Nevanlinna Theory and Rational Values of Meromorphic Functions

by

Taboka Prince Chalebgwa

Dissertation presented for the degree of Doctor of Philosophy in Mathematics in the Faculty of Science at Stellenbosch University

Supervisor: Dr Gareth Boxall

April 2019

Declaration

By submitting this dissertation electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the sole author thereof (save to the extent explicitly otherwise stated), that reproduction and publication thereof by Stellenbosch University will not infringe any third party rights and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

Date: 25/1/2019 ....

Abstract

Nevanlinna Theory and Rational Values of Meromorphic Functions

Taboka Prince Chalebgwa

Department of Mathematical Sciences,
University of Stellenbosch,
Private Bag X1, Matieland 7602, South Africa.

Dissertation: PhD

April 2019

In this thesis, we are concerned with the problem of counting algebraic points of bounded height and degree on graphs of certain transcendental holomorphic and meromorphic functions. Adopting a Nevanlinna theoretic approach for the latter, we attain bounds of the form $C(d)(\log H)^{\beta}$ for the number of algebraic points of height at most H and degree at most d on the restrictions to compact subsets of domains of holomorphy of meromorphic functions with certain growth/decay conditions. In the second half of the thesis, we turn our attention to counting points on graphs of certain analytic functions with growth behaviour stricter than finite order and positive lower order. For these functions, we are able to relax the need to restrict them to compact subsets of $\mathbb{C}$, and indeed, to count points either on the whole graph or nearly all of it. For these functions we also attain a bound of the form $C(d)(\log H)^{\eta}$. We end this work with several pointers towards possible extensions of our results. The results in this thesis can be seen as extensions of the work of Boxall and Jones on algebraic values of certain analytic functions.

Uittreksel

Nevanlinna-Teorie en Rasionale Waardes van Meromorfiese Funksies

("Nevanlinna Theory and Rational Values of Meromorphic Functions")

Taboka Prince Chalebgwa

Departement Wiskundige Wetenskappe,
Universiteit van Stellenbosch,
Privaatsak X1, Matieland 7602, Suid Afrika.

Proefskrif: PhD

April 2019

In hierdie tesis is die probleem om die algebraïese punte van begrensde hoogte en graad op grafieke van sekere transendentale holomorfiese en meromorfiese funksies te tel, van belang. Met behulp van die Nevanlinna-teoretiese benadering vir laasgenoemde, verkry ons grense van die vorm $C(d)(logH)^{\beta}$ vir die getal algebraïese punte waarvan die hoogte op die meeste H en die graad op die meeste d is, met die beperkings tot kompakte deelversamelings van domeine van holomorfie van meromorfiese funksies met sekere groei-/verval-voorwaardes. In die tweede helfte van die tesis vestig ons ons aandag op die tel van punte op grafieke van sekere analitiese funksies met groei-gedrag strenger as eindige orde en positiewe onderorde. Vir hierdie funksies kan ons die beperking tot kompakte deelversamelings van C ophef en, inderdaad, die punte op óf die hele grafiek, óf byna die hele grafiek, tel. Vir hierdie funksies verkry ons ook a grens van die vorm $C(d)(logH)^{\eta}$. Ons sluit hierdie werk af met verkeie aanduidings van moontlike uitbreidings van ons resultate. Die resultate in hierdie tesis kan as uitbreidings van die werk van Boxall en Jones oor algebraïese waardes van sekere analitiese funksies, beskou word.

Acknowledgements

I would first like to thank my advisor, Dr Gareth Boxall, this thesis would not have been possible without his guidance. His extraordinary patience (and it's only the extraordinary kind of patience that couldn't be exhausted by my mathematical sloppiness, occasional unwillingness to work and loss of enthusiasm), insight and generosity resulted in my wonderful stay at SU.

I would also like to thank Dr Gareth Jones for the invite and excellent stay at Manchester in September 2017. It was during this visit that the Gareths suggested the problem that later became Chapter 6 of this work. Apart from Gareth, Gareth has also been an invaluable person to bounce ideas off.

Prof Florian Breuer's guidance and inspiration at the start of my mathematical career shall forever remain influential.

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the author and are not necessarily to be attributed to the NRF.

Contents

Declaration **i**

Abstract **ii**

Uittreksel **iii**

Acknowledgements **iv**

Contents **v**

1 Introduction **1**
1.1 An introduction to algebraic values of analytic functions . . . 1

2 Nevanlinna theoretic background and Masser's polynomial **20**
2.1 Introduction 20
2.2 Some basic properties of $T(r, f)$ 22
2.3 The Nevanlinna characteristic with respect to a point 25
2.4 A discussion of Masser's polynomial 29

3 Functions with rapid decay on a ray **35**
3.1 Introduction 35
3.2 Main Result 37

4 Meromorphic functions of finite order and positive lower order **44**
4.1 Introduction 44
4.2 Main Result 44

5 Algebraic values of the Weierstrass σ-function **51**
5.1 Introduction, lemmas and definitions 51
5.2 Main Result and Proof 54

6 On a question of Miller **59**
6.1 Introduction 59
6.2 Main Result 63

7 Some afterthoughts and current pursuits **67**
7.1 On meromorphic functions and Masser′s polynomial 67
7.2 On the Weierstrass σ- function 69
7.3 On the Weierstrass $\wp$ and $\wp'$ functions 72
7.4 On the question of Miller 74

List of References **76**

Chapter 1

Introduction

1.1 An introduction to algebraic values of analytic functions

The purpose of this section is to place the results of this work within the context of current and more general trends. Indeed, the subject matter of this thesis falls within the theme which can be loosely characterised as "counting rational points on transcendental sets".

In the first subsection, therefore, we give a rather brief and sketchy survey of the techniques and known results in this context. Our particular focus will be on the recent fruitful applications of model theory (mainly via o-minimality) to counting problems in diophantine geometry. We will however not be using o-minimality in this thesis as the kinds of problems we consider are better approached via analytic methods.

In the second subsection, we move from general considerations to the more specific theme of algebraic values of transcendental analytic functions. We will mention a few known results and then end the section with a summary of the main theme and results in this thesis.

1.1.1 A brief survey of counting rational points on transcendental sets

In this subsection we give a brief (and far from exhaustive) survey of a series of results around the application of model theoretic techniques to counting rational points on transcendental sets. The results in our thesis can then be seen as special cases of this general setting. We would like to give particular mention to the Pila-Wilkie theorem as well as Wilkie's conjecture.

1.1.1.1 The Pila-Wilkie theorem

In what follows, given a set $\Gamma \subset \mathbb{R}^n$ and a positive number $t \geq 1$, the homothetic dilation of Γ by t, denoted by $t\Gamma$ is the set:

$$t\Gamma := \{(tx_1, \ldots, tx_n) : (x_1, \ldots, x_n) \in \Gamma\}.$$

The application of the model theoretic notion of o-minimality to number theory can be thought of as having been preceded by the Bombieri-Pila theorem for counting lattice points on graphs of real analytic functions.

More specifically, in [4] Bombieri and Pila considered, amongst several other variants, the following question:

Let $f : [0, 1] \to \mathbb{R}$ be an analytic function, and denote by $X_f \subset \mathbb{R}^2$ the graph of f. Given $t \geq 1$, how does the quantity $|tX_f \cap \mathbb{Z}^2|$ depend on t? For a transcendental function f, the following theorem was proved:

Theorem 1.1.1. *(Bombieri-Pila, [4], Theorem 1)*
Let f be a real analytic function on a closed and bounded interval I, suppose furthermore that f is not algebraic. Let X_f be the graph of f and let $\epsilon > 0$. Then there is a constant $c(f, \epsilon)$ such that

$$|tX_f \cap \mathbb{Z}^2| \leq c(f, \epsilon)t^\epsilon$$

for all $t \geq 1$.

The Pila-Wilkie theorem is a vast generalization of the above theorem to counting rational points on certain subsets of $\mathbb{R}^n$. Before stating the theorem we need preparatory definitions that will provide the model theoretic setting.

The following definitions are taken from van den Dries' classic text, [30].

Given two positive integers m and n with $m \geq n$, the map $\pi : \mathbb{R}^m \to \mathbb{R}^n$ will represent projection onto the first n coordinates.

Definition 1.1.2. *(Semi-algebraic subset of $\mathbb{R}^n$).*
Let S be a subset of $\mathbb{R}^n$. We say S is semi-algebraic if it is defined by a finite sequence of polynomial equations (of the form $P(x_1, \ldots, x_n) = 0$) and inequalities (of the form $Q(x_1, \ldots, x_n) > 0$), or is a finite union of such sets.

Definition 1.1.3. *(A pre-structure over $\mathbb{R}$).*
By a pre-structure we are referring to a sequence $S = \{S_n\}_{n\geq 1}$ where each S_n is a collection of subsets of $\mathbb{R}^n$.

Definition 1.1.4. *A pre-structure S is called a structure (over $\mathbb{R}$) if for all $m, n \geq 1$, the following conditions are fulfilled:*

1. *S_n is a sub-boolean algebra of $\mathcal{P}(\mathbb{R}^n)$, the power set of $\mathbb{R}^n$,*
2. *S_n contains every semi-algebraic subset of $\mathbb{R}^n$,*
3. *if $A \in S_n$ and $B \in S_m$, then $A \times B \in S_{n+m}$,*
4. *if $m \geq n$ and $A \in S_m$, then $\pi[A] \in S_n$.*

Definition 1.1.5. *If S is a structure and $X \subseteq \mathbb{R}^n$, we say X is definable in S if $X \in S_n$. If in addition the boundary (with respect to the Euclidean topology) of every set in S_1 is finite, then S is called an o-minimal structure over $\mathbb{R}$.*

With the preceding definition in mind, we are interested in the diophantine properties of the definable subsets of $\mathbb{R}^n$. The underlying philosophy is that in a certain well-defined sense, a transcendental set should contain "few" rational points. To this end, we recall the following definition from [23]:

Definition 1.1.6. *(Transcendental part of a set X, [23])*
Let $X \subseteq \mathbb{R}^n$. The algebraic part of X, denoted by X^{alg}, is the union of all connected semi-algebraic subsets of X of positive dimension. The transcendental part of X, denoted by X^{trans}, is the set $X \setminus X^{alg}$.

The transcendental part of a definable set X may contain infinitely many rational points. A well known example of such a set is the graph of the function $f(x) = 2^x$. However, a meaningful result is obtained by counting rational points of bounded height. We define the notion of height below:

The height of a (reduced) rational number $\alpha = a/b$ is defined as $H(\alpha) = \max\{|a|, |b|\}$. The height of a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}^n$ is the maximum of all $H(\alpha_j)$ for $j = 1, \ldots, n$.

For a set $X \subset \mathbb{R}^n$, we let $X(\mathbb{Q}) = X \cap \mathbb{Q}^n$. Let $H \geq 1$ be a real number, we set

$$X(\mathbb{Q}, H) = \{x \in X(\mathbb{Q}) : H(x) \leq H\}.$$

The Pila-Wilkie counting theorem is as follows:

Theorem 1.1.7. *(Pila-Wilkie, [23], Theorem 1.8)*
Let $X \subseteq \mathbb{R}^n$ be a definable subset in an o-minimal structure and let $\epsilon > 0$. Then there is a positive constant $C(X, \epsilon)$ such that for all $H \geq 1$

$$|X^{trans}(\mathbb{Q}, H)| \leq C(X, \epsilon) H^{\epsilon}.$$

The Pila-Wilkie theorem immediately prompts two related questions:

- Can one obtain *effective* estimates for the constant $C(X, \epsilon)$ in terms of the parameters with which X is defined?
- Under what conditions (or for which structures) can one obtain better bounds?

Both of these directions have been widely studied in literature. For instance, towards the first point, Jones and Thomas [15] recently gave effective Pila-Wilkie estimates for certain surfaces in $\mathbb{R}^n$ called Pfaffian surfaces.

An immediate corollary of their result applies to a certain class of real analytic functions $f : \mathbb{R}^2 \to \mathbb{R}$ called Pfaffian functions.

For such functions, they proved the existence of a constant C, effectively computable from f and ϵ, such that for all $H \geq 1$, there are at most CH^{ϵ}

rational points of height at most H on the transcendental part of the graph of f.

The conjecture of Wilkie, [23] is related to the second point above. Before discussing the conjecture, we take a short detour to discuss an application of the Pila-Wilkie theorem to diophantine geometry, which yielded a new proof of the Manin-Mumford conjecture.

1.1.1.2 An application to diophantine geometry: Manin-Mumford Conjecture

We briefly discuss the general strategy employed by Pila and Zannier in [24] to give another alternative proof of the Manin-Mumford conjecture (Raynaud's Theorem).

Before stating the theorem (as in [24]) and discussing the proof strategy, we give definitions of some of the terms appearing in the theorem statement. Our definitions and discussion are based on the article by Martin Orr, which appears as Chapter 4 of [16], as well as [12].

Definition 1.1.8. *Let X be an algebraic variety over an algebraically closed field. Then X is said to be complete if, for every algebraic variety Y, the projection $X \times Y \to Y$ is a closed map with respect to the Zariski topologies on $X \times Y$ and Y.*

Definition 1.1.9. *A group variety is an algebraic variety G together with morphisms of varieties $m : G \times G \to G$ (multiplication), $i : G \to G$ (inverse) and a point $e \in G$ (identity) such that the group axioms are satisfied.*

The above two definitions combine to give:

Definition 1.1.10. *A variety G is said to be abelian if it is a complete group variety.*

We also need:

Definition 1.1.11. *Let G be an abelian variety and N a positive integer. Then the set of N-torsion points (of G), denoted by $G[N]$, is the set*

$$\{x \in G : Nx = 0\}$$

under the group law on G.

We are now ready to state the theorem as in [24].

Theorem 1.1.12. *(Raynaud)*
Let A be an abelian variety and X an algebraic subvariety of A, both of which are defined over a number field. Suppose X does not contain any translate of an abelian subvariety of A of positive dimension. Then X contains only finitely many torsion points of A.

Remark 1.1.13. *We remark that in the original theorem of Raynaud, one does not make the assumption that X does not contain any translate of a positive dimensional abelian subvariety, hence the above theorem is a special case.*

Let V be a finite dimensional complex vector space. Viewing V as a group under addition, a *lattice* $\Lambda \subset V$ is a discrete subgroup of V such that the quotient V/Λ is compact. The quotient V/Λ is called a *complex torus*.

We note that if $n \geq 1$ is the dimension of V over $\mathbb{C}$, then $\Lambda \cong \mathbb{Z}^{2n}$.

Theorem 1.1.14. *([16], Theorem 3.1, pp 106)*
Let G be an abelian variety over $\mathbb{C}$. Then G is isomorphic as a complex Lie group to a complex torus.

Proceeding, one can then translate the question about torsion points on an abelian variety A to a question about rational points in $\mathbb{C}^n$:

Let $\mathcal{B} = \{\lambda_1 \ldots, \lambda_{2n}\} \subset \mathbb{C}^n$ be a $\mathbb{Z}$-basis for Λ. So $\Lambda = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_{2n}$.

We note that the N-torsion points in $A = \mathbb{C}^n/\Lambda$ are given by

$$\{c \in \mathbb{C}^n : Nc \in \Lambda\}.$$

For such c, $Nc = a_1\lambda_1 + \cdots a_{2n}\lambda_{2n}$, where $a_1, \ldots, a_{2n} \in \mathbb{Z}$.

Hence

$$c = \frac{1}{N}(a_1\lambda_1 + \cdots a_{2n}\lambda_{2n}),$$

which we identify with $(\frac{a_1}{N}, \ldots, \frac{a_{2n}}{N}) \in \mathbb{Q}^{2n}$.

We thus note that the torsion points on A correspond to the rational points in $\mathbb{R}^{2n}$.

For X an algebraic subvariety of A, the torsion points on X thus correspond to the rational points on the fibres at $x \in X$ of the quotient map $\pi : \mathbb{C}^n \to A$. That is, the rational points of the set

$$Y' = \{(a_1, \dots, a_{2n}) \in \mathbb{R}^{2n} : \pi(a_1\lambda_1 + \dots + a_{2n}\lambda_{2n}) \in X\}.$$

Taking advantage of the fact that the above set is periodic, Pila and Zannier could then restrict their attention to the rational points on the set $Y = Y' \cap [0,1]^{2n}$.

The set Y is then shown to be definable in the o-minimal structure $\mathbb{R}_{an}$, the expansion of the real field with restricted analytic functions. Hence Y is an *analytic variety*, that is, it can be defined locally as the set of common zeros of finitely many analytic functions.

Section 2 of [24] is then devoted to proving a general structure theorem on the algebraic part of analytic varieties. This then led to the conclusion that $Y^{alg} = \emptyset$, for X as in Theorem 1.1.12.

Applying the Pila-Wilkie theorem to Y, it is concluded that for every $\epsilon > 0$, there is a constant $c = c(Y, \epsilon)$ such that there are at most cT^{ϵ} rational points on Y with common denominator at most T.

By the correspondence with torsion points, this implies that there are at most cT^{ϵ} torsion points of order at most T on X.

The following result of Masser provides a lower bound on the number of torsion points that is then used to complete the proof of the theorem:

Theorem 1.1.15. *(Masser, [18])*

Let A be an abelian variety of dimension n over a number field K. Then there exist effective constants $c = c(A, K)$ and $\rho = \rho(n)$ such that for all T-torsion points $P \in A$,

$$[K(P) : \mathbb{Q}] \geq cT^{\rho}.$$

We note that, for large enough T, Masser's lower bound would contradict the Pila-Wilkie estimate. This is because if $\alpha \in X$ is an algebraic point, then all the algebraic conjugates of α over a field of definition for X are also contained in X. Hence, by Masser's result, if X contains a point of order T, then there are at least cT^{ρ} such points. One concludes that the order of torsion points on X is bounded, proving the theorem.

The Pila-Zannier strategy has subsequently been used to prove new results in diophantine geometry. For instance, in [22], Pila used the method to give a proof of the André-Oort conjecture for $\mathbb{C}^n$.

1.1.1.3 Wilkie's Conjecture

As promised at the end of subsection (1.1.1.1), after a brief hiatus we now return to a more in depth discussion of Wilkie's conjecture.

Before giving the precise statement of the conjecture, we need one more definition:

Definition 1.1.16. *Let $\mathcal{L}_{or} = \{+, -, \cdot, 0, 1, <\}$ be the language of ordered rings. Consider the expansion $\mathcal{L}_{\exp} = \mathcal{L}_{or} \cup \{e^x\}$ of $\mathcal{L}_{or}$ by the exponential function. By the structure $\mathbb{R}_{\exp}$ we mean the real numbers $\mathbb{R}$ as an $\mathcal{L}_{\exp}$-structure.*

Remark 1.1.17. *We refer the curious reader to [30] to see how the above model theoretic notion of structure coincides with the set theoretic one provided earlier.*

The structure $\mathbb{R}_{\exp}$ was shown to be o-minimal by Wilkie in [32]. This implies that Theorem 1.1.7 applies to the structure.

However, since the exponential function (and hence $\mathbb{R}_{\exp}$) is very well behaved, one can hope that a bound better than the one provided by the Pila-Wilkie theorem should hold for sets definable in this structure. Wilkie's conjecture is as follows:

Conjecture 1.1.18. *(Wilkie, [23], Conjecture 1.11)*
Let X be a definable set in $\mathbb{R}_{\exp}$. Then there exist positive constants $C_1 = C_1(X)$ and $c_2 = c_2(X)$ such that

$$|X^{trans}(\mathbb{Q}, H)| \leq C_1(\log H)^{c_2} \textit{ for all } H \geq e.$$

Several special cases of Wilkie's conjecture have been proven by many different authors.

In [15], Jones and Thomas and independently Butler, [7], prove the conjecture for one dimensional subsets of $\mathbb{R}^n$ definable in $\mathbb{R}_{\exp}$.

In [21], Pila proved the conjecture for a particular subset of $\mathbb{R}^3$ defined by exponential-algebraic relations. More precisely:

Let $X \subset \mathbb{R}^3$ be defined by

$$X = \{(x, y, z) \in (0, \infty)^3 : \log x \log y = \log z\}.$$

Let $L_x := \{(x, 1, 1) : x > 0\}$ and $L_y := \{(1, y, 1) : y > 0\}$.

Denote by X^0 the region $X \setminus (L_x \cup L_y)$. Pila obtains the following result:

Theorem 1.1.19. *(Pila, [21], Theorem 1.1)*
There exists a constant $c(\epsilon)$ such that for all $H \geq e$

$$|X^0(\mathbb{Q}, H)| \leq c(\epsilon)\, (\log H)^{44+\epsilon}.$$

Before proceeding with our discussion, we will need the notion of height for algebraic numbers. The definition of height is extended to algebraic numbers in the following manner:

Let $P(z) \in \mathbb{C}[z]$ be a polynomial with complex coefficients. Writing $P(z)$ as

$$P(z) = a \prod_{j=1}^{n} (z - \alpha_j),$$

the *Mahler measure* $\mathcal{M}(P)$ of the polynomial P is the quantity

$$\mathcal{M}(P) = |a| \prod_{j=1}^{n} \max\{1, |\alpha_j|\}.$$

If α an algebraic number of degree d, the *logarithmic* height of α, $h(\alpha)$ is defined to be:

$$h(\alpha) = \frac{\log \mathcal{M}(\alpha)}{d},$$

where $\mathcal{M}(\alpha)$ is the Mahler measure of the minimal polynomial of α over $\mathbb{Z}$.

The *absolute Weil* height of α, $H(\alpha)$ is defined as:

$$H(\alpha) = \exp\left\{\frac{\log \mathcal{M}(\alpha)}{d}\right\} = \mathcal{M}(\alpha)^{\frac{1}{d}}.$$

If α and β are algebraic numbers, we use the notation $H(\alpha, \beta)$ to represent the quantity

$$\max\{H(\alpha), H(\beta)\}.$$

Throughout the thesis, unless otherwise stated, when we speak of height we will be referring to the absolute Weil height.

Building on the work of Pila, Butler [7] proved the following more general result:

Theorem 1.1.20. *(Butler, [7], Theorem 1.5)*
Let $d \in \mathbb{N}$ and $a, b, c \in \mathbb{Q}$. Let $X \subset \mathbb{R}^3$ be defined by

$$X = \{(x, y, z) \in (0, \infty)^3 : (\log x)^a (\log y)^b (\log z)^c = 1\}.$$

Then there are constants $C_1 = C_1(a, b, c, d)$ and $c_2 = c_2(a, b, c)$ such that for any number field $\mathbb{K} \subset \mathbb{R}$ of degree d and any $H \geq e$,

$$|X^{trans}(\mathbb{K}, H)| \leq C_1 (\log H)^{c_2}.$$

Perhaps the most general result known is the recent proof by Binyamini and Novikov of Wilkie's conjecture for restricted elementary functions. We define the structure $\mathbb{R}^{RE}$ as

$$\mathbb{R}^{RE} = (\mathbb{R}, <, +, \cdot, \exp|_{[0,1]}, \sin|_{[0,\pi]}).$$

Let $\mathbb{K} \subset \mathbb{C}$ be a number field and suppose that $[\mathbb{K} : \mathbb{Q}] = d$. Binyamini-Novikov prove the following theorem:

Theorem 1.1.21. *(Binyamini-Novikov, [3], Theorem 1)*
Let $X \subset \mathbb{R}^n$ be definable in $\mathbb{R}^{RE}$ and $H \geq e$. Then there exist constants $C_1 = C_1(X, d)$ and $c_2 = c_2(X)$ such that

$$|X^{trans}(\mathbb{K}, H)| \leq C_1(\log H)^{c_2}.$$

Moving away from the generality of definable sets, one can still pursue instances of Wilkie's conjecture for particular examples, even ones not definable in $\mathbb{R}_{an,\exp}$. For instance, one can attempt to prove "Wilkie-like" bounds for rational points on graphs of specific functions, usually analytic transcendental functions.

This is the general theme we will be developing in this thesis. Results in this direction have already been obtained by numerous authors. We take a look at several such results in the next subsection.

1.1.2 An introduction to algebraic values of transcendental functions

Proceeding from the previous subsection, we look at the more specific theme of counting rational or algebraic points on graphs of transcendental complex analytic functions.

Looking at examples of functions such as $f(z) = 2^z$ or $\exp(i\pi z)$, we see that there are transcendental functions with infinitely many rational points. This prompts imposing further restrictions in terms of the degree and height of the inputs and outputs of the functions we are studying. One then endeavours to obtain bounds on the number of points of bounded degree and height.

In what follows, the notation $\overline{\mathbb{Q}}$ represents the field of algebraic numbers.

Given $d \geq 1$ and $H > 1$, Northcott's property says that

$$|\{\alpha \in \overline{\mathbb{Q}} : \deg(\alpha) \leq d, \;\; H(\alpha) \leq H\}| < \infty.$$

In [27], Schmidt showed that the above quantity is bounded above by

$$C(d)H^{d(d+1)}.$$

Hence, given an analytic function f, we trivially have that:

$$|\{(\alpha, f(\alpha)) \in \overline{\mathbb{Q}}^2 : [\mathbb{Q}(\alpha, f(\alpha)) : \mathbb{Q}] \leq d, \;\; H(\alpha, f(\alpha)) \leq H\}|$$

is bounded above by $C(d)H^{2d(d+1)}$.

In light of the Pila-Wilkie theorem and Wilkie's conjecture, *polylogarithmic* bounds are considered very good. These are bounds of the form $C(\log H)^{\beta}$, for some $\beta > 0$ and a constant $C > 0$ depending on d and f.

Additionally, the effective computability of the constants C and β from d and data associated with f is desirable.

In [19], Masser proves the following result for the number of rational points on the graph of the Riemann ζ-function restricted to the interval $(2, 3)$:

Theorem 1.1.22. *(Masser, [19])*
Let ζ be the restriction of the Riemann ζ-function to the interval $(2, 3)$. There is an effective constant $c > 0$ such that for all $H \geq e^e$, the number of rational points of height at most H on the graph of ζ is at most

$$c\left(\frac{\log H}{\log\log H}\right)^2.$$

Masser in fact proved the more general bound $c\left(\frac{d^2\log 4H}{\log(d\log 4H)}\right)^2$ for the number of algebraic points of height at most H and degree at most d on the graph of the ζ-function restricted to the disk $\{z \in \mathbb{C} : |z - \frac{5}{2}| \leq \frac{1}{2}\}$.

In [1], adapting Masser's method, Besson studied the density of algebraic points of bounded degree and height on the graph of the Γ-function restricted to the interval $[n-1, n]$. He obtains the following:

Theorem 1.1.23. *(Besson, [1])*
There exists a positive effective constant c such that for integers $d \geq 1$, $H \geq 3$ and $n \geq 2$, the number of algebraic points of degree at most d and height at most H on the graph of the Γ-function restricted to the interval $[n-1, n]$ is at most

$$c(n^2\log(n))\left(\frac{(d^2\log H)^2}{\log(d\log H)}\right).$$

In [29], assuming only that f is complex analytic and transcendental, Surroca achieves the rather exciting bound of $Cd^3(\log H)^2$ for the number of algebraic points of degree at most d and height at most H on the restriction to a compact set of the graph of f. However, the bound is valid only for infinitely many H. Precisely:

Theorem 1.1.24. *(Surroca, [29])*
Let $\mathcal{U} \subset \mathbb{C}$ be open and connected. Let $\mathcal{K}$ be a compact subset of $\mathcal{U}$. Let f be a transcendental complex analytic function on $\mathcal{U}$. Then for any integer $d \geq 1$, there exists a positive real number $C > 0$ such that there are infinitely many real numbers $H \geq 1$ such that the number of algebraic points of degree at most d and height at most H, with the input belonging to $\mathcal{K}$, is at most

$$Cd^3(\log H)^2.$$

The constant C effectively depends on $\mathcal{U}$, $\mathcal{K}$ and f. It is also shown in the same paper that the theorem cannot be improved any further. That is, one cannot replace the "infinitely many real $H \geq 1$" in the conclusion of the theorem with "for all sufficiently large H".

A theme related to Bombieri-Pila and Surroca's work was recently explored by Gasbarri in [8]. In the paper, Gasbarri studies the density of rational points on certain Riemann surfaces contained in projective varieties. He obtains a version of Surroca's result which holds for arbitrarily large intervals of values for H.

Proceeding, in [5] and [6], motivated by the earlier work of Masser, and in light of Surroca's theorem, Boxall and Jones obtained bounds of the form $C(d)(\log H)^\beta$ for the number of algebraic points of bounded height and degree for certain restrictions of analytic functions which in addition satisfy certain growth/decay assumptions.

The results in these two papers are the main inspiration for our Chapters 3 and 4. We will thus elaborate more on them in the next section.

1.1.3 A brief overview of the results in this work

In this subsection we give a summary of the results in this thesis with a view towards placing them in the general context recounted previously.

The work of Boxall and Jones mentioned above can be viewed through two lenses which then immediately suggest possible paths towards generalization:

- First, algebraic values were counted on *graphs of analytic functions*, one can then ask if analogous results hold for meromorphic functions.
- Even if f was an entire function, the points of bounded degree and height were counted on the *restriction of f to certain subsets* of $\mathbb{C}$. We can then ask when is it possible to enlarge the set to which the function was initially restricted.

The first theme underlies Chapters 3 and 4 whilst Chapters 5 and 6 deal with the second. These form the main body of the thesis.

We begin the thesis with **Chapter 2** wherein we provide the Nevanlinna theoretic background required for the results in Chapters 3 and 4.

Chapter 3 is based on certain results in [5], the first work of Boxall and Jones in this subject. Amongst several other theorems in the paper, they prove a bound of the form $C(\log H)^3(\log\log H)^3$ for the number of rational points $(x, f(x))$ of height at most H and with $x > 0$ on the graph of an entire function satisfying a general growth condition and a decay condition along the positive ray.

The result applies to functions such as $(z-1)(\zeta(z)-1)$ and $\frac{1}{\Gamma}$ and was motivated by Masser's work on rational points on the ζ-function restricted to the interval $(2, 3)$.

Let f be a meromorphic function which rapidly decays on the positive ray with the Nevanlinna characteristic of f explicitly bounded by some nondecreasing function (we specify these conditions later). We prove a bound of the form $C(\log H)^6$ for the number of algebraic points of height at most H

and degree at most d on the restriction to a compact subset of the domain of holomorphy of the graph of f.

As the reader may have already anticipated, an example of a function to which our result applies is $f(z) = \zeta(z) - 1$, which is in particular holomorphic on $B(0, r)$ for $r < 1$, and its restriction to $(1, \infty)$ is dominated by a positive rapidly decreasing function.

We would like to point out that for a meromorphic function f with finitely many poles $a_1, \ldots, a_m$ that are in addition algebraic, one can count the algebraic points on f by using the techniques usually employed for analytic functions:

Indeed, one only needs to count points on the entire function
$g(z) := \prod_{j=1}^{m}(z - a_j)f(z)$. This is the technique employed by Masser (and Boxall-Jones) for the rational points on ζ.

Viewed through this lens, our result is therefore more general since it applies to functions with (possibly infinitely many) not necessarily algebraic poles.

For the discussion of the results in Chapter 4, we need the definition of order and lower order of an entire function f. The corresponding definition for meromorphic functions is given in the next subsection, wherein we discuss Nevanlinna theory, the body of work suitable for studying growth properties of meromorphic functions.

Definition 1.1.25. *Let f be an entire function and let $M(r, f) := \max_{|z| \le r} |f(z)|$. The order and lower order of f are defined as:*

$$\rho := \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r} \textit{ and } \lambda := \liminf_{r \to \infty} \frac{\log \log M(r, f)}{\log r} \textit{ respectively.}$$

Chapter 4 can be viewed as an extension of the results in the second paper by Boxall-Jones, [6]. This paper dealt with algebraic points on graphs of entire functions with finite order ρ and positive lower order λ. A bound of the form $C(\log H)^{\eta}$ was achieved for the number of algebraic points of height at most H and degree at most d on the restriction of f to a compact set. The exponent $\eta = \eta(\lambda, \rho)$ depended only on the order and lower order.

We attain a similar power of log bound for the restriction to a compact set (on which f is holomorphic) of a meromorphic function f of finite order and positive lower order. The definitions that we use for the order and lower order of a meromorphic function are given on page 24.

In Chapters 5 and 6, we change the theme from counting points on graphs of meromorphic functions to counting points on certain analytic functions with stricter growth behaviour. This enables us to relax the need for restricting such functions to compact subsets of $\mathbb{C}$, and indeed, to count points either on the whole graph or nearly all of it:

Chapter 5 deals with counting points on the graph of the Weierstrass σ-function associated to the lattice $\Lambda := \mathbb{Z} + \mathbb{Z}i$. The result in [6] on entire functions of finite order ρ and positive lower order λ applies to the restrictions of σ to compact sets. However, in [25], it was shown that there exist absolute constants K_1 and K_2 such that:

$$K_1 \delta_z e^{r^2} \leq |\sigma(z)| \leq K_2 \delta_z e^{r^2}$$

for all $z = re^{it} \in \mathbb{C}$ where δ_z is the distance from z to the nearest point in $\mathbb{Z}^2$.

Taking advantage of this inequality, and the explicit knowledge about the location of the zeroes of σ, we prove a bound of the form $C(d)(\log H)^{15}$ for the *total* number of algebraic points $(z, \sigma(z))$ of height at most H and degree at most d such that $\sigma(z) \neq 0$.

In [2], Besson proved a bound of the form

$$c(d)R^{10}(\log R)\frac{(\log H)^2}{\log\log H}$$

for the number of algebraic points of height at most H and degree at most d on the graph of the restriction to $\overline{B(0,R)}$ of the σ-function associated to the lattice $\Omega = \mathbb{Z} + \mathbb{Z}\tau$ where $\Im(\tau) > 0$.

Upon specializing his result to the lattice Λ and combining it with our method, we improve the bound obtained for $\sigma(z, \Lambda)$ to $C(\log H)^7$.

In **Chapter 6**, we concern ourselves with counting algebraic points on functions defined by an infinite product of the form

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \tag{1.1}$$

where the zeroes $\{z_n\}$ lie on a ray.

When the sequence $\{z_n\}$ is such that $\sum_{n=1}^{\infty} \frac{1}{|z_n|} < \infty$, then the product in Equation (1.1) defines an entire function of order $\rho \in [0, 1)$ and lower order $\lambda \in [0, \rho]$ which are determined by further properties of the sequence.

In our case, we assume that $z_n \geq 1$ for all n and furthermore, the sequence of zeroes is assumed to be of "moderate" growth in comparison with the function r^{ρ}. We will formalize this notion in the chapter.

The limitations of our tools force us to deal with functions for which $\rho \in (0, \frac{1}{2}]$.

For these functions, we endeavour to count the algebraic points of bounded degree and height outside some fixed sector S_{ϕ} containing the sequence $\{z_n\}$.

For such f, we attain a bound of the form $C(d)(\log H)^{\eta(\lambda,\rho)}$ where $\lambda > 0$. This partially answers a question of Chris Miller about the algebraic points on the graph of functions defined as in Equation (1.1) above.

All our proofs require the construction of a non-zero auxiliary polynomial $P(X, Y) \in \mathbb{Z}[X, Y]$ such that $P(z, f(z)) = 0$ whenever

$$(z, f(z)) \in \overline{\mathbb{Q}}^2, \deg(z, f(z)) \leq d \text{ and } H(z, f(z)) \leq H.$$

This enables us to translate a question about algebraic points on the graph of a function f to a question about the zeroes of another function g, say, the latter for which we then use analytic machinery to answer.

To obtain $P(X,Y)$, we use the following rather crucial lemma of Masser, which appears as Proposition 2 in [19].

Lemma 1.1.26. *(Masser, [19], Prop. 2)*

Let A, Z, T, M, H and d be positive real numbers such that $d, H \geq 1$ and $T \geq \sqrt{8d}$ where d and T are integers. Let f_1, f_2 be functions analytic on an open neighbourhood of $\overline{B(0,2Z)}$, with $\max\{|f_1(z)|, |f_2(z)|\} \leq M$ on this set. Suppose $\mathcal{Z} \subset \mathbb{C}$ is finite and satisfies the following for all $z, w \in \mathcal{Z}$:

- $|z| \leq Z$,
- $|w-z| \leq \frac{1}{A}$,
- $[\mathbb{Q}(f_1(z), f_2(z)) : \mathbb{Q}] \leq d$,
- $H(f_1(z), f_2(z)) \leq H$.

Then there is a nonzero polynomial $P(X,Y)$ of total degree at most T such that $P(f_1(z), f_2(z)) = 0$ for all $z \in \mathcal{Z}$ provided

$$(AZ)^T > (4T)^{\frac{96d^2}{T}}(M+1)^{16d}H^{48d^2}.$$

In [5], Boxall and Jones noted that, from Masser's proof, if $|\mathcal{Z}| \geq \frac{T^2}{8d}$ then $P(X,Y)$ can be chosen to have integer coefficients each with absolute value at most

$$2^{\frac{1}{d}}(T+1)^2H^T.$$

Our applications of the above lemma use the case where $f_1(z) = z$ and $f_2(z) = f(z)$ is entire or holomorphic in some domain, as well as the additional information about the coefficients noted above.

Due to the central role the above lemma plays in all of our results, we give an exposition of its proof in the last section of the next chapter.

We end the thesis with **Chapter 7**, which consists of several remarks concerning some of our current work. These are essentially the possible extensions and generalizations of the results in some of the preceding chapters.

Chapter 2

Nevanlinna theoretic background and Masser's polynomial

2.1 Introduction

This section serves as a brief introduction to Nevanlinna theory. We particularly need this theory in Chapters 3 and 4, where we deal with the question of counting algebraic points on graphs of meromorphic functions. Unless otherwise stated, meromorphic shall be taken to mean meromorphic on $\mathbb{C}$.

We focus on the origin of the classical Nevanlinna characteristic $T(r, f)$, a functional which captures in some sense the growth of a meromorphic function f. After outlining several properties of $T(r, f)$, we proceed to look at what we describe as the "generalized" Nevanlinna characteristic. This comes from a reformulation of the classical Nevanlinna theory wherein the characteristic functional is defined with respect to an arbitrary point in $\mathbb{C}$. We will then prove several lemmas relating the characteristics to each other. As mentioned above, all of these tools will become relevant in Chapters 3 and 4.

The results we are going to discuss concerning classical Nevanlinna theory can be found in any standard text on meromorphic functions. Our main reference nonetheless is the 1964 classic by Hayman, [13].

The origins of Nevanlinna theory can be traced back to the Poisson-Jensen formula:

Lemma 2.1.1. *Let f be a meromorphic function in $B(0, R)$, where $(0 < R < \infty)$. Suppose a_j, $j = 1, \ldots, n$ and b_k, $k = 1, \ldots, m$ are the zeroes and poles of f in $|z| < R$ respectively, with the multiplicity of each zero and pole taken into account. Let $z = re^{i\theta}$ be a point in $|z| < R$ distinct from a_j and b_k. Then:*

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{it})| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} dt + \sum_{j=1}^{n} \log\left|\frac{R(z - a_j)}{R^2 - \overline{a_j}z}\right| - \sum_{k=1}^{m} \log\left|\frac{R(z - b_k)}{R^2 - \overline{b_k}z}\right|.$$

Upon substituting $z = 0$ in the above formula whilst maintaining the rest of the other conditions in the hypothesis of the lemma, one recovers the classical Jensen's formula:

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{it})| dt + \sum_{j=1}^{n} \log\left|\frac{a_j}{R}\right| - \sum_{k=1}^{m} \log\left|\frac{b_k}{R}\right|. \tag{2.1}$$

Classical Nevanlinna theory then arises from a reformulation of the above formula. For a positive real number x, let

$$\log^+ x := \max\{0, \log x\}. \tag{2.2}$$

The non-decreasing function $\log^+$ possesses the following easily verifiable properties:

1. $\log x \leq \log^+ x$.
2. $\log x = \log^+ x - \log^+ \frac{1}{x}$.
3. $\log^+(\prod_{j=1}^n x_j) \leq \sum_{j=1}^n \log^+ x_j$.
4. $\log^+(\sum_{j=1}^n x_j) \leq \log n + \sum_{j=1}^n \log^+ x_j$.

Decomposing the integrand in Jensen's formula via the second property in the list above, the Equation (2.1) can be rewritten as:

$$\log|f(0)| + \sum_{j=1}^{n} \log\left|\frac{R}{a_j}\right| + \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(Re^{it})|}dt = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(Re^{it})|dt$$
$$+ \sum_{k=1}^{m} \log\left|\frac{R}{b_k}\right|.$$

Nevanlinna proceeds to define the following two functionals:

- $m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{it})|dt$, the mean proximity function, which quantifies the size of f on the arcs of $|z| = r$ for which $|f| > 1$.
- $N(r,f) := \sum_{k=1}^{m} \log\left|\frac{r}{b_k}\right|$, the averaged pole counting functional where the b_ks are the poles of f in $|z| < r$.

Remark 2.1.2. *Let $n(r,f)$ be the number of poles of f in $|z| < r$. Then by basic properties of the Riemann-Stieltjes integral, one can show that:*

$$N(r,f) = \int_0^r \frac{n(t,f)}{t}dt.$$

We shall use interchangeably the sum and integral representation of $N(r,f)$ as and when convenient.

The Nevanlinna characteristic $T(r,f)$ is then defined to be:

$$T(r,f) = m(r,f) + N(r,f). \tag{2.3}$$

With the above definition in mind, from Equation (2.1) one can deduce the following:

Lemma 2.1.3. *Let f be a meromorphic function such that $f(0) \neq 0, \infty$, then*

$$T(r,f) = T(r,\frac{1}{f}) + \log|f(0)|.$$

2.2 Some basic properties of $T(r,f)$

The following properties of the Nevanlinna characteristic follow from properties of the $\log^+$ function and the definition of $T(r,f)$.

Let $r > 0$, and suppose $f, f_1, \ldots, f_n$ are meromorphic functions in $\mathbb{C}$. Let $m > 0$ be a positive integer. Suppose $f(0), f_j(0) \neq 0, \infty$ for $j = 1, \ldots, n$. Then:

$$T\left(r, \prod_{j=1}^{n} f_j\right) \leq \sum_{j=1}^{n} T(r, f_j),$$
$$T\left(r, \sum_{j=1}^{n} f_j\right) \leq \sum_{j=1}^{n} T(r, f_j) + \log n,$$
$$T(r, f^m) = mT(r, f).$$

We shall give the other properties as lemmas.

Let f be an entire function. Recall that $M(r, f) := \max_{|z| \leq r} |f(z)|$.

When f is an entire function, $M(r, f)$ and $T(r, f)$ are related by the following lemma:

Lemma 2.2.1. *Let f be an entire function, and $0 \leq r < R$, then*

$$T(r, f) \leq \log^{+} M(r, f) \leq \frac{R + r}{R - r} T(R, f).$$

It turns out that one can bound the number of zeroes (or poles) of f in $|z| < r$ in terms of $M(r, f)$ and $T(r, f)$. More specifically:

Lemma 2.2.2. *(A corollary of Jensen's formula) Let f be a nonconstant entire function such that $f(0) \neq 0$. Let $0 < r < R < \infty$. Then:*

$$n(r, \frac{1}{f}) \leq \frac{1}{\log \frac{R}{r}} \log \left(\frac{M(R, f)}{|f(0)|} \right)$$

For meromorphic f, the corresponding lemma (having let $R = 2r$) is the following:

Lemma 2.2.3. *Let f be a meromorphic function in $\mathbb{C}$, $r > 0$, then*

$$n(r, f) \leq \left(\frac{1}{\log 2} \right) T(2r, f).$$

Proof. First, we note that

$$\int_r^{2r} \frac{n(t,f)}{t} dt \geq n(r,f) \int_r^{2r} \frac{1}{t} dt = n(r,f) \log 2.$$

On the other hand,

$$\int_r^{2r} \frac{n(t,f)}{t} dt \leq \int_0^{2r} \frac{n(t,f)}{t} dt = N(2r,f) \leq T(2r,f).$$

The proof follows. □

We end this subsection with the definition of *order* and *lower order* for meromorphic functions. For the sake of completeness we recall the definition for entire functions given in the previous section.

Recall that the order and lower order of an entire function f are defined as

$$\rho := \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} \text{ and } \lambda := \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r} \text{ respectively.} \tag{2.4}$$

Remark 2.2.4. *If ρ is finite, then ρ is the infimum of the set of all α such that $M(r,f) \leq e^{r^\alpha}$ for sufficiently large r and λ is the supremum of the set of all β such that $e^{r^\beta} \leq M(r,f)$ for sufficiently large r.*

From [9], if f is a meromorphic function, the order of f is defined to be

$$\rho := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}. \tag{2.5}$$

Remark 2.2.5. *In analogy with the case of entire functions, if ρ as defined above is finite, then ρ is the infimum of the set of all α such that $T(r,f) \leq r^\alpha$ for sufficiently large r.*

By abuse of notation, if f is a meromorphic function, we let

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

If the above maximum does not exist, we let $M(r,f) = \infty$.

With the above notation in mind, for our purposes we define the lower order of f to be

$$\lambda := \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}, \text{ if it exists.}$$

Remark 2.2.6. *When λ is finite, then λ is the supremum of the set of all β such that $e^{r^\beta} \leq M(r,f)$ for sufficiently large r.*

2.3 The Nevanlinna characteristic with respect to a point

In this subsection we take a brief look at the work of Grahl, [10]. The reader may have noticed that only Jensen's formula, and not the full Poisson-Jensen formula, played a role in the construction of $T(r,f)$.

In [10], Grahl starts from the more general Poisson-Jensen formula and proceeds to define the Nevanlinna functionals in a fashion analogous to the standard definitions. The immediate consequence is that the Nevanlinna characteristic of a function f can be defined with respect to an arbitrary point a (excluding zeroes or poles) inside a disk centered at the origin, this is captured by the notation $T_a(r,f)$. The classical characteristic is then simply $T_0(r,f)$.

The extra versatility in choosing an arbitrary point a upon which to "center" our characteristic will become crucial as one of the main ingredients in the proofs of the results in Chapters 3 and 4.

We shall mainly concern ourselves with the results that are analogous to the ones we mentioned above for $T(r,f)$. Hence, for a full account of this modification and its applications, we refer the reader to [10].

Recall that for $w,z \in \mathbb{C}$ such that $|w| > |z|$, the *Poisson kernel* $\mathcal{P}(w,z)$ is defined as $\mathcal{P}(w,z) = \Re\left(\frac{w+z}{w-z}\right)$, and this also equals $\frac{|w|^2-|z|^2}{|w-z|^2}$. Letting $w = Re^{it}$ and $z = re^{i\theta}$, we see that:

$$\mathcal{P}(w,z) = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2}.$$

Definition 2.3.1. *For a meromorphic function f in $\mathbb{C}$, let $r > 0$ and $a \in B(0,r)$ such that a is not a pole of f. Let $b_1, \ldots, b_m$ be the poles of f in $B(0,r)$ counted with multiplicity. The (modified) Nevanlinna characteristic of f at a is defined as*

$$T_a(r,f) = N_a(r,f) + m_a(r,f),$$

where

$$m_a(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f(re^{it}) \right| \cdot \mathcal{P}(re^{it}, a)dt,$$

and

$$N_a(r,f) := \sum_{k=1}^{m} \log \left| \frac{r^2 - \overline{b_k}a}{r(a-b_k)} \right|.$$

2.3.1 Properties of the Nevanlinna characteristic $T_a(r,f)$.

Below we outline some of the properties of $T_a(r,f)$ outlined in [10]. We shall present them as lemmas, since some of them are not immediately obvious and need to be derived. However, the reader will recognise many of the properties as analogues of those of $T(r,f)$.

Lemma 2.3.2. *Let $r > 0$, $a \in \mathbb{C}$ such that $|a| < r$ and f a meromorphic function in $\mathbb{C}$ such that $f(a) \neq 0, \infty$. Then*

$$T_a(r,f) = T_a(r, \frac{1}{f}) + \log|f(a)|.$$

Lemma 2.3.3. *Let $r > 0$, $f_1, \ldots, f_n$ be meromorphic functions in $\mathbb{C}$, $a \in \mathbb{C}$ such that $|a| < r$ and $f_j(a) \neq \infty$ for $j = 1, \ldots, n$. Then*

$$T_a\left(r, \prod_{j=1}^{n} f_j\right) \leq \sum_{j=1}^{n} T_a\left(r, f_j\right), \text{and}$$
$$T_a\left(r, \sum_{j=1}^{n} f_j\right) \leq \sum_{j=1}^{n} T_a\left(r, f_j\right) + \log n.$$

Lastly,

Lemma 2.3.4. *Let f be a meromorphic function on $\mathbb{C}$ and $n > 0$. Then*

$$T_a(r, f^n) = nT_a(r,f).$$

Remark 2.3.5. *It follows rather trivially from above that if m, n are integers such that $0 < m \leq n$, then $T_a(r, f^m) \leq T_a(r, f^n)$.*

Lemma 2.3.6. *Let f be an entire function, and $0 \leq r < R$, then*

$$T_a(r, f) \leq \log^+ M(r, f) \leq \frac{R + r}{R - r} T_a(R, f).$$

The next lemma gives a bound on the number of zeroes of f in $|z| < r$ in terms of $T_a(r, \frac{1}{f})$.

Lemma 2.3.7. *Let f be a meromorphic function and $a \in \mathbb{C}$ not a pole or zero of f. Let $r, R > 0$ such that $|a| < r < R$ and denote by $n(r, \frac{1}{f})$ the number of zeroes of f in $|z| < r$. Then*

$$n(r, \frac{1}{f}) \leq C(R, r, a) \cdot T_a(R, \frac{1}{f}),$$

where the constant $C(R, r, a)$ is given by

$$C(R, r, a) = \frac{R^2 + r|a|}{(R - r)(R - |a|)}.$$

Proof. It was shown in [10] Lemma 3 that

$$n(r, f) \cdot \frac{(R - r)(R - |a|)}{R^2 + r|a|} \leq N_a(R, f) - N_a(r, f).$$

Mutatis mutandis, the corresponding inequality holds for $n(r, \frac{1}{f})$. On the other hand, $N_a(r, \frac{1}{f})$ and $m_a(R, \frac{1}{f})$ are both non-negative. Recalling the definition of $T_a(R, \frac{1}{f})$, the lemma follows.

□

A particularly simple yet useful corollary of the above lemma follows when one imposes an explicit relation between the two radii r and R. We consider such an instance below:

Corollary 2.3.8. *Let f satisfy the conditions in the statement of Lemma 2.3.7, and suppose $R = 2r$. Then*

$$n(r, \frac{1}{f}) \leq 5T_a(R, \frac{1}{f}).$$

Proof. One first uses the fact that $|a| < r$ to deduce that $C(R, r, a) \leq \frac{R^2 + r^2}{(R-r)^2}$. Replacing R with $2r$ completes the proof. □

The next two lemmas will give us a way of relating the classical Nevanlinna characteristic $T(r,f)$ with the modified characteristic $T_a(r,f)$. The first lemma appears simply as "Equation (3)" in a comment on Page 102 of [10].

Lemma 2.3.9. *Let f be a meromorphic function and $a \in \mathbb{C}$ not a pole or zero of f. Let $R > 0$ such that $|a| < R$. Then*

$$\frac{R-|a|}{R+|a|} m(R,f) \leq m_a(R,f) \leq \frac{R+|a|}{R-|a|} m(R,f).$$

Lemma 2.3.10. *Let f be a meromorphic function and $a \in \mathbb{C}$ not a pole or zero of f. Let $R > 0$ such that $|a| < R$, and $b_1, \ldots, b_p$ be all the poles of f counting multiplicity in $B(0,R)$, where $f(0) \neq \infty$. Let $\rho = \min\{|a-b_k| : k = 1, \ldots, p\}$. Then*

$$N_a(R,f) \leq p \log\left(\frac{2R}{\rho}\right) + N(R,f).$$

Proof. We note:

$$\left|\frac{R^2 - \overline{b_k}a}{R(a-b_k)}\right| \leq \frac{R^2 + |b_k||a|}{R|a-b_k|} \leq \frac{2R^2}{R\rho} \leq \frac{2R}{\rho} \cdot \frac{R}{|b_k|}.$$

Hence,

$$\begin{aligned} N_a(R,f) = \sum_{k=1}^{p} \log\left|\frac{R^2 - \overline{b_k}a}{R(a-b_k)}\right| &\leq p \log\left(\frac{2R}{\rho}\right) + \sum_{k=1}^{p} \log\left|\frac{R}{b_k}\right| \\ &= p \log\left(\frac{2R}{\rho}\right) + N(R,f). \end{aligned}$$

□

Corollary 2.3.11. *Let f satisfy the conditions in the statement of Lemma 2.3.10, then:*

$$T_a(R,f) \leq p \log\left(\frac{2R}{\rho}\right) + \left(\frac{R+|a|}{R-|a|}\right) \cdot T(R,f).$$

Proof. This is evident by Lemma 2.3.9 and 2.3.10. □

A convenient variant of the above corollary is obtained by replacing R with $2r$ whilst maintaining that $|a| < r$. This yields the following inequality:

$$T_a(2r,f) \leq p \log\left(\frac{4r}{\rho}\right) + 3T(2r,f). \tag{2.6}$$

2.4 A discussion of Masser's polynomial

The auxiliary polynomial of Masser is ubiquitous in the proofs of all our results. In this subsection we are therefore going to give a short exposition of his proof of the proposition.

For the sake of continuity we recall the statement of the proposition here:

Lemma 2.4.1. *(Masser, [19], Prop. 2)*

Let $d \geq 1$ and $T \geq \sqrt{8d}$ be positive integers and A, Z, M and H positive real numbers such that $H \geq 1$. Let f_1, f_2 be functions analytic on an open neighbourhood of $\overline{B(0,2Z)}$, with $\max\{|f_1(z)|, |f_2(z)|\} \leq M$ on this set. Suppose $\mathcal{Z} \subset \mathbb{C}$ is finite and satisfies the following for all $z, w \in \mathcal{Z}$:

- $|z| \leq Z$,
- $|w - z| \leq \frac{1}{A}$,
- $[\mathbb{Q}(f_1(z), f_2(z)) : \mathbb{Q}] \leq d$,
- $H(f_1(z), f_2(z)) \leq H$.

Then there is a nonzero polynomial $P(X, Y)$ of total degree at most T such that $P(f_1(z), f_2(z)) = 0$ for all $z \in \mathcal{Z}$ provided

$$(AZ)^T > (4T)^{\frac{96d^2}{T}}(M+1)^{16d}H^{48d^2}.$$

The proof in [19], like many constructions of auxiliary polynomials in number theory, consists of two parts:

The first part is an algebraic argument using Siegel's lemma to construct a non-zero polynomial $P(w_1, w_2)$ of total degree at most T which satisfies (for some m) the system $P(f_1(z_j), f_2(z_j)) = 0$ for $j = 1, \ldots, m$ where $\{z_1, \ldots, z_m\} \subset \mathcal{Z}$.

The second part is an analytic argument using a Schwarz lemma (or the maximum modulus principle) to show that in fact $P(f_1(z), f_2(z)) = 0$ for all $z_j \in \mathcal{Z}$.

Given a number field K and a system of linear equations with coefficients in K, Siegel's lemma guarantees the existence of a nontrivial solution (in integers) whose height is bounded explicitly in terms of the heights of the coefficients.

The variant of Siegel's lemma that Masser uses is the one proved by Gramain, Mignotte and Waldschmidt in [11]. We give the general statement of the lemma below. During the proof we will quote a special case of the lemma which will be conveniently phrased for the problem at hand.

Lemma 2.4.2. *(GMW, [11], Lemme 1.1)*
Let $N_1, \ldots, N_k$ be positive integers. Consider the following homogeneous linear system:

$$\sum_{\nu_1=0}^{N_1-1} \cdots \sum_{\nu_k=0}^{N_k-1} \alpha_{h,1}^{\nu_1} \cdots \alpha_{h,k}^{\nu_k} x_{\nu_1,\ldots,\nu_k} = 0, \;\; (1 \le h \le \mu),$$

where the $\alpha_{h,r}$ are algebraic numbers. Let

$$K_h := \mathbb{Q}(\alpha_{h,1}, \ldots, \alpha_{h,k}), \;\; d_{h,r} := [\mathbb{Q}(\alpha_{h,r}) : \mathbb{Q}], \;\; \text{for } (1 \le h \le \mu,\ 1 \le r \le k)$$

and

$$d_h = [K_h : \mathbb{Q}], \;\; D = \sum_{1 \le h \le \mu} d_h, \;\; L = \prod_{1 \le r \le k} N_r.$$

For $L > D$, the system has a non-trivial solution in integers given by

$$x_\nu = x_{\nu_1,\ldots,\nu_k} \;\; (0 \le \nu_r < N_r,\ 1 \le r \le k)$$

with

$$|x_\nu| \le (2^{\mu'} \prod_{1 \le h \le \mu} M_h)^{1/(L-D)},$$

where

$$M_h = L^{d_h} \prod_{1 \le r \le k} \mathcal{M}(\alpha_{h,r})^{(N_r-1)d_h/d_{h,r}},$$

$\mathcal{M}(\alpha_{h,r})$ is the Mahler measure of $\alpha_{h,r}$ and $\mu^{'}$ ($\leq \mu$) is the number of fields K_h which do not admit any real embedding.

Given a polynomial $P \in \mathbb{C}[z]$, the length of P, written $\mathcal{L}(P)$, is the sum of the absolute values of the coefficients of P.

The next lemma relates the height of the value of a (multivariate) polynomial at algebraic inputs with the length of the polynomial and the heights of the inputs.

Lemma 2.4.3. *([20], Prop 14.7, pp 174)*

Given $P \in \mathbb{Z}[X_1, \ldots, X_n]$ of degree at most $L_1 \geq 0$ in X_1, ..., $L_n \geq 0$ in X_n and algebraic numbers $\alpha_1, \ldots, \alpha_n$, we have that

$$H(P(\alpha_1, \ldots, \alpha_n)) \leq \mathcal{L}(P)H(\alpha_1)^{L_1} \cdots H(\alpha_n)^{L_n}.$$

The last lemma we need is Liouville's inequality for absolute Weil heights. This follows as an immediate corollary of Equation 3.13 in [31], the logarithmic version of Liouville's inequality.

Lemma 2.4.4. *(Liouville inequality, [31])*
Let α be a non-zero algebraic number with degree d. Then

$$|\alpha| \geq H(\alpha)^{-d}.$$

For $x \in \mathbb{R}$, the notation $[x]$ refers to the integer part of x.

We now have the necessary preamble to give Masser's proof:

Proof. Let $S = \left[\frac{T^2}{8d}\right]$ and note that $S \geq 1$. Pick distinct points $z_1, \ldots, z_S \in \mathcal{Z}$.

We note that if

$$|\{(f_1(z), f_2(z)) : z \in \mathcal{Z}\}| < S,$$

then $S - 1 < \frac{1}{2}(T+1)(T+2)$. There would therefore exist a curve of degree at most T passing through

$$\{(f_1(z), f_2(z)) : z \in \mathcal{Z}\}.$$

We therefore henceforth assume that this is not the case.

Let $L = \left[\frac{T}{2}\right]$ and define the polynomial $P(w_1, w_2)$ as

$$P(w_1, w_2) = \sum_{i=0}^{L} \sum_{j=0}^{L} a_{ij} w_1^i w_2^j$$

where the $a'_{ij}s$ are unknown.

It is clear that the total degree of $P(w_1, w_2)$ is at most T. Consider the system

$$P(f_1(z_k), f_2(z_k)) = 0, \;\; k = 1, \ldots, S \tag{2.7}$$

in the $a'_{ij}s$.

The following particular case of Lemma (2.4.2) (with $\mu = S$) provides the conditions for a suitable solution to exist:

Lemma 2.4.5. *Let $F_k = \mathbb{Q}(f_1(z_k), f_2(z_k))$ and $d_k = [F_k : \mathbb{Q}]$.
Let $d_{k,1} = [\mathbb{Q}(f_1(z_k)) : \mathbb{Q}]$ and $d_{k,2} = [\mathbb{Q}(f_2(z_k)) : \mathbb{Q}]$, where $1 \leq k \leq S$. Let $D := \sum_{1 \leq k \leq S} d_k$.*

If $(L+1)^2 > D$ then the linear system (2.7) has a solution $\boldsymbol{a} = (a_{ij}) \in \mathbb{Z}^{(L+1)^2}$, with

$$\max_{0 \leq i,j < L+1} |a_{ij}| \leq \left(2^S \prod_{k=1}^{S} E_k \right)^{1/((L+1)^2 - D)},$$

where

$$E_k = (L+1)^{2d_k} H(f_1(z_k))^{d_k L} H(f_2(z_k))^{d_k L}.$$

We have $d \geq d_k$ for all $1 \leq k \leq S$, and recalling that $L = \left[\frac{T}{2}\right]$, we note that

$$D \leq dS < 2dS \leq 2d \left[\frac{T^2}{8d}\right] < \frac{T^2}{4} < (L+1)^2.$$

The lemma can thus be applied to the proposition. The following bounds also hold:

$$E_k \leq (T+1)^{2d} H^{dT} \;\text{ and }\; \frac{T^2}{8} < (L+1)^2 - D.$$

Hence:

$$\begin{aligned}|P| := \underset{0\le i,j<L+1}{max} |a_{ij}| &\le \left(2^S \prod_{k=1}^{S} (T+1)^{2d} H^{dT}\right)^{\frac{8}{T^2}},\\ &\le 2^{\frac{1}{d}}(T+1)^2 H^T.\end{aligned}$$

The remaining step is to use the Schwarz lemma to show that the function $F(z) := P(f_1(z), f_2(z))$ vanishes for all $z \in \mathcal{Z}$.

Let $z_0 \in \mathcal{Z}$ be arbitrary and define $\Phi(z)$ as

$$\Phi(z) = \frac{P(f_1(z), f_2(z))}{\prod_{k=1}^{S}(z - z_k)}. \tag{2.8}$$

Note that $\Phi(z)$ is analytic on an open set containing $\overline{B(0,2Z)}$. By the maximum modulus principle

$$|\Phi(z_0)| \le M(2Z, \Phi).$$

Recall that one of the conditions in the hypothesis of the proposition is that $|z| \le Z$ for every $z \in \mathcal{Z}$. Hence, for z on the boundary of $\overline{B(0,2Z)}$, we have that $|z - z_k| \ge Z$ for all $k \in \{1, \ldots, S\}$ and therefore by Equation (2.8),

$$M(2Z, \Phi) \le M(2Z, F) Z^{-S}.$$

Proceeding, since $|z - w| \le \frac{1}{A}$ for all $z, w \in \mathcal{Z}$, we have that

$$|F(z_0)| = |\Phi(z_0)| \prod_{k=1}^{S} |z_0 - z_k| \le (AZ)^{-S} M(2Z, F).$$

Bounding $M(2Z, F)$ in terms of the estimate of the size of the coefficients, we have that

$$M(2Z, F) \le |P|(M+1)^T.$$

Hence

$$|F(z_0)| \le (AZ)^{-S}|P|(M+1)^T. \tag{2.9}$$

By Lemma 2.4.3 we have

$$H(F(z_0)) \leq (T+1)^2|P|H^T. \tag{2.10}$$

We note that if $F(z_0) \neq 0$, by Liouville's Inequality, (Lemma 2.4.4) and Equation (2.10) above, we have that

$$|F(z_0)| \geq H(F(z_0))^{-d} \geq \left(\frac{1}{(T+1)^2|P|H^T}\right)^d.$$

This would contradict Equation (2.9) provided

$$(AZ)^S > (T+1)^{2d}(M+1)^T|P|^{2d}H^{dT},$$

in which case $F(z_0) = 0$.

Recalling that $|P| \leq 2^{\frac{1}{d}}(T+1)^2H^T$, $F(z_0) = 0$ provided

$$(AZ)^S > 4(T+1)^{6d}(M+1)^T H^{3dT}.$$

In particular, we must have that $AZ > 1$. Recall that $T^2 \geq 8d$ and $S = \left[\frac{T^2}{8d}\right]$, hence $S \geq \frac{T^2}{16d}$. Substituting this estimate for S into the above equation, we have that $F(z_0) = 0$ provided

$$(AZ)^T > (4T)^{\frac{96d^2}{T}}(M+1)^{16d}H^{48d^2}.$$

as required.

$\square$

Chapter 3

Functions with rapid decay on a ray

3.1 Introduction

Let f be a meromorphic function which decays rapidly on some ray in $\mathbb{C}$ (in our case $|f(x)| \le ab^{-x}$ for sufficiently large $x > 0$), and suppose that $T(r, f) \le r \log r$ for sufficiently large $r > 0$.

In this chapter we will show that there exists an effective constant $C > 0$ depending on f and $d \ge 1$ such that there are at most $C(\log H)^6$ algebraic points of height at most H and degree at most d on the graph of f restricted to a compact, pole free region.

Although the conditions imposed on the functions we consider here may seem artificial at first, they are just abstractions of properties satisfied by some well-known functions such as the Riemann ζ-function. In particular, our result applies to the function $f(z) = \zeta(z) - 1$.

The motivation for this was an attempt to prove a meromorphic analogue of the following result for analytic functions by Boxall and Jones in [5].

Theorem 3.1.1. *([5], Theorem 3.3)*
Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose $r, a, b, s, t, \theta, \phi \in \mathbb{R}$ are such that $a, s, t > 0$, $b > 1$, $r \ge 0$, $\theta \le 0 \le \phi$ and the following two conditions are satisfied:

- $|f(z)| \le s|z|^{t|z|}$ *for all $z \in \mathbb{C}$ with $|z| \ge r$,*

- $|f(z)| \leq \frac{a}{b^{|z|}}$ *for all* $z \in \mathbb{C}$ *such that* $\theta \leq \arg z \leq \phi$ *and* $|z| \geq r$.

Let d be a positive integer. There exists $C > 0$ such that, for all $H > e^e$, there are at most $C(\log H)^3(\log\log H)^3$ complex numbers z such that the following conditions are satisfied:

- $\theta \leq \arg z \leq \phi$,
- $f(z) \neq 0$,
- $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d$,
- $H(z, f(z)) \leq H$.

In this context, the analogous growth condition is imposed on the Nevanlinna characteristic of the function f, and takes the form $T(r, f) \leq s(r)$, where $s : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is some nonnegative, nondecreasing function.

The next lemma provides a quantitative way of covering the zeroes of a polynomial $P(z)$ with a collection of disks outside of which $|P(z)|$ is bounded below by any desired quantity. We will occasionally use it in subsequent chapters.

Lemma 3.1.2. *(Boutroux-Cartan, [26], Theorem 12.5.7)*
Let

$$P(z) = \prod_{j=1}^{n}(z - z_j).$$

Then for any positive h, the inequality

$$|P(z)| > h^n$$

holds outside at most n disks, the sum of whose radii is at most $2eh$.

We use Masser's proposition, Lemma 1.1.26 in the same form as the application by Boxall and Jones in [5]. The lemma as given here was derived from Masser's proof of Lemma 1.1.26. We quote the statement below:

Lemma 3.1.3. *([5], Lem. 2.1.)*
Let $r, \delta > 0$. Let f be an analytic function on an open set containing $\overline{B(0, 4r(1+\delta))}$. Let d be a positive integer. Then there exists $C > 0$ such that, for all $H > e^e$, there is a nonzero polynomial $P(X, Y) \in \mathbb{Z}[X, Y]$ such that:

- *for all* $z \in \overline{B(0,r)}$*, if* $[\mathbb{Q}(z,f(z)) : \mathbb{Q}] \leq d$ *and* $H(z,f(z)) \leq H$*, then* $P(z,f(z)) = 0$*,*
- $P(X,Y)$ *has total degree at most* $T = C \log H$ *and the coefficient with maximum modulus* $|P|$ *satisfies*

$$|P| \leq 2^{\frac{1}{d}}(T+1)^2 H^T.$$

3.2 Main Result

Theorem 3.2.1. *Let* f *be a non-constant meromorphic function on* $\mathbb{C}$*. Suppose* $a, b, r_0, r_0', r', t \in \mathbb{R}$ *are such that* $r' > 0, r_0 \geq 0, r_0' \geq 1$*,* $a > 0$*,* $b > 1$*,* $t > 0$ *and* f *satisfies the following conditions:*

- f *is analytic on a neighbourhood of* $\overline{B(0,4r')}$*,*
- $|f(x)| \leq ab^{-x}$ *for all* $x \in \mathbb{R}$ *such that* $x \geq r_0$*,*
- $T(r,f) \leq tr \log r$ *for all* $r \geq r_0'$*.*

Let $d \geq 1$ *be a positive integer. Then there exists* $C > 0$ *such that for all* $H > e^e$*, there are at most* $C(\log H)^5(\log\log H)^2$ *complex numbers* z *such that* $|z| \leq r'$*,* $[\mathbb{Q}(z,f(z)) : \mathbb{Q}] \leq d$ *and* $H(z,f(z)) \leq H$*.*

Proof. Let $H > e^e$. Since f is holomorphic on a neighbourhood of $\overline{B(0,4r')}$ Lemma 3.1.3 immediately gives $C_1 > 0$ and a polynomial $P(X,Y) \in \mathbb{Z}[X,Y]$ satisfying the conditions given therein, with C_1 in place of C.

Let $G(z) = P(z,f(z))$. We would like to estimate the number of zeroes of G in $\overline{B(0,r')}$. Let $R(X) = P(X,0)$ and $Q(X,Y) = P(X,Y) - R(X)$. Let us first deal with two minor issues:

- If $P(X,Y) = R(X)$, then the bound we seek simply becomes $C_1 \log H$, the upper bound for the number of solutions of the equation $R(X) = 0$.

- If Y is a factor of every term of $P(X,Y)$, let $k \geq 1$ be the smallest power of Y appearing in $P(X,Y)$. Then the set of algebraic z such that $f(z) = 0$ is simply bounded above by C, for some constant $C > 0$. We could thereafter apply the main argument to $P_1(X,Y) := \frac{1}{Y^k}P(X,Y)$. We can henceforth assume that Y is not a factor of every term of $P(X,Y)$ (hence $R(X) \not\equiv 0$) and proceed with the main argument.

The next step is to utilize the conditions imposed on f to obtain some x_0 on the positive real axis such that x_0 is not too close to a pole and $|G(x_0)| \geq \frac{1}{2}$. It suffices to show that $|Q(x_0, f(x_0))| \leq \frac{1}{2}$ and $|R(x_0)| \geq 1$. The procedure we use to obtain x_0 is an adaptation of the one in [5].

Let $K = \max\{r_0, r_0', \frac{\log a}{\log b}\}$ and $x > 0$. Since Y is a factor of $Q(X,Y)$, in order that $|Q(x, f(x))| \leq \frac{1}{2}$, it suffices to have

$$(T+1)^2|P|ab^{-x}x^T \leq \frac{1}{2} \text{ and } x \geq K.$$

We note that

$$(T+1)^2|P|ab^{-x}x^T \leq \frac{1}{2}$$

if and only if

$$x \log b \geq \log(2a(T+1)^2|P|) + T \log x.$$

The above inequality holds provided that

$$x \log b \geq \max\{2\log(2a(T+1)^2|P|), 2T\log x\}$$

We thus have that

$$|Q(x, f(x))| \leq \frac{1}{2}$$

provided

$$x \geq \max\left\{\frac{2\log(2a(T+1)^2|P|)}{\log b}, e, \left(\frac{2T}{\log b}\right)^{\frac{3}{2}}, K\right\},$$

assuming $C > 0$ was chosen large enough.

We would like to obtain this lower bound on x as a more explicit expression in terms of H. We thus refer back to the bounds in terms of H that we already have for $|P|$ and T. For the remainder of the discussion, $C > 0$ will denote a positive constant not depending on H that may be different at each occurrence.

First, we note that:

$$\log|P| \leq C(\log H)^2$$

and

$$\log T \leq C \log\log H.$$

Therefore,

$$\frac{2\log(2a(T+1)^2|P|)}{\log b} \leq C(\log H)^2$$

We also have:

$$\left(\frac{2T}{\log b}\right)^{\frac{3}{2}} \leq C(\log H)^2$$

Therefore, $|Q(x, f(x))| \leq \frac{1}{2}$ provided $x \geq C(\log H)^2$.

Let $\mu := C(\log H)^2$ and recall that the polynomial $R(X)$ has degree at most $T = C(\log H)$. Also, by Lemma 2.2.3 and the growth condition imposed on $T(r, f)$, the following chain of inequalities holds:

$$n(2\mu, f) \leq \frac{1}{\log 2} T(4\mu, f) \leq C\mu \log \mu.$$

Using the Boutroux-Cartan lemma with $h = 1$, we have that there are at most T disks outside of which $|R(X)| > 1$ and the sum of whose radii is at most $2e$.

There are at most $C\mu \log \mu$ poles of f in $B(0, 2\mu)$. We cover each pole with a disk centered around the pole with radius $\frac{1}{4C \log \mu}$. Hence the sum of the radii of the disks containing the poles as well as the Boutroux-Cartan disks is at most $\frac{\mu}{4}$, for a possibly different choice of $C > 0$.

This implies that the total length of the line segments in $[\mu, 2\mu]$ on which $|R(X)| > 1$ and f has no poles is at least $\frac{\mu}{2}$.

The total number of disks is at most $T + C\mu \log \mu$, which is less than $C\mu \log \mu$ for a larger constant C. Hence the average length of each line segment is at least

$$\rho = \frac{\mu/2}{C\mu \log \mu} = \frac{C}{\log \mu}.$$

This implies that there exists $x_0 \in [\mu, 2\mu]$ such that $|R(x_0)| > 1$ and $|Q(x_0, f(x_0))| \leq \frac{1}{2}$ as desired, and so

$$|G(x_0)| \geq \frac{1}{2}$$

and furthermore, $B(x_0, \rho)$ does not contain any poles of f.

Since throughout our argument we assume that H is always "large enough", we can assume that $r' \leq \mu$. Hence $r' \leq \mu \leq x_0 \leq 2\mu$. We then let $R_0 = 6\mu$, hence, $\overline{B(0, r')} \subset B(x_0, R_0)$. Therefore, to estimate the number of zeroes of G in $\overline{B(0, r')}$, it is enough to estimate the number of zeroes of G in $B(x_0, R_0)$.

This is what we endeavour to do in the rest of the proof and for this we call upon some of the Nevanlinna theory developed in Chapter 2.

First we express $G(z)$ as

$$G(z) = a_1 z^{T_1} (f(z))^{T_1'} + a_2 z^{T_2} (f(z))^{T_2'} + \cdots + a_n z^{T_n} (f(z))^{T_n'},$$

where:

- $|a_j| \leq |P|$ for $j = 1, \ldots, n$,
- $n \leq (T+1)^2$,
- $T_j + T_j' \leq T = C_1 \log H$ for $j = 1, \ldots, n$.

From the second part of Lemma 2.3.3, we get

$$T_{x_0}(R_0, G) \leq \sum_{j=1}^{n} T_{x_0}(R_0, a_j z^{T_j} (f(z))^{T'_j}) + \log n.$$

We apply the first part of Lemma 2.3.3 to the summands on the right hand side of the above inequality to obtain

$$T_{x_0}(R_0, G) \leq \sum_{j=1}^{n} T_{x_0}(R_0, a_j z^{T_j}) + \sum_{j=1}^{n} T_{x_0}(R_0, (f(z))^{T'_j}) + \log n. \tag{3.1}$$

We first note that each $g_j = a_j z^{T_j}$ is an entire function for $j = 1, \ldots, n$, and that $M(r, g_j) \leq M(r, |P| z^T)$ for any $r \geq 1$. We then invoke Lemma 2.3.6 on the first sum of the right hand side of Equation (3.1) to obtain

$$\sum_{j=1}^{n} T_{x_0}(R_0, a_j z^{T_j}) \leq (T+1)^2 \cdot (\log |P| + T \log R_0).$$

Recalling that $T = C \log H$, $R_0 = 6\mu$ where $\mu = C(\log H)^2$ and $\log |P| \leq C(\log H)^2$, one can show first that the right hand side of the above inequality is dominated by a multiple of $T^3 (\log H)^2$. Hence:

$$\sum_{j=1}^{n} T_{x_0}(R_0, a_j z^{T_j}) \leq C(\log H)^5. \tag{3.2}$$

Proceeding, note that since $n \leq (T+1)^2$, the $\log n$ term in Inequality (3.1) also gets absorbed by the above bound.

We now focus on the second sum of the right hand side of Equation (3.1). First, bearing in mind that $T'_j \leq T$, we use Lemma 2.3.4 to conclude that for each $j = 1, \ldots, n$,

$$T_{x_0}(R_0, (f(z))^{T'_j}) \leq T \cdot T_{x_0}(R_0, f(z)).$$

Hence,

$$\sum_{j=1}^{n} T_{x_0}(R_0, (f(z))^{T'_j}) \leq (T+1)^2 T \cdot T_{x_0}(R_0, f(z)).$$

We would now like to exploit the relationship between $T_{x_0}(R_0, f(z))$ and $T(R_0, f(z))$. Recall that f has no poles in $B(x_0, \rho)$ where $\rho = \frac{C}{\log \mu}$.

Let $p = \left(\frac{1}{\log 2}\right) T(2R_0, f)$. By Lemma 2.2.3, this is an upper bound on the number of poles of f in $B(0, R_0)$. We thus have the necessary ingredients to invoke Inequality (2.6) and conclude that:

$$T_{x_0}(R_0, f) \leq \left(\frac{1}{\log 2}\right) \cdot \log\left(\frac{2R_0}{\rho}\right) \cdot T(2R_0, f) + 3T(R_0, f).$$

The above inequality can be written as:

$$T_{x_0}(R_0, f) \leq C \log(R_0 \log \mu) T(2R_0, f).$$

Recalling that $R_0 := 6\mu$, for a possibly different constant $C > 0$ we have that:

$$T_{x_0}(R_0, f) \leq C(\log \mu) T(2R_0, f).$$

We invoke the growth condition imposed on $T(r, f)$ to conclude that

$$\sum_{j=1}^{n} T_{x_0}(R_0, (f(z))^{T_j'}) \leq C \log \mu (T+1)^2 T \cdot (2tR_0 \log(2R_0)).$$

For simplicity, we write the above inequality as

$$\sum_{j=1}^{n} T_{x_0}(R_0, (f(z))^{T_j'}) \leq CT^3 (\log \mu)^2 \mu,$$

where $C > 0$ does not depend on H.

The right hand side of the above inequality is dominated by
$C(\log H)^5 (\log \log H)^2$. Hence

$$\sum_{j=1}^{n} T_{x_0}(R_0, (f(z))^{T_j'}) \leq C(\log H)^5 (\log \log H)^2. \tag{3.3}$$

Therefore, $T_{x_0}(R_0, G) \leq C(\log H)^5 (\log \log H)^2$.

Recalling that $|G(x_0)| \geq \frac{1}{2}$, we first deduce by Lemma 2.3.2 that

$$T_{x_0}\left(R_0, \frac{1}{G}\right) = T_{x_0}(R_0, G) - \log |G(x_0)|.$$

On the other hand, $r' \leq x_0 < \frac{R_0}{2} = 3\mu$, so that by Corollary 2.3.8 we have

$$n(r', \frac{1}{G}) \leq 5 \cdot T_{x_0}(R_0, \frac{1}{G}).$$

We thus conclude that

$$n(r', \frac{1}{G}) \leq C(\log H)^5(\log\log H)^2.$$

This completes the proof.

□

Remark 3.2.2. *At all occurrences, the constant C can be computed effectively from $r', t, r_0, r_0', d, a, b$.*

Chapter 4

Meromorphic functions of finite order and positive lower order

4.1 Introduction

Let f be meromorphic in $\mathbb{C}$ of finite order ρ and positive lower order λ. Suppose furthermore that f is holomorphic in a neighbourhood of $|z| \leq 6s$ for some $s > 0$.

In this chapter we prove a bound of the form $C(\log H)^\eta$ for the number of algebraic points of height at most H and degree at most d on the graph of f restricted to $\overline{B(0,s)}$. The positive constant C depends on λ, ρ, s, d and f whilst $\eta = \eta(\lambda, \rho)$.

This result can be seen as an extension of the corresponding result for algebraic values of entire functions of finite order and positive lower order by Boxall and Jones, [6].

4.2 Main Result

Recall the following definitions of order and lower order of a meromorphic function as defined in Chapter 2.

If f is a meromorphic function, the order of f is defined to be

$$\rho := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}. \tag{4.1}$$

The lower order of f is

$$\lambda := \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

where

$$M(r,f) := \max_{|z|=r} |f(z)|.$$

We prove the following theorem:

Theorem 4.2.1. *Let f be a nonconstant meromorphic function of order ρ and lower order λ. Assume $0 < \lambda \leq \rho < \infty$. Let d, s, α and β be as follows: $d \geq 1$, $s > 0$, $\alpha = \max\{1,\rho\} + \frac{\lambda}{2}$ and $\beta = \frac{\lambda}{2}$. Suppose furthermore that f is holomorphic in a neighbourhood of $\overline{B(0,6s)}$. Then there exists a constant $C > 0$ such that for all $H > e$, there are at most $C(\log H)^{\frac{5\alpha^2}{\beta}}$ complex numbers z such that $|z| \leq s$, $[\mathbb{Q}(z,f(z)) : \mathbb{Q}] \leq d$ and $H(z,f(z)) \leq H$.*

Proof. We would first like to obtain a nonzero polynomial $P(X,Y) \in \mathbb{Z}[X,Y]$ of degree at most $T = (\log H)^\alpha$ such that $|P| \leq 2^{\frac{1}{d}}(T+1)^2 H^T$ and $P(z,f(z)) = 0$ whenever $|z| \leq s$, $[\mathbb{Q}(z,f(z)) : \mathbb{Q}] \leq d$ and $H(z,f(z)) \leq H$.

Recall that $\alpha = \max\{1,\rho\} + \frac{\lambda}{2}$ and $\beta = \frac{\lambda}{2}$.

Let $A = \frac{1}{2s}$, $Z = 3s$, $T = (\log H)^\alpha$, $S = M(6s,f)$ and $M = \max\{6s, S\}$. Then

$$\max\{|z|,|f(z)|\} \leq M \text{ for } |z| \leq 2s \text{ and } (AZ)^T > (4T)^{\frac{96d^2}{T}}(M+1)^{16d}H^{48d^2}$$

for large enough H.

Note that since the bound we set out to prove is worse than $C(\log H)^{2\alpha}$, we can assume that there are at least $\frac{T^2}{8d}$ complex numbers such that $|z| \leq s$, $[\mathbb{Q}(z,f(z)) : \mathbb{Q}] \leq d$ and $H(z,f(z)) \leq H$. Assuming H is large enough, the proof of Lemma 1.1.26 then guarantees us a polynomial $P(X,Y) \in \mathbb{Z}[X,Y]$ satisfying all our requirements.

Let $G(z) = P(z, f(z))$, which we can then express as

$$G(z) = a_1 z^{T_1}(f(z))^{T_1'} + a_2 z^{T_2}(f(z))^{T_2'} + \cdots + a_n z^{T_n}(f(z))^{T_n'},$$

where:

- $|a_j| \leq |P| \leq 2^{\frac{1}{d}}(T+1)^2 H^T$ for $j = 1, \ldots, n$,
- $n \leq (T+1)^2$,
- $T_j + T_j' \leq T = (\log H)^\alpha$ for $j = 1, \ldots, n$.

First we would like to find some point $x_0 = r_0 e^{i\theta}$ such that $1 \leq |G(x_0)| < \infty$.

Let k be the highest power of Y appearing in $P(X, Y)$. Without loss of generality we may assume $k \geq 1$. Let $\tilde{P}(X, Y) = Y^k P(X, \frac{1}{Y})$, $R(X) = \tilde{P}(X, 0)$ and $Q(X, Y) = \tilde{P}(X, Y) - R(X)$. We note that $R(X) \not\equiv 0$. Let $\tilde{Q}(X, Y) = \frac{1}{Y} Q(X, Y)$. Note that

- The degree of X in $\tilde{Q}$ is at most T,
- $|\tilde{Q}| \leq |P| \leq 2^{\frac{1}{d}}(T+1)^2 H^T$ and
- $\tilde{Q}$ has at most $(T+1)^2$ terms.

Let $z = re^{i\theta}$ be such that $1 \leq |f(z)| = M(r, f) < \infty$. This is possible since f has positive lower order.

For such z we have that

$$\left| \tilde{Q}\left(z, \frac{1}{f(z)}\right) \right| \leq 2^{\frac{1}{d}}(T+1)^4 H^T r^T.$$

We note that if

$$2^{\frac{1}{d}}(T+1)^4 H^T r^T \leq \frac{1}{2} M(r, f),$$

then

$$\left| Q\left(z, \frac{1}{f(z)}\right) \right| \leq \frac{1}{2}.$$

Proceeding, if

$$r \geq C(\log H)^{\frac{\alpha+1}{\beta}},$$

then

$$2^{\frac{1}{d}}(T+1)^4 H^T r^T \leq \frac{1}{2} e^{r^\beta}.$$

Since $0 < \beta < \lambda$, where λ is the lower order of f, by Remark 2.2.6 we have that

$$e^{r^\beta} \leq M(r, f) \text{ provided } H \text{ is large enough.}$$

Therefore, for such z, we have that

$$\left| Q\left(z, \frac{1}{f(z)}\right) \right| \leq \frac{1}{2}.$$

Let $\mu = C(\log H)^{\frac{\alpha+1}{\beta}}$ and set $q = \lceil \frac{1}{\log 2} T(4\mu, f) \rceil$, the smallest integer greater than or equal to $\frac{1}{\log 2} T(4\mu, f)$. This is an upper bound on the total number of poles of f in $B(0, 2\mu)$ by Lemma 2.2.3. In particular this is also an upper bound on the number of poles in the annulus $\mu < |z| < 2\mu$, which are the poles we shall henceforth concern ourselves with. Since $T(r, f) \leq r^\alpha$, we have that $q \leq CT(4\mu, f) \leq C\mu^\alpha$.

Recall that the degree of $R(X)$ is at most T. Hence by the Boutroux-Cartan lemma there are at most T disks outside of which $|R(z)| > 1$, the sum of whose diameters is at most $4e$.

Therefore there is an annulus $r_i < |z| < r_j$ passing through the interval $[\mu, 2\mu]$ which does not intersect the Boutroux-Cartan disks, where

$$r_j - r_i = \frac{\mu - 4e}{T+1}.$$

We subdivide this annulus into $q + 1 = C\mu^\alpha$ smaller annuli which cut $[r_i, r_j]$ into equal subintervals. By the pigeonhole principle, at least one of the smaller annuli does not contain any poles of f. The difference between the outer and inner radii of this annulus is

$$\frac{r_j - r_i}{C\mu^\alpha} \geq \frac{1}{CT\mu^{\alpha-1}} \geq \frac{1}{C\mu^{2\alpha-1}}.$$

In summary we have that there is at least one annulus $\tilde{r} < |z| < \tilde{R}$ such that

- f is holomorphic in the annulus.
- $|R(z)| > 1$ for all z in the annulus.
- $M(r_0, f) \geq e^{r_0^\beta}$ where $r_0 := \frac{\tilde{r}+\tilde{R}}{2}$. In particular, there is a $\theta \in [0, 2\pi]$ such that $|f(r_0 e^{i\theta})| = M(r_0, f) \geq e^{r_0^\beta}$. We let $x_0 := r_0 e^{i\theta}$ and $\delta = \frac{1}{C\mu^{2\alpha-1}}$.
- $\left|Q\left(x_0, \frac{1}{f(x_0)}\right)\right| \leq \frac{1}{2}$.

Since $\tilde{P}(X, Y) = Q(X, Y) + R(X)$, we have that

$$\left|\tilde{P}\left(x_0, \frac{1}{f(x_0)}\right)\right| \geq |R(x_0)| - \left|Q\left(x_0, \frac{1}{f(x_0)}\right)\right| \geq \frac{1}{2}.$$

Therefore

$$|G(x_0)| = |P(x_0, f(x_0))| = \left|f(x_0)^k \tilde{P}\left(x_0, \frac{1}{f(x_0)}\right)\right| \geq \frac{1}{2} e^{k r_0^\beta}.$$

We conclude that $|G(x_0)| \geq 1$.

We note that $\overline{B(0,s)} \subset B(x_0, 4\mu)$ and $B(x_0, \delta)$ does not contain any poles of f.

We would like to estimate the number of zeroes of G in $\overline{B(0,s)}$. To do this, it suffices to bound the quantity $n(2\mu, \frac{1}{G})$ from above.

First, by Corollary 2.3.8 and Lemma 2.3.2, we have that:

$$n(2\mu, \frac{1}{G}) \leq 5T_{x_0}(4\mu, \frac{1}{G}) = 5(T_{x_0}(4\mu, G) - \log|G(x_0)|).$$

We now focus on bounding the quantity $T_{x_0}(4\mu, G)$.

By the second part of Lemma 2.3.3, we get

$$T_{x_0}(4\mu, G) \leq \sum_{j=1}^{n} T_{x_0}(4\mu, a_j z^{T_j} (f(z))^{T_j'}) + \log n.$$

Using the first part of Lemma 2.3.3 we split the summand in the above inequality to get:

$$T_{x_0}(4\mu, G) \leq \sum_{j=1}^{n} T_{x_0}(4\mu, a_j z^{T_j}) + \sum_{j=1}^{n} T_{x_0}(4\mu, (f(z))^{T_j'}) + \log n. \tag{4.2}$$

We first note that each $g_j = a_j z^{T_j}$ is an entire function for $j = 1, \ldots, n$, and that $M(r, g_j) \leq M(r, |P|z^T)$ for any $r \geq 1$. Hence:

$$\sum_{j=1}^{n} T_{x_0}(4\mu, a_j z^{T_j}) \leq (T+1)^2 \cdot (\log |P| + T \log 4\mu).$$

The above inequality can be written more explicitly in terms of T and H as:

$$\sum_{j=1}^{n} T_{x_0}(4\mu, a_j z^{T_j}) \leq C(T^3 \log 4\mu + T^2 \log T + T^3 \log H).$$

Recalling that $\mu := C(\log H)^{\frac{\alpha+1}{\beta}}$ and $T = (\log H)^{\alpha}$, the above inequality is dominated by a term of the form:

$$C(\log H)^{4\alpha}. \tag{4.3}$$

We now focus on the second sum of the right hand side of Equation (4.2). Since $T_j' \leq T$ for all $j = 1, \ldots, n$, we have that:

$$T_{x_0}(4\mu, (f(z))^{T_j'}) \leq T \cdot T_{x_0}(4\mu, f).$$

Hence,

$$\sum_{j=1}^{n} T_{x_0}(4\mu, (f(z))^{T_j'}) \leq (T+1)^2 T \cdot T_{x_0}(4\mu, f).$$

Let $p = \left(\frac{1}{\log 2}\right) T(8\mu, f)$, by Lemma 2.2.3 this is an upper bound on the number of poles of f in $B(0, 4\mu)$. We thus have the necessary ingredients to invoke Inequality (2.6) and conclude that:

$$T_{x_0}(4\mu, f) \leq \left[\left(\frac{\log\left(\frac{8\mu}{\delta}\right)}{\log 2} \right) + 3 \right] \cdot T(8\mu, f).$$

We then invoke the growth condition imposed on $T(r, f)$ to conclude that

$$\sum_{j=1}^{n} T_{x_0}(4\mu, (f(z))^{T_j'}) \leq \left[\left(\frac{\log\left(\frac{8\mu}{\delta}\right)}{\log 2} \right) + 3 \right] (T+1)^2 T \cdot (8\mu)^{\alpha}.$$

For simplicity, we can write the above inequality as:

$$\sum_{j=1}^{n} T_{x_0}(4\mu, (f(z))^{T_j'}) \leq C \log\left(\frac{8\mu}{\delta}\right) T^3 \cdot (8\mu)^{\alpha}. \tag{4.4}$$

Recall that $\delta = 1/(C\mu^{2\alpha-1})$. Hence, $\frac{8\mu}{\delta} = C\mu^{3\alpha-1}$. Recalling the definition of μ in terms of H, and then substituting for this, we have that:

$$\log\left(\frac{8\mu}{\delta}\right) \leq C(\log\log H).$$

Upon replacing the rest of the variables appearing on the right hand side of Inequality (4.4) with their respective formulae in terms of H, we note that the sum is dominated by a term of the form:

$$C(\log H)^{\frac{5\alpha^2}{\beta}}. \tag{4.5}$$

Combining Equations (4.3) and (4.5) and noting that $\frac{5\alpha^2}{\beta} > 4\alpha$ (since $\alpha > \beta$), we can thus conclude that:

$$n(2\mu, \frac{1}{G}) \leq C(\log H)^{\frac{5\alpha^2}{\beta}}. \tag{4.6}$$

$\square$

Chapter 5

Algebraic values of the Weierstrass σ-function

5.1 Introduction, lemmas and definitions

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ be the lattice generated by $\omega_1, \omega_2 \in \mathbb{C}$ where $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$. The Weierstrass σ-function associated to the lattice Λ is the entire function

$$\sigma(z, \Lambda) := z \prod_{\omega \in \Lambda'} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right) \tag{5.1}$$

where $\Lambda' = \Lambda \setminus \{0\}$.

For our purposes we take $\omega_1 = 1$ and $\omega_2 = i$ so that $\Lambda = \mathbb{Z} + \mathbb{Z}i = \mathbb{Z}[i]$, the ring of Gaussian integers which we often identify with $\mathbb{Z}^2$.

Henceforth, we shall refer to $\sigma(z, \Lambda)$ simply as $\sigma(z)$.

In this chapter we are concerned with counting *all* algebraic points of height at most H and degree at most d on the graph of the Weierstrass σ-function, excluding the zeroes. The first bound we obtain is $C(\log H)^{15}$. However, an earlier result of Besson in [2] allows us to easily deduce, via our method, a final improved bound $C(\log H)^7$.

The first ingredient we need is the following lemma which follows immediately from the work of Pogány in [25]. In what follows, let $\delta_z := dist(z, \mathbb{Z}^2)$

be the distance from $z \in \mathbb{C}$ to the nearest lattice point (in our case, the nearest Gaussian integer). Pogány proves the following:

Lemma 5.1.1. *There exist positive and absolute constants* $\mathbf{K}_1 < \mathbf{K}_2$ *such that:*

$$\mathbf{K}_1 \delta_z e^{\frac{\pi}{2}|z|^2} \leq |\sigma(z)| \leq \mathbf{K}_2 \delta_z e^{\frac{\pi}{2}|z|^2}, \text{ for all } z \in \mathbb{C}. \tag{5.2}$$

Remark 5.1.2. *Pogány further showed that one can take* $\mathbf{K}_1 = 0.266$ *and* $\mathbf{K}_2 = 1$.

The next ingredient we need is a radius $R_{H,d}$ such that if $|z| \geq R_{H,d}$, $\deg(z) \leq d$, $\text{height}(z) \leq H$ and $\sigma(z) \neq 0$, then $|\sigma(z)|$ is too large to have height at most H and degree at most d.

To this end let us first recall Liouville's inequality from **Chapter 2**:

Lemma 5.1.3. *(Liouville inequality, [31])*
Let α be a non-zero algebraic number with degree d. Then

$$|\alpha| \geq H(\alpha)^{-d}.$$

With the previous inequality, we have:

Lemma 5.1.4. *Let α be a non-zero algebraic number of degree at most d and height at most H. Then*

$$\frac{1}{(2H)^d} \leq |\alpha| \leq (2H)^d. \tag{5.3}$$

Proceeding, we would like to bound δ_z from below in the following sense:

Given d and H, we would like to find a small radius $r_{H,d}$ such that if α has degree at most d and height at most H, and α is not a Gaussian integer, then $\alpha \notin \overline{B(\beta, r_{H,d})}$ for any Gaussian integer β. By Lemma 5.1.4, we know that $|\alpha| \leq (2H)^d$, hence the Gaussian integers we should be concerned with have absolute value at most $(2H)^d + 2$, say.

Note that if $\beta = a + ib$ is a Gaussian integer, then

$$H(\beta) = H(a + ib) \leq 2H(a)H(ib) \leq 2|a||b|.$$

If $\beta = a + ib$ is a Gaussian integer such that $|\beta| \leq (2H)^d + 2$, then:

$$|a| \leq (2H)^d + 2 \text{ and } |b| \leq (2H)^d + 2.$$

Therefore:

$$H(\beta) \leq 2|a||b| \leq 2((2H)^d + 2)^2 < 9(2H)^{2d}.$$

Using basic properties of the multiplicative height function, we can deduce the following:

Lemma 5.1.5. *Let α be an algebraic number such that $\deg(\alpha) \leq d$ and $H(\alpha) \leq H$. Suppose furthermore that α is not a Gaussian integer. Let β be a Gaussian integer with $H(\beta) < 9(2H)^{2d}$. Then*

$$\deg(\alpha - \beta) \leq 2d \text{ and } H(\alpha - \beta) \leq 2H(\alpha)H(\beta) < 9(2H)^{2d+1} \leq 9(2H)^{3d}. \tag{5.4}$$

Using Lemma (5.1.4), we can deduce the following (rather loose) bounds:

$$\frac{1}{(36H)^{6d^2}} < |\alpha - \beta| < (36H)^{6d^2}.$$

We then let:

$$r_{H,d} = \frac{1}{(36H)^{6d^2}}$$

Lemma 5.1.6. *Let $z \in \mathbb{C}$ be such that $\deg(z) \leq d$, $H(z) \leq H$ and $|z - \beta| \geq r_{H,d}$ for all Gaussian integers β with $|\beta| \leq (2H)^d + 2$. If $|z| \geq R_{H,d} := 42d(\log H)^{\frac{1}{2}}$, then $|\sigma(z)| \geq (2H)^{2d}$. This implies that either $H(\sigma(z)) > H$ or $\deg(\sigma(z)) > d$.*

Proof. Say $z = re^{i\theta}$. Consider the inequality

$$K_1 r_{H,d} e^{\frac{\pi}{2}r^2} \geq (2H)^{2d}$$

where K_1 is as in Equation (5.2).

The above inequality holds if

$$K_1 e^{\frac{\pi}{2}r^2} \geq (2H)^{2d}(36H)^{6d^2},$$

which in turn follows if

$$e^{r^2} \geq (8H)^{2d}(36H)^{6d^2}.$$

The previous inequality is guaranteed if

$$r \geq 42d(\log H)^{\frac{1}{2}}.$$

In this case we have that

$$|\sigma(z)| \geq (2H)^{2d}.$$

Lemma (5.1.4) then implies that $\sigma(z)$ cannot be an algebraic number of degree at most d and height at most H.

□

Since σ is an entire function of order and lower order 2, for the rest of our argument we adopt the strategy of Boxall and Jones in [6], the paper on algebraic points of bounded height and degree on restrictions to compact sets of graphs of functions of positive lower order and finite order.

Our main contribution is that we count the total number of such points on the graph of σ, excluding the zeroes. Before stating and proving our theorem we recall Lemma 3.1.2, stated here for the case $h = 1$:

Lemma 5.1.7. *(Boutroux-Cartan)*
Let $P(z) \in \mathbb{C}[z]$ be a monic polynomial with degree $n \geq 1$. Then $|P(z)| > 1$ for all complex z outside a collection of at most n disks the sum of whose radii is $2e$.

5.2 Main Result and Proof

We can now state and prove our main result.

Theorem 5.2.1. *Let $\sigma(z, \Lambda)$ be the Weierstrass σ-function associated to the lattice $\Lambda = \mathbb{Z}[i]$ and suppose $d \geq 1$. Then there is a constant $C > 0$ such that for all $H > e$, there are at most $C(\log H)^{15}$ numbers $z \in \mathbb{C}$ such that $[\mathbb{Q}(z, \sigma(z)) : \mathbb{Q}] \leq d$, $H(z, \sigma(z)) \leq H$ and $\sigma(z) \neq 0$.*

Proof. Let $H > e^e$. In fact, throughout our proof we will always assume H is "large enough". We shall denote by C a positive constant independent of

H. The constant C may not be the same at each occurrence.

We would first like to obtain a non-zero polynomial $P(X,Y) \in \mathbb{Z}[X,Y]$ of degree at most $T = \left(\frac{\log H}{\log\log H}\right)^3$ such that $|P| \leq 2^{\frac{1}{d}}(T+1)^2 H^T$ and $P(z,\sigma(z)) = 0$ whenever $[\mathbb{Q}(z,\sigma(z)) : \mathbb{Q}] \leq d$ and $H(z,\sigma(z)) \leq H$ and $\sigma(z) \neq 0$. To this end, let:

$$A = \frac{1}{2R_{H,d}},\ Z = \frac{\log H}{\log\log H},\ T = \left(\frac{\log H}{\log\log H}\right)^3 \text{ and } M = e^{(2Z)^3}.$$

We then have that $\max\{|z|, |\sigma(z)|\} \leq M$ for all $z \in \overline{B(0,2Z)}$ and,

$$\begin{aligned}\log(AZ)^T &\geq C\left(\frac{\log H}{\log\log H}\right)^3 \log\log H,\\ &> \frac{C\log\log H}{(\log H/\log\log H)^3} + C\left(\frac{\log H}{\log\log H}\right)^3 + C\log H.\end{aligned}$$

Therefore

$$(AZ)^T > (4T)^{\frac{96d^2}{T}}(M+1)^{16d}H^{48d^2}.$$

Since the bound we are trying to prove is worse than $C(\log H)^6$, we can thus assume that there are at least $\frac{T^2}{8d}$ complex numbers such that $[\mathbb{Q}(z,\sigma(z)) : \mathbb{Q}] \leq d$ and $H(z,\sigma(z)) \leq H$. From the proof of Lemma 1.1.26, there is a polynomial $P(X,Y)$ satisfying all our requirements.

Let $G(z) = P(z,\sigma(z))$. We would like to bound the number of zeroes of G for which $\sigma(z) \neq 0$ in $\overline{B(0,R_{H,d})}$. To do this, first let k be the highest power of Y in $P(X,Y)$. We can assume $k \geq 1$. Let $\tilde{P}(X,Y) = Y^k P(X,\frac{1}{Y})$, $R(X) = \tilde{P}(X,0)$, and $Q(X,Y) = \tilde{P}(X,Y) - R(X)$. We note that $R(X)$ is not identically zero. Let $\tilde{Q}(X,Y) = \frac{1}{Y}Q(X,Y)$. The highest power of X in $\tilde{Q}$ is at most T and $|\tilde{Q}| \leq |P| \leq 2^{\frac{1}{d}}(T+1)^2 H^T$. Finally, $\tilde{Q}$ has at most $(T+1)^2$ terms.

We would like to find some $z_i \in \mathbb{C}$ such that $|G(z_i)| = |P(z_i,\sigma(z_i))| \geq 1$.

Let $z = re^{i\theta} \in \mathbb{C}$ be such that $|\sigma(z)| = M(r,\sigma) \geq 1$ and $r \geq 1$. Then

$$\left|\tilde{Q}\left(z, \frac{1}{\sigma(z)}\right)\right| \leq 2^{\frac{1}{d}}(T+1)^4 H^T r^T.$$

Therefore

$$\left|Q\left(z, \frac{1}{\sigma(z)}\right)\right| \leq \frac{1}{2}$$

provided

$$2^{\frac{1}{d}}(T+1)^4 H^T r^T \leq \frac{1}{2} M(r,\sigma).$$

We note that if

$$r \geq C(\log H)^4,$$

then

$$2^{\frac{1}{d}}(T+1)^4 H^T r^T \leq \frac{1}{2} e^r$$

and, by Remark (2.2.4), for r sufficiently large

$$e^r \leq M(r,\sigma).$$

We thus get that

$$\left|Q\left(z, \frac{1}{\sigma(z)}\right)\right| \leq \frac{1}{2}$$

when $r \geq C(\log H)^4$.

Note that the degree of $R(X)$ is also at most T. For $i = 1, \ldots, [T] + 14$, say, let r_i be the ith integer after $C(\log H)^4$. Let z_i be such that $|z_i| = r_i$ and $|\sigma(z_i)| = M(r_i, \sigma)$. By Lemma 5.1.7, there will be at least one i such that $|R(z_i)| > 1$. For such i, we have

$$\left|\tilde{P}\left(z_i, \frac{1}{\sigma(z_i)}\right)\right| \geq \frac{1}{2}.$$

We can (again by Remark (2.2.4)) conclude that

$$|G(z_i)| = |P(z_i, \sigma(z_i))| = \left|\sigma(z_i)^k \tilde{P}\left(z_i, \frac{1}{\sigma(z_i)}\right)\right| \geq \frac{1}{2} e^{k r_i},$$

and therefore

$$|G(z_i)| \geq 1.$$

Recall that $R_{H,d}$ is of the form $C(\log H)^{\frac{1}{2}}$ whilst on the other hand, $r_i \leq C'(\log H)^4 + T + 14$. So, $\overline{B(0, R_{H,d})} \subset \overline{B(z_i, s)}$ where $s = C(\log H)^4$.

By the maximum modulus principle and Lemma 2.2.2, we have that:

$$n(R_{H,d}, \frac{1}{G}) \leq \frac{1}{\log 2} \log\left(\frac{M(3s,G)}{|G(z_i)|}\right) \leq \frac{\log M(3s,G)}{\log 2}.$$

By Remark (2.2.4), for sufficiently large s, we have that

$$M(3s, G) \leq |P|(T+1)^2(3s)^T e^{T(3s)^3}.$$

We can thus deduce that

$$\log M(3s, G) \leq C(\log H)^{15}.$$

Therefore

$$n(R_H, \frac{1}{G}) \leq C(\log H)^{15}$$

as required.

□

5.2.1 Improving the bound with Besson's result

At the time of writing up this thesis we became aware of a 2015 preprint of Besson [2] on algebraic values of the σ-function restricted to compact subsets of $\mathbb{C}$.

The techniques in [2] were an adaptation to the σ-function of Masser's work on rational points on the Riemann ζ-function and hence, except for the use of the auxiliary polynomial $P(X,Y)$, are different from our approach above.

In accordance with the notation in [2], let $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, with $\Im(\tau) > 0$. Let $\sigma(z, \Lambda)$ be the Weierstrass σ-function associated with the lattice Λ.

For $d \geq 1$ an integer, and H and R positive real numbers, let $S(R, d, H)$ be the set

$$S(R,d,H) = \{z \in \mathbb{C} : |z| \leq R,\ [\mathbb{Q}(z,\sigma(z)) : \mathbb{Q}] \leq d,\ H(z,\sigma(z)) \leq H\}.$$

For all $d \geq 1$, $H \geq 3$ and $R \geq 2$, **Théorème 1.1** in [2] says that there exists an effective constant $c > 0$ such that:

$$|S(R,d,H)| \leq cR^{10}(\log R)\frac{d^4(\log H)^2}{\log(d\log H)}. \tag{5.5}$$

Setting $\Lambda = \mathbb{Z}[i]$ and replacing R with $R_{H,d} = 42d(\log H)^{\frac{1}{2}}$ in Equation (5.5), we note that the bound in Theorem 5.2.1 can be improved to $C(\log H)^7$.

Chapter 6

On a question of Miller

In this chapter we count algebraic points on the graphs of certain entire functions of positive lower order λ and order ρ where $0 < \lambda \leq \rho \leq \frac{1}{2}$. These functions are specified by the positive sequence of their zeroes which in addition satisfies certain growth conditions.

This partially addresses a question attributed to Chris Miller about counting algebraic points on the graphs of such functions.

6.1 Introduction

Let $1 \leq z_1 \leq z_2 \leq \ldots$ be an increasing and unbounded sequence of positive real numbers such that $\sum_{n=1}^{\infty} \frac{1}{z_n} < \infty$. Then the infinite product

$$f(z) := \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right)$$

necessarily defines an entire function of order ρ where $0 \leq \rho < 1$.

Let $0 < \phi < \frac{\pi}{2}$ and denote by S_ϕ the sector $S_\phi = \{z \in \mathbb{C} : -\phi \leq \arg z \leq \phi\}$. Then the sequence $\{z_n\}_{n=1}^{\infty} \subset S_\phi$. The main objective of this section is to count the number of algebraic points on the graph of f restricted to $\mathbb{C} \setminus S_\phi$.

One of the crucial ingredients of our argument is deduced from a rather more general example found in ([9], p.66-69).

More specifically, given the sequence $\{a_k\}_{k=1}^{\infty}$ such that $1 \leq a_1 \leq a_2 \leq \dots$ and $\sum_{k=1}^{\infty} \frac{1}{|a_k|^p} < \infty$, Goldberg and Ostrowski were concerned with asymptotically approximating the function

$$g(z) = \prod_{k=1}^{\infty} E\left(\frac{z}{a_k}, p\right)$$

where p is a non-negative integer and

$$E(z,p) := \begin{cases} (1-z) & \text{if } p = 0 \\ (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) & \text{otherwise} \end{cases}$$

is the pth *Weierstrass elementary factor*.

An asymptotic inequality approximating $\log g(z)$ in terms of the function $|z|^\rho$ and certain explicit coefficients was obtained, where $p \leq \rho \leq p+1$. The asymptotic inequality we need is thus a specialization of their result to the case where $p = 0$. We give the specific details in the next lemma.

Let $\{z_n\}_{n=1}^{\infty}$ be the sequence of zeros of f as defined previously and denote by $n(r)$ the number of z_n with modulus less than r. Let

$$\mu := \lim_{r\to\infty} \frac{n(r)}{r^\rho}$$

where $\rho \in (0,1)$ is the order of f. Assume also that $\lambda > 0$ where λ is the lower order of f.

Lemma 6.1.1. *([9])*
Let $0 < \epsilon < 1$ and suppose f, μ, ρ and ϕ are as defined previously. Assume $0 < \mu < \infty$. Then there exists $r_1(\epsilon)$ such that for all $z \in \mathbb{C}$ with $|z| > r_1(\epsilon)$ and $\phi < \arg z < 2\pi - \phi$,

$$\left|\log f(z) - \frac{\mu\pi}{\sin \pi\rho} e^{-i\pi\rho} z^\rho\right| \leq \epsilon A r^\rho \csc\frac{\phi}{2}, \tag{6.1}$$

where $A = 6 + 3\mu\pi\csc(\pi\rho)$.

From the above lemma it follows that

$$\left|\Re\left(\log f(z) - \frac{\mu\pi}{\sin \pi\rho} e^{-i\pi\rho} z^\rho\right)\right| \leq \epsilon A r^\rho \csc\frac{\phi}{2}.$$

More explicitly, writing z as $z = re^{i\theta}$, where $\phi < \theta < 2\pi - \phi$, we deduce from Inequality (6.1) that

$$\left| \log |f(re^{i\theta})| - \frac{\mu\pi}{\sin \pi\rho} \cos \rho(\theta - \pi) r^{\rho} \right| \leq \epsilon A r^{\rho} \csc \frac{\phi}{2}. \tag{6.2}$$

Assuming $\rho \in (0, \frac{1}{2}]$, we have on the one hand that $\rho(\theta - \pi) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So $\cos \frac{1}{2}(\theta - \pi) \leq \cos \rho(\theta - \pi)$. On the other hand, $\cos \frac{1}{2}(\theta - \pi) = \sin \frac{\theta}{2}$. Therefore, from Inequality (6.2), we have that:

$$|f(re^{i\theta})| \geq e^{C(\phi,\rho) r^{\rho}}, \text{ where } C(\phi, \rho) = \frac{\mu\pi \sin \frac{\phi}{2}}{\sin \pi\rho} - \epsilon A \csc \frac{\phi}{2}. \tag{6.3}$$

Given ϕ, ϵ can be chosen such that

$$\epsilon = \min \left\{ \frac{\mu\pi \sin^2 \frac{\phi}{2}}{4A \sin \pi\rho}, \frac{1}{2} \right\}, \text{say.}$$

In this case, we have that

$$C(\phi, \rho) > \frac{\mu\pi \sin \frac{\phi}{2}}{2 \sin \pi\rho} > 0.$$

Recall Lemma 5.1.4 from the previous chapter:

Lemma 6.1.2. *Let α be a non-zero algebraic number of degree at most d and height at most H. Then*

$$\frac{1}{(2H)^d} \leq |\alpha| \leq (2H)^d. \tag{6.4}$$

Using the above lemma we prove the following:

Lemma 6.1.3. *Let $d \geq 1$ and $H \geq e^e$. Let $z = re^{i\theta} \in \mathbb{C}$ such that $\deg(z) \leq d$ and $H(z) \leq H$. Define the constant $K(\phi, \rho, d)$ by*

$$K(\phi, \rho, d) = \left(\frac{2(d+1)}{C(\phi, \rho)} \right)^{\frac{1}{\rho}}.$$

If $r \geq K(\phi, \rho, d)(\log H)^{\frac{1}{\rho}} = R_H$, then $e^{C(\phi,\rho) r^{\rho}} \geq (2H)^{d+1}$.

Hence, for $r \geq \max\{r_1(\epsilon), R_H\}$, we have the following chain of inequalities:

$$|f(re^{i\theta})| \geq e^{C(\phi,\rho) r^{\rho}} \geq (2H)^{d+1}.$$

By Lemma 6.1.2 this implies that either $H(f(z)) > H$ or $\deg(f(z)) > d$.

Proof. We note that $e^{C(\phi,\rho)r^\rho} \geq (2H)^{d+1}$ if

$$C(\phi,\rho)r^\rho \geq (d+1)\log(2H).$$

The above inequality follows if

$$C(\phi,\rho)r^\rho \geq 2(d+1)\log H.$$

And this is true if

$$r \geq K(\phi,\rho,d)(\log H)^{\frac{1}{\rho}}$$

Recalling that $|f(re^{i\theta})| \geq e^{C(\phi,\rho)r^\rho}$ when $r \geq r(\epsilon)$, if $r \geq \max\{r_1(\epsilon), R_H\}$, we obtain the desired chain of inequalities.

□

For the reader's convenience we recall the following:

Lemma 6.1.4. *(Boutroux-Cartan)*
Let $P(z) \in \mathbb{C}[z]$ be a monic polynomial with degree $n \geq 1$. Then $|P(z)| > 1$ for all complex z outside a collection of at most n disks the sum of whose radii is $2e$.

Lemma 6.1.5. *(A corollary of Jensen's formula)*
Let G be a nonconstant entire function such that $G(0) \neq 0$. Let $0 < r < R < \infty$. Then:

$$n(r, \frac{1}{G}) \leq \frac{1}{\log \frac{R}{r}} \log\left(\frac{M(R,G)}{|G(0)|}\right)$$

Since f is an entire function with positive lower order and finite order, we as usual proceed by a slight modification of the strategy of Boxall and Jones in [6].

Our contribution in this case is that we do not need to restrict f to a compact set. We shall in fact bound the number of *all* algebraic points of bounded height and degree on the graph of f except for those inside a sector S_ϕ that excludes the zeroes of f.

6.2 Main Result

We can now state and prove the main result of this section.

Theorem 6.2.1. *Let* $f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)$ *where* $1 \leq z_1 \leq z_2 \leq \ldots$ *and* $\sum_{n=1}^{\infty} \frac{1}{z_n} < \infty$*. Suppose the lower order* λ *and order* ρ *of* f *are such that* $0 < \lambda \leq \rho \leq \frac{1}{2}$*. Let* $0 < \phi < \frac{\pi}{2}$*. Let* d, α, β, γ *be as follows:* $d \geq 1$*,* $\alpha = 1 + \rho$*,* $\beta = \frac{\lambda}{2}$*, and* $\gamma = \frac{2\alpha+\rho}{\beta\rho}$*. Then there is a constant* $C > 0$ *such that for all* $H > e$*, there are at most* $C(\log H)^{\frac{2\alpha(\gamma+1)}{\rho}}$ *numbers* $z \in \mathbb{C} \setminus S_\phi$ *such that* $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d$*,* $H(z, f(z)) \leq H$*.*

Proof. Let $H > e^e$. Throughout our proof the height bound H is assumed to be sufficiently large. We shall denote by C a positive constant independent of H. The constant C may not be the same at each occurrence. Recall that $|P|$ denotes the modulus of the coefficient of the polynomial P with largest absolute value.

We would first like to obtain a non-zero polynomial $P(X, Y) \in \mathbb{Z}[X, Y]$ of degree at most $T = C(\log H)^{\frac{2\alpha}{\rho}}$ such that $|P| \leq 2^{\frac{1}{d}}(T+1)^2 H^T$ and $P(z, f(z)) = 0$ whenever $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d$, $H(z, f(z)) \leq H$ and $z \notin S_\phi$. To this end, let:

$$A = \frac{1}{2R_H}, \;\; Z = C(\log H)^{\frac{1}{\rho}}, \;\; T = C(\log H)^{\frac{2\alpha}{\rho}} \text{ and } M = e^{(2Z)^\alpha}.$$

We then have that $\max\{|z|, |f(z)|\} \leq M$ for all $z \in \overline{B(0, 2Z)}$.

Furthermore, we note that

$$\log(AZ)^T = C(\log H)^{\frac{2\alpha}{\rho}} > C\left(\frac{\log\log H}{(\log H)^{\frac{2\alpha}{\rho}}}\right) + C(\log H)^{\frac{\alpha}{\rho}} + C\log H.$$

Therefore

$$(AZ)^T > (4T)^{\frac{96d^2}{T}}(M+1)^{16d}H^{48d^2}.$$

We note that the bound we are trying to prove is worse than $C(\log H)^{\frac{4\alpha}{\rho}}$. We can thus assume that there are at least $\frac{T^2}{8d}$ complex numbers such that

$[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d$ and $H(z, f(z)) \leq H$. By Lemma 1.1.26 and the comment following it regarding the bound on the size of the coefficients, there is a polynomial $P(X, Y)$ satisfying all our requirements.

Let $G(z) = P(z, f(z))$. We would like to bound the number of zeroes of G in $\overline{B(0, R_H)}$. To do this, first let k be the highest power of Y in $P(X, Y)$. We can assume $k \geq 1$. Let $\tilde{P}(X, Y) = Y^k P(X, \frac{1}{Y})$, $R(X) = \tilde{P}(X, 0)$, and $Q(X, Y) = \tilde{P}(X, Y) - R(X)$. We note that $R(X)$ is not identically zero.

Let $\tilde{Q}(X, Y) = \frac{1}{Y} Q(X, Y)$. The highest power of X in $\tilde{Q}$ is at most T and $|\tilde{Q}| \leq |P| \leq 2^{\frac{1}{d}}(T+1)^2 H^T$. Finally, $\tilde{Q}$ has at most $(T+1)^2$ terms.

We would now like to find some $z_1 \in \mathbb{C}$ such that $|G(z_i)| = |P(z_i, f(z_i))| \geq 1$. To this end, first we would like to find some sufficiently large radius r such that if $|z| \geq r$, then $\left|Q\left(z, \frac{1}{f(z)}\right)\right| \leq \frac{1}{2}$.

Let $z = re^{i\theta} \in \mathbb{C}$ be such that $|f(z)| = M(r, f) \geq 1$. Then:

$$\left|\tilde{Q}\left(z, \frac{1}{f(z)}\right)\right| \leq 2^{\frac{1}{d}}(T+1)^4 H^T r^T.$$

Therefore

$$\left|Q\left(z, \frac{1}{f(z)}\right)\right| \leq \frac{1}{2}$$

provided

$$2^{\frac{1}{d}}(T+1)^4 H^T r^T \leq \frac{1}{2} M(r, f).$$

We note that (for a large enough C) if

$$r \geq C(\log H)^{(2\alpha+\rho)/\beta\rho},$$

then

$$2^{\frac{1}{d}}(T+1)^4 H^T r^T \leq \frac{1}{2} e^{r^\beta}$$

and, by Remark 2.2.4

$$e^{r^\beta} \leq M(r, f).$$

We thus get that

$$\left|Q\left(z, \frac{1}{f(z)}\right)\right| \leq \frac{1}{2}$$

when

$$r \geq C(\log H)^{\frac{(2\alpha+\rho)}{\beta\rho}}.$$

Note that the degree of $R(X)$ is also at most T. For $i = 1, \dots, [T] + 14$, say, let r_i be the ith integer after $C(\log H)^\gamma$ where $\gamma := \frac{2\alpha+\rho}{\beta\rho}$. Let z_i be such that $|z_i| = r_i$ and $|f(z_i)| = M(r_i, f)$. By Lemma 6.1.4, there will be at least one i such that $|R(z_i)| > 1$. For such i, we have:

$$\left|\tilde{P}\left(z_i, \frac{1}{f(z_i)}\right)\right| \geq \frac{1}{2}.$$

We can (again by Remark 2.2.4) conclude that

$$|G(z_i)| = |P(z_i, f(z_i))| = \left|f(z_i)^k \tilde{P}\left(z_i, \frac{1}{f(z_i)}\right)\right| \geq \frac{1}{2}e^{k{r_i}^\beta},$$

and therefore

$$|G(z_i)| \geq 1.$$

Recall that R_H is of the form $C(\log H)^{\frac{1}{\rho}}$ whilst on the other hand,
$r_i \leq C(\log H)^\gamma + T + 14$. So, $\overline{B(0, R_H)} \subset \overline{B(z_i, s)}$ where $s = C(\log H)^\gamma$.

By the maximum modulus principle and Lemma 2.2.2, we have that

$$n(R_H, \frac{1}{G}) \leq \frac{1}{\log 2} \log\left(\frac{M(3s, G)}{|G(z_i)|}\right) \leq \frac{\log M(3s, G)}{\log 2}.$$

By Remark 2.2.4, we have that

$$M(3s, G) \leq |P|(T+1)^2(3s)^T e^{T(3s)^\alpha}.$$

Since $s = C(\log H)^\gamma$ and $T = C(\log H)^{\frac{2\alpha}{\rho}}$, we deduce that

$$\log M(3s, G) \leq C(\log H)^{\frac{2\alpha(\gamma+1)}{\rho}}.$$

Therefore:

$$n(R_H, \frac{1}{G}) \leq C(\log H)^{\frac{2\alpha(\gamma+1)}{\rho}}.$$

as required.

The constant C effectively depends on μ, λ, ρ, ϕ and d.

$\square$

Chapter 7

Some afterthoughts and current pursuits

We take a rather conversational tone in this section wherein we look at several directions in which the results in this thesis could perhaps be improved or extended. As one would expect, our discussion will be open ended, and the subsections will have varying degrees of detailed analysis.

7.1 On meromorphic functions and Masser's polynomial

In **Chapters 3** and **4**, where we deal with counting algebraic points on graphs of certain meromorphic functions, perhaps the most glaring restriction is the requirement on f to be holomorphic in the region in which we count the points.

We would ultimately like to obtain results where the holomorphicity condition is removed and hence obtain the most general counterparts of the results of Boxall and Jones.

This restriction was necessitated by the fact that Masser's polynomial is constructed for functions holomorphic in a neighbourhood of some disk $\overline{B(0, 2R)}$. Hence, in order to remove the restriction, one has to work with a polynomial constructed with meromorphic functions in mind.

For the sake of convenience we remind the reader of the statement of Masser's proposition below:

Lemma 7.1.1. *(Masser, [19], Prop. 2)*

Let $d \geq 1$ and $T \geq \sqrt{8d}$ be positive integers and A, Z, M and H positive real numbers such that $H \geq 1$. Let f_1, f_2 be functions analytic on an open neighbourhood of $\overline{B(0,2Z)}$, with $\max\{|f_1(z)|, |f_2(z)|\} \leq M$ on this set. Suppose $\mathcal{Z} \subset \mathbb{C}$ is finite and satisfies the following for all $z, w \in \mathcal{Z}$:

- $|z| \leq Z$,
- $|w - z| \leq \frac{1}{A}$,
- $[\mathbb{Q}(f_1(z), f_2(z)) : \mathbb{Q}] \leq d$,
- $H(f_1(z), f_2(z)) \leq H$.

Then there is a nonzero polynomial $P(X,Y)$ of total degree at most T such that $P(f_1(z), f_2(z)) = 0$ for all $z \in \mathcal{Z}$ provided

$$(AZ)^T > (4T)^{\frac{96d^2}{T}} (M+1)^{16d} H^{48d^2}.$$

We recall that the proof of the above lemma consisted of two parts:

- An algebraic argument consisting of the use of Siegel's lemma to construct a polynomial $P(X,Y) \in \mathbb{Z}[X,Y]$ such that $P(f_1(z_k), f_2(z_k)) = 0$ where the z_k's belong to a subset of $\mathcal{Z}$.
- An analytic argument using a Schwarz lemma to show that $P(f_1(z), f_2(z)) = 0$ for all $z \in \mathcal{Z}$.

Let f be meromorphic in the neighbourhood of some disk $\overline{B(0,R)}$ and assume that the finite set $\mathcal{Z}$ does not contain any poles of f. Suppose that $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d$ and $H(z, f(z)) \leq H$ for all $z \in \mathcal{Z}$.

Note that the part of the argument using Siegel's lemma does not appeal to the analytic properties of the functions f_1 and f_2. Therefore, for the meromorphic function f, we can still construct a polynomial $P(X,Y) \in \mathbb{Z}[X,Y]$ of total degree at most T such that $P(z_k, f(z_k)) = 0$ where the z_k's belong to a subset of $\mathcal{Z}$.

One is then left with showing that $P(z, f(z)) = 0$ for all $z \in \mathcal{Z}$. We cannot directly adapt the second part of Masser's argument for this case since the maximum modulus principle does not hold for meromorphic functions.

It would thus be nice to find an alternative argument with which to obtain the counterpart of the second part of Masser's argument for meromorphic functions.

7.2 On the Weierstrass σ- function

In **Chapter 5** we studied algebraic points on the Weierstrass σ-function associated with the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}i$.

This restriction to the Gaussian integer lattice stemmed from the fact that the estimates for the σ-function that we used from [25] were proved specifically for $\sigma(\Lambda, z)$ with Λ the lattice above.

Bearing in mind that the bound obtained by Besson in [2] holds for an arbitrary lattice $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$ with $\Im(\tau) > 0$, it would be nice to obtain the $C(\log H)^7$ bound for $\sigma(\Gamma, z)$.

We would thus like to study Pogány's proof in [25] and see whether the restriction can be removed, or otherwise attempt to remove it by an alternative approach.

The level of generality that we are seeking does however come with a cost. Recall that in order to use Pogány's bound, we had to bound from below the parameter δ_z, the distance from z to the nearest lattice point. In this case, the zeroes of σ were algebraic numbers, and in particular, the Gaussian integers.

We were thus able to bound δ_z with the function $r_{H,d} := (36H)^{-6d^2}$, with the additional property that if α is an algebraic number of degree at most d and height at most H and α is not a Gaussian integer, then $\alpha \notin \overline{B(\beta, r_{H,d})}$ where

β is a Gaussian integer.

This assured us that none of the small disks we were excluding contained any algebraic numbers of interest. Thus, *excluding only the zeroes*, we were able to count *all* the algebraic points of height at most H and degree at most d on the graph of σ.

Given an arbitrary lattice $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$, if the lattice points are not algebraic or do not have a "good" transcendence measure, we lose the generality of the previous result as we are no longer assured that the disks around each zero that we have to exclude do not contain any algebraic numbers of height at most H and degree at most d.

In a slightly different direction, in [28], Seip introduced analogues of the Weierstrass σ-function associated to certain discrete sets that are not necessarily lattices in $\mathbb{C}$. Before giving the specific details, we need a few basic definitions from [28].

Definition 7.2.1. *Let $\Gamma = \{z_j\}$ be a discrete set. Then Γ is said to be uniformly discrete if*

$$\mu := \inf_{j \neq k} |z_j - z_k| > 0.$$

Definition 7.2.2. *Let $\Lambda = \{\lambda_{mn}\} \subset \mathbb{C}$ be a lattice. Then Λ is said to be a square lattice if there exists $\beta > 0$ such that*

$$\lambda_{mn} = \sqrt{\pi/\beta}(m + in)$$

for all integers m and n.

The quantity $\frac{\beta}{\pi}$ is referred to as the *density* of Λ.

Definition 7.2.3. *Let Λ be a square lattice and $\Gamma = \{z_{mn}\}$ be a uniformly discrete set. Then Γ is said to be uniformly close to Λ if there exists $Q > 0$ such that*

$$|z_{mn} - \lambda_{mn}| \leq Q$$

for all $m, n \in \mathbb{Z}$.

Let $\Gamma = \{z_{mn}\}$ be uniformly close to a square lattice $\Lambda = \{\lambda_{mn}\}$. Then, to Γ, Seip associates a function $g(z)$ defined by

$$g(z) = (z - z_{00}) \prod_{(m,n)\neq(0,0)} \left(1 - \frac{z}{z_{mn}}\right) \exp\left(\frac{z}{z_{mn}} + \frac{1}{2}\frac{z^2}{{\lambda_{mn}}^2}\right) \tag{7.1}$$

where z_{00} is the point of Γ closest to the origin.

The following theorem is obtained, we only state the first half of the theorem, which is the part relevant to our discussion:

Theorem 7.2.4. *(Seip, [28], Lemma 3.1)*

Let Γ be uniformly close to the square lattice Λ of density β/π. Then there exist constant C_1, C_2 and c, depending only on Q and μ, such that for every $z \in \mathbb{C}$,

$$C_1 \delta_z e^{-c|z|\log|z|} \leq |e^{-\beta|z|^2/2} g(z)| \leq C_2 e^{c|z|\log|z|}$$

where δ_z is the distance from z to the closest point of Γ.

Let $\mathcal{A}$ be the set of all discrete sets Γ that are uniformly close to the square lattice Λ. Let $\mathcal{B}$ be the subset of $\mathcal{A}$ consisting of all $\Gamma \in \mathcal{A}$ such that each $a_{mn} \in \Gamma$ is algebraic and the height and degree of a_{mn} grow slowly as $m, n \to \infty$.

Then to each $\Gamma_\alpha \in \mathcal{B}$ we can associate the function $g(\Gamma_\alpha, z) = g(z)$ as defined in Equation (7.1).

By Theorem 7.2.4 above, we can essentially repeat the arguments in sections 1 and 2 of **Chapter 5** and obtain, for this family of entire functions, a similar bound of the form $C(d)(\log H)^{15}$ for the number of algebraic points of height at most H and degree at most d on the graph of each $g(\Gamma_\alpha, z)$, excluding the zeroes.

This immediately prompts an interesting question. Does the analogue of Besson's bound on σ also hold for the functions g as defined in Equation (7.1)? Analytically speaking, these functions are in some sense "not so different" from the Weierstrass σ-functions. A quick glance at Besson's paper [2] suggests that the techniques might be transferable.

7.3 On the Weierstrass $\wp$ and $\wp'$ functions

Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ be the lattice generated by ω_1 and ω_2 where $\frac{\omega_1}{\omega_2} \in \mathbb{C} \setminus \mathbb{R}$.

The Weierstrass elliptic function associated with Ω is the meromorphic function

$$\wp(z, \Omega) = \frac{1}{z^2} + \sum_{m^2+n^2 \neq 0} \left(\frac{1}{(z + m\omega_1 + n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right).$$

In [14], upon restricting the $\wp$-function to a subset of the fundamental parallelogram with vertices $\{0, \omega_1, \omega_2, \omega_1 + \omega_2\}$, Jones and Thomas prove a bound of the form $c(\log H)^{15}$ for the number of algebraic points of height at most H and degree at most d on the restricted graph. The constant c depends on d and Ω.

Let us fix the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}i$. Then $\wp(z, \Lambda)$ is given by

$$\wp(z, \Lambda) = \frac{1}{z^2} + \sum_{\{m,n\} \neq \{0,0\}} \left(\frac{1}{(z + m + ni)^2} - \frac{1}{(m + ni)^2} \right),$$

where $z \in \mathbb{C} \setminus \Lambda$. The function $\wp(z, \Lambda)$ is thus analytic in $\mathbb{C} \setminus \mathbb{Z}^2$.

Term by term differentiation of $\wp$ yields

$$\wp'(z, \Lambda) = -2 \sum_{m,n} \frac{1}{(z + m + ni)^3}$$

for all $z \in \mathbb{C} \setminus \Lambda$. The two functions are connected by the differential equation

$$\wp'^2(z) = 4\wp^3(z) - \mathfrak{g}_2\wp(z) - \mathfrak{g}_3, \ \ (z \in \mathbb{C} \setminus \mathbb{Z}^2),$$

where

$$\mathfrak{g}_2 = 60 \sum_{\{m,n\} \neq \{0,0\}} \frac{1}{(m + ni)^4} \ \text{ and } \ \mathfrak{g}_3 = 140 \sum_{\{m,n\} \neq \{0,0\}} \frac{1}{(m + ni)^6}.$$

The poles of $\wp'$ lie at $\lambda_{mn} \in \Lambda$, each with multiplicity three. It is known that the zeroes of $\wp'$ are the elements of the lattice $\Gamma := \frac{1}{2}\Lambda - \Lambda$, where $\frac{1}{2}\Lambda$

denotes the set of all points of the form $\lambda_{mn}/2$ for each $\lambda_{mn} \in \Lambda$.

In this subsection, we will sketch a possible alternative strategy of counting the algebraic points of bounded height and degree on the restrictions to any compact set of the graphs of the $\wp(z,\Lambda)$-function and its derivative $\wp'(z,\Lambda)$.

Mutatis mutandis, the same argument works for both $\wp$ and $\wp'$. Hence, for brevity, we shall subsequently refer to both of these functions as $\mathfrak{p}(z)$.

Let $r > 0$ and $\alpha_1, \ldots, \alpha_n$ be the poles of $\mathfrak{p}(z)$ in $\overline{B(0,r)}$, counted with multiplicity. Define the function $g(z)$ as

$$g(z) = \prod_{j=1}^{n}(z - \alpha_j)\mathfrak{p}(z).$$

Then:

- $g(z)$ is holomorphic on an open set containing $\overline{B(0,r)}$.
- The algebraic points on $g(z)$ correspond to the algebraic points on $\mathfrak{p}(z)$.

It is known that $T(r,\mathfrak{p}) = Cr^2 + o(r^2)$ as $r \to \infty$. This implies that the (growth) order and lower order (in the sense of Nevanlinna) of $\mathfrak{p}$ is 2. Since $T(r,q) = \mathcal{O}(\log r)$ when q is a polynomial, we have that for any $\epsilon > 0$, and r sufficiently large,

$$r^{2-\epsilon} \le T(r,g) \le r^{2+\epsilon}.$$

Since $g(z)$ is holomorphic on an open set containing $\overline{B(0,r)}$, the maximum modulus $M(r,g)$ exists. One can then formulate the conditions required for the existence of Masser's polynomial $P(X,Y)$.

The remainder of the argument would be a modification of the one we used in **Chapter 4** for meromorphic functions of positive lower order and finite order for which we obtained a bound of the form $C(\log H)^{5\alpha^2/\beta}$.

Lastly, we also note that the lattice $\Lambda = \mathbb{Z}[i]$ can be replaced with any lattice whose points are algebraic numbers. The constant C will vary with each such lattice.

7.4 On the question of Miller

For the reader's convenience, we start this discussion by first recalling certain details about the function f we studied in **Chapter 6**.

Let $\{z_n\}_{n=1}^{\infty}$ be an increasing and unbounded sequence of positive real numbers with $z_1 \geq 1$. Suppose $\sum_{n=1}^{\infty} \frac{1}{z_n} < \infty$. Then the infinite product

$$f(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_n}\right)$$

defines an entire function of order $\rho \in [0, 1)$.

We restricted ourselves to the case where $\rho \in (0, 1)$. In Equation (6.1) from Lemma 6.1.1, we had a factor of $\cos \rho(\theta - \pi)$ which we required to be positive in order to make the bound meaningful.

This forced us to make the extra assumption that $\rho \in (0, \frac{1}{2}]$.

It would thus be nice to cover the remaining case of $\rho \in (\frac{1}{2}, 1)$.

In another direction, Lemma 6.1.1 also requires us to exclude the sector S_ϕ when counting points on the graph of f since the asymptotic relation does not seem to hold for f in this region. It would be interesting to count points on f restricted to S_ϕ.

A theorem of Levin in [17] gives a promising potential starting point for this endeavour. For this, we will need the following definition:

Definition 7.4.1. *([17], pp 86)*
Let $B = (B(z_j, r_j))$ be a family of disks in the complex plane. Then B is called a C^0-set if

$$\lim_{R \to \infty} \frac{1}{R} \sum_{|z_j| < R} r_j = 0.$$

Recall that $n(r)$ denotes the number of zeroes of f with modulus less than r. The following theorem is then proved:

Theorem 7.4.2. *(Levin, [17], Theorem 7, pp 86)*
Let $\mu = \lim_{t\to\infty} \frac{n(t)}{t^\rho}$. *There exists a* C^0*-set of disks* (B_j) *outside which the asymptotic relation*

$$\log|f(re^{i\theta})| = \frac{\pi\mu}{\sin\pi\rho} r^\rho \cos\rho(\theta-\pi) + o(r^\rho), \quad r\to\infty,$$

holds uniformly with respect to θ*, where* $0 \leq \theta \leq 2\pi$.

The disks B_j contain the zeroes of $f(z)$. We would like to study the proof of the above theorem in order to gain some control on the sizes of the disks.

Ideally, it would be nice if one could relate the height bound H with the radii r_j such that for each H, there will be a family (C_j^H) of exceptional disks outside of which all points of height at most H and degree at most d lie, and the asymptotic relation also holds.

List of References

[1] E. Besson. Points rationnels de la fonction Gamma d'Euler. *Archiv der Mathematik*, 103(1):61–73, 2014.

[2] E. Besson. Points algébriques de la fonction sigma de Weierstrass. *Prepublication*, 2015.

[3] G. Binyamini and D. Novikov. Wilkie's conjecture for restricted elementary functions. *Annals of Mathematics*, 186:237–275, 2017.

[4] E. Bombieri and J. Pila. The number of integral points on arcs and ovals. *Duke Mathematical Journal*, 59(2):337–357, 1989.

[5] G. Boxall and G. Jones. Algebraic values of certain analytic functions. *International Mathematics Research Notices*, rnt239, 2013.

[6] G. Boxall and G. Jones. Rational values of entire functions of finite order. *International Mathematics Research Notices*, rnv052, 2015.

[7] L. Butler. Some cases of Wilkie's conjecture. *Bulletin of the London Mathematical Society*, 44:642–660, 2012.

[8] C. Gasbarri. Rational vs transcendental points on analytic Riemann surfaces. *arXiv preprint*, 2018.

[9] A. Goldberg and I. Ostrovskii. *Value distribution of meromorphic functions*. American Mathematical Society, 2008.

[10] J. Grahl. A modification of the Nevanlinna theory. *Computational Methods and Function Theory*, 10:97–109, 2010.

[11] F. Gramain, M. Mignotte, and M. Waldschmidt. Valuers algébriques de fonctions analytiques. *Acta Arithmetica*, 47:97–121, 1986.

[12] A. Günaydin. The Pila-Wilkie theorem and its consequences, April 2012.

[13] W. Hayman. *Meromorphic functions*. Oxford University Press, 1964.

[14] G. Jones and M. Thomas. Rational values of Weierstrass zeta functions. *Proceedings of the Edinburgh Mathematical Society*, 59:945–958, 2016.

[15] G. Jones and M. Thomas. Effective Pila-Wilkie bounds for unrestricted Pfaffian surfaces. *Arxiv Preprint*, 2018.

[16] G. Jones and A. Wilkie. *O-minimality and diophantine geometry*. Cambridge University Press, 2015.

[17] B. Y. Levin. *Lectures on entire functions*. American Mathematical Society, 1996.

[18] D. Masser. Small values of the quadratic part of the Néron-Tate height of an abelian variety. *Compositio Mathematica*, 53(2):153–170, 1984.

[19] D. Masser. Rational values of the Riemann zeta function. *Journal of Number Theory*, 131(11):2037–2046, 2011.

[20] D. Masser. *Auxiliary polynomials in number theory*. Cambridge University Press, 2016.

[21] J. Pila. Counting rational points on a certain exponential-algebraic surface. *Annales de L'Institut Fourier*, 60(2):489–514, 2010.

[22] J. Pila. O-minimality and the André-Oort conjecture for $\mathbb{C}^n$. *Annals of Mathematics*, 173(3):1779–1840, 2011.

[23] J. Pila and A. Wilkie. The rational points of a definable set. *Duke Mathematical Journal*, 133(3):591–616, 2006.

[24] J. Pila and U. Zannier. Rational points in periodic analytic sets and the Manin-Mumford conjecture. *Rendiconti Lincei-matematica E Applicazioni*, 2008.

[25] T. K. Pogány. Local growth of the Weierstrass σ-function and Whittaker-type derivative sampling. *Georgian Mathematical Journal*, 1:157–164, 2003.

[26] Q. I. Rahman and G. Schmeisser. *Analytic theory of polynomials*. Oxford University Press, 2002.

[27] W. Schmidt. Northcott's theorem on heights I. A general estimate. *Monatshefte f ur Mathematik*, 115(1-2):169–181, 1993.

[28] K. Seip. Density theorems for sampling and interpolation in the Bargmann-Fock space. *Bulletin of the American Mathematical Society*, 26(2):322–328, 1992.

[29] A. Surroca. Sur le nombre de points algébriques où une fonction analytique transcendante prend des valeurs algébriques. *Comptes Rendus Mathématiques. Académie des Sciences. Paris*, 334(9):721–725, 2002.

[30] L. van den Dries. *Tame topology and o-minimal structures*. Cambridge University Press, 1998.

[31] M. Waldschmidt. *Diophantine approximation on linear algebraic groups*. Springer-Verlag Berlin, 2000.

[32] A. Wilkie. Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function. *Journal of the American Mathematical Society*, 9(4):1051–1094, 1996.