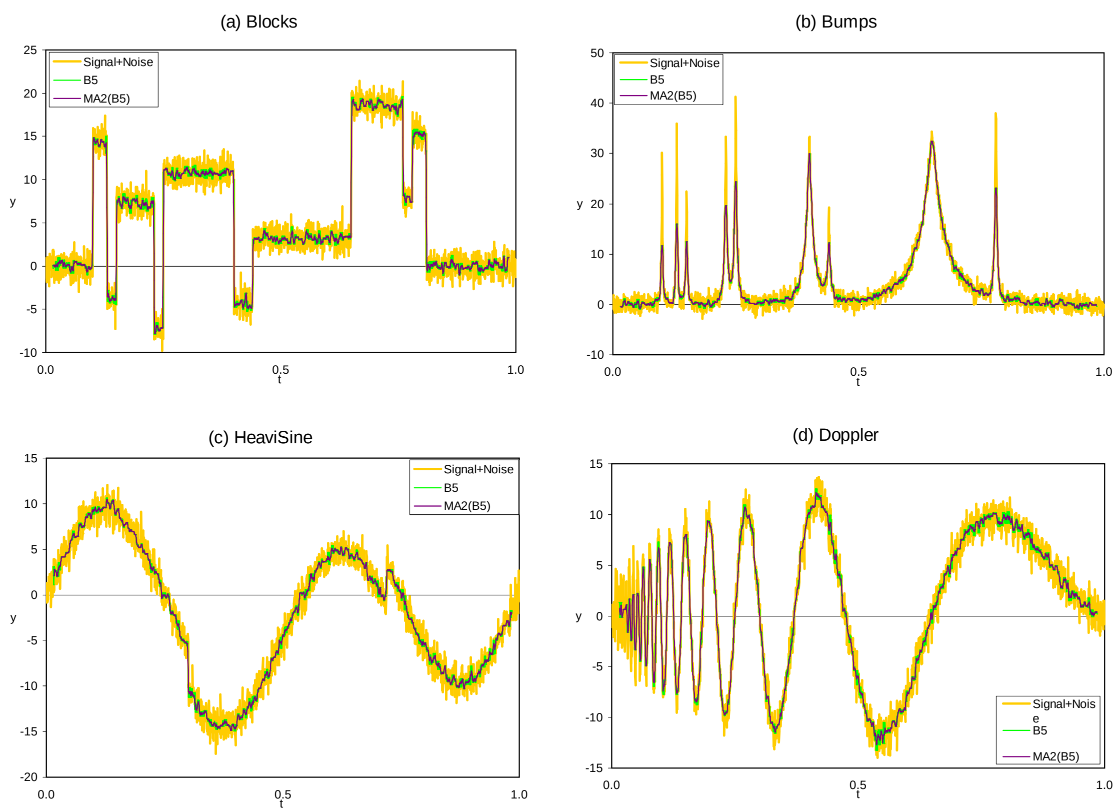

Figure 6.44: B_5 and $MA_2(B_5)$

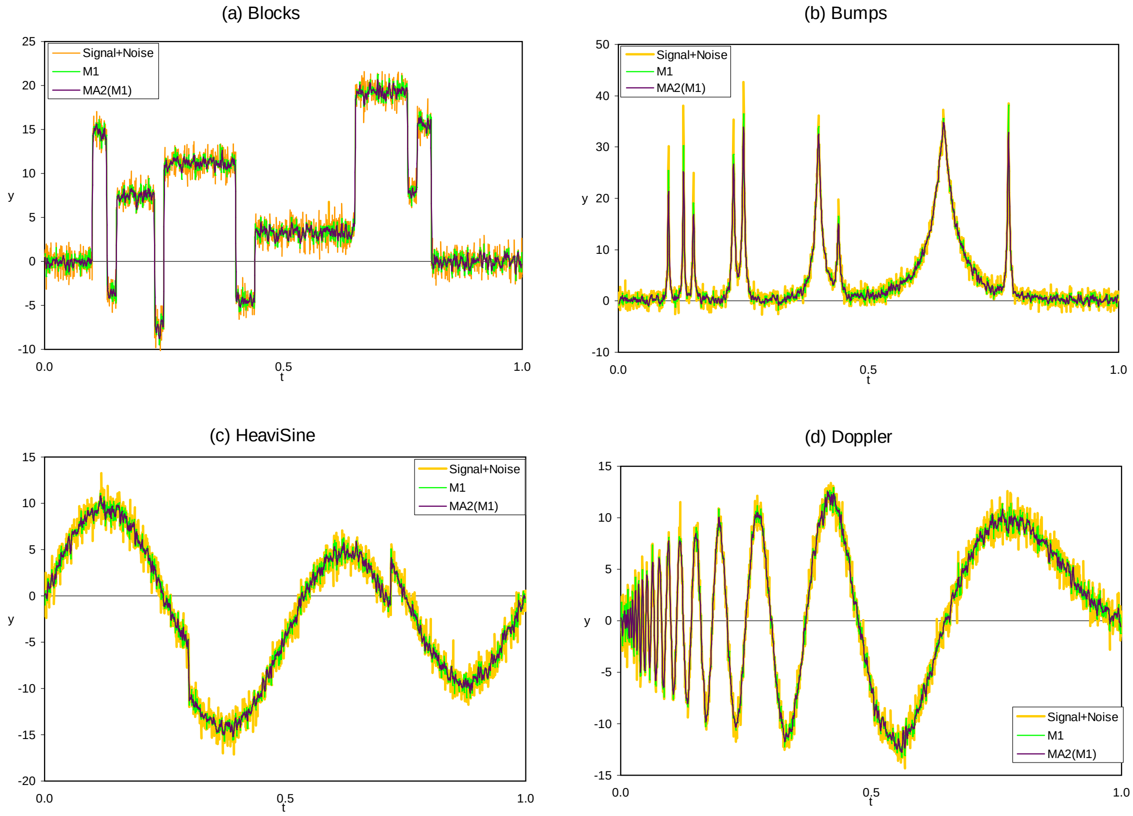

Figure 6.45: M_1 and $MA_2(M_1)$

Since only Gaussian noise was added, there are no spikes to be removed by the nonlinear smoothers. The nonlinear smoothers are used to see how they perform on the sharp features present in some of the functions. It can be seen that the grade of smoothing of the nonlinear smoothers B_1 and M_1 is much the same. Neither of them remove substantial noise. The smoother B_5 produces a much smoother sequence. From all these graphs it may be concluded that the smoothers reconstruct the noisy functions into almost noise-free functions. The sharp features in the signal functions stayed sharp after smoothing.

From Figures 6.46 to 6.49 the following conclusions, summarised in Table 6.5, can be made

(a) Blocks: The $\widehat{EIMS}$ values of the compound LULU smoothers decrease as n increases. B_5 has the smallest value and thus performs best. M_2 performs best of the group of median smoothers. The estimated regression coefficient values of the nonlinear smoothers are of the order of 0,998, very close to one, with B_1 and M_1 performing best.

The $\widehat{EIMS}$ values for linear on nonlinear smoothers follow the same pattern as for the nonlinear smoothers, but with greater values. The reason is that linear smoothers smooth the step edge between blocks to a ramp edge which differs a fair amount from the signal. The smoother with the largest window size, B_5, performed the best according to the $\widehat{EIMS}$ values. The estimated regression coefficients of the linear on nonlinear smoothers are less than for the nonlinear smoothers, but also close to one. The nonlinear smoothers B_1 and M_1 have the largest values.

Since $n = 2048$ data values were used for this function, each block is an impulse consisting of a large number of values. The nonlinear smoothers used in this study have relatively small window sizes compared to the number of data points forming the block. The smoothers will thus never remove the block, but they follow the original signal well.

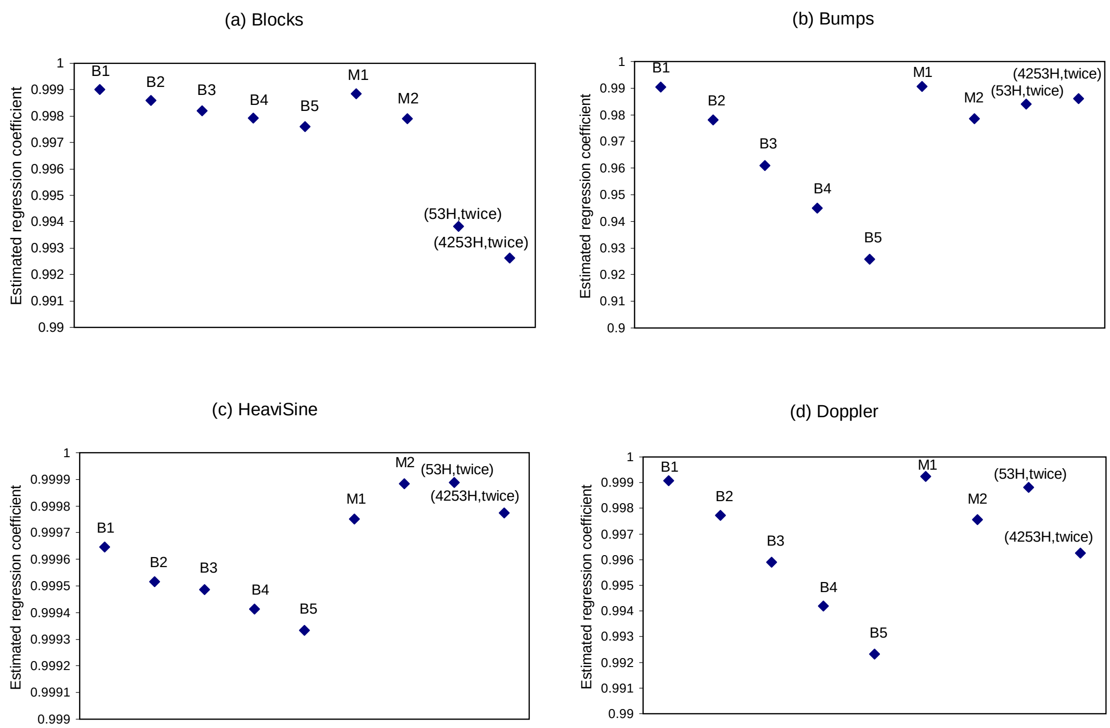

Figure 6.46: Estimated regression coefficient of nonlinear smoothers on rescaled noisy functions

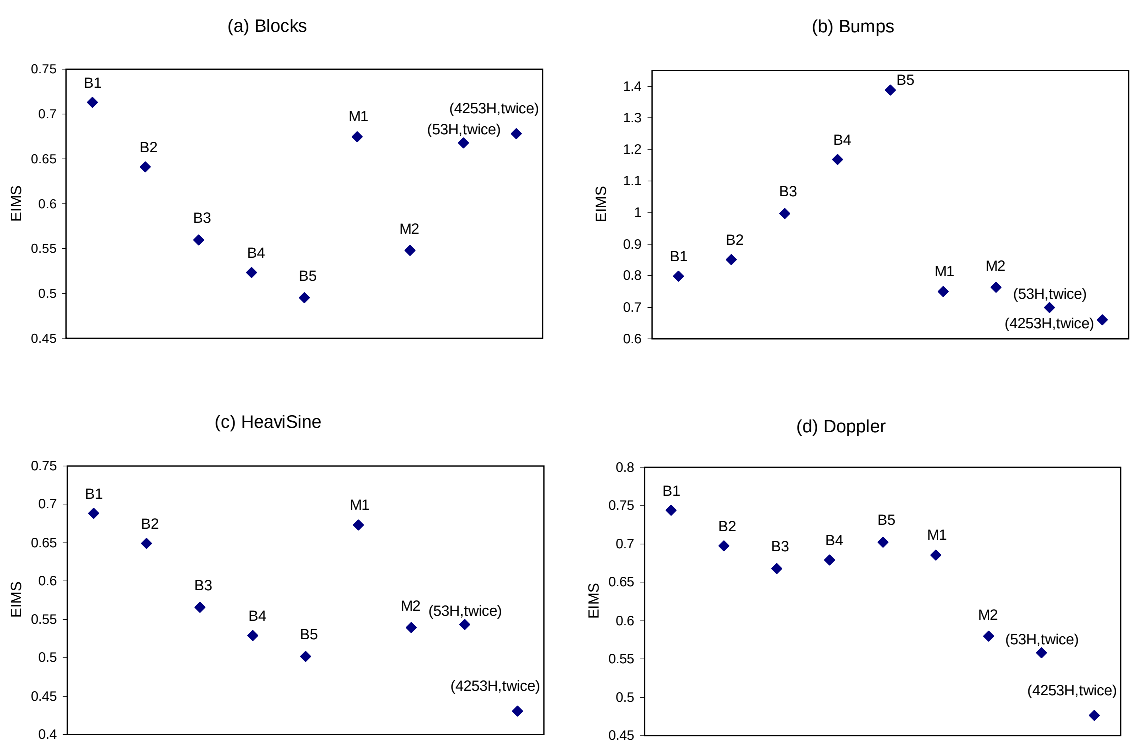

Figure 6.47: $\widehat{EIMS}$ of nonlinear smoothers on rescaled noisy functions

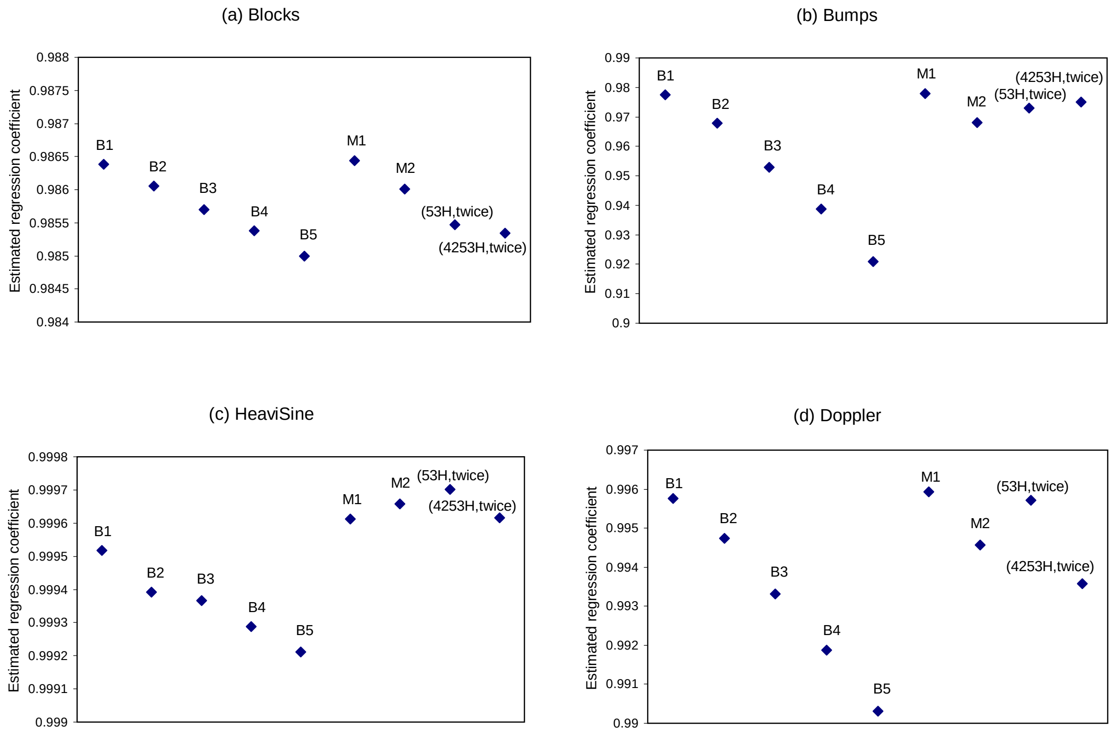

Figure 6.48: Estimated regression coefficient of MA_2 on nonlinear smoothers for rescaled noisy functions

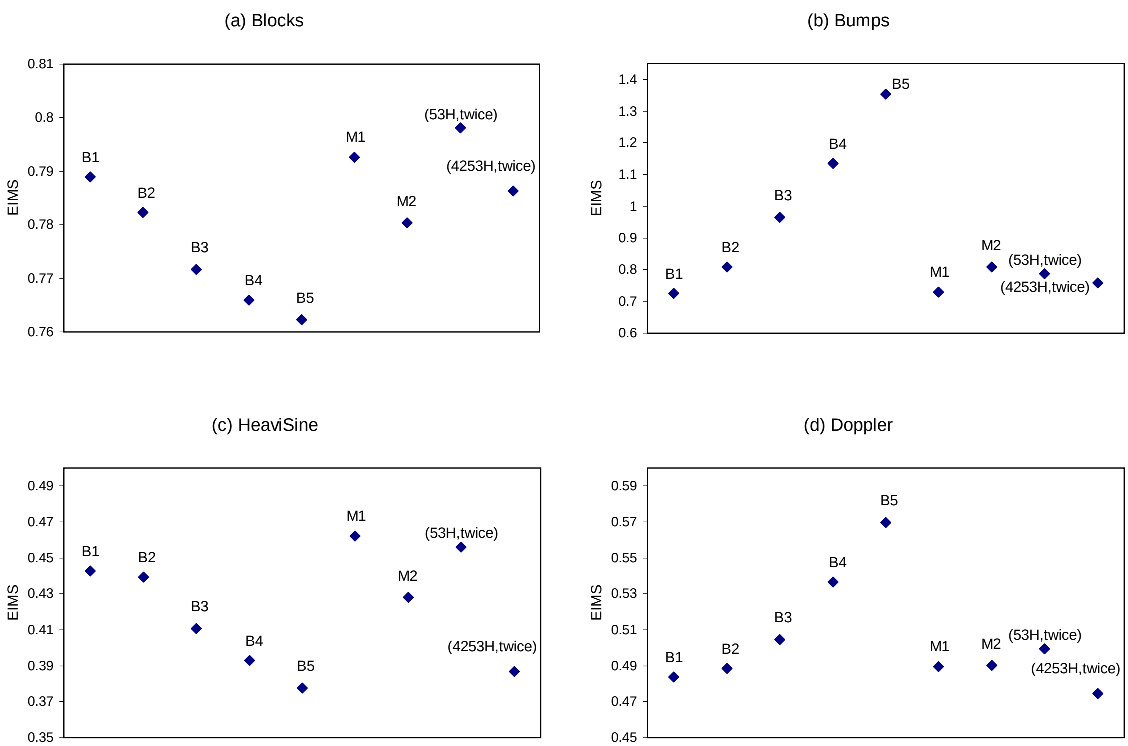

Figure 6.49: $\widehat{EIMS}$ of MA_2 on nonlinear smoothers for rescaled noisy functions

(b) Bumps: From the $\widehat{EIMS}$ values of the nonlinear smoothers it can be seen that all the median smoothers have smaller values than the LULU smoothers with (4253H,twice) being the better smoother. The estimated regression coefficients of the nonlinear smoothers vary over a wider range as for blocks, with values of B_1 and M_1 being closest to one.

For linear on nonlinear smoothers, the $\widehat{EIMS}$ values of B_1 and M_1 are the smallest. The values of the estimated regression coefficients of linear on nonlinear smoothers for these two smoothers confirm that they perform best for smoothing this function.

This function has a number of sharp spikes as the bumps occur. The nonlinear smoothers with the small window sizes follow the signal well.

(c) HeaviSine: From the $\widehat{EIMS}$ values, the median smoother (4253H,twice) has the smallest value for smoothing with nonlinear smoothers. The estimated regression coefficients of the nonlinear smoothers have values closest to one, which mean that they follow the signal best of all the functions. M_2 and (53H,twice) perform best.

For linear on nonlinear smoothers the compound LULU smoother B_5 has an $\widehat{EIMS}$ value slightly less than that of (4253H,twice). There is a small difference in the estimated regression coefficients for the linear on nonlinear smoothers. All the median smoothers have values greater than those of the compound LULU smoothers, which reflect a better fit for this function with (53H,twice) being best.

Of the four special functions, the HeaviSine is the one with the least abnormal features. It follows the pattern of a sine signal with a slow oscillation with two abnormal downward slopes. The first downward slope has a slight dent and the second one has an upward bump on it. Overall it seems that the compound median smoothers perform best in smoothing this function.

(d) Doppler: This function is a sine function that starts with small and fast oscillations which become larger and slower as t increases.

For the $\widehat{EIMS}$ values there is a totally different pattern for the nonlinear smoothers, and the linear on nonlinear smoothers. For nonlinear smoothing, the $\widehat{EIMS}$ values of the compound LULU smoothers are close to each other and the $\widehat{EIMS}$ values of the median smoothers wider spread with (4253H,twice) being the smallest. The estimated regression coefficients of the nonlinear smoothers are close to one with B_1 and M_1 performing best.

For linear on nonlinear, the $\widehat{EIMS}$ values of the compound LULU smoothers form an upward curve with B_1 being the smallest. Here the $\widehat{EIMS}$ values of the linear on the median smoothers are flocked together with (4253H,twice) being overall the smallest. The estimated regression coefficients of linear on nonlinear smoothing are close to one. The smoothers B_1 and M_1 perform best.

6.5 Conclusions

This chapter consisted of four studies. In the first three studies a sine signal at varying frequencies was considered with different noise being added to this signal. In the first study the noisy sine signal was simulated 200 times for 16 different frequencies and smoothed each time. In the smoothing process, firstly a nonlinear smoother was used to remove the non-Gaussian noise, and secondly this smoothed sequence was smoothed by a linear smoother to remove the Gaussian noise. The compound LULU smoothers, B_1 to B_5, the median smoothers, M_1 and M_2, and the compound median smoothers, (53H,twice) and

Table 6.5: Summary of conclusions on special functions

Function	**Smoothing**	**Better smoothers** Regression	EIMS
Blocks	Nonlinear	B_1, M_1	B_5
	Linear on nonlinear	B_1, M_1	B_5
Bumps	Nonlinear	B_1, M_1	(4253H,twice)
	Linear on nonlinear	B_1, M_1	B_1, M_1
HeaviSine	Nonlinear	M_2,(53H,twice)	(4253H,twice)
	Linear on nonlinear	(53H,twice)	B_5
Doppler	Nonlinear	B_1, M_1	(4253H,twice)
	Linear on nonlinear	B_1, M_1	(4253H,twice)

(4253H,twice), were chosen as nonlinear smoothers. The moving average, MA_2, was used as a linear smoother. The estimated regression coefficients and $\widehat{EIMS}$ from the original signal were calculated for 200 simulation runs for each of the smoothers. The results show that the compound LULU smoother B_1 and the median smoother M_1 perform very well, together with the compound median smoother (4253H,twice).

In the second study the same process used in the first study was repeated with an impulse added. Three values of 10 each were added to the sine signal at a constant position throughout the frequency changes. The results showed zigzag patterns, which led to the investigation of the influence of the position and direction of the impulse on the slope of the sine signal.

In the third study positive and negative impulses were added to the positive and negative slopes of the sine signals over all 16 frequencies. The results show that different nonlinear smoothers perform best at different intervals of the frequencies. The boundaries of these intervals differ for each of the four cases investigated. For the smaller frequencies, B_3 performed best because it removed the impulse of length 3. At the larger frequencies, (4253H,twice) performed best for the nonlinear smoothing, while B_1 and M_1 performed best after linear on nonlinear smoothing was applied.

The fourth study was to investigate unusual signals with Gaussian noise added. These signals have been used in well-known literature and are called Blocks, Bumps, HeaviSine and Doppler. The results show that different nonlinear smoothers perform best on each of these signals.

Finally, it may be concluded from the studies in this chapter that the compound LULU smoother B_n is a worthy competitor for the median smoothers. In some cases, especially when blockpulses are present, the compound LULU smoother performs better in the recovery of a sinusoidal signal. Also, for the special functions considered, B_n performs well relative to the median smoothers. The application of an unsophisticated linear smoother after the nonlinear LULU smoothers B_n and median compositions have been applied also seems to be a procedure with merit.

Chapter 7

Practical applications

7.1 Introduction

In the previous chapters the nonlinear smoothers, namely the median and LULU smoothers, were defined and their properties discussed. In this chapter practical applications are used to illustrate how the most important properties of LULU smoothers result in better insight into the interpretation of the smoothing process.

The first data set chosen is the daily closing prices for a time period from the Standard and Poor 500 Index. The application of LULU smoothers to this financial index data was published in Conradie *et al.* (2005). In this chapter the median smoother is also included and compared with the results of the LULU smoothers. The results of the analysis are discussed in Section 7.2.

The second application was on the arterial blood pressure of a patient in intensive care. This data is usually obtained by an online monitoring process. The goal of the smoothing process is to extract the underlying signal. The nursing staff should be alerted to outliers or fluctuations that have a long-term duration. Short-term fluctuations caused by the movement of the patient should be removed. Two data sets of arterial blood pressure, with similar patterns as the data used by Fried *et al.* (2006), are set up and studied. The median and compound LULU smoothers are applied to this data to determine their success in extracting the underlying, relevant signal. The results of the analysis are given in Section 7.3.

7.2 Financial application

7.2.1 Standard and Poor 500

The data used is the daily closing prices of the Standard and Poor 500 Index for the period 1999/01/04 to 2000/10/03 are tabulated in Appendix D. The daily closing prices are illustrated in Figure 7.1. It seems that the closing prices vary from each other within a month with no definite pattern, and no exceptionally large impulses exist.

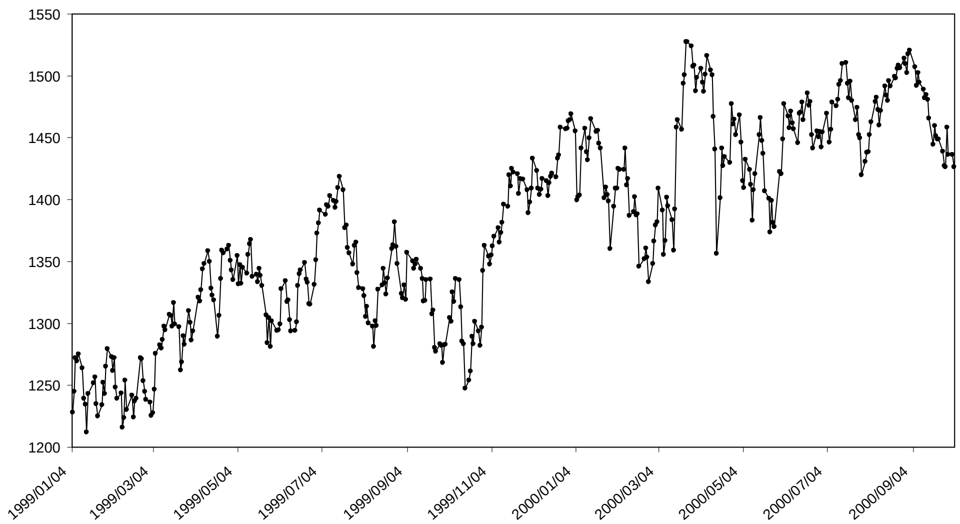

Figure 7.1: Standard and Poor 500 closing prices for 1999/01/04-2000/10/03

The LULU smoothers $F_1, \ldots, F_5$ and $C_1, \ldots, C_5$ were applied to the closing prices. Since the compound LULU smoothers $B_1, \ldots, B_5$ result from these smoothers, only the compound smoothers will be discussed here. From Definition 4.10 the compound LULU smoothers are always between the flooring and ceiling smoothers, or take on either of these values.

The decompositions of the total variation at each level of smoothing for the LULU smoothers $F_1, \ldots, F_5$, $C_1, \ldots, C_5$ and $B_1, \ldots, B_5$ were calculated. All these values were tabulated in Table 4.5. The decomposition of the total variation by B_n, as well as the corresponding percentages at each level, are tabulated in Table 7.1.

Table 7.1: Reduction of total variation for B_n

n	$T(x)$	$T(B_n)$	$T(B_{n-1} - B_n)$	$T(B_{n-2} - B_{n-1})$	$T(B_{n-3} - B_{n-2})$	$T(B_{n-4} - B_{n-3})$	$T(B_{n-5} - B_{n-4})$
1	5 085,45	3 453,33	1 632,12	0	0	0	0
%	100%	67,91%	32,09%	0	0	0	0
2	5 085,45	2 932,65	520,68	1 632,12	0	0	0
%	100%	57,67%	10,24%	32,09%	0	0	0
3	5 085,45	2 555,15	377,5	520,68	1 632,12	0	0
%	100%	50,24%	7,42%	10,24%	32,09%	0	0
4	5 085,45	2 133,57	421,58	377,5	520,68	1 632,12	0
%	100%	41,95%	8,29%	7,42%	10,24%	32,09%	0
5	5 085,45	1 901,11	232,46	421,58	377,5	520,68	1 632,12
%	100%	37,38%	4,57%	8,29%	7,42%	10,24%	32,09%

The total variation of the Standard and Poor 500 series graphed in Figure 7.1 is $T(x) = 5\ 085{,}45$. At the first smoothing step B_1 removes 32,09% of the variation, secondly B_2 removes a further 10,24%, then B_3 removes a further 7,42%, B_4 a further 8,29% and B_5 a further 4,57%. After the application of B_5 a total of 62,62% of the total variation is removed and 37,38% of the total variation is preserved. The additional percentage variation removed by B_n at each smoothing step is illustrated in Figure 7.2.

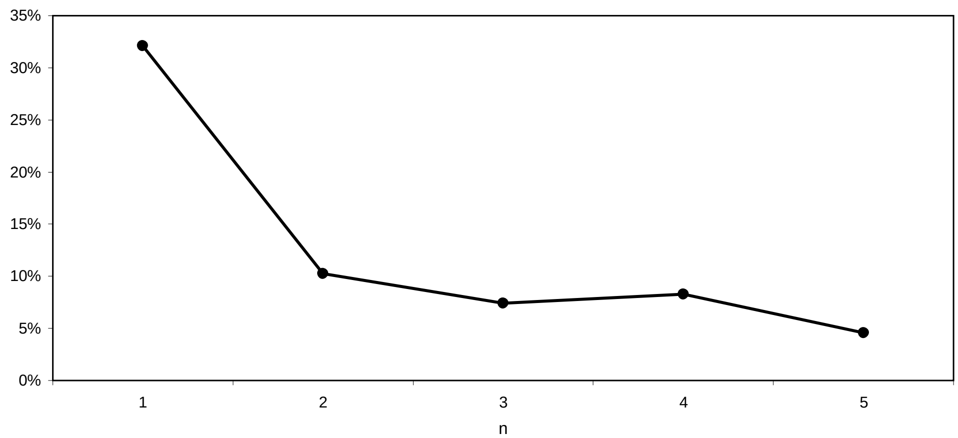

Figure 7.2: Additional percentage variation removed by B_n at each smoothing step for S and P 500

The question of whether to continue with further smoothing steps, or whether to stop, is an open one. The amount of additional variation removed at each level can give an indication of whether it is worth continuing. If the amount of total variation to be preserved for this data is in the order of 40%, it could stop at B_4, since B_5 removes less than a further 5% of the total variation. Each set of data is unique and the analyst should have a feeling for the amount of variation to be removed or preserved.

Figure 7.3: B_1 and M_1 smoothers on S and P 500 closing prices

The median smoothers $M_1, \ldots, M_5$ were also applied to the closing prices. Figure 7.3 illustrates B_1 and M_1 applied to the closing prices, and B_5 and M_5 in Figure 7.4.

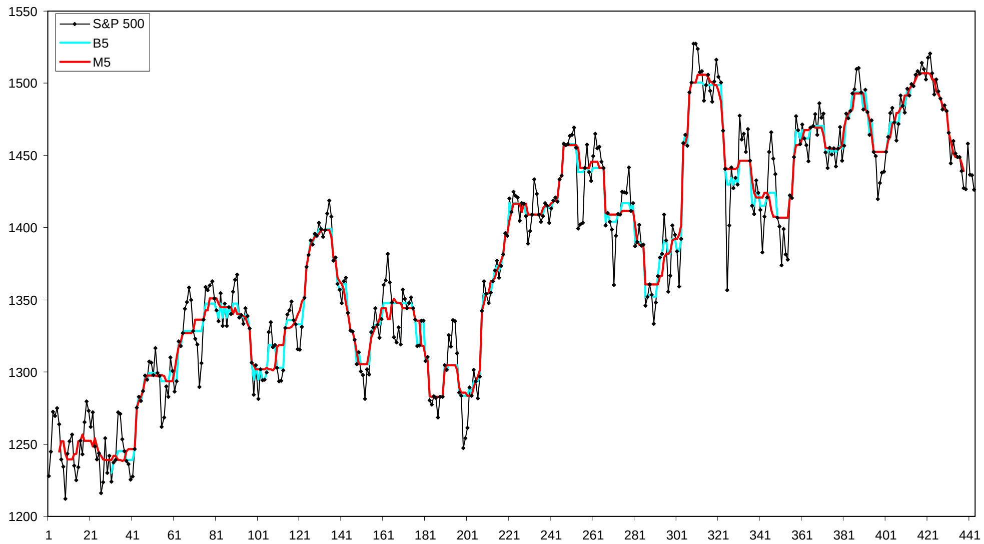

Figure 7.4: B_5 and M_5 smoothers on S and P 500 closing prices

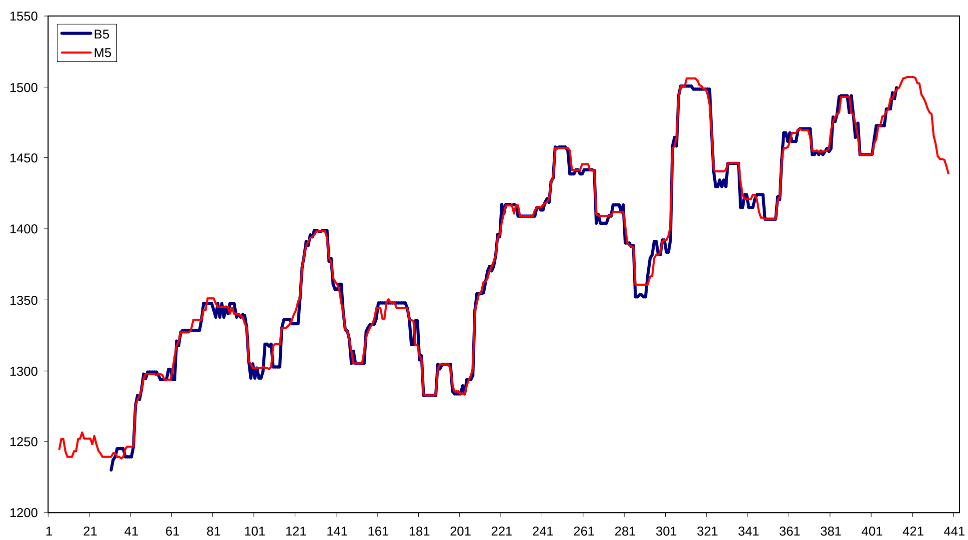

Figure 7.5: B_5 and M_5 output of S and P 500 closing prices

From Figure 7.3 it follows that smoothing by smoothers M_1 and B_1 give very similar results. In Figure 7.4 differences are observed between the output of the smoothers M_5 and B_5. When a short upward (downward) impulse is followed by a short downward (upward) impulse, B_5 removes both impulses, while M_5 results in a small bump. Two places where this is observed are between observations 52 and 57, and between observations 66 and 75. A median smoother with greater window size might delete the impulse, for in Section 3.5 it is stated that the size of the window of the chosen median smoother must be at least twice the length of the impulse to delete the impulse. This is not the case, because when median smoothers with greater window sizes, M_6 and M_9, were applied to the closing prices, the resulting bumps became wider around the two consecutive impulses in opposite directions. It seems that the change in direction of the two impulses which follow on from each other, result in this bump when smoothed by the median smoother.

From Figure 7.5, where the output of B_5 and M_5 are shown, it can be seen that the performance of these smoothers is quite similar.

7.2.2 Standard and Poor 500 plus blockpulses

The way in which LULU smoothers treat blockpulses is illustrated in the Standard and Poor 500 data by arbitrarily adding blockpulses of length 3 and 5. The blockpulses were created by increasing or decreasing arbitrary sets of three and five successive points with a constant of substantial size.

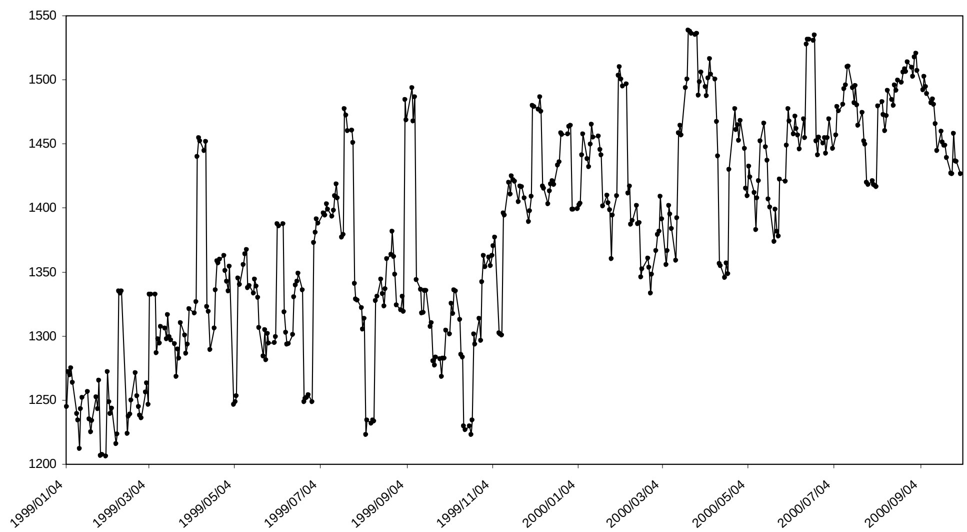

Figure 7.6: Standard and Poor 500 closing prices plus blockpulses

This series where blockpulses were added to the closing prices is illustrated in Figure 7.6. When Figure 7.6 is compared to Figure 7.1, the blockpulses can be clearly distinguished.

The LULU smoothers $F_1, \ldots, F_5$, $C_1, \ldots, C_5$ and $B_1, \ldots, B_5$ were applied to the Standard and Poor 500 closing prices plus blockpulses. The decomposition of the total variation at each level of smoothing for the LULU smoothers $F_1, \ldots, F_5$, $C_1, \ldots, C_5$ and $B_1, \ldots, B_5$ was calculated. The decomposition of the total variation by B_n, as well as the corresponding percentages at each level, are tabulated in Table 7.2.

The total variation of the Standard and Poor 500 plus the blockpulses series graphed in Figure 7.6 is $T(x) = 6\ 901{,}05$. This total variation is expected to be greater than the total variation of the series without blockpulses, due to the blockpulses added. At the first smoothing step B_1 removes 20,54% of the variation, secondly B_2 removes a further 5,84%, then B_3 removes a further 10,69%, B_4 a further 6,25% and B_5 a further 26,38%. B_5 removes the largest portion of the total variation, as a result of the blockpulses of length 5 which are removed in that step. Since B_3 removes the blockpulses of length 3, a large portion of the total variation is also removed at this smoothing level. After the application of B_5 a total of 69,69% of the total variation is removed and 30,31% of the total variation is preserved. The additional percentage variation removed by B_n at each smoothing level is illustrated in Figure 7.7.

Figure 7.8 illustrates which blockpulses added to the Standard and Poor 500 closing prices are removed by the LULU compound smoothers B_4 and B_5. It follows that B_4 and B_5 removed the blockpulses of

Table 7.2: Reduction of total variation for B_n on S and P plus blockpulses

n	$T(x)$	$T(B_n)$	$T(B_{n-1} - B_n)$	$T(B_{n-2} - B_{n-1})$	$T(B_{n-3} - B_{n-2})$	$T(B_{n-4} - B_{n-3})$	$T(B_{n-5} - B_{n-4})$
1	6 901,05	5 483,48	1 417,57	0	0	0	0
%	100%	79,46%	20,54%	0	0	0	0
2	6 901,05	5 080,7	402,78	1 417,57	0	0	0
%	100%	73,62%	5,84%	20,54%	0	0	0
3	6 901,05	4 343,18	737,52	402,78	1 417,57	0	0
%	100%	62,94%	10,69%	5,84%	20,54%	0	0
4	6 901,05	3 911,98	431,2	737,52	402,78	1 417,57	0
%	100%	56,69%	6,25%	10,69%	5,84%	20,54%	0
5	6 901,05	2 091,5	1 820,48	431,2	737,52	402,78	1 417,57
%	100%	30,31%	26,38%	6,25%	10,69%	5,84%	20,54%

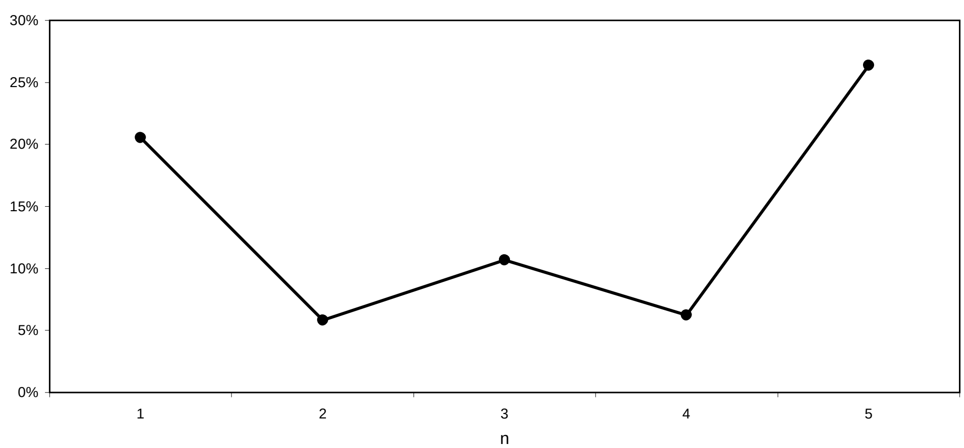

Figure 7.7: Additional percentage variation removed by B_n at each smoothing step for S and P 500 plus blockpulses

length four and shorter. Blockpulses of length 5 are preserved by B_4 but removed by B_5. From the graph it is clear that the compound smoother B_5 removes the largest portion (26,38%) of the total variation.

Figure 7.9 illustrates the smoothers B_5 and $MA_2(B_5)$ on the Standard and Poor 500 closing prices plus blockpulses. The purpose of linear smoothing on nonlinear smoothing is to remove Gaussian noise. However, it is seen here that the linear smoother transforms step edges into ramp edges. This means that some important information in the sequence around the step edges is lost.

The output of the median smoother, M_5, to the Standard and Poor 500 closing prices plus blockpulses compared to the compound LULU smoother, B_5, is illustrated in Figure 7.10. M_5 will also remove the blockpulses of length 5, since the window size of 11 is greater than twice the length of the blockpulse. Differences are observed between the output of the smoothers M_5 and B_5, more or less the same as in

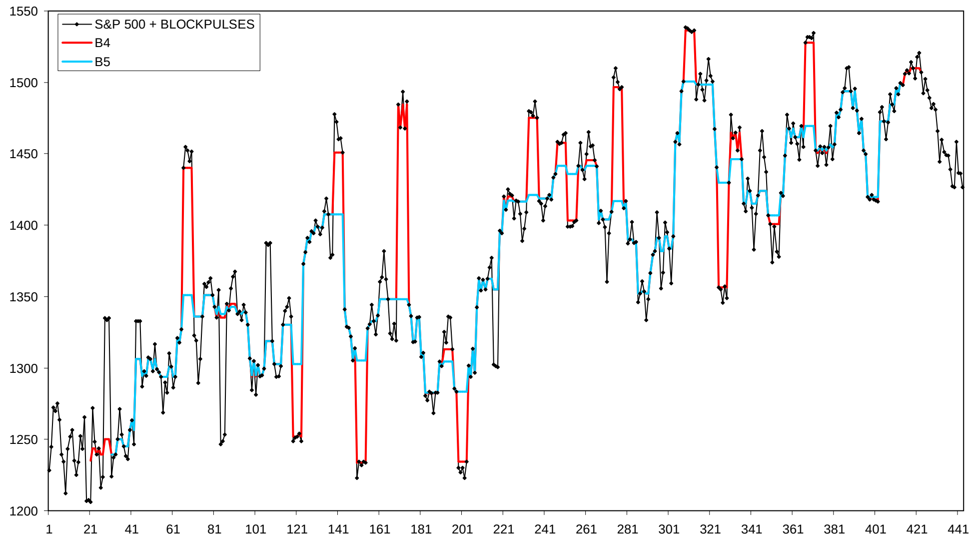

Figure 7.8: B_4 and B_5 smoothers on Standard and Poor 500 plus blockpulses

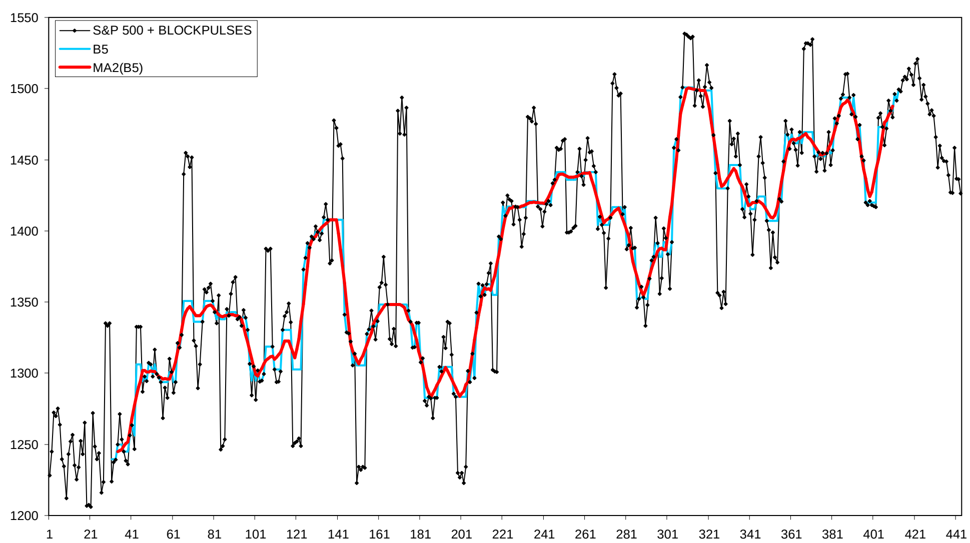

Figure 7.9: B_5 and $MA_2(B_5)$ on Standard and Poor 500 plus blockpulses

Figure 7.4, where M_5 resulted in a small bump, while B_5 produced a smoother output.

To evaluate how the compound LULU smoother B_5 deals with the arbitrarily added blockpulses relative to the mean, the linear regression line of B_5(S & P 500), (B_5 for the original Standard and Poor series), on B_5(S & P 500 + BP), (B_5 for the original Standard and Poor series plus the blockpulses) is fitted. If B_5(S & P 500 + BP) removes the blockpulses successfully, the B_5(S & P 500 + BP) values should be the same as, or very close to the B_5(S & P 500) values, resulting in a regression line with a regression coefficient of one, or very close to one, and an intercept of zero, or very close to zero.

The scatter plot and fitted line for the regression of B_5(S & P 500) on B_5(S & P 500 + BP) are given in Figure

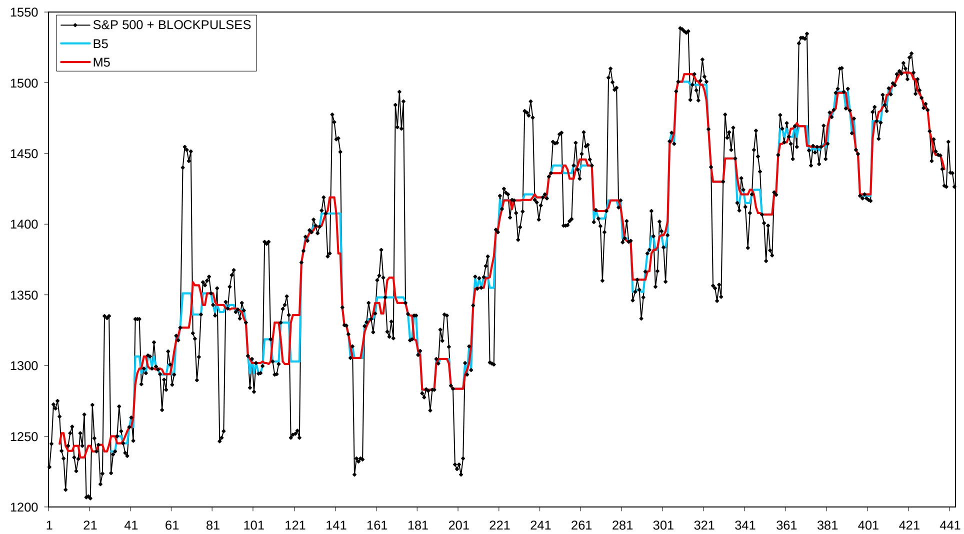

Figure 7.10: B_5 and M_5 smoothers on Standard and Poor 500 plus blockpulses

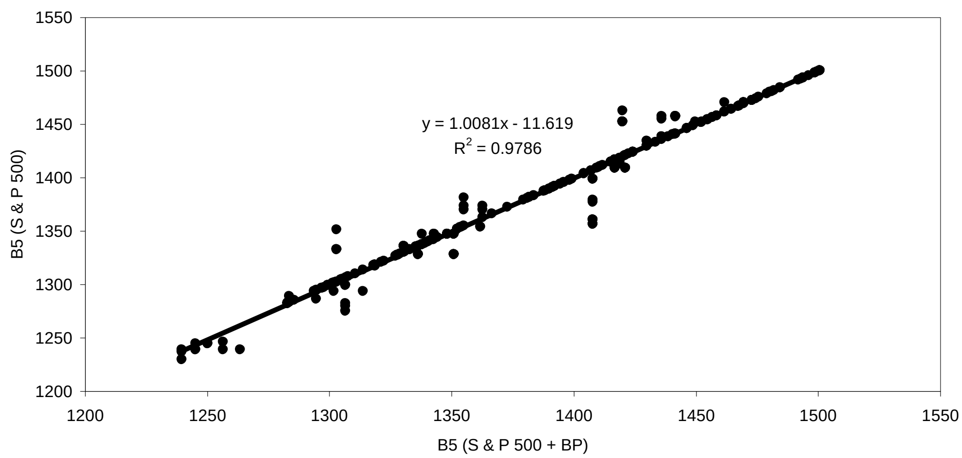

Figure 7.11: Regression of B_5(S & P 500) on B_5(S & P 500 + BP)

7.11. The regression coefficient is 1,0081, the intercept is −11,619 and the coefficient of determination is 97,86%.

The scatter plot and regression line for the median smoother M_5(S & P 500) on M_5(S & P 500 + BP) are given in Figure 7.12. The corresponding results for M_5, the running median with window size 11, are a regression coefficient of 1,0024, an intercept of −3,2617 and a coefficient of determination of 98,6%. The regression coefficient is marginally closer to one, the intercept marginally closer to zero and the coefficient of determination marginally closer to 100% than the corresponding values for the B_5 smoother. The regression lines are compared in Figure 7.14.

For the moving average with window size 11, MA_5, the regression coefficient is 0,9583, the intercept is 54,595 and the coefficient of determination is 94,09%. In Figure 7.13 the scatter plot and fitted regression

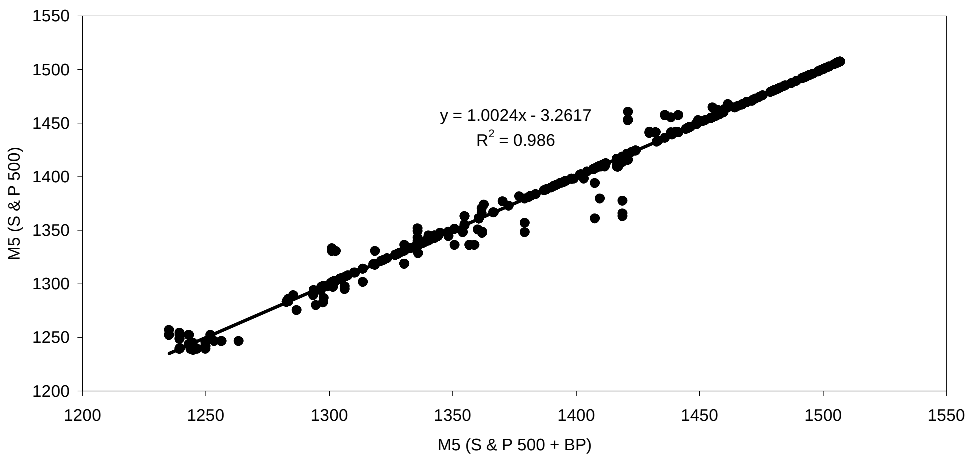

Figure 7.12: Regression of M_5(S & P 500) on M_5(S & P 500 + BP)

line for the moving average, MA_5, are given and in Figure 7.14 all three regression lines are graphed.

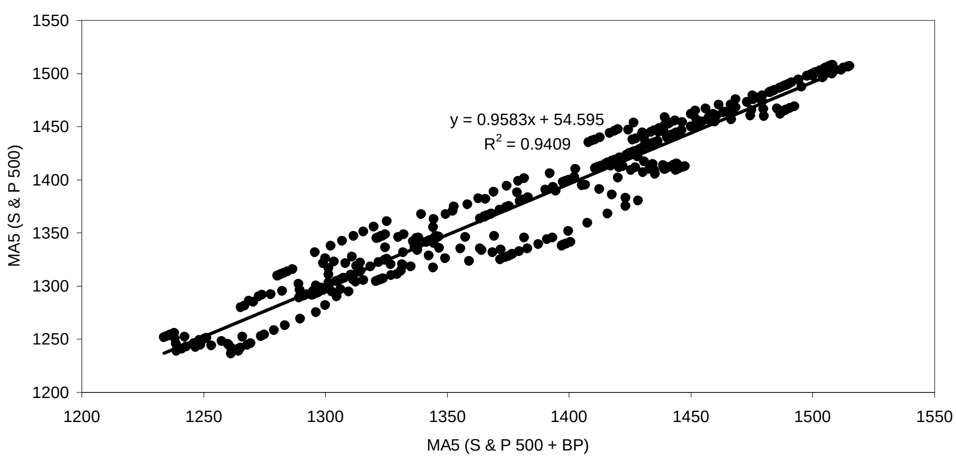

Figure 7.13: Regression of MA_5(S & P 500) on MA_5(S & P 500 + BP)

From Figure 7.14 and the regression analysis it is clear that the behaviour of the compound LULU smoother B_5 and the running median M_5 is very similar. Both perform better than the moving average, MA_5, especially with respect to the intercept, and this is an illustration of the fact that nonlinear smoothers remove the blockpulses better than the linear moving average smoother.

A major advantage of the compound LULU smoother B_5, relative to the running median, however, is its variation decomposition property which provides a monitoring mechanism for the smoothing process. This property is extremely useful from an application point of view.

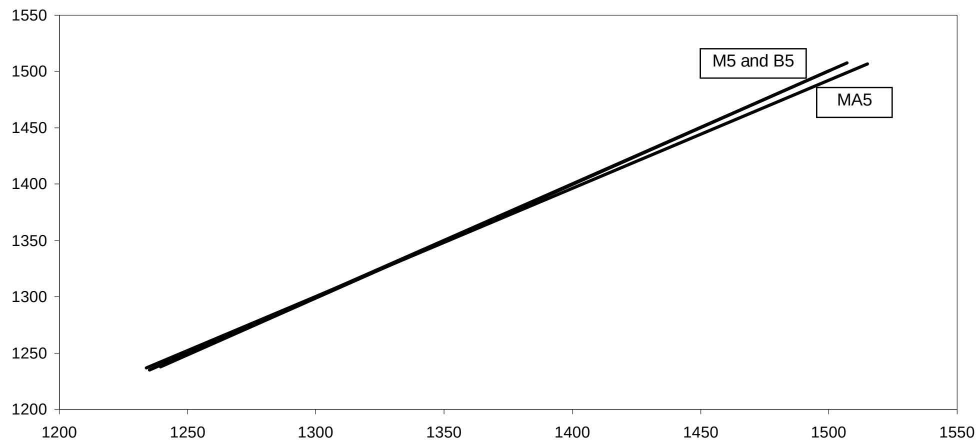

Figure 7.14: Comparison of regression lines

7.2.3 Daily rate, compounded continuously, of Standard and Poor 500 plus blockpulses

Figure 7.15 illustrates the log returns of the Standard and Poor 500 with the blockpulses, i.e. $ln(S_t/S_{t-1})$, where S_t = Standard and Poor 500 with the blockpulses, as well as the output of the compound LULU smoother B_3 and the moving average, MA_2, on B_3.

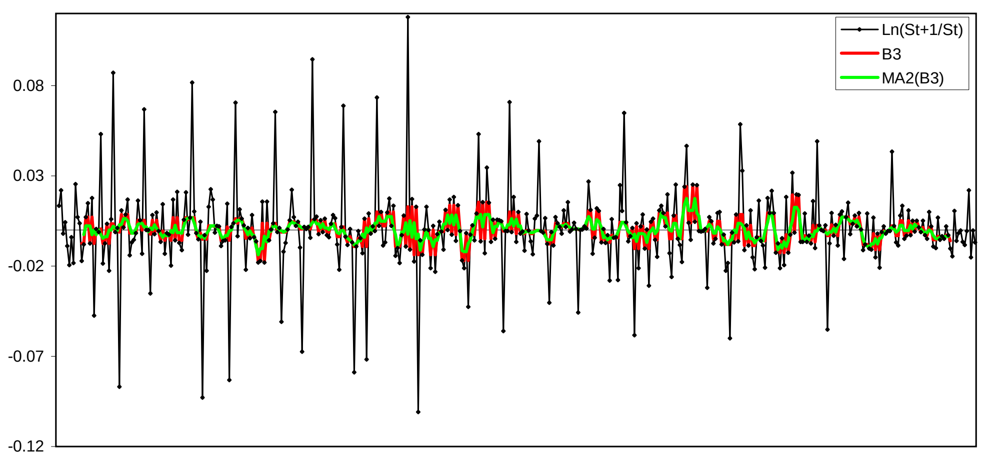

Figure 7.15: $ln(S_t/S_{t-1})$, B_3 and $MA_2(B_3)$

A substantial number of outliers, a blockpulse of length 1 in this case, resulted from the significant increase or decrease at the beginning and end of the blockpulses of the sequence S_t. Since the impulses are of length 1, the greatest portion of the total variation is expected to be removed in the first smoothing step. The reduction of the total variation by B_n is tabulated in Table 7.3 and plotted in Figure 7.16. As expected, the compound LULU smoother B_1 removes most of the total variation in $ln(S_t/S_{t-1})$, namely 59,60%. The smoothing process could stop at this level because a great portion of the total variation has been removed. After the application of B_3, which removed a further 9,17%, 71,57% of the total variation was removed. From Figure 7.15 it is noted that B_3 also preserves some of the variation, which can be considered as well-distributed noise. The output of $MA_2(B_3)$ is smoother than that of B_3 because it removes part of the Gaussian noise.

Table 7.3: Reduction of total variation for B_n on $ln(S_{t+1}/S_t)$

n	$T(x)$	$T(B_n)$	$T(B_{n-1} - B_n)$	$T(B_{n-2} - B_{n-1})$	$T(B_{n-3} - B_{n-2})$	$T(B_{n-4} - B_{n-3})$	$T(B_{n-5} - B_{n-4})$
1	8,270352	3,340934	4,929418	0	0	0	0
%	100%	40,40%	59,60%	0	0	0	0
2	8,270352	3,108993	0,231942	4,929418	0	0	0
%	100%	37,59%	2,80%	59,60%	0	0	0
3	8,270352	2,350839	0,758153	0,231942	4,929418	0	0
%	100%	28,42%	9,17%	2,80%	59,60%	0	0
4	8,270352	2,053127	0,297712	0,758153	0,231942	4,929418	0
%	100%	24,83%	3,60%	9,17%	2,80%	59,60%	0
5	8,270352	1,887798	0,165329	0,297712	0,758153	0,231942	4,929418
%	100%	22,83%	2,00%	3,60%	9,17%	2,80%	59,60%

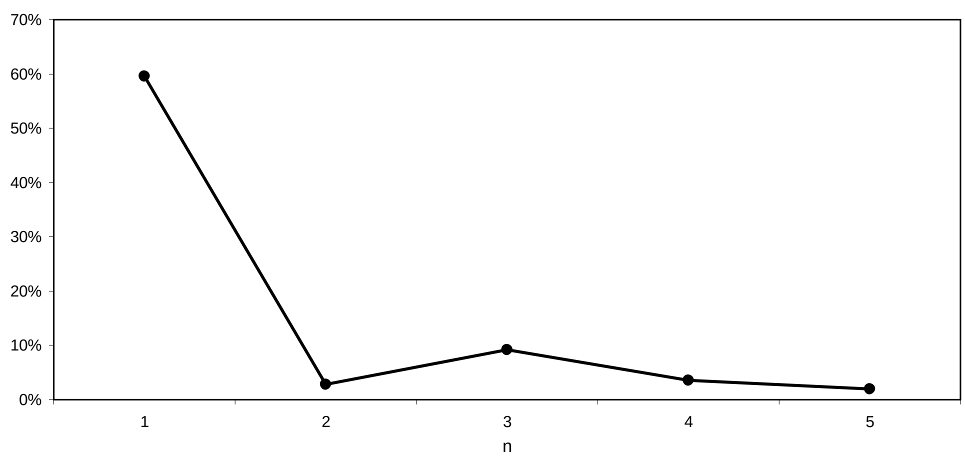

Figure 7.16: Additional percentage variation removed by B_n at each smoothing step for $ln(S_{t+1}/S_t)$

7.3 Medical application

7.3.1 Arterial blood pressure

The smoothing of online monitoring medical data is an important task since important life-related decisions are dependent on the outcome. For example, a patient's heart rate and blood pressure are measured at short time intervals. These data provide important information for medical decision support. The basic goal is to extract the underlying, clinically relevant signal from the observed time series. Short-term fluctuations and outliers caused by, for example, the movement of the patient or measurement errors, should be removed, while sudden level shifts and monotonic trends should be preserved. Medical researchers should know which length of blockpulse of blood pressure can be ignored, or is dangerous to the patient, and hence the smoothing algorithm can be set to correspond to these limits.

Fried *et al.* (2006) analysed two time series representing arterial blood pressure using the repeated median and hybrid smoothers to extract the signal. The first time series has a blockpulse and a step edge as special features, while the second time series has a ramp edge and a few very large outliers. To investigate how the LULU smoothers treat these features, two time series over 300 time units, with similar patterns, were set up. These data sets are given columnwise in Appendix E.

7.3.2 Time series 1 of arterial blood pressure

In the first data set the arterial blood pressure varies between 80 and 120 units with few outliers, then it increases to about 200 units with some significant outliers for about 20 time units. Thereafter it decreases to 80 units whereafter it varies following a downward slope towards a turning point at 20 units. The blood pressure then increases to between 100 and 120 units with a few small and one large outlier. Figure 7.17 shows the data on arterial blood pressure of a patient over time.

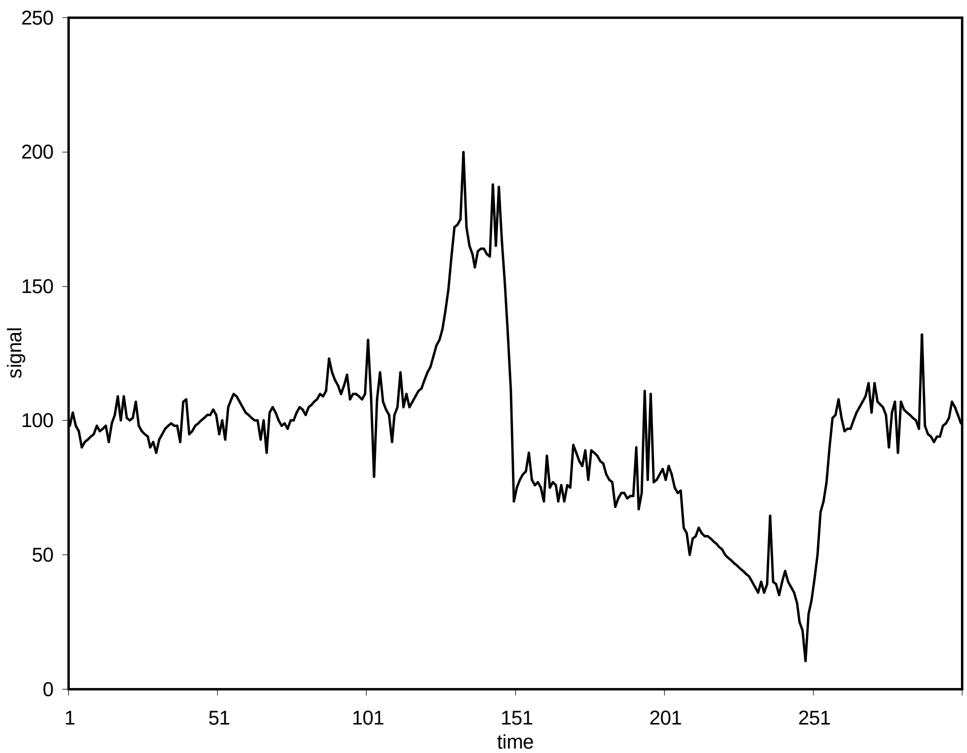

Figure 7.17: Arterial blood pressure for time series 1

The LULU smoothers $F_1, \ldots, F_5$ and $C_1, \ldots, C_5$ and the compound LULU smoothers $B_1, \ldots, B_5$ were applied to this series of arterial blood pressure. The median smoothers, M_1, M_2 and M_5, were also applied to this time series. The step-down end-value rule (cf. Section 2.3.11 (5)) was applied to this example to obtain smoothed values at the ends of the series. If smoothed values of online monitoring data are to be used to support medical decisions, it is important that smoothing starts as soon as possible.

Figure 7.18 shows B_1 and M_1 for time series 1, and Figure 7.19 shows B_5 and M_5 for this series. From both these figures it is seen that the median smoother and the compound LULU smoother give very similar results. All of these smoothers preserve the blockpulse and are not affected by a single impulse or

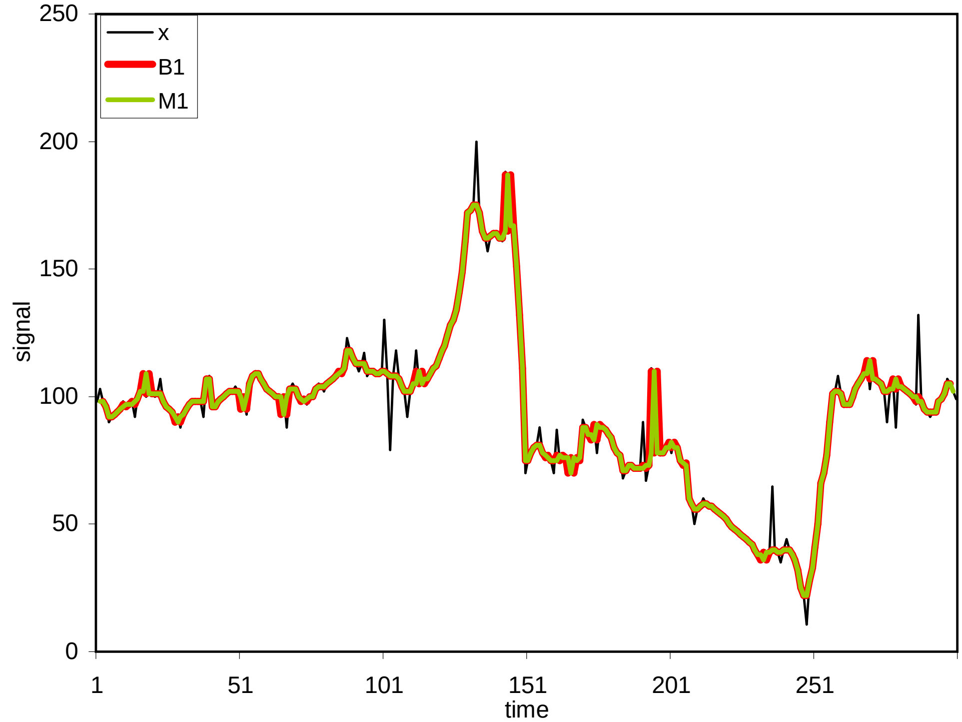

Figure 7.18: B_1 and M_1 smoothers on arterial blood pressure for time series 1

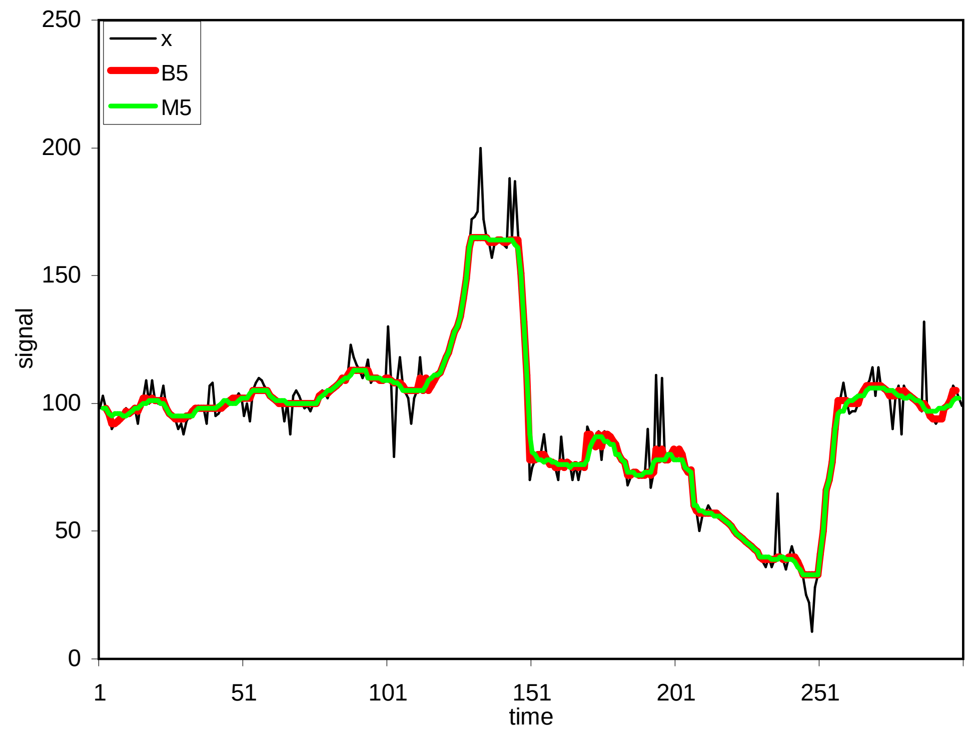

Figure 7.19: B_5 and M_5 smoothers on arterial blood pressure for time series 1

outlier. The advantage that compound LULU smoothers have over median smoothers it that the amount of variation removed at each smoothing level can be calculated. The decomposition of the total variation by B_n, as well as the corresponding percentages at each smoothing level, are tabulated in Table 7.4. The same is calculated for F_n and C_n. The reduction of the total variation at each smoothing level for these smoothers is not tabulated, but the percentage variation removed is shown in Figure 7.20 together with B_n.

Table 7.4: Reduction of total variation for B_n on time series 1

n	$T(x)$	$T(B_n)$	$T(B_{n-1} - B_n)$	$T(B_{n-2} - B_{n-1})$	$T(B_{n-3} - B_{n-2})$	$T(B_{n-4} - B_{n-3})$	$T(B_{n-5} - B_{n-4})$
1	1 432	878	554	0	0	0	0
%	100%	61,31%	38,69%	0	0	0	0
2	1 432	818	60	554	0	0	0
%	100%	57,12%	4,19%	38,69%	0	0	0
3	1 432	496	322	60	554	0	0
%	100%	34,64%	22,49%	4,19%	38,69%	0	0
4	1 432	463	33	322	60	554	0
%	100%	32,33%	2,30%	22,49%	4,19%	38,69%	0
5	1 432	446	17	33	322	60	554
%	100%	31,15%	1,19%	2,30%	22,49%	4,19%	38,69%

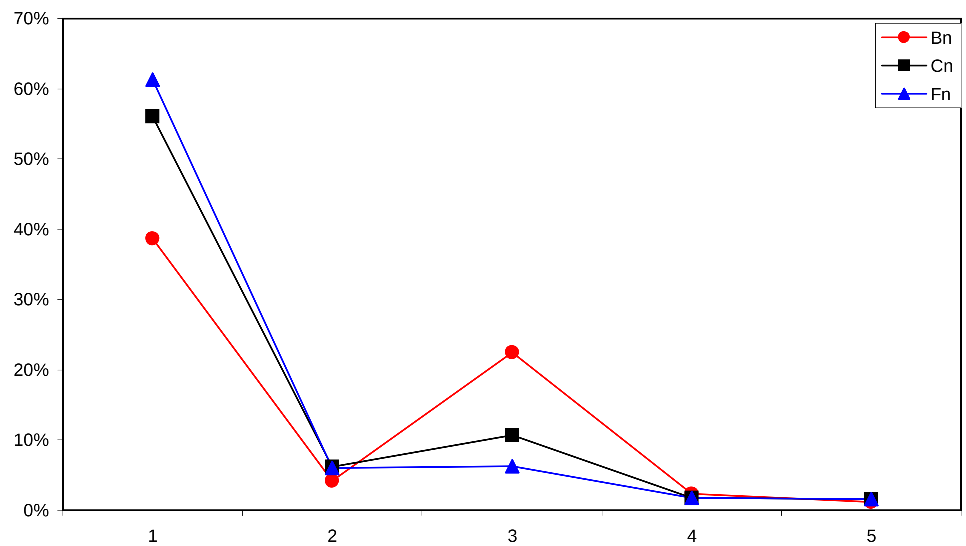

Figure 7.20: Additional percentage variation removed by B_n, C_n and F_n at each smoothing step for time series 1

The total variation of time series 1 graphed in Figure 7.17 is $T(x) = 1\ 432$. From Figure 7.20 it follows that the most variation is removed by the compound LULU smoothers B_1 (38,69%) and B_3 (22,49%). The other compound LULU smoothers remove less than 5% when applied. There is a choice of stopping after B_1 or B_3 has been applied. B_1 would be a good choice if results are needed immediately with no concern for the degree of smoothing, while B_3 will produce a smoother series, removing blockpulses of length three or less.

7.3.3 Time series 2 of arterial blood pressure

For the second data set, Figure 7.21 shows the arterial blood pressure of a patient over time. The observations start with a low blood pressure with little variation around 62, and then increase gradually to about 94 with a few very large outliers (impulses). Thereafter blood pressure gradually drops to 84, followed by the largest outlier of 108. Blood pressure increases to 88 and stays there for a while, whereafter it drops to 78, then it increases to stabilise around 90, with some large outliers when it increases.

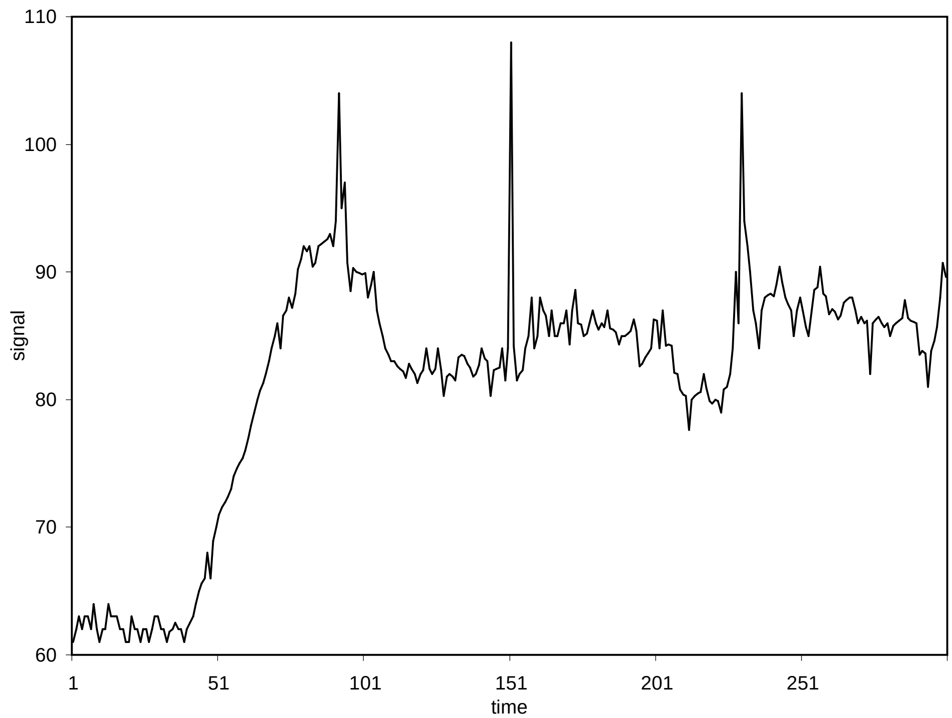

Figure 7.21: Arterial blood pressure for time series 2

The LULU smoothers $F_1, \ldots, F_5$ and $C_1, \ldots, C_5$ were applied to this time series and the compound LULU smoothers $B_1, \ldots, B_5$ were determined from them. The step-down end-value rule (cf. Section 2.3.11 (5)) was again applied to this time series for the same reason as given for time series 1. The median smoothers, M_1, M_2 and M_5, were also applied to this time series.

Figure 7.22 depicts the outcomes of smoothers B_1 and M_1 on time series 2, which give similar results, removing the single impulses and retaining the impulses of length greater than one. From Figure 7.23 where the outcomes of smoothers B_5 and M_5 are shown, it follows that all the impulses are smoothed to form a smoother series than that given in Figure 7.22.

The decompositions of total variation by B_n, F_n and C_n were calculated for this time series. The decomposition of the total variation by B_n, as well as the corresponding percentages at each smoothing level,

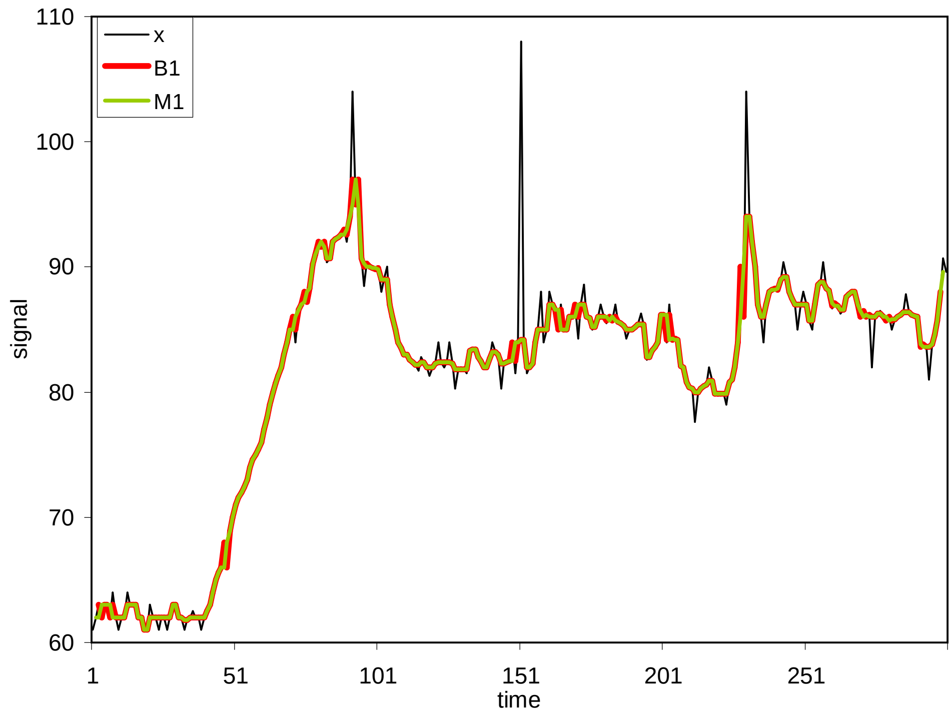

Figure 7.22: B_1 and M_1 smoothers on arterial blood pressure for time series 2

Figure 7.23: B_5 and M_5 smoothers on arterial blood pressure for time series 2

are tabulated in Table 7.5. The reduction of the total variation at each smoothing level for F_n and C_n are not tabulated, but the percentage variation removed is shown in Figure 7.24 together with B_n.

Table 7.5: Reduction of total variation for B_n on time series 2

n	$T(x)$	$T(B_n)$	$T(B_{n-1}-B_n)$	$T(B_{n-2}-B_{n-1})$	$T(B_{n-3}-B_{n-2})$	$T(B_{n-4}-B_{n-3})$	$T(B_{n-5}-B_{n-4})$
1	340,6	164,4	176,2	0	0	0	0
%	100%	48,27%	51,73%	0	0	0	0
2	340,6	144,6	19,8	176,2	0	0	0
%	100%	42,45%	5,81%	51,73%	0	0	0
3	340,6	123,6	21,0	19,8	176,2	0	0
%	100%	36,29%	6,17%	5,81%	51,73%	0	0
4	340,6	109,8	13,8	21,0	19,8	176,2	0
%	100%	32,24%	4,05%	6,17%	5,81%	51,73%	0
5	340,6	105,2	4,6	13,8	21,0	19,8	176,2
%	100%	30,89%	1,35%	4,05%	6,17%	5,81%	51,73%

The total variation of time series 2 graphed in Figure 7.21 is $T(x) = 340{,}6$. This is much less than that of time series 1, since the observations only have a few impulses with no blockpulses. It follows from Figure 7.24 that the most variation is removed by the compound LULU smoother B_1 (51,73%). All the other compound LULU smoothers remove less than 6,20% when applied. For this time series smoothing can be stopped after applying B_1. This differs from the analysis in time series 1, where it could be stopped after applying B_1 or B_3. It is thus clear that a stopping rule is suggested by the nature of the data.

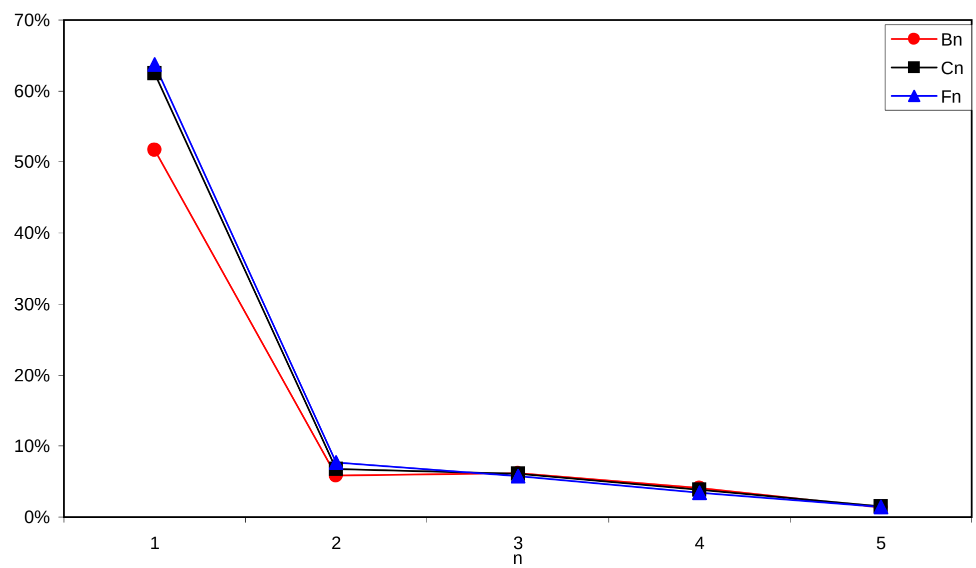

Figure 7.24: Additional percentage variation removed by B_n, C_n and F_n at each smoothing step for time series 2

7.4 Summary

In this chapter the LULU smoothers were applied to financial and medical time series. From these applications it is clear that the LULU smoothers compare well with the median smoothers. The further advantage of the decomposition of the total variation of the compound LULU smoothers makes them even more attractive. Although LULU smoothers do not solve the practical issue of how far to go with variation reduction, they do provide an indication of when there is no further substantial reduction of the total variation. This, along with the practitioner's requirement for *"sufficiently smooth"*, should guide the smoothing process.

Chapter 8

Conclusions

As stated in Chapter 1, one of the purposes of this study was to introduce the class of LULU smoothers to statisticians and to compare their performances with that of other well-known nonlinear smoothers, especially the class of median smoothers. However, during the study, it soon became clear that in order to conduct a meaningful comparison of smoothers, it is in the first place essential to understand smoothing and smoothers in general. This led to the discussion on smoothing and smoothers in general in Chapter 2. That chapter started with a somewhat philosophical discussion of ***the art of smoothing***. The different criteria for evaluating smoothers, as well as concepts related to smoothing, namely effectiveness, consistency, monotonicity, the passes necessary to produce the root of a smoother, stability, dealing with impulsive noise, blockpulses and edges, efficiency, statistical and distributional properties, variation reduction, shape preservation and end-value procedures were defined, discussed and put into perspective. In particular, the concepts of idempotency and co-idempotency were discussed and their usefulness in smoothing were explained. The concepts, "sequence of smoothers" and "compound smoothers", the latter as an extension of the Winsorised smoother, were introduced. The way in which fully trend preserving compound smoothers decompose the total variation of a series was highlighted. This property of the latter smoothers was found to be very informative in the smoothing process and is illustrated in numerous places in the thesis. A number of issues and questions regarding smoothing, listed in Chapter 1, were discussed in general terms in Chapter 2. In Chapters 3, 4, 5 and 6 these issues and questions were then addressed in depth.

An important part of the thesis was to compare the performance of members of the class of LULU smoothers to that of members of the class of median smoothers. For this, an overview of median smoothers and their properties was given in Chapter 3. In particular, monotonicity and the number of passes needed for a median to produce a root, stability, the statistical properties under white and nonwhite noise were given. Furthermore, the edge preservation property of median smoothers was discussed and finally compound median smoothers found in the literature were defined and explained.

The class of LULU smoothers was introduced in Chapter 4. It was not the purpose of that chapter to report on all the details of LULU smoothers found in the literature, but rather to focus on those aspects that are important from a traditional statistical point of view. In this regard, the basic definitions and results from the literature were given. As in Chapter 3, this was done along the lines set out in Chapter 2. Fundamental to LULU smoothers are the smoothers L_n, U_n, L_nU_n and U_nL_n, which are compositions of the extreme selectors, the minimum and the maximum. Of great importance are their monotonicity, idempotency and co-idempotency properties. A further very useful property is that all the LULU smoothers considered in this thesis are fully trend preserving, implying that the total variation of a sequence is the sum of the variation of the smooth sequence plus the variation resulting from the noise removed from the sequence.

The smoothers L_n, U_n, L_nU_n and U_nL_n are used to define the full smoothers. The purpose of the latter is to estimate the unknown signal of a series. In this thesis the Winsorised full smoothers were extended to the so-called compound smoothers, using the smoothers L_nU_n, U_nL_n, C_n and F_n. Some properties of the latter were derived in Chapter 4. The compound smoothers are also fully trend preserving. They have the further advantage that successive smoothing by a compound smoother yields a monotone reduction of the total variation of the series implying a successive trimming of the series that can be monitored by means of the variation that is removed at each window size.

To date, properties of LULU smoothers have only been studied in a deterministic setting and distribution theory based on random sequences has been lacking. In Chapter 4 the exact and asymptotic probability distributions of L_n, U_n, L_nU_n and U_nL_n were derived in the case of independent identically distributed data and for non-identically distributed data. In Chapter 5 these results were used to graph the distributions of the LULU smoothers for different input distributions and to compare them with the distributions of the median smoothers. It is difficult to derive analytically the distributions of LULU smoothers in general and in this regard substantial research still has to be done.

In Chapter 5 the performances of LULU smoothers were compared with those of the class of median smoothers with respect to monotonicity, idempotency, co-idempotency, stability, edge preservation, probability distributions and variation decomposition. Some of these properties were illustrated with simple examples. With regard to monotonicity, stability and edge preservation, LULU smoothers and median smoothers generally behave very similarly. With respect to idempotency and co-idempotency, the LULU smoothers have an advantage in the sense that they produce the root in one pass, whereas the median smoothers need a substantial number of passes before a root is produced. A further useful property of the LULU smoothers, which the median smoothers do not share, is the variation decomposition property, which was discussed on numerous occasions in this thesis.

The comparison of LULU smoothers with nonlinear smoothers based on the median was continued in Chapter 6. A comparison was made of their respective abilities for signal recovery by means of simulation. The success of the smoothers in recovering the signal was measured by the integrated mean square error and the regression coefficient calculated from the least squares regression of the smoothed sequence on the signal. The first series that was studied was formed by constructing a sinusoidal curve and adding different forms of noise to it. In addition to this, LULU smoothers were applied to a number of other interesting examples of series found in the literature. The sequences smoothed by the LULU and median smoothers were further smoothed by a linear smoother. From the studies in this chapter it was found that the compound LULU smoother B_n is a worthy competitor for the median smoothers. In fact, in some cases, especially when blockpulses are present, the compound LULU smoother performed better in the recovery of a sinusoidal signal. For the special functions considered, B_n performed well relative to the median smoothers. The application of an unsophisticated linear smoother, after the nonlinear smoother B_n has been applied, also seems to be a procedure with merit.

In Chapter 7 LULU smoothers were applied practically. Two types of applications were chosen, namely a financial one and a medical one. For the financial application, the closing prices of the Standard and Poor 500 Index for a certain time period were analysed. This series, with blockpulses of different lengths added, as well as the daily growth rate compounded continuously, was also analysed. The compound

LULU and median smoothers were applied to these time series. The decomposition of total variation calculated for each smoothing level of B_n, provided valuable information on each time series. Two arterial blood pressure time series with different patterns, similar to data analysed by Fried *et al.* (2006), were set up to use as a medical application. The compound LULU smoothers managed to extract the underlying signal and remove short impulses in both time series very successfully.

To conclude, the performance of the class of LULU smoothers was studied and compared with that of the class of median smoothers with respect to aspects that are important from a statistical point of view when a time series is smoothed. However, a number of unanswered questions still exist, which can be fruitfully investigated in further research. Some of these are the following:

- Further distribution theory — the probability distributions of some LULU smoothers in general still have to be derived.
- The behaviour of LULU smoothers under dependent data.
- LULU smoothers have been defined in higher dimensions and applied to image processing problems — their statistical properties, however, have not been studied.
- A stopping rule to determine the point where smoothing should stop needs to be defined.

With respect to a stopping rule, the compound LULU smoothers provide a procedure whereby the variation that is removed at each level can be calculated, while a certain part of the original series is preserved. The question is still to what extent "*well-behaved noise*" plus signal is preserved and "*non-well-behaved*" noise has been adequately removed. As soon as large reductions in variation stop, it could be argued that the non-well-behaved noise has been adequately removed. Although the terms *large reductions* and *adequately removed* remain vague, the above does give a procedure that can be explored further.

Finally, from the studies in this thesis, it is clear that the LULU smoothers are worthy competitors to the well-known nonlinear smoothers found in the literature. Their sound underlying theory and exceptional mathematical properties add to their attractiveness. It was shown that they can be applied successfully as a practical smoothing procedure in traditional statistical smoothing problems and are also computationally efficient. It is hoped that this thesis will stimulate further research on the statistical aspects of LULU smoothers and that they will soon become part of statisticians' toolkit of smoothing procedures.

Appendix A

Tables of simulation results

Table A.1: Simulation results for B1

Window size	**n=1**															
Frequency	**0,0625**	**0,1250**	**0,1875**	**0,2500**	**0,3125**	**0,3750**	**0,4375**	**0,5000**	**0,5625**	**0,6250**	**0,6875**	**0,7500**	**0,8125**	**0,8750**	**0,9375**	**1,000**
Normal signal																
B_1(nonlinear)																
$\hat{E}(\beta)$	0.99796	1.00101	1.0012	0.99842	0.99386	0.99605	0.995	0.99181	0.99113	0.98702	0.98588	0.98348	0.98235	0.97773	0.97382	0.97342
$Var(\beta)$	0.00072	0.00042	0.00054	0.00046	0.00043	0.00048	0.00051	0.0004	0.00049	0.0004	0.0004	0.00039	0.00049	0.00057	0.00051	0.00054
$\hat{E}(\beta^*)$	0.01328	-0.01938	-0.01706	0.0121	0.03998	0.02111	0.02938	0.06162	0.0613	0.08383	0.10053	0.11893	0.11841	0.15334	0.17403	0.1798
$Var(\beta^*)$	0.04214	0.02673	0.03824	0.03113	0.0329	0.02882	0.03624	0.02765	0.02931	0.03058	0.02786	0.02695	0.03189	0.0367	0.03704	0.0354
E(IMS)	0.88995	0.87224	0.91158	0.91614	0.9505	0.95323	1.00109	1.00518	1.04029	1.0509	1.07644	1.0913	1.11423	1.11841	1.1385	1.14768
Var(IMS)	0.01936	0.01616	0.01791	0.01432	0.01496	0.01079	0.01638	0.0119	0.01517	0.01246	0.01651	0.01316	0.01216	0.01293	0.01217	0.01273
Mean % total var removed	58.00%	56.59%	54.61%	51.72%	49.61%	46.83%	44.04%	42.03%	40.55%	38.84%	37.27%	36.22%	35.06%	35.73%	35.40%	35.07%
MA_2(linear)																
$\hat{E}(y)$	0.99809	0.9997	0.99768	0.99266	0.98504	0.98411	0.97934	0.97007	0.96245	0.95273	0.94545	0.93747	0.93109	0.91997	0.90663	0.89812
$Var(y)$	0.00072	0.00043	0.00057	0.00047	0.00045	0.00049	0.00055	0.00041	0.00048	0.00037	0.00036	0.00036	0.00044	0.00053	0.00045	0.00047
$\hat{E}(y^*)$	0.0122	-0.01095	0.00545	0.04837	0.10075	0.10602	0.14186	0.21377	0.25826	0.31969	0.38334	0.44657	0.48638	0.56646	0.64354	0.69822
$Var(y^*)$	0.04276	0.02679	0.04003	0.03141	0.03538	0.02964	0.03874	0.0268	0.02823	0.03028	0.02537	0.02509	0.03004	0.03466	0.03394	0.0323
E(IMS)	0.53629	0.5217	0.54148	0.542	0.5621	0.56476	0.60658	0.61568	0.65446	0.6976	0.75768	0.81685	0.86571	0.93034	1.01179	1.08445
Var(IMS)	0.00931	0.00768	0.00903	0.00742	0.00859	0.00724	0.0106	0.0086	0.00811	0.00758	0.00777	0.00656	0.00567	0.00575	0.00597	0.00508
Impulse (size 3)																
B_1(nonlinear)																
$\hat{E}(\beta)$	0.98592	1.00685	0.97891	1.01458	0.97428	1.0164	0.96726	1.01153	0.96907	0.99691	0.97782	0.98415	0.98303	0.97234	0.98654	0.95829
$Var(\beta)$	0.00063	0.00046	0.00046	0.00046	0.0004	0.00052	0.00041	0.00041	0.00037	0.00043	0.00052	0.0005	0.00048	0.00051	0.00053	0.00057
$\hat{E}(\beta^*)$	0.24635	0.10925	0.29764	0.05437	0.33393	0.04512	0.38391	0.07169	0.37058	0.17223	0.3006	0.26464	0.27032	0.32722	0.25026	0.44153
$Var(\beta^*)$	0.04092	0.03111	0.03416	0.03237	0.03208	0.03597	0.02894	0.02768	0.02576	0.03	0.03456	0.03364	0.03216	0.03279	0.03755	0.03347
E(IMS)	1.49844	1.4961	1.49787	1.51027	1.52731	1.55502	1.55657	1.55981	1.56685	1.58807	1.59601	1.60062	1.60704	1.63227	1.64146	1.65941
Var(IMS)	0.01437	0.01348	0.01406	0.01515	0.01387	0.01467	0.01835	0.01526	0.01246	0.01274	0.01545	0.01645	0.01276	0.01433	0.01498	0.01416
Mean % total var removed	54.71%	53.54%	50.77%	49.45%	46.97%	45.05%	43.16%	39.45%	39.12%	37.26%	37.84%	34.63%	34.09%	35.51%	33.94%	35.20%
MA_2(linear)																
$\hat{E}(y)$	0.98543	1.00697	0.97456	1.00935	0.9645	1.00422	0.9507	0.9895	0.94153	0.96079	0.93847	0.93851	0.93206	0.91646	0.91641	0.88819
$Var(y)$	0.00067	0.0005	0.00047	0.00046	0.0004	0.00055	0.00043	0.00042	0.00037	0.00041	0.0005	0.00048	0.00045	0.00048	0.00047	0.00048
$\hat{E}(y^*)$	0.25381	0.10906	0.32942	0.09044	0.4055	0.13361	0.50443	0.23203	0.56318	0.42446	0.57766	0.5934	0.64092	0.72962	0.74188	0.92602
$Var(y^*)$	0.04283	0.03436	0.03474	0.03256	0.03186	0.03775	0.03027	0.02856	0.02496	0.02889	0.03366	0.03279	0.03053	0.03036	0.03439	0.02973
E(IMS)	1.02284	1.01982	1.02801	1.01309	1.06113	1.02048	1.1003	1.00945	1.13356	1.0701	1.16928	1.17298	1.2027	1.30025	1.27929	1.45654
Var(IMS)	0.00933	0.00824	0.01119	0.01023	0.00955	0.00967	0.01161	0.01056	0.0093	0.00997	0.01095	0.01088	0.00773	0.00757	0.00956	0.00833

Table A.2: Simulation results for B2

Window size	**n=2**															
Frequency	**0,0625**	**0,1250**	**0,1875**	**0,2500**	**0,3125**	**0,3750**	**0,4375**	**0,5000**	**0,5625**	**0,6250**	**0,6875**	**0,7500**	**0,8125**	**0,8750**	**0,9375**	**1,000**
Normal signal																
B_2(nonlinear)																
$\hat{E}(\beta)$	0.99721	0.99992	0.99946	0.99519	0.98917	0.99127	0.98955	0.98425	0.98029	0.9737	0.96801	0.96249	0.96007	0.95344	0.94562	0.93838
$Var(\beta)$	0.00072	0.0004	0.00059	0.00047	0.00048	0.0005	0.00062	0.00051	0.00059	0.00045	0.0004	0.00043	0.00057	0.00066	0.00061	0.00061
$\hat{E}(\beta^*)$	0.01909	-0.00791	-0.00896	0.0307	0.0712	0.05414	0.07006	0.11464	0.13304	0.16962	0.22218	0.26752	0.28042	0.33863	0.38099	0.42487
$Var(\beta^*)$	0.04535	0.02682	0.03961	0.03022	0.03655	0.02909	0.04071	0.03099	0.03502	0.03613	0.02681	0.02904	0.03376	0.04249	0.04609	0.04117
E(IMS)	0.76763	0.77896	0.81872	0.84366	0.88726	0.89798	0.94257	0.96509	1.00273	1.01216	1.04434	1.06702	1.09262	1.1162	1.15751	1.1833
Var(IMS)	0.01315	0.01195	0.01197	0.00963	0.01025	0.00874	0.00977	0.01063	0.01443	0.01034	0.01208	0.00893	0.01036	0.00928	0.01102	0.01099
Mean % total var removed	62.98%	60.92%	58.67%	54.81%	52.52%	49.61%	47.20%	45.13%	44.50%	43.72%	42.79%	42.42%	42.63%	43.95%	44.82%	45.65%
MA_2(linear)																
$\hat{E}(y)$	0.99733	0.99888	0.99618	0.98937	0.97984	0.97895	0.97326	0.96374	0.95515	0.94246	0.92936	0.91782	0.90929	0.89699	0.88467	0.8738
$Var(y)$	0.00081	0.00046	0.00062	0.0005	0.00049	0.00052	0.00062	0.00054	0.00059	0.00045	0.00038	0.0004	0.00056	0.00063	0.00057	0.00059
$\hat{E}(y^*)$	0.01844	-0.00106	0.01096	0.06601	0.13195	0.13515	0.1824	0.25729	0.30621	0.37964	0.48197	0.56519	0.62326	0.72854	0.80684	0.88129
$Var(y^*)$	0.04955	0.0299	0.04039	0.03221	0.03732	0.03027	0.04204	0.03272	0.03585	0.03724	0.02693	0.02815	0.03452	0.04087	0.04394	0.03998
E(IMS)	0.48613	0.48483	0.50412	0.51436	0.54334	0.5578	0.60695	0.63531	0.6853	0.74773	0.82283	0.89723	0.97056	1.0548	1.15818	1.24642
Var(IMS)	0.00667	0.00596	0.006	0.00551	0.00647	0.0064	0.00798	0.00749	0.00794	0.00729	0.00701	0.00624	0.00654	0.00511	0.00608	0.00619
Impulse (size 3)																
B_2(nonlinear)																
$\hat{E}(\beta)$	0.98532	1.00818	0.97545	1.01239	0.9668	1.01283	0.95604	1.00688	0.95696	0.98334	0.96116	0.96579	0.96377	0.94792	0.95882	0.92393
$Var(\beta)$	0.00067	0.00052	0.00048	0.00047	0.00043	0.00059	0.00049	0.0005	0.00046	0.00045	0.00057	0.00055	0.00053	0.00062	0.00068	0.0006
$\hat{E}(\beta^*)$	0.25464	0.09605	0.32335	0.06628	0.39012	0.06829	0.46917	0.10656	0.46071	0.26244	0.40798	0.38742	0.41182	0.51049	0.44374	0.69703
$Var(\beta^*)$	0.04302	0.03758	0.03635	0.03231	0.03593	0.04	0.03673	0.03243	0.03076	0.02911	0.03711	0.03752	0.03669	0.0384	0.04453	0.03706
E(IMS)	1.40476	1.40123	1.41863	1.42557	1.45622	1.48799	1.50307	1.48621	1.50933	1.49866	1.52162	1.51935	1.53686	1.58357	1.57487	1.65696
Var(IMS)	0.01961	0.02567	0.02584	0.03023	0.01958	0.0205	0.02324	0.02563	0.01728	0.01953	0.02695	0.02141	0.01736	0.01625	0.0194	0.01834
Mean % total var removed	59.69%	58.38%	54.85%	52.89%	49.91%	48.01%	47.05%	42.99%	43.98%	42.30%	43.45%	41.18%	41.95%	44.20%	43.46%	46.44%
MA_2(linear)																
$\hat{E}(y)$	0.9844	1.00816	0.97073	1.00664	0.95693	0.99939	0.93918	0.98581	0.93226	0.95102	0.92239	0.9204	0.91179	0.89353	0.89502	0.86415
$Var(y)$	0.00078	0.00056	0.00051	0.0005	0.00044	0.00061	0.00051	0.00053	0.0005	0.00045	0.00057	0.00053	0.00052	0.00058	0.00064	0.00054
$\hat{E}(y^*)$	0.26365	0.09875	0.35729	0.1043	0.45908	0.161	0.59034	0.25562	0.63562	0.4846	0.67104	0.69345	0.76685	0.88904	0.8915	1.12261
$Var(y^*)$	0.04809	0.03938	0.03817	0.03487	0.03762	0.04107	0.0387	0.03325	0.03283	0.02953	0.03727	0.03753	0.03514	0.03656	0.041	0.03397
E(IMS)	0.98437	0.97852	0.99866	0.97228	1.04518	0.99204	1.11709	0.98824	1.15976	1.05765	1.18782	1.19111	1.24597	1.38764	1.36182	1.5904
Var(IMS)	0.01279	0.01452	0.01848	0.01892	0.0148	0.01309	0.01689	0.01592	0.01347	0.01339	0.01744	0.01296	0.01017	0.0089	0.0104	0.0104

Table A.3: Simulation results for B3

Window size	**n=3**															
Frequency	**0,0625**	**0,1250**	**0,1875**	**0,2500**	**0,3125**	**0,3750**	**0,4375**	**0,5000**	**0,5625**	**0,6250**	**0,6875**	**0,7500**	**0,8125**	**0,8750**	**0,9375**	**1,000**
Normal signal																
B_3(nonlinear)																
$\hat{E}(\beta)$	0.99729	0.9985	0.99623	0.99043	0.98107	0.98075	0.97695	0.96994	0.96209	0.95757	0.94418	0.9298	0.92127	0.90472	0.88933	0.87506
$Var(\beta)$	0.001	0.00058	0.00069	0.00054	0.00054	0.00053	0.00069	0.00062	0.0007	0.00068	0.00056	0.00065	0.00073	0.00081	0.00095	0.00077
$\hat{E}(\beta^*)$	0.02021	0.00316	0.01281	0.05835	0.12706	0.11935	0.15471	0.20802	0.25927	0.27896	0.39073	0.48587	0.53314	0.65356	0.74561	0.85309
$Var(\beta^*)$	0.05736	0.03668	0.0441	0.03378	0.03873	0.03324	0.0459	0.03932	0.04364	0.05051	0.03426	0.0444	0.04992	0.05455	0.06335	0.05691
E(IMS)	0.65045	0.69199	0.73453	0.77495	0.83028	0.85003	0.89433	0.92066	0.95642	0.99923	1.04602	1.10088	1.15296	1.22518	1.30861	1.3992
Var(IMS)	0.00594	0.00528	0.00358	0.00373	0.00489	0.00428	0.00618	0.00586	0.00758	0.00684	0.00753	0.00635	0.00904	0.00698	0.00716	0.00864
Mean % total var removed	69.52%	65.97%	63.13%	59.28%	57.36%	54.40%	53.15%	52.42%	52.24%	52.21%	52.10%	54.14%	55.89%	58.19%	61.03%	63.88%
MA_2(linear)																
$\hat{E}(y)$	0.9972	0.99664	0.9928	0.98346	0.97014	0.96588	0.95739	0.94566	0.93279	0.92325	0.9053	0.88801	0.87443	0.85257	0.83294	0.81619
$Var(y)$	0.00113	0.00065	0.00071	0.00054	0.00055	0.00053	0.00067	0.00059	0.00064	0.00068	0.00055	0.00063	0.00067	0.00071	0.00084	0.0007
$\hat{E}(y^*)$	0.02162	0.01569	0.03534	0.10366	0.19895	0.21741	0.28413	0.37035	0.46274	0.52189	0.66831	0.78785	0.87222	1.02588	1.14502	1.26257
$Var(y^*)$	0.06348	0.04068	0.04564	0.03379	0.0397	0.03368	0.04497	0.03829	0.04145	0.04904	0.03337	0.04282	0.04674	0.0494	0.05818	0.0526
E(IMS)	0.43241	0.44583	0.46812	0.49167	0.53854	0.56232	0.62962	0.68219	0.75324	0.84833	0.94468	1.0587	1.15698	1.28791	1.4196	1.54359
Var(IMS)	0.00473	0.0041	0.00352	0.00345	0.00521	0.00503	0.00706	0.00673	0.00802	0.00715	0.00698	0.00644	0.00749	0.00586	0.00543	0.0058
Impulse (size 3)																
B_3(nonlinear)																
$\hat{E}(\beta)$	0.99618	1.00287	0.98452	0.9951	0.97342	0.9838	0.95383	0.97602	0.94427	0.95827	0.93911	0.9325	0.92118	0.89912	0.89465	0.86836
$Var(\beta)$	0.00109	0.0006	0.00054	0.00051	0.00051	0.00071	0.0006	0.00065	0.00066	0.00062	0.00074	0.00075	0.00084	0.00076	0.00095	0.00089
$\hat{E}(\beta^*)$	0.06115	0.02754	0.16062	0.07755	0.24546	0.1419	0.40674	0.19636	0.50401	0.33513	0.51645	0.54154	0.61725	0.79156	0.78088	1.03908
$Var(\beta^*)$	0.06478	0.04215	0.04219	0.03722	0.04467	0.04843	0.0494	0.03558	0.04295	0.03869	0.04681	0.05192	0.05316	0.05281	0.05999	0.06514
E(IMS)	0.717	0.77047	0.83388	0.85213	0.92182	0.91501	1.03938	0.96321	1.19162	1.04102	1.22331	1.19849	1.25902	1.39311	1.33781	1.58169
Var(IMS)	0.02539	0.02176	0.0314	0.02623	0.02857	0.02568	0.0232	0.02242	0.00991	0.01768	0.02019	0.01737	0.01432	0.0195	0.01353	0.01737
Mean % total var removed	70.72%	67.77%	64.12%	61.21%	58.93%	56.81%	56.17%	54.31%	54.31%	54.00%	55.13%	55.03%	56.64%	60.57%	61.73%	65.37%
MA_2(linear)																
$\hat{E}(y)$	0.99647	1.00193	0.9805	0.98827	0.96266	0.96874	0.93509	0.95112	0.91607	0.92361	0.90049	0.89037	0.87479	0.84854	0.83649	0.81123
$Var(y)$	0.00121	0.00062	0.00058	0.00053	0.00054	0.0007	0.00059	0.00061	0.00062	0.00062	0.0007	0.0007	0.00078	0.00069	0.00084	0.00077
$\hat{E}(y^*)$	0.06051	0.03422	0.18814	0.12417	0.3162	0.24147	0.53253	0.36362	0.70303	0.58032	0.79372	0.84832	0.95503	1.15447	1.19121	1.43988
$Var(y^*)$	0.07106	0.04392	0.04359	0.03864	0.04639	0.04898	0.04855	0.03444	0.04195	0.03758	0.04454	0.04904	0.05043	0.04845	0.05434	0.05886
E(IMS)	0.49806	0.51939	0.57135	0.55532	0.65099	0.60855	0.60855	0.70005	0.99759	0.86352	1.08838	1.11892	1.22613	1.42555	1.42785	1.68371
Var(IMS)	0.01839	0.01558	0.02541	0.01957	0.02656	0.0192	0.02226	0.01507	0.0102	0.01283	0.01516	0.01289	0.0082	0.0123	0.00951	0.01019

Table A.4: Simulation results for B4

Window size	n=4															
Frequency	**0,0625**	**0,1250**	**0,1875**	**0,2500**	**0,3125**	**0,3750**	**0,4375**	**0,5000**	**0,5625**	**0,6250**	**0,6875**	**0,7500**	**0,8125**	**0,8750**	**0,9375**	**1,000**
Normal signal																
B_4(nonlinear)																
$\hat{E}(\beta)$	0.99796	0.99576	0.99431	0.98596	0.97128	0.96679	0.95665	0.94341	0.92854	0.91382	0.8921	0.87023	0.84847	0.82759	0.80561	0.78641
$Var(\beta)$	0.00187	0.00086	0.00085	0.00067	0.00067	0.00065	0.00078	0.00069	0.00083	0.00088	0.00079	0.00097	0.00125	0.0011	0.00129	0.00145
$\hat{E}(\beta^*)$	0.01914	0.02375	0.02706	0.09443	0.20182	0.22187	0.30118	0.38959	0.49043	0.58196	0.75832	0.89615	1.05043	1.20339	1.34163	1.47108
$Var(\beta^*)$	0.10221	0.04967	0.05367	0.03874	0.04466	0.03934	0.04977	0.04881	0.05804	0.06169	0.05284	0.06227	0.08203	0.07982	0.0894	0.10911
E(IMS)	0.61451	0.66608	0.71763	0.76204	0.81843	0.84231	0.89767	0.93842	0.9967	1.07492	1.16269	1.27442	1.39611	1.52784	1.66231	1.76046
Var(IMS)	0.00506	0.00468	0.0035	0.00383	0.00516	0.00482	0.00665	0.00542	0.00925	0.00731	0.00822	0.00709	0.00874	0.00604	0.00602	0.00425
Mean % total var removed	71.83%	67.65%	64.58%	60.67%	59.28%	56.81%	56.79%	57.09%	58.14%	59.65%	61.64%	65.19%	68.67%	72.60%	77.64%	79.80%
MA_2(linear)																
$\hat{E}(y)$	0.9976	0.99342	0.98998	0.97808	0.96038	0.95112	0.93443	0.91534	0.8953	0.87683	0.8534	0.8303	0.80687	0.78561	0.76333	0.74448
$Var(y)$	0.00219	0.00091	0.00091	0.00071	0.00068	0.00067	0.00076	0.00068	0.00079	0.00082	0.00074	0.0009	0.00127	0.0011	0.00125	0.00138
$\hat{E}(y^*)$	0.02128	0.04028	0.05801	0.15287	0.28439	0.33826	0.46195	0.58699	0.72108	0.83014	1.01168	1.1506	1.31127	1.4635	1.60795	1.73918
$Var(y^*)$	0.11602	0.05112	0.05588	0.03962	0.04499	0.04065	0.04917	0.04776	0.05576	0.06052	0.05321	0.06058	0.08647	0.08524	0.09297	0.1121
E(IMS)	0.41335	0.4323	0.46098	0.49159	0.54821	0.58611	0.67765	0.76177	0.86685	1.00522	1.14462	1.30062	1.45607	1.61605	1.7639	1.8715
Var(IMS)	0.00434	0.00384	0.00365	0.00395	0.00575	0.00574	0.00842	0.00821	0.01074	0.0084	0.0086	0.00728	0.00772	0.00531	0.00499	0.00318
Impulse (size 3)																
B_4(nonlinear)																
$\hat{E}(\beta)$	0.99639	1.0033	0.98231	0.99115	0.96222	0.96853	0.93328	0.9473	0.90661	0.9177	0.88226	0.87196	0.85244	0.82604	0.80701	0.78577
$Var(\beta)$	0.00184	0.00084	0.00072	0.0007	0.00066	0.00083	0.0007	0.00077	0.00078	0.00089	0.00088	0.001	0.00142	0.00108	0.00149	0.00141
$\hat{E}(\beta^*)$	0.06013	0.03127	0.17671	0.10911	0.33168	0.24386	0.56114	0.37838	0.78806	0.61337	0.94144	0.98981	1.12221	1.35299	1.40341	1.65273
$Var(\beta^*)$	0.10528	0.05584	0.04731	0.0455	0.05326	0.05414	0.05132	0.04676	0.05529	0.05483	0.05216	0.06875	0.08288	0.07734	0.08503	0.0912
E(IMS)	0.66048	0.7428	0.79485	0.82052	0.90777	0.89866	1.05336	0.95654	1.25918	1.09688	1.34557	1.35591	1.47527	1.67656	1.66776	1.87815
Var(IMS)	0.01975	0.02268	0.01647	0.01866	0.02563	0.02593	0.01858	0.0165	0.00995	0.01614	0.01776	0.01379	0.01258	0.01267	0.00789	0.00908
Mean % total var removed	73.08%	69.91%	65.82%	62.66%	61.01%	59.83%	59.79%	59.05%	60.49%	61.17%	63.78%	65.74%	68.57%	73.96%	77.23%	79.40%
MA_2(linear)																
$\hat{E}(y)$	0.99662	1.00251	0.97815	0.98429	0.9511	0.95288	0.91232	0.91875	0.87358	0.87958	0.84298	0.83109	0.81028	0.78409	0.76314	0.74328
$Var(y)$	0.00224	0.00092	0.00076	0.00076	0.0007	0.00085	0.00068	0.00078	0.00073	0.00085	0.00082	0.00099	0.00134	0.00111	0.00143	0.00132
$\hat{E}(y^*)$	0.05935	0.03872	0.20876	0.16295	0.41732	0.35976	0.7143	0.57997	1.01998	0.87008	1.20452	1.25258	1.38921	1.61833	1.68026	1.93011
$Var(y^*)$	0.12479	0.05898	0.0491	0.04756	0.05532	0.05449	0.04869	0.047	0.05354	0.05399	0.0517	0.07106	0.08002	0.07933	0.08584	0.08864
E(IMS)	0.46193	0.50509	0.54727	0.53776	0.66233	0.62578	0.86422	0.75876	1.13299	1.00191	1.30064	1.35762	1.50782	1.73705	1.7636	1.96416
Var(IMS)	0.01491	0.01613	0.01404	0.01385	0.02566	0.01936	0.01929	0.01339	0.01086	0.01334	0.01564	0.01179	0.0101	0.00848	0.00561	0.00593

Table A.5: Simulation results for B5

Window size	**n=5**															
Frequency	**0,0625**	**0,1250**	**0,1875**	**0,2500**	**0,3125**	**0,3750**	**0,4375**	**0,5000**	**0,5625**	**0,6250**	**0,6875**	**0,7500**	**0,8125**	**0,8750**	**0,9375**	**1,000**
Normal signal																
B_5(nonlinear)																
$\hat{E}(\beta)$	1.00045	0.99072	0.98609	0.97504	0.95457	0.94869	0.93172	0.90902	0.88246	0.84898	0.81407	0.77255	0.73509	0.72643	0.72758	0.72164
Var(β)	0.00497	0.00128	0.00113	0.00093	0.00095	0.0011	0.00137	0.00148	0.00157	0.00174	0.00168	0.00172	0.00237	0.0021	0.00204	0.00204
$\hat{E}(\beta^*)$	0.0042	0.05935	0.08933	0.17907	0.32607	0.35609	0.46683	0.6331	0.80677	1.02898	1.31995	1.60546	1.87019	1.9379	1.90572	1.91678
$Var(\beta^*)$	0.24156	0.06464	0.06236	0.04656	0.05508	0.06041	0.07918	0.09237	0.10048	0.10086	0.10599	0.10171	0.14423	0.14247	0.15218	0.14673
E(IMS)	0.57922	0.64289	0.70601	0.75248	0.81476	0.84904	0.92836	1.00114	1.10218	1.25052	1.40302	1.584	1.7315	1.81004	1.8027	1.8166
Var(IMS)	0.00476	0.00466	0.00372	0.00392	0.00561	0.00516	0.00738	0.00691	0.00962	0.00877	0.00899	0.00793	0.00587	0.00471	0.00558	0.00549
Mean % total var removed	73.93%	69.29%	65.44%	62.31%	61.88%	60.17%	60.35%	62.78%	65.96%	69.19%	72.86%	77.72%	81.85%	83.73%	84.70%	83.97%
MA_2(linear)																
$\hat{E}(y)$	1.00211	0.98811	0.98089	0.96591	0.94011	0.93226	0.90688	0.87912	0.84656	0.81015	0.77275	0.73162	0.70004	0.69559	0.69021	0.67777
Var(y)	0.00605	0.00128	0.00115	0.00095	0.00096	0.00115	0.00141	0.00143	0.00147	0.00159	0.00153	0.00159	0.00239	0.00215	0.00205	0.00197
$\hat{E}(y^*)$	-0.00648	0.07883	0.12868	0.24612	0.42815	0.46636	0.63662	0.83645	1.05399	1.30649	1.60901	1.87614	2.08465	2.11121	2.13052	2.199
$Var(y^*)$	0.29041	0.0646	0.06286	0.04786	0.05559	0.06406	0.08406	0.09136	0.09612	0.09308	0.09979	0.0972	0.15052	0.15127	0.16266	0.14768
E(IMS)	0.39424	0.42393	0.46109	0.49613	0.58039	0.63106	0.75906	0.88938	1.03969	1.23716	1.4318	1.64276	1.7933	1.86548	1.88613	1.92148
Var(IMS)	0.00431	0.00457	0.00406	0.00502	0.00716	0.00747	0.01067	0.01139	0.01209	0.01104	0.00924	0.00754	0.00499	0.00403	0.00423	0.00413
Impulse (size 3)																
B_5(nonlinear)																
$\hat{E}(\beta)$	0.99617	1.00287	0.9753	0.98307	0.94382	0.9493	0.902	0.90977	0.84894	0.85899	0.8001	0.7758	0.75086	0.73084	0.72304	0.72207
Var(β)	0.00486	0.00129	0.001	0.00112	0.00091	0.00105	0.00119	0.00175	0.00143	0.00184	0.00159	0.00216	0.00184	0.00205	0.00186	0.00193
$\hat{E}(\beta^*)$	0.06111	0.03767	0.23769	0.17454	0.47425	0.37257	0.79305	0.62987	1.23427	1.00812	1.58719	1.73637	1.90569	2.05792	1.98093	2.12432
$Var(\beta^*)$	0.24589	0.06909	0.05865	0.05952	0.06918	0.06083	0.07384	0.10363	0.09546	0.10691	0.10008	0.13257	0.12026	0.13378	0.11461	0.11955
E(IMS)	0.60607	0.70434	0.77341	0.78958	0.90375	0.86816	1.09842	1.00066	1.39319	1.23604	1.57274	1.64449	1.7643	1.8983	1.81001	1.93351
Var(IMS)	0.00487	0.00799	0.00972	0.00605	0.02148	0.00928	0.01708	0.01056	0.01088	0.01044	0.01225	0.01042	0.00711	0.0069	0.00575	0.00801
Mean % total var removed	75.94%	72.13%	67.28%	64.91%	64.53%	63.54%	63.84%	64.93%	68.12%	70.38%	74.02%	77.90%	81.46%	83.44%	84.24%	83.04%
MA_2(linear)																
$\hat{E}(y)$	0.99417	1.00116	0.97002	0.97369	0.92881	0.93293	0.87667	0.87887	0.81601	0.81896	0.75964	0.7343	0.7137	0.69812	0.68578	0.67773
Var(y)	0.00565	0.00129	0.00102	0.00119	0.00094	0.00106	0.00124	0.00169	0.00134	0.00167	0.00145	0.00197	0.00176	0.00206	0.00184	0.00179
$\hat{E}(y^*)$	0.07689	0.05438	0.28092	0.24436	0.58377	0.48009	0.97088	0.83957	1.46608	1.29453	1.87487	2.01582	2.13651	2.25093	2.20398	2.41485
$Var(y^*)$	0.28143	0.06842	0.05988	0.06205	0.07325	0.06132	0.07894	0.10289	0.09153	0.09963	0.09524	0.12394	0.11657	0.13876	0.11729	0.11718
E(IMS)	0.42618	0.48433	0.53921	0.52488	0.6941	0.63798	0.95725	0.88077	1.33002	1.2121	1.58313	1.69443	1.81556	1.94363	1.89004	2.01543
Var(IMS)	0.00527	0.00698	0.01043	0.00572	0.02278	0.01113	0.0208	0.01414	0.01229	0.01236	0.01165	0.00937	0.00646	0.00547	0.00431	0.00584

Table A.6: Simulation results for M1

Window size	**n=1**															
Frequency	**0,0625**	**0,1250**	**0,1875**	**0,2500**	**0,3125**	**0,3750**	**0,4375**	**0,5000**	**0,5625**	**0,6250**	**0,6875**	**0,7500**	**0,8125**	**0,8750**	**0,9375**	**1,000**
Normal signal																
Med_1(nonlinear)																
$\hat{E}(\beta)$	0.9982	1.00119	1.00127	0.99935	0.99486	0.99759	0.99676	0.99419	0.99372	0.99029	0.98865	0.98653	0.9854	0.9805	0.97749	0.97702
$Var(\beta)$	0.0007	0.00041	0.00049	0.00043	0.00042	0.00043	0.00047	0.00036	0.00044	0.00038	0.00039	0.00039	0.00045	0.00054	0.00047	0.00051
$\hat{E}(\beta^*)$	0.01189	-0.02093	-0.01668	0.00558	0.03393	0.01046	0.01595	0.04534	0.04513	0.06081	0.08028	0.09551	0.09468	0.12977	0.14957	0.15785
$Var(\beta^*)$	0.0405	0.02543	0.03518	0.02892	0.03107	0.02737	0.03384	0.02485	0.02807	0.0296	0.0258	0.02681	0.03	0.0352	0.03461	0.03461
E(IMS)	0.82572	0.80625	0.82869	0.82793	0.84834	0.84114	0.87601	0.86597	0.88913	0.8969	0.92391	0.93693	0.95454	0.96565	0.98668	1.00152
Var(IMS)	0.01691	0.01314	0.01444	0.0124	0.01279	0.00907	0.0167	0.01108	0.0109	0.0114	0.01437	0.01225	0.01085	0.01123	0.00905	0.01164
MA_2(linear)																
$\hat{E}(y)$	0.99837	1.00016	0.99779	0.9935	0.98618	0.98561	0.98068	0.97193	0.96509	0.95649	0.94935	0.94179	0.93498	0.92255	0.91034	0.90321
$Var(y)$	0.00069	0.0004	0.00051	0.00044	0.00044	0.00045	0.00052	0.00037	0.00042	0.00036	0.00036	0.00036	0.00041	0.0005	0.00042	0.00046
$\hat{E}(y^*)$	0.01034	-0.01549	0.006	0.0438	0.09441	0.09704	0.13204	0.20029	0.24178	0.29449	0.35847	0.41619	0.45486	0.53748	0.60998	0.66008
$Var(y^*)$	0.04076	0.02514	0.03667	0.02953	0.03311	0.02807	0.03647	0.02505	0.02653	0.02863	0.02422	0.02481	0.02872	0.03314	0.03224	0.03207
E(IMS)	0.5348	0.52248	0.53512	0.53859	0.55443	0.55533	0.5952	0.60108	0.6333	0.67488	0.73675	0.79412	0.84258	0.91324	0.99038	1.06604
Var(IMS)	0.00826	0.00666	0.00764	0.00654	0.0077	0.00617	0.01004	0.00838	0.00691	0.00664	0.00697	0.00617	0.00539	0.00556	0.0055	0.00496
Impulse (size 3)																
Med_1(nonlinear)																
$\hat{E}(\beta)$	0.98628	1.00604	0.98037	1.01529	0.97653	1.01777	0.97011	1.01327	0.97212	0.99978	0.98098	0.98711	0.98541	0.97542	0.98941	0.96186
$Var(\beta)$	0.00056	0.00044	0.00042	0.00043	0.00037	0.00047	0.00038	0.00039	0.00036	0.00042	0.0005	0.00048	0.00047	0.00046	0.00049	0.00052
$\hat{E}(\beta^*)$	0.23973	0.1136	0.28526	0.04558	0.31483	0.03184	0.35998	0.05484	0.34882	0.15179	0.27657	0.24091	0.24895	0.30478	0.23101	0.41413
$Var(\beta^*)$	0.03768	0.02952	0.03217	0.03025	0.02911	0.03296	0.02765	0.02587	0.02473	0.02976	0.03379	0.03221	0.03206	0.03032	0.03564	0.03155
E(IMS)	1.4419	1.43863	1.43458	1.43982	1.45179	1.46678	1.45975	1.45817	1.46551	1.48279	1.48815	1.48496	1.49201	1.51798	1.53134	1.55115
Var(IMS)	0.01532	0.0129	0.01391	0.01456	0.01449	0.01439	0.01734	0.01606	0.01348	0.01447	0.01359	0.01802	0.0131	0.01468	0.01494	0.01414
MA_2(linear)																
$\hat{E}(y)$	0.98598	1.00608	0.9761	1.00983	0.96671	1.00563	0.95364	0.99056	0.94474	0.96411	0.94266	0.94273	0.9353	0.91898	0.91931	0.89294
$Var(y)$	0.0006	0.00047	0.00044	0.00044	0.00038	0.0005	0.0004	0.00039	0.00035	0.0004	0.00047	0.00047	0.00043	0.00044	0.00043	0.00046
$\hat{E}(y^*)$	0.24552	0.11395	0.31689	0.08385	0.38771	0.12153	0.48039	0.21891	0.54097	0.40213	0.54946	0.56243	0.61075	0.7032	0.71407	0.88546
$Var(y^*)$	0.03989	0.03142	0.03307	0.03142	0.02979	0.03446	0.02878	0.02615	0.02414	0.02889	0.03276	0.03191	0.02999	0.02848	0.03261	0.02876
E(IMS)	1.01522	1.01412	1.02227	1.00665	1.05188	1.01067	1.08276	1.00005	1.11565	1.05799	1.15193	1.15412	1.18532	1.28307	1.26403	1.43798
Var(IMS)	0.0114	0.0092	0.01166	0.01056	0.01108	0.01084	0.01237	0.01182	0.01069	0.01079	0.01036	0.01185	0.00808	0.00898	0.00953	0.00938

Table A.7: Simulation results for M2

Window size	**n=2**															
Frequency	**0,0625**	**0,1250**	**0,1875**	**0,2500**	**0,3125**	**0,3750**	**0,4375**	**0,5000**	**0,5625**	**0,6250**	**0,6875**	**0,7500**	**0,8125**	**0,8750**	**0,9375**	**1,000**
Normal signal																
Med_2(nonlinear)																
$\hat{E}(\beta)$	0.99818	1.00083	0.99945	0.99546	0.98921	0.991	0.98725	0.98111	0.97819	0.97323	0.96887	0.96322	0.95912	0.95031	0.944	0.94024
$Var(\beta)$	0.0006	0.00032	0.00043	0.0004	0.00041	0.00037	0.00047	0.00038	0.00045	0.0004	0.00042	0.0004	0.0005	0.00057	0.00054	0.00061
$\hat{E}(\beta^*)$	0.01234	-0.01797	-0.00586	0.03157	0.07311	0.05896	0.0879	0.13721	0.15075	0.17263	0.22266	0.26439	0.28035	0.33959	0.37616	0.40421
$Var(\beta^*)$	0.03654	0.02064	0.03043	0.02552	0.02983	0.02367	0.03195	0.02477	0.03015	0.03146	0.02711	0.02709	0.03197	0.04095	0.0411	0.04217
E(IMS)	0.60161	0.60909	0.62005	0.64328	0.66151	0.67855	0.72289	0.74088	0.77535	0.80489	0.85101	0.87948	0.92376	0.96386	1.01217	1.06386
Var(IMS)	0.00457	0.0048	0.00341	0.00381	0.00394	0.00354	0.00538	0.00624	0.00548	0.00436	0.00556	0.00471	0.00606	0.00544	0.00629	0.00797
MA_2(linear)																
$\hat{E}(y)$	0.9983	0.99977	0.99594	0.98968	0.98066	0.97954	0.97203	0.96034	0.95063	0.94049	0.93077	0.92038	0.91263	0.8995	0.8852	0.87505
$Var(y)$	0.00062	0.00034	0.00045	0.00041	0.00042	0.00039	0.0005	0.00039	0.00043	0.00039	0.00038	0.00037	0.00046	0.00055	0.00048	0.00056
$\hat{E}(y^*)$	0.01155	-0.01152	0.01615	0.06764	0.13082	0.13985	0.19708	0.28322	0.3407	0.39711	0.48782	0.57039	0.61795	0.70927	0.79143	0.85421
$Var(y^*)$	0.03852	0.02127	0.03182	0.02596	0.03123	0.02443	0.0338	0.02485	0.02909	0.03115	0.02482	0.02545	0.03037	0.0399	0.03819	0.03994
E(IMS)	0.45976	0.4612	0.46868	0.48494	0.50724	0.52693	0.58097	0.62366	0.6792	0.74713	0.83284	0.90941	0.98799	1.07342	1.17394	1.27055
Var(IMS)	0.00355	0.00414	0.00312	0.00379	0.00411	0.00432	0.0059	0.00622	0.00618	0.00481	0.00551	0.00485	0.00586	0.00485	0.0054	0.00536
Impulse (size 3)																
Med_2(nonlinear)																
$\hat{E}(\beta)$	0.98634	1.00521	0.97871	1.01153	0.97081	1.01152	0.95997	1.00069	0.95783	0.98198	0.96219	0.96576	0.96179	0.94595	0.95429	0.9245
$Var(\beta)$	0.00048	0.00039	0.00037	0.00037	0.00038	0.00043	0.00038	0.00038	0.00035	0.00042	0.00047	0.00051	0.00047	0.00051	0.00058	0.00057
$\hat{E}(\beta^*)$	0.23896	0.11589	0.2997	0.06715	0.35735	0.07553	0.43531	0.13911	0.45457	0.2715	0.40118	0.38643	0.41197	0.50672	0.45828	0.67221
$Var(\beta^*)$	0.03413	0.02716	0.02989	0.02544	0.02897	0.0283	0.03058	0.0245	0.02654	0.02827	0.03231	0.0343	0.03396	0.03373	0.03901	0.03693
E(IMS)	1.2693	1.26217	1.27276	1.26481	1.29325	1.30792	1.32221	1.30739	1.34507	1.33919	1.36929	1.36066	1.39099	1.4448	1.44928	1.54061
Var(IMS)	0.02364	0.02458	0.02565	0.03228	0.02525	0.02307	0.0235	0.02557	0.02084	0.02137	0.02185	0.02237	0.01812	0.01414	0.01867	0.01814
MA_2(linear)																
$\hat{E}(y)$	0.98559	1.00529	0.97432	1.00614	0.96121	0.99963	0.94396	0.97971	0.9312	0.94741	0.9248	0.9232	0.91529	0.89702	0.89334	0.8646
$Var(y)$	0.00052	0.00042	0.0004	0.00038	0.00039	0.00046	0.0004	0.0004	0.00034	0.00041	0.00045	0.00049	0.00045	0.00049	0.00051	0.00049
$\hat{E}(y^*)$	0.24754	0.11619	0.33154	0.10434	0.42679	0.16168	0.55215	0.29193	0.64105	0.51195	0.66339	0.69374	0.75312	0.8648	0.89066	1.08893
$Var(y^*)$	0.03678	0.0291	0.03138	0.02684	0.03029	0.03006	0.03176	0.02539	0.02572	0.0274	0.03163	0.03354	0.0323	0.03227	0.0364	0.03269
E(IMS)	0.95818	0.95333	0.97411	0.94202	1.01446	0.9554	1.08182	0.96776	1.14275	1.05327	1.18762	1.19267	1.25218	1.38689	1.37452	1.5997
Var(IMS)	0.01667	0.01532	0.01896	0.01957	0.01845	0.01552	0.01749	0.01604	0.01552	0.01281	0.01433	0.01234	0.01028	0.00858	0.00987	0.01154

Table A.8: Simulation results for (53H,twice)

Window size	**n=5;3**															
Frequency	**0,0625**	**0,1250**	**0,1875**	**0,2500**	**0,3125**	**0,3750**	**0,4375**	**0,5000**	**0,5625**	**0,6250**	**0,6875**	**0,7500**	**0,8125**	**0,8750**	**0,9375**	**1,000**
Normal signal																
Med_{53}(nonlinear)																
$\hat{E}(\beta)$	0.99853	1.0024	1.00048	0.99813	0.9925	0.99667	0.99573	0.99169	0.98977	0.9855	0.97959	0.97431	0.97174	0.96508	0.95657	0.95122
$Var(\beta)$	0.00095	0.00045	0.00058	0.00046	0.00054	0.00052	0.00065	0.00057	0.00054	0.00045	0.00043	0.00041	0.00056	0.0006	0.00051	0.00052
$\hat{E}(\beta^*)$	0.01177	-0.02738	-0.01124	0.01141	0.05076	0.01797	0.02397	0.06472	0.06822	0.08211	0.1407	0.17653	0.18729	0.24577	0.29906	0.33888
$Var(\beta^*)$	0.05454	0.02902	0.03906	0.02993	0.03895	0.03195	0.04401	0.03285	0.03403	0.03493	0.02545	0.02753	0.03432	0.04032	0.03744	0.03711
E(IMS)	0.6002	0.60849	0.61037	0.62315	0.63989	0.63665	0.66924	0.67149	0.67852	0.69023	0.71495	0.7323	0.75637	0.78053	0.82054	0.86257
Var(IMS)	0.00688	0.00686	0.00441	0.00477	0.00494	0.00474	0.00738	0.00845	0.00629	0.00508	0.00674	0.00498	0.00629	0.00592	0.00752	0.00953
MA_2(linear)																
$\hat{E}(y)$	0.99783	1.00125	0.99721	0.99231	0.98308	0.98344	0.97836	0.96955	0.96229	0.95287	0.93919	0.9263	0.91685	0.9042	0.89103	0.88114
$Var(y)$	0.00103	0.00049	0.00062	0.0005	0.00056	0.00052	0.00064	0.00059	0.00055	0.00048	0.00042	0.00038	0.00055	0.00057	0.00047	0.00047
$\hat{E}(y^*)$	0.01695	-0.02052	0.00889	0.04608	0.11149	0.10464	0.14239	0.21786	0.25714	0.30489	0.4164	0.4982	0.55383	0.65365	0.74251	0.81976
$Var(y^*)$	0.05797	0.03163	0.04134	0.03179	0.04039	0.03282	0.0438	0.03377	0.03528	0.03694	0.02568	0.02663	0.03469	0.03897	0.03619	0.03366
E(IMS)	0.50564	0.52742	0.56367	0.60562	0.66439	0.71288	0.78846	0.85513	0.93187	1.00443	1.0791	1.17375	1.25423	1.33239	1.41213	1.50743
Var(IMS)	0.00524	0.00628	0.00469	0.00577	0.00636	0.00649	0.00751	0.00872	0.00685	0.00508	0.00482	0.00418	0.00476	0.00396	0.00404	0.00419
Impulse (size 3)																
Med_{53}(nonlinear)																
$\hat{E}(\beta)$	0.98461	1.0107	0.97501	1.01817	0.96792	1.02028	0.95786	1.01743	0.96501	0.99643	0.97133	0.97776	0.97435	0.95657	0.97044	0.93304
$Var(\beta)$	0.00075	0.00061	0.00052	0.00053	0.00049	0.0006	0.00058	0.00055	0.00046	0.00047	0.00061	0.00052	0.00052	0.00055	0.00057	0.00054
$\hat{E}(\beta^*)$	0.26723	0.09087	0.34146	0.03467	0.39747	0.02423	0.46362	0.04208	0.41979	0.18836	0.35401	0.31837	0.34374	0.47114	0.37874	0.64256
$Var(\beta^*)$	0.05035	0.04009	0.04051	0.03783	0.03916	0.03925	0.04398	0.03157	0.03361	0.03409	0.04218	0.03846	0.03506	0.0352	0.03625	0.03382
E(IMS)	1.26579	1.26462	1.2646	1.24799	1.28778	1.28502	1.3043	1.27496	1.31662	1.29233	1.33062	1.30437	1.32018	1.37658	1.36047	1.46211
Var(IMS)	0.02751	0.02643	0.03264	0.03399	0.02661	0.02943	0.02842	0.02884	0.02538	0.02517	0.02608	0.02699	0.02346	0.01995	0.02565	0.02626
MA_2(linear)																
$\hat{E}(y)$	0.98313	1.01097	0.97028	1.0124	0.95784	1.00583	0.94035	0.99391	0.93768	0.96278	0.93084	0.92855	0.91819	0.89734	0.90032	0.86777
$Var(y)$	0.00089	0.00064	0.00054	0.00055	0.00049	0.00061	0.00058	0.00056	0.0005	0.00048	0.00061	0.00051	0.00051	0.00053	0.00051	0.00047
$\hat{E}(y^*)$	0.28052	0.09123	0.37575	0.07333	0.46719	0.12298	0.58755	0.20764	0.613	0.42332	0.63232	0.65165	0.72348	0.87165	0.85662	1.09387
$Var(y^*)$	0.05674	0.0415	0.04212	0.03917	0.03974	0.04085	0.04423	0.03103	0.03612	0.03393	0.04313	0.03918	0.03446	0.03554	0.03305	0.03051
E(IMS)	1.02409	1.0858	1.03332	1.10857	1.1282	1.15991	1.26132	1.16349	1.4176	1.23285	1.53966	1.35921	1.62966	1.54692	1.731	1.77574
Var(IMS)	0.02019	0.02019	0.0226	0.02516	0.01812	0.01984	0.02039	0.01692	0.01541	0.01187	0.01608	0.00976	0.01291	0.00998	0.01148	0.01151

Table A.9: Simulation results for (4253H,twice)

Window size	**n=4253**															
Frequency	**0,0625**	**0,1250**	**0,1875**	**0,2500**	**0,3125**	**0,3750**	**0,4375**	**0,5000**	**0,5625**	**0,6250**	**0,6875**	**0,7500**	**0,8125**	**0,8750**	**0,9375**	**1,000**
Normal signal																
Med_{4253}(nonlinear)																
$\hat{E}(\beta)$	0.99851	1.00239	1.0029	1.00048	0.99358	0.99822	0.99836	0.99362	0.9914	0.98883	0.98249	0.97525	0.97141	0.96204	0.95294	0.94644
Var(β)	0.00106	0.00065	0.0007	0.00053	0.00057	0.00057	0.0007	0.00062	0.0006	0.0006	0.00049	0.00047	0.00061	0.00059	0.00058	0.00054
$\hat{E}(\beta^*)$	0.00971	-0.02704	-0.02763	-0.00727	0.04496	0.0032	0.00262	0.0469	0.05613	0.06214	0.12633	0.16852	0.18883	0.26382	0.31564	0.36419
Var(β^*)	0.06033	0.03764	0.04701	0.03417	0.04291	0.03433	0.04831	0.03712	0.0357	0.04094	0.03016	0.02954	0.0406	0.04115	0.03902	0.0377
E(IMS)	0.58353	0.58196	0.59711	0.6039	0.62413	0.61716	0.64846	0.63643	0.64507	0.65333	0.67577	0.69091	0.71062	0.73236	0.772	0.80929
Var(IMS)	0.00811	0.00833	0.0081	0.00679	0.008	0.00643	0.00978	0.00983	0.00701	0.00695	0.00781	0.007	0.00759	0.00782	0.00808	0.00885
MA_2(linear)																
$\hat{E}(y)$	0.99817	1.00043	0.99945	0.99375	0.98282	0.98324	0.97847	0.96836	0.95991	0.95144	0.93821	0.92478	0.91288	0.89539	0.87863	0.86555
Var(y)	0.00119	0.00072	0.00072	0.00054	0.00059	0.00057	0.00068	0.00058	0.00055	0.0006	0.00047	0.00045	0.00055	0.00054	0.00048	0.00047
$\hat{E}(y^*)$	0.01281	-0.01422	-0.00624	0.03594	0.11545	0.10191	0.13511	0.21746	0.27465	0.32446	0.43851	0.5243	0.59955	0.72927	0.831	0.92338
Var(y^*)	0.06555	0.04181	0.04794	0.03394	0.04374	0.035	0.0473	0.03629	0.03404	0.04065	0.02939	0.02829	0.03739	0.03868	0.03443	0.03389
E(IMS)	0.48707	0.48305	0.49532	0.5019	0.52376	0.52216	0.56307	0.57652	0.6095	0.65984	0.72732	0.79696	0.87106	0.96006	1.06192	1.16183
Var(IMS)	0.00622	0.00674	0.00653	0.00571	0.00691	0.0059	0.00907	0.00854	0.0071	0.0069	0.00637	0.00532	0.00541	0.00586	0.00572	0.00475
Impulse (size 3)																
Med_{4253}(nonlinear)																
$\hat{E}(\beta)$	0.97907	1.01625	0.97104	1.02211	0.9648	1.02373	0.95598	1.0216	0.96053	1.00249	0.97021	0.97696	0.97173	0.95303	0.96553	0.92761
Var(β)	0.00117	0.00063	0.00059	0.00057	0.00048	0.0007	0.00058	0.00062	0.00058	0.00055	0.00069	0.00059	0.00062	0.00057	0.00063	0.00059
$\hat{E}(\beta^*)$	0.3202	0.06815	0.38301	0.02146	0.43462	0.0139	0.4868	0.02936	0.4546	0.15693	0.37371	0.33362	0.37076	0.48986	0.41253	0.66787
Var(β^*)	0.06523	0.04102	0.04318	0.03975	0.03871	0.04645	0.04151	0.03435	0.03885	0.03719	0.04527	0.04075	0.03846	0.03666	0.03908	0.03529
E(IMS)	1.25934	1.25632	1.25393	1.25331	1.26831	1.28962	1.26658	1.26007	1.25847	1.26777	1.27844	1.2649	1.26774	1.29679	1.29365	1.39109
Var(IMS)	0.01824	0.01733	0.02095	0.02123	0.01781	0.01754	0.01958	0.02076	0.01694	0.01811	0.01784	0.02077	0.01536	0.01703	0.0189	0.01823
MA_2(linear)																
$\hat{E}(y)$	0.97714	1.01657	0.96594	1.01618	0.95404	1.0077	0.93779	0.99394	0.93142	0.96314	0.92754	0.92648	0.91373	0.88881	0.88749	0.85246
Var(y)	0.00131	0.00066	0.00065	0.00059	0.00049	0.0007	0.00057	0.00059	0.00056	0.00054	0.00066	0.00055	0.00056	0.00052	0.00055	0.00051
$\hat{E}(y^*)$	0.33822	0.06906	0.41928	0.06469	0.50768	0.12399	0.6124	0.22105	0.66115	0.43674	0.67691	0.69445	0.78314	0.93994	0.95709	1.18928
Var(y^*)	0.07213	0.04228	0.04504	0.04091	0.03981	0.04715	0.04115	0.03361	0.03848	0.03599	0.04378	0.03888	0.03647	0.03362	0.0354	0.03177
E(IMS)	1.05081	1.05113	1.05848	1.03964	1.09013	1.05584	1.11919	1.03142	1.14948	1.08374	1.19341	1.1947	1.23968	1.35034	1.34156	1.5485
Var(IMS)	0.01583	0.01465	0.01807	0.01706	0.01544	0.0157	0.01667	0.01726	0.01429	0.01501	0.01454	0.01425	0.01074	0.01179	0.01134	0.01145

Appendix B

Tables of results on impulse and slope

Table B.1: Results for negative slope and positive impulse

Slope negative, positive impulse

Mean regression coefficients

Frequency	B_1	B_2	B_3	B_4	B_5	M_1	M_2	53Ht	4253Ht	$MA_2(B_1)$	$MA_2(B_2)$	$MA_2(B_3)$	$MA_2(B_4)$	$MA_2(B_5)$	$MA_2(M_1)$	$MA_2(M_2)$	$MA_2(53Ht)$	$MA_2(4253Ht)$
0.0625	0.98807	0.98587	0.99227	0.9898	0.99196	0.9888	0.98744	0.98405	0.98007	0.9868	0.98424	0.99155	0.98882	0.9918	0.9874	0.98671	0.982723	0.97833732
0.125	0.97141	0.96686	0.98723	0.98367	0.98019	0.97288	0.97148	0.96147	0.95425	0.96742	0.9627	0.98533	0.98107	0.97778	0.96898	0.96751	0.956987	0.94929157
0.1875	0.97062	0.96516	0.98076	0.97526	0.9668	0.97213	0.97018	0.96192	0.95584	0.96471	0.959	0.97634	0.96967	0.96178	0.96657	0.96468	0.955731	0.94893557
0.25	0.96034	0.95301	0.97098	0.96411	0.95322	0.96241	0.9582	0.95022	0.94505	0.95193	0.94455	0.96343	0.9556	0.95297	0.95433	0.94997	0.941032	0.93525693
0.3125	0.96073	0.95313	0.96844	0.95782	0.96043	0.96302	0.9574	0.94948	0.94581	0.94939	0.94124	0.95716	0.94641	0.9466	0.9518	0.94622	0.937402	0.93315163
0.375	0.94974	0.93986	0.95474	0.93898	0.94743	0.95313	0.94655	0.93926	0.93254	0.93474	0.92471	0.93953	0.92294	0.93072	0.93829	0.93196	0.923328	0.91633992
0.4375	0.9599	0.94881	0.95915	0.94266	0.93746	0.96346	0.95492	0.9491	0.94519	0.94021	0.92965	0.93855	0.92766	0.91303	0.94386	0.93587	0.928372	0.92273314
0.5	0.94714	0.93258	0.93864	0.91663	0.90743	0.95098	0.93823	0.9331	0.92515	0.92153	0.90815	0.91373	0.89958	0.87705	0.92578	0.91397	0.907224	0.89849483
0.5625	0.94397	0.92872	0.93139	0.90239	0.88039	0.94752	0.93372	0.92967	0.91953	0.91318	0.89905	0.90195	0.88123	0.84474	0.91754	0.90415	0.897612	0.88717995
0.625	0.93958	0.9229	0.91963	0.88676	0.84528	0.94326	0.92699	0.92556	0.91506	0.90387	0.88677	0.88539	0.86129	0.80663	0.90866	0.89261	0.88717	0.87563786
0.6875	0.94212	0.92473	0.91696	0.88386	0.81353	0.94546	0.92707	0.92756	0.91863	0.90087	0.88237	0.87824	0.84889	0.77234	0.90535	0.88798	0.882017	0.87210677
0.75	0.93278	0.91177	0.89198	0.84634	0.77278	0.93693	0.91548	0.91463	0.90522	0.88669	0.86488	0.84986	0.81854	0.73196	0.89177	0.87267	0.863315	0.85274163
0.8125	0.93601	0.91493	0.88669	0.83967	0.74408	0.93982	0.91614	0.91716	0.9086	0.8835	0.86224	0.84071	0.80773	0.7088	0.88791	0.86845	0.859171	0.84880823
0.875	0.92801	0.89994	0.86032	0.80944	0.72224	0.9324	0.9034	0.90262	0.8947	0.86967	0.84307	0.80951	0.77844	0.69299	0.87379	0.85198	0.840501	0.82851836
0.9375	0.93109	0.90072	0.85073	0.79664	0.72614	0.93534	0.90392	0.90174	0.89084	0.86237	0.83651	0.79641	0.76133	0.69125	0.86699	0.84397	0.832321	0.81891057
1	0.92416	0.88744	0.82762	0.77747	0.7168	0.92857	0.8933	0.89146	0.8794	0.85003	0.82093	0.77336	0.73943	0.67478	0.85571	0.82951	0.818368	0.80333394

Expected value of IMS

Frequency	B_1	B_2	B_3	B_4	B_5	M_1	M_2	53Ht	4253Ht	$MA_2(B_1)$	$MA_2(B_2)$	$MA_2(B_3)$	$MA_2(B_4)$	$MA_2(B_5)$	$MA_2(M_1)$	$MA_2(M_2)$	$MA_2(53Ht)$	$MA_2(4253Ht)$
0.0625	1.49639	1.39862	0.74008	0.69107	0.62153	1.44677	1.27967	1.28514	1.26512	1.02578	0.98849	0.51985	0.48793	0.44136	1.02421	0.97332	1.042645	1.06063667
0.125	1.49781	1.39949	0.76151	0.72516	0.67835	1.44445	1.27156	1.27601	1.26499	1.02325	0.97861	0.5146	0.49157	0.46577	1.01951	0.96079	1.026184	1.05955003
0.1875	1.49425	1.3979	0.81013	0.78423	0.75868	1.42551	1.23738	1.24235	1.24339	1.01224	0.96905	0.54329	0.52811	0.51236	1.0013	0.93533	0.999322	1.04255258
0.25	1.51478	1.42829	0.87695	0.8509	0.83037	1.44161	1.26313	1.25872	1.24464	1.02351	0.98868	0.58931	0.57737	0.53044	1.01323	0.95672	1.016745	1.04510638
0.3125	1.53071	1.45352	0.92209	0.90128	0.81015	1.45659	1.29185	1.29814	1.26952	1.0285	0.9984	0.61491	0.61647	0.56748	1.02454	0.97713	1.06089	1.06407259
0.375	1.52548	1.44617	0.98173	0.97632	0.85801	1.44278	1.28003	1.26842	1.24985	1.0397	1.01673	0.69285	0.71643	0.6417	1.03031	0.99003	1.067351	1.06271355
0.4375	1.54511	1.4652	0.98398	0.96556	0.91989	1.45592	1.30005	1.29233	1.26678	1.03604	1.00759	0.6923	0.69446	0.74662	1.02695	0.98695	1.095575	1.0623539
0.5	1.56819	1.49271	1.05434	1.06514	0.99586	1.47066	1.32393	1.30463	1.27971	1.08102	1.06565	0.79864	0.81731	0.88555	1.06784	1.05142	1.158828	1.1097376
0.5625	1.58591	1.50413	1.08648	1.10998	1.11489	1.48235	1.33453	1.31374	1.27974	1.10364	1.09538	0.8698	0.92392	1.06154	1.08682	1.08287	1.211918	1.12505383
0.625	1.57975	1.50557	1.126	1.15747	1.24641	1.47451	1.35068	1.31069	1.27288	1.12739	1.13152	0.95598	1.03838	1.2357	1.11232	1.1316	1.26102	1.14911457
0.6875	1.58975	1.50498	1.15949	1.18234	1.39806	1.48377	1.3519	1.31673	1.26816	1.14469	1.15237	1.03222	1.1534	1.42766	1.12936	1.15439	1.302367	1.16913142
0.75	1.61324	1.54442	1.24887	1.33319	1.58048	1.50259	1.3961	1.34998	1.28006	1.21242	1.24446	1.1849	1.32336	1.64157	1.19381	1.25096	1.400938	1.23817112
0.8125	1.61651	1.53427	1.29137	1.4228	1.73057	1.49996	1.38793	1.33642	1.27345	1.22725	1.26963	1.27246	1.45426	1.79242	1.20963	1.27853	1.455116	1.26462848
0.875	1.60512	1.54843	1.3775	1.56523	1.80918	1.49601	1.42054	1.36577	1.27405	1.28227	1.35921	1.42199	1.61887	1.86372	1.26791	1.36831	1.543368	1.33639001
0.9375	1.63601	1.56955	1.43937	1.67831	1.80479	1.52649	1.44247	1.37477	1.29315	1.32726	1.4114	1.52289	1.76097	1.8827	1.31261	1.41905	1.592986	1.3941261
1	1.65179	1.61241	1.55557	1.76371	1.82435	1.54197	1.49024	1.4087	1.3276	1.40506	1.51722	1.67591	1.86635	1.92487	1.38803	1.52254	1.687362	1.4885988

Table B.2: Results for positive slope and positive impulse

Slope positive, positive impulse

Mean regression coefficients

Frequency	B_1	B_2	B_3	B_4	B_5	M_1	M_2	53Ht	4253Ht	$MA_2(B_1)$	$MA_2(B_2)$	$MA_2(B_3)$	$MA_2(B_4)$	$MA_2(B_5)$	$MA_2(M_1)$	$MA_2(M_2)$	MA_2(53Ht)	MA_2(4253Ht)
0.0625	0.97691	0.97172	0.98862	0.98608	0.97301	0.97833	0.97757	0.96717	0.95856	0.97436	0.96845	0.98797	0.98364	0.96891	0.97618	0.97503	0.963595	0.9534458
0.125	0.96209	0.95641	0.97976	0.9901	0.98871	0.96368	0.96234	0.94981	0.93856	0.9573	0.95148	0.97705	0.99007	0.98617	0.9591	0.95765	0.944127	0.93364019
0.1875	0.97842	0.97445	0.98354	0.9794	0.9727	0.97963	0.97779	0.97369	0.97048	0.97384	0.96953	0.97948	0.97503	0.96701	0.97512	0.97331	0.968538	0.9648628
0.25	0.9759	0.97078	0.97945	0.97313	0.95995	0.97772	0.97415	0.96872	0.9662	0.96849	0.96315	0.97183	0.96461	0.94902	0.96999	0.96679	0.960481	0.95725641
0.3125	0.97275	0.96653	0.97277	0.96191	0.9445	0.97488	0.96998	0.96603	0.96312	0.96216	0.95551	0.96161	0.94985	0.92864	0.96436	0.95943	0.954393	0.95081158
0.375	0.97162	0.96424	0.96731	0.95239	0.9304	0.97373	0.96704	0.96395	0.96192	0.95698	0.94905	0.95202	0.93609	0.91114	0.95933	0.9529	0.948421	0.94536346
0.4375	0.96107	0.95114	0.95374	0.93268	0.89947	0.96415	0.95469	0.95246	0.9482	0.94208	0.93188	0.93364	0.90971	0.87538	0.94512	0.9365	0.931811	0.92687285
0.5	0.95788	0.94655	0.94881	0.92034	0.87537	0.96122	0.94959	0.94676	0.94152	0.93333	0.92247	0.92341	0.89077	0.86486	0.93671	0.92605	0.921058	0.91438637
0.5625	0.94024	0.92486	0.92059	0.87919	0.87344	0.94402	0.92987	0.9252	0.91607	0.91012	0.89563	0.891	0.84422	0.84076	0.91445	0.90078	0.893989	0.88468933
0.625	0.9419	0.9258	0.91776	0.87163	0.84433	0.9452	0.92992	0.92822	0.91909	0.90649	0.8904	0.88307	0.83347	0.80686	0.91084	0.89594	0.890282	0.880831
0.6875	0.94411	0.92767	0.91806	0.8577	0.81004	0.94785	0.93013	0.93327	0.92454	0.90315	0.88614	0.87748	0.81759	0.77004	0.90789	0.89117	0.888299	0.87790578
0.75	0.94532	0.92584	0.90618	0.83726	0.77412	0.94876	0.92768	0.93032	0.92228	0.89822	0.87821	0.86262	0.79967	0.73332	0.90277	0.88389	0.878249	0.86827985
0.8125	0.9416	0.9197	0.89151	0.81539	0.73775	0.94456	0.92022	0.9226	0.91375	0.88837	0.86573	0.84372	0.78244	0.7039	0.89232	0.87192	0.863784	0.85268205
0.875	0.93628	0.91204	0.87596	0.7888	0.71836	0.94025	0.91254	0.91452	0.90512	0.87638	0.8533	0.82204	0.76245	0.68893	0.87999	0.85994	0.850927	0.83658717
0.9375	0.9333	0.90387	0.85173	0.76454	0.71961	0.93798	0.90579	0.90686	0.89636	0.8654	0.84035	0.79461	0.74313	0.68618	0.87014	0.8464	0.838191	0.8214068
1	0.92646	0.89062	0.83197	0.7652	0.72412	0.93066	0.89398	0.89325	0.8831	0.85334	0.82518	0.77375	0.73727	0.68213	0.85886	0.8311	0.821204	0.80372897

Expected value of IMS

Frequency	B_1	B_2	B_3	B_4	B_5	M_1	M_2	53Ht	4253Ht	$MA_2(B_1)$	$MA_2(B_2)$	$MA_2(B_3)$	$MA_2(B_4)$	$MA_2(B_5)$	$MA_2(M_1)$	$MA_2(M_2)$	MA_2(53Ht)	MA_2(4253Ht)
0.0625	1.492	1.39343	0.73888	0.67121	0.61479	1.43918	1.26661	1.26206	1.25713	1.02098	0.98002	0.51497	0.4696	0.43326	1.01523	0.95979	1.051885	1.05428701
0.125	1.5013	1.40333	0.76444	0.67171	0.64173	1.44203	1.26029	1.26948	1.26214	1.02544	0.98124	0.51912	0.44073	0.42708	1.01848	0.95159	1.089399	1.05358132
0.1875	1.50878	1.4162	0.84872	0.82685	0.80916	1.447	1.26994	1.27074	1.2558	1.02885	0.99239	0.58009	0.56853	0.56287	1.02212	0.96281	1.12564	1.05486213
0.25	1.5149	1.42113	0.89641	0.87405	0.86142	1.4498	1.27038	1.28213	1.26584	1.02659	0.98517	0.60379	0.59581	0.5957	1.02108	0.96029	1.174777	1.0632529
0.3125	1.52355	1.43602	0.91254	0.89863	0.8948	1.44637	1.27296	1.27837	1.25517	1.02292	0.98537	0.61592	0.62129	0.64384	1.01329	0.95733	1.204167	1.04685008
0.375	1.5411	1.45182	0.95792	0.94361	0.92768	1.45302	1.2723	1.27186	1.25866	1.02571	0.9867	0.64741	0.66475	0.68677	1.01507	0.95542	1.23429	1.04710644
0.4375	1.53904	1.44489	1.01649	1.01227	1.02721	1.45075	1.27646	1.27075	1.25045	1.03901	1.00534	0.72506	0.76237	0.81489	1.03029	0.98259	1.288932	1.05735786
0.5	1.55542	1.47231	1.04226	1.05668	1.09646	1.466	1.31979	1.31166	1.27822	1.06159	1.03634	0.7771	0.84553	0.89592	1.05351	1.03195	1.367752	1.09364701
0.5625	1.58985	1.51944	1.17363	1.22743	1.11032	1.48518	1.34278	1.31763	1.27395	1.12457	1.12934	0.95179	1.07368	1.04303	1.10797	1.11194	1.422801	1.14019873
0.625	1.5869	1.50197	1.16685	1.23239	1.23654	1.48511	1.34901	1.32072	1.27493	1.12938	1.12882	0.9831	1.1252	1.22802	1.11496	1.12596	1.482708	1.14908665
0.6875	1.59884	1.50886	1.18284	1.28493	1.3912	1.48858	1.35095	1.31234	1.27472	1.14738	1.15082	1.03575	1.22137	1.42237	1.13147	1.15086	1.523665	1.16960492
0.75	1.5986	1.50849	1.23202	1.38288	1.5793	1.48914	1.35326	1.31312	1.26008	1.16785	1.18458	1.13543	1.3537	1.63907	1.1527	1.18947	1.570206	1.18993034
0.8125	1.60511	1.52821	1.26932	1.47656	1.74455	1.4951	1.39081	1.34054	1.27491	1.20881	1.24998	1.23839	1.47644	1.80481	1.1925	1.25955	1.648276	1.24822924
0.875	1.61231	1.53373	1.34154	1.61076	1.82308	1.5017	1.39959	1.33815	1.27392	1.26036	1.31449	1.36517	1.61943	1.87544	1.24698	1.32179	1.702522	1.31183569
0.9375	1.61681	1.56574	1.454	1.75321	1.81553	1.50162	1.43589	1.36129	1.27465	1.3211	1.41222	1.52684	1.76635	1.8901	1.30467	1.42347	1.768661	1.38256314
1	1.62485	1.60125	1.55775	1.78629	1.81081	1.51361	1.47862	1.40219	1.3193	1.40011	1.51765	1.66253	1.8489	1.91404	1.38238	1.5281	1.843646	1.48164771

Table B.3: Results for negative slope and negative impulse

Slope negative, negative impulse

Mean regression coefficients

Frequency	B_1	B_2	B_3	B_4	B_5	M_1	M_2	53Ht	4253Ht	$MA_2(B_1)$	$MA_2(B_2)$	$MA_2(B_3)$	$MA_2(B_4)$	$MA_2(B_5)$	$MA_2(M_1)$	$MA_2(M_2)$	MA_2(53Ht)	MA_2(4253Ht)
0.0625	1.00526	1.00753	1.0008	1.00394	1.00541	1.00471	1.00494	1.00858	1.0129	1.0065	1.00828	1.00176	1.00505	1.00608	1.00579	1.00594	1.010094	1.0156133
0.125	1.02859	1.03275	1.01149	1.01247	1.00922	1.02726	1.02794	1.03923	1.04804	1.02999	1.03442	1.01107	1.01146	1.00721	1.02865	1.02937	1.041227	1.05072001
0.1875	1.02579	1.02741	1.00587	1.00538	1.00095	1.02474	1.0235	1.03408	1.04299	1.02478	1.02651	1.00343	1.00193	0.99595	1.02376	1.02258	1.033929	1.04266959
0.25	1.03405	1.03493	1.00516	1.00461	0.99803	1.03322	1.02954	1.04269	1.05175	1.03111	1.03157	0.99963	0.99859	0.97891	1.03017	1.02659	1.039364	1.04839876
0.3125	1.03278	1.03235	1.00409	0.99899	0.96558	1.03237	1.02738	1.04129	1.04955	1.02701	1.02592	0.99445	0.989	0.9501	1.02651	1.02152	1.034478	1.04139857
0.375	1.04042	1.04098	1.00046	0.98831	0.9505	1.03962	1.0335	1.05256	1.06143	1.03173	1.0316	0.98607	0.97382	0.93022	1.03087	1.02501	1.042081	1.0481554
0.4375	1.02742	1.02589	0.99863	0.98001	0.93208	1.02799	1.0194	1.03874	1.04768	1.01479	1.01366	0.97979	0.94969	0.90669	1.01474	1.00726	1.024868	1.02983853
0.5	1.03232	1.03116	0.99552	0.9714	0.90831	1.033	1.02014	1.0485	1.05906	1.01364	1.01587	0.97152	0.93341	0.87796	1.01298	1.00254	1.030424	1.03494793
0.5625	1.03293	1.02591	0.98936	0.94899	0.87536	1.03511	1.01875	1.04563	1.05883	1.00636	1.00539	0.96074	0.90635	0.83999	1.00782	0.99314	1.023458	1.02897904
0.625	1.03223	1.01988	0.98337	0.93016	0.85209	1.03467	1.01481	1.04081	1.05366	0.99894	0.99213	0.94966	0.885	0.81321	1.00139	0.98288	1.012552	1.01809381
0.6875	1.02914	1.01313	0.97555	0.90747	0.81257	1.03141	1.00946	1.03396	1.04687	0.98963	0.97756	0.93707	0.86066	0.77093	0.99249	0.97207	0.998195	1.00542227
0.75	1.03469	1.01515	0.95789	0.88358	0.77791	1.03693	1.01156	1.03537	1.04612	0.98896	0.97219	0.91617	0.83631	0.73682	0.99231	0.969	0.989668	0.99758191
0.8125	1.02632	1.00512	0.94785	0.85968	0.74231	1.02818	1.00141	1.02545	1.03327	0.97651	0.95722	0.90254	0.81055	0.70698	0.97929	0.95637	0.974199	0.97672643
0.875	1.02764	1.00548	0.93281	0.83125	0.72211	1.02954	0.99726	1.02364	1.02854	0.97011	0.95026	0.88025	0.78302	0.6907	0.97126	0.94686	0.965007	0.96167904
0.9375	1.02266	0.99566	0.92711	0.80952	0.72582	1.02522	0.99004	1.01471	1.0182	0.95675	0.93826	0.86768	0.76438	0.68881	0.95925	0.93253	0.953063	0.94057749
1	1.02431	0.98836	0.91164	0.79621	0.73176	1.02737	0.9873	1.01104	1.01326	0.94859	0.92714	0.84923	0.75226	0.68701	0.95323	0.92189	0.944272	0.92724496

Expected value of IMS

Frequency	B_1	B_2	B_3	B_4	B_5	M_1	M_2	53Ht	4253Ht	$MA_2(B_1)$	$MA_2(B_2)$	$MA_2(B_3)$	$MA_2(B_4)$	$MA_2(B_5)$	$MA_2(M_1)$	$MA_2(M_2)$	MA_2(53Ht)	MA_2(4253Ht)
0.0625	1.47128	1.37301	0.74486	0.6843	0.61852	1.41774	1.24276	1.25121	1.24295	1.00985	0.96895	0.52225	0.48134	0.43811	1.00479	0.94801	1.036573	1.04585494
0.125	1.50329	1.40403	0.77882	0.74145	0.68977	1.44533	1.26442	1.26408	1.26267	1.03131	0.98885	0.53774	0.51617	0.48369	1.0246	0.96337	1.071587	1.06234932
0.1875	1.49942	1.40046	0.82126	0.79695	0.77295	1.4411	1.26168	1.2685	1.25859	1.02674	0.98065	0.55661	0.5435	0.53144	1.02208	0.9611	1.106034	1.06146203
0.25	1.52041	1.43771	0.87769	0.85517	0.8173	1.44893	1.27694	1.26702	1.26323	1.02839	0.99391	0.58705	0.57552	0.52557	1.02088	0.966	1.138189	1.05741385
0.3125	1.50637	1.4223	0.92872	0.9053	0.79684	1.42883	1.25432	1.25294	1.2436	1.02382	0.98896	0.63908	0.63226	0.56157	1.01495	0.96143	1.173997	1.0514262
0.375	1.53886	1.46656	0.93158	0.9169	0.85444	1.45365	1.29784	1.28367	1.27062	1.0327	1.00587	0.63322	0.64708	0.63937	1.02141	0.97832	1.228433	1.06288298
0.4375	1.54453	1.46078	1.02442	1.01409	0.91391	1.45652	1.29573	1.28803	1.25845	1.05722	1.04068	0.75138	0.70644	0.73968	1.04592	1.01855	1.286879	1.08164031
0.5	1.55848	1.49382	1.02123	1.02144	0.99034	1.4557	1.31579	1.28954	1.26736	1.05651	1.03888	0.75659	0.77022	0.87949	1.04158	1.02309	1.328706	1.07742076
0.5625	1.58439	1.51091	1.07481	1.03389	1.11661	1.48432	1.35316	1.32509	1.29351	1.08923	1.07764	0.84301	0.86101	1.05831	1.07659	1.07404	1.407721	1.11486939
0.625	1.57958	1.4819	1.08899	1.07324	1.23377	1.47558	1.32675	1.30342	1.26947	1.09938	1.08666	0.90552	0.9848	1.22284	1.08699	1.08433	1.447535	1.12040645
0.6875	1.58246	1.4968	1.13772	1.14486	1.39007	1.47145	1.33778	1.30312	1.25323	1.12251	1.1289	0.99753	1.11863	1.41751	1.10702	1.13115	1.50457	1.14443728
0.75	1.61312	1.52229	1.14903	1.25766	1.57297	1.50212	1.37038	1.31363	1.26796	1.14912	1.16563	1.07396	1.28543	1.63354	1.13203	1.1735	1.545804	1.17085209
0.8125	1.62198	1.54361	1.2538	1.38886	1.72685	1.50802	1.40098	1.34223	1.27788	1.2131	1.25327	1.22589	1.45187	1.78727	1.19615	1.26412	1.631609	1.24062887
0.875	1.63493	1.56704	1.30292	1.54168	1.81385	1.51693	1.41679	1.33832	1.28378	1.24253	1.30385	1.32793	1.62543	1.87009	1.22678	1.30793	1.671329	1.28592378
0.9375	1.65083	1.58729	1.42945	1.66948	1.80152	1.538	1.45362	1.37003	1.29707	1.3214	1.40673	1.47826	1.77068	1.88275	1.30455	1.41326	1.754418	1.34920027
1	1.64544	1.58345	1.49437	1.76103	1.81378	1.53505	1.46731	1.38282	1.29946	1.36433	1.46003	1.58654	1.87236	1.91845	1.34741	1.47427	1.810893	1.40495991

Table B.4: Results for positive slope and negative impulse

Slope positive, negative impulse

Estimated regression coefficients

Frequency	**B_1**	**B_2**	**B_3**	**B_4**	**B_5**	**M_1**	**M_2**	**53Ht**	**4253Ht**	**$MA_2(B_1)$**	**$MA_2(B_2)$**	**$MA_2(B_3)$**	**$MA_2(B_4)$**	**$MA_2(B_5)$**	**$MA_2(M_1)$**	**$MA_2(M_2)$**	**MA_2(53Ht)**	**MA_2(4253Ht)**
0.0625	1.02073	1.0271	1.01304	1.01747	1.03337	1.01981	1.02124	1.03402	1.04323	1.02369	1.03112	1.01416	1.02083	1.03612	1.02229	1.02399	1.038732	1.04867589
0.125	1.03575	1.04014	1.01775	0.99822	0.9918	1.03451	1.03455	1.04721	1.05904	1.03765	1.04285	1.01674	0.99527	0.98934	1.03626	1.03632	1.050378	1.06049215
0.1875	1.01694	1.01715	1.00337	1.00083	0.99483	1.01703	1.01561	1.02293	1.02814	1.01506	1.01537	1.00013	0.99733	0.98944	1.01512	1.0138	1.021345	1.02601713
0.25	1.0199	1.02079	1.00203	1.00055	0.99426	1.0196	1.01767	1.02812	1.03471	1.01634	1.01758	0.99626	0.994	0.9857	1.01606	1.01402	1.024443	1.03016792
0.3125	1.0203	1.01888	0.99897	0.99154	0.98432	1.02049	1.01619	1.02629	1.03175	1.01351	1.01173	0.98898	0.98192	0.97237	1.01366	1.00945	1.018869	1.02279113
0.375	1.01955	1.01845	0.99991	0.98727	0.97714	1.02026	1.01464	1.02731	1.03332	1.01004	1.00877	0.98608	0.97277	0.96459	1.01048	1.00539	1.01696	1.02035113
0.4375	1.02152	1.02112	0.99866	0.98288	0.97184	1.02198	1.01313	1.03498	1.04241	1.00897	1.00874	0.97998	0.96317	0.9424	1.00871	1.00111	1.020855	1.02434188
0.5	1.02121	1.0176	0.99386	0.97381	0.9403	1.02246	1.00982	1.03271	1.04257	1.00219	1.00144	0.97028	0.94795	0.88865	1.00234	0.99168	1.014517	1.01890124
0.5625	1.03377	1.02686	0.98659	0.95531	0.87625	1.03564	1.01912	1.04803	1.06067	1.00666	1.0058	0.95762	0.92217	0.84084	1.00784	0.99298	1.024988	1.02992232
0.625	1.02733	1.01388	0.97679	0.93949	0.84957	1.02961	1.00903	1.03512	1.04701	0.9937	0.98548	0.94343	0.90123	0.81009	0.99608	0.97671	1.006234	1.01077776
0.6875	1.02489	1.00933	0.97349	0.92893	0.81457	1.02728	1.00621	1.02948	1.04077	0.98569	0.97353	0.93603	0.88596	0.77258	0.98861	0.96889	0.993419	0.99949386
0.75	1.02549	1.00656	0.96491	0.91255	0.77995	1.02751	1.00292	1.02563	1.03347	0.98069	0.96389	0.92391	0.86518	0.73666	0.98394	0.96112	0.981166	0.98554412
0.8125	1.01907	0.99816	0.94835	0.88493	0.74433	1.02051	0.9936	1.01755	1.022	0.96951	0.95021	0.90289	0.832	0.70617	0.97202	0.94857	0.966142	0.96650912
0.875	1.01923	0.9964	0.93843	0.86092	0.73343	1.02188	0.99059	1.01419	1.01696	0.96278	0.94318	0.8869	0.80478	0.69832	0.9647	0.94117	0.957354	0.95266913
0.9375	1.02151	0.99236	0.92522	0.83748	0.73569	1.02406	0.98675	1.01367	1.01354	0.955	0.93498	0.86889	0.78037	0.69655	0.95767	0.92915	0.95073	0.94040789
1	1.0234	0.98593	0.9062	0.80576	0.72064	1.02638	0.98441	1.00971	1.01061	0.94691	0.92376	0.84603	0.75217	0.67676	0.95137	0.91881	0.941021	0.92839676

Expected value of IMS

Frequency	**B_1**	**B_2**	**B_3**	**B_4**	**B_5**	**M_1**	**M_2**	**53Ht**	**4253Ht**	**$MA_2(B_1)$**	**$MA_2(B_2)$**	**$MA_2(B_3)$**	**$MA_2(B_4)$**	**$MA_2(B_5)$**	**$MA_2(M_1)$**	**$MA_2(M_2)$**	**MA_2(53Ht)**	**MA_2(4253Ht)**
0.0625	1.49254	1.39698	0.73037	0.67767	0.61685	1.43858	1.26608	1.27041	1.25739	1.02581	0.98765	0.51225	0.48009	0.44253	1.01856	0.96787	1.023	1.05837647
0.125	1.49953	1.40622	0.81673	0.68622	0.65343	1.4399	1.26404	1.26359	1.2519	1.02519	0.98833	0.56071	0.44382	0.4338	1.01903	0.96168	1.00009	1.04936152
0.1875	1.50046	1.41656	0.85408	0.81566	0.80269	1.43598	1.26524	1.26718	1.24665	1.01615	0.98318	0.57626	0.55248	0.55023	1.00795	0.95606	0.99081	1.0449385
0.25	1.52739	1.43492	0.88467	0.87057	0.8683	1.45836	1.28045	1.28795	1.26139	1.03094	0.99587	0.59679	0.59677	0.60662	1.02272	0.96752	1.011992	1.05674033
0.3125	1.52197	1.43711	0.95922	0.95301	0.95207	1.44876	1.28298	1.2911	1.26332	1.04069	1.00975	0.66512	0.67549	0.69968	1.03312	0.98813	1.04084	1.07158392
0.375	1.51823	1.4411	1.02093	1.01373	1.03511	1.43386	1.27111	1.26509	1.23256	1.03455	1.01718	0.72683	0.74972	0.80421	1.02179	0.98653	1.049237	1.05014702
0.4375	1.55815	1.4761	1.05157	1.05777	1.08792	1.46707	1.30929	1.30815	1.27393	1.06636	1.04062	0.76089	0.80449	0.88258	1.05496	1.02133	1.098164	1.08954821
0.5	1.55338	1.46079	1.07781	1.09215	1.15016	1.45604	1.2852	1.28796	1.25489	1.06483	1.03749	0.8134	0.87814	0.94706	1.05013	1.01653	1.12405	1.08116508
0.5625	1.56698	1.48392	1.03443	1.06217	1.1267	1.46234	1.31346	1.28837	1.2646	1.04983	1.02941	0.80379	0.90685	1.07316	1.03674	1.02156	1.166291	1.07345741
0.625	1.59079	1.48055	1.09254	1.14991	1.2444	1.48377	1.31598	1.29679	1.2595	1.09236	1.06723	0.90302	1.04396	1.22449	1.07747	1.06366	1.210369	1.10407555
0.6875	1.59886	1.51207	1.16929	1.2701	1.40755	1.49075	1.35524	1.331	1.27908	1.1413	1.14294	1.02401	1.21859	1.42887	1.12656	1.14532	1.283127	1.16387906
0.75	1.6142	1.52387	1.24853	1.38796	1.57528	1.50398	1.36808	1.33882	1.28049	1.18118	1.19768	1.14615	1.3897	1.62848	1.16468	1.20195	1.354813	1.20923523
0.8125	1.58814	1.49338	1.26115	1.50074	1.72432	1.47786	1.35629	1.31195	1.24531	1.19768	1.22627	1.2314	1.55092	1.78909	1.18255	1.23617	1.4108	1.23062114
0.875	1.63288	1.56966	1.36059	1.63955	1.79319	1.51403	1.42287	1.35796	1.28801	1.27459	1.34071	1.38066	1.70918	1.8584	1.25593	1.34128	1.500282	1.3218027
0.9375	1.63729	1.56689	1.40793	1.7497	1.79873	1.52325	1.43851	1.36405	1.28474	1.31319	1.39539	1.48073	1.82896	1.88336	1.29541	1.40405	1.570814	1.37212761
1	1.64438	1.58351	1.47426	1.81242	1.81902	1.53831	1.4733	1.3894	1.3076	1.3565	1.45142	1.58345	1.90508	1.92313	1.34054	1.46728	1.643174	1.43606461

Appendix C

Table of results on other examples

Table C.1: Results for Blocks, Bumps, HeaviSine and Doppler functions

Estimated regression coefficients

	B_1	B_2	B_3	B_4	B_5	M_1	M_2	53Ht	4253Ht	$MA_2(B_1)$	$MA_2(B_2)$	$MA_2(B_3)$	$MA_2(B_4)$	$MA_2(B_5)$	$MA_2(M_1)$	$MA_2(M_2)$	MA_2(53Ht)	MA_2(4253Ht)
Blocks	0.99901	0.99859	0.99821	0.99793	0.9976	0.99884	0.99791	0.99383	0.99262	0.98638	0.98605	0.9857	0.98538	0.985	0.98644	0.98601	0.98547	0.98534039
Bumps	0.9903	0.97818	0.96092	0.94494	0.92573	0.99053	0.97847	0.98401	0.98605	0.97759	0.96778	0.95298	0.93865	0.92082	0.97783	0.96811	0.973022	0.97508692
HeaviSine	0.99965	0.99952	0.99949	0.99941	0.99933	0.99975	0.99988	0.99989	0.99977	0.99952	0.99939	0.99937	0.99929	0.99921	0.99961	0.99966	0.999701	0.99961594
Doppler	0.99906	0.99772	0.9959	0.99421	0.99232	0.99923	0.99756	0.99881	0.99625	0.99576	0.99473	0.99331	0.99188	0.99031	0.99593	0.99457	0.995709	0.99358178

Expected IMS

	B_1	B_2	B_3	B_4	B_5	M_1	M_2	53Ht	4253Ht	$MA_2(B_1)$	$MA_2(B_2)$	$MA_2(B_3)$	$MA_2(B_4)$	$MA_2(B_5)$	$MA_2(M_1)$	$MA_2(M_2)$	MA_2(53Ht)	MA_2(4253Ht)
Blocks	0.71288	0.64121	0.55989	0.52316	0.49542	0.67464	0.54797	0.6677	0.67827	0.78895	0.78226	0.77164	0.76598	0.76231	0.79258	0.78037	0.798086	0.78634876
Bumps	0.79854	0.85174	0.99724	1.16886	1.38723	0.75023	0.76341	0.6996	0.66067	0.72508	0.80946	0.96541	1.13459	1.35299	0.72983	0.80772	0.78745	0.75750808
HeaviSine	0.68869	0.64904	0.56543	0.52933	0.50164	0.67299	0.53899	0.54344	0.43076	0.44263	0.43916	0.4107	0.39284	0.3777	0.46219	0.42807	0.455931	0.38693571
Doppler	0.74377	0.69758	0.66794	0.67877	0.70233	0.68534	0.58001	0.55847	0.47644	0.48363	0.48865	0.50465	0.53653	0.56982	0.4895	0.49007	0.499346	0.47443874

Appendix D

Standard and Poor 500 data

Table D.1: U.S. Standard and Poor's 500 Stock Index

Date	Open	High	Low	Close
1999/01/04	1229,23	1248,81	1219,10	1228,10
1999/01/05	1228,10	1246,11	1228,10	1244,78
1999/01/06	1244,78	1272,50	1244,78	1272,34
1999/01/07	1272,34	1272,34	1257,68	1269,73
1999/01/08	1269,73	1278,24	1261,82	1275,09
1999/01/11	1275,09	1276,22	1253,34	1263,88
1999/01/12	1263,88	1264,45	1238,29	1239,51
1999/01/13	1239,51	1247,75	1205,46	1234,40
1999/01/14	1234,40	1236,81	1209,54	1212,19
1999/01/15	1212,19	1243,26	1212,19	1243,26
1999/01/19	1243,26	1253,27	1234,91	1252,00
1999/01/20	1252,00	1274,07	1251,54	1256,62
1999/01/21	1256,62	1256,94	1232,19	1235,16
1999/01/22	1235,16	1236,41	1217,97	1225,19
1999/01/25	1225,19	1233,98	1219,46	1233,98
1999/01/26	1233,98	1253,25	1233,98	1252,31
1999/01/27	1252,31	1262,61	1242,82	1243,17
1999/01/28	1243,17	1266,40	1243,17	1265,37
1999/01/29	1265,37	1280,37	1255,18	1279,64
1999/02/01	1279,64	1283,75	1271,31	1273,00
1999/02/02	1273,00	1273,49	1247,56	1261,99
1999/02/03	1261,99	1276,04	1255,27	1272,07
1999/02/04	1272,07	1272,23	1248,36	1248,49
1999/02/05	1248,49	1251,86	1232,28	1239,40
1999/02/08	1239,40	1246,93	1231,98	1243,77
1999/02/09	1243,77	1243,97	1215,63	1216,14
1999/02/10	1216,14	1226,78	1211,89	1223,55
1999/02/11	1223,55	1254,05	1223,19	1254,04
1999/02/12	1254,04	1254,04	1225,53	1230,13

Date	Open	High	Low	Close
1999/02/16	1230,13	1252,17	1230,13	1241,87
1999/02/17	1241,87	1249,31	1220,92	1224,03
1999/02/18	1224,03	1239,13	1220,70	1237,28
1999/02/19	1237,28	1247,91	1232,03	1239,22
1999/02/22	1239,22	1272,22	1239,22	1272,14
1999/02/23	1272,14	1280,38	1263,36	1271,18
1999/02/24	1271,18	1283,84	1251,94	1253,41
1999/02/25	1253,41	1253,41	1225,01	1245,02
1999/02/26	1245,02	1246,73	1226,24	1238,33
1999/03/01	1238,33	1238,70	1221,88	1236,16
1999/03/02	1236,16	1248,31	1221,87	1225,50
1999/03/03	1225,50	1231,63	1216,03	1227,70
1999/03/04	1227,70	1247,74	1227,70	1246,64
1999/03/05	1246,64	1275,73	1246,64	1275,47
1999/03/08	1275,47	1282,74	1271,58	1282,73
1999/03/09	1282,73	1293,74	1275,11	1279,84
1999/03/10	1279,84	1287,02	1275,16	1286,84
1999/03/11	1286,84	1306,43	1286,84	1297,68
1999/03/12	1297,68	1304,42	1289,17	1294,59
1999/03/15	1294,59	1307,47	1291,03	1307,26
1999/03/16	1307,26	1311,11	1302,29	1306,38
1999/03/17	1306,38	1306,55	1292,63	1297,82
1999/03/18	1297,82	1317,62	1294,75	1316,55
1999/03/19	1316,55	1323,82	1298,92	1299,29
1999/03/22	1299,29	1303,84	1294,26	1297,01
1999/03/23	1297,01	1297,01	1257,46	1262,14
1999/03/24	1262,14	1269,02	1256,43	1268,59
1999/03/25	1268,59	1289,99	1268,59	1289,99
1999/03/26	1289,99	1289,99	1277,25	1282,80
1999/03/29	1282,80	1311,76	1282,80	1310,17
1999/03/30	1310,17	1310,17	1295,47	1300,75
1999/03/31	1300,75	1313,60	1285,87	1286,37
1999/04/01	1286,37	1294,54	1282,56	1293,72
1999/04/05	1293,72	1321,12	1293,72	1321,12
1999/04/06	1321,12	1326,76	1311,07	1317,89
1999/04/07	1317,89	1329,58	1312,59	1326,89
1999/04/08	1326,89	1344,08	1321,60	1343,98
1999/04/09	1343,98	1351,22	1335,24	1348,35
1999/04/12	1348,35	1358,69	1333,48	1358,63
1999/04/13	1358,64	1362,38	1344,03	1349,82
1999/04/14	1349,82	1357,24	1326,41	1328,44
1999/04/15	1328,44	1332,41	1308,38	1322,85
1999/04/16	1322,86	1325,03	1311,40	1319,00
1999/04/19	1319,00	1340,10	1284,48	1289,48
1999/04/20	1289,48	1306,30	1284,21	1306,17
1999/04/21	1306,17	1336,12	1301,84	1336,12
1999/04/22	1336,12	1358,84	1336,12	1358,82
1999/04/23	1358,83	1363,65	1348,45	1356,85
1999/04/26	1356,85	1363,56	1353,72	1360,04

Date	Open	High	Low	Close
1999/04/27	1360,04	1371,56	1356,55	1362,80
1999/04/28	1362,80	1368,62	1348,29	1350,91
1999/04/29	1350,91	1356,75	1336,81	1342,83
1999/04/30	1342,83	1351,83	1314,58	1335,18
1999/05/03	1335,18	1354,63	1329,01	1354,63
1999/05/04	1354,63	1354,64	1330,64	1332,00
1999/05/05	1332,00	1347,32	1317,44	1347,31
1999/05/06	1347,31	1348,36	1322,56	1332,05
1999/05/07	1332,05	1345,99	1332,05	1345,00
1999/05/10	1345,00	1352,01	1334,00	1340,30
1999/05/11	1340,30	1360,00	1340,30	1355,61
1999/05/12	1355,61	1367,36	1333,10	1364,00
1999/05/13	1364,00	1375,98	1364,00	1367,56
1999/05/14	1367,56	1367,56	1332,63	1337,80
1999/05/17	1337,80	1339,95	1321,19	1339,49
1999/05/18	1339,49	1345,44	1323,46	1333,32
1999/05/19	1333,32	1344,23	1327,05	1344,23
1999/05/20	1344,23	1350,49	1338,83	1338,83
1999/05/21	1338,83	1340,88	1326,19	1330,29
1999/05/24	1330,29	1333,02	1303,53	1306,65
1999/05/25	1306,65	1317,52	1284,38	1284,4
1999/05/26	1284,40	1304,85	1278,43	1304,76
1999/05/27	1304,76	1304,76	1277,31	1281,41
1999/05/28	1281,41	1304,00	1281,41	1301,84
1999/06/01	1301,84	1301,84	1281,44	1294,26
1999/06/02	1294,26	1297,10	1277,47	1294,81
1999/06/03	1294,81	1304,15	1294,20	1299,54
1999/06/04	1299,54	1327,75	1299,54	1327,75
1999/06/07	1327,75	1336,42	1325,89	1334,52
1999/06/08	1334,52	1334,52	1312,83	1317,33
1999/06/09	1317,33	1326,01	1314,73	1318,64
1999/06/10	1318,64	1318,64	1293,28	1302,82
1999/06/11	1302,82	1311,97	1287,88	1293,64
1999/06/14	1293,64	1301,99	1292,20	1294,00
1999/06/15	1294,00	1310,76	1294,00	1301,16
1999/06/16	1301,16	1332,83	1301,16	1330,41
1999/06/17	1330,41	1343,54	1322,75	1339,90
1999/06/18	1339,90	1344,48	1333,52	1342,84
1999/06/21	1342,84	1349,06	1337,63	1349,00
1999/06/22	1349,00	1351,12	1335,52	1335,88
1999/06/23	1335,87	1335,88	1322,55	1333,06
1999/06/24	1333,06	1333,06	1308,47	1315,78
1999/06/25	1315,78	1329,13	1312,64	1315,31
1999/06/28	1315,31	1333,68	1315,31	1331,35
1999/06/29	1331,35	1351,51	1328,40	1351,45
1999/06/30	1351,45	1372,93	1338,78	1372,71
1999/07/01	1372,71	1382,80	1360,80	1380,96
1999/07/02	1380,96	1391,22	1379,57	1391,22
1999/07/06	1391,22	1405,29	1387,08	1388,12

Date	Open	High	Low	Close
1999/07/07	1388,12	1395,88	1384,95	1395,86
1999/07/08	1395,86	1403,25	1386,69	1394,42
1999/07/09	1394,42	1403,28	1394,42	1403,28
1999/07/12	1403,28	1406,82	1394,70	1399,10
1999/07/13	1399,10	1399,10	1386,84	1393,56
1999/07/14	1393,56	1400,05	1386,51	1398,17
1999/07/15	1398,17	1409,84	1398,17	1409,62
1999/07/16	1409,62	1418,78	1407,07	1418,78
1999/07/19	1418,78	1420,33	1404,56	1407,65
1999/07/20	1407,65	1407,65	1375,15	1377,10
1999/07/21	1377,10	1386,66	1372,63	1379,29
1999/07/22	1379,29	1379,29	1353,98	1360,97
1999/07/23	1360,97	1367,41	1349,91	1356,94
1999/07/26	1356,94	1358,61	1346,20	1347,76
1999/07/27	1347,75	1368,70	1347,75	1362,84
1999/07/28	1362,84	1370,53	1355,54	1365,40
1999/07/29	1365,40	1365,40	1332,82	1341,03
1999/07/30	1341,03	1350,92	1328,49	1328,72
1999/08/02	1328,72	1344,69	1325,21	1328,05
1999/08/03	1328,05	1336,13	1314,91	1322,18
1999/08/04	1322,18	1330,16	1304,50	1305,33
1999/08/05	1305,33	1313,71	1287,23	1313,71
1999/08/06	1313,71	1316,74	1293,19	1300,29
1999/08/09	1300,29	1306,68	1295,99	1297,80
1999/08/10	1297,80	1298,62	1267,73	1281,43
1999/08/11	1281,43	1301,93	1281,43	1301,93
1999/08/12	1301,93	1313,61	1298,06	1298,16
1999/08/13	1298,16	1327,72	1298,16	1327,68
1999/08/16	1327,68	1331,05	1320,51	1330,77
1999/08/17	1330,77	1344,16	1328,76	1344,16
1999/08/18	1344,16	1344,16	1332,13	1332,84
1999/08/19	1332,84	1332,84	1315,35	1323,59
1999/08/20	1323,59	1336,61	1323,59	1336,61
1999/08/23	1336,61	1360,24	1336,61	1360,22
1999/08/24	1360,22	1373,32	1353,63	1363,50
1999/08/25	1363,50	1382,84	1359,20	1381,79
1999/08/26	1381,79	1381,79	1361,53	1362,01
1999/08/27	1362,01	1365,63	1347,35	1348,27
1999/08/30	1348,27	1350,70	1322,80	1324,02
1999/08/31	1324,02	1333,27	1306,96	1320,41
1999/09/01	1320,41	1331,18	1320,39	1331,07
1999/09/02	1331,07	1331,07	1304,88	1319,11
1999/09/03	1319,11	1357,74	1319,11	1357,24
1999/09/07	1357,24	1361,39	1349,59	1350,45
1999/09/08	1350,45	1355,18	1337,36	1344,15
1999/09/09	1344,15	1347,66	1333,91	1347,66
1999/09/10	1347,66	1357,62	1346,20	1351,66
1999/09/13	1351,66	1351,66	1341,70	1344,13
1999/09/14	1344,13	1344,18	1330,61	1336,29

Date	Open	High	Low	Close
1999/09/15	1336,29	1347,21	1317,97	1317,97
1999/09/16	1317,97	1322,51	1299,97	1318,48
1999/09/17	1318,48	1337,59	1318,48	1335,42
1999/09/20	1335,42	1338,38	1330,61	1335,53
1999/09/21	1335,52	1335,53	1301,97	1307,58
1999/09/22	1307,58	1316,18	1297,81	1310,51
1999/09/23	1310,51	1315,25	1277,30	1280,41
1999/09/24	1280,41	1281,17	1263,84	1277,36
1999/09/27	1277,36	1295,03	1277,36	1283,31
1999/09/28	1283,31	1285,55	1256,26	1282,20
1999/09/29	1282,20	1288,83	1268,16	1268,37
1999/09/30	1268,37	1291,31	1268,37	1282,71
1999/10/01	1282,71	1283,17	1265,78	1282,81
1999/10/04	1282,81	1304,60	1282,81	1304,60
1999/10/05	1304,60	1316,41	1286,44	1301,35
1999/10/06	1301,35	1325,46	1301,35	1325,40
1999/10/07	1325,40	1328,05	1314,13	1317,64
1999/10/08	1317,64	1336,61	1311,88	1336,02
1999/10/11	1336,02	1339,23	1332,96	1335,21
1999/10/12	1335,21	1335,21	1311,80	1313,04
1999/10/13	1313,04	1313,04	1282,80	1285,55
1999/10/14	1285,55	1289,63	1267,62	1283,42
1999/10/15	1283,42	1283,42	1245,39	1247,41
1999/10/18	1247,41	1254,13	1233,70	1254,13
1999/10/19	1254,13	1279,32	1254,13	1261,32
1999/10/20	1261,32	1289,44	1261,32	1289,43
1999/10/21	1289,43	1289,43	1265,61	1283,61
1999/10/22	1283,61	1308,81	1283,61	1301,65
1999/10/25	1301,65	1301,68	1286,07	1293,63
1999/10/26	1293,63	1303,46	1281,86	1281,91
1999/10/27	1281,91	1299,39	1280,48	1296,71
1999/10/28	1296,71	1342,47	1296,71	1342,44
1999/10/29	1342,44	1373,17	1342,44	1362,93
1999/11/01	1362,93	1367,30	1354,05	1354,12
1999/11/02	1354,12	1369,32	1346,41	1347,74
1999/11/03	1347,74	1360,33	1347,74	1354,93
1999/11/04	1354,93	1369,41	1354,93	1362,64
1999/11/05	1362,64	1387,48	1362,64	1370,23
1999/11/08	1370,23	1380,78	1365,87	1377,01
1999/11/09	1377,01	1383,81	1361,45	1365,28
1999/11/10	1365,28	1379,18	1359,98	1373,46
1999/11/11	1373,46	1382,12	1372,19	1381,46
1999/11/12	1381,46	1396,12	1368,54	1396,06
1999/11/15	1396,06	1398,58	1392,28	1394,39
1999/11/16	1394,39	1420,36	1394,39	1420,07
1999/11/17	1420,07	1423,44	1410,69	1410,71
1999/11/18	1410,71	1425,31	1410,71	1424,94
1999/11/19	1424,94	1424,94	1417,54	1422,00
1999/11/22	1422,00	1425,00	1412,40	1420,94

Date	Open	High	Low	Close
1999/11/23	1420,94	1423,91	1402,20	1404,64
1999/11/24	1404,64	1419,71	1399,17	1417,08
1999/11/26	1417,08	1425,24	1416,14	1416,62
1999/11/29	1416,62	1416,62	1404,15	1407,83
1999/11/30	1407,83	1410,59	1386,95	1389,07
1999/12/01	1388,91	1400,12	1387,38	1397,72
1999/12/02	1397,72	1409,04	1397,72	1409,04
1999/12/03	1409,04	1447,42	1409,04	1433,30
1999/12/06	1433,30	1434,15	1418,25	1423,33
1999/12/07	1423,33	1426,81	1409,17	1409,17
1999/12/08	1409,17	1415,66	1403,88	1403,88
1999/12/09	1403,88	1418,43	1391,47	1408,11
1999/12/10	1408,11	1421,58	1405,65	1417,04
1999/12/13	1417,04	1421,58	1410,10	1415,22
1999/12/14	1415,22	1418,30	1401,59	1403,17
1999/12/15	1403,17	1417,40	1396,20	1413,33
1999/12/16	1413,32	1423,11	1408,35	1418,78
1999/12/17	1418,78	1431,77	1418,78	1421,03
1999/12/20	1421,03	1429,16	1411,10	1418,09
1999/12/21	1418,09	1436,47	1414,80	1433,43
1999/12/22	1433,43	1440,02	1429,13	1435,99
1999/12/23	1436,13	1461,44	1436,13	1458,34
1999/12/27	1458,34	1463,19	1450,83	1457,10
1999/12/28	1457,09	1462,68	1452,78	1457,66
1999/12/29	1457,66	1467,47	1457,66	1463,46
1999/12/30	1463,46	1473,10	1462,60	1464,47
1999/12/31	1464,47	1472,42	1458,19	1469,25
2000/01/03	1469,25	1478,00	1438,36	1455,22
2000/01/04	1455,22	1455,22	1397,43	1399,42
2000/01/05	1399,42	1413,27	1377,68	1402,11
2000/01/06	1402,11	1411,90	1392,10	1403,45
2000/01/07	1403,45	1441,47	1400,73	1441,47
2000/01/10	1441,47	1464,36	1441,47	1457,60
2000/01/11	1457,60	1458,66	1434,42	1438,56
2000/01/12	1438,56	1442,60	1427,08	1432,25
2000/01/13	1432,25	1454,20	1432,25	1449,68
2000/01/14	1449,68	1473,00	1449,68	1465,15
2000/01/18	1465,15	1465,15	1451,30	1455,14
2000/01/19	1455,14	1461,39	1448,68	1455,90
2000/01/20	1455,90	1465,71	1438,54	1445,57
2000/01/21	1445,57	1453,18	1439,60	1441,36
2000/01/24	1441,36	1454,09	1395,42	1401,53
2000/01/25	1401,53	1414,26	1388,49	1410,03
2000/01/26	1410,03	1412,73	1400,16	1404,09
2000/01/27	1404,09	1418,86	1370,99	1398,56
2000/01/28	1398,56	1398,56	1356,20	1360,16
2000/01/31	1360,16	1394,48	1350,14	1394,46
2000/02/01	1394,46	1412,49	1384,79	1409,28
2000/02/02	1409,28	1420,61	1403,49	1409,12

Date	Open	High	Low	Close
2000/02/03	1409,12	1425,78	1398,52	1424,97
2000/02/04	1424,97	1435,91	1420,63	1424,37
2000/02/07	1424,37	1427,15	1413,33	1424,24
2000/02/08	1424,24	1441,83	1424,24	1441,72
2000/02/09	1441,72	1444,55	1411,65	1411,71
2000/02/10	1411,70	1422,10	1406,43	1416,83
2000/02/11	1416,83	1416,83	1378,89	1387,12
2000/02/14	1387,12	1394,93	1380,53	1389,94
2000/02/15	1389,94	1407,72	1376,25	1402,05
2000/02/16	1402,05	1404,55	1385,58	1387,67
2000/02/17	1387,67	1399,88	1380,07	1388,26
2000/02/18	1388,26	1388,59	1345,32	1346,09
2000/02/22	1346,09	1358,11	1331,88	1352,17
2000/02/23	1352,17	1370,11	1342,44	1360,69
2000/02/24	1360,69	1364,80	1329,88	1353,43
2000/02/25	1353,43	1362,14	1329,15	1333,36
2000/02/28	1333,36	1360,82	1325,07	1348,05
2000/02/29	1348,05	1369,63	1348,05	1366,42
2000/03/01	1366,42	1383,46	1366,42	1379,19
2000/03/02	1379,19	1386,56	1370,35	1381,76
2000/03/03	1381,76	1410,88	1381,76	1409,17
2000/03/06	1409,17	1409,74	1384,75	1391,28
2000/03/07	1391,28	1399,21	1349,99	1355,62
2000/03/08	1355,62	1373,79	1346,62	1366,70
2000/03/09	1366,70	1401,82	1357,88	1401,69
2000/03/10	1401,69	1413,46	1392,07	1395,07
2000/03/13	1395,07	1398,39	1364,84	1383,62
2000/03/14	1383,62	1395,15	1359,15	1359,15
2000/03/15	1359,15	1397,99	1356,99	1392,14
2000/03/16	1392,15	1458,47	1392,15	1458,47
2000/03/17	1458,47	1477,33	1453,32	1464,47
2000/03/20	1464,47	1470,30	1448,49	1456,63
2000/03/21	1456,63	1493,92	1446,06	1493,87
2000/03/22	1493,87	1505,08	1487,33	1500,64
2000/03/23	1500,64	1532,50	1492,39	1527,35
2000/03/24	1527,35	1552,87	1516,83	1527,46
2000/03/27	1527,46	1534,63	1518,46	1523,86
2000/03/28	1523,86	1527,36	1507,09	1507,73
2000/03/29	1507,73	1521,45	1497,45	1508,52
2000/03/30	1508,52	1517,38	1474,63	1487,92
2000/03/31	1487,92	1519,81	1484,38	1498,58
2000/04/03	1498,58	1507,19	1486,96	1505,97
2000/04/04	1505,98	1526,45	1416,41	1494,73
2000/04/05	1494,73	1506,55	1478,05	1487,37
2000/04/06	1487,37	1511,76	1487,37	1501,34
2000/04/07	1501,34	1518,68	1501,34	1516,35
2000/04/10	1516,35	1527,19	1503,35	1504,46
2000/04/11	1504,46	1512,80	1486,78	1500,59
2000/04/12	1500,59	1509,08	1466,15	1467,17

Date	Open	High	Low	Close
2000/04/13	1467,17	1477,52	1439,34	1440,51
2000/04/14	1440,51	1440,51	1339,40	1356,56
2000/04/17	1356,56	1401,53	1346,50	1401,44
2000/04/18	1401,44	1441,61	1397,81	1441,61
2000/04/19	1441,61	1447,69	1424,26	1427,47
2000/04/20	1427,47	1435,49	1422,08	1434,54
2000/04/24	1434,54	1434,54	1407,13	1429,86
2000/04/25	1429,86	1477,67	1429,86	1477,44
2000/04/26	1477,44	1482,94	1456,98	1460,99
2000/04/27	1460,99	1469,21	1434,81	1464,92
2000/04/28	1464,92	1473,62	1448,15	1452,43
2000/05/01	1452,43	1481,51	1452,43	1468,25
2000/05/02	1468,25	1468,25	1445,22	1446,29
2000/05/03	1446,29	1446,29	1398,36	1415,10
2000/05/04	1415,10	1420,99	1404,94	1409,57
2000/05/05	1409,57	1436,03	1405,08	1432,63
2000/05/08	1432,63	1432,63	1417,05	1424,17
2000/05/09	1424,17	1430,28	1401,85	1412,14
2000/05/10	1412,14	1412,14	1375,14	1383,05
2000/05/11	1383,05	1410,26	1383,05	1407,81
2000/05/12	1407,81	1430,13	1407,81	1420,96
2000/05/15	1420,96	1452,39	1416,54	1452,36
2000/05/16	1452,36	1470,40	1450,76	1466,04
2000/05/17	1466,04	1466,04	1441,67	1447,80
2000/05/18	1447,80	1458,04	1436,59	1437,21
2000/05/19	1437,21	1437,21	1401,74	1406,95
2000/05/22	1406,95	1410,55	1368,73	1400,72
2000/05/23	1400,72	1403,77	1373,43	1373,86
2000/05/24	1373,86	1401,75	1361,09	1399,05
2000/05/25	1399,05	1411,65	1373,93	1381,52
2000/05/26	1381,52	1391,42	1369,75	1378,02
2000/05/30	1378,02	1422,45	1378,02	1422,45
2000/05/31	1422,44	1434,49	1415,50	1420,60
2000/06/01	1420,60	1448,81	1420,60	1448,81
2000/06/02	1448,81	1483,23	1448,81	1477,26
2000/06/05	1477,26	1477,28	1464,68	1467,63
2000/06/06	1467,63	1471,36	1454,74	1457,84
2000/06/07	1457,84	1474,64	1455,06	1471,36
2000/06/08	1471,36	1475,65	1456,49	1461,67
2000/06/09	1461,67	1472,67	1454,96	1456,95
2000/06/12	1456,95	1462,93	1445,99	1446,00
2000/06/13	1446,00	1470,42	1442,38	1469,44
2000/06/14	1469,44	1483,62	1467,71	1470,54
2000/06/15	1470,54	1482,04	1464,62	1478,73
2000/06/16	1478,73	1480,77	1460,42	1464,46
2000/06/19	1464,46	1488,93	1459,05	1486,00
2000/06/20	1486,00	1487,32	1470,18	1475,95
2000/06/21	1475,95	1482,19	1468,00	1479,13
2000/06/22	1479,13	1479,13	1448,03	1452,18

Date	Open	High	Low	Close
2000/06/23	1452,18	1459,94	1438,31	1441,48
2000/06/26	1441,48	1459,66	1441,48	1455,31
2000/06/27	1455,31	1463,35	1450,55	1450,55
2000/06/28	1450,55	1467,63	1450,55	1454,82
2000/06/29	1454,82	1455,14	1434,63	1442,39
2000/06/30	1442,39	1454,68	1438,71	1454,60
2000/07/03	1454,60	1469,58	1450,85	1469,54
2000/07/05	1469,54	1469,54	1442,45	1446,23
2000/07/06	1446,23	1461,65	1439,56	1456,67
2000/07/07	1456,67	1484,12	1456,67	1478,90
2000/07/10	1478,90	1486,56	1474,76	1475,62
2000/07/11	1475,62	1488,77	1470,48	1480,88
2000/07/12	1480,88	1497,69	1480,88	1492,92
2000/07/13	1492,92	1501,39	1489,65	1495,84
2000/07/14	1495,84	1509,99	1494,56	1509,98
2000/07/17	1509,98	1517,32	1505,26	1510,49
2000/07/18	1510,49	1510,49	1491,35	1493,74
2000/07/19	1493,74	1495,63	1479,92	1481,96
2000/07/20	1481,96	1501,92	1481,96	1495,57
2000/07/21	1495,57	1495,57	1477,91	1480,19
2000/07/24	1480,19	1485,88	1463,80	1464,29
2000/07/25	1464,29	1476,23	1464,29	1474,47
2000/07/26	1474,47	1474,47	1452,42	1452,42
2000/07/27	1452,42	1464,91	1445,33	1449,62
2000/07/28	1449,62	1456,68	1413,89	1419,89
2000/07/31	1419,89	1437,65	1418,71	1430,83
2000/08/01	1430,83	1443,54	1428,96	1438,10
2000/08/02	1438,10	1451,59	1433,49	1438,70
2000/08/03	1438,70	1454,19	1425,43	1452,56
2000/08/04	1452,56	1462,93	1451,31	1462,93
2000/08/07	1462,93	1480,80	1460,72	1479,32
2000/08/08	1479,32	1484,52	1472,61	1482,80
2000/08/09	1482,80	1490,33	1471,16	1472,87
2000/08/10	1472,87	1475,15	1459,89	1460,25
2000/08/11	1460,25	1475,72	1453,06	1471,84
2000/08/14	1471,84	1491,64	1468,56	1491,56
2000/08/15	1491,56	1493,12	1482,74	1484,43
2000/08/16	1484,43	1496,09	1475,74	1479,85
2000/08/17	1479,85	1499,32	1479,85	1496,07
2000/08/18	1496,07	1499,47	1488,99	1491,72
2000/08/21	1491,72	1502,84	1491,13	1499,48
2000/08/22	1499,48	1508,45	1497,42	1498,13
2000/08/23	1498,13	1507,20	1489,52	1505,97
2000/08/24	1505,97	1511,16	1501,25	1508,31
2000/08/25	1508,31	1513,47	1505,09	1506,45
2000/08/28	1506,45	1523,95	1506,45	1514,09
2000/08/29	1514,09	1514,81	1505,46	1509,84
2000/08/30	1509,84	1510,49	1500,09	1502,59

Date	Open	High	Low	Close
2000/08/31	1502,59	1525,21	1502,59	1517,68
2000/09/01	1517,68	1530,09	1515,53	1520,77
2000/09/05	1520,77	1520,77	1504,21	1507,08
2000/09/06	1507,08	1512,61	1492,12	1492,25
2000/09/07	1492,25	1505,34	1492,25	1502,51
2000/09/08	1502,51	1502,51	1489,88	1494,50
2000/09/11	1494,50	1506,76	1483,01	1489,26
2000/09/12	1489,26	1496,93	1479,67	1481,99
2000/09/13	1481,99	1487,45	1473,61	1484,91
2000/09/14	1484,91	1494,16	1476,73	1480,87
2000/09/15	1480,87	1480,96	1460,22	1465,81
2000/09/18	1465,81	1467,77	1441,92	1444,51
2000/09/19	1444,51	1461,16	1444,51	1459,90
2000/09/20	1459,90	1460,49	1430,95	1451,34
2000/09/21	1451,34	1452,77	1436,30	1449,05
2000/09/22	1449,05	1449,05	1421,88	1448,72
2000/09/25	1448,72	1457,42	1435,93	1439,03
2000/09/26	1439,03	1448,04	1425,25	1427,21
2000/09/27	1427,21	1437,22	1419,44	1426,57
2000/09/28	1426,57	1461,69	1425,78	1458,29
2000/09/29	1458,29	1458,29	1436,29	1436,51
2000/10/02	1436,52	1445,60	1429,83	1436,23
2000/10/03	1436,23	1454,82	1425,28	1426,46

Appendix E

Arterial blood pressure data

Table E.1: Arterial blood pressure for time series 1

98	93	102	113	120	75	80	57	44	114
103	95	101	110	124	78	78	60	40	107
98	97	100	113	128	80	77	58	38	106
96	98	100	117	130	81	68	57	36	105
90	99	93	108	134	88	71	57	32	102
92	98	100	110	141	78	73	56	25	90
93	98	88	110	149	76	73	55	22	103
94	92	103	109	161	77	71	54	11	107
95	107	105	108	172	75	72	53	28	88
98	108	103	110	173	70	72	52	33	107
96	95	100	130	175	87	90	50	41	104
97	96	98	109	200	75	67	49	50	103
98	98	99	79	172	77	73	48	66	102
92	99	97	108	165	76	111	47	70	101
99	100	100	118	162	70	78	46	77	100
102	101	100	107	157	76	110	45	90	97
109	102	103	104	163	70	77	44	101	132
100	102	105	102	164	76	78	43	102	98
109	104	104	92	164	75	80	42	108	95
101	102	102	102	161	91	82	40	101	94
100	95	105	105	162	88	78	38	96	92
101	100	106	118	161	85	83	36	97	94
107	93	107	105	188	83	80	40	97	94
98	105	108	110	165	89	75	36	100	98
96	108	110	105	187	78	73	39	103	99
95	110	109	107	167	89	74	65	105	101
94	109	111	109	151	88	60	40	107	107
90	107	123	111	132	87	58	39	109	105
92	105	118	112	111	85	50	35	114	102
88	103	115	115	70	84	56	40	103	99

Table E.2: Arterial blood pressure for time series 2

61,0	62,0	77,0	94,0	82,3	108,0	85,5	80,3	88,1	86,5
62,0	62,0	78,0	104,0	84,0	84,2	86,0	77,6	89,0	86,0
63,0	61,0	79,0	95,0	82,4	81,5	85,7	80,0	90,4	86,2
62,0	61,8	80,0	97,0	82,0	82,0	87,0	80,3	89,2	82,0
63,0	62,0	80,7	90,7	82,4	82,3	85,6	80,5	88,0	86,0
63,0	62,5	81,3	88,5	84,0	84,0	85,5	80,6	87,5	86,3
62,0	62,0	82,0	90,3	82,3	85,0	85,3	82,0	87,0	86,5
64,0	62,0	83,0	90,0	80,3	88,0	84,3	80,9	85,0	86,0
62,0	61,0	84,0	89,9	81,8	84,0	85,0	79,9	87,0	85,7
61,0	62,0	85,0	89,8	82,0	85,0	85,0	79,7	88,0	86,0
62,0	62,5	86,0	89,9	81,8	88,0	85,2	80,0	87,0	85,0
62,0	63,0	84,0	88,0	81,5	87,0	85,4	79,9	85,7	85,8
64,0	64,0	86,6	89,0	83,3	86,6	86,3	79,0	85,0	86,0
63,0	65,0	87,0	90,0	83,5	85,0	85,4	80,8	87,0	86,2
63,0	65,6	88,0	87,0	83,4	87,0	82,6	81,0	88,6	86,4
63,0	66,0	87,2	86,0	82,8	85,0	82,8	82,0	88,8	87,8
62,0	68,0	88,3	85,0	82,5	85,0	83,3	84,0	90,4	86,4
62,0	66,0	90,2	84,0	81,8	86,0	83,6	90,0	88,3	86,2
61,0	68,9	91,0	83,5	82,0	86,0	84,0	86,0	88,1	86,1
61,0	70,0	92,0	83,0	82,7	87,0	86,3	104,0	86,7	86,0
63,0	71,0	91,6	83,0	84,0	84,3	86,2	94,0	87,1	83,5
62,0	71,6	92,0	82,6	83,2	87,0	84,0	92,0	86,9	83,8
62,0	72,0	90,4	82,4	83,0	88,6	87,0	90,0	86,3	83,6
61,0	72,4	90,7	82,2	80,3	86,0	84,2	87,0	86,6	81,0
62,0	73,0	92,0	81,7	82,3	85,9	84,3	86,0	87,6	83,8
62,0	74,0	92,2	82,8	82,4	85,0	84,2	84,0	87,8	84,6
61,0	74,6	92,4	82,4	82,5	85,2	82,1	87,0	88,0	85,7
62,0	75,0	92,6	82,0	84,0	86,0	82,0	88,0	88,0	88,0
63,0	75,4	93,0	81,3	81,5	87,0	80,8	88,2	87,0	90,7
63,0	76,0	92,0	82,0	84,0	86,0	80,4	88,3	86,0	89,6

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