The development of pre-service teachers' mathematical knowledge for teaching flexible mental computation

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DECLARATION

By submitting this dissertation electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the sole author thereof (save to the extent explicitly otherwise stated), that reproduction and publication thereof by Stellenbosch University will not infringe any third party rights and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

Date: March 2020

ABSTRACT

This empirical study explored an approach to develop elementary pre-service teachers' mathematical knowledge for teaching flexible mental computation skills in school. The study began by determining pre-service teachers' existing knowledge, beliefs and experience pertaining to how they learned flexible mental computation skills at school. Ways pre-service teachers learned flexible mental computation skills in school shape their beliefs about learning and teaching. Currently, research indicates that pre-service teachers continue to graduate with inadequate skills to do flexible mental computation, and overreliance on calculators and the standard method is prevalent among pre-service teachers. Since teacher knowledge affects learning directly, developing pre-service teachers' mathematical knowledge for teaching flexible mental computation of whole numbers would break the cycle of innumeracy. Whole number computation forms the basis of learning different mathematical topics at school, and most professions and activities in society involve the calculation of whole numbers. To achieve the objective of this study, a purposive sample of 51 pre-service elementary mathematics teachers participated in the study. Both quantitative and qualitative research methods were used to collect data using questionnaires, pre- and post-intervention tests, intervention cycles and interviews. This study used the realistic mathematics education (RME) instructional theory with the backing of design-based research (DBR) to design problems that translated into a hypothetical learning trajectory (HLT) to contribute to the knowledge of how to prepare teachers to teach flexible mental computation. The nature of the HLT makes it suitable for teacher educators to adopt it to develop pre-service teachers for effective teaching. This study found that most pre-service teachers believe that flexible mental computation is important and that concrete objects and the use of the pencil-and-paper method underpins its development. Pre-intervention interviews indicated that PSTs developed flexible mental computation skills through problem solving or memorisation of the multiplication table. Pre-service teachers also indicated that at school they had done mental calculations using strategies prescribed by their teacher. Consequently, during the intervention some PSTs expected a prescription of strategies and had to be persuaded to develop the necessary habits of mind to invent strategies. In the pre-intervention test, 49% of the PSTs' scored above 50% whereas in the post-intervention test 53% scored above 50%. Pre-intervention interviews revealed that although correct answers were provided for specific items, not all answers were calculated in a flexible manner. Interviews revealed the use of fingers and standard algorithms to compute mentally. This occurred during and after the intervention process, and these existing methods were hard to change. Findings confirmed that the invention of flexible strategies demanded considerable effort on the part of the pre-service teachers to solve context-rich problems, discuss, think, imagine reason and justify invented strategies. It was, however, demonstrated that all pre-service teachers could develop fluency and flexibility with mental calculation, provided their knowledge of numbers and operations, ability to generalise patterns, and knowledge of relationships between numbers had improved through practice. Ultimately, a problem solving approach

is recommended as it fosters critical and strategic thinking to re-invent calculation strategies and develop the desire to teach constructively.

OPSOMMING

Hierdie empiriese studie ondersoek ’n benadering tot die ontwikkeling van voordiensonderwysers in aanvangsonderwys se wiskundige kennis van die onderrig van buigsame hoofrekenevaardighede in die skool. Die studie begin met die bepaling van voordiensonderwysers se bestaande kennis, oortuigings en ondervinding wat betref die aanleer van buigsame hoofrekenevaardighede op skool. Maniere waarop voordiensonderwysers buigsame hoofrekenevaardighede aangeleer het op skool vorm hul oortuigings oor leer en onderrig. Tans dui navorsing daarop dat voordiensonderwysers steeds hul studie voltooi met onvoldoende vaardighede om buigsame hoofrekene te kan doen, en dat voordiensonderwysers te veel op sakrekenaars en die standaardmetode vertrou. Aangesien die kennis van onderwysers ’n direkte invloed op leer het, kan die ontwikkeling van voordiensonderwysers se wiskundekennis met die oog op die onderrig van buigsame hoofrekene van heelgetalle die siklus van ongesyferdheid verbreek. Die berekening van heelgetalle vorm die grondslag vir die aanleer van verskillende wiskundige onderwerpe op skool, terwyl die meeste beroepe en aktiwiteite in die samelewing die berekening van heelgetalle verg. Om die doelstelling van hierdie studie te bereik, is ’n doelgerigte steekproefneming uitgevoer waaraan 51 voordienswiskunde-onderwysers in aanvangsonderwys deelgeneem het. Sowel kwantitatiewe as kwalitatiewe navorsingsmetodes is gebruik om data te versamel met behulp van vraelyste, voor- en ná-intervensietoetse, intervensiesiklusse, en onderhoude. Hierdie studie gebruik die realistiese wiskunde-opvoeding (*realisticmathematicseducation*=RME)-teorie vir onderrig, gesteun deur ontwerp-gebaseerde navorsing (*design-basedresearch*=DBR) om probleme te ontwerp wat tot ’n hipotetiese leerbaan (*hypotheticallearningtrajectory*=HLT) verwerk word, om by te dra tot die kennis oor hoe onderwysers voorberei moet word om buigsame hoofrekene te gee. Die aard van die HLT maak dit geskik dat onderwyseropvoeders die HLT kan aanpas by die ontwikkeling van voordiensonderwysers sodat effektiewe onderrig kan plaasvind. Hierdie studie het bevind dat die meeste voordiensonderwysers van mening is dat buigsame hoofrekene belangrik is en dat konkrete voorwerpe en die gebruik van die potlood-en-papier-metode die ontwikkeling daarvan ten grondslag lê. Voorintervensie-onderhoude het aangedui dat voordiensonderwysers (*pre-serviceteachers*=PSTs) buigsame hoofrekenevaardighede deur probleemoplossing of die memorisering van die vermenigvuldigingstabel ontwikkel. Voordiensonderwysers het ook aangedui dat hulle op skool hoofrekene gedoen het met behulp van strategieë wat deur hul onderwyser voorgeskryf is. Gevolglik het sommige PST’s tydens die intervensie ’n voorskrif ten opsigte van strategieë verwag en moes hulle oorreed word om die nodige denkgewoontes te ontwikkel ten einde strategieë te kan bedink. In die voorintervensietoets het 49% van die PST’s meer as 50% behaal, terwyl 53% in die ná-intervensietoets meer as 50% behaal het. Voorintervensie-onderhoude het aan die lig gebring dat alhoewel korrekte antwoorde vir spesifieke items verskaf is, nie alle antwoorde op ’n buigsame manier bereken is nie. In onderhoude is daar aangedui dat vingers en standaardalgoritmes gebruik word om hoofrekene te doen. Dit is tydens en ná die intervensieproses gedoen, en hierdie bestaande metodes was moeilik om te

verander. Uit die bevindings kon daar bevestig word dat die uitdink van buigsame strategieë aansienlike inspanning van die voordiensonderwysers verg wat betref die oplossing van konteksryke probleme, met ander woorde besprekings, denke, voorstelle, redenasies en motiverings rondom strategieë wat uitgedink word. Daar is egter gedemonstreer dat alle voordiensonderwysers vlotheid en buigsaamheid in hoofrekene kan ontwikkel, mits hul kennis van getalle en bewerkings, die vermoë om patrone te veralgemeen, en kennis van verhoudings tussen getalle aan die hand van oefening verbeter kan word. Uiteindelik word ’n probleemoplossingsbenadering aanbeveel, aangesien dit kritiese en strategiese denke bevorder vir die herbedinking van berekeningstrategieë en die ontwikkeling van die begeerte om konstruktief te onderrig.

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DEDICATION

I dedicate this study to my:

husband Abel Luwango,

sons John and Elisio,

mother Arminda,

sister Julia,

brother Jacinto,

nephew Elia, and

niece Secilia

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ABBREVIATIONS

DBR – design-based research

DECLPE – Department of Early Childhood and Lower Primary Education

DfEE - Department for Education and Employment

FMC – flexible mental computation

HLT – hypothetical learning trajectory

MBEC – Ministry of Education and Culture

MKT – mathematical knowledge for teaching

NCTM – National Council of Teachers of Mathematics

NIED – National Institute for Education Development

NIHOBSS – National Institutes of Health Office of Behavioral and Social Sciences

PST – pre-service teacher

RME – realistic mathematics education

TE – teacher educator

TEDS-M – Teacher Education and Development Study in Mathematics

TIMSS - Trends in International Mathematics and Science Study

SACMEQ – Southern and Eastern Africa Consortium for Monitoring Educational Quality

UNUNEFT – University of Namibia, United Nations Educational and Funds-in-Trust

ZPD –zone of proximal development

CHAPTER 1: GENERAL INTRODUCTION TO STUDY

1.1 Background of the study

This study focused on the development of pre-service teachers' mathematical knowledge for teaching flexible mental computation of natural numbers. Flexible mental computation (FMC) is the calculation of algorithms mentally using own invented strategies and not using a calculator, pencil and paper or applying the standard algorithm to calculate in mind (McIntosh, Reys, & Reys,1997; Reys, 1984). The progress of FMC in schools relies heavily on the quality of teacher preparation. The quality of pre-service teachers (PST) graduating from university has raised concern among researchers on PST mathematics content knowledge (Livy, Vale, & Herbert, 2016). Most of the studies conducted in Namibia on teachers and PSTs identified insufficient mathematics content knowledge both in in-service and pre-service teachers (Courtney-Clarke & Wessels, 2014; Kasanda, 2005; Marope, 2005; University of Namibia, United Nations Educational, & Funds-in-Trust (UNUNEFT), 2014). Inadequate mathematics content knowledge has a direct impact on teaching and learning as it forms a barrier to the successful application of an elementary school mathematics curriculum (Copley, 2004). In other words, insufficient subject knowledge impedes the effective implementation of reform ideals regarding the development of FMC skills.

In most cases, mathematics in Namibia is learned without understanding, according to the National Institute for Education Development (NIED, 2010), Ausiku (2014) and Vatilifa (2014). This is a situation similar to that in the United States of America, as reported by the National Council of Teachers of Mathematics (NCTM, 2000). Studies conducted by NIED (2010), Ausiku (2014) and Vatilifa (2014) provide evidence of a persistence in traditional teaching methods that do not promote mathematics learning with understanding. To a certain extent, this persistence of traditional methods of teaching algorithms in schools, as opposed to the invention of FMC strategies, demonstrates the perpetuation of teachers' instrumental understanding of numbers and operations. Literature indicates that teachers who cling to traditional methods of teaching FMC have limited understanding of the relationship between numbers, calculations, addition, subtraction, multiplication and division. Such teachers also lack an understanding of why particular calculation methods work and cannot justify the use of particular formal calculation methods correctly. The teaching of algorithms in isolation –unrelated to other mathematical topics –further proves such teachers' limited relational understanding of mathematical topics and the nature of mathematics. A teacher might understand learners' calculation strategies when a variety of calculation strategies are known to the teacher and the teacher understands why such strategies work. Unless teachers can view calculation strategies from a learner's perspective they will not be able to understand learners' thinking.

Once teachers develop the fundamental approaches for teaching FMC learners are more likely to develop a strong foundation in whole number computation. So, the onus is on teacher education institutions to improve PST knowledge so that they can teach effectively in school. Successful teacher development, however, depends on informed TEs in terms of existing mathematical content knowledge, view of mathematics and their view of MKT.

In teaching, an educator may have a flexible or rigid view of knowledge for teaching mathematics. On one hand, a teacher educator with a flexible view of mathematical knowledge for teaching (MKT) may view strategies for computing algorithms as flexible. On the other hand, a rigid view of MKT leads to a rigid perception of mental computation as comprising fixed strategies. Holm and Kajander (2012, p. 10) assert that "the understanding a teacher [educator] has affects classroom choices as well as beliefs held about learning mathematics". For PSTs to have their knowledge improved, TEs need the appropriate beliefs in the nature of mathematics, view of MKT and the appropriate content knowledge for teaching mathematics as advocated by education reform (Holm & Kajander, 2012). The success of reform ideals pertaining to computations requires educators with flexible views towards mental computation.

Evidently, transformation in teaching cannot be achieved until educators change inappropriate views of mathematics and how mathematics is learned and taught (Emenaker, 1996; Ernest, 1989). Tirosh and Graeber (2003) describe the nature of teacher courses as limited to content knowledge development. Consequently, current research advocates a change in approach to teacher preparation to enable PSTs to learn subject content knowledge without overlooking their "attitude and beliefs" (Sandt, 2007, p. 349). Research shows that in most cases PSTs do not have the opportunity to master the content knowledge they need (Kessel, 2009). As a result, many PSTs continue to graduate without in-depth mathematics content knowledge and knowledge for teaching mathematics (Courtney-Clarke & Wessels, 2014; Kasanda, 2005). This study stresses that if PSTs' mathematical knowledge for teaching FMC is not developed the problem of functional innumeracy among learners and teachers might persist.

From my personal experience as a learner in primary school, the use of flexible calculation strategies was not promoted by my teacher. Instead, a rigid teaching approach to calculations was used where a standard method was used to carry out calculations using the paper-and-pencil method. The standard method was not reconstructed by learners, but was modelled by the teacher. Learners practised calculations similar to the ones provided by the teacher. Currently, despite education reform in Namibia, several observations of lessons and interviews with individual teachers revealed persistence of rote learning in schools (Ausiku, 2014; Junius, 2014; NIED, 2010). This signifies that learners are still learning through memorisation of prescribed facts, as opposed to reconstruction of knowledge. Such teaching approaches continue to contribute to learners' difficulty to learn mathematics (NIED, 2010). The findings of achievement tests on learners indicated that, amid other learning areas, learners performed poorly in whole number calculation (NIED, 2010). Besides other factors, the study attributed the poor performance of learners to a lack of mastery of mathematics basic competencies, teacher knowledge and teaching methods that did not engage learners in classroom discussions (NIED, 2010). Thus, engagement of learners to re-invent calculation methods is necessary.

Currently, the development of learners' mental calculation skills is included in the elementary mathematics school curriculum. However, with reference to findings by NIED (2010), teachers' understanding of how to develop learners' mental calculation skills remains a challenge. In addition, Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ III) outcomes based on learners in Namibia indicated an inability to relate basic calculation skills to real-life situations (Spaull, 2011). Meanwhile, the same study found that teachers performed poorly on achievement tests that were based on content they were expected to teach. A study conducted by Nambira, Kapenda, Tjipueja, and Sichombe (2009) also outlined mental calculation involving the four basic operations as being a difficult learning area in school. This study acknowledges the argument that what is unknown to teachers cannot be taught by such teachers effectively (Spaull, 2011). Therefore, as recommended by researchers, "pre-service training institutions should equip teachers with both subject content knowledge and subject pedagogical knowledge" that learners are expected to learn (NIED, 2010, p.133). A further argument is that incompetency in whole number computation could affect learning of mathematics in higher grades and the ability to solve real-life problems. For PSTs to support learners to solve real-life problems, this study adopted specific theories of learning and approaches to teaching mathematics.

This study is underpinned by the constructivist theory of learning and a realistic mathematics education teaching approach (Bruner, 1977; Freudenthal, 1968; Piaget, 1964; Vygotsky, 1978).The constructivists' belief embraced is that learners can construct meaning through active engagement, and this also applies to PSTs. As emphasised by Gresham (2008), Alkan (2013) and Sawyer (2014), learning through active engagement facilitates recall of constructed strategies. In contrast, rote learning of calculation strategies leads to difficulty in recalling calculation procedures and develops fear of doing mathematics (Alkan, 2013; Gresham, 2008; Sawyer, 2014). A study by Kajander (2010, p.228) confirmed that teachers who participated in the study initially had a fragile understanding of fundamental concepts, but after a strong focus on "specialised mathematical concepts" PSTs' comprehension of key concepts improved.

In addition to the constructivist approach to learning, literature provides a conceptual framework that directed the focus of this study. In order to understand key concepts fundamental to the development of FMC skills, the TE consulted three frameworks. These included the conceptual framework for accurate and flexible mental computation by Heirdsfield (2002), the framework for basic number sense by McIntosh, Reys, and Reys (1992), and the mental computation strategy framework by Hartnett (2007).

The frameworks illuminated the development of research instruments, the intervention process and the hypothetical learning trajectory (HLT).

Moreover, this study inclined towards the constructivist theory of learning to foster meaningful learning through guided reinvention and construction of mental calculation strategies that are flexible. Opting for the constructivist theory of learning was fundamental in this study to prevent employment of a rigid approach to teaching and learning of FMC. A rigid approach to teaching and learning mental calculation would under equip PSTs to participate productively in real-world computational activities (Sawyer, 2014) and create fear for teaching and learning mathematics (Hembree, 1990). Various studies have found the constructivist approach to teaching very effective in promoting meaningful learning and application of knowledge in new contexts. Research has also found rote learning difficult in learning mental computation as it promotes memorisation of the standard method of calculation. If mental calculation is to develop through rote learning, learners may possibly forget calculation strategies easily and make mistakes when computing mentally. Therefore, this study appreciates the need for PSTs and TEs to understand the constructivist theory, as some PSTs and TEs are a product of a traditional teaching approach and have a limited understanding of the constructivist theory (Battista, 1994; Battista, 1999). As such, the problem that triggered this study and what this study aimed to achieve is discussed next.

1.2 Problem statement, objectives of the study and research questions

1.2.1 Problem statement and objectives of the study

Elementary mathematics is key to further learning of mathematics in school, while, computation of whole numbers is the foundation of mathematics learning in the early years of learning (NCTM, 2000). This study argues that for learners to have a strong foundation in whole number computation they need teachers who understand the development of mental computation of whole numbers. However, research conducted both nationally and internationally reveal weak mathematics subject content knowledge in PSTs. Relating to my previous mathematics lessons with different groups of elementary mathematics PSTs, I observed a degree of incompetency in adding, subtracting, multiplying and dividing simple computations involving two-digit numbers. In instance the PSTs were in their third year of a four-year teacher education course, but were unable to mentally calculate simple calculations such as 19 + 13. Both in Namibia and internationally, teacher education has extensively drawn public attention and critique. The quality of teachers graduating has been underrated despite government efforts to educate teachers effectively. Teachers have been found to struggle with teaching particular mathematical concepts (Marope, 2005; UNUNEFT, 2014) and learners continue to perform poorly in mathematics. Specifically, research findings of SACMEQ III (Spaull, 2011) found that most of the teachers who participated in the study were unable to answer most of the test items correctly. Although the SACMEQ IV report (Shigwedha, Nakashole, Auala, Amakutuwa, & Ailonga, 2015) indicates an improvement, the performance of teachers across the nation stills ranks poorly in relation to other countries. The need remains to improve teacher subject content knowledge essential to teaching a specific grade.

A further argument is that a major aspect that is critical to the quality of instruction is comprehension of the mathematics content knowledge that teachers are to teach (Tatto, Schwille, Senk, Ingvarson, Peck, & Rowley, 2008). This implies that awareness of subject content for teaching calls for PSTs to understand the basic skills school learners are expected to learn (Kajander, 2010; Livy, Vale, & Herbert, 2016; Ma, 2010). For example, the skill to calculate 45 + 29 by using compatible numbers such as (44 + 1) + 29 = 44 + 30 is necessary among PSTs. Significantly, the mathematics curriculum for elementary PSTs emphasises the development of PSTs' own ability to calculate mentally. Besides outlining the need for development of PSTs' own ability to calculate mentally, the PST curriculum does not provide a framework that specifies the required knowledge and how PSTs must develop mathematical knowledge for teaching FMC. Similarly, studies conducted on PST mathematical knowledge in Namibia lack precision on how to develop teachers' FMC skills. Such a gap affects the breadth and depth of the development of PSTs' mental computation skills. This study argues that, to develop PSTs' mental computation skills, TEs need to understand how teachers develop FMC skills. Research has found that learners whose teachers have sound MKT perform better compared to learners whose teachers have inadequate MKT (Ball, Hill, & Bass, 2005; NIED, 2010). Therefore, a guiding framework for the development of PSTs' computation skills is imperative.

Despite the state of PST knowledge identified by different studies (Courtney-Clarke & Wessels, 2014; Kasanda, 2005; Marope, 2005; UNUNEFT, 2014), researchers have not developed a guide to assist TEs in teaching a specific skill like FMC and knowledge for teaching FMC (Ball et al., 2005; Duthilleul & Allen, 2005). Such studies are important but more studies are needed to develop domain-specific guides to support TE practice (Gravemeijer, 1994). Since PSTs are assisted by TEs, it is necessary for teacher educators to have a constructivist view of the nature of mathematics and MKT. In devising strategies to address inadequate teaching of FMC in schools, the present study aimed to research the development of flexible calculation skills of first-year PSTs. The findings of this study may form the basis of guidelines that TEs could use to develop PSTs' mental calculation skills.

In addition, outcomes of the study could provide insight into the FMC skills, beliefs and experience of the PSTs involved in the study. A deeper understanding of PSTs' existing mental computation strategies may enable the researcher as a TE to devise effective ways to link and extend PSTs' prior knowledge. Additionally, this study may contribute to PST mathematics curriculum reform to develop mental computation skills effectively, as called for by Bruner (1977). In particular, this study could contribute to approaches for PST knowledge development for teaching FMC by answering the research questions presented next.

1.2.2 Research questions

The primary research question of this study is:

- How can PST's mathematical knowledge for teaching FMC be improved in a real teacher education classroom environment?

To answer the main question the following secondary questions were used:

i. What school experience and mental computation skills do pre-service teachers have upon entering university?
ii. How did pre-service teachers develop flexible mental computation skills and beliefs, as learners at school, about the learning and teaching of flexible mental computation?
iii. How did pre-service teachers develop mathematical knowledge for teaching flexible mental computation skills?
iv. Do PSTs' school experience and beliefs about FMC influence their perception on how to teach FMC in school?

Answers to the questions above will enable this study to:

- Establish pre-service teachers' knowledge, beliefs and experience about mental computation.
- Explore strategies that will enhance flexible mental computation skills in pre-service teachers.
- Design a framework for developing elementary pre-service teachers' mathematical knowledge for teaching flexible mental computation skills.

With the first research question above, this study intended to determine PSTs' understanding of flexible mental computation and to identify existing strategies for performing mental calculations, as well as to discover PSTs' beliefs and experience on how learners learn FMC. This was deemed necessary because PSTs enter university with their own conceptions of what FMC is and how it should be taught based on how they learned mathematics at school. With the first question, preconceived notions about FMC were identified to address any misconceptions based on FMC development. The second research question sought to understand how PSTs developed skills and beliefs in respect to the teaching and learning of FMC. This would support TEs to understand the existing views of PSTs and identify any misconceptions about the development of FMC skills. With an understanding of PSTs' existing knowledge, the TE would be able to establish the range of numbers and operations that PSTs can calculate mentally for extended practice. With the last research question, a domain-specific framework for teaching flexible mental computation could be developed to guide TEs.

1.3 Aim of the study

The purpose of this study was to identify approaches to develop PSTs' mathematical knowledge for teaching FMC by developing a HLT. The ability to do flexible calculations in the mind is critical as FMC is a basic numeracy skill that enables meaningful participation in society. Spaull (2011, p. 3) emphasised that "the basic skills of numeracy...are essential for dignified employment and meaningful participation in society". Meaningful participation in society is enhanced as teachers and learners begin to think and reason mathematically to solve daily life problems that involve simple calculations. This study thus addresses the findings of Spaull (2011), namely a lack of functional numeracy among many learners in Namibia, which is the result of inadequate teacher content knowledge for the teaching of mathematics (Kasanda, 2005; Nambira et al., 2009). Developing PSTs' mathematical knowledge for teaching is imperative as teachers' mathematical knowledge is directly linked to learner achievement (Kilpatrick et al., 2001; Ministry of Education and Culture, Namibia, 1993). If PSTs' knowledge for teaching FMC is not developed, the problem of functional innumeracy among learners and teachers is likely to persist.

1.4 Thesis statement

The development of pre-service teachers' ability to do flexible calculations in the mind requires an in-depth understanding of what FMC is and different types of mathematics problems that can prompt the re-invention of strategies. The development of PSTs' mathematical knowledge for teaching encompasses both mathematics content knowledge, and particular beliefs and attitudes about mathematics teaching and learning. Consequently, this study designed tasks that include the magnitude of numbers reflected in the school elementary curriculum for PSTs to constructively solve calculations they are expected to teach. This study argues that the use of tasks similar to mental calculation tasks for learners are likely to improve PSTs' calculation skills and beliefs about teaching FMC.

As advocated by researchers, this study planned to incorporate intervention experiences that would enable PSTs to "encounter the major and significant issues that arise in teaching mathematics with the aim of developing belief systems that can accommodate new ideas and new approaches" (Herrington, Pence, & Cockcroft, 1992, p. 6). Appropriate intervention experiences could support PSTs to perceive calculation strategies from a learner's perspective and to conceive possible ways of developing learners' FMC skills in school (Whitacre & Nickerson, 2006). This notion is supported by Herrington et al. (1992, p. 6) who stated that "we must seek to incorporate experiences in our teacher education courses that engender beliefs conducive to successful mathematics teaching". In essence, the calculation of problems aligned with the elementary school curriculum may foster PSTs' pedagogical content knowledge and FMC skills through experiencing the process of inventing own calculation strategies.

1.5 Rationale of the study

Many existing studies focus broadly on mathematical knowledge of PSTs (Blomeke, Suhl, & Kaiser, 2011; Kasanda, 2005; Tatto et al., 2008), number sense and FMC of final-year PSTs (Courtney-Clarke & Wessels, 2014; McIntosh et al.,1992; Whitacre & Nickerson, 2006) and FMC of elementary school learners (Heirdsfield, 2002). Having searched through literature, the TE found no study that was conducted on the development of first-year PSTs' mathematical knowledge for teaching FMC using a mixed methods approach. Therefore, this study intends to bridge this gap by exploring ways to develop PSTs' mathematical knowledge for teaching FMC. As a result, outcomes of this study are intended to inform the design of a teaching framework TEs may use to understand how to support PSTs in inventing and understanding different mental calculation strategies constructively.

This study envisages the development of PST knowledge of FMC as well as learners and teaching. As such, this study embraces the idea that, added to subject matter knowledge, is PSTs' knowledge regarding specific FMC aspects that may be of interest to learners and aspects learners may find challenging (Ball & Cohen, 1999). Also, PSTs need knowledge of how to listen to and interpret learners' ideas about FMC (Ball & Cohen, 1999). As a result, Kilpatrick, Swafford, and Findell (2001, p.398) point out the principal role of teacher preparation as being "helping teachers understand the mathematics they teach, how their students learn that mathematics, and how to facilitate that learning". Similarly, the rationale for this study is to enable PSTs to develop FMC skills by inventing their own calculation strategies through problem solving to understand how learners learn FMC. Further understanding is set to manifest through the act of sharing, discussing, justifying and explaining self-invented calculation strategies (Kilpatrick, et al. 2001). Findings from a study by Kind (2014, p.18) indicate that in addition to pedagogical content knowledge "deep knowledge gained from rigorous, advanced academic study provides a sound background for teaching". Evidently, research conducted internationally and nationally indicate that improved teacher knowledge contributes positively to learning (Blomeke et al., 2011; Mullis, Martin, Foy, & Arora, 2012; NIED, 2010). Thus, a constructivist learning experience and in-depth MKT is highly likely to foster a better understanding of how to develop learners' FMC skills. Significantly, the identification of ways to improve PSTs' mathematical knowledge for teaching FMC might culminate in development of knowledgeable PSTs.

1.6 Roles of the researcher

In this study, the researcher assumed the role of a TE to execute the intervention. The researcher had to assume the role of a TE to explore ways to support PSTs' development of FMC skills within a real classroom environment (Given, 2008). Being a researcher and TE at the same time required the researcher to acknowledge that personal values relating to problem solving as an effective approach to the development of FMC skills exist. Personal values of the researcher underpinned the design of appropriate tasks, learning resource material, ways to scaffold tasks and how to manage discourse in the classroom and the HLT. Consequently, the researcher was able to support PST to invent their own calculation strategies, to think aloud by sharing and justifying their calculation strategies. Furthermore, the TE as a researcher had to design research instruments, collect data through tests, questionnaires, interviews, observations and document analysis. Before collecting data, the researcher had to build rapport with the PSTs to enhance their confidence in sharing their perceptions on FMC honestly. After collecting data, the researcher recorded, presented and analysed the information in an honest and more comprehensive way by discussing both positive and negative outcomes. To enhance the trustworthiness of the data the researcher had to read extensively to understand the design and methodology of the study (Wang & Hannafin, 2005). In addition, the researcher had to design quality questionnaires, tests, interview questions and intervention tasks to enhance the credibility of the findings. No data was fabricated as the researcher strived to provide an honest record and interpretation of PST responses to interviews and behaviour demonstrated during the intervention process. All outcomes of the study were recorded without distortion regardless of whether it confirms the hypothesis of the study or not.

1.7 Research design and methodology

The methodology of this study is informed by the pragmatic paradigm. The pragmatic paradigm links theory and practice, and prompted this study to adopt a real-life classroom environment as a research site to find solutions that are realistic and empirical in nature (Given, 2008, p. 673). The chosen paradigm is relevant to this study because it focuses on PSTs' existing experience, knowledge, experience and beliefs concerning FMC (National Institutes of Health - Office of Behavioral and Social Sciences (NIHOBSS), 2018; Wang & Hannafin, 2005). Furthermore, the pragmatic paradigm is relevant since the study seeks to understand how to develop PSTs' knowledge to teach by providing PSTs with a problem solving experience within a real classroom environment using realistic mathematics education (RME) as a theory of instruction. In this study, PSTs' knowledge for teaching mental computation is comprehended and improved using the instructional theory of RME.

Realistic mathematics education is a theory of instruction for mathematics learning (Van den Heuvel-Panhuizen & Drijvers, 2014) that perceives mathematics as a human activity learned through guided re-invention. This is a back-up for design-based research (DBR) (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Cobb & Gravemeijer, 2008) which falls under a developmental paradigm (Trafford & Leshem, 2008) that forms the main methodological approach to this study. DBR was found appropriate as it enables identification of solutions that work for a specific group considering their immediate needs, unlike adopting pre-existing solutions that may not necessarily respond to PSTs' needs. Thus a mixed methods design was used to collect quantitative and qualitative data, since one method may not have served to collect and confirm data from specific instruments. Thus, quantitative data were set to emerge from the questionnaires and baseline test, and qualitative data from semi-structured interviews, observations during the intervention process and field notes.

PSTs' mathematical knowledge for teaching FMC was envisaged as prevailing through the use of a hypothetical learning trajectory (HLT). Considering the absence of an HLT for developing PSTs' mathematical knowledge for teaching FMC, this study sought to use different research methods to design and refine the HLT. An HLT is a domain-specific teaching framework that outlines the basic mathematics operations covered in the intervention, anticipated computation strategies, instructional activities and models for reasoning. In the context of this study, the development of a domain-specific learning trajectory contributed to the identification and design of teaching methods to support PSTs' construction of own calculation strategies and knowledge for teaching. Therefore, the methodology of this study evolved in a cycle as illustrated in Figure 1.1.

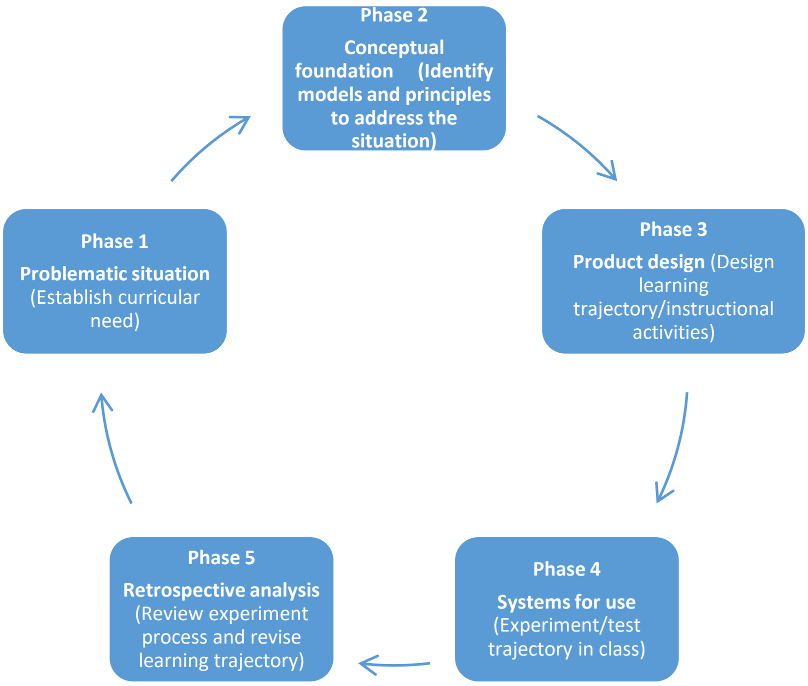

Figure 1.1: Cycle of research process: design informed by Hjalmarson and Lesh (2008, pp. 96-110)

PSTs' beliefs, knowledge and experiences were explored prior to the design of a learning trajectory through a pre-intervention test and semi-structured interviews. The pre-intervention test was developed using existing standardised tests. The envisioned HLT was developed using relevant literature (Hjalmarson & Lesh, 2008) based on PSTs' beliefs and their development of MKT mental computation. RME informed the design of the learning trajectory to include tasks PSTs can imagine or visualise while focusing on the use of contexts, models, students' own productions and constructions, the interactive character of the teaching process, and the intertwinement of various learning strands (Freudenthal, 1968; Treffers, 1987). For example, from a traditional perspective a calculation such as 43+28 is presented as: 'work out the following, 43+28'; whereas from the RME perspective it is presented in the form of a problem such as: 'A farmer has 43 goats in a kraal; 28 more goats are brought into the kraal.

How many goats are in the kraal altogether?' The intention of using problem solving is to engage PSTs in tasks that may elicit the construction of different calculation strategies that are flexible.

1.7.1 Sample

This study was conducted on one of the thirteen university of Namibia campuses. All first-year BEd honours elementary mathematics PSTs of the satellite campus were approached to participate in this study. The participants were PSTs enrolled for early childhood and lower primary teacher education programme. Non-probability sampling was used where a maximum of 15 PSTs were selected purposively for interviews on the basis of their performance in the pre-and post-test results of the baseline test. Interviews were used for the TE to collect in-depth information to understand the calculation strategies PSTs might use to make flexible computations (Teddlie & Yu, 2007). Data were collected using the instruments discussed next.

1.7.2 Instruments and data collection

To gather data, the TE used a written questionnaire, a diagnostic test, an observation schedule, an interview guide, and PSTs' written and oral work. Using the questionnaire, data collection commenced with a survey on PSTs' beliefs about and experience of flexible mental computation. Thereafter, a baseline evaluation of PSTs' mental computation skills was done using a diagnostic pre-test. The baseline tests were followed by semi-structured interviews with the PSTs who were selected purposively, to document their thoughts behind the strategies used during the diagnostic pre-test. After the interviews, the intervention process began, in the form of DBR cycles. Hence, three data sources were exploited in each DBR cycle: observation of PSTs' mental computation strategies used in class; written work; and semi-structured interviews (Cohen, Manion, & Morrison, 2000). After each cycle, the process was reviewed by the TE to improve the learning trajectory. The improved learning trajectory was used again with a subgroup of PSTs, who underperformed in the post-intervention test (Cobb et al., 2003; Bannan-Ritland, 2003). The collected data were analysed as discussed next.

1.7.3 Data analysis

Data analysis was a continuous process from the commencement of the study. Quantitative data on PSTs' existing knowledge, beliefs and experience were analysed descriptively. Data from interviews were reviewed and organised into themes (Creswell, 2013) through coding and summarisation of themes. Data from intervention cycles were interpreted and compared to data that emerged from the questionnaires and the diagnostic tests. Retrospective analysis was used to support the development of a domain-specific learning trajectory using the constructivist theory of learning and the realistic mathematics education teaching approach. Furthermore, the TE recorded how PSTs' mathematical knowledge of teaching flexible mental computation developed (Kelly, Baek, Lesh, & Ritland, 2008). Assurance of the credibility of the findings of the study is outlined next.

1.7.4 Validity

The TE collected data over a period of one month using a variety of instruments to ensure validity and reliability. Internal validity was enhanced through debriefing and member checking to ensure that data were accurately interpreted (Creswell, 2013). The classroom setting, PSTs' views and the intervention lessons were video-recorded. The TE used a researcher journal in which notes pertaining to aspects observed during the intervention were recorded. PSTs' written work was analysed for triangulation of survey findings. Results of this study were compared to existing studies based on PSTs' mathematics learning. External validity was fostered through the use of relevant studies to structure the envisaged classroom activities and experiences and by describing the entire intervention process in detail for reliability and generalizability of the HLT (Cobb & Gravemeijer, 2008). Prior to the intervention process, the HLT was evaluated by an experienced researcher and refined with a small group of PSTs after the intervention process. The scope of this study was limited to a specific mathematics domain and population, as presented in the next section.

1.8 Delineation and limitations

To make this study manageable, the TE focused only on the development of addition, subtraction, multiplication and division of positive whole numbers consisting of one to three-digits. PSTs' existing number knowledge and irregular attendance of intervention sessions hampered PSTs' development of knowledge for teaching FMC. This study excludes PSTs studying upper primary and secondary mathematics, PSTs beyond the first year of study at university, and in-service teachers. The intervention cycles were carried out to enhance growth in PSTs' flexible mental computation skills and knowledge. Specifically, findings of this study cannot be generalised to other contexts as the beliefs, knowledge and experience discussed in this study emerged from one satellite university campus. However, the HLT can be adopted in any teacher education classroom to develop PSTs' mathematical knowledge for teaching FMC.

1.9 Assumptions

This study assumes that PSTs could develop their mathematical knowledge for teaching FMC if instructed as they are expected to teach and by focusing on the computation knowledge involved in the elementary school curriculum. Another assumption is that PSTs' beliefs about the development of FMC skills and mathematics teaching might change once offered the opportunity to learn through a problem solving approach. This study also hypothesises that in-depth comprehension of how to calculate mentally is likely to improve PSTs' ability to compute mentally in order to teach FMC effectively. Various terms were used in this study and are defined in the next section.

1.10 Definitions of terms and concepts

Flexible mental computation: The term flexible mental computation refers to individual calculation strategies emerging from identification of specific features of numbers to solve a problem (Threlfall, 2002). This study has used this term to refer to the process of calculating mentally using a calculation strategy invented personally.

Conceptual framework

In this study conceptual framework is used to refer to all terms used that denote specific mathematical ideas relating to FMC and its development (Cohen, Manion and Morrison, 2000). The concepts are mainly discussed under the literature review section of this study.

Theoretical framework

The term theoretical framework is used in this study to refer to the learning theories (Merriam, 2009), that have informed the development of PSTs' knowledge for teaching FMC. The learning theories suggest effective ways through which meaningful learning may prevail.

Pre-service early childhood and lower primary teachers: The concept is used in this study to refer to first year student teachers who are enrolled at university on a full-time basis to teach pre-primary school and Grades 1 to 3.

Mathematical knowledge for teaching: Mathematical knowledge for teaching refers to the knowledge required to teach flexible mental computation (Ball, Thames & Phelps, 2008). The concept has been used in this study to refer to fundamental skills and concepts pre-service teachers need to develop learners' flexible mental computation skills.

Learners: In this study the concept has been utilised to refer to children in pre-primary and Grades 1 to 3.

Teacher educator: An academic teaching teachers at college or university level. In the context of this study the concept has been used to refer to the researcher conducting the study as well as other academics involved in the training of teachers at tertiary institutions.

Elementary mathematics: Mathematical knowledge developed in pre-primary school and Grades 1 to 3.

1.10 Overview of the study

This study is underpinned by the argument that the development of PSTs' flexible mental computation skills requires an in-depth understanding of the concept of FMC. Thus, in Chapter 2, the TE explores literature underpinning the meaning and importance of FMC. Thereafter, the role of mathematical understanding and thinking in FMC skills development is discussed. Thereafter, an exploration of literature based on the development of learners and PSTs' flexible mental computation skills follows. Since the study focuses on PSTs' mathematical knowledge for teaching, literature pertaining to MKT was reviewed to understand the kinds of knowledge PSTs require for teaching FMC. To understand the current situation concerning PSTs and teacher education, literature based on PST education is discussed. The intention to develop a HLT for PSTs triggered the TE to analyse how FMC skills are reflected in a teacher education programme as presented in a PST elementary mathematics course outline. As a result, the TE analysed the curriculum for PSTs in Namibia, in neighbouring South Africa, and in Singapore, which has one of the most effective teacher education programmes globally. Thereafter, a discussion on the theoretical framework follows.

In Chapter 3 the TE presents the theoretical framework that informed the design of a HLT to develop PSTs' mathematical knowledge for teaching FMC. In Chapter 4 the TE provides a detailed account of the design and methodology that illuminated the data collection process of this study. For validity and reliability of data, this study used a survey to collect data using questionnaires and diagnostic tests before and after the intervention. In addition, for the TE to corroborate data from the questionnaires and diagnostic tests, face-to-face interviews with individual PSTs were conducted. Data collected through questionnaires, diagnostic tests and interviews are presented and interpreted in Chapter 5. In Chapter 6, the TE presents and interprets data from the intervention process. The development of the HLT is discussed in Chapter 7, while also providing a detailed description of the design of the activities used in the intervention. A discussion of the findings, recommendation and conclusion of the whole study is presented in Chapter 8. The next section is Chapter 2 where the conceptual framework of this study is discussed.

CHAPTER 2: LITERATURE REVIEW

2.1 Introduction

In chapter 2 literature pertaining to the development of FMC is discussed. Specifically aspects discussed relate to how FMC evolved, what FMC is, why FMC is important, knowledge that is fundamental to the development of FMC skills, aspects that should be considered when developing skills to compute mentally, contradictions in literature, fundamental knowledge for teaching and PST curriculum. Mainly, literature pertaining to the development of learners' FMC skills is reviewed extensively in chapter 4 as presented next.

2.2 Flexible mental computation

In the past, the concept FMC was not popular in schools in Namibia as the focus was on mental arithmetic and standard algorithms. Within the Namibian context, FMC has received little attention both in school and teacher education programmes. Similar to international trends, a standard algorithm was the main pen-and-paper method used to calculate in the absence of a calculator. The development of mental arithmetic focused on recall of memorised basic number facts for addition and subtraction and on the multiplication table (Battista, 1999; McIntosh, Reys, & Reys, 1992). As to FMC teaching, usually an educator would provide a range of calculation strategies to select from as opposed to the constructivist theory of learning (Kamii & Joseph, 2004; Piaget & Inhelder, 1973; Vygotsky, 1978). However, prescription of calculation strategies presents difficulty of understanding and recalling strategies, culminating in a lack of number sense and innumeracy (McIntosh et al., 1992). Both nationally and internationally, awareness of the development of FMC skills in learners began in 1984 with the work of Kamii and Joseph (2004), as discussed later in this chapter.

In the 1990s, advances in the way that calculations were carried out mentally received increased attention. Consequently, different countries embraced FMC skills development as a better approach to calculating mentally. FMC was embraced by the Netherlands' Realistic Mathematics Education (RME) (Freudenthal, 2002), England's National Numeracy Strategy (Department for Education and Employment - DfEE, 1999), the United States of America's(USA) Principles and Standards for School Mathematics (NCTM, 2000), the Australian National Statement on Mathematics for Australian Schools (Hartnett, 2007). The adoption of FMC emerged in response to learners' struggle to memorise basic number facts and the multiplication table and the challenge of mastering traditional written calculation strategies meaningfully.

To unpack effective ways in which calculation strategies could be developed meaningfully, this study drew on existing literature pertaining to the development of FMC skills (Heirdsfield, 2002; Kamii & Joseph, 2004) and the development of teacher knowledge for teaching FMC, as informed by Ma (2010) and Shulman (1987). With education reform undergone in Namibian schools, the concept ‘mental calculation’ appears in the syllabus but not the concept ‘FMC’. How the concept mental computation is defined and developed remains unknown. Knowledge of the correct definition and the development of FMC is important in teacher education programmes, especially for this study, for teacher knowledge impacts learning. This study posits that PSTs’ development of mathematical knowledge for teaching FMC and their understanding of the concept are imperative in fostering learners’ number sense and ability to invent FMC strategies. According to Shulman (1987, p. 14), PSTs’ understanding is critical in teaching because “to teach is first to understand”. Evidence suggests that understanding and development of FMC skills derive from solving real-life problems using invented calculation methods (Freudenthal, 1968; Kamii & Joseph, 2004). Therefore, to understand the concept FMC and the development of PSTs’ mathematical knowledge for teaching FMC, relevant literature was consulted and is discussed next.

2.3 Defining flexible mental computation

Major changes occurred in the concepts used to denote calculations carried out in the mind and their definition. The literature indicates that the concept used three decades ago was ‘mental arithmetic’ and not FMC (Davis, 1984). Thereafter, the concept mental computation emerged in the literature and has then developed further and is currently referred to as FMC. According to Davis (1984), mental arithmetic refers to rote learning of rules and methods provided by the teacher that are then mastered through ‘drill and practice’, with no attention to the significance and comprehension of such rules. Such methods of learning draw on memorisation of basic addition and multiplication facts to solve routine problems using standard algorithms as modelled by the teacher during lessons. A learning environment where rules and procedures are prescribed by a teacher is referred to as a teacher-centred teaching approach, which has proven ineffective in enhancing learning with understanding. This is why a shift in concepts and meaning occurred, eventually embracing the concepts mental computation and FMC.

This study discovered that the concept mental computation featured more extensively in studies compared to FMC. The literature presents inconsistency in the use and definition of mental computation. In existing studies, in most cases, mental computation has the same meaning as FMC (Koshy & Murray, 2011; McIntosh, 2005). Hazekamp (1986) extends Heirdsfield’s (2011) definition of mental computation to define it as a process that involves calculating precise solutions without applying a traditional method in the head. Hazekamp’s (1986) definition of mental computation excludes the application of a traditional algorithm in the head, a strategy not recognised in FMC. This outlines the need for precision regarding the use of the word ‘flexible’ to distinguish mental computation from FMC. Therefore, this study defined both concepts to outline the meaning and justify the use of the focal concept of this study. The discussion below first defines mental computation and then FMC.

The concept mental computation has been used to denote calculations performed mentally. Heirdsfield (2011, p. 96) defines mental computation as "calculating in the head" whereas other authors define mental computation as a process of calculating a precise answer in the mind without using a calculator or pen and paper (McIntosh, Reys & Reys, 1997; Reys, 1984; Sowder, 1990). An analysis of Heirdsfield's (2011) definition of calculating in the head leads to the inference that mental computation could mean calculating with or without understanding the executed calculation procedure so long as it is calculated in the head. Calculating in the head could also mean applying the standard method to calculate mentally (Hazekamp, 1986). This study asserts that if the definition of mental computation is limited to calculating in the head as suggested by Heirdsfield (2011), mental computation would include the use of standard algorithms to calculate mentally and may result in reinforcement of constrained expertise (Markovits & Sowder, 1994). Although calculating in the head using standard calculation procedures can be accurate, it encourages rote learning and inhibits sensible execution of calculation strategies. Since rote learning impedes the meaningful development of conceptual knowledge in terms of understanding numbers, grasping calculation methods and devising strategies to solve unfamiliar problems, the concept FMC has been introduced to emphasise flexibility in performing calculations. It is important to perform calculations in the mind without applying standard algorithms as flexibility is central to the development of calculation strategies.

FMC is fundamental in developing mathematical understanding and the application of mathematical knowledge in daily life. Therefore, the present study embraced the definition of FMC by Threlfall (2002, p. 42) as "a personal reaction with knowledge, manifested in the subjective sense of what is noticed about the specific problem". The definition by Threlfall (2002) refers to how a person handles information as a result of her/his personal discovery made about a particular problem. Personal discovery of key mathematical information involved in a specific problem evokes active construction of calculation strategies in a flexible way. Application of discovered methods does not imply imposing calculation strategies but does imply the use of one's own methods based on how a problem is perceived from a personal point of view. FMC is not restricted to the ability to calculate fast and correctly in the head (Threlfall, 2002). Calculating fast and correctly in the head can be a result of a skilful application of a standard algorithm in the mind without the ability to work with numbers flexibly. Although the speed and accuracy of a solution are important, these are not enough to facilitate calculations of new tasks involving numbers or to improve understanding of numbers. Speed and accuracy alone do not indicate an ability to calculate flexibly in the mind.

Articulation of the meaning of FMC has led to adjustment in the way that FMC is perceived and developed. Flexibility in performing calculations forms part of computational fluency. Computational fluency signifies performing calculations efficiently, accurately and flexibly. The three ideas refer to calculating using easy and not confusing strategies that allow application of understanding of number knowledge using a variety of invented calculation strategies to solve and verify solutions (Russell, 2000).

Furthermore, high performance in mental computation can be achieved without knowledge of effective strategies and without accompanying number sense (Heirdsfield, 1996; McIntosh & Dole, 2000). In such cases, algorithms are solved using memorised rules of the standard algorithm method as provided and modelled by the teacher during lessons. This refers to a traditional teaching approach that inhibits the capability to solve non-routine types of questions due to a lack of number sense. It also inhibits the development of creative thinking and the ability to solve new problems using any method. Alternatively, FMC contrasts memorisation and rote learning with the ability to carry out calculations meaningfully in the head using self-invented strategies shaped by existing mathematical knowledge.

FMC articulates the use of calculation strategies that are adaptable to new problems and are not rigid. Although the intention to develop mental computation skills in school is outlined in the Namibian elementary school curriculum, FMC is not mentioned in the lower primary school syllabus. The meaning and function of FMC are not well articulated in the curriculum. Educators can translate the concept mental computation as they understand it, possibly leading to direct teaching of calculation strategies. The curriculum for PSTs uses the concept ‘mental mathematics’ and defines it as different ways to find answers to addition, subtraction, division and multiplication (Department of Early Childhood Education (DECE), Namibia, 2014). There are no other examples of possible strategies and how such strategies must be developed in PSTs. However, the inclusion of FMC in the PST curriculum, although the concept mental mathematics is used, demonstrates recognition of the importance of FMC for both PSTs and learners.

2.4 Importance of flexible mental computation

Despite the existence of calculators, FMC of whole numbers plays a major role in real-world situations, in building personal mental capacity to work with numbers and in developing an understanding of numbers and operations in school. Generally, in real-life settings, whole-number calculations are carried out at home, in the workplace, in shops and in open markets. The argument is that the ability to perform calculations at home, in the workplace and when buying items demands a personal capacity to work with numbers. The ability to calculate flexibly in the mind contributes to understanding of numbers, operations and calculation procedures. This culminates in mathematical thinking, mathematical reasoning and the ability to solve problems for which there are no prescribed rules (Russell, 2000). Such use of FMC demonstrates its utility in a real-world setting, as discussed next.

2.4.1 Role of flexible mental computation in a real-world setting

Many calculations embedded in real-life settings can be performed mentally. This study posits that all learners need to develop computation skills as real-life activities involve handling money and numerical information. For example, when purchasing or selling goods, money is exchanged, which in some cases may involve receiving or issuing change. To determine how much change to expect or give, one must be able to quickly work it out mentally. In a process of buying and selling, the absence of FMC skills could result in a loss of money due to the inability to sense the inaccuracy of answers provided by a malfunctioning calculator. Determining the reasonableness of an answer is necessary as, according to Sowder (1990), mental computation provides the basis for technological expertise and for improved dealing with figures encountered in daily life situations. Technological expertise such as establishment of the reasonableness of the answers provided by a calculator is important in daily life.

Daily life mathematics entails marking prices up and down. This involves understanding and calculating the reduced or increased price of an item to determine the difference. For example, to determine the reduced price of an item originally marked N$250.00 when it is on half-price promotion should not demand the use of a calculator. This entails division by two in which FMC can be a quick way to determine the price. Freitag (2014) confirms that FMC is handy in circumstances involving basic calculations. Evidence from an intervention study meant to change beliefs about mathematics indicates acknowledgement of reliance on a calculator to compute a reduced price of an item, as a participant stated that “I would never have even tried to figure out the sale price without a calculator before doing this assignment” (Kalchman, 2009, p. 534). This evidence outlines the role of FMC in liberating the mind from absolute reliance on a calculator. FMC enhances the ability to cope with daily life activities such as determining a best buy in a shop or open market, detecting mistakes in viewpoints and estimating the total cost of groceries without a calculator.

Many calculations are basic and simple to compute and may not require the use of a calculator or the paper-and-pen method. For example, if N$30 is worth 1 000 kwanza (Angolan currency), how many Namibian dollars do you need to buy 4 000 kwanza? This kind of computation is quite basic and can be performed mentally without a calculator or the pen-and-paper method. Reys (1984, p. 550) argues that “real-world mathematics is not always tolerant of a dependence on paper-and-pencil methods”. In support of Reys, this study refers to a real-life situation in Namibia regarding taxi fares that increased from N$10.00 to N$12.00 per person in 2018. A taxi driver was unable to calculate the amount that three people who were travelling together had to pay and what their change would be after issuing him a N$50.00 note. To solve the problem, although a calculator was used, resorting to a cell phone to quickly solve the problem resulted in a delay in the passengers’ programme. In a different scenario, a taxi driver decided to issue a passenger N$90.00 change after the passenger had offered him a N$100.00 note as he could not determine the change without a cell phone or a calculator. Such scenarios confirm the importance of FMC skills and the overreliance on mathematical devices to perform calculations, as also emphasised by McIntosh (2005). In the literature, however, a contradicting argument is that standard computation methods and calculators are available to perform calculations easily and fast.

This study posits that the existence of calculators and the paper-and-pen method should not undermine the ability to carry out calculations mentally. Research indicates that a lack of FMC skills promotes reliance on calculators and standard written algorithms to perform basic calculations (Varol & Farran, 2007). It is crucial to develop the ability to compute mentally to limit reliance on external support except when performing complex calculations. According to Carpenter (1974, p. 127), calculators deprive adults and learners from exercising their “cognitive freedom”. Carpenter (1974) emphasises further that without mathematical knowledge, in this case FMC skills, both adults and learners lack the freedom to solve problems without mathematical devices, resulting in being mathematically handicapped. For instance, there are situations when computing with a calculator can be time-consuming and ineffective when it is used incorrectly. Calculators also undermine individual ability to calculate mentally, which leads to a sense of vulnerability in the absence of a calculator. A further argument is that although the use of calculators cannot be discounted, a distinction is necessary between situations where FMC strategies are more efficient and situations where a calculator or paper-and-pen computation is more efficient.

A lack of FMC skills disadvantages people in many ways. One way as emphasised by Kilpatrick et al. (2001, p. 16) is that “citizens who cannot reason mathematically are cut off from whole realms of human endeavour”. This signifies that the inability to calculate flexibly in the mind denies people the chance and ability to participate in daily life activities involving calculations due to overreliance on a calculator or the paper-and-pen method. Overreliance on calculators leads to underestimation and underutilisation of personal potential and ability to calculate independently and to a lack of confidence. This is why the present study identified FMC as a catalyst towards empowering learners and teachers to utilise and believe in their own ability to perform basic calculations independently in situations involving mathematics.

2.4.2 Role of flexible mental computation in building personal mental capacity

FMC plays a major role in empowering individuals to calculate mentally. This study posits that empowerment of mental calculation capacity emerges in the form of metacognitive skills in the process of inventing diverse calculation strategies. Dawson (2008) defines metacognition as a process involving thinking about individual way of thinking. Metacognition comprises skills that promote learning and thinking. Thinking in order to learn is an ultimate goal of the development of FMC skills. The process of inventing one’s own calculation strategies encourages critical thinking when analysing the magnitude of numbers, determining the relationship between numbers and operations and identifying patterns among numbers. Engagement in the analysis of numbers encourages active engagement in calculations to construct one’s own calculation strategies.

Construction of one’s own strategies empowers personal mental capacity when thinking reflectively, creatively and logically to evaluate one’s own and others’ calculation procedures, when reasoning to justify one's own calculation procedures through communication and when exploring alternative strategies and drawing on informed mathematical judgement to identify procedures that are more efficient. Constant invention of calculation strategies through FMC culminates in personal habits of mind responsible for good problem solving skills to make informed mathematical decisions. The scenario of the taxi driver in Section 2.3.1 demonstrates an inability to compute flexibly. The taxi driver's inability to calculate mentally demonstrate a deficiency in mathematical thinking, logical thinking and mathematical reasoning to realise that adding N$12.00 three times would determine the total amount to be charged. Another aspect missing is thinking in terms of taking away N$30.00 first and then taking away N$6.00 to determine the passengers' change from the N$50.00 note issued as payment. Adding the change and the amount charged must add up to the amount issued, namely N$50.00. This demonstrates mental capacity empowerment emanating from FMC development. Mental capacity empowerment begins at home and should be developed further in school.

2.4.3 Role of flexible mental computation in school

Mathematics is taught in school to enhance learners' numeracy skills. The development of FMC skills is critical in the preparation of functionally numerate people. Askew, Rhodes, Brown, William and Johnson (1997, p. 10) define numeracy as "... the ability to process, communicate and interpret numerical information in a variety of contexts... [it also involves] conceptual understanding of number, a 'feel for number', and the ability to apply arithmetic". Numerate learners have the ability to work with numbers confidently and competently with knowledge of the base-10 number system, knowledge of a variety of calculation strategies and an ability to solve problems in different situations (DfEE, 1999). FMC aims to strengthen the ability to handle calculations mentally in a more comprehensive and creative way. In school, whole-number computation extends over the pre-primary, primary and high school curriculums in topics involving numbers and operations (NCTM, 2000). Algebra, measurement, geometry, data handling and other mathematical topics involve the use of whole numbers. Whole numbers are also used in school subjects such as social sciences or commercial subjects such as accounting and entrepreneurship. Therefore, an in-depth understanding of numbers and flexible calculation of numbers enables understanding of mathematical concepts and learning of mathematics at advanced levels.

Whole-number computation involves understanding numbers and how they operate. FMC strengthens understanding of the base-10 number system and the four basic operations through construction of flexible strategies (Varol & Farran, 2007). Russell(2000, p. 154) argues that "...learning about whole-number computation is a key context for learning to reason about the base ten number system and the operations of addition, subtraction, multiplication, and division". Such reasoning enables the composition and decomposition of numbers coupled with understanding of the relationships among operations to solve non-routine problems using derived facts (Sowder, 1990).In other words, FMC contributes to learners' ability to derive facts and invent strategies for solving mathematics problems both in school and in daily life.

Through FMC, learners also develop knowledge of place value and begin to think and reason mathematically when counting and devising strategies to compute single digits with sums exceeding 10 (Ministry of Education, Ontario, 2003). Place value knowledge is strengthened through FMC when numbers are perceived as "whole quantities instead of discrete columns of digits" (Parrish, 2010, p. 13). For example, when using the 'counting on' strategy to add 8 and 5, learners count beyond 10 and develop place value knowledge as they unitise quantities. Place value is not taught in isolation but develops in the process of solving problems requiring learners to add, subtract, multiply and divide (Kamii & Joseph, 2004). The use of knowledge of numbers and operations to solve problems in a variety of contexts entails the ability to think creatively and analytically. Analytic skills and reasoning impart the ability to make informed decisions and to understand, verify and analyse mathematical arguments (Department of Employment, Education, Training and Youth Affairs, 1997; Maclellan, 2001). Analytic skills develop further through discussion, reasoning, justification and selection of efficient calculation strategies. Heirdsfield and Cooper's (2004) findings indicate that because learners have access to ways that make mathematics meaningful, they gain from efficient calculation strategies and are able to apply different computation methods flexibly in new contexts.

Furthermore, analytical skills and reasoning improve through the process of selecting the most efficient calculation strategy and when numbers and the relationship between numbers and operations are analysed. Reasoning skills improve when calculation methods are justified; for example, when computing 26 + 47, the focus must be on thinking how far 26 is from 30 or how far 47 is from 50 to select the most efficient strategy. Selection of the most efficient strategies promotes efficiency and accuracy that reduce the chance of making mistakes. For example, to compute 46 + 37, if the numbers are decomposed into 40 + 30 = 70 and 6 + 7 = 13, then 70 + 13 = 83 is more likely to form a long string than when the calculation is performed as 46 + 30 = 76 and 76 + 7 = 83, which is why it is necessary to understand how FMC skills regarding positive whole numbers develop. The first method is referred to as partitioning or splitting both numbers while the second method is referred to as sequencing or the jump method. The sequencing or jump method has been found more helpful in Holland and has been used in schools as it facilitates the transition from addition to subtraction (Askew & Brown, 2003; Beishuizen, 1993). In all, FMC yields an experience that mathematics is easy to learn as opposed to the belief that mathematics is difficult to learn and is meant for intelligent learners only.

Generally, mathematics has been considered difficult to learn due to how calculations are developed in the early grades. A participant in an intervention study expressed a feeling of betrayal by the teaching approach used at school, stating, "If I had been allowed to think about and solve problems in ways that make sense to me, I might have really enjoyed mathematics, and I might have been motivated to learn it" (Kalchman, 2009, p. 534). This evidence serves to support the argument that difficulty in understanding mathematical concepts and inability to carry out the four basic operations using the standard method lead to frustration and a lack of interest in mathematics. Alkan (2013) confirms that learners who experience difficulty in understanding mathematical concepts are subjected to frustration and are unsuccessful in learning mathematics.

The literature indicates that the development of mathematical knowledge begins long before formal schooling. Hiebert (1984) asserts that it is possible for learners to develop mathematical understanding on their own, and they really do. The possibility to develop mathematical understanding lies in FMC as it offers learners the opportunity to develop metacognitive skills. As discussed in Section 2.3.2, flexibility in calculations enhances metacognitive skills that lead to understanding mathematics constructively. Metacognitive skills create the opportunity to capitalise on learners' interest in structuring their own learning and making their mathematical world meaningful. However, the problem is that intuitive mathematical knowledge developed before formal schooling is neglected by teachers and abandoned by learners once formal schooling begins due to the emphasis on standard algorithms. An issue raised is that there is a lack of connection between learners' prior knowledge of mathematical concepts and skills and the rules and symbols learned at school (Hiebert, 1984). Consequently, the absence of connections "...induces the shift from intuitive and meaningful problem solving approaches to mechanical and meaningless ones" (Hiebert, 1984, p. 498).

Learning to calculate mentally in a flexible way attaches value and meaning to the learning of mathematical concepts by connecting classroom mathematics and real-world mathematics. FMC in school upholds learners' intuitive knowledge by connecting real-life calculations to classroom mathematics. This does not only facilitate understanding strategies but also develops confidence and interest in learning mathematics, forming the belief that mathematics is easy to learn and enjoyable. This leads to a positive mind-set fundamental to mathematics learning (Cockcroft, 1982). A study conducted by Lovitt and Clarke (1992) provides evidence of a 10-year-old child who was able to compute confidently and efficiently the change he would receive if he paid with a five-dollar note for a chocolate bar that costed45 cents but when 500 – 45 was presented to him, he was not certain whether the operation represented subtraction or division. He was uncomfortable with computing a vertically written calculation that he then calculated wrongly and was unable to explain his calculation procedure. The inability to compute a similar problem that is written vertically provides evidence of a lack of understanding of standard algorithms and the connection between the mathematics that the child uses outside school and the standard algorithms used in the classroom.

Development of FMC skills thus strengthens understanding of standard algorithms and relieves learners from rote learning of computation strategies (Sowder, 1990). Since FMC involves the use of personal calculation strategies, such strategies make more sense to learners and offer a stable foundation for subsequent learning of formal computation methods and written computations (Reys, 1984; McIntosh, 2004). Rote-learned strategies disable critical thinking and the use of personal intuition about number relationships to solve problems and cause fear of learning mathematics (Alkan, 2013). Mathematics anxiety and a lack of self-efficacy resulting from failure to use the standard method effectively could be prevented through FMC development.

The process of inventing calculation strategies involved in FMC allows learners to perceive mathematical knowledge as dynamic and not static as calculations can be performed using different strategies.FMC enhances understanding of why particular procedures work to solve calculations. Grasping of the meaning of mathematical concepts and procedures emerges from development of learners' own computation strategies. FMC involves knowledge of decomposing and recomposing numbers, which enhances understanding of number relationships and basic number facts. This leads to the discovery of the reversibility property of operations without formal instruction in calculation properties. FMC enables learners to solve unfamiliar problems using invented procedures and rules that are easy to execute and recall (Hiebert, 1984). As a result, a sense of ownership of calculation procedures and rules manifests in a mathematics classroom.

Conversely, an inability to calculate flexibly in the mind results in memorisation of prescribed rules without understanding. Memorised rules are easy to forget and difficult to transfer when new problems are encountered. This means that procedures learned instrumentally can be used to solve routine activities only. In most cases, such procedures fade away when prescribed by a teacher without learner involvement. The literature outlines that in most cases, memorised procedures are only recalled until a test is written and are forgotten thereafter (Brophy, 2010). In cases where memorised strategies do not fade away, such strategies cannot be applied to solve non-routine algorithms due to a lack of understanding of procedures and an inability to think creatively.

FMC enhances both the understanding of how procedures work and why such procedures work effectively. This connects to what Skemp (1976) considers as 'relational understanding' as opposed to 'instrumental understanding'. Skemp (1976, p. 89) defines relational understanding as the state of "knowing both what to do and why" while instrumental understanding denotes the use of "rules without reasons". With regard to instrumental understanding, it is easy for learners to memorise rules and apply them as provided by the teacher. The problem is that learners are highly likely to make mistakes once a step is forgotten or when an attempt is made to generalise the rule to solve a different problem. Instrumental understanding involves 'rules without reasons', so learners are unable to explain why the prescribed rules work. Davis (1984) established that teaching mathematics instrumentally made learning situations highly unexciting whereas inventive ways of learning mathematics were stimulating and appealing.

Unlike instrumental understanding that involves strategies that learners cannot justify, FMC enables invention, comprehension, selection, verification and justification of how and why specific strategies work. When learners understand strategies and why such strategies work, they can use these strategies to solve new problems. Relational understanding avoids the memorisation of rules and limits the chances of forgetting rules and calculation procedures. Relational understanding also yields automaticity with basic number facts, such as having knowledge of all pairs of whole numbers that make up 10. Though relational understanding demands ample time to develop, it can foster the instant recall of basic number facts and derived number facts meaningfully. In comparison to instrumental understanding, it is easy to prescribe rules and procedures to learners, which is time-saving but lacks development of a proper understanding of why these rules and procedures work. Moreover, procedures learned instrumentally are easily forgotten and cannot be applied in new situations.

A further argument is that although standard algorithms and calculators are available to perform calculations, learners need to understand how particular standard methods work and must be able to determine the reasonableness of an answer provided by a calculator. Confirmation of the correctness of a calculation is possible when a calculation can be carried out quickly in the mind. Koshy and Murray (2011) affirm that mental computation is helpful as a way to assess the value of a computation procedure and the precision of a solution. Learners must be able to use calculators but should determine the suitable time to use it (Van de Walle, 2007). Once learners are afforded the opportunity to invent and select appropriate strategies for specific problems, they will be able to use their existing knowledge fully and creatively to compute whole numbers meaningfully. The use of standard written algorithms and calculators without mastery of FMC impairs learners' ability to think critically about calculation strategies. Standard algorithms permit the computation of calculations without careful thought about the calculation strategy whereas mental computation emphasises the significance of the numerals in the problem (Maclellan, 2001). The argument is that mastery of the standard method promotes the use of computation procedures with no attention to understanding the strategies that learners are executing and the significance of the numbers in the problem presented to them (Maclellan, 2001). In addition, a study conducted by Blöte, Klein and Beishuizen (2000) found that learners did not completely utilise their mathematical information when a paper-and-pen test was conducted. This shows that the pen-and-paper method employs learners' existing knowledge only partially, causing the other information to remain dormant. Partial utilisation of existing knowledge prevents understanding of the relationship between mathematical knowledge and its application, which could lead to a mental computation skills deficiency and a lack of full utilisation of learners' thinking skills.

FMC enhances conceptual understanding, develops interest in learning mathematics, demonstrates the relevance of mathematics in daily life and enables the application of mathematical knowledge developed in school to solve daily life computations. However, the development of calculation strategies that are flexible requires number sense. Number sense and FMC are interrelated. Figure 2.1 serves to illustrate the relation between number sense and FMC development.

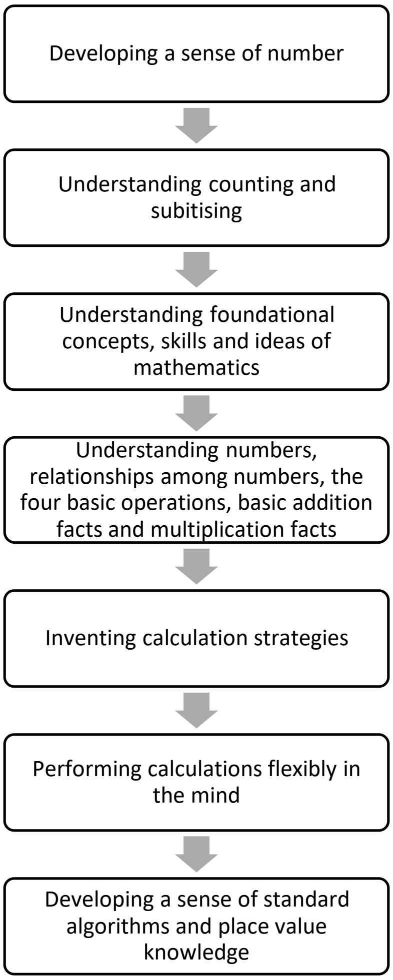

Figure 2.1: Relation between number sense and FMC

Number sense is necessary for the development of FMC, which in turn strengthens number sense. An in-depth discussion of the impact of number sense on the invention of calculation strategies is presented below.

2.5 Foundational knowledge for flexible mental computation skills development

A strong sense of number is fundamental in the learning of mathematics. In mental computation development, number sense is central to the ability to invent strategies to calculate flexibly in the mind. Number sense encompasses five main aspects that lead to the ability to invent one's own calculation strategies. The five key aspects incorporated in the definition of number sense are described by McIntosh et al. (1992, p. 3) as "…a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful strategies for handling numbers and operations". A key concept included in the definition of number sense is 'understanding'. Understanding refers to"… intuitions and ideas about how mathematics works" (Hiebert, 1984, p. 498). Understanding of how mathematics operates is important for the development of FMC skills. Returning to the definition of number sense, this study inferred that understanding should arise from five key aspects. The argument is that a strong development of number sense culminates in understanding how mathematics operates. This calls for a need to understand numbers, to understand operations, to develop the ability and positive attitude to apply knowledge of numbers dynamically, to develop the ability to make mathematical judgements and to develop the ability to devise effective strategies for working with numbers and operations. From the definition of number sense, it is clear that FMC forms part of number sense and strengthens the understanding of standard or written computation (Heirdsfield & Cooper, 2004; Sowder, 1990). It is for this reason that this study found that it was critical to first develop a sense of numbers in a structured way.

2.5.1 Understanding numbers

Efforts to develop number sense must begin with the understanding of numbers. This includes an in-depth comprehension of the magnitude of numbers, the relationships among numbers, different representations of numbers, numeration and the basic number facts (McIntosh et al., 1992; Varol & Farran, 2007). Evidence of the role of number sense in FMC was presented in a study by Markovits and Sowder (1994) in which a learner demonstrated a tendency to employ approaches that were informed by a sense of number to calculate and a persistent use of approaches informed by number sense. The fundamental aspect in understanding the magnitude of numbers and numeration is counting. Counting develops in two forms: rote counting and rational counting. Rote counting and rational counting are the first steps towards understanding numbers. Rote counting develops first to support mastery of the correct counting sequence in the base-10 number system. Rote counting refers to the process of reciting numbers through songs, rhymes, stories and games in the absence of concrete objects. After rote counting, rational counting occurs to foster understanding of quantification and unitising. Rational counting needs to incorporate the use of concrete objects, pictures, symbols, models and number names. Once rational counting forward and backward in ones, twos, threes, fives and tens is mastered, investigation of the relationships among numbers advances the understanding of numbers.

With regard to the relationship between numbers and number facts, comprehension of the relationships among numbers starts with a focus on the spatial relationships of numbers. The literature points out that number relationships and facts are discovered through perceptual and conceptual subitising (Shumway, 2011). This study found that visual understanding of quantities contributed to the instant recall of basic number facts through perceptual and conceptual understanding of quantities, as emphasised by Shumway (2011). Perceptual subitising entails the ability to figure out the total number of elements in a small set without counting the elements one by one whereas conceptual subitising is the ability to perceive and identify smaller quantities and to join smaller quantities to consider them as one entity (Shumway, 2011). Practically, perceptual subitising supports visualization and optical representations of quantities, resulting in quick determination of the total number of items in smaller groups to support conceptual subitising (Shumway, 2011). Conceptual subitising aids automaticity of basic number facts, which also paves the way for derived facts. Perceptual subitising aids the visualisation of the relationships among numbers by learning patterns resulting from grouped objects. Moreover, patterns of grouped objects can advance the ability to compose and decompose numbers, eventually supporting FMC skills (Rechtsteiner-Merz & Rathgeb-Schnierer, 2015). Visualisation facilitates imagination, and imagination is creative thinking that contributes to problem solving. For example, four represented as two pairs of dots supports instant quantification of objects without counting individual dots. It also shows that four is two dots and another two dots or one dot and three dots.

Moreover, through conceptual subitising, two numbers represented with dots on a domino card initiate the composition of numbers. Through conceptual subitising, the ability to add two numbers develops while enhancing part-part-whole-number relationships. It also reflects number combinations that make up another number, leading to the learning of basic number facts. This directly connects to knowledge of breaking down numbers, regrouping numbers and recomposing numbers to calculate flexibly in the mind. Another important aspect is the focus on the one-more, two-more, one-less and two-less relationships of numbers. Freitag (2014) emphasises that fluency with calculations starts with addition and subtraction facts involving one digit. One-digit addition and subtraction begin with counting forward and backwards in ones. Such knowledge is critical in developing sophisticated calculation strategies (Baroody & Wilkins, 1999). Moreover, knowledge of anchors to ten and five is crucial for the development of FMC skills. Understanding how numbers are related to five and ten contributes immensely to the invention of flexible mental calculation strategies. The relationships among numbers guide the selection of calculation methods and the ability to ascertain solutions obtained (Russell, 2000). For example, when calculating 28 – 19, a possible strategy can be to subtract 20 from 28 and add 1 to get 9 while using an empty number line to verify the result (Wright, Martland, & Stafford, 2006). Number relationships linked to number facts promote efficiency and accuracy while fortifying number sense. Finally, in the understanding of numbers, the relationship between numbers and operations is supported by a variety of models.

Different representations of numbers are fundamental to the development of FMC skills. This study found that models played a vital role in enhancing different representations of numbers. Manipulative and representations are useful in creating the opportunity for all learners to develop FMC skills (Ginsburg, Leinwand, Anstrom, Pollock, & Witt, 2005; Harries & Spooner, 2000; Kamii & Joseph, 2004). Manipulatives and representations such as pictures or physical models and mental images are referred to as “tools for thinking about and solving problems” (NCTM, 2000, p. 206). Among other models, dot cards, domino cards, domino blocks, abacuses, number lines, five frames and ten frames are important models that could translate into mental models for FMC development. Mental models enable automaticity with basic number facts in a more meaningful way. For example, the use of models such as dominoes and ten and five frames is important for learners to perceive numbers as groups of objects and to quantify groups of objects without counting objects one by one. The ability to subitise conceptually aids learners to develop a sense of number and computation strategies. Such models eventually become mental models that are highly likely to foster flexibility, fluency, efficiency and accuracy in mental calculations. This study posits that effective development of the aspects fundamental to the development of FMC skills is possible through solving problems linked to daily life activities.

2.5.2 Understanding the four basic operations

The four basic operations are critical in the development of calculation strategies that are flexible. Besides numeration and number facts, competency with whole numbers requires conceptual understanding of operations (Fisher, 2005; Varol & Farran, 2007). This study found that operations should be understood in terms of the practical meaning of each operation and the operation sign used to denote each operation. In my view, as TE, 'understanding' in the context of FMC denotes understanding the connections among the four basic operations: addition, subtraction, multiplication and division. According to Heirdsfield (2002, p. 4), understanding the connection among the four basic operations means understanding the "…interconnected networks of knowledge representations and structures, where access is readily available". Moreover, understanding the four basic operations requires a connection between the symbolic representation of the operations (+, –, × and ÷) and the practical meaning of the concepts addition, subtraction, multiplication and division to comprehend their utility in life. A recent study has indicated that understanding of the four basic operations can prevail through counting sets and manipulating sets (Freitag, 2014). Similarly, a study by McIntosh and Dole (2000) found that learners with good conceptual understanding performed well in mathematics tests and had a better understanding of mental computation and an improved ability to compute mentally.

2.5.3 Developing ability and a positive attitude regarding working with numbers

In addition to understanding the four basic operations, invention of calculation strategies depends on the ability to apply knowledge of numbers dynamically and a positive attitude towards computing mentally. Flexible application of computation strategies requires the use of diverse calculation methods. In this case, a sense of how numbers are made up is important for correct reasoning ability that is necessary for invention of calculation strategies (Fisher, 2005; Kilpatrick et al., 2001). Moreover, a dynamic use of numbers is underpinned by comprehension of number symbols and number size (Sowder, Schapelle, & Lambdin, 1994). Number symbols and number size inform how differently calculations and numbers can be manipulated to construct calculation methods differently. Kamii and Joseph (2004, p. 5) confirm that "children construct logical-mathematical knowledge by putting previously made relationships into new relationships" to work with numbers flexibly. For example, the number 9, understood as 1 + 8 = 9, 5 + 4 = 9, (4 + 4) + 1 = 9, (5 + 5) – 1 = 9, 10 – 1 = 9, 2 + 7 = 9 and 9 = half of 18, can be used to calculate 18 + 9 as 18 + (2 + 7) = (18 + 2) + 7 or as 18 + (10 – 1) = (18 + 10) – 1. Thus, key to the construction of a variety of computational methods is active involvement in computation activities, a critical search for patterns, mathematical comprehension and interest in mathematics (Southwood & Spanneberg, 2000). A positive attitude towards mathematics results from engaging with mathematics that is sensible and interesting. Meanwhile, the ability to work flexibly with numbers requires understanding of numbers and active mental engagement to look for patterns, in terms of compatible numbers, to construct different calculation methods. Once diverse calculation methods are constructed, mathematical judgements are necessary to facilitate the process of selecting calculation strategies that are more efficient.

2.5.4 Developing mathematical judgements

A strong understanding of numbers and operations must culminate in the ability to make mathematical judgements. In the process of calculating flexibly in the mind, the selection of alternative and more efficient strategies is founded on appropriate mathematical judgements of strategies. Appropriate mathematical judgements and computational fluency are underpinned by in-depth understanding of the nature of numbers, the relationships among numbers, operations and the representation of concepts (Kilpatrick et al., 2001). A study conducted by Kazemi (1998) showed that after receiving ample support to engage in conceptual thinking and procedural thinking, learners were able to draw sensible mathematical conclusions. Kazemi's (1998) findings led to the conclusion that for a teacher to promote learners' conceptual thinking, it is necessary to ensure that when learners provide clarifications of calculation procedures, such clarification has to be substantiated with mathematical reasons.

When a wrong answer is provided, a teacher must encourage learners to use such as an opportunity to work further with mathematical concepts and procedures and not to consider wrong answers as an end to their engagement with mathematical ideas. Conceptual analysis of mathematical knowledge and procedures culminates in logical reasoning. A study found that a learner with "... a high level of mental computation can be said to have a high level of mathematical reasoning" (Gurbuz & Erdem, 2016, p. 11). Inability to spot links among mathematical ideas results in underdeveloped mathematical thinking ability. An improved understanding of how numbers work results in the invention and selection of efficient mental computation strategies to solve unfamiliar problems. Effective mathematical judgements lead to the ability to devise effective strategies for working with numbers and operations. The ability to devise effective strategies progresses through three different levels of number sense development, which are discussed in Section 2.7. Now the discussion shifts to a conceptual framework for the development of FMC skills.

2.6 Framework for developing flexible mental computation skills

A focus on whole-number computation calls for the mastery of basic mathematical knowledge underpinning FMC. The literature shows that FMC is influenced by existing knowledge (Threlfall, 2002). For FMC development, existing knowledge refers to the four cognitive aspects fundamental to the invention of calculation skills. These are basic number facts, numeration, the effect of operations on numbers and estimation (Heirdsfield, 2002). These four aspects are incorporated into number sense development and the key aspects are summarised next.

2.6.1 Basic number facts

Knowledge of basic number facts is one of the prerequisites for inventing one's own strategies to calculate proficiently in the mind. Proficiency in mental calculation requires the ability to recall basic addition and multiplication facts instantly (McIntosh, 2004), also referred to as automaticity. Automaticity is the instant recall of basic number facts, which supports the invention of fast strategies to compute mentally. The focus is on the ability to recall basic number facts fast and accurately. Hope and Sherrill (1987, p. 100) confirm that "...basic number facts are the fundamental building blocks of most calculations". Mastery of basic number facts is crucial for the development of mental computation, and its absence leads to mistakes in FMC. To ensure fluency in calculations, Freitag (2014) points to the development of one-digit addition and subtraction facts.

One-digit addition and subtraction facts begin with counting forward and backwards in ones. Heirdsfield and Cooper (2004, p. 52) conducted a study on the causes of inaccuracy in addition and subtraction and found that "poor number facts knowledge, short term recall, understanding of estimation, numeration and effect of operation on number contributed to inaccuracies in mental computation and the inability to access alternative strategies". Similarly, English (2013, p. 15) emphasises that for learners "to use mathematics effectively [they] need to know some facts and be able to recall them instantly." The inability to recall basic number facts when computing mentally hampers the development of sophisticated and efficient calculation strategies by operating on Level 1 using fingers. For example, to compute 38 + 43, a learner should recall basic facts such as 8 + 3 and 30 + 40 and also needs to sense that 38 is only 2 away from 40, leading to 38 + (2 + 41). A lack of automaticity with number facts could result in an inability to invent efficient calculation strategies and to calculate fast and accurately. Heirdsfield and Cooper (2002) found that ineffective calculation methods and a lack of other prerequisites of FMC increased the likelihood of getting the answer wrong.

2.6.2 Numeration

Since FMC involves manoeuvring numbers in the mind, understanding how to breakup numbers differently is important. Numeration includes breaking up numbers in a "canonical and non-canonical" way (Heirdsfield, 2002, p. 10). For example, learners should be able to understand that 25 comprises 2 tens and 5 ones (canonical) and 1 ten and 15 ones (non-canonical). In simpler terms, canonical breakup could be represented as 20 + 5 and non-canonical as 10 + 15. In addition to canonical and non-canonical breakup of numbers, learners need to understand the multiplicative nature of numbers, for instance that 25 is (2 × 10) + (5 × 1). Learners need to view 25 as comprising 2 and 1 in the multiplicative aspect of the base-10 number system, simplified as 25 = 20 + 5. Therefore, tasks must lead to discovery of the relationships among numbers and the properties of numbers such as the associative property, commutative property and distributive property. The knowledge base should include comprehension of operations and how they are related, for example that multiplication is the inverse of division; acquaintance with many relationships among numbers, inclusive of addition and multiplication facts, as well as relationships among calculations, such as 3 × 4 and 3 × 40; and a complete grasp of the base-10 number system and the order of numbers in the base-10 number system. Thus, knowledge of breaking up and building up of numbers is pertinent in the development of fast, accurate and efficient calculation strategies, followed by understanding of the effect of operations on numbers.

2.6.3 Effect of operations on numbers

Further development of efficient, accurate and flexible computation strategies relies on knowledge of the effect of operations on numbers, particularly when adding and subtracting. Learners need to understand the effect of altering the "... addend and the subtrahend..." (Heirdsfield, 2002, p. 11). Knowledge of the effect of varying the addend or subtrahend on numbers could be developed through tasks including computations such as 460 + 70 = 530, leading to the realisation that 460 + 69 = 530 – 1 and that if 130 – 50 = 80, then 130 – 49 = 80 + 1. Also critical to FMC skills development is knowledge of the effect of operations on the place value of a number, for instance 35 + 10 = 45 but 35 × 10 = 350. Therefore, tasks must be designed with the intention to support the discovery of the effect of operations on numbers.

2.6.4 Estimation

Estimation plays an important role in FMC. Estimation is the ability to provide an answer that approximates the accurate answer through a sensible guess. Although the literature provides evidence of the role of mental calculation in estimation (Reys, 1984), discussion of how estimation contributes to FMC is limited. In the limited literature, it is emphasised that the development of FMC skills requires a sense of the size of numbers (Heirdsfield, 2002). The absence of number sense leads to an inability to sense the reasonableness of an answer and inaccuracy of solutions, as evident in Heirdsfield's (2002) study. Similarly, Grade 2 learners who obtained inaccurate answers in Kamii and Joseph's (2004) study demonstrated a lack of number sense and estimation skills. For example, for an algorithm like 7 + 52 + 186, some participants provided answers like 29 or 30. Such answers indicate a lack of a sense of size of numbers that would have directed learners to expect an answer beyond 200 but less than 300 or 250.

Based on number sense knowledge, this study connects numeration to estimation to outline how estimation contributes to accuracy of answers when computing mentally. This study posits that knowledge of the proximity of numbers supports the ability to sense how close a number is to another. For instance, when computing an algorithm like 46 + 18, one would sense the magnitude of 46 as being 4 closer to 50 and 18 as being 2 closer to 20. In finding the sum of the two numbers, a sensible guess would confirm that the solution of 46 + 18 cannot be more than 70 due to awareness of the closeness of numbers to other numbers. Therefore, through estimation, an accurate answer can be obtained.

2.7 Problem solving and flexible mental computation

2.7.1 Developing flexible mental computation through problem solving

This study asserts that problem solving is one of the effective approaches to developing mathematical understanding. Particularly, in the development of FMC, a problem- solving approach serves to provide a context that links computation to real-life mathematics (Heirdsfield, 2011). In the literature, an issue is raised that “many children do not connect the mathematical concepts and skills they possess with the symbols and rules they are taught in school. ...it is the absence of these connections that induces the shift from intuitive and meaningful problem solving approaches to mechanical and meaningless ones” (Hiebert, 1984, p. 498). Literature provides empirical evidence where difficulty to connect daily life calculations and classroom mathematics has manifested in children (Lovitt and Clarke (1992). Generally, a lack of connection could be attributed to the absence of problem solving to link the mathematics used outside school and the standard algorithms used in school.

The issue of a lack of connection between classroom mathematics and real-life mathematics can be addressed through a problem solving teaching approach. Problem solving refers to engaging learners in a task for which they have no awareness of a particular strategy that could lead to the answer, nor hints or memorised rules or approaches to solving the problem (Hiebert et al., 1997). Problem solving promotes the use of existing intuitive knowledge and willingness to imagine things, to explore further, to try new ideas without fear of making mistakes and to participate confidently in class discussions (Fisher, 2005; Polya, 1945). As opposed to prescribed rules and strategies, intuition plays a role in being creative, which enhances the potential to make good inferences using the least information available. Singapore and Taiwan are among the top-performing countries in mathematics globally, both with regard to learners and PSTs. Their education systems use problem solving as a teaching approach, and it has been proven successful (Ginsburg et al., 2005; Lin & Li, 2009). A problem solving teaching approach is also upheld by the NCTM (2000, p. 116), which regards problem solving as a “… hallmark of mathematical activity and a major means of developing mathematical knowledge…” Though problem solving can be effective in enhancing understanding, its meaning and appropriate implementation is crucial.

Problem solving entails solving problems that have no prescribed solutions or strategies that are readily available. It can be argued that through problem solving a chance to build on prior knowledge is created when unfamiliar problems are solved. Problem solving should thus serve to develop new knowledge instead of simply applying prescribed rules. Another benefit is that problem solving creates an opportunity for invention of flexible calculation methods through active mental engagement. This is because a problem solving approach is handy in providing activities that involve skills such as "interpreting, identifying, explaining, questioning, justifying, reflecting, evaluating, reasoning, logic and proof" (Southwood & Spanneberg, 2000, p. 1). The main aim of engaging learners mentally is to develop their mathematical thinking skills and their ability to identify and extract important information necessary to solve a problem. When a solution is found, reasoning skills develop as calculation strategies are discussed and clarified to justify the effectiveness of invented strategies. Problem solving is instrumental not only in the construction of calculation strategies that are flexible but also in outlining the relevance of classroom calculations to daily life situations.

Connection of mathematics to daily life makes learning interesting and enjoyable and improves confidence and reliance on learners' own ability to calculate mentally. It also demonstrates the meaningfulness, practicality and value of FMC in real life. Such a view manifests through an understanding of numbers and the ability to calculate using diverse procedures. This results in a positive mind-set towards the relevance and learning of mathematics. Furthermore, problem solving cultivates a positive attitude and the confidence to persist in calculating when a problem is found challenging. A positive attitude develops a belief that effective calculation strategies can be constructed through constant hard work (Kilpatrick et al., 2001). Persistence in challenging problems requires a positive attitude towards computations, which emanates from problem solving.

Heirdsfield (2011) recommends that FMC instruction should focus on invention of personal calculation methods by means of discovery, collaboration, discourse and reasoning to explain thoughts and answers. Invention of calculation strategies requires creative thinking to generate innovative views that would lead to derived facts, new ways of calculating and innovative mental representations (Fisher, 2005; Kazemi, 1998). Kazemi (1998) asserts that to derive mathematical constructs sensibly and to support creativity, mathematical concepts, experiences, procedures and problems need to be linked. Links should be created among mathematical operations, numbers, prior knowledge, strategies, problem types and daily life mathematics. Problems need to be designed in a way that evokes thinking, invention of different strategies and discussion of constructed methods (Fisher, 2005). Connecting to prior knowledge facilitates understanding and construction of solutions without prescribed rules and strategies. In addition, prior knowledge supports the way in which FMC is perceived in the classroom. To illuminate instruction, the principles of the problem solving teaching process are discussed below.

2.7.2 Guiding principles for a problem solving approach

The effectiveness of developing FMC depends on proper orientation based on research-informed ideas. Kamii and Joseph (2004) provide a guide in the form of principles of teaching. The first principle encourages teachers to introduce real-world problems first, unlike providing context-free algorithms. The second principle discourages teachers from providing learners with examples of strategies to solve problems for learners to reproduce and solve similar exercises. The idea is that learners must be given the opportunity to solve a problem by themselves without any examples. The third principle maintains that the teacher must avoid announcing whether an answer is right or wrong but must rather promote discussion of different strategies and answers for learners to indicate agreement or disagreement.

The fourth principle affirms that after a learner has carried out a calculation, the teacher must persuade the learner to think of other ways to solve the same problem. The fifth principle specifies that the teacher must urge learners to think more than write. Different strategies provided by learners should be written on the blackboard, either by the teacher or the learners, to manage sharing of strategies and for learners to understand place value. As a TE, my view is that the principles of teaching apply to both learners' and PSTs' knowledge development.

2.7.3 Problem solving steps

In problem solving, mathematical thinking is critical to construct calculation strategies that are fast, accurate and fluent. Scusa and Yuma (2008, p. 22) summarise the process of mathematical thinking as involving five key aspects, namely "connections, representation, communication, reasoning and proof, and problem solving". To solve a problem effectively, four different steps should be followed through logical, critical and creative thinking. Most importantly, in a problem solving-oriented learning environment, an explicit outline of the relevant steps involved in problem solving is decisive (Fan & Zhu, 2007). Specifically, the steps involved in the problem solving process may guide the construction of calculation strategies chronologically through "understanding a problem", "devising a plan", "carrying out a plan" and "looking back" (Polya, 1945, p. 23). This study incorporated Scusa and Yuma's (2008) four processes of mathematical thinking under problem solving (Polya, 1945) to outline how mathematical thinking manifested in problem solving.

Firstly, before solving a problem, it is imperative to analyse and discuss the problem to understand it. Understanding a problem requires reading a problem, looking for patterns in numbers used in a problem, thinking of connections between existing knowledge and new knowledge, and finding connections between numbers, representations, operations (reversibility) and reasoning (reversibility) (Hiebert, 1984; Kazemi, 1998). In this case, understanding refers to grasping the real meaning of mathematical concepts to aid invention of one's own computation strategies. Research findings indicate that the ability to recognise patterns among numbers and knowledge of number relationships facilitate calculations (Rechtsteiner-Merz & Rathgeb-Schnierer, 2015). Learning mathematics should include not only calculations but also investigations to discover patterns, proving assumptions and guessing outcomes sensibly (National Research Council [NRC], 1989; Colyvan, 2012). For example, a daily life experience in which three people are sharing a chocolate bar equally among themselves at home connects prior knowledge to classroom mathematics. Another example is the ability to understand that five bags of two oranges and two bags of five oranges comprise the same number of oranges but are different in terms of the number of bags, symbolically meaning that $2 \times 5 = 5 \times 2$.

For learners to reason logically, they need first to think logically about concepts and procedures to formulate sensible and well-informed arguments. Once a problem is understood, spotted connections between concepts and procedures inform the choice of a method to solve the problem (Fisher, 2005). Therefore, when approaching a task, it is essential to think about the information that is available and relevant to the task at hand. Such information contributes to the management and accomplishment of the activity. Before solving a problem, learners need to think of appropriate procedures to solve a problem by considering what they want to achieve at the end of the task. When thinking about concepts and procedures, the focus needs to be on identifying patterns and associations among mathematical ideas (Fisher, 2005). For example, learners should think of number combinations, models and representations that can facilitate calculation of the problem.

Secondly, after understanding a problem, a plan must be devised. Devising a plan involves representing the identified information mathematically using diagrams, numbers or pictures. Planning also involves suggesting a correct way to represent and solve a problem mathematically by using symbols and mental models such as number lines, five frames, ten frames and basic facts. Reasoning about number combination is necessary to compose and decompose numbers to invent calculation strategies and to think of alternative strategies. With regard to calculating mentally, available information needs to be visualised in the form of mental models to solve the problem. Thinking should focus on ways to use the identified patterns and relationships between numbers and operations to solve the problem. Reasoning is critical to suggest a possible strategy to solve the problem. To reason and plan effectively, Fisher (2005, p. 20) suggests the following questions to think of: “Have I thought it through?”, “Have I made a plan, do I know what to do, is there anything more I need?” and “What do I know that will help me?” Answering such questions can serve as a directive towards an attempt to solve the problem.

Thirdly, the identified information should be used to solve the problem as planned. The step of carrying out a plan requires trying out an identified strategy to solve the problem by using available information. Such a step requires communication to oneself and the group through reflection to analyse both the calculation process and the solutions by monitoring one’s own thinking steps. Monitoring and regulation of one’s own thoughts involve thinking that is reflective in order not only to think but also to act upon thoughts to generate all possible solutions to a particular problem (Flavell, 1979). Reflection on individual work encourages the ability to be conscious of the calculations, which then allows the detection of learners' own and others' mistakes in the calculation strategy and solution. Besides, thinking about strategies and solutions helps in ascertaining the reasonableness of one's solutions and in verifying solution strategies. In addition, monitoring through reflection enables checking progress and cognitive approach to problems and ensuring that a cognitive goal is achieved. To take control of individual thinking, self-interrogation must occur by verbally reminding oneself of the steps to be carried out and doing it repeatedly quietly.

Finally, a number of questions must be answered to verify the accuracy of answers and the efficiency of calculation procedures. The step of looking back enables an analysis of one's calculation procedures and answers for identification of one's mistakes and to correct them. A set of questions provided beforehand could serve as a guide for looking back at solutions and procedures. The questions are, "What did you do to get your answer?", "How did you figure out the answer?", "How do you know your answer is correct?" and "Is there another way to solve the same problem? The step of looking back is essential to understand and justify why an answer is correct and to think of alternative strategies that could be more efficient. Therefore, discussion is necessary to enable reflection on calculation procedures and solutions.

2.7.4 Classroom discourse

Once learners have developed their strategies, teachers must encourage learners to reason and prove their solutions, to communicate their ideas to the teacher and the entire class, to make connections among mathematical ideas and to represent mathematical ideas in various ways (NCTM, 2000). Piaget (1973) stated that all knowledge arises from interaction between learners and the learning environment. If learners have limited opportunity to communicate ideas, their mathematical thinking and learning are inhibited. Reasoning aids the observation of relationships among "facts, procedures, concepts and solution methods" (Kilpatrick et al., 2001, p. 129). Both teachers and learners benefit from communication as it is the way through which others' ideas are made known and subjected to analysis and verification.

Through talking, learners benefit by learning how to communicate their ideas to others. Therefore, to promote talking, teachers need to invite learners to explain their solutions to other learners and to use questions that call for interrogation of ideas. Studies indicate that allowing learners to discuss their work before, during and after completing a task fosters focusing of their thoughts on a task. Listening is crucial in the process of communication as it is the only way in which spoken ideas can be heard. Listening is essential to understanding other learners' problem solving strategies and for the presenter to receive corrective feedback. In the process of developing FMC skills, it is important to develop a habit of listening patiently to others to promote ethical behaviour and tolerance in the learning environment. Efforts to assist learners to talk and listen depend on the provision of ample time for learners to think conceptually and to be patient in waiting for an answer.

The findings from a study by Kazemi (1998) reveal that a rigorous promotion of conceptual thinking advances attainment of the ability to solve problems and understand concepts. The study confirms that conceptual thinking is achieved through the creation of "an intellectual climate characterized by argument and justifications" (Kazemi, 1998, p. 414). Conceptual thinking also develops through the opportunity to learn concepts in groups through play and discussion of ideas.

An attitude of cooperation and respect for other learners' ideas also develops. A teacher needs to ensure that a wrong answer does not cause embarrassment for the learner but rather serves as an opportunity to learn further. A teacher needs to model and promote careful judgement through positive feedback in the form of collective analysis of responses. Positive feedback plays a major role in promoting thinking through communication. Interpersonal skills are vital for learners to be considerate of others' emotions. This is necessary in the process of making comments while evaluating other learners' solutions for the refinement of ideas that need to be stated in a better way. Learners need to be willing to work collaboratively and relate to others and the teacher respectfully. Therefore, teachers need to ask appropriate questions and design appropriate tasks that are well organised and can be solved in stages to draw learners' attention and direct their thinking to become logical thinkers.

The process of communicating to share and justify calculation strategies involves the use of language. Therefore, during classroom discourse, the use of appropriate mathematical concepts needs to be encouraged. Verbal expression is necessary to 'think aloud' and make thinking explicit. It is important to read, write and talk about solution procedures, to clarify these in smaller groups and to the entire class, and to listen to other explanations (Fisher, 2005). However, a classroom environment conducive to discussion of strategies with the teacher and among learners is required (Kamii & Joseph, 2004). With regard to writing, when calculating, learners need to present their work in written form. Recording of computation methods is a good way for learners to encapsulate their own thought processes (Duncan, 1996; Hartnett, 2007). In addition, writing encourages learners to arrange and merge the thoughts that surfaced while employing the computational procedure (Pugalee, 2001). Since thinking is an internal process, recording of ideas can aid the representation of thinking processes that cannot be observed. This is why this study found the problem solving approach practical in making the development of FMC skills easy, meaningful, and interesting and enjoyable. The study inferred that through a problem solving approach, learners could invent their own computation strategies easily and meaningfully depending on the types of problems available.

2.7.5 Types of problems

It is critical to have problems properly designed to stimulate thinking and discussion of invented solutions. As discussed in Section 2.6.1, problems must be formulated in a way that stimulates the construction of diverse strategies. Problems should link operations, numbers, daily life experience, prior knowledge, strategies and different problems. A careful selection of problem types and number combinations creates the opportunity to understand and solve addition, subtraction, multiplication and division problems through identification of strategies that are accurate, fast and simple. Developing FMC skills through problem solving requires careful selection of numbers. Different combinations of numbers are necessary to stimulate specific calculation strategies. A variety of problem types is important for understanding the relationships among addition, subtraction, multiplication and division. For example, when 8 friends share 24 sweets equally, either addition, for example 8 + 8 + 8 = 24 (adding up), subtraction, for example 24 – 8 = 16 and 16 – 8 = 8 and 8 – 8 = 0 (taking away equal groups) or multiplication, for example 8 × □ = 24 (reversibility rule) can be used to solve the problem. This advocates for the use of a variety of mathematical terminology and contexts for learners to be exposed to many mathematical concepts and situations. This study drew on Carpenter, Fennema and Franke (1996) to outline 14 different types of problems categorised into five main groups, namely joining problems, separating problems, part-part-whole problems, compare problems, and multiplication and division problems.

Joining problems are subdivided into three types as join: result unknown, join: change unknown and join: start unknown. Separating problems are subdivided into three types as separate: result unknown, separate: change unknown and separate: start unknown. Part-part-whole problems are subdivided into two types as part-part-whole: whole unknown and part-part-whole: part unknown. Compare problems are subdivided into three types as compare: difference unknown, compare: quantity unknown and compare: referent unknown. Multiplication and division problems are subdivided into three types as multiplication, measurement division and partitive division. The problem types outlined by Carpenter et al. (1996) illuminated the choice of types of problems and questions appropriate for the development of FMC skills in the current study.

2.8 Levels of number sense development

When computing mentally, the emphasis should be on achieving proficiency in calculations. Proficiency entails "both flexibility and accuracy" (Heirdsfield, 2002, p. 1), as incorporated in computational fluency (NCTM, 2000). Computational fluency implies accuracy and flexibility in mental computation, which requires automatic recall of number facts. Automatic recall of number facts results in invention of strategies that are fast, precise and proficient (Russell, 2000). A study conducted by Heirdsfield (2003b, p. 59) indicates that "students who were fast and accurate, and solved number facts by recall or derived facts strategies tended to be more proficient mental computers (accurate and used a variety of efficient mental strategies)". In addition, Kilpatrick et al. (2001, p. 129) outline that computational fluency requires "knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately and efficiently". The concepts flexibility, accuracy and efficiently are fundamental in computational fluency. According to the literature, flexibility refers to the ability to solve a problem by using a variety of calculation methods.

It is through flexibility that selection of more efficient strategies can prevail. Knowledge of diverse calculation methods enables the use of one method to find a solution and the use of another method to prove the accuracy of the solution found. Accuracy involves thorough writing of procedures, awareness of fundamental number facts and how numbers are related, and verification of the solution.

With regard to efficiency, reference is made to performing a calculation without wasting time and without becoming confused during the calculation process while observing the calculation steps of the approach employed. An efficient approach is one that is easy to execute with a possibility to detect other issues that form part of the problem and to utilise emerging hints to find a solution. Effective development of FMC skills must yield flexibility, accuracy and efficiency as competencies of mental calculations. However, achievement of mental computation strategies that are flexible, accurate and efficient requires a "...well-organized knowledge of concepts, principles and procedures of inquiry" (Bransford, Brown, & Cocking, 1999, p. 239). The focus is now shifted to the levels of computation that this study found to provide a well-organised structure for developing FMC skills.

2.8.1 Level 1: Counting all

The first step in learning to calculate involves counting concrete objects. Level 1 is a process whereby calculations are carried out by recreating both numbers to add. For example, to calculate 4 + 5, four objects are counted out separately and five objects are counted out separately and are then combined so that all objects are recounted to work out the total. This process is also referred to as direct modelling of a problem using concrete objects or pictures, as evident in Carpenter and Moser (1984). To shift from Level 1 to Level 2, a number line can be used to generate the discovery that the sum of four and one more is the number after four and that four plus five would be four and five ones beyond four (Baroody & Wilkins, 1999). Such discoveries culminate in a discovery of sophisticated strategies. The idea is to indicate the starting point on a number line, then adding the second number by counting on in jumps of one. With regard to counting, Griffin (2003) suggests key issues that learners must understand when counting.

The process of counting needs to develop knowledge of the order of numbers from one to ten and comprehension that every number answers the question of how many items there are in a collection of items. Furthermore, it is necessary for learners to understand that when counting up, a collection of items increases by one and becomes one more than the previous collection of items (Griffin, 2003). As a result, fluency in counting will unfold in stages, starting with fluency in the numbers one to six first then seven to ten through number talks and games. Learners should begin solving simple problems and gradually shift to more complex ones, for example beginning with one-digit number calculations before two-digit number calculations (Kamii & Joseph, 2004). But counting needs to first focus on the two fundamental operation systems of addition and multiplication (Carpenter & Moser, 1984). The development of addition skills must begin with activities involving addition of numbers that add up to

10 and double below 10. Priority should be given to such activities to enable learners to derive facts and to solve problems involving subtraction by using existing addition facts, and division by using multiplication facts, as confirmed in a study by Carpenter and Moser (1984). As for multiplication and division, learners need to add several copies of the same number or count sets having the same number of objects to develop learners' multiplicative reasoning. For division, it is necessary to provide activities that involve distribution of objects into sets with the same number of objects in each set and counting to confirm the number of objects per group (Thompson, 2008). Such activities facilitate the process of advancing to Level 2.

2.8.2 Level 2: Counting on

Drawing from the limited literature, level 2 strategies denote methods whereby one number is kept in mind and not recreated and the other number is recreated. A solution is found by counting on from the number kept in mind. Research shows that models such as the five frames and the ten frames can be useful in creating more efficient calculation strategies (Heirdsfield, 2003a). Research also shows that knowledge of doubles and numbers that make up 10 results in derived strategies of near doubles and compatible numbers (Parrish, 2010; Shumway, 2011). In life, learners first learn to count before they are able to perform calculations mentally. So, knowledge of counting on needs to be linked to addition whereas knowledge of counting back needs to be linked to subtraction to enhance understanding through connection of new and existing knowledge.

2.8.3 Level 3: Breaking down and building up numbers

McIntosh et al. (1992) refer to breaking down and building-up numbers as decomposition and recomposition of numbers. Knowledge about how numbers are composed and decomposed supports learners in solving problems flexibly and efficiently (Parrish, 2010). Level 3 is an advanced level at which numbers are thought of in different ways. This is a strategy anchored in knowledge of decomposing and recomposing numbers. For example, to calculate 16 + 7, 16 can be broken down into 10 + 6 + 7 and further into 10 + 3 + 3 + 7 = 10 + 3 + 10 to find a solution of 23. In the classroom, the focus should not be on providing extra facts for learners to memorise but should be on assisting learners to create links among the bits of knowledge already developed (Kamii & Joseph, 2004). It is the links between existing knowledge and new knowledge that enhance understanding and invention of computation strategies (Kamii & Joseph, 2004). In this case, existing knowledge would refer to knowledge of basic facts developed through perceptual and conceptual subitising, as discussed earlier in Section 2.4.1. Thus, knowledge of breaking-down and building-up numbers is pertinent in the development of fast, accurate and efficient calculation strategies. However, the literature presents opposing ideas on the development of mental calculations that are flexible, as discussed next.

2.9 Contradictions in the literature regarding the development of flexible mental computation

Diverse ideas on the development of computation strategies exist whereby some authors call for direct teaching of FMC while others call for invention of calculation strategies. Some researchers believe that all learners can develop FMC skills provided that they receive the necessary support (Threlfall, 2002). In contrast, Klein and Beishuizen (1998) suggest that learners who are unable to construct their own FMC strategies to solve problems must receive direct instruction to assist them with the construction of FMC strategies. In approval of direct teaching, the NCTM (2000) argues that conveying meaning directly can be very successful in some cases, especially after learners have struggled with a problem independently. In contrast, Threlfall (2002) argues that all learners can develop FMC skills if they are provided with the necessary support. There is agreement in the literature that teaching mental computation strategies openly poses a risk because learners could gain knowledge of such procedures in a traditional way without understanding (Carpenter et al., 1998). Procedures learned in a traditional way could lead to many mistakes and a limited understanding of the number system (Sowder, 1990). In agreement with Threlfall (2002), Hiebert (1984) asserts that it is possible for learners to develop mathematical understanding on their own, and they really do. For instance, a learner giving away two sweets from six sweets that her mother gave her will easily associate giving away with subtraction without anybody's support. A connection to the minus sign should be created for a learner to understand the symbolic representation of the concept subtraction. However, a clear understanding of concepts is required for independent problem solving using invented strategies (Hiebert, 1984; Kilpatrick et al., 2001). With knowledge of concepts, learners could likely construct their own calculation strategies.

Drawing on Hiebert's (1984) discussion, understanding of concepts can appear in two different ways. One way refers to a situation where a learner understands procedures and rules as a result of construction and invention of her/his own rules that are easy to execute and recall. The second way derives from memorising prescribed rules that are easy to forget and difficult to transfer to new situations. This entails that procedures learned instrumentally can be applied to solve certain calculations but fade away if prescribed by a teacher without learner involvement in the construction of such rules. In most cases, rote-memorised information without understanding is only recalled until a test is written and is thereafter forgotten (Brophy, 2010). Therefore, ideas drawn from the literature shaped this study's argument that FMC skills could develop without direct instruction in strategies, provided that appropriate tasks are given and that there is understanding of concepts.

The literature provides evidence that the types of tasks that learners engage in and understanding of the concepts involved in FMC promote the construction of learners' own calculation strategies (Hiebert et al., 1997). Evidence from a study by Rittle-Johnson and Alibali (1999) indicates that learners can benefit from instruction that supports them in inventing their own computation strategies by teachers' focusing on cultivating understanding of the concepts underpinning FMC rather than on direct teaching of computation procedures. The outcomes of the study indicate that instruction that focuses on mathematical concepts increases learners' conceptual understanding and ability to develop correct flexible calculation strategies. Moreover, conceptual instruction enables learners to develop calculation procedures that could be used to solve new tasks. Although direct teaching of calculation strategies yielded reasonable comprehension and calculation competency, the procedures that were taught directly were only used to solve problems that were almost the same as the problems demonstrated in the classroom. Learners who received direct instruction in procedures had difficulty applying the learned procedures to solve problems that were slightly different from the problems used during instruction (Rittle-Johnson & Alibali, 1999). The overall findings indicate that when children are empowered with fundamental concepts, they are able to solve novel problems on their own. Therefore, to facilitate the process of developing FMC skills, it is important to focus on the understanding of fundamental concepts and to support learners to notice number relations and friendly numbers through counting and other approaches.

Instead of teaching mental computation approaches directly, Threlfall (2002) suggests a focus on the types of tasks in which learners should engage. The argument is that the construction of learners' own computation strategies and their choice of the most effective strategies from diverse approaches are underpinned by the nature of the numbers in a problem. Most importantly, teachers need to use a proper combination of numbers to promote the development of diverse calculation strategies rather than prescribe strategies to learners. Threlfall (2002, p. 45.) emphasises that "flexibility cannot be taught as 'a process skill'... [It] arises consequentially through the emphasis on considering possibilities for numbers rather than by focusing on holistic 'strategies'". Again, Threlfall (2002) argues against direct teaching of calculation procedures by advocating for instruction that is directed by a careful selection of numbers when designing tasks.

As suggested by Threlfall (2002), this study endorses the idea that tasks need to engage learners mentally to think and spot the features of numbers and the proximity of numbers to other numbers. Besides, the nature of tasks needs to create the opportunity to decompose numbers or change numbers to facilitate the calculation process. For example, to subtract 16 from 53, a learner can use knowledge of the proximity of a number to a friendly number to do the following: $50 – 13 = 37$ or $57 – 20 = 37$ or $(53 – 20) + 4$. As to the features of numbers, knowledge of the nine-time table could facilitate the calculation of 18×9 by doubling 9×9, resulting in $81 + 81 = 162$ whereas a learner lacking nine-time table knowledge is unlikely to choose the doubling strategy. As a result, the construction of a procedure depends on what the learner is able to detect about the numbers in the problem in relation to what the learner knows and what calculation method he/she finds easy to use. Furthermore, how the learner executes the problem also depends on aspects recognised by the learner, which is informed by the degree of foundation knowledge. Therefore, the invention and selection of the most efficient calculation approaches depends on the numbers involved in a problem and on prior knowledge pertaining to numbers.

The literature also provides instructional suggestions on what to focus on when developing FMC skills. Instead of focusing teaching on direct instruction in alternative approaches or selection of what method to apply, the focus should be on developing “analytic strategies” rather than “solution strategies” (Threlfall, 2002, p. 2). Threlfall discusses the notion of analytic strategies versus solution strategies by emphasising that it is important to develop the knowledge of how to start solving a problem, to imagine and reason about numerals, and to associate numerals with a range of information. For example, when computing 29 + 27, one needs knowledge to separate 29 not only into 20 + 9 but also into 10 + 19 and 27 not only into 20 + 7 but also into 10 + 17, also realising that 29 + 27is close to twice the number 30 or more than twice 25. Heirdsfield (2003a) refers to this analytic knowledge as canonical and non-canonical understanding of numbers. The current study was informed by the idea that analytic strategies are not options to select from but are approaches that could be applied purposefully to solve a problem. PSTs involved in this study engaged in the development of analytic strategies to enhance their calculation skills.

2.10 Pre-service teachers and mathematics teacher education

2.10.1 Mathematics teacher education

Mathematics teacher education is a process through which mathematics teachers develop knowledge, values and skills for teaching mathematics. It is a level where teacher development begins after high school education to progress further when teaching in school. Discussion around mathematics teacher education matters because the quality of teacher education impacts teacher preparation, teacher competency and learning in school. The literature calls for universities to alter the way in which mathematics is taught in school through teacher preparation (Ma, 2010). Teacher preparation must play a major role in changing PSTs’ beliefs that are highly informed by traditional views of teaching.

Efforts to foster a constructivist view of teaching are important as the constructivist view has been found to be more effective than a traditional or mechanistic approach to learning (Battista, 1999). However, the current state of many teacher preparation courses still promotes the traditional view of mathematics that is more inclined to rote acquisition of procedural knowledge (Battista, 1999). Evidence from a study by Goulding (2003) indicates that changing teachers’ perception of mathematics and mathematical knowledge requires adequate time and in-depth discussion of mathematics content knowledge. Various studies indicate that many TEs are the product of an education system that promotes a traditional view of learning mathematics (Battista, 1999; Freudenthal, 2002). This prompts the development of a guide to illustrate ways to develop PSTs’ skills to calculate mentally from a constructivist perspective. Quality teacher education is imperative as teachers’ mathematical knowledge is directly linked to learner achievement (Kilpatrick et al., 2001; Ministry of Education and Culture, Namibia, 1993). Therefore, efforts are necessary to improve PST knowledge for increased learner performance.

A good teacher education programme culminates in good teacher preparation whereas a poor teacher education programme leads to inadequate teacher preparation. The effect of the quality of teacher education on learner performance is evident in two international studies: the Teacher Education and Development Study in Mathematics (TEDS-M) (Blomeke, Suhl, & Kaiser, 2011) and the Trends in International Mathematics and Science Study (TIMSS) (Mullis, Martin, Foy, & Arora, 2012). These studies were both relevant in the context of the current study although the TEDS-M study focused on final-year PSTs' mathematics content knowledge and mathematics pedagogic content knowledge whereas the TIMSS study focused on Grade 4 and Grade 8 learners' performance. The findings are relevant as they provide evidence of the relation between teacher preparation and learner achievement in school.

The outcomes of the TEDS-M indicate that among all the countries that participated in the study, primary school PSTs from Singapore and Taiwan achieved higher test results in mathematics content knowledge and mathematics pedagogical content knowledge than the future teachers from other countries (Blomeke et al., 2011). The impact of the quality of teacher preparation on learners is evident in the findings of the TIMSS in which Grade 4 learners from Singapore outperformed learners from other countries as they came out as top performers, followed by learners from Korea. Taiwan seems not to have participated in the TIMSS. In addition, the outcomes of the TIMSS indicate that mathematics performance was higher in learners whose mathematics teachers had a "… primary education major but not mathematics major…" (Mullis et al., 2012, p. 283). Furthermore, the study indicates that mathematics average performance was the lowest in learners whose teachers had "a major in mathematics but not in primary education" (Mullis et al., 2012, p. 283). Such findings demonstrate the need for PSTs to have in-depth subject content knowledge of elementary school teaching and knowledge of how to teach elementary school mathematics.

The TEDS-M was also relevant to the current study as it directed attention to the need for the PST curriculum to change its focus and approach to mathematics teacher education. TEs and other stakeholders need to consider the content of the mathematics teacher curriculum, the consistency of the way in which the content is delivered and the nature of the tasks that PSTs are engaged in. TEs play a major role in teacher preparation because their approach to the curriculum and PSTs impacts teaching and the chances of PSTs to learn effectively (Tatto, Schwille, Senk, Ingvarson, Peck, & Rowley, 2008). Engaging PSTs in solving problems encountered in elementary school is likely to develop their ability to determine the quality and level of difficulty of a task, to provide feedback and to understand and apply the problem solving teaching approach.

In Namibia, teacher education has been found to produce teachers who are inadequately prepared for teaching. Evidently, a study by Kasanda (2005) found that the content knowledge for PSTs' did not prepare them thoroughly to teach mathematics. The findings of content deficiencies confirm Copley's (2004, p. 2) inference that "a major deterrent to the implementation of a strong mathematics program for children in preschool to third grade is the inadequate preparation of teachers." TEs must reconsider the kind of mathematics content offered in teacher education courses to improve the quality of teacher preparation. Concerns were found among elementary school teachers about the kind and degree of mathematics content that teacher education courses offered (Hart, Oesterle & Swars, 2013). The participants described the content as being on a very advanced level and irrelevant to teaching elementary school learners (Hart et al., 2013). The misalignment of the content offered in teacher education courses and the content necessary for teaching may lead to omission of knowledge imperative for effective teaching. Copley (2004) associates inadequacies in PST knowledge to absence of awareness of important mathematical procedures such as solving problems, how learners reason mathematically, how concepts in mathematics are linked to learners' real-life situation and the form of discussion crucial for a learner's comprehension of mathematical concepts.

Inadequacies in PST knowledge urge the improvement of teacher preparation with an appeal to refocus teacher preparation. Gresham's (2007) and Cardetti and Truxaw's (2014) studies provide evidence that teacher education programmes can improve PSTs' subject matter knowledge. In contrast, Klein (2008, p. 322) doubts that teacher education programmes can entirely improve PSTs' subject matter knowledge, stressing that "certainly, it is unlikely that the program of study at university could successfully overwrite already constituted discursive alienation, even though they may construct some new teaching and mathematical ideas". The current study agrees with the view that teacher development is progressive in that it begins with teacher education courses in tertiary institutions and continues to develop during the teaching process in the classroom (Ma, 2010). It is necessary for TEs to understand that a teacher education programme is the first stage in the professional development of a teacher. Such an understanding is necessary to prevent the temptation of overloading a curriculum with too many topics to be covered within a short time as a long curriculum could compromise quality for the sake of quantity (Altuk, 2010). This urges mathematics education courses to be realistic so that careful selection of content for elementary school teachers is made to include content that is minimal but crucial for elementary school learners.

The teacher education curriculum has to focus on important content knowledge with intentions to rectify misunderstandings concerning mathematics content in the school curriculum. Kind's (2014) study, though conducted on science teachers, indicates that upholding excessive information in a subject of expertise is professionally uplifting while providing teachers with a quality knowledge base and inspiration from advanced knowledge. Once misconceptions are rectified, teachers will have a correct understanding of mathematical concepts, resulting in effective teaching. Effective mathematics teaching has the potential to end the vicious cycle of functional innumeracy among teachers and learners, particularly in schools in Namibia. Ma (2010, p. 149) emphasises that "... in the vicious circle formed by low-quality mathematics education and low-quality teacher knowledge of school mathematics – a third party – teacher preparation may serve as the force to break the circle". Therefore, enrichment of the teacher education curriculum with the important content knowledge is necessary.

Arguing from Kessel (2009) and Ma's (2010) perspective, effective mathematics teacher education develops a profound understanding of the fundamental mathematics that PSTs are going to teach. Opportunities to develop knowledge for teaching are suggested by Gresham (2007) who provides practical experiences of teaching mathematics using manipulatives to enable PSTs to comprehend mathematical concepts, procedures and teaching. Specific attention should be directed to comprehension of both the content that they are going to teach and a variety of teaching methods required in the classroom environment (Gresham, 2007; Ma, 2010; Mundry, 2005). Although ideas on effective teacher preparation exist, research shows that in most cases, PSTs do not have the opportunity to master the content knowledge that they need (Kessel, 2009). Explicitly, Kessel (2009) confirms that PSTs hardly have the chance to develop MKT, both in tertiary institutions and in-service. This is why the current study set out to explore issues pertaining to the development of PSTs' knowledge for teaching specific content as FMC skills. To device appropriate ways to improve PSTs' FMC and pedagogical content knowledge, this study sought to discuss aspects pertaining to teacher education.

2.10.2 Pre-service elementary mathematics teacher programme

Education across the globe is treasured as it is considered a catalyst of cognitive development that contributes towards economic empowerment and nation building. However, successful nation building depends on the quality of the education that citizens receive, which is influenced by the quality of the teachers who are teaching in school. Since the quality of teachers matters, teacher education programmes are developed to support quality teacher development. To address teachers' professional development, the University of Namibia introduced a Bachelor of Education (pre-primary and lower primary) degree programme. A general discussion of the objectives of the teacher education programme is presented below.

The Bachelor of Education (honours) is a four-year Level 8 degree programme offered at the University of Namibia for Grade 1 to Grade 3 teachers. The entry requirements are 25 points in five best subjects of which one must be English (minimum – C symbol) and mathematics (minimum – D symbol) at Grade 12 level. Entry to the course is also possible through a 'mature age entry' programme with its own specific requirements. The admission criteria are set to ensure enrolment of PSTs with a strong mathematics content knowledge base. The course aims to raise the quality of education in schools by affording teachers the necessary knowledge and skills while instilling in them a sense of responsibility and commitment. Information technology is one of the ways in which the programme plans to empower teachers with the necessary skills and knowledge. Teachers' acquisition of the necessary knowledge, skills and values is intended to contribute to their competency to provide learners with high-quality education that is "equitable, relevant and meaningful" (University of Namibia, 2016, p. 87). Central to teacher education is learners' needs, potential and learning abilities.

Learners are individuals with different learning needs that must be addressed by teachers. Therefore, the Bachelor of Education programme aims to enable teachers to be creative in translating the learning objectives outlined in the school syllabus in response to learners' different needs. The programme also aims to enable teachers to skilfully select content and teaching methods suitable for learners according to their needs. At the same time, teachers need to have a broader view of learners, including their daily life encounters outside the school premises. Generally, the programme has different aims in Year 1 up to Year 4 to develop teachers as required by the Namibia National Professional Teacher Standards.

The first course offered under mathematics education is Numeracy and Mathematics Development I. This course is taken by PSTs in their first year as a year course. Among the aims of the course is to develop PSTs' theoretical understanding and competencies regarding number concept development, number sense, mental computation skills and the ability to engage learners in important learning experiences by using the appropriate mode of teaching (University of Namibia, 2016). The course also aims to improve teachers' ability to reflect on mathematics teaching methods and content. One way to cultivate the necessary skills and knowledge is through a practical module called 'Micro-teaching' in which PSTs are expected to prepare a lesson and deliver it to their peers for deliberations and constructive comments.

Through micro-teaching lessons, PSTs model their teaching skills for refinement as they receive positive feedback from the TE and their peers. Bruner (1996, p. 84) states that "no educational reform can get off the ground without an adult actively and honestly participating – a teacher willing and prepared to give and share aid, to comfort and to scaffold". For PSTs to develop learners' FMC skills effectively, they need to develop the appropriate beliefs towards their role as a teacher. Bruner (1996) argues that PSTs need to act as honest and active teachers who are ready to support learners. To support learners in school, PSTs' knowledge needs to be improved through proper teacher preparation programmes, calling for identification of PSTs' existing knowledge and experience.

2.10.3 Pre-service teachers' knowledge and experience

PSTs have been found not sufficiently prepared to teach effectively. Limited knowledge has been attributed to inadequate mathematical knowledge and skills university or college courses offer. A study by Davis (1984) revealed that students learned much less than any person could imagine. Generally, research findings indicate that PSTs' existing knowledge upon entry to university is too shallow and "well below the curricular goals of school mathematics [specifically] ... in elementary arithmetic and basic algebra, as well as basic geometry and measurement" (Jakimovik, 2014, p. 18). This implies that PSTs develop into professionals who are not competent to teach effectively. Recent studies conducted on student knowledge also indicate inadequate knowledge in university students (Klein, 2008). This study argues that detection of PST subject content deficiencies, such as number sense, upon entry to university can create the opportunity to redirect instruction.

With regard to number sense, Sengul (2013) discovered an inadequate sense of number among PSTs, which prompted PSTs involved in the study to be more inclined to use rule-oriented computation strategies than sensible calculation strategies. Similarly, in Namibia, final-year PSTs involved in a study by Courtney-Clarke and Wessels (2014) demonstrated a lack of number sense and flexible number sense strategies to solve problems and to calculate mentally. According to Courtney-Clarke and Wessels, mathematics courses offered in teacher education programmes are inadequate to equip teachers with the necessary knowledge and skills required to teach in school. However deficient PSTs' knowledge may be, though, Sullivan (2008) argues that improvement of such knowledge is possible, depending on the proper disposition and fundamental aspects to develop mathematical knowledge.

In education, the approach used to develop PSTs' mathematical knowledge for teaching is of vital importance. Despite the absence of research on the current state of PSTs' development of mathematical knowledge for teaching FMC skills, the literature outlines aspects fundamental to mathematics learning. Currently, how PSTs are developing mathematical knowledge and skills contradicts reform ideals and efforts. Research evidence shows that the traditional approach to learning subject matter knowledge through memorisation of procedures and facts is still prominent among PSTs (Battista, 1999; Bolden, Barmby & Harries, 2013; Liang, 2013). Research clearly indicates that most PSTs develop calculation strategies through memorisation of basic number facts and the standard algorithms (Battista, 1999; Bjerke, Eriksen, Rodal, Smestad & Solomon, 2013); Stohlmann & Cramer, 2014), which impacts their competency, beliefs and attitudes negatively (Sun & Xin, 2019). A PST in Holm and Kajander's (2012) study admitted that she had learned mathematics through teacher-prescribed formulas and memorisation of calculation strategies that had informed her belief about the learning of mathematics. This evidence illuminates the fact that PSTs' experience, subject content knowledge and beliefs and attitudes regarding mathematics underpin effective teaching. However, a major issue is that new and reformed teaching approaches have not fully permeated teacher education programmes.

The literature indicates that PSTs are developing mathematical knowledge through lectures in which algorithms are presented by the lecturer through a teacher-centred teaching approach that concentrates on solutions, procedures and rules (Alsup, 2003). This indicates that PSTs are not developing FMC skills through investigation and representation of numbers and strategies in diverse ways. Moreover, Ball, Hill and Bass (2005) state that tertiary institutions have yet to contribute meaningfully to the development of PSTs' knowledge about ways to teach mathematics effectively as is evident in a weak knowledge base found in PSTs who are graduating. Similarly, evidence of the low impact of teacher education courses on PSTs' knowledge for teaching effectively shows that PSTs do not possess adequate content knowledge necessary for effective teaching (Livy, Vale, & Herbert, 2016). Research has found that in most cases, PSTs focus on calculation procedures leading to answers and do not listen to learners' justification of their calculation strategies (Copley, 2004). In addition, research findings indicate a lack of knowledge of skills such as solving mathematical problems, reasoning mathematically, establishing connections between learners' real-world experiences and mathematical concepts, and finding ways to communicate effectively for understanding of mathematical concepts (Fennema et al., 1996). Such evidence underscores the low impact of teacher education courses on the development of PSTs' knowledge for teaching.

This study posits that efforts to improve teacher knowledge for teaching should acknowledge that the persistence of a traditional teaching approach in tertiary institutions leads to reliance on factual knowledge developed instrumentally. This eventually results in a fragmented understanding of why specific rules work, and such knowledge is easily forgotten (Battista, 1999). Poor relational understanding of mathematics facts culminates in poor teaching practice which compels TEs to establish PSTs' diverse levels of conceptual understanding for successful implementation of reform ideals (Bolden et al., 2013). Practically, TEs must establish PSTs' level of mathematical knowledge and beliefs early in their studies to address inappropriate beliefs.

Different studies conducted on PSTs indicate a high level of mathematics anxiety, fear or discomfort regarding learning and teaching mathematics among PSTs. It is important to note that fear of mathematics has an impact on teacher confidence to learn and teach mathematics (Henderson, 2012). On a particular university campus, mathematics PSTs were found to have a high degree of mathematics anxiety, especially female PSTs (Hembree, 1990). Mathematics anxiety among elementary school PSTs is still evident two decades following Hembree's (1990) research findings. High levels of anxiety about learning and teaching mathematics amongst PSTs are also evident in Bolden et al. (2013) and Boyd, Foster, Smith and Boyd (2014). Mathematics anxiety is a result of limited conceptual knowledge development in relation to procedural knowledge.

Research indicates that when PSTs' conceptual knowledge is inadequately developed, they are forced to depend on mathematics facts and rules that were developed with little understanding. Limited understanding of concepts results in low self-confidence in doing mathematics. Low self-confidence in doing and teaching mathematics leads to ineffective teaching and in some instances prompts teachers to avoid teaching topics that they do not understand themselves (Kuhs & Ball, 1986; Walshaw, 2012). Elimination of fear of mathematics teaching and learning is possible when PSTs demonstrate a confident disposition after learning mathematics successfully and after learning how learners learn (Copley & Padron, 1998). Consequently, as part of PST knowledge development, understanding how learners learn is important but first TEs have to explore PST existing knowledge and perception on learning FMC. Both misconceptions and beliefs towards FMC can receive appropriate attention once established.

Evidence from qualitative studies conducted in Namibia provides hints about the current situation in some schools although the findings cannot be generalised to all teachers in Namibia. In a study by Vatilifa (2014), PSTs expressed a need to have their mathematical content knowledge and pedagogical content knowledge improved as they found their own content knowledge insufficient for them to teach effectively. Vatilifa outlines that the PSTs involved in the study admitted that they experienced anxiety to teach mathematics as they felt that they lacked strategies to address questions from learners. Another concern was the quality of PSTs' lesson presentation in facilitating understanding among learners (Vatilifa, 2014). One main challenge pointed out by Vatilifa (2014) is the difficulty that PSTs have in teaching a range of mathematics subject matter. Difficulty in teaching specific mathematical concepts outlines a deficit in PSTs' knowledge of how to teach mathematics. This should prompt TEs to focus on the development of PSTs' mathematical knowledge for teaching. However, this demands a proper understanding of how to support PSTs in a way that impacts their experience of learning FMC.

With regard to school mathematics experience, Battista (1994) outlines that most PSTs and TEs have received mathematics teaching conducted from an instrumentalist viewpoint. This outlines the need for TEs to embrace and implement research ideas that aim to improve PSTs' mathematical knowledge development. TEs need to understand how to support PSTs to understand the concepts involved in the mathematics that they are going to teach since only a limited number of PSTs may have learned mathematics through a constructivist approach (NRC, 1989). Developing PST knowledge through appropriate teaching approaches may improve content knowledge and reduce mathematical anxiety. Alsup (2003) argues that in-depth understanding of mathematical concepts can reduce mathematics anxiety as it diminishes memorisation of mathematical procedures and facts without understanding. As a result, low levels of mathematical anxiety may increase PST confidence in teaching and learning mathematics.

Understanding of mathematical concepts promotes understanding of mathematical procedures in mathematical problems. The findings of an intervention study by Gresham (2007) indicate that improving PSTs' understanding of mathematical concepts prior to procedures can help to reduce mathematics anxiety among PSTs. The learning experiences must involve manipulatives and significant tasks that attach real-life meaning to mathematics. Moreover, the use of diverse learning resource materials to promote mathematical understanding is crucial in the knowledge development of PSTs. Mathematics TEs should consider using "visual representations" of mathematical concepts and "a discussion-based approach" (Bolden et al., 2013, p. 80). This approach was found effective in providing improved explanation of mathematical concepts among PSTs. The study by Bolden et al. indicates that the use of visual representations culminates in abstract reasoning or "procedural representations" (Bolden et al., 2013, p. 80). Such findings based on the representation approach and the discussion approach illuminates the development of an instructional approach for developing PSTs' FMC skills.

The development of frameworks for teaching has proven successful in developing PSTs' mathematical knowledge (Hartnett, 2007; Star & Stylianides, 2013). Whitacre and Nickerson's (2006) intervention study shows that the use of a domain-specific HLT can improve elementary school PSTs' mental calculation skills to understand what they are going to teach. A domain-specific HLT is developed once PSTs' existing knowledge is established. The development of the framework needs to incorporate aspects relating to appropriate ways to teach learners at the elementary school stage. Furthermore, the nature of the trajectory must support PSTs to understand that elementary school learners learn "by doing and experiencing" mathematics (Kesicioğlu, 2015, p. 97). TEs need to ensure that PSTs investigate the mathematics that they are expected to teach and the curriculum through problem solving (Ma, 2010). PSTs also need to be assisted to understand the appropriate use of learning resource material, how to incorporate mathematics into different tasks and how to make mathematics pleasant (Kesicioğlu, 2015). Teacher education programmes have a great responsibility to ensure that PSTs learn mathematics with understanding. This will not only reduce their levels of mathematics anxiety but will also improve their MKT.

PSTs must possess the knowledge and skills that learners are expected to learn to ensure effective teaching and learning of mental computation in the classroom. PSTs need skills such as recall of addition, subtraction, multiplication and division facts, conceptual understanding of calculation strategies and improved ability in calculating mentally (Trafton & Suydam, 1975). As summarised by Shulman (1987), to teach effectively, PSTs need MKT in the form of subject matter knowledge (common content knowledge, knowledge at the mathematical horizon and specialised content knowledge) and pedagogical content knowledge (knowledge of content and students, knowledge of content and teaching, and knowledge of the curriculum). However, the existing literature indicates that such knowledge is underdeveloped in PSTs.

Alsup (2003) stresses that teachers will teach elementary school learners effectively, confidently and with autonomy if they are exposed to mathematics education that engages them fully and actively through discussion and logical thinking. PSTs who participated in Vinson's (2001) study indicated that their mathematical understanding improved when mathematical concepts and procedures were taught through pictures or concrete objects. Improved understanding of mathematical concepts leads to improved capability to teach such concepts confidently. When teachers have a poor knowledge base, their confidence to do mathematics is reduced, resulting in increased discomfort with or fear of mathematics (Vinson, 2001). Teachers' knowledge and experience are worth improving as these determine teachers' beliefs towards mathematics learning and teaching.

2.10.4 Pre-service teacher's beliefs

PST's content knowledge, in this case their ability to calculate mentally, is worth improving for effective teaching to prevail in schools. However, content knowledge only is insufficient to address teaching effectiveness in school. Fisher (2005, p. 19) stresses that "all [learners] have the potential [to learn], given the right stimulation and support, to realise their particular talents". To offer the appropriate stimulation and support, a teacher needs not only content knowledge but also appropriate beliefs about effective ways to stimulate and support learning. The knowledge to stimulate and support learning is described by Shulman (1987) as how well the teacher knows the content and the teacher's ability to make such content knowledge more understandable to learners. A teacher's approach towards facilitating the development of calculation strategies in the classroom depends on what the teacher believes is the best way to support learning in the classroom.

It is important to analyse PSTs' perceptions towards FMC to improve their comprehension of the FMC development approach and PSTs' skills to calculate mentally (Yesil-Dagli, Lake, & Jones, 2010). Beliefs can be understood as characteristics that influence PSTs' learning and teaching of mathematics. Teachers' beliefs about what FMC is and what developing calculation skills signify influence their capability to support learners to solve mathematics problems successfully (Emenaker, 1996). With regard to kind of beliefs, PSTs can hold beliefs about learners, the learning process, the learning environment and the content to be taught (Kagan, 1992). Generally, a negative attitude towards mathematics content topics was found among early childhood PSTs in a study conducted by Yesil-Dagli et al. (2010). In their study, negative beliefs emerged from a long period of inappropriate engagement with mathematics and from mathematics teachers. Other studies also indicate that PSTs hold a traditional view of what knowing and doing mathematics entail (Swars, Hart, Smith, Smith, & Tolar, 2007). The view held is that mathematics comprises static concepts to be transferred to learners through direct instruction that is well prepared and organised (Swars et al., 2007). A study conducted in Namibia on PSTs while carrying out their teaching practice lessons indicates that these teachers believe that mathematics is mostly about carrying out calculations and about mastering rules and procedures provided by the teacher. Furthermore, the findings indicate that PSTs believe that learning mathematics signifies following rules as outlined by the teacher coupled with many 'drill and practice' activities as they were taught in school (Vatilifa, 2014). Therefore, how PSTs are going to develop learners' FMC skills is determined by what they believe is the best way to teach mathematics.

Regarding teacher education, it is necessary for teacher education courses to support PSTs to develop a constructivist view towards mathematics to teach effectively. Stigler and Hiebert (1998, p. 6) state that "no matter how good our teachers are, they will only be as effective as the scripts they are using. To improve teaching over the long run, we must improve the script". For PSTs to apply the MKT developed in teacher education courses, their beliefs about the learning and teaching of FMC must be in line with reform ideals. Beliefs exist in various types such as views on the "theory of mathematics, learning mathematics, teaching mathematics... assessment in mathematics ... as well as identifying beliefs on the aims of mathematics education" (Handal, 2003, p. 48). It is crucial to address teachers' beliefs as teachers' perceptions of the teaching and learning of mathematics have been found to influence their approaches towards mathematics teaching (Handal, 2003). Beliefs need to be addressed because beliefs about mathematics and the way of teaching it are directly linked (Zakaria & Maat, 2012). Therefore, teachers' beliefs require attention to transform the way in which they teach.

Teachers across countries have different beliefs about teaching, how learners learn mathematics best and the use of learning resource material. Japanese teachers believe that learners learn by carrying out the activity without any hints from the teacher and by sharing calculation methods with other learners (Stigler & Hiebert, 1998). Such an approach supports learners to understand the advantages and disadvantages of other learners' approaches and to see the connection among different calculation methods. Furthermore, Japanese teachers believe that while solving problems, a sense of irritation and puzzlement is normal as it leads to understanding of knowledge during feedback sessions. Japanese teachers also believe that adequate time is necessary for learners to think creatively and to develop links between approaches and problems. Another belief is that learning can prevail through the trial-and-error and discovery learning methods. In all, such methods are believed to lead to development of the required knowledge and skills at the right stage of the activity (Stigler & Hiebert, 1998). Therefore, TEs need to develop such a belief in PSTs as opposed to modelling calculation strategies for learners to emulate.

In contrast, teachers in the USA believe that learning occurs best when the teacher breaks down the activity into chunks convenient to learners. During the teaching process, a teacher must also present learners with all the basic knowledge and skills necessary to carry out the assigned activity. The belief is that thereafter, tasks must be similar to the examples provided for learners to calculate. To achieve proficiency and understanding, the belief is that more activities must be provided for learners to practise (Stigler & Hiebert, 1998). A teacher is accused of not teaching well if learners end up puzzled or irritated. To mitigate the situation, a teacher intervenes immediately when learners seem not to know what to do by providing the necessary information for them to continue working on the activity.

Generally, the overall view of teachers in Namibia on the teaching of mathematics is similar to that of teachers in the USA although a number of teachers' views reflect those of Japanese teachers (Ausiku, 2014; Junius, 2014). This is evident in one of the studies conducted in Namibia in which a participant stated boldly that "...I teach steps to the learners, they memorize it exactly as I wrote them down on the board and then they apply it" (Junius, 2014, p. 93). As a result, learners develop a similar view of learning, posing a challenge to teachers who prefer to teach constructively. The teacher said, "Learners can very easily say that you are unable to explain the work to them. This happens especially when I try to allow them to arrive on their own at a wonderful mathematical truth" (Junius, 2014, p. 96). This relates to the TE's experience in one of the Grade 9 mathematics lessons when the TE attempted to implement a discovery learning approach for learners to learn through mathematics investigations. However, a male learner objected to the initiative, saying, "But you have not taught us yet – how can you expect us to work that out?" This indicates that a traditional approach to teaching is still prominent among teachers within the Namibian school context, which in turn informs learners' beliefs. Such learner beliefs also affect the effective implementation of new teaching methods. The belief of giving examples first and more similar exercises thereafter for learners to practise is still prevalent in schools. Therefore, PSTs could play a major role in developing a constructivist view of learning mathematics in the early grades if their beliefs are addressed during teacher preparation.

A participant with views similar to the Japanese teachers' beliefs shared with the researcher that implementing the constructivist view of learning was a challenge due to the attitudes of some colleagues whose teaching was more traditionally oriented (Ausiku, 2014). Some teachers in Namibia are aware of ways to teach in a constructive way but resort to traditional teaching methods, saying that these are easy to use considering the situation in schools. Some contributing factors are learners' diversified learning abilities, an overloaded school curriculum, constraints of time and pedagogical content knowledge, inadequate learning resource materials such as textbooks, an examination-driven curriculum, the views of other teachers and parents' expectations (Ausiku, 2014; Junius, 2014), sentiments also echoed by Handal(2003). Such views are contrary to an assertion made by Zakaria and Maat (2012, p. 193) that "...teaching practices should reflect the kind of mathematics beliefs the teachers hold". The argument is that it is not always true that teachers' way of teaching will reveal their own beliefs about mathematics teaching and learning.

In some instances, teachers may hold a constructivist view towards teaching but apply a traditional teaching approach when teaching due to the challenges posed by the learning environment and other factors, as outlined by Handal (2003), Perry (2010), Ausiku (2014) and Junius (2014). Yet, it stands that formation of appropriate beliefs about mathematics could orient teachers towards teaching habits that are constructive and successful (Zakaria & Maat, 2012). In Namibia, teachers' belief systems compromise reform implementation because of the persisting teaching culture amongst teachers. Therefore, transformation of PSTs' beliefs is imperative for proper implementation of Namibian education system reform.

2.10.5 Beliefs about mathematics and the nature of mathematics

Mathematics is a subject that has existed since ancient times, and it emerged because of its usefulness in society. This study argues that how mathematics is defined depends on the period when the subject is being defined and the dominating philosophy within society at the time of defining mathematics. The definition of mathematics has been evolving since its inception. This shows that knowledge is dynamic and not static and is changing constantly. How mathematics was learned, used and performed in the past influenced its definition, which led to the formation of personal beliefs about mathematics. Mathematics is defined in three different ways, referred to as movements, in the philosophy of mathematics.

The first movement is formalism, which perceives mathematics as a subject mainly dealing with "mathematical notation and its manipulation" (Colyvan, 2012, p. 4). This implies that mathematics simply involves manoeuvring of mathematical symbols that have no meaning. Formalism relates mathematical symbols to a kind of game involving parts that can be shifted around by following regulations (Colyvan, 2012). In my view, formalism can be associated with traditional or rote learning of calculation strategies. Research shows that in a traditional way, mathematics is taught in an environment that is too abstract "... using formalism first, removed from authentic contexts and discouraging to the students that do not see its relevance" (Schleicher, 2012, p. 34). An example of how mathematics is learned traditionally is evident in Schleicher's (2012, p. 34) statement that first learners "... are taught the techniques of arithmetic then given lots of arithmetic computations to complete, or they are shown how to solve particular types of equations, then given lots of similar equations to solve". Learning mathematics should include not only calculations but also investigations to discover patterns, prove assumptions and guess outcomes sensibly (NRC, 1989). As a result, learning mathematics is directed towards problem solving, which also leads to logical thinking and reasoning.

The second movement is logicism, which views mathematics as logic. Logicism asserts that the reality about logic makes up mathematics whereas sensible ideas are simple and less strange than mathematical knowledge (Colyvan, 2012). According to logicism, calculation of numbers underpins mathematics (Colyvan, 2012). Arguing from a logicism point of view, the inference can be made that mathematics is all about manipulating numbers to find solutions, which is not the case in reality. Contrary to formalism and logicism, the NRC (1989) indicates that performing mathematics involves more than computation. Though mathematics is a "... science of abstract objects..." the basis of mathematics reality stems from logic, not observation only (NRC, 1989, p. 31). Nevertheless, mathematics employs "... observations, simulation, and ... experimentation" to discover mathematics reality (NRC, 1989, p. 31). The difference the TE observed between logicism and formalism is attention to logic is absent in formalism and that logicism focus mainly on calculation of numbers. Though logicism can be considered under problem solving that aids logical thinking, its confinement of mathematics to mere calculation of numbers obscures the real comprehensive meaning of mathematics.

The third movement is intuitionism, also referred to as constructivism. For intuitionism, "mathematics is about proof and constructions" (Colyvan, 2012, p. 7). According to intuitionism, mathematics is not about the readily available reality of mathematics but mathematical entities must be developed to attach meaning to them. For intuitionism, mathematics is about building mathematical entities and discovering ways to prove such entities in novel, logical ways (Colyvan, 2012). FMC embraces the intuitionist view of mathematics as it promotes the invention of calculation strategies. Ernest (1991) stated that the creation of mathematical facts needs to be a result of verifiable constructions. In other words, PSTs need to justify how and why particular calculation strategies work and understand the order of FMC skills development.

A long-standing traditional view of mathematics learning is that of performing numerous calculations whose procedures are followed mechanically. Many PSTs believe that mathematics is mainly about "symbols and procedures" (Stohlmann & Cramer, 2014, p. 7). Generally, mathematics has been classified as comprising conventions and algorithms that should be learned by heart. Both teachers and learners perceive mathematics in a particular way that in turn forms their belief about what mathematics is and how it is learned. Though a change in teacher belief has been observed (Gravemeijer, 1994), many mathematics educators and learners still believe that teaching and learning mathematics mean performing calculations exactly as taught by the teacher without understanding how and why the method works and its utility in real life. Mathematics is viewed as consisting of compartments of unrelated facts forming a body of knowledge and is learned as such. In other words, it is a subject learned by rote and by adopting calculation procedures presented by the teacher to work out closed-ended questions (Southwood & Spanneberg, 2000), a view associated with formalism. As a result, learners rely heavily on teachers as a source of information.

Another existing belief, observed from my experience as a mathematics teacher, is that mathematics is a challenging subject meant for those possessing mathematical intelligence. As an example of my personal experience as a teacher, a male learner scored 20% on a test and after receiving the test results smiled, saying that he was not disturbed by the outcome because mathematics was very difficult and not everyone could do it and so failing mathematics was normal. Some learners thus believe that they are unsuccessful in mathematics because mathematics is not meant for them or they do not have the ability to succeed in mathematics, an attitude regarded as normal by society. Such a view developed over a long period of learners being unsuccessful in mathematics and is endorsed in many societies. A negative perception of mathematics has come a long way, stemming from the disconnection between classroom mathematics and everyday-life mathematics. However, beliefs about the nature of mathematics can change as a result of appropriate approaches to teaching mathematics in general and FMC in particular.

2.10.6 Role of flexible mental computation in changing pre-service teachers' beliefs

A change of belief is necessary for learners to participate productively in real life through adaptive and creative application of mathematical knowledge. Beliefs are personal philosophies, characteristics, mental constructs or assumptions regarding mathematics learning and teaching that develop both in school and at university. In most cases, both teachers and TEs teach subject content knowledge consciously while teaching beliefs and attitudes unconsciously. Teachers begin to form perceptions about mathematics learning and teaching by considering ways in which they were taught as learners in school and what they experienced while learning mathematics (Handal, 2003). Beliefs also develop when teachers consider how they are taught as PSTs once enrolled in teacher education programmes. In a study conducted by Luwango (2014), a participant stated that the opportunity that the participant had of having very good mathematics teachers in school and university encouraged the participant to emulate a similar approach to teaching.

The participant outlined that the teachers taught mathematics by connecting mathematical concepts to real-life situations. The participant thus sought to teach by emulating a teaching approach as experienced in school and university. Evidently, both positive and negative perceptions about mathematics learning and teaching can manifest in teachers' classrooms as a result of their school experience. However, the current study claims that beliefs can be unlearned since they are learned in school as learners and are perpetuated at university as prospective teachers. The argument is that teachers' beliefs can change during teacher education courses. Research evidence outlines that "some students believe that mathematics is a difficult subject because of the experiences they had during their early school days" (Tsao, 2014, p. 617). Tsao outlines that PSTs require a positive mind-set towards mathematics for them to facilitate mathematics learning competently. For this reason, consciousness of existing beliefs is necessary to prepare learning activities that will promote positive change in PSTs' beliefs and attitudes towards mathematics teaching and learning.

This study found that a positive change in PSTs' beliefs and attitudes towards mathematics learning and teaching could manifest from an understanding of learners' FMC thinking processes. In an attempt to transform PSTs' beliefs about mathematics learning, PSTs need to understand elementary school learners' mathematical thinking (Stohlmann & Cramer, 2014). A study by Ambrose (2004) found that change in beliefs manifested in PSTs who concentrated on learners' mathematical thinking. Specifically, teachers' beliefs can change by engaging them in the process of reinventing calculation strategies through problem solving. TEs need to create learning situations in which PSTs are required to experience the use of various representations in the form of real objects, pictures, language and symbols to change their beliefs (Stohlmann & Cramer, 2014). However, TEs are reminded to choose representations carefully and use representations that are appropriate for FMC skills development and conceptual understanding (Bolden et al., 2013). Research shows that mathematics teacher education programmes that are organised to enhance understanding of mathematical concepts enable transformation of PSTs' beliefs about mathematics (Stohlmann & Cramer, 2014). Fundamental to PSTs' beliefs is the understanding that when teaching, the focus should be on conceptual understanding rather than on rote acquisition of skills (Schoenfeld, 1992). As a result, learning concepts meaningfully and applying knowledge in real-life contexts lead to the belief that learning mathematics is relevant, fun, interesting, innovative and worthwhile. More suggestions are available on effective ways to direct PSTs' beliefs towards appropriate FMC teaching approaches.

The literature emphasises that a problem solving approach is most influential in fostering a belief in PSTs that mathematics can be reinvented and discovered as opposed to direct conveying of facts by a teacher. Research shows that using a problem solving teaching approach to develop PSTs' skills to calculate is a way of supporting them to understand problem solving as a teaching approach (Swars et al., 2007). This requires the use of activities that include problems resembling real-life situations and promoting communication of calculation procedures to the entire class. When developing FMC skills, PSTs have to compare calculation procedures to discover relationships among mathematical ideas.

Once mathematics is learned through discovering the relationship between concepts and procedures, PSTs will understand that mathematics is really about thinking and reasoning to solve problems by using multiple approaches. Communication of ideas is necessary to encourage PSTs to share calculation strategies and to reflect on invented procedures to prove, examine and refine their calculation strategies before they are considered valid. Regarding reflection, evidence shows that PSTs' beliefs can change through reflective thinking and in-depth content knowledge (Stipek, Givvin, Salmon, & MacGyvers, 2001; Swars et al., 2007). The process of engaging PSTs in problem solving, reflection on calculation methods and refinement of calculation strategies changes PSTs' beliefs unconsciously (Ambrose, 2004). Thus, the impact of problem solving on PSTs' beliefs is practical and should be promoted in the development of FMC skills.

Though change in teachers' beliefs is possible, research indicates that beliefs do not fade away completely. Ambrose (2004) emphasises that PSTs need a variety of experiences to develop different beliefs concerning mathematics. Despite changing PSTs' beliefs during teacher education programmes, PSTs' beliefs are not replaced by new ones but old and new beliefs are likely to 'coexist' (Ambrose, 2004). Evidence from Ambrose's study indicates that teachers do not abandon existing beliefs as new beliefs are developed. Instead, extension of teachers' beliefs prevails in the presence of old notions and can eventually lose consideration. Such findings call for stakeholder support and continuous professional development in school to uphold PSTs' transformed beliefs. The argument is that although PSTs' beliefs could change through teacher preparation, many aspects could hamper the application of a constructivist view in their teaching. Implementation of a constructivist view could be constrained by the nature of school learning environments, namely "parental and administrative pressure to follow traditional oriented methods of instruction ... traditional oriented mathematical learning style of the [learners and] a lack of time and materials" (Handal, 2003, p. 53). There is consensus between the factors stated by Handal (2003) and the findings of a study conducted in Namibia by Junius (2014). Findings from studies conducted in Namibia indicate the persistence of a traditional teaching approach.

A research participant with a tendency to cling to a traditional approach to mathematics teaching attributed the choice of approach to its effectiveness in terms of managing to cover an overloaded mathematics curriculum in a short time. The participant alluded to a constructivist approach as being too slow to cover a congested syllabus (Ausiku, 2014). In addition, this study concurs with Junius' (2014, p. 96) argument that "despite the intentions of a learner-centred policy, the Namibian Education System, in my view, still endorses a teacher-centred philosophy as it places so much emphasis on the results of external examinations...[that compel teachers to] prepare... learners for external examination". Such challenges call for teacher education programmes to discuss ways in which a constructivist learning approach could save time through connecting topics and concepts (Fisher, 2005). Understanding how topics and concepts are related is likely to foster a belief that mathematics topics and tasks are connected and not isolated facts.

Although traditional beliefs about learning FMC are persistent, there is evidence that teachers' beliefs are changing gradually and that mathematics is no longer perceived as static but is rather regarded as dynamic. This change is attributed to intervention studies that focus on the use of problem solving activities such as the study by Suurtamm and Vezina (2011) in which teachers admitted that they had memorised algorithms that they could not justify before the intervention but could do so after learning through problem solving. Teachers reported increased understanding of mathematical knowledge and diverse calculation strategies. Learning through problem solving fostered the belief that calculation strategies could be reconstructed by learners through active participation, problem solving and discourse.

Gravemeijer (1994) confirms that many mathematics teachers now believe that the process of developing calculation strategies requires active involvement. Abandonment of the belief that mathematics is mainly about methods of calculation, providing abstract meanings of words, and following prescribed rules and principles is also evident in Gravemeijer (1994). Understanding problem solving as an approach to developing FMC skills might enable PSTs to adopt problem solving as an effective way to teach FMC. Therefore, this study embraces the notion that a change in teachers' and learners' beliefs about mathematics becomes a change in personal theories of teaching and learning mathematics to transform instruction.

If inappropriate beliefs are not changed, such views are going to be transmitted to learners, eventually dissociating them from learning mathematics effectively (Shulman, 1987). Reform ideals will be merely incorporated into learning environments that practise rote learning (Schleicher, 2012), yielding minimal positive outcomes. Studies conducted explain that it is through former encounters and perceptions of teaching and learning that teachers translate novel ideas (Schleicher, 2012). Therefore, the opportunity to encounter reformed teaching practises pertaining to FMC should be explicit in the teacher education curriculum.

2.11 Curriculum content on flexible mental computation

This section focuses on the elementary school mathematics curriculum for first Year PSTs. A curriculum is an outline of content knowledge to be learned with an indication of the order in which the knowledge should be developed and approaches towards acquiring such knowledge (Egan, 2003). A curriculum may include "... goals, core content, learning activities, assessment ... appropriate instructional program, materials used in instruction and ... instructional time..." (Kilpatrick et al., 2001, p. 410). A comprehensive curriculum needs to stipulate subject content knowledge, skills, values, goals, learning activities, assessment, teaching strategies, learning resource material and time. Therefore, knowledge of the key elements of a curriculum informs the design of a HLT for developing PSTs' mathematical knowledge for teaching FMC.

The focal point of this section was FMC with curriculum analysis of three countries. Analysing the curriculum fosters understanding on how the curriculums of other countries develop PSTs' mathematical knowledge for teaching FMC. This study concurs with Kessel (2009) that curriculum knowledge grants TEs the opportunity to improve PSTs' mathematical knowledge for teaching. Additionally, analysing the curriculums of successful countries may refocus teacher development towards enhancing proficiency in PSTs (Kilpatrick et al., 2001). TEs as curriculum developers are likely to identify gaps in the curriculum that will affect curriculum review (Fan & Zhu, 2007). The improvement or development of a curriculum is "... a situated decision, made in knowledge of needs, goals and resources" (Clarke, 2003, p. 157). Once a gap has been identified in a curriculum, review of such a curriculum needs to consider the aim of the content in relation to the education system, the available means through which the curriculum can be adapted and the needs of the society. For a society to be productive, it needs a curriculum that is globally and locally relevant. The curriculum should direct teaching conveniently through clear, specific and accurate information and well-organised topics, and indicate the periods in which to cover such content (Kilpatrick et al., 2001; NCTM, 2000). It is in the light of the ideas set forth above that the mathematics curricula for Namibia, South Africa and Singapore are discussed next. Namibia's curriculum is analysed in relation to that of South Africa to understand what is transpiring in the neighbouring country in terms of teacher preparation. Singapore's discussion is based on the TIMSS results.

2.11.1 Flexible mental computation in the Namibian mathematics curriculum

This study sought to devise ways to develop PSTs' knowledge for teaching FMC. The curriculum for PST states under its course objectives that PSTs have to develop their own mental ability skills (DECE, Namibia, 2014). The concept FMC skills does not feature in the course outline for Namibian elementary school mathematics PSTs enrolled for the Bachelor of Education programme. Under content knowledge, the curriculum refers to mental mathematics as different ways to find answers to addition, subtraction, division and multiplication sums. Under pedagogical knowledge, it states that learners must apply different mental strategies to calculation to provide answers orally and to solve problems in writing (DECE, Namibia, 2014). This study found the objective to develop in PSTs different ways to find answers to addition, subtraction, division and multiplication problems commendable. However, how PSTs ought to develop different calculation strategies is not explicitly outlined.

Mental mathematics is one of the 20 course objectives to be covered in the first year of the Bachelor of Education programme over a 14-week academic year with three lecture hours per week. The course objectives are too many to cover intensively within the available time. Research calls for intensive coverage of critical aspects such as numbers and geometry to lay a solid foundation for further learning of mathematical concepts (NCTM, 2000). This is because the elementary school years are crucial for effective development of FMC skills and limited objectives avoid sacrificing depth for breadth. Different researchers advocate incorporation of less content in the curriculum to enable PSTs to "think about less content in greater depth" (Rowland, Turner, Thwaites & Huckstep, 2009, p. 14).The literature confirms that thorough initial learning of a concept yields double the outcome with half the effort in subsequent learning whereas inadequate initial learning yields half the outcome with double the effort in subsequent learning (Ma, 2010). This study thus suggests a focus on whole-number computation in addition to a few other objectives in the first year of the PST course. The course is explicitly designed to develop PSTs' understanding of content knowledge and pedagogical knowledge, but the link with elementary school mathematics is implicit. The gap created by the implicit link between the elementary school curriculum and the course objectives is evident in the course outline. Therefore, this study found it beneficial to analyse the PST curriculum used in South Africa.

2.11.2 Flexible mental computation in the South African mathematics curriculum

The intention to develop PSTs' ability to calculate mentally is evident in the curriculum of a selected university in South Africa. However, the concept FMC is not used in the course outline for foundation phase teachers enrolled for the Bachelor of Education (foundation phase) programme. Instead, what appears under the module contents is "the development of number concept and computing methods in learners intuitions and limiting constructions in whole number arithmetic" (Curriculum studies Department, 2014, p. 2). For this study, what the two module contents entailed and how the foundation phase teachers had to develop computing methods in learners were implicit. Similar to the Namibian curriculum, aspects of developing foundation phase teachers' own content knowledge with regard to FMC are not explicitly outlined in the module framework. The module objectives pertaining to the development of number concept and computing methods in learners intuition and whole-number arithmetic meant to be covered in the first year of the Bachelor of Education (foundation phase) programme with three lecture hours per week. Unlike the 20 course objectives in the Namibian curriculum, the South Africa course objectives can be covered relatively adequately over an academic year. The course is explicitly designed to develop PSTs' understanding of mathematical concepts and pedagogical content knowledge although its link with elementary school mathematics is also implicit.

2.11.3 Flexible mental computation in the Singaporean mathematics curriculum

Like Namibia and South Africa, the goal to develop mental calculation skills in PSTs is evident in the curriculum for PSTs in Singapore. Selection of Singapore, as a good example, is based on research findings that teacher education system in Singapore is effective with regard to mathematics content knowledge and mathematics pedagogical content knowledge (Blomeke et al., 2011). Consequently, the outstanding performance of Singaporean PSTs prompted an exploration of Singapore's PST mathematics curriculum.

An analysis of the Singaporean curriculum indicates an absence of the term FMC but it contains the term mental computation (Ministry of Education, Singapore, 2012; Wang-Iverson, Myers, & Lim, 2010). Specifically identified under the term mental computation is the fundamental pedagogical content knowledge considered necessary for effective teaching. Whole-number addition and subtraction are first limited to numbers up to 100 by using formal algorithms in which the symbols + and – are to be introduced. With regard to mental computation, the knowledge included is addition and subtraction within 20, addition and subtraction involving a two-digit number and ones without renaming, and addition and subtraction involving a two-digit number and tens. Mental multiplication and division are not mentioned under the theme mental computation. Considering the layout of the themes, it appears that learners are expected to commit addition bonds up to 9 + 9 to memory and carryout addition and subtraction using formal algorithms before learning to calculate mentally (Wang-Iverson et al., 2010). Although multiplication and division are not stipulated under mental computation, the pedagogical content knowledge that PSTs must develop is clearly outlined.

The curriculum stipulates further that PSTs need to support learners to understand multiplication as repeated addition (within 40), the use of the multiplication symbol (x) to write a mathematical statement for a given situation, division of a quantity (not greater than 20) into equal sets given the number of objects in each set and given the number of sets, and solving one-step word problems with pictorial representation, excluding the use of a multiplication table and the use of the division symbol (÷). From the literature, it is evident that the effectiveness of Singapore's mathematics curriculum relies on strong content that is relevant to the Singaporean school context and the model used for teaching problem solving (Kaur et al., 2012). Moreover, Singapore's success is also a result of resources that are produced locally using local expertise and findings of research informed by exceptional systems (Kaur et al., 2012).

With regard to Module 1 of the course 'Curriculum Content Mathematics', one of its five main module objectives is, as stipulated in the curriculum "Operations and number systems; properties of operations:

closure, identity, and inverses [;] extensions of number systems from the natural numbers to integers to rational... to real... [and] more abstract examples, operation tables" (Lim-Teo, 2002, p. 142). Four hours are allocated to the learning of operations and number systems. Lim (2013, p. 3) stresses that the skills and knowledge that PSTs should develop are "... closely aligned with the Ministry of Education's desired outcomes of preparing PSTs to be collaborative learners, confident persons, active contributors and concerned citizens". Such explicit links are also necessary in the two aforementioned curriculums.

2.11.4 A comparison between the three curricula

A curriculum has to be globally and locally relevant to address the needs of the society. As discussed earlier, the curriculum must direct teaching conveniently through objectives that are clear, specific, realistic, accurate and well organised. Significantly, mathematics course objectives and topics need to be limited to aspects fundamental to elementary mathematics and be equally spread out across the duration of a course. This study posits that critical selection and organisation of mathematics topics are necessary to secure ample time for in-depth development of foundational topics such as FMC skills. A glance at Singapore's PST curriculum reveals a need for precision and suggests a cutback in the number of first-year course objectives for PSTs. Although Singaporean PSTs begin their teaching career with a solid knowledge base in mathematics compared to other countries (Ginsburg et al., 2005), precision of the country's curriculum strengthens PSTs' content knowledge and pedagogical content knowledge.

Practically, efforts to improve in Singapore's direction by realignment of PST mathematics courses are necessary. The process of developing PSTs' knowledge for teaching FMC has to offer the essential knowledge, skills, values and attitudes necessary to teach elementary school mathematics (Askew, 2008; Ginsburg et al., 2005). Research suggests the need for an in-depth understanding of elementary school mathematics content knowledge and alignment of PSTs' mathematics curriculum with elementary school learners' daily academic activities (Jakimovik, 2014). Moreover, Singapore's curriculum framework contains fewer topics that recur fewer times compared to other countries (Ginsburg et al., 2005). It is possible that fewer topics provide Singaporean PSTs ample time for intensive development of key content knowledge for teaching.

Although both the Namibian and South African curriculums for PSTs outline the important mathematical domains necessary to teach elementary school mathematics, specific aspects stand out in Singapore curriculum. As stressed in literature, Singapore's mathematics curriculum framework is "...well-defined... highly logical and specific..." (Usiskin & Willmore, 2008, p. 267). Turning to the state of curriculum in Namibia, two important aspects are necessary: curriculum reform to reduce the breadth of objectives in order to increase the depth of content knowledge. Changes in curriculum may positively impact PSTs' understanding the fundamental mathematics content taught in primary school. Furthermore, the Namibian curriculum has mathematics pedagogical content knowledge broadly stated and time for coverage of the lengthy content aspects is inadequate. Such shortcomings in curriculum may contribute to PST incompetency to teach mathematics effectively.

Numerous studies exist where PSTs have admitted uncertainty about how to apply the knowledge developed at university level to execute the school curriculum (Gierdien, 2012; Sapire & Sorto, 2012; Ilukena, Luwango, & Ausiku, 2018). Uncertainty and lack of confidence to teach mathematics in school can be attributed to a mismatch between PST course content experience and elementary school curriculum (Gierden, 2012), a matter requiring urgent attention. A discovery made internationally indicates that "... university curriculum is extremely demanding, with a lot to be achieved in a small amount of time; sessions have to be very concentrated, with many objectives to be achieved in a single session" (Jaworski & Gellert, 2003, p. 867). Due to a demanding curriculum, PSTs have limited time to focus on school mathematics, which is fundamental for teaching elementary mathematics. For this reason, PST course content need to be minimised and aligned with the curriculums of leading countries such as Singapore.

2. 12 Flexible mental computation development of pre-service teachers

In directing the development of FMC skills in PSTs, the literature advocates for the development of analytic strategies rather than solution strategies. Analytic strategies focus on understanding how to go about solving a problem as opposed to solution strategies that focus on the method to be used to solve a specific problem. Although the literature asserts that teacher knowledge development is not rigid or fixed but is flexible (Burgess, 2006), this study posits that some teacher development approaches are more effective than others. The development of the FMC skills of PSTs depends on construction of their own calculation procedures emanating from their own analytic strategies as opposed to direct instruction of solution strategies. Analytic strategies emerge from understanding of numbers, reasoning about numbers and imagination of connections among numbers, which eventually culminate in construction of PSTs' own solution strategies, knowledge that is currently lacking in PSTs (Copley, 2004). Adopting Ma's (2010) view, this study posits that the strategies adopted to develop PSTs' knowledge for teaching FMC must be similar to approaches through which PSTs are expected to develop learners' FMC skills in school. Significantly, it is important for teachers who will teach at the elementary school level to have an in-depth and a basic comprehension of mathematics at the level that they are going to teach (Hart et al., 2013). This is vital in developing PSTs' FMC knowledge concurrently with their teaching skills, as will be discussed later.

A further argument is that to develop learners' FMC skills, teachers should understand how learners develop such skills. To make FMC skills development practical within a university classroom, the TE adopted a "hands-on involvement" approach (Hein, 1991, p. 4) to PSTs' FMC skills development. A hands-on involvement approach is effective as teachers mostly emulate teaching methods that they observed as learners from their teachers (NRC, 1989). It is against this background that TEs need to model appropriate teaching methods when developing PSTs' knowledge for teaching FMC for PSTs to emulate in school. The literature recommends that TEs' instruction should demonstrate how PSTs are expected to teach elementary school learners while linking the university curriculum to the school curriculum (Garet, Birman, Porter, Desimone, Herman, & Yoon, 1999). However, this is not currently the case. As expressed by a PST in a study by Bjerke et al. (2013), most of what the university curriculum covers is disconnected from what is taught in Grade 1 as it is only connected to the grades beyond Grade 1. Connection of the curriculum and modelling lessons can enable PSTs to gain "... experience in doing mathematics – in exploring, guessing, testing, estimating, arguing, and proving" (NRC, 1989, p. 65). Personal construction of PSTs' own calculation strategies could possibly enable them to develop knowledge of how calculation strategies can be constructed and how to teach FMC constructively.

Similarly, evidence from a study by Fennema et al. (1996) indicates that one possible way in which TEs could support PSTs' understanding of different computation strategies to teach FMC effectively in school is to expose PSTs to the ways in which such strategies are developed and to understand the thinking processes of learners. Problem solving has been identified as an appropriate approach that TEs could use to focus PSTs' attention on the thinking process of learners while promoting PSTs' own understanding of FMC. The relevance of problem solving to FMC is that it embeds numbers into a real-life context. Rather than presenting numbers abstractly, problem solving provides a scenario within which the numbers to be calculated represent real-life entities, for example using numbers to represent the cost of an item or to represent the number of books bought or sold. When learning through problem solving, the mathematics content that learners are expected to learn becomes the focus of PSTs' reflection through communication of ideas (Kieboom, 2013). Demonstrating how FMC must be taught in school can develop PSTs' pedagogical content knowledge while focusing on their own analytic skills to invent calculation strategies.

The process standards identify five important aspects through which PSTs could develop analytic strategies to construct their own calculation procedures, namely problem solving, reasoning and proof, communication, connections and representation (NCTM, 2000). Problem solving should be used to provide a real-world context for PSTs to devise a variety of strategies to solve new problems. Such problems need to be contextually enriched to connect real-life mathematics to classroom mathematics (Freudenthal, 2002). For example, a study by Stigler and Hiebert (1998) indicates that Japanese teachers use problem solving to engage learners in discovering new procedures for solving problems. Problem solving enables Japanese teachers not only to maintain learners' attention but also to raise learners' interest in doing mathematics and remain engaged in the task in a meaningful way.

Similarly, to foster PSTs' interest and meaningful engagement in tasks, TEs need to design problem solving activities involving basic number facts, number properties and connections among basic mathematics operations. Therefore, as part of the current study intervention, the researcher as TE designed activities that were likely to integrate FMC and problem solving for PSTs to develop knowledge of basic number facts, number properties and connections among operations. In this way, problem solving would link calculation of numbers to real-life activities while enhancing PSTs' problem solving skills, number sense and calculation proficiency as adults (Threlfall, 2002). PSTs need to carryout investigations to devise their own strategies and evaluate such strategies. The constructed strategies should be communicated clearly to the entire class using correct mathematical language.

Discussion of ideas is necessary as it encourages reasoning and proving strategies by analysing PSTs' own mathematical thinking to explain and justify invented strategies. Problem solving is necessary to create the opportunity for PSTs to discover the connections between addition and subtraction, addition and multiplication, multiplication and division, and division and subtraction, depending on the nature of the problems used by TEs. With regard to representations, PSTs need exposure to diverse representations to model and explain mathematical ideas using concrete objects, pictures, words and symbols (NCTM, 2000). Thus, the process standards for school mathematics illuminate the intervention process to adopt a problem solving teaching approach rather than a traditional teaching approach to avoid rote learning of calculation procedures.

To learn through problem solving, PSTs need to change their belief in traditional approaches to learning and develop an inquisitive mind and a habit of learning (Ball & Cohen, 1999). The persistent traditional views about FMC learning and teaching ought to change. Currently, existing practices relate mainly to standard algorithm, pen-and-paper and calculator usage (Flowers, Kline, & Rubenstein, 2003). It is evident in the literature that flexibility is yet to be attained in school as skills to calculate in the mind are developed through memorisation (Battista, 1999; Bolden et al., 2013). This is a process whereby mental calculation strategies emanate from emulation of calculation strategies provided by a teacher and further calculations are carried out through drill and practice to master the calculation strategies (Battista, 1999; Bolden et al., 2013; Liang, 2013). Changing such traditional views is challenging and requires a transformed learning environment supporting both procedural and conceptual understanding of FMC.

A learning environment for PSTs needs to resemble a socially collaborative society with regulations that encourage PSTs to think mathematically, to think reflectively, to act upon their own learning, to promote discussion, to participate enthusiastically in their own learning and to develop self-efficacy for autonomous thinking (Alsup, 2003). Through problem solving, PSTs are likely to develop as autonomous thinkers, understand how learners develop FMC skills and develop their own strategies for calculating flexibly. In addition, to understand the ways in which learners invent computation strategies, PSTs need to understand learners' solution strategies as they verbalise their thought processes. Sowder (1990) emphasises that for teachers to create a learning environment that promotes the easy development of learners' own calculation methods, teachers need to develop a solid comprehension of a range of calculation strategies that are likely to be employed in problem solving situations.

As emphasised at the beginning of this section, PSTs need to employ analytic strategies to invent their own calculation procedures. Central to the process of inventing calculation strategies is problem solving to develop analytic skills by observing the relationships among numbers. Since construction of calculation strategies through problem solving involves thinking, a TE must encourage appropriate habits of mind. Kessel (2009) stresses that habits of mind and the ability to substantiate arguments underpin the application of mathematical knowledge. Consideration of research findings outlining the inability of final-year PSTs to compute flexibly in the mind prompted the TE to devise ways in which to develop desirable habits of mind to promote flexibility in PSTs (Courtney-Clarke & Wessels, 2014). Therefore, this study has found the 4-6-1 model appropriate as it incorporated the nature of thought, orientation of action and the mind-set necessary to foster learning with understanding.

Most important is that it links with the problem solving steps of understanding a problem, devising a plan, carrying out a plan and looking back. This study posits that for PSTs to understand a problem in a way that informs the use of a particular strategy, PSTs must analyse a problem to identify the numbers used in the problem. Ultimately, this leads to the incorporation of investigative thinking to devise calculation strategies. Once a strategy is devised, a PST must try out the strategy and experiment with the strategy on different problems to establish the efficiency and generalisation of the procedure. Eventually, establishment of the efficiency of a calculation strategy demands reflective thought and reasoning. Reflection and reasoning are fundamental in analysing strategies to justify procedures and to construct calculation strategies that are more efficient.

The relevance of the 4-6-1 model lies in its ability to enhance "attitudes that support powerful learning" (Claxton, Lucas, & Webster, 2010, p. 26) as it has presence of mind at its centre. This study posits that the presence of mind at the centre calls for learning through an active mind because with a dormant mind, calculation strategies are unlikely to be constructed. Important to this study was the notion that PSTs needed to adopt particular ways of operating and thinking to develop the necessary knowledge for teaching FMC. PSTs' way of thinking to perform calculations in the mind could develop in the presence of an active mind at the centre of the problem solving process, as illustrated in Figure 2.2.

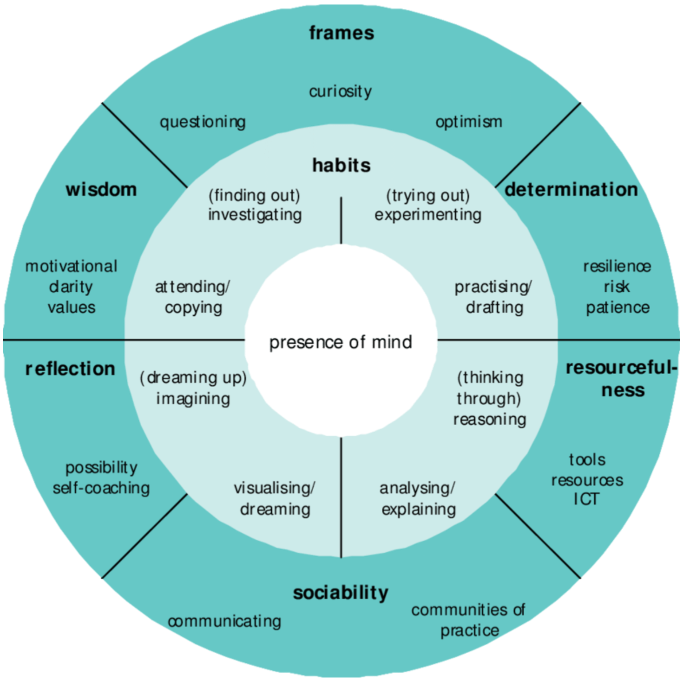

Figure 2.2 The 4-6-1 model Source: Claxton et al. (2010, p. 46)

For the TE, this means that a PST's mind plays a major role in the improvement of own mathematical knowledge. This study posits that knowledge development is possible with the existence of an inquisitive mind and an inclination to learning (Ball & Cohen, 1999). An inquisitive mind is an active mind coupled with the four general habits of mind, namely "investigation, experimentation, imagination and reasoning" (Claxton et al., pp. 26-27). A mind that is thought oriented is strengthened by behaviour that enables learning with self-confidence and action. PSTs need attitudes that support powerful learning, which include "curiosity, determination, resourcefulness, sociability, reflection and wisdom" (Claxton et al., 2010, p. 35). Thus, with an active mind and a positive mind-set, PSTs are likely to carry out investigations and experiments and to use their imagination to invent their own flexible computation strategies that they can justify through logical reasoning and thinking.

With regard to investigation, PSTs need to use their existing knowledge to actively discover connections between numbers and operations through inquiry, observation, attention to the task at hand, thinking carefully, reading, writing, distinguishing effective methods from ineffective ones and observing the computation strategies of other PSTs. Treffers (1987) stresses that the most significant aspect of teaching is a teacher's creating the opportunity for learners to carry out a particular investigation themselves whereas Freudenthal (2002) calls for problem solving and discovery learning. With regard to the focus of investigation, Ma (2010) proposes careful investigation of teaching content, teaching methods and the use of learning resource objects. PSTs' investigations should focus on content, teaching methods and the curriculum, including textbooks.

Connected to investigation is experimentation that requires PSTs to practise by attempting to carry out calculations flexibly to discover effective methods that lead to accurate solutions. Alteration and extension of ways of thinking, reasoning and calculating are mainly possible through experiments that eventually culminate in learning. As discussed in section 2. 12 on MKT, a further argument is that the more PSTs are engaged practically in carrying out a variety of calculations flexibly, the better they will become at FMC. However, ample time and a calm environment are necessary for construction of calculation strategies while PSTs also need to learn attentively and be determined to learn (Claxton et al., 2010). TEs need to scaffold tasks, ask questions, create room for reflection and discussion, and design problems that have specific objectives to confront and extend PSTs' knowledge and skills regarding FMC.

While trying out new ways of calculating flexibly in the mind, PSTs need to use their imagination. Imagination involves thinking, which involves sentiments that influence the performance of tasks, depending on how well a task is understood. PSTs need to use their imagination to decompose and recompose numbers mentally. Through imagination, PSTs may turn physical models such as the open number line, the five frame and the ten frame into mental models to facilitate the process of calculating mentally. For example, thinking of how the ten frame works to calculate 8 + 6 makes it easier as thought is directed towards adding 2 to 8 to fill up one ten frame and 4 more to make 14. Imagination plays a major role in performing such a calculation mentally. PSTs need to think clearly and reason logically to invent strategies and be able to explain clearly how their strategy works and why it is effective. PSTs need to visualise the models to use them to calculate mentally in addition to using their own instincts to create solution strategies. Eventually, with practice, PSTs may develop the ability to manage and guide their imagination towards solving a problem flexibly. Again, as outlined by Claxton et al. (2010), exercise of the habits of mind depends on PSTs' attitude towards learning. The invention of flexible calculation methods is unachievable in the absence of a positive disposition.

Attitudes such as curiosity, determination, resourcefulness, sociability, reflection and wisdom are critical to the development of PSTs' knowledge for teaching. Since the purpose of this study was to develop PSTs' knowledge for teaching FMC, the nature of the intervention tasks need to cultivate in PSTs a habit of being determined to find out how strategies worked. Tasks should create the opportunity for PSTs to use prior knowledge to realise their potential to invent calculation strategies. Consequently, flexibility in thinking and appreciation of the role of existing knowledge in mental calculation may manifest in PSTs. Thus, as a guide, the 4-6-1 model directs TEs towards encouraging PSTs to venture into new ways of calculating without fear and to persist in finding a solution despite unsuccessful attempts.

With regard to resourcefulness, when thinking, PSTs' own intellectual and physical potential amid the resources in the learning environment need to be capitalised on to support learning. Because a classroom is considered a social environment, any intervention should promote learning through communication, sharing of ideas and teamwork. Collaborative learning through discussion requires time for PSTs to carefully draw on existing knowledge for verification of strategies and to consider replacement of inefficient strategies through reflection (Kieboom, 2013). Thinking of what is already known can scaffold the reinvention of calculation strategies to calculate with strategies that PSTs understand.

With regard to wisdom, the 4-6-1 model informed this study that success depended on discovering personal interests. This relates to using preferable calculation strategies and confronting challenges such as struggling to construct PSTs' own calculation procedures and finding the best solutions to such challenges. Discovery of their own potential has great bearing on PSTs' recognition of what counts most in the development of FMC skills. Supporting PSTs through the cultivation of ways of thinking to solve problems may address issues of anxiety in developing FMC skills. The 4-6-1 model provides a skilful approach to developing PSTs' confidence to engage in mathematical thinking processes independently.

2.13 Mathematical knowledge for teaching flexible mental computation

Teacher education has the responsibility to develop a strong and critical reform-oriented mathematical knowledge base necessary for teaching elementary school mathematics. MKT is defined as "the mathematical knowledge that teachers need to carry out their work as teachers of mathematics" (Ball, Thames & Phelps, 2008, p. 1). The development of mathematical knowledge for teaching is critical if PSTs are to teach mathematics effectively because MKT impacts learning directly. Learners whose teachers have sound MKT perform better compared to learners whose teachers have inadequate MKT (Ball et al., 2005). Similarly, studies conducted in England, Turkey, Australia and the USA emphasise the importance of teachers' mathematical content knowledge and pedagogical content knowledge for teaching (Ball, 2000; Walshaw, 2012). The current study echoes Walshaw's (2012) argument that core to successful instruction is the ability and expertise with which a teacher confronts the intellectual requirements of teaching. Therefore, importance of teacher competency prompts a need to focus on PSTs' development of FMC skills so that they will able to respond positively and effectively to the demands of teaching, as is evident below.

Evidence of the impact of improved teacher knowledge is outlined in literature. Reference is made to China, one of the leading nations in learners' mathematics performance with Ma (2010) stating that Chinese learners outperformed USA learners on international mathematics tests. The comparative study by Ma (2010), on Chinese versus USA teachers' subject matter knowledge of mathematics, points to the teachers' qualifications and the kind of knowledge that informs their teaching practice. The study shows that although Chinese elementary school teachers involved in the study did not complete high school, their understanding of elementary mathematics surpassed that of USA teachers with a bachelor's degree and who had access to in-depth content knowledge both in high school and tertiary education. Consequently, numerous researchers call for PSTs to have a thorough understanding of mathematical concepts offered in school and how elementary school learners develop specific skills (Kilpatrick et al., 2001; Kuhs & Ball, 1986; Ma, 2010). These findings suggest that an in-depth understanding of the subject matter to be taught matters more than mere acquisition of advanced mathematics content knowledge.

Although literature on mathematical knowledge for teaching FMC to learners exists, limited studies have critically discussed how to develop PSTs' mathematical knowledge for teaching FMC (Ball et al., 2005; Duthilleul & Allen, 2005). Nevertheless, the existing research could possibly illuminate the initiatives aimed at improving PSTs' mathematical knowledge for teaching FMC. The literature suggests that teachers' content knowledge and their knowledge of how to support learners to learn must develop concurrently. Understanding the subject matter should support PSTs to understand how learners learn. Knowledge of the process that learners follow to construct calculation methods should also support PSTs to develop content knowledge (Ma, 2010). With regard to how such knowledge must be developed, Ma (2010) emphasises the need for PSTs to learn by doing. Evidence from Ma's (2010) study indicates that Chinese teachers who were successful in their teaching often solved in advance the problems that they expected their learners to solve and explored ways to explain and analyse such problems to assist learners. The approach advocated by Ma (2010) is relevant to teacher preparation in Namibia considering the findings of the SAQMEC III study (Spaull, 2011) in which a number of primary school teachers were unable to solve problems that their learners were expected to solve in school. Another study conducted by the University of Namibia in conjunction with the United Nations Educational Funds-in-Trust on elementary teachers (UNUNEFT, 2014) found teachers struggling to teach the four basic operations meaningfully. Therefore, improvement of PST knowledge is critical for this study.

Further recommendation of quality teacher preparation is evident in literature. Despite a record of improved teacher performance in SAQMEC IV study (Shigwedha et al.2015), Namibia considers the achieved progress as minimal. Namibia still finds the situation disturbing as the country ranks eighth among 13 participating countries (Shigwedha et al., 2015). The findings of the SAQMEC IV (Shigwedha et al.2015) study advocate for effective PST mathematics development to improve their content knowledge. This study argues that effective PST support requires exploration of PSTs' computation knowledge upon entry to university to identify their needs and ways to assist them (Bolden et al., 2013; Henderson & Rodrigues, 2008). Existing research has found in PSTs persisting practices such as the use of standard algorithms, the pen-and-paper method involving one strategy only and calculator usage (Flowers et al., 2003). Research has also found PSTs having limited understanding of operations, inability to calculate apply reasoning skills to calculate differently and inaccuracy in justifying calculation strategies (Flowers et al., 2003). As a result, efforts to address the content needs prevalent in PSTs need to focus on the development of knowledge necessary for teaching.

To understand the required MKT, this study was inclined towards the two main knowledge domains by Ball et al. (2008) that originated from Shulman's (1987) seven categories. This study selected the domains by Ball et al. (2008) as it focussed on MKT rather than on knowledge for teaching in general. Recent research findings provide evidence that MKT has a positive impact on teaching (Marshall & Sorto, 2012). Therefore, the meaning of the knowledge categories outlined by Ball et al. (2008) and how this knowledge should develop in PSTs are discussed next.

2.13.1 Subject matter knowledge

Effort to develop PSTs' knowledge for teaching ought to include subject matter knowledge. Subject matter knowledge entails two main aspects: having knowledge of a subject in terms of the mathematical facts, concepts and skills that learners are expected to learn, and having knowledge of the nature of computations. An explicit definition of teacher content knowledge is provided by Shulman (1987, pp. 8-9) in two ways. On the one hand, it refers to "knowledge, understanding, skill, and disposition that are to be learned by school children". On the other hand, it refers to knowledge of how and why such knowledge is structurally organised as it is and the role of each topic in the subject. Subject matter knowledge also includes understanding the reason for teaching such knowledge and why the facts and concepts are of that nature. Consequently, to explain subject matter successfully in school, evidence from Kind's (2014) study proposes that teachers develop precise, profound and rich subject knowledge. For the development of knowledge to teach FMC skills, PSTs require specific subject matter knowledge.

The current study noted that the starting point for PST instruction was comprehension of the calculations elementary school learners were expected to learn, how calculations are structured and why calculation strategies are as they are (Shulman, 1987; Hill, Rowan & Ball, 2005; Ma, 2010). Deficiencies in PSTs' content background and insufficient understanding of school mathematics have been attributed to mathematics courses that do not focus on the content PSTs are expected to teach (Masingila & Olanoff, 2012). Besides, this study acknowledges the fact that although the development of subject matter knowledge cannot begin and end in tertiary institutions, focus should be on conceptual understanding and mental engagement as opposed to factual knowledge and procedural understanding only (Garet, et al., 1999). Therefore, in-depth understanding of the concepts incorporated into FMC is necessary.

Research has indicated that comprehension of content plays a major role in generating novel understanding and confidence in both teachers and learners (Shulman, 1987). Furthermore, the literature indicates that teachers with a strong content knowledge base are better prepared to detect and anticipate learners' mistakes and misunderstanding (Marshall & Sorto, 2012). A classroom setting where knowledge and meaning are reconstructed, teacher identification of learners' mistakes creates meaningful learning opportunities, which in turn fosters a positive attitude and confidence regarding mathematics learning (Ball et al., 2005; Ma, 2010). As to confidence, Emenaker (1996) confirms that mathematics subject matter knowledge has an impact on a teacher's confidence to learn and to teach mathematics. For teachers to teach mathematics confidently, to involve learners actively in learning and to prevent learners from opting out of learning mathematics, teachers need to understand the mathematics that they are going to teach. Hiebert et al. (1997, p. 2) emphasise that "understanding breeds confidence and engagement; not understanding leads to disillusionment and disengagement". To enhance a positive attitude and confidence regarding teaching and learning, teacher courses must focus on the three subcategories of subject matter knowledge.

The three subcategories under the domain of subject matter knowledge are discussed first, followed by the three subcategories under the domain of pedagogical content knowledge. The three subcategories under pedagogical content knowledge are knowledge of content and students, 'knowledge of content and teaching, and knowledge of the curriculum. The three subcategories under subject matter knowledge are common content knowledge, specialised content knowledge and knowledge at the mathematical horizon.

2.12.1.1 Common content knowledge

Development of PSTs' knowledge for teaching must focus on understanding the content to be taught and how to teach it. TEs are encouraged to develop in PSTs an in-depth understanding of the ultimate mathematics for elementary school (Ma, 2010). Comprehension of fundamental mathematics refers to understanding the key concepts imperative for elementary mathematics teaching. PSTs need to be able to do the mathematics meant for learners. For PSTs to support learners to calculate flexibly, PSTs have to be able to calculate flexibly themselves. Common content knowledge refers to knowledge known by both teachers and other individuals. Reference is made specifically to the mathematics content that PSTs must be able to deal with, the calculations that they expect their learners to perform and the knowledge possessed by any person who is mathematically well informed. Common content knowledge is content outlined in the curriculum meant to be mastered by learners. Such knowledge enhances a teacher's ability to detect wrong answers and to discriminate between accurate and inaccurate definitions of FMC encountered in textbooks. Common content knowledge fosters in teachers the use of correct terminology when discussing, explaining in class or writing on the board to represent information that is mathematically correct. Such knowledge is critical for PSTs to effectively perform the calculations that they expect their learners to perform. Research provides evidence that a lack of common content knowledge culminates in wrong pronunciation of mathematical terms, wrong calculations and confusion while attempting to solve a problem (Ball et al., 2008). Underdeveloped content knowledge impact negatively on teaching as time is wasted and understanding is impaired. Therefore, common content knowledge ought to incorporate the development of the four basic operations and comprehension of other basic concepts under FMC.

Before developing learners' FMC skills, PSTs should be able to add, subtract, multiply and divide whole numbers flexibly, fluently and accurately in the absence of a calculator, pen and paper or any other external devices. PSTs must calculate in a way that persons who are not teachers would perform such calculations with intentions other than teaching but to find a correct solution. In addition, PSTs must also have a proper understanding of FMC and what the four basic operations mean. Correct pronunciation and usage of terms such as addition, subtraction, multiplication, division, add, divide, subtract, multiply, minus, plus and other relevant concepts is important. Concerning efforts to improve PSTs' content knowledge, this study drew on the Math to Mastery intervention package provided by Mong and Mong (2012). The package includes preview problems, repeated practice, immediate corrective feedback, formative feedback and self-monitoring of progress. Implementation of the knowledge package is possible if Ma's (2010) recommendation is embraced to engage PSTs in solving whole number computations included in the elementary school syllabus. Basic understanding of content knowledge alone is insufficient for effective teaching; prompting the development of specialised content knowledge.

2.12.1.2 Specialised content knowledge

Becoming effective mathematics teacher demands knowledge unique to the teaching of mathematics. Specialised content knowledge comprises ways to make mathematics more understandable to learners and ways to probe learners' thinking. A contribution to learners' understanding of mathematical concepts is underpinned by the ability to outline critical aspects of particular concepts to make the concepts more clear and comprehensible to learners. While content knowledge relates to distinguishing an incorrect answer from a correct answer, specialised content knowledge entails the ability to identify the source of a mistake and to understand and translate learners' mistakes through the error analysis approach. The error analysis approach, which was adopted by this study, refers to the process of analysing a learner's mistake mathematically to detect the cause of the mistake (Ball et al., 2008). A swift identification of mistakes contributes to a quick response to learners' solutions and accurate teacher intervention by identifying additional strategies to direct learners towards a correct solution. Knowledge of source of mistakes guides a teacher in terms of what to do to promote the accuracy of learners' solutions.

Specialised content knowledge is not necessarily particular to mathematicians in general but is particularly known to teachers. PSTs need knowledge of how learners invent flexible calculation strategies, knowledge of how to evaluate a variety of solutions and effectiveness of strategies in solving other computations (Hill et al., 2005). Also PSTs need knowledge of the correct ways to explain and represent the meaning of operations and how to confirm the accuracy of an answer by outlining the significance of diverse calculation strategies. Comprehension of multiple ways of clarifying concepts enables teachers to accommodate learners with different learning needs and to provoke analytic thought and action in learners. In all, knowledge of concepts, operations and strategies is more critical to a teacher than to any other numerate person.

Specialised content knowledge also includes unique mathematical knowledge required by teachers that is not necessarily taught to learners. For example, knowledge of mathematical resources such as presentation of mathematical ideas, answering when learners want to know why certain aspects work as they do, conveying mathematical ideas using specific examples, noticing the ideas involved in a specific representation, connecting a representation to other representations and specific mathematical ideas, connecting topics with past or upcoming topics, and modifying tasks from textbooks to simplify them or to make them more challenging. Relating to FMC, comprehension of different ways to interpret operations in a way that is not known to learners is part of specialised content knowledge. The findings of a study by Flowers et al. (2003) indicate that teachers with an in-depth understanding of operations are able to justify calculation strategies effectively. Meanwhile, Flowers et al., (2003) discovered that teachers with a superficial understanding of operations were unable to justify why particular strategies worked while others offered justifications that were not clear. As a result, evidence calls for teacher education to focus on the development of PSTs' comprehension of operations and ability to justify their own and others' calculation strategies logically. Recent literature provides an in-depth outline of the specific knowledge required for teaching FMC.

Research provides guidelines as to what knowledge is critical in the development of calculation skills. Ma (2010) refers to the knowledge essential for the teaching of calculation as 'knowledge packages'. The three knowledge package models comprise "subtraction with regrouping, multi-digit multiplication and division by fractions" (Ma, 2010, p. 114). This study excluded the knowledge package of division by fractions as the study focused on whole-number computation only. This is why the subsequent discussion is centred on the knowledge packages of subtraction with regrouping and multi-digit multiplication. The two knowledge packages are adopted because in-depth comprehension of the four basic operations is critical to PST development of skills to teach FMC.

As elaborated by Ma (2010), the knowledge packages develop sequentially, beginning with the subtraction package followed by the multiplication package. The subtraction package contains four structures, starting with "addition and subtraction within 10, followed by addition and subtraction within 20, then subtraction with regrouping of numbers between 20 and 100; and subtraction of large numbers with regrouping" (Ma, 2010, p. 114). The multiplication package consists of three structures, which are "multiplication by one-digit numbers, multiplication by two-digit numbers, and multiplication by three-digit numbers" (Ma, 2010, p. 114). The knowledge package order presented above is strategic in achieving competency in addition, subtraction, multiplication and division (Ma, 2010). It is worth noting that the knowledge packages, though ordered, do not develop in isolation but are linked further in subsequent mathematical domains. The relevance of the knowledge packages to the present study lies in the design of the intervention tasks in terms of the number range and in the development of knowledge at the mathematical horizon.

2.12.1.3 Knowledge at the mathematical horizon

The development of mathematical knowledge for teaching FMC encompasses a broad spectrum. Besides mastering the content of a specific grade, a teacher requires 'horizon content knowledge' (Ball et al., 2008). Horizon content knowledge refers to understanding the connections among topics across the elementary mathematics school curriculum and advanced grades. For this study, understanding how FMC in pre-primary school linked with FMC in grades 2, 3 and beyond enables teachers to teach with the intention to lay a good base for learning FMC in subsequent grades. The absence of horizon content knowledge could limit a teacher's approach to teaching particular topics, eventually inhibiting meaningful learning of a similar topic in later grades. However, a question remains regarding what advanced level is involved, in terms of later grades, and how PSTs could develop horizon content knowledge considering the limited time available during a teacher preparation course.

Incorporating knowledge at the mathematical horizon into FMC points to understanding how the four basic operations and other topics are linked within the elementary mathematics curriculum. Another important aspect for PSTs to develop is knowledge of how mental computation is connected to other subjects offered in elementary school, upper primary school and high school. In addition to content specific to elementary school level, PSTs' high school mathematical knowledge must also be improved if found underdeveloped. Improving PSTs' high school knowledge is instrumental in familiarising PSTs with how the basic operations and FMC support learning in the advanced grades. PSTs need to realise and appreciate the benefit of existing connections in the curriculum to lay a good foundation for learning algebra and other subsequent topics and subjects. Building a solid foundation for further learning require pedagogical content knowledge.

2.13.2 Pedagogical content knowledge

Effective development of FMC skills requires pedagogical content knowledge. Pedagogical content knowledge entails comprehension of the organisation and representation of subject matter and how learning areas can be shaped to address every learner's potential (Shulman, 1987). Existing research has revealed that PSTs need to develop the ability to reason pedagogically and act accordingly even when their subject matter knowledge upon entry to university is at an acceptable level (Battista, 1994). In other words, PSTs must be familiar with particular challenges that learners experience in learning certain mathematics topics (Ball, Lubienski, & Mewborn, 2001). Perceiving numbers from a teacher's perspective only may make it difficult for teachers to predict challenges that learners might experience (Parker & Baldridge, 2003). With knowledge of learning difficulties, mathematics teachers would ensure that learners understand both the concepts and the computation strategies being used and why they produce correct solutions to problems.

The absence of pedagogical content knowledge inhibits learning with understanding. The effect of no pedagogical content knowledge is evident in Battista's (1994) study in which a teacher with advanced mathematics subject matter knowledge was unable to teach for conceptual understanding but rather focused on procedural knowledge that resulted in inadequate learning with understanding. Pedagogical content knowledge is critical for teaching to enable existing content knowledge to be changed and deepened to be in line with principles for teaching school mathematics (Goulding, 2003; Kajander, 2010). At the same time, teachers need to understand the unique combination of subject matter and teaching skills intended for teachers. Therefore, for learners to calculate flexibly in mind, PSTs need mathematical knowledge for teaching FMC skills.

Pedagogical content knowledge is subdivided into three-subcategories which are; knowledge of content and students, knowledge of content and teaching and knowledge of the curriculum (Ball et al., 2008). This study noted that all aspects that promoted learning required teachers' attention, thought and action or reflection before, during and after the lesson. Most importantly, teachers must consider how to provoke analytic thought and action while thinking of what to teach and how to teach it in a more understandable way. Therefore, PSTs need to develop knowledge of content and students.

2.12.2.1 Knowledge of content and students

It is important to understand how to present subject content knowledge in a meaningful way. Knowledge of content and students relates to knowledge of what learners do, what they are likely to struggle with and what their common mistakes are. Knowledge of content and students pertains to having knowledge about learners and knowledge of mathematics. PSTs have been found to have a deficient understanding of how learners learn (Kieboom, 2013). One way to develop knowledge of content and students is through creating opportunities for PSTs to learn by solving and reflecting on problems in the way that learners are expected to calculate. Through such a process, PSTs personally experience how learners think and the kind of mistakes that they commonly make. With knowledge of learning difficulties, PSTs are likely to design problem solving tasks and contexts that will be interesting and motivational to learners. Furthermore, for teachers to identify critical mathematical concepts and skills, teachers need to thoroughly comprehend how mathematical ideas are arranged and understand how learners explore mathematics. Knowledge of content and learners develops PSTs' ability to predict what learners could possibly do with prepared activities and whether problems would be easy or difficult for learners to work out. Another important aspect is PSTs' ability to listen attentively, make sense of learners' calculation procedures and translate learners' promising but incomplete mathematical ideas (Shulman, 1987; Flowers et al., 2003). An attempt to translate learners' ideas requires knowledge of how to communicate mathematical understanding to learners to avoid creating the impression that particular aspects are more important or less important than others. As a result, specific mathematical comprehension and thorough knowledge of learners and understanding of how they think mathematically is required.

In relation to FMC, what is important for knowledge of content and students is consideration of issues that could influence learners' reaction towards various types of depictions and instruction. These are issues such as learners' potential, teachers' and learners' expectations concerning the learning of FMC, comprehension of language and concepts used to formulate problems, cultural traits, level of motivation and what learners understand and can already do. Consequently, PSTs require the ability to recognise learning as facilitated by representation of numbers using appropriate and correct language pertaining to the four basic operations, explanation, pictures, concrete objects and symbols. Moreover, PSTs require comprehension of how the use of pictures and other visual models for quantities such as the ten frame and the five frame improve knowledge of how numbers are built up and decomposed. Another critical aspect regarding content and students is awareness of how visual models for quantities support learners to organise quantities in the mind and their use as mathematical tools to think with and to record and communicate calculation strategies.

In addition to the knowledge packages discussed under specialised content knowledge, PSTs need to understand the number combinations that learners would find very easy to invent mental calculation strategies with, for example problems involving numbers near doubles or close to 10. Knowledge of number combinations leads to awareness of how learners construct calculation strategies and knowledge of which calculation strategies learners are likely to struggle with, for example standard algorithms. Problem solving is an important strategy for exploring the ways in which learners think and calculate. Through problem solving, PSTs could be exposed to effective ways to organise the learning of the four basic operations and to ensure the quality of activities that would motivate the construction of calculation strategies.

Engagement of PSTs with appropriate problems for FMC would develop PSTs' ability to anticipate the level of difficulty of the number combinations in a task and to understand learners' calculation strategies even when the strategy has missing information. Furthermore, knowledge of content and students would allow PSTs to make the connections among the four operations evident to the learners, for example understanding that knowledge of the basic addition facts could facilitate the development of subtraction facts (Flowers et al., 2003). Another example is PSTs' understanding that subtraction can be used to solve a division problem through appropriate FMC tasks. Therefore, it is important for PSTs to understand how their choice of numbers, the context for problem solving and the kind of terminology is influenced by learners' potential, expectations, experience and prior knowledge.

Knowledge of content and students informed this study to establish PSTs' existing knowledge of FMC, their ability to calculate mentally, their experience in terms of how they had learned FMC in school, their learning expectations, their level of motivation and their beliefs about the nature of FMC and how it should be taught in school. Knowledge of how learners develop FMC skills supports the development of knowledge of content and teaching.

2.12.2.2 Knowledge of content and teaching

PSTs need knowledge of content and knowledge of teaching to develop learners' FMC skills. This refers to content knowledge that illuminates the planning of the teaching process, also implying familiarity with crucial aspects of teaching and mathematics. Focusing on teaching, Shulman (1987) defines teaching as action involving understanding and the ability to reason, think and act upon thoughts to effect change of individual actions (Shulman, 1987). Also, Shulman (1987) emphasised that, prior to teaching, understanding of content must prevail. Research conducted in Namibia by Courtney-Clarke and Wessels (2014) found among final-year PSTs a lack of number sense and flexible number sense strategies to solve problems and calculate mentally. Literature indicates that the development of PSTs' knowledge of content include understanding number size, operations, number facts, number relationships, place value and the use of number symbols (Sowder et al., 1994). Whitacre and Nickerson (2006) assert that elementary teachers need to develop number sense to understand elementary school learners' informal calculation strategies. Tasks for PSTs need to include problems that involve both small and large numbers for improved number sense and mental calculation strategies (Flowers et al., 2003). Therefore, the development of PSTs' content knowledge of FMC ought to focus on number size and number relationships.

Besides understanding specific content for teaching, teachers need to understand the use of such knowledge in the classroom. For example, teaching the four basic operations would require analysing similarities between the operations and the images, illustrations, expressions and models that would explain the operations meaningfully. Hill et al. (2005) point out that teachers who understand the subject content very well can teach effectively if they can use their knowledge to execute tasks such as listening to learners, choosing and utilising effective tasks, and handling discourse and learner activity. Similarly, PSTs must understand different approaches to teaching FMC and recognising ways to link FMC to other elementary school mathematics topics and subjects. Furthermore, knowledge of teaching would include the ability to manage and organise learners in a way that promotes the invention of more efficient calculation strategies, skill to probe learners' thinking, being able to choose the kind of feedback to provide and being able to manage feedback sessions, possessing the skill to create the right learning environment, understanding how to navigate the curriculum and how to determine learners' progress. This study asserts that knowledge of the management of teaching is critical if learning is to be effective.

As part of knowledge of teaching, it is important for PSTs to embrace the idea that the manipulation of mathematical signs in writing and the learning of mathematical regulations mechanically do not imply the learning of mathematics (Hiebert et al., 1997). A practical argument clarifies how learning is not memorisation in the statement that "when we memorize names and dates we are not learning history [also] when we memorize titles of books and authors we are not learning literature" (Hiebert et al., 1997, p. 2). The same applies to mathematics in that when we memorise mathematics rules without understanding, we are not learning mathematics. As to FMC, memorisation of calculation strategies inhibits the ability to make mathematical judgements. Research shows that accuracy of solutions stems from understanding of numbers, which informs mathematical judgement (Sowder et al., 1994). Thus, as part of knowledge for teaching, the ability to make mathematical judgements is important for PSTs to efficiency of calculation methods and select appropriate strategies to guide learners when performing mental calculations. In all, knowledge of content and students incorporates aspects relevant to FMC and the teaching process relevant to FMC skills development as supported by knowledge of the curriculum.

2.12.2.3 Knowledge of the curriculum

Knowledge of the curriculum has a direct impact on PST knowledge for teaching FMC skills. Knowledge of the curriculum refers to specific concepts that teachers and learners must engage with in the classroom. This refers to all the mathematics topics included in the elementary school curriculum. Shulman (1987) states that teachers need understanding of the multiple ways in which a mathematical concept can be linked to other mathematical concepts, to real life and to concepts in other school subjects. Linking ideas to other school subjects requires understanding of the concepts incorporated into other subjects, particularly at elementary school level. A link of mathematics to other subjects develops learners' mathematical knowledge for developing competencies to operate in a community with mental freedom and to enhance justice. Specifically, curriculum knowledge must involve understanding of the organisation of the themes covered in an academic year and in subsequent years.

Relating to FMC development of PSTs, the TE used the mathematics syllabus for elementary school to design both test items for the diagnostic test and problem solving questions for the intervention tasks. The number range used in both the diagnostic test and the intervention tasks matched the number range for Grade 1 to Grade 3, encapsulating the number range meant for elementary school level. In addition to knowledge of the number range, PSTs were exposed to the kinds of strategies proposed in the curriculum, the organisation of themes and the kinds of learning resources necessary. Drawing numbers from the curriculum was done to improve PSTs' knowledge of content and learners and to connect FMC skills as offered in all elementary school grades. To have a better understanding of curriculum knowledge this study considered two categories of knowledge of curriculum.

Shulman (1986, p. 10) refers to the two categories as "vertical curriculum knowledge" and "lateral curriculum knowledge". 'Vertical curriculum knowledge' imply understanding the curriculum content previously taught and the forthcoming curriculum content in the mathematics curriculum. Significantly, knowledge of the number range across the grades intensifies PSTs' ability to use available curriculum material such as textbooks and to systematise the teaching of mathematics topics. Recognition of available curriculum material permits establishment of the suitability of such resources to elementary mathematics teaching. For example, aspects of how FMC is linked to algebra, geometry and other topics in elementary mathematics may raise awareness of how PSTs could best lay a solid foundation for learning other mathematics topics (Ma, 2010). Furthermore, such knowledge helps in structuring mathematics to link mathematics topics and concepts.

As to 'lateral curriculum knowledge' PSTs require awareness of how the mathematics curriculum is linked to other subject areas that learners are engaged in (Shulman, 1986, p. 10). In the absence of explicit information pertaining to how PSTs' curriculum knowledge of FMC must be developed, this study sought to use other elementary subjects such as environmental studies to provide a real-life context for problem solving activities. This would facilitate awareness of the connections among FMC, real-world mathematics and other subjects offered at the elementary school level. Connection of FMC to real life provides the opportunity to reinforce the development of FMC skills in other subjects while strengthening learners' competencies to operate in a community with mental freedom.

2.14 Conclusion

The aim of Chapter 2 was to define concepts pertaining to FMC and its development both in learners and PSTs. Relevant literature was discussed to define FMC, to illustrate the importance of FMC and to explain the foundational aspects that underlie the development of FMC skills of both learners and PSTs. Exploring methods to develop PSTs' knowledge of how to teach FMC in school is indispensable considering the impact of mental computation in a real-world setting, in building personal mental capacity and in learning in school. Although extensive literature exists on PSTs' knowledge, experiences and beliefs, the literature search yielded no study specifically on how to develop PSTs' mathematical knowledge for teaching FMC. However, massive research on how to develop learners' FMC skills illuminated the present study. A principal idea in the literature is that an attempt to develop PSTs' mathematical knowledge for teaching FMC is founded on understanding how learners think and learn. As to PSTs' development of subject matter and pedagogical content knowledge, existing studies suggest that a focus on in-depth understanding of the content knowledge that PSTs are expected to teach is necessary. Currently, research findings on PSTs' subject matter knowledge, pedagogical content knowledge and beliefs prompt intensive and appropriate intervention to prepare PSTs adequately for teaching. Several research findings indicate that subject content knowledge, pedagogical content knowledge and beliefs may develop concurrently once PSTs are afforded the opportunity to learn through a problem solving teaching approach.

The literature study established that learning through problem solving was an effective way to engage teachers in the kind of disposition, thinking, reasoning, reflection and discussion comprising successful classroom discourse for developing the FMC skills of learners. Specifically, the literature points out the need to develop PSTs' number knowledge, knowledge of number patterns and knowledge of number relationships that facilitate the invention of calculation strategies and improvement of their ability as prospective teachers to calculate fluently and flexibly and to develop the ability to think tactically. However, a concern relates to mathematics teacher education programmes with a focus on the teaching approaches used by TEs to teach FMC and the relevance of the curriculum for elementary school PSTs. Issues pertaining to a mismatch between the tertiary mathematics curriculum and the elementary school curriculum also emerge in the literature. More studies are needed to establish how PSTs are currently learning to teach mathematics and FMC in particular. Chapter three provides a discussion of the theoretical stance of this study.

CHAPTER 3: THEORETICAL FRAMEWORK

3.1 Introduction

This study was informed by the constructivist theory of teaching and by RME as aligned with Namibian education reform towards learner-centred education (LCE). LCE was adopted following a conclusion that learners experienced difficulties to learn through a teacher-centred approach (Ministry of Education and Culture, Namibia, 1993). LCE views a learner as an active participant whose existing knowledge and experience are critical for further learning (MEC, 1993). LCE is a teaching and learning approach that perceives a learner as key to the learning process. It is an approach whereby a teacher considers learners' backgrounds, prior knowledge, experience, interests and learning strategies. As for this study, constructivism means a process of learning by connecting new knowledge to existing knowledge through active involvement, assimilation and accommodation (Piaget, 1978). The purpose is to guide and actively involve learners in their learning process using their curiosity and passion to learn. LCE relates to the constructivist theory as it also advocates for the active participation of learners in their own learning process. Therefore, the constructivist theory and RME were suitable for this study because learning is not a process of regurgitating facts or mental calculation strategies prescribed by a teacher (Gray & MacBlain, 2012). In this study, the constructivist learning theory and RME were utilised to inform both the design of learning activities and the learning environment envisaged to support the development of PSTs' mathematical knowledge for teaching FMC.

3.2 A constructivist theoretical approach to teaching flexible mental computation

3.2.1 View of knowledge

According to constructivists, knowledge is not something that is readily available to learn mechanically but can be explored and learned with understanding. Hein (1991, p. 2) states that "there is no...knowledge 'out there' independent of the knower, but only knowledge we construct for ourselves as we learn". Once knowledge is presented as facts and strategies readily available for learners to memorise, learning with understanding cannot prevail (Brophy, 2010; Freudenthal, 2002). Hence, when teachers prescribe mental calculation methods, learners may imitate them but are not likely to make connections with their existing knowledge and make sense of the new information. In addition, when knowledge is perceived to be 'out there' and learned as prescribed by a teacher, such knowledge is often not meaningfully applied in contexts other than the one in which it was presented (Maclellan, 2001). For the development of mental calculation, it would mean a situation where a teacher prescribes calculation strategies and number facts to learners without allowing them to discover these personally. Therefore, it is necessary for mathematical facts to be understood through active participation both physically and mentally to construct meaningful mathematical facts and strategies.

The constructivist learning theory perceives learners as individuals with the potential to assemble and organise material both physically and mentally (Bruner, 1996). The constructivist perception advocates for learners to have the opportunity to manipulate mathematical objects to form something and arrange objects systematically. To manipulate objects and mathematical ideas such as number facts and calculation strategies physically and mentally, factors like attention, perception and active use of reasoning skills play a role. Considering the cognitive theory of learning, it is necessary to emphasise that the development of thinking and reasoning skills emerge from a problem solving strategy coupled with comprehensible teaching goals. Thinking and reasoning skills lead to the construction of mathematical facts and strategies.

Construction of knowledge is fostered through problem solving as learners explore solution strategies to specific real-life problems. From a constructivist perspective, knowledge is "...not imitated or from birth, but 'actively constructed' by the child. In this way thought is seen as deriving from action; action is internalized, or carried out mentally in the imagination, and in this way thinking develops" (Nunes, 1995, p. 220). Once the learner is actively engaged in problem solving activities, thinking and reasoning skills develop and culminate in the meaningful construction of number facts and computation strategies. This study considers construction of meaning crucial for learners to make sense of calculation methods and internalise mathematical knowledge to make it their own.

The kind of mathematical knowledge learners have to internalise appears in three forms. Piaget (1971) subdivides mathematical knowledge into innate knowledge (social-conventional knowledge), physical knowledge and logical-mathematical knowledge. The first kind of knowledge is social knowledge that involves words used in mathematics. Kamii and Joseph (2004), in their discussion of Piaget' (1971) ideas, indicate that social knowledge such as words develop partially through agreements among people on what words to use to refer to particular objects or ideas. Consequently, communication is critical for the attainment of social knowledge. In relation to FMC, symbols and knowledge of words such as number, count, one, two, three, addition, subtraction, multiplication, division, operation, product, sum, quotient, difference and so forth relate to the social-conventional knowledge of FMC. Social-conventional knowledge also forms mathematical language of calculation developed through classroom discourse.

The second kind of knowledge is physical knowledge that is constructed from the physical world. Physical knowledge is constructed through abstractions generated from real-life experiences and actions on concrete objects (Piaget, 1971). Initially, a learner focuses on a particular attribute of an item and overlooks the remaining attributes. For example, a learner counting on marbles will at first focus on moving each object while counting without knowledge of the cardinality aspect, quantity or addition structure. It is worth noting in this study that physical knowledge does not translate into logical-mathematical knowledge until the cardinality principle is consciously constructed internally.

The third type of knowledge relates to logical-mathematical knowledge. According to Piaget (1971, p. 15), logical-mathematical knowledge is constructed internally through activities based on the order of actions by way of “reflective abstraction” made on “sensorimotor schemata”, also referred to as “constructive abstraction”. Logical-mathematical knowledge develops in three stages: the stage of exploring physical objects using the senses, the stage of recognising differences and similarities among objects using concrete objects, and the stage of using numbers and symbols or logical abstraction. As explained by Piaget (1973), the sensorimotor schemata evolve as learners explore concrete objects found in their environment physically and purposefully. After sometime, learners construct the idea that objects can be represented abstractly because such objects continue to exist when they are physically absent, a principle Piaget (1971) refers to as ‘object permanence’. During the stage of recognising differences and similarities, learners continue to observe objects to see how one object is similar to another or different from another, to find connections among objects, to see how objects can change in terms of position, substance or quantity and to group objects to form sets of the same size. The logical abstraction stage involves the numerical representation of the items of a set.

Understanding the logical abstraction stage is relevant to the development of FMC skills as representation of numbers is involved in computation. In mental computation, numerical representations involve both counting, which is more linguistic, and the mathematical skill of assigning a numeral to a particular set in order to denote it abstractly. In addition to counting, mental calculations involve combining two different sets or adding them together, subtracting items from a set, adding two or three sets with the same number of objects and grouping items of a particular set or grouping a number of items into sets of specific number of items. As a result, logical-mathematical knowledge develops through the use of symbols and words to represent sets and the number of items in each set as, for example two (2), four (4), two sets of four as 2 × 4, add/plus (+), subtract/minus (–), multiply/reproduce (x) and divide/share (÷). Underpinning logical-mathematical knowledge is the abstract representation emerging from the broad organisation of mathematical activity. With logical-mathematical knowledge, both PSTs and learners can make generalisations to solve problems in new contexts. However, for logical-mathematical knowledge to develop, Piaget (1973) emphasises the need for PSTs to do their thinking themselves to construct logical-mathematical knowledge autonomously. Therefore, it is necessary for educators to have the appropriate view of learning.

3.2.2 View of learning

Educator’s view of what learning is impacts their approach to teaching mathematics. Constructivist theorists such as Bruner (1977) indicate that learning is a process of gaining, transforming and evaluating new information. For learning to occur new facts need to be modified and assessed to establish its value. Once new knowledge is constructed, it must be changed to match a learner’s daily life activities and learners need to establish whether such knowledge is effectively changed to properly address their real-world mathematical encounters. In addition, Cobb (1994, p. 13) argues that the learning of mathematics incorporates individual active construction of meaning through a process of "...enculturation into the mathematical practices of wider society". In other words, learning is a process of cultural transmission and learners must develop diverse mathematical practices and ideas as occur in real life. In addition, Bruner (1977) emphasised that learning occurs through acting on concrete objects, especially at an early childhood stage. Therefore, while developing PSTs' knowledge to teach FMC; TEs need to support PSTs to understand the role of concrete objects in teaching specific concepts.

Research shows that learning occurs as a learner mentally carries out operations that represent concepts and the connections among concepts (Bruner, 1977). In the case of this study, operations would mean the design of problems that involve the use of all four basic operations of mathematics (addition, subtraction, multiplication and division) and that create connections among the four basic operations. It is critical to consider all four basic operations because knowledge development is about the composition of knowledge as a complete set of ideas comprised of "complex cognitive structures" (Piaget, 1964, p. 177). Therefore, appropriate structure for the development of all four basic operations is important.

Structuring the learning process of the four basic operations require designing tasks that are organised into manageable chunks. The learning process must begin with a comprehension of basic concepts and procedures before discussing procedures that are conventionally correct and difficult, for example, the standard method. Basic concepts and procedures have to be developed in connection with prior knowledge and existing experience. New knowledge must be compared and connected with prior knowledge by identifying similarities between the basic concepts and calculation procedures. Kamii and Joseph (2004, p. 17) support the need for connection in the statement that "arithmetic … [is] constructed by each child by making higher-level relationships out of lower-level relationships created before". In FMC development, basic understanding of number facts, number relationships and operations should be linked to advanced ways of perceiving and computing numbers. Disconnection of new knowledge from existing knowledge inhibits meaningful learning. Thus, connection of between existing knowledge and new knowledge is crucial for the development of FMC skills.

Furthermore, to develop the FMC skills of PSTs, first new knowledge should be integrated into existing knowledge (assimilation) and transform existing knowledge to fit new knowledge in case existing knowledge does not match new knowledge (accommodation) (Piaget & Inhelder, 1973). Constructivists point out that as new knowledge is encountered, learners may end up in a state of confusion because of a contradiction that may occur between what they know and the nature of the new knowledge encountered, a mental state referred to as a state of disequilibrium (Piaget, 1964). So, TEs are urged to establish PSTs' existing knowledge to alter any misconceptions in PSTs' prior knowledge. It is important to acknowledge the fact that, just as learners, PSTs learn at a different pace though they all undergo the same stages of development (Piaget, 1964). However, 'schema' and previous experiences as well as memory and motivation were found to promote thinking and perception (Bruner, 1977; Piaget, 1964). Thus all PSTs need to develop FMC according to their pace of learning but need to be exposed to adequate challenge and stimulation of the mind for advancement of thinking (Piaget, 1964). Besides the design of challenging mathematics tasks, a suitable learning environment is necessary to stimulate mathematical thinking and reasoning.

3.2.3 View of the role of learners in a learning environment

Since constructivists believe that learning is an active process, a learner assumes the role of an active participant in the learning process. If learners do not actively participate in the learning process, the knowledge acquired will be meaningless and dormant in real-life situations. Constructivists affirm that "...there is no knowledge independent of the meaning attributed to experience (constructed) by the learner, or community of learners" (Hein, 1991, pp. 1-2). In learning FMC, calculation strategies are likely to have meaning when learners experience the invention of computation methods through problem solving. Eventually, knowledge of computation methods is not perceived as irrelevant to solving real-life problems amid experience of re-constructing strategies through "hands-on involvement" (Hein, 1991, p. 4). In all, it is worth supporting PSTs to perceive learners as active participants and not as passive recipients of number facts and calculation strategies.

To be an active participant, a learner needs cultural tools and a positive attitude to learn effectively. Constructivists indicate that active learning requires participation in discussion, collaboration with other learners to complete tasks, individual mental engagement through metacognition, mathematical reasoning and critical and logical thinking. In addition, Vygotsky (1978) refers to the use of 'elementary mental functions' such as attention, sensation, perception, memory and cultural tools such as language and symbols. With regard to mental engagement, learners must regulate their learning themselves through self-interrogation or 'private speech' to determine the reasonableness of their calculation strategies and solutions. Self-regulation is vital as it enables learners to think about their own work and mental processes. To regulate their own thoughts, learners should be able to harmonise procedures carried out in the mind such as "memory, planning, synthesis, and evaluation" (Schunk, 2012, p. 275). According to Bruner (1996), learners ought to manage their own intellectual action to learn and make meaning of concepts and also to share understanding of concepts through teamwork. Teamwork is important because "mind is inside the head and also with others..." (Bruner, 1996, p. 87). In other words, discussion creates the opportunity to tap from each other's knowledge to develop calculation methods.

The process of inventing FMC strategies relies on discussion of calculation methods, a process that enables learners to evaluate their own strategies and to develop more efficient strategies. When discussing, PSTs have to understand that the role of a learner is to reflect, explain and justify calculation methods they have invented. Core to the process of development is that a teacher must allow learners to construct computation methods individually and in groups without any prior instruction on how to compute mentally. Bruner (1996, p. 84) stresses that "learning in its full complexity involves the creation and negotiation of meaning in a larger culture, and the teacher is the vicar of the culture at large". To ensure active participation, it is a teacher's responsibility to create a classroom culture that may promote commitment, involvement and collaboration. Consequently, it is critical to develop an appropriate view of the role of a teacher in a learning environment.

3.2.4 View of the role of teachers in the learning environment

For effective learning to manifest in a classroom, teachers must have the appropriate understanding of what their role in the classroom environment is. Teachers are significant to learning because they are the main driving force in the teaching process and the main agents of change (Bruner, 1977). This study finds it imperative to support PSTs to perceive themselves as a "communicator, model and identification figure" (Bruner, 1977, p. 91) to initiate and manage discussion clearly, model appropriate attitude towards FMC skills development and use flexible calculation strategies. Moreover, PSTs need to assume the role of developers of an active learning environment, a facilitator, an organiser of classroom experiences, an observer, a listener and a manager. However, PSTs' view of knowledge is likely to influence their view on what their role in the classroom is.

In the context of this study, PSTs need to understand that knowledge is not a duplicate of truth created through possessing a mental representation of truth but is a "natural psychological reality" (Piaget, 1964, p. 177). Furthermore, knowing FMC implies acting on a concept or carrying out calculations derived from a mental representation of reality (Piaget, 1964). Thus, a teacher's role is not to devise calculation strategies for learners but it is to select best learning resource material, to organise the learning environment, create different learning experiences, and provide support when necessary and to design activities that are central to knowledge (Bruner, 1977; Piaget, 1964). Teachers are also expected to have a strong knowledge base to represent reality in their classrooms and to design appropriate activities. Bruner (1977) emphasised that in presenting subject matter, teachers need adequate knowledge of the subject that they are teaching. A lack of sound subject matter knowledge leads to wrong explanations, ineffective instruction and inadequate knowledge development.

It is important that teachers give general and correct explanations for effective development of flexible and efficient calculation strategies. This study notes the emphasis Bruner (1977) made that "a correct and general explanation is often the most interesting of all" (Bruner, 1977, p. 23). In the context of this study, to provide correct explanations and to design appropriate activities, teachers must have a proper understanding of the relations among addition, subtraction, multiplication and division. From Piaget's (1952) perspective, teachers' with a thorough understanding of how the four basic operations function are better equipped to support learners to discover relationships among the four basic mathematics operations. Most important, teachers need to present the "fundamental structure of the subject...

[because]…understanding fundamentals makes a subject more comprehensible" (Bruner, 1977, p. 23), in the case of FMC would mean comprehending operations and the basic number facts. Therefore, teachers require knowledge of number relationships, the meaning of operations and the application of operations in real life, and understand how to present the operations orderly.

The order of presenting the content and language usage is very crucial in a learning environment. Constructivists have found that learning is "provoked by situations" (Piaget, 1964, p. 176) like classroom culture, tasks and assessment processes that are well organised. As a result, a teacher has the responsibility to organise and manage content, the learning environment, material, activities and discussion to stimulate learning. For example, a teacher must adopt a culture of posing appropriate questions orderly to trigger learning (Bruner, 1977). Particularly, the order in which the four basic operations are presented can impact comprehension of calculation strategies.

As to language, the processes of presenting the content, explanation, managing discussions and feedback sessions require the use of appropriate language. It is beneficial for teachers to use appropriate mathematical language and terminology to articulate mathematical ideas. Teachers must also pay attention to underdeveloped mathematical ideas such as concepts and procedures. In some instances, learners' knowledge structures can be partially constructed in previous topics or grades and need to be improved. In the process of improving learners' knowledge, teachers are expected to challenge and assist learners to generate answers but not to provide correct answers or to convey meaning directly. However, improving poorly developed concepts and procedures require appropriate teaching approaches, well developed knowledge of FMC terminology and establishing prior knowledge to inform the planning of activities.

When planning to teach, teachers need to craft activities in a form that relates to learners' existing knowledge, considering that knowledge is constructed meaningfully within a real-life context. The activities should include aspects that are relevant to learners' daily life experiences to aid conceptual understanding. Constructivists acknowledge the significance of meaningful activities as they form a basis for further learning that is in turn founded on existing knowledge (Bruner, 1977). In addition, the quality of tasks need to create the opportunity for more accommodation than assimilation to prevail because learning has been found to occur when more accommodation than assimilation occur (Piaget & Inhelder, 1973). Besides, the level of difficulty of the activity should accommodate learners' diversity in learning style and potential. The process of engaging learners in activities they are familiar with and that are a little advanced aid in conserving the knowledge already constructed (Piaget & Inhelder, 1973). Similarly, Hein (1991, p. 7) encourages teachers to use "different kinds of entry points, various sensory modes, different kinds of stimuli to attract a wide range of learners". For example, teachers may use colourful pictures, diagrams, worksheets, role play, storytelling, real objects, experiments and different kinds of problem solving activities to sustain learners' existing knowledge of FMC. Execution of activities must include different teaching methods such as "... discovery learning, inquiry teaching, peer-assisted learning, discussions and debates, and reflective teaching" (Schunk, 2012, p. 275). In all, the connection of tasks to prior knowledge using diverse forms of stimuli through appropriate teaching methods may have a positive impact on FMC skills development.

3.2.5 Classroom culture

Many factors from the classroom environment stimulate learning in various ways. Bruner (1977) cautions educators that for learning to be effective everything that leads to effective problem solving needs to be developed in learners. The development of FMC skills require considerations of aspects concerning attitude towards learning, degree of mental engagement, social interaction, mutual respect feedback and models used for learning. To start with attitude towards learning, teachers need to promote and model a positive attitude towards FMC and value each learner's existing knowledge and ideas. To cultivate a positive attitude in learners, tasks need to be designed in a way that promotes learning through discovery.

To construct efficient calculation methods, it is important to create a culture of mental engagement by solving problems that culminate in discovery of how addition, subtraction, multiplication and division are similar to and different from each other. According to constructivists a culture of learning through discovery promotes regulation of learners' own thinking processes that generate productive connections between prior and new knowledge and also to develop new knowledge (Piaget, 1971). Research shows that frequent experience with problem solving tasks requiring critical thinking and reasoning promotes meaningful learning. Also, independent discovery of relationships between operations promotes a feeling of self-confidence, resulting in a positive attitude towards learning.

In a constructivist classroom, a culture of working on a task both individually and collaboratively through discussions is indispensable. As a social constructivist, Vygotsky (1962) emphasises that learning prevails within a social context. Similarly, Bruner (1996) suggests that social interaction ought to occur in form of a community of inquiry (Bruner, 1996). Social interaction benefits individual understanding of diverse calculation strategies and mental ability to calculate flexibly. In addition, collaboration benefits learners cognitively as they interact with more knowledgeable learners, teachers and adults within the Zone of Proximal Development (ZPD) (Vygotsky, 1978). Another way social interaction improves learning is through discussion to share and to think about ideas, particularly during group work and feedback sessions (Kamii & Joseph, 2004). However, sharing of ideas require a culture of mutual respect and consideration of each other's ideas has to be promoted through listening and providing constructive comments on constructed computation strategies. Therefore, to learn effectively, a culture of working collaboratively in a respectful way is worth promoting.

In addition to social interaction, the use of models for learning is important in a constructivist classroom. Teachers are encouraged to enrich the learning environment with concrete material (Piaget, 1964) and use simulations to promote a symbolic representation of the four basic operations (Piaget & Inhelder, 1973). Piaget and Inhelder (1973) discovered that symbolic representations and mental images support the process of recognising ideas perceptually. It is also necessary for teachers to incorporate symbolic representations to support the formation of symbols and images for assimilation and accommodation of mathematical ideas. For example, symbolic representations such as operation signs contribute to the development of logical-mathematical knowledge (Bruner, 1977; Piaget, 1964). The use of concrete objects and representations like symbols, words and pictures to represent real objects is necessary to reactivate developed knowledge and to promote further construction of knowledge. Therefore, it is worth cultivating a classroom culture that values prior knowledge, uses appropriate tools for learning, promotes collaborative learning, social interaction and engages learners in appropriate kind of tasks.

3.2.6 The nature of tasks

The type of task used in a constructivist classroom impacts the kind of learning envisaged to transpire. According to Piaget (1964, p. 176), tasks are fundamental to learning because "an operation is ... the essence of knowledge ... [it is] an interiorized action which modifies the object of knowledge". The word 'operation' in the context of my study refers to the activities in which PSTs must be involved. The operations include adding, subtracting, multiplying and dividing numbers. As a way of promoting learning, Piaget (1964) and Bruner (1977) encourage the use of activities that elicit critical thinking. Moreover, Bruner (1996) calls for involvement in solving problems that are key to societal transformation (Bruner, 1996). Specifically, activities must involve partitioning and re-composition of numbers projecting a real-life problem to re-activate existing knowledge and enable generalisation of developed knowledge to different contexts (Piaget & Inhelder, 1973). As a consequence, adequate emphasis on problem solving is crucial for the development of critical thinking, generalisation of knowledge and interest in FMC.

As a way to promote learning, constructivists underscore the role of interest in the subject matter to be learned. Bruner (1977, p. 14) stressed that "…interest in the material to be learned is the best stimulus to learning, rather than such external goals as grades or later competitive advantage". Accordingly, this study embraced context-rich activities to promote PSTs' interest in calculating flexibly in mind. At the same time, interest of PSTs learning at a faster pace must be maintained through tasks that lead development beyond immediate level of understanding. Bruner (1977, p. 39) reports that "experience has shown that it is worth the effort to provide the growing child with problems that tempt him into next stages of development". Although Bruner (1977) used the word 'child' in his statement, extension of learning is also applicable to PSTs. Similarly, activities for PSTs have to promote logical reasoning through interesting activities to address the real needs of learners (Piaget, 1973). Subsequently, careful selection of numbers to include in activities is crucial in the coordination of PSTs' ideas and knowledge development (Piaget, 1951). Therefore, design of tasks that promote learning interest is worth undertaking.

In addition, literature also calls for special attention to maturation level and language usage when designing a task. Maturation must be considered in terms of age and mind in the sense that tasks are not too challenging or too easy to avoid boredom and opting out of the learning process. Mainly, the administration of tasks must begin with easy tasks based on one key concept such as addition and gradually shifting to more challenging tasks and concepts such as subtraction, multiplication and division. Such ordering of tasks is necessary to demonstrate the logic of tasks and to contribute to learning and recall of information (Bruner, 1977). A smooth transition from easy to more challenging questions ensures development of fundamental tools to confront problems at an advanced level. Piaget (1964, p. 184) asserts that "learning is possible if ... more complex structure is base[d] on simpler structures". The links among structures create relationships among concepts that lead to generalisation between mathematical ideas and development of new ideas (Bruner, 1977). Also, tasks should also be designed in a way that connects FMC to mathematics as a whole (Bruner, 1977). Therefore, the incorporation of mental computation skills into different mathematics topics like measurement and other topics is important.

Furthermore, constructivist pointed out to specific features of a calculation task. Bruner (1977) advocates for the design of tasks that may enhance understanding of general statements such as 'reversibility' and 'transivity'. Calculations requiring working backwards are important for understanding reversibility by reasoning that if 4 + 3 = 7, then 7 – 3 = 4 and 7 – 4 = 3. As to transivity, it is important to design tasks that foster the ability to argue that "if A > B and B > C, then A > C" (Harries & Spooner, 2000, p. 15). One way to foster the understanding of reversibility and transivity is through "concrete activity that becomes increasingly formal" (Bruner, 1977). Practically, to foster understanding of reversibility and transivity activities need to incorporate the use of concrete objects first before discussing it abstractly.

3.2.7 Assessment

Assessment is necessary in a constructivist classroom to create an opportunity for learners to recall and learn concepts both abstractly and practically. Although limited explicit discussion on assessment has been done by constructivist theorists, this study has drawn on a few existing ideas relating to assessment. After analysing Piaget's (1951) work, this study inferred that cognitive psychologists view assessment as an opportunity to reactivate knowledge (schema and structures) that was already in existence. Reactivation of learners' prior knowledge or knowledge constructed in the previous lesson serves to support the recall of learned concepts and to support the extension of existing ideas. Literature suggests that a constructivist teacher must use assessment to understand learners' reasoning ability as opposed to focussing only on the correctness of answers (Kamii, 1996). Another aspect to consider is the use of assessment to direct instruction by determining what learners know and what learners have learned from a lesson to know where to start or if it is appropriate to proceed to a new topic or not.

Furthermore, this study argues that a teacher cannot assume that knowledge has been constructed meaningfully until the developed knowledge is applied to solving problems arising in new situations. In the case of PSTs, new situations imply the inclusion of context-rich problems that are in line with daily mathematical encounters. Embedding FMC in real world mathematics can disclose the utility of mental computation skills in life. Hein (1991, p. 4) emphasised that "unless we know the 'reasons why', we may not be very involved in using the knowledge that may be instilled in us even by the most severe and direct teaching". Relating to calculation strategies, PSTs may devise efficient strategies in solving problems only if they discover features of numbers that make specific strategies efficient. Therefore, context-rich problems should be appraised for being instrumental in motivating learners to understand the application of FMC in real-world.

3.2.8 Similarities and differences between the main constructivist theorists

Table 3.1 Comparison of three theorists

Jean Piaget	**Jerome Bruner**	**Lev Vygotsky**
Mainly deals with age related development	Deals with the three modes of representation	Deals with the zone of proximal development (ZPD) and culture
Stages of development	**Modes of representation**	**Zone of proximal development**
Sensorimotor stage Pre-operational stage Concrete operational stage Formal operational stage	Enactive representation (action based) Iconic representation (Image-based) Symbolic representation (Language –based)	There are problems a learner can solve without support, with support and problems that cannot be solved even with an expert's support. Focuses on cultural context and social interaction. Particularly on how culture (values, traditions, beliefs and skills of a social group) progresses on.
	Learning	
Is an internal activity and language skills emerge from cognitive development but also ability to speak reveals level of understanding.	Learning progresses from enactive, iconic to symbolic representation, supported by social interaction. Language facilitates cognitive development.	Learning is an activity that is socially facilitated through scaffolding. Learners require adult support to learn. Learners learn by internalising observed behaviour which first appears externally before it is internalised.
Is a process resulting from biological maturation and interaction with the environment (adaptation to environment where assimilation and accommodation applies to existing schema). Social and emotional impact on learning are excluded. Learners must learn certain concepts only when the right stage of cognitive development is attained. Problem-solving skills cannot be taught and manipulation of tools support learning.	Very young children can learn any material provided that instruction is organized appropriately.	There are tasks learners cannot execute even with the support of a more competent teacher or learner. Advances in thinking capacity results from guided instruction provided by more knowledgeable member within the environment. Cultural tools such as signs, symbols representing words and systems of numbers facilitate learning.

3.3 Principles of realistic mathematics education

For this study, the review of literature has yielded limited research on how PSTs' mathematical knowledge must be developed constructively. Freudenthal (2002) reported that mathematics teacher preparation is new in the field of mathematics education though it is gradually receiving more researcher attention. However, this study was informed by RME considering its articulation of content and context (Freudenthal, 2002). RME is an instructional theory that originated in the Netherlands (Van den Heuvel-Panhuizen & Drijvers, 2014) and provides an instructional theory that is likely to improve the quality of PST preparation. As for the intervention of this study, RME guided the process of designing context-rich problems to develop PSTs' mathematical knowledge for teaching FMC by considering six principles of RME. The six principles are, the activity principle, guidance principle, intertwinement principle, interactivity principle, reality principle and level principle. A detailed discussion of how RME principles relate to learning is provided next.

3.3.1 View of knowledge

Discussing from RME perspective, knowledge is viewed as "form and content" that is organised in the form of linked structures. (Freudenthal, 2002, p. 12). RME define 'form' as the origin of number sequence. Number sequence is perceived to appear in verbal form whereas 'content' refers to applying the verbal form of number sequence to counting objects. Eventually, counted objects become content as mental objects in the form of whole numbers. Also, knowledge is perceived to involve content operations such as addition and subtraction as used in counting models. Understanding of the commutative property of addition develops as meaning is attached to addition in the counting process before engaging with it at an abstract level. Addition is also linked to subtraction through counting before it is approached abstractly.

RME view knowledge as existing in the reality dimension of distributing objects, which then translates into an abstract form as division. The meaning of the four operation signs and the abstract representation of the operations in the form of symbols (–, +, × and÷) also comprise knowledge. Certainly, arguing from RME perspective, the meaning of mathematics symbols emerges from practical use within a specific context. Specifically, knowledge is said to include "… proofs … definitions and notations [and] layout in print and thought" (Freudenthal, 2002, p. 15). In essence, justification of calculation strategies and solutions, the significance of operations signs, written and mental representations of problems constitute knowledge of FMC to be learned in a specific way.

3.3.2 View of learning

From RME perspective, learning is perceived to be a process of guided reinvention of the verbal and abstract representation of calculation procedures and as a process of perceiving mathematical reality. The reality that is to be reinvented is alleged to incorporate context, mental objects and mental activities.

Freudenthal (2002, p. 11) defines learning as "... progress in knowledge and ability… it is a change of viewpoint from content to form… leading to higher levels, by jumps as high as the learner can perform… guided but not lifted by the teacher". Reconstruction of calculation procedures begins with organising and improving mental and physical learning resources to translate them mathematically (Freudenthal, 2002). This study is informed that learning gradually changes from physical manipulation of objects to symbolic representation and mental manipulation of physical objects to transform from content to form. Thus meaningful learning prompts the use of problem solving, ten frames, five frames, abacuses, empty number lines and pictures to bring computation reality to the classroom.

When learning, the basic operations are understood through the act of structuring mathematical concepts rather than defining concepts before developing mathematical structures. The process of structuring mathematics requires guidance through experiments to develop the ability to calculate mentally without direct teaching of FMC skills. Introductory activities can be demonstrated using learning resources such as an abacus. Freudenthal (2002, p. 75) suggested that "… at the start of arithmetic …as on the abacus, it is indispensable as part of the primordial reality". In the absence of an abacus, realists suggest a more accessible manipulative such as PSTs or learners themselves, where they can be required to experiment by counting the members of a class and applying the operations to the number of members available in the class (Freudenthal, 2002). Therefore, direct teaching of operations and calculation strategies may undermine the capacity to reinvent strategies meaningfully.

Learning FMC skills implies the invention of calculation strategies prior to introduction of standard algorithms. Unlike learning mathematics through memorisation of standard algorithms, RME promotes the teaching of mathematics that is significant to learners by allowing them to actively rediscover calculation strategies (Van den Heuvel-Panhuizen & Drijvers, 2014). Early introduction of the standard method discourages analytic and innovative thinking. Thus, in a learning process learning is first embedded in real-world situations then disembodied to deal with abstract mathematics. The first step in learning, from RME perspective, is the provision of quality and realistic learning experiences. Freudenthal (2002, p. 17) asserts that "reality is historically, culturally, environmentally, individually, and subjectively determined" (Freudenthal, 2002, p. 17). The view on how reality is determined provides an insight into the importance of using PSTs' experiences and culture to generate mathematical reality. The use of mathematical learning experiences that connect to historical events, culture and environment connect learners to the reality of mathematics. As a result, early development of "mathematical concepts, tools and procedures" is supported leading to development of advanced mathematics (Van den Heuvel-Panhuizen & Drijvers, 2014, p. 521). However, learning is also considered as a process of visualisation where a mathematical situation is depicted in the head (Van den Heuvel-Panhuizen & Drijvers, 2014). Such learning experiences could be presented in the form of easy problems originating from daily life occurrences, stories about an imagined real-life situation or problems representing real classroom mathematics that learners can picture in the head. Freudenthal

(2002, pp. 166-167) states that "learning simple mathematics at a reasonable level is a more dignified pursuit than learning complex mathematics at no level of understanding at all". Freudenthal substantiates the latter statement with reference to tests conducted on specific university students that most students who learned a great deal of calculus at school lacked mastery of elementary algebra. This study is directed to find ways to support PSTs to develop the elementary mathematics knowledge and understanding required to facilitate learners' FMC skills.

3.3.3 View of classroom culture

Research shows that most PSTs have experienced learning mathematics in a mechanistic or traditional way (Battista, 1999). Realists believe that PSTs emulate their own learning experiences either in school or in teacher education programmes to teach their own learners (Freudenthal, 2002). Evidently, development of teacher education should create an opportunity for them to experience mathematics as a human activity through reinvention of their own calculation strategies (Freudenthal, 2002). Therefore, it is important to change PSTs' attitudes and beliefs regarding mathematics teaching and learning by offering constructive learning experiences that also enable relearning of facts (Freudenthal, 2002). Change of attitude and beliefs would require developing a culture of critical and analytical thinking by way of solving real-life problems. The RME 'interactivity principle' recommends both individual and social activity. To learn how to think critically to re-invent calculation strategies, PSTs need the opportunity to hold discussions with the entire class and in smaller groups. Unlike a culture of focusing only on a correct answer realists call for a focus on processes leading to an answer that has been derived from realistic activities. RME discourages passive observation and active trial and error learning approach that are still persistent in teacher preparation programmes.

PSTs must develop a habit of using correct mathematical terminology to explain calculation strategies. For language impacts thinking correct mathematical terminology must be used to clarify ideas. Realists direct attention to appropriate language usage to facilitate thinking and mathematisation of reality and mathematics. Reinforcement of correct terminology encourages a culture of reflection to clearly represent ideas as part of PSTs' role in a learning environment.

3.3.4 View of the role of a pre-service teacher in a learning environment

Since RME views mathematics as a "private activity" of discovery (Freudenthal, 2002, p. 14), PSTs are viewed as re-inventors of mathematics who are inspired to use their thinking capacity effectively. RME promotes a principle referred to as the 'activity principle'. The activity principle calls for consideration of PSTs as working members of the class. Since mathematics is a human activity, one of the most effective ways to learn mathematics is to do mathematics (Freudenthal, 2002). This calls for individual engagement of PSTs in tasks. To participate actively, PSTs are expected to visualise mathematics structures using drawings and pictures to form mental objects to calculate with.

Considering the idea that mathematics is a private activity, this study is prompted to support PSTs to think creatively, critically and reason logically to reinvent calculation methods. Meanwhile, during the strategy reinvention process, PSTs need to monitor their own thinking to redirect ideas. To promote the skill of regulating thoughts, PSTs ought to reflect critically on their calculation strategies to identify their own and others' strengths and weaknesses. Analysis of strategies elicits constructive judgements about strategies and identification of strategies that are more efficient, fast and accurate. Therefore, PSTs must participate actively with an inquisitive and active mind to develop their own skills to calculate mentally. Realisation of FMC skills development is underpinned by the role that a TE plays within a learning environment.

3.3.5 View of the role of a teacher educator in a learning environment

Within a learning environment, a TE is entrusted with the role of organiser and guide. Because mathematics is a human activity, a TE must organise learning situations through problem solving, the learning environment, and tasks that disclose the links among operations and calculation strategies. A TE must organise and use the necessary tools to develop PSTs' FMC skills and ability to visualise calculation strategies. This refers to tools such as concrete objects, pictures, symbols, diagrams and words. PSTs' development must begin with understanding how the oral sequence of numbers lays a foundation for mental computation and how to deal proficiently with numbers. It should be noted also that PSTs need support to realise that the oral sequence of numbers forms the addition structure referred to as "structure of increasing size" (Freudenthal, 2002, p. 22). The addition structure is a process whereby cardinal counting results in an increase in the number of objects in a set by one more, and adding one more to a number gives the next number in a counting sequence. Overall, knowledge of all the structures among operations is also necessary in developing the FMC skills of PSTs.

As informed by the 'intertwinement principle', TEs have the responsibility to connect mathematics learning areas as opposed to isolating topics. The implication to teaching is preparation of tasks that elicit reversibility to facilitate calculations. For example, to divide 20 by 5, a counting on strategy can be used to count in fives as 5, 10, 15 and 20 to see how many fives add up to 20. Using subtraction, it would be subtracting fives as $20 - 5 = 15$, $15 - 5 = 10$, $10 - 5 = 5$ and $5 - 5 = 0$ and counting the number of times that five is subtracted from 20. With regard to using multiplication, it can be worked out using the reversibility property that if $20 \div 5 = \square$, then $\square \times 5 = 20$. It is a TE's role to prepare learning experiences and guide PSTs to develop understanding of the relationships among operations.

RME, under the 'guidance principle' advocate for scaffolding of work at the onset of the learning process and to prepare activities that can elevate comprehension to greater heights. In the context of RME activities are referred to as 'cognitive processes'. For example, a cognitive process of counting up must derive the meaning of addition whereas a cognitive process of counting back should define subtraction unlike defining the meaning of the words abstractly without discovery. From the realistic perspective, the end purpose of teaching and learning is the manipulation of "mental objects" through "…mental operations" (Freudenthal, 2002, p. 19). For FMC skills development, mental objects would refer to an empty number line, a number chart and the ten and five frames. The use of visual images such as concrete objects and pictures become representations of mental objects; whereas a mental strategy used to manipulate an empty number line in mind forms the mental operations. Therefore, TEs are expected to enrich the learning environment with mental objects to develop mental operations. Cautiously, it should be understood that TE guidance should be minimal to allow PSTs to ascertain their own ability levels to improve through solving appropriate tasks.

3.3.6 View of the nature of tasks

Literature suggests that one way to promote learning interest is to introduce problems requiring the use of concrete objects. Classroom activities should begin with the introduction of basic mathematics structures before introducing advanced structures (Freudenthal, 2002). The implication to TE practice is to first present tasks that involve the use of concrete objects leading to "horizontal mathematization" before "vertical mathematization" that involve representation and manipulation of mathematical ideas symbolically (Freudenthal, 2002, pp. 41-42). Freudenthal (2002, pp. 41-42) defines horizontal mathematisation as the process that "leads from the world of life to the world of symbols… [whereas vertical mathematisation] symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectingly…" For example, representing the number of physical objects in two different sets numerically is horizontal mathematisation whereas adding the two numbers by counting on is vertical mathematisation. In other words, horizontal mathematisation requires representing a real-life problem mathematically while vertical mathematisation involves the mathematical manipulation of real-life situations represented mathematically, and both are of equal importance (Treffers, 1987). RME urge for a systematic introduction of tasks as informed by the 'level principle' that refers to navigation through the use of concrete objects first and smaller numbers before calculating abstractly with large numbers. Structured learning is significant in making learning meaningful.

Another aspect to consider is provision of context. Realists view context as different areas of reality in the form of knowledge that is available to teach and learn specific mathematics. A context can be provided through stories that are real or made up to encourage TEs and PSTs to think through imagination. Another example refers to the use of conflicting explanations to serve as a context for mathematization. Turning to teacher education, there is a need to incorporate tasks that require PSTs to analyse their own and others' calculation strategies with the intention to revise them and to alter inefficient calculation strategies.

3.3.7 View of assessment

As is the case with constructivists, RME does not discuss assessment explicitly. This study has used information pertaining to the 'reality principle' to discuss assessment. The reality principle concerns the application of constructed knowledge to solving problems experienced in the real world (Freudenthal, 2002). From the realistic perspective, the concepts of adding, subtracting, multiplying and dividing are not understood by definition first but by how they derive from the use of mental objects. Therefore, assessment ought to establish PSTs existing knowledge and to uphold meaningful development of concepts. To present evidence of concept formation and understanding, TEs need to encourage PSTs to verbalise and formalise their calculation strategies using language and written symbols. Discussion and writing provide evidence of understanding and concept development and help to identify and address misconceptions. Specifically, assessment must lead to the application of calculation skills to solve new problems and learn new concepts.

3.4 Key components of constructivism and realistic mathematics education

Table 3.2 Key components of constructivism and realistic mathematics education

	View of:						
Theory:	**Knowledge**	**Learning**	**Classroom culture**	**Role of learners /PST**	**The role of a teacher**	**Nature of tasks**	**Assessment**
Constructivism	Meaningful mathematical facts, concepts and strategies.	A process of gaining, transforming and evaluating new information.	Promotes positive attitude towards learning, mental engagement, social interaction, mutual respect, feedback and models used for learning.	Active participant in the learning environment.	Developers of an active learning environment, a facilitator, an organiser of classroom experiences, an observer, a listener, a communicator and a manager.	The use of real-life problems that elicit critical thinking. Task increases in level of difficulty gradually.	A process that reactivates learners' existing knowledge to develop new knowledge and also to recall developed knowledge.
Realistic mathematics education	"Form and content" that is organised in the form of linked structures.	A process of perceiving reality and of guided reinvention of the verbal and abstract representation of calculation procedures.	Creates an opportunity to experience mathematics as a human activity.	Re-inventors of mathematics.	Organiser of learning situations through problem solving, the learning environment, and tasks that disclose the links among operations and calculation strategies.	Tasks begin with the use of concrete objects leading to "horizontal mathematization" before "vertical mathematization" that involve representation and manipulation of mathematical ideas symbolically.	Assessment must establish prior knowledge and must lead to the application of calculation skills to solve new problems and learn new concepts. Must encourage verbalisation and written representation of ideas.

Table 3.2 outlines the link between RME and constructivism that this study embraced. RME activity principle urges consideration of PSTs as working members of a class because mathematics is a human activity. Mathematics as a human activity calls for active participation of PSTs in learning as advocated

for by the constructivists. The need to support PST to develop FMC skills is informed by the principle of guided-reinvention which relates to scaffolding of tasks by constructivists. RME informed the connection of intervention tasks through the intertwinement principle of which constructivists relate to as the structuring of concepts and topics. PSTs were engaged individually and in groups considering RME's interactivity principle and constructivists' theory that learning is both individual and social activity. The nature of problems applied in the intervention incorporated real life problems PSTs are familiar with following the reality principle of RME and in solving reconstruction of knowledge by solving real world problems. The problems PSTs solved were structured to begin with concrete objects and smaller numbers before incorporating abstract calculations with large numbers; as informed by the level principle of RME and the constructivist idea that learning begins with the manipulation of concrete objects.

3.5 Conclusion

Ideas pertaining to the development of a theory of instruction exist. However, there is a need for a specific theory of developing PSTs' mathematical knowledge for teaching FMC. Teaching and learning is a process perceived differently by TEs and PSTs. On the one hand, teaching can be perceived as a process of developing knowledge mechanically and learning as a process involving rote learning of ideas prescribed by a TE. On the other hand, teaching can be perceived as a process of guiding PSTs to invent their own calculation strategies within a real-life context and learning as an active process of reconstructing reality in a community of practice. However, to respond to the main objective of this study which relates to how to develop PSTs' knowledge for teaching FMC, existing theories of learning were consulted.

Theoretical ideas that informed the intervention strategy derived from the constructivist view of learning and the RME instructional theory. An underpinning theoretical assumption is that developing PSTs' mathematical knowledge for teaching FMC requires PSTs to have a particular theoretical view of mental calculation knowledge and also to first understand the content that they are going to teach. This study acknowledges RME's argument that inadequate content knowledge weakens teaching. In addition, PSTs' teaching of FMC skills in school can be effective provided they develop appropriate views of learning, classroom culture, nature of tasks, assessment and views of their role once teaching in a classroom. Fundamental to effective development of FMC is the use of context-rich problems to re-invent strategies actively, consideration of prior knowledge and experience, the use of concrete objects, and the use of discussion to articulate and justify invented strategies.

Inappropriate views of teaching FMC create negative views towards mathematics and undermine learners' capability to calculate flexibly in mind. If rote learning is used to develop PSTs' FMC skills, they will lack understanding of a problem solving teaching approach, efficient computation strategies, the use of representations in the form of pictures or concrete objects, promotion of learning through interaction and the use of mathematical language and reflection to improve calculation strategies. Since pre-conceived notions of TEs impact PST development of appropriate views of teaching and learning, further research into TEs existing views on teaching and learning essential. Considering the theoretical underpinnings of constructivist theory and RME instructional theory this study adopted the design and methodology as discussed in Chapter 4.

CHAPTER 4: RESEARCH DESIGN AND METHODOLOGY

4.1 Introduction

This chapter discusses the design and methodology of this study. A mixed methods design was used with the methodology being informed by the pragmatic paradigm. With a mixed methods design the TE aimed to identify PSTs' beliefs, knowledge and experience concerning flexible mental computation through triangulation of data. Awareness of PSTs' existing beliefs about, and knowledge and experience of the nature of flexible mental computation and how it is learned will inform the intervention process. Design-based research informs the intervention process to explore ways TEs could employ to support the development of PSTs' flexible mental computation strategies of whole numbers. By conducting a mixed methods study, a domain-specific instructional guide was developed to assist TEs in developing PSTs' flexible mental computation strategies. It is possible to identify relevant and effective methods to develop PSTs' flexible mental computation strategies because design-based research and a mixed methods design can provide rich data generated within a real classroom environment.

Therefore, this chapter begins with a discussion of the research design used in this study. The methodology section includes first a discussion of the methods employed in the study. Sampling and a detailed outline of the data collection process are presented thereafter. Towards the end of this chapter, an elaboration on the data analysis process, how reliability and validity of the data were enhanced, the ethical clearance process, limitations of the study, and a summary of the chapter are provided.

4.2 Research design

4.2.1 Pragmatist paradigm

The pragmatic interpretive framework informed the design and methodology of this study. Four beliefs pertaining to the pragmatic framework and implications to methodology are summarised in table 4.1.

Table 4.1 Philosophical beliefs of the pragmatic paradigm (adopted from Creswell, 2013, p.21 & p. 37)

Assumption	Characteristics	Implications for practice
Ontological belief (the nature of reality)	Reality is what is useful, is practical, and "works".	Researcher reports different perspectives as themes develop in the findings.
Epistemological belief (how reality is known)	Reality is known through using many tools of research that reflect both deductive (objective) evidence and inductive (subjective) evidence.	Researcher relies on quotes as evidence from the participant; collaborates, spends time in field with participants and becomes an "insider".
Axiological belief (role of values)	Values are discussed because of the way that knowledge reflects both the researcher's and the participants' views.	Researcher openly discusses values that shape the narrative and includes his or her own interpretation in conjunction with the participants.
Methodological belief (approach to inquiry)	The research process involves both quantitative and qualitative approaches to data collection and analysis.	Researcher works with particulars (details) before generalizations, describes in detail the context of the study, and continually revises questions from experiences in the field.

Considering the philosophical beliefs of the pragmatic paradigm, the researcher recorded and reported strategies that assisted PSTs to calculate flexibly in mind because reality is what works within a particular classroom context (Creswell, 2003). As a result, different instruments were used to collect both quantitative and qualitative data. Prior to data collection, the researcher had to ensure that PST were comfortable with the dual role of the researcher in the classroom. The philosophy also encourages the use quotes as evidence from participants and consideration of personal values that may shape the interpretation of the data. Based on the philosophical assumptions, both quantitative and qualitative designs were found relevant to this study to understand the PSTs as a class and as individuals.

4.2.2 Mixed methods design

This study adopts a mixed methods design employing an explanatory sequential design. Figure 4.1 presents the sequence in which data was collected and analysed. Inclusion of the intervention process illustrates how design-based research fits into the mixed methods design.

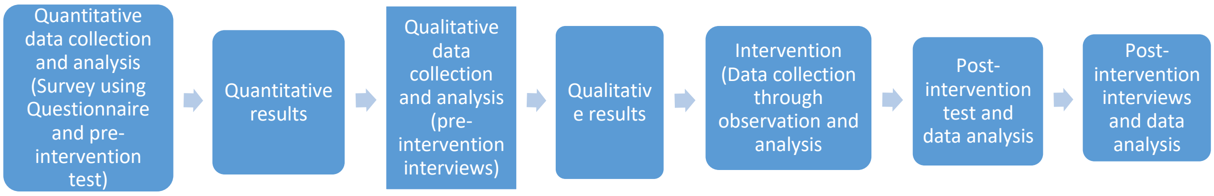

Figure 4.1 Explanatory sequential design (adopted from Creswell, 2015).

The study began with the collection of quantitative data followed by qualitative data. Consideration of mixed methods design studies started in the mid-19th century. In the 1970s and 1980s a debate on research methodology erupted between scholars around the significance of internal and external validity of findings (Tashakkori & Teddie, 1998). As a result, in the late 1990s mixed methods study design became operative. Thus, for this study a mixed methods design is used to confirm and complement data from different sources for improved validity of findings. As presented below, the current mixed methods design study consists of five phases, as proposed by Hjalmarson and Lesh (2008).

4.3 Research methodology

4.3.1 Design-based research context

DBR requires TE to utilize their "knowledge and experience about students, mathematical expertise, personal theories, and information about teaching in order to design a classroom environment that is conducive to learning (Hjalmarson and Lesh, 2008, p.100). Designing a classroom environment conducive to learning prompts the use of a real classroom environment as a research site. The classroom environment used in this study is a lecture hall with fixed seating arrangement. The class has a projector that was not functioning during the intervention process. The TE used the course outline provided by the University of Namibia that reflects the need for the development of PSTs' skills to compute mentally. The participants were PSTs allocated to a part-time lecturer who was appointed only towards the end of March 2018. The TE announced and clarified that a part-time lecturer would be appointed to teach them, thus the researcher's intention was to conduct a study by teaching only the topic on mental computation. The availability of a real classroom environment facilitated the execution of a design-based research methodology.

4.3.2 Design-based research methodology

Design-based research (DBR), which falls in a developmental paradigm (Trafford & Leshem, 2008), formed the methodology of the study (Cobb et al., 2003; Cobb & Gravemeijer, 2008). Design-based research is a "systematic but flexible methodology aimed to improve educational practices through iterative analysis, design, development, and implementation, based on collaboration among researchers and practitioners in real-world settings, and leading to contextually-sensitive design principles and theories"(Wang & Hannafin, 2005, p. 6). Design-based research was used in the present study to improve both theory and practice (Wang & Hannafin, 2005). Design-based research is underpinned by the pragmatist paradigm that illuminates the current study as it connects theory and practice through consideration of solutions that are realistic and empirical in nature (Given, 2008, p. 673). The study is grounded in the phenomena of PSTs' existing knowledge, experience and beliefs about flexible mental computation (NIHOBSS, 2018; Wang & Hannafin, 2005).

The study is further grounded in the theory about how PSTs and learners develop flexible mental computation skills (NIHOBSS, 2018; Wang & Hannafin, 2005). A real-world classroom environment formed the site where an explanatory sequential design was employed by the TE to collect both quantitative and qualitative data types (Creswell, 2015; NIHOBSS, 2018; Wang & Hannafin, 2005). Informed by a pragmatist philosophy, a mixed methods design was found more appropriate for the present study because of its effectiveness in expressing what reality is (Given, 2008; Wang & Hannafin, 2005). In using a mixed methods design, quantitative research could enable findings to be generalised and precise, as it generates wide-ranging inclination and relationships (Creswell, 2015). Also, qualitative research could provide a detailed understanding of specific PSTs' viewpoints or "in-depth personal perspectives" (Creswell, 2015, p. 36). Qualitative and quantitative methods have strengths and weaknesses that could impact the quality of findings.

Since both methods have strengths and weaknesses, one design may make up for the weaknesses of the other when both designs are employed in the same study (Creswell, 2013). Different research methods were used to design and test an HLT for the development of PSTs' mathematical knowledge for teaching FMC effectively. A mixed method design was used to improve confidence in the accuracy of findings by exploring similar issues and to allow for a comparison of findings and also to complement findings. Increased accuracy would then enhance the validity of the findings through quantitative and qualitative type of questions. However, a mixed methods design could be time-consuming, costly and could be problematic if findings did not coincide when checking validity of findings (Creswell, 2009). The study was conducted in phases, as illustrated below, beginning with the articulation of a problem that manifested among the TE's previous elementary mathematics PSTs in a real classroom setting. Quantitative data were collected first to enable the selection of 15 PSTs to participate in the pre-intervention interviews.

In the first phase of the study, the TE established PSTs' curricular needs. The second phase of the study involved the identification of models (Carpenter et al., 1996; Claxton et al., 2010; Van den Heuvel-Panhuizen & Drijvers, 2014) and principles to address the situation (Freudenthal, 2002; Kamii & Joseph, 2004; Treffers, 1987). The design of a learning trajectory or instructional activities occurred in the third phase of the study. Thereafter, in the fourth phase, the experiment was carried out to test the extent to which the trajectory supported the development of PSTs' flexible mental computation skills in class. Outcomes of both experiments, obtained quantitatively and qualitatively, contributed towards the review of the experiment process and revision of the learning trajectory to maximise its impact on PST thinking and reasoning.

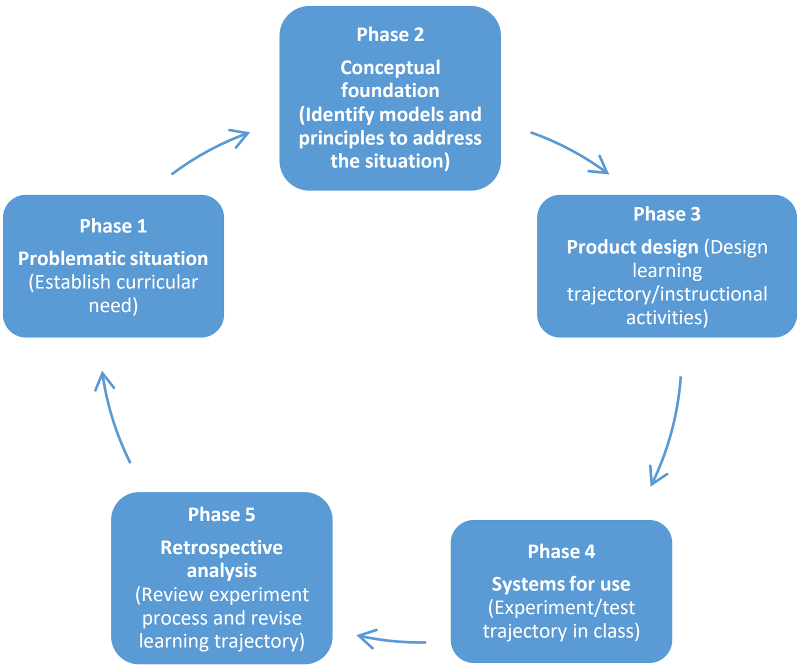

Figure 4.2: Research process: design informed by Hjalmarson and Lesh (2008).

This study was conducted in phases as illustrated in table 4.2 which includes the expected outcomes at each phase of the study.

Table 4.2 Research phases and expected outcomes

Phase	Description	Actions	Expected outcomes
1	Problematic situation	Establish curricular needs	Determine PSTs' beliefs, knowledge and experiences. Identify challenges pertaining to problems used during the intervention lesson.
2	Conceptual foundation	Identify models and principles	Implement the 4-6-1 model of habits of mind, principles of the constructivist theory and realistic mathematics education principles.
3	Product design	Design instructional activities	Refine FMC problems included in intervention tasks.
4	Systems for use	Experiment trajectory in class	Use refined problems in class.
5	Retrospective analysis	Review experiment process and revise learning trajectory	Reflect on lesson and record observed issues in researcher journal to plan for the next intervention.

PST beliefs, knowledge and experiences were explored prior to the intervention process, to make appropriate adjustments to the learning trajectory, using multiple research methods as discussed next.

4.4 Research methods

The empirical study comprised a survey using TE-administered questionnaires; one pre- and post-intervention test; one-on-one, face-to-face pre- and post-intervention interviews; classroom observation; and the intervention process to produce an HLT for elementary PSTs. A survey was conducted as a research strategy that included the entire population of first-year PSTs. A survey is "an approach in which there is empirical research pertaining to a given point in time which aims to incorporate as wide and as inclusive data as possible" (Denscombe, 2007). A survey was used to determine pre-service teachers' FMCs before the intervention and after the intervention to measure the impact of the intervention. According to Kumar (2011, p. 339), "the before-and-after design is technologically sound and appropriate for measuring the impact of an intervention".

Research methods were mixed in this study to increase the trustworthiness of the research findings (Wang & Hannafin, 2005). Quantitative data emerged from the survey, diagnostic tests (pre- and post-intervention test) and classroom observation, while semi-structured interviews, observation of PSTs' written work and field notes produced qualitative data. The baseline evaluation test was developed using existing standardised tests. The envisioned learning trajectory was developed from relevant literature reviewed by the TE (Hjalmarson & Lesh, 2008), based on PSTs' beliefs and their development of MKT flexible mental computation. The philosophy of realistic mathematics education (RME) informed the design of the learning trajectory with a focus on the use of contexts, models, students' own productions and constructions, the interactive character of the teaching process, and the intertwinement of various learning strands (Freudenthal, 1968; Treffers, 1987). Mathematical representations suitable for developing elementary school learners' computation skills were presented to the PSTs and discussed with them to develop their understanding of mental computation development.

4.4.1 Survey: questionnaire

The first approach used in the empirical study was a survey. With reference to Springer (2010, p. 250) a survey is defined as "a self-report measure consisting of questions that can be administered in the form of a personal interview or questionnaire". In this study, a census survey was used, which is "a survey that covers the entire population of interest" (Ary, Jacobs, Sorensen, & Walker, 2014, p. 400). To conduct the survey, this study used a questionnaire where all the 51 first-year PSTs who enrolled for the course agreed to participate in the study. With regard to the process, "a cross-sectional design was used to collect data at one point in time" (Springer, 2010, p. 255). In other words, a questionnaire was only used once before the intervention process. A questionnaire is identified as "a self-report data-collection instrument that each research participant fills out as part of a research study" (Johnson & Christensen, 2012, p. 162). The use of a survey was found to be relevant in this study as it facilitated the process of gathering information from a large group of PSTs easily and rapidly at a specific period (Ary et al., 2014; Best & Kahn, 2006). In addition, questionnaires were used because they can be given directly and in person to PSTs, completed individually at the same time, and collected by the TE immediately. The vulnerability of questionnaires to researcher impact is much less than in interviews where participants are under pressure to find words to express their views correctly (Springer, 2010). Furthermore, a variety of distinctions can be measured with questionnaires.

A questionnaire was used to answer two of the questions of this study, namely: 'What are PSTs' beliefs about ways to develop learners' flexible mental computation skills?' and 'How did PSTs develop flexible mental computation skills and beliefs at school about the learning and teaching of flexible mental computation?' (Johnson & Christensen, 2012). Self-completion questionnaires were administered to all PSTs present in the classroom under the TE's supervision in a period of an hour to ensure a high response rate. Another advantage of a questionnaire is that it is flexible and can be used "to collect quantitative, qualitative and mixed data" (Johnson & Christensen, 2012, p. 163). Both qualitative and quantitative data were collected in terms of PSTs' beliefs, experiences and their definition of what FMC entails. Furthermore, the TE managed to include ample questions and statements in the questionnaire to match the objectives of the study (Johnson & Christensen, 2012). With the questionnaire the TE asked PSTs about their past school experience, their belief about their current ability to compute mentally and their belief about how they would develop learners' FMCs. Statements and questions were presented to them where they had to indicate their level of agreement or disagreement (Johnson & Christensen, 2012). A Likert Scale was used to rate responses from 1 to 4, where 1 indicated strongly disagree, 2 disagree, 3 agree and 4 strongly agree.

There are challenges associated with the use of questionnaires. In this study the questionnaire was printed on paper and distributed. The TE had to bear the cost of duplicating adequate copies for the entire population (51 PSTs). However, the TE had facilities where copies could be made at a lower cost.

Another challenge was that time was needed for duplication, stapling and verification of the copies prior to the survey process. To adhere to the time schedule, the TE ensured that the questionnaires were duplicated and stapled a week before the empirical study. A further challenge is that questionnaires demand thorough preparation, execution and analysis of the information gathered to ensure reliable and valid results (Ary et al., 2014, p. 405).

To increase the validity and reliability of the questionnaire, the TE ensured that the questions matched the objectives of the study. Existing questionnaires about beliefs and experiences of PSTs relating to mathematics were used (Jong & Hodges, 2015; Tarasenkova & Akulenko, 2013). Also, the TE made sure that questions made sense to PSTs by using simple language to formulate the questions. Leading and emotive questions were avoided and statements were brief, understandable and exact, so that they made sense to PSTs (Johnson & Christensen, 2012, p. 164). Since the questionnaire could not be used to determine PSTs' ability to do flexible calculations in the mind, a survey in the form of a diagnostic test was imperative.

4.4.2 Survey: diagnostic test

After the administration of the questionnaire, the TE conducted a diagnostic test where all the 51 PSTs participated voluntarily. For this study a diagnostic test refers to "the use of tests for evaluating aptitude, achievement, and other academically relevant characteristics of students" (Springer, 2010, p. 130). A test designed by the TE, in consultation with existing tests on FMC, was used specifically for this study as it is relevant in a classroom environment to answer the following research question: 'What are the PSTs' existing flexible mental computation skills upon entry to university?' A "criterion-referenced test" was used whereby PSTs' scores were compared to a 50% standard set by the TE with reference to the module pass requirement stipulated in the PST mathematics course outline. The aim of the test was to establish the extent to which PSTs can make flexible computations in the mind and what kind of algorithms they can perform with facility. Outcomes of the test also informed the TE about the types of algorithms PSTs cannot compute in the mind. Such information illuminated the needs of PSTs in terms of FMCs for which the TE had to devise strategies to develop PSTs' skills to compute mentally. Another test was conducted after the intervention to ascertain PSTs' improvement in terms of calculating mentally using personally invented strategies as a result of the intervention (Springer, 2010).

Diagnostic tests pose a couple of challenges, as discussed below. One challenge posed by a test designed by a TE could be the interpretation of data resulting from weak instrument construction, which could be due to inadequate information concerning the reliability and validity of the instrument (Springer, 2010). Therefore, considering the challenge of a researcher-designed test, reliability and validity of the test for this study was determined through a content validity process to ensure that the test items corresponded with the conceptual framework for accurate and flexible mental computation (Heirdsfield, 2002). Meanwhile, the TE also used the mathematics syllabus for elementary schools in Namibia to ensure that PSTs were competent in the mathematics they are expected to teach. Though tests are good at identifying high performance in a particular domain, they could not be used to determine PSTs' skills, strategies and thought processes used in the test to work out answers to algorithms. Since PSTs were expected to provide answers instantly, within 20 seconds, they had no time to explain or justify their answers. The test paper was not handed out to PSTs; rather each question was displayed on the screen for 20 seconds and was read out only once. Thus interviews were found to be more appropriate to grant PSTs an opportunity to demonstrate their solution strategies.

4.4.3 Interview

Following the diagnostic test, face-to-face semi-structured interviews were conducted with individual PSTs. An interview is "a two-way conversation in which the interviewer asks the participant questions to collect data and to learn about the ideas, beliefs, views, opinions and behaviours of the participant" (Maree, 2014, p. 87). This study used interviews to explore the strategies PSTs used to answer computations that were included in the diagnostic test. It is through interviews that the TE could establish whether PSTs used fingers, standard algorithms, derived facts or stripes on their hands to find an answer. Also, interviews were used because of the opportunity they offer to a TE to probe and seek clarification of answers immediately. To probe for clarification, a TE needs to listen carefully to the interviewee's responses that are in line with the topic under discussion, and also to recognise new lines of argument. Interviews were used additionally to confirm PSTs' responses to the questionnaire in terms of the meaning they attached to FMC, their perceptions about what FMC is, what they believed was the best way to develop FMCs, and how they developed FMC skills at school (Punch, 2009).

Furthermore, the purpose of using interviews was "to see the world through the eyes of the participant…to obtain rich descriptive data that will help you to understand the participant's construction of knowledge and social reality" (Maree, 2014, p. 87). Thus a good relationship was developed between the TE and the PSTs to build trust between the TE and the PSTs. Marree (2014, p.87) emphasised that "[if] the persons you are interviewing think the topic is important and they trust you, they will give you information that you will not be able to collect in any other way" (Maree, 2014, p. 87). A semi-structured interview was used that required PSTs to answer a set of predetermined questions. Pre-determination of questions delineates the interview process so that the TE does not to go astray and it also increases the opportunity to repeat the same interview in a different setting. Interview questions excluded questions requiring yes and no answers and leading questions. The questions were short, clear and neutral and the aim of the interview and type of information to be gathered was clarified to increase validity of interviews. Equally, different ways of probing such as "detail-oriented probes, elaboration probes and clarification probes" (Maree, 2014, p. 89) were used. Detail-oriented probes assisted the TE to comprehend the "who, where and what of the answer given by the participant", while "the why" type of questions were avoided so that the interviewees were spared a threatening the PSTs could provide more information about examples and responses which they provided. However, where PSTs were unwilling or unable to answer specific questions, the TE did not press them for answers. Clarification probes were used to verify the TE's understanding of interviewee information for accuracy through paraphrasing of received responses (Maree, 2014).

One of the challenges of interviews is that interviewees might easily stray from the topic under discussion, prompting the TE to be alert to this and to redirect the discussion. The TE needs skills to summarise main aspects quickly in order to maintain a smooth flow in the discussion. If interviews are not backed up with notes and video recordings, viewpoints of interviewees can be forgotten (Maree, 2014). Elaboration probes were used for the PSTs to provide more information on responses provided. However, where PST were not willing or unable to answer specific questions, the TE did not insist on getting answers from them. Clarification probes were used to verify my own understanding of interviewee information for accuracy through paraphrasing of PST responses (Maree, 2014).

One of the challenges of interviews is that interviewees may easily get off track in their discussion prompting the TE to redirect discussion. A TE needs the skills to summarise main aspects quickly to maintain a smooth flow in discussion. Also interviews had to be backed up with notes and video recordings to retain the information provided by the PSTs. The TE also considered the fact that participants were likely not to provide their own views but views they think the TE may want to hear. Thus findings from the interviews were compared to findings of the questionnaire, the diagnostic test and observations carried out during the intervention process.

4.4.4 Observation

During the intervention process, the TE observed factors that impacted on PSTs' development of FMC. Observation is described as "a systematic process of recording the behavioural patterns of PSTs, objects and occurrences without necessarily questioning or communicating with them" (Maree, 2014, pp. 83-84). The TE used only the instincts (sense of sight, touch and hearing) to obtain an in-depth understanding and collect information on PSTs' attitudes towards FMC, written strategies, collaboration among PSTs, and discussion of own strategies in a real classroom setting. Selection of specific aspects for observation revealed the weakness of observations that culminate in high selectivity and subjectivity in terms of the focus of attention. As a result concentration can be on a particular situation or PSTs resulting in isolation of other important occurrences in the entire class. Thus the TE considered issues of bias to focus on the entire class (Maree, 2014). Consideration of issues related to biasness curbed the weakness of observation as a data collection strategy to identify classroom aspects that may improve or hamper the development of FMC skills.

To learn more about how to develop PSTs' FMCs, the TE had to observe PSTs learning in a real classroom environment and reflect on the outcomes of the observation. PSTs' written work was observed to outline the different calculation strategies they used to illuminate the TE's inferences drawn from the kind of methods PSTs are familiar with. From observing PSTs' written work, data were collected directly from a primary source in their original form to "corroborate the evidence from other sources" (Maree, 2014, p. 83). Observations were also made of their performance in solving both easy and challenging problems and included specific number combinations to observe how it motivated PSTs to invent their own mental calculation methods that are flexible (Heirdsfield, 2002). Since the TE assumed the role of a "participant as observer", the TE became immersed in the process of the research by working with the PSTs and shaping the learning process by devising alternative intervention methods for further development of FMC skills.

To increase validity of findings, observations were recorded immediately after the intervention process using "running records" to provide a detailed and accurate account of the events that occurred in the classroom (Maree, 2014, p. 85). A video recorder was used to record both "vocal and non-verbal" conduct of PSTs during the intervention process (Maree, 2014). The TE also recorded own reflections to make meaning of the actions observed in class, as well as find ways to respond positively to specific actions. However, the TE had to be cautious not to misinterpret PSTs' ideas to match a personal line of reasoning (Maree, 2014). A good relationship between the TE and PSTs also developed during observations, which increased the possibility and ease of gathering in-depth data in post-intervention interviews (Maree, 2014). Next is a discussion of how data was analysed.

4.4.5 Document analysis

This study analysed a variety of documents from which important information was generated. Document analysis refers to "… a wide range of written, visual, digital, physical material relevant to the study at hand" (Merriam, 2009). Documents used included a researcher journal, PST journal and PSTs' written calculations. During the study the researcher compiled a descriptive record of information that was found worthwhile pertaining to the intervention process, classroom settings and PST behaviour. The researcher reflected on each lesson immediately after the intervention in a note pad to record aspects that may hinder the development of PSTs' ability to compute flexibly in mind. PSTs instructed PSTs to write their insights and any challenges they encountered during the intervention process.

4.5 Sampling

The entire class of 51 PSTs in the BEd honours first-year elementary mathematics programme of a university satellite campus were approached to participate in the study. Connecting integration was used to connect quantitative data to qualitative data through sampling (NIHOBSS, 2018; Creswell, 2015). A quantitative method was administered to 51 PSTs in the form of a pre-intervention diagnostic test. The results of the test were then used to select PSTs for the interviews (Creswell, Clark, Guttman, & Hanson, 2003). Purposive sampling was used to select a maximum of 15 PSTs as cases for interviews based on their performance in the pre-and post-test results of the diagnostic test. Purposive sampling involves the selection of a "…sample from which the most can be learned" to establish and comprehend key aspects pertaining to a specific problem (Merriam, 2009, p.77). Criterion purposive sampling was used whereby participants were selected to generate data that reflects diverse PST experiences and perspectives. Criterion sampling is a process were a researcher determines the features of the desired participants and the number when the study is designed (Maree, 2014). Therefore, in this study the selection criteria was to involve five PSTs from each group of high, average and below average performers to participate in the interviews. Purposive sampling was used to achieve representativeness and in-depth information (Teddlie & Yu, 2007) from interviews. Again, a quantitative method in the form of a post-intervention diagnostic test was administered to 38 PSTs who were willing to continue participating in the study. Of the 15 PSTs who participated in the pre-intervention interview, only nine PSTs were willing to participate in the post-intervention interviews. An explicit outline of the research sample is provided in table 4.3.

Table 4.3 Summary of the study sample at various stages of the study

Date	Stage of study	Number of participants	Number that opted out of study	Duration of data collection process ±	Type of data collected	Data collection method
16 February 2018	Before intervention	51	0	1h	Quantitative	Questionnaire
20 February 2018	Before intervention	51	0	1h	Quantitative	Pre-intervention test
22 February 2018	Before intervention	15	0	40 min.	Qualitative	Pre-intervention interviews
16 March 2018	After intervention	38	13	1h	Quantitative	Post-intervention test
19 March 2018	After intervention	9	6	40 min.	Qualitative	Post-intervention interviews

Data collection emerged through the process discussed next.

4.6 Data collection

4.6.1 Survey

Data collection commenced when all first-year pre-service elementary teachers enrolled for the BEd honours degree at the selected university satellite campus had completed a survey questionnaire. From the pragmatist point of view, the inquiry process into identifying what works in developing PSTs flexible mental computation skills must begin with an in-depth exploration of both PSTs' and the TE's ideals to locate it directly to classroom situation (Given, 2008). The location of solution strategies to the immediate classroom situation increases the degree to which identified solutions are usable and suitable to a particular learning environment (Given, 2008). Therefore, a survey was conducted to provide a broad overview of the first year of early childhood and lower primary teachers' beliefs, experience and knowledge concerning flexible mental computation.

The design of the survey was informed by the Mathematics Experiences and Conceptions Survey (MECS – M1) by Jong and Hodges (2015) and a beliefs instrument by Tarasenkova and Akulenko (2013). The survey was intended to discover PSTs' knowledge and beliefs upon entry to university. Information from the survey would inform the intervention process to devise proper ways to address the PSTs' flexible mental computation learning needs and beliefs.

The first part of the survey comprised 12 statements covering the PSTs' school experience and the understanding of flexible mental computation. For each statement, PSTs had to rate themselves on the scale by circling the number that best described their school experience (in part 1) and belief (in part 2). The Likert scale comprised four ratings: 1 – strongly disagree, 2 – disagree, 3 – agree and 4 – strongly agree. Part 2 of the survey comprised20 statements based on PSTs' beliefs about how flexible mental computation develops in learners. Information from the survey would inform the intervention process to devise proper ways to address the PSTs' flexible mental computation learning needs and beliefs upon entry to university.

4.6.2 Pre- and post-intervention diagnostic test

A pre- and post-intervention diagnostic test consisted of a survey that was conducted on all the 51 PSTs in the form of a test. The pre-intervention diagnostic test was set to establish PSTs' mental computation skills to support the TE's identification of the PSTs' learning needs and the selection of interview PSTs. The identification of learning needs informed the intervention process in terms of selection of proper instructional approaches and kind of activities for the PSTs. The post-intervention diagnostic test was used to document PSTs' performance in flexible mental computation and understand the intervention aspects that caused the change in performance. Both tests (pre- and post-intervention diagnostic test) comprised 25 items. All test items in both tests were context-free items involving addition, subtraction, multiplication and division of one- to three-digit whole numbers.

The tests were derived from three frameworks, namely the conceptual framework for accurate and flexible mental computation (cognitive aspects) by Heirdsfield (2002); the framework for basic number sense by McIntosh et al. (1992); and the mental computation strategy framework by Hartnett (2007). The instrument was also informed by Courtney-Clarke (2012). The components of number sense included,(a) knowledge of and facility with numbers: place value, relationship between number types, decomposition and comparison to benchmarks, (b) knowledge of and facility with operations: operating on whole numbers, commutativity, associativity, distributivity, identities and inverses, and understanding the relationship between addition and multiplication, subtraction and division, addition and subtraction, and multiplication and division; (c) applying knowledge of and facility with numbers and operations to computational settings: ability to create or invent strategies, ability to apply different strategies, ability to select an efficient strategy, facility with mental calculation, facility with choosing efficient numbers and recognizing reasonableness of calculation. Flexible mental computation strategies represented in the test included count on and back, adjust and compensate (change and fix), double and/or halve, break up numbers and place value.

Each test item was displayed for 20 seconds on the projector screen and PSTs were expected to work out the answers mentally and write them on answer sheets provided by the TE. PSTs were required to provide the answer without using a pen and paper for doing calculations. The tests were marked by the TE immediately after they were written, and 15 PSTs were selected for interviews.

4.6.3 Pre- and post-intervention interviews

Semi-structured interviews in this study were used by the TE to follow up and verify quantitative data collected through insight and an in-depth understanding of PST knowledge, experience and beliefs about flexible mental computation (Heirdsfield, 2002; NIHOBSS, 2018; Wang & Hannafin, 2005). The interviews were conducted by the TE in English, with individual PSTs in person, on campus in their spare time. Interviews were tape-recorded with PSTs' permission to collect information on PSTs' thought processes based on the strategies they used to compute mentally before and after the learning trajectory implementation in the classroom. Survey results on teacher knowledge, experience and beliefs were verified with specific interview responses to detect any contradiction in ideas (Creswell, 2015). At the same time, interviews assisted in generating detailed information concerning PSTs' understanding of flexible mental computation, what flexible mental computation is, and how they developed flexible mental computation skills at school (Creswell, 2015). The interview protocols were informed by the work of various researchers.

Interview guides were informed by interviews conducted by Walker (1999) and Blöte et al. (2000). Interview guide one comprised four computations that PSTs had to carry out one at a time mentally. The TE then asked the PSTs to explain how the computation had been worked out mentally while the TE recorded the PSTs' strategies in writing. The interviews were tape-recorded with PSTs' permission. The tape-recording facilitated the transcription process to document issues that were not written down during the interview process. The second part of the pre-intervention interview guide comprised13 questions based on PSTs' school experiences of learning mathematics and their beliefs about how best flexible mental computation skills were developed.

The post-intervention interview guide consisted of four computations that PSTs carried out one at a time mentally. The TE asked the PSTs to explain how the computation was worked out mentally, while the TE recorded the PSTs' strategies in writing. The interviews were tape-recorded with PSTs' permission. The second part of the post-intervention interview guide consisted of five questions. PSTs were asked to reflect on their ability to compute mentally before the intervention and after the intervention. PSTs were requested to elaborate on the impact that the intervention had had on their knowledge and understanding of flexible mental computation strategies.

4.6.4 Hypothetical learning trajectory

Three data sources were exploited in each design-based research cycle through observations, written work analysis and semi-structured interviews. As informed by Given (2008, p. 673), pragmatism intentionally looks "into the intersubjective interactions between people and the environment". For the TE to analyse how PSTs interacted with each other, with the algorithms and with the TE, observations and analysis of written work were carried out. Meanwhile, semi-structured interviews were conducted to understand PSTs' computation strategies used in the intervention process. After each cycle, the process was reviewed by the TE to improve the learning trajectory. The improved learning trajectory was used again in the subsequent lessons for further improvement (Bannan-Ritland, 2003; Cobb et al., 2003).

The HLT was designed in consultation with three frameworks and books. The frameworks were: the conceptual framework for accurate and flexible mental computation (cognitive aspects) by Heirdsfield (2002); the framework for teaching mathematics from inception to year six (DfEE, 1999); a framework for planning learning experiences and assessment opportunities (Queensland Education Authority, 2004); and a framework for providing learning opportunities in a safe environment (Fraivillig, 2001). Shulman's (1987) types of teacher knowledge framework and additional sources were also used. These included Clements and Sarama (2014), Wright, Martland, Stafford, and Stanger (2006); Wright, Martland, and Stafford, (2006); Fosnot and Dolk (2001a); Fosnot and Dolk (2001b); Shumway (2011); and Parrish (2010). Ideas from the aforementioned sources informed the design of the HLT used in the intervention.

4.6.5 The intervention process

The intervention process was carried out by the author and as a TE at the same time in order to comprehend teaching and learning in a real classroom situation (Kelly, 2003). Seen from the pragmatist philosophical point of view, intervention was the most appropriate approach to the current study as "truth is found in 'what works', and ... truth is relative to the current situation" (Given, 2008, p. 672). For the TE to devise and execute strategies effectively to develop PSTs' flexible mental computation skills, quantitative data needed to underpin the intervention process. In other words, for the TE to create relevant and effective ways to support PST development, such efforts needed to be informed by PSTs' existing knowledge, experience and beliefs. From a TE's perspective, solutions emerging from an in-depth understanding of pre-service teachers' immediate subject domain needs are highly likely to improve pre-service teacher performance.

The 4-6-1 model of learning was used to inform the whole intervention process (Claxton et al., 2010, p. 46). The intervention prevailed where the HLT was used to develop pre-service teachers' flexible mental computation skills and was improved over a series of lessons for a period of one month. In the end, the intervention was finalised, though subject to further improvement by different educators. The post-intervention test was conducted and the same group of PSTs which participated in the first interviews were interviewed after the post-intervention test. Data analysis was carried out right from the onset of the study and was a continuous process.

4.6.6 Post-post intervention test

After the post-intervention test and post intervention interviews, the researcher requested the PSTs who expressed the need for support with multiplication and division to a last intervention class. The improved aspects of the HLT regarding multiplication and division was used to support them (see section 5.9). However, due to time constraints no further interventions were carried out. Section 4.7 indicates how data was analysed.

4.7 Data analysis

4.7.1 Quantitative data analysis

Descriptive statistics were used to analyse quantitative data from questionnaires, and pre and post intervention tests. "Descriptive statistics is a collective name for a number of statistical methods that are used to organise and summarise data in a meaningful way" (Maree, 2007, p.183). Since the study adopted an explanatory sequential design, data analysis began immediately after the survey questionnaires were administered (NIHOBSS, 2018). Responses from the PSTs were analysed manually, as the computer software was not operational yet, and were presented in the form of percentages to gain insight into the experiences of PSTs and the beliefs they held when they entered university. Percentages of the number of PST who agreed or disagreed with a particular statement were interpreted to reflect a specific feature of the population. Integration of data prevailed through collection of data from one open-ended question among closed-ended questions that required PSTs to indicate agreement or disagreement. The open-ended question required PSTs to provided their understanding of FMC is. A pre-intervention diagnostic test was conducted to establish PSTs' flexible mental computation skills upon entry to university. Frequencies and summary statistics were run on numerical data from the diagnostic test using computer software, namely Statistical Package for Social Sciences (SPSS). Frequencies and summary statistics were also run on post-intervention tests and results visualised through frequency tables. Test results were analysed in terms of the distribution of test scores to identify the frequency of the scores obtained by PSTs which represented the level of mental computation skills.

4.7.2 Qualitative data analysis

Data from the questionnaires were followed up using interviews before the intervention (Creswell, 2015). Pre-intervention interviews were audio-recorded, transcribed and verified with the views that emerged in the survey pertaining to student experiences and beliefs related to flexible mental computation skills development (part B of the interviews). Inductive analysis was carried out where interview data collected before and after the intervention were first transcribed, reviewed and organised manually into themes (see appendix J) through coding and summary of codes (Burnard, Gill, Stewart, Treasure, & Chadwick, 2008; Creswell, 2013; Merriam, 2009). Coding refers to "…the process of reading carefully through your transcribed data, line by line, and dividing it into meaningful analytical units" (Maree, 2007, p.105). Thereafter, data were presented in tables and discussed as a whole. Part A of the interviews consisted of algorithms that PSTs had to work out to demonstrate the variety of calculation methods they had developed at school.

Calculation strategies demonstrated by all 15 PSTs were recorded, labelled and summarised by the TE in the form of a table. Observation notes on intervention cycles were recorded in a researcher journal where they were analysed and interpreted to inform the next intervention cycle. Subsequent cycles were also documented in the journal, analysed after each cycle and compared to data which had emerged from preceding experiments in conjunction with literature to understand the research process and progress of the PSTs' development of flexible mental computation skills. Integration of data prevailed as qualitative data was used to explain quantitative data. Retrospective analysis was done to support the development of a "domain-specific instructional theory" by recording the learning trajectory and the support to develop PSTs' flexible mental computation skills (Kelly et al., 2008, p. 8). Reliability and validity were ensured by the research methodology adopted, as well as the provision of a detailed account of the entire intervention process and the HLT.

4.8 Reliability

To ensure repeatability of the current study (Cobb& Gravemeijer, 2008, p. 88), the TE developed an HLT that can be used by other TEs to promote learning effectively in different PST elementary mathematics classrooms. This was enabled through the provision of a detailed outline of the crucial activities and strategies that the TE used and that other educators may use to develop PSTs' flexible mental computation skills (Cobb& Gravemeijer, 2008; Merriam, 2009). For consistency and dependability of quantitative and qualitative data, triangulation through a diagnostic test and survey on PSTs' knowledge, experience and beliefs relating to flexible mental computation were conducted (Merriam, 2009, p. 221). Thereafter, interviews were conducted to verify the survey findings and for PSTs to elaborate further on their experience of flexible mental computation at school and demonstrate their existing knowledge of a variety of flexible mental computation strategies. Questions based on their beliefs relating to the teaching and learning of flexible mental computation were included in the interview.

4.9 Validity

To establish what the instruments would measure, all the instruments were piloted prior to the empirical study. Data collection continued over a period of one month, and multiple methods were used to maximise the "objectivity, validity, and applicability" of the study (Wang & Hannafin, 2005, p. 10). Peer review, debriefing and member checking for accuracy and credibility of data interpretation also occurred to increase internal validity (Creswell, 2013). The classroom setting and PST's views and activities were video-recorded, field notes were developed, PSTs' journal entries were analysed and their work duplicated for in-depth analysis and triangulation (Rasmussen & Stephan, 2008). Results of this study were compared to existing studies based on PSTs' flexible mental computation skills and strategies (Whitacre & Nickerson, 2006, Whitacre, 2015; Whitacre, 2016). Relevant studies were further used to structure the envisaged classroom activities and experiences to enhance generalizability or external validity (Cobb & Gravemeijer, 2008). An experienced researcher evaluated the learning trajectory before and after the experimentation process.

4.10 Ethical procedures

Before commencing with the empirical study, the application for ethical clearance from the ethics committee of the University of Stellenbosch was submitted and the study was approved (see appendix A). The management of the university satellite campus was approached for approval of the empirical study and permission was granted (see appendix B). The objectives of this research were clarified to the PSTs and informed consent from the PSTs was obtained (see appendix C). PSTs were informed that they were not bound to the study and could withdraw at any point of the intervention process. Assurance of proper use of research findings, anonymity, confidentiality, honesty and respect was given. To avoid conflict of interest, the researcher clarified that PST participation was voluntary and would pose no risk to them. PSTs were not deceived but both roles of the researcher, as TE and researcher were clarified before the intervention.

The researcher also announced the need to respect each other's level of FMC as the study intended to devise ways to improve PSTs' knowledge to teach FMC. PST anonymity was ensured through the use of pseudonyms rather than PSTs' real names. During interviews leading questions and personal impressions were avoided to prevent interference with PST perspectives. Discomfort that resulted when the videotaping was positioned in front of the class, was addressed by moving the video recorder to the back of the class. PST participation improved as researcher encouraged participation, mutual respect and also by restating the purpose of the study. The reporting of data included both positive and negative results of the study and falsification or distortion of data was avoided as well as plagiarism. The study was carried out in the shortest time possible to avoid interference with PSTs' learning of other learning exit outcomes, as stipulated in their course outline of the mathematics module.

4.11 Limitations

Owing to the nature of the study that was informed by design-based research methodology, not all satellite campuses of the university were involved in the study, thus limiting the degree to which the study could be generalised. This eventually confined the present study to one classroom environment on one satellite campus as a research site. Limited prior knowledge of the majority of the PSTs, in terms of their number sense and basic number facts, delayed the pace of the study. Struggles with formulation of sentences in English and a lack of appropriate mathematical concepts to express their ideas during interviews limited the extent to which PSTs could clearly express all the possible ideas they had. A mixed methods study provided rich data and was effective in terms of triangulation of findings, but required plentiful time to carry out the quantitative and qualitative data collection and analysis.

4.12 Conclusion

The chapter above outlined the research design and methodology of this study by providing details on data collection tools and processes. This study employed a mixed methods design to collect quantitative and qualitative data (Creswell, 2015). A design-based research methodology was used to design and implement tasks that would improve PSTs' ability to carry out calculations flexibly in the mind without using any external support and mathematical devices (Cobb & Gravemeijer, 2008; Wang & Hannafin, 2005). Quantitative data were collected through a survey questionnaire and pre- and post-intervention tests. Pre- and post-intervention interviews and observation of PSTs actions and written work during the intervention cycles were used to collect qualitative data.

Collecting quantitative and qualitative data consecutively was a challenge as it demanded more time to administer all the data collection instruments and also to carry out the intervention process involving direct teaching of PSTs. Tasks used in the intervention process were in the form of problem solving whereby real-life situations involving mathematics were adopted coupled with the use of different mathematical representations and manipulatives to facilitate the invention of strategies for computing mentally (Freudenthal, 1968; Heirdsfield, 2002; Treffers, 1987). The tasks resulted in the product of this study which is a domain-specific teaching framework for TEs. The argument in this study is that for PSTs to develop their own FMCs and knowledge for teaching, PSTs need to learn the mathematical concepts they are going to teach through a constructivist learning approach. The next chapter presents the data and their interpretation.

CHAPTER5: DATA PRESENTATION AND INTERPRETATION OF SURVEY AND INTERVIEWS

5.1 Introduction

In Chapter 4, the design and methodology of this study is outlined. In Chapter 5, the TE presented and interpreted the findings of this study through an in-depth discussion of the results. The results are presented and discussed in line with the first two aims of this study. This was to:

- Determine skills, beliefs and experience of PSTs about flexible mental computation.
- Establish effective ways to develop PSTs' flexible mental computation skills.

The section below begins with the presentation of quantitative results followed by qualitative results. The quantitative results include an investigation into PST skills to compute mentally, beliefs, school experience, and results of the diagnostic tests. Qualitative findings contain outcomes of the pre- and post-intervention interviews. Findings of the questionnaire present diverse beliefs of PSTs, as discussed next.

5.2 Pre-service teacher beliefs about FMC development

To ensure a high response rate, the TE administered the questionnaires in class during a mathematics teaching slot. All PSTs were given time to complete the form individually. All the questionnaires were collected when all PSTs had completed the forms. Findings concerning PST beliefs are discussed in line with the themes that emerged from the study. A summary of the themes is provided in table 5.1.

Table 5.1 Summary of themes

Themes on beliefs	**Themes on experiences**
Views on what FMC is	FMC development at school
Views on how to do flexible calculations in the mind	How PSTs developed FMC skills at school
Views on how FMC skills develop	

5.2.1 Views on what FMC is

The majority (92%) of the 38 PSTs who agreed to participate in the survey defined the concept 'flexible mental computation' as the ability to use own mind to solve (add, subtract, multiply and divide) mathematics problems using easy methods. PSTs specified that the process of calculating mentally excludes the use of a calculator, counting blocks, sticks, multiplication table, pencil-and-paper or any other calculation device. One respondent believed that flexible mental computation is when calculations are carried out instantly by providing memorised answers. A different belief expressed was that flexible mental computation is when mathematic problems are solved "using stones, fingers and bottle tops to find answers" (Denia, respondent communication, February 16, 2018). Most of the PSTs (61%) did not believe that flexible mental computation is when a learner memorises methods taught by the teacher during lessons to mentally solve problems. Survey results indicated that most of the PSTs (89%) believed that it is important for all learners to develop FMC skills even though calculators exist. The PSTs also outlined their views on ways of carrying out calculations as discussed below.

5.2.2 Views on how to do flexible calculations in the mind

As to calculation strategies, almost all PSTs (92%) believed that flexible mental calculations can be performed in many ways. Eighty-four percent of the class believed that proficiency in FMC is attained by inventing own calculation strategies. More than half of the PSTs (66%) believed that calculation strategies used by learners to perform calculations do not matter as long as the answer is correct; while 79% of the PSTs believed that there are calculation methods that are more efficient than others. Ninety-two percent of the PSTs agreed that how teachers understand FMC determines how they would develop learners' mental computation skills.

5.2.3 Views on how FMC skills develop

Teaching methods PSTs are likely to use in teaching are underpinned by what they believe is the best approach to develop FMC skills. Findings from the questionnaires included aspects pertaining to prescription of calculation strategies by a teacher, solving problems using own calculation strategies, communication, the use of a multiplication table, pen-and-paper method, memorisation of basic number facts, and the use of standard method in mind. Most of the PSTs agreed that calculation methods should be provided by a teacher. This was backed by eighty-two percent of the PSTs who indicated that learners should carry out mental calculations using strategies prescribed by a teacher. PSTs agreed that for learners to develop FMC skills a teacher has to provide examples first and thereafter give plenty of exercises for learners to practise. In pre-intervention interviews one PST confirmed in the statement that "the teacher taught us... in every calculation... two or three best methods to use" (Pobo, respondent communication, February 22, 2018). Another PSTs however indicated that "the teacher gives us two methods, for example… he will give you the first one if you are not managing to get the answer, then you use the second one" (Meni, respondent communication, February 22, 2018). Despite their accounts on teacher approaches, half of the PSTs acknowledged that the ability to calculate without any external support may develop through solving problems mentally without any hints from a teacher.

Survey outcomes also outlined the need for elementary school teachers to encourage learners to communicate their calculation methods with the entire class to develop FMC skills. Eighty-nine percent of the PSTs indicated that communication is necessary for discussion of diverse calculation methods. In addition, 55% of the PSTs indicated that memorisation of the multiplication table aids proficiency in computing mentally. This view surfaced in interviews where a respondent emphasised that "when you know that table, it is easier for you to calculate mentally… we memorise the numbers… know them by heart to get the answer fast" (Andy, respondent communication, February 22, 2018). It is evident that the experience of memorising the multiplication table to perform mental calculations oriented the PSTs toward a specific learning approach. As a result, the PSTs believed that one way to develop learners' flexible mental computation skills is to memorise the multiplication table.

Another belief that surfaced pertained to the role of pen-and-paper method in the development of FMC skills. Sixty-eight percent of the PSTs believed that the use of pencil and paper supports the development of FMC skills. In addition, half (50%) of the PSTs indicated that memorisation of basic number facts, such as 2 + 3 = 5 and number relationships such as 3 objects are 2 more than 1, facilitate FMC skills development. In addition, performance of calculations using standard algorithms also featured among PSTs as a way to perform calculations mentally. Meanwhile, most of the PSTs did not agree that standard algorithms make little contribution towards number sense development. However, 63% of the 38PSTs agreed that it is easier to perform written calculations than to use mental methods to solve a real-life problem, amid expressing a need for FMC skills development.

Although the majority (63%) of the PSTs believed that it is easier to calculate using written calculation methods, 89% of the PSTs agreed that there are some computations that are faster to calculate mentally than using the pencil-and-paper approach. Ninety-five percent of the PSTs believe that FMC contributes immensely towards number sense development. Therefore, to understand how the PSTs developed such beliefs, the next discussion presents information on the way PSTs developed FMC skills at school.

5.3 Pre-service teachers' school experience on FMC development

5.3.1 FMC development at school

Eighty-two percent of the PSTs indicated that they developed flexible mental computation skills at school. Only 16% of the PSTs indicated not having developed FMC skills at school. However, the outcomes of the test before the intervention demonstrate that 51% of the PSTs were unable to calculate more than half of the test items correctly, whereas 49% of the PSTs managed to calculate more than half of the test items correctly. On one hand, the pre-intervention test scores contradict the majority of the PSTs' claims that they developed flexible mental computation skills at school. On the other hand, it is likely that the inability to perform more than half of the test items correctly is because PSTs may have forgotten how to calculate mentally owing to calculator usage in higher grades. Confirmation was provided by one of the PSTs during an interview: "I have forgotten how to calculate mentally... so I would like to have some time to learn to calculate mentally" (Andy, respondent communication, February 16, 2018). Another participant in a separate interview indicated that the ability to compute mentally was disabled in higher grades where calculator usage was permitted, stating that "from [Grade] 6 we started using calculators ... time of calculators... our minds started relaxing... we became lazy" (Mara, respondent communication, February 16, 2018). Though some PSTs attributed poor calculation skills to calculator usage, literature emphasises that a strong number sense and invention of calculation strategies make it difficult for learners to forget strategies, especially when applied to solve daily life problems. However, learning the standard method first may have contributed to PSTs' inability to calculate mentally. Also, evidence emanated from PSTs responses (see section 5.2.3) that strategies were not invented, but were prescribed by a teacher.

5.3.2 How PSTs developed FMC skills at school

Most of the PSTs agreed that, at school, they first learned how to perform calculations using the column method before developing own calculation methods. Despite learning the standard method first at school, most of the PSTs agreed that FMC skills should be developed before standard algorithm or column method. During individual interview sessions, one participant indicated that the only way they could work out the calculations was to use the standard method in mind, whereas other PSTs used a variety of calculation strategies. Some of the existing FMC strategies used were breaking up numbers into place value to add or decomposing into place values, decomposition and friendly numbers or landmarks, using existing canonical and non-canonical understanding of numbers (breaking a number into tens and into ones), using related facts, reversibility, commutative property and making a multiple of ten.

Responses from the questionnaires indicate that 79% of the PSTs were taught a variety of methods to perform mental calculations when they were at school. The prescription of calculation strategies emerged in interview findings indicating that in school a teacher would present learners with a calculation strategy that most learners understood well. Some PSTs pointed out that teachers provided long division methods with which to calculate. In other words, teachers assisted learners to carry out calculations using standard method and not necessarily FMC strategies. Another approach used was where teachers prepared questions for learners to answer quickly and to think fast and improve their own ability to compute mentally. In interviews PSTs stated that some teachers supported learners by organising extra classes where they were provided with more examples of calculation methods and activities. Fluency in calculations was enhanced through activities such as mock exams and unexpected tests. Before proceeding to a new topic, a teacher would test learners' understanding of particular calculation methods so that learners could share their understanding of specific calculation strategies. PSTs also stressed that a teacher would clarify the meaning of the basic operations and the operation signs, and then provide a hint on how to compute before giving equations to work out.

In terms of ability to perform calculations mentally, 66 % of PSTs expressed confidence in performing calculations in mind without applying the standard algorithm. Evidence from pre-intervention interviews confirmed that a large number of interviewees who computed specific calculations correctly used strategies that are flexible and not a standard method. Another aspect that emanated from the questionnaires concerns communication. Eighty-four percent of the PSTs indicated that their teachers encouraged individual learners to communicate calculation methods to the entire class.

Questionnaire findings also revealed the use of manipulatives to develop FMC skills. Eighty-nine percent revealed that they learned FMC skills using manipulatives at school. Examples of manipulatives were provided in interviews where PSTs mentioned the use of manipulatives such as stones, counters and blocks to teach basic number facts and operations. PSTs also outlined that teachers used a multiplication table to replace the use of a calculator. The use of money was also mentioned as a strategy used by teachers to help learners understand the meaning of division. Ultimately, findings from the questionnaires illuminated ways PSTs developed FMC skills at school.

PSTs' responses provided in the questionnaires signified that, in school, calculation strategies were prescribed by a teacher through provision of examples, followed by calculations of a similar nature for PSTs to practise. Prescription of calculation strategies contradicts the constructivist learning approach and the principles of developing FMC skills (Kamii & Joseph, 2004). The use of manipulatives indicates that teachers made efforts to ensure comprehension of the meaning of numbers.

5.4 Pre-intervention test

5.4.1 Analysis of the pre-intervention test items

The test items used in the pre-intervention test are given in Table 5.2, where they are classified according to the four basic operations. The 21 items included mostly addition, subtraction, multiplication and division of numbers involving one, two and three digits. Only one calculation comprised four-digit numbers (item number 20 in Table 5.2). The four-digit number was given to determine PSTs' ability to calculate numbers beyond the three-digit limit stipulated in the elementary school syllabus.

Table 5.2: Flexible mental computation pre-intervention test items

Addition	Subtraction	Multiplication	Division
1. 5 + 7 **2.**19 + 15 **3.**23 + 18 + 37 **13.**59 + ☐ = 82	**4.** 23 – 16 **5.**151 – 98 **6.**563 – 292 **14.** 85 – ☐ = 67 **15.**☐ – 38 = 89 **20.** 2024 –1999 **21.** 799 – 51	**7.** 4 × 50 **8.**32 × 15 **9.**16 × 25 **12.**14 × 8 **18.** 15 × ☐= 45 **19.** 250 × 2	**10.** 16 ÷ 4 **11.** 120 ÷ 6 **16.** 192 ÷ 8 **17.**☐ ÷ 15 = 20

The test items were varied to include one- to three-digit numbers as reflected in the Namibian lower primary syllabus. The items were designed according to the descriptions by McIntosh (2005, p. 9), McIntosh et al. (1992) and Hartnett (2007), to include items that may elicit calculation strategies such as:

- Counting on and back
- Adjusting and compensating (change and fix)
- Doubling and halving
- Using place value to think in multiples of ten to add, subtract, multiply and divide
- Commutative, associative and distributive laws
- Using related facts
- Doubles and near doubles
- Bridging ten/friendly numbers
- Inverse operations
- Skip counting
- Splitting into known parts
- Repeated subtraction/addition

5.4.2 Analysis of possible computation strategies for the test items

The test was answered by just writing down the calculated answer without any written procedures. The purpose was to assess existing calculation skills. To promote mental calculation, the test items were projected in bold on an A4 sheet of paper for only 20 seconds. The formulations of test items included specific number relationships that could encourage the use of particular calculation methods. Table 5.3 outlines possible calculation methods that each test item was designed to elicit.

Table 5.3: Possible flexible mental computation strategies for pre-intervention test

Test items and possible calculation strategies															
Addition				**Subtraction**				**Multiplication**				**Division**			
1. 5 + 7	**Count on to add**	**Use near double**	**Use friendly numbers**	**4.** 23 – 16	**Count on to subtract**	**Count back to subtract**	**Break up one number**	**7.** 4 × 50	**Count on to multiply**	**Double fifty twice**	**Double and halve to adjust numbers**	**10.** 16 ÷ 4	**Half, half to divide by four**	**Adjust two numbers to divide**	
	5,6,7,8,9,1 0,11,12 or 7,8,9,10,11 ,12	5+(5+2)=10+ 2 =12	2+(3+7) =2+10 =12		16…17,18,19 ,20,21,22,23 = 7	23…22,21,20 ,19,18,17,16 = 7	23 – (10+6); 23 – 10=13; 13 – 6=7 so 23 – 16=7		50, 100, 150, 200	2 ×50=100; 2 ×100=200 4=2×2; (2×50)+(2×50) =100+100 =2×100 =200	4 × 50 =2×100 =200		Half 16 is 8 Half 8 is 4	16 ÷ 4 8÷2 = 4÷1	
2.19 + 15	**Friendly number**	**Adjust and compensate**	**Break up one number**	**5.** 151 – 98	**Effect of changing subtrahend**	**Empty number line**		**8.** 32 × 15	**Breaking up one number**	**Distributive property**		**11.** 120 ÷ 6	**Related fact**		
	19+1+14= 20+14 =34	20+15 – 1=35-1 =34	19+(10+5) =29+5 =34		98 is close to 100 151 – 100 =51 so 151 – 98=51+2=53	98 to 100 is 2; 100 to 51 is 51; 2+51=151 – 98			(32×10) + (32×5) =320+ half of 320 =320+ 160 =480	(30+2) × (10+5) [(30× 10) + (30×5)] + [(2× 10) + (2 ×5)] = 300+150+20+10 =480			12÷6=2 120 ÷ 6=20		
3.23 + 18 + 37	**Use commutat ive property/ friendly numbers**	**Breaking up numbers using place value to add**		**6.**563 –292	**Count on to subtract**	**Count back to subtract**		**9.** 16 × 25	**Double and half to multiply**	**Breaking up two numbers**		**16.** 192 ÷ 8	**Half three times to divide by 8**	**Use reversibility property**	**Break up one number**
	(23+37)+1 8 =60+18 =78	20+10+30=6 0; 3+7+8= 60+10 + 8=78			292 to 300 is 8; 563 – 300=263; 263+8=271	563 to 500 is 63; 500 to 300 is 200; 300 to 292 is 8; 200+63+8 =271			16 × 25 8×50 4×100 2×200 1×400=400	10×20 is 200; 10×5 is 50; 6×20 is 120; 6×5 is 30; 200+50+120+30 =400			Half 192 is 96 Half 96 is 48 Half 48 is 24; so 192÷ 8=24	192=8× □; 8×10=80 is 10 eights 20 eights is 160 4 eights is 32; 20+4=24	100÷ 8=12 remainder 4 90÷ 8=11 remainder 2 2÷8=0 remainder 2; (2+2+4) ÷ 8=1 =12+11+1 =24

13. 59 + □ = 82	**Non canonical understanding/ number line**	**Use reversibility property**	**Adjusting two numbers to subtract**	**14.** 85 – □ = 67	**Counting on using canonical understanding**	**Adjust one number and compensate to subtract**		**12.** 14 × 8	**Double, double, double to multiply by 8**	**Decompose one number**		**17.** □ ÷ 15= 20	**Relationship between operations**	**Relationship between multiplication by ten and five**	
	59+1= 60; 60 +22 = 82	82– 59 =82 – 60+1	83 – 60=23		85= 8 tens + 5 ones; From 67 to 85 are 2 tens minus 2 ones=18	85– 70+3=15+3 =18			14×2=28; 28×2=56; 56×2=112	(10+4) × 8 80 + 32 =112			15×20 15×2=30 30×10=300	(10×20)+ (half of 200) =200+100 =300	
				15. □ –38 = 89	**Use reversibility property**	**Adjust one number and compensate**	**Adjust two numbers and compensate**	**18.** 15 × □ = 45	**Count on in 15s**	**Count back in 15s**					
					89+38=90+37 =127	90+38 – 1 =90+37 =127	90+40=130; 130 – 1=129; 129 – 2 =127		15; 30; 45 15×3=45	45; 30; 15 = 3					
				20.2024 – 1999	**Use proximity of numbers**	**Count on to subtract**	**Count back to subtract**	**19.** 250 × 2	**Double to multiply by 2**						
					2 024 – 2000+1=24+1 =25	1999+1=2000; 2000+24=2024; so 2024 – 1999=25	From 2024 to 2 000 is 24; from 2 000 to 1 999 is 1; 24+1=25		250+250=500						
				21.799 – 51	**Proximity of numbers**	**Breaking up numbers using place value**	**Count on in tens to subtract**	**7.**4 × 50	**Count on to multiply**	**Double, double to multiply by four**	**Double and halve**				
					800 – 52=748	799=7 hundreds, 9 tens and 9 ones; 9 tens minus 5 tens equals 4 tens and 9 ones minus one equals 8ones so 799 – 51=748	51,61,71,81,91 and 8 = 10+10+10+10+8 =48 799 – 51=48+700 =748		50, 100, 150, 200	2 ×50=100; 2 ×100=200 4=2×2; (2×50)+(2×50) =100+100 =2×100 =200	4 × 50 =2×100 =200				

The PSTs' performance per test item is provided in Table 5.3. Since the test required PSTs to record only the answer, and not the strategy, this study sought to select any wrong answer randomly to identify the possible strategy that may have led them to the wrong answer. The analysis of wrong answers is provided in section 5.4.3. Data on the strategies that PSTs used in the test were obtained through individual face-to-face interviews with 17 PSTs who were purposively selected according to performance in the pre-intervention test (high, moderate and poor).

5.4.3 Pre-service teacher performance in each pre-intervention test item

The purpose of the test was to determine PSTs' ability to do flexible calculations in mind. The pre-intervention test was administered prior to any discussion related to FMC development. The pre-intervention test was conducted by the TE in a classroom. PST strategies used were established through individual face-to-face interviews after the test. Fifty-one PSTs received an answer sheet with 21 entries for answers. Test items were printed separately on A4 sheets of paper (because of a dysfunctional projector) and projected on the board for 20 seconds before replacing them with the next test item. PSTs were instructed not to use a calculator or any written method, but to use their mind and to record the correct answer only. Test scripts were collected immediately at the end of the test and marked by the TE. Table 5.4 presents the test items, the number and percentage of PSTs who worked out the calculations (correctly or wrongly), and the wrong answers provided by the PSTs. The results are ordered from highest to lowest success rate.

Table 5.4: Pre-intervention test results

Pre-intervention test results							
Test items		**Total number of PSTs who got the answer right or wrong (n = 51).**		**Percentage**		**Correct answer**	**Type of wrong answers provided**
Item number	**Item**	**Right**	**Wrong**	**Right**	**Wrong**		
1	5 + 7	49	2	96	4	**12**	0.4, 13
2	19 + 15	46	5	90	10	**34**	24, 25, 37, 44
7	4 × 50	46	5	90	10	**200**	20, 250,400
19	250 × 2	45	6	88	12	**500**	24,175,300,410,748
10	16 ÷ 4	42	9	82	18	**4**	2, 3, 5, 6, 7, 8, 12, 29
18	15 × □ = 45	41	10	80	20	**3**	2, 4, 5, 20, 124, 205, 500
4	23 - 16	33	18	65	35	**7**	6, 8,9,13,16,17,39
21	799 - 51	29	22	57	43	**748**	14, 25, 31, 48, 149, 301, 548, 728, 735, 738, 739, 741, 744, 747, 749, 759, 761
3	23 + 18 + 37	24	27	47	53	**78**	34, 48, 50, 53, 58, 62, 63, 65, 68, 70, 72, 73, 77, 88, 89
11	120 ÷ 6	23	28	45	55	**20**	3, 4, 12, 15, 6, 18, 20, 23, 24, 30, 60, 104, 120, 172, 832
20	2024 – 1999	23	28	45	55	**25**	3, 19, 21, 23, 24, 26, 34,36, 55, 193, 500, 748, 900, 1023, 1025, 1135, 1290, 1975, 1976
5	151 – 98	20	31	39	61	**53**	5, 13, 20, 36, 38, 43, 45, 49, 52, 55, 57, 60, 61, 69, 88, 93,113, 121, 123, 132, 142, 147,148, 153, 171, 231, 249
13	59 + □ = 82	20	31	39	61	**23**	7, 8, 11, 13, 20, 21, 22, 25, 31, 32, 33, 34, 37, 42, 43, 54,79, 83, 96
17	□ ÷ 15 = 20	19	32	37	63	**300**	3, 5, 11, 24, 120, 125, 130, 150, 200, 220, 225, 250, 280, 360, 400, 500, 600, 1000
14	85 - □ = 67	18	33	35	65	**18**	8, 10, 12, 13, 14, 15, 19, 20, 21, 22, 23, 24, 25, 28, 32, 34, 37, 71, 120, 142, 167
12	14 × 8	13	38	25	75	**112**	2, 6, 8, 17, 23, 33, 40, 42, 44, 52, 54, 55, 84, 88, 92, 96, 100, 102, 114, 116, 125, 136, 139, 144, 148, 204, 244, 332, 402, 600
6	563 – 292	9	42	18	82	**271**	23, 24, 41, 81, 100, 118, 141, 171, 181, 201, 230, 233, 245, 264, 272, 291, 298, 321, 331, 337, 340, 343, 371, 381, 398, 471, 761, 855
9	16 × 25	9	42	18	82	**400**	4, 11, 41, 60, 112, 121, 130, 145, 150, 157, 185, 220, 230, 260, 275, 300, 310, 320, 330, 342, 370, 399, 420, 426, 440, 460, 500, 530, 627, 700, 930, 2132, 2150, 2350, 3000
15	□ - 38 = 89	8	43	16	84	**127**	13, 27, 49, 50, 51, 53, 61, 67, 74, 102, 107, 109, 110, 115, 117, 120, 123, 124, 129, 134, 142, 143
8	32 × 15	4	47	8	92	**480**	8, 40, 43, 47, 63, 85, 87, 108, 120, 130, 133, 145, 207, 300, 310, 316, 320, 330, 340, 345, 364, 369, 420, 460, 470, 525, 526, 530, 570, 682, 828, 920, 3018, 3150, 3160, 3200
16	192 ÷ 8	4	47	8	92	**24**	3, 6, 9, 12, 14, 15, 16, 17, 18, 19, 21, 22, 28, 30, 32, 36, 37, 39, 40, 42, 46, 62, 75, 92, 120, 300

- Item: 5 + 7

  The pre-intervention test results above indicated that 96% of the class could add two one-digit numbers correctly. The two PSTs who failed to calculate 5 + 7 provided answers such as 0.4 and 13. An analysis of the error 0.4 suggests that the error may have been the result of panic as the overall test score of the participant was 62%. The error 13 may have emerged after splitting 5 to add 3 + 2 + 7, but since 2 is next to 7 the PST may have concluded that a 10 was arrived at, plus 3 equals 13. Since both PSTs were not involved in the interviews, the possible reasons provided above could not be verified. Both errors suggest a lack of confidence and ability to regulate own thinking processes, as found in a study by Heirdsfield and Lamb (2007) where inaccurate answers emerged from poor meta-cognitive skills.

- Item: 19 + 15

  Ninety percent of the PSTs managed to add two two-digit numbers both less than 20. However, 10% of the PSTs failed to add the two numbers mentally. Analysis of error 24 suggests that the participant may have attempted to add in the following order (19 + 5) + 10 and omitted the 10. Similarly, error 25 may have resulted from adding (10 +15) + 9 and the nine may have been omitted too. Error 44 could be a result of visualising the standard method to calculate mentally and 2 was carried over. As established in literature (Heirdsfield, 2002), this study infers that a lack of knowledge of proximity of a number to a friendly number and compensation strategy may have led to the making of the errors.

- Item: 4 × 50

  Test results show that only 10% of the PSTs worked out 4 × 50 wrongly to get answers such as 20, 250 and 400. The TE speculates that the answer 20 may have resulted from multiplying 4 by 5 without multiplying by 10 to get 200. The error of 400 may be the result of multiplying by 100 without halving the answer. A lack of number knowledge in terms of the relationship between 50 and 100 is evident, as the PSTs could not imagine that two 100s split into four equal parts equals 50 or four 50s make 200. The errors suggest a lack of regulation of own thinking and the use of alternative strategies to verify accuracy of answers provided (Heirdsfield & Lamb, 2007).

- Item: 250 × 2

  Although 88% of the PSTs calculated the item correctly, an analysis of the erroneous answers for 250 × 2 reflects an inability of 12% of the PSTs to recognise the pattern of 25 × 2 and 250 × 2. A lack of counting on and estimation skills is also projected in the errors, such as 24, 175, 300, 410, 748, as PSTs indicated inability to anticipate a reasonable answer. What may have led to such errors is not very clear and PSTs may have guessed any answer. Also, knowledge of doubles such as 25 + 25 = 50 so 250 + 250 is 500 seems to be poor. Inability to work out the calculation correctly also indicates the impact of more exposure to standard method on PSTs capacity to think mathematically and reason logically to make sense of the calculation.

- Item: 16 ÷ 4

  Despite 82% of the PSTs having divided 16 by 4 correctly, errors such as 2, 3, 5, 6, 7, 8, 12, and 29 were evident. An answer such as 12 seemed to result from 16 – 4, portraying the execution of incorrect action by the memory. Execution of incorrect action may signal poor regulation of own action and thinking. In addition, the errors portray the absence of knowledge of basic number facts, the inability to halve 16 and its quotient to get the correct answer, and inability to work backwards to check solution.

- Item: 15 × □ = 45

  Errors such as 2 and 500 provided by two PSTs confirm the lack of counting-on skills and understanding the effect of multiplication on a number projected in items above.

- Item: 23 – 16

  The error 39 is a result of performing a wrong operation in mind (23 + 16) as identified under the item 16 ÷ 4, whereas errors such as 13 may have resulted from using the standard method in mind and subtracting wrongly, e.g. 2 – 1 = 1, 6– 3 = 3 resulting in 13 as an answer. The errors indicate absence of meta-cognitive skills to monitor own thinking.

- Item: 799 – 51

  Only 57% of the class calculated the item correctly. The error of 749 suggests that 50 was subtracted from 799, but 1 was not taken away, thus indicating inefficient regulation of the calculation process. Meanwhile, the error of 14 reflects inadequate understanding of the magnitude of numbers (Sowder, 1990).

- Item: 23 + 18 + 37

  The majority of the PSTs (53%) struggled to calculate the item mentally. Errors such as 48 and 70 attribute the poor performance to the long chain of numbers to be computed. For example, 48 could result from adding 30 + 18 without the rest, while 70 could result from 23 + 37 without adding 18. The errors herewith signify that PSTs lack number fact knowledge that can lead to more efficient and quick strategies.

- Item: 120 ÷ 6

  Fifty-five per cent of the PSTs provided answers that are not reasonable, such as 3 and 832. PSTs could not reason that 3 × 6 is far from 120. Inability to recognise patterns such as 12 ÷ 6 = 2, so 120 ÷ 6 = 20 is also evident.

- Item: 2024 – 1999

  The error 1975 indicates the use of a standard method wrongly in mind, as follows, 2 – 1= 1, 0 – 9 = 9, 9 – 2 = 7, 9 – 4 = 5, resulting in 1975 as an answer. The rest of the errors also portray incorrect usage of the standard method in mind. A major deficiency in perception and understanding of the magnitude of numbers and proximity to friendly numbers is evident.

- Item: 151 – 98

  The error 153 is possibly a result of using the standard algorithm in mind and borrowing, followed by subtracting 9 from 14 after borrowing from 15. The answer 249 could be a result of adding instead of subtracting. This also shows a lack of regulation of own thinking and mental action. Knowledge of proximity of 98 to 100 could be lacking among the 61% of the PSTs.

- Item: 59 + □ = 82

  Answers such as 37 indicate usage of a column method where 8 – 5 = 3 and 9 – 2 = 7, so the answer is 37. This confirms that the absence of skills to make flexible calculations in mind and a weak understanding of relationship between operations.

- Item: □ ÷ 15 = 20

  Results show that 65% of the PSTs have no knowledge of relationship between operations or working backwards to get the answer.

- Item: 85 – □ = 67

  The error 22 appears to result from wrong application of the standard method in mind as 7 – 5 = 2 and 8 – 6 = 2. As an error, 167 shows a lack of regulation of own thinking and number relationship. PSTs' struggle to solve calculations with unknown initial and unknown change confirms results of a study conducted with elementary school learners where difficulty is posed in terms of thinking inversely (Carpenter et al., 1996).

- Item: 14 × 8

  Many PSTs (75%) struggled to calculate the item mentally. The test results reflect inability of PSTs to work with partial products or doubling and halving to multiply. This also shows little knowledge of the basic multiplication facts.

- Item: 563 – 292

  The error 855 again confirms PSTs' inability to regulate their mental action, as addition was performed instead of subtraction. Analysis of most of the errors confirms wrong usage of a standard method, e.g. the error 331. Answers such as 23 and 331 outline inability to anticipate an approximate answer by estimating a reasonable solution. This also indicates that PSTs are unable to perceive 292 as 300 subtracted from 563 plus 8 more.

- Item: 16×25

  Eighty-two per cent of the PSTs could not calculate the item mentally. Errors like 4 indicate no engagement of mind to calculate correctly. Another error is 2132 which could emerge from $16 + 5 = 21$ and $16 \times 2 = 32$, so the answer 21 connected to 32. Another error is 230, possibly resulting from $20 \times 10 = 200$ and $6 \times 5 = 30$, thus the answer is 200 plus $30 = 230$. The aforementioned answer reveals a very weak ability and understanding regarding mental multiplication. Evidence from the errors discloses no understanding of adjusting numbers to make the calculation easier.

- Item: $\square - 38 = 89$

  In this study, calculation with initial unknown was one of the most difficult calculations for PSTs as 84% of the PSTs were unable to compute it mentally in a flexible way. Answers such as 13, 117 and 143 indicate inability to think inversely to get the correct answer. Research has found that the difficulty of the calculation results from where the unknown is positioned; in this case, at the beginning of the calculation (Carpenter et al., 1996).

- Item: 32×15

  This item was one of the most difficult items for PSTs. Only 8% of the PSTs calculated it correctly, where the standard method was used halfway, such as in errors like 320 where 32 was probably multiplied by 10. PST performance on this item indicates limited knowledge regarding multiplication by 15 as the sum of multiplication by 10 and 5.

- Item: $192 \div 8$

  Ninety-two percent of the PSTs could not calculate $192 \div 8$ correctly in mind. Errors such as 3 and 300 suggest that most PSTs possibly provided answers by guessing without establishing the reasonableness of their answers. Difficulty to divide a three-digit number by a one-digit number greater than 2 is evident in the pre-intervention test results.

Overall findings of the pre-intervention test revealed that PSTs were unable to reason in terms of proximity of numbers to a friendly number such as a multiple of ten. Another aspect is inability to reason in terms of using direct or indirect addition, subtraction, multiplication and division to solve a problem. Similarly, the errors PSTs made indicate incompetency to express a number as a product of prime factors to multiply or divide with smaller or easier numbers. Also, the findings show that PSTs have underdeveloped reasoning pertaining multiplication by 15 and knowledge of doubling and halving to calculate mentally. Errors made in this study are similar to those made by Grade 4 learners in a study by Brumfield and Moore (1985). Carpenter et al. (1998) found that learners who were exposed to standard calculation method before FMC strategies, produced a variety of wrong answers to calculations, as was found in the present study.

Conclusions drawn from the analysis of errors corroborate findings from the questionnaires, as most PSTs visualised the standard method to calculate mentally, which resulted in many incorrect answers displayed in Table 5.4. The errors also confirm the low confidence responses (66%) provided by PSTs on the questionnaires. Furthermore, the kind of errors that were provided by the PSTs also confirm interview expressions of fear of mathematics. It is evident from the study that some PSTs' fear of mathematics is a result of weak conceptual knowledge as found by Hembree (1990) and Vinson (2001). Generally, the unreasonable answers provided by the PSTs portray limited knowledge of the number facts, relationship between numbers and the four basic operations as advocated for by (Heirdsfield, 2002). Such findings prompt educators to determine effective ways to improve PST ability to compute mentally.

Sowder (1990) stated that proficiency in calculating mentally is achieved in situations where strategies taught mechanically are not used. However, inflexible calculation methods can coexist after flexible strategies have been developed (Carpenter et al., 1998), as became evident in the post-intervention test. Individual performance of PSTs on all test items is presented next.

5.4.4 Individual scores on pre-intervention test

Table 5.5 reflects scores of individual PSTs in the test. The cumulative percentage shows that 51% of the PSTs' test scores were below 50%.The highest mark in the pre-intervention test was 86% scored by one participant and the lowest mark was 14% by one participant. Only 49% of the PSTs managed to calculate more than half of the test items correctly. Post-intervention test items and results are discussed next.

Table 5.5: Pre-intervention results of the class

				Total score of individual PSTs		
Mark out of 100	**Frequency of PSTs with specific mark**	**Percentage of PSTs**	**Cumulative percent**	**PST number**	**Items worked out correctly**	**Items struggled with** (All items not listed under correct items, see Addendum M)
14	1	2,0	2,0	1	1,2, 7	
19	1	2,0	3,9	2	1, 2, 4	
24	1	2,0	5,9	3	1, 2, 18, 19, 20	
29	2	3,9	9,8	4	1, 2, 3, 7, 15, 21	
				5	1, 2, 7, 10, 18, 19	
33	4	7,8	17,6	6	1, 7, 10, 11, 17, 18, 19, 21	
				7	1, 2, 4, 7, 11, 18, 19	
				8	1, 2, 4, 7, 10, 19, 20	
				9	1, 2, 4, 7, 10, 18, 19	
38	4	7,8	25,5	10	2, 5, 7, 14, 17, 18, 19, 21	
				11	1, 2, 7, 10, 11, 18, 19, 21	
				12	1, 2, 3, 7, 10, 18, 19, 21	
				13	1, 2, 4, 7, 10, 11, 18, 19,	
43	7	13,7	39,2	14	1, 5, 7, 12, 16, 18, 20, 21	
				15	1, 2, 7, 10, 13, 14, 18, 19, 20	
				16	1, 2, 3, 4, 5, 7, 9, 10, 20	
				17	1, 4, 6, 7, 10, 15, 18, 19, 20	
				18	1, 2, 6, 7, 10, 11, 13, 19, 20	
				19	1, 2, 3, 7, 10, 17, 18, 19	
				20	1, 2, 4, 10, 13, 15, 18, 19, 21	
48	6	11,8	51,0	21	1, 2, 3, 4, 7, 14, 15, 17, 18, 19	
				22	1, 2, 4, 5, 10, 11, 17, 18, 19, 20	
				23	1, 2, 3, 4, 5, 7, 10, 18, 19, 20	
				24	1, 4, 7, 10, 11, 13, 14, 18, 19, 21	
				25	1, 2, 3, 10, 11, 13, 17, 18, 19, 21	
				26	1, 2, 3, 7, 10, 16, 17, 18, 19, 20	
52	9	17,6	68,6	27	(See Addendum N)	5, 6, 8, 9, 10, 12, 13, 14, 16
				28		4, 6, 8,12, 13, 14, 15, 16, 17, 20
				29		4, 6, 8, 9, 11, 12, 13, 15, 16, 20
				30		3, 8, 12, 14, 15, 18, 20
				31		3, 5, 11, 15, 17, 21
				32		5, 6, 8, 9, 11, 15, 16, 17, 20, 21
				33		5, 8, 9, 12, 13, 14, 15, 16, 17, 20
				34		3, 5, 6, 8, 9, 11, 12, 16, 20
				35		2, 3, 6, 8, 9,11, 14, 15, 16, 18,
57	7	13,7	82,4	36		4, 8, 9, 11, 12, 15, 16, 18, 21
				37		3, 5, 6, 8, 9,11, 13, 14

				Total score of individual PSTs		
Mark out of 100	**Frequency of PSTs with specific mark**	**Percentage of PSTs**	**Cumulative percent**	**PST number**	**Items worked out correctly**	**Items struggled with** (All items not listed under correct items, see Addendum M)
				38		5, 6, 8, 9,11, 15, 16, 17, 18
				39		6, 8, 9, 16, 17, 18, 19, 20
				40		3, 8, 9, 13, 14, 15, 16, 17, 20
				41		6, 8, 9, 12, 13, 15, 16, 17, 21
				42		3, 4, 5, 6, 8, 9,15, 16, 17
62	3	5,9	88,2	43		6, 8, 12, 13, 14, 15, 16, 20
				44		3, 5, 6, 8, 9, 16, 17, 20
				45		1, 8, 9, 12, 14, 15, 16, 20
67	2	3,9	92,2	46		5, 6, 8, 12, 14, 15, 16
				47		5, 11, 12, 15, 19
76	1	2,0	94,1	48		6, 8, 15, 16, 17
81	2	3.9	98,0	49		3, 6, 14, 15
				50		6, 13, 15, 16
86	1	2,0	100,0	51		8, 12, 16
Total	51	100,0				

The PSTs' inaccurate answers, reflected in Table 5.5, indicate a possible absence of skills and knowledge, as emphasised by Heirdsfield (2002) and Heirdsfield and Lamb (2007) and which include:

- Number and operation (understanding the four basic operations, numbers, effect of changing subtrahend, effect of changing minuend to choose and implement efficient mental strategy)
- Numeration (knowledge of place value – tens and ones and proximity of numbers)
- Number facts (knowledge of basic facts for fast and accurate implementation of a strategy)
- Metacognition (ability and attitude towards working with numbers for choice and implementation of strategy and checking of solution)
- Memory (for implementation of calculation strategies and developing mathematical judgements)
- Estimation (sense of size of numbers and thinking logically to estimate solutions sensibly)

This study argues that difficulty in calculating mentally may be due to inability to think logically, critically and creatively to derive facts; and lack of number relationships knowledge to recognise patterns between numbers in order to estimate solutions sensibly and check answers (Colyvan, 2012; NRC, 1989; Rechtsteiner-Merz & Rathgeb-Schnierer, 2015). Insight into PSTs' calculation strategies is discussed next.

5.5 Pre-intervention interview

The first interviews were conducted two days after the pre-intervention test, and 15 out of the 51 who wrote the test were interviewed.

The pre-intervention interviews were carried out to identify PSTs' existing flexible mental calculation strategies, as well as their beliefs and experiences pertaining to FMC development. Such information was necessary to validate findings from the questionnaires. The interview consisted of two parts: in the first part PSTs were asked to perform four calculations (see Table 5.6) mentally and justify their strategies. The second part consisted of questions that PSTs had to answer orally. The calculations and number range of calculations are presented next.

5.5.1 Calculations and computation strategies

Table 5.6 outlines pre-intervention interview items and envisaged computation strategies

Table 5.6: Pre-intervention calculations performed by interviewees

Operation	**Addition**	**Subtraction**	**Multiplication**	**Division**
Calculation	1.23 + 18 + 37	2. 151– 98	3.32 × 15	4.120 ÷ 6
Envisioned strategy	**Commutative property/relationship between numbers (reordering numbers to make a friendly number or land mark)**	**Adjust one number and compensate**	**Break up one number or decompose into place value to multiply**	**Breaking one number into its factors**
	(23+37)+18=60+18 =78 **Or** 20+10+30=60; 3+7+8=10 60+18=78	151 – 100 + 2= 51+2 = 53	(32 × 10) + (32 × 5)=320+160 =480	120=12 × 10 120= 2 × 6 × 10 120=20× 6 120 ÷ 6=20

The pre-intervention interview comprised four calculations with the first being 23 + 18 + 37 where all 15 interviewees participated. This item was poorly answered as pre-intervention test results showed that only 47% of the class calculated the item correctly. Out of the 15 interviewees, seven interviewees calculated the item correctly. However, in interviews, 12 PSTs out of 15 carried out the algorithm correctly. Two of the 12 applied the standard method in mind whereas the remaining 10 PSTs used flexible strategies. Among the strategies used were breaking up numbers into place value to add, decomposition, friendly numbers, canonical understanding of sums and commutative property.

The second algorithm was 151 – 98 which was also poorly answered in the pre-intervention test with only 39% correct answers. Five of the 15 interviewees worked out the item correctly in the pre-intervention test, but in interviews 10 out of 15 PSTs worked out the calculation correctly. Two of the PSTs used a standard method, one used both a standard method and own strategy, and seven did a flexible calculation using one of the following methods: breaking up the minuend into a friendly number and decomposing of both numbers.

The third algorithm was 32×15 which was also poorly answered as only 8% of the class provided correct answers. Only two of the 15 interviewees calculated the item correctly in the pre-intervention test. Similarly, out of the 15 PSTs only two managed to successfully work out the calculation mentally in the interview. One of the two PSTs employed the standard method mentally whereas the other participant used knowledge of the distributive property and partial products but had no alternative strategy to perform the calculation.

The fourth algorithm was $120 \div 6$ which was also poorly answered in the pre-intervention test with 45% correct answers. Six interviewees provided a correct answer in the pre-intervention test while only seven PSTs managed to calculate correctly in the interview. Strategies used were counting on in twenties, using a related fact and multiplicative aspect of the quotient, reversibility rule and knowledge of the multiplicative aspect of the dividend coupled with division by prime factors of the divisor. One of the PSTs was unable to explain the own solution strategy applied.

The TE concluded that interviewees' ability to compute the same calculation correctly in interviews after working it out wrongly in the pre-intervention test may have been a result of the relaxed environment as the activity was conducted in the presence of the TE only, enough thinking time, and the familiarity brought about by the second exposure to the same calculation. The next discussion provides information on the semi-structured interviews.

5.5.2 Semi-structured interview

The second part of the interview followed immediately after the first part had been conducted. All 15 PSTs agreed to answer all questions that constituted part B of the interviews, namely:

- Up to what grade did you learn mathematics?
- What comes to mind when you hear the word "flexible mental computation"?
- What is your view about the need to develop learners' flexible mental computation skills?
- What is your view about the need to develop teachers' flexible mental computation skills?
- What is the lowest grade during which you recall learning to calculate in your head?
- How did you learn to perform calculations mentally in the grade mentioned above (fifth bullet point) or in primary school?
- What is the highest grade during which you recall having to learn to calculate in your head?
- How did you learn to calculate mentally in the grade mentioned above (seventh bullet point) or in high school?
- What was the teacher's role when you learned to calculate in your head without a calculator or written algorithms?

- What was your role (responsibility) when you developed flexible mental computation skills?
- What mathematical representations were used to develop flexible mental computation skills?
- What manipulatives were used to develop flexible mental computation skills?
- How best could teachers assist learners to develop flexible mental computation skills?

In interviews, all 12 PSTs indicated that they had learned mathematics from Grade 1 to Grade 12. Generally, PSTs indicated that FMC is about using own mind to calculate easily, think fast and cope with calculations. The main idea that emerged was that FMC is carried out in mind without using calculators, computers, pencil-and-paper method, fingers or sticks. Ten PSTs indicated that FMC improves learners' understanding of numbers, their intellectual capacity and their skills to calculate easily and fast in senior grade examinations without wasting time performing easy calculations on a calculator. Another argument was that FMC eliminates over-reliance on calculators. PSTs also indicated that FMC is useful when shopping as it enables calculating change or the cost price of an item without using a calculator or any other external support. Also, learners' minds are activated to think fast and are relieved from redundancy caused by calculator dependency to become competent mathematicians.

In particular, one participant expressed fear of mathematics owing to incompetency in mental calculations, but was aware that fear could be eliminated through FMC which makes mathematics pleasant, enjoyable and easy. In own words, the participant said: "The first test that you gave us I was frightened so I could not finish things on time also because I did not have any idea on how to get the solutions in the given period of time" (Kibi, respondent communication, February 16, 2018). Also, PSTs found FMC to be a more appropriate approach to calculations than calculators. This was because FMC enhances ability and understanding of the four operations and improves accuracy of calculations.

PSTs also indicated that competency in mental calculations might enable them to support learners to calculate mentally. They argued that learners whose teacher is competent in mental calculations may foster learners' computation skills because a "teacher … must teach by example" (Abu, respondent communication, February 16, 2018). In addition, PSTs indicated that development of FMC skills equips them with deficient skills, develops strategies to calculate differently, changes self-concept and perception about mathematics learning, and enhances understanding of appropriate tasks for learners. Another important aspect is that PSTs will implement similar strategies to teach in school. Although all PSTs indicated that they had developed FMC skills in primary school, they expressed inability to recall clearly how they had learned FMC at school.

In interviews, problem solving emerged as an approach through which PSTs learned FMC skills in primary school. This was presented in short stories, for example, "They went three of them and came back two… one is left…" (Moia, respondent communication, February 16, 2018). Other problems involved activities such as "Fifteen plus an empty box plus the answer. How many do you need to complete this calculation?"(Ambo, respondent communication, February 16, 2018). PSTs also stated that teachers used stones to demonstrate, explain and illustrate the meaning of subtraction like, nine minus five. In this case, nine stones would be counted out of a pile of stones and five stones would be removed from the nine stones and the remaining ones became the answer to nine minus five (Naby, respondent communication, February 16, 2018). Also, representations such as circles drawn on the board or on a poster were used to reflect calculation methods in class.

Numbers and number words such as "sum" and posters of written algorithms such as "one plus and a box… equals five" were displayed in class (Mindo, respondent communication, February 16, 2018). Various PSTs told how teachers provided two or more different calculation methods as examples for learners to use when solving problems (Lacy, respondent communication, February 16, 2018; Meni, respondent communication, February 16, 2018; Pobo, respondent communication, February 16, 2018). Quizzes were used where learners were expected to think and answer in a short time and thereafter learners were invited to provide answers and calculation strategies in class (Andy, respondent communication, February 16, 2018). In addition, learners were required to memorise the multiplication table so that they could recall number facts instantly.

In interviews, PSTs confirmed that memorisation of the multiplication table facilitates mental calculations because "when you know that table, it is easier for you to calculate mentally… we memorise the numbers… know them by heart to get the answer fast" (Andy, respondent communication, February 16, 2018). Besides the multiplication tables, additional material such as bottle tops, counting beads, pencils, rulers, own body, chalk, money, toys, blocks, sticks, pictures of animals, eggs, fingers, stones, beads, marula fruits, number cards, and markings on paper were used to understand numbers and perform calculations, because calculator usage was prohibited (Andy, respondent communication, February 16, 2018). The use of friendly numbers or tens to make calculations simpler also emerged as a strategy that was used in primary school (Noly, respondent communication, February 16, 2018). Also evidence of simple mental calculation in real-life emerged in the interviews as one PST stated that "at home my mom used to give me one dollar… if she gives me… the next day… another one dollar … if I did not use the one dollar she gave me last time… I know that now I have two dollars" (Deto, respondent communication, February 16, 2018). Such examples demonstrate the utility of mental calculations in real life. Concerning feedback, one PST indicated that the type of feedback received in class only indicated whether a learner's answer was right or wrong (Andy, February 16, 2018). Aspects that PSTs discussed in interviews confirmed findings from the questionnaires regarding their school experience and belief about FMC. In section 5.7, findings of the post-intervention test are analysed.

5.6 Post-intervention test

5.6.1 Analysis of the post-intervention test items

The test items used in the post-intervention test are given in Table 5.7 where they are classified according to the four basic operations. The post-intervention test comprised addition, subtraction, multiplication and division of numbers ranging from one-digit to three-digit numbers. Only two calculations consisted of four-digit numbers (item number 11 and 15 in Table 5.7). The four-digit numbers were given to determine PSTs' ability to compute calculations involving four-digit numbers. Similar to the pre-intervention test, the number range used in post-intervention test corresponded with the Namibian lower primary syllabus number range. Next is a classification of calculations according to the four basic operations.

Table 5.7: Flexible mental computation post-intervention test items

Addition	Subtraction	Multiplication	Division
2. 24 + 47 + 16	**1.** 22 – 9	**5.** 14 × 15	**7.** 72 ÷ 6
3. 125 + 38	**4.** 274 – 46	**6.** 16 × 199	**8.** 192 ÷ 32
10. □ + 26 = 63	**9.** 312 – 7	**19.** 16 × 25	**16.** 496 ÷ □ = 62
11. 1 245 + 2 136	**12.** □ – 43 = 39	**20.** 15 × 8	**17.** 368 ÷ 16
	13. 94 – 87	**21.** □ × 12 = 36	**18.** 468 ÷ 12
	14. 800 – 169		
	15. 1005 – 995		

5.6.2 Analysis of possible computation strategies for the test items

The post-intervention test incorporated numbers that outline relationships between specific numbers. Table 5.8 presents the anticipated calculation methods that could be used to calculate each test item. The post-intervention test was administered as the pre-intervention test where only the correct answer was written down and not the calculation strategy. The purpose of the post-intervention test was to establish the impact of the intervention on PSTs' ability to make flexible calculations in mind. The post-intervention test was conducted in class by the TE and each test item was projected in bold on an A4 sheet of paper for 20 seconds. The formulations of test items included specific number relationships that could prompt the use of particular calculation methods. Table 5.8 outlines possible calculation methods each test item was designed to induce.

Table 5.8: Possible flexible mental computation strategies for post-intervention test

Addition		**Subtraction**		**Multiplication**		**Division**	
2. 24 + 47 + 16	**Commutative property (reordering numbers to make a friendly number)**	**1.** 22 - 9	**Count on to subtract and bridging through ten**	**5.** 14 × 15	**Breaking up one number using place value to multiply**	**7.** 72 ÷ 6	**Multiply by a multiple of 10**
	24+16+47 =40+47 =87		9+**1**=10; 22=10+**12,** (1+12) = 13 22-9=13		(14×10)+(14×5)		6×**10**=60;72-60=**12**; 12÷6=**2** =10+2 =12
3. 125 + 38	**Adjust one number and compensate or adjust both numbers**	**4.** 274 - 46	**Break up one number using place value to subtract**	**6.** 16 × 199	**Adjust one number and compensate**	**8.** 192 ÷ 32	**Multiply by a multiple of 10**
	125+40-2 or 123+40 =165-2 =163 =163		274-(40+6) =274-40; 234-6=228		16× (200-1) (16 ×200)-(16 ×1) 3200-16 3 184		32×10=320; half of 320=160; 192-160=32; 160=(5 ×32)+(1×32) =5+1 =6
10.□ + 26 = 63	**Relationship between operations – convert to subtraction and use near double**	**9.** 312 - 7	**Using non-canonical understanding to subtract**	**19.** 16 × 25	**Adjust two numbers (double and halve to multiply)**	**16.**496 ÷□= 62	**Multiply by a multiple of 10**
	63-26 =63-30+4 =33+4 =37		312-7=300+12-7 =300+(7+5)-7 =300+5 =305		8× 50 4 ×100 =400		620÷2=310; 496-310=186; 186=62+62+62 496÷62=(**5**×62)+(**3**×62) = **8**
11. 1 245 + 2 136	**Decompose and use place value knowledge to add**	**12.**□ - 43 = 39	**Relationship between operations – convert to addition and use near double**	**20.** 15 × 8	**Double, double, double to multiply by 8 or decompose one number to multiply**	**17.** 368 ÷ 16	**Multiply by 10, double it and count on, then subtract and count on**
	1 000 + 2 000 = 3 000; 245 + 136 = 240 + 141		43+39=42+40 =82		15×2=30; 30×2=60; 60×2=120 Or (10×80)+(5×8) =80+40		160+160=320; 320=**20**×16; 368-320=48; 48=**3**×16; 368÷16=**23**

Addition		Subtraction		Multiplication		Division	
	= 381; 3 000 +381 = 3381				=120		
		13. 94 - 87	**Counting on to subtract and bridging through ten**	**21.**□× 12 = 36	**Change to division and count on in 12s**	**18.** 468 ÷ 12	**Relationship between operations**
			87+**3**=90; 90+**4**=94 94-87=3+4 =7		36÷12; 12,24,36 =3		120+120+120=360; 108=48+48+4+8 360=**30**×12**;** 108=**9**×12
		14. 800 - 169	**Bridging through multiples of ten and decomposing one number to subtract**				
			800-200=**600**; 200-170=**30**;170-169=**1** 800-169=**631** or 800-100-60-9 =700-60-9 =640-9 =631				
		15. 1005 - 995	**Proximity of numbers and counting on in 5s**				
			995+**5**=1000+**5** =1005; =10				

5.6.3 Pre-service teacher performance on each post-intervention test item

The outcomes of the post-intervention test are analysed in terms of class performance on each test item. As in the pre-intervention test, items of the post-intervention test are ordered as presented during the test in the classroom. Table 5.9 displays the test items, the number of PSTs who calculated specific items correctly or wrongly, and the representation of such performance in percentage. The post-intervention test also involved 21 whole number equations covering all four basic operations as presented next.

Table 5.9: Post-intervention test results

Post-intervention test results							
Test items		**Total PSTs who got the answer right or wrong (N= 38).**		**Percentage**		**Correct answer**	**Type of wrong answers provided**
Item number	**Item**	**Right**	**Wrong**	**Right**	**Wrong**		
1	22 – 9	34	4	89	11	**13**	11, 14
21	□ × 12 = 36	32	6	84	16	**3**	2, 4, 12, 432
3	125 + 38	31	7	82	18	**163**	63, 92, 153, 160, 166
9	312 – 7	30	8	79	21	**305**	3, 31, 35, 39, 304, 315
2	24 + 47 + 16	29	9	76	24	**87**	67, 70, 74, 77, 84, 88, 97
13	94 – 87	29	9	76	24	**7**	8, 11, 13, 16, 17, 612, 631
20	15 × 8	26	12	68	32	**120**	13, 32, 40, 115, 125, 130, 200, 220, 224, 240
15	1005 – 995	25	13	66	34	**10**	1.95, 5, 6, 15, 24, 100, 102, 915
4	274 – 46	19	19	50	50	**228**	28, 34,37, 105, 136, 163, 220, 222, 224, 229, 234, 234, 264
7	72 ÷ 6	19	19	50	50	**12**	4, 8, 9, 11, 13, 14, 16, 18, 22, 30, 192, 1002
12	□ – 43 = 39	19	19	50	50	**82**	-4, 4, 13, 16, 26, 47, 67, 74, 76, 81, 86, 3381
19	16 × 25	17	21	45	55	**400**	15, 23,46,120,131, 158,197,200,230, 300, 312, 320, 408, 450, 480,684,800, 1500, 2155
10	□ + 26 = 63	16	22	42	58	**37**	8, 10, 19, 27, 34, 35, 36, 43, 46, 47, 81, 85, 100
11	1 245 + 2 136	14	24	37	63	**3381**	4, 37, 83, 190, 241, 348, 802, 1381,1384, 2111, 3348, 3360, 3371, 3379, 3379, 3380, 3385, 3391, 3451, 3455, 3481, 3484, 14201
14	800 – 169	14	24	37	63	**631**	6, 31, 99, 100, 69, 531, 611, 629, 641, 659, 675, 710, 729, 730, 731, 749, 769, 781, 3340
5	14 × 15	10	28	26	74	**210**	108, 120, 124, 150, 154, 174, 175, 190, 200, 230, 300, 320, 1450
8	192÷ 32	8	30	21	79	**6**	3, 4, 5, 7, 8, 10, 11, 12, 14, 14.5, 16, 17, 23, 31, 34, 44, 47, 72, 91, 305
6	16 ×199	7	31	18	82	**3 184**	12, 30, 41, 92, 193, 196, 300, 485,584, 587, 809, 1009, 1082, 1640, 1700, 1886, 2095, 2224, 2600, 3000
16	496 ÷ □ = 62	7	31	18	82	**8**	2, 3, 4, 6, 7, 12, 14, 23, 42, 44, 62, 72, 81,180, 731
17	368 ÷ 16	5	33	13	87	**23**	4, 6, 7, 9, 12, 12.75, 14, 15, 16, 17, 20, 24, 25, 26, 30, 31, 32, 36, 50, 68, 69, 83
18	468 ÷ 12	5	33	13	87	**39**	7,10,12, 16, 18,19, 22, 24, 26, 40, 41, 42, 43, 46, 47, 48, 68, 74, 102, 118,120, 124, 229, 818

The total number of PSTs who participated in the post-intervention test reduced from 51 to 38. The reduction in the number of PSTs is attributed to absenteeism as 13 PSTs were absent. The purpose of the post-intervention test was to determine shifts in PSTs' ability to calculate mentally using strategies that are flexible. Since all PSTs could not be interviewed to discuss the strategies used, the TE sought to randomly analyse specific errors to identify the kind of reasoning underdeveloped during the intervention.

- Item: 22– 9

  Eighty-nine percent of the class calculated the item correctly. However, the two errors (11 and 14) demonstrate the inability of the two PSTs to verify their answers using addition. For the two PSTs, more practice is necessary to improve their metacognitive skills that would enable them to check and correct their answers.

- Item: $\square \times 12 = 36$

  Although 84% of the PSTs calculated the item correctly, wrong answers provided by four PSTs reveal an inability to count on to multiply, work backwards to divide 36 by 12 and little knowledge of the relationship between 12 and 36. Also, a lack of number sense is evident in answers like 432.

- Item: 125 + 38

  Errors such as 63 suggest incomplete use of the standard method where 25 and 38 were added, and 100 omitted. Again, inability to monitor own mental action is evident. Errors like 166 reflect the absence of number fact knowledge and inability to recognise the proximity of 38 to 40 to form a friendly number.

- Item: 312 – 7

  Seventy-nine percent of the class provided correct answers, but 3 as an answer suggest guesswork and incompetency in terms of establishing the reasonableness of an answer. The error 315 still shows a persistent use of the standard method were 7 – 2 = 5, so the answer is 315. The answer 315 also demonstrates little knowledge of number facts such as 7 + 5 = 12.

- Item: 24 + 47 + 16

  The majority of the PSTs provided correct answers. Furthermore, the errors are more sensible, as 67 suggests addition of 47 plus 20 without adding 16 and 4.

- Item: 94– 87

  Errors like 612 and 631 show a lack of knowledge of magnitude of number and sensibleness of an answer. Despite the errors, 76% of the class worked out the item correctly.

- Item: 15 × 8

  A shift in reasoning in terms of multiplication by 8 is recorded as 68% of the class managed to calculate this item correctly, unlike in the pre-intervention test where only 25% of the PSTs could calculate 14 × 8 correctly.

- Item: 1005 – 995

  Under this item, a shift in reasoning pertaining to proximity of numbers to a friendly number is evident with a record of 66% of PSTs arriving at a correct answer. However, a lack of ability to check the reasonableness of an answer is still evident in errors like 915.

- Item: 274 – 46

  Improvement is recorded in terms of PSTs' ability to subtract a two-digit number from a three-digit number. Fifty percent of the PSTs managed to work out 274 – 46 correctly, unlike only 39% who could calculate 151 – 98 in the pre-intervention test. The number of errors also reduced in the post-intervention test (see Table 5.4 and Table 5.8).

- Item: 72 ÷ 6

  Fifty percent of the class may have developed the ability to divide using a multiple of 10, and the ability to simplify both numbers to divide by a smaller number or by using partial quotients. However, errors like 192 and 1002 still show the absence of knowledge on how to check the reasonableness of an answer.

- Item: □– 43 = 39

  Improvement has been recorded in terms of reversibility reasoning as 50% of the PSTs calculated correctly in relation to 37% correct answers in the pre-intervention test. The variety of errors also reduced compared to the pre-intervention test, though errors like 3 381 still reflect little knowledge pertaining to number relationship.

- Item: 16 × 25

  An increase from 18% correct responses in the pre-intervention test to 45% in the post-intervention test demonstrates improvement in PSTs' ability to use the double and halve strategy or multiplication by 10 and 6 to work out the answer. The wrong use of the standard method is still evident in errors like 2155.

- Item: □ + 26 = 63

  Post-intervention test results of this item indicate that the majority (58%) of the PSTs did not develop the skill to work backwards or reversibility reasoning to work out the item mentally. In this case the position of the unknown may have increased the difficulty of the item. However, an answer of 3381 indicates no critical mental engagement to provide a sensible answer.

- Item: 1 245 + 2 136

  A drop in performance was recorded under this item. Only 37% of the PSTs correctly calculated two numbers comprising four digits. The drop in performance can be due to subtraction of 1999, which is too close to 2024, unlike the addition of 1 245 and 2 136, which are difficult to calculate with the standard method mentally. The use of the standard method is evident in the error 14201, whereas answers below 3 000 indicate a lack of estimation skills and a sense of magnitude of numbers.

- Item: 800 – 169

  This item proved to be very challenging to calculate mentally as only 37% of the class calculated it correctly. The kind of errors made, e.g. 769, suggest a partial use of the standard method in mind. The ability to perceive 169 in relation to 200 is yet to be achieved by the PSTs.

- Item: 14 × 15

  This item was one of the most difficult items for PSTs. Seventy-four per cent of the class were unable to calculate the item correctly. Errors PSTs made, e.g. 1450 still show steps like 14 × 1 = 14 and 5 × 10 = 50 resulting in 1450 as the answer. This indicates the prominence of the standard method and difficulty to abandon such strategies.

- Item: 192÷ 32

  Seventy-nine percent of the class could not calculate the item correctly. Test results showed that the intervention had little impact (21%) on PST skill to divide a three-digit number by a two-digit number. Knowledge of counting on or subtracting to divide appears not to have been developed fully.

- Item: 16 ×199

  Post-test results also revealed limited impact of the intervention on PSTs' ability to multiply a two-digit number by a three-digit number. Knowledge of the proximity of 199 to 200 is not evident in the test results. PSTs (82%) seem not to have developed knowledge of repeated addition to multiply.

- Item: 496 ÷ □ = 62

  Test results (82 %) of this item reflect PSTs' inability to work backwards, to count on in 62s and subtract repeatedly to divide a three-digit number with an unknown. Errors beyond 9, for example 12 to 731, indicate underdeveloped knowledge of inverse operations and number relationship.

- Item: 368 ÷ 16

  Results show that the intervention had little impact on 87% of the PSTs' reasoning to work with partial quotients or halving each number to divide by smaller numbers.

- Item: 468 ÷ 12

  Only 13% of the PSTs compared to 8% in the pre-test managed to calculate 468 ÷ 12 correctly. The item in the pre-intervention test was 192 ÷ 8 while the item in the post-intervention test involved a large three-digit number divided by a two-digit number. Though the figures in the post-test were large, the increase in performance is an indication that development prevailed in some PSTs and that further increase in performance is possible through intensive practice. Table 5.10 shows a summary of shifts in PSTs' reasoning to compute mentally.

Table 5.10: Shift in PSTs reasoning

Calculation and number range		**Improved reasoning**	**Percentage of correct answers in pre-intervention test (%)**	**Percentage of correct answers in post-intervention test (%)**
1.	24 + 47 + 16	Commutative property (reordering numbers to make a friendly number)	47	76
2.	15 × 8	Double, double, double to multiply by 8 or decompose one number to multiply	25	68
3.	1005 – 995	Proximity of numbers and counting on in 5s	45	66
4.	274 – 46	Break up one number using place value to subtract	39	50
5.	□ – 43 = 39	Relationship between operations – convert to addition and use near double	16	50
6.	□ + 26 = 63	Relationship between operations – convert to subtraction and use near double	16	50

Post-intervention results indicate that major shifts in reasoning prevailed in eight items. Improvement in the items presented in Table 5.10 portray the impact of the intervention on PST's reasoning in terms of multiplication by 15, making friendly numbers to add easily, breaking up a number to subtract, proximity of numbers to a multiple of 10, indirect addition, indirect subtraction, doubling and halving to multiply, and doubling three times to multiply by 8. Next are items that recorded slight improvement and low performance (below 50%).

Table 5.11: Test items PSTs struggled with in the study

Calculation and number range		Intended reasoning	Percentage of correct answers in pre-intervention test (%)	Percentage of correct answers in post-intervention test (%)
1.	14 × 15	Using place value to think in multiples of ten to multiply	8	26
2.	16 ×199	Adjust one number and compensate	No similar item in pre-intervention test	18
3.	192÷ 32	Multiply by a multiple of 10		21
4.	□ + 26 = 63	Relationship between operations – convert to subtraction		42
5.	1 245 + 2 136	Decompose and use place value knowledge to add		37
6.	800 – 169	Bridging through multiples of ten and decomposing one number to subtract		37
7.	496 ÷ □ = 62	Multiply by a multiple of 10		18
8.	368 ÷ 16	Multiply by 10, double it and count on, then subtract and count on		13
9.	468 ÷ 12	Multiply by 10, double it and count on, then subtract and count on		13
10.	16 × 25	Adjust two numbers (double and halve to multiply)	18	45

PSTs struggled to compute the above-mentioned calculations, possibly because of the magnitude of numbers and limited time to develop the knowledge of doubles beyond 100. Although the development recorded is very low, the HLT was improved to include activities that involve counting on with two-digit numbers. The low percentage recorded signifies that PSTs can improve their ability to add, subtract, multiply and divide three-digit and two-digit numbers provided adequate practice takes place. Individual scores on the post-intervention test are discussed in Table 5.12.

5.6.4 Individual scores on post-intervention test

Table 5.12 reflects individual scores and the number of PSTs who scored a specific mark. The cumulative percentage indicates that 47% of the total PSTs' test scores were below 50%. In other words, the majority of the class performed beyond 50%. The highest mark in the post-intervention test is 100% scored by two PSTs compared to one participant in the pre-intervention test with 86% as the highest mark. Such a result reflects increased reasoning in specific individuals and reduced performance in individuals who scored the lowest mark of 10% in the post-intervention test. Details of the results are presented in Table 5.12.

Table 5.12: Pre-service teacher individual performance on test

Total score of individual PSTs						
Mark out of 100	**Frequency of PSTs with specific mark**	**Percentage of PSTs**	**Cumulative percentage**	**PST number**	**Items worked out correctly**	**Items struggled with (All items not listed under correct items, see Addendum M)**
10	1	2,6	2,6		1, 3	
14	1	2,6	5,3		1, 20, 21	
19	1	2,6	7,9		1, 3, 9, 21	
24	3	7,9	15,8		3, 9, 13, 15, 21	
					1, 2, 13, 20, 21	
					1, 3, 4, 15, 21	
29	3	7,9	23,7		1, 7, 9, 12, 13, 21	
					1, 2, 3, 7, 9, 13,	
					1, 2, 3, 7, 9, 13	
33	2	5,3	28,9		1, 4, 9, 12, 16, 19, 21	
					1, 2, 9, 13, 15, 20, 21	
38	3	7,9	36,8		1, 2, 3, 9, 11, 19, 20, 21	
					1, 2, 3, 9, 11, 15, 20, 21	
					1, 2, 4, 9, 13, 15, 20, 21	
43	4	10,5	47,4		2, 3, 4, 9, 11, 13, 15, 20, 21	
					1, 3, 12, 13, 14, 15, 19, 20, 21	
					1, 2, 3, 8, 10, 11, 13, 20, 21	
					1, 2, 3, 4, 9, 10, 13, 19, 21	
52	7	18,4	65,8			4, 6, 7, 8, 9, 10, 16, 17, 18, 19
						3, 4, 5, 6, 7, 8, 14, 16, 18, 19
						1, 5, 6, 8, 11, 13, 14, 16, 17, 18,
						4, 6, 10, 11, 14, 15, 16, 20, 21
						1, 4, 5, 6, 8, 16, 17, 18, 19, 20
						6, 7, 8, 10, 11, 12, 16, 17, 18, 19
						6, 7, 8, 9, 11, 12, 14, 16, 17, 18
57	1	2,6	68,4			5, 6, 8, 11, 12, 14, 17, 18, 19
62	3	7,9	76,3			5, 8, 10, 13, 16, 17, 18, 19
						5, 6, 8, 10, 11, 15, 17, 18,
						5, 6, 8, 10, 16, 17, 18, 21
67	1	2,6	78,9			4, 5, 11, 14, 16, 17, 18
71	3	7,9	86,8			6, 8, 11, 16, 17, 18,
						6, 8, 14, 17, 18, 19

						2, 4, 5, 11, 16, 21
76	1	2,6	89,5			5, 17, 18,
81	1	2,6	92,1			11, 12, 16, 20
86	1	2,6	94,7			16, 17, 18
100	2	5,3	100,0			NONE
						NONE
Total	38	100.0				

A drop in performance from 24% to 10% possibly demonstrate that low performers could not cope with the type of calculations they were subjected to considering their limited knowledge of number relationships and facts. Meanwhile, major improvements from 81% to 100% and 43% to 100% confirmed that PSTs' flexible mental computation can be developed through invention of own calculation strategies. The comparison of items low scorers and high scorers found easy and difficult are presented in Table 5. 13.

Table 5.13: Number of PSTs who found a specific item easy or difficult

		Number of PSTs who found the item easy or difficult											Number of PSTs who found an item easy or difficult							
		Low scorers: mark below fifty (N = 26)				High scorers: mark above fifty (N = 25)							Low scorers: mark below fifty (N = 18)				High scorers: mark above fifty (N = 20)			
Item of pre-intervention test		Easy	%	Difficult	%	Easy	%	Difficult	%		Item of post-intervention test		Easy	%	Difficult	%	Easy	%	Difficult	%
1	5 + 7	25	96	1	4	24	96	1	4		1	22 – 9	16	89	2	11	18	90	2	10
2	19 + 15	21	81	5	19	24	96	1	4		2	24 + 47 + 16	10	56	8	44	19	95	1	5
3	23 + 18 + 37	8	31	18	69	16	64	9	36		3	125 + 38	12	67	16	89	19	95	1	5
4	23 – 16	12	46	14	54	21	84	4	16		4	274 – 46	5	28	13	72	14	70	6	30
5	151 – 98	5	19	21	81	14	56	11	44		5	14 × 15	0	0	18	100	15	75	10	50
6	563 – 292	15	58	11	42	8	32	17	68		6	16 ×199	0	0	18	100	8	40	12	60
7	4 × 50	21	81	5	19	25	100	0	0		7	72 ÷ 6	3	17	15	83	16	80	4	20
8	32 × 15	0	0	26	100	4	16	21	84		8	192÷ 32	1	6	17	94	8	40	12	60
9	16 × 25	1	4	25	96	10	40	15	60		9	312 – 7	12	67	6	33	18	90	2	10
10	16 ÷ 4	18	69	8	31	24	96	1	4		10	□ + 26 = 63	2	11	16	89	14	70	6	30
11	120 ÷ 6	8	31	18	69	16	64	9	36		11	1 245 + 2 136	4	22	14	78	10	50	10	50
12	14 × 8	1	4	25	96	12	48	13	52		12	□ – 43 = 39	3	17	15	83	16	80	4	20
13	59 + □ = 82	5	19	21	81	16	64	9	36		13	94 – 87	11	61	7	39	18	90	2	10
14	85 – □ = 67	4	15	22	85	14	56	11	44		14	800 – 169	1	6	17	94	13	65	7	35
15	□ – 38 = 89	4	15	22	85	6	24	19	76		15	1005 – 995	7	39	11	61	18	90	2	10
16	192 ÷ 8	2	8	24	92	5	20	20	80		16	496 ÷ □ = 62	1	6	17	94	6	30	14	70
17	□ ÷ 15 = 20	7	27	19	73	14	56	11	44		17	368 ÷ 16	0	0	18	100	6	30	14	70
18	15 × □ = 45	20	77	6	23	20	80	5	20		18	468 ÷ 12	0	0	18	100	5	25	15	75
19	250 × 2	21	81	5	19	23	92	2	8		19	16 × 25	3	17	15	83	13	65	7	35
20	2024 – 1999	10	38	16	62	14	56	11	44		20	15 × 8	9	50	9	50	17	85	3	15
21	799 – 51	9	35	17	65	21	84	4	16		21	□ × 12 = 36	15	83	3	17	17	85	3	15

A comparison of low scorers' and high scorers' test results indicate that the intervention had a major positive impact on fast PSTs than on slow PSTs. The disparity in improvement is probably an attribute of PSTs' number knowledge upon entry to university and pace of learning. However, this study infers that slow PSTs may also improve provided extended practice on FMC takes place. Next is a discussion on the post-intervention interviews.

5.7 Post-intervention interview

The post-intervention interviews took place two days after the post-intervention test. After the intervention process, only 9 PSTs were willing to participate voluntarily. Six of the 15 PSTs who participated in the pre-intervention interviews opted out of the study. To ensure the validity of the findings the researcher requested the new interviewees to discuss how their ability to calculate mentally was before the intervention and after. The post-intervention interview consisted of two parts: first PSTs were asked to perform four algorithms mentally and justify their strategies. The second part consisted of questions PSTs had to answer orally. None of the four calculations used in the post-intervention interviews formed part of the post-intervention test. Different calculations were used to check any form of reasoning PSTs had developed as a result of the intervention. The calculations and envisaged calculation strategies are outlined in Table 5.14.

5.7.1 Mental calculations and pre-service teacher computation strategies

Table 5.14: Post-intervention interview items and envisaged calculation strategies

Operation	**Addition**	**Subtraction**	**Multiplication**	**Division**
Calculation	1.37 + 26	2.82 – 43	3.7 × 15 (Initial calculation: 14× 15)	4.100 ÷ 5 (Initial calculation: 468÷12)
Envisioned strategy	**Adjust one number and compensate**	**Adjust both numbers**	**Breaking down one number to multiply**	**Reversibility property or using place value to work with a smaller number**
	37+30 – 4 =67 – 4 =63 or 40+23 =63 Or 40+26 – 3=63	80 – 41=39 or 82 – 40 – 3= 42 – 3 =39	7 × (10+5) =70+ half of 70 =70+35 =70+30+5 =105	100= 5 ×□ 100=(10×10) =(10÷5)×10 =2×10 =20

The first algorithm in the post-intervention interview was 37 + 26, where all 9 interviewees participated. All 9 PSTs worked out the algorithm correctly using flexible mental computation strategies only. None of the PSTs used the standard algorithm. Most of the PSTs used a combination of strategies to work out a particular calculation, while others used only one strategy. The strategies used were compensation strategy, making a 10, breaking each number into its place value (decomposition), using near doubles, and adjusting one or both numbers.

The second algorithm was 82 – 43, where 6 out of the 9 PSTs carried out the calculation correctly. None of the PSTs attempted to calculate mentally using the standard method. Methods employed were canonical understanding of the minuend, decomposition, non-canonical understanding of numbers, adjustment, compensation, a combination of non-canonical understanding of minuend, and decomposition of both the minuend and subtrahend to work with friendly numbers.

The third algorithm was 7×15, which replaced the original calculation involving 14×15. The TE reduced the figures because the PSTs indicated in their journals that they were unable to calculate large numbers mentally, as became evident in the post-intervention test. Reducing the numbers enabled the TE to establish the PSTs' ability to use knowledge regarding multiplication by 15, as discussed in the intervention. Nine PSTs worked out the calculation correctly. Methods used were adding three pairs of 15 and adding one more 15 (multiplication expression), multiplication using known facts, partial products, and making a friendly number.

With regard to the fourth calculation, $100 \div 5$ replaced $468 \div 12$ since PSTs' journal reflections that indicated inability to calculate large numbers mentally. The smaller number magnitude prompted the PSTs to demonstrate the kind of reasoning that facilitated the performance of calculations mentally. Seven of the 9 PSTs performed the calculation correctly in their minds. PSTs used strategies such as counting on in 20s, decomposition and dividing by smaller numbers, using known facts, reversibility, adjusting and compensation. Immediately after the calculation part, semi-structured interviews followed.

5.7.2 Semi-structured interview

All 9 PSTs participated in the one-on-one, face-to-face interviews that comprised the following statements and questions:

- Reflect on your ability to perform flexible calculations before the intervention.
- Reflect on your ability to perform calculations flexibly after the intervention.
- From the intervention, what exactly has had an impact on your knowledge about flexible mental computation skills? Elaborate.
- In what way does the intervention contribute to your preparation as a mathematics teacher? Or Mention the skills you have acquired from the intervention.
- What is your view on the need to develop both teachers' and learners' flexible mental computation skills?

The 9 PSTs who agreed to be interviewed reflected on and described their ability to perform flexible calculations before the intervention in different ways. PSTs described their attempts as: "not good" (Noly, respondent communication, February 19, 2018), "poor" (Mara, respondent communication, February 19, 2018), "a bit hard" (Andy, respondent communication, February 19, 2018), "I tried but not much as I only had one method" (Abu, respondent communication, February 19, 2018), "quite challenging" (Naby, respondent communication, February 19, 2018), "very tough" (Oyu, respondent communication, February 19, 2018), and "I was frightened for having no idea to get the solutions"

(Kibi, respondent communication, February 19, 2018). Such expressions confirm the pre-intervention test results that were below 50%.

After the intervention, PSTs stated that they learned new ways to calculate mentally despite the difficulty to compute three-digit numbers. Although all 9 PSTs acknowledged having progressed in their ability to calculate mentally, one participant admitted to a weakness of being very slow in calculating mentally despite an ability to obtain correct answers using the developed methods. As a result, the participant expressed a desire for more engagement with mental calculations in order to improve individual calculation speed.

As to computation strategies, some PSTs said they had developed a variety of methods to compute mentally as a result of classmates sharing their calculation strategies with the entire class. One participant said, "The lessons helped me because different colleagues came with their methods that I don't know and that helped me" (Naby, respondent communication, March 19, 2018). Other PSTs ascribed their improvement to additional strategies that the TE had shared with the class pertaining to perception of numbers and counting to think more and perform calculations fast in mind. Generally, PSTs expressed comfort in carrying out calculations involving addition and subtraction of two to three-digit numbers but indicated that they had difficulties with multiplication and division. Another participant admitted that "addition and subtraction … I can… [but] multiplication and division I am still struggling" (Abu, respondent communication, March 19, 2018). Meanwhile, other PSTs expressed facility in carrying out calculations involving addition and multiplication, but expressed difficulty in subtraction and division.

One participant declared that "addition and multiplication I am ok… I cannot work out subtraction and division… I am struggling only with the bigger numbers … we just need to focus more on division" (Pobo, respondent communication, March 19, 2018). Another said, "I still need more help… I still find it hard with three-digit numbers (Mara, respondent communication, March 19, 2018). Issues regarding time were also mentioned, where a participant said, "I have gained some new methods… but I do still need time for me to practise that I become better" (Abu, respondent communication, March 19, 2018). Another participant stressed that the intervention affected the level of fear experienced during flexible mental computation, stating that, "When I want to divide and multiply when I looked at big numbers and I got… scared … but later as I learned through the process I came to realise that you can do a division by taking a small number from the big number then you divide that will give you the answer so easily" (Oyu, respondent communication, March 19, 2018). Despite admitting to struggling with subtraction involving two-digit numbers, one participant indicated having improved in "the way we should see numbers… specialised numbers… how you could see them just to break them down into smaller numbers for you to work them easily out… also using familiar numbers" (Kibi, respondent communication, March 19, 2018).

For other PSTs, the interventions served as a refresher course through which they were reminded of FMC strategies they had forgotten. PSTs expressed the desire to apply methods similar to the ones they had developed during the intervention process to assist their learners in school. Also, most PSTs considered FMC important as it makes the learning of mathematics enjoyable, easy and relevant, reduces dependency on mathematical devices such as calculators, and it empowers the mind to increase teacher and learner participation and performance in mathematics. Another sentiment was that if all teachers could effectively assist learners to develop FMC skills at an early stage such learners might develop mentally, eventually viewing the world in a different way, rather than relying only on calculators or computers to calculate.

5.8 PST accounts from reflective journals

With regard to shifts in PSTs' reasoning to calculate mentally, journal reflections outline improvement in specific orientation towards numbers. One form of reasoning that emerged is that of perceiving multiplication as repeated addition (Appendix I, no. 17, 26). Also, PSTs pointed out that calculation with big numbers had been simplified as a result of developed knowledge about friendly numbers (Appendix I, no. 1, 4, 22) and the strategy of arranging numbers to combine numbers that can quickly make a 10 (Appendix I, no. 5, 9). Reasoning of PSTs had also improved in terms of multiplication by 15 as the sum of the product of 10 and 5 (Appendix I, no. 14, 18) and working backwards to multiply or divide (Appendix I, no.19, 22). Other strategies that contributed to PSTs' ability to calculate mentally in a flexible way are compensation, doubling, breaking down numbers or decomposition, and multiplication by 10 and 5 and adjusting both numbers. In addition, the shift in PSTs' reasoning was attributed to PSTs' presentation of diverse calculation strategies on the board during the intervention process after solving the intervention tasks individually, as was made evident in PST journal entries (Appendix I, no. 7, 8, 18, 19, 24), as well as to the opportunity to answer and explain strategies (Appendix I, no. 20, 22). The general perception about FMC was that it is interesting, fun and improves thinking. In view of the expressed need for more practice, a post-post intervention was devised to assist the struggling PSTs.

5.9 Post-post intervention test results

Few PSTs attended the last session as they were already focusing on a mathematics assignment given by the part-time lecturer who resumed duty on 20 March 2018. The assignment was given at the end of the last intervention session, which shifted the PSTs' attention from FMC classes to the assignment. In the last session, one of the PST who had struggled most remarked that she hoped it was all over with the development of their FMCs as she had no intention to engage with FMCs again in year two of her studies. She further wished she had opted to teach languages at the upper primary level to avoid learning mathematics or FMCs that she found too challenging.

Tables 5.15 represent the performance of 14 PSTs who participated in all three tests. Outcomes of the test show significant improvement in struggling teachers' performance. The post-post-intervention focused on repeated addition as connected to multiplication, the distributive property, associative property, commutative property, investigation of patterns that emerge when multiplying by 2, 4 and 8, and patterns such as 2 × 3 = 6 so 2 × 30 = 60 or 4 × 7 = 28 so 4 × 70 = 280. Comparison of PSTs' results of the pre, post and post-post intervention test results is illustrated in figure 5.1.

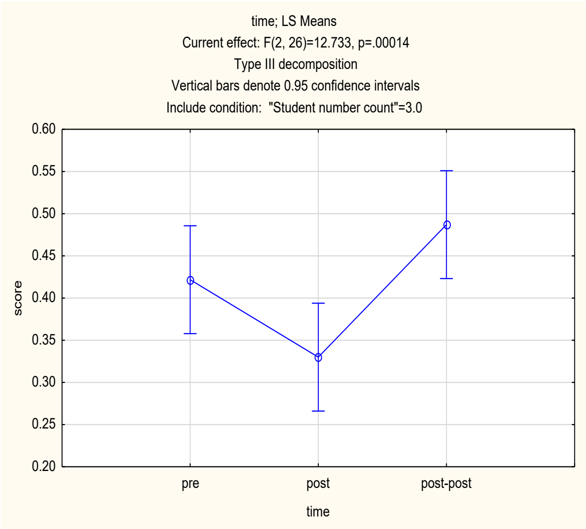

Figure 5.1: Low scorers' performance in pre-, post- and post-post-intervention test

The fourteen PSTs who participated in all three tests showed slight improvement in the last test with 49% average mark compared to 42% in the first test and 33% in the second test. Detailed information pertaining to the three tests is presented in Table 5.15.

Table 5.15: Descriptive statistics of 14 low scorers' performance in three tests

Effect	Descriptive Statistics (Spreadsheet in results.stw) Include condition: "Student number count"=3.0 Level of Factor	N	score Mean	score Std.Dev.
Total		42	0.41	0.13
time	pre	14	0.42	0.12
time	post	14	0.33	0.12
time	post-post	14	0.49	0.11

The descriptive statistics in Table 5.15 outline the impact of the support low performers received after the intervention. The 14 low performers in the post-intervention test showed significant improvement after the HLT was adapted to provide more practice. Their mean score improved from 33% in the post-intervention test to 49% in the post-post-intervention test. However, the 49% improvement recorded means that supporting low scorers requires more time for practice.

5.10 Conclusion

The outcomes of the survey on PST beliefs, conducted before the intervention, indicate that most of the PSTs believe that FMC skills develop through the use of concrete objects and the use of the pencil-paper method. Outcomes of the questionnaires indicate a positive attitude among PSTs in respect of the importance and development of FMC skills. With regards to their existing knowledge, most of the PSTs demonstrated a tendency to calculate using the standard method whereas about half of the PSTs could compute flexibly in mind (49%). After the intervention, 53% of the PSTs could calculate in a flexible manner in their minds. In pre-intervention interviews some PSTs indicated that they had learned to compute mentally through problem solving, whereas others indicated that they had learned through memorisation of the multiplication table. With regard to strategies pertaining to FMC, most of the PSTs indicated that strategies had been prescribed by the teacher for them to select strategies they found suitable to use. The post-post intervention test recorded a significant improvement in PSTs' FMC skills after failing to improve during the main intervention process. A detailed discussion of the intervention process follows in Chapter 6.

CHAPTER 6: DATA PRESENTATION AND ANALYSIS OF THE INTERVENTION PROCESS

6.1 Introduction

In Chapter 5, findings of the survey, pre- and post-intervention tests, and pre- and post-intervention interviews are presented. Chapter 6 serves to present and interpret findings of the intervention process. Thus the enacted HLT was designed prior to the intervention and improved during the intervention process to bridge the gap resulting from the lack of a guiding framework for teaching. Projected in the third objective of this study, the task was to:

- Create and improve a framework to develop elementary pre-service teachers' mathematical knowledge for teaching flexible mental computation skills.

Problem solving activities designed for the intervention were improved during the intervention process to translate into a domain-specific framework to bridge the gap resulting from a lack of a guiding framework for developing PSTs' knowledge for teaching FMC. Developing PSTs' knowledge for teaching FMC is crucial as it determines how PSTs are going to teach. Literature indicates that teachers with in-depth mathematical knowledge for teaching can utilise their knowledge to listen to learners, select and use good tasks, manage learners' tasks and manage classroom discussions effectively (Hill et al., 2005). Research findings reveal the need for PSTs to be involved in investigations of the content they are going to teach, the method to teach it and ways to use learning resources (Ma, 2010). As such, efforts to develop PSTs' mathematical knowledge for teaching FMC included understanding the concepts that PSTs are expected to teach and how to teach them. Consequently, activities designed for the intervention resemble the activities PSTs are expected to use in their classrooms. Similarly, the teaching method used by the TE reflects strategies PSTs are expected to use in school.

This chapter is divided into four main sections beginning with a presentation and discussion of the intervention cycles in the order in which the TE presented the lessons. Then follows a discussion of the means by which learning was supported and the conclusion.

6.2 Detailed description of the learning processes

After establishing PSTs' existing knowledge about FMC through the pre-intervention test and pre-intervention interviews, the researcher assumed the role of a TE to execute the intervention. The intervention comprises classroom events as learning processes to develop PSTs' flexible mental computation skills involving addition, subtraction, multiplication and division of one- to three-digit whole numbers. All intervention cycles were lessons conducted by the TE in a real classroom environment as a research site. Tasks used in the intervention cycles were prepared in advance and submitted to the TE's study promoter for approval. The pre-designed tasks were improved during the intervention process to fit the duration of the lessons and the reasoning level of the PSTs. To connect FMC to real-life activities and to make it more interesting, imaginable and meaningful; the TE formulated story problems that connect to daily life activities, as prompted by Freudenthal (1968; 2002). This was effective in supporting PSTs to think of own methods to solve problems as no specific operation was suggested to them. The process of calculating mentally progresses systematically, as informed by Heirdsfield and Lamb (2007), as follows:

- first recognise a number and operation,
- then choose an efficient strategy,
- implement the strategy, and
- check the accuracy of the answer.

Similarly, Polya (1945, p. 23) relates to the systematic process as:

- understanding a problem,
- devising a plan,
- carrying out a plan, and
- looking back.

During all intervention cycles, the TE encouraged PSTs to record their insights in their reflective journals. Reflective journals enabled PSTs to articulate their progress and challenges. Time slots for mathematics were three hours per week with a two-hour session scheduled on Tuesdays from 16:30 to 18:25 and a one-hour session each Friday from 14:30 to 15:25. The TE recorded PST attendance throughout the intervention process. PSTs in the study were part of a class that was allocated to a part-time lecturer who at the time was yet to be appointed but resumed duty towards the end of the intervention process.

6.2.1 Intervention cycle one

Intervention cycle one began immediately after the pre-intervention interviews. Intervention cycle one was a lesson that comprised 11 story problems based on direct and indirect addition of one- to three-digit numbers. Outcomes of the pre-intervention test revealed that the majority (more than 50%) of PSTs could not compute in mind in a flexible manner, as presented in section 5.4.4. The results prompted the improvement of the initial activities to include problems that would support the PSTs' reasoning to invent own calculation strategies. For intervention cycle one, the TE planned to use 11 questions as drafted before the intervention but due to limited time only five problems were solved in class. Thus the TE decided to split the problems into two parts. Table 6.1 shows how the problems were split.

Table 6.1: Intervention task one

	Task 1	**Number range and equation**	**Mathematical reasoning behind each problem**
1.	Bency has eight apples; Jane gave her nine more apples. How many apples does Bency have now?	8+9	Near doubles and making tens (friendly number)
2.	Betty has 27 bananas, Sandra has 35 bananas and Cathy has 13 bananas. How many bananas do they have altogether?	27+35+13	Use commutativity or combine numbers that make ten
3.	Moiny bought sandals costing N$24.00, a top costing N$47.00 and toothpaste costing N$16.00. What is the cost of all three items together?	24+47+16	Commutativity and making tens (friendly number)
4.	Aunt Sofia had 59 cattle in her kraal and now, after buying more cattle, she has 82 cattle in her kraal. How many cattle did she buy?	59+ □ =82	Use skip counting on a number line and indirect addition
5.	You have some money and your father gives you N$219.00 more. Now you have N$367.00. How much money did you have before?	□ + 219=367	Decompose and reverse + and –
	Homework		
1.	Ann drove some kilometres before resting and drove another 58 km after resting. In total she drove 91 kilometres. What is the distance she drove before resting?	□ + 58=91	Use reversibility property
2.	A dining hall has 118 learners, 17 more learners arrive at the dining hall. How many learners are in the dining hall now?	118+17	Decomposition and near doubles
3.	Eli has attended 24 weeks of his course already. He has some weeks left to complete his 62-weekcourse. How many more weeks are left for him to complete the course?	24+□=62	Reversibility and breaking each number into its place value
4.	A supermarket received 136 bags of potatoes yesterday and 344 bags of potatoes today. What is the sum of the potato bags received over the two days?	136+344	Compensation
5.	Two buses are transporting learners to town. Bus A has 45 learners and bus B has 17 fewer learners than bus A. How many learners are on bus B?	□=45-17	Comparison and adding up in chunks

To improve PSTs' ability to compute calculations mentally, the TE engaged PSTs in addition problems using a ten frame, an empty number line, and an abacus as visual models to have a mental representation of the computation process. The lesson began with a discussion of what flexible mental computation entails. Since the overhead projector in the classroom was damaged, problems were printed on posters and fixed on the chalkboard. Thereafter, the TE and PSTs took turns to read out the questions loudly to the class. PSTs were encouraged to attempt to solve the problems individually, consult each other once in need of assistance and discuss solutions afterwards.

While some PSTs solved the problems immediately, others were reluctant to participate in the class activity. Although PSTs were reluctant to compute mentally at the beginning of the lesson, participation increased gradually as the lesson progressed. The TE monitored PSTs' progress by observing the shift in their calculation strategies from one problem to the next. As PSTs solved problems, the TE sparked their thinking with the following questions:

- What did you do to get your answer? How did you figure out the answer?
- How do you know your answer is correct?
- Is there another way to solve the same problem?

When PSTs provided solutions, the TE allowed individual PSTs to write their strategies on the board one at a time. Thereafter, the TE asked each PST to explain to the entire class the strategies they wrote on the board, without labelling solutions right or wrong. Instead, the TE encouraged PSTs to express agreement or disagreement with a particular solution. During the lesson, the TE informed the class to announce immediately if they needed clarity on a particular strategy and to record their challenges in their reflective journals. After a variety of calculation strategies were written on the board, the TE asked PSTs to indicate how the different strategies related to each other. However, a negative attitude among PSTs was observed as some PSTs opted to laugh at strategies they found inefficient or wrong. The TE condemned such attitudes immediately and encouraged respect among PSTs. At the end of the intervention process PSTs were required to design their own number chart as homework. Discussions held in the classroom revealed PST progress on using flexible strategies to compute mentally, despite various challenges.

During the lesson a couple of challenges emerged. One challenge related to PSTs turning up late for class and attending class without a note book or pen to write with. This prompted the TE to encourage PSTs to act responsibly. Another challenge concerned insufficient time to see all PSTs' calculation strategies and the immovable seats that prevented access to every PSTs' solutions. Also, the seating arrangements hindered the implementation of a cooperative learning approach. Although many PSTs mastered addition of two two-digit numbers, addition of two one-digit numbers, and addition of three numbers comprising only two-digit numbers, many PSTs struggled to solve story problems involving three-digit numbers. To facilitate mental calculation, the TE invited PSTs to demonstrate the use of models to create mental images for mental computation. As to communication, few PSTs were willing to compute mentally and share their answers with the class. To motivate PSTs, the TE planned to implement the ZPD (Vygotsky, 1978) approach by grouping both less active and more active PSTs together. More knowledgeable PSTs and less knowledgeable PSTs were selected based on the outcome of the pre-intervention test and the calculation methods invented by PSTs in class. In addition to TE observations, PSTs recorded own challenges in their reflective journals.

In their journals, PSTs indicated that they had struggled to compute three- and four-digit numbers mentally. They also pointed out a lack of understanding of how to use the abacus and five frames to calculate mentally. With regard to the selection of more efficient strategies, a couple of journal entries indicated PSTs being against the use of long solution methods. Others entries conveyed that PSTs' lack of practice during their spare time delayed their development of flexible mental computation skills. A couple of PSTs indicated that they were still struggling to compute numbers from 10 to 100 mentally as some still relied on the use of fingers for computation. In one journal entry, a PST mentioned that the following question was too challenging: 'Two buses are transporting learners to town. Bus A has 45 learners and bus B has 17 fewer learners than bus A. How many learners are on bus B?' Some PSTs indicated that they were unable to determine what operation they could use to solve this problem. With regard to communication of solution strategies, the TE encouraged PSTs to write their answers on the board first before explaining and justifying their solution strategies. Among other strategies shared in class, many PSTs used the commutative property and strategy of making friendly numbers or tens to compute easily, e.g. 7+5+3 = 7+3+5, which leads to 10+5=15. When some PSTs noticed that the lesson was being recorded, they showed discomfort as the video recorder was stationed in front of the class. Thus the TE considered repositioning the video recorder in the next lesson.

To contain issues of reluctance to solve problems and fear of communicating solution strategies, the TE planned to continue implementing a cooperative learning approach while encouraging mutual respect. For motivation, the TE considered emphasising the purpose of each lesson by outlining the lesson content at the beginning of the lesson. In addition, the TE also planned to use a story to demonstrate methods of counting on and back innovatively and to connect intervention cycle one to intervention cycle two.

6.2.2 Intervention cycle two

Intervention cycle two began with the announcement of the lesson objective followed by a discussion of what FMC entails. PSTs summarised the meaning of FMC as a way of computing using calculation strategies that are quick, easy and straight forward, accurate and not complicated or confusing. Journal comments on lesson one were addressed in lesson two and also in subsequent lessons. The issue of absenteeism and turning up late for classes was also discussed with the class before checking the homework that was given in lesson one. The TE observed that only a few PSTs attended to the homework. Again, the TE encouraged PSTs always to commit themselves to tasks given to them to improve their computation skills as adults and prospective teachers. Next, PSTs were instructed to count backwards in twos, fives and tens. Since most PSTs indicated facility with addition of two-digit whole numbers, the TE decided to conduct a quiz out of ten marks to confirm their journal reflections. The quiz was held before the beginning of intervention cycle two to establish PST's progress on intervention cycle one.

After the quiz the TE collected the answer scripts and discussed the test items with the entire class. PSTs participated in the lesson, though some were still reluctant to share their answers with the whole class. The TE continued to encourage participation among PSTs, which eventually improved. To answer one of the questions raised in the reflective journals pertaining to the use of a ten frame for addition, the TE used pictures of tomatoes on two separate ten frames to demonstrate addition of 6 and 8 using a ten frame. Subsequently, the TE used a story to create a context for counting back and to introduce subtraction of whole numbers. Owing to time constraints, the one-hour lesson did not involve any other task and was concluded with the TE highlighting the role of storytelling in connecting counting to reality, while drawing attention to the lesson. In all, the TE observed that the process of sharing and justifying computation strategies requires time, so tasks need to be short. As such, the TE envisaged inclusion of content that could be covered within the available time. The quiz was marked after the lesson and feedback provided during intervention cycle three.

6.2.3 Intervention cycle three

The goal of intervention cycle three was again to assist PSTs to solve subtraction problems in multiple ways. The empty number was used to support PSTs to visualise calculation strategies in mind. The problems used in the learning process are presented in the Table 6.2.

Table 6.2: Intervention task two

	Task 2	**Number range and equation**	**Mathematical intent of each problem**
1.	John has 14 minutes to complete a task. He uses up 7 minutes. How many more minutes are left for him to complete the task?	7+7=14	Add up
2.	Andrew bought a shirt costing N$32.00. He had N$65.00 in his purse. How much money remained in his purse after paying for the shirt?	65 – 32	Remove or count back
3.	Jacob and Alfonsina raised funds for their year-end party. Jacob raised N$96.00 and Alfonsina raisedN$151. How much more money does Jacob need to earn to catch up with Alfonsina?	151 – 96	Adjust one number to create a friendly number or adjust both numbers
4.	A teacher had some books on his table. He now has 73 books after handing out 59 books. How many books were on his table before?	□ – 59=73	Reverse, compensate, decompose and use friendly numbers
5.	Karen owes Nambu N$999.00. Karen pays back an amount of money but there is still N$654.00 outstanding. How much money did Karen pay?	999 – □=654	Adjust both numbers
	Homework		
1.	A school had 127 learners at the beginning of the year, of which some were Grade 12 learners. All the Grade 12 learners passed their year-end examinations and now the school has 89 learners. How many learners were in Grade 12?	127 – □=89	Reverse, use friendly numbers and decompose
2.	A store has 123 tables and sells some of the tables. Now it has 64 tables left in store. How many tables were sold?	123 – 64	Adjust both numbers to create an easier number
3.	A farmer has 800 goats and sells 169 goats. What is the total number of goats he has now?	800 – 169	Adjust one number to create an easier number

4.	The reading on a water meter is 563 after 271 units had been consumed. What is the previous water meter reading?	563 – 271	Adjust one number to create an easier number
5.	Two trucks are full of watermelons. The white truck has 96 watermelons more than the green truck with 256 watermelons. How many watermelons are on the white truck?	□ – 96=256	Work backwards using addition

The intervention process started with the announcement of the lesson objective followed by a counting backwards exercise where all PSTs counted together as a class. PSTs were then divided into groups of four to six members per group. The TE started by clarifying a question posed by a PST in the previous lesson about how to determine the operation sign to use when solving a problem. In response to the question, PSTs were encouraged to read and understand a problem to identify and use any sign they may find appropriate to use. Thereafter, PSTs counted backwards in twos from 100, in fives from 100, in tens from 100 and in hundreds from 1 000 to increase PST number knowledge. To count backwards easily, the TE asked PSTs to visualise the number chart in their mind to keep track of the counting process. The pattern of 2, 4, 6, 8, and 10 was highlighted to aid the process of counting backwards as 10, 8, 6, 4 and 2. The importance of developing backward counting skills was discussed to project the connection between counting and subtraction. PSTs worked cooperatively in mixed-ability groups of five using the strategy: 'Think-Pair-Share'. The activity involved direct and indirect subtraction where the TE displayed one question at a time to give adequate thinking and writing time. While PSTs were writing, the TE visited groups to observe their progress. Once all PST were done with their writing, the TE encouraged PSTs who were reluctant to participate in the previous lesson to share their strategies with the entire class. Improvement in participation was evident as many PSTs raised their hands to share their strategy with the class. Each time a question was solved the TE posed questions such as:

- What did you do to get your answer? How did you figure out the answer?
- How do you know your answer is correct?
- Is there another way to solve the same problem?

The TE continued to promote mutual respect by encouraging PSTs not to laugh at or denigrate any calculation method. By the conclusion of the lesson, the PSTs had gathered a summary of numerous FMC strategies that had been invented during the lesson. After the lesson some PSTs indicated in their reflective journals that subtraction strategies were confusing. Some PSTs indicated that they were unable to understand strategies invented by others in class. This confirms the argument that it is difficult for learners to understand strategies imposed on them by teachers. For increased understanding, PSTs were encouraged to do more practice and to persist in inventing own calculation methods in intervention cycle four.

6.2.4 Intervention cycle four

Intervention cycle four focused on supporting PSTs to solve multiplication problems in multiple ways. The type of problems and intent of each question are elaborated upon next.

Table 6.3: Intervention task three

	Task 3	**Number range and equation**	**Mathematical intent of each problem**
1.	There are 9 dogs in a house. How many legs are there altogether?	9 × 4	Double (double, double to multiply by 4)
2.	Jane sold 14 packets of oranges. There are 8 oranges in each packet. How many oranges did she sell altogether?	14 × 8	Decompose or double three times when multiplying by eight
3.	A mini bus transported a group of learners from school to town. It carried 15 learners per trip for 6 trips. How many learners were transported altogether?	6×15	Decompose and use partial quotients
4.	Mimi bought a number of boxes of apples. There are 56 apples in each box. She has 224 apples in total, how many boxes of apples did Mimi buy?	224 ÷ 56	Do repeated addition or decomposition
5.	A school which had 15 classrooms allocated an equal number of learners to each class after registering 420 learners. How many learners were allocated to one classroom?	15 × □ = 420	Make use of partial quotients (using friendly multipliers: 10 & 5)
	Homework		
1.	We have 9 boxes a dozen eggs. How many eggs do we have in total?	9×12	Partial products (using friendly multipliers: 10 & 2)
2.	Nangura sold 16 bags of potatoes weighing 5 kg each at N$25.00 per bag. What is the total amount of money he collected for the 16 bags?	16 × 25	Double and halve
3.	Hausiku bought 14 books, each costing the same amount of money. He paid N$210.00 for all the books. Calculate the price he paid for each book.	14×□=210	Apply partial quotients (use friendly multipliers: 10& 5) or express 14 as a product of prime factors to divide
4.	A particular region has 16 pre-primary classrooms with 25 learners per classroom. How many pre-primary learners are there in the 16 classrooms altogether?	16 × 25	Break a number into smaller factors (create a one-digit multiplier)
5.	A number of learners each paid N$199.00 for a class party. A total amount of N$995.00 was collected. How many learners have paid for the party?	□ × 199 = 995	Use friendly numbers

PSTs were encouraged to model calculation strategies using picture cards of dogs and oranges packaged in groups of four, a picture of a dozen eggs, and number line and dot cards as visual models. The lesson objective was announced at the beginning of the lesson. Then, PSTs were asked to think of a variety of contexts representing multiplication. Contexts such as pairs of shoes, cases of cool drink, a bus with six wheels and the number of ears of a particular number of people. PSTs worked out four questions individually and shared their solution strategies with the entire class. The problems included in the activity intended to develop PSTs' reasoning in terms of doubling twice to multiply by 4, doubling three times to multiply by 8, decomposing one number into tens and fives to calculate easily (friendly multipliers), decomposing a number into its prime factors to divide or multiply, using partial quotients, doubling and halving, using repeated addition to multiply or divide, and using known facts to multiply or divide. Participation increased but PSTs struggled to multiply 6 by 15. To advance PSTs thinking to multiply by 15 the TE planned an activity involving investigations carried out during intervention cycle five.

6.2.5 Intervention cycle five

Intervention cycle five was still based on multiplication following the PSTs' challenges observed during intervention cycle four. In class, PSTs struggled to multiply by 15 as, in the pre-intervention test, 92% of the class could not multiply by 15. As a result, the purpose of intervention cycle five was to engage PSTs in an investigation that could improve their ability to analyse numbers critically through investigation, experimentation, imagination and reasoning. PSTs explored multiplication of 15 by 2, 3, 4, 10, 12, 16 and 17 to identify patterns that could facilitate multiplication by 15. The investigation permitted PSTs to experiment with multiplication by 15 and use their imagination to split 15 to calculate easily. As a result, PSTs were able to understand the reason why the procedure worked effectively. Discovery of an easy way to calculate by 15 illuminated both their conceptual and relational understanding (Skemp, 1978) of the calculation procedure involved in multiplying by 15.

Table 6.4: Investigation of a pattern from products of fifteen

	× 10	× 5	× 15
2			
3			
4			
10			
12			
16			
17			

The lesson started off actively, but some PSTs were again too reluctant to discuss their ideas with their colleagues in groups. To actively engage PSTs in the activity, a cooperative learning approach called 'Think-Pair-Share' was used again. The approach requires PSTs first to listen to the context of the question, then to think and calculate individually, and then to discuss solutions in pairs, before sharing their strategy with the group members and the entire class. Group leaders were selected by members of the group to manage the process of sharing calculation strategies within the group. PSTs worked collaboratively and recorded calculation methods in their note books before they were written on the board for discussion with entire class. PSTs struggled to multiply some numbers by 10, 5 and 15 as they could not see the relationship or pattern between the products of 10, 5 and 15 as it took them long to complete the task. After the investigation had been carried out, PSTs were invited to discuss the pattern that emerged when multiplying by 15. This prompted an explicit articulation of the relationship and pattern between products. PSTs were asked to discuss the efficiency of each solution strategy to identify methods that are fast, accurate and proficient. Again, various PSTs indicated the need for support to understand various multiplication strategies developed by others.

PSTs' views regarding the first intervention lessons involving addition and subtraction indicated that tasks facilitated their invention of strategies to solve problems related to multiplication. PSTs had a specific interest in particular strategies, but were unable to understand how to use multiplication where indirect division was involved. This demonstrated that specific PSTs were unable to carry out indirect division requiring knowledge of reversibility of operations, while other PSTs found it challenging to multiply large numbers. However, PSTs developed strategies such as doubling and halving, using friendly numbers, using partial products, skip counting, using known facts, doing repeated addition and breaking factors into smaller factors. Eventually, as a reinforcement exercise, PSTs were given an activity comprising three story problems to solve individually. The conclusion of the lesson involved highlighting computation strategies that were invented.

For more practice with multiplication, the TE prepared homework to investigate multiplication by large numbers such as $16 \times 36 = (10\times 36) + (5 \times 36) + (1 \times 36)$; $360 + 180 + 36 = 576$; 16×21 and 16×47. With regard to large numbers, PSTs had a problem with manipulation of a long string of numbers in their minds. It emerged during the intervention process that it takes time and more practice to achieve fluency in computing mentally. More time for practice is a challenge as, on average, a TE has only two hours to spend on one learning exit outcome. This follows the three contact hours per week meant to cover 19 learning exist outcomes over a period of 14 weeks. This study spent 12 hours (four weeks) and an additional three hours (one week) to cover both fast and struggling PSTs. The time constraint limited in-depth discussion of strategies and collaborative activities. PSTs expressed the desire to practise multiplication involving three-digit numbers to improve their calculation skills. As time was a constraint, intervention cycle six was implemented.

6.2.6 Intervention cycle six

Intervention cycle six was set to enable PSTs to solve direct and indirect division problems using own calculation strategies. Detailed information pertaining to the problems PSTs were engaged in, namely the number range and equation, the mathematical intent of each problem and illustration of possible strategy to be invented, is presented in Table 6.5, followed by a description of intervention six learning process.

Table 6.5: Intervention task four

	Task 4	**Number range and equation**	**Mathematical intent of each problem**
1.	Hakwa has 30 tomatoes. He plans to put 5 tomatoes into separate plastic bags. How many plastic bags does Hakwa need?	30 ÷ 5	Do repeated subtraction
2.	Nguni has 120 sweets. He plans to share the sweets equally among his friends. If he has 6 friends, how many sweets will each friend receive?	120 ÷ 6	Share/deal out (fair sharing)
3.	A farmer has 96 seedlings to plant in 6 rows. How many seedlings will each row have?	96÷6	Multiply up (multiply by tens and twos)
4.	A farmer has 112 seedlings to plant in a number of rows. She wants to plant 7 seedlings in each row. How many rows will she have?	112÷□=7	Multiply by 10 and use known facts
5.	You have N$3 000.00 and you want to buy fabric costing N$500.00 per package. How many packages are you going to buy?	3 000 ÷ 500	Count on in 500s
	Homework		
1.	If you have 76 sweets to distribute equally among 6 learners, how many sweets will each learner receive and how many sweets will remain?	76÷6	Decompose into friendly numbers
2.	Muronga had an amount of money that he shared with his 19 team members. If each member received N$5.00, what is the total amount shared?	□÷19=5	Reverse and count on
3.	A total of 243 sheep were herded into 3 camps. How many sheep did each camp hold?	243÷3	Use known facts and a friendly number ten
4.	Martha has 192 school calendars to distribute among 32 classrooms. How many school calendars will each class receive?	192 ÷ 32	Use proportional reasoning
5.	You saved N$108.00 in a period of 9 months. On average, how much did you save per month?	108=9×□	Decompose, using known facts and friendly numbers

Pictures of tomatoes, a poster of a dozen eggs, an empty number line and dot cards were used again as visual models. The lesson began with a recap of the investigation about multiplication by 5, 10 and 15 carried out in the previous lesson. Then, PSTs engaged in an activity comprising five story problems. The problems were presented to trigger PSTs' thoughts in terms of division as repeated subtraction, sharing or dealing out, multiplying up by 10, 5 or 2, using known facts, employing reversibility, counting on to divide, breaking down numbers into friendly numbers, and proportional reasoning. Problems were solved one at a time as the lesson progressed.

In pairs, PSTs worked on one question at a time before discussing their methods with the whole class. Gradually, PSTs gained more interest in sharing their calculation methods with the class. This was evident as many PSTs raised hands to communicate their strategies with the class. As in previous lessons, the TE posed probing questions to promote reflective thought through critical thinking and reasoning, as presented next:

- What did you do to get your answer? How did you figure out the answer?
- How do you know your answer is correct?
- Is there another way to solve the same problem?

PSTs were thoughtful, they invented own calculation strategies and communicated solution methods with the whole class. Division of a three-digit number by a two-digit number was too challenging for some PSTs, so homework was given. The homework items included 32 × 15, 16 × 25, 120 ÷ 6, 14 × 8, 198 ÷ 8, 85 – □ = 67, 59 + □ = 82, 461 – 289, 27 + 28 and □ ÷ 15 = 20. At the end of intervention cycle six, the class discussed the four basic operations to highlight the connections they discovered between the operations and recorded their reflections in their journals too. Despite time constraints, all four basic operations were covered. A major breakthrough of the entire intervention process was its contribution to PST competency and belief in the teaching of FMC despite the differences in the degrees of achievement. Disparity in achievement could possibly be attributed to PSTs' difference in pace of learning, existing knowledge, absenteeism and low intrinsic motivation. After intervention six, the post-intervention test and interviews were conducted.

6.3 Means by which learning was supported

6.3.1 Classroom activities

This study aimed to develop PSTs' mathematical knowledge for teaching FMC through a problem solving teaching approach as found effective by Freudenthal (2002), Whitacre and Nickerson (2006), Heirdsfield (2011) and Rechtsteiner-Merz and Rathgeb-Schnierer (2015). The activities laid the foundation for supporting the development of FMC strategies in PSTs. Research affirms that the degree to which PSTs are prepared in one operation will assist them to excel in subsequent operations (Bruner, 1977; Kamii & Joseph, 2004). In other words, what is learned next is built on what was learned previously. To sequence the operations of the intervention, the TE conducted activities involving both addition and subtraction. Both task one and two combined calculations that involve direct and indirect addition and subtraction. Task three and four included calculations relating to direct and indirect multiplication and division. However, the sequencing of the operations was not meant to compartmentalise operations but to present the tasks in an orderly fashion and allow PSTs to discover the connection between operations.

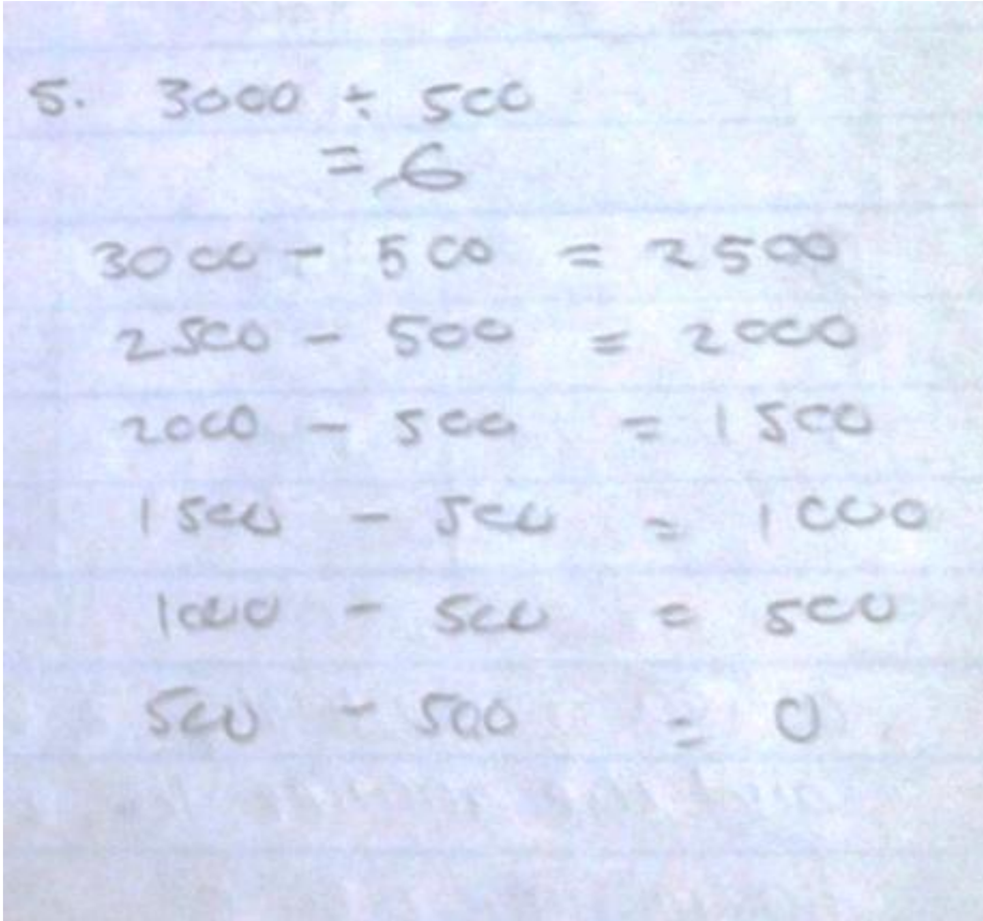

Figure 6.1: Division as repeated subtraction

Figure 6.1 (question five) illustrates how subtraction facilitates calculations involving division. Tasks one to four are presented in the tables where the number range used in problems and the mathematical reasoning behind each problem are outlined.

6.3.2 Classroom discussion

During the intervention process, PSTs were granted the opportunity to write their calculation strategies on the board and to explain how and why the strategy worked. Illustrations of PSTs' strategies written on the board appear in Figure 6.2.

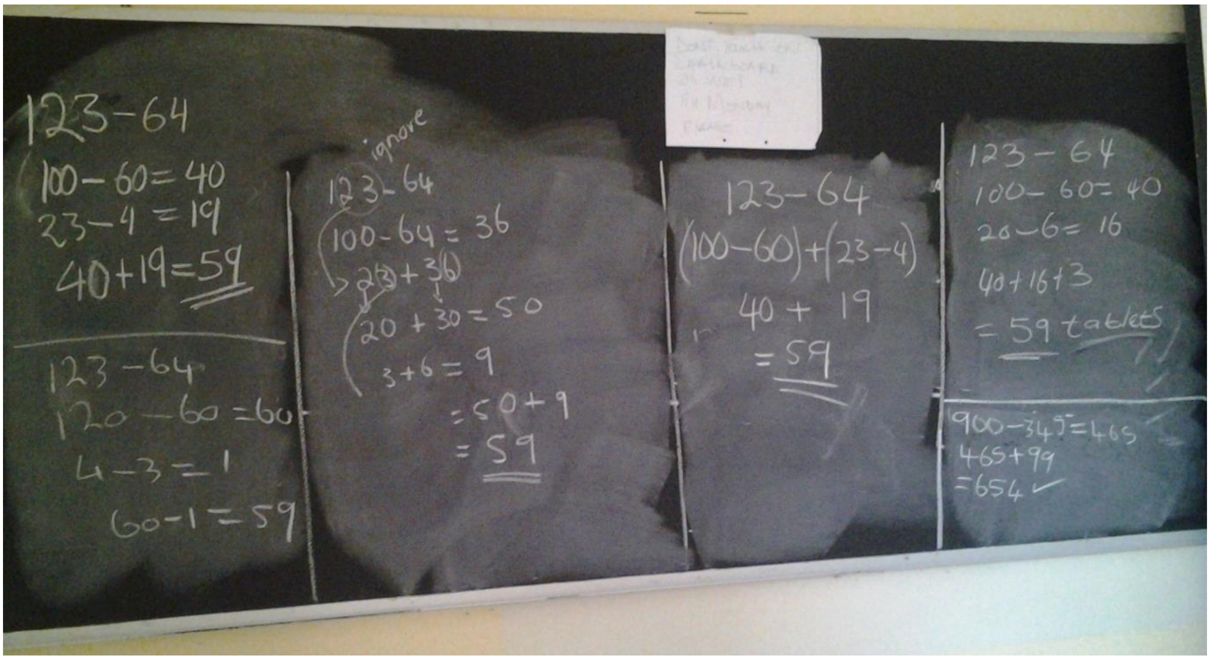

Figure 6.2: Examples of PSTs' invented strategies

PSTs outlined that the process of discussion and justification of strategies, as well as TE intervention to clarify specific strategies, facilitated understanding, detection of mistakes in reasoning and skills to compute flexibly in mind. Ultimately, the intervention reminded them of methods they had learned in primary school. Reference was made to strategies such as decomposition and the use of friendly numbers. PSTs described the tasks as helpful in improving their ability to compute mentally. Specific development occurred in terms of ability to discover short and straightforward calculation methods. With regard to time, the TE observed that the discussion of calculation strategies demands ample time, thus calling for a reduction in the number of problems to be solved in class. In addition, there is a need to develop a teaching framework for slow-learning PSTs and for quick-learning PSTs to accommodate PSTs according to their entry levels of FMC knowledge. The argument is that PSTs with 100% achievement found the problems easy as they matched their potential to calculate mentally when others struggled. One of the top performers confirmed that FMC is fascinating especially when the degree of difficulty increases.

6.3.3 Tools that supported learning

During the intervention process, diverse tools were used to support PSTs to develop their own mental calculation strategies. The intervention used five different types of tools to improve reasoning when calculating numbers mentally. These included an abacus, number chart, open number line, ten frame and picture cards. Among the five tools, PSTs found the ten frames very helpful in devising flexible ways to computing in the mind (Appendix I, no. 4, 15, 26). Figure 6.3 projects an example of a ten frame and picture cards.

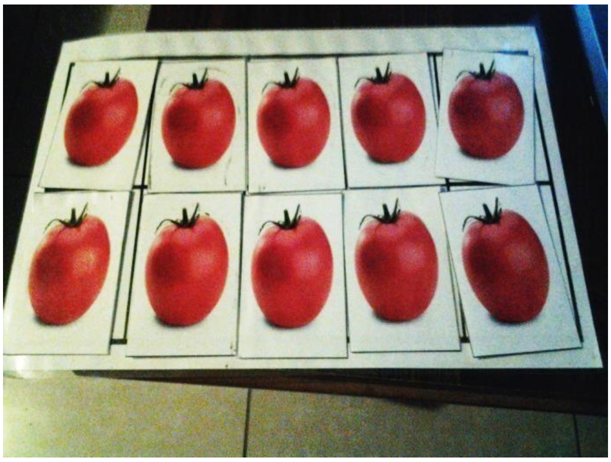

Figure 6.3: A ten frame and picture cards

The ten frame was helpful in developing PST knowledge for teaching friendly numbers and visualising groups of ten in the mind. To extend number knowledge to hundred, 10 ten frames were used to visualise ten groups of ten in a hundred. Visualisation of groups of ten in a hundred lays the foundation for multiplication and division. A summary of Chapter 6 is provided in section 6.4.

6.4 Conclusion

Chapter 6 has presented a detailed discussion of the intervention cycles. The purpose of the intervention was to devise methods to develop PSTs' knowledge for the purpose of teaching FMC. Class activities, discussion of solutions and tools facilitated PSTs' development of FMC skills. Besides expressions of fear and a lack of confidence to calculate mentally before the intervention, PSTs managed to discover their potential to develop their own calculation strategies. During the intervention, most PSTs expressed difficulty in multiplying and dividing three-digit numbers mentally. Challenges experienced during the intervention included PSTs' negative reaction towards colleagues' inefficient strategies and inaccurate solutions, a lack of intrinsic motivation to compute mentally, absenteeism, slow learning pace, limited time for more practice, and inadequate existing knowledge and experience. As a result, PSTs reached different developmental levels after the intervention process. Consequently, challenges encountered in this study have raised questions such as: How can a TE minimise absenteeism? How can a TE navigate the overloaded PST mathematics curriculum without compromising quality amid PSTs' pace of learning and existing knowledge? Which ways are more effective for improving PST intrinsic motivation levels and behaviour towards each other? All this prompts a reconsideration of teaching methods, and the amount and type of content offered in teacher education courses.

Generally, the intervention was successful in developing PSTs' understanding of what FMC is, ability to reason to invent own computation strategies, knowledge on how to support learners to do flexible computing in the mind and changing PSTs' beliefs about mathematics as a subject, and best approaches for teaching FMC. Results from the pre- and post-intervention test, pre- and post-intervention interviews, and PSTs' reflections and strategies written on the board during the intervention process indicate that the intervention yielded positive results mainly for PSTs with a fast learning pace. Major improvement relates to PSTs who developed 100% mastery of all operations in the post-intervention test and others who demonstrated slight improvement in one or two operations. As informed by constructivist theory (Piaget, 1971; Bruner, 1977; Vygotsky, 1978) and philosophy of realistic mathematics education (Freudenthal, 1968; Treffers, 1987), the subsequent discussion outlines how the initial learning trajectory was improved.

CHAPTER 7: REFINING THE HYPOTHETICAL LEARNING TRAJECTORY

7.1 Introduction

Chapter 6 provided a detailed account of the intervention process. Chapter 7 discusses the development of an HLT designed in this study through a design research approach. An HLT is a developmental path that specifies activities, learning objectives and materials to be used for developing PST knowledge from one level of understanding to another in specific ways. The purpose of design research is to craft and refine a medium to support learning and also to understand how the learning resource functions (Cobb & Gravemeijer, 2008). The HLT comprises four entries: the four basic operations, the anticipated computation strategies, all the instructional activities used during the intervention process, and the models that supported reasoning of PSTs. The HLT was refined in response to difficulties that transpired during the intervention, while attempting to support learning.

The design of the HLT began with the development of tasks in the form of problem solving. The questions were formulated in consultation with trajectories for counting, and a learning trajectory for addition and subtraction developed by Clements and Sarama (2014) for learners. In addition, a learning trajectory by Whitacre and Nickerson (2006) that is based on number sense development of PSTs was used. This chapter presents the design of the initial HLT; activities as initially designed; a discussion of the aspects from the intervention sessions that led to the adaptation of the initial trajectory; the stages of activity improvement; and a discussion of the final enacted trajectory.

7.2 Design of the initial hypothetical learning trajectory

In particular, six RME principles informed the design of the activities (See chapter 3, section 3.3). These included: the activity principle, reality principle, level principle, intertwinement principle, interactivity principle and guidance principle. The activity principle directed the design of the intervention activities to engage PSTs as active PSTs in the entire intervention process. To develop fluency and flexibility with mental calculations, PSTs had to invent their own calculation methods through physical and mental engagement (Freudenthal, 2002). The reality principle informed the incorporation of problems that reflected daily life mathematics for PSTs to apply their knowledge in solving practical problems. Inclusion of real-life problems in the HLT were meant to support PSTs in making meaning of invented computation strategies and eventually understanding standard algorithms. Since FMC is best developed through invention of calculation strategies, PSTs' ability to think strategically had to be enhanced using the 4-6-1 model of teaching (Claxton et al., 2010), (see figure 2.2). The model suggests ways of ensuring active mental engagement of PSTs in the learning process. As a result, the TE encouraged the development of the four habits of mind through investigations, experimentation with new strategies, reasoning to clarify calculation strategies, and the use of models and representations for visualisation and imagination of calculation procedures.

As to existing number knowledge, the level principle guided the design of the activity to include both easy and challenging problems, and small and large numbers to accommodate all PSTs (Piaget, 1964; Bruner, 1977). Problems were crafted considering fundamental knowledge that underlies the effective development of FMC skills. These include knowledge of numbers, the ability to recognise and generalise patterns and relationships between numbers, calculating fluently and flexibly, and the ability to think strategically. Rechtsteiner-Merz and Rathgeb-Schnierer (2015) found that calculations are facilitated by the ability to identify patterns between numbers and to understand the relationship between numbers. Numbers should be presented in a context of problem solving to connect FMC to real-world settings (Freudenthal, 2002; Heirdsfield, 2011). Problems ought to create an opportunity for generalisation of patterns, such as $1 + 2 = 3$, $10 + 20 = 30$, $11 + 12 = 23$, $211 + 312 = 523$, and the relationships between operations and computation strategies. Intentionally designed problems serve to enable PSTs to transcend to a different developmental level by thinking critically about developing strategies that are straightforward, accurate and fast.

During the intervention process, the guidance principle evoked the provision of guidance for PSTs to represent and solve problems. Guidance was given in the form of questions to model the situation in a problem, using pictures and writing calculations to represent the problem mathematically. Guidance was meant to shift thinking from 'horizontal mathematics' to 'vertical mathematics' (Treffers, 1987; Freudenthal, 2002; Van den Heuvel-Panhuizen & Drijvers, 2014) and to connect the four basic operations.

Guided by the intertwinement principle, instead of compartmentalizing topics, the HLT included all the four basic operations for PSTs to understand the connection between addition, subtraction, multiplication and division, and algebra, e.g. a problem where an answer is known and part of the initial information unknown. All four operations were arranged for the activities, to foster a smooth transformation in PSTs' thinking and perceptions, ensuring proper reconstruction of information in memory (Piaget & Inhelder, 1973; Bruner, 1977). Thinking and reasoning was planned to develop through individual mental engagement and as a group to embrace the interactivity principle.

Based on the interactivity principle PSTs were allowed to develop computation strategies individually as well as in groups. Sharing of solution strategies was encouraged through presentation of individual approaches on the chalkboard. Strategies were subjected to scrutiny by the entire class for refinement of personal strategies and understanding of diverse approaches. Constructivists state that advancement of knowledge occurs when learning with others who are more experienced, collaboratively rather than individually (Bruner, 1996; Vygotsky, 1978). Thus, discussion in groups and as a class was encouraged. The designed activities are presented next.

7.4 Aspects that triggered the improvement of the initial trajectory

7.4.1 Changes to activity one

As illustrated in Table 7.1, class activity one comprised 11 problems. The table only includes the number range and equation, and mathematical reasoning behind each problem. Details regarding the context of the equations and envisioned strategies are presented in Appendix H. During intervention one, the TE discovered that 11 problems were too many to solve and provide feedback for within a one-hour learning slot. This is because PSTs needed more time to analyse problems, and to think and invent a calculation strategy. Another issue was the amount of time consumed by discussions around a specific problem. The TE discovered that feedback sessions demanded adequate time, especially when diverse strategies are developed and presented on the chalk board. As a result, only five out of the 11 problems were solved and discussed in class. The remaining six questions were then given as homework for PSTs to practise in their spare time.

Another aspect relates to the small number range used. The number range had to be increased to include three-digit numbers, and more direct and indirect addition and subtraction problems relating to reversibility. Also, in the final trajectory, one problem was omitted to reduce the number of homework problems to five. The change in number range was necessary to include both easy and challenging problems for the purpose of extending PSTs' understanding of the relationship between numbers and operations; to improve the ability to compute large numbers; to think strategically in order to invent own calculation strategies that are fluent and flexible; and to identify the pattern that emerges when multiplying by numbers such as ten, five and fifteen.

7.4.2 Changes to activity two

Initially, activity two comprised six questions to be solved by the PSTs in the classroom, and contained no homework. Based on the number of items PSTs could answer in class during the first intervention, the TE designed more problems to include as homework. Since activity two comprised both direct and indirect addition and subtraction, the development of calculation strategies evolved at a faster pace in relation to activity one. This was probably due to skills developed in intervention cycle one and two. The number range was increased to include three-digit numbers and types of problems relating to reversibility and relationship between numbers. Four more problems were designed to provide homework to extend thinking. Specific reasoning promoted by each problem is outlined in table7.1.

7.4.3 Changes to activity three

Activity three comprised six problems pertaining to direct and indirect multiplication and division. The number range changed to include three-digit numbers to extend PSTs' reasoning in terms of multiplying and dividing by a three-digit number. However, many PSTs reported inability to devise strategies to multiply large numbers. As a result, a task involving investigation into multiplication of numbers by 10, 5 and 15 was designed and carried out in a subsequent intervention cycle.

7.4.4 Changes to activity four

Initially, activity three involved four questions which incorporated only direct division, as shown in Table 7.1. The activity had no homework and the type of questions did not involve indirect division and multiplication. The types of problems were improved to include indirect division and multiplication. The TE had to increase the number of problems to 10 in order to have five questions for class work and five problems for homework. The number range was increased to include two-digit numbers above fifty and three-digit numbers below 300 to extend PST calculation skills.

7.5 Stages of hypothetical learning trajectory design

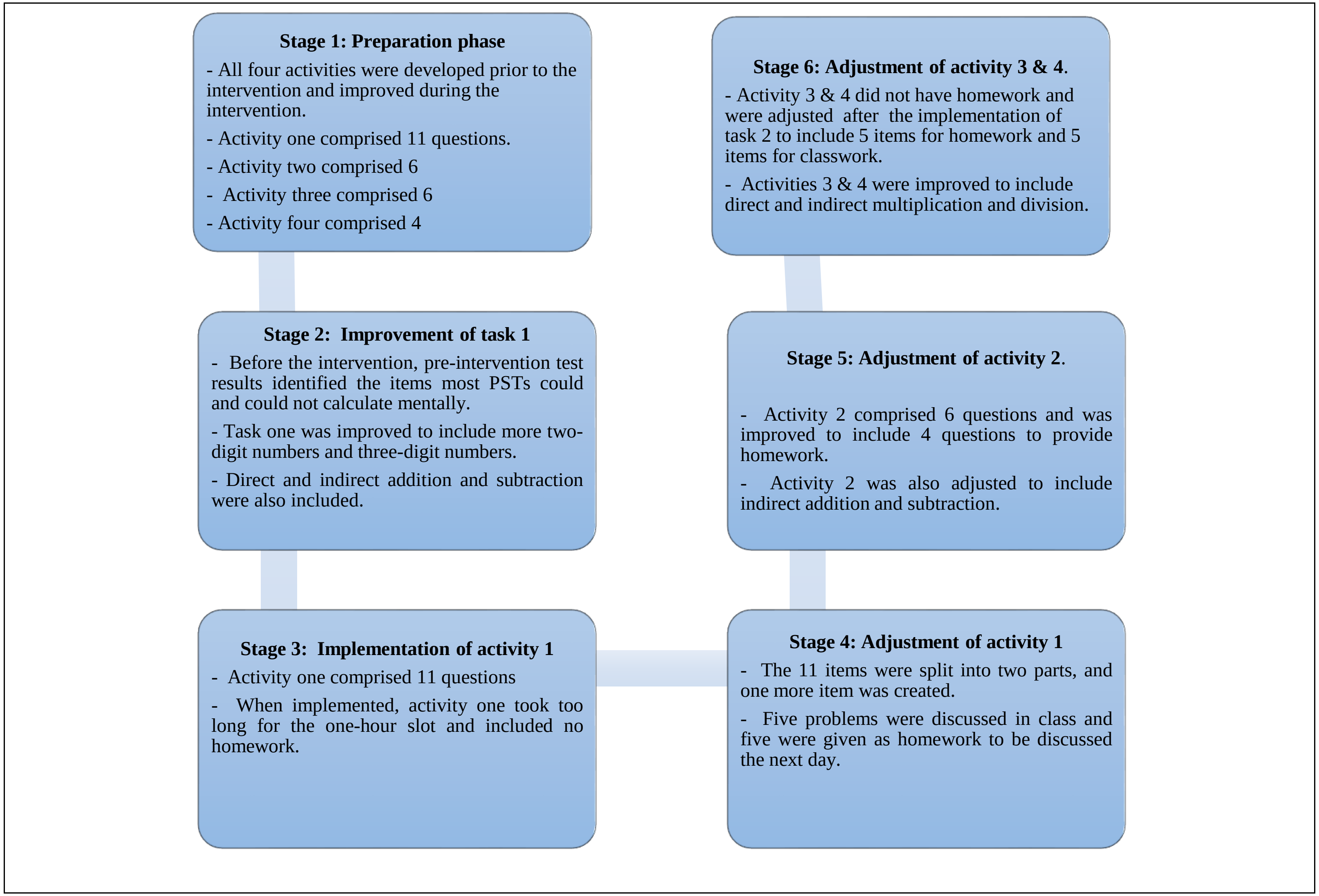

Figure 7.1: Stages of hypothetical learning trajectory design

7.6 The final enacted hypothetical learning trajectory

Initially the study planned to use tools such as a number chart, abacus, open number line, ten frame, array model and dot images. Due to time constraints, it was impossible to use all these tools to support learning. Specific tools that were used to support learning are discussed in the next sections.

7.3 Activities as initially designed and incorporated in the hypothetical learning trajectory

Table 7.1: Activities as initially designed for the intervention

Class activity 1			Class activity 2			Class activity 3			Class activity 4		
	Number range and equation	**Mathematical reasoning behind each problem**		**Number range and equation**	**Mathematical reasoning behind each problem**		**Number range and equation**	**Mathematical reasoning behind each problem**		**Number range and equation**	**Mathematical reasoning behind each problem**
1.	6+7	Counting all and counting on	1.	14 – 7	Adding up	1.	2 × 4	Doubling to multiply by four	1.	30 ÷ 5	Repeated subtraction
2.	7+5+3	Commutativity	2.	65 – 32	Removal or counting down in tens	2.	6×15	Using known facts	2.	20÷5	Sharing/dealing out (fair sharing)
3.	8+9	Doubles and near doubles	3.	96+□=151	Adjusting one number to create an easier number	3.	3×□=36	Repeated addition or skip counting in threes	3.	384 ÷ 16	Multiplying up (multiplying by tens and twos)or proportional reasoning using common factors
4.	9 + 7	Double and two more/ less	4.	123-59	Place value and negative numbers	4.	9×12	Partial products or using friendly numbers	4.	3 000 ÷500	Counting on in 500s
5.	9 +7	Making tens	5.	999 – 345	Place value and negative numbers	5.	8 × 25	Doubling and halving or double, double, double to multiply by eight			
6.	33+58	Making friendly numbers	6.	123 – □=64	Adjusting both numbers to make an easier number	6.	12 × 30	Breaking factors into smaller factors (create a one-digit multiplier)			
7.	24+38	Decomposition									
8.	8+6	Compensation to make a double									
9.	□ + 18=41	Friendly number and constant difference									

Class activity 1			Class activity 2			Class activity 3			Class activity 4		
	Number range and equation	**Mathematical reasoning behind each problem**		**Number range and equation**	**Mathematical reasoning behind each problem**		**Number range and equation**	**Mathematical reasoning behind each problem**		**Number range and equation**	**Mathematical reasoning behind each problem**
10.	36+9	Compensation to make a ten									
11.	45+28	Adding up in chunks									

7.6.1 Hypothetical learning trajectory part one: Direct and indirect addition

The study started by using the number chart for PSTs to discover how counting forward and backwards in twos, fives, tens and hundreds from any number can contribute to FMC skills. As a result, counting was used as an introductory task to intervention cycle one. PSTs were encouraged to have a mental image of the number chart to support calculations in mind. The HLT included four different number charts that illustrate multiples of 2, 5, 10 and 100. The charts were meant to help PSTs visualise different multiples of a particular number. However, considering limitation of time, the number chart was not used in other interventions but counting forward and backwards prevailed. The HLT was improved to include an investigation to demonstrate different whole number combinations that make 20. The empty number line was mostly used for addition and subtraction to demonstrate proximity of numbers to a friendly number or multiple of 10. Also, the refined HLT included a table that displayed counting on to connect repeated addition to multiplication. The use of the abacus for calculations was discontinued as most PSTs could not easily understand how to calculate with it. The empty number line and ten frame were most effective in supporting PSTs in visualising the proximity of numbers to multiples of 10 or friendly numbers. The HLT was improved further to include a diagnostic test comprising 10 items. The second part of the HLT comprised items that linked addition and subtraction in the form of direct and indirect subtraction.

7.6.2 Hypothetical learning trajectory part two: Direct and indirect subtraction

Part two of the improved HLT included an introductory activity, the second intervention task and a diagnostic test. The introductory activity involved counting back in twos, fives, tens and hundreds to support PSTs' reasoning in terms of combining the counting back process with subtraction. The introductory activity was displayed in the form of a modified number chart. The transformed chart displayed numbers in descending order to create a mental image of the counting sequence when counting back to subtract. The third part of the HLT linked part three to part two through the use of repeated addition to multiply. HLT part three comprised direct and indirect multiplication to connect multiplication and division.

7.6.3 Hypothetical learning trajectory part three: Direct and indirect multiplication

Since PSTs indicated that they had a problem with multiplication and division, the TE created a table for PSTs to investigate and discover patterns when a number is multiplied by 2, 3, 4, 6, 8, 9, 20 and 25. Discovery of patterns supported PSTs to understand the strategies of 'double, double to multiply by 4' and 'double, double, double to multiply by 8'. In addition, reasoning in terms of multiplication by 5, 10 and 15 was promoted using a table to display the products and analysing them to discover a pattern. An activity to explore a pattern between a product 2 and 3 and a product of 20 and 30 was also included in the improved HLT. Similarly, part three of the HLT include class activity three in its improved form as explained in section 7.3.2, and a diagnostic test out of 10.

7.6.4 Hypothetical learning trajectory part four: Direct and indirect division

Part three of the HLT was improved to include an introductory task involving doubling and halving. In response to PSTs' requests for more support in terms of reasoning to multiply and divide large numbers easily in the mind, the TE displayed specific two-digit numbers from 35 to 97 for PSTs to double. Also, even numbers from 62 to 100 were displayed in a table form for PSTs to halve them. The activity was meant to improve PSTs' reasoning in terms of doubling large two-digit numbers and halving large two-digit numbers. This was necessary to address the PSTs' challenge, demonstrated during the intervention, concerning multiplication and division of large numbers. Another investigation involved observing the relationship between 5×2=10, 10÷2=5 and 10÷5=2. Improved class activity four forms part of HLT part four as outlined in 7.3.4. HLT part four also includes a diagnostic test out of 10 to test PSTs' ability to divide mentally.

7.7 Designing the hypothetical learning trajectory

Figure 7.2 presents the connection between the initial trajectory and the final enacted trajectory

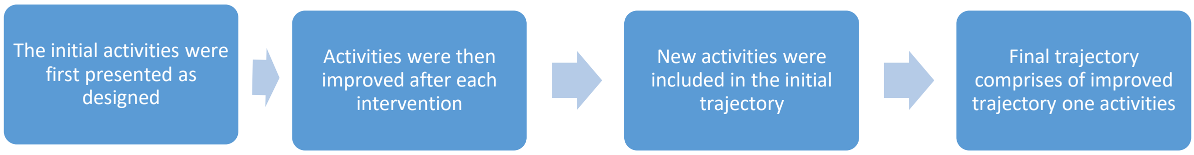

Figure 7.2 Correlation between initial and the final enacted trajectory

7.7.1 Conceptual framework

The refinement of the HLT was informed by conceptual outcomes of the intervention cycles in response to difficulties that arose during all learning processes. During the intervention, major changes were made to the initial activities in terms of the magnitude of numbers and the mathematical reasoning each problem intended to develop (McIntosh et al., 1992; Hiebert, 1984). All activities were improved to include large numbers and reversibility reasoning, and to include homework and diagnostic tests. All four class activities were part of the improved HLT and the numbers used involved numbers close to 10, near doubles and friendly numbers to enable PSTs to discover the relationship between numbers to invent own calculation strategies (Parrish, 2010; Shumway, 2011). Since many PSTs reported the need for more support to multiply and divide large numbers, the TE improved the initial HLT to include investigations for PSTs to discover patterns between numbers (Rechtsteiner-Merz & Rathgeb-Schnierer, 2015; Southwood & Spanneberg, 2000). Homework was also given for PSTs to calculate mentally after the lesson. Such improvements supported the PSTs' perception of numbers and how large numbers can be changed to multiply or divide by a smaller number. Theoretical aspects that informed the refinement of the hypothetical trajectory are discussed next.

7.7.2 Theoretical framework

The development of the HLT was informed by the constructivist theory and the realistic mathematics education principles. The constructivist theory emphasise active participation for learning to occur (Bruner, 1996; Freudenthal, 2002). The activities were design to transcend from small to large numbers and to illustrate the relationship between numbers and operations (Freudenthal, 2002). The principles of RME guided the design of activities to engage PSTs actively, individually and in groups. Scaffolding of tasks prevailed to improve PSTs' ability to invent calculation strategies (Freudenthal, 2002). As to the application of invented strategies, problems were design to reflect real world mathematical situations. In addition, the problems were designed to develop understanding of the connection between operations in terms of reversibility. Generally, the refinement of the hypothetical learning trajectory was informed by the constructivist theory and realistic mathematics education theory. The next section presents a conclusion to chapter 7.

7.8 Conclusion

Chapter 7 presents a discussion concerning the development and refinement of the HLT. The developed HLT includes learning activities, objectives and learning resources TE may utilize to improve PSTs' knowledge to teach FMC. Tasks used were in form of problem solving and were developed prior to the intervention. Aspects that emerged during the intervention informed the refinement of the HLT. These aspects include number magnitude, number range and context of problems. Other aspects discussed relate to the stages of the HLT design, activities as initially designed and the final HLT. Though the HLT was designed within a single classroom context its application can be generalised to other contexts presenting similar needs to improve PSTs' level of reasoning and fluency and flexibility with mental computation. Chapter 8 presents a discussion, contribution and recommendations of this study.

CHAPTER 8: DISCUSSION, CONTRIBUTION AND RECOMMENDATION

8.1 Introduction

The aim of this mixed methods study was to explore ways to develop PSTs' mathematical knowledge for teaching FMC. Chapter 1 provides a general introduction to the study by discussing the background, objectives and design of the study. Chapter 2 includes a discussion of the literature that outlines the content involved in FMC and the ways how to develop PSTs' knowledge for teaching FMC. Chapter 3 presents the theories that guided the development of the FMC activities and the intervention process. In chapter 4 the research design and methodology of this study is discussed. Presentation and interpretation of data from the survey and interviews is provided in chapter 5. Chapter 6 includes data presentation and analysis of the intervention process. A discussion of the design and refinement of the activities that constitute the HLT is available in chapter 7. Chapter 8 discusses the impact of the findings of this study on the consulted theoretical and conceptual body of knowledge relevant to FMC, on the profession of teaching elementary mathematics teachers, and on pedagogical implications pertaining to PST knowledge for teaching FMC skills. In the present chapter, discussion of findings begin with the impact of this study on PSTs' change in beliefs about approaches for developing FMC skills, and on beliefs about own ability to compute without any external aid or applying the standard method in mind and by calculator usage. Discussion also focuses on changes in PSTs' experience pertaining to how FMC develops and changes PSTs' ability to do flexible computation in mind. In conclusion, Chapter 8 summarises the limitations and contributions of this study, and provides recommendations for further research. Discussion of findings and recommendations for further research are made in response to the following questions:

i. What are the pre-service teachers' school experience and existing mental computation skills upon entry to university?
ii. How did pre-service teachers develop flexible mental computation skills and beliefs at school about the learning and teaching of flexible mental computation?
iii. How do pre-service teachers in a teacher education course develop mathematical knowledge for teaching flexible mental computation skills?
iv. Do PSTs' school experience and beliefs about FMC influence their development of FMC in a teacher education course?

Major findings of this study pertain to PST beliefs, experience, fluency and flexibility to calculate mentally. The section 8.2 begins with change in PSTs' beliefs about the teaching of FMC.

8.2 Changes in pre-service teacher beliefs

8.2.1 Change in beliefs about the FMC skills teaching approach

One of the major findings revealed in this study relates to PSTs' belief about appropriate approaches to develop learners' FMC skills. The survey on PSTs' existing knowledge, beliefs and experiences revealed teaching methods that are prominent in schools and identified possible strategies PSTs used to calculate mentally. Findings of this study confirm teaching practices discussed by Stigler and Hiebert (1998) and Junius (2014), where mathematics teaching was found to begin with an example provided by a teacher followed by an activity requiring learners to practise. Results from the questionnaires provide evidence that 82% of the PSTs agreed that a teacher would provide examples first before instructing learners to solve problems (see section 5.2 and section 5.6.2). This implies that FMC skills may have been developed through memorisation of facts or multiplication tables and through drill and practice of strategies, i.e. they were not invented by learners but provided by a teacher. As a result, the inference is drawn that the way in which PSTs developed FMC at school influenced their current belief about the FMC teaching approach and their ability to compute mentally – as is evident in the kind of errors they made in both tests (see Table 5.4 and Table 5.9). Provision of a different experience during the intervention afforded PSTs insight into constructive approaches to the development of FMC skills. Handal (2003) emphasised that experience of learning mathematics in a specific way may trigger PSTs to teach in a similar way. Similarly, evidence from PSTs' expression in post-intervention interviews showed that they are prepared to develop learners' FMCs exactly the way they developed FMC skills during the intervention (see section 5.7.2).

Another belief PSTs developed was that, when learners used strategies invented by them to solve problems, they achieved proficiency in performing calculations mentally without fear, and that overreliance on calculators may diminish (see section 5.7.2). Varol and Farran (2007) found that a lack of FMC skills promotes reliance on calculators and standard written algorithms to perform basic calculations. Reliance on calculators for calculations was expressed in PST reflection journals before the intervention (see Appendix I), but such views changed as PSTs stated in post-intervention interviews that calculator usage in a lower primary classroom was not necessary. Furthermore, change in PST beliefs is also underpinned by constructivist theory (Ernest, 2002; Piaget, 1951; Piaget & Inhelder, 1973) which advocates learning through active mental engagement. Interview findings indicate that PSTs' active reconstruction and justification of calculation strategies may improve their ability to calculate mentally and belief in the teaching of FMC (see section 5.7.2). In addition, this study confirms that independent mental engagement of PSTs to devise own calculation strategies requires motivation and guidance in using own imagination, especially when PSTs have been exposed to the standard method and have used strategies provided by a teacher (Colyvan, 2012; Schleicher, 2012). Moreover, findings on PST beliefs enhance understanding about the impact of active individual engagement on

PST belief about appropriate ways to develop learners' FMC skills and beliefs about individual potential to make flexible calculations in the mind.

8.2.2 Beliefs about individual ability to make flexible calculations in the mind and the use of calculators

Interview findings and reflective journal data have revealed fear and little confidence in PSTs in their ability to calculate mentally. Research conducted on elementary mathematics PSTs indicate that fear of mathematics reduces confidence in learning and teaching mathematics (Boyd, et al., 2014; Bolden, et al., 2013; Tsao, 2014). Reluctance to calculate without prior examples during the intervention confirmed the little confidence that PSTs have in their ability to calculate mentally. Reluctance to invent strategies could also suggest that learning to calculate mentally using teacher prescribed calculation methods contribute to disengagement of mind and reduced effort to identify patterns and relationships between numbers. Findings from reflective journals confirmed PSTs' fear about doing mental calculations before the intervention, but as the interventions progressed, a degree of confidence was demonstrated (Appendix I, no. 18, 23 & 26). Additional evidence for fear and confidence in own ability to compute mentally also emerged in pre-intervention interviews (see section 5.6.2) and post-intervention interviews (5.7.2). Moreover, 66% of articulation about confidence in the ability to do mental calculations, as evident in questionnaires, signifies that fear of mental calculation is still persistent among PSTs and needs to be addressed in teacher education programmes. If not addressed, FMC is unlikely to be developed effectively in school. However, outcomes of the intervention suggest that PSTs could develop confidence in their own ability to do mental calculations if supported in inventing their own calculation strategies, as advocated by Heirdsfield (2011) and Kazemi (1998). In all, findings of this study confirm that calculation methods can be invented by means of discovery, collaboration, discourse and reasoning to explain thoughts and answers.

Kazemi (1998) asserted that to derive mathematical constructs sensibly and to support creative mathematical concepts, experiences, procedures and problems need to be linked. Links should be created between mathematical operations, numbers, prior knowledge, strategies, problem types and daily life mathematics. PSTs' support emanated from intervention problems that connected mathematics to daily life, which eventually made FMC skill development interesting, easy, enjoyable and empowering (see section 5.7.2). Furthermore, this study has confirmed theoretical assertions that learning requires time (Piaget & Inhelder, 1973). In this study, the process of inventing own calculation strategies to compute fluently and in a flexible manner in the mind took time, yet even more time was necessary to assist PSTs who could not compute calculations involving three-digit numbers (see Appendix I, no. 2, 3, 5, 7). From this study, a conclusion is made that many PSTs had more confidence in adding, subtracting, multiplying and dividing two-digit numbers, but expressed inability to multiply and divide three-digit numbers (see Appendix I).

8.3 Changes in pre-service teachers' experience of mental calculation

This study offered PSTs a chance to develop MKT flexible mental computation through a constructivist approach by solving problems using own calculation methods. The constructivist approach has been found to be more effective (Battista, 1999) as through active engagement and effort PSTs might remember FMC skills; unlike instrumental learning that may cause PSTs to easily forget what they have learned and develop fear of doing mathematics (Alkan, 2013; Gresham, 2008; Sawyer, 2014). In this study, change in PSTs' reasoning stemmed from tasks that incorporated number combinations that directed PSTs towards recognising specific number relationships and different ways of perceiving numbers. Outcomes of this study indicate that such tasks enable PSTs to actively experience how decomposition and recomposition of numbers facilitate FMC skills development. To change PSTs' experience of memorising multiplication tables, a practice also evident in literature (Holm & Kajander, 2012), this study set out to provide PSTs with a different learning experience.

Guided by the definition of problem solving held by Hiebert et al. (1997) and the guiding principles on the implementation of a problem solving teaching approach by Kamii and Joseph (2004), PSTs had the experience of solving context-rich problems without prior examples or calculation strategies. These results refute the practice of developing FMC skills by providing examples first as outlined in interviews and questionnaires referred to in 8.2.1. Also, the intervention proved to be successful in creating the opportunity for PSTs to develop FMC skills in ways other than the standard method (see Chapter 6), for example, through understanding relationships between numbers and operations within a social environment (Van den Heuvel-Panhuizen & Drijvers, 2014) and through discussion of strategies (Kamii & Joseph, 2004). Literature advocates discussion on calculation strategies to develop an understanding of diverse calculation strategies and why strategies work (Kamii & Joseph, 2004). Though Kamii and Joseph (2004) conducted their study with children, discussion has equally proved to be effective in developing PSTs' flexible mental computation skills (see Appendix I, no. 7). This study's intervention results demonstrates that classroom norms such as respect for each other's invented strategies need reinforcement to prevent underrating of invented strategies such as level one methods (see section 6.2.1). As shown by Kind (2014), such experience not only improved PSTs' ability to do flexible computations in the mind but it also changed their beliefs, as discussed in section 8.2.1. Therefore, the practical implication of PSTs' experience in inventing own calculation strategies relates to development of better understanding of how learners develop FMC skills.

Results of this study confirm that the use of problem solving provides a background for a meaningful use of numbers and a connection to context-free calculations; a connection between the four basic operations; and a connection between classroom mathematics and real-life mathematics. The absence of connection prevents representation of daily life calculations using classroom mathematics (Lovitt and Clarke (1992), an ability referred to as 'horizontal' and 'vertical' mathematisation (Freundenthal, 2002). The experience to create own calculation strategies also changed the experience of PSTs relating to the possibility of finding solutions in the absence of any mathematical devices. This confirms the assertion of Polya (1945) and Fisher (2005) that problem solving can motivate participation in solving a problem as it triggers imagination and exploration of solution methods.

8.4 Changes in pre-service teachers' flexible mental computation skills

Despite changes in PSTs' experience of learning mathematics without prior examples, results of the post-intervention test indicate an improvement in PSTs' ability to make flexible calculations in the mind. Although the overall performance of the entire class after the intervention was 53%, major improvements were recorded in terms of individual performance on specific test items (see Table 5.8, 5.9 and 5.10). On one hand, post-intervention test results mean that PSTs can develop specific skills to do mental computation through individual mental engagement to analyse relationships and patterns between numbers, as specified in section 5.5.2. Furthermore, the improvement of PSTs' skills to calculate in a flexible manner in their minds is a response to identifying ways to strengthen what Kajander (2010) found as fragile mathematical knowledge among PSTs upon entry into tertiary institutions of teacher education. Generally, a careful selection of numbers used in the tasks triggered the invention of own calculation strategies. In addition, PSTs' ability to think and reason mathematically to invent own calculation strategies emanated from executing problem solving steps emphasised by Polya (1945). Such steps involve understanding a problem, devising a plan, carrying out a plan and looking back to reflect on steps carried out. On the other hand, although problem solving steps were carried out, some PSTs performed below average in the post-intervention test.

Post-intervention test results also do indicate that the TE had a challenge in supporting 47% of the class whose performance was below 50% (see Table 5.10). This study attributes poor performance of specific PSTs to prior knowledge of numbers and degree of reasoning with numbers (see Table 5.4 and Table 5.5), inability to calculate three-digit numbers mentally (see section 5.7.2) and limited time to respond to specific PSTs' slow pace of learning (see section 5.7.2). The TE is confident that performance in FMC could improve with intensive practice, as found in a recent study involving fourth-year PSTs whose skills improved over four weeks of practising multiplication and division strategies (Mutawah, 2016). The disparity that this study has recorded in PSTs' performance calls for more practice to achieve fluency in FMC. As a result, development of an HLT for PSTs with a slow learning pace is highly recommended.

All in all, discussion in the classroom (Gee, 2008; Kamii & Joseph, 2004; Vygotsky, 1978), prior knowledge (Bruner, 1977; Piaget, 1971; Vygotsky, 1978), conceptual and procedural understanding of FMC through problem solving (Polya, 1945; Skemp, 1976), and motivation and attitude or orientation towards learning (Bruner, 1977; Claxton et al., 2010) contributed to the development of PST mathematical knowledge (see Table 5.4, Table 5.8 and section 5.7.2). Also, the use of pictures (Kamii & Joseph, 2004; Ma, 2010; Piaget, 1964;), the use of a variety of mathematical representations (Freudenthal, 2002; Kamii & Joseph, 2004;), consideration of mathematical language or tools (Vygotsky, 1978), the use of the 4-6-1 model for encouraging PSTs to think strategically and to engage PSTs mentally (Piaget, 1951; Ernest, 1991) and the use of well-structured, contextual and realistic mathematics tasks (Bruner, 1977; Piaget & Inhelder,1973) contributed to the development of PSTs' ability to compute mentally and to teach FMC (see Appendix I).

8.5 Conclusion in terms of research questions.

i. What are the pre-service teachers' school experience and existing mental computation skills upon entry to university?

Findings of this study indicate that PSTs have different experiences in terms of how they developed FMC skills in school. Evidence from interviews indicate experience were a teacher would provide an example of how to solve a particular problem followed by an activity requiring learners to solve similar problems (see section 5.5.2). In other cases, experience of learning through problem solving was expressed (see section 5.5.2). As to existing mental computation upon entry to university, pre-intervention test results indicate that 51 % of PSTs who participated in this study have underdeveloped FMC skills upon entry to university.

ii. How did pre-service teachers develop flexible mental computation skills and beliefs at school about the learning and teaching of flexible mental computation?

Findings of this study indicate that PSTs' developed FMC skills through direct teaching of mental computation strategies prescribed by a teacher (see section 5.2 and section 5.6.2). Eventually, direct teaching of FMC strategies resulted in the belief that FMC develops through teaching FMC strategies directly (see section 5.7.2).

iii. How can pre-service teachers develop mathematical knowledge for teaching flexible mental computation skills?

Efforts to improve pre-service teachers' FMC skills were based on the hypothesis that if PSTs understand the fundamentals of calculating mentally, their computation skills and knowledge for teaching could improve. Also, effective development of PST knowledge for teaching requires establishing their existing knowledge, experience and belief about FMC teaching and learning so that misconceptions are addressed effectively. In addition, PSTs need to develop appropriate habits of mind (see section 2.12). Furthermore, teacher education programme must create the opportunity for PSTs to develop appropriate knowledge for teaching mathematics as discussed in section 2.10.

iv. Do PSTs' school experience and beliefs about FMC influence their development of FMC in a teacher education course?

Findings of this study indicate that how PSTs developed FMC in school influences their response towards innovative ideas of FMC. The researcher's general observation of reluctance to think critically to invent their own strategies during the intervention process, imply that PST expect to be taught as in school were examples were provided before carrying out a similar task (see section 6.2.1). Limitations of this study are discussed in section 8.6.

8.6 Limitations of study

Findings of this study are limited to the development of FMC skills of first-year elementary mathematics teachers based on one university satellite campus. Findings therefore cannot be generalised to all PSTs, but the HLT may be used with PSTs having underdeveloped FMC skills. Immovable seats in the learning environment used in this study restricted the application of a cooperative learning approach. Researchers need to investigate how university fixed seating arrangements for elementary mathematics teachers could be adjusted to promote the constructivist learning approach. Practically, DBR required more time than anticipated as the one-month period allocated to the study restricted further redesign of the trajectory to improve PSTs' multiplicative reasoning which is more challenging than additive reasoning. Extension of the duration of the study may have further enriched the findings of this study as found in a longitudinal study by Mutawah (2016) though conducted on final-year PSTs.

8.7 Summary of contributions

8.7.1 Theoretical contribution

This study contributes to educational research through an in-depth discussion of how problem solving teaching approach may change PST beliefs and knowledge about FMC skills development concurrently. The intervention process has demonstrated to be an effective way for PSTs to experience mathematics as a human activity as advocated for by Freudenthal, 2002 and Van den Heuvel-Panhuizen, M., & Drijvers, P., 2014. Generally, the disparity in PST performance after the intervention confirms that it takes time to learn, prior knowledge is necessary for further learning, mental engagement is critical for re-invention of calculation strategies, positive attitude towards FMC is important, modelling of appropriate teaching strategies is important as PSTs desire to teach the way they were taught, communication, individual and collaborative work is necessary for refinement of calculation strategies. The study also shows that the quality of problems PSTs are engaged in facilitate the re-invention of calculation strategies.

8.7.2 Practice-related contribution

Practically, study has contributed to the domain of FMC development at teacher education level by designing a HLT teacher educators may use to develop PSTs' knowledge for teaching FMC.

Particular contribution relates to knowledge how to promote PSTs' number knowledge, how to spark PSTs' habits of mind to think critically to invent calculation strategies, and how to use discussion to encourage PSTs to share, justify and refine calculation strategies. The study exposes different calculation strategies, materials and representations of numbers that may facilitate computation. Furthermore, the study contributes to knowledge pertaining to mathematics education curriculum of successful countries such as Singapore which may contribute to reviewing the curriculum that was used in this study.

Another contribution has been made to the elementary mathematics teacher education fraternity in terms of an HLT comprising tasks, computation strategies and models for reasoning that TEs may use to teach FMC skills constructively. This study also contributes to existing instruments that can be used to test PSTs' skills to calculate flexibly in mind and a questionnaire that TE may use to establish PSTs' beliefs and school experience. With regards to content knowledge, this study discovered that multiplication and division are operations PSTs struggled to calculate mentally just as elementary school learners do (Carpenter, Ansell, Franke, Fennema & Weisbeck, 1993). Also, problems with initial unknown proved difficult for PSTs to solve like in the study of Carpenter, Fennema, Peterson, & Carey (1988). As to in-service elementary mathematics teachers, this study has developed activities they may use to develop elementary school learners. This study also contributes to the limited discussion available pertaining to how TE could positively influence PSTs' beliefs and use their existing knowledge to teach FMC.

Generally, as a major contribution, the HLT attempts to minimise the existing disparity in teaching practice of FMCs across university campuses and other teacher preparation programmes intending to develop PSTs' FMCs (Tatto et al., 2008). Meanwhile, the study also illuminates the need for precision and focus on PST curricula to avoid compromising depth for breadth of content (Rowland et al., 2009). As to the HLT, the teaching framework may be refined further to suit specific PSTs' cognitive demands locally, nationally and internationally.

8.7.3 Methodological contribution

The use of both quantitative and qualitative research design; and DBR methodology contribute to the knowledge how a mixed methods research design connect to DBR methodology in higher education context. Access to this study could serve as a guide to researchers who may intend to use DBR as a research methodology to improve PST knowledge. Locally, attention is drawn to the need to refocus PST curriculum, how to establish PST existing perceptions and a guide TE may use to develop PSTs' knowledge for teaching FMC.

8.7 Recommendations for further research

The findings of this study strongly support the notion that TEs should explore first-year PSTs' flexible mental computation experiences, beliefs and skills, at the beginning of their mathematics course, so that TEs can respond to PST needs upon entry to university, rather than identifying their challenges in their final year of studies. Another recommendation is that educators should extend the current study to explore PSTs' flexible mental computation skills of basic calculations involving decimals, percentages and fractions to determine the extent to which PSTs may meaningfully solve problems requiring the computation of numbers in those areas. By extension, there is also a need to establish first-year PSTs' skills, beliefs and experiences of other mathematics learning domains such as algebra, geometry, measurement and data-handling. Furthermore, this study has identified the need for research on the teaching methods currently used by TEs to develop PSTs' mathematical knowledge for teaching. Also, further research is necessary to develop distinct frameworks that would support struggling PSTs to develop FMC skills according to individual pace of learning.

Further aspects to consider relate to how both school and university curriculum content can be reduced to focus on fundamental concepts and provide adequate time for mastery of key concepts to lay a strong foundation for further learning. Since PSTs are expected to apply the developed knowledge of teaching FMC, this study recommends the promotion of classroom environments appropriate for FMC development. So far, limited research on teacher education (Ball et al., 2005; Duthilleul & Allen, 2005) accentuates the need to explore the different strategies that teacher educators are currently using to develop PSTs' mathematical knowledge for teaching, and how the lecturing and teaching method (Battista, 1999; Junius, 2014) can be improved to model and implement a constructivist teaching approach.

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APPENDIX A: ETHICAL CLEARANCE LETTER - STELLENBOSCH UNIVERSITY

UNIVERSITEIT·STELLENBOSCH·UNIVERSITY
jou kennisvennoot • your knowledge partner

Approval Notice

New Application

16-May-2017
Luwango, Luiya L

Proposal #: SU-HSD-004257
Title: The development of pre service teachers mathematical knowledge for teaching flexible mental computation.

Dear Ms Luiya Luwango,

Your **New Application** received on **04-May-2017**, was reviewed
Please note the following information about your approved research proposal:

Proposal Approval Period: **16-May-2017 -15-May-2020**

Please take note of the general Investigator Responsibilities attached to this letter. You may commence with your research after complying fully with these guidelines.

Please remember to use your **proposal number** (**SU-HSD-004257**) on any documents or correspondence with the REC concerning your research proposal.

Please note that the REC has the prerogative and authority to ask further questions, seek additional information, require further modifications, or monitor the conduct of your research and the consent process.

Also note that a progress report should be submitted to the Committee before the approval period has expired if a continuation is required. The Committee will then consider the continuation of the project for a further year (if necessary).

This committee abides by the ethical norms and principles for research, established by the Declaration of Helsinki and the Guidelines for Ethical Research: Principles Structures and Processes 2004 (Department of Health). Annually a number of projects may be selected randomly for an external audit.

National Health Research Ethics Committee (NHREC) registration number REC-050411-032.

We wish you the best as you conduct your research.

If you have any questions or need further help, please contact the REC office at 218089183.

Included Documents:

Sincerely,

Clarissa Graham
REC Coordinator
Research Ethics Committee: Human Research (Humanities)

Investigator Responsibilities

Protection of Human Research Participants

Some of the general responsibilities investigators have when conducting research involving human participants are listed below:

1.Conducting the Research. You are responsible for making sure that the research is conducted according to the REC approved research protocol. You are also responsible for the actions of all your co-investigators and research staff involved with this research. You must also ensure that the research is conducted within the standards of your field of research.

2.Participant Enrollment. You may not recruit or enroll participants prior to the REC approval date or after the expiration date of REC approval. All recruitment materials for any form of media must be approved by the REC prior to their use. If you need to recruit more participants than was noted in your REC approval letter, you must submit an amendment requesting an increase in the number of participants.

3.Informed Consent. You are responsible for obtaining and documenting effective informed consent using **only** the REC-approved consent documents, and for ensuring that no human participants are involved in research prior to obtaining their informed consent. Please give all participants copies of the signed informed consent documents. Keep the originals in your secured research files for at least five (5) years.

4.Continuing Review. The REC must review and approve all REC-approved research proposals at intervals appropriate to the degree of risk but not less than once per year. There is **no grace period.** Prior to the date on which the REC approval of the research expires, **it is your responsibility to submit the continuing review report in a timely fashion to ensure a lapse in REC approval does not occur.** If REC approval of your research lapses, you must stop new participant enrollment, and contact the REC office immediately.

5.Amendments and Changes. If you wish to amend or change any aspect of your research (such as research design, interventions or procedures, number of participants, participant population, informed consent document, instruments, surveys or recruiting material), you must submit the amendment to the REC for review using the current Amendment Form. You **may not initiate** any amendments or changes to your research without first obtaining written REC review and approval. The **only exception** is when it is necessary to eliminate apparent immediate hazards to participants and the REC should be immediately informed of this necessity.

6.Adverse or Unanticipated Events. Any serious adverse events, participant complaints, and all unanticipated problems that involve risks to participants or others, as well as any research related injuries, occurring at this institution or at other performance sites must be reported to Malene Fouch within **five (5) days** of discovery of the incident. You must also report any instances of serious or continuing problems, or non-compliance with the RECs requirements for protecting human research participants. The only exception to this policy is that the death of a research participant must be reported in accordance with the Stellenbosch Universtiy Research Ethics Committee Standard Operating Procedures. All reportable events should be submitted to the REC using the Serious Adverse Event Report Form.

7.Research Record Keeping. You must keep the following research related records, at a minimum, in a secure location for a minimum of five years: the REC approved research proposal and all amendments; all informed consent documents; recruiting materials; continuing review reports; adverse or unanticipated events; and all correspondence from the REC

8.Provision of Counselling or emergency support. When a dedicated counsellor or psychologist provides support to a participant without prior REC review and approval, to the extent permitted by law, such activities will not be recognised as research nor the data used in support of research. Such cases should be indicated in the progress report or final report.

9.Final reports. When you have completed (no further participant enrollment, interactions, interventions or data analysis) or stopped work on your research, you must submit a Final Report to the REC.

10.On-Site Evaluations, Inspections, or Audits. If you are notified that your research will be reviewed or audited by the sponsor or any other external agency or any internal group, you must inform the REC immediately of the impending audit/evaluation.

APPENDIX B: ETHICAL CLEARANCE LETTER – UNIVERSITY OF NAMIBIA

UNAM
UNIVERSITY OF NAMIBIA

ETHICAL CLEARANCE CERTIFICATE

Ethical Clearance Reference Number: FOE /269/2017 **Date: 10 October, 2017**

This Ethical Clearance Certificate is issued by the University of Namibia Research Ethics Committee (UREC) in accordance with the University of Namibia's Research Ethics Policy and Guidelines. Ethical approval is given in respect of undertakings contained in the Research Project outlined below. This Certificate is issued on the recommendations of the ethical evaluation done by the Faculty/Centre/Campus Research & Publications Committee sitting with the Postgraduate Studies Committee.

Title of Project: The development of pre-service teachers' mathematical knowledge for teaching flexible mental computation.

Researcher: Luiya Luwango

Faculty: Faculty of Education /RUNDU CAMPUS

Take note of the following:

(a) Any significant changes in the conditions or undertakings outlined in the approved Proposal must be communicated to the UREC. An application to make amendments may be necessary.

(b) Any breaches of ethical undertakings or practices that have an impact on ethical conduct of the research must be reported to the UREC.

(c) The Principal Researcher must report issues of ethical compliance to the UREC (through the Chairperson of the Faculty/Centre/Campus Research & Publications Committee) at the end of the Project or as may be requested by UREC.

(d) The UREC retains the right to:

(i) Withdraw or amend this Ethical Clearance if any unethical practices (as outlined in the Research Ethics Policy) have been detected or suspected,

(ii) Request for an ethical compliance report at any point during the course of the research.

UREC wishes you the best in your research.

Prof. P. Odonkor: UREC Chairperson

Ms. P. Claassen: UREC Secretary

APPENDIX C: STELLENBOSCH CONSENT TO PARTICIPATE IN RESEARCH

UNIVERSITEIT•STELLENBOSCH•UNIVERSITY
jou kennisvennoot • your knowledge partner

STELLENBOSCH UNIVERSITY
CONSENT TO PARTICIPATE IN RESEARCH

Research Title: **The development of pre-service teachers' mathematical knowledge for teaching flexible mental computation**

You are asked to participate in a research study conducted by Mrs. Luiya Luwango possessing the qualifications: Master degree in mathematics education (MEd), Honours degree in mathematics education (Bed-hons), Further Diploma in mathematics education (FDE) with Rhodes University and a Basic Education Teachers' Diploma (BETD) with Rundu College of Education (RCE); currently in the department of curriculum studies at Stellenbosch University. The results of the study will contribute to the finalization of my PhD studies. You were selected as a possible participant in this study because you have enrolled for the BEd honours course in early childhood and lower primary education; and part of your course is to understand how to develop learners' mental computation skills.

1. PURPOSE OF THE STUDY

To establish pre-service teachers' mathematical knowledge, beliefs and experiences upon entry to university; understand how teacher educators could develop pre-service teachers' mathematical knowledge to teach flexible mental computation and to develop a framework for developing mental computation skills of early childhood pre-service teachers.

2. PROCEDURES

If you volunteer to participate in this study, we would ask you to do the following things:

Participate in a survey based on your school mathematics experience and beliefs on how mathematics is best taught and learned. You will also participate in a baseline test that will not count for anything else other than to establish your understanding of flexible mental computation upon entry to university and another test after the intervention. The test outcomes will inform the design of a flexible mental computation hypothetical learning trajectory for teaching pre-service teachers. Fifteen students will be selected to participate in one interview before the intervention and one interview after the intervention in which you could possibly participate.

The entire process will prevail on Rundu Campus during your normal class hours except for the interviews that will be conducted in your spare time. The interviews will be tape recorded with your permission and will not be used against you but for the purpose of the study and your name will not be disclosed to anyone else. The intervention is planned to prevail over a month between April and May 2017, where you will be engaged in normal classroom activities as stipulated in your course outline.

3. POTENTIAL RISKS AND DISCOMFORTS

Although the empirical study will be conducted during normal classes, it is planned not to interfere with students' further learning of the other learning exit outcomes outlined in the course outline. The study will be limited in time (one month) to allow students to cover the rest of the learning outcomes stipulated in the course outline.

4. POTENTIAL BENEFITS TO SUBJECTS AND/OR TO SOCIETY

The research will benefit you, as a prospective teacher, in terms of improving your ability to compute mentally in a flexible way; also with skills how to develop learners' flexible mental computation skills in school. This study will also benefit other UNAM students and lecturers in the early childhood and lower primary department.

5. PAYMENT FOR PARTICIPATION

There is no payment involved.

6. CONFIDENTIALITY

Any information that is obtained in connection with this study and that can be identified with you will remain confidential and will be disclosed only with your permission or as required by law. Confidentiality will be maintained by means of proper recording, reporting and safe keeping of the data by the researcher. To benefit the society, data will be stored in the researcher's file and findings will be shared with the supervisor in charge of the study, the Stellenbosch examiners and other academics without identifying any student by name.

The intervention is to be video-taped and the interviews audio-recorded to assist the researcher when transcribing the data, and you have the right to review and edit the tapes. The video tape and interview tapes will only be accessible to the researcher, and will be erased as per Stellenbosch University requirement.

7. PARTICIPATION AND WITHDRAWAL

You can choose whether to be in this study or not. If you volunteer to be in this study, you may withdraw at any time without consequences of any kind. You may also refuse to answer any questions you don't want to answer and still remain in the study. The investigator may withdraw you from this research if circumstances arise which warrant doing so.

8. IDENTIFICATION OF INVESTIGATORS

If you have any questions or concerns about the research, please feel free to contact Mrs. Luiya Luwango at [lluwango@unam.na; 0812881377] or Dr. Hellena Wessels at [hwessels@su.ac.za; 0027218082286].

9. RIGHTS OF RESEARCH SUBJECTS

You may withdraw your consent at any time and discontinue participation without penalty. You are not waiving any legal claims, rights or remedies because of your participation in this research study. If you have questions regarding your rights as a research subject, contact MsMaléneFouché [mfouche@sun.ac.za; 021 808 4622] at the Division for Research Development.

SIGNATURE OF RESEARCH SUBJECT OR LEGAL REPRESENTATIVE

The information above was described to [*me/the subject/the participant]* by [*name of relevant person]* in [*Afrikaans/English/Xhosa/other]* and [*I am/the subject is/the participant is]* in command of this language or it was satisfactorily translated to [*me/him/her*]. [*I/the participant/the subject]* was given the opportunity to ask questions and these questions were answered to *[my/his/her]* satisfaction.

[I hereby consent voluntarily to participate in this study/I hereby consent that the subject/participant may participate in this study.] I have been given a copy of this form.

____
Name of Subject/Participant

____
Name of Legal Representative (if applicable)

____ ______________
Signature of Subject/Participant or Legal Representative **Date**

SIGNATURE OF INVESTIGATOR

I declare that I explained the information given in this document to __________________ [*name of the subject/participant]* and/or [his/her] representative ____________________ [*name of the representative]. [He*/she] was encouraged and given ample time to ask me any questions. This conversation was conducted in [*Afrikaans/*English/*Xhosa/*Other*] and [*no translator was used/this conversation was translated into* ___________ by ______________________].

____ ______________
Signature of Investigator **Date**

APPENDIX D: QUESTIONNAIRE

Questionnaire

Name: **Duration: 30 minutes**

Student number:

Note:
- Please provide your name and student number correctly (use your student card to verify your student number) for further consultations.
- This is not a test but a collection of information on your school experience and belief on learning of flexible mental computation.
- Please rate yourself honestly on the scale below by crossing the box that best describes your school experience (in part 1) and your belief (in part 2).
- Listen to the example indicating how you should complete the questionnaire.
- Please read each statement carefully.

1. PRE-SERVICE TEACHERS' SCHOOL EXPERIENCE AND UNDERSTANDING OF FLEXIBLE MENTAL COMPUTATION (Cross in the appropriate box below).

i) I am a:

1	2
male	female

ii) My age falls within the following age category:

1	2	3	4	5
Below 25	25-30	31-40	41-50	51-60

iii) Please complete the following sentence using your own understanding.
Flexible mental computation (mental calculation) is:

....

....

....

....

		Strongly disagree	**Disagree**	**Agree**	**Strongly agree**
A.	I developed flexible mental computation skills in school.	**1**	**2**	**3**	**4**

B.	I understood how to perform calculations using the column method before using own calculation methods.	1	2	3	4
C.	At school I was taught different methods to perform mental calculations.	1	2	3	4
D.	I did not develop flexible mental computation skills in school.	1	2	3	4
E.	Should flexible mental computation skills be developed before standard algorithm (column method)?	1	2	3	4
F.	I am able to perform calculations in my mind without applying standard algorithm.	1	2	3	4
G.	My mathematics teachers encouraged individual learners to communicate their calculation methods with the entire class.	1	2	3	4
H.	My school teachers used counters, blocks and other teaching material (manipulatives) to teach us basic number facts.	1	2	3	4

2. PRE-SERVICE TEACHERS' BELIEFS ON HOW FLEXIBLE MENTAL COMPUTATION DEVELOPS IN LEARNERS (Cross in the appropriate box below).

		Strongly disagree	Disagree	Agree	Strongly agree
A.	Learners should carry out mental calculations using strategies prescribed by the teacher.	1	2	3	4
B.	Flexible mental computation is when a learner memorises methods taught by the teacher during lessons to mentally solve problems.	1	2	3	4
C.	How a teacher understands flexible mental computation determines how he/she would develop learners' mental computation skills.	1	2	3	4
D.	It is not important for all learners to develop flexible mental computation skills since calculators are available.	1	2	3	4
E.	For learners to develop flexible mental computation skills a teacher must provide examples first and then give plenty of exercises for learners to practice.	1	2	3	4
F.	Primary school teachers should encourage the class to communicate their calculation methods with the entire class to develop flexible mental computation strategies.	1	2	3	4
		Strongly disagree	Disagree	Agree	Strongly agree
G.	The ability to perform calculations without any external support develops through:				
	i) Memorising the multiplication table.	1	2	3	4
	ii) Calculating with pencil and paper.	1	2	3	4

	iii) Solving problems mentally without any hints from the teacher.	1	2	3	4
H.	Some computations are faster to calculate mentally than using pencil and paper approach.	1	2	3	4
I.	Flexible mental computation contributes substantially to number sense development.	1	2	3	4
J.	Flexible mental calculations can be performed in many ways.	1	2	3	4
K.	The ability to work out answers correctly mentally has a great contribution towards number sense development.	1	2	3	4
L.	It does not matter which calculation strategy you use as long as the answer is correct.	1	2	3	4
M.	There are calculation methods that are more efficient than others.	1	2	3	4
N.	Mental computation using standard methods contribute only a little to number sense development.	1	2	3	4
O.	It is easier to calculate using written calculation methods than to calculate mentally to solve a real life problem.	1	2	3	4
P.	Effective development of flexible mental computation requires memorization of basic number facts such as 2+3=5.	1	2	3	4
Q.	Effective development of flexible mental computation requires memorization of number relationships such as 3 is 2 more than 1.	1	2	3	4
R.	Learners attain proficiency in mental calculation when they perform calculations using procedures invented by themselves.	1	2	3	4
S.	Mental computation, where a standard algorithm is performed in mind, has little contribution towards number sense development.	1	2	3	4

APPENDIX E PRE-INTERVIEW GUIDE

PRE-INTERVIEW GUIDE

PRE-SERVICE TEACHERS' EXPERIENCES ON FLEXIBLE MENTAL COMPUTATION

Name:

Student number:

Part A:

I have four questions below that I selected from the first test you wrote. I would like you to work on one question at a time. With these I would like you to articulate your thinking and reasoning to clarify the steps you carried out in your head when you calculated the questions in the test. The main idea is to clarify your thoughts and understanding in written form. I wish to tape record our conversation that will serve the purpose of informing the development of a learning trajectory for elementary school pre-service teachers and not for any other reasons not stated here; do you permit me to tape record our conversation? The tape is going to be played after the interviews for you to listen to it.

Instruction:

Researcher: Please work out the following question mentally. Explain how you worked out the answer. Explain why you worked it out that way. Apart from the strategy you have used now, how else would you have calculated the same question?

1. $23 + 18 + 37$
2. $151 - 98$
3. 32×15
4. $120 \div 6$

Part B: (THE PURPOSE IS TO CLARIFY STUDENTS' SURVEY ANSWERS AND RESEARCHER WILL USE STUDENTS' SURVEY ANSWERS AS REFERENCE)

1. Up to what grade did you learn mathematics?
2. What comes to your mind when you hear the word "flexible mental computation"?
3. What is your view towards the need to develop learners' flexible mental computation skills?
4. What is your view towards the need to develop teachers' flexible mental computation skills?
5. What is the lowest grade you recall learning to calculate in your head?
6. How did you learn to perform calculations mentally in the grade mentioned above (5.) or in primary school?
7. What is the highest grade you recall to have learned to calculate in your head?
8. How did you learn to calculate mentally in the grade mentioned above (7.) or in high school?
9. What was the teacher's role when you learned to calculate in your head without a calculator or written algorithms?
10. What was your role when you developed flexible mental computation skills?
11. What mathematical representations were used to develop flexible mental computation skills?
12. What manipulatives were used to develop flexible mental computation skills?
13. How best could teachers assist learners to develop flexible mental computation skills?

APPENDIX F POST-INTERVIEW GUIDE

POST-INTERVIEW GUIDE

PRE-SERVICE TEACHERS' KNOWLEDGE AND EXPERIENCE OF FLEXIBLE MENTAL COMPUTATION

Name:

Student number:

Part A:

I have four questions below that I selected from the second test you wrote. I would like you to work on one question at a time. With these I would like you to express your thinking and reasoning to clarify the steps you carried out in your head when you calculated the questions in the test. The main idea is to clarify your thoughts and understanding in written form. I wish to tape record our conversation that is going to be used to understand the impact of the intervention on your knowledge concerning flexible mental computation and your experience of the intervention. Such information will confirm the reliability and validity of the learning trajectory developed for teacher educators to develop pre-service teachers' mathematical knowledge to teach flexible mental computation skills. The information will not serve for reasons other than the ones stated here. Do you permit me to tape record our conversation? You may listen to the tape after the interviews.

Instruction:

Researcher: Please work out the following question mentally. Write down your answer. Explain how you worked that out in your head. Explain why you worked it out that way. Apart from the strategy you have used now, how else would you have calculated the same question?

1. $37 + 26$
2. $82 - 43$
3. 14×15
4. $468 \div 12$

Part B:

1. Reflect on your ability to perform calculations flexibly before the intervention?
2. Reflect on your ability to perform calculations flexibly after the intervention?
3. From the intervention, what exactly has impacted your knowledge about flexible mental computation skills? Elaborate.
4. In what way does the intervention contribute towards your preparation as a mathematics teacher? Or (mention the skills you have acquired from the intervention).
5. What is your view towards the need to develop both teachers and learners' flexible mental computation skills?

APPENDIX G: PRE- AND POST-INTERVENTION TEST

PRE- INTERVENTION TEST

PRE-SERVICE TEACHERS' EXISTING MENTAL COMPUTATION SKILLS

(This copy will not be provided to the students but is for the researcher to work from).

Mentally calculate the following without using a calculator or pen and paper method. Write down the answer only, in the space provided on the answer sheet. Please remember to provide your name and student number on your test paper. The test paper will not be handed out to you rather each question will be displayed on the screen for 20 seconds and read out once only. Your results will be kept anonymous and will not be reported in a derogatory way.

1. $5 + 7$
2. $19 + 15$
3. $23 + 18 + 37$
4. $23 - 16$
5. $151 – 98$
6. $563 – 292$
7. 4×50
8. 32×15
9. 16×25
10. $16 \div 4$
11. $120 \div 6$
12. 14×8
13. $59 + \square = 82$
14. $85 - \square = 67$
15. $\square - 38 = 89$
16. $192 \div 8$
17. $\square \div 15 = 20$
18. $15 \times \square = 48$
19. 250×2
20. $2024 – 1999$
21. $799 - 51$

POST-INTERVENTION TEST

PRE-SERVICE TEACHERS' MENTAL COMPUTATION SKILLS

(This copy will not be provided to the students but is for the researcher to work from).

Mentally calculate the following without using a calculator or pen and paper method. Write down the answer only, in the space provided on the answer sheet. Please remember to provide your name and student number on your test paper. The test paper will not be handed out to you rather each question will be displayed on the screen for 20 seconds and read out once only. Your results will be kept anonymous and will not be reported in a derogatory way.

1. 22 - 9
2. 24 + 47 + 16
3. 125 + 38
4. 274 - 46
5. 14 × 15
6. 16 × 199
7. 72 ÷ 6
8. 192 ÷ 32
9. 312 - 7
10. □ + 26 = 63
11. 1 245 + 2 136
12. □ - 43 = 39
13. 94 - 87
14. 800 -169
15. 1005 - 995
16. 496 ÷ □ = 62
17. 368 ÷ 16
18. 468 ÷ 12
19. 16 × 25
20. 15 × 9
21. □ × 12 = 36

APPENDIX H: A HYPOTHETICAL LEARNING TRAJECTORYFOR DEVELOPING NUMBER COMPUTATION FLUENCY IN PRE-SERVICE TEACHERS

<table>
<tr><th>OPERATION</th><th colspan="2">Anticipated computation strategies</th><th>INSTRUCTIONAL ACTIVITIES</th><th>Models for reasoning</th></tr>
<tr><td>ADDITION</td><td>Counting on and backwards in 2, 5, 10 and 100.</td><td><table>
<tr><td>2</td><td>4</td><td>6</td><td>8</td><td>10</td></tr>
<tr><td>12</td><td>14</td><td>16</td><td>18</td><td>20</td></tr>
<tr><td>22</td><td>24</td><td>26</td><td>28</td><td>30</td></tr>
<tr><td>32</td><td>34</td><td>36</td><td>38</td><td>40</td></tr>
<tr><td>42</td><td>44</td><td>46</td><td>48</td><td>50</td></tr>
<tr><td>52</td><td>54</td><td>56</td><td>58</td><td>60</td></tr>
<tr><td>62</td><td>64</td><td>66</td><td>68</td><td>70</td></tr>
<tr><td>72</td><td>74</td><td>76</td><td>78</td><td>80</td></tr>
<tr><td>82</td><td>84</td><td>86</td><td>88</td><td>90</td></tr>
<tr><td>92</td><td>94</td><td>96</td><td>98</td><td>100</td></tr>
</table></td><td>a) Count forward in 2s to 100; and from any given number (Develop a number chart of 2s to 100).

b) Count forward in 5s to 100; and from any given number (Develop a number chart of 5s to 100).

c) Count forward in 10s to 100; and from any given number (Develop a number chart of 10s to 100).

d) Count forward in 100s from 100 to 2 000; and from any given number (Develop a number chart of 100s to 2000).</td><td>Number chart</td></tr>
</table>

100	200	300	400	500
600	700	800	900	1000
1100	1200	1300	1400	1500
1600	1700	1800	1900	2000

5	10	15	20	25
30	35	40	45	50
55	60	65	70	75
80	85	90	95	100

10	20	30	40	50
60	70	80	90	100

Investigation

Plastic one	Plastic two	Plastic one	Plastic two
1	19	11	9
2	18	12	8
3	17	13	7
4	16	14	6
5	15	15	5

Investigation

Suzy has twenty books that she wants to put in two separate plastic bags. What is the possible number of books she can put in each plastic bag?

Number display

6	14	16	4
7	13	17	3
8	12	18	2
9	11	19	1
10	10		

Counting on in: Connecting repeated addition to multiplication

×1	**×2**	×3	×4	×5	×6	×7	×8	×9	×10	×11
11	22	33	44	55	66	77	88	99	110	121
12	24	36	48	60	72	84	96	108	120	
13	26	39	52	65	78	91	104	117	130	
14	28	42	56	70	84	98	112	126		
15	30	45	60	75	90	105	120			
16	32	64	80	96	112	128				
17	34	51	68	85	102	119				
18	36	54	72	90	108	126				
19	38	57	76	95	114	133				
20	40	60	80	100	120					
21	42	63	84	105	126					

Counting on to 130

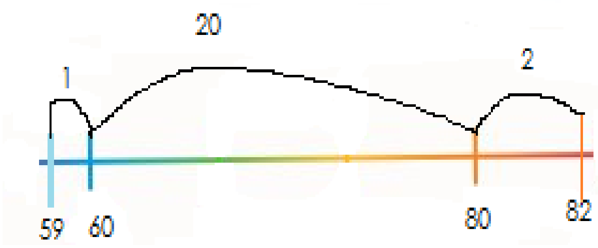
Repeated addition connected to multiplication.

Number chart

22	44	66	88	110						
23	46	69	92	115						
24	48	72	96	120						
25	50	75	100	125						
26	52	78	104	130						
27	54	81	108							
28	56	84	112	140						
29	58	87	116							
30	60	90	120							
31	62	93	124							
32	64	96	128							

Topic	Activity	Task	Resources
Counting all, Counting on, near doubles and making tens	**Task** **orNear doubles** **Counting all** 8+9=8+8+1 8+9 = 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17 =16+1 **or** (9+9)-1=18-1 **Counting on** =17	**Task** 1. Bency has eight apples; Jane gave her nine more apples. How many apples does Bency have now?	-Double ten frame Empty number line

	extend to 15+14, 25+24, 75+74, 105+104	8...9,10,11,12,13,14,15,16,17 or (8+2)+(9+1) Or 10+10=20 9...10,11,12,13,14,15,16,17 20-3=17 or 8=7+1; 8+9=7+(1+9) or 9=7+2; 8+9=7+(2+8) = 7+10 = 7+10 = 17 =17		
	Commutativity	27+35+13=27+13+35 = 40+35 =40+30+5 =75	2. Betty has 27 bananas, Sandra has 35 bananas and Cathy has 13 bananas. How many bananas do they have altogether?	-Abacus
	Commutativity **and** **Making tens**	24+47+16 47+24+16 =47+40 =40+40+7 =87	3. Moiny bought sandals costing N$ 24.00, a top costing N$ 47.00 and toothpaste costing N$ 16.00. What is the cost of all three items together?	Empty number line

	Skip counting		4. Aunt Sofia had 59 cattle in her kraal and now she has 82 cattle in her kraal after buying more cattle. How many more cattle did she buy afterwards?	Empty number line
	Decomposition method and reversibility relation between + and -	$\square$ + 219=367 or 367-(200+19) 367-219=(300+60+7)-(200+10+9) 167-19 =(300-200)+(60-10)+(7-9) (160+7)-19 =100+50+(-2) (160-19)+7 =100+48 141+7=148 =148	5. You have some money and dad gives you N$ 219.00 more; now you have N$ 367.00. How much money did you have before?	Empty number line

		Homework	Homework	
	Making landmarks or friendly numbers (e.g. 5, multiples of 10 & 25) and using reversibility property	☐ + 58=91 or (91+2)-(58+2) or from 58 to 60=2 91-58=(91-60)+2 = 93-60 from 60 to 90=30 =31+2 =33 from 90 to 91=1 =33 91-58=2+30+1 =33	1. Ann drove some kilometres before resting and drove another 58 km after resting; in total she drove 91 kilometres. What is the total distance she drove before resting?	Empty number line
	Using decomposition and doubles anchor with 1 less	118+17=100+18+17 =100+(18+18)-1 =100+36-1 =136-1; =135	2. 118 learners are in a dining hall, 17 more learners arrive in the dining hall. How many learners are in the dining hall now?	Empty number line
	Breaking each number into its place value	24+□=62 62-24=(60+2)-(20+2+2) =(60+2-2)-(20+2) =60-(20+2) 60-20=40 40-2=38	3. Eli has attended 24 weeks of his course already; he has some weeks left to complete his 62 week course. How many more weeks are left for him to complete the course?	Empty number line
	Compensation	136+344 = (135+1)+ 344	4. A supermarket received 136 bags of potatoes yesterday and 344 bags of potatoes today. What is the sum of	Empty number line

		= 135+(1+344) =135+345 =(100+35)+(300+45) =100+300+35+45 =400+35+5+40 =400+40+40 =480	the potato bags received over the two days?	
	Adding up in chunks	17+□=45 =45-17 =45-(10+7) (40+5)-10=35 35-7=30+5-7 =23+5 =28	5. Two buses are transporting learners to town. Bus A has 45 learners and bus B has 17 learners less than bus A. How many learners are on bus B?	Empty number line

		Diagnostic test Test item 1. 6 + 8 2. 16 + 5 3. 17 + 15 4. 38 + 37 5. □ + 42=71 6. 19 + 18 7. 53 + 28 8. 48 + □=73 9. 78 + 23 + 12		

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<tr><td>10.</td><td>27 + 14 + 13 + 26</td></tr>
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<tr>
<td>SUBTRACTION</td>
<td>Subtracting 2, 5, 10 and 100 by counting back from specific numbers</td>
<td>
<table>
<tr><td>100</td><td>98</td><td>96</td><td>94</td><td>92</td></tr>
<tr><td>90</td><td>88</td><td>86</td><td>84</td><td>82</td></tr>
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<table>
<tr><td>100</td><td>95</td><td>90</td><td>85</td><td>80</td></tr>
<tr><td>75</td><td>70</td><td>65</td><td>60</td><td>55</td></tr>
</table>
</td>
<td>a) Count backwards in 2s from 100 to 0; and from any given number.

b) Count backwards in 5s from 100 to 0; and from any given number.

c) Count backwards in 10s from 1000 to 0; and from any given number.

d) Count backwards in 100s from 2000 to 0; and from any given number.</td>
<td>Number chart</td>
</tr>
</table>

50	45	40	35	30
25	20	15	10	5

100	90	80	70	60
50	40	30	20	10

2000	1900	1800	1700	1600
1500	1400	1300	1200	1100
1000	900	800	700	600
500	400	300	200	100

	Adding up	**Task** Adding from the subtrahend(7) to the minuend(14) to work out 14-7 Using a number line 7...8,9,10,11,12,13,14 (1+1+1+1+1+1+1)=7 or 7+**3**=10 10+**4**=14 **3**+**4**=**7** **or** 7+7=14	1. John has 14 minutes to complete a task, he uses up 7 minutes. How many more minutes are left for him to complete the task?	Empty number line.
	Removal or counting	65-32 = 65-(10+10+10+2)(on a number line) = 33 or 65-32 65-(30+2) 65-30=35 35-2=33 or 65-32 (10+10+10+10+10+10	2. Andrew bought a shirt costing N$32.00; he had N$ 65.00 in his purse. How much money remained in his purse after paying for the shirt?	Empty number line.

		+1+1+1+1+1) (10+10+10+~~10+10+10~~ +1+1+1+~~1+1~~) 10+10+10+3=33		
	Adjusting one number to create an easier number	151-96 151-(96+4) 151-100=51 51+4=55	3. Jacob and Alfonsina raised funds for their end year party. Jacob raised N$ 96.00 and Alfonsina raised N$ 151. How much more money does Jacob need to earn to catch up with Alfonsina?	Empty number line.
	Compensation, decomposition and friendly numbers	□ – 59=73 73+59=(72+1)+59 =72+60 =70+2+60 or 70+2+30+30	4. A teacher had some books on her table; he now has 73 books after giving out 59	Empty number line

		=60+10+2+60 70+30+2+30 =60+60+10+2 100+32=132 =120+10+2 =132	books. How many books were on his table before?	
	Keeping a constant difference (Adjusting both numbers)	999-□=654 999-654 (999+1)-(654+1) 1000-655=345	5. Karen owes Nambu N$ 999.00 and he pays back an amount of money and has N$ 654.00 outstanding. How much money did Karen pay?	Empty number line

		Homework	**Homework**	
	Using landmarks or friendly numbers and decomposition	127-□=89 127-89=(127-90)+1 =(100+27)-90+1 =100-90+27+1 =10+27+1 =38	1. A school had 127 learners at the beginning of the year of which some were grade 12 learners. All the grade 12 learners passed their end year examinations and now the school has 89 learners. How many learners were in grade 12?	
	Keeping a constant difference	123-64 119-60 =59	2. A store has 123 tables and sells some of the tables; now it has 64 tables left in store. How many tables were sold?	
	Adjusting one number to create an easier number	800-169 (800-170)+1 800=600+200; 600+(200-170)+1 =600+30+1 =631	3. A farmer has 800 goats and slaughters 169 goats. What is the total number of goats he has now?	

	Adjusting one number to create an easier number	563-271 =563-200-70-1 =363-70-1 =362-70 =302-10 =300-10+2 =290+2 =292	4. The reading on a water meter is 563 after 271 units were consumed. What is the previous water meter reading?	
		☐-96=256 256+96=(250+2+4)+96 250+2+100 =352	5. Two trucks are full of water melons; a white truck has some water melons which is 96 water melons more than the green truck with 256 water melons. How many water melons are on the white truck?	
		Diagnostic test 1. 94-8 2. □-43=39 3. 56-□=18 4. 47-28 5. 316-154 6. 185-67 7. 281-148 8. 563-292 9. 450-226 10. 2024-1999		
Multiplication	**Multiplying by:** **2, 3, 4, 6 and 8**	**Investigation**	**Investigation**	Number chart

<table>
<tr>
<td>Building on foundation
×3, ×4, ×6, ×9, ×8, ×7</td>
<td>Discovering patterns</td>
<td>
<table>
<tr><td></td><td>× 2</td><td>×3</td><td>×4</td><td>×6</td><td>×8</td><td>×9</td><td>×20</td><td>×25</td></tr>
<tr><td>5</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>6</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>7</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>8</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>9</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>12</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>15</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>16</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>19</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>
</td>
<td>Distributive property:
$9 \times 5 = (5+4)\times5$
$=(5\times5)+(4\times5)$
Associative property:
$6\times5=2\times3\times5$
$=(2\times3)\times5$ or $=2\times(3\times5)$
Commutative property:
$7\times3 = 3\times7$
$3+3+3+3+3+3+3=7+7+7$</td>
<td></td>
</tr>
</table>

<table>
<tr>
<td>Foundation facts (skip counting)
×2, ×10, ×5, ×1, ×0</td>
<td>Multiplying by:
10, 5 & 15

Discovering patterns</td>
<td>
<table>
<tr><th></th><th>× 10</th><th>× 5</th><th>× 15</th></tr>
<tr><td>2</td><td></td><td></td><td></td></tr>
<tr><td>3</td><td></td><td></td><td></td></tr>
<tr><td>4</td><td></td><td></td><td></td></tr>
<tr><td>10</td><td></td><td></td><td></td></tr>
<tr><td>12</td><td></td><td></td><td></td></tr>
<tr><td>16</td><td></td><td></td><td></td></tr>
<tr><td>17</td><td></td><td></td><td></td></tr>
</table>
Explore the following:
2 × 3=
20 ×30=
4×7=
40×70
8×9=
80×90=
</td>
<td>- Multiply the following numbers by 10, 5 and 15;
- Analyse the products and write down the relationship you have discovered between the products.

Explore the following:
2 × 3=
20 ×30=
4×7=
40×70
8×9=
80×90=
25×4=
250×40=</td>
<td>Ten and five frame; Pictures of tomatoes and oranges.</td>
</tr>
</table>

		25×4= 250×40=		
	Doubling (two equal groups of objects)	**Task** 9 × 4 =4×9 (9+9)+(9+9)=18+18 double, double to multiply by 4. or 10×4-4 =40-4 36	1. There are 9 dogs in a house. How many legs are there altogether?	Pictures of dogs.
	Decomposition or Doubling three times when multiplying by eight	14=10+4; (10×8) + (4×8) = 80 + 32 = 112 or 14 × 8 = 14 × (2×2×2) 14 ×2= 28 28×2=56 56×2=112	2. Jane sold 14 packets of oranges. There are 8 oranges in each packet. How many oranges did she sell altogether?	Pictures of bags of four oranges.
	Using friendly numbers or partial quotients or decomposition into prime factors or known facts	(6×10)+(6×5)=60+30=90 Or 6×15 = 2×3×15; 2×15=30; 30×3=90 6×15 = (15+15)+(15+15)+(15+15) =(2×15)+ (2×15)+(2×15) = 3(30) =90	3. A mini bus transported a group of learners from school to town. It carried 15 learners per trip for 6 trips. How many learners were transported altogether?	6×15 Array Model

	Repeated addition	56 + 56 = 112; 112 + 56 = 168; 168 + 56 = 224 or 56 + 56 = 112; 112 + 112 = 224	4. Mimi bought a number of boxes of apples; there are 56 apples in each box. She has 224 apples in total, how many boxes of apples did Mimi buy?	Number chart
	Decomposition	200 + 24 50 + 6 200 ÷ 50 = 4; 24÷6=4 224 ÷ 56 = 4 boxes of apples		
	Partial quotients (using friendly multipliers – 10, 5 or 2)	15 × □ = 420 15×10=150 or 15×10=150; 15×10=150; 15×10=150 15×10=150 150+150+150=450 420-300=120 450 – 30 = 420 120÷15=8 = 10+10+ 8 10+10+8=28 learners per class = 28 learners per class	5. A school having 15 classrooms allocated an equal number of learners to each class after registering 420 learners. How many learners were allocated to one classroom?	Array Model

		Homework	**Homework**	
	Partial products or **Friendly numbers**	$9 \times 12 = (9 \times 10) + (9 \times 2)$ $= 108$ $9 \times 12 = (10 \times 12) - 12$ $= 120 - 12$ $= 108$	1. We have nine dozen eggs. How many eggs do we have in total?	Dozen egg box picture
	Doubling and halving **16×25**	16×25 $\div 2$ $\times 2$ 8×50 $\div 2$ $\times 2$	2. Comaly sold sixteen 5Kg bags of potatoes at N$ 25.00 per bag. What is the total amount of money he collected for the sixteen bags?	

		4×100 $\div 2$ { } $\times 2$ 2×200 $=400$		
	Partial quotients (using friendly multipliers – 10, 5 or 2)	14×□=210; 14×10=140 14×<u>5=70</u> 14×15=210	3. Hausiku bought 14 books each costing the same amount of money. He paid N$ 210.00 for all the books; calculate the price he paid for each book.	
	Breaking factors into smaller factors (create a one-digit multiplier)	16×25 $(4 \times 25) + (4 \times 25) + (4 \times 25) + (4 \times 25) = 100+100+100+100$ $= 400$ Also $16 \times 25 = 4 \times (4 \times 25)$ $= 4 \times 100$ $= 400$ Count forward in 100s four times.	4. A particular region has 16 pre-primary classrooms with 25 learners per classroom. How many pre-primary learners are in the 16 classrooms altogether?	

<table>
<tr>
<td></td>
<td>Friendly numbers and known facts</td>
<td>□ × 199 = 995
5×200=1000</td>
<td>5. A number of learners paid each N$ 199.00 for a class party. A total amount of N$ 995.00 is collected, how many learners have paid for the party?</td>
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<tr>
<td></td>
<td></td>
<td>Diagnostic test
1. 7×4
2. 12×9
3. 19×3
4. □×9=72
5. 31×□=310
6. 24×5
7. 14×8
8. 13×12
9. 4×56
10. 28×15</td>
<td></td>
<td></td>
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<tr>
<td>Division</td>
<td>Doubling and Halving</td>
<td>Investigation
Number relationship and relationship between multiplication and division.
<table><tr><td></td><td>Doubling</td><td></td><td>Doubling</td><td></td><td>Halving</td><td></td><td>Halving</td></tr></table></td>
<td>Investigation
Relationship between multiplication and division
5×2=10</td>
<td>Number chart</td>
</tr>
</table>

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<table>
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<tr><td>65</td><td></td><td>97</td><td></td><td>82</td><td></td><td>62</td><td></td></tr>
</table>
</td>
<td>10÷2=5

10÷5=2</td>
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<td></td>
<td>Repeated subtraction

30÷5</td>
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<tr><td>30 ÷ 5
30-5=25
25-5=20
20-5=15
15-5=10
10-5=5
5-5=0</td><td><u>Quicker way</u>

30-10=20
20-10=10
10-10=0</td><td>
Groups of five
2
2
<u>2</u>
<u>6</u></td></tr>
</table>
</td>
<td>1. Hakwa has 30 tomatoes. He plans to put 5 tomatoes in a plastic bag. How many plastic bags does Hakwa need?</td>
<td>Pictures of tomatoes pasted on three A3 size ten frames.</td>
</tr>
</table>

	Sharing/dealing out (fair sharing)	120 = (10 + 10 +10 + 10 + 10 + 10) + (10 + 10 +10 + 10 + 10 + 10) 120= 20 + 20 + 20 + 20+ 20 + 20 20 sweets per person	2. Nguni has 120 sweets. He plans to share the sweets with his friends. How many sweets will each person receive if they are 6 altogether?	Number line
	Multiplying up (multiplying by tens and twos)	10×6=60 5×6=30 1×6=6 10+5+1=16	3. A farmer has 96 seedlings to plant in 6 rows, how many seedlings will each row have?	Number chart
	Proportional reasoning	112=70+42 70=7×10 42=7×6	4. A farmer has 112 seedlings to plant in a number of rows; she wants to plant 7 seedlings in each row. How many rows will she have?	Empty number line

		112÷7=16		
	Counting on in 500s	500, 1000, 1500, 2000, 2500, 3000 = 6 packages	5. You have N$ 3000.00 and you want to buy fabric costing N $ 500.00 per package, how many packages are you going to buy?	Empty number line
		Homework	**Homework**	
	Decomposition into friendly numbers	76=60+10+6 60÷6=10 10÷6=1 remainder 4 6÷6=1	1. If you have 76 sweets to distribute equally among 6 learners, how many sweets will each learner receive and how many sweets will remain.	

		76÷6= 12 remainder 4		
	Counting on	□÷19=5 19+19+19+19+19=95	2. Muronga had an amount of money that he shared with his 19 team members. If each member received N$ 5.00; what is the total amount shared?	
	Using known facts and a friendly number ten	243÷3 (24×10)+3 24÷3=8 8×10=80 3÷3=1 243÷3=81	3. A total of 243 sheep were distributed among three households. How many sheep did each household receive?	

	Proportional reasoning **Multiplying up (multiplying by tens and fives)**	192 ÷ 32 ÷2↓ ↓÷2 96 ÷ 16 ÷2 ↓ ↓÷2 48 ÷ 8 ÷2 ↓ ↓ ÷2 24 ÷ 4 ÷2 ↓ ↓ ÷2 12 ÷ 2 = 6 32×10=320 32×5=160 192=160+32 32×6=192 6 calendars per class.	4. Martha has 192 school calendars to distribute among 32 classrooms. How many school calendars will each class receive?	

	Decomposition using known facts and friendly numbers	9×□=108 108=90+18 90=9×10 18=9×2 108÷9=12	5. You saved N$108.00 in a period of 9 months, on average, how much did you save per month?	
		Diagnostic test 1. 8÷2= 2. 18÷6= 3. 28 ÷ □ =7 4. 99 ÷ 9 = 5. □ ÷ 3 =25 6. 48 ÷ 12= 7. 72 ÷24= 8. 56÷□=14 9. □÷17=5 10. 320 ÷ 3		

APPENDIX I: JOURNAL ENTRIES BEFORE AND DURING THE INTERVENTION PROCESS

	Views before the intervention	Views on impact of the intervention
1.	… My initial thoughts were “primary mathematics is easy” but then as more problems were presented I got to learn that I was not as good as I thought I was. Multiplication and division of large numbers … gave me a little trouble but the math and addition were good.	The lesson was very helpful for me because I solve most my math calculations in a standard form. From what we discussed I picked up a favourite method which to me is the best for mental computation. The method involves subtracting a number from one and then adding it to the next number, don’t know the official name of the process and also doubling. Mental computation with big numbers have been made easy. Friendly numbers and deconstruction of numbers can be very helpful.
2.	It is hard when you don’t know how to calculate without calculator.	So far with addition and subtraction I have no problem. I understand more now.[Multiplication and division] I don’t understand at all, I find it hard. Just hoping it is not the last day. Understanding bit by bit will practice home more and find out from my fellow students.
3.	Calculating without a pen or a paper I’m not good but I try my best. Not good. So I really need help on this.	The problem solving topic really help me a lot now at least I can solve some problem by decomposing number and doubling them. But the most method I enjoy was doubling number on both side. I only have a few problem on solving problem by subtraction. But addition I’m fine but I’m getting there slow by slow. The first number solving … I tried but those last big number I still have a problem solving those mentally. I still need clarification with big number.
4.	I am not that good but with the help of multiplication time table that I go through every time it’s helping a lot.	I came to become more familiar with a ten frame as an easier method to use too as it helps me work out while doing or experiencing. I observed that collecting like or friendly numbers can give quick answer. Therefore, when we started mental ability calculation it looked quite tuff and confusing but after learning some methods on how to mental compute my experience has changed. I enjoyed and like the outcome of the quiz.
5.	I am able to calculate some without a calculator but I really struggle a lot. This means I’m really not that good.	The lesson supported me by helping me realize friendly numbers and be able to carry out calculations. The lesson is supporting me in a very good way because now I can do some calculations very fast using my mind only. Today’s lesson made me understand the fact that one has to arrange numbers when calculating them, as it helps in dealing with huge numbers. I really find it hard to deal with big numbers using multiplication but I’m happy to know more. I have realized that I still struggle with division especially when it comes to dealing with huge numbers.
6.	I’m really not good in solving mathematics problems mentally so I really need your help for me to improve.	It was very good as I developed new strategies in working out problems mentally. I am at least coping well in solving problems mentally. The lesson went well but I still need more examples as I am struggling to solve some of the mathematics problems.

7.	I am not good in calculating without a calculator or pencil-paper method.	The lesson supported my understanding of what flexible mental computation is when my fellow students were writing their method on the board. The lesson was enjoyable, I think I am getting to know mental computation better. All I need for now is more activities for practice. I was a bit confused. I just need some last explanation on division and multiplication.
8.	Honestly I am not good to work out mathematics calculations without a calculator.	The lesson has supported me in a very large way to help me carry out mentally calculations through the variety strategies by gaining other methods which I did not know at first and also helped me develop a fastest way of mental computing calculations. It helped me understand that mental flexible computation is a process of calculating using your mind to work out calculations instead of a calculator or using models. The activities we did in the activity have helped me but not that well. I at least gained mental knowledge on how to calculate division. I now understand ways of flexible mental computing calculations in a short and straight forward way. I am developing. It has helped me to improve my mental computing skills to another level. The problem hindering me is the long numbers to work or calculate out.
9.	It's time consuming and I might not be accurate.	It helped me a lot, I developed fast working with bigger numbers. I am fine with the lessons. Multiplication is very simple compared to other operation signs. The lesson was good, I'm getting the method so easily and fast.
10.	I am not that good but I always try and test myself for what I know.	The lesson was very nice, it has increased my mental computation although somehow some method where slight challenging. But I am almost there I'm catching up something in every minute counted. Multiplication was not easy for me which means that the lesson was not good for me as I was trying to recall from what I know. The lesson helped me to recall some method which were using back to primary school.
11.	Most of the time I prefer using calculators since it's fast and always gives right answer.	The lesson helped me out to gain somehow methods of calculating mentally. I am now better off compared to when I wasn't taught by the lecturer. Multiplication is tough for me. I am trying by all means but it's taking time for me to calculate mentally.
12.	I am not good at calculating without a calculator or pencil-paper method.	The lesson has supported me psychologically as I can able to count numbers by adding and subtracting numbers without using a calculator. The lesson has helped me to use common numbers when using flexible mental computation. I really enjoyed the lesson. I still need mental computing help as somehow I am struggling to get answers I am confused with some other question especial the ones for subtraction.
13.	I am not that good because I only did it 15 years ago all these years I used a calculator.	I was not that good in calculating without a calculator but after the practice I am catching up and getting used to it.
14.	I am not good in calculation without a calculator.	Multiplication it was easy because I learn more about multiplication table. The lesson went well, I learned a lot how to decompose, add and multiply. I know about multiplication on 5 and 10 is so easy.

15.	Mental computation is a challenge to me, I only do few easy calculation mental. With a lot of work I will manage to get good at mental computation.	The lesson helped me to improve my ability to calculate mentally. It gave me the clear understanding of using number frames to calculate mentally. With the clear understanding of how to use friendly numbers I understood what flexible mental computation is. My brain is slow at calculating or processing the answers, it's making it difficult for me to calculate mentally. I have been bedridden for about two weeks. I missed out a lot...
16.	It is not easy calculating without a calculator but I use my brain to think and apply my mind on a particular question asked.	The lesson was good, it really helped me a lot because I have learned and used a variety of methods that help me to count and calculate well. The lesson was great, at least I have learned on how to deal with my learners when it comes to counting whereby I will use something in the classroom as counting object and the story was great.
17.	I am not good at using mental commutation. I find it easy when using a paper even if I do not have a calculator.	We used different methods like doubling and decomposing where we went through each method process how to work varieties of calculation. I got that multiplication is almost the same as addition. I did not get any idea on division. I discovered new methods of division. We used different methods.
18.	Not good.It was something that was scaring at the start but in the end it was an exciting experience. But it was the best ever.	The lesson was fantastic, exciting as learning new methods was the most interesting lesson. I am now confident enough to solve some mathematics problems. The strategies from different students were fascinating as methods were so mind opening. The lesson was really great as we discovered the relationship between multiplication of 10 with 5 gives the total as multiplication by 15. Working with friendly numbers are really great when dealing with problems e.g. subtraction and division or multiplying.
19.	I am not that good because I did not struggle with some of the questions.	It give me a clue on how to calculate mentally. I have learned some different strategies of mentally calculation. Everyday new learning techniques comes from my fellows. Multiplication is difficult but I have learned that you have to solve it with division. It sometimes deals with the multiplication table. Too reluctant on calculating mentally, not practicing on own spare time.
20.	Not that good because it requires me time.	The lesson improved my understanding because I was able to carry out the calculation on my own. Participation is getting better in the class as students were having freedom in answering questions. Addition we understood, we need different sign.
21.	Mental computation is good even though I'm not that good. I am willing to practice more so that I will be smart enough.	The lesson has supported me in a good way because after this lesson I can calculate using mental and even very fast. The lesson was really supportive to me to understand what mental computation really is. Now I know that mental computation is the ability of calculating without using a calculator. Very supportive, I learned how to work with large numbers by decomposition and working with familiar numbers to work out the calculation. It has been really supportive because I can calculate division, addition, subtraction and multiplication. Very supportive, I understand how to calculate mental and I can teach someone. I have understood almost everything now. Very helpful.
22.	It is very difficult, since I am used of using calculator.	It helped me understand how to calculate complicated numbers for instance using the given number to find out the missing number whereby I just needed to subtract from the other given number. Addition method is very interesting to deal with friendly numbers and converting complicated numbers back to friendly numbers to make it easy to add. My lecturer… give us more freedom how to express our own formulas in adding up numbers. The multiplication method is very easy to tackle as it deals with doubling numbers. Mathematics is very interesting and I love dealing

		with figures and whole numbers. The method of division is very complicated at times when dealing with big numbers which is not friendly numbers. I need a clear method to deal with division.
23.	It was really challenging calculating big numbers for few seconds without a calculator.	The lesson was so good so nice. I enjoyed to think critically and get the answer. It's really keeping me to go smarter and smarter and the negative attitudes that I have for mathematics are fading away slowly but surely.
24.	I am fair at calculating without a calculator.	The lesson was interesting and straight forward. I find it simple to calculate with my mind because I came across new method on how to calculate very fast. I learned what flexible mental computation is when I tried out one method on how to compute mental without the help of a calculator. Division was really too much difficult for me at first but I came to understand the process through the discussion.
25.	I am just good at addition and subtraction and I do struggle much with multiplication and division.	At least I learned some new methods on how to carryout calculations mentally without a calculator. I find it hard to work with division and multiplication mentally. Strategies for multiplication are real affective than for division. I cannot work with big numbers coming to division.
26.	I felt shocked about answering the questions without using calculator. At the same time I feel good because it will help to sharpen my mind. I faced a challenge to answer without calculator since I am used with using calculator.	The lesson was very nice. The things was amazing is the method of using ten frame. It made me feel like I am in front of learners teaching them. For me this mental computation is making me to think fast, I am getting to understand it well. The lesson for today was multiplication. So it was a bit tough for me at the beginning but at the end I understood that you can just add to answer a multiplication question.

APPENDIX J: COLOUR CODED INTERVIEW TRANSCRIPTS

Experience & Shaping belief

Participants		8. How did you learn to calculate mentally in the grade mentioned above (7.) or in high school?	
1.	(Pobo)	The teacher taught us ... in every calculation two or three best methods to use.	
2.	(Noly)	I developed by looking at the numbers... to calculate in my mind. I was looking for tens how many tens are there in the numbers. to make them to be tens.... It was the easier way for me.	
3.	(Andy)	The teacher was just like give us a question paper so we need to write answers like in a group and then some of our friends or colleagues taught us the way they answer things mentally, using hands, stones and tally marks.	from friends and colleagues
4.	(Lacy)	We were really given methods on how to do it mentally but then now I can't really recall them nicely all those methods [given by] our teacher.	Provided methods.
5.	(Pano)	you have to have a piece of papers... a piece of paper then you like read the question and at the same time you start counting if it is like five let me say if it is two plus three and then you like count one, two then plus that three then you just count it altogether then you will get different answers. (Counting on.)	Piece of paper and Counting all.
6.	(Mola)	you are being told on how to calculate especially given those multiplication table things for you people just to memorize and then understand how things work so like that mind also develop and doing that continuously the mind also becomes like very activated to think fast and then answer the question.	Memorize multiplication table.
7.	(Femo)	I learned from my fellow learners and what my teachers was teaching us. Actually I used my skill from grade seven... eight [Cannot remember the rest].	
8.	(Abu)	I cannot recall.	
9.	(Mara)	We were using a pen and a paper that will help you to calculate where you will be doing the subtraction, division... where you can draw sticks or you draw circles but using a pen and a paper. also we memorized the multiplication table?	
10.	(Meni)	The teacher gives us two methods for example he will give you the first one if you are not managing to get the answer then you use the second one. Unfortunately I followed the one the teacher gives us.	
11.	(Mando)	Grade three only.	
12.	(Ambo)	Working with powers five to the power two... is like five times five.	
13.	(Ndile)	The teacher had to take us out and line us take four learners put them aside she asks maybe two learners then will ask a learner how many students are there after adding the two learners. methods... ah I can't remember first she could show us how to add problems and she always encouraged us that if you can't work out a problem now look for your own methods to solve the problem. encouraged as	Using lnrs as manipulatives for number sense. Own method alternative to teacher method.
14.	(Deto) (cited)	we were not told to use any calculators we were told we should know about the... the multiplication table so we were all ask to know about it from one up to twelve. Some of us if you are not getting it then we have to memorize that is what our teacher told us. It is best because it helped us in many ways. By practicing. multiplication table. You have to ask yourself some question then you have to work it out yourself maybe you just go in the textbook you look for question you just get a question then you test yourself. After I am done with testing myself I have to go and get back to the book and see what the answers are again.	No calculator. Memorize Self administered activities.
15.	(Naby)	Only emphasized in lower primary.	

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Quote experience shaping belief

cited

Quote Formulation of teacher belief through experience

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